Black holes and Dirichlet branes in the theory of strings

Jason C. Breckenridge

Department of Physics

McGill University, Montréal

A thesis submitted to the
Faculty of Graduate Studies and Research
in partial fulfillment of
the requirements of the degree of
Doctor of Philosophy

July 1997

0-612-36960-9

Abstract

The research presented here is carried out along two related fronts. Calculational techniques which derive from the symmetry properties of the theory of strings are used to generate new solutions corresponding to five dimensional rotating black holes. The Dirichlet brane (D-brane) content of these black holes is then identified and this information is used to compute the microscopic statistical entropy, which is then shown to be identical to the classical Bekenstein-Hawking entropy. The symmetry techniques are then further exploited to create new low-energy background solutions describing different supersymmetric bound states of D-branes. In one case these D-brane bound states have constituent D-branes which differ in dimension by two. In the second case these bound states represent arbitrary numbers of D-branes which intersect at non-trivial angles.

Résumé

La recherche présentée ici comprend deux orientations intimement liées. Des techniques de calcul qui se dérivent des caractéristiques de symétrie de la théorie de cordes sont utilisées pour créer de nouvelles solutions qui représentent des trous noirs en cinq dimensions qui ont un moment angulaire non-nul. Le contenu de ces trous noirs en terme d'hypermembranes de Dirichlet (D-branes) est identifié et utilisé pour calculer l'entropie microscopique et statistique. Cette entropie est identique à l'entropie classique de Bekenstein et Hawking. Les techniques de calcul sont également exploitées pour créer de nouvelles solutions de basse énergie qui décrivent des états liés de D-branes. Un calcul se consacre à produire des solutions d'états liés dans lesquelles les D-branes ont une différence de dimension de deux. L'autre calcul produit des états liés dans lesquelles des D-branes s'intersectent à angles arbitraires.

Acknowledgements

To my supervisor, Robert Myers, I owe much. I thank him for his guidance, encouragement and patience. I have profited enormously from his tuition. I express also my gratitude to Nemanja Kaloper and Ramzi Khuri for many stimulating and useful discussions, and for their willingness to share their ideas. The courses of Professors C.S. Lam and R. Sharp were also very profitable.

I would like also to thank those with whom I have collaborated, David Lowe, Andy Strominger, Cumrun Vafa, and in particular Amanda Peet and Guy Michaud for useful discussions, and for grace under questioning.

My office mates Mohammed Garousi, Omid Hamidi-Ravari, Yongyian Feng and Patrick Girard have made office life enjoyable, even if there are no windows. Thanks also to Paul Mercure and Juan Gallego for keeping the computers running.

For the other inhabitants of the third and fourth floors, who have made both the city of Montréal in general and the department of physics in particular interesting places to exist, there doesn't seem to be any simple way to express how I feel. Thus I will simply say "thank you for everything" to Martin Kamela, Alexander Marini, Chris Roderick, Rainer Ullmann, May Chiao, Peter Bamert, Rakie Cham, and others whose names I either forget, or never learned in the first place.

Special thanks are owed to those who found the patience to withstand my efforts to learn a second language, among them Philippe Page, Rahma Tabti, Denis Michaud, Harald Roussel, Hélène Nadeau and others too numerous to mention.

The financial support of NSERC and FCAR du Québec is also appreciated.

Official soundtrack by Jimi Hendrix, Stevie Ray Vaughan, Miles Davis, Wes Montgomery, and Ethel.

Finally, I would like to say to my family that I am at last "out of school".

Contents

List of figures

I

Introduction and Outline

This thesis is based on research done in the context of the theory of strings. Recent developments in this field of high energy physics, particularly progress in the understanding of its non-perturbative aspects have shown that string theory is rather more than just a theory of strings; that is, other (extended) objects, such as Dirichlet branes (D-branes) and supergravity p-branes must be included in the theory. These additional objects fill out multiplets of states that are connected by special symmetries, known as dualities, which relate different parts of string theories, or even entire string theories, to one another.

In addition, the network of dualities of string theory has been interpreted as evidence [1-3] that the four consistent string theories are really different parts of a larger theory which has been dubbed M-theory. We will not be concerned here with this larger theory, but it is clear that these dualities, several of which we will use to great effect, have a large role to play. Here we wish merely to set up the context for this work.

Black holes are extremely interesting objects predicted by Einstein's general theory of relativity [4-5]. These are objects so massive, possessing gravitational fields so powerful that not even light is able to escape, hence the term "black hole". It has been shown that these objects possess a thermodynamic entropy [6-7], for which one would ideally like to have a microscopic and statistical interpretation. Classical general relativity offers no clues as to what this interpretation might be.

At the same time, classical general relativity predicts that singularities may form, points where the curvature of spacetime grows without bound. It is thus to be expected that quantum gravity is required to explain both the microscopic entropy, and to resolve these curvature singularities, to discover the deep structure of these objects [8].

String theory is at present a strong candidate for a theory of quantum gravity [9]. It is logical, therefore to study black holes in the context of string theory, in the hope that some light may be shed on the physics of black holes, and also in an attempt to validate the theory of strings as a physical theory.

Some progress has in fact been made on this front. One species of additional object demanded by the dualities of the theory of strings which have been used to great effect are the D-branes. These are objects extended in zero or more dimensions, and have been recently used to compute, for the first time, the statistical entropy of black holes [10-13]. Thus it is clear that D-branes represent useful probes into the non-perturbative regime of string theory, in addition to their role in filling multiplets as required by duality. We will see examples of calculations of this kind in chapter VI.

In addition, D-branes are interesting objects in their own right [14-15]. There are many questions to be asked and, hopefully, answered with regard to how D-branes interact with each other, for example. In chapter VII we will see many examples of D-branes forming supersymmetric bound states, in which one D-brane can be considered to have "dissolved" in its companion, or in which the D-branes in question intersect at non-trivial angles.

1.1. Outline

String theory is a large subject, and the size of this thesis reflects this. The desire was that this thesis provide a self-contained introduction for non-specialists. This necessitates the inclusion of much review material which string theorists may choose not to read. Here we present a short guide.

Chapters II through V represent introductions to the theory of strings, duality symmetries in string theory, p- and D-branes, and black holes in string theory, respectively. Chapter II contains material on the basics of string quantization, supersymmetry and the superstring, the process of compactification of higher dimensions, then systematically presents the known consistent superstring theories, including details of their low-energy effective space-time actions.

The third chapter presents the duality symmetries of the theory of strings, beginning with the $O(d, d, \mathbb{R})$ symmetry of the low-energy string equations of motion, continuing with target-space or T-duality and then finally one very small part of the string duality family, Type IIA–Heterotic string/string duality in six dimensions.

Chapter IV presents the theory of Dirichlet branes, both from the world-sheet point of view, as well as the space time point of view, where D-branes are related to a class of p-brane solutions of supergravity theories. The material here may be of interest even to specialists, given the recent rise of D-branes in string theory.

Black holes in the context of string theory are the subject of chapter V. A catalog of black hole solutions to the vacuum Einstein equation as well as the Einstein-Maxwell system is given, along with generalizations to higher dimensions. The string-theory analogs of these solutions are also discussed. Short treatments of the thermodynamics of black holes and the connection between black holes and D-branes are also given, leading to the research which is presented in chapters VI and VII.

The original research which is presented in this thesis is to be found in chapters VI and VII and was carried out on two related fronts. One is the construction of new black hole solutions in string theory. The second is the construction of new supersymmetric solutions representing bound states of D-branes. In chapter VI, new solutions for a five-dimensional rotating supersymmetric black hole and a six-dimensional rotating black string are presented. Analysis of these solutions, including a microscopic counting of the black hole entropy, is then carried out. A third example, that of the construction of a non-rotating dyonic black hole is also presented. The results presented here have been published in [12,13] or will be published [16]. Here, we present the research in greater detail.

In chapter VII the emphasis switches to D-branes, with the construction of supersymmetric D-brane bound states the subject. In the first case we have simple bound states, but in the second part of chapter VII we construct bound states which intersect non-orthogonally. The publications in which this research appears are [17-18].

Chapter VIII is the final chapter, which presents a summary of the work completed and discusses some avenues of future research. The appendices gather information on notation and conventions, as well as some topics that would require too long a digression to present in the main text.

Those unfamiliar with string theory and in no particular hurry will hopefully find it profitable to read the entire thesis. Those who wish only to understand chapters VI and VII can skip most of chapter II except section 2.3. Although most of chapter III should be read, since the symmetry properties of the theory of strings play a major role in this work, sections 3.2.2, 3.3.1 and section 3.5.2 can be safely left aside.

For chapter IV, it is possible to leave aside section 4.1 on p-branes as well as section 4.4.1. As far as chapter V is concerned, sections 5.1.4, 4.2 can be safely omitted.

Those familiar with the theory of strings are no doubt perfectly capable of choosing which sections to read or to omit. It should be restated, however, that the symmetry properties of string theory play a crucial role in this work, and it is recommended that chapter III should at least be skimmed quickly before moving on to chapters VI and VII.

II

The Theory of Strings

Quantum field theory based upon the notion of a point particle has enjoyed unparalleled success in the description of nature at the subatomic level [19]. The standard model of particle physics [20] represents the final achievement of this programme, combining the notion of a field which is to be quantized with the ideas of renormalization, which both constrains the available physical quantum fields and provides a set of techniques to extract answers which can be compared to experiment [21].

However, despite the success of this theory, there remain several less than satisfying aspects of the standard techniques of quantum field theory, as well as one outright problem. An example of the first is the large number of free parameters in the standard model. The second is the resistance of gravitation to all and any attempts at quantization within this theoretical framework.

The theory which has come to be loosely known as "string theory"[1] has emerged in recent years as one of few serious candidates for a quantum theory of gravity. This development was based on what one might call a radical departure from the established formalism of quantum field theory: the generalization from the idea of the point particle to objects of one dimension (called strings), and even to objects of higher dimensionality. This departure results in a theory which inescapably contains gravitation, as well as having room to contain all known physics. At the present moment, no one knows just how the standard model will fall out of string theory, but nonetheless progress continues to be made.

[1] We will see why I say "loosely" later on.

2.1. The basics

When one attempts to change a feature such as the dimensionality of the fundamental constituents of an entire theoretical framework, the task must be done with care and with full understanding of the roles played by the objects under scrutiny. Thus to generalize a point particle to a higher dimensional object, it is necessary to consider issues not commonly dealt with in standard treatments of quantum field theory. To describe how this generalization is made, it is useful to begin with a description of point particle theory, something with which the reader is no doubt familiar.

2.1.1. From point particles to strings

If we consider a massless point particle without spin moving in a Minkowski spacetime, a field theory action S suitable for describing this system is often written as

$$S = \int d^n x \left(\partial_\mu \phi\right) \left(\partial^\mu \phi\right) \tag{2.1.1}$$

where ϕ is the scalar function of the n spacetime coordinates which determines how the particle behaves. Thus, this action is written in the spacetime formalism, in which the action is calculated as an integral over all space, the equation of motion which is obtained by varying this action with respect to ϕ turns out to be

$$\Box \phi = 0 \tag{2.1.2}$$

where $\Box = \partial_\mu \partial^\mu$ is the Minkowski space d'Alembertian operator. The solutions of (2.1.2) will show that the particle is required to exist only on the light cone, as is natural for a particle without mass.

The above action and equation of motion are what most think of when the words "massless scalar particle" are mentioned. The quantity ϕ is a quantum field composed of creation and annihilation operators, which in turn give rise to states with any number of particles, hence the name *quantum field theory* given to this formalism.

However, there exists, in the case of a single-particle state another way to formulate this system, in terms of the world-line along which the particle propagates. In this case we may write the action as

$$S_w = \frac{\mathcal{T}_0}{2} \int d\tau \, e(\tau) \, \eta_{\mu\nu} \left(\partial_\tau X^\mu\right) \left(\partial_\tau X^\nu\right) \tag{2.1.3}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, the indices μ and ν running over the dimensions of the spacetime, τ is an arbitrary parameter along the trajectory, $X^\mu(\tau)$ is the position of the particle, and $\mathcal{T}_0$ is a constant required to make the action dimensionless (when $\hbar = c = 1$). The object $e(\tau)$ is a metric or measure along the world-line and $\partial_\tau = \frac{\partial}{\partial \tau}$. The role played by $e(\tau)$ is to guarantee that S remains invariant under reparametrizations of the world-line. It can be shown that (2.1.3) is invariant under $\tau \to \hat{\tau}(\tau)$. Hence it is clear that the physics of (2.1.3) should have no dependence on how the world-line is parameterized. We can use the reparametrization freedom to make a choice in which $e = 1$, for which (2.1.3) becomes simply

$$S = \frac{\mathcal{T}_0}{2} \int d\tau \, \eta_{\mu\nu} \left(\partial_\tau X^\mu\right) \left(\partial_\tau X^\nu\right) . \tag{2.1.4}$$

Varying S with respect to X^μ results in the equation of motion

$$\partial_\tau^2 X^\mu = 0 \tag{2.1.5}$$

of which the solutions are, obviously, straight lines $X^\mu = p^\mu \tau + X_0^\mu$ in the Minkowski spacetime. However, one must remember that not all straight lines in Minkowski space are permissible as solutions of the theory described by (2.1.4). In writing (2.1.4) we have made a particular choice of world-line parameterization, and this choice has consequences. Since S must be invariant under reparametrizations of the world-line, we must impose constraints on our point particle to ensure that this is always true. The appropriate constraint in this case, which derives from the requirement that $\frac{\delta S}{\delta e} = 0$ is the vanishing of the quantity

$$\mathfrak{T} = \eta_{\mu\nu} \left(\partial_\tau X^\mu\right) \left(\partial_\tau X^\nu\right) . \tag{2.1.6}$$

This restricts our straight-line solutions to be the lightlike geodesics in our spacetime, the light cones, i.e., $\eta_{\mu\nu} p^\mu p^\nu = 0$.

The spacetime and world-line formulations are indeed quite different in content. One distinction, to which we alluded earlier, is usually denoted in the literature as the distinction between *first-quantization* and *second-quantization*. The distinction between these two is that first-quantization indicates the quantization of a particle, and second-quantization the quantization of a field from which particles may be created, thus producing multi-particle states.

If, for the purposes of illustration, we restrict our attention to states with a fixed number of particles, then we can consider the spacetime and world-line formulations as complementary. On the one hand, in the spacetime description the particle is considered to move in an n-dimensional spacetime which may be considered as a "container" for the particle[2] whereas in the world-line description the spacetime coordinates are fields which exist on the world-line. The world-line formulation of the theory is the natural framework in which to begin the generalization of the point particle to extended objects such as strings. Although much effort has been put into the development of what is known as *string field theory*, a second-quantized version of string theory, it is not yet clear that this avenue of research will produce anything useful [22-23].

In the simplest generalization of the point particle, about which much has been written [9,23,24-27] one begins with a zero-dimensional object and generalizes it to a one-dimensional object. As mentioned, for this task one begins with the world-line description of the point particle. This is due to the fact that the dimensionality of the world-volume of the particle appears directly in the action, as the number of dimensions over which it is necessary to integrate when calculating the action.

A one-dimensional particle, henceforth known as a string, sweeps out a two-dimensional world-volume as it moves through spacetime, which we will call the world sheet. This is in contrast to the one-dimensional world-volume, the world-line, of the point particle. Strings can be of two types, depicted in Fig. 2.1.1. In the case that the strings are intervals, or segments, they are known as *open* strings. The other possibility is that the strings form loops, in which case they are called *closed* strings. In either case we need two world sheet coordinates to describe the motion.

[2] Or, for that matter, the fields from which the particles are created.

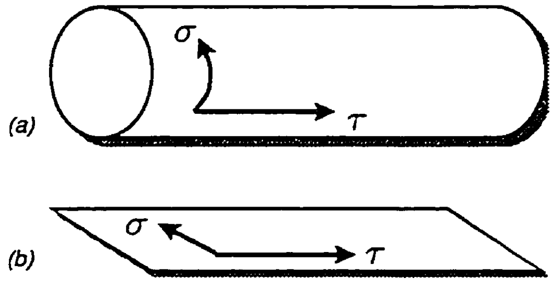


Figure 2.1.1: Schematic depiction of *(a)* closed string and *(b)* open string world sheets.

The action in this case may be written:

$$S = \frac{\mathcal{T}_1}{2}\int d^2\sigma\sqrt{-e}e^{ab}\eta_{\mu\nu}\left(\partial_a X^\mu\right)\left(\partial_b X^\nu\right) \tag{2.1.7}$$

where e_{ab} is the metric on the world sheet ($e^{ab} = (e_{ab})^{-1}$ and $e = \det e_{ab}$). In general we will use Minkowski signature, both in spacetime and on the world sheet. The X^μ are the coordinate "fields" on the world sheet. We are using $d^2\sigma$ to denote $d\tau d\sigma$, integration over the world sheet. Here τ denotes the time coordinate, which has the range $-\infty \leq \tau \leq \infty$, while the spatial coordinate σ extends over $0 \leq \sigma \leq \pi$, running from one end of the string to the other in the case of open strings, or representing one complete revolution in the case of closed strings. The constant $\mathcal{T}_1$ makes the action dimensionless when $\hbar = c = 1$. Equation (2.1.7) is a string theory action written in what is commonly known as the "sigma model" or "Polyakov" form of the action [28].

The equation of motion derived from (2.1.7) by variation with respect to X^μ is written

$$e^{ab}\partial_a\partial_b X^\mu = 0 \tag{2.1.8}$$

which can be seen to be the simple linear wave equation $(\partial_\tau^2 - \partial_\sigma^2)X^\mu = 0$ if[3]

$$e^{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.1.9}$$

which must, just as in the case of the point particle, be supplemented by constraints derived by requiring that the variation of the action with respect to the world sheet metric, $\frac{\delta S}{\delta e_{ab}}$, vanish, in order to guarantee that the action does not depend on the

[3] We will see later that the world sheet metric may always be put in this form, at least locally.

coordinate system used on the world sheet. The energy momentum tensor in $(1+1)$-dimensional field theory is normally defined as

$$T_{ab} = \frac{2\pi}{\sqrt{-e}} \frac{\delta S}{\delta e^{ab}} \tag{2.1.10}$$

and thus the constraint is simply $T_{ab} = 0$. It is also worth noting that by solving for the constraints of e_{ab} and then eliminating them one arrives at the "Nambu-Goto" form of the action [29]:

$$S = \frac{\mathcal{T}_1}{2} \int d\sigma d\tau \sqrt{(\partial_\tau X)^2 (\partial_\sigma X)^2 - (\partial_\tau X \cdot \partial_\sigma X)^2} \tag{2.1.11}$$

which has an interpretation in terms of the area of the world sheet swept out by the string.

Up to now we have been discussing free point particles and the generalization to free strings. What can be said about interactions? On the world-line of a point particle the natural coupling [30], which preserves Poincaré and gauge invariance,[4] is that of a gauge field. Thus, one can generalize the action (2.1.4) to be

$$S = \frac{\mathcal{T}_0}{2} \int d\tau \left\{ g_{\mu\nu} \left(\partial_\tau X^\mu\right) \left(\partial_\tau X^\nu\right) + A_\mu \left(\partial_\tau X^\mu\right) \right\} \tag{2.1.12}$$

where $g_{\mu\nu}$ is the (general) metric and A_μ is a gauge field. This action describes the point particle subject to external forces, gravity and electromagnetism, as it moves along its world-line. This interaction with *background* fields described by $g_{\mu\nu}$ and A_μ should be distinguished from interactions between different point particles.

It is then evident that we can generalize (2.1.12) to the case of the string as

$$S = \frac{\mathcal{T}_1}{2} \int d^2\sigma \left\{ \sqrt{-e} e^{ab} G_{\mu\nu} \left(\partial_a X^\mu\right) \left(\partial_b X^\nu\right) + \epsilon^{ab} B_{\mu\nu} \left(\partial_a X^\mu\right) \left(\partial_b X^\nu\right) \right\} \tag{2.1.13}$$

where ϵ^{ab} is the Levi-Civita tensor on the world sheet, $B_{\mu\nu}$ is an antisymmetric tensor (2-form) potential, the generalization of A_μ of (2.1.12).

---

[4] Which really means that the indices on the various objects involved get contracted in the appropriate manner.

2.1.2. From strings to ...

Now that we have given up the idea that the fundamental objects in our theory are points, we may seem to be on a slippery slope. What is to prevent us from taking membranes or higher dimensional objects as our starting point? Of course, there are limits imposed by the dimensionality of the spacetime in which one lives, but other than that there seems to be no obvious restriction.

The action for any of these systems can be written down simply by generalizing from the world sheet indices a,b in (2.1.7) to world-volume indices which run over a greater range. The world-volume metric maintains its Minkowski signature. The action corresponding to (2.1.7) for an object of $n + 1$ dimensions, an n-brane, can be written [30]

$$S = \frac{\mathcal{T}_n}{2} \int d^{n+1}\sigma \sqrt{-e} e^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \tag{2.1.14}$$

where we have generalized our parameter $\mathcal{T}_1$ to $\mathcal{T}_n$, known as the tension, which is again necessary to make S dimensionless. The spacetime metric $G_{\mu\nu}$ represents a general, possibly curved, spacetime background. Just as in (2.1.12) we can add a gauge field which in this case will be an $n+1$-form, which has a natural coupling in an $n + 1$-dimensional world-volume. If this is done the action appears as

$$\begin{aligned} S = \frac{\mathcal{T}_n}{2} \int d^{n+1}\sigma \Big\{ &\sqrt{-e} e^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \\ &- \frac{2}{(n+1)!} \epsilon^{a_1 \cdots a_{n+1}} \partial_{a_1} X^{\mu_1} \cdots \partial_{a_{n+1}} X^{\mu_{n+1}} A^{(n+1)}_{\mu_1 \cdots \mu_{n+1}} \Big\} \end{aligned} \tag{2.1.15}$$

where $A^{(n+1)}$ is the $n + 1$-form gauge potential and ϵ is the totally antisymmetric tensor on the world sheet.

Recall that in the case of the point particle we noted that a reparametrization choice had been made in writing (2.1.4). The fact that one can always choose a reparametrization in which the action appears in the form (2.1.4) is one reason why the action makes sense physically. That is, the physics has no dependence upon how we choose to parameterize the world-line of our particle. A similar feature is at play in the case of the string. For a general $(n + 1)$-dimensional object, the metric on the world-volume e_{ab} will have $\frac{1}{2}(n + 1)(n + 2)$ components constrained by the $n + 1$ independent reparametrization (diffeomorphism) invariances of the world-volume coordinates. Thus $\frac{1}{2}n(n+1)$ components remain and e cannot be eliminated for $n > 0$. However, under a local Weyl rescaling of the world-volume metric

$$e_{ab} \rightarrow \Omega(\sigma)\, e_{ab}, \tag{2.1.16}$$

the combination $\sqrt{-e}\, e^{ab}$ appearing in the action transforms as

$$\sqrt{-e}\, e^{ab} \rightarrow \Omega^{\frac{1}{2}(n+1)-1} \sqrt{-e}\, e^{ab}. \tag{2.1.17}$$

and we then see why the string ($n = 1$) has a special place amongst all of these objects. The string displays *conformal invariance* [31], i.e., the action is independent of the scale.

Thus for the string, and only for the string it is always possible to transform the world sheet metric in such a way that it is the two-dimensional Minkowski metric (2.1.9). We first use the diffeomorphism invariance to put the world sheet metric in the *conformal gauge:*

$$e_{ab} = e^{2\varphi} \eta_{ab}, \tag{2.1.18}$$

then follow by conformally rescaling such that $\varphi = 0$. We note also that this property of the string is connected with the fact that the energy-momentum tensor (2.1.10) is traceless [9,31] which will have consequences of extreme importance when we come to quantize the string. These features differentiate a string theory from, say, a membrane theory.

Membranes and higher dimensional objects have also other difficulties. Equation (2.1.14) defines a quantum field theory which is renormalizable by power counting for $n = 1$ and non-renormalizable for $n > 1$.

More recent results in the theory of objects with higher dimensionality than strings have indeed removed or shown the promise of removing some of the barriers that stand in their way of being on an equal footing with the string. At this moment in history the string still reigns, but the advent of supermembranes [30] and eleven-dimensional M-theory [32], seems to indicate that certain higher-dimensional branes, as well as certain point particles should be regarded on an equal footing with strings.

2.1.3. The bosonic string

In this section we will develop the bosonic string in its open and closed versions. Our discussion will follow most closely that of [9]. To begin we write the action of the bosonic string from (2.1.15) with, of course, $n = 1$, and putting $\mathcal{T}_1 = (2\pi\alpha')^{-1}$ where α' is the Regge slope, or inverse string tension:

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-e}e^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}. \tag{2.1.19}$$

Here we have set $G_{\mu\nu} = \eta_{\mu\nu}$ and $B^{(2)}_{\mu\nu} = 0$, to begin with a development of the free bosonic string in Minkowski space, a complete understanding of which is necessary before moving on to more general spacetimes.

It is proper to note that the action (2.1.19) is not the most general action possible even when restricted to flat backgrounds. There are two other possible terms which are consistent both with Poincaré invariance in a D-dimensional spacetime and with renormalizability by power counting. These terms are:

$$\begin{aligned} S_1 &= \lambda\int d^2\sigma\sqrt{-e} \\ S_2 &= \frac{\lambda'}{2\pi}\int d^2\sigma\sqrt{-e}\mathcal{R} \end{aligned} \tag{2.1.20}$$

where $\mathcal{R}$ is the scalar curvature of the world sheet, or the Ricci scalar in two dimensions, and λ and λ' are arbitrary constants.

The S_1 term is a world-volume cosmological constant term. It does not have Weyl symmetry and therefore leads to inconsistent classical field equations. If we take the action $S+S_1$, the trace of the equation of motion for e_{ab} implies that $e_{ab} = 0$, unless $\lambda = 0$. On the other hand, S_2 will not concern us here either, since it is a topological invariant which is fixed by the global topology of the string world sheet. Since the equations of motion are local, this topological invariant has no effect and can be dropped from the action with no loss in generality.

The symmetries of (2.1.19) are the following: reparametrization invariance:

$$\begin{aligned} \delta X^\mu &= \xi^a\partial_a X^\mu \\ \delta e^{ab} &= \xi^c\partial_c e^{ab} - \partial_c\xi^a e^{cb} - \partial_c\xi^b e^{ac} \\ \delta\sqrt{-e} &= \partial_a\left(\xi^a\sqrt{-e}\right), \end{aligned} \tag{2.1.21}$$

Weyl scaling invariance:

$$\begin{aligned} \delta e^{ab} &= \Omega e^{ab} \\ \delta X^{\mu} &= 0, \end{aligned} \tag{2.1.22}$$

and in addition the global symmetries of the background space in which the string is propagating. Here we are in Minkowski space and therefore we simply have the Poincaré invariance:

$$\begin{aligned} \delta X^{\mu} &= \Theta^{\mu}{}_{\nu} X^{\nu} + \kappa^{\mu} \\ \delta e^{ab} &= 0. \end{aligned} \tag{2.1.23}$$

Note that in the above symmetries, ξ^a and Ω are arbitrary functions of the world sheet coordinates σ^a, whereas $\Theta^{\mu}{}_{\nu}$ ($\Theta_{\mu\nu}$ antisymmetric) and κ^{μ} are constants.

As permitted by the invariances we can write $e_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \eta_{ab}$ by using the two reparametrizations and Weyl scaling. The action then simplifies to:

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma\, \eta^{ab} \partial_a X^{\mu} \partial_b X_{\mu} \tag{2.1.24}$$

from which the equation of motion we obtain is the free wave equation in two dimensions,

$$(\partial_{\sigma}^2 - \partial_{\tau}^2) X^{\mu} = 0 \tag{2.1.25}$$

For open strings, it is necessary to ensure that the action (2.1.24) is invariant under a general variation in X^{μ} of $X^{\mu} \to X^{\mu} + \delta X^{\mu}$ which gives rise to a volume term proportional to the equation of motion (2.1.25) *and* to a surface term, the vanishing of which tells us the boundary conditions to be imposed at the edges ($\sigma = 0$ and $\sigma = \pi$) of the string world sheet. This surface term appears as

$$\frac{1}{2\pi\alpha'} \int d\tau \left\{ \partial_{\sigma} X^{\mu} \delta X_{\mu} \Big|_{\sigma=\pi} - \partial_{\sigma} X^{\mu} \delta X_{\mu} \Big|_{\sigma=0} \right\} = 0 \tag{2.1.26}$$

and it can be made to disappear with the imposition of the *Neumann* boundary condition $\partial_{\sigma} X^{\mu}|_{\sigma=0,\pi} = 0$ or the *Dirichlet* boundary condition $X^{\mu}|_{\sigma=0,\pi} =$ constant. We will say more about these conditions later.

In the case of closed strings, on the other hand, functions X^{μ} being periodic in σ and obeying (2.1.25) ensure that (2.1.24) is stationary.

As is the usual case in a two dimensional system, the general solution to a wave equation like (2.1.25) can be given in terms of a sum of two arbitrary functions

$$X^{\mu}(\sigma, \tau) = X_r^{\mu}(\tau - \sigma) + X_l^{\mu}(\tau + \sigma)\,. \tag{2.1.27}$$

The functions have been labeled r and l since they are the "right-moving" and "left-moving" modes of the coordinate fields. It is useful to rewrite the action in terms of *light cone coordinates* for which the definitions and the derivatives are written

$$\begin{aligned} \sigma^{\pm} &= \tau \pm \sigma \\ \partial_{\pm} &= \tfrac{1}{2}(\partial_{\tau} \pm \partial_{\sigma}), \end{aligned} \tag{2.1.28}$$

and the world sheet metric is given by

$$\eta_{+-} = \eta_{-+} = -\frac{1}{2}, \eta_{++} = \eta_{--} = 0. \tag{2.1.29}$$

With this choice the action appears as

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \, \partial_+ X^{\mu} \partial_- X_{\mu}. \tag{2.1.30}$$

The utility of the light-cone coordinates is evident, since X_r^{μ} is only a function of σ^-, and likewise only σ^+ appears in X_l^{μ}.

As mentioned earlier, varying the metric with respect to e^{ab} gives us the world sheet energy momentum tensor T_{ab} defined as

$$T_{ab} = 4\pi\alpha' \frac{1}{\sqrt{-e}} \frac{\delta S}{\delta e^{ab}} \tag{2.1.31}$$

which one finds upon calculation to be

$$T_{ab} = \partial_a X^{\mu} \partial_b X_{\mu} - \frac{1}{2} e_{ab} e^{cd} \partial_c X^{\mu} \partial_d X_{\mu}. \tag{2.1.32}$$

Note that one has the identity $T_{ab}\eta^{ab} = 0$ which is a result of conformal invariance and holds in general. Now, the wave equation must be supplemented by the constraint $T_{ab} = 0$. These constraints take the form

$$\begin{aligned} T_{10} &= T_{01} = \partial_{\tau} X^{\mu} \partial_{\sigma} X_{\mu} = 0, \\ T_{00} &= T_{11} = \frac{1}{2}\left((\partial_{\tau} X)^2 + (\partial_{\sigma} X)^2\right) = 0 \end{aligned} \tag{2.1.33}$$

or, in the light-cone coordinates, of the form

$$\begin{aligned} T_{++} &= \frac{1}{2}(T_{00} + T_{01}) = \partial_+ X^{\mu} \partial_+ X_{\mu} = 0, \\ T_{--} &= \frac{1}{2}(T_{00} - T_{01}) = \partial_- X^{\mu} \partial_- X_{\mu} = 0, \\ T_{+-} &= T_{-+} = 0. \end{aligned} \tag{2.1.34}$$

where the third line here follows from $T_{ab}\eta^{ab} = 0$. In two-dimensional quantum field theory, energy-momentum conservation $\partial_a T^{ab} = 0$ may be expressed as $\partial_- T_{++} + \partial_+ T_{-+} = 0$. However, one has that $T_{-+} = 0$ automatically, and thus the conservation law becomes $\partial_- T_{++} = 0$, which is a powerful statement, implying an infinite set of conserved quantities.

To see these conserved charges, let $f(\sigma^+)$ be an arbitrary function of σ^+. Then $\partial_- f = 0$ and by extension $\partial_-(fT_{++}) = 0$, so the charge $Q_f = \int d\sigma f T_{++}$ is conserved. Since we can choose any $f(\sigma^+)$ that we want, we have an infinite set of these conserved quantities. This property is unique to the case of two dimensions, and thus to string theory (with $n = 1$). Physically, these conserved quantities represent residual symmetries which remain after the imposition of conformal gauge on the world sheet metric. Consider that a reparametrization ξ^a which obeys

$$\partial^a \xi^b + \partial^b \xi^a = \Omega\, \eta^{ab} \tag{2.1.35}$$

preserves the choice of gauge of the world sheet metric. If we then define "light cone" versions of the reparametrizations as $\xi^\pm(\sigma^\pm) = (\xi^0 \pm \xi^1)$ and consider world sheet reparametrizations to be generated by the operator

$$\mathcal{V} = \xi^a \partial_a \tag{2.1.36}$$

we find that the generators of the residual symmetries can be written

$$\mathcal{V}^\pm = \xi^\pm \partial_\pm \tag{2.1.37}$$

which are the generators of the conformal group in two dimensions. In the case that $f \sim \xi^+$ the conserved charges generate (2.1.37). Only in two dimensions is the group of conformal transformations infinite dimensional.

In solving (2.1.27) for the case of closed string, we need only consider the periodicity requirement of $X^\mu(\sigma, \tau)$ in σ, which gives us the general solution as:

$$X_r^\mu = \frac{1}{2}\Big(x^\mu + \ell^2 p^\mu \sigma^- + i\ell \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu \, e^{-2in\sigma^-}\Big), \tag{2.1.38a}$$

$$X_l^\mu = \frac{1}{2}\Big(x^\mu + \ell^2 p^\mu \sigma^+ + i\ell \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^\mu \, e^{-2in\sigma^+}\Big) \tag{2.1.38b}$$

where the α_n^μ are the Fourier components, which will be interpreted as oscillator[5] components describing the excitations of the string, i.e., the α_n^μ are annihilation operators, while the α_{-n}^μ are creation operators. Also a length parameter ℓ has been introduced, related to the inverse string tension α' according to $\ell = \sqrt{2\alpha'}$. The variables x^μ and p^μ are the center of mass position and momentum of the string. Note that adding (2.1.38a) and (2.1.38b) together cancels the term linear in σ, consistent with the requirement of periodicity. The requirement that $X^\mu(\tau,\sigma)$ should be real implies that $\alpha_{-n}^\mu = (\alpha_n^\mu)^\dagger$ and that x^μ and p^μ are themselves real.

The classical Poisson brackets for closed strings from (2.1.24) are

$$\begin{aligned} \left[X^\mu(\sigma), X^\nu(\sigma')\right]_{pb} &= \left[\partial_\tau X^\mu(\sigma), \partial_\tau X^\nu(\sigma')\right]_{pb} = 0, \\ \left[\partial_\tau X^\mu(\sigma), X^\mu(\sigma')\right]_{pb} &= 2\pi\alpha'\delta(\sigma - \sigma')\,\eta^{\mu\nu}. \end{aligned} \tag{2.1.39}$$

Insertion of the solution (2.1.38) gives the Poisson brackets of the oscillator components as

$$\begin{aligned} \left[\alpha_m^\mu, \alpha_n^\nu\right]_{pb} &= \left[\tilde\alpha_m^\mu, \tilde\alpha_n^\nu\right]_{pb} = im\,\delta_{m+n,0}\,\eta^{\mu\nu}, \\ \left[\alpha_m^\mu, \tilde\alpha_n^\nu\right]_{pb} &= 0. \end{aligned} \tag{2.1.40}$$

Adopting the convention that the zero modes $\tilde\alpha_0^\mu$ and α_0^μ are defined as $\tilde\alpha_0^\mu = \alpha_0^\mu = \frac{1}{2}\ell p^\mu$ then gives us the useful result that (2.1.40) remains valid when $m = 0$ or $n = 0$ or both. Note also that we have $[p^\mu, x^\mu]_{pb} = \eta^{\mu\nu}$ as we should expect.

The case of the open string is slightly different, for as we have mentioned we must choose boundary conditions at $\sigma = 0$ and at $\sigma = \pi$ such that (2.1.26) vanishes. The first choice which accomplishes this is $\partial_\sigma X^\mu = 0$ at the endpoints of the string, i.e., the normal derivative vanishes at the string boundary. This condition is a "free boundary condition" in the sense that momentum cannot flow off the end of the string, and the end of the string is free to move about in spacetime. The general solution of the wave equation with these boundary conditions is

$$X^\mu(\sigma,\tau) = x^\mu + \ell^2 p^\mu \tau + i\ell \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos n\sigma. \tag{2.1.41}$$

---

[5] It should be noted that the α_m are related to conventionally normalized oscillator components by $\alpha_m^\mu = \sqrt{m}\,a_m^\mu$, etc..

The boundary conditions cause the left-moving and right-moving modes to form standing waves, meaning that we now have $\alpha = \tilde{\alpha}$. We can therefore write

$$2\partial_{\pm}X^{\mu} = \partial_{\tau}X^{\mu} \pm \partial_{\sigma}X^{\mu} = \ell \sum_{-\infty}^{\infty} \alpha^{\mu}_{\nu}\, \mathrm{e}^{-in\left(\sigma^{\pm}\right)}, \tag{2.1.42}$$

where we have set the zero mode $\alpha^{\mu}_{0} = \ell p^{\mu}$.

It is now necessary to implement the constraints $T_{ab} = 0$. To do so we consider the mode expansions of the constraints (2.1.34). For closed strings it can be shown that these reduce to $(\partial_{\tau}X_{r})^{2} = (\partial_{\tau}X_{l})^{2} = 0$ thus we find

$$\begin{aligned} L_m &= \frac{1}{4\pi\alpha'} \int_0^{\pi} d\sigma\, \mathrm{e}^{-2im\sigma}\, T_{--} = \frac{1}{4\pi\alpha'} \int_0^{\pi} d\sigma\, \mathrm{e}^{-2im\sigma}\, (\partial_{\tau}X_{r})^{2} \\ &= \frac{1}{2} \sum_{-\infty}^{\infty} \alpha^{\mu}_{m-n}\alpha_{n\mu} = 0 \\ \tilde{L}_m &= \frac{1}{4\pi\alpha'} \int_0^{\pi} d\sigma\, \mathrm{e}^{2im\sigma}\, T_{++} = \frac{1}{4\pi\alpha'} \int_0^{\pi} d\sigma\, \mathrm{e}^{2im\sigma}\, (\partial_{\tau}X_{l})^{2} \\ &= \frac{1}{2} \sum_{-\infty}^{\infty} \tilde{\alpha}^{\mu}_{m-n}\tilde{\alpha}_{n\mu} = 0. \end{aligned} \tag{2.1.43}$$

The Fourier modes of the constraints can be identified with the infinite set of conserved quantities which exist in the theory. We do not, therefore, expect the constraints to change as the system evolves. We emphasize that $L_m = 0$ and $\tilde{L}_m = 0$ are independent constraints for the closed string.

In the open string case things become much more convenient if we extend formally the definitions of X^{μ}_{r} and X^{μ}_{l} beyond the usual range $0 \leq \sigma \leq \pi$ by arranging that $X_r(\sigma + \pi) = X_l(\sigma)$ and $X_l(\sigma + \pi) = X_r(\sigma)$. If this is done then the open string boundary conditions imply that X_r and X_l are periodic functions of σ with a period 2π. These choices are made to get around the fact that $\mathrm{e}^{in\sigma}$ is not a periodic function on $0 \leq \sigma \leq \pi$. The constraints in this case amount to the vanishing of T_{++} on the range $-\pi \leq \sigma \leq \pi$, or at the same time the vanishing of the

Fourier components

$$\begin{aligned} L_m &= \frac{1}{2\pi\alpha'}\int_0^{\pi} d\sigma \left(e^{im\sigma} T_{++} + e^{-im\sigma} T_{--}\right) \\ &= \frac{1}{8\pi\alpha'}\int_{-\pi}^{\pi} d\sigma\, e^{im\sigma}\{(\partial_\tau X)^2 + (\partial_\sigma X)^2\} \\ &= \frac{1}{2}\sum_{-\infty}^{\infty} \alpha^\mu_{m-n}\alpha_{n\mu} = 0. \end{aligned} \tag{2.1.44}$$

These are the infinite set of conserved quantities for the open string.

The Hamiltonian on the world sheet is given by

$$\mathcal{H} = \frac{1}{4\pi\alpha'}\int_0^{\pi} d\sigma\{(\partial_\tau X)^2 + (\partial_\sigma X)^2\} \tag{2.1.45}$$

which upon substitution of (2.1.38) for closed strings or (2.1.41) for open strings gives us

$$\mathcal{H} = \frac{1}{2}\sum_{-\infty}^{\infty} \alpha^\mu_{-n}\alpha_{n\mu} \qquad \text{(open strings)} \tag{2.1.46a}$$

$$\mathcal{H} = \frac{1}{2}\sum_{-\infty}^{\infty} \left(\alpha^\mu_{-n}\alpha_{n\mu} + \tilde{\alpha}^\mu_{-n}\tilde{\alpha}_{n\mu}\right) \qquad \text{(closed strings)} \tag{2.1.46b}$$

while noting that we have $\mathcal{H} = L_0$ for open strings and $\mathcal{H} = L_0 + \tilde{L}_0$ for the closed ones. Note that we have not considered quantum normal ordering effects in (2.1.46) which will play a role when we come to consider the mass spectrum.

The constraint $L_0 = 0$ gives an important formula for the mass in terms of the internal modes of oscillation of the open string, while for the closed string this is given by $L_0 + \tilde{L}_0 = 0$. These formulae are

$$M^2 = \frac{1}{\alpha'}\sum_{n=1}^{\infty} \alpha^\mu_{-n}\alpha_{n\mu} \qquad \text{(open strings)} \tag{2.1.47a}$$

$$M^2 = \frac{2}{\alpha'}\sum_{n=1}^{\infty} \left(\alpha^\mu_{-n}\alpha_{n\mu} + \tilde{\alpha}^\mu_{n}\tilde{\alpha}_{n\mu}\right) \qquad \text{(closed strings)} \tag{2.1.47b}$$

and are the mass shell conditions for the two different types of strings. For closed strings we also point out that the set of constraints have the additional condition that $L_0 - \tilde{L}_0 = 0$, which comes from the fact that the combination $L_0 - \tilde{L}_0$ generates rigid rotations of the string [9], $\sigma \to \sigma +$ constant, which should have no physical effect since σ is periodic. Therefore the two terms in (2.1.47b) will give equal contributions. This is commonly termed *the level matching condition,* that is we must always excite equal numbers of left-moving and right-moving modes, i.e.,

$$\sum_{n=1}^{\infty} \alpha_{-n}^{\mu} \alpha_{n}^{\mu} = \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^{\mu} \tilde{\alpha}_{n}^{\mu}. \tag{2.1.48}$$

The various Fourier modes of the energy momentum tensor, L_m and $\tilde{L}_m$ are known as the Virasoro operators. The Poisson brackets of the Virasoro operators can be computed to be

$$\begin{aligned} [L_m, L_n]_{pb} &= i\,(m-n)\,L_{m+n} \\ \left[\tilde{L}_m, \tilde{L}_n\right]_{pb} &= i\,(m-n)\,\tilde{L}_{m+n} \end{aligned} \tag{2.1.49}$$

which is known as the *Virasoro algebra.* This algebra is fundamental in the theory of strings. The Virasoro operators allow us to define the physical states of the theory.

The Fock space built up through application of the creation operators α_{-n}^{μ} and $\tilde{\alpha}_{-n}^{\mu}$ to the ground state $|0\rangle$ is not positive definite due to the Minkowski metric signature. The Virasoro operators allow us to eliminate the unphysical states. Since we cannot simply impose $T_{ab} \mid \text{phys}\rangle = 0$, we must impose the weaker condition $L_m \,|0\rangle = 0$ for $m \geq 0$.

The version of the Virasoro algebra given above is classical. However, a complete quantum computation of this algebra, which includes a detailed consideration of the effects of normal ordering of the oscillator components is beyond the scope of this brief introduction. It will suffice to say that when quantum effects are taken into account the result is that the Virasoro algebra gains an anomaly term, or central charge, meaning that the algebra no longer closes, becoming:

$$[L_m, L_n] = (m-n)\,L_{m+n} + \left(\frac{D-26}{12} m^3 + \frac{26-D}{12} m\right) \delta_{m+n\,0} \tag{2.1.50}$$

where D is the number of bosonic fields X^{μ}, and the number 26 derives from the contribution of certain ghost, or negative norm, fields which appear in the Fadeev-Popov gauge-fixing procedure [28,33]. Thus we arrive at the famous result that the "critical dimension" of the bosonic string theory is 26. This is the number of spacetime fields for which the Virasoro anomaly vanishes, leaving a Virasoro algebra which closes. The Virasoro algebra is tied to the conformal symmetry of the bosonic string action (2.1.19). Classically, the action is conformally invariant, but quantum effects create a *conformal anomaly* [31] thereby breaking the conformal invariance. Therefore, for $D = 26$ the conformal anomaly vanishes, and the bosonic string theory is consistent.

Having 26 dimensions with Minkowski signature we will have, in general, states forming representations of $SO(1, 25)$. However, elimination of the non-physical negative-norm states through application of the Virasoro operators implies that in general the states of the theory will form massive representations of $SO(25)$. In addition the massless states, being constrained to lie on the light cone will form representations of $SO(24)$.

When normal-ordering effects are included, the mass-shell condition for the open bosonic string becomes

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty} \alpha^{\mu}_{-n}\alpha_{n\,\mu} - 1\right). \tag{2.1.51}$$

Thus the spectrum of states of the open bosonic string includes, as the ground state, a scalar tachyon with $\alpha' M^2 = -1$ and a massless vector boson with 24 independent polarizations as the first excited state. An indication that $SO(25)$ is in fact the correct group can be obtained from the fact that we have 324 states with $\alpha' M^2 = 1$ which is precisely the dimension of the symmetric traceless representation of $SO(25)$ [9]. It is interesting to note that the spin at a given mass level is constrained by the formula $J \leq \alpha' M^2 + 1$.

For the closed string, the normal-ordered mass formula is

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty}\left(\alpha^{\mu}_{-n}\alpha_{n\mu} + \tilde{\alpha}^{\mu}_{-n}\tilde{\alpha}_{n\mu}\right) - 4\right). \tag{2.1.52}$$

and the states are more numerous given that we have both left- and right-moving oscillators. The ground state is again a tachyon with $\alpha' M^2 = -4$. At the first excited state level, which is massless, we have the first indication that we are on the right track. We find a set of massless states which again have $SO(24)$ quantum numbers resulting from the tensor product $SO(24)_r \otimes SO(24)_l$ of the left- and right-moving modes. The symmetric and traceless part of this tensor product is a massless spin-two particle: the graviton, the quantum of the gravitational field. At the same time, the antisymmetric part is an antisymmetric tensor field, commonly known as the Kalb-Ramond potential, while the trace part is a scalar field which has been called the dilaton.

Although we have found a graviton state in the spectrum, the bosonic string has some deficiencies. The presence of tachyons in the spectrum is under normal circumstances considered a bad thing. Another glaring defect is the complete lack of fermions. Thus it is necessary to incorporate the concept of supersymmetry into string theory to form the superstring theory which as we will see is successful in correcting both of these defects.

2.2. The superstring

In this section we will overview the elements which come together to form the superstring theory. The concept of supersymmetry, a symmetry between bosonic and fermionic degrees of freedom, was developed as a means of circumventing the Coleman-Mandula theorem [34] which states that the maximal set of symmetries a physical theory may possess are those of Poincaré invariance, internal global symmetries, whose generators are Lorentz scalars (charge, isospin) and the discrete symmetries C, P and T. One of the assumptions of Coleman and Mandula was that the symmetry algebra of the S-matrix contains no anticommutators. By including anti-commuting generators which transform under spinor representations of the Lorentz group, the Poincaré spacetime symmetries can be extended.

Below, supersymmetry is briefly presented using the formalism of superspace and superfields [35-39]. Then supersymmetry is installed on the string world sheet. This necessitates a treatment of the boundary conditions which are to be imposed on the fermionic fields on the world sheet. From this point, the computation of the supersymmetric version of the Virasoro algebra and its anomaly can be carried out in a similar manner to that of the bosonic string.

We will demonstrate that the incorporation of $N = 1$ supersymmetry[6] on the world sheet results in a spacetime theory *with* fermions and with critical dimension $D = 10$. Later we will see that world sheet supersymmetry can be extended to spacetime supersymmetry through truncation of the spectrum in a consistent manner [40].

2.2.1. Superfields on the world sheet

The introduction of supersymmetry on the world sheet requires the pairing of a fermionic degree of freedom $\psi^\mu(\sigma, \tau)$ with each bosonic degree of freedom $X^\mu(\sigma, \tau)$. The ψ^μ are two-component world sheet spinors.

The simplest way to install supersymmetry on the world sheet is through the introduction of *superspace.* In superspace, the ordinary coordinates are supplemented with the addition of a number of Grassman-valued (anti-commuting) coordinates θ^α which in two dimensions, for example, form two-component Majorana spinors. Thus superspace is composed of both bosonic and fermionic coordinates [41]. Here we will work with a superspace in which there are equal numbers of bosonic and fermionic coordinates. A general function $\mathcal{K}^\mu$ in superspace can be expanded in a Taylor series in the Grassman coordinates as

$$\mathcal{K}^\mu(\sigma, \theta) = X^\mu(\sigma) + \bar{\theta}\psi^\mu(\sigma) + \frac{1}{2}\bar{\theta}\theta\mathcal{U}^\mu(\sigma) \tag{2.2.1}$$

where $\bar{\theta} = \theta^\dagger\gamma^0$. The series terminates due to the anti-commuting nature of the Grassman coordinates θ^α. $\mathcal{K}^\mu$ here is known as a *superfield* which contains the bosonic scalar field X^μ, the fermionic field ψ^μ_α as well as yet another bosonic scalar field $\mathcal{U}^\mu$ which plays an essential role as an auxiliary field, allowing the supersymmetry algebra to close *without* resorting to the use of the equations of motion, often referred to as putting particles "on shell".[7]

The supersymmetry generators are represented on superspace by the operators

$$Q_\alpha = \frac{\partial}{\partial\bar{\theta}^\alpha} + i\left(\gamma^a\theta\right)_\alpha \partial_a \tag{2.2.2}$$

---

[6] N is a parameter that counts the number of supersymmetry generator pairs.

[7] For a discussion of on-shell versus off-shell supersymmetry one is invited to consult [9] or [25].

where γ^a are the two-dimensional Dirac matrices,[8] which change fermionic degrees of freedom into bosonic degrees of freedom, and vice versa. If we introduce an anti-commuting parameter ϵ_α as the infinitesimal parameter of a supersymmetry transformation, then the combination $\bar{\epsilon}Q$ generates the transformation:

$$\begin{aligned} \delta\theta^\alpha &= \left[\bar{\epsilon}Q, \theta^\alpha\right] = \epsilon^\alpha, \\ \delta\sigma^a &= \left[\bar{\epsilon}Q, \sigma^a\right] = i\bar{\epsilon}\gamma^a\theta \end{aligned} \tag{2.2.3}$$

which can be used to define transformations on superfields as

$$\delta\mathcal{K}^\mu = \left[\bar{\epsilon}Q, \mathcal{K}^\mu\right] = \bar{\epsilon}Q\mathcal{K}^\mu, \tag{2.2.4}$$

and it can then be shown that the effect of this supersymmetry transformation on the component fields of the superfield $\mathcal{K}^\mu$ is

$$\begin{aligned} \delta X^\mu &= \bar{\epsilon}\psi^\mu, \\ \delta\psi^\mu &= -i\gamma^a\epsilon\partial_a X^\mu + \epsilon\mathcal{U}^\mu, \\ \delta\mathcal{U}^\mu &= -i\bar{\epsilon}\gamma^a\partial_a\psi^\mu. \end{aligned} \tag{2.2.5}$$

The presence of the auxiliary field $\mathcal{U}^\mu$ allows the supersymmetry algebra to close *without the use of the equations of motion.* Therefore we see that supersymmetry can be interpreted as a geometrical transformation in superspace, i.e., one which "rotates" fermions into bosons and vice versa. We also mention that any function of superfields is also a superfield and hence will transform according to (2.2.4).

The task now is to learn how to write supersymmetry invariant Lagrangians in two dimensions, such that we can find that which corresponds to the "superstring". For this we will need two things: (1) a covariant derivative on superspace and (2) rules for integrating over Grassman valued coordinates. The former is supplied by

$$\mathcal{D} = \frac{\partial}{\partial\bar{\theta}} - i\gamma^a\theta\partial_a \tag{2.2.6}$$

which has the useful property that $\{\mathcal{D}_\alpha, Q_\beta\} = 0$. Therefore if an object $\mathcal{K}$ transforms as $\delta\mathcal{K} = \bar{\epsilon}Q\mathcal{K}$ then so does $\mathcal{D}_\alpha\mathcal{K}$, and thus $\mathcal{D}_\alpha$ behaves as a superspace covariant derivative. The rules for (2) are handled by saying that the natural integral

[8] See appendix A for the explicit representation.

over all of superspace is given by $\int d^2\sigma d^2\theta$ where $d^2\theta \equiv d\theta^1 d\theta^2$ and $\int d^2\theta$ dealt with through the definition

$$\int d^2\theta \left(a + \theta^1 b_1 + \theta^2 b_2 + \theta^1\theta^2 c\right) \equiv c. \tag{2.2.7}$$

Thus given a Lagrangian of the form

$$S = \int d^2\sigma d^2\theta\, \mathcal{K} \tag{2.2.8}$$

where $\mathcal{K}$ is some superfield, and it can be shown that

$$\begin{aligned} \delta S &= \int d^2\sigma d^2\theta \bar{\epsilon} Q\, \mathcal{K} \\ &= 0 \end{aligned} \tag{2.2.9}$$

through integration by parts in both the Grassman variables, and the ordinary variables. Therefore, any action of the form (2.2.8) is invariant under supersymmetry transformations.

We then want to find a Lagrangian in which the elementary superfields are the superfield analogs of the spacetime coordinate fields on the world sheet as in (2.1.24). One which comes readily to mind from this analogy is

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma d^2\theta \bar{\mathcal{D}}\mathcal{K}^\mu \mathcal{D}\mathcal{K}_\mu. \tag{2.2.10}$$

The expansions of the superspace covariant derivatives will be

$$\begin{aligned} \mathcal{D}\mathcal{K}^\mu &= \psi^\mu + \theta\mathcal{U}^\mu - i\gamma^a\theta\partial_a X^\mu + \frac{i}{2}\bar{\theta}\theta\gamma^a\partial_a\psi^\mu, \\ \bar{\mathcal{D}}\mathcal{K}^\mu &= \bar{\psi}^\mu + \mathcal{U}^\mu\bar{\theta} + i\partial_a X^\mu\bar{\theta}\gamma^a - \frac{i}{2}\bar{\theta}\theta\partial_a\bar{\psi}^\mu\gamma^a, \end{aligned} \tag{2.2.11}$$

and their substitution into (2.2.10) and execution of the integrals over the Grassman coordinates θ results in a supersymmetric Lagrangian of the form:

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \left(\partial_a X^\mu \partial^a X_\mu - i\bar{\psi}^\mu\gamma^a\partial_a\psi_\mu - \mathcal{U}^\mu\mathcal{U}_\mu\right). \tag{2.2.12}$$

The equations of motion then imply that the auxiliary field $\mathcal{U}^\mu$ vanishes, and henceforth it will be ignored.

2.2.2. Constraints and boundary conditions

The fermionic fields ψ^μ that we have introduced on the world sheet must be complemented with a set of constraints and boundary conditions, just as in the bosonic case we examined earlier. The fermionic equation of motion derived from (2.2.12) will be the Dirac equation in two dimensions $\gamma^a\partial_a\psi^\mu = 0$. An appropriate basis with which to work for the Dirac matrices on the world sheet is

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{2.2.13}$$

The fields ψ^μ can be decomposed using this basis into

$$\psi^\mu = \begin{pmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{pmatrix} \tag{2.2.14}$$

where ψ_- are the right-moving and ψ_+ are the left-moving modes of ψ, which are also eigenstates of the chirality operator $\gamma^3 = \gamma^0\gamma^1$, that is

$$\gamma^3\psi_\pm = \mp\psi_\pm \tag{2.2.15}$$

and we can write decoupled equations for the right- and left-moving modes as

$$(\partial_\sigma \pm \partial_\tau)\,\psi^\mu_\mp = 0 \tag{2.2.16}$$

indicating that $\psi_\pm = \psi_\pm(\sigma^\mp)$, paralleling those we have seen for the bosonic coordinates X^μ. In the light-cone coordinates we have used previously the fermionic part of the action can be written

$$S_f = \frac{i}{2\pi\alpha'}\int d^2\sigma\,\left(\psi^\mu_-\partial_+\psi_{-\mu} + \psi^\mu_+\partial_-\psi_{+\mu}\right) \tag{2.2.17}$$

in which form the decoupling between right- and left-moving fields is rather apparent.

In this case as well, the X^μ satisfy the same free wave equation as in the purely bosonic case, and the mode expansions (2.1.38) carry over unchanged. The fermionic coordinates will have their own surface terms from the variation of the Lagrangian (2.2.12), the vanishing of which requires that $\psi_+\delta\psi_+ - \psi_-\delta\psi_-$ vanish at each end of the open string, which can be satisfied by putting $\psi_+ = \pm\psi_-$ at each end. The relative sign at $\sigma = 0$ (say) can be chosen with no loss of generality to be positive, i.e.,

$$\psi_+^\mu(0,\tau) = \psi_-^\mu(0,\tau) \tag{2.2.18}$$

which leaves us with two meaningful choices of sign at the $\sigma = \pi$ end of the string. Choosing

$$\psi_+^\mu(\pi,\tau) = \psi_-^\mu(\pi,\tau) \tag{2.2.19}$$

are known as *Ramond* (R) boundary conditions which have the mode expansions

$$\begin{aligned} \psi_-^\mu(\sigma,\tau) &= \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} \rho_n^\mu \, \mathrm{e}^{-in(\tau-\sigma)} \\ \psi_+^\mu(\sigma,\tau) &= \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} \rho_n^\mu \, \mathrm{e}^{-in(\tau+\sigma)} \end{aligned} \tag{2.2.20}$$

where $n \in \mathbb{Z}$ implies summation over all integers and the ρ are the oscillator components. On the other hand, the choice

$$\psi_+^\mu(\pi,\tau) = -\psi_-^\mu(\pi,\tau) \tag{2.2.21}$$

are known as *Neveu-Schwarz* (NS) boundary conditions which have the different mode expansions

$$\begin{aligned} \psi_-^\mu(\sigma,\tau) &= \frac{1}{\sqrt{2}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} \eta_r^\mu \, \mathrm{e}^{-ir(\tau-\sigma)} \\ \psi_+^\mu(\sigma,\tau) &= \frac{1}{\sqrt{2}} \sum_{r\in\mathbb{Z}+\frac{1}{2}} \eta_r^\mu \, \mathrm{e}^{-ir(\tau+\sigma)} \end{aligned} \tag{2.2.22}$$

where $r \in \mathbb{Z} + \frac{1}{2}$ implies a sum over half-integers r, and again the η are the oscillator components. The Ramond boundary conditions and the integral mode expansions (2.2.20) describe string states that are spacetime fermions, while the Neveu-Schwarz boundary conditions and half-integral mode expansions (2.2.22) describe string states that are spacetime bosons.

For closed strings, on the other hand, the surface terms vanish if the boundary conditions are periodic *or* antiperiodic *separately* for the left-moving and the right-moving fields.

This means that we can have

$$\psi_-^\mu = \sum_{n\in\mathbb{Z}} \rho_n^\mu \, \mathrm{e}^{-2in(\tau-\sigma)} \quad \textbf{or} \quad \psi_-^\mu = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \eta_r^\mu \, \mathrm{e}^{-2ir(\tau-\sigma)} \qquad (2.2.23a)$$

AND

$$\psi_+^\mu = \sum_{n\in\mathbb{Z}} \tilde{\rho}_n^\mu \, \mathrm{e}^{-2in(\tau+\sigma)} \quad \textbf{or} \quad \psi_+^\mu = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \tilde{\eta}_r^\mu \, \mathrm{e}^{-2ir(\tau+\sigma)} \qquad (2.2.23b)$$

which correspond to the closed string having four distinct sectors, or sets of states, which can be referred to as NS-NS, NS-R, R-NS, R-R. The NS-NS and R-R sectors describe spacetime bosons, while the NS-R and R-NS describe the spacetime fermions.

How does this come about? We begin by imposing the canonical commutation relations on the fermionic coordinates, as in

$$\left\{\psi_a^\mu(\sigma,\tau), \psi_b^\nu(\sigma',\tau)\right\} = \pi\delta\left(\sigma-\sigma'\right)\eta^{\mu\nu}\delta_{ab} \qquad (2.2.24)$$

which upon substitution of the mode expansions (2.2.20) or (2.2.22) for the open string and (2.2.23) for the closed string give us the anticommutation relations for the modes ρ_n^μ and η_r^μ as

$$\begin{aligned} \left\{\eta_r^\mu, \eta_s^\nu\right\} &= \eta^{\mu\nu}\delta_{r+s}, \\ \left\{\rho_n^\mu, \rho_m^\nu\right\} &= \eta^{\mu\nu}\delta_{n+m}. \end{aligned} \qquad (2.2.25)$$

Note that for the closed string we will have another set of relations involving $\tilde{\eta}_r^\mu$ and $\tilde{\rho}_n^\mu$. Consider now a ground state of the Fock space such that

$$\alpha_n^\mu|0\rangle = \tilde{\alpha}_n^\mu|0\rangle = \rho_n^\mu|0\rangle = 0 \qquad n>0 \qquad (2.2.26)$$

for the NS boundary condition, or for the Ramond boundary condition

$$\alpha_n^\mu|0\rangle = \tilde{\alpha}_n^\mu|0\rangle = \eta_r^\mu|0\rangle = 0 \qquad n,r>0. \qquad (2.2.27)$$

For the half-integer modes η_r^μ, it is possible to identify a unique non-degenerate ground state, which therefore is a scalar (spin zero). For the case of the integer modes ρ_n^μ this is not possible since we have the zero modes ρ_0^μ. These will obey the algebra

$$\left\{\rho_0^\mu, \rho_0^\nu\right\} = \eta^{\mu\nu} \qquad (2.2.28)$$

which is the Dirac algebra, and thus up to a normalization the zero modes ρ_0^μ just are the Dirac matrices. If we demand that the Dirac matrices satisfy $\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}$ then we find that $\gamma^\mu = i\sqrt{2}\rho_0^\mu$.

If we write the mass shell relation for the superstring, generalizing (2.1.47), we find it to be for the NS condition

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty}\alpha_{-n}^{\mu}\alpha_{n\mu} + \sum_{r=\frac{1}{2}}^{\infty} r\eta_{-r}^{\mu}\eta_{r\mu} - \frac{1}{2}\right) \qquad \text{(open strings)} \quad (2.2.29a)$$

$$M^2 = \frac{1}{\alpha'}\left(2\sum_{n=1}^{\infty}\left(\alpha_{-n}^{\mu}\alpha_{n\mu} + \tilde{\alpha}_{n}^{\mu}\tilde{\alpha}_{n\mu}\right)\right.$$
$$\left. + \sum_{r=\frac{1}{2}}^{\infty} r\left(\eta_{-r}^{\mu}\eta_{r\mu} + \tilde{\eta}_{-r}^{\mu}\tilde{\eta}_{r\mu}\right) - 1\right) \qquad \text{(closed strings)} \quad (2.2.29b)$$

and for the Ramond boundary condition

$$M^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty}\alpha_{-n}^{\mu}\alpha_{n\mu} + \tfrac{1}{2}\sum_{n\in\mathbb{Z}} n\rho_{-n}^{\mu}\rho_{n\mu}\right) \qquad \text{(open strings)} \quad (2.2.30a)$$

$$M^2 = \frac{1}{\alpha'}\left(2\sum_{n=1}^{\infty}\left(\alpha_{-n}^{\mu}\alpha_{n\mu} + \tilde{\alpha}_{n}^{\mu}\tilde{\alpha}_{n\mu}\right)\right.$$
$$\left. + \tfrac{1}{2}\sum_{n\in\mathbb{Z}} n\left(\rho_{-n}^{\mu}\rho_{n\mu} + \tilde{\rho}_{-n}^{\mu}\tilde{\rho}_{n\mu}\right)\right). \qquad \text{(closed strings)} \quad (2.2.30b)$$

At each mass level we expect that for the Ramond boundary the states will fill out representations of the Dirac algebra. As is well known, the irreducible representations of the Dirac algebra are in fact the spinor representations of $SO(1, D-1)$ in D dimensions.

We now need to find the constraints analogous to the vanishing of the Virasoro operators for the bosonic string, which allowed us to eliminate the non-physical states of the theory, leaving behind the physical spectrum. Recall that for the bosonic string the Virasoro operators (2.1.43) for the closed string and (2.1.44) for the open string had their origin in the variation of the Lagrangian with respect to the world sheet metric ((2.1.31)). Thus the removal of non-physical states for the bosonic string is dependent upon the reparametrization independent form of the action (2.1.30). This action can be thought of formally as D scalar fields X^μ coupled to gravity in two dimensions, described by e^{ab}. From this we reason that in the case in which there are also fermionic fields coupled to 2-dimensional gravity, as in the superstring action (2.2.12), the proper course of action is to treat the X^μ and ψ^μ as superpartners coupled to 2-dimensional supergravity. In this case the parameter describing the supersymmetry transformations ϵ becomes dependent on the world sheet coordinates $\epsilon(\sigma, \tau)$. This local supersymmetry algebra will give us the infinite number of super-Virasoro operators we will need to eliminate the non-physical states for the superstring.

The super-Virasoro constraints have, therefore, contributions from the fermionic fields. When the parameter of the supersymmetry transformation ϵ becomes local, the supersymmetric variation of (2.2.12) no longer vanishes, but rather is given by

$$\delta S = \frac{1}{2\pi\alpha'} \int d^2\sigma \, (\partial_a \bar{\epsilon}) \, J^a \tag{2.2.31}$$

where J_a is the world sheet supercurrent given by

$$J_a = \frac{1}{2}\gamma^b \gamma_a \psi^\mu \partial_b X_\mu . \tag{2.2.32}$$

The supercurrent can be decomposed into positive- and negative-helicity components as

$$\begin{aligned} J_+ &= \psi_+^\mu \partial_+ X_\mu , \\ J_- &= \psi_-^\mu \partial_- X_\mu . \end{aligned} \tag{2.2.33}$$

We also write the world-sheet energy-momentum tensor for the superstring as

$$T_{ab} = \partial_a X^\mu \partial_b X_\mu + \frac{i}{4}\bar{\psi}^\mu \gamma_a \partial_b \psi_\mu + \frac{i}{4}\bar{\psi}^\mu \gamma_b \partial_a \psi_\mu - \text{(trace)} \tag{2.2.34}$$

which is re-expressed in light cone coordinates as

$$\begin{aligned} T_{++} &= \partial_+ X^\mu \partial_+ X_\mu + \frac{i}{2}\psi_+^\mu \partial_+ \psi_{+\,\mu} , \\ T_{--} &= \partial_- X^\mu \partial_- X_\mu + \frac{i}{2}\psi_-^\mu \partial_- \psi_{-\,\mu} , \\ T_{+-} &= T_{-+} = 0. \end{aligned} \tag{2.2.35}$$

The supercurrent components $J_\pm$ are connected to T_{++} and T_{--} through an algebra

$$\begin{aligned} \{J_+(\sigma), J_+(\sigma')\} &= \pi\delta(\sigma - \sigma')T_{++}(\sigma), \\ \{J_-(\sigma), J_-(\sigma')\} &= \pi\delta(\sigma - \sigma')T_{--}(\sigma), \\ \{J_+(\sigma), J_-(\sigma')\} &= 0, \end{aligned} \tag{2.2.36}$$

and therefore, if we wish to set $T_{++} = T_{--} = 0$ as in the case of the bosonic string, we must set $J_+ = J_- = 0$ as well.

Then one is able to write the super-Virasoro constraints for the closed string as

$$L_m = \frac{1}{4\pi\alpha'}\int_0^\pi d\sigma\, e^{-2im\sigma}\, T_{--}$$

$$= \frac{1}{2}\begin{cases} \displaystyle\sum_{n=-\infty}^{\infty} \alpha^\mu_{m-n}\alpha_{n\mu} + \sum_{r=-\infty}^{\infty}\left(r+\frac{m}{2}\right)\eta^\mu_{m-r}\eta_{r\mu} & \text{Neveu-Schwarz} \\ \displaystyle\sum_{n=-\infty}^{\infty} \alpha^\mu_{m-n}\alpha_{n\mu} + \sum_{n=-\infty}^{\infty}\left(n+\frac{m}{2}\right)\rho^\mu_{m-n}\rho_{n\mu} & \text{Ramond} \end{cases} \tag{2.2.37a}$$

$$G_r = \frac{\sqrt{2}}{4\pi\alpha'}\int_0^\pi d\sigma\, e^{-2ir\sigma}\, J_- \tag{2.2.37b}$$

$$= \sum_{n=-\infty}^{\infty} \eta^\mu_{r-n}\alpha_{n\mu} \qquad \text{Neveu-Schwarz}$$

$$F_m = \frac{\sqrt{2}}{4\pi\alpha'}\int_0^\pi d\sigma\, e^{-2im\sigma}\, J_- \tag{2.2.37c}$$

$$= \sum_{n=-\infty}^{\infty} \rho^\mu_{m-n}\alpha_{n\mu} \qquad \text{Ramond}$$

for the right-movers and similarly for the left-movers as

$$\tilde{L}_m = \frac{1}{4\pi\alpha'}\int_0^\pi d\sigma\, e^{2im\sigma}\, T_{++}$$

$$= \frac{1}{2}\begin{cases} \displaystyle\sum_{n=-\infty}^{\infty} \tilde{\alpha}^\mu_{m-n}\tilde{\alpha}_{n\mu} + \sum_{r=-\infty}^{\infty}\left(r+\frac{m}{2}\right)\tilde{\eta}^\mu_{m-r}\tilde{\eta}_{r\mu} & \text{Neveu-Schwarz} \\ \displaystyle\sum_{n=-\infty}^{\infty} \tilde{\alpha}^\mu_{m-n}\tilde{\alpha}_{n\mu} + \sum_{n=-\infty}^{\infty}\left(n+\frac{m}{2}\right)\tilde{\rho}^\mu_{m-n}\tilde{\rho}_{n\mu} & \text{Ramond} \end{cases} \tag{2.2.38a}$$

$$\tilde{G}_r = \frac{\sqrt{2}}{4\pi\alpha'}\int_0^\pi d\sigma\, e^{2ir\sigma}\, J_+ \tag{2.2.38b}$$

$$= \sum_{n=-\infty}^{\infty} \tilde{\eta}^\mu_{r-n}\tilde{\alpha}_{n\mu} \qquad \text{Neveu-Schwarz}$$

$$\tilde{F}_m = \frac{\sqrt{2}}{4\pi\alpha'}\int_0^\pi d\sigma\, e^{2im\sigma}\, J_+ \tag{2.2.38c}$$

$$= \sum_{n=-\infty}^{\infty} \tilde{\rho}^\mu_{m-n}\tilde{\alpha}_{n\mu}. \qquad \text{Ramond}$$

For the open string case we have

$$
\begin{aligned}
L_m &= \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma\{\mathrm{e}^{im\sigma}T_{++} + \mathrm{e}^{-im\sigma}T_{--}\} \\
&= \frac{1}{2\pi\alpha'}\int_{-\pi}^\pi d\sigma\,\mathrm{e}^{im\sigma}T_{++} = 0 \\
&= 2\begin{cases} \displaystyle\sum_{n=-\infty}^{\infty} \alpha^\mu_{m-n}\alpha_{n\mu} + \sum_{r=-\infty}^{\infty}\left(r+\frac{m}{2}\right)\eta^\mu_{m-r}\eta_{r\mu} & \text{Neveu-Schwarz} \\ \displaystyle\sum_{n=-\infty}^{\infty} \alpha^\mu_{m-n}\alpha_{n\mu} + \sum_{n=-\infty}^{\infty}\left(n+\frac{m}{2}\right)\rho^\mu_{m-n}\rho_{n\mu} & \text{Ramond} \end{cases}
\end{aligned}
\tag{2.2.39a}
$$

$$
\begin{aligned}
G_r &= \frac{\sqrt{2}}{2\pi\alpha'}\int_0^\pi d\sigma\{\mathrm{e}^{ir\sigma}J_+ + \mathrm{e}^{-ir\sigma}J_-\} \\
&= \frac{\sqrt{2}}{2\pi\alpha'}\int_{-\pi}^\pi d\sigma\,\mathrm{e}^{ir\sigma}J_+ = 2\sum_{n=-\infty}^{\infty}\eta^\mu_{r-n}\alpha_{n\mu} \qquad \text{Neveu-Schwarz}
\end{aligned}
\tag{2.2.39b}
$$

$$
\begin{aligned}
F_m &= \frac{\sqrt{2}}{2\pi\alpha'}\int_0^\pi d\sigma\{\mathrm{e}^{im\sigma}J_+ + \mathrm{e}^{-im\sigma}J_-\} \\
&= \frac{\sqrt{2}}{2\pi\alpha'}\int_{-\pi}^\pi d\sigma\,\mathrm{e}^{im\sigma}J_+ = 2\sum_{n=-\infty}^{\infty}\rho^\mu_{m-n}\alpha_{n\mu} \qquad \text{Ramond}
\end{aligned}
\tag{2.2.39c}
$$

exactly similar to the right-moving closed string.

The methods for dealing with these operators parallel those used for the bosonic string, and the reader interested in the specific details is recommended to consult the references, in particular [9] and [25].

It is reasonably clear, however, that what is going to result is a supersymmetric generalization of the Virasoro algebra appearing in (2.1.50). This generalization has, in fact, two parts corresponding to the NS and R sector. In the NS sector the super-Virasoro algebra is

$$
\begin{aligned}
[L_m, L_n] &= (m-n)\,L_{m+n} + \frac{D}{8}\left(m^3 - m\right)\delta_{m+n,0} \\
[L_m, G_r] &= \left(\frac{m}{2} - r\right)G_{m+r} \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{D}{2}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}
\end{aligned}
\tag{2.2.40}
$$

while that of the Ramond sector is very similar, appearing as

$$\begin{aligned}
[L_m, L_n] &= (m-n)\, L_{m+n} + \frac{D}{8} m^3 \delta_{m+n,0} \\
[L_m, F_n] &= \left(\frac{m}{2} - n\right) F_{m+n} \\
\{F_m, F_n\} &= 2L_{m+n} + \frac{D}{2} m^2 \delta_{m+n,0}
\end{aligned} \tag{2.2.41}$$

where as in the case of the bosonic string, the anomaly terms derive from the normal ordering of the oscillator components. A physical bosonic state is then identified by requiring that

$$\begin{aligned}
G_r \mid \text{phys}\rangle &= 0, \qquad r > 0 \\
L_m \mid \text{phys}\rangle &= 0, \qquad m > 0 \\
(L_0 - a) \mid \text{phys}\rangle &= 0
\end{aligned} \tag{2.2.42}$$

where a is a normal-ordering constant for bosonic states. For physical fermionic states the conditions are

$$\begin{aligned}
F_m \mid \text{phys}\rangle &= 0 \qquad r > 0 \\
L_m \mid \text{phys}\rangle &= 0 \qquad r > 0 \\
(F_0 - a') \mid \text{phys}\rangle &= 0
\end{aligned} \tag{2.2.43}$$

where a' is the normal ordering constant for the fermionic states. The normal ordering constants are $a = \frac{1}{2}$ and $a' = 0$ for the open string, and $a = 1$ and $a' = 0$ for the closed string. These are reflected in the mass formulae (2.2.29) and (2.2.30).

It can be shown [25] that the critical dimension D that results in a super-Virasoro algebra without anomalies is 10. In this case the supersymmetric generalization of the conformal anomaly, the *superconformal anomaly* will vanish, and thus the superstring theory is consistent[9] quantum theory in ten spacetime dimensions. One might have been hoping for $D = 4$, but this is not the case. The question then arises as to how one arrives at a theory in a more reasonable number of spacetime dimensions, i.e., four.

Another question concerns the existence, although we will not demonstrate this explicitly, of a tachyon in the spectrum. The tachyon can be understood from the normal ordering term in (2.2.29). There is the further question of the relationship

---

[9] Well, it is almost consistent at this point, see section 2.2.2.1.

between world-sheet and spacetime supersymmetry. That is, given that we have supersymmetry on the world sheet, do we then have supersymmetry in spacetime as well? If not automatically, is it possible to arrange this to be the case? These are topics that we will explore in the next sections.

2.2.2.1. Spacetime supersymmetry

The superstring model above, even with critical dimension $D = 10$ is still not yet completely consistent. The spectrum still includes a tachyon. The tachyon can, however, be eliminated when the spectrum of string states is truncated in a very specific manner, and when this is done, the number of fermionic and bosonic degrees of freedom at each mass level are the same, and we have spacetime supersymmetry.

The truncation of the string spectrum necessary for creating spacetime supersymmetry is known as the GSO projection, after the originators Gliozzi, Scherk and Olive [40]. It should be noted, however, that the desire to produce a theory which is spacetime supersymmetric is not the only argument in favor of making this modification to the spectrum. There is the aforementioned tachyon that we would like to eliminate.

Further, one sees a problem in that the theory has certain anticommuting operators ψ^μ that map bosons to bosons [9]. That is, if $|\phi\rangle$ is said to be a bosonic state, then $\psi^\mu\,|\phi\rangle$ is a state of integer spin that has been created by an anticommuting operator acting on $|\phi\rangle$. This is not normally encountered in physics. Further, consider n such operators acting on our state:

$$\psi^{\mu_1}\psi^{\mu_2}\cdots\psi^{\mu_n}\,|\phi\rangle\,. \tag{2.2.44}$$

If n is even, there is no real difficulty, since the product of an even number of anticommuting operators is commuting. For n odd, however, it is tempting to propose eliminating the states. The GSO projection consists of the proposal that all states of the form (2.2.44) with n odd are to be eliminated from the spectrum of states, while those of even n are kept. This is done by formally introducing a quantum number $(-1)^F$, called G-parity for historical reasons and under which the Fermi fields ψ^μ are odd and the bosonic fields X^μ even.

Thus we define for the NS and R sectors the GSO projection operators

$$P = \frac{1 + (-1)^{F+1}}{2} \qquad F = \sum_{r=\frac{1}{2}}^{\infty} \eta^{\mu}_{-r}\eta^{\mu}_{r} \qquad \text{NS sector} \tag{2.2.45a}$$

$$P = \frac{1 + k\,(-1)^{F+1}}{2} \qquad F = \sum_{n=1}^{\infty} \rho^{\mu}_{-n}\rho^{\mu}_{n} \qquad \text{R sector} \tag{2.2.45b}$$

where the zero-mode contribution is counted by writing $F + 1$ and k can be chosen to be either +1 or -1 independently for the right-moving and left-moving fields. Note that for left-moving fields F in (2.2.45a) or (2.2.45b) is replaced by

$$F = \sum_{r=\frac{1}{2}}^{\infty} \tilde{\eta}^{\mu}_{-r}\tilde{\eta}^{\mu}_{r} \qquad \text{or} \qquad F = \sum_{n=1}^{\infty} \tilde{\rho}^{\mu}_{-n}\tilde{\rho}^{\mu}_{n}. \tag{2.2.46}$$

Note that the GSO projection is carried out separately on right- and left-moving states. In this way, the GSO projection eliminates our unruly tachyonic scalar, for which $F = 0$, since only states with $(-1)^F = -1$ survive the projection in the NS sector, and gives us a theory which can be shown to have ten-dimensional spacetime supersymmetry [42].

Historically, closed string theories in which k has the same value for both right- and left-moving fields are known as type IIB theories whereas k chosen to have opposite values for the left- and right-movers leads to type IIA theories. We will describe these theories in more detail in a later section.

2.2.3. Compactification

The procedure which allows us to take a theory which is formulated in 10 spacetime dimensions (or 26 for the purely bosonic string, if we wish) and effectively reduce the number of spacetime dimensions is known as compactification. In the same way as there are two ways of thinking about string theory, from the point of view of the world sheet and from the point of view of spacetime, one can consider compactification from these two complementary points of view.

Compactification from the spacetime perspective, usually called *Kaluza-Klein* compactification, is an ansatz which tells us how higher dimensional fields, such as the metric, appear to a four-dimensional physicist for example. Since this compactification procedure will be of great utility in our work, we will explain it in sufficient detail. However, we will begin with a discussion of compactification from the world sheet perspective.

In the following, we will restrict our attention to compactification of higher dimensions on tori, that is we make the extra coordinates periodic. This is only the simplest possibility. There exists an extensive literature which considers other possibilities, such as orbifolds [43] or more general Calabi-Yau manifolds [44].

2.2.3.1. Compactification of world-sheet fields

As the reader has undoubtedly already noted, the addition of fermionic fields on the world sheet was responsible for lowering the critical dimension of the superstring theory to ten from the twenty-six of the bosonic string theory. This gives one the idea that the addition of other fields on the world sheet, not necessarily fermionic ones, may have a similar effect.

In fact this can be brought about in conjunction with the elimination of the higher dimensions for the bosonic string in a process sometimes called *toroidal compactification.* On the world sheet, the higher dimensions beyond say 10, the critical dimension of the superstring, are just a selection from among the twenty-six which live there, with the exception of time. Thus we choose 16 coordinate fields on the world sheet, and we enforce on them periodic boundary conditions in a spacetime sense. These fields are already periodic on the world sheet, but we now add a condition such as $X^i = X^i + 2\pi R^i \cdot m_i$ where i runs over the 16 "compact" coordinates, R^i represents the radii, or periods, of these coordinates and m_i are arbitrary integers. Thus we have formed a torus.

For simplicity, let us consider a single coordinate X of the closed bosonic string on the world sheet that is compactified into a circle. We then have $X \equiv X + 2\pi Rm$, and the mode expansion of X is modified to reflect that the center of mass momentum along the compact direction is quantized, giving

$$X(\sigma,\tau) = x + \alpha' p\tau + 2L\sigma + \frac{i\ell}{2}\sum_{n\neq 0}\frac{1}{n}\{\alpha_n\, \mathrm{e}^{-2in(\tau-\sigma)} + \tilde{\alpha}_n\, \mathrm{e}^{-2in(\tau+\sigma)}\}. \quad (2.2.47)$$

Note that the momentum zero mode is quantized as $p = \frac{k}{R}$ with $k \in \mathbb{Z}$.This ensures any momentum eigenfunctions along the compact direction are single valued. Also,

$L = mR$ with $m \in \mathbb{Z}$, and m describes the number of times that the string wraps around the compact coordinate. This wrapping cannot occur in the uncompactified case, since the energy will diverge as $R \to \infty$. Note also that wrapping around compact coordinates does not occur for open strings[10]

As usual the mode expansion (2.2.47) can be decomposed into right- and left-moving parts

$$\begin{aligned} X_r(\tau-\sigma) &= x_r + \frac{\ell}{2}\left(\sqrt{\alpha'}\,p + \frac{L}{\sqrt{\alpha'}}\right)(\tau-\sigma) + \frac{i\ell}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n\, e^{-2in(\tau-\sigma)} \\ X_l(\tau+\sigma) &= x_l + \frac{\ell}{2}\left(\sqrt{\alpha'}\,p - \frac{L}{\sqrt{\alpha'}}\right)(\tau+\sigma) + \frac{i\ell}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n\, e^{-2in(\tau+\sigma)} \end{aligned} \tag{2.2.48}$$

where x_r and x_l are the center of mass position of the right- and left-moving modes respectively. Recall that $\ell = \sqrt{2\alpha'}$. Substituting this into the zero-mode Virasoro constraints gives us, for the compact coordinate

$$\begin{aligned} L_0 &= \frac{1}{4}\left(\sqrt{\alpha'}\,p + \frac{L}{\sqrt{\alpha'}}\right)^2 + N + \frac{\alpha'}{4}(p_\mu)^2 \\ \tilde{L}_0 &= \frac{1}{4}\left(\sqrt{\alpha'}\,p - \frac{L}{\sqrt{\alpha'}}\right)^2 + \tilde{N} + \frac{\alpha'}{4}(p_\mu)^2 \end{aligned} \tag{2.2.49}$$

which give a formula for the 25-dimensional mass of the form

$$\alpha' M^2 = 2(N + \tilde{N} - 2) + \alpha'\frac{k^2}{R^2} + \frac{m^2 R^2}{\alpha'}. \tag{2.2.50}$$

where

$$\begin{aligned} N &= \sum_{n=1}^{\infty}\left(\alpha^{\mu}_{-n}\alpha_{n\,\mu} + \alpha^{25}_{-n}\alpha_{n\,25}\right) \\ \tilde{N} &= \sum_{n=1}^{\infty}\left(\tilde{\alpha}^{\mu}_{-n}\tilde{\alpha}_{n\,\mu} + \tilde{\alpha}^{25}_{-n}\tilde{\alpha}_{n\,25}\right) \end{aligned} \tag{2.2.51}$$

are the contributions of the oscillator components. Thus we see that the 25-dimensional mass has a contribution from the momentum of the center of mass of the string along the 26'th dimension.[11] Note here that we are labelling the

---

10 We will see in chapter IV that considerations of open versus closed strings in conjunction with compactification lead us to new objects in string theory, the D-branes.

11 From the mass formula one may be led to expect that changing $R \leftrightarrow \frac{\alpha'}{R}$ is a symmetry and this is in fact true as we will see in chapter III.

dimensions $\mu = 0 \ldots 24$. The level matching condition $L_0 - \tilde{L}_0 = 0$ gives the relation

$$N - \tilde{N} = pL = km. \tag{2.2.52}$$

2.2.3.2. World sheet current algebra

The closed string has no points of preference like the endpoints of an open string. Therefore, if one contemplates the addition of charges to a closed string, one must consider that the charge is distributed over the string. This can be carried out through the addition of bosonic fields on the world sheet, which are Lorentz singlets, but carry internal quantum numbers.

The consistency of string theory is quite fragile, but it turns out that consistency can be maintained as long as certain conditions are met. First of all, the total number of bosonic fields must always add up to 26. Thus the number n of these charge fields that one must have if there are D coordinate fields X^μ is given by $n = 26 - D$. It can be shown that with suitable, special conditions [9] on the zero modes of these charge fields, they can generate an $SO(2n)_r \otimes SO(2n)_l$ internal symmetry group, while satisfying the quantum consistency conditions. At the same time, the zero mode conditions on the charge fields prevent the extension of the Lorentz group of the theory beyond $SO(1, D-1)$. Thus, by adding a current algebra, one can construct a consistent completely bosonic string theory in D dimensions with $SO(2n) \otimes SO(2n)$ symmetry. In a sense, in this way we can "adjust" the critical dimension of the bosonic string.

Let us now return to toroidal compactification. Consider again the single compact coordinate of section 2.2.3.1. Let $|k, m\rangle$ denote the ground state of a Fock space which has internal momentum number k and winding number[12] m. Let us now construct massless vector states. Two are given by

$$(\alpha^\mu_{-1}\tilde{a}_{-125} \pm a_{-125}\tilde{\alpha}^\mu_{-1})\,|0,0\rangle \tag{2.2.53}$$

[12] We suppress the 25-momentum.

since they have $N = \tilde{N} = 1$ and $p = L = 0$, $M^2 = 0$. These two vectors can be considered to result from the decomposition of the graviton $G_{\mu\nu}$ and the antisymmetric tensor field $B_{\mu\nu}$ with respect to the 25-dimensional Lorentz subgroup.

More massless fields can be found when $p, L \neq 0$. If we take $pL = km = 1$ for example, setting $p = L/\alpha'$ then we can have zero mass if $N = 1$, $\tilde{N} = 0$, and if $R^2 = \alpha'$ from (2.2.49) and (2.2.50). Since $km = 1$ requires $k = m = \pm 1$, for this special value of the compactification radius, we have four massless vector fields given by

$$\alpha^{\mu}_{-1}\,|1,1\rangle\,, \quad \alpha^{\mu}_{-1}\,|-1,-1\rangle\,, \quad \tilde{\alpha}^{\mu}_{-1}\,|1,-1\rangle\,, \quad \tilde{\alpha}^{\mu}_{-1}\,|-1,1\rangle \tag{2.2.54}$$

in addition to those, (2.2.53), that are present at any compactification radius.

Thus we see a rough outline of how compactification gives rise to the fields needed to form representations of $SO(2d)_r \otimes SO(2d)_l$, at least at special compactification radii. Thus nonabelian symmetry can arise from this procedure. This procedure is of particular interest in the case of the *heterotic* string [45], where 16 (right-moving) dimensions are compactified to produce a nonabelian symmetry group of $SO(32)$. It can also be arranged to form the group $E_8 \otimes E_8$, since the exceptional group E_8 has 248 generators whereas $SO(32)$ has 496 generators.

2.2.3.3. Kaluza-Klein compactification

The Kaluza-Klein programme is, as we have noted, an approach to compactification from the spacetime perspective as it is unnecessary to even mention the existence of a world sheet when carrying it through. In fact, this programme predates string theory [46] and in its simplest terms it is an ansatz that tells us how the physics of fields in a spacetime of dimension d appear when observed in $n < d$ dimensions, where $d - n$ dimensions have been compactified. The fields themselves only depend on non-compact coordinates.

The compactified dimensions are considered to have a radius small enough that at large length scales they are indistinguishable from points. An example of this effect is a cylinder appearing as a line from a great distance. When they are very small, very high energies are necessary to probe these compact dimensions, and they are thus completely hidden from low energy physicists. Kaluza-Klein theory is a subject unto itself [47] as well as having applications in string theory.

Let us first consider a simple example, one which we will have occasion to use during the course of this work. Let us consider that we have the set of spacetime fields that were mentioned in section 2.1.3, that is in spacetime with a background metric $G_{\mu\nu}$, and a dilaton field ϕ. We wish to compactify one coordinate, that is we want to carry out Kaluza-Klein compactification from $d+1$ dimensions to d dimensions. The Kaluza-Klein compactification ansatz for this situation is, in matrix form

$$G_{(d+1)\hat{\mu}\hat{\nu}} = \begin{pmatrix} G_{d\,\mu\nu} + \mathrm{e}^{2\sigma}\,\Lambda^{(1)}_{(d)G\,\mu}\Lambda^{(1)}_{(d)G\,\nu} & \mathrm{e}^{2\sigma}\,\Lambda^{(1)}_{(d)G\,\mu} \\ \mathrm{e}^{2\sigma}\,\Lambda^{(1)}_{(d)G\,\nu} & \mathrm{e}^{2\sigma} \end{pmatrix} \tag{2.2.55}$$

where we have written $G_{(d+1)\,x^{(d+1)}x^{(d+1)}}$ as the exponential of a field, labelled σ, which transforms as a scalar under under the d-dimensional Lorentz group. The field $\Lambda^{(1)}_{(d)G\,\nu}$ transforms as a vector in d dimensions. In this case the metric can be simply written in line-element form as

$$\begin{aligned} ds^2_{(d+1)} &= G_{(d)\,\mu\nu}dx^\mu dx^\nu + \mathrm{e}^{2\sigma}\left(dx^6 + \Lambda^{(1)}_{(d)G\,\mu}dx^\mu\right)^2 \\ &= \left(G_{(d)\,\mu\nu} + \mathrm{e}^{2\sigma}\,\Lambda^{(1)}_{(d)G\,\mu}\Lambda^{(1)}_{(d)G\,\nu}\right)dx^\mu dx^\nu \\ &\qquad + \mathrm{e}^{2\sigma}\left(dx^6\right)^2 + 2\,\mathrm{e}^{2\sigma}\,\Lambda^{(1)}_{(d)G\,\mu}dx^\mu dx^6. \end{aligned} \tag{2.2.56}$$

Here it is thus very clear that in d dimensions the $(d+1)$-dimensional metric appears as a d-dimensional metric plus a vector field, denoted $\Lambda^{(1)}_G$, and a scalar. As we have mentioned, there is also a scalar field called the dilaton. Under such a compactification it is transformed as

$$\phi_{(d+1)} = \phi_d - \sigma. \tag{2.2.57}$$

It is also clear that exactly the same sort of procedure takes place for the other fields in the action. For example, in the case of the antisymmetric tensor B, one has the ansatz

$$\begin{aligned} B_{(d+1)\,\mu\nu} &= \frac{1}{2}\left[B_{(d)\,\mu\nu} - \frac{1}{2}\left(\Lambda^{(1)}_{(d)G\,\mu}\Lambda^{(1)}_{(d)B\,\nu} - \Lambda^{(1)}_{(d)B\,\mu}\Lambda^{(1)}_{(d)G\,\nu}\right)\right]dx^\mu \wedge dx^\nu \\ &\quad + \Lambda^{(1)}_{(d)B\,\mu}dx^\mu \wedge dy, \end{aligned} \tag{2.2.58}$$

where here the vector field coming from the compactification of B_6 are denoted $\Lambda_B^{(1)}$. Note, however that the ansatz for B_6 also involves the gauge fields $\Lambda_G^{(1)}$ coming from the compactification of the metric.

Let us now briefly generalize this idea to compactification of multiple coordinates. What does the $D = 10$ metric look like in, say, $D = 6$? The answer is the following

$$G_{10\,\hat{\mu}\hat{\nu}} = \begin{pmatrix} G_{6\,\mu\nu} + \tilde{G}_{\tilde{\mu}\tilde{\nu}}\Lambda^{(1)\,\tilde{\mu}}_{6G\mu}\Lambda^{(1)\,\tilde{\nu}}_{6G\nu} & \tilde{G}_{\tilde{\mu}\tilde{\nu}}\Lambda^{(1)\,\tilde{\mu}}_{6G\mu} \\ \tilde{G}_{\tilde{\mu}\tilde{\nu}}\Lambda^{(1)\,\tilde{\nu}}_{6G\nu} & \tilde{G}_{\tilde{\mu}\tilde{\nu}} \end{pmatrix} \tag{2.2.59}$$

where G_{10} is the ten-dimensional metric, G_6 is the six-dimensional metric, the fields $\Lambda_{6G}^{(1)}$ are $U(1)$ six-dimensional gauge fields, ϕ_6 is the dilaton[13] in six dimensions. $\tilde{G}$ is a collection of ten six-dimensional scalars, also called moduli in the literature. Note here that the hatted indices run from 0 to 9, the indices topped with a $\tilde{}$ run from 6 to 9 and indices μ, ν by themselves run from 0 to 5. From this we see that the ten-dimensional metric appears in six dimensions as a metric, four vector fields and ten scalars.

It is common in the literature to define $e^{2\sigma} = \det \tilde{G}$ which is a measure of the "size" of the compact space. Also note that the six-dimensional dilaton and that in ten dimensions are related by the formula

$$\phi_6 = \phi_{10} - \frac{1}{2}\log\det \tilde{G}_{\tilde{\mu}\tilde{\nu}}. \tag{2.2.60}$$

There is also the necessity of rescaling the Newton constant depending on the dimension in which one is working, so as to maintain a dimensionless action. If we define a constant κ which is related to the familiar Newton constant G_N through $\kappa^2 = 8\pi G_N$, then the six-dimensional and the ten-dimensional constants κ are related by

$$\kappa_6 = \frac{\kappa}{\sqrt{V_4}} \tag{2.2.61}$$

where V_4 is the volume of the four coordinates that were compactified, that is $V_4 = \int d^4x\sqrt{-\det \tilde{G}}$.

It should be apparent that compactification on the world sheet as discussed in the previous section, and the Kaluza-Klein method are, after all, equivalent. We can

[13] We will, for the most part in this work, follow the convention that the dimensionality of an object is given as a subscript when there is any chance of confusion. See appendix A.

easily imagine constructing string theory from the beginning on a manifold with a number of compact dimensions, in which case the spacetime description of the theory would be identical to that which would be produced by the Kaluza-Klein procedure on the same manifold. There is thus no essential difference between them. Compactification on the world sheet is convenient for introducing nonabelian symmetries into a string theory. Kaluza-Klein is appropriate for working with the spacetime description of a string theory. Let us now move on to discuss these spacetime descriptions of superstring theories and how they are obtained.

2.3. The four superstring theories and their low energy limits

In the previous sections we have given an overview of the basics of string theory. We now wish to present the four consistent superstring theories in a systematic fashion, as well as write down spacetime actions which represent the low-energy limits of the bosonic sector of these theories, since it is with these low energy spacetime descriptions that we will be most concerned in this work. First, however, we describe the methods used to arrive at the low-energy spacetime actions that we will be writing down.

2.3.1. Equations of motion for the spacetime fields

Let us begin with the world sheet action in conformal gauge (2.1.24) in which we will replace the flat-space background $\eta_{\mu\nu}$ with general background $G_{\mu\nu}$. This gives us the action

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\, \partial_a X^\mu \partial^a X^\nu G_{\mu\nu}. \tag{2.3.1}$$

It turns out that there is no way to regularize the theory described by (2.3.1) without breaking the world sheet scale or conformal invariance. For example, it is clear that the use of Pauli-Villars regularization, with the introduction of massive regulator fields, will violate the scale invariance. Dimensional regularization methods turn out not to help since the non-linear sigma model (2.3.1) is only scale invariant in precisely two (world sheet) dimensions.

The breakdown of scale invariance in a quantum field theory is characterized in terms of what is known as the β-function. In different ways which depend on the model under study, and the way in which the β-function is defined, a non-vanishing β-function is created by the appearance of ultraviolet divergences in Feynman diagrams. In string theory the problem is slightly different. The question is still of divergences but the appearance of these divergences is tightly linked to whether or not the quantum field theory defined by the action (2.3.1) is Weyl invariant on a curved world sheet. Weyl invariance implies global scale invariance which in its turn implies a vanishing β-function and therefore ultraviolet finiteness [48]. If the β-function is computed, and is non-zero, then setting it to zero results in a set of constraints which must hold for the quantum theory to be Weyl invariant.

Here we sketch the computation of the β-functions. We make an expansion of the action in powers of α' by writing the coordinate fields as quantum fluctuations around some vacuum expectation value, $X^\mu(\sigma,\tau) = X_0^\mu + \hat{X}^\mu(\sigma,\tau)$. At such a point in the spacetime, the metric can be expanded in what are known as *Riemann normal coordinates* [4] as

$$G_{\mu\nu}(X_0) = \eta_{\mu\nu} - \frac{1}{3}R_{\mu\lambda\nu\rho}\hat{X}^\lambda\hat{X}^\rho - \frac{1}{6}\nabla_\kappa R_{\mu\lambda\nu\rho}\hat{X}^\kappa\hat{X}^\lambda\hat{X}^\rho + O\left(\hat{X}\right)^4 \qquad (2.3.2)$$

where $R_{\mu\lambda\nu\rho}$ the Riemann tensor at the point X_0. We then choose a reparametrization of the world sheet such that $e^{ab} = \mathrm{e}^{-2\varphi}\,\eta^{ab}$. From this $e_{ab} = \mathrm{e}^{2\varphi}\,\eta_{ab}$, and therefore, since we are using dimensional regularization in $2+\epsilon$ dimensions, we will have $\sqrt{-e} = \mathrm{e}^{(2+\epsilon)\varphi}$.

Substituting our expansion (2.3.2) and the expansion $\mathrm{e}^{\epsilon\varphi} = 1 + \epsilon\varphi + \ldots$ (2.3.1) becomes

$$\begin{aligned}\hat{S} = \frac{1}{4\pi\alpha'}\int d^{2+\epsilon}\sigma\Big\{ &\left(\partial_a\hat{X}^\mu\partial^a\hat{X}^\nu\right)(1+\epsilon\varphi)\,\eta_{\mu\nu} \\ &- \frac{1}{3}R_{\mu\lambda\nu\rho}\hat{X}^\lambda\hat{X}^\rho\partial_a\hat{X}^\mu\partial^a\hat{X}^\nu(1+\epsilon\varphi) + O\left(\hat{X}\right)^5\Big\}.\end{aligned} \qquad (2.3.3)$$

Now, the Feynman diagrams which contribute to the β-function at the one-loop level are given in Fig. 2.3.1. Diagram *(a)* of Fig. 2.3.1 is obtained simply making the contraction $\langle\hat{X}^\lambda\hat{X}^\rho\rangle$ with two of the $\hat{X}$'s that appear in the quartic term of (2.3.3) whereas the diagram *(b)* comes from the insertion of a kinetic term $\epsilon\varphi\partial_a\hat{X}^\mu\partial^a\hat{X}^\nu$

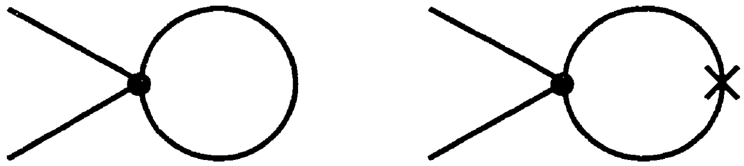


Figure 2.3.1: The diagrams which contribute to the β-function at one loop. The cross in *(b)* represents insertion of a kinetic term of coefficient $\epsilon\varphi$.

in with the quartic term. We have that

$$\begin{aligned}\left\langle \hat{X}^{\lambda}(\sigma)\,\hat{X}^{\rho}(\sigma')\right\rangle &= \pi\eta^{\lambda\rho}\lim_{\sigma\to\sigma'}\int\frac{d^{2+\epsilon}k}{(2\pi)^{2+\epsilon}}\frac{e^{ik(\sigma-\sigma')}}{k^2}\\ &\sim\frac{\eta^{\lambda\rho}}{2\epsilon}\end{aligned} \tag{2.3.4}$$

since it is logarithmically divergent[14] and thus the factor of ϵ in the denominator of (2.3.4) cancels the factor of ϵ in the numerator, resulting in $\hat{S}$ having φ dependence. It turns out that the dependence on φ of the sum of the two diagrams in Fig. 2.3.1 vanishes.

However, there are additional diagrams like that of Fig. 2.3.1 *(b)* with $\hat{X}^{\mu}\partial_a\hat{X}^{\nu}$ and $\partial_a\hat{X}^{\mu}\partial^a\hat{X}^{\nu}$ on the external legs that are proportional to $\partial_a\varphi$. We can integrate these diagrams by parts and drop terms proportional to $\partial_a\partial^a\hat{X}^{\mu}$ which vanish by the equations of motion, and what is left is a net φ dependence which can be absorbed into the wavefunction and spacetime metric renormalizations [49]

$$\begin{aligned}\hat{X}^{\mu} &\to \hat{X}^{\mu}+\frac{1}{6\epsilon}R^{\mu}_{\nu}(X_0)\,\hat{X}^{\nu}+O\left(\hat{X}^2\right)\\ G_{\mu\nu} &\to G_{\mu\nu}-\frac{1}{2\epsilon}R_{\mu\nu}(X_0)\end{aligned} \tag{2.3.5}$$

where $R_{\mu\nu}$ is the Ricci tensor ($R_{\mu\nu}=g^{\lambda\rho}R_{\lambda\mu\rho\nu}$). These renormalizations are an important feature of non-linear sigma models. This absorption, however, causes in turn a reappearance of an effective action which has dependence on φ, due to cancellation of ϵ, given by

$$S_{\beta}=\frac{1}{8\pi\alpha'}\int d^2\sigma\,\varphi R_{\mu\nu}(X)\,\partial_a X^{\mu}\partial^a X^{\nu} \tag{2.3.6}$$

---

[14] In dimensional regularization at one loop divergences manifest themselves as simple poles. Also, the contraction $\langle\partial_a\hat{X}^{\mu}\partial^a\hat{X}^{\nu}\rangle$ is a quadratically divergent massless tadpole and is discarded in this technique.

where we write $X = X_0 + \hat{X}(\sigma, \tau)$. Thus, to one loop order in α' (2.3.1) produces a Weyl invariant theory, that is to say a theory independent of the scaling parameter φ, if and only if

$$R_{\mu\nu}(X) = 0 \tag{2.3.7}$$

which are precisely the vacuum Einstein equations. Thus by demanding Weyl invariance, an intrinsic property of the string, be maintained in a curved spacetime we arrive at a well-known dynamical equation for the spacetime metric. We can then write our β-functional as

$$\beta_{\mu\nu}(X) = -\frac{1}{2\pi} R_{\mu\nu}(X)\,. \tag{2.3.8}$$

The condition for the vanishing of the β-function must coincide with the equations of motion, if we are to give them a sensible physical interpretation. Thus we see that here the β-function represents a long-wavelength or low-energy approximation to the equation of motion for the gravitational field.

To be sure, (2.3.1) is not the most general action for which the β-functional can be computed. There are in fact two other terms which can be added to the Lagrangian which are invariant under reparametrizations of the world sheet and which are renormalizable by power counting. These are:

$$\begin{aligned} S_1 &= -\frac{1}{4\pi\alpha'} \int d^2\sigma\, \epsilon^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu} \\ S_2 &= \frac{1}{4\pi} \int d^2\sigma \sqrt{-e}\, \phi \mathcal{R} \end{aligned} \tag{2.3.9}$$

where $B_{\mu\nu}$ is the antisymmetric Kalb-Ramond field, ϵ^{ab} is the Levi-Civita antisymmetric tensor on the world sheet, ϕ is the dilaton field and $\mathcal{R}$ is the Ricci scalar on the world sheet. The origin of $B_{\mu\nu}$ and ϕ in terms of the string spectrum was discussed in section 2.1.3. Note also that S_2 is almost the topological invariant of the world sheet which we met in (2.1.20), but now we take advantage of the fact that in two dimensions scalar fields are dimensionless to generalize it to include the dilaton field ϕ explicitly as we are considering world sheet metrics that are not necessarily flat. However, we note that it comes in with a different power of α' and thus only contributes to the β-function at a higher-loop order than S or S_1.

Carrying out an analysis similar to, but of course more involved than that sketched above since we compare tree-level terms of S_2 with one-loop terms of $S + S_1$, leads us to the equations of motion for the spacetime degrees of freedom:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{4} H_\mu{}^{\rho\sigma} H_{\nu\rho\sigma} + 2\nabla_\mu \nabla_\nu \phi &= 0 \\ \nabla_\lambda H^\lambda{}_{\mu\nu} - 2\,(\nabla_\lambda \phi)\, H^\lambda{}_{\mu\nu} &= 0 \\ 4(\nabla_\mu \phi)^2 + 4\nabla_\mu \nabla^\mu \phi + R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} &= 0 \end{aligned} \tag{2.3.10}$$

where $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\rho B_{\mu\nu} + \partial_\nu B_{\rho\mu}$ or in the language of forms [50] $H = dB$.

We are then led, on the basis of the equations of motion (2.3.10) to formulate the action in spacetime as

$$S_{26} = \frac{1}{2\kappa^2} \int d^{26}x \sqrt{-G}\, \mathrm{e}^{-2\phi} \left(R + 4(\nabla\phi)^2 - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right) \tag{2.3.11}$$

where $\kappa = \sqrt{8\pi G_N}$ contains the 26-dimensional Newton constant. This action can be verified to reproduce the equations of motion (2.3.10). Equation (2.3.11) then describes the low-energy or long-wavelength limit of the massless degrees of freedom of the bosonic string.

It is worth noting here that the explicit presence of the dilaton in (2.3.11) gives this action a different appearance from the standard action for Einstein gravity coupled to various matter fields [51], which in D dimensions is

$$S = \frac{1}{16\pi G_N} \int d^D x \sqrt{-g} (R + \mathscr{L}_{\text{matter}}). \tag{2.3.12}$$

If we rescale the metric $G_{\mu\nu}$ in (2.3.11) through the relation[15]

$$g_{\mu\nu} = \mathrm{e}^{-\phi/6}\, G_{\mu\nu}, \tag{2.3.13}$$

then we obtain the *Einstein-frame metric*, $g_{\mu\nu}$, so named because in terms of this metric the action appears as

$$S_{26} = \frac{1}{2\kappa^2} \int d^{26}x \sqrt{-g} \left(R - \frac{1}{6} (\nabla\phi)^2 - \frac{\mathrm{e}^{-\phi/3}}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right) \tag{2.3.14}$$

vhere the Ricci scalar term is similar to that in (2.3.12).

[15] The information in appendix B is useful here.

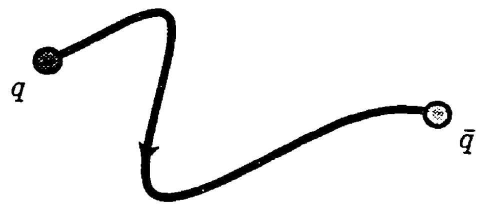


Figure 2.3.2: A schematic of the Chan-Paton method with charges q and $\bar{q}$ transforming under a symmetry group at the ends of the open string.

This completes our sketch of the methods for obtaining the low-energy equations of motion and the corresponding spacetime action from the world sheet by requiring the β-function to vanish at the one-loop level. It is clear that it is possible to continue to higher loops, or to obtain the corresponding equations for the massive modes of the string as well [52], but that is another story. We now move on to describe the four known consistent string theories and their low energy effective actions in spacetime, which turn out to be supergravity actions.

2.3.2. The Type I superstring

The superstring theory which is based on open strings is known as Type I string theory due to the fact that it has one supersymmetry in ten dimensions. The reason for this can be stated intuitively as follows. For open strings the left- and right-moving modes, as we have seen, are not independent as they are in the closed-string case, but rather combine into standing waves, which places an additional constraint on the spectrum of states and breaks half of the supersymmetry.

It is possible in the case of the open superstring theory to add a Yang-Mills gauge field sector to the theory through the addition of charges transforming under some internal symmetry group to the ends of the string. This procedure is known as the Chan-Paton method [53] and is shown schematically in Fig. 2.3.2. The symmetry groups that one can add to the open string in this manner are quite varied at the classical level, but when constraints arising from quantum anomalies are taken into account, it is found that only the group $SO(32)$ is possible [54]. In this case the charges q and $\bar{q}$ of Fig. 2.3.2 lie in a real representation, and the string is said to be *unoriented.*[16] There is thus one unique consistent Type I superstring theory.

The low-energy effective action of the Type I theory, commonly referred to as the Type I supergravity action, contains massless fields from both the unoriented closed and open string sectors. It is necessary to include unoriented closed strings since the Chan-Paton charges can combine to form Yang-Mills singlets. Thus the type I supergravity action contains a metric $\mathbb{G}$ and a dilaton $\phi^{(i)}$ from the closed-string NS-NS sector; an antisymmetric tensor $\mathbb{A}^{(2)}$ from the closed string R-R sector, and from the open string NS sector we have a set of $SO(32)$ gauge fields $\mathbb{A}^{(1)}$. The action is written as

$$\begin{aligned} S_I = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-\mathbb{G}}\Big\{ & e^{-2\phi^{(i)}}\left(R + 4(\nabla\phi^{(i)})^2\right) - \frac{1}{12}\left(\mathbb{F}^{(3)}\right)^2 \\ & - \frac{e^{-\phi^{(i)}}}{4}\,\mathrm{Tr}\left(\mathbb{F}^{(2)}\right)^2\Big\} \end{aligned} \tag{2.3.15}$$

where R is the Ricci scalar, $\mathbb{F}^{(3)} = d\mathbb{A}^{(2)}$ is the field strength of the Ramond-Ramond two-form potential. Note that $\mathbb{G}$ is the string-frame metric. The two-form field strength of the open string gauge fields $\mathbb{F}^{(2)}$ is defined as $\mathbb{F}^{(2)} = d\mathbb{A}^{(1)} + \mathbb{A}^{(1)} \wedge \mathbb{A}^{(1)}$.

2.3.3. The Type II superstring

The Type II superstring theories are based on closed strings only. Since open strings must be unoriented, removing them allows us to work with oriented closed strings. There are two Type II superstring theories, both of which have $N = 2$ supersymmetry in spacetime, hence the name "Type II".

In our discussion of the GSO projection and spacetime supersymmetry in section 2.2.2.1 we defined the GSO projection operators

$$P = \frac{1 + (-1)^{F+1}}{2} \qquad F = \sum_{r=\frac{1}{2}}^{\infty} \eta^{\mu}_{-r}\eta^{\mu}_{r} \qquad \text{NS sector} \tag{2.3.16a}$$

$$P = \frac{1 + k\,(-1)^{F+1}}{2} \qquad F = \sum_{n=1}^{\infty} \rho^{\mu}_{-n}\rho^{\mu}_{n} \qquad \text{R sector} \tag{2.3.16b}$$

[16] An *oriented* string is not invariant under world sheet parity, $\sigma \to \pi - \sigma$ (for the open string), which exchanges the ends of the string, but an unoriented string is invariant under this transformation.

where for left-moving fields F in (2.3.16b) is replaced by

$$F = \sum_{r=\frac{1}{2}}^{\infty} \tilde{\eta}^{\mu}_{-r}\tilde{\eta}^{\mu}_{r} \qquad \text{or} \qquad F = \sum_{n=1}^{\infty} \tilde{\rho}^{\mu}_{-n}\tilde{\rho}^{\mu}_{n}, \tag{2.3.17}$$

and where k can be chosen to be either $+1$ or -1 independently for the right-moving and left-moving fields, i.e., the GSO projection is carried out separately on right- and left-moving states. We also briefly noted that for type IIA theories, k was chosen to have different signs for the right- and left-moving fields, whereas for the type IIB theory k was chosen to be the same for both directions.

One can distinquish more clearly between the states of the type IIA and the type IIB theories by considering the zero-mode contribution to $(-1)^F$. Recall from section 2.2.2 that the ground state for the Ramond sector is degenerate, forming a spinor representation of $SO(1, D-1)$. After negative-norm states have been removed, using methods such as covariant [9] or lightcone gauge quantization [9,25], in ten dimensions the Ramond sector zero modes will form a massless $SO(8)$ spinor.

Let us focus for the moment on the right-moving Ramond ground state. The 16 independent components of this ground state can be chosen to be

$$|s_\alpha\rangle = \prod_{\alpha=1}^{4} \left(f^\dagger_\alpha\right)^{s_\alpha} |0\rangle_R \qquad s_\alpha = 0 \text{ or } 1 \tag{2.3.18}$$

where $f_\alpha = \rho_0^{2\alpha-1} + i\rho_0^{2\alpha}$. If we introduce the chirality operator for $SO(8)$

$$\chi = \prod_{i=1}^{8} \gamma^i \tag{2.3.19}$$

then the ground state spinor splits into two spinors, one each of positive and negative chirality.

To see this, notice first that

$$\sum_{\alpha=1}^{4} f^\dagger_\alpha f_\alpha = \sum_{i=1}^{8} \rho^i_0 \rho^i_0. \tag{2.3.20}$$

Then it may be shown that $f^\dagger_\alpha$ anticommutes with χ, and therefore that $|0\rangle_R$ and $f^\dagger_\alpha\, |0\rangle_R$ have opposite chirality. In addition

$$[\chi, f^\dagger_\alpha] = 2if^\dagger_\alpha \tag{2.3.21}$$

from whence it follows that the chirality of $|0\rangle_R$ and $f_\alpha^\dagger|0\rangle_R$ can be chosen to be $(-i)^4 = 1$ and $i(-i)^3 = -1$ respectively. As a result of all this, our state $|s_\alpha\rangle_R$ has the chirality

$$\chi = (-1)^{\sum_\alpha s_\alpha} = (-1)^{\rho_0^i \rho_0^i} \tag{2.3.22}$$

which means that $(-1)^F$ of the zero modes gives the chirality of the $SO(8)$ spinor ground state. Thus, since k is chosen to be opposite for right- and left-movers in the case of the type IIA theory, consequently the right- and left-moving ground state spinors have opposite chirality. In the type IIB case the right- and left-moving ground states have the same chirality since k is the same for both sets of fields.

Another characteristic of the type II string theories is that they do not have the freedom to introduce a Yang-Mills gauge group. The reason is simple. With closed strings there are no free ends on which one can attach the charges, as in the Chan-Paton method of the type I string. No charges transforming under a symmetry group in turn implies no Yang-Mills gauge symmetry.

2.3.3.1. Type IIA superstring

If we consider that right-moving and left-moving fields have opposite chirality, then the theory has two conserved supersymmetries of opposite handedness. This theory is called Type IIA, and it is left-right symmetric, or non-chiral as a result of the choice of opposite chirality for the right-moving and left-moving fields as was discussed in the last section.

The low-energy type IIA supergravity effective action for the massless bosonic states of the string spectrum is given by [55]

$$\begin{aligned} S_{IIA} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}\Big\{ & e^{-2\phi^{(a)}}\left(R + 4(\nabla\phi^{(a)})^2 - \frac{1}{12}\left(H^{(a)}\right)^2\right) \\ & -\frac{1}{4}\left(F^{(2)}\right)^2 - \frac{1}{48}\left(F^{(4)}\right)^2\Big\} - \frac{1}{4\kappa^2}\int B^{(a)} dA^{(3)} dA^{(3)} \end{aligned} \tag{2.3.23}$$

where $G_{\mu\nu}$ is the string frame metric, $H^{(a)} = dB^{(a)}$ is the field strength of the Kalb-Ramond field, $F^{(2)} = dA^{(1)}$ and $F^{(4)} = dA^{(3)} - H^{(a)}A^{(1)}$ are the Ramond-Ramond field strengths of the one-form potential $A^{(1)}$ and the three-form potential $A^{(3)}$ respectively, and finally $\phi^{(a)}$ is the dilaton. Assuming that the dilaton vanishes asymptotically, Newton's constant is given by $\kappa^2 = 8\pi G_N$.[17] The reader is reminded that in 10 dimensions the Einstein metric $g_{\mu\nu}$ is obtained from the string frame metric $G_{\mu\nu}$ through the equation[18]

$$g_{\mu\nu} = \mathrm{e}^{-\phi^{(a)}/2}\, G_{\mu\nu}. \tag{2.3.24}$$

Of special note here is the field content of the Ramond-Ramond sector, that is the gauge potentials $A^{(1)}$ and $A^{(3)}$. Of course, the Hodge duals of the field strengths of these potentials, defined in D-dimensions by

$$\begin{aligned} \tilde{F}^{(D-n)} &= {}^{*}F^{(n)} \\ F^{(D-n)}_{\mu_1\cdots\mu_{D-n}} &= \sqrt{-G}\,\epsilon_{\mu_1\cdots\mu_{D-n}\mu_{D-n+1}\cdots\mu_D} F^{(n)\,\mu_{D-n+1}\cdots\mu_D} \end{aligned} \tag{2.3.25}$$

where $\epsilon_{\mu_1\cdots\mu_D}$ is the Levi-Civita totally antisymmetric tensor density, will represent an alternate way to describe the physical content of the Ramond-Ramond potentials in the Type IIA supergravity. Also worth noting is that we could add to (2.3.23) a scalar (0-form) field strength which would represent a non-propagating instanton field. However, since we will not need it, we refrain from writing this field explicitly.

2.3.3.2. Type IIB superstring

The other possibility, as we have seen, for a closed superstring theory is to use the same chirality for the right- and left-moving fields. It is in this case permissible to symmetrize the left- and right-moving modes or one can choose not to do so. If one decides to carry out such a symmetrization, one arrives at the unoriented closed-string sector of the Type I string and one is subsequently forced by quantum consistency conditions to include $SO(32)$ open strings in order to obtain a consistent superstring theory, one which is already known.

On the other hand, if one makes no demands of symmetrization, one has a theory of oriented closed strings which has two supersymmetries of the same chirality. This is the type IIB superstring theory, which is left-right asymmetric or chiral.

---

[17] In the case that the dilaton tends to the value $\phi_0^{(a)}$ at asymptotic infinity, we have $\kappa^2 \to \kappa^2\,\mathrm{e}^{-2\phi_0^{(a)}}$.
[18] For D dimensions the transformation is $g_{\mu\nu} = \mathrm{e}^{-4\phi^{(a)}/(D-2)}\, G_{\mu\nu}$.

The Type IIB supergravity action is given by [56]

$$S_{IIB} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-J}\Big\{ e^{-2\phi^{(b)}}\left(R + 4(\nabla\phi^{(b)})^2 - \frac{1}{12}\left(H^{(b)}\right)^2\right) - \frac{1}{2}(\nabla\chi)^2 - \frac{1}{12}\left(F^{(3)} + \chi H^{(b)}\right)^2 - \frac{1}{480}\left(F^{(5)}\right)^2\Big\} + \frac{1}{4\kappa^2}\int A^{(4)}F^{(3)}H^{(b)} \tag{2.3.26}$$

where $J_{\mu\nu}$ is the string-frame metric[19] , $H^{(b)} = dB^{(b)}$ is the field strength of the Kalb-Ramond field, $F^{(3)} = dA^{(2)}$ and $F^{(5)} = dA^{(4)} - \frac{1}{2}(B^{(b)}F^{(3)} - A^{(2)}H^{(b)})$ are RR field strengths, while $\chi = A^{(0)}$ is the RR scalar, and $\phi^{(b)}$ is the dilaton.

It is to be noted here that strictly speaking there is no covariant action we can write down for $A^{(4)}$. The Hodge dual of a five-form field strength in ten dimensions is again a five-form field strength. The kinetic term of the five-form field strength in (2.3.26) describes both a self-dual ($F^{(5)} = {}^*F^{(5)}$) and an anti-self-dual ($F^{(5)} = -{}^*F^{(5)}$) field strength and there is no simple way of modifying this action such that the physical degrees of freedom correspond to the self-dual part and at the same time the anti-self-dual part vanishes. Thus to eliminate the anti-self-dual part of $F^{(5)}$ we will impose the constraint $F^{(5)} = {}^*F^{(5)}$ "by hand" at the level of the equations of motion [57].

Note also here that the Ramond-Ramond field content here is distinct from that of equation (2.3.23).

2.3.4. The heterotic superstring

There exists still another possibility. Since for closed string theories the left- and right moving modes are not coupled, it is possible to imagine handling each set of modes in a different way [45]. This is exactly what is done in the case of the heterotic string. Simply put, the right-moving modes are handled in a supersymmetric fashion, that is one introduces right-moving superfields on the world sheet, while the left-moving modes remain strictly bosonic. However this introduces a problem in that the critical dimension of the right-moving modes is now 10 rather than the 26 of the bosonic string.

---

19 The string frame metric is again related in ten dimensions to the Type IIB Einstein frame metric through $j_{\mu\nu} = e^{-\phi^{(b)}/2} J_{\mu\nu}$.

This means that there are only 10 right-moving X_r^μ to pair with 10 of the 26 left-moving X_l^ν, and in order to form a true spacetime coordinate one must have both X_r^μ and X_l^μ. This means that there are 16 extra left-moving bosonic fields, which cannot be simply removed in order to maintain a vanishing Virasoro anomaly or conformal central charge. We can, however take advantage of the procedures explained in section 2.2.3.1 to add a current algebra to the world sheet, in conjunction with toroidal compactification.

As previously mentioned, this produces a string theory with critical dimension 10, which at the same time contains a Yang-Mills gauge group with $SO(32)$ or $E_8 \otimes E_8$ symmetry. The heterotic string thus has, in a consistent closed string theory, both fermions from the spacetime supersymmetry of the right-moving modes as well as a Yang-Mills gauge group from the left moving modes. The heterotic string is often thought to be the superstring theory with the most phenomenological relevance.

The bosonic part of the low-energy heterotic supergravity action can be written as

$$S_{het} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-\mathcal{G}}\,\mathrm{e}^{-2\phi^{(h)}}\left(R + 4(\nabla\phi^{(h)})^2 - \frac{1}{12}\left(H^{(h)}\right)^2 - \frac{1}{4}\mathrm{Tr}\left(\mathcal{F}^{(2)}\right)^2\right) \tag{2.3.27}$$

where $\mathcal{G}$ is the heterotic string-frame metric, $\phi^{(h)}$ the dilaton as usual and $\mathcal{F}^{(2)} = d\mathcal{A}^{(1)}$ is the field strength of the Yang-Mills one-form $\mathcal{A}^{(1)}$ which takes values in the Lie Algebra of $SO(32)$ or $E_8 \otimes E_8$ depending on the case at hand [32].

We end this section by stating that this work will be primarily concerned with properties and solutions of the Type II string theories, although we will have also use for the heterotic string as we will see in the next chapter. We have included a short description of the Type I string mainly in an attempt to give a more complete overview of the basics of string theory. Those who find the choice of symbols non-standard should be reminded that effort is being expended to develop a clear notational system, and that further information on notation and conventions can be found in appendix A.

2.3.5. A compactified action

In this section we will briefly put together some of the ideas that we have introduced in the previous sections, namely compactification according to the Kaluza-Klein procedure, Hodge dualization, and the low energy effective action of a superstring theory, in order to see how these things might work together.

Let us take as an example the low energy effective supergravity action in 10 dimensions of the type IIA superstring, given in (2.3.23) and repeated here as

$$\begin{aligned} S_{IIA} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}\Big\{ & e^{-2\phi^{(a)}}\left(R + 4(\nabla\phi^{(a)})^2 - \frac{1}{12}\left(H^{(a)}\right)^2\right) \\ & - \frac{1}{4}\left(F^{(2)}\right)^2 - \frac{1}{48}\left(F^{(4)}\right)^2\Big\} - \frac{1}{4\kappa^2}\int B^{(a)} dA^{(3)} dA^{(3)} \end{aligned} \tag{2.3.28}$$

where to remind the reader $G_{\mu\nu}$ is the string frame metric, $H^{(a)} = dB^{(a)}$ is the field strength of the Kalb-Ramond field, $F^{(2)} = dA^{(1)}$ and $F^{(4)} = dA^{(3)} - H^{(a)}A^{(1)}$ are the Ramond-Ramond field strengths, and $\phi^{(a)}$ is the dilaton. Recall also that Newton's constant is given by $\kappa^2 = 8\pi G_N$.

If we apply the Kaluza-Klein procedure to this action, choosing the four-dimensional manifold T^4, the four-torus, we have the result

$$\begin{aligned} S_{6\,IIA} = \frac{1}{2\kappa_6^2}\int d^6x\sqrt{-G_6}\Big\{ & e^{-2\phi_6^{(a)}}\left(R_6 + 4(\nabla\phi_6^{(a)})^2 - (\nabla\sigma)^2 - \frac{1}{12}\left(H_6^{(a)}\right)^2\right) \\ & - \frac{e^{2\sigma}}{4}\left(F_6^{(2)}\right)^2 - \frac{e^{-2\sigma}}{4}\left(F_6^{(4)}\right)^2\Big\} - \frac{1}{8\kappa_6^2}\int B_6^{(a)} F_6^{(2)} F_6^{(4)} \end{aligned} \tag{2.3.29}$$

where

$$\kappa_6 = \kappa/\sqrt{V_{T^4}} \tag{2.3.30}$$

allows us to compute the six-dimensional Newton's constant and V_{T^4}, the volume of the compact manifold T^4 and where we note the appearance of the additional scalar field σ as a result of the compactification. It is important to note that this is a truncated action, i.e., we are setting many scalar and vector fields which arise as a result of the compactification procedure to zero, namely the components of the fields in (2.3.28) in the compact directions.[20]

---

[20] A complete compactification of the type IIA low energy effective action will be presented in chapter III.

This action can be made easier to work with if we use Hodge duality in six dimensions. We take the Hodge dual of the four-form field strength $F^{(4)}$ as

$$\tilde{F}^{(2)}_{6\,\mu\nu} = \sqrt{-G_6}\,\epsilon_{\mu\nu\rho\sigma\gamma\delta}F_6^{(4)\rho\sigma\gamma\delta} \tag{2.3.31}$$

and with this replacement for $F_6^{(4)}$ the action appears as

$$S_{6\,IIA} = \frac{1}{2\kappa_6^2}\int d^6x\sqrt{-G_6}\Big\{ \mathrm{e}^{-2\phi_6^{(a)}}\left(R_6 + 4(\nabla\phi_6^{(a)})^2 - (\nabla\sigma)^2 - \frac{1}{12}\left(H_6^{(a)}\right)^2\right) - \frac{\mathrm{e}^{2\sigma}}{4}\left(F_6^{(2)}\right)^2 - \frac{\mathrm{e}^{-2\sigma}}{4}\left(\tilde{F}_6^{(2)}\right)^2\Big\} - \frac{1}{8\kappa_6^2}\int B_6^{(a)}F_6^{(2)}\tilde{F}_6^{(2)}. \tag{2.3.32}$$

This action is easier to deal with, since we have two two-form field strengths coupling in different ways, and therefore two one-form gauge potentials, rather than one one-form and one three-form potential. This gives us a small example of how the ideas of compactification and Hodge duality work for the low energy actions we will be considering later in this work.

There is one final note to be added. Often in the literature on low-energy string theory one speaks of the "string coupling" constant, often given the symbol g. This coupling constant arises in string perturbation theory, and is related to the dilaton field as $\bar{g} = \mathrm{e}^{\phi}$. The string coupling can be related to the Newton constant G_N and the inverse string tension α' in ten dimensions as

$$\bar{g}^2 = \frac{G_N}{8\pi^6\alpha'^4}. \tag{2.3.33}$$

Of course, when compactification is carried out, G_N must be adjusted as above, by factors of the square root of the volume of the compact manifold. We will avoid using the symbol g for the string coupling from this point, due to the possibility of confusion with the determinant of the Einstein frame metric of type IIA superstring theory. We will rather use the symbol "g-bar" $\bar{g}$ as above. Thus completes chapter II, an overview of the basics of the theory of strings. It is hoped that the reader has benefited from reading it and that it has prepared him for the chapters that follow.

III

The symmetries of the theory of strings

One of the most fascinating aspects of the theory of strings is the number of symmetries [58] which can relate different regimes of a given string theory to each other, or relate one string theory to another. Recently, the latter has had a crucial role to play in advances in our understanding of the structure of string theory, for example the belief [1,2,3] that the different string theories are each a different ten-dimensional description of a more fundamental eleven-dimensional "M-theory". This new understanding has led to the development of techniques which can be used to construct new solutions of the low-energy supergravity equations of motion. This leads, more or less directly, to improvements in our comprehension of the non-perturbative regimes of string theory by allowing construction and analysis of larger families of non-perturbative solutions.

What are these symmetries? In this work we will divide these symmetries, which often in the literature are known by the name *dualities* into three main groups:

1. $O(d, d)$ symmetries are symmetries of the low-energy equations of motion which result from independence of solutions of various dimensions. In a sense we can "rotate" and/or "boost" a solution in such a way that it becomes a different solution.
2. Target-space (T-) duality. This is the symmetry we alluded to when we wrote equation (2.2.50) in which, the radius of a compact dimension is inverted, i.e., $R \to \alpha'/R$.
3. S-duality, which can be said to relate weak- and strong-coupling regimes of a string theory, or even different string theories.

In what follows we will attempt to remove at least some of the mysteries from each of these symmetries or dualities, in its turn.

3.1. General Remarks on Symmetry

In the previous chapter, we developed string theory by starting with a world-sheet action and then ending with a space-time description in terms of low-energy effective actions which turn out to be supergravity actions. It is a similar route that we will follow in the development of the symmetries of string theory, beginning with an introduction to how these symmetries manifest themselves on the world sheet before we move on to explore their space-time formulations.

Field theories in two dimensions which display conformal symmetry, whether or not they are related to the string world sheet, are known as Conformal Field Theories (CFT) [31]. There is a large literature on CFT, but it is enough for the reader to understand that world-sheet string actions are indeed CFT's. Within CFT there are certain classes of operators, sometimes denoted *truly marginal,* for which, as their coupling constants change value, the CFT continuously traces out a space known as a *moduli space* $\mathcal{M}$ which has dimension equal to the number of these truly marginal operators. Thus $\mathcal{M}$ describes an infinite collection of continuously related conformal field theories.

It sometimes occurs that one can span (at least a neighborhood of) the moduli space of the CFT by acting on the coupling constants with some continuous group which we will denote $\mathscr{G}$. In addition, one sometimes finds that there exists some subgroup $\mathscr{G}_s$ of $\mathscr{G}$ which is a physical symmetry of the CFT. An element $g \in \mathscr{G}$ transforms the CFT $\mathscr{L}_{\mathcal{M}_1}$ at some point $\mathcal{M}_1$ in the moduli space into another theory $\mathscr{L}_{\mathcal{M}_2}$ corresponding to a different point $\mathcal{M}_2$ of $\mathcal{M}$. When g is an element of $\mathscr{G}_s$, then $\mathscr{L}_{\mathcal{M}_1}$ and $\mathscr{L}_{\mathcal{M}_2}$ are physically equivalent. By extension, all CFT's in the moduli space which are related by $\mathscr{G}_s$, called the *orbit* of $\mathscr{G}_s$, are physically equivalent. It is evident that the group $\mathscr{G}$ will depend in detail on the moduli space $\mathcal{M}$.

To translate what may be a confusing foray into conformal field theories into the language of string theory, the couplings of the truly marginal operators mentioned above are usually grouped into the metric $G_{\mu\nu}(X^\sigma)$, antisymmetric tensor (Kalb-Ramond) field $B_{\mu\nu}(X^\sigma)$, and the dilaton $\phi(X^\sigma)$, with all of which we have made our acquaintance in the second chapter. There are also other fields but we will discuss fields specific to particular superstring theories later.

Our objective in discussing the symmetries of string theory is to find the groups $\mathcal{G}_s$ under which the world sheet action is physically unchanged. The case of most relevance is that case in which d dimensions have been compactified.

3.2. $O(d, d)$ symmetry of the string

A world sheet action for string theory which describes a number d of coordinates which are compactified into a d-torus T^d can be written [9]

$$S = \frac{1}{4\pi}\int d^2\sigma\{\sqrt{-e}\, e^{ab}G_{ij} + \epsilon^{ab}B_{ij}\}\partial_a X^i \partial_b X^j \tag{3.2.1}$$

where $1 \leq i, j \leq d$ and the X^i are coordinates that have been made periodic,[1] as in section 2.2.3.1, that is

$$X^i \equiv X^i + 2\pi m^i \tag{3.2.2}$$

and where we have made the metric G_{ij} and the antisymmetric tensor B_{ij} dimensionless by dividing out the string tension as in

$$G_{ij} \to \frac{G_{ij}}{\alpha'}, \qquad B_{ij} \to \frac{B_{ij}}{\alpha'}. \tag{3.2.3}$$

It is again drawn to the reader's attention that (3.2.1) describes *only* the parts of the metric and antisymmetric tensor fields which lie in the set of compactified coordinates. Note also that we are leaving out the dilaton field from this discussion.

From the conformal field theory perspective, the number of truly marginal operators for a generic d-dimensional background is d^2. In the case of (3.2.1) we have, due to the symmetry of G_{ij}, $d(d+1)/2$ operators

$$\sqrt{-e}e^{ab}\partial_a X^i \partial_b X^j \tag{3.2.4}$$

and from the antisymmetry of B_{ij} the $d(d-1)/2$ operators

$$\epsilon^{ab}\partial_a X^i \partial_b X^j \tag{3.2.5}$$

---

[1] Here we set the $R_i = 1$ for simplicity of presentation.

with their corresponding couplings G_{ij} and B_{ij}, which add together to form the necessary set of d^2 operators and their associated couplings. From the CFT point of view we have a d^2-dimensional moduli space for which the task is to find the symmetry group $\mathscr{G}$.

3.2.1. The bosonic string

We quantize the theory just as in chapter II but for the presence of the antisymmetric tensor field B_{ij}. In this case the coordinate $X^i(\sigma,\tau)$ and its associated canonical momentum P_i are given by

$$
\begin{aligned}
X^i(\tau,\sigma) &= x^i + m^i\sigma + G^{ij}(p_j - B_{jk}m^k) \\
&\quad + \frac{i}{\sqrt{2}}\sum_{n\neq 0}\frac{1}{n}\left[\alpha_n^i\,\mathrm{e}^{-in(\tau-\sigma)} + \tilde{\alpha}_n^i\,\mathrm{e}^{-in(\tau+\sigma)}\right] \\
2\pi P_i &= G_{ij}\partial_\tau X^j + B_{ij}\partial_\sigma X^j \\
&= p_i + \frac{1}{\sqrt{2}}\sum_{n\neq 0}\left[(G_{ji} - B_{ij})\alpha_n^j\,\mathrm{e}^{-in(\tau-\sigma)} + (G_{ij}+B_{ij})\tilde{\alpha}_n^j\,\mathrm{e}^{-in(\tau+\sigma)}\right]
\end{aligned} \tag{3.2.6}
$$

where p_i is the center of mass momentum and the oscillator components α^i and $\tilde{\alpha}^i$ are functions of G_{ij} and B_{ij}. Of course, since the X^i are compact as per (3.2.2) we will have p_i quantized in integer modes.

The Hamiltonian and the Virasoro constraints then take the form

$$
\begin{aligned}
\mathcal{H} &= L_0 + \tilde{L}_0 \\
&= \frac{1}{4\pi}\int_0^{2\pi} d\sigma\Big\{4\pi^2 P_i P_j G^{ij} + \partial_\sigma X^i\left(G_{ij} - B_{ik}B_{lj}G^{kl}\right)\partial_\sigma X^j \\
&\qquad + 4\pi\partial_\sigma X^i P_j B_{ik} G^{kj}\Big\} \\
&= \frac{1}{4\pi}\int_0^{2\pi} d\sigma\left(P^2 + \tilde{P}^2\right)
\end{aligned} \tag{3.2.7}
$$

where

$$
\begin{aligned}
P_a &= \left(2\pi P_i - \left(G_{ij} + B_{ij}\right)\partial_\sigma X^j\right)\bar{\varpi}_a^i \\
\tilde{P}_a &= \left(2\pi P_i + \left(G_{ij} - B_{ij}\right)\partial_\sigma X^j\right)\bar{\varpi}_a^i
\end{aligned} \tag{3.2.8}
$$

where[2] ϖ_i are a basis of the compactification lattice[3] Λ^d. That is, the process of compactification can be considered to be "division" of a non-compact manifold

---

2 ϖ and $\bar{\varpi}$ are defined by: $\sum_{a=1}^d \varpi_i^a\varpi_j^a = 2G_{ij}$, $\sum_{a=1}^d \varpi_i^a\bar{\varpi}_a^j = \delta_i{}^j$ and $\sum_{a=1}^d \bar{\varpi}_a^i\bar{\varpi}_a^j = \frac{1}{2}G^{ij}$.

3 In terms of a basis $(\varepsilon_1\cdots\varepsilon_d)$ of d-dimensional Euclidean space $\mathbb{R}^d$ a lattice is the set of all points whose expansion coefficients in the basis are all integers.

by a periodic lattice. In the case at hand, the torus in d-dimensions is obtained by dividing

$$T^d = \frac{\mathbb{R}^d}{\pi \Lambda^d} \tag{3.2.9}$$

Even in the presence of B_{ij}, P and $\tilde{P}$ decouple, describing independently right- and left-moving modes. The Hamiltonian consists of a part which describes the zero modes, along with an oscillator contribution. We will consider how the oscillator components behave under symmetry transformations later, for now we will concentrate on the zero mode part of the spectrum. Substituting (3.2.6) into (3.2.7) gives the Hamiltonian of the zero modes as

$$\begin{aligned} \mathcal{H} &= L_0 + \bar{L}_0 \\ &= \frac{1}{2}\left(p^2 + \bar{p}^2\right) \\ &= \frac{1}{2}\left(k_i k_j G^{ij} + m^i m^j \left(G_{ij} - B_{ik} B_{lj} G^{kl}\right) + 2 m^i k_j B_{ik} G^{kj}\right) \end{aligned} \tag{3.2.10}$$

where k_i and m_j are integers. The integers k_i give the momentum eigenvalue of the center of mass of the string along the i'th direction. In a similar way, since a closed string can wrap around a compact direction an integer number of times, m_i gives the winding number (the number of wraps) around the i'th compactified coordinate. We also have the zero mode momenta

$$\begin{aligned} p_a &= \left(k_i + m^j \left(G_{ij} + B_{ij}\right)\right) \bar{\varpi}^i_a, \\ \bar{p}_a &= \left(k_i - m^j \left(G_{ij} - B_{ij}\right)\right) \bar{\varpi}^i_a. \end{aligned} \tag{3.2.11}$$

We must now identify the group $\mathcal{G}$ which generates the moduli space of the whole set of Lagrangians, i.e., the whole set of the CFT's as well as the subgroup $\mathcal{G}_d$ in the d-dimensional space under which the physics is invariant. It turns out that, sparing the reader many details, the moduli space for toroidal compactifications is isomorphic to the coset space[4] [59-60]

$$\frac{O(d,d,\mathbb{R})}{O(d,\mathbb{R}) \otimes O(d,\mathbb{R})} \tag{3.2.12}$$

[4] Let $\mathbb{R}^{(a,b)}$ be an $(a+b)$-dimensional space with inner product of signature (a,b). $O(a,b,\mathbb{R})$ is then the orthogonal group on $\mathbb{R}^{(a,b)}$. Then $\mathscr{G}^{(a,b)} = \frac{O(a,b,\mathbb{R})}{(O(a,\mathbb{R}) \otimes O(b,\mathbb{R}))}$ can be identified as the set of space-like a-dimensional hyperplanes in $\mathbb{R}^{(a,b)}$, that is hyperplanes upon which the inner product is positive definite.

where $O(d, d, \mathbb{R})$ is the non-compact orthogonal group in $d + d$ dimensions. [5] A convenient manner of representing elements of $\mathfrak{g} \in O(d, d, \mathbb{R})$ is to group them into matrices of the form

$$\mathfrak{g} = \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \mathfrak{C} & \mathfrak{D} \end{pmatrix} \tag{3.2.13}$$

where $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ are each $d \times d$ matrices such that $\mathfrak{g}^t \mathcal{J} \mathfrak{g} = \mathcal{J}$ where

$$\mathcal{J} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{3.2.14}$$

where I is the $d \times d$ identity matrix. This then implies that

$$\begin{aligned} \mathfrak{A}^t \mathfrak{C} + \mathfrak{C}^t \mathfrak{A} &= 0 \\ \mathfrak{B}^t \mathfrak{D} + \mathfrak{D}^t \mathfrak{B} &= 0\,. \\ \mathfrak{A}^t \mathfrak{D} + \mathfrak{C}^t \mathfrak{B} &= I \end{aligned} \tag{3.2.15}$$

Note also that with this representation, $\mathfrak{g}^t \in O(d, d, \mathbb{R})$ as well.

If we take the zero-mode momenta, p_a and $\tilde{p}_a$, they form an even self-dual Lorentzian lattice $\Gamma^{(d,d)}$ [59]. Here *even* means that the Lorentzian length is an even integer, that is

$$\tilde{p}^2 - p^2 = 2m^i k_i \in 2\mathbb{Z}. \tag{3.2.16}$$

It is known that all even self-dual (d, d) Lorentzian lattices are related to one another by $O(d, d, \mathbb{R})$ rotations [61]. Thus any $O(d, d, \mathbb{R})$ rotation of the lattice $\Gamma^{(d,d)}$ returns an even self-dual lattice. In addition, to any such lattice there exists a corresponding toroidal background.

The momenta $(p, \tilde{p})$ transform as vectors under $O(d, d, \mathbb{R})$. At the same time, the Hamiltonian, and thus the spectrum of zero modes, is invariant under the maximal compact subgroup $O(d, \mathbb{R}) \otimes O(d, \mathbb{R})$, i.e., invariant under rotations of p and $\tilde{p}$ separately. Therefore, we have identified the solution-generating group $\mathcal{G}$ as $O(d, d, \mathbb{R})$, and the moduli space is locally isomorphic to the coset manifold (3.2.12).

We recall from (3.2.8) that the momenta p and $\tilde{p}$ are specified by G and B. Therefore, the manner in which the solution generating group $\mathcal{G}$ acts upon G and B is defined by its action on the momentum vectors $(p, \tilde{p})$. We can write the

[5] The Lorentz group, for example, is $SO(3, 1)$.

Hamiltonian (3.2.10) in the form

$$\mathcal{H} = \tfrac{1}{2}\mathcal{Z}^t\mathcal{M}\mathcal{Z} \tag{3.2.17}$$

where $\mathcal{M}$ is a $2d \times 2d$ matrix given by

$$\mathcal{M}(G, B) = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} \tag{3.2.18}$$

and where $\mathcal{Z} = (m_i, k_j)$ is a vector of integers which count the winding number and the momentum modes. Under an $O(d, d, \mathbb{R})$ rotation defined by g (3.2.13) the matrix $\mathcal{M}$ transforms as

$$\mathcal{M}' = \mathrm{g}\mathcal{M}\mathrm{g}^t. \tag{3.2.19}$$

Thus, we have identified the group of transformations $\mathcal{G} = O(d, d, \mathbb{R})$ which carries a CFT containing a set of d^2 couplings defining a d^2-dimensional moduli space into a related CFT with a like number of couplings, in the case of the bosonic string. The space-time version of this formalism is not hard to write down, since the components in the space-time metric and antisymmetric tensor fields just *are* these couplings.

Therefore, in space-time one is allowed to write down transformation matrices of the form given in (3.2.13) which transform the space-time fields. This is, as we will discover later, of great utility.

3.2.2. The heterotic string

As we saw in section 2.3.4, the heterotic string treats left- and right-moving modes differently. Thus the symmetry group on the moduli space of the heterotic string is slightly different, and here we will make note of these differences.

The world sheet bosonic action for the heterotic string can be written

$$\begin{aligned} S = \frac{1}{2\pi}\int d^2\sigma \Big\{ &\left(\sqrt{-e}\, e^{ab}\mathcal{G}_{\mu\nu} + \epsilon^{ab}B^{(h)}_{\mu\nu}\right)\partial_a X^\mu \partial_b X^\nu \\ &+ e^{ab}\mathcal{A}_{\mu\alpha}\partial_a X^\mu \partial_b X^\alpha + \left(\sqrt{-e}\, e^{ab}\mathcal{G}_{\alpha\beta} + \epsilon^{ab}B^{(h)}_{\alpha\beta}\right)\partial_a X^\alpha \partial_b X^\beta \Big\} \end{aligned} \tag{3.2.20}$$

where the indices $1 \leq \alpha, \beta \leq 16$ run over the space of the chiral bosons[6] X^α. Of course, the vector fields $\mathcal{A}_{\mu\alpha}$ are the couplings of the truly marginal operators $\epsilon^{ab}\partial_a X^\mu \partial_b X^\alpha$ of the corresponding conformal field theory.

As mentioned in section 2.2.3.1 the X^α are used to construct the Yang-Mills group of the heterotic string. Thus the internal coordinates must live on the weight lattice of $E_8 \otimes E_8$ or $SO(32)$. The indices (μ, α) label a $(16 + d)$-dimensional orthonormal basis, in which $\mathcal{G}_{\mu\nu} = \delta_{\mu\nu}$, $\mathcal{G}_{\alpha\beta} = \delta_{\alpha\beta}$. From this we can plausibly argue that the moduli space of the heterotic string, with a d-dimensional toroidal background (as in the previous subsection on the bosonic string) is locally isomorphic to the space [59,60]

$$\frac{O\left(d+16, d, \mathbb{R}\right)}{O\left(d+16, \mathbb{R}\right) \otimes O\left(d, \mathbb{R}\right)}. \tag{3.2.21}$$

Thus, for the heterotic string, the relevant symmetry group is $O(d+16, d, \mathbb{R})$, which of course has a similar space-time interpretation to that of (3.2.18). However, for the space time interpretation in the case of the heterotic string, the configuration of the gauge fields $\mathcal{A}_{\mu\alpha}$ must lie in a subgroup of $SO(32)$ or $E_8 \otimes E_8$ that commutes with a set of the $U(1)$ generators of the gauge group. Thus if the subgroup in question commutes with $\hat{p}$ of the $U(1)$ generators, then we will have $O(d+\hat{p}, d, \mathbb{R})$ symmetry.

To close this section, we remark that the $O(d, \mathbb{R}) \otimes O(d, \mathbb{R})$ or $O(d+16, \mathbb{R}) \otimes O(d, \mathbb{R})$ symmetry of the Hamiltonians that we have derived was done only with the zero modes. Therefore, it is only a symmetry of the low-energy or long-wavelength part of string theory. The action of the solution generating groups $O(d, d, \mathbb{R})$ and $O(d, d+16, \mathbb{R})$ is to map a conformal background onto *the leading order of* a conformal background. From the spacetime perspective, these symmetries and solution generating groups hold only for the low-energy effective actions of string theories. However, as we will see in the next section, there is a subgroup of $O(d, d, \mathbb{R})$, namely $O(d, d, \mathbb{Z})$ that is an exact symmetry of the theory of strings.

---

[6] The X^α are the left-moving bosonic fields used to construct the world-sheet current algebra as discussed in section 2.2.3.1.

3.3. T-duality symmetry of the string

Loosely said, T-duality implies that the string cannot "tell" whether it is propagating on a compact coordinate with radius R, or on a coordinate with radius $\frac{\alpha'}{R}$. T-duality is a property of string theory that has no analog in point particle field theory. One can understand this by noting that a one-dimensional extended object can wrap around a compact dimension. The winding number gives a contribution to the energy since the string must stretch somewhat each time it winds. Let us now consider in general terms this symmetry of string theory. Let us explain this symmetry in detail in the simplest circumstances possible, a single bosonic string coordinate compactified on a circle of radius R. Some of what we say here is repetition of things already mentioned in section 2.2.3.1, but done with more attention to the symmetry that was hinted at in (2.2.50).

We write the world sheet action of our single bosonic coordinate as[7]

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma \partial_a X \partial^a X \tag{3.3.1}$$

where the usual compactification

$$X \equiv X + 2\pi R m \qquad (m \in \mathbb{Z}) \tag{3.3.2}$$

is imposed. $X(\sigma,\tau)$, since it satisfies as usual a free wave equation, is decomposed into left- and right-moving modes as

$$\begin{aligned} X_r\left(\sigma^-\right) &= x_r + \sqrt{\frac{\alpha'}{2}} p_r\, \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \alpha_n\, e^{-in\sigma^-}, \\ X_l\left(\sigma^+\right) &= x_l + \sqrt{\frac{\alpha'}{2}} p_l\, \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}_n\, e^{-in\sigma^+}, \end{aligned} \tag{3.3.3}$$

where x_r and x_l are the center of mass position. The dimensionless center of mass momenta p_l and p_r are defined to be

$$\begin{aligned} p_r &= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\alpha'}}{R} n + \frac{R}{\sqrt{\alpha'}} m\right), \\ p_l &= \frac{1}{\sqrt{2}}\left(\frac{\sqrt{\alpha'}}{R} n - \frac{R}{\sqrt{\alpha'}} m\right). \end{aligned} \tag{3.3.4}$$

---

7 We use here the orthonormal reparametrization (see section 3.1).

The canonical momentum conjugate to X is

$$P = \frac{1}{2\pi\sqrt{2\alpha'}}\left[p_l + p_r + \sum_{n\neq 0} \alpha_n \, e^{in\sigma^-} + \sum_{n\neq 0} \tilde{\alpha}_n \, e^{-in\sigma^+}\right] \tag{3.3.5}$$

and the total momentum conjugate to the center of mass coordinate $x = x_l + x_r$ is

$$p = \frac{1}{\sqrt{2\alpha'}}(p_l + p_r)\,. \tag{3.3.6}$$

The Hamiltonian then reads

$$\mathcal{H} = L_0 + \tilde{L}_0 \tag{3.3.7}$$

where

$$\begin{aligned} L_0 &= \tfrac{1}{2}p_r^2 + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n \\ \tilde{L}_0 &= \tfrac{1}{2}p_l^2 + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}\tilde{\alpha}_n \end{aligned} \tag{3.3.8}$$

are, as we have seen before, the zero modes of the Virasoro operators.

We now note that L_0 and $\tilde{L}_0$ are invariant under the transformation

$$\begin{gathered} \frac{R}{\sqrt{\alpha'}} \to \frac{\sqrt{\alpha'}}{R}\,. \\ m \leftrightarrow n \end{gathered} \tag{3.3.9}$$

That is to say, if we invert the radius of our compact coordinate while at the same time interchanging winding and momentum quantum numbers, the Virasoro constraints remain satisfied and therefore the spectrum is unchanged. The oscillator components are also transformed under (3.3.9) according to

$$\alpha_n \to \alpha_n \qquad \tilde{\alpha}_n \to -\tilde{\alpha}_n \tag{3.3.10}$$

which implies that $\partial_\tau X_r \to \partial_\tau X_r$ and $\partial_\tau X_l \to -\partial_\tau X_l$ under the action of T-duality.

The one-loop partition function of the compactified bosonic string is

$$Z = \int_\Gamma d^2\varrho Z'(\varrho, \bar{\varrho}) \sum_{p_l, p_r} \mathrm{Tr} \exp\left(i\pi\varrho L_0 - i\pi\bar{\varrho}\tilde{L}_0\right) \tag{3.3.11}$$

where ϱ is a modular parameter which describes conformally inequivalent tori, Γ is the region of the ϱ-plane which covers the set of inequivalent tori, and where

$Z'(\varrho, \bar{\varrho})$ represents the contribution of all coordinates other than our compact one. The trace is taken over the space spanned by the oscillators α, $\tilde{\alpha}$ and we sum over all momenta according to (3.3.4). Symmetry of the partition function under (3.3.9) follows immediately upon recognition that the mode and winding number integers are dummy variables.

Thus we have shown that target space duality is a symmetry of the one-loop string partition function, and therefore of the free string spectrum. We claim, as a result, that compactification on a small radius ($\frac{R}{\sqrt{\alpha'}} << 1$) is completely equivalent to compactification on a large radius ($\frac{\sqrt{\alpha'}}{R} >> 1$). This claim can be proved by demonstrating T-duality of the higher-genus (higher-loop) contribution to the partition function. We will not go through the details here, the interested reader can find the relevant material in [58] and the references therein.

It will be found, however, that for the higher-genus partition function we will have [62]

$$Z\left(\phi + 2\log R, \frac{1}{R}\right) = Z(\phi, R) \tag{3.3.12}$$

which is to say that in order for the T-duality symmetry to hold to all orders in perturbation theory, the dilaton field must be transformed as well, according to $\phi' = \phi + 2\log R$.

We remark here that T-duality in this form has been often regarded as evidence of the existence of a minimal length in string theory [63]. This question, although interesting, takes us well beyond the scope of this work.

As in the case of the $O(d, d, \mathbb{R})$ symmetry, we can construct for T-duality a mapping between space-time fields of a low energy effective action, rather than between couplings in a world sheet conformal field theory. When this is done, certain very useful methods of transforming known solutions into new solutions can be obtained. We will see use of this later.

3.3.1. Relation to $O(d, d)$ symmetry

It turns out that the symmetry group $O(d, d, \mathbb{R})$ contains within it the T-duality symmetry. If we write down a matrix $\mathfrak{g}$ (3.2.13) of the form

$$\mathfrak{g}_{\varepsilon} = \begin{pmatrix} I - \varepsilon_i & \varepsilon_i \\ \varepsilon_i & I - \varepsilon_i \end{pmatrix} \tag{3.3.13}$$

where ε_i is zero, save for the $i - i$'th component, which is unity. I is again the d-dimensional identity matrix. The duality generated by (3.3.13) can be shown to be a generalization to several compact coordinates of the T-duality symmetry. For example, $\mathfrak{g}_{\varepsilon_i}$ takes $R_i \to \frac{1}{R_i}$ in the case where the d-dimensional background is a direct product of a circle of radius R_i and some $(d-1)$-dimensional background.

Thus T-duality lies in the $O(d, d, \mathbb{Z})$ subgroup of $O(d, d, \mathbb{R})$. This is then evidence that $O(d, d, \mathbb{Z})$ is an exact symmetry of string theory. Showing that this is true requires consideration of the transformation of the oscillator components in addition to the zero modes.

We begin by writing down the equal-time canonical commutation relations

$$[X^i(\sigma), P_j(\sigma')] = i\delta^i{}_j \delta(\sigma - \sigma'). \tag{3.3.14}$$

We then substitute the mode expansions for X^i and P^i from (3.2.6) to obtain the commutation relations of the oscillator components, of which the non-vanishing ones are

$$\begin{aligned} [x^i, p_j] &= i\delta^i{}_j \\ [\alpha^i_n, \alpha^j_m] &= [\tilde{\alpha}^i_n, \tilde{\alpha}^j_m] = mG^{ij}\delta_{m+n\,0} \end{aligned} \tag{3.3.15}$$

where G_{ij} is the background metric of the d compact directions of section 3.2. Recall that the oscillator components are themselves functions of G_{ij} and B_{ij}. The Hamiltonian can again be computed by inserting the mode expansions into (3.2.7) generalizing (3.2.17) to

$$\mathcal{H} = \tfrac{1}{2}\mathcal{Z}^t \mathcal{M} \mathcal{Z} + N + \tilde{N} \tag{3.3.16}$$

where the oscillator numbers are now given by

$$N = \sum_{n=1}^{\infty} G_{ij}\alpha^i_{-n}\alpha^j_n, \qquad \tilde{N} = \sum_{n=1}^{\infty} G_{ij}\tilde{\alpha}^i_{-n}\tilde{\alpha}^j_n. \tag{3.3.17}$$

From (3.2.13) and (3.2.18) we can compute that the transformation of G_{ij} and B_{ij} under a group element $\mathfrak{g} \in O(d, d, \mathbb{Z})$ is given by

$$K' = (\mathfrak{A}K + \mathfrak{B})(\mathfrak{C}K + \mathfrak{D})^{-1} \tag{3.3.18}$$

where $K = G + B$ and from which G and B can be retrieved by taking the decomposition into symmetric and antisymmetric parts. In this way a pair of relations between the original and transformed metrics can be found as

$$(\mathfrak{D} + \mathfrak{C}K)^t G' (\mathfrak{D} + \mathfrak{C}K) = G = (\mathfrak{D} - \mathfrak{C}K)^t G' (\mathfrak{D} - \mathfrak{C}K)\,. \tag{3.3.19}$$

Now, to be a symmetry of the string, the group element $\mathfrak{g}$ must preserve the commutation relations of all the oscillators (3.3.15). Preservation of the commutation relations in conjunction with the mode expansions (3.2.6) uniquely fixes the action of duality on the oscillators to be [64]

$$\alpha_n \to (\mathfrak{D} + \mathfrak{C}K)^{-1}\,\alpha_n, \qquad \tilde{\alpha}_n \to (\mathfrak{D} - \mathfrak{C}K)^{-1}\,\tilde{\alpha}_n. \tag{3.3.20}$$

One can see that with (3.3.19) and (3.3.20) the oscillator numbers (3.3.17) are invariant, thus the entire Hamilton is invariant under $O(d, d, \mathbb{Z})$. Also from (3.3.20) one can demonstrate that under T-duality transformations given by (3.3.13) the left-moving oscillators $\tilde{\alpha}_n$ behave as in (3.3.10). To be sure, $O(d, d, \mathbb{Z})$ is more general than just T-duality. For example, it includes transformations which change the basis of the compactification lattice Λ^d [58].

3.3.2. T-duality of type II superstrings

In the case of the type II string there is more to T-duality than inverting the radii of compactified directions. When applied to a type II string theory, T-duality has the effect of reversing the relative chiralities of right- and left-moving ground states [65]. Since, as we have seen, type IIA has opposite chiralities for the two sets of ground states, while for the type IIB theory they have the same chirality, T-duality has the effect of exchanging the type IIA and type IIB theories.

Let us sketch an argument. Consider one dimension x^9 which is compactified. From (3.3.4) we see that in the $R \to \infty$ limit we have $p_r^9 = -p_l^9$ while in the

$R \to 0$ limit we will have $p_r^9 = p_l^9$. Both type II theories are $SO(9,1)$ invariant, but under different representations of $SO(9,1)$. T-duality reverses[8] the sign of the left-moving X^9, and by supersymmetry also of the left-moving ψ^9 [66].

If we then separate the Lorentz generators into their right- and left-moving components as $\mathcal{L}^{\mu\nu} = \mathcal{L}_r^{\mu\nu} + \mathcal{L}_l^{\mu\nu}$ then T-duality flips the sign of all the terms in $\mathcal{L}_l^{\mu 9}$ so that the Lorentz generators of the T-dual theory are $\tilde{\mathcal{L}}^{\mu 9} = \mathcal{L}_r^{\mu 9} - \mathcal{L}_l^{\mu 9}$. This reverses the sign of the helicity for all the states, and switches the chirality of the left-moving zero modes. Essentially, one of the s_α in the analog of (2.3.18) for left-moving modes changes, resulting in a change of chirality of the ground state according to (2.3.22). Therefore, the relative chiralities of the right- and left-moving ground states are reversed, since the right-movers maintain their chirality while the chirality of the left-movers is switched.

We also know that the type IIA and type IIB string theories contain different Ramond-Ramond fields, and therefore T-duality, to be consistent, must transform one set into another. The action of T-duality of x^9 on the spinor fields can be written

$$S_{r\,\alpha}(\sigma^-) \to S_{r\,\alpha}(\sigma^-) \qquad S_{l\,\alpha}(\sigma^+) \to \mathcal{P}_9 S_{l\,\alpha}(\sigma^+)$$

for a matrix $\mathcal{P}_9$ which represents the parity transformation in x^9 on the spinors. This must be consistent with $\psi_l^9 \to -\psi_l^9$, thus $\mathcal{P}_9$ must anticommute with γ^9 and commute with all the other γ^μ. A definition of $\mathcal{P}_9$ which accomplishes this is $\mathcal{P}_9 = \gamma^9\gamma^{11}$ where γ^{11} is the chirality operator. By the γ-algebra identity

$$\gamma^\nu \gamma^{[\mu_1} \cdots \gamma^{\mu_n]} = \gamma^{[\nu} \cdots \gamma^{\mu_n]} + \sum_{\text{perms}} \delta^{\nu\mu_1} \gamma^{[\mu_2} \cdots \gamma^{\mu_n]}, \tag{3.3.21}$$

the effect on the type IIA one-form potential $A^{(1)}$, say, is to add a 9-index if there isn't one, and to remove one if there is. That is to say that we have

$$\mathcal{P}_9 A_\mu^{(1)} = \begin{cases} \chi & \mu = 9 \\ A_{\mu 9}^{(2)} & \mu \neq 9. \end{cases} \tag{3.3.22}$$

The remaining components needed to fill out the type IIB $A_{\mu\nu}^{(2)}$ will of course come from the type IIA $A_{\mu\nu 9}^{(3)}$ and so on.

---

8 We will see this effect explained in detail in Chapter IV.

As a final remark, due to the special way in which fields of the Ramond-Ramond sector of type II theories transform under T-duality, one cannot utilize the solution generating transformation $O(d, d, \mathbb{R})$ in the type II case unless the Ramond-Ramond fields vanish.

3.4. String duality

String duality is the most recently discovered symmetry of string theory. String duality refers to a collection of symmetries that continues to grow as our knowledge improves, rather than to a single symmetry. The major reason for which string duality has remained undiscovered for so long, even though string theory itself goes back some twenty-five years, has to do with the fact that string duality is not a manifest feature of the perturbation expansion of string theory, as is T-duality, but is rather a property of the exact theory [3,67].

As a result, string duality is in a position to provide us with clues about string theory in the strong coupling regime. At least in those cases in which the background has enough supersymmetry, we can gather with the help of string duality much useful information about strongly coupled strings. This is important since it has become clear that purely perturbative string theory is not sufficient to solve problems such as that of the value of the cosmological constant, how string theory selects its vacuum, or how supersymmetry is broken. To place string duality in its proper context, before we move on to the details of the precise string duality symmetries we will need for the work at hand, we offer an introduction in the next section.

3.4.1. Introduction to string duality

The basic idea that is at play in string duality is that the strong-coupling limit of one string theory is equivalent physically to the weak-coupling limit of a different theory. This "different" theory may contain, in general, objects other than strings [1,68]. For example, the multiplets of string duality contain, in addition to the vibrating strings which are the basic quanta of string theory, classical objects such as solitons.

By way of illustration, in string duality there is a conjectured duality [67] which relates strings to fivebranes in ten dimensions[9] (a fivebrane has five spatial and one time dimension). There are as yet many mysteries associated with higher membrane theories, thus we set these aside for the purpose of this work to concentrate our attention on a small subset of string duality.

In order to introduce the idea of a relation between strong-coupling and weak coupling limits, let us consider for a moment Maxwell's equations:

$$\nabla \cdot \vec{E} = \rho_e \qquad \nabla \times \vec{E} + \partial_t \vec{B} = 0 \tag{3.4.1a}$$

$$\nabla \cdot \vec{B} = \rho_m \qquad \nabla \times \vec{B} - \partial_t \vec{E} = 0 \tag{3.4.1b}$$

where we have added a magnetic source term, ρ_m for symmetry. These equations have a symmetry under the transformation

$$\vec{E} \to \vec{B}, \qquad \vec{B} \to -\vec{E}, \qquad \rho_e \leftrightarrow \rho_m. \tag{3.4.2}$$

Dirac [69] calculated the consequences of the existence of a magnetic charge, or magnetic monopole and found that single-valuedness of a quantum mechanical wavefunction was dependent on having

$$q^e q^m = 2\pi\hbar n \tag{3.4.3}$$

where n is an integer, which is known as the Dirac quantization condition.

The quantization condition (3.4.3) tells us that in the case that electrically charged objects in a theory are weakly coupled, ($q^e << 1$) the magnetic objects in the theory are strongly coupled ($q^m >> 1$) and vice versa. Magnetic monopoles have been shown to exist in grand unified theories [70]. They are classical solutions without singularities, with a characteristic size set by the scale of the spontaneous symmetry breaking.

In a regime which is electrically weakly coupled, therefore, the electrically charged objects differ greatly from the magnetically charged ones. Electrically charged objects have pointlike interactions since the coupling is not strong enough to resolve the structure of the interaction. This is akin to the Fermi theory of the weak

[9] See section 2.1.2 for a short discussion of these objects or refer to [30].

interaction. The magnetically charged objects are strongly coupled and have finite size. On this basis and on the basis of further evidence, it was conjectured [71] that there exists a regime of the theory where the roles of electric (weak) and magnetic (strong) coupling are reversed. Here the electrically charged objects would have finite size and the magnetically charged ones would be pointlike.

In string theory the basic idea is the same, but the execution, due to the increased complexity of the theory is vastly more complicated. We will reduce this complexity by focussing our attention on a particular string duality, usually given the name string/string duality, which will play a substantial role in this work.

3.5. String/string duality

String/string duality, as the name might suggest, relates one string theory to another. In contrast to the T-duality studied in the last section, string/string duality is, like $O(d,d,\mathbb{R})$ symmetry, known to be exact only for the low energy effective actions of string theories. It is conjectured to be generally true for complete string theories, but while these conjectures remain to be proved, one can make good use of string/string duality in the context of low energy string theory.

The specific string/string duality that we will discuss here is that which exists between the low-energy effective action of the type IIA superstring compactified to six dimensions on the Calabi-Yau manifold[10] [72-73] $K3$ and the low energy effective action of the heterotic superstring compactified of the four-torus T^4. This conjecture was first put forward in [74-76] and is considered in detail in [77].

The evidence upon which a conjecture of this sort is normally based is the equivalence of the low energy effective actions, that is the space-time degrees of freedom can be mapped one-to-one from one theory to the other, as well as the identification of the moduli spaces. This second means that the group which moves the couplings of the corresponding CFT through the moduli space as well as the subgroups which form orbits of physically identical theories are the same. This evidence is quite compelling. There remains, however, one other aspect of the

[10] We state that those who do not know what a Calabi-Yau manifold is will not, in fact, be required to know anything other than that $K3$ is compact and four-dimensional.

two theories which must be ascertained to coincide: the spectrum of Bogomol'nyi-Prasad-Sommerfeld (BPS) saturated states. The reader unfamiliar with BPS states need not be concerned, as we digress in the next section to their explanation.

3.5.1. BPS saturated states

To understand BPS saturated states, it is first necessary to understand slightly more about supersymmetry. We first introduce supersymmetry generators Q and $\bar{Q} = Q^\dagger$ which are generalizations in spacetime of those given in (2.2.2). In four space-time dimensions, for example [36] we can write the anticommutator of the supersymmetry generators as

$$\{Q^i_\alpha, \bar{Q}^j_{\dot{\beta}}\} = 2\sigma^\mu_{\alpha\dot{\beta}} P_\mu \delta^{ij} \tag{3.5.1a}$$

$$\{Q^i_\alpha, Q^j_\beta\} = \{\bar{Q}^i_{\dot{\alpha}}, \bar{Q}^j_{\dot{\beta}}\} = 0 \tag{3.5.1b}$$

where the σ^μ are the Pauli matrices, $(\dot{\beta})\alpha$ are (anti)spinor indices and P_μ the momentum. Taking the trace of (3.5.1a) gives [25]

$$P^0 = \mathcal{H} = \frac{1}{4} \sum_{\alpha=1,2} \left(Q^i_\alpha (Q^i_\alpha)^\dagger + (Q^i_\alpha)^\dagger Q^i_\alpha \right) \geq 0 \tag{3.5.2}$$

which is a positive semi-definite operator, giving energy eigenvalues which are non-negative. This means that a vacuum state which is invariant under supersymmetry transformations has zero energy.

In section 2.2 we noted that one of the symmetries of physical processes was an internal global symmetry, whose generators are Lorentz scalars. These generators $\mathcal{B}$ will form a Lie algebra,

$$[\mathcal{B}_i, \mathcal{B}_j] = iC_{ij}{}^k \mathcal{B}_k \tag{3.5.3}$$

where the $C_{ij}{}^k$ are the structure constants. Consider that the Hamiltonian $\mathcal{H}$ is invariant under this symmetry group. In this case the supersymmetry algebra can gain terms known as *central charges* Γ, as in

$$\begin{aligned} \{Q^i, Q^j\} &= \Gamma^{ij}, \\ \{\bar{Q}^i, \bar{Q}^j\} &= (\Gamma^{ij})^\dagger, \end{aligned} \tag{3.5.4}$$

where we suppress the spinor indices. The Γ commute with all the Q and $\bar{Q}$ and generate an Abelian invariant subalgebra of the internal symmetry group generated by the $\mathcal{B}$. The presence of central charges necessitates a rediagonalization of the basis of the Q and when this is carried out one has a new set of supersymmetry generators whose anticommutation relations can be written schematically as

$$\{Q_{1\,i}, \bar{Q}_{1\,j}\} = \left(2M + \mathcal{Z}_j\right)\delta_{ij}, \tag{3.5.5a}$$

$$\{Q_{2\,i}, \bar{Q}_{2\,j}\} = \left(2M - \mathcal{Z}_j\right)\delta_{ij}, \tag{3.5.5b}$$

where the $\mathcal{Z} \geq 0$ are the eigenvalues of the central charges in the new basis. Since $\{Q_{1\,i}, \bar{Q}_{1\,j}\}$ and $\{Q_{2\,i}, \bar{Q}_{2\,j}\}$ are positive definite operators and $\mathcal{Z} \geq 0$, then all the central charges lie in the range $0 \leq \mathcal{Z} \leq 2M$. When one or more of the $\mathcal{Z} = 2M$, then we say that we have a *saturated* state. When this happens the anticommutation relation (3.5.5b) vanishes for some i, and thus the multiplets of supersymmetric states become *short*, i.e., they contain fewer states than in the unsaturated case [36]. In the case that all the $\mathcal{Z}$ are saturated, the multiplets contain one-half of the states of the most general multiplet.

Now, consider a one-particle state $|\psi\rangle$ which is annihilated by the supersymmetry generators Q_2 from (3.5.5). Schematically, we have

$$\begin{aligned} \langle\psi|\,\{Q_2, Q_2\}\,|\psi\rangle &= \langle\psi|\,2\mathcal{H} - \mathcal{Z}\,|\psi\rangle \\ 0 &= 2\,\langle\psi|\,\mathcal{H}\,|\psi\rangle - \langle\psi|\,\mathcal{Z}\,|\psi\rangle \end{aligned} \tag{3.5.6}$$

which gives us the mass of the particle and its charge under the internal symmetry group (3.5.3). Therefore, for any particle in a BPS state, the mass is entirely determined by its charge. It has the largest possible ratio of the charge to the mass, and is said to be in an extremal state.[11]

Since the supersymmetry algebra contains the Hamiltonian, we can uncover much more information about the dynamics of a supersymmetric theory than is possible with ordinary internal symmetries. With furthur analysis, strong constraints can be placed on the interactions and on the phases of the theory [3].

The importance of this is that this result is a consequence of supersymmetry and the dynamics of the theory have no effect on the mass, i.e., the mass is free of radiative corrections and therefore the mass remains known even when the coupling

[11] We will have more to say in later chapters about extremal solutions of the string equations of motion.

becomes large. These solutions are also stable, as no decay into lower energy states is possible.

Thus the utility of BPS states is that our knowledge of their properties is independent of whether or not we have a perturbative description at a given value of the coupling. This explains why these states are so important in studies of duality, as it gives us a way to compute a spectrum of states at strong coupling and compare this to the conjectured dual weakly coupled theory.

3.5.2. Type IIA — heterotic string/string duality

Let us now develop the evidence for the string/string duality that exists between the type IIA superstring compactified on $K3$ and the heterotic superstring compactified on T^4.

The low-energy effective action describing the heterotic string compactified on T^4 is [77]

$$S_h = \frac{1}{2\kappa_6^2}\int d^6x\sqrt{-\mathcal{G}}\,e^{-2\phi^{(h)}}\Big\{R + 4(\nabla\phi^{(h)})^2 - \frac{1}{12}\left(H^{(h)}\right)^2 - \mathcal{F}^{(2)\,\alpha}_{\mu\nu}(L\mathcal{M}L)_{\alpha\beta}\,\mathcal{F}^{(2)\,\beta\,\mu\nu} + \frac{1}{8}\mathcal{G}^{\mu\nu}\,\mathrm{Tr}\left[(\partial_\mu\mathcal{M})\,L\,(\partial_\nu\mathcal{M})\,L\right]\Big\}, \tag{3.5.7}$$

where $\mathcal{G}_{\mu\nu}$ is the heterotic metric in the string frame, the $U(1)$ gauge fields $\mathcal{A}^{(1)\,\alpha}$ are represented by their field strengths $\mathcal{F}^{(2)\,\alpha} = d\mathcal{A}^{(1)\,\alpha}$ and $H^{(h)} = dB^{(h)} + 2L_{\alpha\beta}\mathcal{A}^{(1)\,\alpha}\wedge\mathcal{F}^{(2)\beta}$ is the field strength of the heterotic antisymmetric tensor field $B^{(h)}$ (which includes Chern-Simons terms of the gauge fields). Note that the space time indices μ, ν run $0 \le \mu,\nu \le 5$ and the gauge indices α, β run $1 \le \alpha,\beta \le 24$. $\phi^{(h)}$ is the dilaton and $\mathcal{M}$ is a 24×24 matrix of scalars representing an element of the coset space[12] $\frac{O(4,20)}{O(4)\otimes O(20)}$ which satisfies

$$\mathcal{M}^t = \mathcal{M} \qquad \mathcal{M}L\mathcal{M}^t = L \tag{3.5.8}$$

where

$$L = \begin{pmatrix} -I_{20} & \\ & I_4 \end{pmatrix} \tag{3.5.9}$$

The 80 scalars originate from the internal components of the ten-dimensional metric, antisymmetric tensor field, and the gauge fields. Since we have compactified four

12 As discussed in section 3.2 with $d = 4$.

dimensions, according to section 2.2.3.3 we will have four gauge fields from the metric, four from the antisymetric tensor field and, of course, the 16 that existed in the 10-dimensional theory reduced to six dimensions.

The other theory in question here, the type IIA theory compactified on $K3$ is given by

$$S_{IIA} = \frac{1}{2\kappa_6^2} \int d^6x \sqrt{-G}\, e^{-2\phi^{(a)}} \Big\{ R + 4(\nabla\phi^{(a)})^2 - \frac{1}{12}\left(H^{(a)}\right)^2 - F^{(2)\,\alpha}_{\mu\nu} (LML)_{\alpha\beta}\, F^{(2)\,\beta\,\mu\nu} + \frac{1}{8} G^{\mu\nu}\,\mathrm{Tr}\left[(\partial_\mu M)\, L\, (\partial_\nu M)\, L\right] - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma\kappa\delta} B^{(a)}_{\mu\nu} F^{(2)\,\alpha}_{\rho\sigma} L_{\alpha\beta} F^{(2)\,\beta}_{\kappa\delta} \Big\}, \tag{3.5.10}$$

where $0 \leq \mu, \nu, \rho, \sigma, \kappa, \delta \leq 5$ and as before $1 \leq \alpha, \beta \leq 24$. Here $G_{\mu\nu}$ is the type IIA metric, $H^{(a)} = dB^{(a)}$ is the field strength of the antisymmetric tensor field, $\phi^{(a)}$ is the dilaton, $A^{(1)\,\alpha}$ are the 24 abelian gauge fields, again with their field strengths $F^{(2)\,\alpha} = dA^{(1)\,\alpha}$ and M is a 24×24 matrix of scalars that finds itself in the same coset space as in the heterotic string above and also obeys the same relations

$$M^t = M \qquad MLM^t = L \tag{3.5.11}$$

as in (3.5.8). The scalars come from the components of the metric and the antisymmetric tensor field along the tangent space of $K3$. Note that the Chern-Simons terms for $H^{(a)}$ are absent, although they are replaced by the final term in (3.5.10). Also note that $\epsilon^{\mu\cdots\delta}$ denotes the totally antisymmetric tensor. The gauge fields are descended from the three-form potential $A^{(3)}$ and the one-form potential $A^{(1)}$ from the Ramond-Ramond sector. The $A^{(3)}$ potential provides 23 gauge fields, 22 of which come in the form $A^{(3)}_{ij\mu}$ where i and j denote tangent space directions on the compactification manifold $K3$, and another from dualization of the three form in 6 dimensions. $A^{(1)}$ of course provides the remaining gauge field.

It can be shown that the equations of motion obtained by variation of the actions (3.5.7) and (3.5.10) are identical if one maps the fields type IIA $\leftrightarrow$ heterotic according to

$$\phi^{(a)} = -\,\phi^{(h)}, \qquad G_{\mu\nu} = \mathrm{e}^{-2\phi^{(h)}}\,\mathcal{G}_{\mu\nu}, \tag{3.5.12a}$$

$$A^{(1)\,\alpha}_{\mu} = \mathcal{A}^{(1)\,\alpha}_{\mu}, \qquad M = \ \mathcal{M}, \tag{3.5.12b}$$

$$H^{(a)}_{\mu\nu\rho} = \frac{1}{6}\epsilon_{\mu\nu\rho\sigma\kappa\delta}\ \mathrm{e}^{-2\phi^{(h)}}\ H^{(h)\,\sigma\kappa\delta}. \tag{3.5.12c}$$

Thus we have, for the case of the duality between the type IIA superstring compactified on $K3$ and the heterotic string compactified on T^4 a map for the low-energy space-time fields, giving us the means to take a solution of one theory and convert it easily into a solution of the other theory. We shall make use of this mapping in later chapters.

We mentioned above that three things have to be done to support such a duality conjecture. The first we have just done, showing that the low energy equations of motion are identical under an appropriate mapping of fields. The second, showing that the moduli spaces are identical is a highly non-trivial undertaking, involving a foray into the mathematics of algebraic geometry, and thus we will content ourselves with a few remarks.

We noted above that both the matrix of scalar fields $\mathcal{M}$ from the heterotic string and that, M from the type IIA string were elements of the coset space (3.2.12). This makes plausible the idea that the moduli spaces are identical. Also in [78] it was demonstrated that the moduli space of conformal field theories on the Calabi-Yau manifold $K3$ was in fact locally of the form $\frac{O(4,20)}{O(4)\otimes O(20)}$. We recognize this as the same as that of the heterotic string compactified on T^4 from section 3.2, therefore that the moduli spaces would be identical is not surprising. We see from this the reason for which the type IIA string must be compactified on $K3$ in order for this duality to hold.

The third criterion that we noted above, the identity of the spectra of BPS saturated states in the two theories is also a subject that will take us too far afield, and we will make do with some short plausibility arguments. To begin we note that a fundamental string solution of the heterotic string equations of motion will be electrically charged [75] from the $B^{(h)}_{\mu\nu}$, and hence magnetically charged under the type IIA $B^{(a)}_{\mu\nu}$ since they are Hodge duals of one another ((3.5.12c)). Thus the fundamental type IIA string will have be charged conversely, electrically under $B^{(a)}_{\mu\nu}$ and magnetically under $B^{(h)}_{\mu\nu}$. The question, as motivated by the duality conjecture, is the existence of solutions which are fundamental (solitonic) in the heterotic (type IIA) variables while at the same time being solitonic (fundamental) in the type IIA (heterotic) variables.

We have not yet discussed the construction of classical solutions for the low energy actions of string theory, which is a topic with which we will concerned in the remaining chapters. As a modest attempt at completeness, however, let us write down a classical spacetime solution corresponding to a fundamental heterotic string in six dimensions [79], which is

$$\begin{aligned} ds_h^2 &= \left(1+\frac{\mathcal{C}}{r^2}\right)^{-1}(-dt^2+(dx^5)^2)+dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2) \\ \mathcal{A}^{(1)\,\alpha} &= 0 \\ B^{(h)} &= -\frac{\mathcal{C}}{\mathcal{C}+r^2}dt\wedge dx^5 \\ \mathcal{M} &= I_{24} \\ e^{-2\phi^{(h)}} &= 1+\frac{\mathcal{C}}{r^2} \end{aligned} \tag{3.5.13}$$

where $\mathcal{C}$ is a constant which is determined by the tension of the heterotic string. This solution has the required electric charge under $B^{(h)}$ and can be shown to have an essential singularity at $r=0$.

If we map this solution to a type IIA solution using (3.5.12), it takes the form

$$\begin{aligned} ds_a^2 &= -dt^2+(dx^5)^2+\left(1+\frac{\mathcal{C}}{r^2}\right)dr^2+(\mathcal{C}+r^2)(d\theta^2+\sin^2\theta d\varphi^2) \\ A^{(1)\,\alpha} &= 0 \\ H^{(a)} &= -2\mathcal{C}\,dx^i\wedge dx^j\wedge dx^k \\ M &= I_{24} \\ e^{-2\phi^{(a)}} &= \left(1+\frac{\mathcal{C}}{r^2}\right)^{-1} \end{aligned} \tag{3.5.14}$$

where we have given the $H^{(a)}$ resulting from the dualization due to compactness of expression. This is evidently magnetic under $B^{(a)}$ and it can be seen that $r=0$ in this case is merely a coordinate rather than an essential singularity by defining a new coordinate $\rho = \log r$ near $r = 0$. Thus we have written down a singular electric fundamental solution for the heterotic string and shown that under string/string duality it becomes a non-singular magnetic solitonic solution of the type IIA supergavity [80].

Now we must do the converse, a fundamental solution of the type IIA string must become a solitonic solution of the heterotic string. It can be shown that the fundamental solution of the type IIA string is [77]

$$\begin{aligned} ds_a^2 &= \left(1+\frac{C'}{r^2}\right)^{-1}(-dt^2+(dx^5)^2)+dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2) \\ A^{(1)\alpha} &= 0 \\ B^{(a)} &= -\frac{C'}{C'+r^2}dt\wedge dx^5 \\ M &= I_{24} \\ e^{-2\phi^{(a)}} &= 1+\frac{C'}{r^2} \end{aligned} \tag{3.5.15}$$

which is, in fact, identical to (3.5.13) except for the fact that the constant C' is determined by the type IIA string tension. It is then evident that the corresponding solitonic solution of the heterotic superstring will be identical to (3.5.14) and we write it here as

$$\begin{aligned} ds_h^2 &= -dt^2+(dx^5)^2+\left(1+\frac{C'}{r^2}\right)dr^2+(C'+r^2)(d\theta^2+\sin^2\theta d\varphi^2) \\ \mathcal{A}^{(1)\alpha} &= 0 \\ H^{(h)} &= -2C'\,dx^i\wedge dx^j\wedge dx^k \\ \mathcal{M} &= I_{24} \\ e^{-2\phi^{(h)}} &= \left(1+\frac{C'}{r^2}\right)^{-1}. \end{aligned} \tag{3.5.16}$$

Thus we have the expected duality structure, analogous to that between electric charges and magnetic monopoles in field theory, occuring between the type IIA string compactified on $K3$ and the heterotic string compactified on T^4. There is more work required [77] to show that these solutions (3.5.13), (3.5.14), (3.5.15) and (3.5.16) have all the properties expected of heterotic and type IIA fundamental strings and solitons. However, since we have not yet studied how such solutions are constructed, I propose to leave this subject for another day.

IV

The theory of Dirichlet branes

As a result of the study of non-perturbative dualities, such as T-duality, about which we wrote in chapter III, Dirichlet-branes, or D-branes for short have been developed [14,65,81]. Their role in string theory is, as we shall see, rather important, particularly in the Type I and Type II theories. D-branes also play an essential role in the duality symmetries, as the carriers of fundamental Ramond-Ramond charge. Certain of the multiplets exchanged by various S-dualities interchange the Neveu-Schwarz–Neveu-Schwarz and Ramond-Ramond sectors of type II string theories. Since fundamental strings do not carry Ramond-Ramond charges, other objects, the D-branes, must fulfill this role.

In this chapter we will develop the theory of D-branes from two points of view, as particular types of supergravity solutions known as p-branes, and from a more specifically string-theoretic standpoint as *topological defects* in type I and type II superstring theory, leading to the extension of T-duality to the type I string.

4.1. p-branes as solutions to supergravity

p-branes are classical solutions to low energy supergravity equations of motion, such as those associated with the five consistent string theories introduced in chapter II [82-83]. They take the form, in the simplest case, of Poincaré invariant hyperplanes in spacetime.

To begin, let us write down a general supergravity action which can be considered a subset of the bosonic sectors of any of the supergravity actions introduced previously, since it includes a metric and a dilaton. To this we couple a $\hat{d}$-form potential, the action appearing as

$$S = \frac{1}{2\kappa^2}\int d^D x\sqrt{-g}\left(R - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2(\hat{d}+1)!}\,e^{-a\phi}\left(F^{(\hat{d}+1)}\right)^2\right) \tag{4.1.1}$$

where $g_{\mu\nu}$ is the Einstein-frame metric, ϕ is the dilaton, and $F^{(\hat{d}+1)}$ is the $(\hat{d}+1)$-form field strength of the $\hat{d}$-form potential, a is a constant depending on $\hat{d}$, and D is the spacetime dimension. Evidently, the precise values of $\hat{d}$ (and therefore of a) will depend on the specific supergravity we are considering. For example, in the case of the type IIA supergravity, $\hat{d} = 1$ and $\hat{d} = 3$ and their Hodge duals are the relevant possibilities.

We now add a coupling to an elementary $\hat{d}$-dimensional object [83], known as a $(p = \hat{d} - 1)$-brane described by a world-volume action similar to that given in (2.1.15)

$$\begin{aligned} S_{\hat{d}} = \frac{\mathcal{T}_{\hat{d}}}{2}\int d^{\hat{d}}\sigma\Big\{&\sqrt{-e}\,e^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}\,e^{a\phi/\hat{d}} + (\hat{d}-2)\sqrt{-e} \\ &- \frac{2}{(\hat{d})!}\epsilon^{a_1\cdots a_{\hat{d}}}\partial_{a_1}X^{\mu_1}\cdots\partial_{a_{\hat{d}}}X^{\mu_{\hat{d}}}A^{(\hat{d})}_{\mu_1\cdots\mu_{\hat{d}}}\Big\}\end{aligned} \tag{4.1.2}$$

where e_{ab} is the world-volume metric with its determinant e and ϵ is the totally antisymmetric tensor on the world sheet. The σ^μ are the world-volume coordinates of the p-brane. Note two changes between this world-volume action and that of (2.1.15). The first is that the dilaton now appears in the kinetic term and the second is the inclusion of the world-volume cosmological constant term $(\hat{d}-2)\sqrt{-e}$. As noted in section 2.1.3, this term is not Weyl invariant and leads to inconsistent classical equations for the string ($\hat{d} = 2$). However, when $\hat{d} \neq 2$ we no longer have Weyl invariance to be broken and thus the inclusion of this term is necessary for full generality. The choice of coupling for this term is thus explained.

The dependence on the dilaton of (4.1.2) is chosen such that under the rescaling

$$\begin{aligned} g_{\mu\nu} &\to \lambda^{\frac{2\hat{d}}{D-2}} g_{\mu\nu} \\ A^{(\hat{d})}_{\mu_1\cdots\mu_{\hat{d}}} &\to \lambda^{\hat{d}} A^{(\hat{d})}_{\mu_1\cdots\mu_{\hat{d}}} \\ e^{\phi} &\to \lambda^{\frac{2\hat{d}\tilde{d}a}{D-2}}\, e^{\phi} \\ e^{ab} &\to \lambda^2 e^{ab} \end{aligned} \tag{4.1.3}$$

where we have introduced the "dual" world-volume dimension $\tilde{d} = D - \hat{d} - 2$, both S and $S_{\hat{d}}$ scale the same way, namely

$$S \to \lambda^{\tilde{d}} S, \qquad S_{\hat{d}} \to \lambda^{\hat{d}} S_{\hat{d}}. \tag{4.1.4}$$

The field equation and the Bianchi identity of the $\hat{d}$-form potential may be written

$$d\,{}^*(\mathrm{e}^{-a\phi}\, F^{(\hat{d}+1)}) = 2\kappa^2\,(-1)^{\hat{d}^2}\ {}^*\mathbb{J} \tag{4.1.5a}$$

$$dF^{(\hat{d}+1)} = 0 \tag{4.1.5b}$$

where $\mathbb{J}$ is a rank $\hat{d}$ tensor source given by

$$\mathbb{J}^{\mu_1\cdots\mu_{\hat{d}}} = \mathcal{T}_{\hat{d}} \int d^{\hat{d}}\sigma \epsilon^{a_1\cdots a_{\hat{d}}} \partial_{a_1} X^{\mu_1} \cdots \partial_{a_{\hat{d}}} X^{\mu_{\hat{d}}} \frac{1}{\sqrt{-g}} \prod_{\nu=0}^{D-1} \delta\left(x^\nu - X^\nu\right) \tag{4.1.6}$$

Variation of the action $S + S_{\hat{d}}$ results in the following equations of motion: for the Einstein equation we have

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{1}{2}\left((\partial_\mu\phi)(\partial_\nu\phi) - \frac{1}{2} g_{\mu\nu}(\nabla\phi)^2\right) \\ - \frac{1}{2(\hat{d}!)}\,\mathrm{e}^{-a\phi}\left(\left(F^{(\hat{d}+1)}\right)^2_{\mu\nu} - \frac{1}{n(\hat{d}+1)} g_{\mu\nu}\left(F^{(\hat{d}+1)}\right)^2\right) = \kappa^2 T^{\hat{d}}_{\mu\nu} \end{aligned} \tag{4.1.7}$$

where the energy momentum tensor $T^{\hat{d}\,\mu\nu}$ and $(F^{(\hat{d}+1)})^2_{\mu\nu}$ are given by

$$\begin{aligned} T^{\hat{d}\,\mu\nu} &= -\,\mathcal{T}_{\hat{d}} \int d^{\hat{d}}\sigma \sqrt{-e}\, e^{ab} \partial_a X^\mu \partial_b X^\nu\, \mathrm{e}^{a\phi/\hat{d}} \frac{1}{\sqrt{-g}} \prod_{\nu=0}^{D-1} \delta\left(x^\nu - X^\nu\right), \\ \left(F^{(\hat{d}+1)}\right)^2_{\mu\nu} &= F^{(\hat{d}+1)}_{\mu\alpha_1\cdots\alpha_{n-1}} {F^{(\hat{d}+1)}_{\nu}}^{\alpha_1\cdots\alpha_{n-1}}, \end{aligned} \tag{4.1.8}$$

the equation of motion of the $\hat{d}$-form potential is

$$\begin{aligned} &\partial_{\mu_1}\left(\sqrt{-g}\,\mathrm{e}^{a\phi}\, F^{(\hat{d}+1)\mu_1\cdots\mu_n}\right) \\ &\qquad = 2\kappa^2 \mathcal{T}_{\hat{d}} \int d^{\hat{d}}\sigma \epsilon^{a_1\cdots a_{\hat{d}}} \partial_{a_1} X^{\mu_1} \cdots \partial_{a_{\hat{d}}} X^{\mu_{\hat{d}}} \prod_{\nu=0}^{D-1} \delta\left(x^\nu - X^\nu\right), \end{aligned} \tag{4.1.9}$$

and finally the dilaton equation is written

$$\begin{aligned} &\partial_\mu\left(\sqrt{-g}\partial^\mu\phi\right) + \frac{a}{2(\hat{d}+1)!}\sqrt{-g}\,\mathrm{e}^{-a\phi}\left(F^{(\hat{d}+1)}\right)^2 \\ &\qquad = \frac{a\kappa^2\mathcal{T}_{\hat{d}}}{\hat{d}} \int d^{\hat{d}}\sigma \sqrt{-e}\, e^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}\, \mathrm{e}^{a\phi/\hat{d}} \prod_{\nu=0}^{D-1} \delta\left(x^\nu - X^\nu\right). \end{aligned} \tag{4.1.10}$$

In addition, the field equations of the p-brane, obtained by varying $S + S_{\hat{d}}$ with respect to X^μ and e^{ab} respectively, are written

$$\partial_a\left(\sqrt{-e}e^{ab}\partial_b X^\mu g_{\mu\nu}\,e^{a\phi/\hat{d}}\right) - \frac{1}{2}\sqrt{-e}e^{ab}\partial_a X^\mu \partial_b X^\nu \partial_\rho\left(g_{\rho\mu}\,e^{a\phi/\hat{d}}\right) - \frac{1}{\hat{d}!}\epsilon^{a_1\cdots a_{\hat{d}}}\partial_{a_1}X^{\mu_1}\cdots\partial_{a_{\hat{d}}}X^{\mu_{\hat{d}}}F^{(\hat{d}+1)}_{\rho\mu_1\cdots\mu_{\hat{d}}} = 0 \tag{4.1.11}$$

and

$$e_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}\,e^{a\phi/\hat{d}}\,. \tag{4.1.12}$$

It is evident that these equations are rather complex and difficult to solve. However, we can make progress in finding a solution by imposing a simplifying ansatz.

We begin this ansatz by assuming that the world-volume of our object will be invariant under Poincaré transformations. We also, for the sake of simplicity, demand that the solution be isotropic in the coordinates orthogonal to the translationally invariant ones. For the solutions for which we are looking, then, spacetime is divided into two sets. A set of translationally invariant coordinates of the solution, which are the world-volume coordinates, and a set of coordinates in which the solution is isotropic, normally termed the *transverse* space. It is worth noting that this last restriction can be relaxed in various generalizations of the prototype p-brane solutions for which we now search.

Poincaré invariance is thus imposed on the $\hat{d}$ world-volume coordinates, whereas the isotropic nature of the transverse space can be assured by imposing $SO(D-\hat{d})$ invariance. Thus our ansatz imposes (Poincaré)($\hat{d}$) $\otimes$ $SO(D-\hat{d})$ symmetry. We therefore divide the coordinates x^μ into two ranges, as $x^\mu = \{x^{\hat{\mu}}, y^i\}$ where $\hat{\mu} = 0, 1, \cdots, \hat{d}-1$ and $i = \hat{d}, \cdots, D-1$.

An ansatz for a metric which realizes this symmetry is given by [79]

$$ds^2 = e^{2f_1(r)}\,\eta_{\hat{\mu}\hat{\nu}}dx^{\hat{\mu}}dx^{\hat{\nu}} + e^{2f_2(r)}\delta_{ij}dy^i dy^j \tag{4.1.13}$$

where f_1 and f_2 are functions to be determined and $r = \sqrt{\vec{y}^2}$ is the radial coordinate in the transverse space. From this ansatz, it is easy to see that translational invariance in the world-volume directions $x^{\hat{\mu}}$ as well as $SO(D-\hat{d})$ invariance in the transverse space is guaranteed by the metric elements having only dependence of the transverse radial vector r. The accompanying ansatz for the scalar dilaton field is simply $\phi(x^\mu) = \phi(r)$.

We now come to the ansatz for the $\hat{d}$ form potential. As we mentioned in section 2.1.1 the natural coupling to the world-line of a point particle is a one-form potential. To the world sheet of a string couples a two-form potential. Thus as in section 2.1.2 we have an $\hat{d}$-form potential coupling to the $\hat{d}$-dimensional world volume of an extended $\hat{d}-1$-dimensional object which carries charge.

The charges of objects like the ones we are constructing are defined in terms of Gauss'-law type integrals of the corresponding field strength over the surface at asymptotic infinity. The simplest ansatz that gives a non-zero electric charge q^e can be written

$$A^{(\hat{d})} = -\,\mathrm{e}^{f_3(r)}\, dx^{\hat{\mu}_1} \wedge dx^{\hat{\mu}_2} \wedge \cdots \wedge dx^{\hat{\mu}_{\hat{d}}} \tag{4.1.14}$$

for which the field strength is

$$F^{(\hat{d}+1)} = \partial_i\, \mathrm{e}^{f_3(r)}\, dy^i \wedge dx^{\hat{\mu}_1} \wedge dx^{\hat{\mu}_2} \wedge \cdots \wedge dx^{\hat{\mu}_{n-1}} \tag{4.1.15}$$

This ansatz is called the *electric ansatz* since it is an ansatz for a generalization of the one-form potential familiar from the Maxwell theory. We restate in the interest of clarity that the hatted indicies, e.g., $\hat{\mu}$ represent coordinates in the world-volume of the p-brane, while i, etc., represent transverse coordinates.

For the p-brane we also make the coordinate split $X^\mu = \{X^{\hat{\mu}}, Y^i\}$ and make the *static gauge* choice which identifies the world-volume coordinates of the p-brane and their spacetime counterparts, that is $X^{\hat{\mu}} = \sigma^{\hat{\mu}}$. We also impose the condition that Y^i = constant for which we will see the significance later.

Now that our ansatz has been specified, it is time to insert it into our equations of motion. Substitution into (4.1.12) results in

$$e_{ab} = \mathrm{e}^{2f_1 + a\phi/\hat{d}}\, \eta_{ab} \tag{4.1.16}$$

while the world-volume $\hat{\mu}, \hat{\nu}$ components of the Einstein equation reduce to a single equation

$$
\begin{aligned}
\mathrm{e}^{(\hat{d}-2)f_1+\tilde{d}f_2}\,\delta^{ij}\Big((\hat{d}-1)\partial_i\partial_j f_1 &+ \frac{\hat{d}(\hat{d}-1)}{2}\partial_i f_1\partial_j f_1 + (\tilde{d}+1)\partial_i\partial_j f_2 \\
&+ \frac{\tilde{d}(\tilde{d}+1)}{2}\partial_i f_2\partial_j f_2 + \tilde{d}(\hat{d}-1)\partial_i f_1\partial_j f_2 \\
&+ \frac{1}{4}\,\mathrm{e}^{-2\hat{d}f_1+2f_3-a\phi}\,\partial_i f_3\partial_j f_3 + \frac{1}{4}\partial_i\phi\partial_j\phi\Big) \\
&= -\kappa^2 \mathcal{T}_{\hat{d}} e^{(\hat{d}-2)f_1+a\phi/2}\prod_{i=D-\hat{d}}^{D-1}\delta\left(y^i\right)
\end{aligned}
\tag{4.1.17}
$$

where again $\tilde{d} = D - \hat{d} - 2$, the dimension "dual" to $\hat{d}$. For the transverse space components i, j we have

$$
\begin{aligned}
\mathrm{e}^{\hat{d}f_1+(\tilde{d}-2)f_2}\Big(\tilde{d}\partial^i\partial^j f_2 &- \delta^{ij}\tilde{d}\delta^{kl}\partial_k\partial_l f_2 + \hat{d}\partial^i\partial^j f_1 - \hat{d}\delta^{ij}\delta^{kl}\partial_k\partial_l f_1 \\
&+ \hat{d}\partial^i f_1\partial^j f_1 - \frac{\hat{d}(\hat{d}+1)}{2}\delta^{ij}\delta^{kl}\partial_k f_1\partial_l f_1 - \hat{d}(\partial^i f_1\partial^j f_2 + \partial^i f_2\partial^j f_1 \\
&+ (\tilde{d}-1)\delta^{ij}\delta^{kl}\partial_k f_1\partial_l f_2) + \frac{1}{2}\partial^i\phi\partial^j\phi - \frac{1}{4}\delta^{ij}\delta^{kl}\partial_k\phi\partial_l\phi\Big) \\
&- \frac{1}{2}\,\mathrm{e}^{-\hat{d}f_1+(\tilde{d}-2)f_2+2f_3-a\phi}\left(\partial^i f_3\partial^j f_3 - \frac{1}{2}\delta^{ij}\delta^{kl}\partial_k f_3\partial_l f_3\right) = 0,
\end{aligned}
\tag{4.1.18}
$$

for the potential equation (4.1.11) we have

$$
\delta^{ij}\partial_i\left(\mathrm{e}^{-\hat{d}f_1+\tilde{d}f_2-a\phi}\,\partial_j f_3\right) = 2\kappa^2\mathcal{T}_{\hat{d}}\prod_{i=D-\hat{d}}^{D-1}\delta\left(y^i\right), \tag{4.1.19}
$$

and for the dilaton (Eq. (4.1.10) we obtain

$$
\begin{aligned}
\delta^{ij}\partial_i\left(\mathrm{e}^{\hat{d}f_1+\tilde{d}f_2}\,\partial_j\phi\right) &- \frac{a}{2}\,\mathrm{e}^{-\hat{d}f_1+\tilde{d}f_2+2f_3-a\phi}\,\delta^{ij}\partial_i f_2\partial_j f_3 \\
&= a\kappa^2\mathcal{T}_{\hat{d}} e^{\hat{d}f_1+a\phi/2}\prod_{i=D-\hat{d}}^{D-1}\delta\left(y^i\right)
\end{aligned}
\tag{4.1.20}
$$

while finally, for the p-brane equation we find

$$
\partial_i\left(\mathrm{e}^{\hat{d}f_1+a\phi/2} - e^{f_3}\right) = 0. \tag{4.1.21}
$$

Equations (4.1.17), (4.1.18), (4.1.19), (4.1.20) and (4.1.21) thus provide the system of five equations needed to determine the four unknown functions f_1, f_2, f_3, ϕ and the fixed coupling a.

The unique solution, assuming that the metric is asymptotically flat, i.e., $g_{\mu\nu} \to \eta_{\mu\nu}$ as $r \to \infty$ can be found to be [82,83]

$$\begin{aligned} f_1 &= \frac{\tilde{d}}{2(\hat{d}+\tilde{d})}\left(f_3 - \bar{f}_3\right), \\ f_2 &= -\frac{\hat{d}}{2(\hat{d}_{\tilde{d}})}\left(f_3 - \bar{f}_3\right), \\ \frac{a\phi}{2} &= \frac{a^2}{4}\left(f_3 - \bar{f}_3\right) + \bar{f}_3 \end{aligned} \tag{4.1.22}$$

where $\bar{f}_3 = a\phi_0/2$ and ϕ_0 is the vacuum expectation value of the dilaton. The function f_3 is then given by

$$e^{-f_3} = \mathcal{H}(\vec{y}) = e^{-\bar{f}_3} + \mathcal{G}(\vec{y}) \tag{4.1.23}$$

where $\mathcal{G}$ is given by

$$\mathcal{G} = \begin{cases} \frac{1}{\tilde{d}}\frac{\ell}{r^{\tilde{d}}} & \tilde{d} \geq 1 \qquad (4.1.24a) \\ -\ell \log(r) & \tilde{d} = 0 \qquad (4.1.24b) \\ -\frac{r}{\ell} & \tilde{d} = -1 \qquad (4.1.24c) \end{cases}$$

where we have

$$\ell = \begin{cases} 2\kappa^2 \mathcal{T}_{\hat{d}} \mathcal{A}_{\tilde{d}+1} & \tilde{d} \geq 1 \\ \frac{\kappa^2 \mathcal{T}_{\hat{d}}}{\pi} & \tilde{d} = 0 \end{cases} \tag{4.1.25}$$

where in turn $\mathcal{A}_{\tilde{d}+1}$ is the area of the unit $(\tilde{d}+1)$-sphere. The parameter a is given as

$$a^2 = 4 - \frac{2\hat{d}\tilde{d}}{(D-2)}. \tag{4.1.26}$$

We thus arrive at our basic solution for p-branes, or Poincaré invariant hyperplanes, which is given for the general action (4.1.1) and (4.1.2) in the case that the dilaton vanishes asymptotically as

$$\begin{aligned} ds^2 &= \mathcal{H}^{\alpha} dx^{\hat{\mu}} dx^{\hat{\nu}} \eta_{\hat{\mu}\hat{\nu}} + \mathcal{H}^{\beta} dy^i dy^j \delta_{ij}, \\ A^{(\hat{d})} &= \pm \frac{1}{\mathcal{H}(\vec{y})} dx^{\hat{\mu}_1} \wedge dx^{\hat{\mu}_2} \wedge \cdots \wedge dx^{\hat{\mu}_{\hat{d}}}, \\ e^{\phi} &= \mathcal{H}^{\gamma}, \end{aligned} \tag{4.1.27}$$

where

$$\alpha = -\frac{\tilde{d}}{(D-2)}, \quad \beta = \frac{\hat{d}}{(D-2)}, \quad \gamma = -\frac{a}{2}. \tag{4.1.28}$$

Note that it is evident from the form of $\mathcal{H}$ from (4.1.23) that the solution approaches flat space at asymptotic infinity in the transverse space. Note also that $g = e^{\phi} \to 0$ as $r \to 0$, and thus these solutions may correspond to the exterior of perturbative states.

4.1.1. Masses and charges for p-branes

The definition of the Arnowitz-Deser-Misner (ADM) mass [4] for a given solution, such as a p-brane, is given in the case of asymptotic Cartesian coordinates by the formula

$$M = \frac{1}{2\kappa^2} \oint \sum_{i=1}^{D-\hat{d}} n^i \left[\sum_{j=1}^{D-\hat{d}} \left(\partial_j h_{ij} - \partial_i h_{jj} \right) - \sum_{\hat{\mu}=1}^{\hat{d}-1} \partial_i h_{\hat{\mu}\hat{\mu}} \right] r^{\tilde{d}+1} d\Omega \qquad (4.1.29)$$

where n^i is a radial unit vector in the transverse space and $h_{\mu\nu}$ is the deformation of the *Einstein-frame* metric

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \qquad (4.1.30)$$

from flat space in the asymptotic region. It is thus a measure of how quickly a spacetime approaches flat space. In equation (4.1.29), the indices i and j denote the $9-p$ transverse coordinates, while $\hat{\mu}$ labels the p spatial coordinates parallel to the world-volume.

Application of this formula gives for the prototype p-brane solution (4.1.27) the result[1]

$$M = \mathcal{T}_{\hat{d}} e^{\bar{f}_3} \mathcal{A}_{\tilde{d}+1} \qquad (4.1.31)$$

where we have again made ϕ_0 arbitrary.

The charges are defined in a similar manner to the ADM mass as Gaussian integrals, that is, we define the "electric" charge q^e and the "magnetic" charge q^m as [83]

$$q^e = \frac{1}{\sqrt{2}\kappa} \oint {}^*F^{(\hat{d}+1)} , \qquad q^m = \frac{1}{\sqrt{2}\kappa} \oint F^{(\hat{d}+1)} \qquad (4.1.32)$$

[1] It should be stated that the ADM mass formula (4.1.29) is only appropriate when $\tilde{d} \geq 1$ since for $\tilde{d} < 1$ this measure of the mass diverges.

where $\kappa = \sqrt{8\pi G_N}$ is related to the Newton constant G_N in D dimensions. Note that Hodge duality in the q^e formula is performed with respect to the string-frame metric. Their application to the solution (4.1.27) results in the charges

$$\begin{aligned} q^e &= \sqrt{2}\kappa \mathcal{T}_{\hat{d}} (-)^{(\hat{d}+1)(D-\hat{d})} \mathcal{A}_{\tilde{d}+1} \\ q^m &= 0. \end{aligned} \tag{4.1.33}$$

From (4.1.33) and (4.1.31) one sees that the charge q^e and the mass M obey the relation

$$M = \frac{1}{\sqrt{2}\kappa} |q^e| \, e^{\bar{f}_3} \tag{4.1.34}$$

which looks very much like the mass-charge relation of a BPS saturated state, although here no supersymmetry has been assumed.

4.1.2. Multi-centre p-brane solutions

It is, of course, evident from the form of the basic p-brane solution that we can replace the harmonic function $\mathcal{H}$ of (4.1.26) with a more general version. One standard generalization is that of an array of parallel p-branes. Thus we can have

$$\mathcal{H}(\vec{y}) = 1 + \sum_i \mathcal{G}_i \left(\frac{\ell_i}{|\vec{r} - \vec{r}_i|} \right) \tag{4.1.35}$$

where $\vec{r}_i$ is a constant vector defining the centre of the p-brane. This harmonic function represents an array of parallel p-branes, each having the same dimension and transverse space. The ability to superpose solutions in this manner is related to the *zero force condition*, in which the static force between two parallel p-branes vanishes. This implies that two parallel p-branes can remain in equilibrium. This condition is closely related to supersymmetry.

We have thus succeeded in constructing elementary p-brane solutions to the supergravity action (4.1.1) coupled to a $\hat{d}$-dimensional object. We label these solutions elementary because they have non-vanishing electric charge and exhibit δ-function singularities at $r = 0$. This is as we discussed in chapter III on string duality. Let us move on to consider briefly solitonic solutions to supergravity.

4.1.3. Supergravity solitons

In this section we seek the solutions dual to those of the last section, namely solitonic solutions. As we noted in chapter III, these should be regular at the origin, rather than singular, and should carry a non-vanishing magnetic charge.[2]

To begin, we consider the action (4.1.1) alone. We construct an ansatz, this time with Poincaré invariance in a $\tilde{d}$-dimensional world-volume[3] and with $SO(D-\tilde{d})$ invariance in the now $(D-\tilde{d})$-dimensional transverse space, thus the group is Poincaré$(\tilde{d}) \otimes SO(D-\tilde{d})$. A split of the coordinates into world-volume and transverse is again made, but this time $x^\mu = \{x^{\hat{\mu}}, y^i\}$ where $\hat{\mu} = 0, 1, \cdots, \tilde{d}-1$ and $i = \tilde{d}, \cdots, D-1$. For the $\tilde{d}$-form potential, however, we write the ansatz for its field strength rather than for the potential itself. To obtain a non-vanishing magnetic charge from the definition given in (4.1.32) we write the ansatz as

$$F^{(\hat{d}+1)} = \sqrt{2}\kappa q^m dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_{\hat{d}+1}} \tag{4.1.36}$$

where q^m is the magnetic charge. Since this is a harmonic form there is no globally valid potential which can be written down which gives $F^{(\hat{d}+1)}$ when the exterior derivative is taken, but it does satisfy the Bianchi identities.[4]

It is not difficult to show that the field equations of the action (4.1.1) are satisfied when the replacements $\hat{d} \to \tilde{d}$ and $a \to -a$ are made in equations (4.1.17) – (4.1.21) with the source terms set equal to zero. Thus we can write the solution (with $\phi_0 = 0$) in this case as

$$\begin{aligned} ds^2 &= \mathcal{H}^{\alpha} dx^{\hat{\mu}} dx^{\hat{\nu}} \eta_{\hat{\mu}\hat{\nu}} + \mathcal{H}^{\beta} dy^i dy^j \delta_{ij}, \\ F^{(\hat{d}+1)} &= \sqrt{2}\kappa q^m dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_{\hat{d}+1}} \\ e^{\phi} &= \mathcal{H}^{\gamma} \end{aligned} \tag{4.1.37}$$

where

$$\alpha = -\frac{\tilde{d}}{(D-2)}, \quad \beta = \frac{\hat{d}}{(D-2)}, \quad \gamma = \frac{a}{2}. \tag{4.1.38}$$

---

[2] As the name "soliton" suggests, at least certain classes of these solutions exhibit scattering behavior normally associated with particular solutions of non-linear wave equations [84].

[3] Thus we see why we called $\tilde{d}$ the dimension dual to $\hat{d}$.

[4] According to the Hodge decomposition theorem one can decompose a general p-form ω as $\omega = d\alpha + {}^*d({}^*\beta) + \gamma$. Here α is a $(p-1)$-form, and β a $(p+1)$-form. The form γ satisfies the Laplace equation $\nabla^2\gamma = 0$ and is thus harmonic. Such a form cannot, in general, be written as the exterior derivative of a $(p-1)$-form.

and where

$$\mathcal{H}(\vec{y}) = 1 + \mathcal{G}(\vec{y}) \tag{4.1.39}$$

where $\mathcal{G}$ is given by

$$\mathcal{G} = \begin{cases} \frac{1}{\hat{d}} \frac{\ell'}{r^{\hat{d}}} & \hat{d} \geq 1 \quad (4.1.40a) \\ -\ell' \log(r) & \hat{d} = 0 \quad (4.1.40b) \\ -\frac{r}{\ell'} & \hat{d} = -1 \quad (4.1.40c) \end{cases}$$

where we now have

$$\ell' = \begin{cases} 2\kappa^2 \mathcal{T}_{\tilde{d}} \mathcal{A}_{\tilde{d}+1} & \tilde{d} \geq 1, \\ \frac{\kappa^2 \mathcal{T}_{\tilde{d}}}{\pi} & \tilde{d} = 0. \end{cases} \tag{4.1.41}$$

In setting the source terms to zero one avoids the δ-function singularities, and thus these solutions are considered regular. However, they are regular only in the string frame. Since our supergravity action (4.1.1) is written in the Einstein frame, the solutions are singular. Also note that these solutions are nonperturbative, since the string coupling $g = e^{\phi} \to \infty$ as $r \to 0$.

For these solutions the mass and charges are given by

$$\begin{aligned} M &= \mathcal{T}_{\tilde{d}} e^{\bar{f}_3} \mathcal{A}_{\hat{d}+1} \\ q^e &= 0, \\ q^m &= \sqrt{2}\kappa \mathcal{T}_{\tilde{d}} (-)^{(\tilde{d}+1)(D-\tilde{d})} \mathcal{A}_{\hat{d}+1}. \end{aligned} \tag{4.1.42}$$

where again $\bar{f}_3 = a\phi_0/2$, and therefore we have an exactly similar relation between the mass and the charge as in the elementary case, i.e.,

$$M = \frac{1}{\sqrt{2}\kappa} |q^m| e^{\bar{f}_3}. \tag{4.1.43}$$

As a final remark, we will state without proof that the electric charge of the elementary solution and the magnetic charge of the solitonic solution obey a Dirac quantization condition [85-87] which is written

$$q^e q^m = 2\pi n, \qquad n \in \mathbb{Z}. \tag{4.1.44}$$

This completes our short survey of the supergravity solutions known as p-branes, where incidentally $p = \hat{d} - 1$ = the number of *spatial* dimensions in the world-volume. These solutions we constructed by means of an ansatz that separates the

D spacetime dimensions into world-volume and transverse coordinates. We found two classes of these solutions, corresponding to elementary (singular, electric) and solitonic (regular, magnetic) cases. We shall move on in the next section to develop what are known as Dirichlet Branes or D-branes and show that they are closely related to the p-branes.

4.2. Dirichlet branes

D-branes [14,65,81] are extended objects that are closely related to the p-branes of the previous section. It will turn out that D-branes can be described in spacetime by Bogomol'nyi-Prasad-Sommerfeld (BPS) saturated versions of a subset of the possible p-brane solutions. The necessity of their presence in string theory can be argued on the basis of T-duality, which was presented in detail in chapter III. In certain limits, T-duality results in what might be termed paradoxical behavior. Let us examine this paradox further, demonstrating how it leads us to the D-branes.

4.2.1. T-duality for open strings

There seems to be a paradox that arises in the Type I string in the limit that the radius of a compact dimension vanishes. Let us write the mass spectrum for a closed string where one dimension has been compactified. From (2.2.50) one has

$$\alpha' M^2 = 2(N + \tilde{N} - 2) + \alpha' \frac{k^2}{R^2} + \frac{m^2 R^2}{\alpha'}. \tag{4.2.1}$$

where N and $\tilde{N}$ are the contribution to the mass of the oscillators. From this we see that when the radius of the compact coordinate goes to zero, any non zero mode ($k \neq 0$) of the center of mass momentum will have infinite mass, while at the same time a continuum of masses are produced from the winding number m. Thus the dimension does not disappear, but becomes uncompactified.

However, open strings cannot wind around the periodic dimension and thus they have no quantum number which plays the role of m in (4.2.1), thus when $R \to 0$ the $k \neq 0$ states go to infinite momentum as expected, but there is no new continuum of states. Due to this the compactified dimension disappears, leaving behind a theory in $D - 1$ spacetime dimensions.

What is it that brings about the seeming paradox? It is the fact that the open string theory always contains a closed-string sector. Thus we have, in the $R \to 0$ limit a string theory in which the closed strings live in D spacetime dimensions, but at the same time the open strings live in $D-1$ dimensions. This seems slightly bizarre at first view.

It is possible, however, to puzzle out what is going on [65]. Between its endpoints the open string is exactly the same as the closed string, and thus should still be moving in D dimensions. The thing that is different with the open string is thus only the endpoints, which are seemingly restricted to lie on a $D-1$ dimensional hyperplane.

Indeed, that this is the case can be demonstrated from the T-duality transformations. Recall from chapter II, equation (3.3.4) that the right- and left-moving zero-mode momenta of the closed string are

$$
\begin{aligned}
p_r = \alpha_0^{25} &= \sqrt{\frac{\alpha'}{2}}\left(\frac{k}{R} + \frac{Rm}{\alpha'}\right), \\
p_l = \tilde{\alpha}_0^{25} &= \sqrt{\frac{\alpha'}{2}}\left(\frac{k}{R} - \frac{Rm}{\alpha'}\right).
\end{aligned}
\tag{4.2.2}
$$

where we have chosen to label the compact coordinate X^{25}. Under the transformation

$$
R \to \frac{\alpha'}{R} \qquad k \leftrightarrow m \tag{4.2.3}
$$

it is easy to see that

$$
\alpha_0^{25} \to \alpha_0^{25}, \qquad \tilde{\alpha}_0^{25} \to -\tilde{\alpha}_0^{25}. \tag{4.2.4}
$$

We can straightforwardly generalize this to several compactified dimensions by defining a vector of momentum-mode integers $\vec{k}$ and a similar vector of winding mode integers $\vec{m}$, and we will have the zero-mode momenta as

$$
\begin{aligned}
p_r^i = \alpha_0^i &= \sqrt{\frac{\alpha'}{2}}\left(\frac{k_i}{R_i} + \frac{R_i m_i}{\alpha'}\right), \\
p_l^i = \tilde{\alpha}_0^i &= \sqrt{\frac{\alpha'}{2}}\left(\frac{k_i}{R_i} - \frac{R_i m_i}{\alpha'}\right).
\end{aligned}
\tag{4.2.5}
$$

where the index i indicates the compact dimension. It is then clear that the state denoted by $(\vec{k}, \vec{m})$ with radii R_i is the same as the stated denoted by $(\vec{m}, \vec{k})$ at the dual radii $R_i' = \alpha'/R_i$.

Let us now define coordinates Y dual to X, which describe the theory after the tranformation (4.2.3). We will then have, due to a suitable generalization of (4.2.4) the relationship between X and Y as

$$Y_r^\mu\left(\sigma^-\right) = X_r^\mu\left(\sigma^-\right) \qquad Y_l^\mu\left(\sigma^+\right) = (-1)^{S_\mu} X_l^\mu\left(\sigma^+\right) \tag{4.2.6}$$

where $S_\mu = 0$ for a coordinate along which T-duality is not being carried out, and $S_\mu = 1$ for coordinates which are being T-dualized.

It is clear that the change in sign of the left-moving coordinate makes very little[5] difference in the case of the closed string, but let us return to the open string case. Recall that the boundary condition at the end of the string must be chosen so that the surface term (2.1.26) vanishes. The condition that is chosen is normally the Neumann condition $\partial_\sigma X = 0$. The action of T-duality on this boundary condition is as follows:

$$\begin{aligned}
\partial_\sigma X^\mu &= \left(\partial_\sigma \sigma^-\right)\partial_- X^\mu + \left(\partial_\sigma \sigma^+\right)\partial_+ X \\
&= -\,\partial_- X_r^\mu + \partial_+ X_l^\mu \\
&= -\,\left(\partial_- Y_r^\mu + \partial_+ Y_l^\mu\right) \\
&= -\,\partial_\tau Y^\mu
\end{aligned} \tag{4.2.7}$$

which implies that in the T-dual theory the compact Y^μ are constant along each world sheet boundary, that is they do not move. This is the Dirichlet boundary condition, Y^μ = constant. Note also that this effect is completely reversible. T-duality applied to a Dirichlet boundary condition changes it to a Neumann boundary condition.

Further, the compact Y^i coordinates are the same on every world sheet boundary [66]. To see this, we write the open string mode expansions as

$$X_r^\mu = \frac{1}{2}\left(x^\mu + \ell^2 p^\mu \sigma^- + i\ell \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu\, \mathrm{e}^{-in\sigma^-}\right) \tag{4.2.8a}$$

$$X_l^\mu = \frac{1}{2}\left(x^\mu + \ell^2 p^\mu \sigma^+ + i\ell \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu\, \mathrm{e}^{-in\sigma^+}\right) \tag{4.2.8b}$$

---

5 Except for a change in chirality for the type II string as we saw in chapter II.

where $\ell = \sqrt{2\alpha'}$ and where for the compact coordinates $p^i = \frac{n_i}{R_i}$. The dual compact coordinates $Y^i = X_r^i - X_l^i$ then have

$$\begin{aligned} Y^i(\sigma = 0) - Y^i(\sigma = 0) &= -\pi\ell^2 p^i \\ &= \frac{2\pi\alpha' n_i}{R_i} \\ &= 2\pi n_i R_i'. \end{aligned} \tag{4.2.9}$$

which indicates that $Y^i(\sigma = \pi)$ and $Y^i(\sigma = 0)$ are identical points on the dual torus of coordinates. The ends of the open string are attached to the hyperplane while at the same time the string can wind around the compact dimension. Thus for the open string, T-duality has again interchanged momentum modes and winding modes. The ends of the string are, of course, still free to move in all dimensions that have not undergone a T-duality transformation. One can think of these hyperplanes as defects in spacetime. Such a topological defect with p spatial dimensions is in general described by the combination of Neumann and Dirichlet boundary conditions

$$\partial_n X^0 = \partial_n X^1 = \cdots = \partial_n X^p = X^{p+1} = X^{p+2} = \cdots = X^{D-1} = 0 \tag{4.2.10}$$

These hyperplanes have been given the name Dirichlet or D-branes, and we will see that they are dynamical objects.

4.2.2. An action for D-branes

Now that we have good evidence that the ends of open strings become trapped on D-branes under T-duality, how do we describe them? As objects which constitute part of string theory, they should be described, as are strings themselves, by a conformally invariant field theory. One can write a σ-model action including Dirichlet boundary conditions which represents a D-brane moving in an arbitrary massless background (metric, antisymmetric tensor and dilaton), then by carrying out a computation paralleling that of section 2.3.1 the vanishing of the β-function gives us field equations. From these equations we can derive an effective action [88] for a Dp-brane with p spatial dimensions (and thus $\hat{d} = p+1$-dimensional world-volume) to be

$$\hat{S} = \mathcal{T}_p \int d^{\hat{d}}\sigma\, e^{-\phi} \sqrt{-\det(\hat{G}_{ab} + \hat{B}_{ab} + 2\pi\alpha' \mathbb{F}^{(2)})} + \mu_p \int A^{(\hat{d})} \tag{4.2.11}$$

where $\hat{G}_{ab} = G^{\mu\nu}\partial_a X_\mu \partial_b X_\nu$ is the induced metric on the D-brane world-volume, $\hat{B}_{ab}$ the induced antisymmetric tensor and similarly for $\hat{\mathbb{F}}^{(2)}$, the field strength of the world-volume $U(1)$ gauge field $\mathbb{A}^{(1)}$ of the open string. The parameters $\mathcal{T}_p$ and μ_p are the D-brane tension (= mass density) and charge density under the Ramond-Ramond $p+1 = \hat{d}$-form $A^{(\hat{d})}$. The dilaton dependence comes about from the fact that this is a tree-level open string action. It is also possible to have several D-branes whose world-volumes coincide. In this case the $U(1)$ gauge field of the single brane is generalized to a $U(n)$ gauge field for n coincident branes. If one of the n D-branes is displaced, then this breaks the $U(n)$ symmetry to $U(n-1)\otimes U(1)$.

That the combination $\hat{B}_{ab} + 2\pi\alpha'\hat{\mathbb{F}}^{(2)}$ appears in the action is not an accident. If we carry out a gauge transformation on $\hat{B}_{ab}$ we find

$$\hat{B}'_{ab} = \hat{B}_{ab} + \partial_a \hat{\chi}_b - \partial_b \hat{\chi}_a \tag{4.2.12}$$

which gives rise to a surface term that must be cancelled by assiging to $\hat{\mathbb{A}}^{(1)}$ the transformation

$$\hat{\mathbb{A}}_a^{(1)\prime} = \hat{\mathbb{A}}_a^{(1)} - \hat{\chi}_a. \tag{4.2.13}$$

From T-duality we can determine that there should be additional terms in the Ramond-Ramond part of the action (4.2.11) [65]. One way to see this is as follows. Consider the example of a $D1$-brane (D-string) in the x^1-x^2 plane. For this D-string there are only two non-zero terms in the potential to which it couples, and these are $A^{(2)}_{tx^1}$ and $A^{(2)}_{tx^2}$. Let us use world-volume coordinates σ^0 and σ^1, choosing a gauge in which $\partial_{\sigma^0} X^t = 1$ and $\partial_{\sigma^1} X^{x^1} = 1$. Evidently $\sigma^0 = t$ and $\sigma^1 = x^1$ so we can write the action for this D-string as

$$S'_{D\text{-string}} = \mu_1 \int dx^1 \left(A^{(2)}_{tx^1} + \partial_{x^1} X^{x^2} A^{(2)}_{tx^2} \right). \tag{4.2.14}$$

If we choose the gauge $\mathbb{A}^{(1)}_{x^2} = X_{x^2}$ for the world-volume $U(1)$ gauge field of the D-brane and carry out T-duality in the x^2 direction using the results of section 3.3.2 we obtain

$$S'_{D\text{-string}} = \mu_1 \int dx^1 \left(A^{(3)}_{tx^1x^2} + 2\pi\alpha' \mathbb{F}^{(2)}_{x^1x^2}\chi \right) \tag{4.2.15}$$

where $\mathbb{F}^{(2)} = d\mathbb{A}^{(1)}$, and χ is the type IIB RR scalar. The generalization of this can be shown to be [89]

$$S'_{D\text{-brane}} = \mu_p \int \mathrm{Tr}\left(e^{2\pi\alpha' \mathbb{F}^{(2)}+B}\right) \sum_{\hat{d}} A^{(\hat{d})}. \tag{4.2.16}$$

The Ramond-Ramond term in (4.2.11) thus represents the lowest order in an expansion of (4.2.16). The integration picks out the $\hat{d}$-form of the expansion, and the trace is taken over the n-dimensional representation of $U(n)$, in the case that there are n coincident D-branes. In (4.2.16) all spacetime form fields are implicitly written as induced fields on the D-brane world-volume.

The massless closed type II strings coupling to the D-brane have the spacetime action given in equations (2.3.23) and (2.3.26) and rewritten here generically as

$$\begin{aligned} S_{II} = \frac{1}{2\kappa^2} \int d^{10}x\sqrt{-G}\Big\{ & e^{-2\phi}\left(R + 4(\nabla\phi)^2 - \frac{1}{12}H^2\right) \\ & - \sum \frac{1}{2(\hat{d}+1)!}\left(F^{(\hat{d}+1)}\right)^2 \Big\} \end{aligned} \tag{4.2.17}$$

where the fields are as in section 2.3.3 and where $\hat{d} = 1, 3$ for the type IIA theory and $\hat{d} = 0, 2, 4$ in the type IIB theory. Note that the case $\hat{d} = 4$ is problematic as explained in chapter II, giving rise to a self-dual 5-form field strength. Also, here we ignore the Chern-Simons terms.

4.2.3. *D*-brane tension and charge

Let us now try to discover some of the properties of these D-branes given the action (4.2.11). The first thing we will calculate is the D-brane tension $\mathcal{T}_p$, for which the simplest calculation is that illustrated in Fig. 4.2.1, the exchange of a closed string between two D-branes.

We begin by parameterizing the world sheet as given in Fig. 4.2.1, where $0 \leq \tau \leq \pi$ runs along the world sheet from one D-brane to the other and $0 \leq \sigma \leq 2\pi t$ is a periodic coordinate with modulus $0 \leq t \leq \infty$. With our new understanding of duality in the open string, we have two ways of interpreting this graph. The first way is to make time run horizontally along τ, in which sense we have a tree-level closed string exchange. If we make time run vertically instead, then what we see

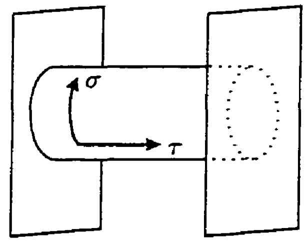


Figure 4.2.1: Schematic depiction of a closed string being exchanged between two D-branes.

is an open string, with one end fixed to each of the D-branes, appearing out of the vacuum, splitting, rejoining, and then disappearing, i.e., it is one-loop open string graph.

Consider taking the $t \to 0$ limit of the open-string loop amplitude. This effectively takes the area of the loop to zero, but unlike a closed string loop, which is topologically a torus, there is no requirement of modular invariance[6] to cut off the range of integration hence preventing a divergence. However, since we have a dual description, by taking time horizontally we find that the limit $t \to 0$ is dominated by the lowest modes of the closed string spectrum. Thus the $t \to 0$ limit may be interpreted as a closed string infrared divergence and the string folklore of no ultraviolet divergences in string theory and that all divergences are controlled by long-distance (lightest modes) physics, is upheld.

One-loop vacuum amplitudes can be computed with the Coleman-Weinberg formula [90] which is a sum over the zero-point energies of all the modes, as in

$$A = -\frac{V_{\hat{d}}}{2} \int \frac{d^{\hat{d}}k}{(2\pi)^{\hat{d}}} \int_0^\infty \frac{dt}{t} \sum_i e^{-\pi\alpha' t \frac{\vec{k}^2 + M_i^2}{2}} \tag{4.2.18}$$

where we are performing a sum over the physical spectrum of the string M_i^2 transverse to the D-brane, and an integral over the momentum $\vec{k}$ in the $p+1 = \hat{d}$ extended directions of the D-brane world-volume. The mass spectrum is given by

$$M^2 = \frac{y^j y_j}{4\pi^2\alpha'^2} + \frac{1}{\alpha'} \begin{cases} \sum_{n=1}^{\infty} \alpha_{-n}^{\mu}\alpha_{n\mu} + \sum_{r=1/2}^{\infty} r\eta_{-r}^{\mu}\eta_{r\mu} - \frac{1}{2} & \text{(NS)} \\ \sum_{n=1}^{\infty} \alpha_{-n}^{\mu}\alpha_{n\mu} + \frac{1}{2}\sum_{n\in\mathbb{Z}} n\rho_{-n}^{\mu}\rho_{n\mu} & \text{(R)} \end{cases} \tag{4.2.19}$$

[6] The modular group is the symmetry group of tori.

where $y^j = x_1^j - x_2^j$ is the separation distance of the branes, thus the presence of D-branes modifies the mass spectrum to include a contribution from the open strings stretching between them.

Carrying through the computation [68] gives the result

$$A = V_{\hat{d}} \int_0^{\infty} \frac{dt}{t} (2\pi t)^{-\hat{d}/2} \, e^{-tr^2/8\pi\alpha'^2} \prod_{n=1}^{\infty} \left(1 - q^{2n}\right)^{-8}$$
$$\times \frac{1}{2} \left(-16 \prod_{n=1}^{\infty} \left(1 + q^{2n}\right)^8 + \frac{1}{q} \prod_{n=1}^{\infty} \left(1 + q^{2n-1}\right)^8 - \frac{1}{q} \prod_{n=1}^{\infty} \left(1 - q^{2n-1}\right)^8 \right) \tag{4.2.20}$$

where $q = e^{-\pi t}$ and we have included an overall factor of two from exchange of the ends of the string, which is a symmetry of unoriented strings. At this point our work is done, as the second line of equation (4.2.20) vanishes due to the "obscure identity" of Jacobi Θ-functions [91], thus

$$A = 0.$$

This indicates that two separated D-branes with the same dimension exert no forces on each other. This also indicates that D-branes are supersymmetric states, with the net forces from the NS-NS and R-R sectors of the closed superstring exactly cancelling.

At this point, it is possible to make a field-theoretical calculation where one computes the exchange between the D-branes of not a string, but rather of the various background fields, i.e., the graviton and dilaton. Due to the mixing between the graviton and the dilaton, this calculation is best carried out in the Einstein frame [92] which decouples these propagators.

Changing to the Einstein frame involves writing

$$g_{\mu\nu} = e^{\phi/2} G_{\mu\nu} \tag{4.2.21}$$

in terms of which our generic type II action is written

$$S_{II} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-g} \left\{ R - \frac{1}{2} (\nabla\phi)^2 - \frac{e^{-\phi}}{12} H^2 - \sum \frac{e^{\left(4-\hat{d}\right)\phi/2}}{2(\hat{d}+1)!} \left(F^{(\hat{d}+1)}\right)^2 \right\} \tag{4.2.22}$$

and our D-brane action as

$$\hat{S} = \mathcal{T}_p \int d^{\hat{d}}\sigma \, e^{(\hat{d}-4)\phi/4} \sqrt{-\det \hat{g}_{ab}} + \frac{\mu_p}{\sqrt{2}\kappa} \int A^{(\hat{d})} \tag{4.2.23}$$

where $B = \mathbb{F}^{(2)} = 0$.

To leading order in the coupling, the energy of interaction of the two D-branes comes from the exchange of a single graviton, a single dilaton or a single RR field. The graviton propagator in D dimensions is written [93]

$$\Delta_D^{\mu\nu,\rho\sigma} = \left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \frac{2}{D-2}\eta^{\mu\nu}\eta^{\rho\sigma}\right)\Delta_D \tag{4.2.24}$$

where the scalar propagator $\Delta_D(x)$ is given by

$$\Delta_D\left(x\right) = \int \frac{d^D k}{(2\pi)^D} \frac{e^{ikx}}{k^2} \tag{4.2.25}$$

while the sources N_ϕ, $N_{A^{(\hat{d})}}$, and $N_{\hat{g}\,\mu\nu}$ are those obtained by linearizing the world-volume action of each D-brane, which we will label simply 1 and 2. Since only one component of $A^{(\hat{d})}$ couples to a static planar Dp-brane, we can treat in this computation the tensor RR potential as a scalar. The sources then take the rather simple form

$$\begin{aligned} N_\phi &= \frac{\hat{d}-4}{4}\mathcal{T}_p \prod_i \delta\left(y^i\right) \\ N_{A^{(\hat{d})}} &= \frac{\mu_p}{\sqrt{2}\kappa} \prod_i \delta\left(y^i\right) \\ N_{\hat{g}\,\mu\nu} &= \frac{1}{2}\mathcal{T}_p \prod_i \delta\left(y^i\right) \times \begin{cases} \eta_{\mu\nu} & \text{if}\mu,\nu \leq \hat{d}-1 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{4.2.26}$$

where the product of δ-functions serves to localize the D-brane in the transverse space. The amplitude is then given by the integral

$$\begin{aligned} A_f = -2\kappa^2 \int d^{10}x \, d^{10}z \Big\{ & N_\phi^1 \Delta_{10}\left(x-z\right) N_\phi^2 - N^1_{A^{(\hat{d})}} \Delta_{10}\left(x-z\right) N^2_{A^{(\hat{d})}} \\ & + N^1_{\hat{g}\,\mu\nu} \Delta_{10}^{\mu\nu,\rho\sigma}\left(x-z\right) N^2_{\hat{g}\,\rho\sigma} \Big\} \end{aligned} \tag{4.2.27}$$

where the integration variables x and z are associated with D-branes 1 and 2 respectively. Making the required computations the result comes out to be

$$A_f = V_{\hat{d}}\left(\mu_p^2 - 2\kappa^2\mathcal{T}_p^2\right)\Delta^e_{10-\hat{d}}\left(r\right) \tag{4.2.28}$$

where $\Delta^e_{10-\hat{d}}(r)$ is the scalar propagator in the $10 - \hat{d}$-dimensional Euclidean space transverse to the D-branes. The form is as expected, replusion due to the Ramond-Ramond charge in addition to an attraction due to the graviton and the dilaton.

Comparing (4.2.28) and our string result, we conclude that

$$\mathcal{T}_p = \frac{\mu_p}{\sqrt{2}\kappa} \tag{4.2.29}$$

and thus the RR repulsion exactly balances the gravitational attraction for these objects. Also, (4.2.29) is nothing other than the mass-charge relation for a BPS saturated state. D-branes, therefore, are supersymmetric.

It is possible to compute the exactly value of $\mathcal{T}_p$ (and therefore of μ_p) by separating the contributions of the RR and NS-NS closed string sectors to the string amplitude (4.2.20). To do this we take the separations to be large ($\sqrt{\mathcal{Y}^2} = r \to \infty$) and expand the integrand for $t \sim 0$. The second line of (4.2.20) is then $\sim (8-8)\frac{t^4}{16} + O(\mathrm{e}^{-1/t})$, and using the representation

$$\Delta^e_D(r) = \frac{\pi}{2}\int_0^\infty ds(2\pi^2 s)^{-D/2}\,\mathrm{e}^{-r^2/2\pi s} \tag{4.2.30}$$

of the propagator we have the result

$$A = V_{\hat{d}}(1-1)\,2\pi(4\pi^2\alpha')^{3-p}\Delta^e_{10-\hat{d}}(r) + O\left(\mathrm{e}^{-r/\sqrt{\alpha'}}\right) \tag{4.2.31}$$

from which we can extract, through comparison with the field theory result (4.2.28) the charge density and tension of D-branes as

$$2\kappa^2\mathcal{T}_p^2 = \mu_p^2 = 2\pi(4\pi^2\alpha')^{3-p} \tag{4.2.32}$$

which, as might be expected for intrinsic modes of a fundamental theory, are fixed by the value of the inverse string tension. Note also that if we write the charge and tension in terms of $\hat{d}$, the dimension of the world-volume, it is written as

$$\hat{\mu}^2_{\hat{d}} = 2\kappa^2\mathcal{T}^2_{\hat{d}} = 2\pi(4\pi^2\alpha')^{\frac{7-2\hat{d}}{2}}. \tag{4.2.33}$$

4.2.3.1. Charge quantization

Dirac's quantization condition [69], (3.4.3) can be extended to higher dimensions [85,86,87]. Let us consider a Dp-brane located at the origin and perform the integral

$$\oint {}^*F^{(\hat{d}+1)} = \mu_{\hat{d}} \tag{4.2.34}$$

Now recall that when we take the Hodge dual of $F^{(\hat{d}+1)}$ we obtain a field strength $\tilde{F}^{(9-\hat{d})}$ which can be written as the exterior derivative of the potential $\tilde{A}^{(8-\hat{d})}$, with the caveat that the potential in this case is not globally defined, so we write

$${}^*F^{(\hat{d}+1)} = \pm\tilde{F}^{(9-\hat{d})} \simeq d\tilde{A}^{(8-\hat{d})}$$

and we are allowed, in the same manner as Dirac, to define a smooth potential everywhere except along a singular hyperstring cutting the $(9-\hat{d})$-dimensional sphere $S_{(9-\hat{d})}$ on the hypersphere $S_{(8-\hat{d})}$. The wave function of a $(7-\hat{d})$ brane obtains a phase shift when transported around such a singular hyperstring given by

$$\tilde{\phi} = \mu_{(7-\hat{d})} \oint \tilde{F}^{(9-\hat{d})} = \pm\mu_{\hat{d}}\mu_{(7-\hat{d})} \tag{4.2.35}$$

which must be an integer multiple of 2π for the wave function to remain single-valued. For the D-brane charge given by (4.2.33) this can be seen to be the case with the integer being unity. Thus the charge of D-branes as computed by closed string exchange is consistent with the charge quantization rule.

4.2.4. D-brane excitations

The ground state of an open superstring is a massless spacetime vector and its fermionic superpartner. As demonstrated in [94] the various components of an open string describe the excitation of the D-branes to which they are attached. In particular, for the massless vector, polarizations parallel to the world-volume of a single D-brane describe $U(1)$ gauge fields living in the world-volume. Components transverse to the brane describe the transverse oscillations of the D-brane, that is deformations in its shape. This is entirely expected, since it is difficult to imagine that D-branes, as massive objects, could remain perfectly rigid in a theory which contains gravity.

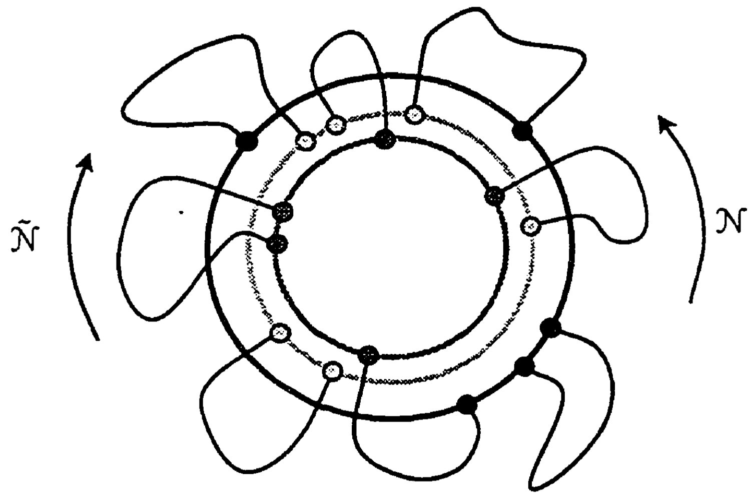


Figure 4.2.2: Schematic depiction of three coincident D-strings wrapped around a compact coordinate with right- ($\mathcal{N}$) and left-moving ($\tilde{\mathcal{N}}$) open strings, forming a "string gas" attached.

The perturbative excitations of a Dp-brane, therefore, are described by a full-fledged open superstring theory, which possesses a low energy limit which is an Abelian supersymmetric Yang-Mills theory dimensionally reduced from ten to $\hat{d}$ dimensions, where $\hat{d}$ is the world-volume dimension of the brane. The presence of D-branes necessitates a new analysis of the open string, which in the absence of D-branes was unavoidably *unoriented,* and for which the Kalb-Ramond field does not appear since it is not symmetric under exchange of the ends of the string, known as world sheet parity. This reanalysis shows that in the presence of D-branes *oriented* open strings are quite normal, and even required. We now understand slightly better the origin of the $B_{\mu\nu} + 2\pi\alpha'\mathbb{F}^{(2)}_{\mu\nu}$ component of the action (4.2.11).

Thus we see that we can consider an D-brane in an excited state as possessing a "gas" of open strings attached to the brane. Let us consider the energy of a D-string with such a collection of open strings attached to it. Recall from (4.2.19) that the mass of an individual open string was

$$M^2 = \frac{y^j y_j}{4\pi^2\alpha'^2} + \frac{1}{\alpha'}\begin{cases} \displaystyle\sum_{n=1}^{\infty} \alpha^{\mu}_{-n}\alpha_{n\mu} + \sum_{r=1/2}^{\infty} r\eta^{\mu}_{-r}\eta_{r\mu} - \frac{1}{2} & \text{(NS)} \\ \displaystyle\sum_{n=1}^{\infty} \alpha^{\mu}_{-n}\alpha_{n\mu} + \tfrac{1}{2}\sum_{n\in\mathbf{Z}} n\rho^{\mu}_{-n}\rho_{n\mu} & \text{(R)} \end{cases} \tag{4.2.36}$$

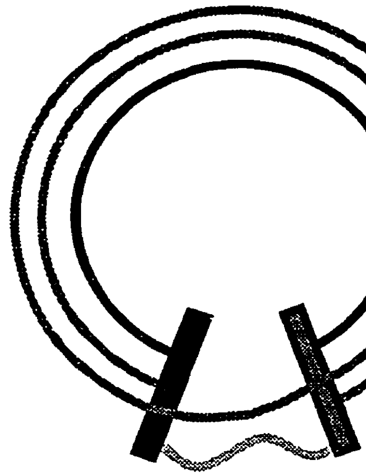


Figure 4.2.3: Schematic depiction of three strings attached between two D-branes. Two strings have winding number $m = 0$ and one has $m = 1$ around the compact coordinate.

where now we have generalized y^j to include the possibility that an open string stretching between two separated D-branes may also wrap around a compact direction an integer number of times (see Fig. 4.2.3). Thus in general we have $y^j = x_1^j - x_2^j + 2\pi R^j m$. If we want the low energy excitations of D-branes, we focus on massless open string states. For simplicity, let us consider a single D-brane. In this case the separation $y^j y_j = 0$ and we are left only with the oscillator contribution of (4.2.36). We have then $M^2 = 0$ for the ground state $r = 1/2$ in the Neveu-Schwarz sector, as well as for $n = 0$ in the Ramond sector. Concentrating on the NS sector, we form the corresponding vertex operators

$$V = \zeta_{\hat{\mu}} \partial_\tau X^{\hat{\mu}} \, \mathrm{e}^{iP\cdot X}, \qquad \text{(World-volume)} \qquad (4.2.37a)$$

$$V = \zeta_i \partial_\tau X^i \, \mathrm{e}^{iP\cdot X} \;\to\; \zeta_i \partial_\sigma Y^i \, \mathrm{e}^{iP\cdot X}, \qquad \text{(Transverse)} \qquad (4.2.37b)$$

where Y^i is the T-dual coordinate to X^i in the space transverse to the D-brane. We then interpret, from the vertex operators in the world volume, (4.2.37a), that there exists a gauge field in the world volume of the D-brane. For transverse polarizations ζ_i, T duality transforms the vertex operator into that of the transverse position of the brane, and we then see that the transverse polarizations do indeed describe the transverse ondulations of the D-brane. Of course, the NS bosons have their companion R fermions.

Let us now specialize to the relatively simple case of a D-string which is wrapped around a compact coordinate with compactification radius R (see Fig. 4.2.2). As we saw in section 2.2.3.1, in the case of a compact radius, the center of mass momentum of the string as it travels along the compactified D-string will be quantized in integer units of $1/R$. The energy will be a sum of the zero point energy of the D-string and the contribution from the gas of open strings. The zero point energy is simply a product of the D-string tension $\mathcal{T}_1$ from (4.2.32) multiplied by the distance around the compact coordinate, $2\pi R$. For the contribution of the "string gas", let us suppose that we have a state in which there are $\mathfrak{N}_k$ right-moving open strings with momentum eigenvalue k, and similarly $\tilde{\mathfrak{N}}_k$ for the left-moving strings. We can then write the mass of the state as

$$M = 2\pi\mathcal{T}_1 R + \frac{\mathcal{N} + \tilde{\mathcal{N}}}{R} \tag{4.2.38}$$

where

$$\mathcal{N} = \sum_{k=1}^{\infty} k\,\mathfrak{N}_k, \qquad \tilde{\mathcal{N}} = \sum_{k=1}^{\infty} k\,\tilde{\mathfrak{N}}_k. \tag{4.2.39}$$

Since for the D-string there are $10 - 2 = 8$ transverse directions, we will have 8 massless bosonic and 8 massless fermionic modes associated with each of the strings traveling around the D-string.

Of course, all these things become more complicated when one has more than one coincident brane present [65]. In this case one can have the open strings attaching themselves to the same brane, or to different branes, depending on the Chan-Paton charge. The number of massless states then increases rapidly with the number of coincident branes. It is also clear that it is possible to generalize to other D-branes.

4.2.5. Supersymmetry of D-branes

As we noted earlier, the interaction energy of two similar D-branes, (4.2.20) vanishes, indicating that the D-branes are supersymmetric. The results for the D-brane tension and charge also indicate that they are BPS saturated states. The only question which remains to be answered is the number of supersymmetries that are left unbroken by these states.

This question is not difficult to answer. The action of T-duality in the open string sector of the type I string theory produces D-branes as we have seen. Away from a D-brane only closed strings can propagate, and therefore the physics is locally that of the type II theory. As we have already noted, the type II theory has two supersymmetries, one each for the left- and right-moving fields, whereas the type I theory has only one supersymmetry, a result of the left- and right-moving fields combining to form standing waves. Thus any state containing a D-brane to which open strings can couple must break half of the supersymmetries.

4.3. Classical D-brane solutions

Now that we have identified D-branes as the carriers of RR charge in string theory, and we have written down an world-volume action (4.2.11) and (4.2.16) for them and have succeeded in computing certain of their properties, we would like to find background field solutions to the supergravity equations of motion which correspond to these objects in spacetime.

The likely candidates were developed in section 4.1. These p-branes have the same symmetry properties as D-branes, that is, a D-brane has Poincaré invariance in its world-volume, since there are Neuman boundary conditions imposed here, and at the same time $SO(D-\hat{d})$ invariance in the transverse space, as a result of the imposition of the Dirichlet boundary conditions which localize the D-brane in spacetime.

Also, it is possible to compute the amplitudes of closed strings scattering from a D-brane [95] through the techniques of conformal field theory, meaning from a world sheet perspective. From these amplitudes it is possible to extract information regarding the long range background fields of D-branes. The results show that these fields correspond exactly to those of our elementary p-brane solutions (4.1.27). This is yet more evidence suggesting an identification of elementary supergravity p-branes with classical background field representation of D-branes.

We also saw that D-branes are supersymmetric, although they break half of the supersymmetries, and have a tension, or mass, and a charge related by BPS saturation. Thus, if we can find p-brane solutions which satisfy these properties, we will have found the classical low-energy spacetime solutions corresponding to our D-branes.

4.3.1. Classical supersymmetry

As we saw in the opening section of this chapter, the p-brane solutions, both elementary and solitonic were found to have a relation very much like that of (4.2.29) which is a hint that the p-brane solutions are supersymmetric, but does not in itself constitute a proof. For the case of p-branes, this is a consequence of choosing the coefficients in the p-brane world-volume action to be as in (4.1.2) [83]. What we will undertake in this section is to make plausible the fact that the elementary p-brane solutions in $D = 10$ preserve, in fact, one-half of the supersymmetries and therefore can be identified as our classical background versions of D-branes.

If we consider our p-brane solutions, we note that there are no fermionic fields present. Thus, when the supersymmetry transformations are applied to such a solution, the supersymmetric variation of the bosonic field vanishes, since it is proportional to the fermionic fields. However, the supersymmetric variation of the fermionic fields, even when they are vanishing themselves need not vanish as it is proportional to the bosonic fields. To demonstrate supersymmetry, there must exist covariantly constant Killing spinors such that the supersymmetry transformations of the gravitino, the superpartner of the graviton, and the dilatino, that of the dilaton vanish identically [79].

We work in the context of type II supergravity. Our strategy, following [83] will be to assume an ansatz for a solution as in (4.1.13), substitute this into the supersymmetry transformation rules of the gravitino and dilatino, demand that there be unbroken supersymmetry[7] and compare the results to those of section 4.1. The motivation for this strategy is the fact that the supersymmetry transformation rules are first-order equations rather than second order as are the equations of motion.

To facilitate this comparison we will work in the Einstein frame. The type IIA supergravity action is written in the Einstein frame as

$$
\begin{aligned}
S_{IIA}^{E} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}\Big\{ R - \frac{1}{2}(\nabla\phi^{(a)})^2 - \frac{e^{-\phi^{(a)}}}{12}\left(H^{(a)}\right)^2 \\
- \frac{e^{3\phi^{(a)}/2}}{4}\left(F^{(2)}\right)^2 - \frac{e^{\phi^{(a)}/2}}{48}\left(F^{(4)}\right)^2\Big\} - \frac{1}{4\kappa^2}\int B^{(a)} dA^{(3)} dA^{(3)}
\end{aligned}
\tag{4.3.1}
$$

where $g_{\mu\nu}$ is the Einstein frame metric, $H^{(a)} = dB^{(a)}$ is the field strength of the Kalb-Ramond field, $F^{(2)} = dA^{(1)}$ and $F^{(4)} = dA^{(3)} - H^{(a)}A^{(1)}$ are the Ramond-Ramond field strengths of the one-form potential $A^{(1)}$ and the three-form potential $A^{(3)}$ respectively, and finally $\phi^{(a)}$ is, as always, the dilaton.

For the type IIA theory, the supersymmetry transformation for the gravitino is

$$
\begin{aligned}
\delta\psi_{\mu}^{(a)} = \mathcal{D}_{\mu}\varepsilon + \frac{e^{3\phi^{(a)}/4}}{64}\left(\gamma_{\mu}{}^{\nu\rho} - 14\delta_{\mu}{}^{\nu}\gamma^{\rho}\right)\gamma^{11}F^{(2)}_{\nu\rho}\varepsilon \\
+ \frac{e^{-\phi^{(a)}/2}}{96}\left(\gamma_{\mu}{}^{\nu\rho\sigma} - 9\delta_{\mu}{}^{\nu}\gamma^{\rho\sigma}\right)\gamma^{11}H^{(a)}_{\nu\rho\sigma}\varepsilon \\
+ \frac{i\, e^{\phi^{(a)}/4}}{256}\left(\gamma_{\mu}{}^{\nu\rho\sigma\lambda} - \frac{20}{3}\delta_{\mu}{}^{\nu}\gamma^{\rho\sigma\lambda}\right)F^{(4)}_{\nu\rho\sigma\lambda}\varepsilon
\end{aligned}
\tag{4.3.2}
$$

and for the type IIA dilatino we have

$$
\begin{aligned}
\delta\lambda^{(a)} = \Big(\frac{\sqrt{2}}{4}\mathcal{D}_{\mu}\phi^{(a)}\gamma^{\mu}\gamma^{11} + \frac{3\, e^{3\phi^{(a)}/4}}{16\sqrt{2}}\gamma^{\nu\rho}F^{(2)}_{\nu\rho} \\
+ \frac{i\, e^{-\phi^{(a)}/2}}{24\sqrt{2}}\gamma^{\nu\rho\sigma}H^{(a)}_{\nu\rho\sigma} - \frac{i\, e^{\phi^{(a)}/4}}{192\sqrt{2}}\gamma^{\nu\rho\sigma\lambda}F^{(4)}_{\nu\rho\sigma\lambda}\Big)\varepsilon
\end{aligned}
\tag{4.3.3}
$$

where the γ^{μ} are the Dirac matrices in $D = 10$, with

$$
\gamma^{\mu_1\mu_2\cdots\mu_n} = \gamma^{[\mu_1}\gamma^{\mu_2}\ldots\gamma^{\mu_n]} \tag{4.3.4}
$$

and

$$
\gamma^{11} = i\gamma^0\gamma^1\cdots\gamma^9. \tag{4.3.5}
$$

---

[7] Only the vacuum retains all of the supersymmetries.

Also, here the covariant derivative $\mathcal{D}$ is written

$$\mathcal{D}_\mu = \partial_\mu + \frac{1}{4}\omega_{\mu\nu\rho}\gamma^{\nu\rho} \tag{4.3.6}$$

where $\omega_{\mu\nu\rho}$ is the spin connection [51] obtained by solving the equation[8]

$$\nabla_\mu e^i_\nu = \partial_\mu e^i_\nu + \omega^i_{\mu j} e^j_\nu - \Gamma^\rho_{\mu\nu} e^i_\rho = 0 \tag{4.3.7}$$

where in turn the e^i_μ and $\Gamma^\rho_{\mu\nu}$ are the usual vielbein and affine connection. Explicit formulas for $\omega_\mu{}^{ij}$ and $\Gamma_{\mu\nu\lambda}$ are

$$\begin{aligned}\omega_\mu{}^{ij} =& \frac{1}{2}e^{\nu i}(\partial_\mu e^j_\nu - \partial_\nu e^j_\mu) - \frac{1}{2}e^{\nu j}(\partial_\mu e^i_\nu - \partial_\nu e^i_\mu)\\ &- \frac{1}{2}e^{\rho i}e^{\sigma j}e^k_\mu(\partial_\rho e_{\sigma k} - \partial_\sigma e_{\rho k}),\end{aligned} \tag{4.3.8a}$$

$$\Gamma_{\mu\nu\lambda} = \frac{1}{2}(\partial_\lambda g_{\mu\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}). \tag{4.3.8b}$$

On the type IIB side, the supergravity action in the Einstein frame is given by

$$\begin{aligned}S^E_{IIB} =& \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-j}\Big\{R - \frac{1}{2}(\nabla\phi^{(b)})^2 - \frac{e^{-\phi^{(b)}}}{12}\left(H^{(b)}\right)^2 - \frac{e^{-5\phi^{(b)}/2}}{2}(\nabla\chi)^2\\ &- \frac{e^{\phi^{(b)}}}{12}\left(F^{(3)} + \chi H^{(b)}\right)^2 - \frac{1}{480}\left(F^{(5)}\right)^2\Big\} + \frac{1}{4\kappa^2}\int A^{(4)}F^{(3)}H^{(b)}\end{aligned} \tag{4.3.9}$$

where $j_{\mu\nu}$ is the Einstein frame metric, $H^{(b)} = dB^{(b)}$ is the field strength of the Kalb-Ramond field, $F^{(3)} = dA^{(2)}$ and $F^{(5)} = dA^{(4)} - \frac{1}{2}(B^{(b)}F^{(3)} - A^{(2)}H^{(b)})$ are RR field strengths, while $\chi = A^{(0)}$ is the RR scalar, and $\phi^{(b)}$ is the dilaton.

It is known [9] that the format used here to present the type IIB supergravity is not the most efficient for the presentation of the type IIB supersymmetry transformations[9] for the gravitino and dilatino. The way to make it more efficient is as follows. We first recognize that the type IIB supergravity possesses a global $SU(1,1)$ symmetry, a non-compact version of $SU(2)$. The maximal compact subgroup of $SU(1,1)$ is $U(1)$. Then we combine the dilaton $\phi^{(b)}$ and the RR scalar χ into a 2×2 matrix $V^a_\pm$ where $a = 1, 2$, $V^a_- = (V^a_+)^*$ and impose the condition

$$\epsilon_{ab}V^a_- V^b_+ = \det V = 1 \tag{4.3.10}$$

---

[8] In the case of vanishing torsion, i.e., $T^i_{\mu\nu} = \nabla_\mu e^i_\nu - \nabla_\nu e^i_\mu = 0$. Note that when an object has multiple indicies we add a connection term for each, as in $\nabla_\mu e^i_\nu = \partial_\mu e^i_\nu + \omega_\mu{}^i{}_j e^j_\nu - \Gamma^\lambda_{\mu\nu} e^i_\lambda$.

[9] There is no simple way to write the action in the new form, so we will omit it.

so that V transforms under $SU(1,1)$ with $U(1)$ charges ± 1. We then form the $SU(1,1)$-invariant quantity

$$\mathcal{P}_\mu = -\epsilon_{ab} V_+^a \partial_\mu V_+^b. \tag{4.3.11}$$

For the two two-form potentials of the type IIB theory we combine their respective field strengths as

$$\Xi^1 = H^{(b)} \qquad \Xi^2 = F^{(2)} \tag{4.3.12}$$

then we define a new complex three-form field strength as

$$\mathcal{F}^{(3)} = -\epsilon_{ab} V_+^a \Xi^b \tag{4.3.13}$$

with the definition of a covariant derivative that maintains the charge U of a field under $U(1)$ which is

$$\hat{\mathcal{D}}_\mu = \left(\partial_\mu - U \epsilon_{ab} V_-^a \partial_\mu V_+^b\right). \tag{4.3.14}$$

After these definitions, we can write the type IIB supersymmetry transformations as

$$\delta\psi_\mu^{(b)} = \hat{\mathcal{D}}_\mu \varepsilon + \frac{i}{4\,(5!)} \gamma^{\nu\rho\sigma\lambda\delta} \gamma_\mu F^{(5)}_{\nu\rho\sigma\lambda\delta} \varepsilon + \frac{1}{96} \left(\gamma_\mu{}^{\nu\rho\sigma} - 9\delta_\mu{}^\nu \gamma^{\rho\sigma}\right) \mathcal{F}^{(3)}_{\nu\rho\sigma} \varepsilon^* \tag{4.3.15}$$

for the gravitino and

$$\delta\lambda^{(b)} = i\gamma^\mu \varepsilon^* \mathcal{P}_\mu - \frac{i}{24} \gamma^{\mu\nu\rho} \mathcal{F}^{(3)}_{\mu\nu\rho} \varepsilon \tag{4.3.16}$$

for the dilatino since in the type IIB case, ε is chiral $\gamma^{11}\varepsilon = \varepsilon$.

It is found, upon substituting the ansatz of (4.1.13) into the supersymmetry transformations that for $D = 10$ and world-volume dimension $1 \leq \hat{d} \leq 7$ that Killing spinors exist for which the transformations (4.3.2) and (4.3.3) or (4.3.15) and (4.3.16) vanish. However, this requirement reduces the four unknown functions f_1, f_2, f_3, and ϕ to one, exactly as occured when the ansatz was substituted into the equations of motion. The result of all this is that we may interpret the elementary p-brane solutions of §4.1 as the low energy background solutions corresponding to D-branes.

That this works should not, after all, be considered a miracle. The supersymmetry variation of an equation of motion is something that should vanish by the equations of motion. Supersymmetry involves enough constraints that if the supersymmetry transformations are known, as well as one equation of motion, the remaining equations of motion can be deduced. Therefore that an ansatz that is supersymmetric also solves the equations of motion is not surprising. Of course, we had no reason in this chapter to choose a different ansatz when we constructed the p-branes.

There exist, of course, many other variants of the p-brane solutions. Some are called "black" p-branes because they display the phenomenon of an event horizon. Other types of solutions exist which break more than half, but not all, of the supersymmetries. These other types of solutions are also BPS saturated states and play a role in many string dualities. However, a systematic exposition of these solutions is beyond the scope of this work. Let us move on to dicuss in the next chapter black holes in the theory of strings.

V

The theory of black holes

The existence of completely collapsed objects, known as black holes by the fact that a region of spacetime from which not even light can escape is formed, is quite likely the most intriguing prediction of Einstein's general theory of relativity. Add to this the accompanying prediction of spacetime points within this "region of no escape" where the spacetime curvature becomes unboundedly large, suggestively labeled the singularity. Black holes thus present an enigma even at this level, suggesting strongly that a quantized theory of gravity is necessary to determine the physics of the singularity.

In addition, the laws of classical black hole mechanics, which we will discuss in more detail later, take such a form as to suggest a direct connection with thermodynamics, leading Bekenstein [6] to posit that black holes possess an intrinsic entropy. At that point, the lack of a non-zero temperature made the thermodynamic analogy seem unlikely to be anything other than just that, an analogy. After all, if a black hole is in some way defined by this region of no escape, then the temperature of a black hole should be zero, since it is a perfect absorber. This piece of the puzzle was discovered by Hawking [7] when through application of quantum mechanics he demonstrated that black holes emit thermal radiation, and thus possess a non-zero temperature, completing the thermodynamic structure.

This is, of course, not the end, but rather the beginning of the story since new questions then surfaced. The first was the statistical interpretation of the entropy. For ordinary thermodynamic systems, the entropy can be computed by statistical analysis of the fundamental degrees of freedom of the system. What are these degrees of freedom in the case of black holes?

A second question raised by the work of Hawking is known as the information loss paradox of black holes [96-98]. If a black hole is allowed to radiate away all of its mass, thus evaporating completely, where does the information about the quantum states that formed it go? Such a process is non-unitary, thus violating the principles of the quantum mechanics that were required to (theoretically) produce the process itself.

It is clear, then, that a quantum theory of gravity is called for and that black holes bring sharply into focus this need. String theory, as a serious candidate for a theory containing quantum gravity is a logical framework in which to attempt a response to the questions posed by black hole physics.

The study of black holes in string theory is normally begun within the framework of classical, exact solutions to the low energy effective string equations of motion. When fields such as the dilaton and the various gauge fields vanish, the classical solutions of the Einstein equation, $R_{\mu\nu} = 0$, such as the Schwarzschild or Kerr solutions are solutions of the low energy effective string equations of motion, for example (2.3.10). While the black hole solutions of string theory form a much wider class than those of general relativity, it is worthwhile to begin with a short review of the black holes of general relativity.

5.1. Black holes

A spherical body of matter which is sufficiently cold and contains sufficient mass cannot, according to general relativity, exist in hydrostatic equilibrium [5]. It must, therefore, suffer complete collapse under its own gravity and form a spacetime known as a black hole.

One rather interesting characteristic ascribed to the process of gravitational collapse is called the *cosmic censorship conjecture* [99]. This conjecture states that the curvature singularities produced are always shrouded by an event horizon, i.e., a black hole. In other words, the singularity can never be observed at asymptotic infinity. To be precise, this conjecture must specify conditions on the matter fields in question, for example it has been shown that in the gravitational collapse of a perfect fluid "naked" singularities can occur [100]. Despite substantial theoretical effort this conjecture remains exactly that.

Let us pause briefly to consider the definition of a black hole. The notion of a region of no escape due to strong gravitational fields is not defined with sufficient precision by: The region of spacetime such that every timelike (or null) worldline with at least one point in the region is completely contained in the region. With a definition such as this, everyone's causal future is a black hole. An appropriate definition of a black hole in the case of asymptotically flat spacetimes is *that region from which it is impossible to escape to future null infinity.*[1] With this definition one is essentially confined to finite distance from the origin r, and we choose to begin our study of black holes with the confinement to finite r described by the Schwarzschild solution.

5.1.1. The Schwarzschild metric

The Schwarzschild solution, a solution of the source-free (vacuum) Einstein equation was first written down in 1916 [101] and for which appears the classic graphical representation in Fig. 5.1.1. The solution is written in $D = 4$ in the usual spherical coordinates $\{t, r, \theta, \varphi\}$ in spacetime with Minkowski signature as the line element

$$ds^2 = -\left(1 - \frac{2G_N M}{r}\right) dt^2 + \left(1 - \frac{2G_N M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \quad (5.1.1)$$

where M is the mass.

One notes immediately the existence of four points at which (5.1.1) is singular, $r = 2G_N M$ and $r = 0$, $\theta = 0$ and $\theta = \pi$. Such singularities can be due to either of a) a failure of the system of coordinates used to describe the solution or, b) the existence of an essential singularity in the spacetime. Here $r = 2G_N M$ is an example of the first, defining the boundary of the region of no escape called the *event horizon*. This boundary occurs at the Schwarzschild radius and is the radius at which the escape velocity equals that of light. The points $\theta = 0, \pi$ are the trivial singularities of polar coordinates. On the other hand, the point $r = 0$ is the true singularity where the curvature grows without bound.

1 These concepts are the subject of appendix C.

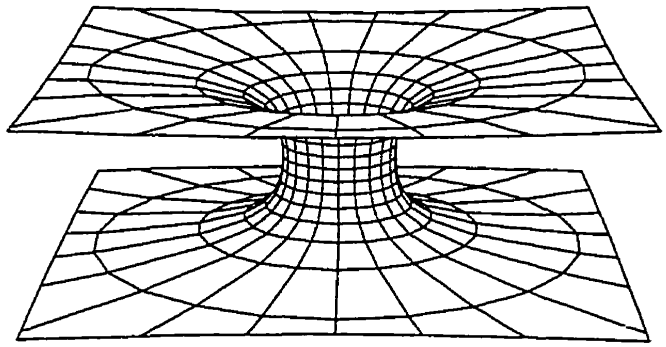


Figure 5.1.1: The spacetime geometry described by the Schwarzschild solution. The figure represents a time-slice $t = 0$ with one degree of rotational freedom suppressed, i.e., circles at radius r are actually spheres of area $4\pi r^2$.

Represented in Fig. 5.1.2 is a Penrose diagram of the maximally analytically extended[2] Schwarzschild solution. Since the Schwarzschild solution is invariant under time reversal, the maximally extended solution contains a "white hole" (region IV) which emerges from i^- in the infinite timelike past as well as the black hole, region III which extends toward i^+ in the infinite timelike future, as well as two asymptotically flat regions (I, II). One thing to note here is that the singularity is spacelike, and thus the hapless adventurer who stumbles into a Schwarzschild black hole has no choice but to collide with the singularity. H^+ and H^- denote respectively the future and past event horizons. For black holes formed by the collapse of infalling matter, only regions I and III are expected to be relevant physically.

Schwarzschild black holes have been shown to be stable under small perturbations [102], which indicates that it is classically impossible to extract energy from such a black hole. We can then identify a Schwarzschild black hole as the ultimate ground state of a heavy mass.

The Schwarzschild solution has, of course, higher dimensional generalizations. The causal structure remains identical with each point in Fig. 5.1.2 representing a

[2] These concepts are explained in appendix C.

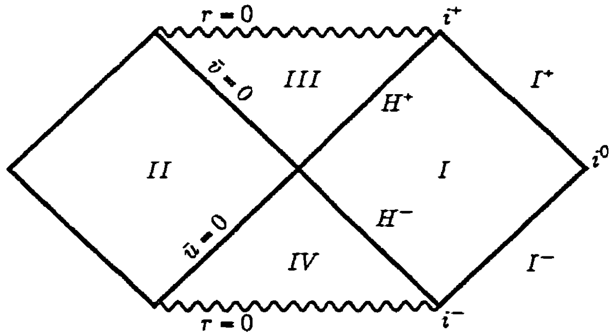


Figure 5.1.2: Penrose diagram of the maximal analytic extension of the Schwarzschild black hole.

manifold of dimension $D-2$. These solutions are written [103]

$$ds^2 = -\left(1 - \frac{\beta}{r^{D-3}}\right) dt^2 + \left(1 - \frac{\beta}{r^{D-3}}\right)^{-1} dr^2 + r^2 d\Omega^{D-2} \tag{5.1.2}$$

where $d\Omega^{D-2}$ is the line element on the unit $(D-2)$-sphere and the parameter β is related to the mass by

$$M = \frac{(D-2)\mathcal{A}_{D-2}}{2\kappa^2}\beta \tag{5.1.3}$$

where $\kappa = \sqrt{8\pi G_N}$ was defined in chapter II, and $\mathcal{A}_{D-2}$, the area of the $(D-2)$-sphere is given by

$$\mathcal{A}_{D-2} = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)}. \tag{5.1.4}$$

Another way to obtain a higher dimensional generalization is to form a metric of the form

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2G_N M}{r}\right) dt^2 + \left(1 - \frac{2G_N M}{r}\right)^{-1} dr^2 \\ &+ r^2(d\theta^2 + \sin^2\theta d\varphi) + \delta_{ij} dx^i dx^j \end{aligned} \tag{5.1.5}$$

where $D-5 \leq i, j \leq D-1$. This is particularly useful when one wishes to study solutions which have a four dimensional compactification. It is evident that one can combine the two generalizations to form, for example in six dimensions, objects like a five-dimensional Schwarzschild metric, adding a flat coordinate to lift it to $D = 6$.

5.1.2. Reissner-Nordstrom solution

The Reissner-Nordstrom [104] solution represents a charged black hole solution to the Einstein-Maxwell equations, which are given by

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \left(F_{\mu\rho} F_{\nu}{}^{\rho} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right) \tag{5.1.6}$$

where $F = dA_M$ is the field strength of the Maxwell gauge field A_M, which satisfies the Maxwell equations which are written in terms of forms as

$$dF = 0, \qquad d({}^*F) = \mathcal{A}_{D-2}\ {}^*\mathbb{J} \tag{5.1.7}$$

where $\mathbb{J}$ is the one-form Maxwell current source.

The Reissner-Nordstrom solution includes, therefore, a gauge field as well as a metric and is written

$$\begin{aligned} ds^2 = &- \left(1 - \frac{2G_N M}{r} + \frac{G_N^2 q^2}{r^2}\right) dt^2 \\ &+ \left(1 - \frac{2G_N M}{r} + \frac{G_N^2 q^2}{r^2}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2) \\ A_M = &\sqrt{\frac{G_N}{4\pi}} \left(\frac{q^e}{r} dt + q^m \cos\theta\, d\varphi \right) \end{aligned} \tag{5.1.8}$$

where $q = \sqrt{q^{e2} + q^{m2}}$ and in turn q^e and q^m are the electric and magnetic charge, respectively, as defined in equation (4.1.32).

The causal structure of the Reissner-Nordstrom metric is quite different than that of the Schwarzschild solution and even changes rather drastically depending on the relative values of q and M. If $0 < |q| < M$ the metric coefficient can be factored into two real roots as

$$\left(1 - \frac{2G_N M}{r} + \frac{G_N^2 q^2}{r^2}\right) = \left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right) \tag{5.1.9}$$

where

$$r_{\pm} = G_N M \pm G_N \sqrt{M^2 - q^2} \tag{5.1.10}$$

and thus the spacetime exhibits two horizons, an inner horizon at $r = r_-$ and an outer one at $r = r_+$. A Penrose diagram of the maximal analytic extension of this spacetime is to be found in Fig. 5.1.3. It consists of an infinite chain of asymptotically

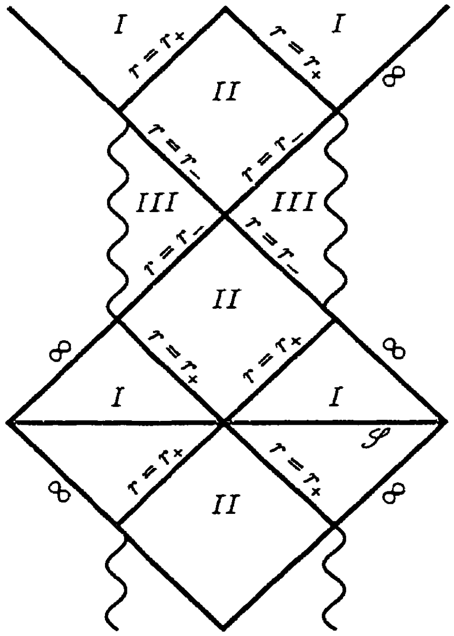


Figure 5.1.3: Penrose diagram of the Reissner-Nordstrom black hole for $q < M$.

flat regions I connected by regions II and III. Each region III is bounded by a timelike singularity. The significance of the inner horizon is that the trajectory of an infalling observer is only unique up to $r = r_-$, at which point it becomes dependent upon boundary conditions at the singularity. To see this, consider the surface $\mathscr{S}$ of Fig. 5.1.3. $\mathscr{S}$ is a Cauchy surface for the two regions I and the neighboring regions II. However, in the neighboring regions III there are timelike curves travelling backward in time (past-directed) which approach the singularity and do not cross $r = r_-$. Thus $r = r_-$ is the future Cauchy horizon for $\mathscr{S}$, and the conditions on $\mathscr{S}$ do not determine the continuation of the curve beyond $r = r_-$. Further, one is not forced to collide with the singularity in contrast with the Schwarzschild case, since it is now timelike. In fact, our infalling observer must actually exert himself to reach the singularity as freely falling observers avoid it, continuing through the regions II, III, II and into another asymptotically flat region I of the spacetime.

When $q = M$, however, the picture changes to that of Fig. 5.1.4 in which the inner and outer horizons coincide, $r_+ = r_- = G_N M$. Note that the surface marked

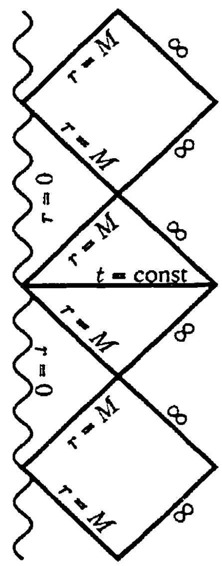


Figure 5.1.4: Penrose diagram of the Reissner-Nordstrom black hole for $q = M$.

t = const appears to make contact with the singularity, but this is nothing more than an artifact of the conformal rescaling used to construct the diagram. To see this, we compute the proper distance to the horizon $r = G_N M$ from some point at a radius $r_0 > G_N M$ along a radial curve at fixed t, which is given by

$$L = \int_{G_N M}^{r_0} \frac{dr}{\left(1 - \frac{G_N M}{r}\right)} = \infty. \tag{5.1.11}$$

Thus, as $r \to G_N M$ the fixed-t surface takes on the geometry of a infinite cylinder, so we have what is called an "infinite throat" (see Fig. 5.1.5). It seems that the horizon has been pushed away to infinity, though one can still fall into the black hole in finite proper time since the horizon is still a finite distance away in timelike or null directions.

The final case to consider is that of $q > M$. Here the Reissner-Nordstrom solution describes a naked singularity, shown in Fig. 5.1.6. It is widely considered that $q > M$ is impossible to achieve given the cosmic censorship conjecture, as well as other considerations. For example, a black hole with $q \sim M$ would exert a repulsive electrostatic force on protons that is greater than its gravitational pull on them by a factor of $\frac{eq}{mM} \sim \frac{e}{\mu} \sim 10^{18}$, and such a differential in forces is likely

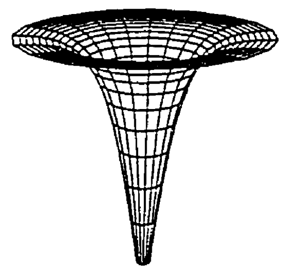


Figure 5.1.5: Representation of a time-slice for the extremal Reissner-Nordstrom black hole ($q = M$).

to pull in neutralizing charge. When Hawking radiation is in effect, then, a black hole will preferentially radiate away its charge, depending of course on the charge to mass ratio of the particles in the theory. A small charge to mass ratio can result in the charge remaining essentially constant, which is likely to be true for any possible realistic magnetically charged black holes. Thus $q = M$ is the largest possible charge to mass ratio for the Reissner-Nordstrom solution. In this case the black hole is called *extremal.*

Again we can generalize the Reissner-Nordstrom solution to higher dimensions by writing the metric in a higher dimensional form as in [103]

$$\begin{aligned} ds^2 &= -\left(1 - \frac{\beta}{r^{D-3}} + \frac{\lambda^2}{r^{2(D-2)}}\right) dt^2 \\ &\quad + \left(1 - \frac{\beta}{r^{D-3}} + \frac{\lambda^2}{r^{2(D-2)}}\right)^{-1} dr^2 + r^2 d\Omega^{D-2} \\ A_M &= \pm\sqrt{\frac{D-2}{\kappa^2 (D-3)}}\,\frac{\lambda}{r^{D-3}} \end{aligned} \tag{5.1.12}$$

where β and the mass are related as in (5.1.3) and the electric charge q^e is given in terms of λ by

$$q^e = \pm\lambda \frac{\mathcal{A}_{D-2}}{\kappa^2}\sqrt{2(D-2)(D-3)} \tag{5.1.13}$$

One may ask what has happened to the magnetic charge of the higher dimensional Reissner-Nordstrom solution. The answer is that there is no magnetic charge in higher dimensions. q^m is defined by integrating the Maxwell two-form field

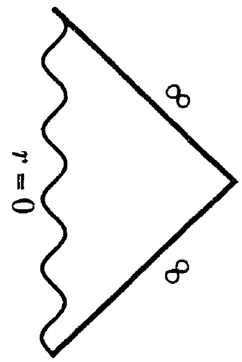


Figure 5.1.6: Penrose diagram of the Reissner-Nordstrom black hole for $q > M$.

strength F over the surface at infinity, and only in two dimensions is this sphere two-dimensional, leading to a nonvanishing result. If one includes more general form fields, such as those of the theory of strings, then one can find black hole solutions in higher dimensions have magnetic-like charges, as well as being electrically charged under these fields. Also, in higher dimensions, we can compactify some of the higher dimensions such that the surface at infinity is a two sphere, with each point on this surface being the compact manifold.

5.1.3. Kerr-Newman solution

The Schwarzschild and the Reissner-Nordstrom solutions were both known shortly after Einstein published his general theory of relativity. The addition of angular momentum to the Schwarzschild solution, giving us what is known as the Kerr metric, took until 1963 [105]. Shortly thereafter a charged generalization of the Kerr metric was found by Newman *et al.* [106].

The Kerr-Newman solution appears as

$$
\begin{aligned}
ds^2 &= -\frac{\Delta}{\rho^2}(dt - a\sin^2\theta d\varphi)^2 + \frac{\sin^2\theta}{\rho^2}((r^2+a^2)d\varphi - adt)^2 \\
&\qquad + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \qquad (5.1.14) \\
A_M &= \sqrt{\frac{G_N}{4\pi}}\frac{q^e r}{\rho^2}\left(dt - a\sin^2\theta d\varphi\right)
\end{aligned}
$$

where

$$\rho^2 = r^2 + a^2 \cos^2\theta,$$
$$\Delta = r^2 + a^2 + \left(G_N q^e\right)^2 - 2G_N M r \tag{5.1.15}$$

in Boyer-Lindquist coordinates [107]. The mass M, charge q^e and the angular momentum per unit mass $a = J/M$ are the three parameters of this solution. For $a = 0$ we recover the Reissner-Nordstrom solution, $q^e = 0$ gives the Kerr metric and $a = q^e = 0$ reduces to the Schwarzschild solution. Due to a remarkable series of theorems by Israel, Carter, Hawking, and Robinson, collectively known as the "no hair" theorem [108-109], which means roughly that in $D = 4$ any complete gravitational collapse settles down to an endpoint uniquely determined by three parameters: the mass, angular momentum, and charge, the Kerr-Newman solution represents an exhaustive family of black holes in four dimensions.

The causal structure of the Kerr-Newman solution is similar to that of the Reissner-Nordstrom metric. Of course, the addition of angular momentum brings about certain changes. We again have both inner and outer horizons with radii given by

$$r_{\pm} = G_N M \pm G_N \sqrt{M^2 - \left(\frac{J}{G_N M}\right)^2 - (q^e)^2}, \tag{5.1.16}$$

but the structure of the singularity is modified. Computation of the scalar curvature shows that

$$\rho^2 = r^2 + a^2 \cos^2\theta = 0 \tag{5.1.17}$$

is a true curvature singularity when[3] $M \neq 0$. This gives the impression that there is a singularity at the origin only for $\theta = \pi/2$. The true nature of the singularity can be found to be that of a ring, through which one may pass to negative r, denoted in Fig. 5.1.7 by the two asymptotically flat regions labelled $r < 0$.

It is possible to classically extract energy from a black hole [110]. This process can occur as long as the black hole rotates or has charge, and the process of energy extraction subtracts angular momentum, or charge, or both, from the black hole. These processes ultimately result in a Schwarzschild black hole from which it is impossible, classically, to extract energy. There is thus a limit to the amount of

[3] We assume here that the effect of charge is negligible for an astrophysical body [51].

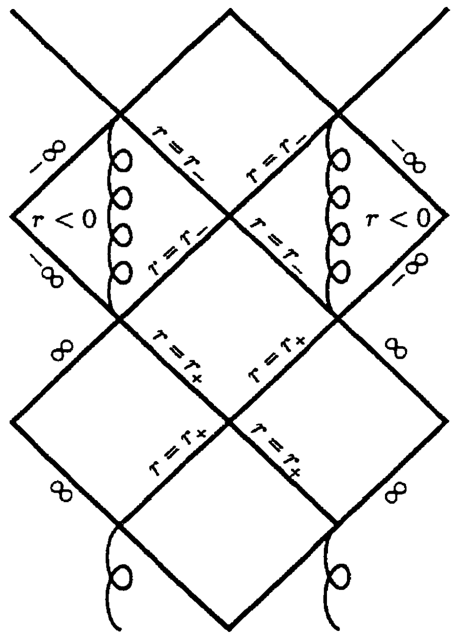


Figure 5.1.7: Penrose diagram of the Kerr black hole for $\sqrt{a^2 + (G_N q^e)^2} < G_N M$.

energy that one can extract in this way. Given a black hole of mass M, angular momentum J and charge q^e, the total mass energy of the black hole can be written [4]

$$M^2 = \left(M_{\rm ir} + \frac{(q^e)^2}{4M_{\rm ir}}\right)^2 + \frac{J^2}{4G_N^2 M_{\rm ir}^2} \qquad (5.1.18)$$

where $M_{\rm ir}$ is the *irreducible mass*, that is the mass that the Schwarzschild black hole which remains after all of the charge and angular momentum have been removed by maximally efficient energy extraction processes. This formula shows that one can think of the mass energy of a black hole as made up of contributions from the irreducible mass, an electromagnetic mass energy, and a rotational energy.

As with other black hole solutions to the vacuum Einstein equation, it is possible to add flat directions to the Kerr metric in the manner of (5.1.5), as well a generalizing the solution to higher dimensions. In higher dimensions, of course, more parameters are required to describe this group of solutions, for example each pair of additional dimensions brings with it the possibility of a new plane of rotation, and thus an additional angular momentum parameter. One can see, then that there will be cases for odd- and even-D. The higher dimensional generalization of the Kerr metric in odd dimensions in Boyer-Lindquist coordinates [107] is [103]

$$
\begin{aligned}
ds^2 = &- dt^2 + \sum_i^{\langle \frac{D-1}{2} \rangle} \left\{ (r^2 + {a_i}^2)((d\mu_i)^2 + {\mu_i}^2 d\varphi_i^2) \right\} \\
&+ \frac{\beta r^2}{\Pi\Sigma} \left(dt + \sum_i^{\langle \frac{D-1}{2} \rangle} a_i {\mu_i}^2 d\varphi_i \right)^2 + \frac{\Pi\Sigma}{\Pi - \beta r^2} dr^2
\end{aligned}
\tag{5.1.19}
$$

where

$$
\begin{aligned}
\Pi &= \prod_{i=1}^{\langle \frac{D-1}{2} \rangle} (r^2 + a_i^2) \\
1 - \Sigma &= \sum_{i=1}^{\langle \frac{D-1}{2} \rangle} \frac{a_i^2 \mu_i^2}{r^2 + a_i^2} \\
1 &= \sum_{i=1}^{\langle \frac{D-1}{2} \rangle} \mu_i^2
\end{aligned}
\tag{5.1.20}
$$

and where $\langle \frac{D-1}{2} \rangle$ indicates the greatest integer, that is the greatest integer $\leq \frac{D-1}{2}$, and the μ_i are the direction cosines which specify the direction of the radial vector. Note that $0 \leq \mu_i \leq 1$ since for any i, the pair (μ_i, φ_i) and $(-\mu_i, \varphi_i + \pi)$ are the same direction. The a_i are, of course, the angular momenta per unit mass in the i-th plane of rotation.

When the dimension of the spacetime D is even, possessing therefore an odd number of spatial directions, the solution is given as

$$
\begin{aligned}
ds^2 = &- dt^2 + r^2 d\bar{\varphi}^2 + \sum_i^{\langle \frac{D-2}{2} \rangle} \left\{ (r^2 + {a_i}^2)((d\mu_i)^2 + {\mu_i}^2 d\varphi_i^2) \right\} \\
&+ \frac{\beta r}{\Pi\Sigma} \left(dt + \sum_i^{\langle \frac{D-2}{2} \rangle} a_i {\mu_i}^2 d\varphi_i \right)^2 + \frac{\Pi\Sigma}{\Pi - \beta r} dr^2
\end{aligned}
\tag{5.1.21}
$$

where Π and Σ are as in (5.1.20), and $\bar{\varphi}$ represents the direction cosine of the spatial coordinate which is not paired up. The constraint for the direction cosines in this case is then

$$
\sum_{i=1}^{\langle \frac{D-2}{2} \rangle} \mu_i^2 + \bar{\varphi}^2 = 1. \tag{5.1.22}
$$

Again the mass is given by (5.1.3). For these solutions, the angular momenta are obtained by transforming to Cartesian coordinates at large radius according to

$$x^i = r\mu_i \cos\varphi_i \qquad y^i = r\mu_i \sin\varphi_i \tag{5.1.23}$$

giving the off diagonal parts of the metric as

$$2\frac{\beta a_i \sin^2\theta}{r^{D-3}}\mu_i^2 dt d\varphi_i = -2\frac{\beta a_i}{r^{D-1}} dt(y^i dx^i - x^i dy^i) \tag{5.1.24}$$

which allows us to identify the angular momentum in each plane of rotation as

$$\begin{aligned} J_{\varphi_i} = J^{y^i x^i} &= \frac{\mathcal{A}_{D-2}\beta a_i}{\kappa^2} \\ &= \frac{2Ma_i}{D-2}. \end{aligned} \tag{5.1.25}$$

Let us now discuss some further properties of these solutions. In Boyer-Lindquist coordinates, the event horizons appear where $g^{rr} = (g_{rr})^{-1} = 0$. Thus we find the horizons by setting

$$0 = \begin{cases} \Pi - \beta r & (D \text{ even}) \quad (5.1.26a) \\ \Pi - \beta r^2 & (D \text{ odd}) \quad (5.1.26b) \end{cases}$$

for which, in general, analytic solutions can not be found.

For D even, positive mass ensures that any existing horizons will be located at positive r, thus avoiding naked singularities. In general there are three possibilities, no horizons, one degenerate horizon, or two horizons. When none of the angular momentum parameters vanish, (5.1.26a) is a polynomial of degree $(D-2)$. As shown by Galois, these polynomials are soluble in terms of radicals only for $D = 4, 6$, although the polynomials are not completely general since they contain $D/2$ free parameters. This coincides with our discussion of the Kerr-Newman solution previously. One finds, however, that the vanishing of at least one of the angular momentum parameters is sufficient to guarantee the existence of a horizon.

For odd dimension, the case D=5 is quadratic in r and the roots can be written as [103]

$$2r_\pm^2 = \beta - a_1^2 - a_2^2 \pm \sqrt{(\beta - a_1^2 - a_2^2)^2 - 4a_1^2 a_2^2} \tag{5.1.27}$$

for which the existence of a horizon requires

$$\begin{aligned}\beta &\geq a_1^2 + a_2^2 + 2|a_1 a_2| \\ M^3 &\geq \frac{27\pi}{32G_N}(J_1^2 + J_2^2 + 2|J_1 J_2|)\end{aligned} \tag{5.1.28}$$

For general D odd, two vanishing spin parameters are required to guarantee the existence of a horizon. Also, for D odd it is possible to find horizons at positive radius even in the case of negative mass, however these solutions are rather pathological, containing regions of causality violation, which allows such black holes to evade the positive energy theorem for black holes of [111].

Singularities exist for the higher dimensional solutions when the deviation from flat space becomes infinite. If we write the metric in the form $g_{\mu\nu} = \eta_{\mu\nu} + hk_\mu k_\nu$ where k_μ is a vector field, then $h \to \infty$ implies that the metric coefficient of dt^2 diverges, which in Boyer-Lindquist coordinates means that

$$\left.\begin{array}{ll}\dfrac{\beta r}{\Pi\Sigma} & \qquad (D \text{ even}) \qquad (5.1.29a) \\[2ex] \dfrac{\beta r^2}{\Pi\Sigma} & \qquad (D \text{ odd}) \qquad (5.1.29b)\end{array}\right\} \to \infty.$$

For D even, when any of the angular momentum parameters a_i vanish, then Π contains overall factors of r^2 which cause a divergence at $r = 0$. If none of the a_i vanish, then Π is everywhere finite and $\Sigma = 0$ is required to have a singularity. This latter condition can be shown to occur only on the surface of a $(D-3)$-sphere with radii in the rotation planes given by the angular momentum parameters. Thus for $D = 4$ we recover the ring geometry of the singularity of the Kerr metric.

In the case that D is odd, and all $a_i \neq 0$, again Π is finite everywhere. It can be shown [103] that for D odd, Σ can be written as

$$\Sigma = \sum_i \frac{r^2 {\mu_i}^2}{r^2 + a_i^2} \tag{5.1.30}$$

which contains an overall factor of r^2 to cancel that of the numerator. Thus, in order to obtain a divergence, at least one of the angular momenta must vanish. When one of the a_i goes to zero, there is an overall factor of r^2 from Π which cancels the numerator of (5.1.29b), but a cancellation of a factor of r^2 also occurs in Σ, in which case it is necessary to have the direction cosine μ_j, corresponding to the vanishing angular momentum parameter a_j, vanish as well. Σ will then diverge on a $(D-4)$-sphere, again with radii in the rotation planes given by the corresponding angular momentum parameters. It can also be shown [103], that the singularities discussed here correspond to curvature singularities.

5.1.4. Taub-NUT metric

If one is prepared to put up with "pathological" behavior of a solution of the Einstein equations, it is possible to find other vacuum solutions. Here we will write down one of the simplest of such solutions, which has two parameters. These are the Taub-Newman-Unti-Tamburino (Taub-NUT) metrics [112]. These are written in $D = 4$ as

$$\begin{aligned} ds^2 = &- f\,(dt + 2\ell \cos\theta d\phi)^2 + f^{-1} dr^2 \\ &+ (r^2 + \ell^2)(d\theta^2 + \sin^2\theta d\varphi^2) \end{aligned} \tag{5.1.31}$$

where

$$f = 1 - 2\frac{G_N M r + \ell^2}{r^2 + \ell^2}. \tag{5.1.32}$$

where ℓ is the Taub-NUT parameter. This metric is singular at $r = r_\pm = G_N M \pm \sqrt{G_N^2 M^2 + \ell^2}$ where $f = 0$, but can be extended across these surfaces. It is found that this metric exhibits a line singularity [113] at $\theta = 0, \pi$, which can only be avoided if the time coordinate is made periodic with period $2\sqrt{G_N^2 M^2 - \ell^2}$. This metric is widely considered not to represent a physically realizable spacetime.

Of course, there is a generalization of the Taub-NUT metric with non-zero angular momentum [114] which we will write down here as

$$ds^2 = \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) - \frac{\Delta}{\Sigma}(dt - P_\theta d\varphi)^2 + \frac{\sin^2\theta}{\Sigma}(adt - P_r d\varphi)^2 \tag{5.1.33}$$

where

$$\begin{aligned} \Delta &= r^2 - 2G_N M r - \ell^2 + a^2, \\ \Sigma &= r^2 + (\ell - a\cos\theta)^2, \\ P_r &= r^2 + a^2 + \ell^2, \\ P_\theta &= 2\ell\cos\theta + a\sin^2\theta. \end{aligned} \tag{5.1.34}$$

The parameter a is as usual the angular momentum per unit mass.

5.2. The thermodynamics of black holes

As we mentioned earlier, the laws of black hole mechanics are analogous with those of thermodynamics. In this section we describe this analogy in more detail.

One of the anchors of the thermodynamic analogy is the *black hole area theorem* [109], which states that classically in a closed system, the area of all black holes in the universe can never decrease, $\delta\mathscr{A} \geq 0$. This resembles greatly the second law of thermodynamics, that in any physically allowed process the total entropy $\mathcal{S}$ of an isolated system cannot decrease.

Following [51], we wish to define a quantity ϑ on the horizon of an arbitrary stationary black hole. We do this with the help of Killing vectors ξ^μ which generates a one parameter group of isometries of a given spacetime, as is given by Killing's equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \tag{5.2.1}$$

where ∇_μ is the covariant derivative associated with the metric in question. This gives a necessary and sufficient condition to ensure that all lengths are preserved by the displacement $\epsilon\xi^\mu$. Thus the Killing vector field allows us to construct conservation laws from symmetries in a differential geometric context. One property of the Killing field is that its contraction with the tangent of a geodesic, $\xi_\mu u^\mu$ is constant along that geodesic. For a stationary black hole, there exists a Killing field χ^μ which is normal to the horizon. If χ^μ does not coincide with the stationary Killing vector ξ^μ, which generates the isometry of the solution as time evolves,[4] then we can form an axial Killing field ψ^μ from a linear combination of ξ^μ and χ^μ such as

$$\chi^\mu = \xi^\mu + \Omega_H \psi^\mu \tag{5.2.2}$$

where Ω_H is the angular velocity of the horizon, which is given by

$$\Omega_H = \frac{a}{r_+^2 + a^2} \tag{5.2.3}$$

for the case of the Kerr metric. Due to the fact that the horizon is a null surface and that χ^μ is normal to it, then we know that $\chi^\mu \chi_\mu|_{\text{ horizon}} = 0$ and we can therefore

[4] ξ^μ expresses the fact that the time t is a cyclic coordinate.

write

$$\nabla^\mu \left(\chi^\nu \chi_\nu\right) = -2\vartheta \chi^\mu. \tag{5.2.4}$$

It can be shown that ϑ is constant over the horizon, and can be computed to be [51]

$$\vartheta^2 = -\frac{1}{2}\left(\nabla^\mu \chi^\nu\right)\left(\nabla_\mu \chi_\nu\right) \vert_{\text{ horizon}} \tag{5.2.5}$$

which for a static black hole has the physical interpretation of the force at infinity that is necessary to hold a unit test mass in place on the horizon, called the *surface gravity.*

From the constancy of ϑ on the horizon the following simple formula for the mass of a stationary axisymmetric spacetime may be derived,

$$M = 2\int_\sigma dV \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) n^\mu \xi^\nu + \frac{1}{4\pi G_N} \vartheta \mathscr{A} + 2\Omega_H J \tag{5.2.6}$$

where σ denotes a spacelike hypersurface which intersects the horizon and J represents the angular momentum, and $\mathscr{A}$ is the area of the event horizon. From this in turn we derive a formula for the variation of the mass, which in the vacuum case comes out to be

$$\delta M = \frac{1}{4\pi}\left(\mathscr{A}\delta\vartheta + \vartheta\delta\mathscr{A}\right) + 2\left(J\delta\Omega_H + \Omega_H \delta J\right). \tag{5.2.7}$$

A second formula for the mass can be derived by considering perturbations of the metric [115] which is given as

$$\delta M = -\frac{1}{4\pi G_N} \mathscr{A}\delta\vartheta - 2J\delta\Omega_H \tag{5.2.8}$$

which we add to (5.2.7) to finally obtain

$$\delta M = \frac{1}{8\pi G_N}\vartheta\delta\mathscr{A} + \Omega_H \delta J. \tag{5.2.9}$$

Thus we see that the surface gravity ϑ plays the role of a temperature in the black hole when we make the following comparison between thermodynamics and black hole mechanics:

Law	Context	
	Thermodynamics	**Black holes**
0th	T = constant throughout body in thermal equilibrium	ϑ = constant over horizon of stationary black hole
1st	$dE = TdS + dW$	$dM = \frac{1}{8\pi G_N}\vartheta d\mathscr{A} + \Omega_H dJ$
2nd	$\delta S \geq 0$	$\delta\mathscr{A} \geq 0$
3rd	Impossible to achieve $T = 0$ in a physical process	Impossible to achieve $\vartheta = 0$ in a physical process

It turns out that general expressions for the surface gravity and horizon area of the metrics described in (5.1.19) and (5.1.21) can be obtained [103]. For the area of the horizon we have

$$\mathscr{A} = \frac{A_{D-2}\beta}{2\vartheta}\left(D - 3 - 2\sum_{i=1}^{\langle\frac{D-2}{2}\rangle}\frac{a_i^2}{r_+^2 + a_i^2}\right) \tag{5.2.10}$$

and for the surface gravity

$$\vartheta = \begin{cases} \left.\dfrac{\partial_r\Pi \; - \; \beta}{2\beta r}\right|_{r=r_+} & (D \text{ even}) \quad (5.2.11a) \\ \left.\dfrac{\partial_r\Pi - 2\beta r}{2\beta r^2}\right|_{r=r_+}. & (D \text{ odd}) \quad (5.2.11b) \end{cases}$$

The indication that the relationship between the laws of thermodynamics and those of black hole mechanics may not be simply an analogy comes from the fact that the thermodynamic energy E and the black hole mass M are not just analogs of one another, but rather describe the same physical quantity: the total energy. As noted earlier, one naïvely sets the temperature of a black hole to zero, since it is a perfect absorber, which would seem to ruin the identification completely. However, as shown by Hawking [7], quantum effects in the region of the event horizon result in the emission of a blackbody spectrum of particles at a non-zero temperature. The thermodynamic entropy S and temperature T are then related to the area of the horizon and the surface gravity as ($c = \hbar = 1$)

$$S = \frac{k_B\mathscr{A}}{4G_N}, \tag{5.2.12a}$$

$$T = \frac{\vartheta}{2\pi k_B} \tag{5.2.12b}$$

where k_B is the Boltzmann constant, which henceforth will be set to unity. It is also possible to obtain a formula for the irreducible mass of a higher dimensional black hole as [103]

$$M_{\text{ir}} = \frac{\vartheta \mathscr{A}}{8\pi G_N} \frac{D-2}{D-3}. \tag{5.2.13}$$

The thermodynamic analogy is thus complete, and the search for a microscopic understanding of the entropy of a black hole can begin in earnest.

There are a few things to note here. For the Kerr metric, the entropy and temperature are given by

$$\mathcal{S} = \frac{\mathscr{A}}{4G_N} = 2\pi G_N \left(\left(M + \sqrt{M^2 - \left(\frac{J}{G_N M}\right)^2 - (q^e)^2} \right)^2 + \left(\frac{J}{G_N M}\right)^2 \right),$$

$$T = \frac{\vartheta}{2\pi} = \frac{\sqrt{M^2 - \left(\frac{J}{G_N M}\right)^2 - (q^e)^2}}{4\pi G_N M \left(M + \sqrt{M^2 - \left(\frac{J}{G_N M}\right)^2 - (q^e)^2} \right) - (q^e)^2}. \tag{5.2.14}$$

Therefore, when $G_N^2 (q^e)^2 + a^2 = G_N^2 M^2$, or when the black hole is extremal, the temperature vanishes, and from (5.2.14) we see that the entropy reduces to

$$\mathcal{S} = 2\pi G_N \left(2 \left(\frac{J}{G_N M}\right)^2 + (q^e)^2 \right) \tag{5.2.15}$$

and therefore $\mathcal{S} \neq 0$ when $T = 0$. It has been shown that, just as in attempts to attain very low thermodynamic temperatures, the closer a black hole approaches $\vartheta = 0$, the more difficult it is to get still closer [116]. Also, there is another formulation of the third law, known as the Nernst theorem, which states that the entropy S of a system must tend to zero[5] as the temperature does. For black holes, however, it is possible that the area remain finite as the surface gravity vanishes. Recently, arguments have been made to produce counter examples to the Nernst version of the third law other than in black hole physics [117].

---

[5] Or to a "universal constant".

5.3. Black holes of string theory

It is evident from chapter II and III that the string equations of motion are more complicated than those of general relativity, even in $D = 4$. There are more fields to consider, and thus the class of black hole solutions in string theory is very much more general than the Kerr-Newman solution [118].

Since the dilaton couples to the gauge field strength[6] in (2.3.23) the charged black hole solutions of string theory are not those of the Einstein-Maxwell equations (5.1.6). Thus we must begin anew, attempting to solve the low energy string equations of motion, in order to find the string analog of the Reissner-Nordstrom solution. This was done in [119]. The result for the string-frame metric and associated fields was also found more recently through the solution generating techniques described in chapter III. The specific procedure used is discussed in [120] and similar techniques will be explained in detail in chapter VI. For now the result is

$$\begin{aligned} ds^2 &= -\left(1-\frac{\beta}{r}\right)\left(1+\frac{\beta(x^2-1)}{r}\right)^{-2} dt^2 + \left(1-\frac{\beta}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \\ \mathcal{A}^{(1)} &= \Lambda_G^{(1)} = \frac{-\beta x\sqrt{x^2-1}}{2\sqrt{2}\left(r+\beta(x^2-1)\right)} dt \\ e^{-2\phi^{(h)}} &= e^{-2\psi^{(a)}} = 1 + \frac{\beta(x^2-1)}{r} \end{aligned} \tag{5.3.1}$$

where β is again a mass parameter related to the physical mass of the Schwarzschild solution that was the starting point of the solution generating by (5.1.3), and where $x \geq 1$ is a parameter used in generating the solution. Here we have a non-zero dilaton as well as a gauge field. Note that this is a solution of both the Type IIA and heterotic string, since it involves only fields which are common to both of these theories.

The causal structure of this solution is identical to that of the Schwarzschild solution (see Fig. 5.1.2). One has the event horizon (coordinate singularity) at $r = \beta$ and a curvature (essential) singularity at $r = 0$. The absence of an inner horizon is particularly noteworthy. Also, as $r \to 0$, the string coupling e^{ϕ} becomes weak. It is

[6] This can be seen explicitly when the action is written in the Einstein frame.

difficult to speculate on what this might mean, however, since we have no reason to trust this solution close to the singularity.

The physical mass M depends upon the frame, Einstein or string, used to compute it. Rescaling to the Einstein metric through $g_{\mu\nu} = e^{-2\phi} G_{\mu\nu}$ (for $D = 4$) and comparing to Schwarzschild asymptotically we obtain the mass, while the charge is obtained from a similar expansion of the gauge field and comparing to (5.1.8). The results are

$$M = \frac{\beta x^2}{2G_N} \qquad\qquad q^e = \sqrt{\frac{\pi}{2}}\,\frac{\beta x\sqrt{x^2-1}}{G_N} \tag{5.3.2}$$

from which we see that the charge to mass ratio depends only on the parameter x, as

$$\frac{(q^e)^2}{M^2} = 2\pi\left(\frac{x^2-1}{x^2}\right). \tag{5.3.3}$$

For a given mass M, therefore, the amount of charge can be augmented by increasing x and decreasing β, which reduces the area of the event horizon. If we take the *extremal limit,* which is done by taking $\beta \to 0$ and $x \to \infty$ simultaneously in such a way as to maintain a fixed mass, we find that the largest possible charge to mass ratio is $|q^e| = \sqrt{2\pi}M$, at which point the horizon has shrunk onto the singularity. The metric now appears as

$$ds^2 = -\left(1 + \frac{2G_N M}{r}\right)^{-2} dt^2 + dr^2 + r^2 d\Omega^{D-2} \tag{5.3.4}$$

which is often called an *extremal charged black hole,* even though strictly speaking it is not a black hole. Its Penrose diagram is to be found in Fig. 5.3.1 where we see that the singularities are null. Notice also that the spatial part is completely flat. This process of extremization will be used later to create BPS saturated black hole solutions. As we saw in chapter III, BPS states have the maximum ratio of charge to mass, thus this process is useful for creating black hole solutions with this property. Extremization is, however, no guarantee of supersymmetry, as it remains possible to create extremal solutions which are not supersymmetric. We will see an example of such a solution in section 5.4.

Rescaling to the Einstein frame metric makes it easier to compare (5.3.1) with the black holes of general relativity. Using (2.3.24) and applying also the transformation

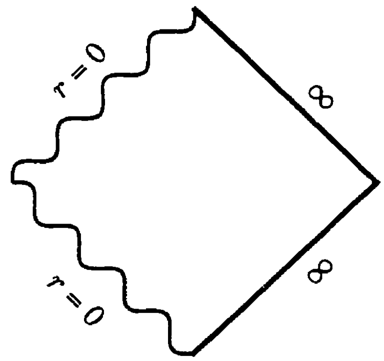


Figure 5.3.1: Penrose diagram of the extremal black hole with non-zero dilaton.

$R = r + \beta(x^2 - 1)$ results in a solution appearing as

$$\begin{aligned} ds_E^2 &= -\left(1 - \frac{2G_N M}{R}\right) dt^2 + \left(1 - \frac{2G_N M}{R}\right)^{-1} dR^2 \\ &\quad + R\left(R - \frac{G_N (q^e)^2}{\pi M}\right) d\Omega^2 \\ \mathcal{A}^{(1)} &= \Lambda_G^{(1)} = -\sqrt{\frac{G_N}{4\pi}}\frac{q^e}{R} dt \\ e^{2\phi^{(h)}} &= e^{2\phi^{(a)}} = 1 - \frac{G_N (q^e)^2}{\pi M R} \end{aligned} \tag{5.3.5}$$

where we see that the geometry is that of the Schwarzschild solution, but with the area of the spheres (Ω^2) reduced. When $R \to 0$ the area goes to zero and this surface is singular. In the $R - t$ plane the causal structure is independent of q^e and thus it is given by Fig. 5.3.1 in the Einstein frame also.

From the area theorem of black holes and from the fact that the area of the extremal black hole (5.3.4) is zero, we see that there is no classical process which could produce an extremal black hole from a non-extremal one. The auxiliary conditions upon which the area theorem depends are satisfied by the low energy effective actions of string theory [118].

5.3.1. Magnetically charged black holes

Let us now begin with the action of the heterotic string, (2.3.27) and perform a compactification on $K3 \otimes T^2$ down to four dimensions. Further, we set all fields except the metric, dilaton and one gauge field to zero, and finally we transform to the Einstein frame. The result can be written

$$S_h = \frac{1}{2\kappa_4^2}\int d^4x\sqrt{-\mathcal{g}}\left(R - 2(\nabla\phi^{(h)})^2 - e^{-2\phi^{(h)}}\left(\mathcal{F}^{(2)}\right)^2\right) \tag{5.3.6}$$

which can be seen to be invariant under the four-dimensional transformation

$$\begin{aligned} &\mathcal{g}_{\mu\nu} \rightarrow \mathcal{g}_{\mu\nu}, \\ &\phi^{(h)} \rightarrow -\phi^{(h)}, \\ &\mathcal{F}^{(2)} \rightarrow \tilde{\mathcal{F}}^{(2)} = {}^*\mathcal{F}^{(2)}, \end{aligned} \tag{5.3.7}$$

which is a manifestation of a self-duality of the heterotic string in $D = 4$ [121].

If we take our "stringy" Reissner-Nordstrom solution, (5.3.5) interpreted as a $D = 4$ heterotic solution and apply to it this duality transformation, we obtain, due to the dualization, a magnetically charged black hole (with $q^m = q^e$), which can be written

$$\begin{aligned} ds_E^2 &= -\left(1 - \frac{2G_NM}{R}\right)dt^2 + \left(1 - \frac{2G_NM}{R}\right)^{-1}dR^2 \\ &\quad + R\left(R - \frac{G_N(q^m)^2}{\pi M}\right)d\Omega^2, \\ \tilde{\mathcal{A}}^{(1)} &= \sqrt{\frac{G_N}{4\pi}}q^m\cos\theta d\varphi, \\ e^{-2\phi^{(h)}} &= \left(1 - \frac{G_N(q^m)^2}{\pi MR}\right). \end{aligned} \tag{5.3.8}$$

Here we note that the string coupling becomes strong near the curvature singularity, due to the change in sign of the dilaton. Since the metric is invariant under this transformation, the Penrose diagram is again that of the Schwarzschild metric for $|q^m| < \sqrt{2\pi}M$. However, the metric in the string frame is altered by the transformation, and is written

$$\begin{aligned} ds_E^2 = &-\left(1 - \frac{2G_NM}{R}\right)\left(1 - \frac{G_N(q^m)^2}{\pi MR}\right)^{-1}dt^2 \\ &+ \left(\left(1 - \frac{2G_NM}{R}\right)\left(1 - \frac{G_N(q^m)^2}{\pi MR}\right)\right)^{-1}dR^2 \\ &+ R^2d\Omega^2, \end{aligned} \tag{5.3.9}$$

The first thing we note is that the area of the two-spheres does not vanish as one approaches the singularity at $r = G_N(q^m)^2/(\pi M)$. Furthermore, the extremal limit of this metric,

$$ds^2 = -dt^2 + \left(1 - \frac{2G_N M}{r}\right)^{-2} dr^2 + r^2 d\Omega^2 \qquad (5.3.10)$$

is for $r > 2G_N M$ without curvature singularities. Here a time slice t = const. has the infinite throat geometry of the extremal Reissner-Nordstrom solution (Fig. 5.1.5), that is there is an infinite proper distance between $r = 2G_N M$ and $r > 2G_N M$. But, the surface $r = 2G_N M$ is also an infinite distance away in timelike and null directions as well as spacelike directions. The horizon has moved off to infinity, taking with it the singularity, thus effectively neither of these exist since they could not be discovered by mass or charge probes.

5.3.2. Asymptotically nonvanishing dilaton

The action (5.3.6) can be easily verified to be invariant under the transformation

$$\begin{aligned} \mathscr{g}_{\mu\nu} &\to \mathscr{g}_{\mu\nu}, \\ \phi^{(h)} &\to \phi^{(h)} + \phi_0^{(h)}, \\ \mathcal{F}^{(2)} &\to \bar{\mathcal{F}}^{(2)} = \mathrm{e}^{\phi_0^{(h)}} \mathcal{F}^{(2)}, \end{aligned} \qquad (5.3.11)$$

where $\phi_0^{(h)}$ is the value of the dilaton at asymptotic infinity. Although the Einstein frame metric is at first glance unaffected by such a transformation, it will have dependence on $\phi_0^{(h)}$ through the charge, which is rescaled by a factor $\mathrm{e}^{\phi_0^{(h)}}$ which results in the solution appearing as

$$\begin{aligned} ds_E^2 = &-\left(1 - \frac{2G_N M}{R}\right) dt^2 + \left(1 - \frac{2G_N M}{R}\right)^{-1} dR^2 \\ &+ R\left(R - \frac{G_N (q^m)^2\, \mathrm{e}^{-2\phi_0^{(h)}}}{\pi M}\right) d\Omega^2, \\ \bar{\mathcal{A}}^{(1)} = &\sqrt{\frac{G_N}{4\pi}} q^m \cos\theta d\varphi, \\ \mathrm{e}^{-2\phi^{(h)}} = &\ \mathrm{e}^{-2\phi_0^{(h)}} \left(1 - \frac{G_N (q^m)^2\, \mathrm{e}^{-2\phi_0^{(h)}}}{\pi M R}\right). \end{aligned} \qquad (5.3.12)$$

In this case the extremal limit is now $|q^m| = \sqrt{2\pi} M e^{\phi_0^{(h)}}$ and thus when $\phi_0^{(h)}$ is large, we may indeed have black holes with q^m or $q^e >> M$.

Thus we see at play here the same sort of weak/strong, electric/magnetic, singular/solitonic structure that we saw in Chapter III. There is much more to say here. We can also imagine dyonic black holes, with both electric *and* magnetic charge, and string theory generalizations of black holes with angular momentum. Some of these things we will discuss in later chapters, others we leave for the references.

5.4. D-branes and black holes

The similarity between the forms of the black hole metrics that we have written down here and those which we wrote down in the previous chapter for the D-branes will not have gone unnoticed.

It is not hard to imagine the creation of black hole solutions which have, in higher dimensions, non-vanishing higher form potentials and thus have, in the type II string theories, the Ramond-Ramond charges of which the D-branes are the sources, as we have seen. In this scenario, a black hole can be said to be composed of a number of D-branes, forming in fact a bound state of these objects. This is particularly true of supersymmetric black holes, which are composed of supersymmetric bound states of D-branes, and for which the technology now exists, thanks to D-branes, to compute the entropy from a fundamental statistical viewpoint.

Let us consider here a simple example of a black hole with non-trivial D-brane content. The following is an extremal string-frame black hole solution of the type IIA effective action in ten dimensions:

$$ds^2 = -f_1^{-1}dt^2 + f_1\left(dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right) + f_2\sum_{i=4}^{9}(dx^i)^2, \tag{5.4.1a}$$

$$A^{(1)} = \frac{2}{\sqrt{u+v}}\left(u^{\frac{3}{2}}\cos\theta d\varphi + \frac{v^{\frac{3}{2}}\left(1+\frac{u}{r}\right)}{r\left(\mathcal{A}+\mathcal{B}\right)}dt\right), \tag{5.4.1b}$$

$$e^{2\phi^{(a)}} = (f_2)^{\frac{3}{2}}, \tag{5.4.1c}$$

where

$$f_1 = \sqrt{\mathcal{A}^2 - \mathcal{B}^2}, \qquad f_2 = \sqrt{\frac{\mathcal{A}+\mathcal{B}}{\mathcal{A}-\mathcal{B}}},$$
$$\mathcal{A} = \left(1+\frac{u}{r}\right)\left(1+\frac{v}{r}\right), \qquad \mathcal{B} = \frac{u-v}{u+v}(1-\mathcal{A}). \tag{5.4.2}$$

Here $u, v \geq 0$ are magnetic and electric charge parameters, respectively. The directions $x^4, x^5, \cdots, x^9$ are considered to be compact with radii $R_4, R_5, \cdots, R_9$. As can be seen from the field content, in the Ramond-Ramond sector we have a dyonic one-form potential. As shown in chapter IV, for an electric D-brane, a $\hat{d}$-form potential couples to an object with a $\hat{d}$-dimensional world-volume. In the magnetic case, a $\hat{d}$-form potential couples to the world volume of a ($\tilde{d} = D - \hat{d} - 2$)-dimensional object. Thus the content of this black hole in terms of D-branes is an electric D-point, and a magnetic $D6$-brane.

One can compute the physical ADM mass and charges as defined in (4.1.29) and (4.1.32) to be

$$\begin{aligned} M &= \frac{4\pi}{\kappa^2}(u+v), \\ q^e &= -\frac{8\pi v^{\frac{3}{2}}}{\kappa\sqrt{2(u+v)}}(2\pi)^6 \prod_{i=4}^{9} R_i, \\ q^m &= -\frac{8\pi u^{\frac{3}{2}}}{\kappa\sqrt{2(u+v)}}. \end{aligned} \tag{5.4.3}$$

In order to compare charges and masses, we should divide out the volume of the compact 6-torus, thus obtaining an electric charge per unit 6-volume of the six-torus T^6, which we write as

$$\tilde{q^e} = \frac{q^e}{(2\pi)^6 \prod_{i=4}^{9} R_i} = -\frac{8\pi v^{\frac{3}{2}}}{\kappa\sqrt{2(u+v)}}. \tag{5.4.4}$$

We note that if $u = 0$, then $q^m = 0$, so the magnetic $D6$-brane has vanished. Then we note that $2\kappa^2 M_0^2 = (\tilde{q^e})^2$, indicating as in (4.1.34) that alone the D-point is a BPS saturated state, as expected. For $v = 0$, then the D-point vanishes, and likewise $2\kappa^2 M_6^2 = (q^m)^2$, which indicates that, as in (4.1.43) the magnetic $D6$-brane is by itself a BPS state.

However, if we compute the mass charge relation of the bound state we find that

$$2\kappa^2 M^2 - (\tilde{q^e})^2 - (q^m)^2 = \frac{6(4\pi)^2}{\kappa} uv \geq 0 \tag{5.4.5}$$

thus the bound state does not saturate the bound for both $u, v \neq 0$. We can conclude that this particular black hole is not supersymmetric [16]. If one computes the mass of the bound state in terms of the mass of its constituents, one finds that

$$M - M_0 - M_6 = \frac{4\pi}{\kappa^2}(u + v - v - u) = 0 \tag{5.4.6}$$

which indicates that although the long range potential between a D-point and a $D6$-brane is repulsive,[7] they can form threshold bound states. In [122]. an approximate D-brane count of the entropy was carried out for a less general version of this solution with a single charge parameter. The construction of the solution (5.4.1) will be considered in detail in chapter VI.

[7] This is discussed in chapter VII.

VI

Spinning black holes and their entropy

Recently, significant progress has been made in understanding the degrees of freedom giving rise to the entropy of certain black holes in string theory [10]. String theory has thus demonstrated a remarkable and detailed knowledge of black hole thermodynamics. In [10] it was found that the newly-understood rules [66,123-128] for counting degeneracies of BPS-saturated, D-brane bound states precisely reproduces the Bekenstein-Hawking entropy for a certain five-dimensional extremal Reissner-Nordstrom black hole. These results were extended to leading order above extremality in [11,129] solidifying the identification of the microscopic states responsible for the entropy.

Here we will begin with an elementary introduction to the method of counting of the entropy for a non-spinning black hole. In the sections which follow we construct two different classes of black holes in five spacetime dimensions. In the first section we construct a spinning generalization of the static black hole given in [10]. We detail the methods used to construct the solution, which make use of the symmetry properties of string theory as discussed in chapter III. The microscopic entropy will then be computed from a counting of the degeneracies of the D-brane bound state associated with the black hole.

In section 2 we will construct another five dimensional black hole, which we will represent in six dimensions as a black string. This will also have non-zero angular momentum. Here, however, we will compute the entropy of not only the extremal limit, but of the near-extremal solution as well, generalizing the result of [11] to the spinning case. It is found that the D-brane techniques of counting the microscopic entropy reproduce the exact result both at and to leading order away from the extremal state. Thus in these cases it is shown that the stringy degeneracies continue to match the extremal Bekenstein-Hawking entropy when rotation is added.

We also, in section 6.4, demonstrate that one can overcome the tendency of solution generating techniques to produce non-zero Taub-NUT charges when used to generated black hole solutions that are dyonic. The reader will also note that in this chapter, the Newton constant G_N has been set to unity

6.1. Counting entropy with D-branes

Our task in this section is to demonstrate the method of computing the entropy of BPS saturated black holes using the D-brane technology as first developed in [10]. In section 4.2.4, we sketched how D-strings could be excited, and that these excitations could be interpreted as a gas of open strings attached to a D-string. This picture can clearly be extended to more general D-branes.

The basic idea behind the counting of black hole entropy[1] is rather simple. Begin with a BPS saturated black hole in some number of dimensions, usually four or five. This black hole is then embedded into a type II superstring theory compactified on some manifold, for example $K3 \otimes T^2$. The embedded black hole solution then carries charge under the Ramond-Ramond sector of the type II theory in question. The solution can also carry charge under the Neveu-Schwarz–Neveu-Schwarz sector, which is interpreted as momentum along compact directions. As an aside, this also gives us NS-NS charge quantization. One can perform the usual classical computations of Bekenstein-Hawking entropy for the black hole interpreted as a type II BPS state.

Black holes are objects which are strongly coupled. This means that the string coupling g is strong enough that the string length is much less than the Schwarzschild radius of the black hole. In normal circumstances, this means that trying to probe the interior of such an object by means of perturbation theory, or weak-coupling expansions, is futile. However, the BPS nature of the state under consideration comes to the rescue. Since BPS states are free of quantum corrections, we are

[1] Strictly speaking, of course, it is degeneracy that we are counting, the logarithm of which gives the entropy.

allowed to reduce the coupling to the regime where we can consider the bound state of D-branes which carry the RR charge of the black hole to be weakly bound. In this situation, the string length becomes larger than the Schwarzschild radius of the black hole. An image one might have is that the black hole has "unfolded" into its constituent parts, such as a bound state of D-branes. Since D-branes carry integer units of a fundamental RR charge, we are able to compute the precise number of D-branes which form the bound state.

The NS-NS charge, interpreted as the total momentum of the gas of open strings which exists on and between the various D-branes of the bound state, then comes into play. The logarithm of the number of distinct ways in which the total momentum can be distributed amongst the constituent branes of the bound state gives us precisely the entropy. We then rely on the characteristics of BPS saturated states to protect the degeneracy count from quantum corrections as the coupling is returned to its original value.

After this overview of the method, a concrete example is in order. For simplicity, let us consider a static black hole in five dimensions, which is a solution of low energy type IIB superstring effective action, compactified on T^5. The configuration that we will consider contains a number of $D5$-branes which are wrapped over the entire compact manifold, as well as a number of D-strings which are wrapped on one of the compact coordinates of T^5. We also consider a NS-NS charge to be a momentum along the compact direction around which the D-strings are wrapped.

As mentioned, we will have, as a result of the NS-NS charge or momentum, a gas of open strings that travel along the D-branes. We will have strings for which both ends connect to the D-strings, and strings which attach solely to the $D5$-branes. We will also have two other sets of open strings. One set starts at the D-string and ends on the $D5$-brane, for the other set the orientation is reversed. It is important to treat these strings as distinguishable. As mentioned, we want to excite these strings up to the maximum of the NS-NS charge.

As discussed in section 4.2.4, sometimes we wind up with massive excitations, thus we search for the way to distribute the momentum such that the maximum number of excitations remain massless. Consider the string action (2.2.12). In the case that we have a string which begins on a $D5$-brane and terminates on a D-string, we will have the following boundary conditions:

$$
\begin{array}{ccc}
 & D5\text{-brane} & D\text{-string} \\
t & \mathrm{N} & \mathrm{N} \\
x^1 & \mathrm{D} & \mathrm{D} \\
x^2 & \mathrm{D} & \mathrm{D} \\
x^3 & \mathrm{D} & \mathrm{D} \\
x^4 & \mathrm{D} & \mathrm{D} \\
x^5 & \mathrm{N} & \mathrm{N} \\
x^6 & \mathrm{N} & \mathrm{D} \\
x^7 & \mathrm{N} & \mathrm{D} \\
x^8 & \mathrm{N} & \mathrm{D} \\
x^9 & \mathrm{N} & \mathrm{D}.
\end{array}
\tag{6.1.1}
$$

From the combinations of boundary conditions we see that we have two N-N, four D-D and four N-D. Now, the mode expansions of the open string will obviously depend upon the set of boundary conditions imposed. Recall from (2.1.41) the mode expansion of X^μ for the open string. We will have slightly different mode expansions in the case of Dirichlet boundary conditions at the ends of the string, or for mixed boundary conditions [65]. We write them here as

$$X^\mu(\sigma,\tau) = x^\mu + \ell^2 p^\mu \tau + \frac{i\ell}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu(\mathrm{e}^{-in\sigma^+} + \mathrm{e}^{-in\sigma^-}), \quad \text{(NN)} \tag{6.1.2a}$$

$$X^\mu(\sigma,\tau) = \frac{i\ell}{2}\sum_{r\in\mathbb{Z}+\frac{1}{2}}\frac{1}{r}\alpha_r^\mu(\mathrm{e}^{-ir\sigma^+} + \mathrm{e}^{-ir\sigma^-}), \quad \text{(DN,ND)} \tag{6.1.2b}$$

$$X^\mu(\sigma,\tau) = \frac{\delta X^\mu}{2\pi}\sigma + \frac{i\ell}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu(\mathrm{e}^{-in\sigma^+} - \mathrm{e}^{-in\sigma^-}). \quad \text{(DD)} \tag{6.1.2c}$$

Thus, when we have mixed boundary conditions we have half-integer modes just as for the NS boundary condition in the case of the fermionic world sheet fields.

As usual, in the Ramond sector the zero point energy, i.e., the normal ordering constant in the mass vanishes, whereas in the Neveu-Schwarz sector we have

$$E_0 = (8 - N_{ND})\left(-\frac{1}{24} - \frac{1}{48}\right) + N_{ND}\left(\frac{1}{48} + \frac{1}{24}\right) = \frac{N_{ND}}{8} - \frac{1}{2} \tag{6.1.3}$$

where N_{ND} is the number of coordinates with mixed boundary conditions. Since in our case we have $N_{ND} = 4$, the zero point energy vanishes. The moding of the fermionic fields is the same as that of the bosonic fields in the Ramond sector, and opposite in the NS sector. Now, the Neveu-Schwarz fermionic vacuum state of the four ND coordinates is a spinor under the group $SO(4)$, and is thus multiplied by the product of Dirac matrices $\gamma^6\gamma^7\gamma^8\gamma^9$. From the GSO projection it will have definite chirality, i.e., $\gamma^6\gamma^7\gamma^8\gamma^9\chi = \chi$ and therefore what remains is in a two dimensional representation. Since the open strings are oriented, we have a second table like (6.1.1) and each string can attach to any of the branes at each end. Thus the count for the NS sector is

$$2 \cdot 2 \cdot \mathfrak{n}_5 \cdot \mathfrak{n}_1 \tag{6.1.4}$$

where $\mathfrak{n}_5$, $\mathfrak{n}_1$ are the numbers of fivebranes and D-strings respectively.

Now we examine the Ramond sector. The Ramond sector fermions which are transverse to the string and in the world volume of the fivebrane are half-integer moded. Again, $N_{ND} = 4$ so the vacuum has zero energy. This time, the vacuum state is in the spinor representation of $SO(1,5)$, with the GSO projection removing the representation with negative chirality. In the case that the positive chirality representation is composed only of left-moving modes, then the two-dimensional representation which has positive chirality under *both* $SO(1,1)$ from the NN coordinates and the $SO(4)$ from the coordinates $x^1, \cdots, x^4$, forming the group $SO(1,1) \otimes SO(4)$, survives the GSO projection. Again we have oriented strings so the result for the Ramond sector is therefore

$$2 \cdot 2 \cdot \mathfrak{n}_5 \cdot \mathfrak{n}_1. \tag{6.1.5}$$

The NS-NS charge $\mathcal{Q}$ will be quantized in integer multiples of $1/R_5$ where R_5 is the radius of the compact direction x^5, that is the total momentum $P_5 = \mathcal{Q}/R_5$. Now, we have for each momentum value $4\mathfrak{n}_5 \cdot \mathfrak{n}_1$ bosons and $4\mathfrak{n}_5 \cdot \mathfrak{n}_1$ fermions.

Now, the task that we face is determining the number of ways of dividing up the NS momentum number $\mathcal{Q}$ amongst the fermionic and bosonic ground states. The number of ways of distributing $\mathcal{Q}$ amongst $4n_1n_5$ fermions and $4n_1n_5$ bosons is given by the partition function [130]

$$\sum d(\mathcal{Q})s^{\mathcal{Q}} = \left(\prod_{w=1}^{\infty}\frac{1+s^w}{1-s^w}\right)^{4n_1n_5} . \tag{6.1.6}$$

Here the origin of the numerator of the expansion are the fermions, meaning that only one fermion can be excited at a time, and the denominator, which is really just a shorthand way of writing an expansion in powers of s, has its origin in the bosonic degrees of freedom. For large $\mathcal{Q}$, the coefficients on the left hand side can be approximated by [9]

$$d(\mathcal{Q}) \sim e^{2\pi\sqrt{n_1n_5\mathcal{Q}}} \tag{6.1.7}$$

and thus we have the entropy, finally as

$$S_{\text{micro}} = \log d(\mathcal{Q}) = 2\pi\sqrt{n_1\, n_5\, \mathcal{Q}}. \tag{6.1.8}$$

Now let us compare this result with the classical entropy of a five dimensional black hole which is charged under RR $D5$-branes and D-strings. Such a solution has been given in [129]. It may be written in the string frame as

$$\begin{aligned}
ds^2 =& (f_1f_2)^{-1/2}\Big(-dt^2+(dx^5)^2+f_3(dt-dx^5)\Big)^2\\
&+f_1^{1/2}\left(f_2^{1/2}\sum_{i=1}^{4}(dx^i)^2+f_2^{-1/2}\sum_{i=6}^{9}(dx^i)^2\right)\\
B^{(b)} =& \frac{1}{2}\left(\frac{1}{f_1}-1\right)dt\wedge dx^5\\
F^{(3)} =& \frac{1}{2}dx^i\wedge dx^j\wedge dx^k\wedge\epsilon_{ijkl}\partial^l f_2\\
e^{-2\phi^{(b)}} =& \frac{f_2}{f_1}
\end{aligned} \tag{6.1.9}$$

where

$$f_1 = 1+\frac{a_1n_1}{r^2}, \qquad f_2 = 1+\frac{a_2n_5}{r^2}, \qquad f_3 = \frac{a_3\mathcal{Q}}{r^2}, \tag{6.1.10}$$

and where in turn the constants a_1, a_2, and a_3 normalize the D-brane charges and NS momentum into integer units. In terms of the five-dimensional Newton constant

$G_N^{(5)}$ they are given as

$$a_1 = \frac{4G_N^{(5)} R_5 \, e^{\phi^{(6)}}}{\pi\alpha'}, \qquad a_2 = e^{\phi^{(6)}}\alpha', \qquad a_3 = \frac{4G_N^{(5)}}{\pi R_5}. \tag{6.1.11}$$

Computing the classical Bekenstein-Hawking entropy for this solution one obtains

$$\mathcal{S} = \frac{\mathcal{A}}{4G_N^{(5)}} = 2\pi\sqrt{\mathrm{n}_1\mathrm{n}_5\mathcal{Q}} \tag{6.1.12}$$

in complete agreement with (6.1.8).

Thus we have seen how to compute the entropy of a black hole by means of a particular implementation of the D-brane technology [129]. There exist other methods. For example, the method used in [10] uses a more sophisticated approach based on cohomology of instanton moduli spaces. We will use this alternative method to count the entropy of a spinning five dimensional black hole in the next section. For the example at hand, the result is exactly the same.

6.2. Solution generating

Before we explain our method for generating the final solution, let us review some salient features of low–energy actions for the heterotic and Type II theories in six and five dimensions. We will work with simplified versions of the six dimensional actions found in equations (3.5.7), the heterotic string compactified on T^4 and (3.5.10), the type IIA string compactified on the Calabi-Yau manifold $K3$. All abelian gauge fields except one $F^{(2)} = dA^{(1)}$ have been set to zero, and all scalars resulting from compactification, the moduli, are vanishing. The one remaining gauge field is taken to be a right–handed[2] internal gauge field on the heterotic side, and a field of Ramond–Ramond origin on the Type II side.

We then write on the heterotic side, to lowest order in α' [131] the action which we will use as:

$$S_h\ (T^4) = \int d^6x\sqrt{-\mathcal{G}_6}e^{-2\phi_6^{(h)}}\left\{R + 4(\nabla\phi_6^{(h)})^2 - \frac{1}{12}\left(H_6^{(h)}\right)^2 - \frac{1}{4}\left(\mathcal{F}_6^{(2)}\right)^2\right\} \tag{6.2.1}$$

[2] We take the field to be right–handed, or of positive chirality, so that the extremal configuration is supersymmetric.

where

$$H^{(h)}_{6\,\mu\nu\lambda} = \partial_\mu B^{(h)}_{6\,\nu\lambda} - \frac{1}{2}\mathcal{A}^{(1)}_{6\,\mu}\mathcal{F}^{(2)}_{6\,\nu\lambda} + \text{(cyclic)} \tag{6.2.2}$$

Note that the Chern–Simons terms come from the gauge fields arising from compactification. For the Type IIA side, we write the simplified six dimensional action as

$$S_{IIA\ (K3)} = \int d^6x \Big[\sqrt{-G_6} \Big[e^{-2\phi_6^{(a)}} \left(R + 4(\nabla\phi_6^{(a)})^2 - \frac{1}{12}\left(H_6^{(a)}\right)^2 \right) \\ - \frac{1}{4}\left(F_6^{(2)}\right)^2 \Big] + \frac{1}{16}\epsilon^{\mu\nu\lambda\rho\alpha\beta} B^{(a)}_{6\,\mu\nu} F^{(2)}_{6\,\lambda\rho} F^{(2)}_{6\,\alpha\beta} \Big] \tag{6.2.3}$$

where

$$H^{(a)}_{6\,\mu\nu\lambda} = \partial_\mu B^{(a)}_{6\,\nu\lambda} + \text{(cyclic)} \tag{6.2.4}$$

These two actions are related by the string/string duality relation given in section 3.5(equation (3.5.12)), which we repeat here as

$$\phi_6^{(a)} = -\,\phi_6^{(h)} \qquad\qquad G_{6\mu\nu} = e^{-2\phi_6^{(h)}}\,\mathcal{G}_{6\mu\nu} \tag{6.2.5a}$$

$$A^{(1)}_{6\mu} = \mathcal{A}^{(1)}_{6\mu} \tag{6.2.5b}$$

$$H^{(a)}_{6\mu\nu\rho} = \frac{1}{6}\epsilon_{\mu\nu\rho\sigma\kappa\delta}\; e^{-2\phi_6^{(h)}}\; H_6^{(h)\,\sigma\kappa\delta}. \tag{6.2.5c}$$

Since we will need also the five-dimensional type IIA action compactified on $K3 \otimes S^1$, we give here the standard Kaluza–Klein reduction on the circle S^1 with coordinate labelling $y = x^5$, for the type IIA case as

$$\begin{aligned} ds_6^2 &= G_{5\mu\nu}dx^\mu dx^\nu + e^{2\sigma}\left(dy + \Lambda^{(1)}_{G\mu}dx^\mu\right)^2 \\ \phi_6 &= \phi + \frac{1}{2}\sigma \\ B_6 &= \frac{1}{2}\left[B_{\mu\nu} - \frac{1}{2}\left(\Lambda^{(1)}_{G\mu}\Lambda^{(1)}_{B\nu} - \Lambda^{(1)}_{B\mu}\Lambda^{(1)}_{G\nu}\right)\right] dx^\mu \wedge dx^\nu + \Lambda^{(1)}_{B\mu}dx^\mu \wedge dy \end{aligned} \tag{6.2.6}$$

where $\Lambda^{(1)}_{G\mu}$ and $\Lambda^{(1)}_{B\mu}$ are the gauge fields coming from the compactification of the metric and antisymmetric tensor fields respectively.

The five dimensional type IIA action (in the sector with $A_y^{(1)} = 0$) is expressed in the string frame as[3]

$$S_{IIA\ (K3\otimes S^1)} = \int d^5x \left[\sqrt{-G}\left[e^{-2\phi^{(a)}}\left(R + 4(\nabla\phi^{(a)})^2 - \frac{1}{12}\left(H^{(a)}\right)^2 - \left(\partial_\mu\sigma\right)^2 - \frac{1}{4}e^{2\sigma}\left(\Xi_G^{(2)}\right)^2 - \frac{1}{4}e^{-2\sigma}\left(\Xi_B^{(2)}\right)^2\right) - \frac{1}{4}e^{\sigma}\left(F^{(2)}\right)^2\right] + \frac{1}{8}\epsilon^{\mu\nu\lambda\alpha\beta}\Lambda_{B\mu}^{(1)}F_{\nu\lambda}^{(2)}F_{\alpha\beta}^{(2)}\right] \tag{6.2.7}$$

where $\Xi_G^{(2)} = d\Lambda_G^{(1)}$, $\Xi_B^{(2)} = d\Lambda_B^{(1)}$ and

$$H_{\mu\nu\lambda}^{(a)} = \partial_\mu B_{\nu\lambda}^{(a)} - \frac{1}{2}\Lambda_{G\mu}^{(1)}\Xi_{B\nu\lambda}^{(2)} - \frac{1}{2}\Lambda_{B\mu}^{(1)}\Xi_{G\mu\nu}^{(2)} + \text{(cyclic)} \tag{6.2.8}$$

In the five dimensional Einstein frame, the transformation to which is defined by $g_{\mu\nu} = e^{-4\phi/3}G_{\mu\nu}$, the action appears as

$$S_{IIA\ (K3\otimes S^1)} = \int d^5x \left[\sqrt{-g}\left(R - \frac{4}{3}(\nabla\phi^{(a)})^2 - \left(\partial_\mu\sigma\right)^2 - \frac{1}{4}e^{2\sigma-4\phi^{(a)}/3}\left(\Xi_G^{(2)}\right)^2 - \frac{1}{4}e^{-2\sigma-4\phi^{(a)}/3}\left(\Xi_B^{(2)}\right)^2 - \frac{1}{4}e^{8\phi^{(a)}/3}\left(V^{(2)}\right)^2 - \frac{1}{4}e^{\sigma+2\phi^{(a)}/3}\left(F^{(2)}\right)^2\right) + \frac{1}{8}\epsilon^{\sigma\rho\mu\nu\lambda}\left(\mathcal{V}_\sigma^{(1)}\Xi_{G\rho\mu}^{(2)}\Xi_{B\nu\lambda}^{(2)} + \Lambda_{B\sigma}^{(1)}F_{\rho\mu}^{(2)}F_{\nu\lambda}^{(2)}\right)\right] \tag{6.2.9}$$

where in this action we have Hodge–dualized the three–form $H^{(a)}$ via

$$\begin{aligned} H_{\mu\nu\lambda}^{(a)} &= \frac{1}{2}e^{8\phi^{(a)}/3}\sqrt{-g}\epsilon_{\sigma\rho\mu\nu\lambda}V^{(2)\sigma\rho} \\ &= \partial_\mu B_{\nu\lambda}^{(a)} - \frac{1}{2}\Lambda_{G\mu}^{(1)}\Xi_{B\nu\lambda}^{(2)} - \frac{1}{2}\Lambda_{B\mu}^{(1)}\Xi_{G\nu\lambda}^{(2)} + \text{(cyclic)}\,. \end{aligned} \tag{6.2.10}$$

Note also that $V^{(2)} = d\mathcal{V}^{(1)}$.

Having completed the exposition of the actions we will use here, we now turn to the black hole solution of the five dimensional theory with which we will begin. This is a five dimensional black hole which spins in a single plane, which is a solution of the five–dimensional Einstein equations, and which can be found in general form in

[3] Note that we omit for simplicity the subscripts on the five-dimensional fields. In this chapter we work only with five- and six-dimensional actions.

equation (5.1.19). We add a trivial flat compact dimension with coordinate y, and the metric is then

$$\begin{aligned} ds_6^2 &= G_{6\,\mu\nu} dx^\mu dx^\nu \\ &= -dt^2 + \left(r^2 + a^2\right) \sin^2\theta d\varphi^2 + \frac{\beta}{\rho^2}\left(dt + a\sin^2\theta d\varphi\right)^2 \\ &\quad + \frac{\rho^2}{r^2 + a^2 - \beta} dr^2 + \rho^2 d\theta^2 + r^2\cos^2\theta d\psi^2 + dy^2 \end{aligned} \tag{6.2.11}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, and β and a are the mass and angular momentum parameters as defined in chapter V. The coordinate system we are using here is that of spherical polar coordinates in five dimensions, $r, \theta, \varphi, \psi, t$ with the additional flat coordinate y. This black hole can be thought of as a solution of the six dimensional low energy action of type IIA string theory. It is a solution which has only the metric excited but no gauge fields, antisymmetric tensor, dilaton, or moduli fields turned on. From it, we will obtain a charged spinning black hole solution of the Type II theory in five dimensions. This black hole will be a spinning generalization of the solution in [10].

6.2.1. Generating techniques

Our method for generating the desired black hole solution is to use a series of transformations, namely $O(6, 6, \mathbb{R})$ boosts involving the time t and the circle coordinate y, and string/string duality. String/string duality is implemented simply by computing the mapping given in equation (6.2.5). For the $O(d, d, \mathbb{R})$ transformations, the procedure we have implemented is that outlined in [120] which functions as follows:

Let us consider first the procedure in the case of the heterotic string with one non-zero gauge field, the extension to more than one gauge field is straightforward. One first forms, from the fields making up the solution which one wishes to transform, the linear combinations

$$K_{\pm\mu\nu} = -B^{(h)}_{\mu\nu} - \mathcal{G}_{\mu\nu} - \frac{1}{4}\mathcal{A}^{(1)}_\mu \mathcal{A}^{(1)}_\nu \pm \eta_{\mu\nu} \tag{6.2.12}$$

from which the following matrix is formed

$$\mathcal{M} = \begin{pmatrix} K_-^t \mathcal{G}^{-1} K_- & K_-^t \mathcal{G}^{-1} K_+ & -K_-^t \mathcal{G}^{-1} \mathcal{A}^{(1)} \\ K_+^t \mathcal{G}^{-1} K_- & K_+^t \mathcal{G}^{-1} K_+ & -K_+^t \mathcal{G}^{-1} \mathcal{A}^{(1)} \\ -(\mathcal{A}^{(1)})^t \mathcal{G}^{-1} K_- & -(\mathcal{A}^{(1)})^t \mathcal{G}^{-1} K_+ & (\mathcal{A}^{(1)})^t \mathcal{G}^{-1} \mathcal{A}^{(1)} \end{pmatrix} \tag{6.2.13}$$

where the superscript ^{t} indicates the transpose. At this point the effect of an $O(d, d, \mathbb{R})$ transformation on the solution in question is contained in the relation

$$\mathcal{M} \rightarrow \tilde{\mathcal{M}} = \Omega_d \mathcal{M} \Omega_d^t \tag{6.2.14}$$

where the transformation matrix $\Omega_d \in O(d, d, \mathbb{R})$. Specific examples of transformation matrices will be given later.

After the $O(d, d, \mathbb{R})$ transformation has been carried out, there remains the question of extracting the new metric, antisymmetric tensor field, and gauge fields from the new matrix $\tilde{\mathcal{M}}$. The way to do this is also found in [120] and consists of forming the matrix

$$V = \begin{pmatrix} \eta/2 & -\eta/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1/\sqrt{2} \end{pmatrix} \tag{6.2.15}$$

which leaves $\tilde{\mathcal{M}}$ in a state looking like

$$\tilde{\mathcal{M}}' = V\tilde{\mathcal{M}}V^t = \begin{pmatrix} \mathcal{G}'^{-1} & -\mathcal{G}'^{-1}K' & \mathcal{G}'^{-1}\mathcal{A}^{(1)'}/\sqrt{2} \\ -K'^t\mathcal{G}'^{-1} & K'^t\mathcal{G}'^{-1}K' & -K'^t\mathcal{G}'^{-1}\mathcal{A}^{(1)'}/\sqrt{2} \\ \mathcal{A}^{(1)'^t}\mathcal{G}'^{-1}/\sqrt{2} & -\mathcal{A}^{(1)'^t}\mathcal{G}'^{-1}K'/\sqrt{2} & \mathcal{A}^{(1)'^t}\mathcal{G}'^{-1}\mathcal{A}^{(1)}/2 \end{pmatrix} \tag{6.2.16}$$

where $K' = (K'_\pm \mp \eta)$ and therefore the new metric, new K', and gauge field $A^{(1)}$ can be extracted from the upper left, upper center, and upper right parts respectively of $\tilde{\mathcal{M}}'$. Then the antisymmetric tensor field is the antisymmetric part of K' as in

$$\begin{aligned} \mathcal{G}' &= \left(\mathcal{G}'^{-1}\right)^{-1} \\ B^{(h)'} &= -\frac{1}{2}\left(K' - K'^t\right). \\ e^{2\phi^{(h)'}} &= \frac{\det \mathcal{G}'}{\det \mathcal{G}} e^{2\phi^{(h)}} \end{aligned} \tag{6.2.17}$$

Note also that the dilaton field is transformed in accord with our discussion in chapter III.

As was also noted in chapter III, in the case of the heterotic string the group under which the equations of motion of the low energy effective action are invariant is $O(d, d+\hat{p}, \mathbb{R})$ where d is the number of Killing coordinates, that is the number of coordinates in the full ten dimensional theory with respect to which the solution is independent, and $\hat{p}$ can be thought of as the number of gauge fields in the solution.[4]

[4] See section 3.2.2 for further clarification.

In the present case, we have a five-dimensional stationary solution, which is therefore independent of time plus five other dimensions, only one of which is represented in the six-dimensional versions of the action, equations (6.2.1) and (6.2.3). So in the present context, on the heterotic side the group in question is $O(6,7,\mathbb{R})$.

What of the type IIA side of the story? Under string/string duality, the (six-dimensional) gauge fields present in the heterotic solution are mapped into Ramond-Ramond one-form gauge potentials. Recall also from section 3.3.2 that T-duality when applied to the type II superstring changes the chirality of the solution, and alters as well the field content of the Ramond-Ramond sector. Since T-duality is in fact, as mentioned, in the $O(d,d,\mathbb{Z})$ subgroup of $O(d,d,\mathbb{R})$, it is not possible to carry out the more general $O(d,d,\mathbb{R})$ transformations on the RR sector. If one did so then the result would be a solution which had one "foot" so to speak in each of the type IIA and type IIB theories and the interpretation of such a solution is completely unclear. Therefore, one can apply this technique to the type II theories only when the Ramond-Ramond fields all vanish. In this case the formulae (6.2.12) and (6.2.13) apply when all $A^{(1)\,\alpha}=0$.

Having described the procedure with which we will apply the $O(d,d)$ transformations to our solution, let us outline the series of steps that will lead us to our new solution. We begin with the metric (6.2.11) as a string-frame[5] type IIA solution in six dimensions. We apply an $O(6,6)$ boost mixing the (t,y) directions, following the five dimensional black hole construction of [132]. The boost matrix Ω_d used for this first transformation is

$$\Omega_d = \begin{pmatrix} \begin{matrix} I_4 & 0 & 0 \\ 0 & x & \sqrt{x^2-1} \\ 0 & \sqrt{x^2-1} & x \end{matrix} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} I_4 & 0 & 0 \\ 0 & x & -\sqrt{x^2-1} \\ 0 & -\sqrt{x^2-1} & x \end{matrix} \end{pmatrix} \qquad (6.2.18)$$

where $1 \le x < \infty$ is the boost parameter. Note the difference in sign of the off-diagonal parts of Ω_d between the blocks. This causes the resulting six–dimensional solution to have no $G_{6\,y\mu}$ for $\mu < 5$, but has a $B^{(a)}_{6yt}$ and a $\phi^{(a)}_6$.

[5] Since the dilaton is zero string frame and Einstein frame metrics are identical.

The next step is to apply string/string duality to create a heterotic solution from the type IIA solution. This is done by applying the mapping (6.2.5), after which the new $B^{(h)}$ is computed by integrating the field strength $H^{(h)}$ according to

$$B^{(h)}_{6\,\mu\nu} = \int dx^\rho H^{(h)}_{6\,\rho\mu\nu} + f_\rho(x^\sigma) \tag{6.2.19}$$

where here there is *not* a sum over ρ, but rather this is carried out for each x^ρ upon which the solution depends. The arbitrary integration functions f_ρ are then functions of all variables except the variable of integration x^ρ. Comparison of the set of results permits computation of a unique $B^{(h)}$ for which $H^{(h)}$ is the field strength, up to gauge transformations.

Taking the heterotic solution produced by the string/string duality and using the $O(6,7)$ symmetry we can thus apply a second boost, mixing the time t and the internal direction involving $A^{(1)}_6$, with parameter z. String/string duality is then applied a second time to convert the heterotic solution back to a Type IIA solution, followed by the standard Kaluza–Klein reduction to five dimensions as given in (6.2.6), which in turn is followed by the change to the Einstein frame.

The above boost parameters x and z are carefully chosen to satisfy $z = 2x^2 - 1$ which reduces the five dimensional dilaton to a constant. Note that this is consistent with the range $1 \le z < \infty$ throughout the range of x. The resulting configuration is a charged spinning five dimensional black hole with constant dilaton and constant moduli, written as

$$\begin{aligned}
ds_5^2 = &-\frac{\rho^2\left(\rho^2-\beta\right)}{\left[\rho^2+\beta(x^2-1)\right]^2}dt^2 + \frac{\left[\rho^2+\beta(x^2-1)\right]}{\left(r^2+a^2-\beta\right)}dr^2 \\
&+ \left[\rho^2+\beta(x^2-1)\right]d\theta^2 + \frac{\beta a x^3 \sin^2\theta\left(\rho^2\right)}{\left[\rho^2+\beta(x^2-1)\right]^2}dt d\varphi \\
&- \cos^2\theta\left[\frac{\beta a\sqrt{x^2-1}^3\left(\rho^2-\beta\right)}{\left[\rho^2+\beta(x^2-1)\right]^2}dt + \frac{\beta^2 a^2 x^3\sqrt{x^2-1}^3\sin^2\theta}{\left[\rho^2+\beta(x^2-1)\right]^2}d\varphi\right]d\psi \\
&+ \sin^2\theta\left[r^2+a^2+\beta(x^2-1)+\frac{\beta a^2 x^2\sin^2\theta\left[\rho^2-\beta(x^2-1)^2\right]}{\left[\rho^2+\beta(x^2-1)\right]^2}\right]d\varphi^2 \\
&+ \cos^2\theta\left[r^2+\beta(x^2-1)-\frac{\beta a^2(x^2-1)\cos^2\theta\left[\rho^2+\beta\left(x^4-1\right)\right]}{\left[\rho^2+\beta(x^2-1)\right]^2}\right]d\psi^2
\end{aligned}$$

$$
\begin{aligned}
B^{(a)} &= -\frac{\beta x\sqrt{x^2-1}\sin^2\theta\left[r^2+\beta(x^2-1)\right]}{\left[\rho^2+\beta(x^2-1)\right]}d\varphi\wedge d\psi \\
&\quad + \frac{-\beta a x(x^2-1)}{\left[\rho^2+\beta(x^2-1)\right]}dt\wedge\left(\sin^2\theta d\varphi+\frac{x\cos^2\theta}{\sqrt{x^2-1}}d\psi\right) \\
A^{(1)} &= \frac{\beta x\sqrt{2(x^2-1)}}{\left[\rho^2+\beta(x^2-1)\right]}\left(dt+ax\sin^2\theta d\varphi-a\sqrt{x^2-1}\cos^2\theta d\psi\right) \\
e^{\phi^{(a)}} &= 1 = e^{\sigma}
\end{aligned}
\tag{6.2.20}
$$

where as in the initial solution (6.2.11) we have $\rho^2 \equiv r^2 + a^2\cos^2\theta$, β is the mass parameter and a the angular momentum parameter from (6.2.11). This solution appears quite complicated, but when the extremal limit is taken, it will simplify substantially.

6.2.2. The extremal black hole

Here we will exhibit the extremal limit of the black hole written in (6.2.20). This is done by taking the boost parameter x off to infinity, and simultaneously the mass parameter β and the angular momentum parameter a to zero such that the quantities

$$
\begin{aligned}
&\lim_{\substack{x\to\infty\\ \beta\to 0}} \beta x^2 \equiv \mu, \\
&\lim_{\substack{x\to\infty\\ a\to 0}} a x \equiv \omega,
\end{aligned}
\tag{6.2.21}
$$

remain finite, where β, a are the quantities appearing in the metric (6.2.11). After doing a coordinate transformation to match with [10], $r^2 \to r^2 + \mu$, we obtain for the extremal metric and gauge fields

$$
\begin{aligned}
ds_5^2 &= -\left(1-\frac{\mu}{r^2}\right)^2\left[dt-\frac{\mu\omega\sin^2\theta}{(r^2-\mu)}d\varphi+\frac{\mu\omega\cos^2\theta}{(r^2-\mu)}d\psi\right]^2 \\
&\quad + \left(1-\frac{\mu}{r^2}\right)^{-2}dr^2+r^2(d\theta^2+\sin^2\theta d\varphi^2+\cos^2\theta d\psi^2)
\end{aligned}
\tag{6.2.22a}
$$

$$
A^{(1)} = \frac{\sqrt{2}}{\lambda}\frac{\mu}{r^2}\left(dt+\omega\sin^2\theta d\varphi-\omega\cos^2\theta d\psi\right) \tag{6.2.22b}
$$

$$
\Lambda_B^{(1)} = \frac{\lambda^3}{\sqrt{2}}A^{(1)} \tag{6.2.22c}
$$

$$
e^{\sigma+2\phi^{(a)}/3} = \lambda^2. \tag{6.2.22d}
$$

We recall for clarity that this solution is an extremal type IIA solution which has one non-zero Ramond-Ramond one-form potential $A^{(1)}$, one non-zero Neveu-Schwarz–Neveu-Schwarz gauge field $\Lambda_B^{(1)}$ and, while the result of our solution generating procedure yields zero dilaton and modulus $\phi^{(a)} = 0 = \sigma$, we have shifted these scalars by a constant to (6.2.22*d*), which introduces the scaling of the gauge fields by λ given above [10]. The above fields are the only ones excited in this black hole background. Notice that when we take $\omega \to 0$, we recover the solution of [10], and thus the solution in this limit is very similar to that considered in section 6.1.

6.2.3. Properties of the solution

From the asymptotic metric, applying equation (5.1.25), we obtain for the angular momentum, in the independent planes defined by φ, ψ,

$$\begin{aligned} J_1 &\equiv J_\varphi = +\frac{\pi}{4}\mu\omega, \\ J_2 &\equiv J_\psi = -\frac{\pi}{4}\mu\omega \end{aligned} \tag{6.2.23}$$

and for the mass, computed according to the formula (4.1.29) we find

$$M_{ADM} = \frac{3\pi\mu}{4}, \tag{6.2.24}$$

while the charges under the form fields $\Xi^{(2)}$ and $F^{(2)}$ are[6]

$$\begin{aligned} q_\Xi &\equiv \frac{1}{4\pi^2}\int_{S^3} {}^\star\left(e^{-2\sigma-4\phi^{(a)}/3}\,\Xi_B^{(2)}\right) = \mu/\lambda^2, \\ q_F &\equiv \frac{1}{16\pi}\int_{S^3} {}^\star\left(e^{\sigma+2\phi^{(a)}/3}F^{(2)}\right) = -\frac{\pi}{2\sqrt{2}}\mu\,\lambda. \end{aligned} \tag{6.2.25}$$

Note that this black hole, although a solution of the low–energy string theory equations, is not a solution of the Einstein–Maxwell equations in five dimensions. In the spinning configuration, the magnetic dipole field combines with the electric monopole field so that the Chern–Simons contributions to the equations of motion are nontrivial.

Let us now obtain the classical entropy of this extremal spinning black hole. In the above coordinates, the horizon is at $r = r_0 = \sqrt{\mu}$, and its entropy is found to be

[6] The sphere S^3 is at infinity, so we can ignore the effects of the Chern–Simons terms.

$(|J_1| = |J_2| \equiv J)$

$$S = \frac{1}{2}\pi^2 \mu \sqrt{\mu - \omega^2} \\ = 2\pi \sqrt{\frac{q_\Xi q_F^2}{2} - J^2} \tag{6.2.26}$$

where in the second line we have written the classical entropy in terms of the charges and angular momenta. Note that both of these expressions are independent of λ.

This extremal rotating charged black hole has a horizon with finite area, a feature not easy to find. Ordinarily the addition of rotation (without energy) to an extremal Reissner-Nordstrom black hole destabilizes the horizon and yields a naked singularity. However string theory, in order to avoid a conflict with the microscopic counting, cleverly stabilizes the horizon with the help of a Chern-Simons coupling in the low-energy field theory. In this process a qualitatively new class of supersymmetric spinning black hole solutions was found [12]. Also note that the angular momentum is bounded above: $J^2_{max} = q_\Xi q_F^2/2$ (in going beyond this limit, closed timelike curves develop).

6.2.4. *D*-brane counting of the microscopic entropy

Let us examine the D-brane states that are responsible for the degeneracy of the extremal black holes that we are considering. Due to our method of construction, the RR gauge field $A^{(1)}$ in (6.2.22) conceals equal numbers of rotating D-points and $D4$-branes. The $D4$-branes are wrapped on the four-cycle of $K3$. As a reminder, we are considering the type IIA string compactified on $K3 \otimes S^1$ down to five dimensions.

It is easier to do the counting if we first carry out a T-duality transformation along the S^1 direction, converting the solution to a type IIB solution. On the type IIB side, applying the information from chapter IV, we will have D-strings wrapped on S^1 bound to $D5$-branes wrapped on $K3 \otimes S^1$. It was also mentioned in chapter IV that the dynamics of D-branes is described by open oriented string theories dimensionally reduced to the world-volume. Thus for the D-point we have an $N = 1$, $U(\frac{q_F}{\sqrt{2}})$ Yang-Mills theory reduced from ten to two dimensions, and for the $D5$-brane we have a similar theory reduced from ten to six dimensions. These two Yang-Mills theories interact on the common volume $\mathbb{R} \otimes S^1$, where the $\mathbb{R}$ is the time coordinate. If we take the size of S^1 to be much larger than that of the $K3$, then we are justified in ignoring the dynamics of the six dimensional theory and thus the complete D-brane effective field theory will be a theory on the world-volume of the D-string–$D5$-brane intersection.

The $D5$-brane charge can be viewed as an element of the $K3$ cohomology $H^*(K3, \mathbb{Z})$ which is identified with how the internal part of the $D5$-brane wraps around $K3$.[7] Note that the dot product $\frac{1}{2} q_F \cdot q_F$ is the same as the intersection of cycles in the $K3$ cohomology.

In [133] relations were obtained between the cohomology of symmetric products of certain hyper-Kähler manifolds and the partition functions of the bosonic and supersymmetric strings. On the basis of these results it was conjectured [126] that the bound states we are considering here can be identified with a sigma model on the symmetric product of $(\frac{1}{2} q_F \cdot q_F + 1)$ copies of $K3$ i.e., on the quotient space

$$M = \frac{(K3)^{\otimes \frac{1}{2} q_F^2 + 1}}{S_{\frac{1}{2} q_F^2 + 1}}$$

where S_n denotes the symmetric group, the group of permutations of n objects. This conjecture has been verified in essentially all cases, at least up to T-duality [127,128]. The strategy is to use the cohomology of $K3$ to count the possible ways of forming our bound state.

The light-cone helicity of the six dimensional theory can be obtained in a manner similar to that of [126,133], by introducing helicity operators $\tilde{F}$, F for the left- and right-moving states, respectively. We will have $SU(2)_L \otimes SU(2)_R$ or $O(4)$ holonomy, but only the $U(1)_L \otimes U(1)_R$ subgroup will enter consideration in our analysis. The charges of the states under the $U(1)_L \otimes U(1)_R$ subgroup are thus given by $(\tilde{F}, F)$. Let J_1 and J_2 be currents associated with the commuting left- and

---

[7] The arguments for compactification on $T^4 \otimes S^1$ are (essentially) identical with the replacement of T^4 for $K3$ in the following discussions. Only the dimension of the manifold enters in the asymptotic growth below.

right-moving $U(1)$ elements of $O(4)$. These can be related to the helicities through

$$\begin{aligned} J_1 &= \frac{1}{2}\left(\tilde{F}+F\right), \\ J_2 &= \frac{1}{2}\left(\tilde{F}-F\right). \end{aligned} \tag{6.2.27}$$

Consider, for example, the case where the D-branes vanish, $q_F = 0$. The ground states of the sigma model are identified with the $K3$ cohomology, which has dimension 24. The helicities $\tilde{F}$ and F for the ground states run over the values $\{-1, 0, 1\}$, so we have the (J_1, J_2) spectrum consisting of 20 states with (0,0), two states with $(\pm 1, 0)$, and two states with $(0, \pm 1)$. These we recognize as the light-cone oscillator quantum numbers of bosonic strings in 26 dimensions.

The D-brane BPS states considered in [10] correspond to Ramond-Ramond states of this sigma model which are ground states on the right-moving side, while the left-moving states are excited to a level $\tilde{n}$ determined by the NS-NS charge, i.e., $\tilde{n} = q_\Xi$. Recall that there is a bound for the $\tilde{F}$ and F with respect to $\tilde{L}_0$ and L_0 [134]. This can be seen through bosonization of the $U(1)$ currents. Let $J_1 = \sqrt{\hat{c}}\,\partial\phi$ with $\hat{c}$ the complex dimension of the manifold M, in our case $\hat{c} = q_F^2 + 2$. A state with charge $\tilde{F}$ will then be represented by an operator

$$\exp\left(\frac{i\tilde{F}\phi}{\sqrt{\hat{c}}}\right)\cdot\Phi \tag{6.2.28}$$

where Φ is an operator which can contain any other state in the sigma model as well as oscillator mode factors of the $U(1)$ current, but *not* the $U(1)$ momentum modes. An exactly analagous construction holds for F. In particular, note that since the Φ are of positive dimension, the dimensions of the operators are restricted by

$$\tilde{L}_0 \geq \frac{\tilde{F}^2}{2\hat{c}}, \qquad L_0 \geq \frac{F^2}{2\hat{c}}. \tag{6.2.29}$$

We wish to count the entropy in a regime where q_F is macroscopic but held fixed. Moreover, we take the NS-NS charge q_Ξ to be arbitrarily large. We are also interested in a region with similarly macroscopic angular momenta $|J_1|, |J_2| >> 1$. Let us consider the case in which the system is a right-moving ground state with fixed F. Then we can consider arbitrarily large values of $\tilde{F}$ to make both J_1 and

J_2 large with the same sign[8]. Fixing $\tilde{F}$, and thus the angular momenta, imposes constraints on the left-moving Hilbert space. Since this is where the entropy comes from, we must therefore make an estimate of the number of left-moving states are still available when $\tilde{F}$ is fixed. Considering a regime[9] where $(q_\Xi - \tilde{F}^2/2\hat{c}) >> 1$ as well as $q_\Xi/q_F^2 >> 1$, the answer is supplied by the bosonization discussed above. Since the total eigenvalue is $\tilde{L}_0 = \tilde{n} = q_\Xi$, and we have used up $\frac{\tilde{F}^2}{2\hat{c}} = \frac{\tilde{F}^2}{2q_F^2+4}$ for the states we are interested in, the $\tilde{L}_0$ eigenvalue of the extra operator Φ is given by

$$\tilde{L}_0(\Phi) = n = \tilde{n} - \frac{\tilde{F}^2}{2\hat{c}} = q_\Xi - \frac{\tilde{F}^2}{2q_F^2 + 4} \tag{6.2.30}$$

Since the oscillatory states make the maximum contribution to degeneracy of string states, we can effectively take n, the oscillator number remaining once the angular momenta have been fixed, as the available oscillator number. From [133] we have a formula for the generating function for the dimension of the cohomology of the symmetric product of k manifolds W as

$$\sum s^k \dim\left(H^*\left(\frac{W^{\otimes k}}{S_k}\right)\right) = \frac{\prod_{k=1}^{\infty}(1+s^k)^{b_-}}{\prod_{k=1}^{\infty}(1-s^k)^{b_+}} \tag{6.2.31}$$

where b_- and b_+ are the dimensions of the fermionic and bosonic subspaces[10] of $H^*(W)$.[11] Using methods of computing the upper bound on the growth of such a generating function which can be found in [135] we obtain a degeneracy growth of

$$\begin{aligned} d \sim \exp\left(2\pi\sqrt{\frac{n\hat{c}}{2}}\right) &= \exp\left(2\pi\sqrt{\left(q_\Xi - \frac{\tilde{F}^2/4}{\frac{1}{2}q_F^2+1}\right)\left(\frac{1}{2}q_F^2+1\right)}\right) \\ &\sim \exp\left(2\pi\sqrt{q_\Xi\left(\frac{1}{2}q_F^2+1\right) - \frac{1}{4}\left(|J_1|+|J_2|\right)^2}\right) \end{aligned} \tag{6.2.32}$$

where we have substituted $\tilde{F} = J_1 + J_2$ and use absolute value signs for the angular momenta in order to write the final answer in its most general form, independently

---

[8] To consider J_1 and J_2 with the opposite sign, the entropy would have come from the right-movers and we would be considering large values of F while the left-movers were in a ground state.

[9] It may be that our final results are valid beyond this regime of charges. Further note that the given regime does not exclude the possibility that the ratio of $\tilde{F}^2/2\hat{c}$ to q_Ξ is only slightly less than one.

[10] In other words the cohomology classes of odd and even dimension respectively.

[11] The similarity between (6.2.31) and the partition function (6.1.6) used in the alternate D-brane counting will not have gone unoticed.

of whether or not J_1 and J_2 have the same sign. The entropy is thus

$$S_{micro} \sim 2\pi \sqrt{q_\Xi \left(\frac{1}{2}q_F^2 + 1\right) - \frac{1}{4}\left(|J_1| + |J_2|\right)^2} \tag{6.2.33}$$

Taking $|J_1| = |J_2| = J$, we see that this formula agrees with what we found for the classical entropy (6.2.26) of the spinning black hole. For the record we mention that this computation also sharpens the computation in [10] where, in principle, one should have counted only the spin–zero D-branes to make the comparison with the non-rotating black hole. It is satisfying that the classical and D-brane methods give the same result, and we regard this as additional evidence for the D-brane picture of [14].

One would also like to consider the possibility of the two angular momenta, J_1 and J_2, unequal. We see that for the right-moving ground state $|F|$ is bounded as [134]

$$|F| = |J_1 - J_2| \leq \frac{\hat{c}}{2} = \frac{1}{2}q_F^2 + 1, \tag{6.2.34}$$

and therefore the difference between the spins cannot be arbitrarily large. This bound can be combined with the previously noted relation $q_\Xi/q_F^2 >> 1$, to demonstrate that these calculations are valid for $|F|/|\tilde{F}| = |J_1 - J_2|/|J_1 + J_2| << 1$. As a result, one would not expect to see a difference in the angular momenta at the macroscopic level of the black hole computations. Constructions of extremal black holes analogous to that presented here in which the starting point is a five dimensional Kerr solution with two independent angular momenta confirm this finding. In these cases, demanding that the extremal or supersymmetric limit be nonsingular requires setting $|J_1| = |J_2|$ [13]. Hence the D-brane and black hole results are also in perfect agreement on this further aspect of the calculation. An example of such a calculation will be presented in the next section.

6.3. The non-extremal case

In this section we will combine the analyses of [11] and the previous section to consider the entropy of a spinning black hole solution just above extremality. Again we will find perfect agreement - a seven parameter fit - between the detailed thermodynamic behavior predicted by the Bekenstein-Hawking entropy and by the microscopic state counting.

6.3.1. A rotating nonextremal black hole

We begin with a simplified low-energy action for six-dimensional type IIB string theory which contains only the following terms ($G_N = 1$),

$$S_{IIB\ (K3)} = \frac{1}{16\pi} \int d^6x \sqrt{-J_6} \left(R - (\nabla\sigma)^2 - \frac{1}{12} e^{2\sigma} \left(F_6^{(3)}\right)^2 \right) \qquad (6.3.1)$$

in the six-dimensional Einstein frame. $F^{(3)}$ as usual denotes the RR three form field strength. We adopt conventions in which $G_N = 1$. The scalar σ here is the logarithm of the volume of the internal four-manifold in the string frame. The ten-dimensional string dilaton $\phi^{(b)}$ is an arbitrary constant for our solutions and will be omitted. We will further compactify to five dimensions using the Kaluza-Klein ansatz (6.2.6), where again y will be used to denote the fifth spatial coordinate. We will also take the asymptotic length L of the compact y coordinate to be very large.

The solutions of interest to us are most simply represented as six-dimensional black string solutions to (6.3.1), which wind around the y direction and hence are black holes in five dimensions. The six-dimensional black string can can carry both electric and magnetic charge with respect to $F^{(3)}$:

$$\begin{aligned} q^e &\equiv \frac{1}{8} \int_{S^3} e^{2\sigma} \, {}^*F^{(3)}, \\ q^m &\equiv \frac{1}{4\pi^2} \int_{S^3} F^{(3)} \ . \end{aligned} \qquad (6.3.2)$$

It may also carry total ADM momentum P along the y direction which appears in five dimensions as an electric charge of the Kaluza-Klein gauge field $A_G^{(1)}$ coming from compactification of the metric [11]:

$$P \equiv \frac{2\pi n}{L}. \qquad (6.3.3)$$

We have chosen our conventions so that n and $q^m q^e \equiv \frac{1}{2}\Omega^2$ are integers. In five spacetime dimensions the spatial rotation group is $SO(4) = SU(2) \otimes SU(2)$. Hence solutions are in addition labeled by two independent angular momenta.

Black string solutions are also characterized by the asymptotic value of σ. We are primarily interested in the entropy which cannot depend on the asymptotic value of σ [136-140]. For a special asymptotic value $\bar{\sigma}$, the sources for σ cancel exactly and the equations of motion imply σ is constant everywhere. This special value is

$$e^{2\bar{\sigma}} = \frac{2q^e}{\pi^2 q^m}. \tag{6.3.4}$$

In order to compute the entropy it is sufficient to consider the solutions with $\sigma = \bar{\sigma}$.

Reduction from six to five dimensions yields in the usual way a second five dimensional scalar field whose asymptotic value is L, the size of the S^1 parameterized by y. This scalar could also be frozen to a value which would be proportional to n/Ω. However it is important not to freeze this field because we will need to compute how the entropy varies as a function of both the energy and n with all other quantities - in particular the asymptotic values of the fields - held fixed. This is impossible to do if the value of the scalar field is tied to n/Ω. This problem does not arise for the scalar σ because, once the behavior of the entropy is known for any value of the ratio q^e/q^m, it is determined for any other value by duality which implies that it can depend only on the product $\Omega^2/2$.

The solutions in which we are interested can be generated by methods exactly analogous to those which were used to obtain (6.2.20) starting from (6.2.11) in the previous section. Beginning with a slightly more general version of the five dimensional Kerr solution which now spins in two independent planes, written as:

$$\begin{aligned} ds_6^2 &= G_{6\,\mu\nu} dx^\mu dx^\nu \\ &= -dt^2 + (r^2+a^2)\sin^2\theta d\varphi^2 + \frac{\beta}{\rho^2}\left(dt + a\sin^2\theta d\varphi + b\cos^2\theta d\psi\right)^2 \\ &\quad + \frac{r^2\rho^2}{\left(r^2+a^2\right)\left(r^2+b^2\right) - \beta r^2} dr^2 + \rho^2 d\theta^2 + (r^2+b^2)\cos^2\theta d\psi^2 + dy^2 \end{aligned} \tag{6.3.5}$$

where a and b are the angular momentum parameters, and $\rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta$. The coordinate system is the same as that in (6.2.11). We consider this as a heterotic string solution in five dimensions. Again we may lift this solution to be a black string solution of heterotic string theory in six dimensions by adding a trivial flat direction y. As before we begin with a boost which mixes the time direction t with the compact internal direction y to yield a nontrivial right-handed gauge field. Next, string-string duality, (6.2.5), is applied, converting the solution to a type IIA solution, followed by a T-duality transformation along the compact coordinate y which produces a black string solution of Type IIB string theory in six dimensions. We note here that due to T-duality being a subgroup of $O(d,d)$, the same formalism can be applied to implement this duality transformation, using a matrix of the form

$$\Omega_T = \begin{pmatrix} I_5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & I_6 \end{pmatrix}. \qquad (6.3.6)$$

Lastly, a boost is performed along the string yielding the following metric:

$$\begin{aligned} ds_6^2 = &- \left[1 - \frac{(r_+^2 x^2 - r_-^2(x^2-1))}{\rho^2}\right] dt^2 + \left[1 - \frac{(r_-^2 x^2 - r_+^2(x^2-1))}{\rho^2}\right] dy^2 \\ &+ \sin^2\theta \left[r^2 + a^2 + \frac{(a^2 r_+^2 - b^2 r_-^2)\sin^2\theta}{\rho^2}\right] d\varphi^2 \\ &+ \cos^2\theta \left[r^2 + b^2 + \frac{(b^2 r_+^2 - a^2 r_-^2)\cos^2\theta}{\rho^2}\right] d\psi^2 \\ &+ \rho^2 d\theta^2 + \frac{\rho^2}{r^2}\left[\left(1 - \frac{r_-^2}{r^2}\right)\left(1 - \frac{r_+^2 - a^2 - b^2}{r^2}\right) + \frac{a^2 b^2}{r^4}\right]^{-1} dr^2 \\ &+ \frac{2\sin^2\theta}{\rho^2}\left[\left(a r_+^2 x - b r_-^2\sqrt{x^2-1}\right)dt + \left(a r_+^2\sqrt{x^2-1} - b r_-^2 x\right)dy\right]d\varphi \\ &+ \frac{2\cos^2\theta}{\rho^2}\left[\left(b r_+^2 x - a r_-^2\sqrt{x^2-1}\right)dt + \left(b r_+^2\sqrt{x^2-1} - a r_-^2 x\right)dy\right]d\psi \\ &+ 2x\sqrt{x^2-1}\frac{(r_+^2 - r_-^2)}{\rho^2}dt\,dy + 2\cos^2\theta\sin^2\theta\frac{ab\,(r_+^2 - r_-^2)}{\rho^2}d\varphi\, d\psi, \\ \sigma = &\ \bar{\sigma}, \end{aligned} \qquad (6.3.7)$$

where $\rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta$ has not changed, x is the boost parameter, and a,b are components of the angular momentum per unit mass of the original Kerr solution. A nontrivial RR three-form field strength is present, but its precise form will not be needed in the following. The parameters $r_\pm$ are related to the charge by $Q^2 \equiv 2q^e q^m = (\pi r_+ r_-)^2$. The outer and inner event horizons are located at

$$r^2 = \frac{1}{2}\left(r_+^2 + r_-^2 - a^2 - b^2 \pm \sqrt{\left(r_+^2 - r_-^2 - a^2 - b^2\right)^2 - 4a^2 b^2}\right). \qquad (6.3.8)$$

The six-dimensional ADM energy of this solution is

$$E = \frac{L\pi}{8}\left(2\left(r_+^2 + r_-^2\right) + \left(r_+^2 - r_-^2\right)\left(2x^2 - 1\right)\right), \tag{6.3.9}$$

and the ADM momentum along the string is given by

$$P = \frac{\pi L}{4} x\sqrt{x^2 - 1}\left(r_+^2 - r_-^2\right). \tag{6.3.10}$$

The angular momenta in the independent planes defined by φ and ψ are

$$\begin{aligned} J_1 \equiv J_\varphi &= \frac{\pi L}{4}\left(a r_+^2 x - b r_-^2 \sqrt{x^2 - 1}\right), \\ J_2 \equiv J_\psi &= \frac{\pi L}{4}\left(b r_+^2 x - a r_-^2 \sqrt{x^2 - 1}\right). \end{aligned} \tag{6.3.11}$$

Following [11], we expect the Bekenstein-Hawking entropy to agree with the D-brane counting away from the supersymmetric extremal limit [12] provided the momentum density P/L and the excitation energy density $\delta E/L$ are small. To study this limit we expand

$$r_\pm = r_0 \pm \epsilon\ , \tag{6.3.12}$$

with $\epsilon \ll 1$, and x finite. Note we need to take the limit in such a way that $r_+^2 - (|a| + |b|)^2 > r_-^2$ in order to avoid naked singularities. This implies that a^2 and b^2 are of order ϵ. The longitudinal size of the string near the horizon is finite in this limit. To first order in ϵ, the excitation energy is

$$\delta E = \frac{L\pi r_0 \epsilon}{2}\left(2x^2 - 1\right), \tag{6.3.13}$$

and the classical entropy is given by

$$\begin{aligned} \mathcal{S} = \frac{1}{2} L\pi^2 r_0^2 \Big(& \left(4r_0\epsilon - a^2 - b^2\right) x^2 + 2abx\sqrt{x^2 - 1} - \frac{1}{2}\left(4r_0\epsilon - a^2 - b^2\right) \\ & + \frac{1}{2}\sqrt{\left(4r_0\epsilon - a^2 - b^2 - 2ab\right)\left(4r_0\epsilon - a^2 - b^2 + 2ab\right)}\Big)^{1/2} . \end{aligned} \tag{6.3.14}$$

Now define the following quantities

$$\begin{aligned} \tilde{n}_R &= \frac{L}{4\pi}(\delta E + P) - \frac{(J_1 - J_2)^2}{2\Omega^2}, \\ \tilde{n}_L &= \frac{L}{4\pi}(\delta E - P) - \frac{(J_1 + J_2)^2}{2\Omega^2}, \end{aligned} \tag{6.3.15}$$

in terms of which the entropy (6.3.14) is given by the simple formula

$$\mathcal{S} = \pi\Omega\left(\sqrt{2\tilde{n}_L} + \sqrt{2\tilde{n}_R}\right) \tag{6.3.16}$$

which has the form of a sum of entropy from right-moving and left-moving strings on the D-branes. We will now see how this comes about.

6.3.2. D-brane count in non-extremal case

As discussed in [10] and in section 6.2.4 the $P = 0$ black hole ground state is a bound state of q^e Ramond-Ramond D-strings wound around the S^1 in the y direction with q^m RR fivebranes wound around both the S^1 and the internal four-manifold $K3$. In the limit of large radius L for S^1, the excitations of this system are described by a supersymmetric sigma model on a manifold of real dimension $2\Omega^2$ [126,127,128]. In the regime of charges we are interested in, to leading order the degeneracy comes from string modes with short wavelengths and hence the curvature of the manifold is irrelevant. Thus we have the same leading degeneracy as the excitations of $2\Omega^2$ species of massless bosons and $2\Omega^2$ species of massless fermions which move around the S^1. Ignoring for the moment the angular momentum, the entropy of N_B (N_F) species of right-moving bosons (fermions) with total energy E_R in a box of length L is given by the standard thermodynamic formula

$$\mathcal{S} = \sqrt{\frac{\pi\,(2N_B + N_F)\,E_R L}{6}}. \tag{6.3.17}$$

At low energies and large L the system is dilute, meaning that interactions can be ignored, and the entropy from right-moving modes and left-moving modes is additive. Hence, using $N_F = N_B = 2\Omega^2$ and $E_{R,L} = 2\pi n_{R,L}/L$, (6.3.17) becomes [10], [11]

$$\mathcal{S} = \pi\Omega\left(\sqrt{2n_L} + \sqrt{2n_R}\right) , \tag{6.3.18}$$

where $n_{L,R}$ are given by (6.3.15) with $J_1 = J_2 = 0$.

Now we must make a correction for the angular momentum. As argued in the previous section of this chapter, $J_1 + J_2$ is carried by left-movers, while $J_1 - J_2$ is carried by right movers. Fixing the total angular momentum carried by the right movers decreases the number of states available for a fixed energy. As previously shown, the effect of this on the entropy for left-movers only is to replace n_L with $\tilde{n}_L$. However since the entropy of left and right movers is additive we have simply

$$\mathcal{S} = \pi\Omega\left(\sqrt{2\tilde{n}_L} + \sqrt{2\tilde{n}_R}\right) \quad , \tag{6.3.19}$$

in agreement with the black hole calculation (6.3.16).

We have thus shown that the D-brane method of counting the entropy of BPS saturated rotating black holes gives perfect agreement, including numerical factors, with the classical Bekenstein-Hawking entropy. The agreement, as we have just seen continues, to hold to leading order in a small parameter ϵ away from the extremal supersymmetric state. These are remarkable results and can be interpreted to indicate that string theory "understands" black hole entropy at a very deep level.

Of course, the current state of the art in the counting of the entropy of black holes through the D-brane methods is still rather limited, applying only to extremal or very near-extremal black holes. At the same time, advances are being made in the understanding of the entropy of non-supersymmetric black holes [141].

6.4. A dyonic black hole

Let us turn now to the construction of a black hole which, like the solutions considered thus far, possesses both electric and magnetic charge, but these charges are combined into a gauge field of the solution in a more restrictive manner. The definition of dyonic that we will impose here is the following: A given n-form potential is said to be dyonic if both magnetic and electric components are found for a fixed value of $n-1$ of the indicies. Thus, for a three-form, for example, one must have a magnetic charge at $A^{(3)}_{\mu\nu\varphi}$ and an electric charge at $A^{(3)}_{\mu\nu t}$.

It has been found that constructing dyonic solutions using the solution generating techniques has the unfortunate side effect of generating a non-zero Taub-NUT charge [142]. The time coordinate of the resulting spacetime must be made periodic in order to avoid a line singularity as the azimuthal angle vanishes. Since Taub-NUT-like metrics are widely believed to not represent physically realizable spacetimes, the appearance of this parameter in the solution generating process is unfortunate.

It is evident that this restriction can be overcome by beginning the solution generating process, not with the Schwarzschild metric, but rather with the Taub-NUT solution (5.1.31). This gives us an extra parameter, the initial Taub-NUT charge, with which to cancel the Taub-NUT charge appearing in the new solution.

To generate a dyonic solution, the following steps are carried out. First, one interprets the metric (5.1.31) as a type IIA string frame solution in six dimensions by adding two flat compact coordinates, the action is given by (2.3.28). In particular the Ramond-Ramond sector is completely null. An $O(d, d, \mathbb{R})$ boost mixing time and the sixth coordinate, x^5 is performed to create a non-zero term in the Kalb-Ramond field $B^{(a)}_{6\,tx^5} \neq 0$. The solution is still type IIA since the $O(d, d, \mathbb{R})$ transformations remain within a particular theory.

Now we carry out string/string duality according to equation (3.5.12), which converts our non-zero $B^{(a)}_{6\,tx^5} \to B^{(a)}_{6\,\varphi x^4}$, as well as converting the solution to a heterotic solution. We then perform a T-duality transformation, which changes $B^{(a)}_{6\,\varphi x^4} \to G_{6\,\varphi x^4}$. Solution remains heterotic. A second boost is then performed, this time an $O(d, d+\hat{p}, \mathbb{R})$ boost of the heterotic string, mixing time and the coordinate x^4. This creates a non-zero $G_{6\,tx^4}$ term, giving us two components of $G_{6\,\mu x^4}$ that will become our dyonic gauge field.

To convert these off-diagonal components of G_6 into a Ramond-Ramond gauge field, we first lift the solution to seven dimensions, by adding yet another flat coordinate. Then, the new dimension and the x^4 dimension, along which we find the terms of interest, are interchanged, and the solution is re-compactified to six dimensions. We now have a six dimensional metric, a six dimensional heterotic gauge field and an additional scalar field.

Finally, string/string duality is again applied to convert the heterotic solution to a type IIA solution, where the dyonic one-form is in the Ramond-Ramond sector. The result is a rather complicated metric, G_6, RR field $A^{(1)}_6$, dilaton $\phi^{(a)}_6$ and a scalar σ. The Taub-NUT charge of the metric can at this point be made to vanish by setting

$$\ell = \frac{\beta\sqrt{x^2-1}\sqrt{w^2-1}}{w} \tag{6.4.1}$$

where x and w are the parameters used in the first and second boosts, respectively. The solution may be written [16]

$$ds^2 = -\frac{\mathcal{C}}{f_1}dt^2 + f_1\left(\frac{dr^2}{\mathcal{C}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right) + f_2\left((dx^4)^2 + (dx^5)^2\right), \tag{6.4.2a}$$

$$A_6^{(1)} = \left(\frac{(u+\beta)\sqrt{u}}{\sqrt{u+\beta+v}}\cos\theta d\varphi + \frac{2\sqrt{v}\,(\beta+v)\left(1+\frac{u}{r}\right)}{(\mathcal{A}+\mathcal{B})\sqrt{u+\beta+v}\,\;r}dt,\right) \tag{6.4.2b}$$

$$e^{2\phi_6^{(a)}} = f_2 = e^{\sigma}, \tag{6.4.2c}$$

where

$$\begin{aligned} f_1 &= \sqrt{\mathcal{A}^2 - \mathcal{B}^2}, \\ f_2 &= \sqrt{\frac{\mathcal{A}+\mathcal{B}}{\mathcal{A}-\mathcal{B}}}, \\ \mathcal{A} &= \left(1+\frac{u}{r}\right)\left(1+\frac{v}{r}\right), \\ \mathcal{B} &= \frac{v-u}{r} - \frac{uv\,(u-v)}{(u+\beta+v)\;\,r^2}, \\ \mathcal{C} &= 1 - \frac{2\beta}{r} - \frac{vu\beta}{(v+u+\beta)\;\,r^2}, \end{aligned} \tag{6.4.3}$$

where β is a mass parameter, and u and v are the charge parameters of the magnetic and electric charges, respectively. The solution has been written in a form in which one obtains the extremal limit by setting $\beta = 0$. The parameters u, v and β are related to the original boost parameters x and w by

$$x = \sqrt{\frac{u}{\beta}}, \qquad w = \sqrt{\frac{u+v}{u}}. \tag{6.4.4}$$

For this solution the horizon is found to be located at

$$r_H = \beta \pm \sqrt{\frac{\beta(u+\beta)(v+\beta)}{u+v+\beta}} \tag{6.4.5}$$

and both roots are seen to be real and positive for $u, v \geq 0$ and $\beta \geq 0$. In the extremal limit, $\beta \to 0$, it is zero regardless of the values of the charge parameters u, v. The consequences are not entirely grave. Even with the horizon at $r_H = 0$ in the extremal limit, the area does not vanish. The formula for the area[12] is

$$\mathscr{A} = 16\pi^3 R_4 R_5\sqrt{(r_H+u)^2(r_H+v)^2 - \left(r_H(v-u) - \frac{uv(u-v)}{u+v+\beta}\right)^2} \tag{6.4.6}$$

[12] Area is computed in the six-dimensional Einstein frame.

which in the extremal limit, $r_H \to 0$, $\beta \to 0$ results in the classical entropy

$$\mathcal{S} = 4\pi^3 R_4 R_5 \frac{(uv)^{\frac{3}{2}}}{u+v}. \tag{6.4.7}$$

Note here that we have integrated over the compact coordinates x^4 and x^5, which have ranges $2\pi R_4$ and $2\pi R_5$ respectively, as well as over θ and φ. Further analysis of this solution, including analysis of BPS limits and D-brane content was carried out in chapter V.

In the previous sections, we created three black hole solutions. Two of these solutions represent rotating charged black holes in five (section 6.2) and six (section 6.3) dimensions. These black holes had event horizons which had non-zero area in the extremal limit, and thus non-zero entropy in this limit. Furthermore, they were BPS saturated states carrying Ramond-Ramond charge in the type II superstring theories, which allowed us to calculate their entropy from a microscopic counting of D-brane degrees of freedom.

In the last section, we illustrated techniques for solution generating which can be used to create dyonic black holes, overcoming the tendancy to create unwanted Taub-NUT charge. The resulting black hole was unfortunately not supersymmetric, although the area of the event horizon is non-vanishing in the extremal limit. The D-brane counting techniques can not, therefore, be used for this black hole. However, it is clear that more computations can be done to create a dyonic black hole amenable to D-brane analysis.

One other thing is clear as well. There is a close relationship between the bound states of D-branes and black holes, we spoke of this briefly at the end of chapter V. This indicates that advances in the understanding of black hole entropy can be made by means of advances in the knowledge of how D-branes form bound states. What sorts of bound states are possible, what are their properties, and what is required of them in order that they form BPS saturated states. We turn to questions such as these in the next chapter.

VII

Bound states of D-branes

Understanding of non-perturbative aspects of string theory has advanced rapidly during the past two years [1,2,14]. In the case of the Type II (and I) superstrings, of particular non-perturbative interest are the Dirichlet branes (D-branes) which carry charges of the Ramond-Ramond (RR) potentials, as discussed in chapter IV.

As we have seen in chapter VI, D-branes have also proven to be valuable tools from a calculational standpoint, leading to the computation of the entropy of black holes from a counting of the underlying microscopic degrees of freedom. In analyses such as these, the bound states are required to be supersymmetric in order that the counting, which can only be done at weak coupling, is protected from loop corrections by BPS saturation as the coupling is increased to where the bound state forms a black hole. Thus supersymmetric D-brane bound states are of particular interest.

In this chapter, we will extend the known exact low-energy supergravity solutions which describe bound state configurations in two ways. In section 1 we will construct supersymmetric bound states of D-branes in which the dimensions of the D-branes involved differ by two rather than four. In section 2 we will extend the class of known low-energy background field solutions to those which intersect at non-trivial angles.

A great deal of effort has gone into generating the low-energy background field solutions corresponding to various D-brane bound states [143]. These solutions have so far been restricted to those describing p-branes which are either parallel or intersect orthogonally. It has been shown [144], from the world-sheet standpoint however, that there exist supersymmetric configurations where the angles between the D-branes are *other* than zero or $\pi/2$. Preserving supersymmetry in such multiple D-brane configurations requires that the angles are restricted to lie in an $SU(N)$ subgroup of rotations. The corresponding background field configurations remain largely unexplored. In section 2, however, we will present one such class of solutions. Our basic solution describes any number of D-membranes whose relative orientations are given by certain $SU(2)$ rotations.

7.1. Bound states of $(p, p-2)$ D-branes

For the most part, the attention of researchers has been focussed on examples of D-brane bound states in which the difference in the dimension of the D-branes involved is a multiple of four. This preference comes about since it is the well-known requirement for supersymmetry in a configuration of two *separated* D-branes [65,81].

This feature is also revealed by an examination of the static (long-range) potential between separated D-branes. Supersymmetry implies stability or a precise cancelation of the inter-brane forces. For example, let us consider a $D0$-brane separated a distance r from a Dp-brane, where we will allow $p = 0, 2, 4$, or 6. There are three contributions to the static potential: gravitational, dilatonic and vector[1]

$$\begin{aligned} U_{grav} &= -\frac{\kappa^2}{8\mathcal{A}_{8-p}}\frac{M_0\,M_p}{r^{7-p}} \\ U_{dila} &= -\frac{1}{2\,(7-p)\,\mathcal{A}_{8-p}}\frac{\beta_0\,\beta_p}{r^{7-p}} \\ U_{vect} &= +\frac{1}{(7-p)\,\mathcal{A}_{8-p}}\frac{q_0\,q_p}{r^{7-p}}\,\delta_{0,p}. \end{aligned} \tag{7.1.1}$$

The Kronecker delta appears in the gauge field potential because only $D0$-branes carry electric charge under the RR vector. Using the relations relating the various charges – which may be determined by examining the explicit low-energy solutions (see chapter IV and below) – i.e., $q_0 = \sqrt{2}\kappa\,M_0$ and $\beta_p = \frac{3-p}{2}\kappa\,M_p$, we may sum

[1] The normalization of the mass and charge densities (i.e., M_p and q_p) in these potentials will be discussed in section 7.1.2.1. The 'charge' density for dilaton is chosen such that the asymptotic field around a p-brane takes the form: $\phi \simeq \frac{1}{(7-p)\mathcal{A}_{8-p}}\frac{\beta_p}{r^{7-p}}$. In these formulae, $\mathcal{A}_n$ is the area of a unit n-sphere, see (5.1.4).

these potentials to find

$$U_{total} = -\frac{\kappa^2}{2(7-p)\mathcal{A}_{8-p}}\frac{M_0\,M_p}{r^{7-p}}\left(4-p-4\delta_{0,p}\right) \quad . \tag{7.1.2}$$

Hence we see that the three forces precisely balance for two $D0$-branes, resulting in a constant (vanishing) potential. Even in the absence of the gauge potential, however there is a similar cancelation for the $D0$- and $D4$-brane system. In this case, the two branes carry dilaton charges of opposite signs so that the dilatonic repulsion precisely balances the gravitational attraction.[2] The vanishing potential or stability of these two configurations is a reflection of the supersymmetry which is preserved. In the former, 1/2 of the supersymmetries are preserved, while 1/4 are preserved in the latter.

If we consider the case of a $D0$-brane with a $D2$-brane, we see that total potential is attractive and so this configuration is unstable. Hence at the same time, it fails to preserve any supersymmetries. However, since the potential is attractive (i.e., $U_{total} < 0$), the $D0$-brane would presumably be drawn into the Dirichlet membrane and eventually the combined system would settle into a stable bound state configuration. While supersymmetry implies stability, the converse is not necessarily true. However we will be able to show by an explicit construction that in fact the stable ground state configuration is supersymmetric, preserving 1/2 of the supersymmetries. In general, our construction allows for the construction of supersymmetric bound states involving D-branes with dimensions differing by two.

7.1.1. Some preliminaries

Let us recall here the actions we will be working with in this chapter. We will be concerned here with the type II actions exclusively. These actions were detailed in chapter II, but for the convenience of the reader are repeated here.

[2] This mechanism was also observed for the multicenter solutions constructed in [145].

The bosonic part of the low-energy action for type IIA string theory in ten dimensions is from (2.3.23)

$$S_{IIA} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}\left\{e^{-2\phi^{(a)}}\left[R + 4(\nabla\phi^{(a)})^2 - \frac{1}{12}\left(H^{(a)}\right)^2\right] - \frac{1}{4}\left(F^{(2)}\right)^2 - \frac{1}{48}\left(F^{(4)}\right)^2\right\} - \frac{1}{4\kappa^2}\int B^{(a)}dA^{(3)}dA^{(3)} \tag{7.1.3}$$

where $G_{\mu\nu}$ is the string-frame metric, $H^{(a)} = dB^{(a)}$ is the field strength of the Kalb-Ramond field, $F^{(2)} = dA^{(1)}$ and $F^{(4)} = dA^{(3)} - H^{(a)}A^{(1)}$ are the Ramond-Ramond field strengths, and finally $\phi^{(a)}$ is the dilaton. Assuming the dilaton vanishes asymptotically, Newton's constant is given by $\kappa^2 = 8\pi G_N$. For the type IIB case, we write the action as ((2.3.26))

$$S_{IIB} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-J}\left\{e^{-2\phi^{(b)}}\left[R + 4(\nabla\phi^{(b)})^2 - \frac{1}{12}\left(H^{(b)}\right)^2\right] - \frac{1}{2}(\partial\chi)^2 - \frac{1}{12}\left(F^{(3)} + \chi H^{(b)}\right)^2 - \frac{1}{480}\left(F^{(5)}\right)^2\right\} + \frac{1}{4\kappa^2}\int A^{(4)}\,F^{(3)}\,H^{(b)} \tag{7.1.4}$$

where $J_{\mu\nu}$ is the string-frame metric, $H^{(b)} = dB^{(b)}$ is the field strength of the Kalb-Ramond field, $F^{(3)} = dA^{(2)}$ and $F^{(5)} = dA^{(4)} - \frac{1}{2}(B^{(b)}F^{(3)} - A^{(2)}H^{(b)})$ are RR field strengths, while $\chi = A^{(0)}$ is the RR scalar, and $\phi^{(b)}$ is the dilaton. As mentioned in chapter II, we are following the convention that the the self duality constraint $F^{(5)} =^* F^{(5)}$ be applied by hand at the level of the equations of motion. All of the solutions in the following will be presented in terms of the string-frame metric, however, conversion to the Einstein-frame metric would be accomplished using:

$$g_{\mu\nu} = e^{-\phi^{(a)}/2}\,G_{\mu\nu}\,, \qquad j_{\mu\nu} = e^{-\phi^{(b)}/2}\,J_{\mu\nu}\,. \tag{7.1.5}$$

The low energy background field solutions describing a single Dp-brane were given in chapter IV ((4.1.27)) for the case of D spacetime dimensions (recall that $\hat{d} = p+1$). Here we specialize these solutions to $D = 10$, which contain only a nontrivial metric, dilaton and a single RR potential, $A^{(p+1)}$, on which we have made a gauge transformation to ensure that it vanishes asymptotically rather than going to unity:

$$
\begin{aligned}
ds^2 &= \sqrt{\mathcal{H}(\vec{y})}\left(\frac{-dt^2+d\vec{x}^2}{\mathcal{H}(\vec{y})}+d\vec{y}^2\right)\\
A^{(p+1)} &= \pm\left(\frac{1}{\mathcal{H}(\vec{y})}-1\right)dt\wedge dx^1\wedge\cdots\wedge dx^p\\
e^{2\phi} &= \mathcal{H}(\vec{y})^{\frac{3-p}{2}}\ .
\end{aligned}
\tag{7.1.6}
$$

Here, the p spatial coordinates $x^{\hat{\mu}}$ run parallel to the worldvolume of the brane, while the orthogonal subspace is covered by the $9-p$ coordinates y^i. As we saw in chapter IV, the solution is completely specified by a single function which may be written as

$$
\mathcal{H} = 1+\frac{\mu}{7-p}\left(\frac{\ell}{r}\right)^{7-p}\ . \tag{7.1.7}
$$

for[3] $p = 0, 1, \ldots, 6$. Here, μ is some dimensionless constant, ℓ is an arbitrary length scale and $r^2 = \sum_{i=1}^{9-p}(y^i)^2$. The RR field strength for this configuration is

$$
F^{(p+2)} = \mp\mathcal{H}^{-2}\partial_j\mathcal{H}\, dy^j\wedge dt\wedge dx^1\wedge\cdots\wedge dx^p\ . \tag{7.1.8}
$$

For $p > 3$, the D-branes are actually magnetically charged in terms of the RR fields appearing in the above low energy actions, (7.1.3) and (7.1.4). In this case, eq. (7.1.8) describes the Hodge dual of the magnetic field

$$
F^{(8-p)} = \pm\partial_j\mathcal{H}\, i_{\hat{y}^j}\left(dy^1\wedge\cdots\wedge dy^{9-p}\right) \tag{7.1.9}
$$

where $i_{\hat{y}^j}$ denotes the interior product with a unit vector pointing in the y^j direction. For $p = 3$, the five-form field strength should be self-dual. In this case, the correct solution may be constructed by replacing the electric five-form (7.1.8) by $(F^{(5)} +^* F^{(5)})/2$ to produce[4]

$$
F^{(5)} = \mp\frac{\partial_j\mathcal{H}}{2}\left(\frac{1}{\mathcal{H}^2}dy^j\wedge dt\wedge dx^1\wedge dx^2\wedge dx^3 - i_{\hat{y}^j}\left(dy^1\wedge\cdots\wedge dy^6\right)\right) \tag{7.1.10}
$$

while the dilaton remains constant (i.e., $e^\phi = 1$) in accord with eq. (7.1.6).

---

[3] As noted in chapter IV, this solution is also valid for $p = 8$, while $\mathcal{H} = 1-\mu\log(r/\ell)$ for $p = 7$. These solutions can also be extended to the D-instanton with $p = -1$, for which the metric becomes euclidean without t or $x^{\hat{\mu}}$ [146].

[4] This is not quite a duality rotation because the kinetic term for $F^{(5)}$ in the IIB action (7.1.4) has the unconventional normalization $1/(4\cdot 5!)$, – which simplifies the T-duality transformation – rather than $1/(2\cdot 5!)$ which is implicit in producing eq. (7.1.6).

7.1.2. Bound state of $p = 0, 2$ D-branes

As discussed in chapter IV, at the world-sheet level a Dp-brane is described by imposing a combination of Neumann and Dirichlet boundary conditions on the string world-sheet boundaries. Neumann conditions are imposed on the coordinate fields associated with the $p+1$ directions parallel to the D-brane's world-volume, i.e., $\partial_{normal} X^{\mu} = 0$. The fields associated with the remaining $9-p$ coordinates orthogonal to the D-brane satisfy Dirichlet boundary conditions, i.e., $X^{\mu} =$ constant, which fixes the world-sheet boundaries to the brane.

These objects were originally discovered by considering the action of T-duality in the toroidal compactification of open (bosonic) theories, as was discussed in chapter IV. In this context, T-duality trades the standard Neumann condition for the Dirichlet boundary condition, written as $\partial_{tangent} X^{\mu} = 0$. Hence if T-duality is implemented along one of the world-volume coordinates of a Dp-brane, one of the Neumann boundary conditions is replaced by a Dirichlet condition to produce a D($p-1$)-brane [89,147]. Alternatively, applying T-duality to a coordinate in the transverse space will replace a Dirichlet condition with a Neumann condition extending the Dp-brane to a $D(p+1)$-brane. For the present purposes, we wish to consider a Dp-brane which is oriented at an angle with respect to some orthogonal coordinate axes, e.g., tilted by an angle ζ in the (X^1,X^2)-plane. This requires imposing Neumann and Dirichlet boundary conditions on linear combinations of these coordinates

$$\begin{aligned} \partial_n \left(X^1 + \tan\zeta\, X^2 \right) &= 0 \\ \partial_t \left(X^1 - \cot\zeta\, X^2 \right) &= 0 \end{aligned} \tag{7.1.11}$$

Now consider implementing the T-duality on X^2 in this example. The interchange of the Neumann and Dirichlet conditions results in mixed boundary conditions which may be expressed as

$$\begin{aligned} \partial_n X^1 + i\tan\zeta\, \partial_t X^2 &= 0 \\ \partial_n X^2 - i\tan\zeta\, \partial_t X^1 &= 0\,. \end{aligned} \tag{7.1.12}$$

Here the factor of i appears since we are considering a euclidean world-sheet. Now these mixed boundary conditions can be recognized as an example of the compatible boundary conditions arising when the Kalb-Ramond potential $B_{\mu\nu}$ and/or the world-volume gauge field strength $F_{\mu\nu}$ acquire a nonvanishing expectation value [148], i.e.,

$$\partial_n X^\mu - i\,\mathcal{F}^\mu{}_\nu\,\partial_t X^\nu = 0 \tag{7.1.13}$$

where $\mathcal{F}_{\mu\nu} = B_{\mu\nu} + 2\pi\alpha'\mathbb{F}^{(2)}_{\mu\nu}$. Recall from chapter IV that $B_{\mu\nu}$ and $\mathbb{F}^{(2)}_{\mu\nu}$ always appear in this combination due to gauge invariance. In the present situation then, T-duality has induced $\mathcal{F}_{12} = -\tan\zeta$.

Now a nonvanishing $\mathcal{F}_{\mu\nu}$ will induce new couplings of the D-brane to the RR form potentials [149]. From chapter IV, the full coupling of the RR fields to a Dp-brane is given by the following integral over the world-volume

$$\int \mathrm{Tr}\left[e^{\mathcal{F}}\sum A^{(n)}\right] . \tag{7.1.14}$$

Hence in the above example if we begin with a Dp-brane angled in the (X^1,X^2)-plane, the result is a $D(p+1)$-brane with a nonvanishing flux $\mathcal{F}_{12}$. This final brane would then couple to both $A^{(p+2)}$ and $A^{(p)}$, and so should be regarded as a bound state of a $D(p-1)$-brane with a $D(p+1)$-brane.

While the above description is formulated at the level of the string world-sheet, we can easily lift the discussion to one of background fields. We begin by constructing the solution for a (delocalized) Dp-brane oriented at an angle in the (X^1,X^2)-plane, and apply T-duality on X^2 to find a solution describing the bound state of a $D(p-1)$-brane and a $D(p+1)$-brane. This will be our approach to building the background field solutions for these bound states. We illustrate the procedure in this section by considering in detail the construction of a bound state solution for $p=0$ and 2 branes.

We begin with the low energy Type IIB solution describing a D-string

$$\begin{aligned} ds^2 &= \sqrt{\mathcal{H}}\left(\frac{-dt^2+dx^2}{\mathcal{H}} + dy^2 + \sum_{i=2}^{8}\left(dy^i\right)^2\right) \\ A^{(2)} &= \pm\left(\frac{1}{\mathcal{H}} - 1\right) dt\wedge dx \\ e^{2\phi^{(b)}} &= \mathcal{H} \end{aligned} \tag{7.1.15}$$

where x is the coordinate parallel to the D-string, and we have singled out one of the transverse coordinates as $y = y^1$, for later convenience. Now $\mathcal{H}$ is a harmonic function in the transverse coordinates. Normally, we would choose $\mathcal{H} = 1 + \frac{\mu}{6}(\ell/r)^6$ as in equation (7.1.7). For our present purposes, however, we need a slightly different harmonic function in that we want to delocalize the D-string in one of the transverse directions, in order to implement T-duality on the background fields along this direction.

There are at least two different ways to do this. The harmonic function $\mathcal{H}$ is a solution of (the flat-space) Poisson's equation in the transverse coordinates, with some delta-function source. For example in (7.1.7), the source is chosen so that $\partial^i\partial_i\mathcal{H} = -\mu\ell^6\mathcal{A}_7\prod_{i=1}^{8}\delta(y^i)$. The first way to accomplish a delocalization of the string is to follow the 'vertical reduction' approach [150]. An infinite number of identical sources are added in a periodic array along the y-axis. Then a smeared solution may be extracted from the long range fields, for which the y-dependence is exponentially suppressed. An easier approach, which might be termed 'vertical oxidation', is to simply replace the above eight-dimensional δ-function source by that of a line source extending along y, i.e., $\partial^i\partial_i\mathcal{H} = -\mu\ell^5\mathcal{A}_6\prod_{i=2}^{8}\delta(y^i)$. This construction produces one of the anisotropic (p,q)-branes considered in [151].

In any event, the number of dimensions transverse to our smeared-out D-string is effectively only 7, rather than 8, and the solution may be taken as in (7.1.7) with $p = 2$:

$$\mathcal{H} = 1 + \frac{\mu}{5}\left(\frac{\ell}{r}\right)^5 \tag{7.1.16}$$

where here $r^2 = \sum_{i=2}^{8}(y^i)^2$. Note that the form of the RR potential in eq. (7.1.15) tells us that we have a D-string oriented along x and smeared out in y, rather than the other way around.

Now we perform a rotation on our delocalized D-string, in the y-x plane:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sin\zeta & \cos\zeta \\ \cos\zeta & -\sin\zeta \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \tag{7.1.17}$$

where ζ will be the angle between the $\tilde{x}$-axis and axis of the D-string, i.e., the x-axis. We then have,

$$\begin{aligned} dx &= \cos\zeta\, d\tilde{x} + \sin\zeta\, d\tilde{y} \\ dy &= \cos\zeta\, d\tilde{y} - \sin\zeta\, d\tilde{x} \end{aligned} \tag{7.1.18}$$

and after the rotation, the solution (7.1.15) becomes

$$
\begin{aligned}
ds^2 &= \sqrt{\mathcal{H}}\left\{\frac{-dt^2}{\mathcal{H}} + \left(\frac{\cos^2\zeta}{\mathcal{H}} + \sin^2\zeta\right) d\tilde{x}^2 + \left(\frac{\sin^2\zeta}{\mathcal{H}} + \cos^2\zeta\right) d\tilde{y}^2 \right.\\
&\qquad \left. + 2\cos\zeta\sin\zeta\left(\frac{1}{\mathcal{H}} - 1\right) d\tilde{x}d\tilde{y} + \sum_{i=2}^{8}\left(dx^i\right)^2\right\}\\
A^{(2)} &= \pm\left(\frac{1}{\mathcal{H}} - 1\right) dt\wedge(\cos\zeta\, d\tilde{x} + \sin\zeta\, d\tilde{y})\\
e^{2\phi^{(b)}} &= \mathcal{H}.
\end{aligned}
\tag{7.1.19}
$$

Following the discussion at the beginning of this section, we apply T-duality in the $\tilde{y}$ direction on our delocalized and rotated D-string. The resulting solution should then describe a bound state of a D-point ($p = 0$) and a D-membrane ($p = 2$). The ten-dimensional T-duality map between the spacetime degrees of freedom of the type IIA and the type IIB string theories was given in [56]. Using our notation and conventions, the map from the IIB to the IIA theory reads as

$$
\begin{aligned}
G_{\tilde{z}\tilde{z}} &= \frac{1}{J_{\tilde{z}\tilde{z}}} & e^{2\phi^{(a)}} &= \frac{e^{2\phi^{(b)}}}{J_{\tilde{z}\tilde{z}}}\\
G_{\mu\nu} &= J_{\mu\nu} - \frac{J_{\tilde{z}\mu}J_{\tilde{z}\nu} - B^{(b)}_{\tilde{z}\mu}B^{(b)}_{\tilde{z}\nu}}{J_{\tilde{z}\tilde{z}}} & G_{\tilde{z}\mu} &= -\frac{B^{(b)}_{\tilde{z}\mu}}{J_{\tilde{z}\tilde{z}}}\\
B^{(a)}_{\mu\nu} &= B^{(b)}_{\mu\nu} + 2\frac{B^{(b)}_{\tilde{z}[\mu}J_{\nu]\tilde{z}}}{J_{\tilde{z}\tilde{z}}} & B^{(a)}_{\tilde{z}\mu} &= -\frac{J_{\tilde{z}\mu}}{J_{\tilde{z}\tilde{z}}}\\
A^{(1)}_{\mu} &= A^{(2)}_{\tilde{z}\mu} + \chi B^{(b)}_{\tilde{z}\mu} & A^{(1)}_{\tilde{z}} &= -\chi\\
A^{(3)}_{\tilde{z}\mu\nu} &= A^{(2)}_{\mu\nu} + 2\frac{A^{(2)}_{\tilde{z}[\mu}J_{\nu]\tilde{z}}}{J_{\tilde{z}\tilde{z}}} & &\\
A^{(3)}_{\mu\nu\rho} &= A^{(4)}_{\mu\nu\rho\tilde{z}} + \frac{3}{2}\left(A^{(2)}_{\tilde{z}[\mu}B^{(b)}_{\nu\rho]} - B^{(b)}_{\tilde{z}[\mu}A^{(2)}_{\nu\rho]} - 4\frac{B^{(b)}_{\tilde{z}[\mu}A^{(2)}_{|\tilde{z}|\nu}J_{\rho]\tilde{z}}}{J_{\tilde{z}\tilde{z}}}\right) & &
\end{aligned}
\tag{7.1.20}
$$

where the fields are as described in section 7.1.1. Here $\tilde{z}$ denotes the Killing coordinate with respect to which the T-dualization is applied, while μ, ν, ρ denote any coordinates other than $\tilde{z}$.

A straightforward application of the T-duality map (7.1.20) to the solution (7.1.19) yields

$$
ds^2 = \sqrt{\mathcal{H}}\left\{\frac{-dt^2}{\mathcal{H}} + \frac{d\tilde{x}^2 + d\tilde{y}^2}{1 + (\mathcal{H} - 1)\cos^2\zeta} + \sum_{i=2}^{8}\left(dy^i\right)^2\right\}
$$

$$
\begin{aligned}
A^{(3)} &= \pm \frac{(1-\mathcal{H})\cos\zeta}{1+(\mathcal{H}-1)\cos^2\zeta}\, dt\wedge d\tilde{x}\wedge d\tilde{y} \\
A^{(1)} &= \pm \frac{\mathcal{H}-1}{\mathcal{H}}\sin\zeta\, dt \\
B^{(a)} &= \frac{(1-\mathcal{H})\cos\zeta\,\sin\zeta}{1+(\mathcal{H}-1)\cos^2\zeta}\, d\tilde{x}\wedge d\tilde{y}. \\
e^{2\phi^{(a)}} &= \frac{\mathcal{H}^{\frac{3}{2}}}{1+(\mathcal{H}-1)\cos^2\zeta}
\end{aligned}
\tag{7.1.21}
$$

Hence as expected this solution involves both $A^{(3)}$ and $A^{(1)}$ indicating the presence of a $D2$-brane and a $D0$-brane, respectively, in the $(\tilde{x},\tilde{y})$-plane. Since the bound state solution only depends on $r^2 = \sum_{i=2}^{8}(y^i)^2$ as in (7.1.16), the $D0$-brane is delocalized in world-volume of the D-membrane. Remarkably T-duality has produced $G_{\tilde{x}\tilde{x}} = G_{\tilde{y}\tilde{y}}$ so that the bound state is spatially isotropic, even though it has lost the usual world-volume Lorentz invariance which characterizes the single D-brane solutions (7.1.6). Note that the off-diagonal term in the metric (7.1.19), which was produced by the rotation (7.1.17), has disappeared. Instead a Kalb-Ramond field has been generated, as is required by the Kalb-Ramond coupling appearing in $F^{(4)}$ and by the presence of both $A^{(3)}$ and $A^{(1)}$ in this solution. One can verify that with $\zeta = 0$, the T-dual solution reduces to a D-membrane with $A^{(1)} = 0 = B^{(a)}$, as expected. Similarly with $\zeta = \pi/2$, $A^{(3)}$ and $B^{(a)}$ vanish leaving a single $D0$-brane delocalized in the $(\tilde{x},\tilde{y})$-plane. We should also note that this solution (7.1.21) for a bound state of D0- and $D2$-branes appears in [152].

7.1.2.1. Mass and Charge Relations

In this section, we consider some of the physical characteristics of the above bound state solution (7.1.21). The physical charge densities associated with the various RR fields were defined in chapter IV, equation (4.1.32). We arrange that in our solutions the form potentials vanish asymptotically so that the charge formulae yield the correct results while ignoring the interactions between the different potentials. In using the definitions of (4.1.32) in this section, we drop the use of the ^{e} and ^{m} superscripts and instead label the charges with subscripts indicating the spatial dimension of the brane to which they belong. The D-particle and D-membrane carry charges for $A^{(1)}$ and $A^{(3)}$, respectively, which for the above solution yields

$$\begin{aligned} q_0 &= \mp \frac{(2\pi)^2 R_{\tilde{x}} R_{\tilde{y}}}{\sqrt{2}\kappa} \mu \ell^5 \sin\zeta\, \mathcal{A}_6 \\ q_2 &= \pm \frac{1}{\sqrt{2}\kappa} \mu \ell^5 \cos\zeta\, \mathcal{A}_6 \end{aligned} \tag{7.1.22}$$

where in calculating q_0 we have set $\tilde{x}$ ($\tilde{y}$) to have a range of $2\pi R_{\tilde{x}}$ ($2\pi R_{\tilde{y}}$). Here q_2 is a charge per unit area while q_0 is the total charge. The corresponding charge density associated with the delocalized $D0$-branes is then

$$\tilde{q}_0 = \frac{q_0}{(2\pi)^2 R_{\tilde{x}} R_{\tilde{y}}} = \mp \frac{1}{\sqrt{2}\kappa} \mu \ell^5 \sin\zeta\, \mathcal{A}_6 \,. \tag{7.1.23}$$

For a p-brane, the ADM mass per unit p-volume is also defined in chapter IV, (4.1.29). The ADM mass density of the bound state (7.1.21), which for the present purposes is effectively a membrane with $p = 2$, is then

$$M_{0,2} = \frac{1}{2\kappa^2} \mu \ell^5 \mathcal{A}_6 \,.$$

Therefore we have

$$\left(M_{0,2}\right)^2 = \frac{1}{2\kappa^2}\left(\tilde{q}_0^2 + q_2^2\right) \,. \tag{7.1.24}$$

This relation indicates that this bound state saturates the BPS bound for this system [65].

It is interesting to consider the ratio of the charge densities

$$\frac{\tilde{q}_0}{q_2} = -\tan\zeta \,. \tag{7.1.25}$$

We also know that the source for $\tilde{q}_0$ is spread over the ($\tilde{x}$,$\tilde{y}$)-plane, and so in the stringy discussion surrounding (7.1.14), we would expect that the D-membrane carries a flux[5] $\mathcal{F}_{\tilde{x}\tilde{y}} = -\tan\zeta$. In fact, this flux precisely agrees with that arising in the preceding discussion given the identification: $X^1 = \tilde{y}$, $X^2 = \tilde{x}$. Further, we might consider the limit

$$\lim_{r\to 0} B^{(a)}_{\tilde{x}\tilde{y}} = -\tan\zeta \,. \tag{7.1.26}$$

This suggests that the Kalb-Ramond field accounts for the total flux in $\mathcal{F}$, and so the world-volume gauge field should vanish, i.e., $\mathbb{F}^{(2)}_{\mu\nu} = 0$. Of course, $B^{(a)}_{\tilde{x}\tilde{y}}$ can

[5] The orientation for $\mathcal{F}$ is in keeping with that used to calculate q_0.

be shifted by a constant via a gauge transformation, which at the same time would induce a nonvanishing $\mathbb{F}^{(2)}_{\tilde{x}\tilde{y}}$. This has no physical consequences for the bound state solution, but it is amusing to show that in this case the T-dual solution is a rotated D-string in a background where the $\tilde{x}$ and $\tilde{y}$ axes are also tilted.

It is also interesting to see that the results for the charges are consistent with the appropriate charge quantization rules (4.2.35), namely

$$q_p = n_p \mu_p = n_p (2\pi)^{\frac{7}{2}-p} \left(\alpha'\right)^{\frac{1}{2}(3-p)} \tag{7.1.27}$$

where μ_p is the charge density of a fundamental Dp-brane and n_p is an integer. If one begins with a D-string with $q_1 = n_1 \mu_1$, then the charges in the T-dual bound state satisfy $q_0 = n_0 \mu_0$ and $q_2 = n_2 \mu_2$ with $n_1 = -(n_0 + n_2)$. This requires taking into account that the range of $\tilde{y}$ in the original solution before T-duality solution is $R'_{\tilde{y}} = \alpha'/R_{\tilde{y}}$, and similarly the gravitational couplings of the T-dual theories are related by $\kappa' = \kappa\sqrt{\alpha'}/R_{\tilde{y}}$. Further, one notes that the rotation angle is quantized as $\tan\zeta = \frac{m}{n}\frac{R'_{\tilde{y}}}{R_{\tilde{x}}}$.

7.1.3. More bound state solutions

In the preceding section, we presented in detail the procedure for constructing the solution for a $D0$-brane bound to a D-membrane by beginning with a D-string. It is now a simple matter to construct other bound state solutions by simply changing the starting point of the construction. In general if we begin with a Dp-brane, the resulting solution describes a $D(p{-}1)$-brane bound to a $D(p{+}1)$-brane. In the following, we present the results for $p = 2, 3, 4$ and 5. We also give a solution describing a bound state of a $D4$-brane, $D0$-brane, and two different $D2$-branes, which results from applying our procedure twice on a certain D-membrane solution.

In general, the resulting bound state solutions are anisotropic in that the full Lorentz invariance in the world-volume of the $D(p{+}1)$-brane is lost. The invariance that remains is Euclidean invariance in the plane in which the $D(p{-}1)$-brane is delocalized, i.e., $(\tilde{x},\tilde{y})$-plane in (7.1.21), and Lorentz invariance in the remaining world-volume directions of the $D(p{+}1)$-brane.

As p is varied in these examples, the relevant T-duality alternates between mapping IIB fields to IIA fields, and vice versa. The former transformation is given in (7.1.20). Using our conventions, the T-duality map from type IIA theory to the type IIB theory [56] is explicitly:

$$
\begin{aligned}
J_{\tilde{z}\tilde{z}} &= \frac{1}{G_{\tilde{z}\tilde{z}}} & e^{2\phi^{(b)}} &= \frac{e^{2\phi^{(a)}}}{G_{\tilde{z}\tilde{z}}} \\
J_{\mu\nu} &= G_{\mu\nu} - \frac{G_{\tilde{z}\mu}G_{\tilde{z}\nu} - B^{(a)}_{\tilde{z}\mu}B^{(a)}_{\tilde{z}\nu}}{G_{\tilde{z}\tilde{z}}} & J_{\tilde{z}\mu} &= -\frac{B^{(a)}_{\tilde{z}\mu}}{G_{\tilde{z}\tilde{z}}} \\
B^{(b)}_{\mu\nu} &= B^{(a)}_{\mu\nu} + 2\frac{G_{\tilde{z}[\mu}B^{(a)}_{\nu]\tilde{z}}}{G_{\tilde{z}\tilde{z}}} & B^{(b)}_{\tilde{z}\mu} &= -\frac{G_{\tilde{z}\mu}}{G_{\tilde{z}\tilde{z}}} \\
A^{(2)}_{\mu\nu} &= A^{(3)}_{\mu\nu\tilde{z}} - 2A^{(1)}_{[\mu}B^{(a)}_{\nu]\tilde{z}} + 2\frac{G_{\tilde{z}[\mu}B^{(a)}_{\nu]\tilde{z}}A^{(1)}_{\tilde{z}}}{G_{\tilde{z}\tilde{z}}} & \chi &= -A^{(1)}_{\tilde{z}} \\
A^{(2)}_{\tilde{z}\mu} &= A^{(1)}_{\mu} - \frac{A^{(1)}_{\tilde{z}}G_{\tilde{z}\mu}}{G_{\tilde{z}\tilde{z}}} \\
A^{(4)}_{\mu\nu\rho\tilde{z}} &= A^{(3)}_{\mu\nu\rho} - \frac{3}{2}\left(A^{(1)}_{[\mu}B^{(a)}_{\nu\rho]} - \frac{G_{\tilde{z}[\mu}B^{(a)}_{\nu\rho]}A^{(1)}_{\tilde{z}}}{G_{\tilde{z}\tilde{z}}} + \frac{G_{\tilde{z}[\mu}A^{(3)}_{\nu\rho]\tilde{z}}}{G_{\tilde{z}\tilde{z}}}\right)
\end{aligned}
\tag{7.1.28}
$$

The field definitions are again given in section 7.1.1, and $\tilde{z}$ is the Killing coordinate which is T-dualized (while $\mu, \nu, \rho \neq \tilde{z}$). Note that in this map only the elements of the four-form RR potential involving $\tilde{z}$ are given. The remaining components are determined by requiring that the corresponding five-form field strength is self-dual.

7.1.3.1. $p = 3, 1$ branes

Here our approach is to begin with the D-membrane solution (7.1.6) carrying electric charge from $A^{(3)}$. We single out $y = y^1$ and delocalize the solution in this transverse direction. Then we rotate by an angle ζ as in (7.1.17) where we set $x = x^1$. The resulting solution is

$$
\begin{aligned}
ds^2 &= \sqrt{\mathcal{H}}\Big\{\frac{-dt^2 + (dx^2)^2}{\mathcal{H}} + \left(\frac{\cos^2\zeta}{\mathcal{H}} + \sin^2\zeta\right)d\tilde{x}^2 + \left(\frac{\sin^2\zeta}{\mathcal{H}} + \cos^2\zeta\right)d\tilde{y}^2 \\
&\quad + 2\cos\zeta\sin\zeta\left(\frac{1}{\mathcal{H}} - 1\right)d\tilde{x}d\tilde{y} + dr^2 + r^2\Big(d\theta^2 \\
&\quad + \sin^2\theta\Big(d\varphi_1^2 + \sin^2\varphi_1\Big(d\varphi_2^2 + \sin^2\varphi_2\Big(d\varphi_3^2 + \sin^2\varphi_3 d\varphi_4^2\Big)\Big)\Big)\Big)\Big\} \\
A^{(3)} &= \pm\left(\frac{1}{\mathcal{H}} - 1\right)dt \wedge (\cos\zeta\, d\tilde{x} + \sin\zeta\, d\tilde{y}) \wedge dx^2 \\
e^{2\phi^{(a)}} &= \sqrt{\mathcal{H}}.
\end{aligned}
\tag{7.1.29}
$$

where $\mathcal{H} = 1 + \frac{\mu}{4}(\ell/r)^4$. We have also introduced polar coordinates on the effective transverse space (originally described by y^i with $i = 2, \ldots, 7$). This facilitates writing the magnetic contribution to the four-form RR potential which appears after T-dualizing.

Now applying T-duality with respect to $\tilde{y}$ as in (7.1.28), we obtain the following solution:

$$\begin{aligned}
ds^2 &= \sqrt{\mathcal{H}}\Big\{ \frac{-dt^2 + (dx^2)^2}{\mathcal{H}} + \frac{d\tilde{x}^2 + d\tilde{y}^2}{1 + (\mathcal{H}-1)\cos^2\zeta} + dr^2 + r^2\Big(d\theta^2 \\
&\quad + \sin^2\theta\left(d\varphi_1^2 + \sin^2\varphi_1\left(d\varphi_2^2 + \sin^2\varphi_2\left(d\varphi_3^2 + \sin^2\varphi_3 d\varphi_4^2\right)\right)\right)\Big)\Big\} \\
A^{(4)} &= \mp\frac{\cos\zeta}{2}\frac{\mathcal{H}-1}{\mathcal{H}}\left(1 + \frac{\mathcal{H}}{1 + (\mathcal{H}-1)\cos^2\zeta}\right) dt \wedge d\tilde{x} \wedge dx^2 \wedge d\tilde{y} \\
&\quad \mp \mu\ell^4 \cos\zeta\, \sin^4\theta \sin^3\varphi_1 \sin^2\varphi_2 \cos\varphi_3\, d\theta \wedge d\varphi_1 \wedge d\varphi_2 \wedge d\varphi_4 \\
A^{(2)} &= \pm\frac{\mathcal{H}-1}{\mathcal{H}}\sin\zeta\, dt \wedge dx^2 \\
B^{(b)} &= \frac{(1-\mathcal{H})\cos\zeta\,\sin\zeta}{1 + (\mathcal{H}-1)\cos^2\zeta} d\tilde{x} \wedge d\tilde{y} \\
e^{2\phi^{(b)}} &= \frac{\mathcal{H}}{1 + (\mathcal{H}-1)\cos^2\zeta}.
\end{aligned} \tag{7.1.30}$$

Note that the T-duality map (7.1.28) explicitly produced the electric component of the potential $A^{(4)}$, and the magnetic component was determined by demanding that $F^{(5)}$ be self-dual. As evidenced by the presence of the four-form and two-form RR potentials, we have a bound state of a D-three-brane and a D-string.

7.1.3.2. $p = 4, 2$ branes

Once again we apply the same procedure of delocalization and rotation on a $D3$-brane, followed by T-duality. This case is slightly more complicated, as the $D3$-brane is charged by the self-dual five-form field strength. Thus one must use the linear combination of electric and magnetic fields given in (7.1.10).

The rotated solution is

$$
\begin{aligned}
ds^2 &= \sqrt{\mathcal{H}}\Bigg\{ \frac{-dt^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2}{\mathcal{H}} + \left(\frac{\cos^2\zeta}{\mathcal{H}} + \sin^2\zeta\right) d\tilde{x}^2 \\
&\quad + \left(\frac{\sin^2\zeta}{\mathcal{H}} + \cos^2\zeta\right) d\tilde{y}^2 + 2\cos\zeta\sin\zeta\left(\frac{1}{\mathcal{H}} - 1\right) d\tilde{x}d\tilde{y} \\
&\quad + dr^2 + r^2\left(d\theta^2 + \sin^2\theta\left(d\varphi_1^2 + \sin^2\varphi_1\left(d\varphi_2^2 + \sin^2\varphi_2 d\varphi_3^2\right)\right)\right)\Bigg\} \\
A^{(4)} &= \pm\frac{1}{2}\left(\frac{1}{\mathcal{H}} - 1\right) dt \wedge (\cos\zeta\, d\tilde{x} + \sin\zeta\, d\tilde{y}) \wedge dx^2 \wedge dx^3 \\
&\quad \pm \frac{1}{2}\mu\ell^3 \sin^3\theta \sin^2\varphi_1 \cos\varphi_2\, (\sin\zeta\, d\tilde{x} - \cos\zeta\, d\tilde{y}) \wedge d\theta \wedge d\varphi_1 \wedge d\varphi_3 \\
e^{2\phi^{(b)}} &= 1
\end{aligned}
\tag{7.1.31}
$$

where $\mathcal{H} = 1 + \frac{\mu}{3}(\ell/r)^3$. Note also that the dilaton here is a constant which has been set equal to zero.

Applying the duality map (7.1.20) gives us the result:

$$
\begin{aligned}
ds^2 &= \sqrt{\mathcal{H}}\Bigg\{ \frac{-dt^2 + \left(dx^2\right)^2 + \left(dx^3\right)^2}{\mathcal{H}} + \frac{d\tilde{x}^2 + d\tilde{y}^2}{1 + (\mathcal{H} - 1)\cos^2\zeta} \\
&\quad + dr^2 + r^2\left(d\theta^2 + \sin^2\theta\left(d\varphi_1^2 + \sin^2\varphi_1\left(d\varphi_2^2 + \sin^2\varphi_2 d\varphi_3^2\right)\right)\right)\Bigg\} \\
A^{(3)} &= \mp\frac{1}{2}\frac{\mathcal{H} - 1}{\mathcal{H}} \sin\zeta\, dt \wedge dx^2 \wedge dx^3 \\
&\quad \pm \frac{\mu\ell^3\cos\zeta}{2} \sin^3\theta \sin^2\varphi_1 \cos\varphi_2\, d\theta \wedge d\varphi_1 \wedge d\varphi_3 \\
B^{(a)} &= \frac{(1 - \mathcal{H})\cos\zeta\sin\zeta}{1 + (\mathcal{H} - 1)\cos^2\zeta} d\tilde{x} \wedge d\tilde{y} \\
e^{2\phi^{(a)}} &= \frac{\sqrt{\mathcal{H}}}{1 + (\mathcal{H} - 1)\cos^2\zeta}
\end{aligned}
\tag{7.1.32}
$$

Here the interpretation is that of a D-membrane, associated with the electric component of the three-form potential, $A^{(3)}_{tx^2x^3}$, in a bound state with a $D4$-brane carrying a magnetic field with $A^{(3)}_{\theta\varphi_1\varphi_3}$. This is consistent with the dyonic nature of the initial five-form self dual field strength.

In [153], the authors give a solution of a bound state of a D-membrane with a $D4$-brane. Their solution, obtained from compactification of $D = 11$ supergravity, agrees precisely with the solution (7.1.32) given above.

7.1.3.3. $p = 5, 3$ branes

Here the starting point is a $D4$-brane which would carry an electric six-form field strength according to (7.1.6), so we must Hodge dualize to the magnetic four-form field strength (7.1.9). The magnetic potential is again most easily expressed using polar coordinates in the transverse space around the delocalized $D4$-brane. Applying our standard construction, the final solution, as the reader can easily verify, is

$$
\begin{aligned}
ds^2 &= \sqrt{\mathcal{H}}\left\{\frac{-dt^2+\sum_{i=2}^{4}\left(dx^i\right)^2}{\mathcal{H}}+\frac{d\tilde{x}^2+d\tilde{y}^2}{1+(\mathcal{H}-1)\cos^2\zeta}\right. \\
&\qquad\left.+dr^2+r^2\left(d\theta^2+\sin^2\theta\left(d\varphi_1^2+\sin^2\varphi_1 d\varphi_2^2\right)\right)\right\} \\
A^{(4)} &= \mp\,\mu\ell^2\sin\zeta\left(\frac{1+\frac{1}{2}(\mathcal{H}-1)\cos^2\zeta}{1+(\mathcal{H}-1)\cos^2\zeta}\right)\sin^2\theta\cos\varphi_1\, d\tilde{x}\wedge d\tilde{y}\wedge d\theta\wedge d\varphi_2 \\
&\qquad \pm\frac{\sin\zeta}{\mathcal{H}}\,dt\wedge dx^2\wedge dx^3\wedge dx^4 \\
A^{(2)} &= \pm\,\mu\ell^2\cos\zeta\sin^2\theta\cos\varphi_1 d\theta\wedge d\varphi_2 \\
B^{(b)} &= \frac{(1-\mathcal{H})\cos\zeta\sin\zeta}{1+(\mathcal{H}-1)\cos^2\zeta}d\tilde{x}\wedge d\tilde{y} \\
e^{2\phi^{(b)}} &= \frac{1}{1+(\mathcal{H}-1)\cos^2\zeta}
\end{aligned}
\tag{7.1.33}
$$

where $\mathcal{H} = 1+\frac{\mu}{2}(\ell/r)^2$. In this case the bound state is made up of dyonic $D3$-branes and magnetically charged $D5$-branes.

7.1.3.4. $p = 6, 4$ branes

Beginning with a $D5$-brane, we dualize the associated electric seven-form field strength to a magnetic three-form field strength and compute the two-form magnetic potential in polar coordinates. After repeating the usual steps once again, the final result is

$$
\begin{aligned}
ds^2 &= \sqrt{\mathcal{H}}\left\{\frac{-dt^2+\sum_{i=2}^{5}\left(dx^i\right)^2}{\mathcal{H}}+\frac{d\tilde{x}^2+d\tilde{y}^2}{1+(\mathcal{H}-1)\cos^2\zeta}\right. \\
&\qquad\left.+dr^2+r^2\left(d\theta^2+\sin^2\theta d\varphi_1^2\right)\right\} \\
A^{(3)} &= \mp\,\frac{\mu\ell\sin\zeta}{1+(\mathcal{H}-1)\cos^2\zeta}\cos\theta d\tilde{x}\wedge d\tilde{y}\wedge d\varphi_1 \\
A^{(1)} &= \mp\,\mu\ell\cos\zeta\cos\theta\, d\varphi_1
\end{aligned}
$$

$$
\begin{aligned}
B^{(a)} &= \frac{(1-\mathcal{H})\cos\zeta\sin\zeta}{1+(\mathcal{H}-1)\cos^2\zeta} d\tilde{x}\wedge d\tilde{y} \\
e^{2\phi^{(a)}} &= \frac{1}{\sqrt{\mathcal{H}}\left(1+(\mathcal{H}-1)\cos^2\zeta\right)}
\end{aligned}
\tag{7.1.34}
$$

where $\mathcal{H} = 1 + \mu\ell/r$. The bound state here contains a $D4$-brane and a $D6$-brane, which are both magnetically charged.

7.1.3.5. $p = 4, 2, 2, 0$ branes

It is a simple exercise to apply our procedure involving delocalization, rotation and T-duality with respect to more than just one of the transverse coordinates of the original D-brane solutions. The resulting solution describes a bound state involving more than just two types of D-branes. To illustrate this idea, we considered the following example: Beginning with the D-membrane solution (7.1.6), we singled out two orthogonal planes: (x^1, y^1) and (x^2, y^2). Applying the procedure in the (x^1, y^1)-plane – with a rotation angle ζ to $(\tilde{x}, \tilde{y})$ – produces a bound state of $p = 3$ and 1 D-branes, as in part (i) above. Repeating the procedure a second time in the (x^2, y^2)-plane – rotating by ψ to $(\hat{x}, \hat{y})$ – yields the following solution

$$
\begin{aligned}
ds^2 &= \sqrt{\mathcal{H}}\left\{\frac{-dt^2}{\mathcal{H}} + \frac{d\tilde{x}^2+d\tilde{y}^2}{1+(\mathcal{H}-1)\cos^2\zeta} + \frac{d\hat{x}^2+d\hat{y}^2}{1+(\mathcal{H}-1)\cos^2\psi}\right. \\
&\qquad \left. +dr^2 + r^2\left(d\theta^2+\sin^2\theta\left(d\varphi_1^2+\sin^2\varphi_1\left(d\varphi_2^2+\sin^2\varphi_2 d\varphi_3^2\right)\right)\right)\right\} \\
A^{(3)} &= \pm\frac{(\mathcal{H}-1)\cos\zeta\sin\psi}{1+(\mathcal{H}-1)\cos^2\zeta}\, dt\wedge d\tilde{x}\wedge d\tilde{y} \pm \frac{(\mathcal{H}-1)\cos\psi\sin\zeta}{1+(\mathcal{H}-1)\cos^2\psi}\, dt\wedge d\hat{x}\wedge d\hat{y} \\
&\qquad \pm\mu\ell^3\cos\zeta\cos\psi\,\sin^3\theta\sin^2\varphi_1\cos\varphi_2\, d\theta\wedge d\varphi_1\wedge d\varphi_3 \\
A^{(1)} &= \mp\frac{\mathcal{H}-1}{\mathcal{H}}\sin\zeta\sin\psi\, dt \\
B^{(a)} &= \frac{(1-\mathcal{H})\cos\zeta\,\sin\zeta}{1+(1-\mathcal{H})\cos^2\zeta}\, d\tilde{x}\wedge d\tilde{y} \\
&\qquad + \frac{(1-\mathcal{H})\cos\psi\,\sin\psi}{1+(\mathcal{H}-1)\cos^2\psi}\, d\hat{x}\wedge d\hat{y} \\
e^{2\phi^{(a)}} &= \frac{\mathcal{H}^{\frac{3}{2}}}{\left(1+(\mathcal{H}-1)\cos^2\zeta\right)\left(1+(\mathcal{H}-1)\cos^2\psi\right)}
\end{aligned}
\tag{7.1.35}
$$

where $\mathcal{H} = 1 + \frac{\mu}{3}(\ell/r)^3$. The electric potential $A^{(1)}$ indicates the presence of $D0$-branes, while the magnetic component of $A^{(3)}$ arises from $D4$-branes. Meanwhile the two electric components of $A^{(3)}$ indicates that there are two kinds of D-membranes, one in the $(\tilde{x},\tilde{y})$-plane and another in the $(\hat{x},\hat{y})$-plane.

7.1.4. Discussion

Using T-duality, we have provided a straightforward construction of low-energy background field solutions corresponding to D-brane bound states for which the difference in dimension is two. We have also presented a number of explicit examples of such solutions. Since supersymmetry is preserved by T-duality, the bound state solutions retain the supersymmetric properties of the initial configuration which involves a single D-brane. Hence these bound states preserve one half of the supersymmetries. Our discussion of the background fields complements that of Polchinski, who recently gave a string world-sheet description of these bound states [65]. Indeed (7.1.24) explicitly shows that the bound state of $p = 0, 2$ branes saturates the BPS bound given there. Similarly extending the calculations of section 7.1.2.1 to the other examples, we find

$$\left(M_{p-1,p+1}\right)^2 = \frac{1}{2\kappa^2}\left(\tilde{q}_{p-1}^2 + q_{p+1}^2\right) \tag{7.1.36}$$

with $M_{p-1,p+1} = \frac{\mu\ell^{6-p}}{2\kappa^2}A_{7-p}$. In close analogy to equation (7.1.23), we defined the charge density of the $D(p-1)$-brane as $\tilde{q}_{p-1} = ((2\pi)^2 R_{\tilde{x}} R_{\tilde{y}})^{-1} q_{p-1}$. For the dyonic $D3$-branes, the charge density that enters this formula can be written as the sum of the electric and magnetic contributions:

$$q_3 = \frac{1}{2}\left(q_3^e + q_3^m\right).$$

Note, of course, that $q_3^e = q_3^m$. In the last example with a bound state of four kinds of branes, this relation extends in the obvious way with a sum of squares of all of the charge densities.

While we have explicitly given all the bound state solutions with asymptotically flat Minkowski-signature geometries, one could also apply our procedure to constructing more exotic solutions involving instantons, strings, or domain walls

– i.e., Dp-branes with $p = -1, 7$ and 8. For example, a *euclidean* $p = 0$ solution in the type IIA theory would correspond to an instantonic string. Applying our construction would lead to a 'bound state' solution with an instantonic membrane ($p = 1$) and a delocalized instanton ($p = -1$). One could also further explore the possibilities arising from multiple applications of our construction, as considered in the example of section 7.1.3.5. Another obvious extension would be to begin with the multiple D-brane solutions discussed in chapter IV. The harmonic function (7.1.7) appearing in the original solutions (7.1.6) was chosen to solve Poisson's equation with a single delta-function source. It is straightforward to introduce more sources producing solutions which describe several separated parallel D-branes. Used as the starting point for the construction given here, these solutions would yield multiple bound states resting in static equilibrium — a possibility which arises due to their supersymmetric character.

It would also be of interest to examine in more detail the correspondance of our low energy background field solutions with the stringy description of these bound states. The charge and mass densities can in principle be extracted from a one-loop string amplitude describing the interaction of two D-branes (see e.g., [65]). This approach was in fact recently considered for the present D-brane bound states by Lifschytz [154]. Alternatively, by examining the scattering of closed strings from D-branes, one can also extract all of their long-range fields [95]. Applying this technique to the D-brane bound states, one again finds a precise agreement between these long-range fields and the corresponding low energy solutions [155].

7.2. Membranes at angles

We begin by writing down the solution describing an arbitrary number n of D-membranes, each of which is rotated by certain $SU(2)$ angle, in the type IIA low energy effective string theory. The solution contains only a nontrivial (string-frame) metric, three-form RR potential and dilaton:

$$
\begin{aligned}
ds^2 &= \sqrt{1+\mathcal{X}}\Bigg[\frac{1}{1+\mathcal{X}}\Big(-dt^2+\sum_{j=1}^{4}(dx^j)^2 \\
&\qquad + \sum_{a=1}^{n}\mathcal{X}_a\left\{[(R_a)^1{}_i dx^i]^2+[(R_a)^3{}_j dx^j]^2\right\}\Big)+\sum_{i=5}^{9}(dy^i)^2\Bigg] \\
A^{(3)} &= \frac{dt}{1+\mathcal{X}}\wedge\Big\{\sum_{a=1}^{n}\mathcal{X}_a\ (R_a)^2{}_i dx^i\wedge(R_a)^4{}_j dx^j \\
&\qquad - \sum_{a<b}^{n}\mathcal{X}_a\mathcal{X}_b\sin^2(\alpha_a-\alpha_b)\ \left(dx^1\wedge dx^3-dx^2\wedge dx^4\right)\Big\} \\
e^{2\phi^{(a)}} &= \sqrt{1+\mathcal{X}}
\end{aligned}
\tag{7.2.1}
$$

where

$$
\mathcal{X}=\sum_{a=1}^{n}\mathcal{X}_a+\sum_{a<b}^{n}\mathcal{X}_a\mathcal{X}_b\sin^2(\alpha_a-\alpha_b)\ .
$$

Above, the rotation matrix R_a associated with the a'th D-membrane is given by

$$
R_a=\begin{pmatrix}\cos\alpha_a & -\sin\alpha_a & & \\ \sin\alpha_a & \cos\alpha_a & & \mathbf{0} \\ & \mathbf{0} & \cos\alpha_a & \sin\alpha_a \\ & & -\sin\alpha_a & \cos\alpha_a\end{pmatrix}
\tag{7.2.2}
$$

The matrices acting in the space of x^i's are easily recognized as $SU(2)$ rotations as follows: one defines the complex coordinates $z^1 = x^1 + ix^2$ and $z^2 = x^3 + ix^4$. Then the above rotations are given by $(z^1, z^2) \rightarrow (e^{i\alpha_a}z^1, e^{-i\alpha_a}z^2)$, or $z^i \rightarrow \left[\exp(i\alpha_a\sigma_3)\right]^i{}_j z^j$. One expects from [144] that restricting the relative orientation of the membranes in this way will preserve some of the supersymmetry, and we confirm this fact in the following.

The functions $\mathcal{X}_a$ are harmonic functions in the transverse space of y^i's. That is, they solve the flat-space Poisson's equation in the transverse space, e.g.,

$$
\delta^{ij}\partial_i\partial_j\mathcal{X}_a=-\ell_a^3\mathcal{A}_4\prod_{k=5}^{9}\delta\left(y^k-y_a^k\right)\ . \tag{7.2.3}
$$

yielding the solutions

$$
\mathcal{X}_a(\vec{y})=\frac{1}{3}\left(\frac{\ell_a}{|\vec{y}-\vec{y}_a|}\right)^3\ . \tag{7.2.4}
$$

Above, ℓ_a are arbitrary positive parameters which have the dimension of length, and we use $\mathcal{A}_4$ to denote the volume of a unit four-sphere, given in (5.1.4). In fact, one may introduce any number of delta-function sources at arbitrary positions on the right hand side of equation (7.2.3), and the corresponding solution would describe a system of parallel branes.

A few words are in order as to the origin of this solution. It is in effect an interpolation between the known solutions for parallel D-membranes, and that [156] for orthogonal D-membranes intersecting over a point. It is straightforward to verify that when the angles are all set to $\alpha_a = 0$, the solution reduces to that of n parallel branes lying in the (y^2, y^4) plane. Note that in this case the membranes have also been delocalized or smeared out in the x^1 and x^3 directions. One may also verify that choosing all $\alpha_a = \alpha_o$ simply corresponds to an overall $SU(2)$ rotation of the previous solution. Similarly the known configuration of orthogonally oriented membranes is reproduced by choosing α_a's to be either zero or $\pi/2$. Further with the α_a set to either α_o and $\pi/2 + \alpha_o$, equation (7.2.1) corresponds to a rotation of this solution. Finally, one may verify that making a further $SU(2)$ rotation of the entire solution simply corresponds shifting all of the angles α_a by the same constant. For this to work, it is important that the second term in $A^{(3)}$ is proportional to $dt \wedge \mathrm{Re}(dz^1 \wedge dz^2)$, which is invariant under $SU(2)$ rotations. Verifying that (7.2.1) solves the low-energy field equations of type IIA string theory was done with the aid of a computer.

To remind the reader of our notation: we refer to x^i and y^i as world-volume and transverse coordinates, respectively, as was the case in chapter IV. Here however, for a given brane, a particular (linear combination of) x^i may actually still correspond to a transverse direction, although it will be one in which the brane is delocalized. Hence in the next section, when we smear out the solution in some y^i making the solution independent of this coordinate, the designation for the coordinate is changed to x^i. We will also assume that the x^i coordinates are all compact with a range of $2\pi L_i$.

7.2.1. Mass and Charge Relations

In this section, we consider some of the physical characteristics of the above configuration (7.2.1). In particular, we calculate the mass and charge densities of our solution. The latter densities are calculated using asymptotic flux integrals, and so they are completely determined by the leading-order behavior of the asymptotic fields. In examining the solution, one sees that these leading order fields are simply linear superpositions of the asymptotic fields generated by the individual rotated membranes. Hence we generalize the rotation appearing in these linearized fields by replacing α_a by an independent angle β_a in the lower two-by-two block of the rotation matrices (7.2.2). Such a configuration would only solve the linearized asymptotic equations of motion, and not the full nonlinear supergravity equations, but this generalization does yield some interesting insight when examining the BPS bound.

The ADM mass per unit p-volume is defined in (4.1.29). Calculating the mass per unit four-volume (of the internal space of x^i's) for our angled system gives us the result

$$M = \frac{A_4}{2\kappa^2} \sum_{a=1}^{n} \ell_a^3 \ . \tag{7.2.5}$$

Thus the mass density is simply the sum of the mass densities of the constituent branes, which was to be entirely expected. Note then that this result is completely independent of the rotation angles.

The membranes carry an electric RR four-form field strength and the corresponding physical charge density is given by (4.1.32)

$$q^e = \frac{1}{\sqrt{2}\kappa} \oint {}^*F^{(4)} \ . \tag{7.2.6}$$

Hodge duality produces a six-form which is then integrated over the asymptotic four-sphere in the transverse space and some two-torus in (x^1, x^2, x^3, x^4). Thus given, the three-form potential in (7.2.1), in applying (7.2.6) we obtain a number of independent charges related to the choice of asymptotic surface over which one integrates. For example the term in $A^{(3)}$ proportional $dt \wedge dx^2 \wedge dx^4$ yields a term in ${}^*F^{(4)}$ to be integrated over the compact coordinates x^1 and x^3 as well as the four-sphere at infinity. We use the following notation to write the resulting charge

$$q^e_{13} = -q^e_{31} = -\frac{\mathcal{A}_4}{\sqrt{2}\kappa}\left(4\pi^2 L_1 L_3\right)\sum_{a=1}^{n}\ell_a^3 \cos\alpha_a \,\cos\beta_a \tag{7.2.7}$$

where the antisymmetric matrix notation will be useful later on. This result gives the charge per unit area in the (x^2, x^4) plane, i.e., the plane in which the branes lie for $\alpha_a = \beta_a = 0$. In order to compare the charges, however, we should divide out the area of the orthogonal (x^1, x^3) torus in order to produce a charge per four-volume in the entire compact space. Hence we define $\tilde{q^e}_{13} = q^e_{13}/(4\pi^2 L_1 L_3)$. In a like manner all the charge densities $\tilde{q^e}_{ij}$ can be calculated and we list the nonvanishing contributions

$$\begin{aligned}
\tilde{q^e}_{13} &= -\frac{\mathcal{A}_4}{\sqrt{2}\kappa}\sum_{a=1}^{n}\ell_a^3 \cos\alpha_a \,\cos\beta_a, \\
\tilde{q^e}_{14} &= -\frac{\mathcal{A}_4}{\sqrt{2}\kappa}\sum_{a=1}^{n}\ell_a^3 \cos\alpha_a \,\sin\beta_a, \\
\tilde{q^e}_{23} &= \frac{\mathcal{A}_4}{\sqrt{2}\kappa}\sum_{a=1}^{n}\ell_a^3 \sin\alpha_a \,\cos\beta_a, \\
\tilde{q^e}_{24} &= \frac{\mathcal{A}_4}{\sqrt{2}\kappa}\sum_{a=1}^{n}\ell_a^3 \sin\alpha_a \sin\beta_a\,.
\end{aligned} \tag{7.2.8}$$

Of course these charge densities are dependent on the rotation angles which orient the various D-membranes. Note that if $\alpha_a = \beta_a = 0$ we recover the expected charge configuration of a collection of parallel membranes lying in the (x^2, x^4) plane, i.e.,

$$\tilde{q^e}_{13} = -\frac{\mathcal{A}_4}{\sqrt{2}\kappa}\sum_{a=1}^{n}\ell_a^3, \qquad \tilde{q^e}_{14} = \tilde{q^e}_{23} = \tilde{q^e}_{24} = 0$$

where the single nonvanishing charge density is simply the sum of that for the individual branes.

Having calculated these physical characteristics of our configuration of D-membranes with angles, we would like to examine the BPS bound. The latter may be determined from the eigenvalues of the Bogomol'nyi matrix, which is derived using both the supersymmetry algebra and the asymptotic form of the background fields [79,157]. Unbroken supersymmetries arise when this matrix has eigenspinors with a vanishing eigenvalue. In the present problem, the Bogomol'nyi matrix is [158]

$$\mathrm{M} = M + \frac{1}{\sqrt{2}\kappa}\tilde{q^e}_{ij}\Gamma_{0ij} \tag{7.2.9}$$

for which the distinct eigenvalues are

$$M \pm \frac{1}{\sqrt{2}\kappa}\sqrt{\tilde{q^e}_{ij}\tilde{q^e}_{ij} \pm \frac{1}{2}\epsilon_{ijkl}\tilde{q^e}_{ij}\tilde{q^e}_{kl}}\ . \tag{7.2.10}$$

In these formulae, the implicit sums all run from 1 to 4, and we use the antisymmetric notation $\tilde{q^e}_{ij} = -\tilde{q^e}_{ji}$ introduced above. Also note that the two signs in the eigenvalues are chosen independently. Since the mass is positive, the eigenvalues for which the first sign is positive cannot vanish, and hence at least half of the supersymmetries are broken by our solution. The vanishing of the remaining eigenvalues can be expressed in terms of a BPS mass limit

$$M_\pm^2 = \frac{1}{2\kappa^2}\left(\tilde{q^e}_{ij}\tilde{q^e}_{ij} \pm \frac{1}{2}\epsilon_{ijkl}\tilde{q^e}_{ij}\tilde{q^e}_{kl}\right)\ . \tag{7.2.11}$$

Substituting our values for the charge densities (7.2.8) results in

$$M_\pm^2 = \left(\frac{\mathcal{A}_4}{2\kappa^2}\right)^2\left[\left(\sum_{a=1}^{n}\ell_a^3\cos(\alpha_a \mp \beta_a)\right)^2 + \left(\sum_{a=1}^{n}\ell_a^3\sin(\alpha_a \mp \beta_a)\right)^2\right]\ . \tag{7.2.12}$$

In comparing these BPS bounds (7.2.12) with the mass (7.2.5), we find that in general the mass exceeds the former bounds. To make this more apparent, one may introduce complex variables $Z_{\pm,a} = (\mathcal{A}_4/2\kappa^2)\ell_a^3\ \exp[i(\alpha_a \mp \beta_a)]$. Now it is clear that generically $M^2 = (\sum_a |Z_{\pm,a}|)^2$ exceeds $M_\pm^2 = |\sum_a Z_{\pm,a}|^2$. It is also clear that the only way to lower the mass to one of the bounds is to chose all of the phases to be equal, i.e., $\alpha_a - \beta_a = 2\theta$ or $\alpha_a + \beta_a = 2\theta'$. There are only two distinct choices here up to an overall rotation. If we set $\alpha_1 = \beta_1 = 0$ to fix the overall orientation of the configuration, we must choose the remaining angles with $\beta_a = \alpha_a$ or $\beta_a = -\alpha_a$. The former corresponds to the choice made in our solution (7.2.1), and for which we then have $M = M_+$ and one-quarter of the supersymmetries being preserved. The latter choice, for which $M = M_-$, would yield a slightly different configuration. Complex $SU(2)$ rotations are again relevant in this case, but now the $SU(2)$ acts on $(z^1, \bar{z}^2)$=$(x^1 + ix^2, x^3 - ix^4)$. Our solution would be modified by changing the sign of α_a in the lower two-by-two block of the rotation matrices (7.2.2), and the sign of $dx^2 \wedge dx^4$ would be reversed in the last term in $A^{(3)}$. As expected, our results here are entirely consistent with the analysis of [144] mentioned earlier which is formulated at the level of the string world-sheet and provides an independent confirmation of their results when applied to D-membranes.

7.2.2. T-Duality

The ten-dimensional T-duality map between the type IIA and IIB string theories was given in equations (7.1.20) and (7.1.28). In the next subsection, we consider the effect of T-duality along coordinates that are in the transverse space. The effect of these transformations is to extend the dimension of the D-branes. The results then are new solutions describing Dp-branes with relative $SU(2)$ angles and so remaining parallel over a (p-2)-brane. In subsection 7.2.2.2, we consider the effect of T-duality transformations along world-volume coordinates. The results here involve more exotic bound state configurations of D-branes, as found in the previous section.

7.2.2.1. Transverse directions

In order to apply T-duality along one of the transverse coordinates, e.g., y^5, we must first delocalize the solution in this direction, which we then denote as x^5. This amounts to replacing the sources in (7.2.3) by four-dimensional delta-functions, producing solutions of the form

$$\mathcal{X}_a(\vec{y}) = \frac{1}{2}\left(\frac{\ell_a}{|\vec{y}-\vec{y}_a|}\right)^2$$

where now $\vec{y} = (y^6, y^7, y^8, y^9)$. A straightforward application of the T-duality map (7.1.28) from the type IIA to the type IIB theory along x^5 in this smeared out solution yields

$$ds^2 = \sqrt{1+\mathcal{X}}\left[\frac{1}{1+\mathcal{X}}\left(-dt^2 + \sum_{i=1}^{5}(dx^i)^2 \right.\right.$$
$$\left.\left. + \sum_{a=1}^{n}\mathcal{X}_a\left\{[(R_a)^1{}_i dx^i]^2 + [(R_a)^3{}_j dx^j]^2\right\}\right) + \sum_{i=6}^{9}(dy^i)^2\right]$$

$$F^{(5)} = dt \wedge dx^5 \wedge dy^k \wedge \partial_k \left\{ \frac{1}{1+\mathcal{X}} \left[\sum_{a=1}^{n} \mathcal{X}_a \, (R_a)^2{}_i dx^i \wedge (R_a)^4{}_j dx^j \right.\right.$$
$$\left.\left. - \sum_{a<b}^{n} \mathcal{X}_a \mathcal{X}_b \, \sin^2(\alpha_a - \alpha_b) \, \left(dx^1 \wedge dx^3 - dx^2 \wedge dx^4 \right) \right] \right\}$$
$$+ dx^h \wedge dy^i \wedge dy^j \wedge \, \epsilon_{hijk} \partial_k \left\{ \sum_{a=1}^{n} \mathcal{X}_a \, (R_a)^1{}_l dx^l \wedge (R_a)^3{}_m dx^m \right\}$$
$$e^{2\phi^{(b)}} = 1 \, . \qquad (7.2.13)$$

This solution obviously describes a system of angled $D3$-branes, as indicated by the presence of the nontrivial five-form RR field strength. We have written the solution in terms of the self-dual field strength, rather than the potential $A^{(4)}$, because the magnetic part of the latter is rather unwieldy when the $D3$-branes are centered at arbitrary positions $\vec{y}_a$. If one sets $\vec{y}_a = 0$, the potential can be given in a fairly compact form using polar coordinates on the transverse space. Note also that ϵ_{hijk} is the antisymmetric Levi-Civita symbol on the transverse space with $h, i, j, k = 6 \ldots 9$ and $\epsilon_{6789} = +1$.

One can carry this process further by delocalizing the above solution in another transverse coordinate y^6 (which we then denote x^6 — also, note that one now has $\mathcal{X}_a = \ell_a / |\vec{y} - \vec{y}_a|$), and applying T-duality along this direction to produce a system of $D4$-branes with $SU(2)$ angles. Here, the T-duality map from type IIB to type IIA generates a magnetic three-form potential through $A^{(3)}_{\mu\nu\rho} = A^{(4)}_{\mu\nu\rho 6}$ (the remaining terms in this relation vanish in the present case). This part of the transformation is equivalent to mapping the field strengths $F^{(4)}_{\mu\nu\rho\sigma} = F^{(5)}_{\mu\nu\rho\sigma 6}$, since the delocalized solution is independent of x^6. Hence the T-dual solution may be expressed as

$$ds^2 = \sqrt{1+\mathcal{X}} \left[\frac{1}{1+\mathcal{X}} \left(- dt^2 + \sum_{i=1}^{6} \left(dx^i \right)^2 \right.\right.$$
$$\left.\left. + \sum_{a=1}^{n} \mathcal{X}_a \left\{ \left[(R_a)^1{}_j dx^j \right]^2 + \left[(R_a)^3{}_j dx^j \right]^2 \right\} \right) + \sum_{i=7}^{9} \left(dy^i \right)^2 \right]$$
$$F^{(4)} = dy^i \wedge dy^j \wedge \, \epsilon_{ijk} \partial_k \left\{ \sum_{a=1}^{n} \mathcal{X}_a \, (R_a)^1{}_l dx^l \wedge (R_a)^3{}_m dx^m \right\}$$
$$e^{2\phi^{(a)}} = \frac{1}{\sqrt{1+\mathcal{X}}} . \qquad (7.2.14)$$

Again the magnetic field strength takes a much more compact form than the corresponding potential for the multi-center solution. One sees that this solution describes a system of $D4$-branes since the magnetic $F^{(4)}$ is the only nontrivial RR field.

Of course, this procedure of T-dualizing in the transverse space can be continued to produce configurations of higher dimensional D-branes with angles. Since the $SU(2)$ rotations effectively extend the dimension of the world-volume by two, the remaining solutions will have a transverse space of dimension lower than three, and hence will not be asymptotically flat. For example, the solution describing angled $D6$-branes would have a transverse space of dimension one, and thus would have the appearance of an anisotropic domain wall.

7.2.2.2. World-volume directions

An alternative to the above procedure is to apply T-duality in the world volume directions of the original solution (7.2.1). Since the membranes are rotated in these directions, T-dual configurations will involve D-brane bound states for which the difference in dimension is two, as discussed in section one. To simplify the procedure we specialize the general solution to the case of two D-membranes and also set the rotation angles $(\alpha_1, \alpha_2) = (0, \alpha)$. With these simplifications, equation (7.2.1) reduces to

$$\begin{aligned}
ds^2 &= \sqrt{1+\mathcal{X}}\left\{\frac{1}{1+\mathcal{X}}\Big(-dt^2 + (1+\mathcal{X}_1)\Big[(dx^1)^2 + (dx^3)^2\Big] + (dx^2)^2 + (dx^4)^2\right.\\
&\qquad + \mathcal{X}_2\left[\left(\cos\alpha dx^1 - \sin\alpha dx^2\right)^2 + \left(\cos\alpha dx^3 + \sin\alpha dx^4\right)^2\right]\Big)\\
&\qquad \left. + \sum_{i=5}^{9}\left(dy^i\right)^2\right\}\\
A^{(3)} &= \frac{dt}{1+\mathcal{X}}\wedge\Big\{-(\mathcal{X}_2 + \mathcal{X}_1\mathcal{X}_2)\sin^2\alpha\, dx^1\wedge dx^3 + \mathcal{X}_2\sin\alpha\cos\alpha\, dx^1\wedge dx^4\\
&\qquad - \mathcal{X}_2\cos\alpha\sin\alpha\, dx^2\wedge dx^3\\
&\qquad + \left(\mathcal{X}_1 + \mathcal{X}_2\cos^2\alpha + \mathcal{X}_1\mathcal{X}_2\sin^2\alpha\right)\, dx^2\wedge dx^4\Big\}\\
e^{2\phi^{(a)}} &= \sqrt{1+\mathcal{X}} \qquad\qquad (7.2.15)
\end{aligned}$$

and $\mathcal{X}$ is given by

$$\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_2 + \mathcal{X}_1\mathcal{X}_2\sin^2\alpha.$$

We also simplify the following results by positioning the second membrane at the origin, i.e., we set $\vec{y}_2 = 0$, but leave $\vec{y}_1$ arbitrary.

As the first example, we apply T-duality along the x^4 direction — note that this direction is tangent to the world-volume of the a=1 membrane, but is angled with respect to the second. We find that

$$
\begin{aligned}
ds^2 = \sqrt{1+\mathcal{X}}\Big\{ & \frac{1}{1+\mathcal{X}}\Big(-dt^2 + (1+\mathcal{X}_1)\left(dx^1\right)^2 + \left(dx^2\right)^2 \\
& + \mathcal{X}_2\left(\cos\alpha dx^1 - \sin\alpha dx^2\right)^2\Big) + \frac{\left(dx^3\right)^2 + \left(dx^4\right)^2}{1+\mathcal{X}_2\sin^2\alpha} \\
& + dr^2 + r^2\left(d\theta^2 + \sin^2\theta\left(d\varphi_1^2 + \sin^2\varphi_1\left(d\varphi_2^2 + \sin^2\varphi_2 d\varphi_3^2\right)\right)\right)\Big\} \\
A^{(4)} = & -\frac{1}{2}\mathcal{X}_2\sin^2\alpha\left\{\frac{1+\mathcal{X}_1}{1+\mathcal{X}} + \frac{1}{1+\mathcal{X}_2\sin^2\alpha}\right\}dt\wedge dx^1\wedge dx^3\wedge dx^4 \\
& -\frac{1}{2}\mathcal{X}_2\cos\alpha\sin\alpha\left\{\frac{1}{1+\mathcal{X}_2\sin^2\alpha} + \frac{1}{1+\mathcal{X}}\right\}dt\wedge dx^2\wedge dx^3\wedge dx^4 \\
& + \ell_2^3\sin\alpha\sin^3\theta\sin^2\varphi_1\cos\varphi_2\Big(\cos\alpha\, dx^1 \\
& \qquad - \sin\alpha\, dx^2\Big)\wedge d\theta\wedge d\varphi_1\wedge d\varphi_3 \\
A^{(2)} = & \frac{dt}{1+\mathcal{X}}\wedge\left\{\mathcal{X}_2\cos\alpha\sin\alpha dx^1 + \left(\mathcal{X} - \mathcal{X}_2\sin^2\alpha\right)dx^2\right\} \\
B^{(b)} = & \frac{\mathcal{X}_2\cos\alpha\sin\alpha}{1+\mathcal{X}_2\sin^2\alpha}dx^3\wedge dx^4 \\
e^{2\phi^{(b)}} = & \frac{1+\mathcal{X}}{1+\mathcal{X}_2\sin^2\alpha}
\end{aligned}
\tag{7.2.16}
$$

where we have transformed the coordinates transverse to the system into spherical coordinates to facilitate the computations of the four-form RR potential. Setting $\mathcal{X}_2 = 0$, one can verify that this solution reduces to that of a D-string lying parallel to x^2 and at the same time delocalized in x^1, x^3 and x^4. Setting $\mathcal{X}_1 = 0$ and comparing with the solutions of the first section of this chapter, one finds that the solution is precisely that of a D(3,1)-brane bound state. There has been a rotation of this bound state so that it lies in $(\cos\alpha\, x^2 + \sin\alpha\, x^1, x^3, x^4)$ with the D-strings oriented along the first direction. The bound state is also delocalized in the orthogonal $\cos\alpha\, x^1 - \sin\alpha\, x^2$ direction. The angle α also determines the relative charge densities of the D-strings and D3-branes—in section 7.1.3.1 $\zeta = \pi/2 - \alpha$.

Next, applying T-duality in the y^2 direction produces a solution of the form

$$
\begin{aligned}
ds^2 &= \sqrt{1+\mathcal{X}}\left\{\frac{-dt^2}{1+\mathcal{X}} + \frac{(dx^1)^2+(dx^2)^2+(dx^3)^2+(dx^4)^2}{1+\mathcal{X}_2\sin^2\alpha} + \sum_{i=5}^{9}(dy^i)^2\right\}\\
A^{(3)} &= \frac{\mathcal{X}_2\cos\alpha\sin\alpha}{1+\mathcal{X}_2\sin^2\alpha}dt\wedge\left(dx^1\wedge dx^2 - dx^3\wedge dx^4\right)\\
&\qquad + \ell_2^3\sin^2\alpha\,\sin^3\theta\,\sin^2\varphi_1\,\cos\varphi_2\,d\theta\wedge d\varphi_1\wedge d\varphi_3\\
A^{(1)} &= \left\{\frac{1+\mathcal{X}_2\sin^2\alpha}{1+\mathcal{X}} - 1\right\}dt\\
B^{(a)} &= \frac{\mathcal{X}_2\cos\alpha\,\sin\alpha}{1+\mathcal{X}_2\sin^2\alpha}\left(dx^3\wedge dx^4 - dx^1\wedge dx^2\right)\\
e^{2\phi^{(a)}} &= \frac{(1+\mathcal{X})^{\frac{3}{2}}}{\left(1+\mathcal{X}_2\sin^2\alpha\right)^2}.
\end{aligned}
\tag{7.2.17}
$$

In this case setting $\mathcal{X}_2 = 0$ reduces the solution to that of a D-particle positioned at $\vec{y}_1$ and delocalized in the x^i directions. Setting $\mathcal{X}_1 = 0$ reproduces a special case of the D(4,2,2,0)-brane bound state given in (7.1.35). Here the two angles ζ,ψ of equation (7.1.35) are related, and given in terms of α as $\zeta = -\psi = \pi/2 - \alpha$.

As a final example, we perform T-duality along x^3 in the two membrane solution (7.2.15) with the resulting solution

$$
\begin{aligned}
ds^2 &= \sqrt{1+\mathcal{X}}\left\{\frac{-dt^2 + \left(1+\mathcal{X}_1+\mathcal{X}_2\cos^2\alpha\right)(dx^1)^2 + \left(1+\mathcal{X}_2\sin^2\alpha\right)(dx^2)^2}{1+\mathcal{X}}\right.\\
&\qquad\left. - \frac{2\mathcal{X}_2\cos\alpha\sin\alpha\,dx^1dx^2}{1+\mathcal{X}} + \frac{(dx^3)^2+(dx^4)^2}{1+\mathcal{X}_1+\mathcal{X}_2\cos^2\alpha} + \sum_{i=5}^{9}(dy^i)^2\right\}\\
A^{(4)} &= -\frac{\mathcal{X}_2\cos\alpha\,\sin\alpha}{2}\left\{\frac{1}{1+\mathcal{X}} + \frac{1}{1+\mathcal{X}_1+\mathcal{X}_2\cos^2\alpha}\right\}dt\wedge dx^1\wedge dx^3\wedge dx^4\\
&\quad + \left\{\frac{1-2\mathcal{X}_1}{2\mathcal{X}_1} + \frac{1}{2\left(1+\mathcal{X}_1+\mathcal{X}_2\cos^2\alpha\right)}\right.\\
&\qquad\qquad \left. - \frac{1+\mathcal{X}_2}{2\mathcal{X}_1(1+\mathcal{X})}\right\}dt\wedge dx^2\wedge dx^3\wedge dx^4\\
&\quad + \ell_1^3\sin^3\theta\,\sin^2\varphi_1\,\cos\varphi_2\,dx^1\wedge d\theta\wedge d\varphi_1\wedge d\varphi_3\\
&\quad + \ell_2^3\cos\alpha\,\sin^3\theta\,\sin^2\varphi_1\,\cos\varphi_2\left(\cos\alpha\,dx^1\right.\\
&\qquad\qquad\qquad \left. - \sin\alpha\,dx^2\right)\wedge d\theta\wedge d\varphi_1\wedge d\varphi_2\\
A^{(2)} &= -\frac{\mathcal{X}_2\sin\alpha}{1+\mathcal{X}}\,dt\wedge\left\{\sin\alpha\,(1+\mathcal{X}_1)\,dx^1 + \cos\alpha\,dx^2\right\}\\
B^{(b)} &= -\frac{\mathcal{X}_2\cos\alpha\,\sin\alpha}{1+\mathcal{X}_1+\mathcal{X}_2\cos^2\alpha}dx^3\wedge dx^4
\end{aligned}
$$

$$e^{2\phi^{(b)}} = \frac{1+\mathcal{X}}{1+\mathcal{X}_1+\mathcal{X}_2\cos^2\alpha}\,. \tag{7.2.18}$$

where we have also put $\vec{y}_1 = 0$ here for simplicity. With $\mathcal{X}_2 = 0$, we have a single $D3$-brane filling (x^2, x^3, x^4) and delocalized in x^1. With $\mathcal{X}_1 = 0$, one may verify that the result describes a $D(3,1)$-brane bound state parallel to $(\sin\alpha\, x^1 + \cos\alpha\, x^2, x^3, x^4)$ with the $D1$-branes lying in the first of these directions. Again the relative charge densities of the bound state are determined by the rotation angle.

7.2.3. Discussion

In this section we presented a new low-energy solution (7.2.1) describing an arbitrary number n of D-membranes oriented at angles with respect to one another. We were also able to show that this configuration saturated the BPS bound because the relative rotations between the membranes are in an $SU(2)$ subgroup. As a result, the system preserves one-quarter of the supersymmetries.

Our solution provides the most general supersymmetric configuration containing (only) two D-membranes. One might think of extending the rotations considered here to an arbitrary $SU(2)$ rotation, but this generalization would only change the overall orientation of our solution. Following the analysis of [144], with three D-membranes one might make $SU(3)$ rotations while still preserving one-eighth of the supersymmetries. This would extend the space in which the rotations act to produce an effective seven-dimensional world volume. It would be interesting to find the corresponding background field solution. For general n, one might consider $SU(n)$ rotations [144], however, in practice one would be limited to $SU(4)$ by the fact that the spacetime is ten-dimensional.

By applying T-duality to the membrane solution (7.2.1), we produced solutions describing systems of higher dimensional D-branes oriented at angles, and also configurations involving $D(p{+}1,p{-}1)$-brane bound states. Since supersymmetry is preserved by T-duality, these other new solutions also preserve one-quarter of the supersymmetries. By explicit construction, we have confirmed the existence of a supersymmetric configuration including $D0$-branes and $D(4,2,2,0)$-bound states. These supersymmetric solutions were conjectured in [154], where it was shown that the interaction potential precisely vanished between these two objects.

VIII

Conclusions

String theory has become an important area of research in recent years. As eloquently argued by Polchinski [3] this is due in large part to string theory being the only way we have yet found to soften the divergences of quantum gravity while remaining consistent with Lorentz invariance. This feature alone makes string theory a worthy candidate for study.

The study of black holes in the context of string theory is important for two reasons. First, the existence of long-standing theoretical questions such as the microscopic interpretation of the entropy of a black hole, and the black hole information paradox make it clear that a quantum theory of gravity is necessary. If string theory pretends to be a quantum theory of gravity, then it should provide answers where other approaches fail. As we have seen in chapter IV, string theory has made progress on this front.

The second reason for the study of black holes in string theory is that black holes are part of the nonperturbative regime of the theory. It has become clear that knowledge of the nonperturbative regime of string theory is necessary to describe the microscopic entropy of a black hole, or to shed light on cosmological questions. Thus black holes can aid us in discerning the nonperturbative structure of the theory of strings.

During the past two years, progress in this area has advanced dramatically. With this has come the detailing of the various dualities which relate different parts of string theories to other parts, or different string theories to other string theories. This in turn has led to the discovery that there are other objects than strings contained in the theory of strings. Dirichlet branes (D-branes) provide certain states that are necessary to fill out multiplets of states which are related by the system of duality symmetries of string theory, which otherwise would be incomplete. With the inclusion of D-branes, the system of dualities points to the astonishing, yet natural, conclusion that the four consistent string theories in ten dimensions are different ways of describing a more fundamental eleven dimensional theory, which has been dubbed M-theory.

The Dirichlet branes have also proven themselves very useful as probes into the nonperturbative behavior of string theory, as it is the degrees of freedom of a bound state of D-branes that we counted when we computed the microscopic entropy of two different black holes in chapter VI. In the first section of chapter IV, the microscopic entropy of a five-dimensional extremal, supersymmetric and rotating black hole was computed using the D-brane technique. This calculation extended the validity of the then-nascent "D-brane technology" to the case of rotating black holes in five dimensions.

The extension of this technique to non-extremal, and thus non-supersymmetric black holes was the subject of the second section of chapter VI. Again this research represents the first time such a computation was done for the case of non-zero angular momentum, and as such was an important test of the D-brane technique. The fact that computations of this type are successful is substantial evidence that string theory "knows" about the microscopic degrees of freedom underlying the thermodynamics of black holes. Thus the theory of strings is making its first successes as a candidate for a quantum theory of gravity.

These computations also make it clear that D-branes and their bound states and black holes are closely related. It is then logical that to advance the state of the art of one is to do so for the other, and that by using them together we have powerful tools for the study of nonperturbative string theory. This is not to mention that D-branes are interesting objects in their own right.

In chapter VII, the knowledge of the properties of bound states of D-branes was improved in two ways. In section one it was demonstrated that by means of a simple construction combining rotation with one of the duality symmetries of the theory of strings, that of T-duality, the set of known D-brane bound states can be augmented considerably. More specifically, it was shown that two D-branes, which differ in dimension by two, for example a D-membrane and a D-point can form a bound state which is supersymmetric. In the final bound state, the D-brane of smaller dimension is effectively "delocalized" in its partner, i.e., one could think of it as having "dissolved" in the other D-brane. Formerly, the search for supersymmetric bound states of D-branes was biased toward those in which the dimension of the constituents differed by four. The reason for this prejudice is that in this type of bound state the D-branes can have arbitrary separation. Thus a large class of D-brane bound states which had been previously overlooked were brought to the attention of the string theory community.

The second part of chapter VII concerns a further extension in the set of known low-energy background solutions which represent D-brane bound states. Here, bound states which are composed of an arbitrary number of D-membranes which intersect at arbitrary angles were constructed. These existence of these solutions had been demonstrated by other researchers [144], however the explicit solution had not been previously written down. It was also demonstrated that enlargement of the collection of bound states could be carried out by applying the duality symmetries of string theory to the basic solution.

8.1. Future directions

String theory has begun to deliver on its promise as a possible theory of quantum gravity. However, it is apparent that there is much that remains to be done, even in the area of the physics of black holes. The techniques of the D-brane technology of entropy counting are still restricted to only a tiny class of possible black hole solutions, namely those black holes which are extremal or very close to extremal. It is, however, possible to make very good estimates of the entropy of non-extremal black holes [141].

For example, it may be possible to exploit the symmetries of string theory to decode the microscopic physics of non-extremal black holes by studying the ways in which the symmetries alter the microscopic physics of the black holes for which the counting works when the symmetries are applied. This will require more detailed knowledge of how symmetries work, which in turn may require more information about the eleven dimensional M-theory, which as yet remains mysterious.

There exist also a large number of supersymmetric, and non-supersymmetric black holes and D-brane bound states which remain to be discovered. Perhaps with time a pattern will emerge which will lead to formulation of a structure which includes both of these as well as links between them. Conceivably, such a structure may reveal clues as to the mechanism of supersymmetry breaking, leading to progress on models of superstring grand unification.

It is clear that much has been learned about the theory of strings since the importance of dualities and the nonperturbative regime of the theory have been recognized. It is equally clear that we are only beginning to uncover its secrets.

Appendix A.
Notation and conventions

In this work we will be using the "East-Coast" [36] metric diag$(-1, 1, 1, \ldots)$ with the number of 1's depending on the number of dimensions we are working in. We use units where $c = \hbar = 1$. Note that the Newton constant $G_N \neq 1$, *except* in chapter VI. Uppercase X, Y will denote the coordinate fields on the world-sheet whereas lowercase x, y will denote coordinates in space-time. The representation of the Dirac matrices on the world sheet is

$$\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{A.1}$$

with $\gamma^3 = \gamma^0\gamma^1$. We also work with

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu} \tag{A.2}$$

For the Ricci tensor and Ricci scalar we have $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$ and $R = g^{\mu\nu}R_{\mu\nu}$.

String theory is a large subject, and many symbols must be used to denote different physical quantities. This is made even more of a problem when string dualities are used to related the same quantities from different string theories to each other, and/or when one begins to compactify fields from one dimension to another. Thus one is pressed to invent a notational scheme which is clear without being cumbersome. In this work we will stress clarity at the expense of occasionally having to use a few more symbols. A detailed list of symbols follows this section. Let us begin by stating some general guidelines:

1. Indicies on the world-sheet are indicated by roman characters, a, b, etc., while indicies in space time are denoted μ, ν, and so on. Occasionally we use α, β, to indicate an index in the space of $U(1)$ gauge fields of the heterotic string, for example in (3.2.20).
2. The same symbol with a different type of index will in general denote the same object (or similar objects) in different circumstances. Thus η^{ab} denotes the Minkowski metric on the world-sheet, whereas $\eta^{\mu\nu}$ denotes the Minkowski metric in spacetime.

3. String-frame metrics will be denoted by uppercase letters, for example G for the Type IIA supergravity, J for its type IIB counterpart, etc..
4. Einstein-frame metrics will be denoted by the lowercase counterparts of the string frame metrics, i.e., for the type II theory g and j for type IIA and type IIB respectively.
5. When there is the possibility for confusion, the dimensionality will be placed on the object in question as a subscript, for example ϕ_6 for the six-dimensional dilaton field.
6. In the case of compactifications, space-time indices will be arranged according to:

    6.1. The indices μ, ν, etc., are used for the target space, the space to which one is compactifying. Thus in the case of compactifying from ten to six dimensions μ, ν, etc., will run from 0 to 5.

    6.2. Hatted indices $\hat{\mu}$, $\hat{\nu}$, are used for the space from which one is compactifying. Thus for the example of compactifying from ten to six dimensions, $\hat{\mu}$, $\hat{\nu}$, run from 0 to 9.

    6.3. The compactified space will be indicated with indices, $\tilde{\mu}$, $\tilde{\nu}$, etc., which evidently in our $D = 10$ to $D = 6$ example run from 6 to 9.

7. n-form potentials and their $n+1$-form field strengths will always be denoted by $A^{(n)}$ and $F^{(n+1)}$ respectively in the Ramond-Ramond sector, whereas in the Neveu-Schwarz-Neveu-Schwarz sector the Kalb-Ramond field will be B and its field strength H.
8. Kaluza-Klein gauge fields coming from compactification are denoted $\Lambda_G^{(1)}$ when derived from the type IIA string frame metric, $\Lambda_{B(a)}^{(1)}$ when derived from the type IIA Kalb-Ramond field, and so on. The field strength of the $\Lambda^{(1)}$ is given the symbol $\Xi^{(2)}$. The logical extension is that $\Lambda_J^{(1)}$ denotes such a gauge field in the type IIB string theory.

8.1. We use the notation of forms wherever appropriate, for example $H = dB$, $dx \wedge dy$, etc..

8.2. Hodge dualization in the notation of forms is denoted by the standard "Hodge star", while the result of this procedure is denoted with a $\tilde{}$, i.e., we have $\tilde{F}^{(D-n)} = {}^{*}F^{(n)}$.

8.3. When there is possibility of confusion we will use superscripts $^{(i)}$, $^{(h)}$, $^{(a)}$ or $^{(b)}$ to denote to which string theory, type I, heterotic, type IIA or type IIB, respectively a particular field belongs. Of course the n of the Ramond-Ramond fields make clear to which type II theory they belong.

9. It will, despite our best efforts to the contrary, sometimes occur that two different quantities will be denoted by the same symbol. Care has been taken to ensure that the usages are sufficiently different that context is more than adequate to distinguish them.

A.1. List of symbols

S	Action
$\mathcal{S}$	Entropy
$\mathcal{T}_0$, $\mathcal{T}_1$	Normalization constants for point particles, strings
$\mathcal{T}_n$	Tension of n-brane
α'	Inverse string tension or Regge slope
$\eta_{\mu\nu}$	Minkowski metric in spacetime
η_{ab}	Minkowski metric on world-volume
τ, σ	World-sheet coordinates
$\sigma^{\pm} = \tau \pm \sigma$	World-sheet light cone coordinates
X, Y	World-sheet coordinate fields
ψ	World-sheet fermionic fields
a, b, c, etc.	World-sheet indices, tangent space indices

e^{ab}	World-sheet metric
ϵ^{ab}	World-sheet antisymmetric tensor
T_{ab}	World-sheet energy momentum tensor
Ω	World-sheet conformal factor
γ^a	Dirac matrices on world-volume, on tangent space
γ^μ	Dirac matrices in spacetime
$\mathcal{R}$	World-sheet Ricci scalar
α_n^μ, $\tilde{\alpha}_n^\mu$	Right- and left-moving bosonic oscillator components
η_r^μ, $\tilde{\eta}_r^\mu$	Neveu-Schwarz oscillator components
ρ_n^μ, $\tilde{\rho}_n^\mu$	Ramond oscillator components
N, $\tilde{N}$	Right- and left-moving oscillator number
$\mathcal{H}$	Hamiltonian
θ^α	World-sheet superspace coordinate
$\mathcal{K}$	World-sheet superfield
$\mathcal{D}$	Superspace covariant derivative
J_a	World-sheet supercurrent
L_m, $\tilde{L}_m$	Right- and left-moving Virasoro operators
G_r	Neveu-Schwarz Virasoro operators
F_m	Ramond Virasoro operators
$[\ ]_{pb}$	Classical Poisson brackets
$SO(n)$	Special orthogonal group
E_8	Exceptional group
D	Spacetime dimension
G_N	Newton constant
κ	Supergravity coupling constant
x, y	Space-time coordinates
R	Ricci scalar, also radius of compact coordinates
$\mathbb{G}$	Type I string frame metric
${}_{i}g$	Type I Einstein frame metric
$\mathcal{G}$	Heterotic string frame metric
$\mathscr{g}$	Heterotic Einstein frame metric

G	Type IIA string frame metric
g	Type IIA Einstein frame metric
J	Type IIB string frame metric
j	Type IIB Einstein frame metric
$\phi^{(i)}$	Type I dilaton field
$\phi^{(h)}$	Heterotic dilaton field
$\phi^{(a)}$	Type IIA dilaton field
$\phi^{(b)}$	Type IIB dilaton field
$\phi_0^{(?)}$	Asymptotic value of dilaton
$B^{(h)}$	Heterotic Kalb-Ramond field
$H^{(h)}$	Heterotic NS-NS three-form field strength
$B^{(a)}$	Type IIA Kalb-Ramond field
$H^{(a)}$	Type IIA NS-NS three-form field strength
$B^{(b)}$	Type IIB Kalb-Ramond field
$H^{(b)}$	Type IIB NS-NS three-form field strength
$\mathbb{A}^{(1)}$	Type I Yang-Mills gauge field
$\mathbb{F}^{(2)}$	Type I Yang-Mills field strength
$\mathcal{A}^{(1)}$	Heterotic Yang-Mills gauge field
$\mathcal{F}^{(2)}$	Heterotic Yang-Mills field strength
$A^{(n)}$	Ramond-Ramond n-form potentials
$F^{(n)}$	Ramond-Ramond n-form field strengths
$\chi = A^{(0)}$	Ramond-Ramond type IIB scalar
$\Lambda_G^{(1)}$	Kaluza-Klein gauge field from metric
$\Lambda_{B^{(a)}}^{(1)}$	Kaluza-Klein gauge field from Kalb-Ramond field
$\Xi^{(2)}$	Field strengths of Kaluza-Klein gauge fields
e^{σ}	Kaluza-Klein scalar
$\tilde{G}$	Kaluza-Klein moduli
$\mathcal{V}^{(1)}$	$D = 5$ Hodge dual of Kalb-Ramond field
$V^{(2)}$	$D = 5$ Hodge dual of NS-NS field strength
$\psi^{(a)}$	Type IIA gravitino
$\psi^{(b)}$	Type IIB gravitino

$\lambda^{(a)}$	Type IIA dilatino
$\lambda^{(b)}$	Type IIB dilatino
Q	Supersymmetry generators
q^e, q^m	Electric, magnetic charge
$\mathcal{H}$	Harmonic functions in space transverse to p-branes
$\hat{d}$	World volume dimension of p-branes
$\tilde{d}$	World volume dimension "dual" to $\hat{d}$, $\tilde{d} = D - \hat{d} - 2$
$\hat{G}_{ab}$	Metric induced on Dp-brane world-volume
$\hat{B}_{ab}$	AS tensor field induced on Dp-brane world volume
Ω_H	Angular velocity of black hole horizon
χ^μ	Killing field normal to black hole horizon
ψ^μ	Axial Killing field
ξ^μ	Stationary Killing field
ϑ	Surface gravity of black hole
μ_i	Direction cosines
β	Mass parameter for black holes
$\mathscr{A}$	Area of event horizon of black hole
T	Thermodynamic temperature
λ	Generic coupling constant
$\mathcal{A}_n$	Area of unit n-sphere
$\mathscr{S}$	Cauchy surface
$\Theta_{\mu\nu}$	Lorentz transformation
κ^μ	Poincaré translation
$\mathcal{N}$, $\tilde{\mathcal{N}}$	Total D-brane string gas occupation number
$\mathfrak{N}_k$	Occupation number at momentum level k (right-moving)
$\tilde{\mathfrak{N}}_k$	Occupation number at momentum level k (left-moving)
$\mathcal{L}$	Generators of Lorentz transformations
h	Deviation of a metric from flat space
Ω^D	Line element of sphere S^D
e^a_μ	Vielbein
${\omega_\mu}^i{}_j$	Spin connection

$\Gamma^{\alpha}_{\mu\nu}$	Affine connection
∇	Covariant derivative (spacetime)
$\mathscr{V}$	Conformal group generator

A.2. Relation to supergravity conventions

In this section we give the relationship between the conventions used here and those commonly used by the supergravity community. For this purpose, we take the conventions of [56] as indicative of those in use by the supergravity community.

To begin, there are general differences such as the metric signature. As stated, here we use the "East coast" metric diag$(-,+,+,\ldots)$ whereas in supergravity the "West coast" metric diag$(+,-,-,\ldots)$ is still, unfortunately, popular. This change brings in a change of sign for each index contraction. There is also an overall sign difference of the action.

Another difference of a general nature concerns the definition of field strengths. In the notation of [56], the field strength of a rank-n antisymmetric tensor potential $C^{(n)}$ is given by

$$\begin{aligned} \partial C^{(n)} &= \partial_{[\mu_1} C^{(n)}_{\mu_2\cdots\mu_{n+1}]} \\ &= \frac{1}{n+1}\left(\partial_{\mu_1} C^{(n)}_{\mu_2\cdots\mu_{n+1}} + \text{cyclic}\right). \end{aligned} \tag{A.2.1}$$

When written in the notation of forms, as used here, a factor of $\frac{1}{n!}$ is absorbed into the definition of the form, thus giving

$$d\tilde{C}^{(n)} = \partial_{\mu_1} \tilde{C}^{(n)}_{\mu_2\cdots\mu_{n+1}} + \text{cyclic}. \tag{A.2.2}$$

Similar consideration will apply to antisymmetric products, i.e., in the notation of [56] we write the product of a rank-n antisymmetric tensor with another of rank m as

$$\begin{aligned} C^{(n)}C^{(m)} &= C^{(n)}_{[\mu_1\cdots\mu_n} C^{(m)}_{\mu_{n+1}\cdots\mu_{n+m}]} \\ &= \frac{n!m!}{(n+m)!}\left(C^{(n)}_{\mu_1\cdots\mu_n} C^{(m)}_{\mu_{n+1}\cdots\mu_{n+m}} + \text{cyclic}\right) \end{aligned}$$

whereas with the form notation used here we have

$$\tilde{C}^{(n)} \wedge \tilde{C}^{(m)} = C^{(n)}_{\mu_1\cdots\mu_n} C^{(m)}_{\mu_{n+1}\cdots\mu_{n+m}} + \text{cyclic}.$$

In the above note that "cyclic" *includes* considerations of the change in sign due to the number of permutations between canonical order and the cyclic order in question.

With such generalities aside, we present the mapping between our fields and those of [56]. Here "Superstring" refers to our notation, while "Supergravity" refers to that of [56]. All of the following refer to ten dimensional fields.

Superstring	**Supergravity**	**Description**
G	$\hat{g}$	string frame metric†
ϕ	$\hat{\phi}$	dilaton
B	$\hat{B}^{(1)}$	Kalb-Ramond field
H	$3\hat{\mathcal{H}}^{(1)}$	NS-NS field strength
χ	$\hat{\ell}$	RR scalar
$A^{(1)}$	$-\hat{A}^{(1)}$	RR one-form potential
$F^{(2)}$	$\hat{F}^{(1)}$	RR two-form field strength
$A^{(2)}$	$\hat{B}^{(2)}$	RR two-form potential
$F^{(3)}$	$3\hat{\mathcal{H}}^{(2)}$	RR three-form field strength
$A^{(3)}$	$\frac{3}{2}\hat{C}$	RR three-form potential
$F^{(4)}$	$6\hat{G}$	RR four-form field strength
$A^{(4)}$	$4\hat{B}$	RR four-form potential
$F^{(5)}$	$20\hat{F}$	RR five-form field strength

Note that the unconventional normalization of the five-form field strength $F^{(5)}$, as mentioned in chapter II is duly reflected in the last line of the above.

† Thus there is no change in normalization, only the change in metric signature mentioned previously.

Appendix B.
Some useful mathematical tools

Here we group together some mathematical relations which are useful when deriving equations of motion for string theories, and of course in gravitation theory.

B.1. Calculus of variations

The calculation of the supergravity equations of motion is made simpler with the use of

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu} = -\frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\delta g_{\mu\nu} \qquad (B.1.1a)$$

$$\delta(g^{\mu\nu}) = -\,g^{\mu\alpha}g^{\mu\beta}\delta g_{\alpha\beta} \qquad (B.1.1b)$$

$$\delta\Gamma^{\mu}{}_{\nu\rho} = \frac{1}{2}\left(\nabla_\rho\delta g_\nu{}^\mu + \nabla_\nu\delta g_\rho{}^\mu - \nabla^\mu\delta g_{\nu\rho}\right) \qquad (B.1.1c)$$

$$\delta R_{\mu\nu\rho}{}^{\sigma} = \nabla_{[\mu}\nabla^{\sigma}\delta g_{\nu]\rho} - \nabla_{[\mu|}\nabla_\rho\delta g_{|\nu]}{}^{\sigma} \qquad (B.1.1d)$$

$$\delta R = \nabla_\mu\nabla_\nu\delta g^{\mu\nu} - \nabla_\mu\nabla^\mu\delta g_\nu{}^\nu - R_{\mu\nu}\delta g^{\mu\nu} \qquad (B.1.1e)$$

Where square brackets indicate antisymmetrization, that is

$$\nabla_{[\mu}V_{\nu]} = \frac{1}{2}(\nabla_\mu V_\nu - \nabla_\nu V_\mu) \qquad (B.1.2)$$

and the vertical bars | | indicate that the indicies they enclose are to be excluded from the antisymmetrization.

In varying an action, we need to remove the derivatives of the variations. Doing this, we perform integrations by parts and we can use the following relations:

$$\int d^Dx\sqrt{-g}F(\phi)\nabla^2(g_{\mu\nu}\delta g^{\mu\nu}) = \int d^Dx\sqrt{-g}g_{\mu\nu}(\nabla^2F(\phi))\delta g^{\mu\nu} \qquad (B.1.3a)$$

$$+\int d^Dx\sqrt{-g}\nabla_\rho\left[F(\phi)g_{\mu\nu}\nabla^\rho\delta g^{\mu\nu} - g_{\mu\nu}\delta g^{\mu\nu}\nabla^\rho F(\phi)\right] \qquad (B.1.3b)$$

$$\int d^Dx\sqrt{-g}F(\phi)\nabla_\mu\nabla_\nu\delta g^{\mu\nu} = \int d^Dx\sqrt{-g}\left[\partial_\mu\partial_\nu F(\phi) - \Gamma^\rho{}_{\mu\nu}\partial_\rho F(\phi)\right]\delta g^{\mu\nu}$$

$$+\int d^Dx\sqrt{-g}\nabla_\mu\left[F(\phi)\nabla_\nu\delta g^{\mu\nu} - \nabla_\nu F(\phi)\delta g^{\mu\nu}\right] \qquad (B.1.3c)$$

where we have kept the boundary terms as total derivatives.

B.2. Conformally related spacetimes

For conformally related spacetimes $g_{\mu\nu} = e^{\alpha\phi}G_{\mu\nu}$, where we denote the covariant derivative with respect to the metric G by ∇ and that with respect to the conformally related metric g by $\hat{\nabla}$ we have,

$$\begin{aligned} R_{\mu\nu}(g) = R_{\mu\nu}(G) &- \frac{\alpha}{2}(D-2)\nabla_\mu\nabla_\nu\phi - \frac{\alpha}{2}G_{\mu\nu}\nabla^2\phi \\ &+ \frac{\alpha^2}{4}(D-2)\nabla_\mu\phi\nabla_\nu\phi - \frac{\alpha^2}{4}(D-2)G_{\mu\nu}(\nabla\phi)^2 \qquad (B.2.1a) \\ R(g) = e^{-\alpha\phi}\Big\{ &R(G) - \alpha(D-1)\nabla^2\phi \\ &- \frac{\alpha^2}{4}(D-1)(D-2)(\nabla\phi)^2\Big\} \qquad (B.2.1b) \\ (\hat{\nabla}\phi)^2 = e^{-\alpha\phi}&(\nabla\phi) \qquad (B.2.1c) \end{aligned}$$

where D is the dimension of the spacetime.

For superstring theories in D dimensions, the conformal transformation from the string frame to the Einstein frame is

$$g_{\mu\nu} = \mathrm{e}^{-4\phi/(D-2)}\, G_{\mu\nu}. \qquad (B.2.2)$$

B.3. Miscellaneous useful formulas

Here we gather a few formulas which are sometimes useful in the course of computations.

$$\Gamma^\mu_{\nu\mu} = \partial_\nu(\log\sqrt{-g}) \qquad (B.3.1a)$$

$$R_{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\alpha(\sqrt{-g}\Gamma^\alpha_{\mu\nu}) - \partial_\mu\partial_\nu(\log\sqrt{-g}) - \Gamma^\alpha_{\beta\mu}\Gamma^\beta_{\nu\alpha} \qquad (B.3.1b)$$

$$\nabla_\mu A^{(1)\,\mu} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}A^{(1)\,\mu}) \qquad (B.3.1c)$$

$$\nabla_\mu F^{(2)\,\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}F^{(2)\,\mu\nu}) \qquad (B.3.1d)$$

Appendix C.
Penrose diagrams

The most fundamental thing concerning two separated space-time points is their causal relation. Is point a inside, outside or exactly on the future or past light cone of point b. When it comes to discussing black holes, causal structure becomes rather subtle and particularly important. *Penrose diagrams* [51,96,113,159] are a very useful aid to the graphical depiction of the causal structure of spacetime. Here we introduce the ideas involved.

We begin with Minkowski space in spherical polar coordinates. The line element is written

$$\begin{aligned} ds^2 &= -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &= -dt^2 + dr^2 + r^2 d\Omega_2^2 \end{aligned} \tag{C.1}$$

and therefore at each point (r, t) where $-\infty \leq t \leq \infty$ and $0 \leq r \leq \infty$ there is a two-sphere S^2 of area $4\pi r^2$. We then introduce *light-cone* coordinates

$$u = t - r \qquad v = t + r \tag{C.2}$$

in terms of which the line element is written

$$ds^2 = -du\,dv + \frac{1}{4}(v-u)^2 d\Omega_2^2 \tag{C.3}$$

The relation between the original coordinates r and t and the light cone coordintes in various asymptotic regions of spacetime is needed to create Penrose diagrams. These relationships are expressed in Fig. C.1. Indicated are

- $i^+ = \{t \to \infty, r \text{ fixed }\}$ = future timelike infinity,
- $i^- = \{t \to -\infty, r \text{ fixed }\}$ = past timelike infinity,
- $i^0 = \{r \to \infty, t \text{ fixed }\}$ = spacelike infinity,
- $I^+ = \{v \to \infty, u \text{ fixed }\}$ = future null infinity,
- $I^- = \{u \to -\infty, v \text{ fixed }\}$ = past null infinity.

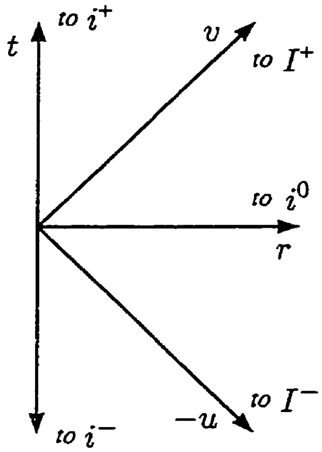


Figure C.1: Relation between Minkowski spherical polar coordinates and the light cone coordinates.

Past and future null infinity (along the light cone) are very useful concepts. For example the ADM formula used to compute the mass of p-branes in chapter III is based on the deviation of the metric from flat space as large distances from the object. In the case of an object which is emitting gravitational radiation, if a pulse is emitted at time t, then we must wait a period of time $t \geq r$ for the pulse to pass before we can make a measurement of the object after the pulse. As $r \to \infty$ we make such measurements at I^+.

However, in the light cone coordinates I^+ is at an infinite value of v. We would, however, like to be able to draw a diagram on a finite sheet of paper. We therefore introduce a conformal factor ω^2 given by

$$\omega^2 = \frac{4}{(1+v^2)(1+u^2)} \tag{C.4}$$

and coordinates ψ and ζ related to the lightcone coordinates by the transformations

$$\begin{aligned} \psi &= \tan^{-1} v + \tan^{-1} u \\ \zeta &= \tan^{-1} v - \tan^{-1} u \end{aligned} \tag{C.5}$$

and in these coordinates the Minkowski line element takes on the appearance

$$ds^2 = -d\psi^2 + d\zeta^2 + \sin^2 \zeta d\Omega_2^2 \tag{C.6}$$

where the new coordinates ψ and ζ then range over the half-diamond $\zeta \pm \psi < \pi$, $\zeta > 0$.

$$\begin{aligned} -\pi &< \psi + \zeta < \pi \\ -\pi &< \psi - \zeta < \pi \qquad (C.7) \\ 0 &\leq \zeta \end{aligned}$$

Equation (C.6) is simply the natural Lorentz metric in spherical polar coordinates $S^3 \otimes \mathbb{R}$. We can think of it as an unphysical pseudometric $\bar{g}_{\mu\nu}$ which is related to the physical metric by

$$\bar{g}_{\mu\nu} = \omega^2 g_{\mu\nu} \qquad (C.8)$$

where ω^2, the conformal factor, may or may not be infinite. The fact that $\bar{g}_{\mu\nu}$ is conformal to the physical metric means that the causal relation between two points is the same in both of the metrics, due to the angle-preserving nature of a conformal transformation. The pseudometric is finite at values of ψ, ζ that correspond to the asymptotic regions of Minkowski space, thus the asymptotic points are mapped to finite ones.

Further, statements about the asymptotic behavior of the physical metric can be translated into statements about the behavior of the pseudometric at the points i^0, etc., as long as the physical metric under discussion is asymptotically flat. The conditions for asymptotic flatness of a curved space-time are complicated [4,51] but they essentially mean that we can perform the conformal mapping of infinity to a finite point as was just done.

A Penrose diagram of Minkowski space is given in Fig. C.2. The upper and lower triangles represent the future and past light cones, each point is actually a two-sphere.

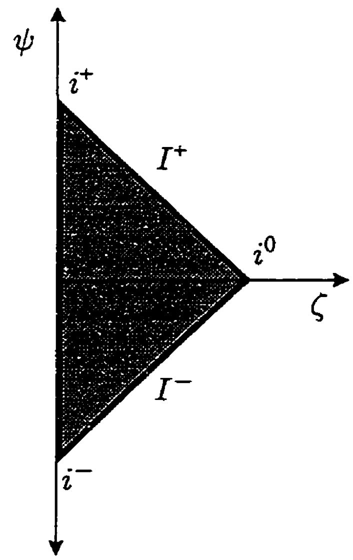


Figure C.2: Minkowski space represented as a Penrose diagram

C.1. A black hole example

Let us move on to a more complicated example, that of the Schwarzschild black hole. The Penrose diagram for the maximal analytic extension of the Schwarzschild metric was given in Fig. 5.1.2 and is reproduced in Fig. C.3 for the convenience of the reader. What is meant by *maximal analytic extension*? Our discussion follows that of [113]. Consider the Schwarzschild metric, which we write (putting $G_N = 1$ in this section)

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \qquad (C.9)$$

where M is the gravitational mass as measured from infinity, or ADM mass. It can be shown that any spherically symmetric solution of the vacuum Einstein equation is locally isometric to the Schwarzschild metric. One normally regards the Schwarzschild metric as being the solution outside some spherical body of radius $r_0 > 2M$, while the metric inside the body has a different form determined by the energy-momentum tensor of the matter of which the body is composed.

The Schwarzschild metric is singular when $r = 0$, and when $r = 2M$ (as well as possessing the standard trivial singularities of spherical polar coordinates at $\theta = 0$

and $\theta = \pi$). One must therefore cut $r = 0$ and $r = 2M$ out of the manifold, which divides the spacetime into two disconnected regions, defined by $0 < r < 2M$ and $2M < r < \infty$. If we wish our spacetime to be connected, i.e., to be able to go everywhere, in principle, in the spacetime in starting from anywhere, then we must choose either one, or the other, but not both. The obvious choice is to take the region with $r > 2M$. Thus, $(C.9)$ corresponds to the region I of Fig. C.3.

The question posed is then "is this manifold with Schwarzschild metric $(C.9)$ for $r > 2M$ *extendible*?" That is, can we embed this manifold into a larger manifold with a new metric which coincides with $(C.9)$ for $r > 2M$? The obvious place to look for such an extension is where $r \to 2M$. To make the extension, consider defining a new coordinate $\hat{r}$ as

$$\hat{r} \equiv \int \frac{dr}{1 - \frac{2M}{r}} = r + 2M \log(r - 2M). \qquad (C.10)$$

Then we take our light-cone coordinates in terms of $\hat{r}$,

$$u = t - \hat{r}, \qquad v = t + \hat{r}$$

and we can write the metric $(C.9)$ in the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dv\,dr + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \qquad (C.11)$$

This new metric is non-singular on the larger manifold for which $0 < r < \infty$, and when $2M < r < \infty$ it is isometric to the $2M < r < \infty$ region of the original Schwarzschild metric.

There is, however, the feature of $(C.11)$ is not time symmetric. The surface $r = 2M$, upon which $t \to \infty$ acts as a one-way membrane. Future-directed timelike and null curves can cross the surface $r = 2M$ only from the outside ($r > 2M$) to the inside ($r < 2M$). Past-directed timelike or null curves in the outside region cannot cross into the inside. Also, the future directed timelike or null curve which crosses the surface at $r = 2M$ approaches $r = 0$ within a finite affine distance. Thus the extension to $(C.11)$ is represented by regions I and III in Fig. C.3.

If one uses the coordinate u rather than v, then the metric takes the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) du^2 - 2dudr + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \qquad (C.12)$$

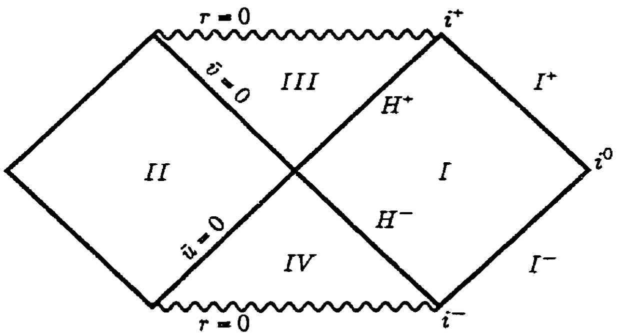


Figure C.3: Penrose diagram of the maximal analytic extension of the Schwarzschild black hole.

which is again non-singular for $0 < r < \infty$ and isometric to ($C.9$) in the region $r > 2M$. For this extension, however, the direction of time is reversed. In this case only past-directed timelike or null curves may pass through the $r = 2M$ surface from the outside to the inside. This extension is represented by regions I and IV of Fig. C.3.

It is possible to make both extensions simultaneously. One can find a still larger manifold in which to embed the manifolds defined by ($C.12$) and ($C.11$) such that they coincide with each other and with ($C.9$) in the region $r > 2M$. The construction of this manifold was carried out by Kruskal [160] and in terms of both light-cone coordinates the metric takes the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dudv + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \qquad (C.13)$$

where r is given by

$$r + 2M\log(r - 2M) = \frac{1}{2}(v - u). \qquad (C.14)$$

It is possible to go further still. If we apply the most general coordinate transformation under which the Kruskal metric ($C.13$) retains its form, we find

$$ds^2 = -\left(1 - \frac{2M}{r}\right) \frac{du}{du'}\frac{dv}{dv'} du'dv' + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \qquad (C.15)$$

where $u' = u'(u)$ and $v' = v'(v)$ are arbitrary differentiable functions. If we choose u' and v' as

$$x = \frac{1}{2}(v' - u'), \qquad \tau = \frac{1}{2}(u' + v') \qquad (C.16)$$

then the metric has the final form

$$ds^2 = K^2(\tau, x)(-d\tau^2 + dx^2) + r^2(\tau, x)(d\theta^2 + \sin^2\theta d\varphi^2) \qquad (C.17)$$

If we then choose

$$u' = -\,\mathrm{e}^{-\frac{u}{4M}}, \qquad v' = \mathrm{e}^{\frac{v}{4M}} \qquad (C.18)$$

then r is determined implicitly by

$$\tau^2 - x^2 = (2M - r)\,\mathrm{e}^{\frac{r}{2M}} \qquad (C.19)$$

and K is found to be

$$K^2 = \frac{16M^2}{r}\,\mathrm{e}^{\frac{-r}{2M}}\,. \qquad (C.20)$$

Here regions I through IV refer to Fig. C.3. The metric $(C.17)$ for $x > |\tau|$ is represented by region I, isometric to the Schwarzschild metric for $r > 2M$. The region defined by $x > -\tau$ is represented by regions I and III, and is isometric to the extension $(C.11)$. Similary the region defined by $x > \tau$ is isometric to $(C.12)$, corresponding to regions I and IV. There is yet another region, that defined by $x < -|\tau|$, which is represented by region II. This is again isometric with the exterior Schwarzschild metric ($r > 2M$), and can be regarded as another asymptotically flat universe lying on the far side of the Schwarzschild "throat" (see figure Fig. 5.1.1 of chapter V).

There are no timelike or null curves which travel from region I to region II. All such curves which cross the surface $r = 2M$ approach the singularity at $r = 0$ where they terminate. Thus we have found the maximal analytic extension of the Schwarzschild metric. It is clear that the same sorts of analyses can be made in the case of other exact solutions of both the Einstein equation and the string theory equations of motion.

References

1. M.J. Duff, Int. J. Mod. Phys. **A11** (1996) 5623 [hep-th/9608117].
2. J.H. Schwarz, Nucl. Phys. Proc. Suppl. **55B** (1997) 1 [hep-th/9607201].
3. J. Polchinski, Rev. Mod. Phys. **68** (1996) 1245 [hep-th/9607050].
4. C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation*, W.H. Freeman, San Francisco, 1973.
5. S. Chandrasekhar, *The Mathematical Theory of Black Holes*, Clarendon Press, Oxford, 1983.
6. J.D. Bekenstein, Phys. Rev. **D7** (1973) 2333.
7. S.W. Hawking, Commmun. Math. Phys. **43** (1975) 199.
8. G.T. Horowitz, D. Marolf, Phys. Rev. **D52** (1995) 5670 [gr-qc/9504028]; G.W. Gibbons and G.T. Horowitz, Class. Quant. Grav. **12** (1995) 297 [hep-th/9410073].
9. M. Green, J.H. Schwarz, and E.Witten, *Superstring Theory (Two Volumes)*, Cambridge University Press, Cambridge, 1986.
10. A. Strominger and C. Vafa, Phys. Lett. **B379** (1996) 99 [hep-th/9601029].
11. G.T. Horowitz and A. Strominger, Phys. Rev. Lett. **77** (1996) 2368 [hep-th/9602051];
    G.T. Horowitz, J.M. Maldacena and A. Strominger, Phys. Lett. **B383** (1996) 151 [hep-th/9603109].
12. J.C. Breckenridge, R.C. Myers, A.W. Peet and C. Vafa, Phys. Lett. **B391** (1997) 93 [hep-th/9602065].
13. J.C. Breckenridge, D.A. Lowe, R.C. Myers, A.W. Peet, A. Strominger and C. Vafa, Phys. Lett. **B381** (1996) 423 [hep-th/9603078].
14. J. Polchinski, Phys. Rev. Lett. **75** (1995) 4724 [hep-th/9510017].
15. C.V. Johnson, Nucl. Phys. Proc. Suppl. **52A** (1997) 326 [hep-th/9606196].
16. J.C. Breckenridge, R.R. Khuri and R.C. Myers, in preparation.

17. J.C. Breckenridge, G. Michaud, and R.C. Myers, Phys. Rev. **D55** (1997) 6438 [hep-th/9611174].
18. J.C. Breckenridge, G. Michaud, and R.C. Myers, *New angles on D-branes,* to appear in Physical Review D, e-print hep-th/9703041.
19. L.H. Ryder *Quantum Field Theory,* Cambridge University Press, Cambridge, 1985;
    C. Itzykson and J. Zuber *Quantum Field Theory,* McGraw-Hill, New York, 1980;
    T. Cheng and L. Li, *Gauge theory of elementary particle physics,* Oxford University Press, Oxford, 1991.
20. J.F. Donoghue, E. Golowich and B.R. Holstein, *Dynamics of the Standard Model,* Cambridge University. Press, Cambridge, 1992;
    C.P. Burgess, *A primer on the standard model of particle physics,* (Unpublished).
21. R.N. Cahn and G. Goldhaber, *The Experimental Foundations of Particle Physics,* Cambridge University Press, Cambridge, 1989;
    W. de Boer, Prog. Part. Nucl. Phys. **33** (1994) 201 [hep-ph/9402266].
22. M. Kaku and K. Kikkawa, Phys. Rev. **D10** (1974) 1110;
    M. Kaku and K. Kikkawa, Phys. Rev. **D10** (1974) 1823.
23. M. Kaku *Introduction to Superstrings,* Springer-Verlag, New York, 1988.
24. D. Lust and S. Theisen, *Lectures on String Theory,* Springer-Verlag, New York, 1989.
25. D. Bailin and A. Love, *Supersymmetric Gauge Field Theory and String Theory,* Institute of Physics Publishing, Philadelphia, 1994.
26. J. Polchinski, *What is string theory?,* lectures at the Les Houches Summer School 1994, e-print hep-th/9411028.
27. H. Ooguri and Z. Yin, *TASI lectures on perturbative string theories,* lectures at TASI-96, Boulder, Colorado, 1996 e-print hep-th/9612254.
28. A.M. Polyakov, *Gauge fields and strings,* Harwood Academic, 1987.
29. Y. Nambu, *Lectures at the Copenhagen Conference,* 1970;

T. Goto, Prog. Theor. Phys. **46** (1971) 1560.

30. M. Duff, *Supermembranes,* lectures at the TASI-96, Boulder, Colorado, e-print hep-th/9611203..
31. P. Di Francesco, P. Mathieu and D. Sénégal, *Conformal Field Theory,* Springer-Verlag, New York, 1996.
32. P.K. Townsend, *Four Lectures on M-Theory,* lectures given at ICTP Summer School 1996, Trieste, Italy, e-print hep-th/9612121.
33. L.D. Fadeev and V.N. Popov, Phys. Lett. **B25** (1967) 29.
34. S. Coleman and J. Mandula, Phys. Rev. **159** (1967) 1251.
35. J. Wess and J. Bagger, *Supersymmetry and Supergravity, Second Edition*, Princeton University Press, Princeton,1992.
36. J. Lykken, *Introduction to Supersymmetry,* lectures given at TASI-96, Boulder, Colorado, e-print hep-th/9612114.
37. P. West, *Introduction to Supersymmetry and Supergravity,* World Scientific, London, 1990.
38. P. Fayet and S. Ferrara, Phys. Rep. **32** (1977) 249.
39. M.F. Sohnius, Phys. Rep. **128** (1985) 39.
40. F. Gliozzi, J. Scherk and D. Olive, Phys. Lett. **B75** (1976) 282;
    F. Gliozzi, J. Scherk and D. Olive, Phys. Lett. **B122** (1977) 253.
41. A. Salam and J. Strathdee Nucl. Phys. **B76** (1974) 477.
42. P. Ramond and J.H. Schwarz Phys. Lett. **B64** (1976) 75.
43. L.J. Dixon, Nucl. Phys. **B261** (1985) 678;
    L.J. Dixon, Nucl. Phys. **B274** (1986) 285.
44. B.R. Greene, *String theory on Calabi-Yau manifolds,* lectures given at TASI-96, Boulder, Colorado, 1996, e-print hep-th/9702155.
45. D.J. Gross, J.A. Harvey, E. Martinec and R. Rohm, Phys. Rev. Lett. **54** (1985) 502;
    D.J. Gross, J.A. Harvey, E. Martinec and R. Rohm, Nucl. Phys. **B256** (1985) 253;
    D.J. Gross, J.A. Harvey, E. Martinec and R. Rohm, Nucl. Phys. **B267** (1986) 75.

46. T. Kaluza, Sitz. Preuss. Akad. Wiss. **K1** (1921) 966;
O. Klein, Helv. Phys. Acta Suppl. **IV** (1956) 58.

47. T. Appelquist, A. Chodos, P.G.O. Freund, eds., *Modern Kaluza-Klein theories*, Addison-Wesley, Menlo Park, 1987;
A. Salam and J. Strathdee, Ann. Phys. **141** (1982) 316.

48. C. Callan, D. Friedan, E. Martinec and M. Perry, Nucl. Phys. **B262** (1985) 593.

49. S. Tanimura, Phys. Lett. **B340** (1994) 57 [hep-th/9408096];
L. Alvarez-Gaume, D.Z. Freedman, Commun. Math. Phys. **80** (1981) 443.

50. B.F. Schutz, *Geometrical Methods of Mathematical Physics*, Cambridge University Press, Cambridge, 1980.

51. R.M. Wald *General Relativity*, University of Chicago Press, Chicago, 1984.

52. O. Alvarez, Nucl. Phys. **B216** (1983) 125;
O. Alvarez, *Differential Geometry in String Models*, in *Workshop on Unified String Theories, 29 July - 16 August, 1985*, eds. M. Green and D. Gross, World Scientific, Singapore, 1986.

53. J.E. Paton and H.M. Chan, Nucl. Phys. **B10** (1969) 516.

54. N. Marcus and A. Sagnotti, Phys. Lett. **B119** (1982) 97;
M.B. Green and J.H. Schwarz Nucl. Phys. **B255** (1985) 93.

55. R.R. Khuri and T. Ortín, Nucl. Phys. **B467** (1996) 355.

56. E. Bergshoeff, C. Hull and T. Ortin, Nucl. Phys. **B451** (1995) 547 [hep-th/9504081].

57. E. Bergshoeff, H.J. Boonstra and T. Ortín, Phys. Rev. **D53** (1996) 7206 [hep-th/9508091].

58. A. Giveon, M. Porrati and E. Rabinovici,Phys. Rep. **244** (1994) 77 [hep-th/9401139].

59. K.S. Narain, Nucl. Phys. Lett. **B169** (1986) 369.

60. K.S. Narain, M.H. Sharmadi and E. Witten, Nucl. Phys. **B289** (1987) 414.

61. J.P. Serre, *A Course in Arithmetic,* Springer-Verlag, New York, 1973.
62. P. Ginsparg and C. Vafa, Nucl. Phys. **B289** (1987) 414.
63. L.J. Garay, Int. J. Mod. Phys. **A10** (1995) 145 [gr-qc/9403008];
M.B. Green, Nucl. Phys. Proc. Suppl. **49** (1996) 123 [hep-th/9602061].
64. T. Kugo and B. Zwiebach, Prog. Theor. Phys. **87** (1992) 801.
65. J. Polchinski, *TASI Lectures on D-branes,* lectures at TASI-96, Boulder, Colorado, e-print hep-th/9611050.
66. J. Dai, R.G. Leigh, J. Polchinski, Mod. Phys. Lett. **A4** (1989) 2073.
67. R.R. Khuri, *Solitons, Black Holes and Duality in String Theory,* e-print hep-th/9704110.
68. C.V. Johnson, N. Kaloper, R.R. Khuri and R.C. Myers, Phys. Lett. **B368** (1996) 71 [hep-th/9509070].
69. P.A.M. Dirac Proc. Roy. Soc. **A133** (1931) 60.
70. G. 't Hooft Nucl. Phys. **B79** (1974) 276;
A.M. Polyakov JETP Lett. **20** (1974) 194.
71. C. Montonen and D. Olive Phys. Lett. **B72** (1977) 117.
72. E. Calabi, "On Kähler manifolds with vanishing canonical class, algebraic geometry and topology," in *Algebraic Geometry and Topology: A Symposium in Honor of S. Lefschetz*, Princeton University Press, 1955.
73. S.T. Yau, Proc. Natl. Acad. Sci. **74** (1977) 1798.
74. C.M. Hull and P.K. Townsend, Nucl. Phys. **B438** (1995) 109 [hep-th/9410167];
M.J. Duff, Nucl. Phys. **B442** (1995) 47 [hep-th/9501030].
75. M.J. Duff and R.R. Khuri, Nucl. Phys. **B411** (1994) 473 [hep-th/9305142].
76. E. Witten, Nucl. Phys. **B443** (1995) 85 [hep-th/9503124].
77. A. Sen, Nucl. Phys. **B450** (1995) 103 [hep-th/9504027].
78. N. Seiberg, Nucl. Phys. **B303** (1988) 369.
79. A. Dabholkar, G. Gibbons, J. Harvey and F. Ruiz Nucl. Phys. **B340** (1990) 33.

80. J.A. Harvey and A. Strominger, Nucl. Phys. **B449** (1995) 535 [hep-th/9504047].

81. J. Polchinski, S. Chaudhuri and C.V. Johnson, *Notes on D-branes*, e-print hep-th/9602052.

82. K.S. Stelle, *Lectures on Supergravity p-branes*, lectures at ICTP 1996, Trieste, Italy, e-print hep-th/9701088;
E. Bergshoeff, *p-Branes, D-branes and M-branes*, e-print hep-th/9611099;
M.J. Duff and K.S. Stelle, Phys. Lett. **B253** (1991) 113.

83. M.J. Duff, R.R. Khuri and J.X. Lu, Phys. Rep. **259** (1995) 213 [hep-th/9412184].

84. R.R. Khuri, Phys. Lett. **B294** (1992) 325 [hep-th/9205052].

85. R.I. Nepomechie, Phys. Rev. **D31** (1985) 1921.

86. C. Teitelboim, Phys. Lett. **B167** (1986) 63.

87. C. Teitelboim, Phys. Lett. **B167** (1986) 69.

88. R.G. Leigh, Mod. Phys. Lett. **A4** (1989) 2767.

89. M.B. Green, C.M. Hull and P.K. Townsend, Phys. Lett. **B382** (1996) 65 [hep-th/9604119].

90. S. Coleman and E. Weinberg Phys. Rev. **D7** (1973) 1888.

91. E.T. Whittaker and G.N. Watson, *A Course of Modern Analysis*, Fourth Edition, Cambridge University Press, Cambridge, 1958.

92. C. Bachas, *(Half) A Lecture on D-branes*, e-print hep-th/9701019.

93. M. Veltman in *Methods in Field Theory*, eds. R. Balian and J. Zinn-Justin, North Holland, Amsterdam, 1975.

94. R.G. Leigh, Mod.Phys.Lett. **A4** (1989) 2767.

95. M.R. Garousi and R.M. Myers, Nucl. Phys. **B475** (1996) 193 [hep-th/9603194].

96. A. Strominger, *Les Houches Lectures on Black Holes*, lectures at the Les Houches summer school, 1994, e-print hep-th/9501071;
A. Strominger and S.P. Trivedi, Phys. Rev. **D48** (1993) 5778 [hep-th/9302080].

97. T. Banks, Nucl. Phys. Proc. Suppl. **41** (1995) 21 [hep-th/9412131].

98. D.N. Page Int. J. Mod. Phys. **D3** (1994) 93.

99. R. Penrose, Phys. Rev. Lett. **14** (1965) 57;
P.S. Jang and R.M. Wald. J. Math. Phys. **18** (1977) 41.

100. P. Yodis, H.-J. Seifert and H. Müller zum Hagen, Commun. Math. Phys. **34** (1973) 135;
P. Yodis, H.-J. Seifert and H. Müller zum Hagen, Commun. Math. Phys. **37** (1974) 29.

101. K. Schwarzschild, Sitzber. Deut. Akad. Wiss. Berlin, Kl. Math.-Phys. Tech. (1916) 189;
K. Schwarzschild, Sitzber. Deut. Akad. Wiss. Berlin, Kl. Math.-Phys. Tech. (1916) 424.

102. C.V. Vishveshwara, Phys. Rev. **D1** (1970) 2870.

103. R.C. Myers and M.J. Perry, Ann. Phys. **12** (1986) 304.

104. R. Reissner, Ann. Physik **50** (1916) 810;
G. Nordstrom, Proc. Kon. Ned. Akad. Wet. **20** (1918) 1238.

105. R.P. Kerr, Phys. Rev. Lett. **11** (1963) 237;
R.P. Kerr and A. Schild, "A New Class of Vacuum Solutions of the Einstein Field Equations," in G. Barbera, ed., *Proceedings of the Galileo Centenary Meeting on General Relativity, Problems of Energy and Gravitational Waves,* Comitato Nazionale per le Manifestazione Celebrative, Florence, 1965.

106. E.T. Newman, E. Couch, K. Chinnapared, A. Exton, A. Prakash, and R. Torrence J. Math. Phys. **6** (1965) 918.

107. R.H. Boyer and R.W. Lindquist, J. Math. Phys. **8** (1967) 265.

108. W. Israel, Phys. Rev. **164** (1967) 1776;
W. Israel, Commun. Math. Phys. **8** (1968) 245;
B. Carter, Phys. Rev. Lett. **26** (1971) 331.

109. S.W. Hawking, Phys. Rev. Lett. **26** (1971) 1344;
S.W. Hawking, Commun. Math. Phys. **25** (1972) 152;
D.C. Robinson, Phys. Rev. Lett. **34** (1975) 905.

110. R. Penrose, Rev. del Nuovo Cimento **1** (1969) 252.

111. G.W. Gibbons, S.W. Hawking, G.T. Horowitz and M.J. Perry, Commun. Math. Phys. **88** (1983) 295.

112. A.H. Taub, Ann. Math. **53** (1951) 472;
E.T. Newman, L. Tamburino and T.Unti, J. Math. Phys. **4** (1963) 915.

113. S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Space-time*, Cambridge University Press, Cambridge, 1973.

114. D.N. Page, Phys. Lett. **B78** (1978) 249.

115. J.M. Bardeen, B. Carter and S.W. Hawking, Commun. Math. Phys. **31** (1973) 161.

116. R.M. Wald, Ann. Phys. **82** (1974) 548.

117. R.M. Wald, *The 'Nernst Theorem' and Black Hole Thermodynamics,* e-print gr-qc/9704008.

118. G.T. Horowitz, *The dark side of string theory,* Proceedings of String Theory and Quantum Gravity 1992, Trieste, Italy, e-print hep-th/9210119.

119. G. Gibbons, Nucl. Phys. **B207** (1982) 337.

120. S.F. Hassan and A. Sen Nucl. Phys. **B375** (1992) 103 [hep-th/9109038].

121. G. Aldazabal, A. Font, L.E. Ibanez and F. Quevedo, Phys. Lett. **B380** (1996) 33 [hep-th/9602097].

122. H.J. Sheinblatt, *Statistical entropy of an extremal black hole with 0- and 6-brane charge,* e-print hep-th/9705054.

123. P. Horava, Phys. Lett. **B231** (1989) 251.

124. E. Witten, Nucl. Phys. **B460** (1996) 335 [hep-th/9510135].

125. A. Sen, Phys. Rev. **D54** (1996) 2964 [hep-th/9510229];
A. Sen, Phys. Rev. **D53** (1996) 2874 [hep-th/9511026].

126. C. Vafa, Nucl. Phys. **B463** (1996) 415 [hep-th/9511088].

127. M. Bershadsky, V. Sadov and C. Vafa, Nucl. Phys. **B463** (1996) 420 [hep-th/9511222].

128. C. Vafa, Nucl. Phys. **B463** (1996) 435 [hep-th/9512078].

129. C. Callan and J. Maldacena, Nucl. Phys. **B472** (1996) 591 [hep-th/9602043].

130. C.V. Johnson, R.R Khuri and R.C. Myers, Phys. Lett. **B378** (1996) 78 [hep-th/9603061].

131. A. Sen, Int. J. Mod. Phys. **A9** (1994) 3707 [hep-th/9402002].

132. G.T. Horowitz and A. Sen, Phys. Rev. **D53** (1996) 808 [hep-th/9509108].

133. C. Vafa and E. Witten, Nucl. Phys. **B431** (1994) 3 [hep-th/9408074].

134. W. Boucher, D. Friedan and A. Kent, Phys. Lett. **B172** (1986) 316.

135. T.M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, 1976.

136. S. Ferrara, R. Kallosh and A. Strominger, Phys. Rev. **D52** (1995) 5412 [hep-th/9508072].

137. M. Cvetic and D. Youm, Phys. Rev. **D53** (1996) 584 [hep-th/9507090].

138. M. Cvetic and A. Tseytlin, Phys. Rev. **D53** (1996) 5619 [hep-th/9512031].

139. R. Kallosh, A. Linde, T. Ortin, A. Peet and A. van Proeyen, Phys. Rev. **D46** (1992) 5278 [hep-th/9205027].

140. F. Larsen and F. Wilczek, Phys. Lett. **B375** (1996) 37 [hep-th/9511064].

141. G.T. Horowitz and J. Polchinski, Phys. Rev. **D55** (1997) 6189 [hep-th/9612146].

142. M. Cvetic and D. Youm, Phys. Rev. Lett. **75** (1995) 4165 [hep-th/9503082].

143. A.A. Tseytlin, *Composite BPS Configurations of P-Branes in 10 and and 11 Dimensions*, e-print hep-th/9702163.

144. M. Berkooz, M.R. Douglas and R.G. Leigh, Nucl. Phys. **B480** (1996) 265 [hep-th/9606139].

145. J. Rahmfeld, Phys. Lett. **B372** (1996) 198 [hep-th/9512089]; M.J. Duff and J. Rahmfeld, Nucl. Phys. **B481** (1996) 332 [hep-th/9605085].

146. G.W. Gibbons, M.B. Green and M.J. Perry, Phys. Lett. **B370** (1996) 37 [hep-th/9511080].

147. E. Bergshoeff, M. de Roo, M.B. Green, G. Papadopoulos and P.K. Townsend, Nucl. Phys. **B470** (1996) 113 [hep-th/9601150];

E. Bergshoeff and M. de Roo, Phys. Lett. **B380** (1996) 265 [hep-th/9603123];
J. Borlaf and Y. Lozano, Nucl. Phys. **B480** (1996) 239 [hep-th/9607051];
E. Alvarez, J.L.F. Barbon and J. Borlaf, Nucl. Phys. **B479** (1996) 218 [hep-th/9603089].

148. R.G. Leigh, Mod. Phys. Lett. **A4** (1989) 2767.

149. M. Li, Nucl. Phys. **B460** (1996) 351 [hep-th/9510161];
M.R. Douglas, *Branes within Branes,* e-print hep-th/9512077.

150. H. Lu, C.N. Pope, and K.S. Stelle, Nucl. Phys. **B481** (1996) 313 [hep-th/9605082];
R.C. Myers, Phys. Rev. **D35** (1987) 455.

151. R.R. Khuri and R.C. Myers, Nucl. Phys. **B466** (1996) 60 [hep-th/9512061].

152. J.G. Russo and A.A. Tseytlin, Nucl. Phys. **B490** (1997) 121 [hep-th/9611047].

153. M.B. Green, N.D. Lambert, G. Papadopoulos and P.K. Townsend, Phys. Lett. **B384** (1996) 86 [hep-th/9605146].

154. G. Lifschytz, *Probing Bound States of D-Branes,* e-print hep-th/9610125.

155. M. Garousi and R.C. Myers, in preparation.

156. A.A. Tseytlin, Nucl. Phys. **B475** (1996) 149 [hep-th/9604035];
J.P. Gauntlett, D.A. Kastor, J. Traschen, Nucl. Phys. **B478** (1996) 544 [hep-th/9604179].

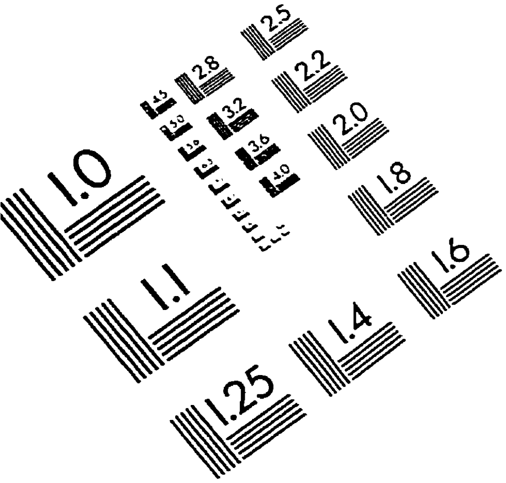
157. D.Z Freedman and G.W. Gibbons, Nucl. Phys. **B233** (1984) 24;
C.M. Hull, Nucl. Phys. **B239** (1984) 541.

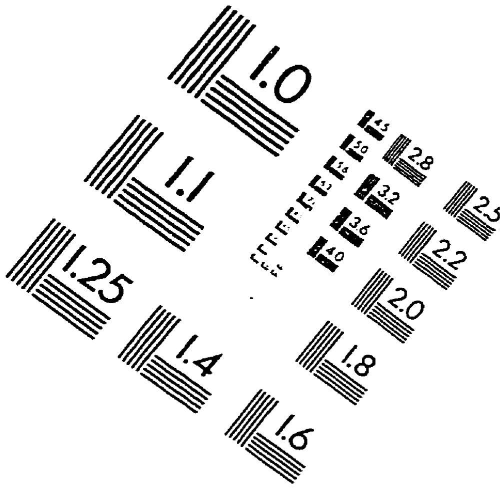
158. H. Lu and C. Pope, Nucl. Phys. **B465** (1996) 127 [hep-th/9512012].

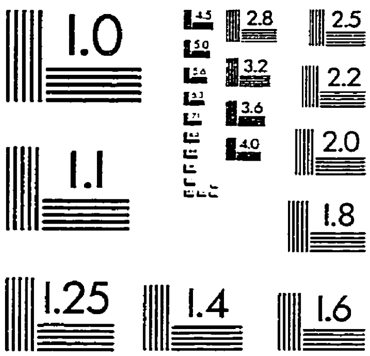
159. R. Penrose, *Conformal treatment of infinity,* in C. DeWitt and B.S. DeWitt, eds., *Relativity, Groups and Topology,* Gordon and Breach, New York,1964.

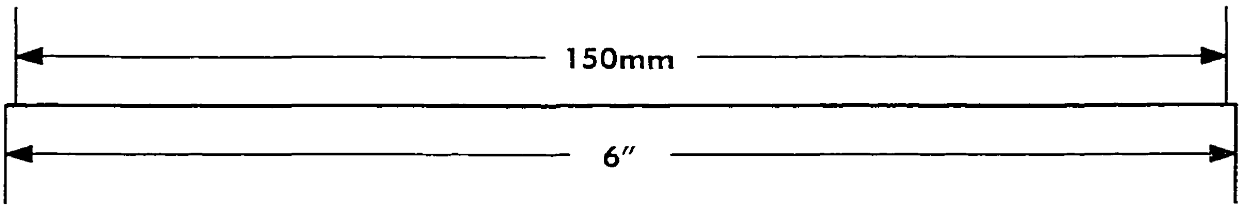
160. M.D. Kruskal, Phys. Rev. **119** (1960) 1743.

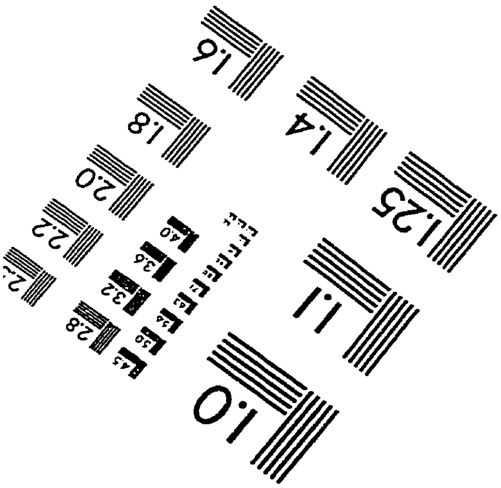
IMAGE EVALUATION TEST TARGET (QA-3)

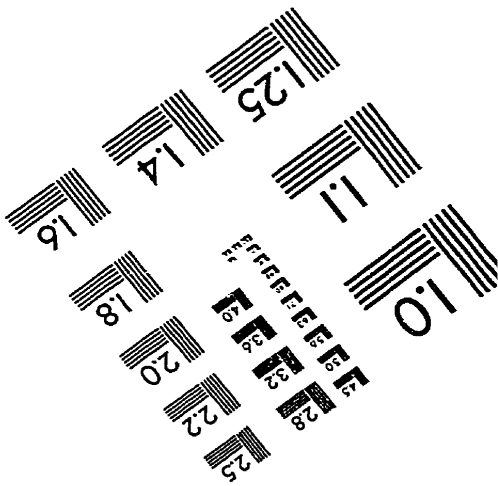












APPLIED IMAGE, Inc
1653 East Main Street
Rochester, NY 14609 USA
Phone: 716/482-0300
Fax: 716/288-5989