Intersection Theory on Surfaces

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Abstract

This thesis studies intersection theory on projective surfaces with isolated singularities. We review the classical intersection theory on a nonsingular surface, proceed to an overview of types of singularity that may arise, and then discuss the intersection theory of Snapper-Kleiman, that of Reeve-Tyrrell, and a modification of the latter that we propose.

The intersection theory of Snapper-Kleiman applies to varieties of any dimension but is restricted to locally principal divisors; that of Reeve-Tyrrell applies to arbitrary divisors but is restricted to surfaces. Our modification has the same domain of application as the theory of Reeve-Tyrrell but simplifies computations: it allows us to prove the theories are all equivalent on normal surfaces. We finish by developing generalizations of the main theorems on nonsingular surfaces.

Résumé

Le but de cette thèse est d'étudier la théorie d'intersection sur les surfaces projectives ayant des singularités isolées. Nous passons en revue la théorie classique d'intersection sur une surface non-singulière, ensuite nous verrons quelques types de singularités, puis nous discutons de la théorie d'intersection de Snapper-Kleiman, de celle de Reeve-Tyrrell, et d'une modification de cette dernière que nous suggerons.

La théorie d'intersection de Snapper-Kleiman s'applique aux variétés de dimensions quelconques mais elle est limitée aux diviseurs principaux. Celle de Reeve-Tyrrell s'applique aux diviseurs quelconques mais elle est limitée aux surfaces. Notre modification s'applique au même domaine que la théorie de Reeve-Tyrrell mais elle simplifie les calculs: elle nous permet de prouver que les trois théories sont équivalentes sur les surfaces normales. Pour finir, nous developpons les generalizations des théoremes principaux sur les surfaces non-singulières.

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Introduction

The basic question of intersection theory is "how many times do two curves intersect?" Classically, the question has a satisfactory answer once it is placed in the right setting. Let C and D be curves, of degrees m and n respectively, on the projective plane over an algebraically closed field. Then we have Bezout's theorem: counting multiplicity, C and D intersect in mn points. It is natural to want to generalize this theorem.

The simplest generalization is to look at curves on a nonsingular surface X, that is, a nonsingular projective variety of dimension two over an algebraically closed field. In this case, which is the subject of Section 1.2, we obtain a version of Bezout's theorem that is somewhat more complicated. Instead of each curve having an integer degree, it has a value in a particular finitely generated free abelian group $\operatorname{Num} X$. The intersection pairing is then a nondegenerate pairing on $\operatorname{Num} X$ whose form is described by the Hodge index theorem. Once established, this intersection theory allows one to answer a variety of questions. It provides the Nakai-Moishezon criterion for finding ample divisors, it provides the adjunction formula for computing the genus of a curve on a surface, and it provides the Riemann-Roch theorem as a way of measuring the set of rational functions with specified poles and zeros.

Very general intersection theories exist, applying to varieties of n dimensions that may be singular. However, many complications appear in this general setting that are absent in the case of curves on surfaces. As a simple example, two distinct curves on a surface will intersect in a finite number of points. One can quite adequately describe this situation by simply giving the number of points. However, in higher dimensions, subschemes of codimension one will generally intersect in a variety of codimension two; to adequately describe this situation, one must have some information about the subscheme. We present a discussion of the intersection theory that appears when the underlying variety is projective space in Section 1.1. In the case when the underlying scheme is more general but nonsingular, one defines the Chow ring of formal linear combinations of subschemes of any codimension. The multiplication in this ring then takes the place of the intersection pairing, with the intersection of two subschemes being a formal linear combination of subschemes of various dimensions. The intersection theory is quite complicated; while versions of (for example) the Riemann-Roch theorem can be proven, their interpretation is not at all straightforward. When the underlying variety is singular, one no longer has a well-defined multiplication pairing of arbitrary subschemes, and one has even less algebraic structure to work with. On can still intersect arbitrary subschemes with a locally principal divisor, yielding operations on the Chow space. In this case the theory quite difficult.

As a result of this complexity, we will focus on surfaces. Moreover, while a surface may have singularities in codimension one, it is no longer practical to use the language of curves on such a surface. It is then necessary to restrict oneself to Cartier divisors. The theory of Section 2.5 does in fact apply on such a surface, but it no longer has a simple interpretation in terms of counting intersections of curves. We will focus on surfaces with isolated singularities, and in fact we will occasionally make additional restrictions on the nature of the singularities.

Unfortunately, the classical intersection theory on a nonsingular surface does not generalize in an obvious way to the case of singular surfaces, even when the singularities are isolated and normal. Several new phenomena arise when the surface may have singularities. The first is that divisors cannot necessarily be moved around by linear equivalence: on a nonsingular surface, given any divisor and a fixed finite set of points, we can always find a linearly equivalent divisor which avoids the given points. If the surface is singular, on the other hand, some divisors that pass through the singularity may fail to be locally principal. Such divisors are not linearly equivalent to any divisor which avoids the singularity. A second phenomenon which arises on singular surfaces is that some intersection numbers may be fractional. Two families of examples which will serve to illustrate these and other behaviors are presented in Sections 1.4.1 and 1.4.2.

Singular surfaces also arise naturally in a number of contexts. For example, Hilbert modular surfaces are constructed as a quotient of $\mathfrak{h}^2$ (where $\mathfrak{h}$ is the complex upper half-plane) by certain subgroups of $SL_2(L)$, where L is a real quadratic field. This quotient yields singular and non-projective surfaces. Compactifying them yields projective surfaces with further singularities. Curves on these surfaces parameterize certain families of abelian varieties, and the intersection numbers of these curves yield information about Hilbert modular forms. Resolutions of the singularities of Hilbert modular surfaces are known, but the theory is cumbersome; perhaps a more convenient algebraic theory of intersections on surfaces would be useful in the study of these objects.

Two different approaches have been taken to generalize intersection theory on surfaces to the situation of surfaces with isolated singularities. The first approach, taken in the paper by Reeve and Tyrrell ([**RT62**]), relies on the fact that singularities on surfaces can be resolved. For any surface X, we know that the normalization yields a unique normal surface mapping birationally to the original surface. This normal surface then has isolated singularities which can be resolved by a sequence of blow-ups and normalizations. The surface X^* thus obtained, called a *resolved model* of X, is not unique, but any two resolved models of X can be blown up repeatedly to yield a common model mapping birationally to both. The idea of [**RT62**] is to define an intersection theory on a surface in terms of intersections on a resolved model of the surface. We discuss this intersection theory in Section 2.3. If one can find a resolved model, this intersection theory is well-suited to computations; all intersection calculations are reduced to intersection calculations on the resolved model, which is a nonsingular surface. Such a resolved model is guaranteed to exist, and may be produced algorithmically by a suitable sequence of normalizations and blow-ups, all of which can be described in quite concrete terms.

A second approach to defining an intersection theory on a singular surface is cohomological. On a normal surface, locally principal divisors correspond to invertible sheaves, so intersection problems can be rewritten as problems about invertible sheaves. On a nonsingular surface, it turns out that the intersection number can be calculated from the Euler characteristics of several sheaves. Simply applying this definition to invertible sheaves on a singular surface yields an intersection number with appropriate properties. This is the approach of Snapper ([**Sna60**]), Kleiman ([**Kle66**]), and Bădescu ([**Băd01**]), which we discuss in Section 2.5. It proceeds by constructing a polynomial analogous to the Hilbert polynomial (as used in Section 1.1 to define intersection numbers) and then defining the intersection number in terms of a coefficient of this polynomial. Some of the necessary background in cohomology is reviewed in the Appendix. This intersection theory generalizes well to n dimensions, and is more commonly found in the literature than that of Reeve and Tyrrell. However, it is restricted to locally principal divisors. We will see in Proposition 3.3.1 that any divisor that is locally principal can be replaced with a linearly equivalent divisor that avoids the singularities, so in some sense this intersection theory evades the difficult questions. It is also not at all clear how to go about computing this intersection number, as it is not obtained from local quantities; one may need to calculate Euler characteristics of a number of sheaves to obtain the intersection number. Some results (such as Proposition 2.5.14) are available to ease computation of this intersection number.

For nonsingular surfaces, of course, both these approaches coincide with the classical intersection number. If a surface is normal and the divisors involved are locally principal, then both intersection theories are defined. Do they agree?

In Section 2.4, we develop a third intersection theory, based on work by Andreatta and Goren in **[AG02]**, valid on any surface with isolated singularities. It defines the intersection number by making reference to a resolved model, but it makes use of the fact that locally principal divisors correspond to invertible sheaves, and that we have a well-defined way to pull an invertible sheaf back from a surface to its resolved model. This theory can also be applied to divisors that are not locally principal. A relatively straightforward calculation shows that it agrees with the theory of **[RT62]**. A theorem about maps of surfaces and the intersection number of **[Sna60]** shows that for locally principal divisors on normal surfaces, all three definitions of intersection number agree.

Finally, now that we have a definition of intersection number on a surface with isolated singularities, in Chapter 3 we will generalize some of the theory of intersections on nonsingular surfaces. Versions of the adjunction formula and the Nakai-Moishezon criterion exist in the literature, and we discuss these. We also present some further results about numerical equivalence and a version of the Riemann-Roch theorem that corresponds quite closely to the Riemann-Roch theorem on a nonsingular surface.

As far as notation goes, we shall generally follow **[Har77]**, although we will need to introduce some notation of our own. An important exception is that when we use the words "curve" and "surface" we do not assume nonsingularity.

Throughout this thesis, $\Bbbk$ will denote an algebraically closed field. We will write n-dimensional affine space over $\Bbbk$ as $\mathbb{A}^n$, and we will denote n-dimensional projective space with coordinates $(x_0 : \cdots : x_n)$ by $\mathbb{P}^n_{(x_0 : \cdots : x_n)}$. If X is a scheme over $\Bbbk$, we will write the sheaf of differentials as $\Omega_{X/\Bbbk}$, and if X is n-dimensional, we will write $\wedge^n \Omega_{X/\Bbbk}$ as ω_X. If X is nonsingular, then ω_X is invertible and we can obtain a canonical divisor, which we will denote K_X. If I is an ideal in $\Bbbk[X_1, \ldots, X_n]$ we will write $Z(I)$ to mean the variety of common zeros of I and if M is a $\Bbbk[X_1, \ldots, X_n]$-module then M_I will denote the localization of M by the inverses of all the elements in I. If X is a scheme, the structure sheaf of X will be $\mathcal{O}_X$, the sheaf of (locally) invertible elements will be written $\mathcal{O}_X^*$, the sheaf of total quotient rings will be written $\mathcal{K}_X$, and the sheaf of invertible elements of this will be written $\mathcal{K}_X^*$. If $\mathcal{F}$ is a sheaf on X, then $\mathcal{F}_x$ will denote the stalk of $\mathcal{F}$ at x.

A curve will be an integral separated projective scheme of dimension 1 over $\Bbbk$. A surface will be an integral separated projective scheme of dimension 2 over $\Bbbk$ that is nonsingular in codimension 1. The field of rational functions on a surface X will be denoted $\mathscr{K}_X$ or just $\mathscr{K}$; if X and X^* are birational, then $\mathscr{K}_X$ is canonically isomorphic to $\mathscr{K}_{X^*}$ so we will generally identify them. We will also use $\mathscr{K}$ to denote the constant sheaf of total quotient rings on an integral scheme; if the scheme is not integral, we will write the sheaf of total quotient rings as $\mathcal{K}$.

Since we have several different intersection theories on a singular surface, we have different notation for each: $(C.D)_{\mathrm{RT}}$ denotes the intersection pairing of **[RT62]**, $(C.D)_*$ denotes the intersection pairing of Section 2.4, and $(C.D)_{\mathrm{Sn}}$ denotes the intersection pairing of **[Sna60]**. Once we have shown that they are equal, we will sometimes use $C.D$ to mean any of them.

CHAPTER 1

Intersection Theory on Nonsingular Surfaces

1.1. Intersection Theory in $\mathbb{P}^n$

We will first discuss an intersection theory on projective n-space. Unlike the rest of this work, this will apply to objects of various dimensions, not only curves on surfaces. For the proofs of everything in this section, see [**Har77**, Sec. I.7]. Throughout this thesis $\mathbb{k}$ will denote an algebraically closed field.

To begin with, we need to know something about the dimensionality of the intersection of two varieties:

THEOREM 1.1.1. *Let Y and Z be varieties, of dimensions r and s respectively, in $\mathbb{P}^n$. Then every irreducible component of $Y \cap Z$ has dimension at least $r+s-n$. Further, if $r+s-n \geq 0$ then $Y \cap Z$ is nonempty.*

EXAMPLE 1.1.2. Consider the two surfaces in $\mathbb{P}^4_{(x_0:\cdots:x_n)}$ given by $Z(x_1, x_2)$ and $Z(x_1^2 + x_0x_3, x_2^2 + x_0x_4)$. The intersection of these two surfaces is $Z(x_1, x_2, x_0x_3, x_0x_4)$. This consists of the points $(0:0:0:x_3:x_4)$ and the point $(1:0:0:0:0)$. So in particular, one of the irreducible components is a point but the other is a line, of strictly higher dimension, showing that we may in fact have an inequality. However, if we restrict ourselves to (irreducible) curves in $\mathbb{P}^2$, this cannot happen and the inequality is an equality: all the components of the intersection are points.

The approach taken in this theory is to some extent a global one, associating certain polynomials to each variety and then extracting degree information. Intersection numbers, however, are extracted from local data.

DEFINITION 1.1.3. Let $M = \oplus_{\ell=0}^{\infty} M_\ell$ be a graded module over the polynomial ring $\mathbb{k}[x_1, \ldots, x_n]$. Then the *Hilbert function* ϕ_M of M is given by $\phi_M(\ell) = \dim_k M_\ell$ for each $\ell \in \mathbb{Z}$. Recall that the annihilator $\mathrm{Ann}(M)$ of an R-module M is the set $\{r \in R | rM = 0\}$.

THEOREM 1.1.4. *Let M be a finitely generated module over $\mathbb{k}[x_0, \ldots, x_n]$. Then there is a unique (integer-valued) polynomial $P_M(z) \in \mathbb{Q}[z]$ such that $\phi_M(\ell) = P_M(\ell)$ for all $\ell \gg 0$. Furthermore,* $\deg P_M(z) = \dim Z(\mathrm{Ann}\, M)$.

DEFINITION 1.1.5. The polynomial $P_M(z)$ of the theorem is the *Hilbert polynomial* of M. If $Y \subseteq \mathbb{P}^n$ is an algebraic set of dimension r, we define the *Hilbert polynomial* of Y to be the Hilbert polynomial of its homogeneous coordinate ring $S(Y)$ as a $\Bbbk[x_0, \ldots, x_n]$-module (a polynomial of degree r). We define the *degree* of Y to be $r!$ times the leading coefficient of this polynomial.

EXAMPLE 1.1.6. Consider the hyperplane $H = Z(x_0)$ in $\mathbb{P}^2$. Then our module M is $\Bbbk[x_0, x_1, x_2]/x_0\Bbbk[x_0, x_1, x_2]$. In this case M_ℓ is the set of homogeneous polynomials of degree ℓ in three variables, not containing x_0. The monomials $x_1^i x_2^{\ell-i}$ form a basis for M_ℓ over $\Bbbk$, so it is $\ell+1$-dimensional. Thus $\phi(\ell) = \ell + 1$ and the degree of $Z(x_0)$ is 1.

EXAMPLE 1.1.7. Next consider the curve $C = Z(y^2z - x^3) \subset \mathbb{P}^2$. The module M we have to examine is $\Bbbk[x,y,z]/(y^2z - x^3)\Bbbk[x,y,z]$. Let M' denote $\Bbbk[x,y,z]$ and M'' denote $(y^2z - x^3)\Bbbk[x,y,z]$, where the degree of an element $(y^2z - x^3)f$ is just its degree as a polynomial, 3 plus the degree of f. Then we have an exact sequence of graded modules:

$$0 \to M'' \to M' \to M \to 0.$$

So for each ℓ, the degree ℓ parts form an exact sequence. In particular, this means that the $\dim M'_\ell = \dim M_\ell + \dim M''_\ell$. There are $\binom{\ell+2}{2}$ monomials of degree ℓ in $\Bbbk[x,y,z]$, so M'_ℓ has dimension $\binom{\ell+2}{2}$. Clearly M'' is isomorphic to M' with the degree shifted by three, so M''_ℓ has dimension $\binom{(\ell-3)+2}{2}$ for $\ell \geq 3$. So the Hilbert polynomial is given by

$$\phi(\ell) = \binom{\ell+2}{2} - \binom{(\ell-3)+2}{2} = 3l,$$

and the degree of this curve is 3.

It is obvious how this computation generalizes to show that for a curve in $\mathbb{P}^2$ whose defining (homogeneous) polynomial has degree d, the Hilbert polynomial is

$$\phi(\ell) = \binom{\ell+2}{2} - \binom{(\ell-d)+2}{2} = d\ell + 1 - (d-1)(d-2)/2,$$

so that the degree of the curve is d. Note also that the genus appears in the constant coefficient of the Hilbert polynomial.

REMARK 1.1.8. It turns out that the Hilbert polynomial is in a particular sense universal. Say a function f on modules is *additive* if whenever we have a short exact sequence $0 \to M' \to M \to M'' \to 0$ we have $f(M) = f(M') + f(M'')$. The *length* of a module is the length of any composition series, that is, any chain of submodules whose successive quotients are nonzero simple modules (if no such finite chain exists, the module is said to have infinite length). Then the Hilbert polynomial is the universal additive function on graded modules that vanishes on graded modules of finite length. See [**Eis95**, p. 487] for more details.

PROPOSITION 1.1.9. *The degree function has the following properties:*

(1) *If* $Y \subseteq \mathbb{P}^n$, $Y \neq \emptyset$, *then the degree of* Y *is a positive integer.*
(2) *Let* $Y = Y_1 \cup Y_2$, *where* Y_1 *and* Y_2 *have the same dimension* r *and where* $\dim(Y_1 \cap Y_2) < r$ *then* $\deg Y = \deg Y_1 + \deg Y_2$.
(3) $\deg \mathbb{P}^n = 1$.
(4) *If* $H \subseteq \mathbb{P}^n$ *is a hypersurface whose ideal is generated by a homogeneous polynomial of degree* d, *then* $\deg H = d$.

DEFINITION 1.1.10. Let $\mathfrak{p}_j$ be the homogeneous prime ideal of Z_j. Let S be any ring. Then if $\mathfrak{p}$ is a minimal prime containing the annihilator of a graded S-module M, then we define the multiplicity of M at $\mathfrak{p}$ to be the length of $M_{\mathfrak{p}}$ over $S_{\mathfrak{p}}$.

Let $Y \subseteq \mathbb{P}^n$ be a projective variety of dimension r. Let H be a hypersurface not containing Y. Then $Y \cap H = Z_1 \cup \cdots \cup Z_s$ where Z_j are varieties of dimension $r - 1$.

We define the *intersection multiplicity* $i(Y, H; Z_j)$ of Y and H along Z_j to be $\mu_{\mathfrak{p}_j}(S/(I_Y + I_H))$.

REMARK 1.1.11. Here I_Y and I_H are the homogeneous ideals of Y and H. The module $M = S/(I_Y + I_H)$ has annihilator $I_Y + I_H$, and $Z(I_Y + I_H) = Y \cap H$, so $\mathfrak{p}_j$ is a minimal prime of M.

With these definitions, we get a version of Bezout's theorem:

THEOREM 1.1.12. *Let* Y *be a variety of dimension* r *in* $\mathbb{P}^n$, *and let* H *be a hypersurface not containing* Y. *Let* $Z_1, \ldots Z_s$ *be the irreducible components of* $Y \cap H$. *Then*

$$\sum_{j=1}^{s} i(Y, H; Z_j) \cdot \deg Z_j = (\deg Y)(\deg H).$$

COROLLARY 1.1.13. *Let* Y, Z *be distinct curves in* $\mathbb{P}^2$, *having degrees* d, e. *Let* $Y \cap Z = \{P_1, \ldots, P_s\}$. *Then*

$$\sum_{j=1}^{s} i(Y, H; P_j) = de.$$

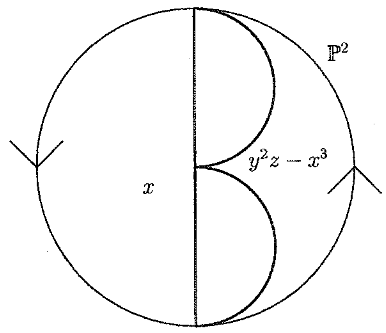

FIGURE 1.1.1. The curves H and C in $\mathbb{P}^2$

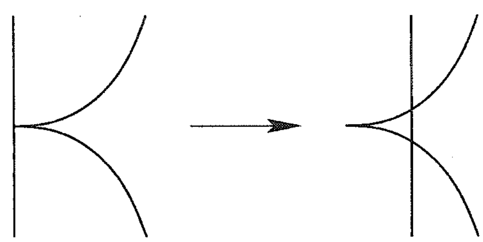

FIGURE 1.1.2. Moving curves into general position in $\mathbb{P}^2$

REMARK 1.1.14. The number $\sum_{j=1}^{s} i(Y, H; P_j)$ is a global function of the objects Y and H. We can see that if we move the objects Y and H independently by any "smooth" transformation (that is, one that does not change the degree, such as translation or rotation) this number will not change, although all the local intersection numbers will. So we will denote this number $Y.H$, and it will be this number that has the most natural generalization on surfaces.

EXAMPLE 1.1.15. Consider again our two curves from Examples 1.1.6 and 1.1.7, $H = Z(x)$ and $C = Z(y^2z - x^3)$. They intersect at the points $(0:1:0)$ and $(0:0:1)$. Adding the ideals, $S/(I_H + I_C) = \mathbb{k}[x, y, z]/\langle x, y^2z\rangle \cong \mathbb{k}[y, z]/\langle y^2z\rangle$. At $(0:1:0)$, the relevant prime is $\langle x, z\rangle$. Localizing, we get the $\mathbb{k}[x, y, z]_{\langle x,z\rangle}$-module $k(y)$ (where x and z annihilate the module). This has length 1. At $(0:0:1)$, the relevant prime is $\langle x, y\rangle$, and we get the $\mathbb{k}[x, y, z]_{\langle x,y\rangle}$-module $k(z)[y]/\langle y^2\rangle$. This contains the length-1 module $yk(z)$, and the quotient is $k(z)$, so it has length 2. Thus the total number of intersections is 3, as predicted by the theorem.

REMARK 1.1.16. When intersection theory was being developed, a number of different approaches to defining intersection numbers were tried. One approach observes that in some cases it is easy to tell what the intersection numbers should be; if the curves intersect simply enough, we can simply count their intersections. We know

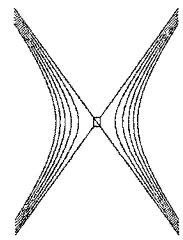

FIGURE 1.1.3. Deforming a quadratic curve into a pair of lines

that the intersections will be simple enough if the curves are in "general position". Defining exactly what "general position" and "simple enough" mean is not so simple; we will see a way to do this for nonsingular surfaces in Section 1.2.4. So we could simply apply rigid transformations to the objects, moving them until they are in general position, and then we would simply count the intersections.

REMARK 1.1.17. A more sophisticated approach would be to observe that intersection numbers are discrete; if we had a reasonable family of curves, say obtained by varying the defining equations smoothly, we would expect the intersection numbers with a fixed curve to remain constant. Of course, defining exactly what families are "reasonable" is not so simple, but given such a definition, we would simply prove that a curve of degree d can be deformed into a family of d lines. We will develop several kinds of equivalence on surfaces in Section 1.2.1 and Section 1.2.7. We then know that a curve of degree d and a curve of degree e are equivalent to a family of d lines and a family of e lines respectively, and it is clear that these two families will intersect in de points.

EXAMPLE 1.1.18. To apply this theory to some classical questions, first consider the space of lines in $\mathbb{P}^2$. This is isomorphic to $\mathbb{P}^2$: map the line $a_0x_0 + a_1x_1 + a_2x_2 = 0$ to $(a_0 : a_1 : a_2)$. Then the set of lines through $(\alpha_0 : \alpha_1 : \alpha_2)$ is the variety

$$Z(\alpha_0a_0 + \alpha_1a_1 + \alpha_2a_2) \subset \mathbb{P}^2_{(a_0 : a_1 : a_2)}.$$

If β is another point, then the set of lines through these two points is the intersection of $Z(\alpha_0a_0 + \alpha_1a_1 + \alpha_2a_2)$ and $Z(\beta_0a_0 + \beta_1a_1 + \beta_2a_2)$; using the theory above we see that if $\alpha \neq \beta$ then

$$Z(\alpha_0a_0 + \alpha_1a_1 + \alpha_2a_2).Z(\beta_0a_0 + \beta_1a_1 + \beta_2a_2) = 1,$$

which is simply a restatement of the fact that through any two distinct points there is a unique line.

More interestingly, we can ask how many conics pass through 4 or 5 points in the plane. Without loss of generality we may assume that the first three points are $(1:0:0)$, $(0:1:0)$, and $(0:0:1)$. The set of conics passing through these three points may be parameterized by taking $a_0x_1x_2+a_1x_0x_2+a_2x_0x_1$ to $(a_0:a_1:a_2)$. Then if α is a point, the set of conics passing through it is $Z(a_0\alpha_1\alpha_2+a_1\alpha_0\alpha_2+a_2\alpha_0\alpha_1)$, and we obtain a one-parameter family of conics. Given another point β not equal to any of the four given points, there is a unique conic passing through α, β, and our original three points.

1.2. Intersection Theory on Nonsingular Surfaces

We will give a brief overview of intersection theory on nonsingular surfaces. For more details, see [**Har77**], particularly Section V. The general approach taken here is to consider some curves "equivalent", allowing one to replace the curves with equivalent curves so that their intersection is simple. We will see later that when dealing with singular surfaces, some curves passing through the singularities are different in essential ways from curves not passing through the singularities. Thus we cannot hope to simply replace them with curves that have simple intersections.

DEFINITION 1.2.1. A *nise scheme* is a *n*oetherian *i*ntegral *se*parated scheme over $\Bbbk$ which is regular in codimension one.

DEFINITION 1.2.2. A *surface* will be a projective nise scheme of dimension 2 over $\Bbbk$.

In this section, the surfaces will generally be nonsingular, but this will be specified when appropriate.

We will have several examples of surfaces that will recur throughout the text.

EXAMPLE 1.2.3. The projective plane, $\mathbb{P}^2$. This is a nonsingular surface, and on it we already have an intersection theory as discussed in Section 1.1. This will allow us to check that our definitions are reasonable.

EXAMPLE 1.2.4. The surface $\mathbb{P}^1\times\mathbb{P}^1$. Using the Segre embedding, this can be viewed as a quadric surface, the zeros of $x_0x_1-x_2x_3$ in $\mathbb{P}^3_{(x_0:\cdots:x_3)}$. While still very simple, this surface is different enough from $\mathbb{P}^2$ that it is worth considering.

EXAMPLE 1.2.5. Let $m>1$ be an integer such that the characteristic of $\Bbbk$ does not divide m. Then the Fermat curve, $M_F=Z(x_1^m+x_2^m-x_3^m)\subset\mathbb{P}^2_{(x_1:\cdots:x_3)}$ is irreducible and nonsingular. In Section 1.4.1 we will show that the surface $X_F=$

$Z(x_1^m + x_2^m - x_3^m) \subset \mathbb{P}^3_{(x_0 : \cdots : x_3)}$ is an irreducible surface with a unique singular point at $(1 : 0 : 0 : 0)$. We will then construct another surface

$$\widetilde{X}_F = Z\Big(\{x_i y_j - x_j y_i\}_{i,j}, y_1^m + y_2^m - y_3^m\Big) \subset \mathbb{P}^3_{(x_0 : \cdots : x_3)} \times \mathbb{P}^2_{(y_1 : \cdots : y_3)},$$

the blow-up of the cone on the Fermat curve. See Section 1.3 for a description of the operation of blowing-up. We will show that this latter surface is nonsingular, and we will use these two surfaces as nontrivial examples throughout this work.

1.2.1. Divisors. We will generally talk about the intersections of *divisors*, that is, formal sums of curves, rather than the intersections of curves, as the algebraic structure of divisors makes them more convenient to work with.

DEFINITION 1.2.6. A *prime divisor* on a scheme X is a closed integral subscheme of codimension 1. A *Weil divisor* on a scheme X is an element of the free abelian group $\operatorname{Div} X$ generated by the prime divisors. A Weil divisor $\sum n_i Y_i$ is *effective* if $n_i \geq 0$ for all i. The support $\operatorname{Supp} D$ of a Weil divisor D is the set of all prime divisors whose coefficient is nonzero in D. If C and D are two divisors on X, then we say $C \geq D$ if $C - D$ is effective.

REMARK 1.2.7. Normally one makes certain restrictions on the scheme X when defining Weil divisors: the scheme X should be a nise scheme.[1] This implies in particular that for every point Y with codimension one, the local ring $\mathcal{O}_{X,Y}$ is a discrete valuation ring. We denote the corresponding valuation on the ring of rational functions $\mathscr{K}(X)$ by v_Y. For a rational function f, $v_Y(f)$ is called the order of vanishing of f along Y. We will always talk about Weil divisors on surfaces that are nonsingular in codimension 1 (where they are formal sums of curves) or on nonsingular curves over $\Bbbk$ (where they are formal sums of points).

LEMMA 1.2.8. *Let X be a nise scheme, and let $f \in \mathscr{K}^*$ be a nonzero function on X. Then $v_Y(f) = 0$ for all but finitely many prime divisors Y.*

See [**Har77**, II.6.1] for a proof.

REMARK 1.2.9. Observe that this applies to the two situations we have discussed above, a nonsingular surface and a nonsingular curve, but that it also applies to some kinds of singular surfaces, so that it will be useful to discuss Weil divisors in later sections as well.

[1] It is possible to make sense of Weil divisors on schemes that are singular in codimension one. In this case, one does not have a valuation, but one can instead use the length of certain modules to compute the appropriate coefficients of prime divisors. This is discussed briefly in [**Eis95**, Sec. 11.5].

DEFINITION 1.2.10. Let X be a nise scheme and let $f \in \mathscr{K}^*$. The *divisor of f*, denoted (f), is

$$(f) = \sum_Y v_Y(f) \cdot Y,$$

where the summation is over all prime divisors Y on X. Any divisor that can be written in this way is said to be a *principal divisor*. Two divisors D and D' are said to be *linearly equivalent* if their difference is a principal divisor. The group of divisors modulo the group of principal divisors is called the *divisor class group* of X and is denoted $\operatorname{Cl} X$.

REMARK 1.2.11. If $X = \operatorname{Spec}(R)$ is affine and R is a Dedekind domain, then the divisor class group of X is just the class group of R in the sense of algebraic number theory, which measures how far R is from having unique factorization.

DEFINITION 1.2.12. A divisor Y on a nise scheme X is said to be *locally principal at x*, for some point x on X, if there exists a neighborhood U of x on which Y is principal. The divisor Y is said to be *locally principal* if it is locally principal at every point of X.

REMARK 1.2.13. For every point x on X, a divisor Y on X yields a divisor Y' on $\operatorname{Spec} \mathcal{O}_{X,x}$. It is clear that Y will be locally principal at x if and only if Y' is principal. If R is a noetherian integral domain, $\operatorname{Cl}(\operatorname{Spec} R) = 0$ if and only if R is a unique factorization domain ([**Har77**, Prop. II.6.2]). So every divisor on X will be locally principal at x if and only if $\mathcal{O}_{X,x}$ is a unique factorization domain. Every regular local ring is a unique factorization domain, so if X is nonsingular at x, every divisor on X is locally principal at x. If X is singular at x, $\mathcal{O}_{X,x}$ may still be a unique factorization domain; such points x are called factorial singularities and are described in more detail in Section 2.1.8.

REMARK 1.2.14. In particular, on a nonsingular surface, every divisor is locally principal.

There is a limited converse to this remark:

PROPOSITION 1.2.15. *Let X be an algebraic $\Bbbk$-scheme of dimension n and let x be a closed point on X. If there exists a nonsingular prime divisor Z which is locally principal at x, then X is nonsingular at x.*

PROOF. This can be found in more detail in [**Mum99**, III.7 Prop. 2]. Recall the definition of the cotangent space $T^*_{X,x}$ at x of a scheme X, $T^*_{X,x} = \mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}$. By

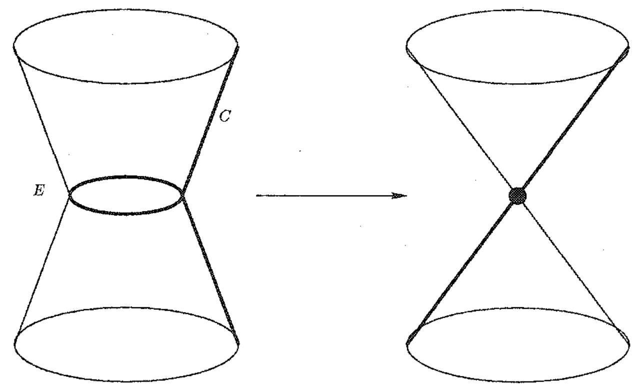

FIGURE 1.2.1. The divisors C and E on the blow-up of the cone over the Fermat curve

definition, the local ring $\mathcal{O}_{X,x}$ is a regular local ring if and only if $T^*_{X,x}$ has dimension n over $\Bbbk$. Now suppose we have an algebraic $\Bbbk$-scheme X and a closed subscheme Z which is nonsingular at x and which is also locally principal at x, say defined by $f \in \mathcal{O}_{X,x}$. Then we have $T^*_{Z,x} \cong T^*_{X,x}/k \cdot df$. So we have $\dim_{\Bbbk} T^*_{X,x} \leq \dim_{\Bbbk} T^*_{Z,x} + 1$. But Z is nonsingular by assumption and therefore $\dim_{\Bbbk} T^*_{Z,x} = \dim X - 1$. Thus we get $\dim_{\Bbbk} T^*_{X,x} \leq \dim X$. But we always have $\dim_{\Bbbk} T^*_{X,x} \geq \dim X$, so $\dim_{\Bbbk} T^*_{X,x} = \dim X$ and X is nonsingular at x. $\square$

See Section 2.1.9 for more information about divisors with locally principal multiples.

EXAMPLE 1.2.16. Consider Example 1.2.5, the blow-up of the cone over the Fermat curve. The nonsingular surface $\widetilde{X}_F$ is

$$Z\Big(\{x_i y_j - x_j y_i\}_{i,j}, y_1^m + y_2^m - y_3^m\Big) \subset \mathbb{P}^3_{(x_0 : \cdots : x_3)} \times \mathbb{P}^2_{(y_1 : \cdots : y_3)}.$$

The curve C defined by $y_1 = 0$ and $y_2 = y_3$ is a prime divisor corresponding to a ruling of the cone. We know that it is supposed to be locally principal, but we can see this directly. Take a neighborhood that excludes the curves $Z(y_1, y_2 - \mu y_3)$ for each $\mu \neq 1$ an mth root of unity, and the curve $Z(y_2)$ (note that y_1 and y_2 are never both zero at the same point). Then take the rational function y_1/y_2. This has valuation 1 on our curve and zero elsewhere in our neighborhood.

Consider the rational function $(y_2 - y_3)/(y_1 - y_3)$. On $C = Z(y_1, y_2 - y_3)$, $y_1 - y_3$ is a unit, as is $(y_2^m - y_3^m)/(y_2 - y_3)$. But $y_1^m + y_2^m - y_3^m = 0$ there, so $(y_2 - y_3)/(y_1 - y_3) = u y_1^m$ for some unit u, meaning that it has valuation m. Similarly, its valuation on $C' = Z(y_2, y_1 - y_3)$ is $-m$, so mC is linearly equivalent to mC'.

A second prime divisor we can take on the same surface is $E = Z(x_1, x_2, x_3)$. This curve is a projective curve isomorphic to the Fermat curve: $Z(y_1^m + y_2^m - y_3^m) \subset \mathbb{P}^2$. It corresponds to the exceptional fiber of the blow-up.

There is a second type of divisor, useful in more general situations (schemes singular in codimension 1, for example). Let X be a scheme, and let $\mathcal{K}$ be the sheaf of total quotient rings (for integral schemes, this is just the constant sheaf of rational functions), let $\mathcal{K}^*$ be the sheaf of invertible elements of $\mathcal{K}$, and let $\mathcal{O}_X^*$ be the sheaf of invertible elements of $\mathcal{O}_X$.

DEFINITION 1.2.17. A *Cartier divisor* is a global section of the sheaf $\mathcal{K}^*/\mathcal{O}_X^*$. A Cartier divisor is *principal* if it is the image of a global section of $\mathcal{K}^*$ under the canonical map. We denote the abelian group of Cartier divisors $\operatorname{CaDiv} X$, and the quotient of this by the abelian group of principal Cartier divisors we call the *Cartier class group* of X and denote $\operatorname{CaCl} X$.

REMARK 1.2.18. In more concrete terms, a Cartier divisor is specified by giving an open cover $\{U_i\}$ and a set of functions $\{f_i \in \mathcal{K}^*(U_i)\}$ such that f_i/f_j is an invertible element of $\mathcal{O}_X(U_i \cap U_j)$. Two such descriptions are equivalent if their ratio is everywhere an invertible element of $\mathcal{O}_X$. A Cartier divisor is principal if we can take the open cover to be $\{X\}$.

PROPOSITION 1.2.19. *Let X be a normal nise scheme. Then we have a one-to-one correspondence between locally principal Weil divisors and Cartier divisors.*

See [**Har77**, Rem. 6.11.2] for a proof. The correspondence associates a Cartier divisor $\{(U_i, f_i)\}$ with the Weil divisor that is the divisor of f_i on U_i.

REMARK 1.2.20. What is the role of normality in Proposition 1.2.19? Suppose X is nonsingular in codimension 1 but not normal at x. Then let D be a Weil divisor D which is locally principal on some neighborhood U of x, and suppose that $D|_U$ is the divisor associated to f and f'. Then if X is normal, we know that $f/f' \in \mathcal{O}_X(U)^*$. If X is not normal, this may not be the case, and we may have several nonequivalent Cartier divisors corresponding to the same Weil divisor.

DEFINITION 1.2.21. A Cartier divisor specified by $\{(U_i, f_i)\}$ is *effective* if for every i we have $f_i \in \mathcal{O}_X(U_i)$.

REMARK 1.2.22. Clearly under the correspondence of Proposition 1.2.19 these are precisely the Cartier divisors that correspond to effective locally principal Weil divisors.

1.2.2. Divisors, invertible sheaves and line bundles. Divisors are intimately related to invertible sheaves. On a scheme X that is either projective over $\Bbbk$ or integral, every invertible sheaf is isomorphic to a subsheaf of the sheaf $\mathcal{K}$ of rational functions on the scheme (see [**Har77**, Rem. II.6.14.1]). Each such subsheaf is locally generated on a cover $\{U_i\}$ by rational functions $\{f_i\}$. The collection of f_i^{-1} clearly satisfy compatibility relations, so they give a Cartier divisor. Had we chosen different generators, say $\{g_j\}$ on $\{V_j\}$, we would nevertheless have $g_j/f_i \in \mathcal{O}_X(U_i \cap V_j)^*$, so they would have yielded the same Cartier divisor.

Conversely, given a Cartier divisor D, we can construct a sheaf[2] $\mathcal{L}_X(D)$ by taking $\mathcal{L}_X(D)(U_i) = (f_i^{-1})\mathcal{O}_X(U_i)$ where (f_i, U_i) are the local equations of D. If D has a zero of order m along Y, then every section of this sheaf will have poles of order at most m along Y. As before, this is independent of the choice of local equations for D. When X is clear from context we will often write $\mathcal{L}_X(D)$ as $\mathcal{L}(D)$.

If X is a nise scheme and D corresponds to a Weil divisor $\sum_Y k_Y Y$, then the $\mathcal{O}_X(U)$-module $\mathcal{L}_X(D)(U)$ is just all the rational functions f that satisfy $v_Y(f) \geq -k_Y$ for every prime divisor Y intersecting U. In particular, if D is a prime divisor, $\mathcal{L}(-D)$ is the ideal sheaf of D.

DEFINITION 1.2.23. Let D be an effective divisor. Define a closed subscheme structure on $\operatorname{Supp} D$ using the exact sequence

$$0 \to \mathcal{L}(-D) \to \mathcal{O}_X \to \mathcal{F} \to 0 \tag{1.2.1}$$

by defining $\mathcal{O}_D$ to be $\mathcal{F}$. We will call this the *canonical closed subscheme structure defined by D.*

Each abstract invertible sheaf is isomorphic to many different subsheaves of $\mathcal{K}$, but the Cartier divisors that this process yields are all linearly equivalent ([**Har77**, Prop. II.6.13]). So isomorphism classes of invertible sheaves correspond bijectively to linear equivalence classes of Cartier divisors.

There is a third way of looking at locally principal divisors and invertible sheaves; one can use the language of line bundles. See [**Sha94**, Sec. VI.1.4] for a more detailed exposition.

DEFINITION 1.2.24. A *line bundle* on a scheme X over $\Bbbk$ is a scheme Y, a morphism $\pi : Y \to X$, and a collection $\{(U_i, \psi_i)\}$ such that:

[2]The $\mathcal{L}$ stands for "line bundle", because as we will show later in this section, invertible sheaves correspond naturally to line bundles, and one often mixes the two languages.

(1) $\{U_i\}$ is an open cover of X,
(2) ψ_i is an isomorphism from $\pi^{-1}(U_i)$ to $U_i \times \mathbb{A}^1$,
(3) $\pi \circ \psi_i^{-1}$ is the identity on U_i, and
(4) $\psi_i \circ \psi_j^{-1}$ is multiplication by a scalar on each fiber.

A *section* of a line bundle (Y, π) is a morphism $f : X \to Y$ such that $\pi \circ f$ is the identity on X. A *meromorphic section* is a section on some open dense subset of X.

EXAMPLE 1.2.25. The simplest line bundle over X is just $X \times \mathbb{A}^1$. This is called the trivial line bundle. A section of this line bundle is then any regular function on X; a meromorphic section is any rational function on X.

REMARK 1.2.26. Suppose that we have some line bundle (Y, π) with a section f such that f is never zero. Then we can form an isomorphism from Y to $X \times \mathbb{A}^1$ by taking $\psi_i^{-1}(x,t)$ to $(x, t/\psi_i(f(x)))$, and we see that Y is isomorphic to the trivial bundle.

EXAMPLE 1.2.27. Let X be $\mathbb{P}^1$, with the open sets $U_0 = \{(x_0 : x_1) | x_0 \neq 0\}$ and $U_1 = \{(x_0 : x_1) | x_1 \neq 0\}$. Construct Y by gluing $U_0 \times \mathbb{A}^1$ and $U_1 \times \mathbb{A}^1$ by identifying $((x_0 : x_1), t) \in U_0 \times \mathbb{A}^1$ with $((x_0 : x_1), tx_0/x_1) \in U_1 \times \mathbb{A}^1$. This defines a line bundle over $\mathbb{P}^1$. Suppose now we define a map taking $(x_0 : x_1)$ to $((x_0 : x_1), x_1/x_0) \in U_0 \times \mathbb{A}^1$ and another taking $(x_0 : x_1)$ to $((x_0 : x_1), 1) \in U_1 \times \mathbb{A}^1$. These maps clearly glue to give a section of this line bundle that has exactly one zero. But there are no regular functions on $\mathbb{P}^1$ except the constants, so this is clearly not isomorphic to the trivial line bundle.

REMARK 1.2.28. Note that to construct this line bundle, the essential data we needed was a regular function f_{ij} on each $U_i \cap U_j$. This then allowed us to glue together the $U_i \times \mathbb{A}^1$, provided that the f_{ij} satisfied a compatibility criterion, namely $f_{ij} f_{jk} = f_{ik}$ (with $f_{ii} = 1$). We call such a collection of f_{ij} *transition functions*, and we have exactly one such collection for every line bundle. Identifying the compatibility criteria as a cochain condition, we see that the set of line bundles on X is in bijection with $H^1(X, \mathcal{O}_X^*)$.

Now suppose we have a Cartier divisor D on a nise scheme X. Let U_i be an open cover of X such that on U_i, D is the divisor associated to f_i. Now, on $U_i \cap U_j$ the functions f_i and f_j have the same divisor, so f_j/f_i can have neither poles nor zeros. They clearly satisfy the compatibility criterion for a set of transition functions, so we can construct a line bundle using them.

If we have a line bundle and a meromorphic section f, then on each U_i, f is a rational function. By the compatibility criterion, the divisor of f on U_i and the divisor of f on U_j are equal on $U_i \cap U_j$, so we can piece together all of these to yield a Cartier divisor D on X. If g is another meromorphic section, then f/g is a rational function on X, and so the Cartier divisor associated to g is linearly equivalent to the Cartier divisor associated to f.

These two constructions are inverse to one another. To see this, let D be a Cartier divisor and (U_i, f_i) be chosen so that on U_i the divisor D is the divisor of f_i and the U_i cover X. Now, this method gives us a line bundle with transition functions f_j/f_i. The divisor associated to this line bundle can be obtained by taking the divisor of any meromorphic section, so construct the section that is f_i on U_i; by construction, this satisfies the compatibility criteria, and its divisor is clearly D. One can similarly show that going from line bundle to divisor and back yields the same line bundle.

Now that we have this correspondence between line bundles, sheaves and Cartier divisors, we will often use them interchangeably. For example, a Cartier divisor is *ample* if it corresponds to an ample invertible sheaf (see [**Har77**, Sec. II.7]).

1.2.3. The canonical sheaf. On a nonsingular surface X, there is a particular sheaf that is of interest. Recall that since X is nonsingular, the sheaf of differentials $\Omega_{X/\Bbbk}$ is locally free of rank two (in fact, this is equivalent to the nonsingularity of X). So the sheaf $\omega_X = \Omega_{X/\Bbbk} \wedge \Omega_{X/\Bbbk}$ is an invertible sheaf. We call ω_X the *canonical sheaf* of X. Any associated Cartier divisor K_X we call a *canonical divisor* on X.

EXAMPLE 1.2.29. As an example, we will calculate the canonical sheaves of both $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$. A general result ([**Mum99**, III.1 Ex. B]) gives a description of $\Omega_{X/\Bbbk}$ in the case where $R = \Bbbk[x_1, \ldots, x_n]/\langle f_1, \ldots, f_m\rangle$ and $X = \operatorname{Spec} R$. In this case, $\Omega_{X/\Bbbk}$ is generated as an R-module by $dx_1, \ldots, dx_n$ and has the relations $df_1 = \cdots = df_m = 0$. Thus on $\mathbb{A}^2$, the canonical sheaf is freely generated by $dx \wedge dy$.

Cover $\mathbb{P}^2$ with three copies of $\mathbb{A}^2$ in the usual way, $U_i = \{(x_0 : x_1 : x_2) | x_i \neq 0\}$. On U_0, the canonical sheaf is generated by $d(x_1/x_0) \wedge d(x_2/x_0)$. On U_1, it is generated by $d(x_2/x_1) \wedge d(x_0/x_1)$. On $U_0 \cap U_1$, the canonical sheaf is generated by either, so their ratio must be a regular function with no zeros. In the coordinate system on U_0,

$$\begin{aligned} d(x_2/x_1) \wedge d(x_0/x_1) &= d((x_2/x_0)/(x_1/x_0)) \wedge d(x_0/x_1) \\ &= (x_1/x_0)^{-3} d(x_2/x_0) \wedge d(x_1/x_0). \end{aligned}$$

So this sheaf is isomorphic to the sheaf generated on U_0 by 1, on U_1 by $(x_0/x_1)^{-3}$ and on U_2 by $(x_0/x_2)^{-3}$. The divisor associated to this is just $-3Z(x_0)$. Thus we find that $\omega_{\mathbb{P}^2} = \mathcal{O}(3)$. More generally, $\omega_{\mathbb{P}^n} = \mathcal{O}(n+1)$.

We can approach $\mathbb{P}^1 \times \mathbb{P}^1$ in the same way, covering it with four open sets $U_{ij} = \{(x_0 : x_1, y_0 : y_1) | x_i \neq 0, y_j \neq 0\}$. Then each U_{ij} is isomorphic to $\mathbb{A}^2$. On U_{00}, the canonical sheaf is generated by $d(x_1/x_0) \wedge d(y_1/y_0)$. On U_{01}, the canonical sheaf is generated by $d(x_1/x_0) \wedge d(y_0/y_1)$. On $U_{00} \cap U_{01}$ this second generator is

$$d(x_1/x_0) \wedge d(y_0/y_1) = -(y_1/y_0)^{-2} d(x_1/x_0) \wedge d(y_1/y_0).$$

Similar computations hold for the other U_{ij}. So this sheaf is isomorphic to the sheaf generated on U_{00} by 1, and generated on U_{ij} by $(-1)^{i+j}(x_0/x_i)^{-2}(y_0/y_j)^{-2}$. This has divisor $-2Z(x_0) - 2Z(y_0)$. We note that $\omega_{\mathbb{P}^1 \otimes \mathbb{P}^1} \cong p_1^* \omega_{\mathbb{P}^1} \otimes p_2^* \omega_{\mathbb{P}^1}$ where p_i is the projection on the i-th component. In fact, for any two curves, $\omega_{X \times Y} \cong p_1^* \omega_X \otimes p_2^* \omega_Y$ (see [**Har77**, Exer. II.8.3]).

1.2.4. The intersection pairing. The general approach to intersection theory taken here is to address the simplest case, intersections with multiplicity one, and then to get at the other cases by replacing the divisors with linearly equivalent divisors that intersect more simply. The reason for considering principal divisors trivial for the purposes of intersection comes from the behavior of divisors on projective curves. If we have a projective curve C on X, then (if C and f are in sufficiently general position) we can look at the intersections of C with (f) as the number of zeros of f on C minus the number of poles of f on C; we know from the theory of curves that this will always be zero. So we expect a principal divisor to have zero intersection with any divisor.

DEFINITION 1.2.30. Two prime divisors Y and Y', locally defined by f and f', *intersect transversally at* P if f and f' together generate the maximal ideal in $\mathcal{O}_{X,P}$.

EXAMPLE 1.2.31. Consider again the curves from Example 1.1.15, $Y = Z(x)$ and $H = Z(y^2 z - x^3)$, now considered as divisors on $\mathbb{P}^2$. At $(0:1:0)$, $\mathcal{O}_{X,P} = \Bbbk[x,y,z]_{\langle x,z \rangle}$ and $\langle x, y^2 z - x^3 \rangle = \langle x, z \rangle$ so the curves intersect transversally. However, at $(0:0:1)$, $\mathcal{O}_{X,P} = \Bbbk[x,y,z]_{\langle x,y \rangle}$ and $\langle x, y^2 z - x^3 \rangle = \langle x, y^2 \rangle \neq \langle x, y \rangle$, so the curves do not intersect transversally (as we expect from our calculations in Example 1.1.15).

Alternatively, refer back to Example 1.2.16. Consider our two curves, C and E on the surface $\widetilde{X}_F = Z\left(\{x_i y_j - x_j y_i\}_{i,j}, y_0^m + y_1^m - y_2^m\right)$. These intersect at the point $P = (1:0:0:0, 0:1:1)$. There C has local equation $y_1 = 0$ and E has local equation

$x_2 = 0$. To see this, observe that $Z(x_2) = Z(y_2) \cup Z(x_1, x_2, x_3)$. Since y_2 has no zeros near $(1:0:0:0, 0:1:1)$, this yields an adequate local equation for E. We then have:

$$\mathcal{O}_{X,P} = \left(\Bbbk[x_0, x_1, x_2, x_3, y_1, y_2, y_3]/\left\langle \{x_i y_j - x_j y_i\}_{i,j}, y_1^m + y_2^m - y_3^m \right\rangle\right)_{\langle x_1, x_2, x_3, y_1\rangle}$$

and the maximal ideal is $\langle x_1, x_2, x_3, y_1\rangle$. Consider $\langle x_2, y_1\rangle$. It contains $x_2 y_i = x_i y_2$ so it contains x_i for $i = 1, 2, 3$. Thus it equals the maximal ideal and the intersection is transverse.

THEOREM 1.2.32. *There exists a unique pairing*

$$C.D : \operatorname{Cl} X \times \operatorname{Cl} X \to \mathbb{Z},$$

such that

(1) $C.D = D.C$,
(2) $C.(D + D') = C.D + C.D'$, *and*
(3) *if* C *and* D *intersect transversally everywhere then* $C.D = \#(C \cap D)$.

See [**Har77**, V.1.1] for a proof. Given two divisors, it is first shown that each can be written as the difference of two nonsingular curves, and further that these curves can be chosen to intersect transversally. Then since the intersection number depends only on the linear equivalence class, it is completely determined by the properties in Theorem 1.2.32. The essential part of this argument, showing that divisors can be written as the difference of two well-behaved curves proceeds by first writing the divisor's sheaf $\mathcal{L}(D)$ as the difference of two very ample invertible sheaves. Each very ample invertible sheaf $\mathcal{L}(D')$ yields an embedding of X into projective space in which the divisor D' arises (up to linear equivalence) as a hyperplane section and in particular is an effective divisor. A version of Bertini's theorem valid in any characteristic ([**Har77**, II.8.18]) then shows that this hyperplane can be chosen so that D' is a nonsingular curve which intersects a list of other curves transversally.

An alternative method of proving this theorem would be through the use of the formula in Theorem 1.2.38, which gives an explicit method for computing the intersection number. This is the approach taken in (for example) [**Mum66**, Chap. 12].

EXAMPLE 1.2.33. The divisor class group of $\mathbb{P}^2$ is $\mathbb{Z}$, generated by a line ([**Har77**, II.6.4]. So a curve of degree m is equivalent to m lines, and a family of m lines intersects a family of n lines in mn points. This is the classical Bezout's theorem.

It is often impractical to compute the intersection form directly, by moving the curves, so one often uses a formula that can be computed from local data.

DEFINITION 1.2.34. Suppose C and D are curves that share no common irreducible component, and suppose that they intersect at P. Then if f and g are local equations of C and D, we define $(C.D)_P$ to be the dimension of $\mathcal{O}_{X,P}/\langle f, g\rangle$ as a $\Bbbk$-vector space.

EXAMPLE 1.2.35. The curves $x = 0$ and $y^2z = x^3$ in $\mathbb{P}^2_{(x:y:z)}$ from Example 1.1.15 intersect at $(0:0:1)$. The module $\mathcal{O}_{\mathbb{P}^2,0}$ is just the ring of rational functions with no pole at 0, which is isomorphic to $\Bbbk[x,y]_{\langle x,y\rangle}$. Now $\Bbbk[x,y]_{\langle x,y\rangle}/\langle x, y^2 - x^3\rangle$ is isomorphic to $\Bbbk[y]_{\langle y\rangle}/\langle y^2\rangle$. But we know that $\Bbbk[y]_{\langle y\rangle}/\langle y^2\rangle$ is generated as a $\Bbbk$-vector space by 1, y, and the set of all rational functions $1/(y-a)$ for $a \neq 0$. But $1/(y-a) = (y+a)/(y^2-a^2) = -(y+a)/a^2$, so 1 and y form a basis for $\Bbbk[y]_{\langle y\rangle}/\langle y^2\rangle$ over $\Bbbk$, so it is two-dimensional and the intersection has multiplicity two, as we would expect from Example 1.1.15.

REMARK 1.2.36. How does this definition of intersection number compare with the definition given in Section 1.1? Consider the case of two curves X and Y, locally defined by f and g, in $\mathbb{P}^2$, intersecting at some point x. Using the definition from Section 1.1, $i(X,Y;x) = \mu_{\mathfrak{m}_x}(S/(f,g)S)$ where (recall) $S = \Bbbk[x_0, x_1, x_2]$. Recalling the definition of $\mu_{\mathfrak{m}_x}$, this is just the length of $(S/(f,g)S)_{\mathfrak{m}_x} = S_{\mathfrak{m}_x}/(f,g)S_{\mathfrak{m}_x}$ as a $S_{\mathfrak{m}_x}$-module. Since $S_{\mathfrak{m}_x}$ is a local ring with residue field $\Bbbk$, any simple module over $S_{\mathfrak{m}_x}$ is isomorphic to $\Bbbk$, and so the length of a composition series is exactly the dimension as a $\Bbbk$-vector space. So

$$i(X,Y;x) = \dim_{\Bbbk} S_{\mathfrak{m}_x}/(f,g)S_{\mathfrak{m}_x} = \dim_{\Bbbk} \mathcal{O}_{\mathbb{P}^2,x}/(f,g)\mathcal{O}_{\mathbb{P}^2,x} = (X.Y)_x.$$

These local intersection numbers are useful, of course, only if they allow us to calculate the global intersection number.

THEOREM 1.2.37. *Suppose C and D are curves with no common irreducible component. Then*

$$C.D = \sum_{P \in C \cap D} (C.D)_P.$$

See **[Har77**, V.1.4].

1.2.5. Cohomology and the intersection pairing. There is another formula for computing the intersection number of two divisors. In this case it is based on cohomology. Recall that the Euler characteristic $\chi(\mathcal{F})$ of a sheaf $\mathcal{F}$ on a surface X is defined to by

$$\chi(\mathcal{F}) = \dim H^0(X,\mathcal{F}) - \dim H^1(X,\mathcal{F}) + \dim H^2(X,\mathcal{F}).$$

See Section A.3 for more details.

THEOREM 1.2.38. *Let C and D be divisors on a nonsingular surface X. Then*

$$C.D = \chi(\mathcal{O}_X) - \chi(\mathcal{L}(-C)) - \chi(\mathcal{L}(-D)) + \chi(\mathcal{L}(-C-D)).$$

For a full proof see [**Mum66**, Chap. 12], where it is shown that the right-hand side satisfies all the criteria for our intersection form; then uniqueness yields the result. We will see in Corollary 1.2.68 that this follows immediately from the Riemann-Roch theorem. We will cover a part of the proof here. This appears as [**Mum66**, Prop. 12.1].

LEMMA 1.2.39. *Let C and D be effective divisors having no prime divisors in common. Then*

$$\chi(\mathcal{O}_X) - \chi(\mathcal{L}(-C)) - \chi(\mathcal{L}(-D)) + \chi(\mathcal{L}(-C-D)) = \sum_{x \in \operatorname{Supp}(C) \cap \operatorname{Supp}(D)} (C.D)_x.$$

PROOF. We have the exact sequences

$$0 \to \mathcal{L}(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$$

and

$$0 \to \mathcal{L}(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$

Since all the sheaves listed are locally free, these are locally free resolutions of $\mathcal{O}_C$ and $\mathcal{O}_D$. Now, a priori, we could compute $\operatorname{Tor}^{\mathcal{O}_X}(\mathcal{O}_C, \mathcal{O}_D)$ using either of these resolutions, as the cohomology of (for example)

$$0 \to \mathcal{L}(-C) \otimes \mathcal{O}_D \to \mathcal{O}_X \otimes \mathcal{O}_D \to 0.$$

However, using [**Rot79**, Thm. 11.21], we can compute the cohomology using the double complex

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{L}(-C) \otimes \mathcal{L}(-D) & \longrightarrow & \mathcal{L}(-D) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{L}(-C) & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array} .$$

The total complex of this double complex is

$$0 \to \mathcal{L}(-C-D) \to \mathcal{L}(-C) \oplus \mathcal{L}(-D) \to \mathcal{O}_X \to 0, \tag{1.2.2}$$

and the cohomology of (1.2.2) at the i-th term is $\operatorname{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_C, \mathcal{O}_D)$.

Let $x \in X$ and let f and g be the local equations of C and D near x respectively. Then $\mathcal{O}_{C,x} = \mathcal{O}_{X,x}/f\mathcal{O}_{X,x}$ and $\mathcal{O}_{D,x} = \mathcal{O}_{X,x}/g\mathcal{O}_{X,x}$. If x is not on $\operatorname{Supp} C \cap \operatorname{Supp} D$, then one of f or g will be a unit and the corresponding stalk will be zero and $\operatorname{Tor}_i^{\mathcal{O}_{X,x}}(\mathcal{O}_{C,x}, \mathcal{O}_{D,x}) = 0$ for $i \geq 0$. Now assume $x \in \operatorname{Supp} C \cap \operatorname{Supp} D$. We know f is not a zero divisor in $\mathcal{O}_{X,x}$, so we have the exact sequence

$$0 \longrightarrow \mathcal{O}_{X,x} \xrightarrow{f} \mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{C,x} \longrightarrow 0\ ,$$

where the left-hand map is multiplication by f. Applying the functor $- \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{D,x}$ to this complex gives the complex

$$0 \longrightarrow \mathcal{O}_{D,x} \xrightarrow{f} \mathcal{O}_{D,x} \longrightarrow \mathcal{O}_{C,x} \otimes \mathcal{O}_{D,x} \longrightarrow 0\ ,$$

which is also exact since f is not a zero divisor in $\mathcal{O}_{X,x}/g\mathcal{O}_{X,x} = \mathcal{O}_{D,x}$. But we have the long exact sequence of $\operatorname{Tor}^{\mathcal{O}_{X,x}}$,

$$\begin{aligned} \operatorname{Tor}_1^{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}, \mathcal{O}_{D,x}) \longrightarrow \operatorname{Tor}_1^{\mathcal{O}_{X,x}}(\mathcal{O}_{C,x}, \mathcal{O}_{D,x}) \longrightarrow \mathcal{O}_{D,x} \xrightarrow{f} \mathcal{O}_{D,x} \\ \longrightarrow \mathcal{O}_{C,x} \otimes \mathcal{O}_{D,x} \longrightarrow 0\ . \end{aligned} \tag{1.2.3}$$

By construction, $\mathcal{O}_{X,x} \otimes -$ is just the identity functor, which clearly has zero derived functors, so for $n > 0$ we have $\operatorname{Tor}_n^{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}, \mathcal{O}_{D,x}) = 0$, which implies in particular that $\operatorname{Tor}_1^{\mathcal{O}_{X,x}}(\mathcal{O}_{C,x}, \mathcal{O}_{D,x}) = 0$. Looking further along the long exact sequence, we also have

$$\operatorname{Tor}_{n+1}^{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}, \mathcal{O}_{D,x}) \longrightarrow \operatorname{Tor}_{n+1}^{\mathcal{O}_{X,x}}(\mathcal{O}_{C,x}, \mathcal{O}_{D,x}) \longrightarrow \operatorname{Tor}_n^{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}, \mathcal{O}_{D,x}) \longrightarrow 0\ .$$

As above, for $n > 0$ we have $\operatorname{Tor}_n^{\mathcal{O}_{X,x}}(\mathcal{O}_{X,x}, \mathcal{O}_{D,x}) = 0$, so $\operatorname{Tor}_i^{\mathcal{O}_{X,x}}(\mathcal{O}_{C,x}, \mathcal{O}_{D,x}) = 0$ for all $i > 1$.

Thus we have $\operatorname{Tor}_i^{\mathcal{O}_{X,x}}(\mathcal{O}_{C,x}, \mathcal{O}_{D,x}) = 0$ for $i > 0$ and for all x. Thus (1.2.2) is a resolution of $\mathcal{O}_C \otimes \mathcal{O}_D$, so the complex

$$0 \to \mathcal{L}(-C-D) \to \mathcal{L}(-C) \oplus \mathcal{L}(-D) \to \mathcal{O}_X \to \mathcal{O}_C \otimes \mathcal{O}_D \to 0,$$

is exact, and therefore the Euler characteristic is additive on it. We have also shown that the sheaf $\mathcal{O}_C \otimes \mathcal{O}_D$ is supported only on $\operatorname{Supp} C \cap \operatorname{Supp} D$, and at x it is isomorphic to $\mathcal{O}_{X,x}/(f,g)\mathcal{O}_{X,x}$, which has $\Bbbk$-dimension $(C.D)_x$. Thus:

$$\begin{aligned}
C.D &= \sum_{x \in \operatorname{Supp} C \cap \operatorname{Supp} D} (C.D)_x \\
&= \dim H^0(X, \mathcal{O}_C \otimes \mathcal{O}_D) \\
&= \chi(\mathcal{O}_C \otimes \mathcal{O}_D) \\
&= \chi(\mathcal{O}_X) - \chi(\mathcal{L}(-C) \oplus \mathcal{L}(-D)) + \chi(\mathcal{L}(-C-D)) \\
&= \chi(\mathcal{O}_X) - \chi(\mathcal{L}(-C)) - \chi(\mathcal{L}(-D)) + \chi(\mathcal{L}(-C-D)).
\end{aligned}$$

□

We will see that the formula from Theorem 1.2.38 is also true when using the intersection theory from Section 2.5 with Cartier divisors on a singular surface.

This same formula can be viewed as a formula for the Euler characteristic of a tensor product: if $\mathcal{F}$ and $\mathcal{G}$ are invertible sheaves, we can write $\mathcal{F}.\mathcal{G}$ to mean the intersection number of the associated divisors. We then have

$$\chi(\mathcal{F} \otimes \mathcal{G}) = \chi(\mathcal{F}) + \chi(\mathcal{G}) - \mathcal{F}.\mathcal{G} - \chi(\mathcal{O}_X).$$

1.2.6. Arithmetic genus.

DEFINITION 1.2.40. Let X be a projective scheme of dimension r over $\Bbbk$. Then the *arithmetic genus* $p_a(X)$ is defined to be

$$p_a(X) = (-1)^r(\chi(\mathcal{O}_X) - 1).$$

See [**Har77**, Exer. III.5.3]).

Now suppose D is an effective Cartier divisor on the surface X. Then using Equation (1.2.1) D defines a subscheme Y of X (which will not be reduced unless all the coefficients in D are at most one, and which will not be integral unless D is a prime divisor). We can then compute the arithmetic genus $p_a(Y)$ of Y.

By the definition of Y, we have an exact sequence

$$0 \to \mathcal{L}(-D) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0.$$

As a result, $\chi(\mathcal{O}_X) = \chi(\mathcal{L}(-D)) + \chi(\mathcal{O}_Y)$. Thus we have proven:

PROPOSITION 1.2.41. *Let X be a surface and let D be an effective Cartier divisor on X. Then D defines a subscheme Y of X, and we have*

$$\begin{aligned} p_a(Y) &= 1 - \chi(\mathcal{O}_X) + \chi(\mathcal{L}(-D)) \\ &= \chi(\mathcal{L}(-D)) - p_a(X). \end{aligned}$$

This suggests the following definition:

DEFINITION 1.2.42. Let X be a surface and let D be any Cartier divisor on X. Then define the *arithmetic genus $p_a(D)$ of D* by

$$p_a(D) = \chi(\mathcal{L}(-D)) - p_a(X).$$

REMARK 1.2.43. This gives the zero (empty) divisor arithmetic genus 1, and it depends only on the linear equivalence class of the divisor. When D is effective, this definition gives D the same arithmetic genus as the subscheme Y it defines.

If we suppose C is another Cartier divisor on X, then what is the relation between $p_a(C)$, $p_a(D)$ and $p_a(C+D)$?

PROPOSITION 1.2.44. *Let C, $\{C_i\}_{1 \le i \le n}$, and D be Cartier divisors on a surface X. Then*

$$p_a(C+D) = p_a(C) + p_a(D) + C.D - 1,$$

and

$$p_a\left(\sum_{i=1}^{n} C_i\right) = \sum_{i=1}^{n} p_a(C_i) + \sum_{i=1}^{n} \sum_{j=i+1}^{n} (C_i.C_j) - (n-1).$$

PROOF. Apply Theorem 1.2.38:

$$\begin{aligned} C.D &= \chi(\mathcal{O}_X) - \chi(\mathcal{L}(-C)) - \chi(\mathcal{L}(-D)) + \chi(\mathcal{L}(-C-D)) \\ &= -\big(\chi(\mathcal{L}(-C)) + 1 - \chi(\mathcal{O}_X)\big) - \big(\chi(\mathcal{L}(-D)) + 1 - \chi(\mathcal{O}_X)\big) \\ &\qquad + \big(\chi(\mathcal{L}(-C-D)) + 1 - \chi(\mathcal{O}_X)\big) + 1 \\ &= -p_a(C) - p_a(D) + p_a(C+D) + 1. \end{aligned}$$

The more general formula follows by an easy induction. □

REMARK 1.2.45. We will see in Lemma 2.1.34 that if D is effective and the associated subscheme Y is connected and reduced, then $p_a(Y) \ge 0$. However, we have $p_a(2D) = 2p_a(D) - D.D + 1$, which may be negative, depending on the self-intersection of D. Thus nonreduced schemes may have negative arithmetic genus. Further, if D

and D' have disjoint support, then $p_a(D + D') = p_a(D) + p_a(D') - 1$, so disconnected schemes may also have negative arithmetic genera. This is not just an artifact of our definition of the arithmetic genus of a divisor, since for effective divisors we are simply computing the standard arithmetic genus of the subscheme induced by Equation (1.2.1), which may well be negative.

1.2.7. Equivalence of divisors. Observe that intersection numbers are in $\mathbb{Z}$, so any torsion divisor must have zero intersection with everything. This suggests that the divisor class group is finer than it needs to be for the purposes of intersection theory.

DEFINITION 1.2.46. Let X be a nonsingular surface. We say a divisor C is *numerically equivalent to zero* if $C.D = 0$ for all divisors D. We say D_1 and D_2 are *numerically equivalent* if $D_1 - D_2$ is numerically equivalent to zero. The group of divisors on X modulo numerical equivalence we denote by $\operatorname{Num} X$.

REMARK 1.2.47. Here we have a version of Bezout's theorem for nonsingular surfaces: the intersections of a curve are completely described by its value in $\operatorname{Num}(X)$, which we will see in Remark 1.2.56 is a finitely-generated free abelian group. So choosing a basis for $\operatorname{Num}(X)$, from each divisor D_1 we obtain a finite list of numbers v_1; the intersection of two divisors D_1 and D_2 is computed using the intersection matrix M of the surface as ${v_1}^T \mathrm{M} v_2$. So the vector v_1 or the image of D_1 in $\operatorname{Num}(X)$ serves as a sort of degree with which we can compute the intersections of a curve. Theorem 1.2.70 (the Hodge index theorem) will give a description of the matrix M.

REMARK 1.2.48. It is not only torsion divisors that are numerically equivalent to zero. For example, if we consider $\mathbb{P}^1 \times C$ for some elliptic curve C, the divisor class group will be isomorphic to $\mathbb{Z} \oplus \operatorname{Cl} C$, while the numerical equivalence class group will be $\mathbb{Z} \oplus \mathbb{Z}$ ([**Har77**, V.2.3]). The theory of Jacobians tells us that $\operatorname{Cl} C$ is just $\mathbb{Z} \oplus C$, where C is interpreted as the group of closed points $C(k)$, and C need not be a torsion group. However, there is a concrete description of which divisors are numerically equivalent to zero in terms of a more flexible form of equivalence, namely algebraic equivalence. See Remark 1.2.58 for the result.

EXAMPLE 1.2.49. For a concrete example where numerical and linear equivalence differ, we can look to our by now standard list of examples. Both $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are so simple that $\operatorname{Num} X \cong \operatorname{Cl} X$. So look at Example 1.2.5, the blow-up of the cone on the Fermat curve. We will compute its class group and its numerical equivalence class group in Section 1.4.1. There we will see that $\mathrm{Cl}\,X$ is a direct sum of $\mathbb{Z}$ and an image of the class group on the Fermat curve, which will typically have some smooth component. On the other hand, we will see that $\mathrm{Num}\,X$ is $\mathbb{Z} \oplus \mathbb{Z}$; we will see that $\mathrm{Num}\,X$ can never have a smooth component.

Between linear equivalence and numerical equivalence, there lies a third concept, that of algebraic equivalence.

Let X be a surface possibly with isolated singularities. Let T be a nonsingular curve, and let D be a nonzero effective Weil divisor on $X \times T$. Then D defines a subscheme Z of $X \times T$ having pure dimension 2 and having ideal sheaf $\mathcal{L}_{X \times T}(-D)$. Let $t \in T$. We know $X \times \{t\}$ is a closed subscheme of $X \times T$. Call its ideal sheaf $\mathcal{I}$. Then we can compute the subscheme $Z \cap X \times \{x\}$ of $X \times T$. This subscheme has ideal sheaf $\mathcal{L}_{X \times T}(-D) + \mathcal{I}$. This ideal sheaf leads to an ideal sheaf in $\mathcal{O}_X$, namely $\mathcal{L}_{X \times T}(-D)/\mathcal{I}$. This ideal sheaf defines a subscheme of X. We know Z has codimension one, and $X \times \{t\}$ is defined by a single equation, so their intersection, if nonempty, cannot have codimension more than two in $X \times T$. If their intersection has a component of dimension 2, then it contains all of X and in particular, $X \times \{t\}$ is in the support of D. If there is no $t \in T$ for which $\mathrm{Supp}\,D$ contains $X \times \{t\}$, then the image of Z under the projection to T must have dimension one. Since it is closed and T is irreducible, it must be all of T, so the intersection of Z and $X \times \{t\}$ is never empty. Then for each t, D defines a nonzero effective divisor on X. Say in this case that D defines an *algebraic family of effective divisors*, and let D_t denote the divisor on X corresponding to $t \in T$.

For every two points 0 and 1 of T, we say the corresponding divisors D_0 and D_1 are *prealgebraically equivalent*. Two not necessarily effective divisors are prealgebraically equivalent if they can each be written as the differences of prealgebraically equivalent effective divisors.

REMARK 1.2.50. The condition that the dimension of D_t is constant is equivalent to saying that Z is flat over T, and this is in fact how the definition is often stated.

DEFINITION 1.2.51. Two divisors D and D' are *algebraically equivalent*[3] if there exists a chain of divisors $D = D_0, D_1, \ldots, D_n = D'$ such that D_i is prealgebraically equivalent to D_{i+1} for all i.

REMARK 1.2.52. The divisors algebraically equivalent to zero form a subgroup G_a of the group of divisors.

REMARK 1.2.53. We saw in the proof that a nonzero effective divisor cannot be prealgebraically equivalent to the empty divisor; later we will see that it cannot be algebraically equivalent to zero since it has positive intersection with some ample divisor.

REMARK 1.2.54. Linearly equivalent divisors are algebraically equivalent. In this case, the parameterizing family is $\mathbb{P}^1$ and the divisor comes from a linear combination of the two linearly equivalent divisors.

REMARK 1.2.55. One can show that algebraically equivalent divisors are numerically equivalent ([**Har77**, Exer. V.1.7]). This fits with the intuition of algebraically equivalent divisors being smoothly deformable into each other, while intersection numbers are a discrete quantity. This condition is a natural consequence of flatness, similar to the fact that the fibers of a flat morphism all have the same dimension.

REMARK 1.2.56. The group of Weil divisors on X modulo algebraic equivalence is called the *Néron-Severi group* $\operatorname{NS} X$. The Néron-Severi theorem states that on any projective variety that is nonsingular in codimension 1 the Néron-Severi group is a finitely-generated abelian group. This is in contrast with the divisor class group, which typically has some continuous component (for curves, for example, it has the Jacobian of the curve). Observe that this implies that the numerical equivalence class group is a finitely-generated free abelian group.

For more on algebraic equivalence and proofs of most of these results, see [**Har77**, III.9.8.5] and [**Har77**, Ex. V.1.7]. For a proof of the Néron-Severi theorem see [**Har77**,

[3]There are several different definitions of algebraic equivalence in the literature. In [**Har77**, Ex. III.9.8.5], the definition is very close to the one presented here when restricted to Cartier divisors. In [**Har73**] and most texts on intersection theory ([**Ful98**], for example), the definition of "prealgebraically equivalent" is unnecessary because the curve T is replaced by a variety of any dimension. The divisor D is then replaced by a cycle of appropriate codimension and the divisor (or cycle, in general) D_t is computed as the intersection of D with $X \times \{t\}$. Such an intersection of objects of high codimension requires much more machinery than we are covering in this work. In any case, all these definitions of algebraic equivalence yield the same results when applied to our situation.

App. B, Sec. 5] in the complex case, and see [**LN59**] and [**Har73**] for fields of arbitrary characteristic.

DEFINITION 1.2.57. We say C is equivalent to D in the sense of *algebraic equivalence with division* if there exists some nonzero n such that nC is algebraically equivalent to nD.

REMARK 1.2.58. In [**Mat57**], it is shown that on a nonsingular surface X, a divisor D is numerically equivalent to zero if and only if some multiple nD of it is algebraically equivalent to zero. That is, the concepts of numerical equivalence and algebraic equivalence with division are identical on a nonsingular surface. On a singular surface, we do not yet have an intersection theory, so we cannot yet test this theorem. In Section 3.1 we will see that given a suitable definition of intersection number, the same theorem holds on any surface with isolated singularities. Denote the group of divisors algebraically equivalent to zero $G_a(X)$ and the group of divisors numerically equivalent to zero $G_n(X)$. Let $G_\tau(X)$ be the group of divisors with a multiple algebraically equivalent to zero. The theorem just stated amounts to saying $G_\tau(X) = G_n(X)$. By the Néron-Severi theorem, the group of divisors modulo algebraic equivalence is finitely generated, and so the group $G_\tau(X)/G_a(X)$ is finite.

COROLLARY 1.2.59. *There exists a finite base for the group of divisors modulo algebraic equivalence with division. That is, every divisor is expressible to within this kind of equivalence as an integer linear combination of elements of this base in a unique way.*

1.2.8. The adjunction formula. Given an easy-to-compute intersection form, it is natural to apply it to get results about preexisting divisors, as in the adjunction formula:

THEOREM 1.2.60 (Adjunction Formula). *Let C be a nonsingular curve of arithmetic genus $p_a(C)$ on the nonsingular surface X, and let K be a canonical divisor on X, as defined in Section 1.2.3. Then we have*

$$2p_a(C) - 2 = C.(C + K).$$

See [**Har77**, V.1.5].

EXAMPLE 1.2.61. On $\mathbb{P}^1_{(x_0:x_1)} \times \mathbb{P}^1_{(y_0:y_1)}$, the divisor class group is $\mathbb{Z} \oplus \mathbb{Z}$, generated by $\alpha = \{\mathrm{pt}\} \times \mathbb{P}^1$ and $\beta = \mathbb{P}^1 \times \{\mathrm{pt}\}$. Thus any divisor is equivalent to $a\alpha + b\beta$ for some integers a and b. We will say that such a divisor has type (a, b).

The curves α and β intersect transversally, so clearly $\alpha.\beta = 1$. What is the self-intersection of, say, α? If we consider the rational function $(a_1x_0 - a_0x_1)/(b_1x_0 - b_0x_0)$ we see that α is linearly equivalent to any divisor of the form $\{\mathrm{pt}\} \times \mathbb{P}^1$. Two such divisors do not intersect, so $\alpha.\alpha = 0$. Similarly, $\beta.\beta = 0$. Thus using the basis (α, β), the intersection form has matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

The curves α and β are isomorphic to $\mathbb{P}^1$, so they are clearly of genus 0. Suppose that the canonical divisor K has type (a, b). If we take the first curve C, we have that $-2 = C.K = b$. Similarly, $-2 = a$ and the canonical divisor has type $(-2, -2)$, as we computed in Example 1.2.29.

EXAMPLE 1.2.62. On $\mathbb{P}^2$, consider the curve $H = Z(x_0)$. This is clearly isomorphic to $\mathbb{P}^1$, hence has genus zero. If we write the canonical divisor as n times our curve, we get that $-2 = 1 + n$ or $n = -3$. This implies that the canonical divisor is linearly equivalent to -3 times H, as we computed in Example 1.2.29. So if C is a nonsingular curve of degree d on $\mathbb{P}^2$ of genus g, then $2g - 2 = d(d - 3)$ or

$$g = \frac{1}{2}(d-1)(d-2).$$

EXAMPLE 1.2.63. In Section 1.4.1 we will see that the numerical equivalence class group of the blow-up of the cone on the Fermat curve is $\mathbb{Z} \oplus \mathbb{Z}$, generated by the two divisors C and E described in Example 1.2.16. Unfortunately, the divisor class group is now more complicated than the numerical equivalence class group, but the canonical divisor is primarily important when intersecting it with other divisors, so it will be sufficient for our purposes to find its numerical equivalence class. Consider the divisor C. In that same example, we found another divisor C' such that mC and mC' were linearly equivalent. Then $m(C.C) = C.(mC) = C.(mC') = m(C.C')$. But C and C' did not intersect, so $C.C = 0$. The divisor E, the exceptional fiber, has self-intersection $-m$ (see Section 1.4.1 for an explanation). So any divisor on the surface is numerically equivalent to $aC + bE$ for some integers a and b. We have $(aC + bE).C = b$ and $(aC + bE).D = a - bm$. Thus the intersection matrix in the basis (C, E) has the form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & -m \end{smallmatrix}\right)$.

Now, C is a projective line, so its genus is 0. Then $-2 = C.K$. The curve E is the Fermat curve, so its genus is given by the formula computed in Example 1.2.62: $g = \frac{1}{2}(m-1)(m-2)$. So $(m-1)m = E.E + E.K$. In this case, the self-intersection number of E is $-m$, so we have $m^2 = E.K$. This implies that K is numerically equivalent to $m(m-2)C - 2E$.

1.2.9. The Riemann-Roch theorem. We now come to the Riemann-Roch theorem. This theorem allows one to describe the space of global sections of an invertible sheaf or of a line bundle. It can also be applied in many other situations; we will see some useful corollaries below.

DEFINITION 1.2.64. For any divisor D on a nonsingular surface X, we let

$$\ell(D) = \dim_k H^0(X, \mathcal{L}(D)).$$

We define the *superabundance* $s(D)$ to be $\dim_k H^1(X, \mathcal{L}(D))$.

THEOREM 1.2.65 (Riemann-Roch). *If D is any divisor on a nonsingular surface X and K is a canonical divisor on X, then*

$$\ell(D) - s(D) + \ell(K - D) = \frac{1}{2} D.(D - K) + 1 + p_a(X).$$

See [**Har77**, V.1.6] for a full proof; we will go over a short version here.

PROOF. First observe that $\ell(K - D) = \dim H^0(X, \mathcal{L}(-D) \otimes \omega_X)$, where ω_X denotes the canonical sheaf. By Serre duality (see the Appendix), this is just equal to $\dim H^2(X, \mathcal{L}(D))$. Hence the left-hand side is just the Euler characteristic $\chi(\mathcal{L}(D))$; we need to show that for any D

$$\chi(\mathcal{L}(D)) = \frac{1}{2} D.(D - K) + 1 + p_a.$$

Both sides depend only on the linear equivalence class of D, so we can look for a linearly equivalent divisor that is more convenient. We will find nonsingular curves C and E so that D is linearly equivalent to $C - E$. To see that this is possible, fix an ample divisor H (recall that a divisor H is ample if and only if $\mathcal{L}(H)$ is ample). Then by the definition of ampleness, for k large enough, $\mathcal{L}(D + kH)$ and $\mathcal{L}(kH)$ will be generated by global sections. If we then take ℓ large enough so that ℓH is very ample, we will have $D + (k + \ell)H$ and $(k + \ell)H$ very ample by [**Har77**, Exer. II.7.5]. Now, if F is any very ample divisor, F gives an embedding of our surface into a projective space in which F is cut out by a hyperplane up to linear equivalence. But Bertini's theorem ([**Har77**, II.8.18 and III.7.9.1]) states that we can find another hyperplane which cuts the surface in a nonsingular curve. These two hyperplanes give linearly equivalent divisors, so we see that any very ample divisor is linearly equivalent to a nonsingular curve. Thus, any divisor is linearly equivalent to the difference between two nonsingular curves.

REMARK 1.2.66. On a surface that has isolated singularities, a partial version of this result holds. Since all our surfaces are by definition projective, there is some divisor H that is ample. Then by the definition of ampleness, any divisor D yields $\mathcal{L}(D+kH)$ and $\mathcal{L}(kH)$ that are generated by global sections; however if D is not locally principal, neither is $D+nH$ for any n, and so it can never arise as a hyperplane section. If D is locally principal, however, we can find n so that $D+nH$ and nH are very ample as above. We can then use Bertini's theorem to find a nonsingular curve linearly equivalent to $D+nH$ (or nH) that also avoids the singularities of the surface.

Of course, this is not the only difficulty involved in obtaining an analogue of the Riemann-Roch theorem for singular surfaces; for example, Serre duality becomes more complicated as the canonical sheaf may fail to be locally free, or fails to equal the dualizing sheaf, or in some cases, Serre duality may simply fail to hold. See Section 3.3 and the Appendix for a more thorough discussion.

Now, the ideal sheaf of C is $\mathcal{L}(-C)$ by [**Har77**, II.6.18], and similarly the ideal sheaf of E is $\mathcal{L}(-E)$. This yields the short exact sequences

$$0 \to \mathcal{L}(-E) \to \mathcal{O}_X \to \mathcal{O}_E \to 0,$$

and

$$0 \to \mathcal{L}(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0.$$

Tensoring with $\mathcal{L}(C)$ we get

$$0 \to \mathcal{L}(C-E) \to \mathcal{L}(C) \to \mathcal{L}(C) \otimes O_E \to 0,$$

and

$$0 \to \mathcal{O}_X \to \mathcal{L}(C) \to \mathcal{L}(C) \otimes \mathcal{O}_C \to 0.$$

Now, χ is additive on short exact sequences (See Proposition A.3.4) so we get

$$\chi(\mathcal{L}(C-E)) = \chi(\mathcal{O}_X) + \chi(\mathcal{L}(C) \otimes \mathcal{O}_C) - \chi(\mathcal{L}(C) \otimes \mathcal{O}_E).$$

In Definition 1.2.6 we defined $p_a(X)$ so that $\chi(\mathcal{O}_X) = 1 + p_a(X)$. Since C and E are nonsingular curves, we can apply the Riemann-Roch theorem for nonsingular curves ([**Har77**, IV.1.3]). This yields

$$\chi(\mathcal{L}(C) \otimes \mathcal{O}_C) = \deg_C(\mathcal{L}(C)) + 1 - g_C,$$

and

$$\chi(\mathcal{L}(C) \otimes \mathcal{O}_E) = \deg_E(\mathcal{L}(C)) + 1 - g_E.$$

Now, if F is an irreducible nonsingular curve on our nonsingular surface, and G is any curve on our surface intersecting F transversally, we have [**Har77**, Lem. V.1.3] that asserts that

$$\#(F \cap G) = \deg_F(\mathcal{L}(G) \otimes \mathcal{O}_F).$$

If G does not intersect F transversally, we can always find a linearly equivalent divisor that does, so the formula becomes

$$F.G = \deg_F(\mathcal{L}(G) \otimes \mathcal{O}_F).$$

Substituting this into the results from the Riemann-Roch theorem for curves, we get

$$\chi(\mathcal{L}(C) \otimes \mathcal{O}_C) = C.C + 1 - g_C,$$

and

$$\chi(\mathcal{L}(C) \otimes \mathcal{O}_E) = C.E + 1 - g_E.$$

We can use the adjunction formula to compute the genera of C and E:

$$g_C = \frac{1}{2}C.(C+K) + 1, \quad \text{and} \quad g_E = \frac{1}{2}E.(E+K) + 1.$$

Substituting and expanding, we get:

$$\chi(\mathcal{L}(C-E)) = \frac{1}{2}(C-E).(C-E-K) + 1 + p_a(X),$$

as required. □

This theorem has many applications. We will see several.

EXAMPLE 1.2.67. If a divisor D is ample, then $H^i(X, \mathcal{L}(nD)) = 0$ for all $n \gg 0$ and $i > 0$ (see Proposition A.3.2). Thus $\chi(\mathcal{L}(nD)) = \dim_{\Bbbk}(H^0(X, \mathcal{L}(nD)))$ and for $n \gg 0$ we get

$$\dim_{\Bbbk}(H^0(X, \mathcal{L}(nD))) = \frac{1}{2}(n^2 D.D - nD.K) + 1 + p_a(X).$$

COROLLARY 1.2.68. *Let C and D be divisors on a nonsingular surface X. Then*

$$C.D = \chi(\mathcal{O}_X) - \chi(\mathcal{L}(-C)) - \chi(\mathcal{L}(-D)) + \chi(\mathcal{L}(-C-D)).$$

PROOF. Evaluate the right hand side using the Riemann-Roch theorem:

$$\begin{aligned}
\text{RHS} &= \chi(\mathcal{O}_X) - \chi(\mathcal{L}(-C)) - \chi(\mathcal{L}(-D)) + \chi(\mathcal{L}(-C-D)) \\
&= \chi(\mathcal{O}_X) - \frac{1}{2}(-C).(-C-K) - \chi(\mathcal{O}_X) \\
&\quad - \frac{1}{2}(-D).(-D-K) - \chi(\mathcal{O}_X) \\
&\quad + \frac{1}{2}(-C-D).(-C-D-K) + \chi(\mathcal{O}_X) \\
&= C.D.
\end{aligned}$$

□

COROLLARY 1.2.69 (Adjunction Formula Redux). *Let D be any Cartier divisor. Then the adjunction formula holds for D:*

$$2p_a(D) - 2 = D.(D+K).$$

PROOF. We apply the Riemann-Roch theorem to $-D$:

$$\chi(\mathcal{L}(-D)) = \frac{1}{2}D.(D+K) + 1 + p_a(X).$$

Recall that $p_a(D)$ is defined to be $\chi(\mathcal{L}(-D)) - p_a(X)$ and the result follows immediately. □

1.2.10. Applications of the Riemann-Roch theorem.

THEOREM 1.2.70 (Hodge Index Theorem). *Let H be an ample divisor on the nonsingular surface X and suppose that D is a divisor, D not numerically equivalent to zero, with $D.H = 0$. Then $D^2 < 0$.*

See [**Har77**, V.1.9].

Recall that the Néron-Severi theorem has as a corollary that $\operatorname{Num} X$ is a finitely generated free abelian group. So we can consider the intersection pairing as a bilinear form on $\operatorname{Num} X \otimes_{\mathbb{Z}} \mathbb{R}$. The Hodge Index Theorem then says that in diagonal form, this has one 1 on the diagonal, corresponding to a real multiple of H, and the rest -1.

REMARK 1.2.71. Suppose D is ample. What is $D.D$? Well, let n be such that nH is very ample. Then we have an embedding such that nD is linearly equivalent to a hyperplane section H. In particular, H is effective. By Bertini's theorem, we can find another hyperplane such that its section H' does not share any irreducible components with H. We can be sure the divisors H and H' intersect because the hyperplanes are objects of codimension one, so their intersection will have codimension two; this is guaranteed to have nontrivial intersection with our surface, an object of dimension two. But H and H' are both effective divisors, so their intersection will be positive. Thus $nD.nD > 0$, and so $D.D > 0$. Now let C be any irreducible curve on our surface X. Then we can find a hyperplane section which does not contain C. Since the hyperplane will have codimension one and the curve will have dimension one, they will certainly intersect, and so $nD.C > 0$ and therefore $D.C > 0$. It turns out these two intersection conditions are also sufficient for D to be ample.

THEOREM 1.2.72 (Nakai-Moishezon Criterion). *A divisor D on the nonsingular surface X is ample if and only if $D^2 > 0$ and $D.C > 0$ for all irreducible curves C on X.*

See [**Har77**, V.1.10].

EXAMPLE 1.2.73. On $\mathbb{P}^2$, the hyperplane $H = Z(x)$ is ample, as is any divisor of positive degree; all others have zero or negative intersection with H.

EXAMPLE 1.2.74. On $\mathbb{P}^1 \times \mathbb{P}^1$, the sum of the two generators is ample, for example. More generally, when is the divisor $D = a\alpha + b\beta$ ample? We have $D.\alpha = b$ and $D.\beta = a$, so we must have a and b positive. Then $D.D = 2ab$ is also positive. We have seen in Example 1.2.61 that every curve C is linearly equivalent to $a'\alpha + b'\beta$ for some integers a' and b'. But every curve is an effective divisor, which must have nonnegative intersection with the effective divisors α and β, and we know $b' = \alpha.C \geq 0$ and $a' = \beta.C \geq 0$. If both a' and b' are zero, then C is linearly equivalent to zero, that is, is the divisor of some rational function. But this rational function could be restricted to $\mathbb{P}^1$ to give a rational function with zeros but no poles, which is impossible. So if $D = a\alpha + b\beta$ with a and b positive then $C.D = ab' + ba' > 0$ for every curve C, so D is ample if and only if a and b are both positive.

EXAMPLE 1.2.75. On the blow-up of the cone over the Fermat curve, discussed in Example 1.2.63, the situation is more complicated. We have two divisors, C, which is derived from a ruling on the cone, and E, which is the exceptional fiber.

Suppose the divisor $aC+bE$ is ample. Then in particular, $(aC+bE).C = b$ must be positive. We must also have $(aC + bE).E = a - bm > 0$. Finally, the self-intersection

$$(aC + bE).(aC + bE) = 2ab - b^2m = b(a + a - bm)$$

must be positive. This follows from the first two conditions. Now suppose we have a divisor $aC+bE$ such that $b>0$ and $a>bm$. We have already seen that it must have positive self-intersection. Let D be any curve on the surface. Recalling the definition of our intersection pairing, we see that any two effective divisors have nonnegative intersection number; if their supports have nonempty intersection, their intersection number must be positive. So it is only necessary to show that an arbitrary curve D intersects one of the curves C or E. But we have a projection map from our surface to the Fermat curve, taking $(x_0:x_1:x_2:x_3, y_1:y_2:y_3)$ to $(y_1:y_2:y_3)$. The image of every curve C is either a point or the whole Fermat curve. If the image is a point, then the curve must be the preimage of that point under the projection; in particular, it must intersect E. If the image of the curve is the whole Fermat curve, then in particular, some point gets sent to $(0:1:1)$; the preimage of that point must lie on C. Thus the divisor $aC+bD$ is ample if and only if $b>0$ and $a>bm$.

Theorem 1.2.72 is also true for singular surfaces (or even varieties), provided that one has an adequate definition of all the objects involved. This is provided by the cohomological intersection theory discussed in Section 2.5. See Section 3.4 for details of this more general version.

1.3. Blow-up

When dealing with singular surfaces, it is often very useful to have a tool for producing a birationally equivalent surface that is less singular. For examples of the results in this section, see Section 1.4.1.

To begin with, we define the blow-up of $\mathbb{A}^n$. Consider the variety $\mathbb{A}^n \times \mathbb{P}^{n-1}$ with coordinates $x_1, \ldots, x_n$ and $y_1 : \cdots : y_n$ (note the unusual numbering of the coordinates).

DEFINITION 1.3.1. The *blow-up of* $\mathbb{A}^n$ *at* 0 is the closed set X of $\mathbb{A}^n \times \mathbb{P}^{n-1}$ given by the equations $x_i y_j = x_j y_i$, $i, j = 1, \ldots, n$. The morphism $\phi : X \to \mathbb{A}^n$ associated with this blow-up is simply the restriction of the projection map.

THEOREM 1.3.2. *The morphism ϕ has the following properties:*

(1) *$\phi : X \setminus \phi^{-1}(0) \to \mathbb{A}^n \setminus \{0\}$ is an isomorphism.*

(2) *$\phi^{-1}(0)$ is canonically isomorphic to $\mathbb{P}^{n-1}$ and there is a natural bijection between lines through the origin in $\mathbb{A}^n$ and points in the* exceptional fiber *$\phi^{-1}(0)$. If ℓ is a line and p_ℓ the corresponding point in $\mathbb{P}^{n-1}$ then*

$$\overline{\phi^{-1}(\ell \setminus \{0\})} \cap \phi^{-1}(0) = p_\ell.$$

(3) *X is irreducible.*

See [**Har77**, I.4].

DEFINITION 1.3.3. Let $Y \subset \mathbb{A}^n$ be a variety with $0 \in Y$ and such that $\dim(Y) > 0$. The *blow-up of* Y *at* 0 is $\widetilde{Y} = \overline{\phi^{-1}(Y \setminus \{0\})}$. We call $\widetilde{Y} \cap \phi^{-1}(0)$ the exceptional fiber.

REMARK 1.3.4. The blow-up is also often called the *strict transform*, particularly when discussing a curve on a surface.

Given a set $S \subseteq \mathbb{P}^{n-1}$, let $T = \overline{f^{-1}(S)}$ where $f : \mathbb{A}^n \setminus \{0\} \to \mathbb{P}^{n-1}$ is the canonical morphism. We call T the *cone over* S.

THEOREM 1.3.5. *In the notation above we have:*

(1) $I(\widetilde{Y}) = \langle \{f(x_1, \dots, x_n, y_1 : \dots : y_n) | f(x_1, \dots, x_n, x_1, \dots, x_n) \in I(Y)\}\rangle$,

(2) *Let C be the cone over the projective set $\widetilde{Y} \cap (\{0\} \times \mathbb{P}^{n-1})$ and let $C_{Y,0}$ be the tangent cone to Y at 0. Then $C \cong C_{Y,0}$.*

See [**Gor02**, Thm. 6.3.3].

There is a particularly explicit representation for the ideal of the blow-up.

COROLLARY 1.3.6. *Let Y be a positive dimensional affine variety containing 0. Let $I(Y)^*$ be the ideal generated by the leading forms (the homogeneous parts of lowest degree) of all the polynomials in $I(Y)$. Choose generators $g_1, \dots, g_t$ for $I(Y)$ such that their leading forms generate $I(Y)^*$. For every polynomial $h \in \Bbbk[x_1, \dots, x_n]$ we define a polynomial $H_h(x_1, \dots, x_n, y_1, \dots, y_n)$ as follows: write $h = h_r + \sum_i h_i'$ where h_r is homogeneous of degree r and h_i' is a term of degree $> r$. Then write each h_i' as $p_i q_i$ where q_i is homogeneous of degree r (this can be done in many ways; the choice is arbitrary). Define then $H_h = h_r(y_1, \dots, y_n) + \sum_i p_i(x_1, \dots, x_n) q_i(y_1, \dots, y_n)$.*

Let $I = \left\langle \{x_i y_j - x_j y_i\}_{i,j} \cup \{H_{g_i} | i = 1, \dots, t\} \right\rangle$. Then $Z(I) = \widetilde{Y}$.

See [**Gor02**, Cor. 6.3.4].

Since a blow-up is so closely related to the original surface, one might expect the intersection theory to be related. Suppose for the remainder of this section we have a *nonsingular* surface X and its blow-up $\widetilde{X}$, with projection map ϕ and exceptional fiber E. Then we have a number of results (see [**Har77**, Sec. V.3]):

PROPOSITION 1.3.7. *E is isomorphic to $\mathbb{P}^1$ and $E.E = -1$.*

REMARK 1.3.8. There is a converse to this: if we have a nonsingular surface X' containing a curve E isomorphic to $\mathbb{P}^1$ and having self-intersection -1, then X' is the blow-up of some nonsingular surface (see [**Har77**, Thm. V.5.7], due to Castelnuovo).

We will see in Proposition 2.2.16 that blowing up a singular surface enough to yield a nonsingular surface results in an exceptional manifold with more than one integral component, but that the intersection matrix of these components is always negative definite. In **[Băd01]** there is much discussion of when curves on a nonsingular surface can be blown down to yield a singular surface.

Since we are assuming that our surface X is nonsingular, we know that every divisor is a Cartier divisor, and corresponds to an invertible sheaf. We can pull this invertible sheaf back to $\widetilde{X}$ to get an invertible sheaf there, which corresponds to a Cartier divisor on $\widetilde{X}$. We denote this map from the group of Cartier divisors on X to the group of Cartier divisors on $\widetilde{X}$ by ϕ^*. It takes principal divisors into principal divisors, so it induces a map from $\operatorname{Cl} X$ to $\operatorname{Cl} \widetilde{X}$. Suppose we have a divisor $D = \sum a_P P$ on X. Let $\widetilde{P}$ denote the strict transform of the prime divisor P. We know that ϕ is an isomorphism away from E, so on $\widetilde{X} \setminus E$, the divisor $\phi^*(D)$ must be equal to $\sum a_P \widetilde{P}$. However, we may obtain some multiple of E as well. We will see in Section 2.4 that a similar process is useful on singular surfaces.

PROPOSITION 1.3.9. *The natural maps $\phi^* : \operatorname{Cl} X \to \operatorname{Cl} \widetilde{X}$ and $\mathbb{Z} \to \operatorname{Cl} \widetilde{X}$ given by $1 \mapsto 1 \cdot E$ give rise to an isomorphism $\operatorname{Cl} \widetilde{X} \cong \operatorname{Cl} X \oplus \mathbb{Z}$. The intersection theory on $\widetilde{X}$ is given by:*

1. *If $C, D \in \operatorname{Cl} X$ then $(\phi^* C).(\phi^* D) = C.D$,*
2. *if $C \in \operatorname{Cl} X$, then $(\phi^* C).E = 0$,*
3. *$E.E = -1$, and finally*
4. *if $\phi_* : \operatorname{Cl} \widetilde{X} \to \operatorname{Cl} X$ denotes projection on the first factor, then if $C \in \operatorname{Cl} X$ and $D \in \operatorname{Cl} \widetilde{X}$ then $(\phi^* C).D = C.(\phi * D)$.*

Finally, we have a result about the canonical divisor of $\widetilde{X}$:

PROPOSITION 1.3.10. *The canonical divisor $K_{\widetilde{X}}$ is given by $K_{\widetilde{X}} = \phi^* K_X + E$. Therefore $K_{\widetilde{X}}^2 = K_X^2 - 1$.*

REMARK 1.3.11. Observe that if $D = \sum_P a_P P$ then $\phi^*(D) = \sum_P a_P P + n_D E$ for some n_D uniquely determined by condition 2. We will see that an analogous criterion allows us to define the pullback of a divisor in the singular case as well.

If the original surface is singular, then we do not yet have any intersection theory on it. But one approach would be to use these results to construct an intersection theory on a singular surface by relating it to a sufficiently blown-up version. This is precisely what we do in Section 2.3 and in Section 2.4.

Once such a theory is constructed, there will in fact be a relation between the intersection theories on a singular surface and on its resolved model, but the relationship will be significantly more complicated, since the precise nature of the singularity will affect the exceptional fiber.

1.4. Examples

In order to illustrate intersection theory on surfaces with isolated singularities, it is necessary (or at least extremely useful) to have a handful of examples in which one can work out explicit solutions. So far we have used examples such as $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, which are simple enough that they require no particular computation. However, our examples of singular surfaces will be more complicated, so it makes sense to pause and develop some useful facts about them here.

1.4.1. The cone on a curve. Our first family of examples will be the cones on nonsingular curves. Let K be a nonsingular curve (that is, a nise projective scheme of dimension one) embedded in $\mathbb{P}^n_{(y_1 : \cdots : y_{n+1})}$, defined by the homogeneous polynomials $f_1, \ldots, f_k$.

DEFINITION 1.4.1. The *cone C on K* is the subset $Z(f_1, \ldots, f_k)$ of $\mathbb{A}^{n+1}$. The *projective cone X on K* is the projectivization of C, that is, the closure in $\mathbb{P}^{n+1}_{(y_0 : \cdots : y_{n+1})}$ of the image of the cone under $(y_1, \ldots, y_{n+1}) \mapsto (1 : y_1 : \ldots : y_{n+1})$.

REMARK 1.4.2. The projectivization X of C is in fact defined by the same polynomials $f_1, \ldots, f_k$, now interpreted as homogeneous polynomials in $n+2$ variables. To see this, we need only check that there are no surplus points at infinity creeping in. So take $(0 : x_1 : \cdots : x_n)$ satisfying all f_i. Since f_i does not mention x_0, $(t : x_1 : \cdots : x_n)$ is on the cone for all $t \neq 0$, and $(0 : x_1 : \cdots : x_n)$ must be on the closure.

For the purposes of demonstrating the intersection theory of Section 2.3, we will need to find a resolved model (see Definition 2.2.10) of the cone. We will see that a single blow-up suffices to accomplish this. Let $\widetilde{C}$ denote the blow-up of the cone C at the origin, and let $\widetilde{X}$ denote the blow-up of X at the origin.

PROPOSITION 1.4.3. *The surfaces C and X are nonsingular except at the origin. At the origin, they are singular unless K is a projective line in $\mathbb{P}^n$. The surface $\widetilde{C}$ is nonsingular and given by*

$$\widetilde{C} = Z\Big(\{x_i y_j - x_j y_i\}_{i,j}, \{f_i(y_1, \ldots, y_{n+1})\}_i\Big) \subset \mathbb{A}^{n+1}_{(x_1, \ldots, x_{n+1})} \times \mathbb{P}^n_{(y_1 : \cdots : y_{n+1})}.$$

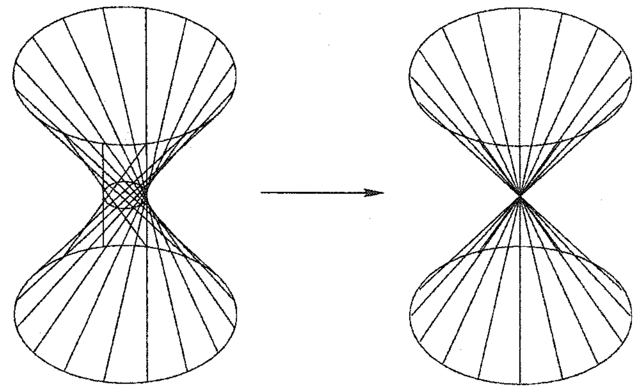

FIGURE 1.4.1. The blow-up $\widetilde{C}$ of the cone C on $x_0^2 + x_1^2 - x_2^2$

The surface $\widetilde{X}$ is nonsingular, projective, and is given by

$$\widetilde{X} = Z\Big(\{x_iy_j - x_jy_i\}_{i,j>0}, \{f_i(y_1, \ldots, y_{n+1})\}_i\Big) \subset \mathbb{P}^{n+1}_{(x_0 : \cdots : x_{n+1})} \times \mathbb{P}^n_{(y_1 : \cdots : y_{n+1})}.$$

Finally, $\widetilde{X}$ is the projectivization of $\widetilde{C}$. In both cases, the projection map ϕ onto C or X respectively is given by taking the projection on the first component, and the exceptional fiber $\phi^{-1}(0)$ is isomorphic to K.

PROOF. Let us begin by examining the blow-up $\widetilde{C}$. Using Corollary 1.3.6, we see that

$$\widetilde{C} = Z\Big(\{x_iy_j - x_jy_i\}_{i,j}, \{f_i(y_1, \ldots, y_{n+1})\}_i\Big) \subset \mathbb{A}^{n+1} \times \mathbb{P}^n.$$

We also note that the cone over the special fiber is isomorphic to the tangent cone to C at the origin. This is just C, the cone over K, so that the special fiber is just K.

The points on $\widetilde{C}$ are just $(ty_1, \ldots, ty_{n+1}, y_1 : \cdots : y_{n+1})$ for $(y_1 : \cdots : y_{n+1}) \in K$ and $t \in k$. Fix t and y and compute the Jacobian matrix with respect to the polynomials $\{f_i(y_1, \ldots, y_{n+1})\}_i$ and $\{x_iy_j - x_jy_i\}_{i,j}$ we gave above. We obtain a matrix that looks like

$$\begin{pmatrix} \mathbf{M}_b & * \\ 0 & \mathbf{M}_k \end{pmatrix},$$

where $\mathbf{M}_k$ is the Jacobian matrix we have for K at y, which has rank $n - m$ and where $\mathbf{M}_b$ is the matrix of row vectors

$$\begin{aligned} v_{ij} &= \left(\frac{\partial}{\partial x_1}(x_iy_j - x_jy_i), \ldots, \frac{\partial}{\partial x_{n+1}}(x_iy_j - x_jy_i)\right) \\ &= (0, \ldots, y_j, \ldots, -y_i, \ldots, 0). \end{aligned}$$

Now, the y_i are not all zero; in fact, we can change coordinates so that $y_1 = 1$ and all the other y_i are zero (the polynomials $x_i y_j - x_j y_i$ can still be used because they have the same zeros as the polynomials we get when we change coordinates). Then we are left with n independent row vectors, giving our whole matrix a rank of $2n-m = (2n+1)-(m+1)$. This implies that $\widetilde{C}$ is nonsingular. Since $\widetilde{C}$ is isomorphic to C away from the exceptional fiber, this also implies that C is nonsingular except possibly at the origin. At the origin, we know that the tangent cone is isomorphic to C; the tangent space will be the linear span of C. C will be nonsingular at the origin only if the tangent space is two-dimensional, which can happen only if C lies in a plane, which can happen only if K is a line in $\mathbb{P}^n$.

Recall that X is the projectivization of C, given by the same equations; a very similar argument, together with the fact that the blow-up is a purely local operation, implies that $\widetilde{X}$ is the projectivization of $\widetilde{C}$, and is given by the same equations, and is in particular nonsingular. □

We have the following commutative diagram:

$$\begin{tikzcd}
\widetilde{C} \arrow[rr, hook] \arrow[d] \arrow[dddr] & & \widetilde{X} \arrow[d, "\pi_1"] \arrow[dddl, "\pi_2"'] \\
C \arrow[rr, hook] & & X \\
C \setminus \{0\} \arrow[u, hook] \arrow[rr, hook] \arrow[dr] & & X \setminus \{0\} \arrow[u, hook] \arrow[dl, "\psi"] \\
& K &
\end{tikzcd}$$

PROPOSITION 1.4.4. *If the curve K is a curve of degree d, then the blow-up $\widetilde{C}$ of C is a line bundle over K with degree $-d$. It is equivalent to the divisor of y_1 on K. Similarly, $\widetilde{X}$ is a $\mathbb{P}^1$-bundle over K, that is, a ruled surface over K.*

PROOF. If we fix i and look at those points y on $\widetilde{C}$ where the ith coordinate is one, we get a trivialization: take

$$(ty_1, \ldots, t, \ldots, ty_{n+1}, y_1 : \cdots : 1 : \cdots : y_{n+1}) \mapsto (t, y_1, \ldots, y_{n+1}).$$

This exhibits $\widetilde{C}$ as a line bundle over K. Let us try to determine what divisor on the curve this corresponds to by constructing a meromorphic section. Take this chart and begin with $i = 1$, setting $t = 1$. Then for $j \neq i$, we have

$$(1, y_2, \ldots, y_{n+1}, 1 : y_2 : \cdots : y_{n+1}) = \left(ty'_1, \ldots, t, \ldots, ty'_{n+1}, y'_1 : \cdots : 1 : \cdots : y'_{n+1}\right),$$

so $y'_k = (1/y_j)y_k$ and $ty'_1 = 1$ so $t = y_j = 1/y'_1$. So this section has a pole of order k when $K \subset \mathbb{P}^n_{(y_1 : \cdots : y_{n+1})}$ intersects $Z(y_1)$ with multiplicity k. But we have Bezout's theorem (Theorem 1.1.12), which asserts that if K is a curve of degree d, this happens d times, so we have a line bundle of degree $-d$.

If we now look at $\widetilde{X}$, we see that the same trivializations work:

$$(s : ty_1 : \cdots : t : \cdots : ty_{n+1}, y_1 : \cdots : 1 : \cdots : y_{n+1}) \mapsto ((s : t), y_1, \ldots, y_{n+1}).$$

$\square$

Let us first describe the behavior of divisors on the cone X. We will not be able to describe the intersection theory on X since we have as yet no intersection theory even defined on X, but we can still describe the divisor class group. In [**Har77**, Exer. II.6.3], the relationship between a variety V, the cone on V, and the projective cone on V are discussed. Applying this to our situation we extract several results.

PROPOSITION 1.4.5. *Let ψ be the projection map from $X \setminus \{0\}$ to K. Then:*

(1) $\operatorname{Cl} X = \operatorname{Cl} K$, *with the isomorphism induced by* ψ^*,

(2) *we have an exact sequence*

$$0 \to \mathbb{Z} \to \operatorname{Cl} K \to \operatorname{Cl} C \to 0 \tag{1.4.1}$$

where the first arrow takes 1 *to the intersection of any hyperplane with* K, *and the second arrow is* ψ^*, *and*

(3) *if* $\mathcal{O}_P$ *is the local ring of the vertex of the cone,* $\operatorname{Cl} C = \operatorname{Cl} \operatorname{Spec} \mathcal{O}_P$.

REMARK 1.4.6. Recall that K is a curve. Thus we know $\operatorname{Cl} K$ is an extension of $\mathbb{Z}$ by the Jacobian $\operatorname{Jac} K$ of the curve K, an abelian variety of dimension equal to the genus of K.

REMARK 1.4.7. In the exact sequence (1.4.1), the first map takes 1 to the intersection of any hyperplane with K. But if K is a curve of degree d, Bezout's theorem (Theorem 1.1.12) implies that this is a divisor of degree d on K. This is of particular interest because we know that on a nonsingular surface, the local class group of any point is trivial; here we see some behavior peculiar to singular surfaces. In particular, we have a divisor that is not locally principal. Some multiple of it may be locally principal. If for example K is $\mathbb{P}^1$ embedded with degree 2 in $\mathbb{P}^2$, then there is exactly one divisor (up to linear equivalence) that is not locally principal, and twice that divisor is locally principal. If on the other hand K is an elliptic curve and $\Bbbk$ is the field of complex numbers, the Jacobian of K will again be K, and in particular it will have a point of infinite order.

What can we say about the divisors on the nonsingular surface $\widetilde{X}$? We have seen that $\widetilde{X}$ is a ruled surface over K. From [**Har77**, V.2.3] we obtain:

PROPOSITION 1.4.8. *Let π_2 be the projection from $\widetilde{X}$ to K. Then*

$$\operatorname{Cl}\widetilde{X} \cong \mathbb{Z} \oplus \pi_2^* \operatorname{Cl} K,$$

and

$$\operatorname{Num}\widetilde{X} = \mathbb{Z} \oplus \mathbb{Z}.$$

The class group $\operatorname{Cl}\widetilde{X}$ is generated by the exceptional fiber E and by $\{\pi_2^*(D_i)\}$ for a collection $\{D_i\}$ of generators of $\operatorname{Cl} K$. The numerical equivalence class group $\operatorname{Num}\widetilde{X}$ is generated by E and any ruling of the cone.

Further, we can describe the intersection pairing on $\widetilde{X}$.

PROPOSITION 1.4.9. *Let E be the exceptional fiber and let R be a ruling on the cone. If K is a curve of degree m, then*

(1) $E.E = -m$,
(2) $E.R = 1$, *and*
(3) $R.R = 0$.

LEMMA 1.4.10. *Let $\mathfrak{L}$ be a line bundle on a variety X, and let S be the zero section. Let $\mathcal{N}_{S/\mathfrak{L}}$ be the normal sheaf of S in $\mathfrak{L}$ (if S is defined as a closed subscheme of $\mathfrak{L}$ by the sheaf of ideals $\mathcal{I}$, then $\mathcal{N}_{S/\mathfrak{L}}$ is $\mathcal{H}om_{\mathcal{O}_\mathfrak{L}}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_\mathfrak{L})$; see* [**Har77**, Sec. 8]*). Then the line bundle corresponding to the invertible sheaf $\mathcal{N}_{S/\mathfrak{L}}$ is isomorphic to $\mathfrak{L}$.*

PROOF. Let $\{(U_i, \phi_i\}$ be a local trivialization for $\mathfrak{L}$, and let π be the projection function for $\mathfrak{L}$. Let the transition functions for $\mathfrak{L}$ be (f_{ij}). Observe first that S is isomorphic to X.

On $\pi^{-1}(U_i)$, ϕ_i establishes an isomorphism between $\pi^{-1}(U_i)$ and $\mathbb{A}^1 \times U_i$. Writing an arbitrary point of $\mathbb{A}^1 \times U_i$ as (t, x), the zero section S is locally defined by the polynomial t. On $U_j \cap U_i$, (t, x) is mapped to $(f_{ij}t, x)$, so this defining polynomial for S is taken to $f_{ij}^{-1}t$. Thus if $\mathcal{I}$ is the ideal sheaf defining S, then the line bundle associated to $\mathcal{I}/\mathcal{I}^2$ has transition functions f_{ij}^{-1}. So its dual, which is by definition the line bundle associated to $\mathcal{N}_{S/\mathcal{L}}$, has transition functions f_{ij} and is therefore isomorphic to $\mathcal{L}$. ☐

PROOF OF PROPOSITION 1.4.9. The ruling R on $\widetilde{X}$ is the set of common zeros of $\langle y_i - c_i, i = 1, \dots, n\rangle$ for some set of constants c_i. The exceptional fiber E is the set of zeros of $\langle x_1, \dots, x_n\rangle$. They intersect at $(1:0:\cdots:0, c_1:\cdots:c_n)$, and clearly they generate the maximal ideal there, so these two divisors intersect transversally and $E.R = 1$.

Recall that we have a projection map π_2 to the original curve K. Any ruling on the blow-up of the cone is the pullback of a point on the original curve. This map shows that all the rulings on the cone are algebraically equivalent. But two different rulings do not intersect, so each ruling must have self-intersection 0.

Let us compute the self-intersection of the exceptional fiber E. By **[Har77**, Ex. V.1.4.1], $E.E = \deg_E \mathcal{N}_{E/X}$. But as a line bundle $\mathcal{N}_{E/X}$ is isomorphic to $\widetilde{C}$ by Lemma 1.4.10, which has degree $-m$ by Proposition 1.4.4. ☐

1.4.2. Toric varieties. A particularly manageable class of singular varieties is the class of toric varieties. These are all birational to the torus $(\Bbbk^*)^n$ for some n, but they can be singular. When they are, the resolution of those singularities is relatively straightforward. The singularities are always Cohen-Macaulay (see Section 2.1.4 for what this means). Every toric variety is acted on in a natural way by the torus, and it turns out that every variety with the torus as a dense open subset that is acted on by the torus is a toric variety (see **[Oda78**, Sec. I.4]). In studying divisors on a toric variety, one may focus on those invariant under the torus action; such divisors have a simple description.

Toric varieties are manageable chiefly because they are constructed from discrete objects, in particular fans of cones (which will be defined later). Properties of the toric varieties usually arise from straightforward properties of the fans, and relations between different fans of cones yield relations between the toric varieties they yield.

1.4.2.1. *Basic definitions.* Our treatment will follow **[Ful93]** closely; refer there for more detail.

DEFINITION 1.4.11. Let N be a finitely generated free abelian group. Then N is a lattice in $N \otimes \mathbb{R}$ which we denote $N_{\mathbb{R}}$. Say σ is a *rational polyhedral cone* if there exist $v_1, \dots, v_m \in N$ such that

$$\sigma = \langle v_1, \dots, v_m\rangle = \{r_1 v_1 + \cdots + r_m v_m | r_i \geq 0 \ \forall i\}.$$

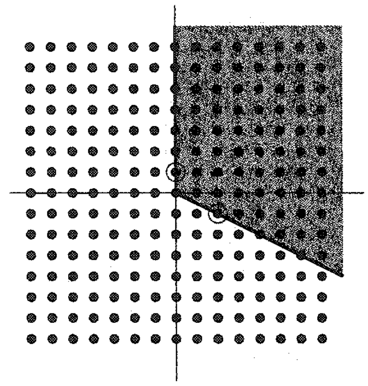

FIGURE 1.4.2. The cone σ_1 and its generators in N

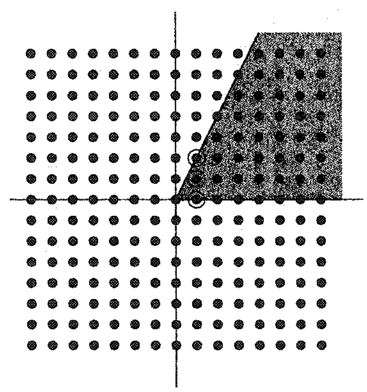

FIGURE 1.4.3. The cone $\sigma_1^\vee$ and its generators in M

σ is *strongly convex* if $v \in \sigma$ implies $-v \notin \sigma$. We will often say simply *cone* instead of "strongly convex rational polyhedral cone" when no confusion seems likely to result.

EXAMPLE 1.4.12. Take N generated by e_1 and e_2. Let $\sigma_1 = \langle e_2, 2e_1 - e_2 \rangle$ as in Figure 1.4.2. This is a strongly convex rational polyhedral cone as defined above.

DEFINITION 1.4.13. Let $M = \mathrm{Hom}\,(N, \mathbb{Z})$. If σ is a cone, then the *dual cone* $\sigma^\vee$ is defined by

$$\sigma^\vee = \{u \in M_\mathbb{R} | \forall v \in N_\mathbb{R}\ u(v) \geq 0\}.$$

REMARK 1.4.14. One easily shows that $\sigma^\vee$ is a rational polyhedral cone, and $\sigma^\vee \cap M$ is a finitely generated semigroup. As we would expect, $(\sigma^\vee)^\vee = \sigma$.

EXAMPLE 1.4.15. Looking again at the cone from Example 1.4.12, we see that

$$\sigma_1^\vee = \langle e_1^*, e_1^* + 2e_2^* \rangle,$$

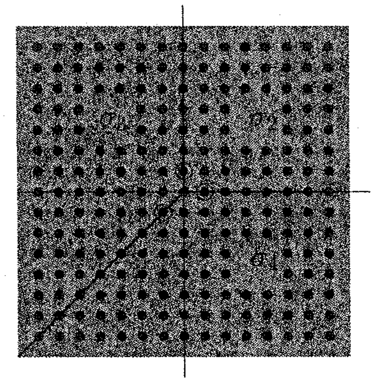

FIGURE 1.4.4. The fan Δ_2 and the generators of its cones in N

where $\{e_1^*, e_2^*\}$ is the dual basis, so that $e_i^*(e_j) = \delta_{ij}$. Observe that $\sigma_1^\vee \cap M$ is generated by $\{e_1^*, e_1^* + e_2^*, e_1^* + 2e_2^*\}$ but not by any subset of these.

DEFINITION 1.4.16. Let u be any vector in $M_\mathbb{R}$. Define $u^\perp = \{v \in N_\mathbb{R} | u(v) = 0\}$. Then a subset τ is a *face* of σ if $\tau = \sigma \cap u^\perp$ for some u in $\sigma^\vee$. A *fan of cones* is a collection Δ of cones such that every face of a cone in Δ is a cone in Δ and such that if σ and σ' are cones in Δ, then $\sigma \cap \sigma'$ is a face of both σ and σ'.

EXAMPLE 1.4.17. The faces of σ_1 are σ_1 itself, $\langle e_1 \rangle$, $\langle 2e_1 - e_2 \rangle$, and $\langle 0 \rangle$. This forms a fan of cones; in fact, taking all the faces of any cone yields a fan Δ_1.

EXAMPLE 1.4.18. For a less trivial fan, let $\sigma_2 = \langle e_1, e_2 \rangle$, $\sigma_3 = \langle e_2, -e_1 - e_2 \rangle$, and $\sigma_4 = \langle -e_1 - e_2, e_1 \rangle$ as in Figure 1.4.4. Take Δ_2 to be the collection of all the faces of σ_2, σ_3, and σ_4.

Having defined fans of cones, we can now begin to construct varieties, our original goal.

DEFINITION 1.4.19. Let σ be a cone. Define S_σ to be the (finitely generated) semigroup $\sigma^\vee \cap M$. Let $\Bbbk[S_\sigma]$ be the semigroup algebra, that is, $\Bbbk$ extended by monomials $\chi(u)$ for every u in S_σ with the relations $\chi(u)\chi(v) = \chi(u + v)$. We define the *affine toric variety* U_σ to be $\operatorname{Spec} \Bbbk[S_\sigma]$.

EXAMPLE 1.4.20. Taking again the cone σ_1, recall that the semigroup S_{σ_1} is equal to $\mathbb{Z}e_1^* + \mathbb{Z}(e_1^* + e_2^*) + \mathbb{Z}(e_1^* + 2e_2^*)$, so

$$\Bbbk[S_{\sigma_1}] = \Bbbk[X, XY, XY^2] = \Bbbk[U, V, W]/\langle UW - V^2 \rangle.$$

Thus U_{σ_1} is the quadric cone.

REMARK 1.4.21. If τ is a face of σ, then we have $S_\sigma \subset S_\tau$, so we have a morphism from U_τ into U_σ. This embeds U_τ as a principal open subset of U_σ.

DEFINITION 1.4.22. Let Δ be a fan of cones. Then the cones of Δ form a set that is partially ordered by the relation "is a face of". Whenever σ is a face of σ', one has a morphism $f_{\sigma,\sigma'}$ embedding U_σ into $U_{\sigma'}$. Construct the scheme $X(\Delta)$ from the disjoint union of all the U_σ by gluing each $x \in U_\sigma$ to $f_{\sigma,\sigma'}(x)$. Then we say $X(\Delta)$ is the *toric variety* associated to Δ.

EXAMPLE 1.4.23. If Δ is the fan of faces of a single cone σ, then $X(\Delta)$ is just U_σ. So for the cone σ_1, the toric variety we obtain is just the quadric cone.

EXAMPLE 1.4.24. Now consider Δ_2, the fan from Example 1.4.18. $U_{\sigma_2} = \Bbbk[X, Y]$, $U_{\sigma_3} = \Bbbk[X^{-1}, X^{-1}Y]$, and $U_{\sigma_4} = \Bbbk[XY^{-1}, Y^{-1}]$. All three are isomorphic to $\mathbb{A}^2$. I claim these glue together in the usual way to give $\mathbb{P}^2$: if (T_0, T_1, T_2) are the homogeneous coordinates on $\mathbb{P}^2$, then $X = T_1/T_0$ and $Y = T_2/T_0$.

REMARK 1.4.25. Notice that every cone contains the face $\{0\}$. If the lattice N is n-dimensional, then $U_{\{0\}} = \operatorname{Spec} \Bbbk[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}] = (\Bbbk^*)^n$, the torus of algebraic groups. Thus every toric variety contains the torus as a dense open subset (hence the name). It is also clear that a toric variety will have the same dimension as N, the lattice on which it is built.

1.4.2.2. *Points on a toric variety.* There is a fairly simple description of the closed points on an affine toric variety. A closed point on the affine variety U_σ is by definition a maximal ideal in $\Bbbk[S_\sigma]$, which corresponds to a homomorphism from $\Bbbk[S_\sigma]$ to $\Bbbk$. Such a homomorphism is just given by a semigroup homomorphism from S_σ to $\Bbbk$, where $\Bbbk$ is considered as a semigroup with respect to multiplication.

This description leads to a natural action of the torus on any toric variety. In particular, a point t on the torus contained in U_σ is a map of semigroups from $S_{\{0\}} = M$ to $\Bbbk^*$. If x is a point on U_σ, we define the multiplication $t \cdot x$ so that

$$(t \cdot x)(u) = t(u)x(u).$$

EXAMPLE 1.4.26. Let $X(\Delta_2)$ be the toric variety of Examples 1.4.18 and 1.4.24. We have seen that $X(\Delta_2)$ is isomorphic to $\mathbb{P}^2_{(x_0 : x_1 : x_2)}$. What is the torus action on $X(\Delta_2)$? We have $N = \mathbb{Z}e_1 + \mathbb{Z}e_2$ so write $M = \mathbb{Z}e_1^* + \mathbb{Z}e_2^*$. Then any point of the torus is (k_1, k_2) for $k_i \in \Bbbk^*$; we can identify this with the semigroup homomorphism $(n_1e_1^* + n_2e_2^*) \mapsto k_1^{n_1}k_2^{n_2}$. Consider first $U_{\sigma_2} \cong \mathbb{A}^2$. Then the point (x, y) corresponds to the map $(n_1e_1^* + n_2e_2^*) \mapsto x^{n_1}y^{n_2}$. We can see that the torus action here is

$$(k_1, k_2) \cdot (x, y) \mapsto (k_1x, k_2y).$$

By continuity we can infer that the torus action on $\mathbb{P}^2_{(x_0:x_1:x_2)}$ is given by

$$(k_1, k_2) \cdot (x_0 : x_1 : x_2) \mapsto (x_0 : k_1x_1 : k_2x_2).$$

If $u \in S_\sigma$, then from the monomial χ^u we can obtain a regular function on U_σ: its value at a point x is $x(u)$.

DEFINITION 1.4.27. Let U_σ be an affine toric variety. There is a canonical distinguished point x_σ in U_σ defined by

$$x_\sigma(u) = \begin{cases} 1 & \text{if } u \in \sigma^\perp \\ 0 & \text{otherwise.} \end{cases}$$

If σ spans $N_\mathbb{R}$, this is the unique fixed point of the torus action; if σ does not span $N_\mathbb{R}$, there are no fixed points (see [**Ful93**, Sec. 2.1]).

For each cone σ we also obtain a closed subset of $X(\Delta)$ by letting V_σ be the closure of the orbit of x_σ. Each V_σ is then a complete integral scheme with an open dense torus embedding, and it turns out that V_σ is again a toric variety whose fan can be explicitly computed. We can see from the definition that the V_σ are the only irreducible closed subsets of $X(\Delta)$ invariant under the torus action.

We have an inclusion-reversing correspondence between these V_σ and the cones σ in the fan Δ; in particular, if σ spans $N_\mathbb{R}$, V_σ is just the point x_σ, if $\sigma = 0$ then $V_\sigma = X(\Delta)$, and if σ has dimension one we call it an *edge* and V_σ has codimension one.

EXAMPLE 1.4.28. Returning to our toric variety $X(\Delta_2)$ from Example 1.4.26, what is V_σ for the various cones σ of Δ_2? Looking at the torus action, we see that the fixed points are $(1:0:0) \in U_{\sigma_2}$, $(0:1:0) \in U_{\sigma_3}$, and $(0:0:1) \in U_{\sigma_4}$. Thus, for example, $V_{\sigma_2} = \{x_{\sigma_2}\} = (1:0:0)$. Suppose we let σ be the edge generated by e_1. Then S_σ is generated by e_1^*, e_2^* and $-e_2^*$ and $\sigma^\perp$ is generated by e_2^* and $-e_2^*$. Then x_σ is defined by

$$(n_1e_1^* + n_2e_2^*) \mapsto \begin{cases} 1 & \text{if } n_2 = 0 \\ 0 & \text{otherwise} \end{cases} = 1^{n_1}0^{n_2}.$$

We can recognize this as the point $(1,0)$ in $\mathbb{A}^2$, which corresponds to the point $(1:1:0)$ in $\mathbb{P}^2$. The orbit of this point is $\{(1:k:0) | k \in \Bbbk^*\}$, so $V_\sigma = Z(x_2) \subset \mathbb{P}^2_{(x_0:x_1:x_2)}$.

We have already seen (in Example 1.4.12) that toric varieties may be singular. In [**Ful93**, Sec. 2.1] we find a simple characterization of when this happens:

PROPOSITION 1.4.29. *An affine toric variety U_σ is nonsingular if and only if $\sigma \cap N$ is generated by part of a basis of the lattice N. In this case $U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$ where* $\Bbbk$ *is the dimension of σ as a cone in $N_\mathbb{R}$. If on the other hand U_σ is singular, it will be singular at exactly the point x_σ.*

In Section 2.1 we will describe many different kinds of singularity. The singularities that occur in the category of toric varieties are rather limited. In particular, every toric variety is normal and Cohen-Macaulay.

1.4.2.3. *Resolution of singularities.* In view of the simple characterization of singularities given above, it is quite straightforward to take a two-dimensional affine toric variety and produce a nonsingular toric variety which is closely related. One simply introduces new edges subdividing the cone until the generators of each cone generate the lattice. We will give a more explicit description of this process that shows that this is always possible.

Let Δ and Δ' be fans on N and N', and suppose we have a map f from N to N' such that if σ is a cone of Δ then $f(\sigma)$ is contained in some cone σ' of Δ'. Then f induces a map from each $S_{\sigma'}$ to S_σ, and we get a morphism from $X(\Delta)$ to $X(\Delta')$. Now suppose both fans are two-dimensional and Δ is obtained by subdividing some two-dimensional cone σ' of Δ', and further that f is just the identity. Then the map of toric varieties is an isomorphism on the toric variety obtained by deleting σ' and its preimage. But since σ' is two-dimensional, deleting it removes the single point $x_{\sigma'}$. Thus we have an isomorphism except for a single point, whose preimage is V_τ for every new edge τ, just as in Section 1.4.1. See Example 1.4.30 for a worked example.

Let σ be a two-dimensional cone. Then any minimal generator e_2 along an edge is part of a basis $\{e_1, e_2\}$ for N. The other generator is $me_1 - ke_2$ for m some positive integer and k any integer. Taking an automorphism of the lattice of the form $\left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$, we can transform this to $(m, cm + k)$, so we can take $0 \leq k < m$ without loss of generality. Then the cone σ is generated by e_2 and $me_1 - ke_2$. Since we chose a minimal generator along the edge, m and k will be relatively prime. Now, the group G of m-th roots of unity acts on $\mathbb{A}^2$ by $\zeta(u, v) = (\zeta u, \zeta^k v)$, and it turns out that $U_\sigma \cong \mathbb{A}^2/G$. So all singularities of toric surfaces (and in fact toric varieties of any dimension) arise as quotients of a nonsingular variety by a finite group.

Now suppose we have a singular affine toric variety generated by the cone σ, and that σ is generated as above by $(0, 1)$ and $(m, -k)$. The goal is to produce

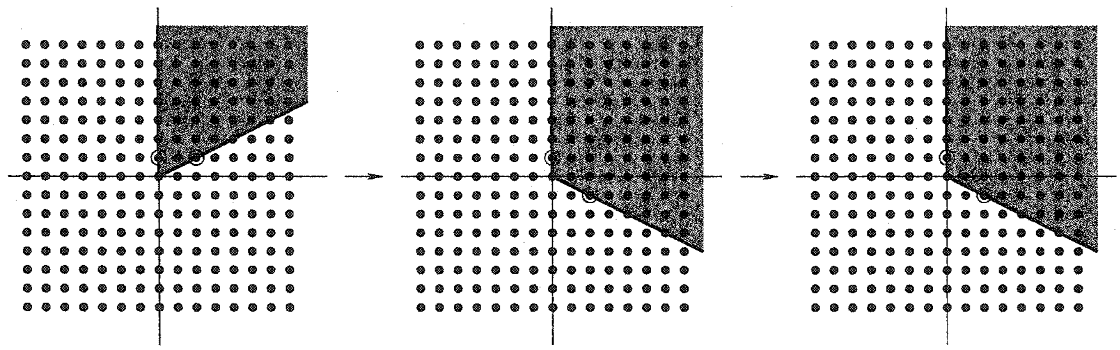

FIGURE 1.4.5. The process of resolving the singularity of an affine toric variety

a nonsingular toric variety by subdividing σ; this will yield a map of just the sort we need to produce a resolved model in the sense of Section 2.3. So insert the line through e_1. Then we have two cones $\sigma' = \langle e_1, e_2\rangle$ and $\sigma'' = \langle e_1, me_1 - ke_2\rangle$. But σ'' is "less singular" than the initial cone σ: rotate it though ninety degrees, so that one generator is e_2 and the other is $ke_1 + me_2$, then translate this point as before so we obtain $ke_1 - k'e_2$ for some $0 \leq k' < k$, $k' = m - a_1k$. We can carry on this process just like the Euclidean algorithm until we obtain a nonsingular cone. One can view this as the construction of the continued fraction

$$\frac{m}{k} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\cdots - \cfrac{1}{a_r}}}.$$

In this way we obtain a resolved model of the original affine toric variety. The exceptional fiber is a chain of $\mathbb{P}^1$s intersecting transversally and having self-intersection $-a_i$.

EXAMPLE 1.4.30. Consider the affine toric variety U_{σ_1} of Example 1.4.12. We defined σ_1 to be the cone in $N_{\mathbb{R}}$ generated by e_2 and $2e_1 - e_2$. These clearly do not generate the lattice, so we see that U_{σ_1} is singular, which we knew already. Let us follow the algorithm described above to resolve this singularity.

We must first choose generators for the lattice N so that the generator along one side is $(0,1)$ and the generator along the other is $(m, -k)$ for $0 < k < m$. For this cone, e_1 and e_2 will do. Then we are supposed to subdivide the cone by introducing the edge generated by e_1. Then we obtain two cones, σ_5, generated by e_1 and e_2, and σ_6 generated by e_1 and $2e_1 - e_2$. In this case both cones are nonsingular, so we stop.

The continued fraction we constructed was particularly simple:

$$\frac{1}{2} = \frac{1}{-(-2)}.$$

As a result we obtain a nonsingular toric variety $X(\Delta_3)$ covered by two affine toric varieties; the exceptional fiber has self-intersection -2. We see that this is the same self-intersection we would have obtained by simply blowing up the quadric cone at its singular point, and in fact this is the same variety (this can be seen using the theory of minimal models, since neither contains a rational curve with self-intersection -1; see [**Har77**, Sec. V.5]).

We have seen that the local behavior of toric varieties is fairly simple to compute from the structure of the cones that make them up. The global structure is also relatively easy to compute from the fan of cones. For example:

PROPOSITION 1.4.31. *The toric variety defined by Δ is proper over $\Bbbk$ if and only if the cones of Δ together cover all of $N_{\mathbb{R}}$.*

See [**Ful93**, Sec. 2.4].

1.4.2.4. *Divisors and the torus action.* Since toric varieties have a natural torus action, it is natural to focus on divisors that are invariant under the torus action. We say that a divisor is a T-Weil divisor if it is invariant under the torus action. This is particularly useful because of:

PROPOSITION 1.4.32. *Every Weil divisor on a toric variety is linearly equivalent to a T-Weil divisor.*

See [**Ful93**, Sec. 3.3] for this and for the next few remarks.

Recall that we have a description of the orbits of the torus action, so we know that every T-Weil divisor is of the form $\sum a_\tau V_\tau$ as τ ranges over all the edges (cones of dimension 1) in Δ.

Let U_σ be an affine toric variety. Then in [**Ful93**, Sec. 3.3] it is shown that every T-Cartier divisor is the divisor of $\chi(u)$ for some u in M. Let τ be any edge of σ. If v_τ is the first lattice point along τ, then the divisor of $\chi(u)$ is $\sum \langle u, v_\tau \rangle V_\tau$. Clearly the divisor of u will equal the divisor of u' if and only if $\langle u - u', v \rangle = 0$ for every v in σ. So the group of T-Cartier divisors is isomorphic to $M/(\sigma^\perp \cap M)$.

Now let Δ be a fan and let $X(\Delta)$ be the associated toric variety. Then a T-Cartier divisor on $X(\Delta)$ is given by a T-Cartier divisor $\chi(-u_\sigma)$ on each cone σ in Δ with the criterion that if $\sigma \subset \sigma'$, $u_\sigma - u_{\sigma'} \in \sigma^\perp \cap M$.

The map $\phi : u \mapsto \chi(u)$ gives a map from M to the group G of T-Cartier divisors. Recall that the group of Cartier divisors modulo the group of principal divisors is denoted $\operatorname{CaCl} X$. Using ϕ, we get the following:

PROPOSITION 1.4.33. *Let $X = X(\Delta)$ be a toric variety, defined by an n-dimensional fan Δ that is not contained in any proper subspace of $N_{\mathbb{R}}$. Then there is a commutative diagram with exact rows:*

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "\phi"] \arrow[d, equal] & G \arrow[r] \arrow[d, hook] & \operatorname{CaCl} X \arrow[r] \arrow[d, hook] & 0 . \\ 0 \arrow[r] & M \arrow[r, "\phi"] & \oplus_\tau \mathbb{Z} V_\tau \arrow[r] & \operatorname{Cl} X \arrow[r] & 0 \end{tikzcd}$$

Moreover, $\operatorname{CaCl} X(\Delta)$ *is free abelian, and we have*

$$\operatorname{rank}(\operatorname{CaCl} X(\Delta)) \leq \operatorname{rank}(\operatorname{Cl} X(\Delta)) = d - n$$

where d is the number of edges in Δ and $\operatorname{rank}(A)$ *is the cardinality of the largest $\mathbb{Z}$-linearly independent set of elements in A.*

REMARK 1.4.34. If Δ is a single two-dimensional cone σ, then σ has two edges and is two-dimensional, so $\operatorname{Cl}(X(\Delta))$ has rank zero and is therefore torsion and in fact finite. So there exists some constant k_σ such that for any Weil divisor D we have $k_\sigma D$ principal on σ. If Δ is any two-dimensional fan, then we can clearly construct some constant k such that for any Weil divisor D, kD is principal on U_σ for every maximal cone σ in Δ. This implies that every Weil divisor has a multiple that is locally principal.

More generally, it is not hard to show that on an n-dimensional toric variety, every Weil divisor has a locally principal multiple if and only if every n-dimensional cone has exactly n edges.

Since toric varieties are Cohen-Macaulay, there is a version of Serre duality that applies to them (see [**Ful93**, Sec. 4.4]). It relies on a dualizing sheaf (which may not be the canonical sheaf if the toric variety is singular) which can be computed explicitly. It is the sheaf associated to the divisor $-\sum V_\tau$ as τ ranges over all edges in Δ. The local sections of this sheaf on U are rational functions with at least simple zeros along each V_τ for every V_τ intersecting U. If the toric variety is nonsingular, this sheaf will be equal to the canonical sheaf.

Suppose we have a toric variety $X(\Delta')$ in which the singularities of $X(\Delta)$ are resolved. Then we can take the canonical divisor K on $X(\Delta')$; we know it will be a

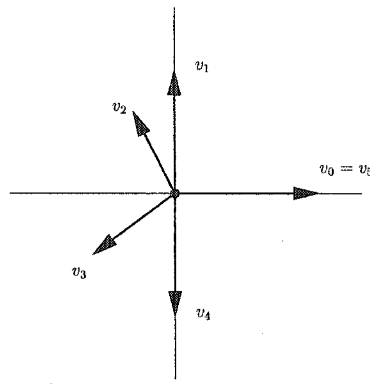

FIGURE 1.4.6. The edges v_i in the fan of a nonsingular toric surface

formal sum of all the edges in Δ' because $X(\Delta')$ is nonsingular. Then taking the push-forward K_* of this (see Definition 2.3.4) we obtain the dualizing divisor on $X(\Delta)$. So in the case of toric varieties, we have a simple description of the dualizing sheaf even when this sheaf is not invertible. In Section 3.3 we will see that the same is true for any variety that is a local complete intersection. This is a more restrictive conclusion, since toric varieties need not be local complete intersections.

Suppose $X(\Delta)$ is a nonsingular toric variety of dimension 2, and let τ and τ' be different edges of the same two-dimensional cone. Then $\tau = u^\perp$ and $\tau' = (u')^\perp$ for some u and u' in M. Then the divisor of $\chi(u)$ is V_τ, and the divisor of $\chi(u')$ is $V_{\tau'}$. Further, $\chi(u)$ and $\chi(u')$ together generate the maximal ideal at x_σ, so the two curves intersect transversally. Since we already know that two edges that do not share a cone do not intersect, we can determine how any two different divisors intersect. In [**Ful93**, Sec. 2.5], the nonsingular toric surfaces are completely described. We will summarize this discussion.

Let Δ be a two-dimensional fan of cones, and suppose that $X(\Delta)$ is complete. Then we know that Δ spans all of $N_\mathbb{R}$, so Δ can be described by giving the first point along each edge in counterclockwise order around the origin. Let these points be $v_0, \ldots, v_d = v_0$. Now suppose $X(\Delta)$ is nonsingular. Then v_0 and v_1 must together generate the lattice, and v_1 and v_2 must also generate the lattice. Since v_2 must be in the second or third quadrants, we must have $v_2 = -v_0 + a_1 v_1$. In general, we must have $a_i v_i = v_{i-1} + v_{i+1}$ for all i. One can show that these a_i must satisfy $a_1 + \cdots + a_d = 3d - 12$, and that conversely, any sequence satisfying these criteria defines a nonsingular toric variety. Then one shows that the intersection form on

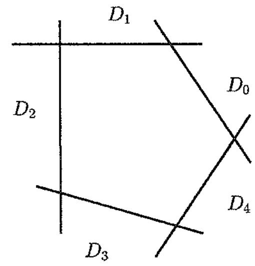

FIGURE 1.4.7. The curves D_i on a nonsingular toric surface intersect in a loop

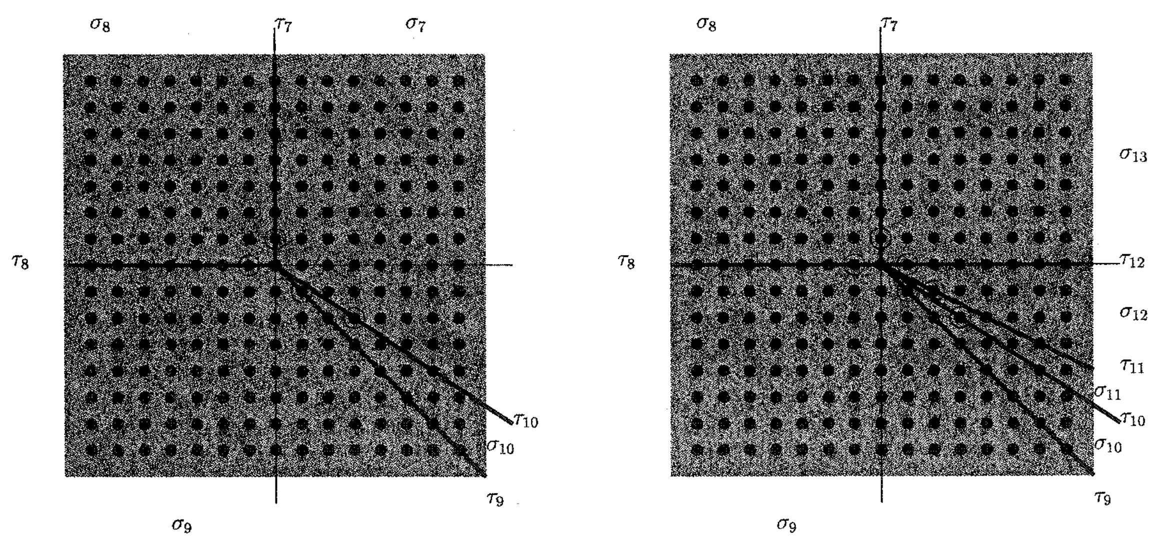

FIGURE 1.4.8. The fan Δ_4 and a fan Δ_5 resolving its singularities

$X(\Delta)$ is given by

$$D_i.D_j = \begin{cases} -a_i & \text{if } i = j \\ 1 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

EXAMPLE 1.4.35. Consider the toric surfaces $X(\Delta_4)$ and $X(\Delta_5)$ defined by the fans Δ_4 and Δ_5 as shown in Figure 1.4.8. It is easy to see that in $X(\Delta_4)$ only the cone σ_7 leads to a singular affine toric variety and that $X(\Delta_5)$ is a nonsingular toric variety mapping to $X(\Delta_4)$.

Examining Δ_5, we see that if v_i is the generator for τ_i, we have

$$\begin{aligned} v_{12} + v_8 &= 0v_7 \\ v_7 + v_9 &= -1v_8 \\ v_8 + v_{10} &= 2v_9 \\ v_9 + v_{11} &= 1v_{10} \\ v_{10} + v_{12} &= 2v_{11} \\ v_{11} + v_7 &= 2v_{12}. \end{aligned}$$

Let V_i' denote the closure of the orbit of x_{τ_i} on $X(\Delta_5)$. Then we see that the intersection matrix on $X(\Delta_5)$ looks like

$$(V_i'.V_j')_{i,j} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 1 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

Let V_i denote the closure of the orbit of x_{τ_i} on $X(\Delta_4)$. Then we know that the canonical divisor on $X(\Delta)$ is $V_7 + V_8 + V_9 + V_{10}$. We know $V_8 + V_9$ is locally principal, since it does not pass through the singular point x_{σ_7}. Is $V_7 + V_{10}$? Consider the affine toric variety U_{σ_7}. We know the principal divisors are images of elements of the lattice M under the map ϕ defined above. Let e_1^* and e_2^* be the dual basis to e_1 and e_2. Then

$$\begin{aligned} \phi(e_1^*) &= \langle e_1^*, e_2\rangle V_7 + \langle e_1^*, 3e_1 - 2e_2\rangle V_{10} \\ &= 3V_{10} \\ \phi(e_2^*) &= \langle e_2^*, e_2\rangle V_7 + \langle e_2^*, 3e_1 - 2e_2\rangle V_{10} \\ &= V_7 - 2V_{10}, \end{aligned}$$

and we see that $V_7 + V_{10}$ is principal on U_{σ_7}. Thus the dualizing sheaf is invertible.

CHAPTER 2

Intersection Theory on Surfaces with Isolated Singularities

2.1. A Bestiary of Singularities

The object of this thesis is to study singular surfaces, so it seems wise to present a description of several types of singularities that might arise on a surface. Unlike the rest of this work, this section will focus on local properties of a surface, that is, given a point $x \in X$, properties of the scheme $\operatorname{Spec} \mathcal{O}_{X,x}$ and to a lesser extent its completion $\operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$. For that reason, in this section we relax the restriction that a surface be complete.

2.1.1. General singularities.

DEFINITION 2.1.1. A variety X is *nonsingular* at a point x if the local ring $\mathcal{O}_{X,x}$ is a regular local ring.

Recall that a regular local ring of dimension n is one in which the maximal ideal modulo its square has dimension n (as a vector space over the residue field). Equivalently, a scheme X of dimension n is nonsingular at a point x if the sheaf of differentials $\Omega_{X/\mathbb{k}}$ is a locally free sheaf of rank n at x. See [**Har77**, Thm. 8.15] and the surrounding text for a proof and further discussion.

If X is a variety of dimension r and we have a representation of X as

$$Z(\langle f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)\rangle),$$

then X is nonsingular at x if and only if the rank of the $m \times n$ matrix

$$\left(\frac{\partial f_i}{\partial x_j}(x)\right)_{i,j}$$

is $n - r$.

In general, the set of points at which a variety is singular is a proper closed subset ([**Har77**, I.5.3]). If this closed set has a component of codimension one, say C, then the local ring $\mathcal{O}_{X,C}$ is not a regular local ring, so the maximal ideal $\mathfrak{m}_{X,C}$ is not principal, so $\mathcal{O}_{X,C}$ is not a discrete valuation ring. This means that we cannot compute the valuation of a rational function at such a generic point. As a result, Weil divisors are not very useful on such surfaces. Cartier divisors, on the other hand, are well-behaved, and in fact the intersection theory of Section 2.5 can be applied. We will see, however, that the objects of interest in intersection theory are often Weil divisors that are not locally principal, so the theory has been formulated in those terms.

EXAMPLE 2.1.2. Consider the variety in $\mathbb{A}^3$ given by $Z(x^2(x+1)-y^2)$. This will be singular whenever $y=x=0$, that is along a line.

2.1.2. Nonsingular in codimension 1.

DEFINITION 2.1.3. If a surface is nonsingular at every point of codimension 1, then it is said to be *nonsingular in codimension* 1.

Equivalently, if the set of singular points has no component of codimension 1, then the surface is nonsingular in codimension 1.

On a surface that is nonsingular in codimension 1, the local ring of every prime divisor is a discrete valuation ring, so Weil divisors are well-defined and useful. This is the kind of surface we will focus on; only a few sections of this work (Section 2.5 and a few others) can be applied to surfaces that are singular in codimension 1. A surface which is nonsingular in codimension 1 has only finitely many singular points, so it is frequently convenient to examine each such point individually.

EXAMPLE 2.1.4. Suppose the characteristic of $\mathbb{k}$ is not 2 or 3. Then consider the surface $\mathbb{k}[u,v,w,y]/\langle u^3-v^2, w^2-uy^2, w^3-vy^3, uw-vy\rangle$. This has Jacobian

$$\begin{pmatrix} 3u^2 & -2v & 0 & 0 \\ -y^2 & 0 & 2w & -2uy \\ 0 & -y^3 & 3w^2 & -3vy^2 \\ w & -y & u & -v \end{pmatrix},$$

which has rank two except when $y=u=v=w=0$. So this surface has a singularity at the origin and no other singularities.

2.1.3. Normal singularities. Recall that the integral closure of a ring R in a ring S is the set of elements $y \in S$ satisfying a monic polynomial with coefficients in R. A ring R is integrally closed when the integral closure of R in its field of fractions is again R.

DEFINITION 2.1.5. A point x on a surface X is *normal* if the local ring $\mathcal{O}_{X,x}$ is integrally closed. A surface is *normal* if all its points are normal.

EXAMPLE 2.1.6. Let $R = \Bbbk[x,y,z]_{\langle x,y,z\rangle}/\langle y^2 - x^3\rangle$. Consider the element y/x in the quotient field of R. This satisfies the monic polynomial $X^2 - x$ but is not in R. So R is not integrally closed and we see that the surface of Example 2.1.2 is not normal at $(0,0,0)$.

REMARK 2.1.7. Every regular local ring is a unique factorization domain, and a unique factorization domain is always integrally closed. So every nonsingular point of a scheme is a normal point. In particular, this implies that the normal points are an open dense subset of any surface.

If a surface is normal at all its points, then it is nonsingular in codimension 1 ([**AM69**, Prop. 9.2]). The converse is not true, as we will see in Example 2.1.14.

Every surface X possesses a unique normalization, that is, a normal surface Y and a finite morphism $f : Y \to X$ that is an isomorphism away from the non-normal points of X. In particular, what this means is that any surface which is singular in codimension one has a unique normal surface mapping birationally to it which is nonsingular in codimension one. So problems on such a surface can (at least in principle) be resolved by referring to the unique normalization. In contrast, while every surface has a nonsingular surface mapping to it in a similar way (see Section 2.2, particularly Proposition 2.2.6), this surface is not unique, which complicates the theory greatly. When dealing with curves, of course, the normalization resolves all the singularities in a unique way.

Given any affine scheme $\operatorname{Spec} R$, the normalization is $\operatorname{Spec} S$, where S is the integral closure of R in its field of fractions, along with the morphism induced by the natural inclusion. If $\operatorname{Spec} R$ is of finite type over a field, then this morphism is finite. If now X is any scheme, then it can be shown that for any affine cover, the normalizations of all the affine neighborhoods glue to give a scheme that is noetherian if X is, and the morphisms glue to give a morphism that is finite whenever X is of finite type over a field.

Suppose now that X is projective. Is the normalization of X also projective?

PROPOSITION 2.1.8. *Let X be a noetherian scheme of finite type over $\Bbbk$, and let (Y, f) be the normalization of X. Let $\mathcal{L}$ be a line bundle on Y such that $f_*(\mathcal{L})$ is an ample line bundle on X. Then $\mathcal{L}$ is ample on Y.*

PROOF. Observe that f is affine by the definition of normalization, so for any quasi-coherent sheaf $\mathcal{M}$ we can apply [**Har77**, Ex. III.4.1] to obtain isomorphisms $H^i(Y, \mathcal{M}) \cong H^i(X, f_*(\mathcal{M}))$ for all $i \geq 0$.

Now let $\mathcal{F}$ be any coherent sheaf on Y. Then since $f_*(\mathcal{L})$ is ample on X, by the cohomological criterion of ampleness, $H^i(X, f_*(\mathcal{L}) \otimes f_*(\mathcal{F}^n)) = 0$ for $i > 0$ and n large enough. But then we obtain $H^i(Y, \mathcal{L} \otimes \mathcal{F}^n) = 0$ for $i > 0$ and n large enough, and we see that $\mathcal{L}$ is ample on Y. □

COROLLARY 2.1.9. *Let X be a surface that is nonsingular in codimension one. Then its normalization is also a surface; in particular, the normalization of such a surface is projective. Let $\mathcal{L}$ be any ample invertible sheaf on X, and let (Y, f) be the normalization of X. Then $f^*(\mathcal{L})$ is ample on Y.*

PROOF. We have $f_*(f^*(\mathcal{L})) = \mathcal{L}$, as we can see by considering the corresponding locally principal divisor: it is clearly equal up to a set of codimension two. □

In order to deal with normality, it is useful to have a more tangible criterion for normality of a surface. This is provided by Serre's criterion.

DEFINITION 2.1.10. Let A be a ring and M an A-module. A sequence of $x_1, \ldots, x_r$ of elements of A is called a *regular sequence* for M if $(x_1, \ldots, x_r)M \neq M$, x_1 is not a zerodivisor on M and x_k is not a zerodivisor on $M/(x_1, \ldots, x_{k-1})M$ for all k. If A is a local ring with maximal ideal $\mathfrak{m}$ then the *depth* of M is the maximal length of a regular sequence for M with elements taken from $\mathfrak{m}$.

REMARK 2.1.11. In [**Eis95**, Chap. 18], a more general definition of depth is used, allowing depth to be computed on rings that are not necessarily local; what we call depth M on a local ring A with maximal ideal $\mathfrak{m}$ would there be denoted depth$(\mathfrak{m}, M)$; if $M = A$, this would be abbreviated (confusingly!) to depth $\mathfrak{m}$. However, in the special case where A is local with maximal ideal $\mathfrak{m}$, the same reference also abbreviates depth$(\mathfrak{m}, M)$ to depth M, yielding two quite different interpretations when M is an ideal. "However," Eisenbud asserts, "confusion does not really arise in practice." ([**Eis95**, p.425]).

DEFINITION 2.1.12. We say a noetherian ring A satisfies condition *S2* (the S is for J.-P. Serre) if for every prime ideal $\mathfrak{p}$ of codimension ≥ 2 we have depth $A_{\mathfrak{p}} \geq 2$.

THEOREM 2.1.13 (Serre). *A noetherian ring A is a product of integrally closed domains if and only if A is nonsingular in codimension* 1 *and satisfies condition S2.*

See [**Eis95**, Thm. 11.5] and the proof of [**Eis95**, Thm. 18.15] for more information.

If A is the local ring of a point x on a surface that is nonsingular in codimension 1, then A has dimension 2 and is an integral domain. Then there is exactly one ideal with codimension ≥ 2, namely the maximal ideal $\mathfrak{m}$. Thus we need only show that $\operatorname{depth} A_{\mathfrak{m}} \geq 2$. We can do this by producing a regular sequence of length 2. Such a sequence is just a pair of elements f and g in $\mathfrak{m}$ such that g is not a zerodivisor in A/fA. If no such sequence exists, then the surface is not normal.

If x is a normal singularity on X, then by definition the local ring $\mathcal{O}_{X,x}$ is integrally closed. In [**Bǎd01**, Lem. 4.2], we see that the ring $\widehat{\mathcal{O}}_{X,x}$ is also integrally closed. This may occasionally be easier to investigate.

EXAMPLE 2.1.14. Let $Y = \operatorname{Spec} S$ be a normal affine surface. Then let R be a subring of S such that S/R is finite dimensional as a vector space over $\Bbbk$. Then S and R will have the same quotient field, and S will be the integral closure of R. So $X = \operatorname{Spec} R$ will not be normal. Further, the inclusion $R \hookrightarrow S$ defines the finite birational morphism $f : Y \to X$. Then Y is the normalization of X, and there is a finite set of points $x_1, \ldots, x_n$ such that f provides an isomorphism of $Y \setminus f^{-1}(\{x_1, \ldots, x_n\})$ with $X \setminus \{x_1, \ldots, x_n\}$. For the proof see [**Mum99**, Ex. III.8.K].

EXAMPLE 2.1.15. For a specific example of a surface, assume the characteristic of $\Bbbk$ is not 2 or 3, and take $S = \Bbbk[x, y]$ and $R = \Bbbk[x^2, x^3, xy, y]$. Then any element of S can be written as $a + bx + f(x)x^2 + yg(x, y) = a + bx + r$ for some $r \in R$, and we see that S/R is a finite-dimensional $\Bbbk$-vector space as required in Example 2.1.14. Now write $u = x^2$, $v = x^3$, $w = xy$, and we get R mapping into $\Bbbk[u, v, w, y]/\langle u^3 - v^2, w^2 - uy^2, w^3 - vy^3, uw - vy\rangle$. This map corresponds to the morphism of varieties $f(x, y) = (x^2, x^3, xy, y)$. Consider the set-theoretic map

$$(u, v, w, y) \mapsto \begin{cases} (0, y) & \text{if } u = 0 \\ (v/u, y) & \text{otherwise.} \end{cases}$$

This is a set-theoretic inverse to f, which implies that f is surjective. Since R is an integral domain, R is irreducible and so is $f(R)$. We can use the computer algebra package Macaulay 2 ([**GS**]) to verify that the ideal of this scheme is equal to its radical. Thus R is isomorphic to $\Bbbk[u, v, w, y]/\langle u^3 - v^2, w^2 - uy^2, w^3 - vy^3, uw - vy\rangle$, the coordinate ring of the surface from Example 2.1.4 on which the point 0 is an isolated non-normal singularity. The normalization of this surface is just $\operatorname{Spec} S$ with the map f.

2.1.4. Cohen-Macaulay singularities.

DEFINITION 2.1.16. Let A be a local noetherian integrally closed domain. We then say A is *Cohen-Macaulay* if $\operatorname{depth} A = \dim A$, considering A as an A-module.

We say a point x on a surface X is *Cohen-Macaulay* if the local ring $\mathcal{O}_{X,x}$ is Cohen-Macaulay. A surface X is *Cohen-Macaulay* if all its points are Cohen-Macaulay.

REMARK 2.1.17. For general rings A, we have an inequality $\operatorname{depth} A \leq \dim A$ (translating [**Eis95**, Prop. 18.2] to our language). A Cohen-Macaulay ring is one for which the depth is as large as possible.

This is definition of a Cohen-Macaulay point on a surface is rather awkward-looking, but fortunately we have:

PROPOSITION 2.1.18. *Let X be an algebraic $\Bbbk$-scheme of dimension 2 that is nonsingular in codimension 1. Then X is Cohen-Macaulay at x if and only if X is normal at x.*

PROOF. Let A be the local ring of a point x on a surface X and suppose x is normal. We then know that A is nonsingular in codimension 1. Further, every prime ideal has codimension one or two, so we need only consider prime ideals of codimension 2, and there is only one of these, the maximal ideal $\mathfrak{m}$. We know from Serre's criterion (Theorem 2.1.13) that $\operatorname{depth} A_{\mathfrak{m}} = \operatorname{depth} A \geq 2$. Recall that for any ring R, $\operatorname{depth} R \leq \dim R$. In particular, we have $\operatorname{depth} A \leq 2$, so $\operatorname{depth} A = 2 = \dim A$ so A is Cohen-Macaulay.

Now suppose instead that x is Cohen-Macaulay. Then for every prime $\mathfrak{p}$,

$$\operatorname{depth} A_{\mathfrak{p}} = \dim A_{\mathfrak{p}} = \operatorname{codim} \mathfrak{p},$$

so by Serre's criterion x is normal. □

REMARK 2.1.19. The fact that a Cohen-Macaulay ring that is nonsingular in codimension 1 is therefore normal is true for varieties of any dimension.

If a projective variety is Cohen-Macaulay, then we have a version of Serre duality (see A.4). This will be essential for the Riemann-Roch theorem in Section 3.3.

EXAMPLE 2.1.20. Toric varieties are always Cohen-Macaulay and always have isolated singularities, and therefore toric varieties are always normal.

2.1.5. Gorenstein singularities.

DEFINITION 2.1.21. A local noetherian ring R is *Gorenstein* if it has a finite injective resolution as an R-module. A point x on a surface X is *Gorenstein* if the local ring $\mathcal{O}_{X,x}$ is Gorenstein. A surface is *Gorenstein* if all its points are Gorenstein.

A Gorenstein ring is always Cohen-Macaulay (see [**Fos73**, Chap. 12]). If a surface X is Cohen-Macaulay, then the dualizing sheaf ω_X is invertible in a neighborhood of a point x if and only if x is a Gorenstein point on X (see [**Bǎd01**, 3.11]). Comparing with Theorem A.4.9, we see that a local complete intersection is Gorenstein everywhere.

Generally, unless it is a local complete intersection, it is awkward to tell if a ring is Gorenstein. However, we have a few results that are useful for this purpose.

THEOREM 2.1.22 (Murthy). *Let R be Cohen-Macaulay, a quotient of a Gorenstein ring, and a unique factorization domain. Then R is Gorenstein.*

See [**Fos73**, Thm. 12.3]. Applied to surfaces, this implies that every factorial singularity (see Definition 2.1.8) is Gorenstein.

THEOREM 2.1.23 (M. Artin). *Let X be a nonsingular projective surface and let $E \subset X$ be a connected closed subscheme of dimension 1 with integral components $E_1, \ldots, E_n$. Then the following conditions are equivalent:*

(1) *There exists a morphism $f : X \to Y$ with the following properties: Y is a normal projective surface, $f(E) = y$ is a Gorenstein point on Y, f is an isomorphism between $X \setminus E$ and $Y \setminus \{y\}$, and $f^*(\omega_Y^\circ) \cong \omega_X^\circ$.*
(2) *The intersection matrix $(E_i.E_j)_{i,j}$ is negative definite, the E_i are nonsingular rational curves, and $E_i.E_i = -2$ for all i.*

REMARK 2.1.24. We will see in Example 2.1.29 that it is quite possible to satisfy all the conditions of condition 1 except for $f^*(\omega_Y^\circ) \cong \omega_X^\circ$; thus this theorem is not as useful for proving that a point is *not* Gorenstein as one might hope.

EXAMPLE 2.1.25. We saw in Example 1.4.35 that the surface $X(\Delta_4)$ has an invertible canonical sheaf and is therefore Gorenstein.

2.1.6. Local complete intersections.

DEFINITION 2.1.26. Let X be a surface and x be a point on X. Let U be an open affine neighborhood of x, so that U is isomorphic to an open subset of a variety cut out of affine n-space by some ideal I. We know that the ideal I cannot be generated by less than $n-2$ elements; if for some U the ideal I can be generated by exactly $n-2$ elements, then we say X is a *local complete intersection* at x. We say X is a local complete intersection if it is a local complete intersection at every point.

It turns out ([**Har77**, Rem. II.8.22.2]) that being a local complete intersection is an intrinsic property of a scheme. In particular, it does not depend on the embedding into affine space.

A local complete intersection is Cohen-Macaulay, and it is normal if and only if it is nonsingular in codimension 1 ([**Har77**, Prop. II.8.23]). A local complete intersection is Gorenstein ([**Eis95**, Cor. 21.19]) and so the dualizing sheaf is invertible; if we have a description of the projective surface as a closed subset of $\mathbb{P}^N$, we have a quite explicit description of the dualizing sheaf (see Theorem A.4.9).

EXAMPLE 2.1.27. Every nonsingular variety is a local complete intersection (see [**Har77**, Thm. II.8.17]).

EXAMPLE 2.1.28. Any hypersurface is a local complete intersection. Thus a hypersurface is always Cohen-Macaulay and Gorenstein, and it is normal if and only if it is nonsingular in codimension one.

EXAMPLE 2.1.29. Consider Example 1.2.5, the (projective) cone on the Fermat curve $x^m + y^m = z^m$ of degree m. In Section 1.4.1 we have described a resolved model, achieved by a single blow-up. The cone is a hypersurface, so it is a local complete intersection.

2.1.7. Rational singularities. Rational singularities are not simple to define, but they are much more tractable than more general sorts of singularities; the sorts of exceptional fiber that can arise are well-understood.

Recall from Section 1.2.6 the definition of the *arithmetic genus* p_a of an effective divisor D:

$$p_a(D) = (-1)^r(\chi(\mathcal{O}_D) - 1).$$

DEFINITION 2.1.30. Let x be a normal singularity on the surface X, and let (Y, f) be a resolved model of X (see Section 2.3). Suppose that the exceptional manifold has integral components $\{E_i\}$. Then x is *rational* if any of the following equivalent conditions holds:

(1) $R^1 f_*(\mathcal{O}_Y) = 0$,

(2) $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y)$,
(3) $p_a(Z) \leq 0$ for every positive divisor Z with support contained in E,[1]
(4) $H^1(\mathcal{O}_Z) = 0$ for every positive divisor Z with support contained in E, and
(5) for each $Z > 0$ with support equal to E, the canonical homomorphism $d : H^1(\mathcal{O}_Z^*) \to \mathbb{Z}^n$ which associates the n-tuple $\left(\deg_{E_1}(L|_{E_1}), \ldots, \deg_{E_n}(L|_{E_n})\right)$ to each invertible $\mathcal{O}_X$-module L is an isomorphism.

EXAMPLE 2.1.31. If we consider our familiar example from Section 1.4.1 and Example 2.1.29, the cone on the Fermat curve, we see that the singularity is always normal since the cone is a hypersurface which is nonsingular in codimension one. The singularity is resolved by a single blow-up, yielding an exceptional fiber isomorphic to the original curve. Thus the singularity will be rational if and only if the curve has arithmetic genus zero — that is, if and only if the curve is isomorphic to $\mathbb{P}^1$. But recall that the Fermat curve of degree m has genus $(m-1)(m-2)/2$, so (for example) the cone on the Fermat curve of degree 3 is a local complete intersection but the singularity at the origin is not rational.

EXAMPLE 2.1.32. Consider the toric variety Example 1.4.35. Here the exceptional fiber is two copies of $\mathbb{P}^1$ with intersection matrix $\left(\begin{smallmatrix} -2 & 1 \\ 1 & -2 \end{smallmatrix}\right)$; let $Z = aV'_{11} + bV'_{12}$ be a nonzero effective divisor. Then by Proposition 1.2.44,

$$\begin{aligned} p_a(Z) &= ab - 1 - 2a(a-1)/2 - (a-1) - 2b(b-1)/2 - (b-1) \\ &= ab - (a^2 + b^2) + 1 \\ &= -(a-b)^2 - ab + 1, \end{aligned}$$

which is clearly less than or equal to zero.

So Example 1.4.35 is rational.

None of the equivalent conditions for a singularity to be rational is especially tractable. However, we have a more explicit description.

Given a list of prime divisors $\{E_i\}$ on a nonsingular surface, by the local formula for the intersection number (Theorem 1.2.37) we know that the intersection $E_i.E_j$ is a nonnegative integer for $i \neq j$. So construct a graph[2] whose nodes are the E_i and having $E_i.E_j$ edges connecting E_i and E_j when $i \neq j$. Observe that this graph is

[1] There is a misprint in the statement of this theorem/definition in **[Băd01]**. There condition 3 is written as "$p_a(Z) = 0$ for every positive divisor Z with support contained in E". However, examining the proof and **[Băd01**, Lem. 3.3], we see that the correct condition is as stated here.

[2] In the literature, there are many slightly different definitions of "graph", and the objects described here are not graphs in the most common sense. These objects could more accurately be described as

connected if and only if the underlying topological space $\bigcup_i E_i$ is connected. A cycle in a graph is a list of vertices $v_0, \ldots, v_n$ and edges $e_0, \ldots, e_n$ such that e_i connects v_i and $v_{(i+1) \bmod n}$ and such that no vertex and no edge appears twice. In particular, if a pair of vertices is connected by more than one edge, there is a cycle in the graph. Recall also that the graph is a tree if and only if it is connected and it contains no cycles, or equivalently, if the number of edges is exactly one less than the number of vertices. Say the E_i form a tree if the associated graph is a tree.

PROPOSITION 2.1.33. *Let X be a normal surface singular at x, and let $(\widetilde{X}, f)$ be a resolved model of X. Then if x is a rational singularity, the exceptional fiber (whose integral components we will label $\{E_i\}$) is a tree of copies of $\mathbb{P}^1$.*

We will require a lemma:

LEMMA 2.1.34. *Let Y be a reduced connected subscheme of a surface X, and let Y have pure dimension 1 with n irreducible components $\{C_i\}$. Then*

$$p_a(Y) \geq 0.$$

If $p_a(Y) = 0$, then the graph associated to the $\{C_i\}$ has no cycles and $p_a(\{C_i\}) = 0$ for all i.

PROOF. Recall the formula from Section 1.2.6:

$$p_a(\sum_{i=1}^n C_i) = \sum_{i=1}^n p_a(C_i) + \sum_{i=1}^n \sum_{j=i+1}^n (C_i.C_j) - (n-1).$$

We know that for $i \neq j$ we have $C_i.C_j \geq 0$. In fact, the term $\sum_{i=1}^n \sum_{j=i+1}^n (C_i.C_j)$ is precisely the number of edges in the graph. Every curve has nonnegative arithmetic genus ([**Har77**, Exer. III.5.3]) so if the graph has m edges, this inequality yields $p_a(Y) \geq m - (n-1)$. But since Y is connected, the graph must be connected; since it has n vertices, it must have $n-1$ edges, so we obtain $p_a(Y) \geq 0$. If the graph contains a cycle, it must contain at least n edges, so the inequality will be strict. If any curve has nonzero arithmetic genus, the inequality will clearly also be strict. □

PROOF OF PROPOSITION. Assume x is a rational singularity. Then consider the divisor $\sum_i E_i$. This is supported on $f^{-1}(x)$ and so we must have $p_a(\sum_i E_i) \leq 0$. But since x is a normal singularity, we know that $f^{-1}(x)$ is connected by Zariski's main theorem (see [**Har77**, Thm. V.5.2]). Applying Lemma 2.1.34 to $\sum_i E_i$ yields

weighted multigraphs with no self-edges, but the reader should have no trouble understanding what is meant.

$p_a(\sum_i E_i) = 0$ and further every E_i must have arithmetic genus zero. Since $\widetilde{X}$ is a resolved model of X, all the E_i are nonsingular, hence isomorphic to $\mathbb{P}^1$ by [**Har77**, Cor. I.6.12]. Finally the lemma also tells us that the graph associated to the $\{E_i\}$ has no cycles, thus is a tree. □

2.1.8. Factorial singularities.

DEFINITION 2.1.35. A point x on a surface X is *factorial* if the local ring $\mathcal{O}_{X,x}$ is a unique factorization domain. A surface is factorial if every point on it is factorial.

This implies that x is also a normal point, since every unique factorization domain is integrally closed ([**Eis95**, Prop. 4.10]). Further, since the local class group is trivial, every divisor is locally principal. As we saw in Proposition 1.2.15, if there exists even one prime divisor nonsingular at x that is a locally principal divisor, then X is nonsingular at x. This implies that at a factorial singularity, every prime divisor through the singularity is itself singular.

EXAMPLE 2.1.36. The surface $X = Z(x^2 + y^3 + z^5)$ is factorial ([**Har77**, Exer. V.5.8]). In fact, if the characteristic of $\Bbbk$ is not 2, 3, or 5, the completion $\widehat{\mathcal{O}}_{X,0}$ of the local ring $\mathcal{O}_{X,0}$ at the origin is also factorial. This is quite unusual; in these characteristics, $\Bbbk[[X,Y,Z]]/\langle X^2 + Y^3 + Z^5\rangle$ is the only nonregular normal complete 2-dimensional local ring which is a unique factorization domain (see [**Lip69**]).

EXAMPLE 2.1.37. Recall Example 1.2.5: here a ruling on the cone is nonsingular and passes through the singularity; hence this surface is not factorial.

2.1.9. When does a divisor have a locally principal multiple? Intersection theory on singular surfaces is much simpler for locally principal divisors than for arbitrary divisors. For example, we have the correspondence between locally principal divisors, line bundles, and invertible sheaves. Since the intersection form one obtains is linear, it is quite straightforward to extend the theory to cover divisors whose multiples are locally principal. However, as we will see in Example 2.1.42, not all divisors are of this form. So the goal in this section is to study $\operatorname{Cl}\mathcal{O}_{X,x}$ and determine when it is a torsion group.

A noetherian topological ring is a *Zariski ring* if the topology on it is generated by an ideal contained in the Jacobson radical. Every local ring with the usual topology is a Zariski ring.

An integral domain A is a *Krull ring* if it has an associated family of valuations $\{v_i\}$ on its quotient field such that A is the intersection of all the valuation rings and such that every element of the quotient field has zero valuation in almost all v_i.

A noetherian integral domain that is integrally closed is always a Krull ring (see [**Fos73**, Chap. 1]). Furthermore, the completion of a local noetherian integrally closed integral domain is again a local noetherian integrally closed integral domain (see [**Băd01**, Lem. 4.2]). Thus if we have a normal surface, we may apply the following:

PROPOSITION 2.1.38 (Mori). *Let A be a Zariski ring whose completion $\widehat{A}$ is a Krull ring. Then the class group of A injects into the class group of $\widehat{A}$.*

See [**Sam61**] for a proof.

PROPOSITION 2.1.39. *Let y be a normal singularity on the affine surface Y. Then y is a rational singularity if and only if the $\mathfrak{m}_y$-adic completion $\widehat{\mathcal{O}}_{Y,y}$ of the local ring $\mathcal{O}_{Y,y}$ has a finite class group.*

Let (Y^*, ϕ) be a resolved model of Y with exceptional manifold E having integral components $\{E_i\}$. If y is a rational singularity, the order of the divisor class group $\mathrm{Cl}\,\widehat{\mathcal{O}}_{Y,y}$ is equal to the absolute value d of the determinant of the intersection matrix $(E_i \cdot E_j)_{i,j}$.

See [**Băd01**, Thm. 4.6] for a proof.

REMARK 2.1.40. We see immediately that for any rational singularity, every divisor has a locally principal multiple. Moreover, there is a global constant d such that for every divisor D, the divisor dD is locally principal.

EXAMPLE 2.1.41. Consider our toric variety Example 1.4.35. We have shown in Example 2.1.32 that its singularities are rational, so every divisor must have a locally principal multiple. In fact, we showed that this was true in general for toric varieties in Remark 1.4.34.

EXAMPLE 2.1.42. Consider again the surface from Example 2.1.31, the cone C on the Fermat curve K, letting P be the vertex of the cone. Now, we saw in Section 1.4.1 that $\mathrm{Cl}\,\mathrm{Spec}\,\mathcal{O}_P$ is a quotient of $\mathrm{Cl}\,K$ by $\mathbb{Z}$. If K is not $\mathbb{P}^1$, then this will be infinite, and we see immediately that the singularity cannot be rational. If our ground field is the complex numbers, then for any surface with positive genus we will have an element of infinite order, that is, we will have a divisor none of whose multiples is locally principal.

2.2. Resolution of Singularities for Surfaces

We have seen many kinds of surface singularity, with a variety of behaviour. In order to do computations on a surface, it will very often be preferable to pass to a nonsingular surface that is birational to the surface in question, where we can apply the results of Section 1.2. However, in order to relate results on the nonsingular surface to results on our original, singular, surface, we will need the birational map to have a special form. We will see that such a surface and birational map can be produced in an algorithmic way.

DEFINITION 2.2.1. Let X be a surface. Then (X^*, ϕ) is a *desingularization* if X^* is nonsingular and ϕ is a birational morphism from X^* to X.

We have already seen two operations that take a surface X and yield a (possibly singular) surface X^* and a birational morphism ϕ in this way. The first, discussed in Section 1.3, is the blow-up of a surface. We saw in Section 1.4.1 that this can lead to a non-singular surface; however it does not always do so, by any means. In fact, if X is singular and normal, then the blow-up $\widetilde{X}$ may fail to be normal. This brings us to the second process we have seen: in Section 2.1.3, we described the normalization of a surface. This normalization yields a surface which is guaranteed to be normal and has a birational morphism to the original surface. So in some sense, it reduces the severity of the singularities on a surface.

PROPOSITION 2.2.2. *Let X be a surface with singular locus Σ, and let (X^*, ϕ) be a desingularization of X. Then there exists a finite set M of points of X such that ϕ is an isomorphism from $X^* \setminus \phi^{-1}(\Sigma \cup M)$ to $X \setminus (\Sigma \cup M)$.*

PROOF. Since X^* is by definition nonsingular, it is normal, and therefore ϕ factors through the normalization of X. The normalization is an isomorphism outside the singular locus of X, and the map from X^* to the normalization of X is a birational map of normal spaces, so we can apply [**Har77**, Lem. V.5.1] to its inverse to see that this map introduces at most finitely many additional points where ϕ is not an isomorphism. □

THEOREM 2.2.3. *Let X be a surface, and let (X_0, ϕ_0) be the normalization of X. Then let (X_{i+1}, ϕ_{i+1}) be obtained from (X_i, ϕ_i) by first letting (Y_i, ψ_i) be obtained from X_i by blowing up a singular point of X_i, and then letting (Z_i, γ_i) be the normalization of Y_i. We define $\phi_{i+1} = \phi_i \circ \psi_i \circ \gamma_i$. Then for some n we have X_n nonsingular and the process terminates. In this case (X_n, ϕ_n) is a desingularization of X.*

PROOF. By definition, our surfaces are all of finite type over $\Bbbk$. Thus they are *excellent* (see [**Gro67**, Sec. 7.8]). We then apply [**Lip69**, Thm. 2.1] and obtain the desired result. $\square$

REMARK 2.2.4. Observe that we have produced a desingularization that is an isomorphism away from the singular points of X, showing that this is always possible. In particular, if X is nonsingular in codimension one, we can find a desingularization which is an isomorphism except for a finite set of points on X.

REMARK 2.2.5. Observe that the desingularization obtained depends on the projective embedding of X. In fact, it can be shown that there is, in a particular sense, a minimal desingularization of X, but that this procedure does not in general yield it.

Because desingularizations are not unique, it will be necessary to have some way of relating any two desingularizations of the same surface.

THEOREM 2.2.6. *Let X be a surface, and let (X_1^*, ϕ_1) and (X_2^*, ϕ_2) be desingularizations of X. Then there exist surfaces Y_1 and Y_2 as well as birational morphisms ψ_1 and ψ_2 such that $(Y_1, \phi_1 \circ \psi_1)$ and $(Y_2, \phi_2 \circ \psi_2)$ are desingularizations of X, and such that there is an isomorphism $\gamma : Y_1 \to Y_2$ making the following diagram commute:*

$$\begin{tikzcd}
Y_1 \arrow[rr, "\gamma"'] \arrow[d, "\psi_1"'] & & Y_2 \arrow[d, "\psi_2"] \\
X_1 \arrow[dr, "\phi_1"'] & & X_2 \arrow[dl, "\phi_2"] \\
& X &
\end{tikzcd}$$

PROOF. In the proof of [**Har77**, Thm. V.5.5], it is shown that the birational transformation $\phi_2^{-1} \circ \phi_1$ can be factored by constructing a nonsingular surface Y and sequences of blow-ups $f_1 : Y \to X_1$ and $f_2 : Y \to X_2$ such that $\phi_2^{-1} \circ \phi_1 = f_2 \circ f_1^{-1}$. Then set $Y_1 = Y$, $\psi_1 = f_1$, $Y_2 = Y$, and $\psi_2 = f_2$, and the result is proved. $\square$

DEFINITION 2.2.7. Let X be a surface with isolated singular points Σ, and let (X^*, ϕ) be a desingularization. Then (X^*, ϕ) is a *monoidal model* if ϕ is an isomorphism of $X \setminus \Sigma$ with $X^* \setminus \phi^{-1}(\Sigma)$ and if for all $x \in \Sigma$ the set $\phi^{-1}(x)$ has pure dimension one. We also say that the monoidal model has *exceptional manifold*[3] $\phi^{-1}(\Sigma)$.

[3]In [**RT62**], the term "fundamental manifold" is used instead of "exceptional manifold". We have used the latter term because it is essentially the generalization of the exceptional fiber of a blow-up.

REMARK 2.2.8. Any desingularization can be made into a monoidal model by blowing down (possibly more than once) any preimage of a nonsingular point and then blowing up any isolated point in the preimage of Σ.

PROPOSITION 2.2.9. *If X is a normal surface with finite singular locus Σ and (X^*, ϕ) is a desingularization such that ϕ is an isomorphism of $X \setminus \Sigma$ with $X^* \setminus \phi^{-1}(\Sigma)$, then (X^*, ϕ) is a monoidal model.*

PROOF. The only condition we need to verify is that every singular point has a preimage of pure dimension one. This follows from Zariski's main theorem as presented in [**Har77**, Thm. V.5.2]. □

DEFINITION 2.2.10. Let X be a surface, and let (X^*, ϕ) be a monoidal model. If Σ is the set of singular points of X, then $\phi^{-1}(\Sigma)$ forms a closed (but not necessarily reduced) subscheme Ω^* of X^*. If:

(1) Ω^* is purely of dimension one,
(2) each irreducible component of Ω^* is nonsingular,
(3) no two components of Ω^* have intersections that are not transverse, and
(4) no three components of Ω^* have a common point,

then we shall say that X^* *is a resolved model of X* with exceptional manifold Ω^*.

REMARK 2.2.11. Any blow-up of a resolved model is again a resolved model. As a result, given any two resolved models of the same surface, we can find a resolved model that is, up to isomorphism, obtained from each by a succession of blow-ups.

THEOREM 2.2.12. *Let X be a surface, and let (X^*, ϕ) be a desingularization of X. Then there exists a resolved model (X^{**}, ψ) of X which is obtained from (X^*, ϕ) by a succession of blow-ups.*

PROOF. Let Σ be the set of singular points of X, and let $Y = \phi^{-1}(\Sigma)$. Then [**Har77**, Thm. V.3.9] shows that we can find (X^{**}, ψ) such that $\psi^{-1}(\Sigma)$ has normal crossings, that is, each irreducible component is nonsingular, and when r components meet at P, their defining equations are linearly independent modulo $\mathfrak{m}_{X^{**},P}$. In particular, this means that no more than 2 can meet at any point. So we see that (X^{**}, ψ) is a resolved model of X, as required. □

REMARK 2.2.13. While a resolved model is technically more convenient, it may take many more blow-ups to obtain a resolved model than a simple desingularization. Since, naively implemented, a blowup doubles the number of variables in a problem, this could be a problem from a computational point of view.

EXAMPLE 2.2.14. The surface we discussed in Example 1.2.75, the blow-up of the cone on the Fermat curve, is a resolved model for the cone on the Fermat curve. We saw that the exceptional divisor E, a curve isomorphic to the Fermat curve itself, was the exceptional manifold.

EXAMPLE 2.2.15. In Example 1.4.35, we constructed two toric varieties $X(\Delta_4)$ and $X(\Delta_5)$. Because they are constructed on the same lattice and the second is a subdivision of the first, we can construct a map ϕ from $X(\Delta_5)$ to $X(\Delta_4)$ making $(X(\Delta_5), \phi)$ a resolved model of $X(\Delta_4)$. The exceptional manifold in this case is two curves which we called V_{10}' and V_{11}'.

We will use the symbol $C.D$ to denote the standard intersection pairing on a nonsingular surface (and it will always be clear from context which surface is meant).

Let X^* be a monoidal model of the surface X with exceptional manifold Ω^*. Then Ω^* can be partitioned into disjoint connected components, one or more corresponding to each singular point of X. Let $\mu_1, \ldots, \mu_s$ be the integral components of Ω^*.

PROPOSITION 2.2.16. *We have:*

(1) *No non-trivial linear combination of the* μ_i *is algebraically equivalent with division to zero.*
(2) *The intersection matrix* $\mathbf{d} = (d_{ij}) = (\mu_i.\mu_j)$ *is nonsingular, symmetric, negative definite and has no negative elements except on the diagonal.*
(3) *The matrix* $\mathbf{k} = -\mathbf{d}^{-1}$ *has no negative elements.*

The first two are theorems of DuVal ([**DV44**]) and the third follows from the second by virtue of a theorem of Coxeter ([**Cox34**]). Since the μ_i can be classified into sets so that no prime divisor from one set intersects a prime divisor from any other set, $\mathbf{d}$ can be written in block-diagonal form, as can $\mathbf{k}$. However, while $\mathbf{d}$ is a matrix with integer entries, $\mathbf{k}$ may not be.

REMARK 2.2.17. Let $Z = \sum_i k_i E_i$ be any divisor supported on the exceptional manifold. Then $Z.Z = \sum_{i,j} k_i k_j (E_i.E_j)$. Thus the intersection matrix associated to the $\{E_i\}$ is negative definite if and only if every nonzero divisor supported on the exceptional manifold has negative self-intersection.

EXAMPLE 2.2.18. Taking Example 1.2.5, the cone on the Fermat curve and its blow-up, there is only one component to the exceptional manifold, namely the exceptional fiber $x_1 = x_2 = x_3 = 0$. We showed in Proposition 1.4.9 that the exceptional fiber has self-intersection $-m$, so the matrix $\mathbf{d} = (-m)$, a one-by-one matrix. The matrix $\mathbf{k}$ is then $(1/m)$.

EXAMPLE 2.2.19. Returning to the toric surfaces discussed in Example 2.2.15, we saw that the exceptional manifold is made up of two curves V'_{10} and V'_{11}, each isomorphic to $\mathbb{P}^1$. Referring back to Example 1.4.35, we see that they have intersection matrix

$$\mathbf{d} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}.$$

This matrix is negative definite and has negative inverse

$$\mathbf{k} = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}.$$

Observe also that the preimage of the unique singular point is connected.

2.3. Intersection theory using resolution of singularities — the method of Reeve and Tyrrell

In this section we will follow the approach taken in [**RT62**]. The basic principle is to exploit the resolution of singularities described in Section 2.2 to define intersection numbers that are independent of the particular resolution chosen.

2.3.1. Equivalence on surfaces.

Recall that a Weil divisor on a surface X is a formal sum of prime divisors, that is, closed integral subschemes of codimension 1. On a singular surface, Weil divisors that pass through the singularities may not be locally principal. Hence such divisors do not correspond to invertible sheaves, and are not linearly equivalent to divisors that do not pass through the singularities.

DEFINITION 2.3.1. Given a finite set Ω^* of prime divisors on a nonsingular surface X^*, we will say that $C \equiv D \mod \Omega^*$ if $C - D$ is algebraically equivalent with division (Definition 1.2.57) to some divisor in the span of Ω^*.

DEFINITION 2.3.2. If P is a prime divisor on a surface X, and if (X^*, ϕ) is a monoidal model of X, define the *strict transform* $\widetilde{P}$ of P to be the prime divisor whose generic point is the preimage of the generic point of P under ϕ. We extend this to arbitrary divisors by linearity.

REMARK 2.3.3. Observe that if Ω^* is the exceptional manifold, then $\widetilde{P}$ will not be a component of Ω^*.

DEFINITION 2.3.4. Define the *push-forward* $\phi_*(P)$ of a prime divisor P on X^* to be closure of the image of P under $\phi|_{X^*\setminus\Omega^*}$ or 0 if P is an integral component of Ω^*. Extend this map by linearity to all divisors.

REMARK 2.3.5. Clearly $\phi_*(\widetilde{C}) = C$.

PROPOSITION 2.3.6. *Let X be a surface, let (X^*, ϕ) be a resolved model, and let Ω^* be the exceptional manifold. Then there is a natural one-to-one correspondence (given by taking the strict transform) between divisors on X and divisors on X^* with no components taken from Ω^*.*

EXAMPLE 2.3.7. Returning to Example 2.2.14, if we consider a ruling on the cone on the Fermat curve, say $x_1 = x_3 = 0$, the strict transform of this will simply be a ruling on the blow-up, $y_1 = y_3 = 0$, with no component of the exceptional fiber.

This correspondence now allows us to translate questions about divisors on a singular surface into more tractable questions about divisors on a nonsingular surface.

Rather than attempt to deal with thorny issues of algebraic equivalence on a singular surface, **[RT62]** defines a concept of equivalence that makes reference to a resolved model. We will show that it is independent of the choice of resolved model, and that it is in fact the same as algebraic equivalence.

DEFINITION 2.3.8. Let (X^*, ϕ) be a resolved model of X with exceptional manifold Ω^*. Let D_1 and D_2 be divisors on X and $\widetilde{D_1}$ and $\widetilde{D_2}$ be their strict transforms. Then we say D_1 is equivalent to D_2 on X relative to X^* if $\widetilde{D_1} \equiv \widetilde{D_2} \mod \Omega^*$.

LEMMA 2.3.9. *If X_1^* is adjacent to X_2^*, then equivalence of divisors on X relative to X_1^* and equivalence of divisors on X relative to X_2^* are the same.*

We can reduce this to the case of projective adjacency, where X_2^* is a blow-up of X_1^*. Given this simple relationship, it is easy to show the two concepts are the same. Using the existence of a common resolved model dominating any pair of resolved models, this gives:

PROPOSITION 2.3.10. *Let X be a singular surface. Then we have a well-defined concept of equivalence on X, independent of the choice of resolved model.*

DEFINITION 2.3.11. If X is a singular surface and (X^*, ϕ) is any resolved model of X, then we say D_1 and D_2 are *relatively equivalent on X* if they are equivalent on X relative to X^*.

EXAMPLE 2.3.12. For Example 1.2.5, the cone on the Fermat curve, all the rulings on the blow-up are algebraically equivalent since it is a $\mathbb{P}^1$-bundle over the Fermat curve, and the Fermat curve itself serves to parameterize a family of divisors. So we can deduce that all the rulings on the cone are relatively equivalent. Further, since we know the class group for the blow-up, we see that every curve is relatively equivalent to a multiple of a single ruling on the cone.

PROPOSITION 2.3.13. *Let X be a surface and let (X^*, ϕ) be a monoidal model of X.*

(1) *Let D_1 and D_2 be divisors on X, and let $\widetilde{D_1}$ and $\widetilde{D_2}$ denote their strict transforms. Then D_1 and D_2 are algebraically equivalent if and only if $\widetilde{D_1}$ and $\widetilde{D_2}$ are algebraically equivalent.*
(2) *If F_1 and F_2 are algebraically equivalent divisors on X^*, then $\phi_*(F_1)$ and $\phi_*(F_2)$ are algebraically equivalent divisors on X.*

PROOF. Let Σ be the set of singular points of X and Ω^* be the exceptional manifold. Then we know that ϕ provides an isomorphism between $X \setminus \Sigma$ and $X^* \setminus \Omega^*$.

Let T be a nonsingular curve. Then $\phi \times \mathrm{id}_T$ is a birational morphism ψ from $X^* \times T$ to $X \times T$. The restriction of ψ is an isomorphism from $(X^* \setminus \Omega^*) \times T$ to $(X \setminus \Sigma) \times T$.

Let D be a divisor on $X \times T$. Then we can use ψ to obtain a divisor D' on $(X^* \setminus \Omega^*) \times T$. If we just take the closures of all the prime divisors, this defines $\widetilde{D}$ as a divisor on $X^* \times T$. Conversely, if we have a divisor C on $X^* \times T$, we can use ϕ to get a divisor on $(X \setminus \Sigma) \times T$. Since Σ has no components of codimension one, this completely defines a divisor $\psi_*(C)$. We see that $\psi_*(\widetilde{D}) = D$ and that $C - \widetilde{\psi_*(C)}$ is supported only on $\Omega^* \times T$. Clearly ψ_* and $\widetilde{\ }$ take effective divisors to effective divisors. Clearly also the only effective divisors whose image is zero under ψ_* are those supported entirely on the exceptional fiber.

Recall the definition of algebraic equivalence from Definition 1.2.51. Let $t \in T$. Then $X \times \{t\}$ is a prime divisor on $X \times T$. Since ψ_* maps divisors of the form $X^* \times \{t\}$ isomorphically onto divisors of the form $X \times \{t\}$, it is clear that the support of C will contain $X^* \times \{t\}$ if and only if the support of $\psi_*(C)$ contains $X \times \{t\}$. Similarly, $\operatorname{Supp} \widetilde{D}$ will contain $X^* \times \{t\}$ if and only if $\operatorname{Supp} D$ contains $X \times \{t\}$.

Let C and C' be prealgebraically equivalent effective divisors on X. Then there exists a divisor D on $X \times T$ such that $C = D_0$ and $C' = D_1$ and which forms an algebraic family of effective divisors on X. Then $\widetilde{D}$ forms an algebraic family of effective divisors on X^*, and $\widetilde{D}_0 = \widetilde{C}$ and $\widetilde{D}_1 = \widetilde{C'}$ so $\widetilde{C}$ and $\widetilde{C'}$ are prealgebraically equivalent as well.

Suppose F and F' are effective divisors on X^* that are prealgebraically equivalent. Then there exists a divisor D on $X \times T$ such that $F = D_0$ and $F' = D_1$ and which forms an algebraic family of effective divisors on X^*. Then $\psi_* D$ forms an algebraic family of effective divisors on X, and $\psi_*(D_0) = \phi_*(F)$ and $\widetilde{D}_1 = \phi_*(F')$ so F and F' are prealgebraically equivalent as well. Taking $F = \widetilde{C}$ and $F' = \widetilde{C'}$ we see that C and C' are prealgebraically equivalent whenever $\widetilde{C}$ and $\widetilde{C'}$ are. □

COROLLARY 2.3.14. *Let D_1 and D_2 be divisors on a surface X. Then D_1 and D_2 are relatively equivalent if and only if they are algebraically equivalent with division.*

PROOF. Let (X^*, ϕ) be a resolved model with exceptional manifold Ω^*. We have shown in Proposition 2.3.13 that D_1 and D_2 are algebraically equivalent with division if and only if their strict transforms are algebraically equivalent with division. By definition, D_1 and D_2 are relatively equivalent if and only if their strict transforms are algebraically equivalent with division modulo Ω^*. The reverse implication is then immediate. So assume that D_1 and D_2 are relatively equivalent. Then we know $\widetilde{D_1}$ is algebraically equivalent with division to $\widetilde{D_2} + E$ for some divisor E supported on Ω^*, or E is algebraically equivalent with division to $\widetilde{D_1 - D_2}$. But then we know that $\phi_*(E)$ is algebraically equivalent with division to $\phi_*(\widetilde{D_1 - D_2})$. But $\phi_*(E) = 0$, so $D_1 - D_2$ is algebraically equivalent with division to zero as required. □

2.3.2. Local intersection multiplicities. For the intersection theory of Reeve and Tyrrell, we shall require a resolved model. In fact, we shall require even more from our resolved models. If C_1 and C_2 are prime divisors on X, and $\widetilde{C_1}$ and $\widetilde{C_2}$ are their strict transforms, then:

DEFINITION 2.3.15. A resolved model X^* of X is *simply adapted* to C_1 and C_2 if

(1) $\widetilde{C_1}$ and $\widetilde{C_2}$ have no common point in Ω^*,
(2) neither $\widetilde{C_1}$ nor $\widetilde{C_2}$ pass through any point common to two prime divisors μ_i and μ_j, and
(3) Every intersection between $\widetilde{C_1}$ or $\widetilde{C_2}$ and any μ_i is transverse.

REMARK 2.3.16. Given a resolved model, we can always produce a resolved model simply adapted to a particular pair of prime divisors by blowing up a finite number of times.

REMARK 2.3.17. We will see later (Section 2.4.21) that resolved models are not necessary for computations; for the moment, however, they simplify the proofs.

DEFINITION 2.3.18. Let O be a singularity of a singular surface X and let C_1 and C_2 be distinct prime divisors passing through O. Let (X^*, ϕ) be a resolved model of X simply adapted to C_1 and C_2 with exceptional manifold Ω^* having irreducible components $\mu_1, \ldots, \mu_s$. Let $\widetilde{C_1}$ and $\widetilde{C_1}$ be the strict transforms of C_1 and C_2. Let $\mathbf{d} = (d_{ij})$ be the intersection matrix $(\mu_i \cdot \mu_j)$. Then we define the *local multiplicity of intersection of C_1 and C_2 at O on X relative to X^** to be:

$$I_{\mathrm{RT}}(C_1 \cdot C_2; O; X \text{ rel } X^*) = \sum k_{ij}(\widetilde{C_1} \cdot \mu_i)(\widetilde{C_2} \cdot \mu_j),$$

where the summation is taken over all i, j where μ_i and μ_j are contained in $\phi^{-1}(O)$, and where $\mathbf{k} = (k_{ij}) = -\mathbf{d}^{-1}$. We label this intersection number with RT to distinguish it from the other intersection numbers we discuss.

PROPOSITION 2.3.19. *Let X be a singular surface with a singular point O, and let C_1 and C_2 be distinct prime divisors passing through O. Let X_1^* and X_2^* be two resolved models simply adapted to C_1 and C_2. Then:*

$$I_{\mathrm{RT}}(C_1 \cdot C_2; O; X \textit{ rel } X_1^*) = I(C_1 \cdot C_2; O; X \textit{ rel } X_2^*).$$

See [**RT62**, Prop. 3.4] for the proof. The situation reduces to showing that the local intersection number is unchanged by a single blow-up; the proof proceeds by considering the different possible locations of the center of the blow-up (at an intersection of one curve with a μ_i, at the intersection of two μ_i, or elsewhere) and calculating the effect on the matrix $\mathbf{k}$.

REMARK 2.3.20. In view of this last proposition, we can speak of local intersection numbers without reference to particular resolved model; we will abbreviate $I_{\mathrm{RT}}(C_1 \cdot C_2; O; X \text{ rel } X_1^*)$ to $I_{\mathrm{RT}}(C_1 \cdot C_2; O; X)$.

REMARK 2.3.21. We have defined local intersection numbers in terms of prime divisors, but we can extend them to divisors by linearity, always with the exception that the two divisors may contain no common prime divisor, and the model must be simply adapted to all the pairs of prime divisors they contain.

EXAMPLE 2.3.22. Consider Example 1.2.5, and take two rulings on the cone on the Fermat curve, D_1 given by $x_1 = x_3 = 0$ and D_2 given by $x_2 = x_3 = 0$. The blow-up is simply adapted to this pair of divisors, so we may use the formula. It is straightforward to see that each ruling intersects the exceptional fiber transversally, so using the intersection matrix from Example 2.2.18 we see that the total intersection number at the origin is $1/m$.

EXAMPLE 2.3.23. Return to the toric varieties discussed in Example 2.2.19 and defined in Example 1.4.35. We have two curves V_7 and V_{10} on $X(\Delta_4)$. Their strict transforms are clearly V_7' and V_{10}' on $X(\Delta_5)$. We can compute

$$\begin{aligned} I_{\mathrm{RT}}(V_7, V_{10}; x_{\sigma_7}; X(\Delta_4)) &= \sum_{i,j\in\{11,12\}} k_{ij}\left(\widetilde{V_7}.V_i'\right)\left(\widetilde{V_{10}}.V_j'\right) \\ &= \sum_{i,j\in\{11,12\}} k_{ij}(V_7'.V_i')(V_{10}'.V_j') \\ &= k_{11\,12}(V_7'.V_{11}')(V_{10}'.V_{12}') \\ &= 1/3. \end{aligned}$$

2.3.3. Global intersection numbers.

DEFINITION 2.3.24. Let C_1 and C_2 be two distinct prime divisors on a singular surface X. Then the *total intersection number* $(C_1.C_2)_{\mathrm{RT}}$ is defined to be the sum of all the local intersection numbers. At a singular point we take the local intersection number just defined, and at a nonsingular point we take the classical local intersection number. We extend this definition to divisors that share no prime divisors by linearity.

Now that we have a global intersection number, we can begin moving curves; one would hope that the intersection number would not be affected by this, and indeed it is not:

THEOREM 2.3.25. *The total intersection number $(D_1.D_2)_{\mathrm{RT}}$ of two divisors on a surface X remains unchanged if either curve is replaced by a relatively equivalent curve. Moreover, if D_1 and D_2 are given in terms of a base $g_1, \dots, g_s$ for relative equivalence on X, so that the coordinates of D_1 is $\boldsymbol{a}_1$ and those of D_2 are $\boldsymbol{a}_2$ then*

$$(D_1.D_2)_{\mathrm{RT}} = \boldsymbol{a}_1{}^T \boldsymbol{f} \boldsymbol{a}_2,$$

where $\boldsymbol{f}$ is the intersection matrix of the curves g_i on X.

For the proof, see [**RT62**, Thm. 4.2].

For notational convenience, to any divisor D we can associate the vector D^{Ω^*} whose i-th component is $\widetilde{D}.\mu_i$. Then we can write an explicit formula for the global intersection number:

$$\begin{aligned}(D_1.D_2)_{\mathrm{RT}} &= \widetilde{D_1}.\widetilde{D_2} + \sum_{i,j} k_{ij}(\widetilde{D_1}.\mu_i)(\widetilde{D_2}.\mu_j) \\ &= \widetilde{D_1}.\widetilde{D_2} + \left(D_1^{\Omega^*}\right)^T \mathbf{k} D_2^{\Omega^*}\end{aligned}$$

REMARK 2.3.26. Now that we are free to move curves by way of algebraic equivalence, we may safely intersect curves that share prime divisors using these formulas.

2.4. Intersection theory using resolution of singularities — pullbacks and divisors

The approach discussed in Section 2.3 leads to fairly straightforward calculations but is definitionally rather awkward. In this section we will take another approach which yields the same results, but which is in some ways more natural, as it takes into account the relationship between locally principal divisors and invertible sheaves. Moreover, its technical requirements will be weaker, so that computations may be done with fewer blow-ups: we will require only a monoidal model rather than a resolved model. In fact, this theory could readily be applied to any desingularization of a surface with isolated singularities, but we have chosen to describe only the case of monoidal models; in almost all practical circumstances, this is the case that arises. Two modifications would be required. The first would be to allow nonsingular points to be blown up, which would present no technical difficulties beyond extending the exceptional manifold to include the preimages of all the "fundamental points", that is, points where the resolution fails to be an isomorphism. The second modification would be to allow the exceptional manifold to have isolated points; these would not contribute to the pullback of a divisor and so would not present a technical problem.

It is natural to associate a divisor on a singular surface to a divisor on a monoidal model (in the sense of Section 2.2). But there is more than one way to do this. The simplest is to take the strict transform (as we did in Section 2.3). This results in a divisor which has no parts taken from the exceptional manifold. For a more useful operation, we will choose an appropriate divisor supported on the exceptional manifold and add it as well.

First let us establish some notation we shall use throughout this section.

DEFINITION 2.4.1. Let X be a surface with isolated singular points Σ, and let (X^*, ϕ) be a monoidal model of X with exceptional fiber Ω^*. Let the integral components of Ω^* be $\{\mu_i\}$. Then if E is any divisor on X^*, we let E^{Ω^*} denote the column vector of intersection numbers $E.\mu_i$. Let $\mathbf{d} = (d_{ij}) = (\mu_i.\mu_j)_{i,j}$ be the intersection matrix of the μ_i, and let $\mathbf{k} = -\mathbf{d}^{-1}$.

REMARK 2.4.2. The idea of associating a $\mathbb{Q}$-divisor D^* on a monoidal model to any divisor D on X comes from [**AG02**, Sec. 7]. The basic intersection properties satisfied by such a divisor are proved there, and a version of the adjunction formula is derived. We will elaborate on the results presented there and use them to define an intersection theory.

2.4.1. Pullbacks of divisors.

PROPOSITION 2.4.3. *Let $\mathcal{F}$ be an invertible subsheaf of $\mathscr{K}$ on X, locally generated on U_i by f_i. Then $\phi^*(\mathcal{F})$ is an invertible subsheaf of $\mathscr{K}$ on X^*, locally generated on $\phi^{-1}(U_i)$ by f_i.*

PROOF. We have [**Har77**, Exer. II.6.8], which shows that f^* provides a homomorphism from $\operatorname{CaCl} X$ to $\operatorname{CaCl} X^*$. So in particular, every invertible sheaf pulls back to give an invertible sheaf, and every trivial invertible sheaf pulls back to give a trivial invertible sheaf.

If U is any open set on X^* not intersecting Ω^*, then $\phi|_U$ is an isomorphism, and the result is obvious. Since the result is local, let U be an open set on X on which $\mathcal{F}$ is trivial. Write $\mathcal{F}|_U \cong \mathcal{O}_X|_U$. Then, using [**Har77**, Exer. II.6.8] again, we know that $\phi^*(\mathcal{F}|_U) \cong \mathcal{O}_{X^*}|_{\phi^{-1}(U)}$.

The remaining question is to identify the sheaf $\phi^*(\mathcal{F})$ as a subsheaf of $\mathscr{K}$. The invertible sheaf $\mathcal{F}$ is naturally a subsheaf of $\mathscr{K}$, which means that it comes with an injective sheaf homomorphism $\iota : \mathcal{F} \hookrightarrow \mathscr{K}$. Then we apply the functor ϕ^* to this diagram and get $\phi^*(\iota) : \phi^*(\mathcal{F}) \to \phi^*(\mathscr{K})$. But ϕ^* is the identity on $\mathscr{K}$, so this gives us $\phi^*(\iota) : \phi^*(\mathcal{F}) \to \mathscr{K}$. We see that $\phi^*\iota$ is just $\iota \circ \phi_*$, where ϕ_* is the push-forward functor from sheaves on X^* to sheaves on X. The question then is, when does ϕ_* fail to be injective? In the situation of a monoidal model, ϕ_* is clearly injective.

Let U be an open neighborhood on which $\mathcal{F}$ is isomorphic to $\mathcal{O}_X|_U$. Let f be $\iota(U)(1)$. Then $\mathcal{F} = f\mathcal{O}_X|_U$. We have shown that $\phi^*(\mathcal{F})|_{\phi^{-1}(U)} \cong \mathcal{O}_{X^*}|_{\phi^{-1}(U)}$, so let $g = \iota^*(\phi^{-1}(U))(1)$. Then $\phi^*(\mathcal{F})|_{\phi^{-1}(U)} = g\mathcal{O}_{X^*}|_{\phi^{-1}(U)}$. But since ϕ^* is the identity on $\mathscr{K}$, $g = f$. □

DEFINITION 2.4.4. Suppose that D is a locally principal divisor on the normal surface X. Then $\mathcal{L}(D)$ is a subsheaf of $\mathscr{K}$, the constant sheaf of rational functions on X. Then we have shown that $\phi^*(\mathcal{L}(D))$ defines a locally principal divisor D^* on X^*, which we will call the *pullback* $\phi^*(D)$.

PROPOSITION 2.4.5. *With definitions as in Definition 2.4.1, we have*

$$D^* = \widetilde{D} + \sum_i \gamma_i(D)\mu_i,$$

where $\widetilde{D}$ denotes the strict transform of D and the coefficients $\gamma_i(D)$ are given by

$$\gamma_i(D) = \sum_j k_{ij}\widetilde{D}.\mu_j,$$

and so

$$D^* = \widetilde{D} + \sum_{i,j} k_{ij}\left(\widetilde{D}.\mu_i\right)\mu_j. \tag{$\star$}$$

PROOF. Suppose first that D is a principal divisor on the surface X. Then it is the divisor associated to some function f. Now, $\phi^* f$ is a function on X^*; let us compute the divisor associated with it. Recall that away from the exceptional manifold ϕ is an isomorphism, so the divisor D^* of $\phi^* f$ must be $\widetilde{D} + \sum_i \gamma_i(D)\mu_i$ for some integral coefficients $\gamma_i(D)$. But D^* is principal, so D^* must have zero intersection with every divisor. In particular:

$$\begin{aligned} 0 &= D^*.\mu_i \\ &= \widetilde{D}.\mu_i + \sum_j \gamma_j(D)(\mu_i.\mu_j) \\ &= \widetilde{D}.\mu_i + \sum_j \gamma_j(D)d_{ij}, \end{aligned}$$

or, letting $\gamma(D)$ be the column vector with ith element $\gamma_i(D)$,

$$\left(\widetilde{D}\right)^{\Omega^*} = -\mathbf{d}\gamma(D).$$

Thus we have

$$\gamma(D) = \mathbf{k}\left(\widetilde{D}\right)^{\Omega^*}$$

and therefore

$$D^* = \widetilde{D} + \sum_{i,j} k_{ij}\left(\widetilde{D}.\mu_i\right)\mu_j.$$

If we let Ω^* denote the "column vector" whose ith element is the prime divisor μ_i, then we can write this as

$$D^* = \widetilde{D} + \Omega^{*T}\mathbf{k}\left(\widetilde{D}\right)^{\Omega^*}.$$

Note that although the entries in $\mathbf{k}$ may not be integers, the $\gamma_i(D)$ must be, since we are guaranteed to obtain a divisor (in the usual sense) when we take the divisor of a rational function.

Suppose now that we have a divisor D that is only locally principal. It is defined by an invertible sheaf $\mathcal{L}(D)$, which we can pull back to X^* to get an invertible sheaf, from which we can extract a divisor D^*. Now, we know that on a suitable open set U, the invertible sheaf $\mathcal{L}(D)$ is generated by a (global) rational function f, and on U, D is equal to the divisor associated to f. As a result, on $\phi^{-1}(U)$, the divisor D^* will be equal to the divisor associated to $\phi^* f$. We have just derived a formula, $(\star)$, for computing the divisor associated to f. In fact, the formula $(\star)$ does not depend on the open set U or the function f, so we can simply apply $(\star)$ to get the divisor associated to the pullback of $\mathcal{L}(D)$. Once again the coefficients turn out to be integral. $\square$

What can we do with divisors that are not locally principal? We can simply take the formula $(\star)$ above as a definition of the pullback D^* of a divisor D. Of course, we no longer expect the coefficients to be integral, necessarily, but we obtain something which we can abuse notation by calling a divisor.

DEFINITION 2.4.6. Let D be any Weil divisor on X, where X has isolated singularities but is not necessarily normal. Then we define a $\mathbb{Q}$-Weil divisor $D^* = \phi^*(D)$ on X^* by setting

$$D^* = \widetilde{D} + \sum_{i,j} k_{ij}\left(\widetilde{D}.\mu_i\right)\mu_j.$$

Symbolically,

$$D^* = \widetilde{D} + \Omega^{*T}\mathbf{k}\left(\widetilde{D}\right)^{\Omega^*}.$$

REMARK 2.4.7. Every locally principal divisor on X yields a divisor with integral coefficients on X^*. The converse, however, is not true. Recall from Section 2.1.9 that when we have the cone on a curve of positive genus, if the ground field is the complex numbers, we have a local divisor of infinite order. This divisor may not yield a divisor with integral coefficients above, but it will yield one with rational coefficients, in fact bounded by a global constant (the common denominator of the intersection matrix of the exceptional manifold) on the surface. So a sufficiently large multiple of this divisor will yield a divisor whose pullback has integral coefficients but which is not locally principal.

EXAMPLE 2.4.8. Let us return to our toric surfaces from Example 1.4.35 and compute the pullbacks of all the T-Weil divisors. In Example 2.2.19 we computed $\mathbf{d}$ and $\mathbf{k}$, and the strict transform of V_i is always just V_i', so all we need to do is evaluate

$(V_i')^{\Omega^*}$. But the intersection matrix on $X(\Delta_5)$ has a very simple form, so we obtain:

$$\begin{aligned}
V_7^* &= V_7' + \frac{1}{3}V_{11}' + \frac{2}{3}V_{12}' \\
V_8^* &= V_8' \\
V_9^* &= V_9' \\
V_{10}^* &= V_{10}' + \frac{2}{3}V_{11}' + \frac{1}{3}V_{12}'.
\end{aligned}$$

REMARK 2.4.9. In Proposition 1.3.9, we had a number of results about $\phi^*(D)$ for a divisor D on a nonsingular surface. Examining Definition 2.4.4, we see that we have generalized the pullback operation on a nonsingular surface.

Recall that for any divisor C on X^* we defined the push-forward $\phi_*(C)$ in Definition 2.3.4.

PROPOSITION 2.4.10. *Let D be a divisor on X, and μ_i be any integral component of the exceptional fiber. Then*

$$D^*.\mu_i = 0.$$

Conversely, let C be any divisor on X^*. If $C.\mu_i = 0$ for all i, then $C = (\phi_*(C))^*$.

PROOF. Observe that

$$\begin{aligned}
D^*.\mu_i &= \widetilde{D}.\mu_i + \sum_j \gamma_j(D)(\mu_i.\mu_j) \\
(D^*)^{\Omega^*} &= \widetilde{D}^{\Omega^*} + \mathbf{d}\gamma(D) \\
&= \widetilde{D}^{\Omega^*} + \mathbf{dk}\widetilde{D}^{\Omega^*} \\
&= \widetilde{D}^{\Omega^*} - \widetilde{D}^{\Omega^*} \\
&= 0.
\end{aligned}$$

We know that $C - (\phi_*(C))^*$ will be supported only on the exceptional fiber. But by assumption $(C - (\phi_*(C))^*).\mu_i = 0$ for all i. We know that the intersection matrix $\mathbf{d} = (\mu_i.\mu_j)_{i,j}$ is negative definite, so this implies $C - (\phi_*(C))^* = 0$. ☐

COROLLARY 2.4.11. *Let D be any divisor on X. Then D^* is the unique $\mathbb{Q}$-divisor C on X^* such that:*

(1) *$\phi_*(C) = D$ and*
(2) *$C.\mu_i = 0$ for every i.*

REMARK 2.4.12. If D is an effective divisor on X, then $\widetilde{D}$ will be an effective divisor on X^*. But recall from Proposition 2.2.17 that the matrix $\mathbf{k}$ has positive entries. Since each μ_i is also effective, $\widetilde{D}.\mu_i$ is nonnegative and thus $\mathbf{k}\widetilde{D}^{\Omega^*}$ is a vector of nonnegative rationals, and we see that D^* is effective. It is clear that if C is an effective divisor on X^*, then $\phi_*(C)$ is an effective divisor on X.

PROPOSITION 2.4.13. *Let D_1 and D_2 be divisors on X. Then D_1 and D_2 are algebraically equivalent if and only if D_1^* and D_2^* are algebraically equivalent.*

PROOF. Suppose first that D_1 and D_2 are algebraically equivalent. Then we have shown in Proposition 2.3.13 that $\widetilde{D_1}$ and $\widetilde{D_2}$ are algebraically equivalent. Thus in particular if μ_i is any integral component of Ω^* we have $\widetilde{D_1}.\mu_i = \widetilde{D_2}.\mu_i$. Write

$$D_j^* = \widetilde{D_j} + \sum_i \gamma_i(D_j)\mu_i,$$

and recall that Equation $(\star)$ expresses the $\gamma_i(D_j)$ in terms of $\widetilde{D_j}.\mu_j$. This immediately gives $\gamma_i(D_1) = \gamma_i(D_2)$. Write $E = \sum_i \gamma_i(D_1)\mu_i$ so that $D_1^* = \widetilde{D_1} + E$ and $D_2^* = \widetilde{D_2} + E$. Then by assumption $\widetilde{D_1}$ is algebraically equivalent to $\widetilde{D_2}$ and E is obviously algebraically equivalent to itself, so we see that D_1^* is algebraically equivalent to D_2^*.

Now suppose that D_1^* is algebraically equivalent to D_2^*. We showed in Proposition 2.3.13 that this implies that $\phi_*(D_1^*)$ is algebraically equivalent to $\phi_*(D_2^*)$, so D_1 is algebraically equivalent to D_2 as required. □

2.4.2. Intersection numbers. We now have a second natural way to take divisors on a singular surface and obtain divisors on a monoidal model of it. We can use this operation to define an intersection number on X.

DEFINITION 2.4.14. Let D_1 and D_2 be divisors on X, and let (X^*, ϕ) be a monoidal model (as is the case throughout this section). Then we define

$$(D_1.D_2)_* = D_1^*.D_2^*$$

REMARK 2.4.15. The value $(D_1.D_2)_*$ will not necessarily be an integer (although it will whenever both divisors are locally principal) but this is to be expected.

EXAMPLE 2.4.16. Let us return to our toric surfaces from Example 1.4.35 and use the pullbacks we computed in Example 2.4.8 to compute the intersection matrix for the surface:

$$(V_i.V_j)_* = \begin{pmatrix} \frac{2}{3} & 1 & 0 & \frac{1}{3} \\ 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ \frac{1}{3} & 0 & 1 & -\frac{1}{3} \end{pmatrix}.$$

Observe that $(V_7.V_{10})_* = \frac{1}{3}$, which agrees with our computation in Example 2.3.23.

PROPOSITION 2.4.17. *Let D_1 and D_2 be divisors on X. Let (X^{**}, ψ) be a second monoidal model of X, and let $(D_1.D_2)_{**}$ denote the intersection pairing defined as in Definition 2.4.14 relative to X^{**}. Then*

$$(D_1.D_2)_* = (D_1.D_2)_{**}.$$

PROOF. Recalling Proposition 2.2.6, we see that it is sufficient to check this result when X^{**} is a blow-up of X^*, that is, when we have a commutative diagram

$$\begin{tikzcd} & X^{**} \arrow[dl, "\pi"'] \arrow[ddl, "\psi"] \\ X^* \arrow[d, "\phi"'] & \\ X & \end{tikzcd}$$

where π is a simple blow-up of nonsingular surfaces.

First let C be any divisor on X. Clearly $\psi_*(\pi^*(\phi^*(C))) = C$. Let μ_i be a component of the exceptional manifold of X^{**}. What is $\mu_i.\pi^*(\phi^*(C))$? Since π is a simple blowup, we can apply Proposition 1.3.9. If μ_i is the exceptional fiber of π, then $\mu_i.\pi^*(\phi^*(C)) = 0$. Otherwise, $\pi_*(\mu_i)$ is a component of the exceptional manifold of X^*; then $\mu_i.\pi^*(\phi^*(C)) = \pi_*(\mu_i).(\phi^*(C)) = 0$. So $\mu_i.\pi^*(\phi^*(C)) = 0$ for all i and Proposition 2.4.10 that $\psi^*(C) = \pi^*(\phi^*(C))$.

What is $(D_1.D_2)_{**}$? By definition, it is just $\psi^*(D_1).\psi^*(D_2)$. But we have shown that this is equal to $\pi^*(\phi^*(D_1)).\pi^*(\phi^*(D_2))$. Using Proposition 1.3.9, this is equal to $\phi^*(D_1).\phi^*(D_2)$. By definition this is equal to $(D_1.D_2)_*$, as required. $\square$

Thus our new definition of intersection number is independent of the monoidal model.

2.4.3. Relationship with Section 2.3.

PROPOSITION 2.4.18. *Suppose that D is a divisor on X and E is a divisor on X^*. Then*

$$D^*.E = (D.\phi_*(E))_*.$$

PROOF. Simply write $E = \phi_*(E)^* + \sum e_i\mu_i$ for some coefficients e_i. □

REMARK 2.4.19. In particular, this means that $(D_1.D_2)_* = D_1^*.D_2^* = D_1^*.\widetilde{D_2}$.

We can now make an extremely useful computation to obtain:

THEOREM 2.4.20. *Let D_1 and D_2 be divisors on a surface X. Then*

$$(D_1.D_2)_{\mathrm{RT}} = (D_1.D_2)_*.$$

PROOF. We simply choose (X^*, ϕ) to be a resolved model well-adapted to D_1 and D_2. Then:

$$\begin{aligned}
(D_1.D_2)_* &= D_1^*.D_2^* \\
&= D_1^*.\widetilde{D_2} \\
&= \left(\widetilde{D_1} + \sum_i \gamma_i(D_1)\mu_i\right).\widetilde{D_2} \\
&= \widetilde{D_1}.\widetilde{D_2} + \sum_i \gamma_i(D_1)\left(\widetilde{D_2}.\mu_i\right) \\
&= \widetilde{D_1}.\widetilde{D_2} + \left(\widetilde{D_2}^{\Omega^*}\right)^T \left(\mathbf{k}\left(\widetilde{D_1}\right)^{\Omega^*}\right) \\
&= \widetilde{D_1}.\widetilde{D_2} + \left(\widetilde{D_1}^{\Omega^*}\right)^T \mathbf{k}\left(\widetilde{D_2}\right)^{\Omega^*} \\
&= (D_1.D_2)_{\mathrm{RT}}.
\end{aligned}$$

□

We see that the intersection theory of this section yields the same results as the theory of Reeve and Tyrrell.

REMARK 2.4.21. Observe that the formula for the global intersection number produced in the theory of Reeve and Tyrrell is well-defined even when the resolved model is not well-adapted to the two divisors, and in fact when we have only a monoidal model. By comparing with this intersection theory, we see that it is correct in such a case as well: we can apply the theory of Reeve and Tyrrell with any monoidal model.

EXAMPLE 2.4.22. Consider Example 1.2.5, the cone on the Fermat curve, and choose the divisor D to be given by $x_1 = x_3 = 0$. Now, in Example 2.3.7 we computed the strict transform $\widetilde{D}$ to be a ruling on the ruled surface we obtain. In Section 1.4.1 we have also computed the self-intersection number of the exceptional fiber to be $-m$, so $\mathbf{k}$ is the one-by-one matrix $1/m$, so $D^* = \widetilde{D} + \frac{1}{m}\mu$. Observe that D has self-intersection $1/m$, as we computed in Example 2.3.22.

2.5. Cohomological Intersection Theory

We have seen that it is possible to define an intersection theory on a singular surface by making reference to an appropriate resolution of singularities. However, it would often be desirable to have a more intrinsic formulation of intersection theory. In **[Sna59]** and **[Sna60]**, it was shown that by examining the Euler characteristics of suitable sheaves one can obtain an intersection theory. We will see that this new theory yields the same results as those we have already seen whenever the divisors involved are locally principal and the surface is normal.

This theory was put forward in **[Sna59]** and **[Sna60]** and developed further in **[Kle66]**. The proofs in this section follow **[Bǎd01]**, with the exception of some parts of Section 2.5.3; intersection theory based on resolution of singularities is not discussed in **[Bǎd01]**.

This cohomological approach has several drawbacks: first, it cannot readily be extended to handle divisors that are not locally principal, and second, it is not in any sense a local theory; one cannot assign intersection numbers at a point.

To motivate these definitions somewhat, recall the Riemann-Roch theorem on a nonsingular surface. Letting L_1 and L_2 be invertible sheaves associated to the divisors D_1 and D_2, it asserts:

$$\begin{aligned}\chi(L_1^{\otimes n_1} \otimes L_2^{\otimes n_2}) = \frac{1}{2}(n_1^2(D_1.D_1) + 2n_1n_2(D_1.D_2) + n_2^2(D_2.D_2)) \\ - \frac{1}{2}(n_1(D_1.K) + n_2(D_2.K)) + 1 + p_a(X)\end{aligned}$$

where K is the canonical divisor. Observe that the coefficient of n_1n_2 is $D_1.D_2$, so we can obtain the intersection number by evaluating the Euler characteristic. Since the Euler characteristic of an invertible sheaf is a well-defined concept on any projective algebraic $\Bbbk$-scheme, one might try to work from the Euler characteristic to define the intersection number. This is precisely how this cohomological intersection theory works, but a few results will be necessary before the intersection form can be defined.

2.5.1. A theorem of Snapper. The essential result is that there is a numerical polynomial analogous to the Hilbert polynomial.

Recall that if $\mathcal{F}$ is a sheaf on a topological space X, then $\operatorname{Supp}\mathcal{F}$ denotes the support of $\mathcal{F}$, that is, the set of points x such that the stalk $\mathcal{F}_x$ is nonzero. If X is a noetherian scheme and $\mathcal{F}$ is coherent, then $\operatorname{Supp}\mathcal{F}$ is closed.

THEOREM 2.5.1 (Snapper). *Let $\mathcal{F}$ be a coherent sheaf of $\mathcal{O}_V$-modules on the complete algebraic $\mathbb{k}$-scheme V, and let $\mathcal{L}_1, \ldots, \mathcal{L}_t$ be t invertible $\mathcal{O}_V$ sheaves for some $t \geq 0$. Then the function*

$$f_{\mathcal{F}}(n_1, \ldots, n_t) = \chi(\mathcal{F} \otimes \mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_t^{n_t})$$

is a numerical polynomial in $n_1, \ldots, n_t$, of degree $\leq s = \dim(\operatorname{Supp}\mathcal{F})$.

This theorem was originally proved in [**Sna59**], but we will follow the exposition in [**Băd01**, Chap. 1].

We first need a lemma from [**Gro61**, 3.1.2], the *Dévissage lemma*:

LEMMA 2.5.2 (Dévissage). *Let X be a noetherian scheme, and let $\mathfrak{K}$ denote the abelian category of coherent $\mathcal{O}_X$-modules. Let $\mathfrak{K}'$ be a subclass of $\mathfrak{K}$ such that $0 \in \mathfrak{K}'$ and whenever we have an exact sequence $0 \to \mathcal{A}' \to \mathcal{A} \to \mathcal{A}'' \to 0$ in $\mathfrak{K}$, if two of $\mathcal{A}$, $\mathcal{A}'$, and $\mathcal{A}''$ are in $\mathfrak{K}'$, then the third is. Let X' be a closed subset of the underlying space of X. Suppose that for each irreducible closed subset Y of X' with generic point y there is an $\mathcal{O}_X$-module $\mathcal{G} \in \mathfrak{K}'$ such that the stalk $\mathcal{G}_y$ is a vector space of dimension 1 over the residue field $k(y)$. Then every coherent $\mathcal{O}_X$-module with support contained in X' is in $\mathfrak{K}'$. In particular, if $X' = X$ then $\mathfrak{K}' = \mathfrak{K}$.*

For a proof of this lemma, see [**Gro61**, 3.1.2].

We will use a quite specific version of this lemma: $X' = X$, and the module $\mathcal{G}$ will be $\mathcal{O}_Y$, since if y is the generic point of Y, $\mathcal{O}_{Y,y}$ is just the function field of Y, which is also $k(y)$.

PROOF OF THEOREM 2.5.1. First observe that we need to assume that V is complete to ensure that the cohomology groups are all finite-dimensional and thus that χ is well-defined.

We will proceed by induction on s, the dimension of the support of $\mathcal{F}$. Clearly if $\mathcal{F} = 0$, the result holds. If $s = 0$, then $\mathcal{F}$ is supported on a finite set of points. But then each $\mathcal{L}_i$ is free on the support of $\mathcal{F}$, and since $\mathcal{L}_i$ is generated by a single element, $\mathcal{F} \otimes \mathcal{L}_i \cong \mathcal{F}$ and $f_{\mathcal{F}}$ is a constant function as required.

If the support of $\mathcal{F}$ is not all of V, we may take V to be the support of $\mathcal{F}$ with the subscheme structure given by the annihilator ideal $\operatorname{Ann}(\mathcal{F}) \subset \mathcal{O}_V$ ($\operatorname{Ann}(\mathcal{F})(U) = \{x \in \mathcal{O}_V(U) | x\mathcal{F}(U) = 0\}$). So assume V is the support of $\mathcal{F}$.

Let $\mathfrak{K}$ be the category of all coherent $\mathcal{O}_V$-modules, and let $\mathfrak{K}'$ be the subclass of $\mathfrak{K}$ consisting of those modules satisfying the conditions of the theorem. Recall that the Euler characteristic is additive in exact sequences, so that if we have $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ exact, then if two of $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{F}''$ are in $\mathfrak{K}'$, then all three are. Using the Dévissage lemma (2.5.2), it will be sufficient to show that for every closed integral subscheme X of V we have $\mathcal{O}_X \in \mathfrak{K}'$. So assume X is a closed integral subscheme of V.

We now prove this last statement by another induction, this time on t. Again we may assume $V = X$. Since X is integral we may assume without loss of generality that every invertible sheaf is a subsheaf of $\mathcal{K}$, the sheaf of rational functions on X ([**Har77**, II.6.15]). Recall that $\mathcal{L}_1^\vee$ denotes the inverse of the invertible sheaf $\mathcal{L}_1$, and define the following sheaves:

$$\mathcal{J} = \mathcal{L}_1^\vee \cap \mathcal{O}_X, \qquad \mathcal{I} = \mathcal{J} \cdot \mathcal{L}_1 = \mathcal{J} \otimes \mathcal{L}_1$$
$$\mathcal{G} = \mathcal{O}_X/\mathcal{J}, \qquad \mathcal{H} = (\mathcal{O}_X/\mathcal{I}) \otimes \mathcal{L}_1^\vee.$$

Then we have exact sequences

$$0 \to \mathcal{J} \to \mathcal{O}_X \to \mathcal{G} \to 0$$

and

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{H} \otimes \mathcal{L}_1 \to 0.$$

We tensor these with $\mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_t^{n_t}$ and $\mathcal{L}_1^{n_1-1} \otimes \cdots \otimes \mathcal{L}_t^{n_t}$ respectively, and obtain the following exact sequences:

$$0 \to \mathcal{J} \otimes \mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_t^{n_t} \to \mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_t^{n_t} \to \mathcal{G} \otimes \mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_t^{n_t} \to 0$$

and

$$0 \to \mathcal{I} \otimes L_1^{n_1-1} \otimes \cdots \otimes \mathcal{L}_t^{n_t} \to \mathcal{L}_1^{n_1-1} \otimes \cdots \otimes \mathcal{L}_t^{n_t} \to \mathcal{H} \otimes \mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_t^{n_t} \to 0.$$

But $\mathcal{I} \otimes \mathcal{L}_1^{n_1-1} \otimes \cdots \otimes \mathcal{L}_t^{n_t} = \mathcal{J} \otimes \mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_t^{n_t}$ so we have

$$\chi(\mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_t^{n_t}) - \chi(\mathcal{L}_1^{n_1-1} \otimes \cdots \otimes \mathcal{L}_t^{n_t}) = \chi(\mathcal{G} \otimes \mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_t^{n_t}) - \chi(\mathcal{H} \otimes \mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_t^{n_t}).$$

But $\mathcal{H}$ and $\mathcal{G}$ have support properly contained in X, since they are supported on the divisors and poles of the Cartier divisor associated to $\mathcal{L}_1$. So by the inductive hypothesis on s, the right-hand side of the equality is a numerical polynomial of degree less than s. But by the inductive hypothesis on t, $\chi(\mathcal{L}_2^{n_2} \otimes \cdots \otimes \mathcal{L}_t^{n_t})$ is a numerical polynomial of degree at most s; since the first difference had n_1-degree less than s, $\chi(\mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_t^{n_t})$ is a numerical polynomial of degree at most s, as required. $\square$

EXAMPLE 2.5.3. Let X be the cone on the Fermat curve (Example 1.2.5) of degree 2, and let D be a ruling on the cone. Then in **[Sna59**, Ex. 10.2] it is shown that $\chi(\mathcal{L}(n(2D))) = 1 + 2n + n^2$ and so if Theorem 2.5.1 applied to D we would have $\chi(\mathcal{L}(nD)) = 1 + n + \frac{1}{4}n^2$ which is impossible (see Section 3.3 for an explanation of $\mathcal{L}(D)$ when D is not locally principal).

2.5.2. Cohomological intersection numbers.

DEFINITION 2.5.4. The *(Snapper) intersection number*

$$(\mathcal{L}_1 \ldots \mathcal{L}_t; \mathcal{F})_{\mathrm{Sn}}$$

on a complete $\Bbbk$-scheme V is the coefficient of $n_1 \cdots n_t$ in $\chi(\mathcal{F} \otimes \mathcal{L}_1^{n_1} \otimes \cdots \otimes \mathcal{L}_t^{n_t})$. When $\mathcal{F} = \mathcal{O}_W$ for some closed subscheme W of V, we will simply write

$$(\mathcal{L}_1 \cdots \mathcal{L}_t; W)_{\mathrm{Sn}}.$$

In the most common case for our purposes, when $W = V$, that is, $\mathcal{F} = \mathcal{O}_V$, we will write

$$(\mathcal{L}_1 \cdots \mathcal{L}_t)_{\mathrm{Sn}}.$$

REMARK 2.5.5. As usual, when V is normal we will treat invertible sheaves, locally principal divisors and line bundles interchangeably. Thus, in particular, we will occasionally take the intersection number of locally principal divisors without further comment.

REMARK 2.5.6. Let us evaluate this intersection form in the case where V is a nonsingular surface. Applying the Riemann-Roch theorem we get:

$$\chi(\mathcal{L}(n_1 D_1 + n_2 D_2)) = \frac{1}{2}(n_1^2(D_1.D_1) + 2n_1 n_2(D_1.D_2) + n_2^2(D_2.D_2)) \\ - \frac{1}{2}(n_1(D_1.K) + n_2(D_2.K)) + 1 + p_a(X)$$

where K is the canonical divisor. Observe that the coefficient of n_1n_2 is $D_1.D_2$, so this new intersection form reduces to the classical intersection form when the surface is nonsingular.

REMARK 2.5.7. What is the purpose of this apparently surplus sheaf $\mathcal{F}$? Let us suppose that V is a nonsingular surface and that $\mathcal{F}$ is an invertible sheaf corresponding to the divisor D, while $\mathcal{L}$ corresponds to E. Then we may use Riemann-Roch to obtain

$$\chi(\mathcal{F}\otimes\mathcal{L}^n)=\frac{1}{2}((D.D)+2n(D.E)+n^2(E.E))-\frac{1}{2}((D.K)+n(E.K))+1+p_a(X).$$

Here the coefficient of n is $(D.E)+\frac{1}{2}(E.K)$. So $(\mathcal{L}\cdot\mathcal{F})_{\mathrm{Sn}}\neq(\mathcal{L};\mathcal{F})_{\mathrm{Sn}}$, that is, the "extra" sheaf does not serve the same purpose as the invertible sheaves.

REMARK 2.5.8. This apparently defines an intersection number for any number of invertible sheaves. However, if t is greater then the dimension of the support of $\mathcal{F}$ (or in particular, if it is greater than the dimension of V), then the degree of $\chi(\mathcal{F}\otimes\mathcal{L}_1^{n_1}\otimes\cdots\otimes\mathcal{L}_t^{n_t})$ is too small for the term $n_1\cdots n_t$ to occur, so the intersection number $(\mathcal{L}_1\ldots\mathcal{L}_t;\mathcal{F})_{\mathrm{Sn}}$ will be zero. If $t=0$, then the intersection number $(;\mathcal{F})_{\mathrm{Sn}}$ is just $\chi(\mathcal{F})$. If t is less than the dimension of the support of $\mathcal{F}$, there is in general no simple formula. If V is a nonsingular surface, the intersection number $(\mathcal{L})_{\mathrm{Sn}}$ gives us information about the canonical divisor:

$$\chi(\mathcal{L}(nD))=\frac{1}{2}D.Dn^2+\frac{1}{2}D.Kn+\chi(\mathcal{O}_X).$$

PROPOSITION 2.5.9. *The intersection form $(\mathcal{L}_1\ldots\mathcal{L}_t;\mathcal{F})_{\mathrm{Sn}}$ is trivial on principal divisors, symmetric and multilinear in the $\mathcal{L}_i$.*

PROOF. Symmetry and triviality on principal divisors are immediate. Let $\mathcal{M}$ and $\mathcal{N}$ be invertible sheaves, and evaluate

$$(\mathcal{M}\otimes\mathcal{N}^{-1}\cdot\mathcal{L}_2\cdots\mathcal{L}_t;\mathcal{F})_{\mathrm{Sn}}.$$

Compute

$$\begin{aligned}&\chi(\mathcal{M}^m\otimes\mathcal{N}^{-n}\otimes\mathcal{L}_2^{n_2}\otimes\cdots\otimes\mathcal{L}_t^{n_t};\mathcal{F})=\\&\qquad(\mathcal{M}\cdot\mathcal{L}_2\cdots\mathcal{L}_t;\mathcal{F})_{\mathrm{Sn}}mn_2\cdots n_t-(\mathcal{N}\cdot\mathcal{L}_2\cdots\mathcal{L}_t;\mathcal{F})_{\mathrm{Sn}}nn_2\cdots n_t+\cdots,\end{aligned}$$

by setting m and n in turn to zero. Then set $m=n=n_1$ and obtain

$$\begin{aligned}&\chi\big((\mathcal{M}\otimes\mathcal{N}^{-1})^{n_1}\otimes\mathcal{L}_2^{n_2}\otimes\cdots\otimes\mathcal{L}_t^{n_t};\mathcal{F}\big)=\\&\qquad((\mathcal{M}\cdot\mathcal{L}_2\cdots\mathcal{L}_t;\mathcal{F})_{\mathrm{Sn}}-(\mathcal{N}\cdot\mathcal{L}_2\cdots\mathcal{L}_t;\mathcal{F})_{\mathrm{Sn}})n_1n_2\cdots n_t+\cdots.\end{aligned}$$

□

REMARK 2.5.10. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be invertible sheaves on a surface X. Let $f(n_1, n_2)$ be $\chi(\mathcal{L}_1^{n_1} \otimes \mathcal{L}_2^{n_2})$. Then we know f has degree two, so $f(0,0) - f(1,0) - f(0,1) + f(1,1)$ is just the $n_1 n_2$ coefficient of f. In other words:

$$(\mathcal{L}_1.\mathcal{L}_2)_{\mathrm{Sn}} = \chi(\mathcal{O}_X) - \chi(\mathcal{L}_1) - \chi(\mathcal{L}_2) + \chi(\mathcal{L}_1 \otimes \mathcal{L}_2).$$

Our definition in terms of coefficients of numerical polynomials is clearly also equivalent to using the formula of Lemma 1.2.38 as our definition for the intersection number. However, our construction of the numerical polynomial $\chi(\mathcal{F} \otimes \mathcal{L}_1^n \cdots \mathcal{L}_t^n)$ has the advantage that various coefficients of this polynomial have geometric significance. For example, arithmetic genera can be obtained from them; **[Sna60]** goes into details.

REMARK 2.5.11. Let us return to the question of the "surplus sheaf" in the intersection number. Let X be any surface, and let Z be a locally principal effective divisor. Then we have an exact sequence

$$0 \to \mathcal{L}(-Z) \to \mathcal{O}_X \to \mathcal{O}_Z \to 0.$$

Letting $\mathcal{F}$ be an invertible sheaf, this leads to an exact sequence

$$0 \to \mathcal{F}^n \otimes \mathcal{L}(-Z) \to \mathcal{F}^n \to \mathcal{F}^n \otimes \mathcal{O}_Z \to 0.$$

Since the Euler characteristic is additive on exact sequences, we get

$$\chi(\mathcal{F}^n \otimes \mathcal{O}_Z) = \chi(\mathcal{F}^n) - \chi(\mathcal{F}^n \otimes \mathcal{L}(-Z)).$$

We know from Theorem 2.5.1 that this must be at most a linear polynomial, so we can extract the coefficient of n by evaluating it at 1 and 0 to get

$$\begin{aligned}(\mathcal{F}; \mathcal{O}_Z)_{\mathrm{Sn}} &= \chi(\mathcal{F}) - \chi(\mathcal{F} \otimes \mathcal{L}(-Z)) - \chi(\mathcal{O}_X) + \chi(\mathcal{L}(-Z)) \\ &= (\mathcal{F} \cdot (-Z))_{\mathrm{Sn}}\end{aligned}$$

2.5.3. Comparison with results from Sections 2.3 and 2.4. We have seen in Remark 2.5.10 that the intersection form of Snapper is a candidate for a generalization of the intersection form on a nonsingular surface. We will establish that it agrees with the intersection forms defined in Section 2.3 and Section 2.4.

DEFINITION 2.5.12. Let $f : V' \to V$ be a morphism between two complete and irreducible algebraic $\Bbbk$-schemes with generic points x' and x. The *degree* of f is defined to be:

$$\deg(f) = \begin{cases} \frac{\mathrm{length}_{\mathcal{O}_x}(\mathcal{O}_{x'})}{\mathrm{length}_{\mathcal{O}_x}(\mathcal{O}_x)} & \text{if } \dim(V') = \dim(V) = \dim f(V') \\ 0 & \text{otherwise.} \end{cases}$$

EXAMPLE 2.5.13. Let f be birational. Then $f(x') = x$ and f yields an isomorphism from $\mathcal{O}_x$ to $\mathcal{O}_{x'}$, so the degree of f is one.

PROPOSITION 2.5.14. *Let $f : V' \to V$ be a morphism between two complete and irreducible algebraic $\mathbb{k}$-schemes, and assume that $t \geq \dim(V), \dim(V')$. Let $\mathcal{L}_1, \ldots, \mathcal{L}_t$ be invertible $\mathcal{O}_V$-modules and $\mathcal{L}'_i = f^*(\mathcal{L}_i)$ for all i. Then:*

$$(\mathcal{L}'_1 \ldots \mathcal{L}'_t)_{V'} = \deg(f) \cdot (\mathcal{L}_1 \ldots \mathcal{L}_t)_V$$

The proof of this requires a number of lemmas and some knowledge of spectral sequences, so we will not reproduce it here; see [**Băd01**, Lem. 1.18].

THEOREM 2.5.15. *Let X be a normal surface and let D_1 and D_2 be locally principal divisors on X. If $(D_1.D_2)_{\mathrm{RT}}$ denotes the intersection pairing of Section 2.3 and $(D_1.D_2)_{\mathrm{Sn}}$ denotes the intersection pairing of this section, then we have*

$$(D_1.D_2)_{\mathrm{RT}} = (D_1.D_2)_* = (D_1.D_2)_{\mathrm{Sn}}.$$

PROOF. Let (X^*, ϕ) be a resolved model of X. Then if $\mathcal{L}_i$ is the invertible sheaf associated to D_i, we recall that the divisor associated to $\phi^*(\mathcal{L}_i)$ is denoted D_i^*. Then since ϕ is birational, we have $((D_1.D_2)_{\mathrm{Sn}})_X = ((D_1^*.D_2^*)_{\mathrm{Sn}})_{X^*}$. But X^* is a nonsingular surface, and we showed in Remark 2.5.6 that $((D_1^*.D_2^*)_{\mathrm{Sn}})_{X^*}$ is just the classical intersection number $D_1^*.D_2^*$. On the other hand, in Section 2.4 we showed that $D_1^*.D_2^*$ is equal to $(D_1.D_2)_{\mathrm{RT}}$, the intersection form of Reeve and Tyrrell (Section 2.3). $\square$

CHAPTER 3

Classical Results on a Singular Surface

On any normal surface, the three intersection pairings we defined in Chapter 2 agree whenever they are all defined; when the surface is nonsingular, they agree with the standard intersection pairing. As a result, we will hereafter write simply $C.D$ to mean any of the above intersection pairings.

In Section 1.2 we proved a number of results about divisors on nonsingular surfaces using the intersection pairing. In this chapter, we will review some similar results on singular surfaces from the literature, and we will endeavor to prove several generalizations of the results from Section 1.2 to the singular case.

3.1. Numerical and Algebraic Equivalence

What does the Néron-Severi group of a singular surface look like?

PROPOSITION 3.1.1. *Let X be a surface with isolated singularities, and let (X^*, ϕ) be a resolved model of X. Recall that the Néron-Severi group* $\mathrm{NS}\,X$ *is the group of divisors modulo algebraic equivalence. Let $\{\mu_i\}$ be the integral components of the exceptional manifold, and let G be the subgroup of* $\mathrm{NS}\,X^*$ *generated by the $\{\mu_i\}$. Then we have an exact sequence*

$$0 \longrightarrow G \longrightarrow \mathrm{NS}\,X^* \xrightarrow{\phi_*} \mathrm{NS}\,X \longrightarrow 0\,.$$

PROOF. We have shown in Proposition 2.3.13 that both ϕ_* and the operation of taking the strict transform are well-defined on the Néron-Severi groups. We also saw that for any C on X, $\phi_*(\widetilde{C}) = C$, so ϕ_* is surjective.

Suppose P is a prime divisor not contained in Ω^*, then by the definition of ϕ_* acting on $\mathrm{Div}\,X^*$, the image $\phi_*(P)$ is a nonzero prime divisor on X. Conversely, if P is contained in Ω^*, then $\phi_*(P)$ is nonzero. So the kernel of $\phi_* : \mathrm{Div}\,X^* \to \mathrm{Div}\,X$ is precisely the set of divisors supported on the exceptional manifold, and in particular it is generated by the $\{\mu_i\}$. Thus when we pass to the Néron-Severi group, the kernel of ϕ_* is precisely the subgroup of $\mathrm{NS}\,X^*$ generated by the $\{\mu_i\}$. $\square$

By analogy with Definition 1.2.7, we define:

DEFINITION 3.1.2. Let D_1 and D_2 be divisors on the surface X. Then we say D_1 and D_2 are *numerically equivalent* if for every divisor C on X we have $D_1.C = D_2.C$. The group of divisors modulo numerical equivalence we call the *numerical equivalence class group* of X and denote $\operatorname{Num} X$.

PROPOSITION 3.1.3. *Let X be a surface and let (X^*, ϕ) be a resolved model of X with fundamental manifold Ω^* having integral components $\{\mu_i\}_{i=1}^n$. Then the μ_i are linearly independent in* $\operatorname{Num} X^* \otimes \mathbb{Q}$. *We can choose $\{\nu_j\}_{j=1}^m \subset \operatorname{Num} X \otimes \mathbb{Q}$ so that $\{\mu_i\}_{i=1}^n \cup \{\nu_j^*\}_{j=1}^m$ is a basis for* $\operatorname{Num} X^* \otimes \mathbb{Q}$. *Then $\{\nu_j\}_{j=1}^m$ are a basis for* $\operatorname{Num} X \otimes \mathbb{Q}$.

In the basis $\{\mu_i\}_{i=1}^n \cup \{\nu_j^*\}_{j=1}^m$, the intersection matrix on X^* looks like

$$\begin{pmatrix} \mathbf{m} & 0 \\ 0 & \mathbf{d} \end{pmatrix},$$

where $\mathbf{m}$ is some matrix with rational entries and $\mathbf{d}$ is the intersection matrix of the μ_i whose properties were discussed in Proposition 2.2.16. Then the intersection matrix on X in the basis $\{\nu_j\}_{j=1}^n$ is $\mathbf{m}$.

PROOF. We saw in Proposition 2.2.16 that the intersection matrix $(\mu_i.\mu_j)$ is negative definite. In particular, this means that the μ_i are linearly independent elements of $\operatorname{Num} X^* \otimes \mathbb{Q}$. Let $D_1, \ldots, D_m$ extend the μ_i to a basis for $\operatorname{Num} X^* \otimes \mathbb{Q}$. Now let $a_1, \ldots, a_n$ and $b_1, \ldots, b_m$ be rational numbers, and suppose that

$$\sum_i a_i \mu_i + \sum_j b_j (\phi_*(D_j))^*$$

is numerically equivalent to zero. Then in particular,

$$\mu_k . \left(\sum_i a_i \mu_i + \sum_j b_j (\phi_*(D_j))^* \right) = 0.$$

But recall that $\mu_k.C^* = 0$ for any divisor C on X. So

$$\mu_k . \left(\sum_i a_i \mu_i \right) = 0,$$

and we know that $\mathbf{d}$ is negative definite, so $a_i = 0$ for every i. But then we have

$$\sum_j b_j (\phi_*(D_j))^* = 0,$$

and as a result

$$\sum_j b_j (D_j - (\phi_*(D_j))^*) = \sum_j b_j D_j.$$

By construction of ϕ_* and $\cdot^*$, the left-hand side is in the span of the μ_i, while the right-hand side is in the span of the D_j. But by construction $\{\mu_i\}_{i=1}^n \cup \{D_j\}_{j=1}^m$ is a basis, so this implies that $b_j = 0$ for all j. Let $\nu_j = \phi_*(D_j)$. Then we have shown that $\{\mu_i\}_{i=1}^n \cup \{\nu_j^*\}_{j=1}^m$ is a basis for $\operatorname{Num} X^* \otimes \mathbb{Q}$.

Now let $b_1, \ldots, b_m$ be rational numbers and suppose that $\sum_j b_j \nu_j$ is numerically equivalent to zero. Let D be a divisor on X^*. We have shown that

$$D.\phi^*\left(\sum_j b_j \nu_j\right) = \phi_*(D).\left(\sum_j b_j \nu_j\right),$$

which implies that

$$D.\phi^*\left(\sum_j b_j \nu_j\right) = 0$$

for all divisors D on X^*. This in turn implies that $b_j = 0$ for all j. Let C be any divisor on X. Then we can write $C^* = \sum_j b_j \nu_j^*$. Applying ϕ_* to both sides we get $C = \sum_j b_j \nu_j$.

The intersection matrix of the μ_i is by definition $\mathbf{d}$. We showed in Proposition 2.4.10 that $\nu_j^*.\mu_i = 0$, and by definition $\nu_j.\nu_k = \nu_j^*.\nu_k^*$. □

REMARK 3.1.4. We see that for the $\{\nu_i\}$ any basis of $\operatorname{Num} X \otimes \mathbb{Q}$ would have sufficed.

COROLLARY 3.1.5. *Let X be a normal surface and let (X^*, ϕ) be a resolved model of X. Let D_1 and D_2 be divisors on X. Then D_1 and D_2 are numerically equivalent if and only if D_1^* and D_2^* are.*

PROOF. Let $\{\nu_i\}$ be a basis for $\operatorname{Num} X \otimes \mathbb{Q}$ as above. Then $\sum_i a_i \nu_i$ is numerically equivalent to zero if and only if all the a_i are zero, which is true if and only if $\sum_i a_i \nu_i^*$ is numerically equivalent to zero. □

As discussed in Remark 1.2.58, a theorem of Matsusaka states that on a nonsingular surface X, a divisor D is numerically equivalent to zero if and only if some multiple nD of it is algebraically equivalent to zero. Denote the group of divisors algebraically equivalent to zero $G_a(X)$ and the group of divisors numerically equivalent to zero $G_n(X)$. Let $G_\tau(X)$ be the group of divisors with a multiple algebraically equivalent to zero. The theorem just stated amounts to saying $G_\tau(X) = G_n(X)$. By the Néron-Severi theorem, the group of divisors modulo algebraic equivalence is finitely generated, and so the group $G_\tau(X)/G_a(X)$ is finite. Now that we have an intersection theory on a singular surface, can we determine the relation between these two forms of equivalence on such a surface?

THEOREM 3.1.6. *Let X be a surface. Then $G_\tau(X) = G_n(X)$.*

PROOF. Let D be any divisor on X. Let (X^*, ϕ) be a resolved model of X. Then Corollary 3.1.5 shows that D is numerically equivalent to zero if and only if D^* is. The theorem of Matsusaka shows that D^* is numerically equivalent to zero if and only if some multiple nD^* is algebraically equivalent to zero. Proposition 2.4.13 shows that nD^* is algebraically equivalent to zero if and only if nD is algebraically equivalent to zero. $\square$

3.2. The Adjunction Formula

On a nonsingular surface, the adjunction formula was extremely useful, both in its own right and in the proof of the Riemann-Roch theorem. So it would be desirable to prove a version on a singular surface. The direct approach, of generalizing the proof used on a nonsingular surface, does not work well, but if the surface is normal, then we have a resolved model and a way of pulling divisors back to the resolved model. In [**AG02**, Sec. 7], this approach is used to prove a version of the adjunction formula essentially equivalent to the one proven here for the special case of nonsingular curves.

Recall that the adjunction formula (Theorem 1.2.60) allows us to compute the genus of a curve (or, using Corollary 1.2.69, of a divisor) on a nonsingular surface, given the intersections of the curve with the canonical divisor on the nonsingular surface. The canonical divisor is the divisor associated to the sheaf $\Omega_{X/\Bbbk} \wedge \Omega_{X/\Bbbk}$. This sheaf is invertible because $\Omega_{X/\Bbbk}$ is a locally free sheaf of rank two, which is true if and only if X is nonsingular. So on a singular surface, the canonical sheaf is not invertible everywhere. So it is not obvious how to obtain a canonical divisor from this sheaf. In fact, we will see that there are several candidates for a "canonical" divisor, but all of them come from sheaves that agree with this sheaf on the nonsingular points of X.

We shall first construct a candidate for the role of canonical sheaf directly. Let X be a surface and let Σ be the set of its singular points. Recall that by the definition of surface we know Σ has pure codimension two. Choose a Weil divisor $K_{X\setminus\Sigma}$ on $X \setminus \Sigma$ which corresponds to the sheaf $\Omega_{X/\Bbbk} \wedge \Omega_{X/\Bbbk}|_{X\setminus\Sigma}$. We can extend $K_{X\setminus\Sigma}$ in a unique way by taking the closures of its prime divisors to get a divisor K_* defined on all of X. It is clear that if we had taken some other canonical divisor $K'_{X\setminus\Sigma} = K_{X\setminus\Sigma} + D$ for some principal divisor D, the extension K'_* would equal $K_* + D$. Now, if (X^*, ϕ) is some resolved model of X with exceptional manifold Ω^*, we know that ϕ is an isomorphism from $X^* \backslash \Omega^*$ to $X \backslash \Sigma$. So in particular, K_* agrees with some canonical divisor $\phi_*(K_{X^*})$ outside Σ. So we could have equivalently defined K_* to be the push-forward of any canonical divisor on X^*.

Recall that intersection of divisors on X amounts to intersection of the pullback of the divisors on X^*. Since X^* is nonsingular, the adjunction formula holds there. So if we take some nonsingular curve C on the original surface, its pullback C^* will be a curve $\widetilde{C}$ on the resolved model, plus some components along the exceptional fiber. Since C is nonsingular, $\widetilde{C}$ will also be nonsingular, and since they are isomorphic except at finitely many points, they will be isomorphic, and in particular, they will have the same genus. Intersecting C^* with the canonical divisor above yields a term arising from $\widetilde{C}$ itself, plus a term arising from the components along the exceptional fiber. The curve itself satisfies the adjunction formula, but these extra terms will throw off our calculation.

PROPOSITION 3.2.1. *let X be a surface and let (X^*, ϕ) be a resolved model of X with exceptional manifold Ω^*. Let C be a nonsingular curve on X with genus g. Let K be the canonical divisor on X^*, and let $K_* = \phi_*(K)$. Then*

$$C.(C + K_*) = 2g - 2 + \left(\widetilde{C}^{\Omega^*}\right)^T \mathbf{k}(\widetilde{C}^{\Omega^*} + K^{\Omega^*}). \tag{3.2.1}$$

PROOF. Let $C^* = \widetilde{C} + \sum_i \gamma_i(C)\mu_i$, and let $(K_*)^* = K + \sum_i \kappa_i \mu_i$. Consider $\mu_j.(K_*)^* = 0$ to get $\kappa = \mathbf{k}K^{\Omega^*}$. Then:

$$\begin{aligned}
(C.(C + K_*))_* &= C^*.(C^* + (K_*)^*) \\
&= \widetilde{C}.(C^* + (K_*)^*) \\
&= \widetilde{C}.(\widetilde{C} + K) + \sum_i (\gamma_i(C) + \kappa_i)\left(\widetilde{C} \cdot \mu_i\right) \\
&= 2g(\widetilde{C}) - 2 + \left(\widetilde{C}^{\Omega^*}\right)^T \gamma(C) + \left(\widetilde{C}^{\Omega^*}\right)^T \kappa \\
&= 2g(\widetilde{C}) - 2 + \left(\widetilde{C}^{\Omega^*}\right)^T \mathbf{k}\widetilde{C}^{\Omega^*} + \left(\widetilde{C}^{\Omega^*}\right)^T \mathbf{k}K^{\Omega^*} \\
&= 2g(\widetilde{C}) - 2 + \left(\widetilde{C}^{\Omega^*}\right)^T \mathbf{k}(\widetilde{C}^{\Omega^*} + K^{\Omega^*}).
\end{aligned}$$

□

Thus we have a version of the adjunction formula that holds on a singular surface. Unfortunately, it makes reference to a resolution of singularities, so that an understanding of how divisors intersect on the singular surface does not suffice to apply this formula. Nevertheless, it does show that for a curve which does not pass through any singular points, adjunction holds as per normal.

COROLLARY 3.2.2. *Let C be any curve on X not passing through a singularity of X. Then:*

$$C.(C + K_*) = 2p_a(C) - 2.$$

PROOF. Recall that we have Corollary 1.2.69, which implies that we can apply the adjunction formula to arbitrary divisors on a nonsingular surface. Suppose C is a singular curve which does not pass through any singularity of X. Then $\widetilde{C} \cong C$ and in particular $p_a(\widetilde{C}) = p_a(C)$. Then our proof above goes through unchanged, yielding

$$C.(C + K_*) = 2p_a(C) - 2 + \left(\widetilde{C}^{\Omega^*}\right)^T \mathbf{k}(\widetilde{C}^{\Omega^*} + K^{\Omega^*}).$$

Since we are assuming that C does not pass through the singularities of X, we can rewrite this as

$$C.(C + K_*) = 2p_a(C) - 2.$$

□

3.3. The Riemann-Roch Problem

The Riemann-Roch problem is the following: Given a divisor D on a surface X, how many rational functions are there with poles at most as bad as D? On a nonsingular surface, D is always locally principal, so this amounts to asking about global sections of $\mathfrak{L}(D)$. The answer is then $\dim_{\Bbbk} H^0(X, \mathfrak{L}(D))$, and the Riemann-Roch theorem gives us a way to calculate $\chi(\mathfrak{L}(D))$, which can often be induced to yield the information we want.

On a singular surface the question is more complicated. First of all, a divisor may not be locally principal; in such a case it is not obvious what sheaf to associate to it, and the most obvious choice of sheaf to associate to it will not be invertible. Even for locally principal divisors, it is not clear what divisor to use in place of the canonical divisor; we have a canonical divisor of sorts which we used in Proposition 3.2.1, but we also have a dualizing sheaf, and these may not correspond.

We will restrict ourselves to the case of locally principal divisors and normal surfaces. We will see that locally principal divisors can always be moved away from the singularities, making them unusually simple.

PROPOSITION 3.3.1. *Let X be a normal surface with singular locus Σ, and let D be a divisor on X. Then D is locally principal if and only if D is linearly equivalent to a divisor whose support does not contain Σ.*

PROOF. If the support of D does not contain Σ, then D is a divisor on the nonsingular scheme $X \setminus \Sigma$, so it is locally principal. Any divisor linearly equivalent to a locally principal divisor is also locally principal.

Suppose D is locally principal. By assumption X is projective, so using a version of Bertini's theorem from [**Har77**, Rem. II.8.18.1], we can find a hyperplane H such that $H \cap X$ is a nonsingular curve. By construction $H \cap X$ is also a very ample divisor C on X. But then $D + kC$ is generated by global sections for large enough k by the definition of ampleness. By [**Har77**, Exer. II.7.5] $D + (k+1)C$ is very ample. Applying Bertini's theorem again, we can find a divisor C' that is a nonsingular curve linearly equivalent to $D+(k+1)C$ and which does not intersect Σ. Then D is linearly equivalent to $C' - (k+1)C$, whose support does not contain Σ. □

THEOREM 3.3.2 (Riemann-Roch for Cartier divisors). *Let X be a normal surface, and let D be any locally principal divisor on X. Let (X^*, ϕ) be a resolved model of X and let $K_* = \phi_*(K_{X^*})$ be the push-forward of a canonical divisor on X. Then*

$$\chi(\mathcal{L}(D)) = \frac{1}{2}D.(D - K_*) + p_a(X). \tag{3.3.1}$$

The proof proceeds by observing that the Riemann-Roch theorem is essentially a generalization of the adjunction formula, and we have already found a version of the adjunction formula for singular surfaces.

LEMMA 3.3.3. *Let C and C' be locally principal divisors on X satisfying Equation* (3.3.1). *Then $-C$ and $C + C'$ satisfy Equation* (3.3.1).

PROOF. Suppose that C is a locally principal divisor on X satisfying the Riemann-Roch theorem. Then from Remark 2.5.10 we have:

$$-C.C = \chi(\mathcal{O}_X) - \chi(\mathcal{L}(C)) - \chi(\mathcal{L}(-C)) + \chi(\mathcal{O}_X),$$

which gives

$$\chi(\mathcal{L}(-C)) = C.C - \chi(\mathcal{L}(C)) + 2\chi(\mathcal{O}_X).$$

By assumption C satisfies Equation (3.3.1), so substitute, obtaining

$$\begin{aligned}\chi(\mathcal{L}(-C)) &= C.C + \chi(\mathcal{O}_X) - \frac{1}{2}C.(C - K_*) \\ &= \frac{1}{2}(-C).((-C) - K_*) + \chi(\mathcal{O}_X).\end{aligned}$$

So $-C$ also satisfies Equation (3.3.1).

Now suppose C and C' both satisfy Equation (3.3.1). Again using Remark 2.5.10 we obtain

$$\begin{aligned}\chi(\mathcal{L}(C + C')) &= \chi(\mathcal{L}(C)) + \chi(\mathcal{L}(C')) - \chi(\mathcal{O}_X) + C.C' \\ &= \frac{1}{2}C.(C - K_*) + \frac{1}{2}C'.(C' - K_*) - \chi(\mathcal{O}_X) + C.C' \\ &= \frac{1}{2}(C + C').((C + C') - K_*)\chi(\mathcal{O}_X).\end{aligned}$$

So $C + C'$ satisfy Equation (3.3.1). □

PROOF OF THEOREM. Recall from Proposition 3.3.1 that if D is locally principal we can find a linearly equivalent divisor that avoids the singularities of X. So assume that the support of D does not contain a singularity of X.

Let P be any prime divisor in the support of D. We can rewrite the adjunction formula from Corollary 3.2.1,

$$P.(P + K_*) = 2p_a(P) - 2,$$

as

$$p_a(P) = \frac{1}{2}P.(P + K_*) + 1.$$

Recall Definition 1.2.42:

$$p_a(P) = \chi(\mathcal{L}(-P)) - p_a(X).$$

Substituting, we obtain

$$\chi(\mathcal{L}(-P)) = \frac{1}{2}(-P).((-P) - K_*) + \chi(\mathcal{O}_X),$$

that is, $-P$ satisfies Equation (3.3.1). Then D is an integer linear combination of divisors satisfying Equation (3.3.1), so by Lemma 3.3.3, D satisfies Equation (3.3.1). □

REMARK 3.3.4. This theorem has exactly the same form as the classical Riemann-Roch theorem for surfaces. However, it is not as successful at providing an answer to the Riemann-Roch problem, since we are primarily interested in $\dim_{\Bbbk} H^0(X, \mathcal{L}(D))$

but we instead obtain

$$\chi(\mathcal{L}(D)) = \dim_{\Bbbk} H^0(X, \mathcal{L}(D)) - \dim_{\Bbbk} H^1(X, \mathcal{L}(D)) + \dim_{\Bbbk} H^2(X, \mathcal{L}(D)).$$

If X is nonsingular, then the canonical sheaf on X is also a dualizing sheaf for X and if K is a canonical divisor we can write $H^2(X, \mathcal{L}(D)) = H^0(X, \mathcal{L}(K - D))$, a group of global sections. Then the only term which is difficult to interpret is $\dim_{\Bbbk} H^1(X, \mathcal{L}(D))$ and we at least know its sign. If X has normal singularities, then we know X has a dualizing sheaf, but it may not be invertible and it may be difficult to compute.

We can take another approach to proving a version of the Riemann-Roch theorem. First observe that the Riemann-Roch theorem expresses the Euler characteristic in terms of intersection numbers. On the other hand, the intersection theory of Section 2.5 expresses intersection numbers in terms of Euler characteristics. So it would be sufficient to begin with the formula from Remark 2.5.10 and "invert" it.

PROPOSITION 3.3.5. *Suppose X is a normal surface with dualizing sheaf ω. Then let $\mathcal{F}$ be any locally free sheaf of finite rank on X. We have:*

$$\chi(\mathcal{F}) = \chi(\mathcal{F}^{\vee} \otimes \omega).$$

PROOF.

$$\begin{aligned}\chi(\mathcal{F}) &= \dim_{\Bbbk} H^0(X, \mathcal{F}) - \dim_{\Bbbk} H^1(X, \mathcal{F}) + \dim_{\Bbbk} H^2(X, \mathcal{F}) \\ &= \dim_{\Bbbk} H^2(X, \mathcal{F}^{\vee} \otimes \omega) - \dim_{\Bbbk} H^1(X, \mathcal{F}^{\vee} \otimes \omega) + \dim_{\Bbbk} H^0(X, \mathcal{F}^{\vee} \otimes \omega) \\ &= \chi(\mathcal{F}^{\vee} \otimes \omega).\end{aligned}$$

□

For example, we have $\chi(\omega) = \chi(\mathcal{O}_X)$.

THEOREM 3.3.6 (Riemann-Roch for Cartier divisors on a Gorenstein surface). *Let X be a Gorenstein surface. Then the dualizing sheaf ω is invertible, and corresponds to some locally principal divisor K°. Let D be any locally principal divisor on X. Then*

$$\chi(\mathcal{L}(D)) = \frac{1}{2} D.(D - K^{\circ}) + 1 + p_a(X),$$

or

$$\begin{aligned}\dim_{\Bbbk} H^0(X, \mathcal{L}(D)) - \dim_{\Bbbk} H^1(X, \mathcal{L}(D)) + \dim_{\Bbbk} H^0(X, \mathcal{L}(K^{\circ} - D)) \\ = \frac{1}{2} D.(D - K^{\circ}) + 1 + p_a(X).\end{aligned}$$

PROOF. Let $\mathcal{F}$ be an invertible sheaf, and let us use the intersection theory of Section 2.5 to compute $\mathcal{F}.(\mathcal{F}^\vee \otimes \omega)$:

$$\begin{aligned}\mathcal{F}.(\mathcal{F}^\vee \otimes \omega) &= \chi(\mathcal{O}_X) - \chi(\mathcal{F}) - \chi(\mathcal{F}^\vee \otimes \omega) + \chi(\mathcal{F} \otimes \mathcal{F}^\vee \otimes \omega)\\ &= 2\chi(\mathcal{O}_X) - 2\chi(\mathcal{F}).\end{aligned}$$

Upon rearrangement and use of the formula $p_a(X) = \chi(\mathcal{O}_X) - 1$, this yields:

$$\chi(\mathcal{F}) = \frac{1}{2}\mathcal{F}.(\mathcal{F} \otimes \omega^\vee) + 1 + p_a(X).$$

□

This version of the Riemann-Roch theorem relies on the dualizing sheaf of X, which is a global object, whereas the canonical sheaf K_* can be computed from local data or from a resolved model. However, if we want to express $H^2(X, \mathcal{L}(D))$ as $H^0(X, \mathcal{L}(K^\circ - D))$, we will need to compute K° in any case, so this may be a more appropriate version of the theorem. We will see that the distinction disappears when the surface is a local complete intersection.

PROPOSITION 3.3.7. *Let X be a local complete intersection with isolated singular points Σ. Let ω_X° be the dualizing sheaf on X, and let $\omega_{X/\Bbbk}$ be $\wedge^2\Omega_{X/\Bbbk}$. Then*

$$\omega^\circ|_{X\setminus\Sigma} \cong \omega_X|_{X\setminus\Sigma}.$$

PROOF. By assumption, X is a local complete intersection. So let it be selected from $\mathbb{P}^n$ by the sheaf of ideals $\mathcal{I}$. The scheme $X \setminus \Sigma$ is a closed, nonsingular subscheme of the nonsingular scheme $\mathbb{P}^n \setminus \Sigma$. So we can apply [**Har77**, Thm. II.8.17] to show that we have an exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{(\mathbb{P}^n\setminus\Sigma)/\Bbbk} \otimes \mathcal{O}_{X\setminus\Sigma} \to \Omega_{(X\setminus\Sigma)/\Bbbk} \to 0.$$

Now, $\Omega_{(\mathbb{P}^n\setminus\Sigma)/\Bbbk} = \Omega_{\mathbb{P}^n/\Bbbk}|_{\mathbb{P}^n\setminus\Sigma}$, $\Omega_{(X\setminus\Sigma)/\Bbbk} = \Omega_{X/\Bbbk}|_{X\setminus\Sigma}$, and $\mathcal{O}_{X\setminus\Sigma} = \mathcal{O}_X|_{X\setminus\Sigma}$, so we can rewrite this as

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathbb{P}^n/\Bbbk} \otimes \mathcal{O}_X|_{X\setminus\Sigma} \to \Omega_{X/\Bbbk}|_{X\setminus\Sigma} \to 0.$$

If we take the exterior algebra of this exact sequence, then [**Har77**, Exer. II.5.16d] gives an isomorphism

$$\wedge^n\left(\Omega_{\mathbb{P}^n/\Bbbk} \otimes \mathcal{O}_X|_{X\setminus\Sigma}\right) \cong \left(\wedge^2\Omega_{X/\Bbbk}|_{X\setminus\Sigma}\right) \otimes \left(\wedge^{n-2}\mathcal{I}/\mathcal{I}^2\right).$$

We recognize $\wedge^2\Omega_{X/\Bbbk}|_{X\setminus\Sigma}$ as $\omega_{X/\Bbbk}|_{X\setminus\Sigma}$, from linear algebra we know

$$\wedge^n\left(\Omega_{\mathbb{P}^n/\Bbbk} \otimes \mathcal{O}_X|_{X\setminus\Sigma}\right) \cong \left(\wedge^n\Omega_{\mathbb{P}^n/\Bbbk}\right) \otimes \left(\mathcal{O}_X|_{X\setminus\Sigma}\right),$$

and $\wedge^n \Omega_{\mathbb{P}^n/\Bbbk}$ is $\omega_{\mathbb{P}^n/\Bbbk}$. Now, X is nonsingular on $X \setminus \Sigma$, so it is a local complete intersection there; this implies that $\mathcal{J}$ is a locally free $\mathcal{O}_{\mathbb{P}^n}$-module of rank $n-2$, so $\wedge^{n-2}\mathcal{J}/\mathcal{J}^2$ is an invertible sheaf. Taking the dual commutes with taking the exterior product, so tensoring both sides with $(\wedge^{n-2}\mathcal{J}/\mathcal{J}^2)^\vee$ gives the equation

$$\omega_{X/\Bbbk}|_{X\setminus\Sigma} \cong \omega_{\mathbb{P}^n/\Bbbk} \otimes \wedge^{n-2}\big(\mathcal{J}/\mathcal{J}^2\big)^\vee|_{X\setminus\Sigma}.$$

Now, we have also assumed that X is a local complete intersection. By Theorem A.4.9,

$$\omega_X^\circ \cong \omega_{\mathbb{P}^n/\Bbbk} \otimes \wedge^{n-2}(\mathcal{J}/\mathcal{J}^2)^\vee.$$

Putting these together, we obtain

$$\omega_X^\circ|_{X\setminus\Sigma} \cong \omega_{X/\Bbbk}|_{X\setminus\Sigma}.$$

□

PROPOSITION 3.3.8. *Let X be a local complete intersection that is nonsingular in codimension* 1. *Let (X^*, ϕ) be a resolved model of X, and let K_{X^*} be a canonical divisor for X^*. Let K° be the divisor associated to a dualizing sheaf on X. Then $K_* = \phi_*(K_{X^*})$ is linearly equivalent to K°.*

PROOF. Recall that ϕ is by definition an isomorphism between $X^* \setminus \phi^{-1}(\Sigma)$ and $X \setminus \Sigma$. So if ω_{X^*} is the canonical sheaf on X^* and ω_X is the canonical sheaf on X, then $\phi_*(\omega_{X^*})|_{X\setminus\Sigma} \cong \omega_X|_{X\setminus\Sigma}$. We showed in Proposition 3.3.7 that this implies $\phi_*(\omega_{X^*})|_{X\setminus\Sigma} \cong \omega_X^\circ|_{X\setminus\Sigma}$.

Now let P be any prime divisor on X. Then the generic point of P is contained in $X \setminus \Sigma$, so the divisor associated to an invertible sheaf $\mathcal{F}$ depends only on $\mathcal{F}|_{X\setminus\Sigma}$. Thus K° is linearly equivalent to the divisor associated to $\phi_*(\omega_{X^*})|_{X\setminus\Sigma}$. From the definition of the push-forward, this is clearly just $\phi_*(K_{X^*})$. □

This shows that when X is a local complete intersection, we have a version of the Riemann-Roch theorem that is essentially the same as the Riemann-Roch theorem for nonsingular surfaces.

When X is singular we have a resolved model (X^*, ϕ). It may in fact be easier to do computations directly on the surface X^* than on X, even given the versions we have of the Riemann-Roch theorem. In particular, we do not have a version of the Riemann-Roch theorem that can tell us anything about functions whose poles are specified in terms of a divisor that is not locally principal. Thus we will relate such functions to functions on the resolved model.

DEFINITION 3.3.9. Let X be a surface with isolated singularities and let D be a divisor on X. Define $\mathcal{L}'_X(D)$ to be the sheaf

$$\mathcal{L}'_X(D)(U) = \{f \in \mathscr{K} | (f) \geq -D\},$$

where (f) denotes the divisor associated to f.

REMARK 3.3.10. It is clear that $\mathcal{L}'_X(D)$ is in fact a sheaf, and that if D is locally principal $\mathcal{L}'_X(D) = \mathcal{L}_X(D)$. It is *not* clear that D can somehow be extracted from $\mathcal{L}'_X(D)$. Nevertheless, since $\mathcal{L}'_X$ is a generalization of $\mathcal{L}_X$, we will write it as $\mathcal{L}_X$.

PROPOSITION 3.3.11. *The sheaf $\mathcal{L}_X(D)$ is coherent.*

PROOF. Let $D = \sum_P a_P P$ be a Weil divisor on X, and let U be an open affine neighborhood on X. We know that only finitely many prime divisors P have nonzero a_P; let $\{P_i\}$ be the collection of prime divisors P_i intersecting U with $a_{P_i} < 0$. Then for each i, let f_i be an element of $\mathcal{O}_X(U)$ with $v_{P_i}(f) \geq -a_{P_i}$. Then we see that $(\prod_i f_i)\mathcal{L}(D)(U)$ is an ideal in $\mathcal{O}_X(U)$. Since X is noetherian, we know $(\prod_i f_i)\mathcal{L}(D)(U)$ is finitely generated, and therefore $\mathcal{L}(D)(U)$ is a finitely generated $\mathcal{O}_X(U)$-module. Thus, since it is clearly quasi-coherent, $\mathcal{L}(D)$ is a coherent subsheaf of $\mathcal{K}$. □

PROPOSITION 3.3.12. *Let X be a surface and let D be a divisor on X. Let (X^*, ϕ) be a resolved model of X having exceptional manifold Ω^* with integral components μ_i. Write $D^* = \widetilde{D} + \sum_i a_i\mu_i$ and let b_i be the largest integer less than or equal to a_i. Define $D^- = \widetilde{D} + \sum_i b_i\mu_i$. Then*

$$H^0(X, \mathcal{L}_X(D)) = H^0(X^*, \mathcal{L}_{X^*}(D^-)).$$

PROOF. By construction, $H^0(X, \mathcal{F}) = \Gamma(X, \mathcal{F}) = \mathcal{F}(X)$. Since $\mathcal{L}_X(D)$ is a subsheaf of $\mathscr{K}(X)$, $\mathcal{L}_{X^*}(D^-)$ is a subsheaf of $\mathscr{K}(X^*)$, and X and X^* are birational so $\mathscr{K}(X) = \mathscr{K}(X^*)$, it is reasonable to talk about equality of these two groups.

Let f be a global section of $\mathcal{L}_X(D)$. Then by definition, $(f) \geq -D$. Now f is also a rational function on X^*. If P is any prime divisor not contained in Ω^*, then the generic point of P is contained $X^* \setminus \Omega^*$, which is isomorphic by ϕ to $X \setminus \phi(\Omega^*)$. So $(f) \geq \widetilde{D}$ on $X^* \setminus \Omega^*$. Write $(f) = -D^* + C^* + E$ for some effective divisor C on X and some $\mathbb{Q}$-divisor E supported on Ω^*. Then since (f) is principal, in particular $(f).\mu_i = 0$ for every i. But then by Proposition 2.4.10, $(f).\mu_i = E.\mu_i = 0$ for every i, which implies $E = 0$. So $(f) = -D^* + C^*$, where C is some effective divisor on X. But Proposition 2.4.12 implies that C^* is an effective divisor on X^*, so we have $(f) \geq -D^*$. Since (f) is a $\mathbb{Z}$-divisor, this implies that $(f) \geq -D^-$.

Now suppose that $(f) \geq -D^{-}$. Then $(f) \geq -D^{*}$, so write $(f) = -D^{*} + C$ for some effective divisor C. As before, $(f).\mu_i = 0$ for all i, so $C.\mu_i = 0$ for all i, so by Proposition 2.4.10 $C = (\phi_*(C))^*$. Clearly $\phi_*(C)$ is effective, so $(f) = (-D + C)^*$ so on X we have $(f) = -D + C \geq -D$. □

3.4. The Nakai-Moishezon Criterion

On a nonsingular surface, the Nakai-Moishezon criterion allows us to tell when a divisor is ample. Ampleness for divisors is defined in terms of ampleness of the associated invertible sheaf. On a nonsingular surface, ampleness is perfectly well-defined for invertible sheaves. However, if one wants to extend it to divisors that are not locally principal, difficulties arise. Thus we will restrict ourselves to Cartier divisors when talking about ampleness.

PROPOSITION 3.4.1. *Let X be a normal surface, and suppose that H is an ample Cartier divisor on X. Then $H.H > 0$, and if P is any prime Weil divisor on X, then $P.H > 0$.*

PROOF. Since H is ample, there exists some n such that nH is very ample. Then nH gives a projective embedding of X, and nH is linearly equivalent to any hyperplane section of X. By a version of Bertini's theorem ([**Har77**, Rem. II.8.18.1]), we can choose a hyperplane H_1 such that $H_1 \cap X$ is a nonsingular curve avoiding all the singularities of X. Choose a second hyperplane H_2 not passing though any singularity of X. Then H_2 has codimension one and $H_1 \cap X$ has dimension one in $\mathbb{P}^n$, so their intersection must be a nonempty finite set of points. In particular, the divisors $H_1 \cap X$ and $H_2 \cap X$ must have positive intersection number. Both of these are linearly equivalent to nH, so $nH.nH > 0$ and therefore $H.H > 0$. If P is a prime divisor on X, then P has dimension one, so it must intersect H_1 in a finite nonempty set of points, and we see that $nH.P > 0$ so $H.P > 0$. □

REMARK 3.4.2. Suppose D is a Weil divisor such that nD is locally principal and $\mathcal{L}(nD)$ is very ample. Then the arguments above apply to D, and we see that $D.D > 0$ and $D.P > 0$ for every prime divisor P.

Such a situation can occur: let D be any divisor which has a locally principal multiple kD. Since X is projective, there is an a very ample divisor H on X. Then for ℓ large enough, $kD + k\ell H$ is very ample ([**Har77**, Sec. II.7]) for large enough ℓ. But then $k(D + \ell H)$ is very ample, even though $D + \ell H$ cannot be locally principal unless D is.

One might hope that, like numerical or algebraic equivalence, ampleness could be straightforwardly related to ampleness of the pullback of a divisor. This is not, unfortunately, the case.

PROPOSITION 3.4.3. *Let D be a locally principal divisor on the normal surface X. Let (X^*, ϕ) be a resolved model of X. Then D^* is not ample on X^*.*

PROOF. Let μ be an integral component of the exceptional manifold. Then we showed in Proposition 2.4.10 that $\mu.D^* = 0$; by the Nakai-Moishezon criterion on nonsingular surfaces (Theorem 1.2.72) this implies D^* is not ample. □

REMARK 3.4.4. This is also a consequence of Proposition 3.3.12: here $D^* = D^-$, and since the global sections of D and D^* are the same, every map defined by nD^* factors through X and the map defined by nD.

Using the intersection theory of Section 2.5 it is possible to prove a very general form of the Nakai-Moishezon criterion:

THEOREM 3.4.5. *Let V be a complete algebraic $\Bbbk$-scheme and let $\mathcal{L}$ be an invertible sheaf of $\mathcal{O}_V$-modules. Then $\mathcal{L}$ is ample if and only if for every integral closed subscheme $W \subseteq V$ of dimension $t > 0$ we have $(L^{\cdot t}; \mathcal{O}_W) > 0$ (where $(L^{\cdot t}; \mathcal{O}_W)$ denotes the t-fold intersection of L with itself relative to $\mathcal{O}_W$).*

This is proven in [**Bǎd01**, Thm. 1.22].

This has a particular corollary that is useful to us:

THEOREM 3.4.6 (Nakai-Moishezon criterion for singular surfaces). *If V is a complete surface over $\Bbbk$ and L is an invertible $\mathcal{O}_V$-module, then L is ample if and only if $L.L > 0$ and $(L; \mathcal{O}_C) > 0$ for every integral curve $C \subset V$. If in addition $H^0(L) \neq 0$, then the condition $L.L > 0$ is not needed.*

REMARK 3.4.7. Observe that if C is a locally principal prime divisor, then we have shown in Remark 2.5.11 that $(L; \mathcal{O}_C)$ is just $L.C$. However, on any surface that has non-factorial singularities, there will be some prime divisors that are not locally principal, and so verification of this criterion will require evaluation of some Euler characteristics.

Conclusion

We now have several different methods for computing intersections of divisors on surfaces. In Chapter 3 we saw that combining the different theories was very fruitful, yielding in particular a version of the Riemann-Roch theorem. This is possible because of Theorem 2.5.15, which shows that all three different theories agree when applied to locally principal divisors on a normal surface.

In this text we have discussed only one very small part of the field of intersection theory. The general intersection theory on potentially singular varieties of any dimension is discussed in great detail in [**Ful98**]. The theory of algebraic surfaces, including much discussion of when a surface may be blown down to give various kinds of singularity, can be found in [**Bǎd01**].

The material in this thesis suggests a number of questions which could warrant further investigation.

- For many purposes, it is useful to know when a divisor is locally principal. Can this be determined in terms of its intersection behavior? If so, one could take the quotient of $\mathrm{NS}\, X$ by the locally principal divisors to obtain a sort of local Néron-Severi group.
- The Nakai-Moishezon theorem for a singular surface is not presented in a very satisfying form: a version which is more readily computable would be nice, and would probably lead to a straightforward proof of the Hodge index theorem. The simple relationship between $\mathrm{Num}\, X$ and $\mathrm{Num}\, X^*$ suggest that a proof of both these theorems should be reasonable.
- We have seen that if $\mathcal{F}$ is an invertible sheaf on X then $H^i(X, \mathcal{F})$ and $H^i(X^*, \phi^*(\mathcal{F}))$ are isomorphic for $i = 0$; how are they related for $i > 0$? The Leray spectral sequence (see [**God58**, II.4.17]) may yield some results here.
- On a normal surface, it is known that a dualizing sheaf exists, but actually computing it is nontrivial. We have seen that if the surface is a local complete

intersection, the dualizing sheaf can be straightforwardly computed from the canonical sheaf. It seems plausible that this relationship can be generalized at least to the situation where the surface is merely Gorenstein, and possibly further.

- When the field $\Bbbk$ is the complex numbers, there is another kind of equivalence of divisors that we have not discussed, namely homological equivalence. For nonsingular surfaces, homological equivalence is the same as numerical equivalence and algebraic equivalence with division (see [**Har73**]). Is this the case also for singular surfaces?
- When the $\Bbbk$ is the complex numbers, one also has a way of dealing with intersections called intersection homology theory. One can reformulate this in purely sheaf-theoretic terms to obtain a theory which is valid in any characteristic (see [**Kir88**, Sec. 5.5]). Does this intersection theory agree with the ones we discuss here?
- Singular surfaces occasionally arise in arithmetic, as relative curves over the spectrum of a Dedekind domain. In this situation, one does not have an algebraically closed ground field, so none of the theory described in this thesis can be applied directly, but similar ideas may well be applicable. However, since the Dedekind domain is affine, the surface will not be complete. If the Dedkind domain is a ring of integers, one can apply Arakelov theory; if the Dedekind domain is the function field of a curve, one can complete it, at which point one can have a complete surface and attempt to do some intersection theory.

APPENDIX A

Cohomology

We have needed a number of cohomological results in the text. This section briefly recalls the most important results without proof. The reader is assumed to have some understanding of cohomology and in particular derived functor cohomology. For a good exposition of these results and the surrounding theory, see [**Har77**, Chap. III] and [**Eis95**, Part III].

A.1. Sheaf Cohomology

DEFINITION A.1.1. Let X be a topological space, let $\mathfrak{Ab}(X)$ be the category of sheaves of abelian groups on X, and let $\mathfrak{Ab}$ be the category of abelian groups. Then we have the global sections functor $\Gamma(X,\cdot) : \mathfrak{Ab}(X) \to \mathfrak{Ab}$ given by taking a sheaf $\mathcal{F}$ to $\Gamma(X,\mathcal{F}) = \mathcal{F}(X)$. Define the i-th cohomology functor of X to be the i-th right derived functor of $\Gamma(X,\cdot)$, that is, $H^i(X,\cdot) = R^i\Gamma(X,\cdot)$.

REMARK A.1.2. When X or $\mathcal{F}$ have additional structure (say X is a noetherian scheme and X is a quasi-coherent $\mathcal{O}_X$-module), we will nevertheless take cohomology in this sense, "forgetting" the extra structure. This is not as confusing as one might think because of the following proposition.

PROPOSITION A.1.3. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathfrak{Mod}$ denote the category of $\mathcal{O}_X$-modules. Then the right derived functors of the functor $\Gamma(X,\cdot)$ from $\mathfrak{Mod}(X)$ to $\mathfrak{Ab}$ coincide with the functors $H^i(X,\cdot)$.*

REMARK A.1.4. This proposition follows from the following useful fact: If $\mathcal{F}$ is a flasque sheaf (sometimes called *flabby*) on a topological space X then

$$H^i(X,\mathcal{F}) = 0, \quad \forall i > 0.$$

As a result we can calculate cohomology using a flasque resolution rather than requiring an injective resolution.

REMARK A.1.5. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, then each cohomology group $H^i(X,\mathcal{F})$ is in fact an $\mathcal{O}_X(X)$-module. If X is a scheme over $\Bbbk$, then this implies that $H^i(X,\mathcal{F})$ is a vector space over $\Bbbk$.

THEOREM A.1.6 (Grothendieck). *Let X be a noetherian topological space of dimension n. Then for all $i > n$ and all sheaves of abelian groups $\mathcal{F}$ on X, we have $H^i(X, \mathcal{F}) = 0$.*

THEOREM A.1.7. *Let $X = \operatorname{Spec} A$ be the spectrum of a noetherian ring A. Then for all quasi-coherent sheaves $\mathcal{F}$ on X and for all $i > 0$, we have $H^i(X, \mathcal{F}) = 0$*

A.2. Čech Cohomology

The machinery of derived functors provides a concise definition for sheaf cohomology, but it does not seem to provide a feasible algorithm for actually computing cohomology groups. As a result, we introduce Čech cohomology, which is definitionally much more awkward but computationally easier.

Let X be a topological space, and let $\mathfrak{U} = \{U_i | i \in I\}$ be an open cover of X for some well-ordered index set I. For any finite set of indices $i_0, \ldots, i_p \in I$ we denote the intersection $U_{i_0} \cap \cdots \cap U_{i_p}$ by $U_{i_0,\ldots,i_p}$. Now let $\mathcal{F}$ be a sheaf of abelian groups on X. We define a complex

$$C^\bullet(\mathfrak{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0,\ldots,i_p}).$$

An element of $C^p(\mathfrak{U}, \mathcal{F})$ is determined by giving an element $\alpha_{i_0,\ldots,i_p} \in \mathcal{F}(U_{i_0,\ldots,i_p})$ for each $(p+1)$-tuple of indices $i_0 < \cdots < i_p$. We define the map $d : C^p \to C^{p+1}$ by

$$(d\alpha)_{i_0,\ldots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\ldots,\widehat{i_k},\ldots,i_{p+1}}|_{U_{i_0,\ldots,i_{p+1}}}.$$

The notation $\widehat{i_k}$ means omit i_k.

DEFINITION A.2.1. Let X be a topological space and $\mathfrak{U}$ be an open cover of X. For any sheaf of abelian groups $\mathcal{F}$ on X, we define the p-th *Čech cohomology group of $\mathcal{F}$ with respect to the covering* $\mathfrak{U}$ to be

$$\check{H}^p(\mathfrak{U}, \mathcal{F}) = h^p(C^\bullet(\mathfrak{U}, \mathcal{F})) = \ker(d : C^p \to C^{p+1}) / \operatorname{im}(d : C^{p-1} \to C^p).$$

REMARK A.2.2. For a general cover $\mathfrak{U}$ short exact sequences of sheaves may not lead to long exact sequences of Čech cohomology groups. However, the following theorem explains the utility of Čech cohomology.

THEOREM A.2.3. *Let X be a finite noetherian separated scheme, let $\mathfrak{U}$ be an open affine cover of X, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then for all $p \geq 0$, we* *have natural isomorphisms*

$$\check{H}^p(\mathfrak{U}, \mathcal{F}) \cong H^p(X, \mathcal{F}).$$

Given the machinery of Čech cohomology, it is fairly straightforward to compute the cohomology of $\mathbb{P}^r$ by taking the usual affine cover of $\mathbb{P}^r$ ([**Har77**, Sec. I.2]):

THEOREM A.2.4. *Let A be a noetherian ring, let $X = \mathbb{P}^r_A$, with $r \geq 1$, and let $S = A[x_0, \ldots, x_r]$. Then:*

(1) *The natural map $S \to \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{O}(n))$ is an isomorphism of graded S-modules;*
(2) *$H^i(X, \mathcal{O}_X(n)) = 0$ for $0 < i < r$ and all $n \in \mathbb{Z}$;*
(3) *$H^r(X, \mathcal{O}_X(-r-1)) \cong A$; and*
(4) *the natural map*

$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \to H^r(X, \mathcal{O}_X(-r-1)) \cong A$$

is a perfect pairing of finitely generated free A-modules, for each $n \in \mathbb{Z}$.

A.3. Cohomological Properties of Coherent Sheaves

The cohomology groups $H^i(X, \mathcal{F})$ are frequently too inconvenient to calculate, so a simpler invariant is often useful. One often uses the Euler characteristic $\chi(\mathcal{F})$ because, as we will see, it is additive on exact sequences, and also because it is constant on flat families, unlike the dimensions of $H^i(X, \mathcal{F})$.

THEOREM A.3.1 (Serre). *Let X be a projective scheme over a noetherian ring A, and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X over* Spec A. *Let $\mathcal{F}$ be a coherent sheaf on X. Then:*

(1) *for each $i \geq 0$, $H^i(X, \mathcal{F})$ is a finitely generated A-module;*
(2) *there is an integer n_0, depending on $\mathcal{F}$, such that for each $i > 0$ and each $n \geq n_0$, $H^i(X, \mathcal{F} \otimes (\mathcal{O}_X(1))^n) = 0$.*

In particular, if X is projective over a field $\Bbbk$, each $H^i(X, \mathcal{F})$ is a finite-dimensional $\Bbbk$-vector space.

PROPOSITION A.3.2. *Let A be a noetherian ring and let X be a proper scheme over* Spec A. *Let $\mathcal{L}$ be an invertible sheaf on X. Then the following conditions are equivalent:*

(1) *$\mathcal{L}$ is ample;*

(2) *For each coherent sheaf* $\mathcal{F}$ *on* X*, there is an integer* n_0*, depending on* $\mathcal{F}$*, such that for each* $i > 0$ *and each* $n \geq n_0$*, we have* $H^i(X, \mathcal{F} \otimes \mathcal{L}^n) = 0$.

DEFINITION A.3.3. Let X be a projective scheme over $\Bbbk$. Let $\mathcal{F}$ be a coherent sheaf on X. The *Euler characteristic* $\chi(\mathcal{F})$ is defined by

$$\chi(\mathcal{F}) = \sum_{i \geq 0} (-1)^i \dim_{\Bbbk} H^i(X, \mathcal{F}).$$

PROPOSITION A.3.4. *Let*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

be a short exact sequence of coherent sheaves on X*. Then*

$$\chi(\mathcal{F}) = \chi(\mathcal{F}') + \chi(\mathcal{F}'').$$

A.4. Serre Duality

Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Given two sheaves of $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ in X, define a new sheaf of $\mathcal{O}_X$-modules $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ by $U \mapsto \operatorname{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$. Then $\operatorname{Hom}(\mathcal{F}, \cdot)$ is a left-exact covariant functor from $\mathfrak{Mod}(X)$ to $\mathfrak{Ab}$, and similarly $\mathcal{H}om(\mathcal{F}, \cdot)$ is a left-exact covariant functor from $\mathfrak{Mod}(X)$ to $\mathfrak{Mod}(X)$.

DEFINITION A.4.1. The functor $\operatorname{Ext}^i(\mathcal{F}, \cdot)$ is the i-th right derived functor of $\operatorname{Hom}(\mathcal{F}, \cdot)$. Similarly, $\mathcal{E}xt^i(\mathcal{F}, \cdot)$ is the i-th right derived functor of $\mathcal{H}om(\mathcal{F}, \cdot)$.

PROPOSITION A.4.2. *For any* $\mathcal{G} \in \mathfrak{Mod}(X)$*, we have*

$$\operatorname{Ext}^i(\mathcal{O}_X, \mathcal{G}) \cong H^i(X, \mathcal{G}),$$

for all $i \geq 0$*. If* $\mathcal{F}, \mathcal{L} \in \mathfrak{Mod}\, X$ *and* $\mathcal{L}$ *is locally free of finite rank,* $\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$ *its dual, then*

$$\operatorname{Ext}^i(\mathcal{F} \otimes \mathcal{L}, \mathcal{G}) \cong \operatorname{Ext}^i(\mathcal{F}, \mathcal{L}^\vee \otimes \mathcal{G}),$$

and, in particular,

$$\operatorname{Ext}^i(\mathcal{L}, \mathcal{G}) \cong \operatorname{Ext}^i(\mathcal{O}_X, \mathcal{L}^\vee \otimes \mathcal{G}) \cong H^i(X, \mathcal{L}^\vee \otimes \mathcal{G}).$$

DEFINITION A.4.3. Let X be a proper scheme of dimension n over $\Bbbk$. A *dualizing sheaf* for X is a coherent sheaf ω_X° on X, together with a *trace* morphism $t : H^n(X, \omega_X^\circ) \to k$, such that for all coherent sheaves $\mathcal{F}$ on X, the natural pairing

$$\operatorname{Hom}(\mathcal{F}, \omega_X^\circ) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X^\circ),$$

followed by t, gives an *isomorphism*

$$\operatorname{Hom}(\mathcal{F}, \omega_X^\circ) \cong H^n(X, \mathcal{F})',$$

where V' denotes the dual $\Bbbk$-vector space to V.

PROPOSITION A.4.4. *Let X be a proper scheme over $\Bbbk$. Then a dualizing sheaf for X, if it exists, is unique up to a unique isomorphism.*

PROPOSITION A.4.5. *Let X be a projective scheme over a field $\Bbbk$. Then X has a dualizing sheaf.*

REMARK A.4.6. The proof is constructive. We first embed X into $P = \mathbb{P}^N$ for some N. One can explicitly construct the dualizing sheaf on $\mathbb{P}^N$; it is

$$\omega_P^\circ = \wedge^N \Omega_{P/\Bbbk} \cong \mathcal{O}_P(-N-1).$$

Then if X has dimension n and codimension r, we let

$$\omega_X^\circ = \mathcal{E}xt_P^r(\mathcal{O}_X, \omega_P^\circ).$$

One then shows that this yields a functorial isomorphism, for $\mathcal{F} \in \mathfrak{Coh}(X)$,

$$\operatorname{Hom}_X(\mathcal{F}, \omega_X^\circ) \cong H^n(X, \mathcal{F})'.$$

Taking $\mathcal{F} = \omega_X^\circ$, the identity element $1 \in \operatorname{Hom}(\omega_X^\circ, \omega_X^\circ)$ gives us a homomorphism $t : H^n(X, \mathcal{F}) \to k$. Then (ω_X°, t) is a dualizing sheaf for X.

The importance of dualizing sheaves is that they provide a relation between cohomology groups of different dimensions on well-behaved schemes.

THEOREM A.4.7 (Duality for a Projective Scheme). *Let X be a projective scheme of dimension n over $\Bbbk$. Let ω_X° be a dualizing sheaf on X, and let $\mathcal{O}(1)$ be a very ample sheaf on X. Then:*

(1) *for all $i \geq 0$ and $\mathcal{F}$ coherent on X, there are natural functorial morphisms*

$$\theta^i : \operatorname{Ext}^i(\mathcal{F}, \omega_X^\circ) \to H^{n-i}(X, \mathcal{F})'$$

such that θ^0 is the trace map for ω_X°;

(2) *The following conditions are equivalent:*
   (a) *X is Cohen-Macaulay and equidimensional (i.e., all irreducible components have the same dimension);*
   (b) *for any $\mathcal{F}$ locally free on X, we have $H^i(X, \mathcal{F}(-q)) = 0$ for $i < n$ and $q \gg 0$;*

(c) *the maps θ^i of (1) are isomorphisms for all $i \geq 0$ and all $\mathcal{F}$ coherent on X.*

In particular, we have the following important result:

COROLLARY A.4.8. *Let X be a projective Cohen-Macaulay scheme of equidimension n over $\Bbbk$. Then for any locally free sheaf $\mathcal{F}$ of finite rank on X there are natural isomorphisms*

$$H^i(X, \mathcal{F}) \cong H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X^\circ)'.$$

In some cases we have a description of the dualizing sheaf:

THEOREM A.4.9. *Let X be a closed subscheme of $P = \mathbb{P}^N$ which is a local complete intersection (and hence Cohen-Macaulay) of codimension r. Let $\mathcal{I}$ be the ideal sheaf of X. Then*

$$\omega_X^\circ \cong \omega_P \otimes \wedge^r (\mathcal{I}/\mathcal{I}^2)^\vee.$$

In particular, ω_X° is an invertible sheaf on X.

PROPOSITION A.4.10. *If X is a projective nonsingular variety over an algebraically closed field $\Bbbk$, then the dualizing sheaf ω_X° is isomorphic to the canonical sheaf ω_X.*

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