Extended Frobenius Manifolds and the Open WDVV Equations

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Abstract

In this thesis, we give a geometric setting for the open Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations. We generalize the notion of a Frobenius manifold, which provides a geometric setting for the original WDVV equations. In particular, we define the notion of an extension morphism, and show that the open WDVV equations arise as the associativity of this extension. The generalized notion of a Frobenius manifold we give is an F-manifold with compatible flat structure, which we call a Frob manifold. We show that Frob manifolds have many properties analogous to Frobenius manifolds. For example, there is a relation between semisimple Frob manifolds and solutions to a generalization of the Darboux-Egoroff equations. We also show that Frob manifolds parametrize isomonodromic deformations. We characterize extensions in terms of both flat coordinates and canonical coordinates, and give a theorem for specifying an extension. We show examples of extensions of Frobenius manifolds, including the quantum cohomology of $\mathbb{P}^n$, and the A_n singularity.

Abrégé

Dans cette thèse, nous donnons un cadre géométrique pour les équations Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) ouvertes. Nous généralisons la notion d'une variété Frobenius, laquelle donne un cadre géométrique pour les équations WDVV originales. En particulier, nous définissons un morphisme dextension et montrons que les équations WDVV ouvertes se manifestent comme la condition dassociativité de cette extension. La notion généralisée que nous donnons est une variété-F avec une connexion plate compatible, que nous appelons une variété Frob. Nous démontrons des propriétés des variétés Frobs qui sont analogues aux propriétés des variétés Frobenius. Par example, il y a une relation entre les variétés Frobs semisimples et une généralisation des équations Darboux-Egoroff. Nous montrons aussi que les variétés Frobs paramètrent les déformations isomonodromiques. Nous caractérisons les extensions du point de vue des coordonnées plates aussi bien que les coordonnées canoniques, et donnons un théorème qui spécifie une extension. Nous montrons des exemples des extensions des variétés Frobenius, y compris la cohomolologie quantum de $\mathbb{P}^n$ et la singularité A_n.

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CHAPTER 1
Introduction: Motivation and Summary

Frobenius manifolds, introduced by Dubrovin [3], appear in a wide variety of mathematical contexts, including enumerative geometry, unfolding of singularities, [5], and isomonodromic deformations [8].

A Frobenius manifold is a differential manifold having the structure of a Frobenius algebra on the tangent spaces, along with a certain potential condition for the multiplication. Recall that a Frobenius algebra is an algebra with a symmetric, nondegenerate bilinear form g compatible with the multiplication $\circ$ in the sense that

$$g(a \circ b, c) = g(a, b \circ c). \tag{1.1}$$

The Frobenius manifold has a metric, which we also denote by g, that restricts to the compatible bilinear form at each tangent space. We do not impose any positivity condition on g. We require that g be flat, and the potential condition is that locally there exists a function Φ, unsurprisingly called the potential, such that

$$g(\partial_\alpha \circ \partial_\beta, \partial_\gamma) = g(\partial_\alpha, \partial_\beta \circ \partial_\gamma) := A_{\alpha\beta\gamma} = \partial_\alpha \partial_\beta \partial_\gamma \Phi, \tag{1.2}$$

where the derivatives are respect to local flat coordinates. If the multiplication is potential, then associativity is encoded in the famous Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations, which are partial differential equations for Φ.

In many situations, the potential carries some interesting information, and the WDVV equations can be used to recover this potential. A famous example is when Φ is the generating function for Gromov-Witten invariants of $\mathbb{P}^2$. In [7], Kontsevich showed that the WDVV equations can be used to compute all of these invariants recursively, solving a long standing problem in enumerative geometry.

In [6], Solomon introduced some equations, similar to the WDVV equations, which he called the open WDVV (OWDVV) equations. These are equations satisfied by the generating function of the so-called open Gromov-Witten (OGW) invariants. The motivation for this thesis was to find a geometric setting for these OWDVV equations and interpret them as some associativity constraint.

To find this geometric setting, we need to generalize the notion of a Frobenius manifold. This is due to the absence of a symmetric pairing (the metric on the Frobenius manifold) in the context of OGW invariants. Once we define the correct generalization, we define a certain notion of extension where the OWDVV equations arise as the associativity condition of this extension. We depart from our original motivation of OGW to study extensions in general and to see where else they may appear.

In Chapter 2, we introduce the generalization of a Frobenius manifold we will use, which we call a Frob manifold. This has been previously introduced by Manin in [9], where he called it an F-manifold with compatible flat structure. Because we do not wish to talk about F-manifolds ([5]), we will call it simply a Frob manifold. We note the following inclusions

$$\text{Frobenius manifolds} \subset \text{Frob manifolds} \subset F\text{-manifolds}.$$

A Frob manifold should be thought of as a Frobenius manifold without a compatible metric. Instead of a potential function, the potential condition we impose is the existence of a local potential vector with components Φ^α such that

$$\partial_\alpha \circ \partial_\beta := \sum_\gamma A_{\alpha\beta}{}^\gamma \partial_\gamma = \sum_\gamma \partial_\alpha \partial_\beta \Phi^\gamma \partial_\gamma. \tag{1.3}$$

In the absence of a compatible metric, we cannot lower the upper index γ on Φ. The components $\partial_\alpha \partial_\beta \Phi^\gamma$ transform in the correct way for a multiplication tensor, as long as we restrict to transformations which are affine. Therefore, instead of having a metric, we only require an affine structure or equivalently, a flat, torsion-free connection ∇.

Similar to Frobenius manifolds, the potential and associativity conditions of a Frob manifold can be encoded in the flatness of another connection called the first structure connection. This is shown in Theorem 1.

After defining a Frob manifold, we define the notion of a morphism which we will use later to define extensions. We generalize the notions of the identity and Euler field (which are essential ingredients in the study of Frobenius manifolds) to Frob manifolds.

We also look at semisimple Frob manifolds and in Proposition 2 show how they are related to a generalization of the Darboux-Egoroff equations. These generalized Darboux-Egoroff equations are important in proving the flatness of another connection, called the second structure connection, which is related to isomonodromic deformations. The key to proving the flatness is exploiting a symmetry of the generalized Darboux-Egoroff equations. Although the idea of the proof is similar in the case of Frobenius manifolds, there are technical differences which arise in the absence of a compatible metric.

It has been shown before that solutions to these generalized Darboux-Egoroff equations lead to F-manifolds with compatible flat structure (Frob manifolds) [1]. Our point of view, which involves the introduction of functions called connection potentials, shows that solutions to generalized Darboux-Egoroff equations and semisimple Frob manifolds are in fact equivalent. The connection potentials along with grading by an Euler field are key in defining the second structure connection.

In Chapter 3, we introduce extensions. An extension of a Frob manifold M is an exact sequence

$$0 \to I \to N \to M \to 0. \tag{1.4}$$

We define this more precisely, and return to our original motivation by showing how the OWDVV equations arise as the associativity of a rank-1 extension.

We also look at different algebraic and potential aspects of rank-1 extensions. We can divide extensions into two types. In the first type, the algebra is essentially that of an extension by a module, and the associativity is related to Hochschild cohomology. The second type is of more interest to us, and we call it an auxiliary extension. We give a theorem about the potentiality of these extensions (Theorem 3), which we use in Chapter 4 to show the existence of an extension of the Frobenius manifold associated to the space of polynomials (or the A_n singularity). Another result for auxiliary extensions is that an auxiliary extension of a semisimple Frob manifold is again semisimple (Proposition 13). We also look at the behaviour of the second structure connection and isomonodromic deformations under an extension.

Chapter 4 is devoted to examples. After exhaustively classifying extensions of 1- and 2-dimensional Frobenius manifolds, we turn to extensions of quantum cohomology and extensions of the space of polynomials (or the A_n singularity).

Quantum cohomology is a deformation of the classical cohomology ring of a symplectic manifold. It is a Frobenius manifold where the potential is the generating function for the Gromov-Witten (GW) invariants. We conjecture the existence of an extension of the quantum cohomology ring of $\mathbb{P}^n$ for all $n > 0$ (the case $n = 1$ can be checked explicitly, and $n = 2$ is in [6]). The coefficients of the extended potential, which we compute by recursively solving the ODWVV equations, seem to have some enumerative interpretation. For example, they match with the real Gromov-Witten invariants shown in [4] up to sign, although our solutions come with other coefficients which do not have a known interpretation as an enumerative invariant.

We also do some computations for the extension of $\mathbb{P}^1 \times \mathbb{P}^1$, and the Grassmannian $G(2,4)$. Interestingly, the coefficients for the extension of $G(2,4)$ are sometimes imaginary.

The other important example we look at is the A_n singularity. This singularity has the ring structure

$$\mathbb{C}[x]/\langle x^{n+1}\rangle. \tag{1.5}$$

The unfolding of this singularity comes with the structure of a Frobenius manifold [3], [5], [8]. This Frobenius manifold can also be viewed as the space of polynomials of degree $n + 1$. We prove the existence of a certain extension of this Frobenius manifold using the previously mentioned Theorem 3.

CHAPTER 2
Frob manifolds

In this chapter, we introduce a generalization of a Frobenius manifold called a Frob manifold. It is the same as an F-manifold with compatible flat structure, which was introduced in [9]. It can be thought of as almost a Frobenius manifold, but it is lacking a compatible metric. A Frob manifold with a compatible metric becomes a Frobenius manifold.

The important thing about having no metric is to have a more general notion of morphism, without needing to respect the metric structure. We define morphisms, and will use this definition later in Chapter 3 to define extensions.

Similar to Frobenius manifolds, we can have an identity and Euler field on a Frob manifold. We introduce these and give important properties. We also study semisimple Frob manifolds and show how they are related to a generalization of the Darboux-Egoroff equations.

In the last section, we introduce two structure connections on a Frob manifold, which are analogous to those given to a Frobenius manifolds. The second structure connection gives rise to isomonodromic deformations.

Although many results in this chapter are analogous to the case Frobenius manifolds, there are technical differences which come from not having a metric.

2.1 Definition

Definition 1. *Let M be a manifold equipped with a flat, torsion-free connection ∇. Let*

$$\circ : TM \otimes TM \to TM \tag{2.1}$$

be a symmetric, bi-linear multiplication.

- *A triple $(M, \nabla, \circ)$ as above is called a pre-Frob manifold.*
- *A pre-Frob manifold is called potential if locally there exists a vector Φ, called a vector potential, such that*

  $$X \circ Y = [X, [Y, \Phi]] \tag{2.2}$$

  for all flat vector fields X and Y.
- *A pre-Frob manifold is called associative if $\circ$ is associative.*
- *A pre-Frob manifold is called a Frob manifold if it is potential and associative.*

The expression $[X, [Y, \Phi]]$ is symmetric in X, Y by Jacobi identity and the fact that the connection is torsion free. That is,

$$\begin{aligned} [X, [Y, \Phi]] &= [[X, Y], \Phi] + [Y, [X, \Phi]] \\ &= [Y, [X, \Phi]] \end{aligned} \tag{2.3}$$

since $[X, Y] = \nabla_X Y - \nabla_Y X = 0$ for all flat X, Y. So the potential condition is compatible with $\circ$ being symmetric. If $\circ$ is associative, then it equips the tangent spaces of M with the structure of a commutative algebra.

A manifold with a flat, torsion-free connection is called an *affine* manifold. It can be covered by charts whose transition maps are affine linear. We denote the sheaf of flat vector fields (the vector fields whose covariant derivative vanishes) by TM^f. The full tangent sheaf is the tensor product of the sheaf of flat vector fields with the structure sheaf, $TM = TM^f \otimes \mathcal{O}_M$.

In flat coordinates t^α, the potential condition says that $A_{\alpha\beta}{}^\gamma = \partial_\alpha \partial_\beta \Phi^\gamma$, where $A_{\alpha\beta}{}^\gamma = dt^\gamma(\partial_\alpha \circ \partial_\beta)$ are the coefficients of the multiplication tensor $\circ$, and Φ^γ are the components of Φ. The object $\partial_\alpha \partial_\beta \Phi^\gamma$ transforms like a tensor under affine transformations. The associativity condition for a potential pre-Frob manifold is

$$\sum_\eta \partial_\alpha \partial_\beta \Phi^\eta \partial_\eta \partial_\gamma \Phi^\delta = \sum_\eta \partial_\alpha \partial_\gamma \Phi^\eta \partial_\eta \partial_\beta \Phi^\delta, \text{ for all } \alpha, \beta, \gamma, \delta. \tag{2.4}$$

We will sometimes use the shorthand $\Phi_{\alpha\beta}{}^\gamma$ for $\partial_\alpha \partial_\beta \Phi^\gamma$. The components Φ^γ are only determined up to linear polynomials in t^α, since taking two derivatives will kill these terms. A vector field whose components in flat coordinates are linear is called affine. So the vector potential Φ is defined up to an affine vector field.

A vector field V is affine if and only if $[V, TM^f] \subset TM^f$. Since the connection is torsion-free, we have that $[TM^f, TM^f] = 0$, so if V is affine, then for $X, Y \in TM^f$, $[X, [Y, V]] = 0$. This shows again that Φ is defined only up to the addition of an affine vector field.

The conditions for a pre-Frob manifold to be Frob can be conveniently rephrased in terms of the so-called structure connection (sometimes called the first structure connection):

Definition 2. *The structure connection ∇^λ is defined by*

$$\nabla^\lambda_X Y = \nabla_X Y + \lambda X \circ Y, \tag{2.5}$$

where λ is a complex parameter.

Theorem 1. *Let M be pre-Frob manifold, and let $R^\lambda = \lambda R_1 + \lambda^2 R_2$ be the curvature of ∇^λ (since ∇ is flat, there is no term constant in λ). Then:*

1. *M is potential if and only if $R_1 = 0$.*
2. *M is associative if and only if $R_2 = 0$.*

In other words, M is Frob if and only if $R^\lambda = 0$ for all λ.

Proof. We look at associativity and R_2 first. If

$$R_2(X,Y)Z = X \circ (Y \circ Z) - Y \circ (X \circ Z) = 0, \tag{2.6}$$

then

$$\begin{aligned} X \circ (Z \circ Y) &= X \circ (Y \circ Z), \text{ by commutativity,} && (2.7)\\ &= Y \circ (X \circ Z), \text{ by } R_2 = 0, && (2.8)\\ &= (X \circ Z) \circ Y, \text{ by commutativity.} && (2.9) \end{aligned}$$

Conversely, if $\circ$ is associative, then the above expression for R_2 vanishes.

Now we look at R_1. Let t^α be flat coordinates for ∇, and write $\partial_\alpha = \frac{\partial}{\partial t^\alpha}$.

$$R_1(\partial_\alpha, \partial_\beta)\partial_\gamma = \nabla_\alpha(\partial_\beta \circ \partial_\gamma) - \nabla_\beta(\partial_\alpha \circ \partial_\gamma) + \partial_\alpha \circ (\nabla_\beta \partial_\gamma) - \partial_\beta \circ (\nabla_\alpha \partial_\gamma). \tag{2.10}$$

The last two terms of R_1 are zero, so all that remains is

$$\nabla_\alpha(\partial_\beta \circ \partial_\gamma) - \nabla_\beta(\partial_\alpha \circ \partial_\gamma) = \left(\partial_\alpha A_{\beta\gamma}{}^\delta - \partial_\beta A_{\alpha\gamma}{}^\delta\right) \partial_\delta. \tag{2.11}$$

If this vanishes, then $\partial_\alpha {A_{\beta\gamma}}^\delta = \partial_\beta {A_{\alpha\gamma}}^\delta$, so the 1-form defined by $\sum_\beta {A_{\beta\gamma}}^\delta dt^\beta$ is closed, and therefore locally exact by the Poincaré lemma, that is

$$ {A_{\beta\gamma}}^\delta = \partial_\beta B_\gamma^\delta, \tag{2.12} $$

for some functions B_γ^δ. Since ${A_{\beta\gamma}}^\delta$ is symmetric in β and γ, the 1-form $\sum_\gamma B_\gamma^\delta dt^\gamma$ is closed, and again by Poincaré lemma

$$ B_\gamma^\delta = \partial_\gamma \Phi^\delta. \tag{2.13} $$

So in conclusion, we have

$$ {A_{\alpha\beta}}^\gamma = \partial_\alpha \partial_\beta \Phi^\gamma. \tag{2.14} $$

Conversely, if there exists such a Φ, then R_1 vanishes. □

Now we give a notion for morphisms of Frob manifolds.

Definition 3. *A morphism of Frob manifolds is a map of affine manifolds $\phi : N \to M$ such that*

$$ \phi_*(X \circ_p Y) = \phi_*(X) \circ_{\phi(p)} \phi_*(Y), \tag{2.15} $$

for all $X, Y \in T_pN$.

By map of affine manifolds, we mean that ϕ can be written locally in affine charts as an affine transformation.

We could also formulate (2.15) by saying we have a morphism of sheaves of algebras on N

$$ TN \to \phi^* TM. \tag{2.16} $$

Proposition 1. *Let $\phi : N \to M$ is a morphism of Frob manifolds with local vector potentials Ψ and Φ, respectively. Then locally there exists an affine vector field A such that pointwise the vector potentials are related by*

$$\phi_* \left(\Psi|_p\right) = \Phi|_{\phi(p)} + A|_{\phi(p)}. \tag{2.17}$$

Proof. Let $X, Y \in TN^f$.

$$\phi_*(X \circ_p Y) = \phi_*[X, [Y, \Psi]] \tag{2.18}$$

$$= [\phi_* X, [\phi_* Y, \phi_* \Psi]]. \tag{2.19}$$

Since ϕ_* is an algebra homomorphism, this is equal to

$$\phi_* X \circ_{\phi(p)} \phi_* Y = [\phi_* X, [\phi_* Y, \Phi]]. \tag{2.20}$$

Recalling that an affine vector field is killed by taking two Lie derivatives, comparing the last lines of (2.19) and (2.20) completes the proof. □

2.2 Semisimple Frob manifolds

Definition 4. *A pre-Frob manifold M is called semisimple if there is a basis $e_1, \ldots, e_n$ for TM such that $e_i \circ e_j = \delta_{ij} e_i$.*

Theorem 2. *A semisimple pre-Frob manifold is a Frob manifold if and only if the following conditions are satisfied:*

1. *Locally there exist coordinates y^i, called canonical coordinates, such that $e_i = \frac{\partial}{\partial y^i}$.*

2. *The connection coefficients* Γ^i_{jk} *with respect to these canonical coordinates* y^i *satisfy*

$$\Gamma^i_{jk} = 0, \text{ for } i, j, k \text{ all distinct,} \tag{2.21}$$

$$\Gamma^i_{jj} = -\Gamma^i_{ij}, \text{ for } i \neq j. \tag{2.22}$$

Proof. By Theorem 1, we need to check the vanishing of the structure connection ∇^λ. This vanishes if and only if

$$\left[\nabla^\lambda_{e_i}, \nabla^\lambda_{e_j}\right] e_k = \nabla^\lambda_{[e_i,e_j]} e_k. \tag{2.23}$$

It suffices to look at the λ-linear terms since we assume $\nabla = \nabla^0$ is flat, and $\circ$ is associative by the existence of the semisimple basis. In other words, we just need to check the potential condition.

The equation for the λ-linear terms is

$$e_i \circ \nabla_{e_j} e_k + \nabla_{e_i}(e_j \circ e_k) - e_j \circ \nabla_{e_i} e_k - \nabla_{e_j}(e_i \circ e_k) = [e_i, e_j] \circ e_k, \tag{2.24}$$

which more explicitly is

$$\Gamma^i_{jk} e_i + \delta_{jk}\left(\sum_r \Gamma^r_{ik} e_r\right) - \Gamma^j_{ik} e_j - \delta_{ik}\left(\sum_r \Gamma^r_{jk} e_r\right) = f^k_{ij} e_k, \tag{2.25}$$

where $[e_i, e_j] = \sum_r f^r_{ij} e_r$ (we will indicate summation explicitly, so there is no implied summation over repeated indices). The coefficient of e_k on the left side is zero, so f^k_{ij} vanishes. This means the flows of the e_i all commute, so there exist coordinates y^i such that $e_i = \frac{\partial}{\partial y^i}$.

Now we have that $e_i = \partial_i := \frac{\partial}{\partial y^i}$, and so the right-hand side of (2.25) vanishes. We need the left side to vanish as well. If we take i, j, k distinct, we get $\Gamma^i_{jk}\partial_i - \Gamma^j_{ik}\partial_j = 0$ (no summation implied), then the coefficients must vanish since $i \neq j$.

Now using the preceding result and taking $j = k \neq i$, we get (no summation implied)

$$\Gamma^i_{jj}\partial_i + \Gamma^i_{ij}\partial_i + \Gamma^j_{ij}\partial_j - \Gamma^j_{ij}\partial_j = 0. \tag{2.26}$$

The ∂_j terms cancel identically, and the ∂_i terms cancel if and only if $\Gamma^i_{jj} = -\Gamma^i_{ij}$. □

Remark 1. *Pick any coordinate basis $e_i = \partial_i$, and a point p in M. Assume that M is a manifold over, say, $\mathbb{C}$. The $\mathbb{C}$-algebra T_pM is a module over itself, and the vanishing of the left-hand side of* (2.24) *says that $\nabla : T_pM \otimes_{\mathbb{C}} T_pM \to T_pM$, which is given by extending $\nabla_{e_i} e_j$ $\mathbb{C}$-bilinearily, is a cocycle is the sense of Hochschild cohomology (see [11] for Hochschild cohomology).*

That is, the element of $\mathrm{Hom}_{\mathbb{C}}(T_pM \otimes_{\mathbb{C}} T_pM, T_pM)$ *we get by extending*

$$(e_i, e_j) \mapsto \sum_k \Gamma^k_{ij} e_k \tag{2.27}$$

$\mathbb{C}$-bilinearly represents a class in $HH^2(T_pM, T_pM)$.

We have supposed so far that we are given a flat connection ∇. We would now like to make explicit the conditions on Γ^i_{jk} so that this connection is indeed flat. In this way, we will see what data is necessary to give a Frob manifold in canonical coordinates.

Lemma 1. *A torsion-free connection* ∇, *with connection coefficients satisfying* (2.21) *and* (2.22), *is flat if and only if the following expressions vanish:*

$$\partial_i \Gamma^a_{ja} - \partial_j \Gamma^a_{ia}, \tag{2.28}$$

$$\partial_i \Gamma^a_{aa} - \partial_a \Gamma^a_{ia}, \tag{2.29}$$

$$\partial_i \Gamma^j_{aj} + \Gamma^j_{ij}\Gamma^j_{aj} - \Gamma^j_{ij}\Gamma^i_{ai} - \Gamma^j_{aj}\Gamma^a_{ia}, \tag{2.30}$$

$$-\partial_i \Gamma^i_{ji} - \partial_j \Gamma^i_{ji} - \Gamma^i_{ii}\Gamma^i_{ji} + \Gamma^i_{ji}\Gamma^j_{jj} - \Gamma^i_{ji}\Gamma^i_{ji} + \Gamma^i_{ji}\Gamma^j_{ij} - \sum_{r\neq i,j} \Gamma^i_{ri}\Gamma^r_{jr}, \tag{2.31}$$

where indices with different letters are all distinct (and no summation is implied over repeated indices unless explicitly indicated).

Proof. This is by a direct computation of the non-vanishing components $R^a{}_{bij}$ of the curvature tensor, with the use of (2.21) and (2.22) to simplify the expressions. □

The expressions (2.28) and (2.29) vanish if and only if there is a collection of functions f_i, each defined up to a constant, such that, for all i, j,

$$\Gamma^i_{ji} = \partial_j f_i. \tag{2.32}$$

Locally, we take f_j to be of the particular form

$$f_i = \frac{1}{2}\log(\phi_i), \tag{2.33}$$

where ϕ_i is a collection of functions whose image lies in $\mathbb{C}^*$. We will call the ϕ_i the connection potentials. Now we define the rotation coefficients:

$$\gamma_{ij} := \frac{1}{2}\frac{\partial_i \phi_j}{\sqrt{\phi_i \phi_j}}. \tag{2.34}$$

Note that the rotation coefficients are not necessarily symmetric in i, j.

Proposition 2. *Let ϕ_i be connection potentials, and let ∇ be the connection whose connection coefficients are specified by*

$$\Gamma^i_{ji} = \partial_j \left[\frac{1}{2}\log(\phi_i)\right] = \frac{1}{2}\frac{\partial_j \phi_i}{\phi_i}, \tag{2.35}$$

along with the conditions

$$\Gamma^i_{ji} = \Gamma^i_{ij}, \tag{2.36}$$

$$\Gamma^i_{jk} = 0, \text{ for } i, j, k \text{ all distinct}, \tag{2.37}$$

$$\Gamma^i_{jj} = -\Gamma^i_{ij}, \text{ for } i \neq j. \tag{2.38}$$

Let

$$\gamma_{ij} := \frac{1}{2}\frac{\partial_i \phi_j}{\sqrt{\phi_i \phi_j}}. \tag{2.39}$$

Then ∇ is flat if and only if, for all $i \neq j \neq k \neq i$,

$$e_k \gamma_{ij} = \gamma_{ik}\gamma_{kj}, \tag{2.40}$$

and

$$e\gamma_{ij} = 0. \tag{2.41}$$

When γ_{ij} are the rotation coefficients of a metric, the equations (2.40), (2.41) are a called the Darboux-Egoroff equations. So we will refer to these equations as the generalized Darboux-Egoroff equations.

Proof. We need to show that all the equations in Lemma 1 are satisfied. Since we are given ϕ_i, equations (2.28), and (2.29) are already satisfied. Next we show that

(2.40) is equivalent to (2.30). Expand

$$\partial_k\gamma_{ij} - \gamma_{ik}\gamma_{kj} = \frac{1}{2}\frac{\partial_k\partial_i\phi_j}{\sqrt{\phi_i\phi_j}} - \frac{1}{4}\left(\frac{\partial_i\phi_j\partial_k\phi_i}{\sqrt{\phi_i^3\phi_j}} + \frac{\partial_i\phi_j\partial_k\phi_j}{\sqrt{\phi_i\phi_j^3}} + \frac{\partial_i\phi_k\partial_k\phi_j}{\sqrt{\phi_i\phi_k^2\phi_j}}\right). \tag{2.42}$$

Setting this equal to zero and multiplying by $2\sqrt{\phi_i\phi_j}$, we obtain

$$\partial_k\partial_i\phi_j - \frac{1}{2}\left(\frac{\partial_i\phi_j\partial_k\phi_i}{\phi_i} + \frac{\partial_i\phi_j\partial_k\phi_j}{\phi_j} + \frac{\partial_i\phi_k\partial_k\phi_j}{\phi_k}\right) = 0. \tag{2.43}$$

On the other hand, we expand

$$\begin{aligned}\partial_k\Gamma^j_{ij} + \Gamma^j_{kj}\Gamma^j_{ij} - \Gamma^j_{kj}\Gamma^k_{ik} - \Gamma^j_{ij}\Gamma^i_{ki} = \\ \frac{1}{2}\left(\frac{\partial_k\partial_i\phi_j}{\phi_j} - \frac{\partial_i\phi_j\partial_k\phi_j}{\phi_j^2}\right) + \frac{1}{4}\left(\frac{\partial_k\phi_j\partial_i\phi_j}{\phi_j^2} - \frac{\partial_k\phi_j\partial_i\phi_k}{\phi_j\phi_k} - \frac{\partial_i\phi_j\partial_k\phi_i}{\phi_j\phi_i}\right) = \\ \frac{1}{2}\frac{\partial_k\partial_i\phi_j}{\phi_j} - \frac{1}{4}\left(\frac{\partial_k\phi_j\partial_i\phi_j}{\phi_j^2} + \frac{\partial_k\phi_j\partial_i\phi_k}{\phi_j\phi_k} + \frac{\partial_i\phi_j\partial_k\phi_i}{\phi_j\phi_i}\right).\end{aligned} \tag{2.44}$$

Setting this equal to zero and multiplying by $2\phi_j$, we obtain (2.43).

Now we assume (2.40) and show that (2.41) is equivalent to the vanishing of (2.31). By use of (2.40), the terms which appear in the sum in (2.31) are

$$\Gamma^i_{ri}\Gamma^r_{jr} = \frac{1}{4}\frac{\partial_r\phi_i\partial_j\phi_r}{\phi_i\phi_r} = \sqrt{\frac{\phi_j}{\phi_i}}\gamma_{jr}\gamma_{ri} = \sqrt{\frac{\phi_j}{\phi_i}}e_r\gamma_{ji}. \tag{2.45}$$

Also, we can check that

$$\partial_i\Gamma^i_{ji} + \Gamma^i_{ii}\Gamma^i_{ji} - \Gamma^i_{ji}\Gamma^j_{ij} = \sqrt{\frac{\phi_j}{\phi_i}}e_i\gamma_{ji}, \tag{2.46}$$

and

$$\partial_j\Gamma^i_{ji} + \Gamma^i_{ji}\Gamma^i_{ji} - \Gamma^i_{ji}\Gamma^j_{jj} = \sqrt{\frac{\phi_j}{\phi_i}}e_j\gamma_{ji}, \tag{2.47}$$

so that the vanishing of (2.31) becomes $\sum_k e_k \gamma_{ji} = 0$. □

So we have seen that connection potentials ϕ satisfying the generalized Darboux-Egoroff equations specify a semisimple Frob manifold, and conversely for any semisimple Frob manifold, there exist connection potentials satisfying the generalized Darboux-Egoroff equations.

2.3 Identity field

Definition 5. *Let $(M, \nabla, \circ)$ be a Frob manifold. A vector field e such that $e \circ X = X$ for all X is called an identity.*

We are interested in the case when e is a flat vector field, so without loss of generality, we can assume it is the 0-th coordinate vector field,

$$e = \frac{\partial}{\partial t^0}. \tag{2.48}$$

In terms of the vector potential Φ, for all flat X, we must have

$$[e, [X, \Phi]] = X. \tag{2.49}$$

In flat coordinates coordinates, this is equivalent to

$$\partial_0 \partial_\alpha \Phi^\beta = \delta_\alpha^\beta. \tag{2.50}$$

If our Frob manifold is semisimple with canonical coordinates y^i, then $e = e_1 + \ldots + e_n$, where $e_i = \frac{\partial}{\partial y^i}$.

Proposition 3. *For a semisimple Frob manifold, e is flat if and only if*

$$\sum_q \Gamma^i_{iq} = 0, \tag{2.51}$$

which is equivalent to

$$e(\phi_i) = 0, \tag{2.52}$$

where ϕ_i are the connection potentials.

Proof.

$$\nabla_{e_i}(e_1 + \ldots + e_n) = \sum_{q,r} \Gamma^r_{iq} \partial_r. \tag{2.53}$$

For this to vanish for all i, we must have

$$\sum_q \Gamma^r_{iq} = 0, \tag{2.54}$$

for all i, r. If $i \neq r$, then by (2.21) and (2.22), this reduces to to $\Gamma^r_{ii} + \Gamma^r_{ir} = -\Gamma^r_{ir} + \Gamma^r_{ir} = 0$. We are then left with the case when $r = i$, which gives

$$\sum_q \Gamma^i_{iq} = 0 = \frac{1}{2}\frac{1}{\phi_i} \sum_q \partial_q \phi_i = \frac{1}{2}\frac{e(\phi_i)}{\phi_i}. \tag{2.55}$$

□

2.4 Euler field

Definition 6. *An affine vector field E such that $\mathcal{L}_E(\circ) = d_0(\circ)$ is called an Euler field.*

Recall that an affine vector field is one which can be written, in flat coordinates, with coefficients that are linear polynomials. The flows of these vector fields give affine transformations. An affine vector field E can also be characterized by the property that it preserves flat vector fields under the Lie bracket, $[E, TM^f] \subset TM^f$. We can therefore consider the spectrum of $\operatorname{Ad} E$ acting on TM^f. We are interested in the case when $\operatorname{Ad} E$ is diagonalizable. In this case, we can chose flat coordinates such that

$$E = \sum_{d_\alpha \neq 0} d_\alpha t^\alpha \partial_\alpha + \sum_{d_\alpha = 0} r_\alpha \partial_\alpha. \tag{2.56}$$

The coordinates t^α are called homogeneous, and d_α is called the degree of t^α, i.e. $E(t^\alpha) = d_\alpha t^\alpha$. We allow some of the degrees to be 0, in which case $E\left(e^{t^\beta}\right) = r_\alpha e^{t^\beta}$. In this way, E induces a grading of all the tensorial objects on M. For example, we have $[E, \partial_\alpha] = -d_\alpha \partial_i$.

The fact that d_0 appeared before in the definition of an Euler field is no coincidence: if t_0 is the identity coordinate, the degree of t^0 must agree with the degree of $\circ$. Explicitly, if we write a for the degree of $\circ$ temporarily, we have

$$\mathcal{L}_E(\partial_0 \circ \partial_\alpha) = (-d_0 + a - d_\alpha)\partial_\alpha = \mathcal{L}_E(\partial_\alpha) = -d_\alpha \partial_\alpha. \tag{2.57}$$

it follows that $a = d_0$.

Proposition 4. *Let E be an affine vector field on a Frob manifold (M, Φ). Then E is an Euler field if and only if*

$$[E, \Phi] = d_0 \Phi + \textit{an affine vector field term}, \tag{2.58}$$

or equivalently in flat coordinates,

$$E(\Phi^k) = (d_0 + d_k)\Phi^k + \textit{linear terms.} \tag{2.59}$$

Such a vector potential Φ is called quasihomogeneous.

Proof. First notice that

$$[E, \Phi] = \sum_\alpha (E(\Phi^\alpha)\partial_\alpha + \Phi^\alpha [E, \partial_\alpha]) = \sum_\alpha \left((E - d_\alpha)\Phi^\alpha\right)\partial_\alpha, \tag{2.60}$$

so that the coordinate expression is indeed equivalent to the coordinate-free one.

Now, the condition $\mathcal{L}_E \circ = d_0 \circ$ is equivalent to

$$[E, X \circ Y] - [E, X] \circ Y - X \circ [E, Y] = d_0(X \circ Y), \tag{2.61}$$

for all vector fields X, Y. But if it holds for flat vector fields then it holds for all vector fields by tensorality. So assume X, Y are flat, and use the vector potential to write the left side as

$$[E, [X, [Y, \Phi]]] - [[E, X], [Y, \Phi]] - [X, [[E, Y], \Phi]. \tag{2.62}$$

By the Jacobi identity, this equals $[X, [Y, [E, \Phi]]]$. So the condition becomes

$$[X, [Y, [E, \Phi]]] = d_0[X, [Y, \Phi]], \tag{2.63}$$

which holds if and only if $[E, \Phi] = d_0\Phi$, to up an affine vector field. □

Now we consider when M is semisimple, and look at the Euler field in the canonical coordinate description.

Proposition 5. *Let y^i be canonical coordinates, and $e_i = \frac{\partial}{\partial y^i}$ the coordinate vector fields. E satisfies $\mathcal{L}_E \circ = d_0 \circ$ if and only if*

$$E = d_0 \sum (y_i + c_i) \partial_i. \tag{2.64}$$

Proof. Recall that $\mathcal{L}_E \circ = d_0 \circ$is equivalent to

$$[E, X \circ Y] - [E, X] \circ Y - X \circ [E, Y] = d_0(X \circ Y). \tag{2.65}$$

Write $E = \sum_i E^i e_i$ and take $X = e_k$ and $Y = e_l$. Since $[E, e_k] = -\sum_i e_k(E^i)e_i$. The condition is

$$-\delta_{kl}\sum_i e_k(E^i)e_i + e_k(E^l)e_l + e_l(E^k)e_k = d_0\delta_{kl}e_k. \tag{2.66}$$

This is equivalent to $e_k(E^i) = d_0\delta_{ik}$, so $E^i = d_0(y^i + c^i)$. □

By translating the coordinates, we can set c_i to zero.

Proposition 6. *Let $E = d_0(y^1\partial_1 + \ldots + y^n\partial_n)$, then E is affine if and only if the connection coefficients in canonical coordinates are homogeneous of degree $-d_0$. That is,*

$$E(\Gamma^i_{jk}) = -d_0\Gamma^i_{jk}. \tag{2.67}$$

Proof. E is affine if and only $[E, B]$ is flat for all flat vector fields B. In other words,

$$\nabla_l[E, B] = 0. \tag{2.68}$$

for all l. Since the connection is torsion free and B is flat, we need $\nabla_l[E, B] = -\nabla_l\nabla_B E = 0$. By a direct computation using that $\partial_l B^j = -\sum_a B^a\Gamma^j_{la}$, we find the i-th component of $\nabla_l\nabla_B E = 0$ to be

$$d_0\sum_j B^j\left(\Gamma^i_{jl} + \sum_k y^k\left[\partial_l\Gamma^i_{jk} + \Gamma^i_{la}\Gamma^a_{jk} - \Gamma^i_{ak}\Gamma^a_{lj}\right]\right) = 0. \tag{2.69}$$

We recognize the coefficient of y^k as being part of the component R^i_{jlk} of the vanishing curvature tensor. The expression becomes

$$d_0\sum_j B^j(\Gamma^i_{jl} + \sum_k y^k\partial_k\Gamma^i_{jl}) = 0. \tag{2.70}$$

Since B is arbitrary, we see that E is affine if and only if

$$E(\Gamma^i_{jk}) = -d_0\Gamma^i_{jk}. \tag{2.71}$$

□

In terms of the connection potentials ϕ_i, where $\Gamma^i_{ji} = \frac{1}{2}\partial_j \log(\phi_i)$, we have that

$$\begin{aligned} E\left(\partial_j \log(\phi_i)\right) &= \partial_j \left(E \log(\phi_i)\right) + [E, \partial_j] \log(\phi_i) \\ &= \partial_j \left(E - d_0\right) \log(\phi_i). \end{aligned} \tag{2.72}$$

By Proposition 6, this must be to equal to $-d_0\partial_j \log(\phi_i)$, which implies that $\partial_j \left(E \log(\phi_i)\right) = 0$. Since this holds for all j we get that $E \log(\phi_i)$ is equal to a constant. We normalize these constants so that $E\phi_i = (D_i - 2d_0)\phi_i$, where D_i is a constant. We thus obtain the corollary:

Corollary 1. *E is affine if and only if the connection potentials ϕ_i are homogeneous. We choose constants D_i so that*

$$E\phi_i = (D_i - 2d_0)\phi_i. \tag{2.73}$$

2.5 Compatible metric and Frobenius manifolds

Definition 7. *Let $(M, \nabla, \circ)$ be a Frob manifold. We call a symmetric bilinear form g on TM a metric g (we do not impose any positivity condition). A metric g is called compatible with $(M, \nabla, \circ)$ if*

1. *∇ is the Levi-Civita connection for g.*
2. *$g(X \circ Y, Z) = g(X, Y \circ Z)$ for all $X, Y \in TM$.*

The second condition says that the tangent spaces of M are Frobenius algebras. A Frob manifold with a compatible metric g is called a Frobenius manifold.

In flat coordinates, define $\Phi_\gamma := \sum_\delta g(\Phi^\delta \partial_\delta, \partial_\gamma)$. Taking two partial derivatives, and using the flatness of g, we find that

$$\partial_\alpha \partial_\beta \Phi_\gamma = \partial_\alpha \partial_\beta \sum_\delta g(\Phi^\delta \partial_\delta, \partial_\gamma) = \sum_\delta g(\partial_\alpha \partial_\beta \Phi^\delta \partial_\delta, \partial_\gamma) = g(\partial_\alpha \circ \partial_\beta, \partial_\gamma). \tag{2.74}$$

Using the Frobenius algebra property, we see that $\partial_\alpha \partial_\beta \Phi_\gamma$ is symmetric in α, β, γ. In particular $\partial_\alpha \Phi_\beta$ and $\partial_\beta \Phi_\alpha$ differ by a constant. By adding linear terms to Φ_α, we can make this constant 0. So we see that the 1-form $\sum_\alpha \Phi_\alpha dt^\alpha$ is closed, which gives locally the existence of a potential function Φ so that $\partial_\alpha \partial_\beta \partial_\gamma \Phi = g(\partial_\alpha \circ \partial_\beta, \partial_\gamma) = g(\partial_\alpha, \partial_\beta \circ \partial_\gamma)$.

If there is a flat identity, ∂_0, then we have

$$\partial_0 \partial_\alpha \partial_\beta \Phi = g_{\alpha\beta}. \tag{2.75}$$

We impose that the Euler field be conformal with respect to g. A conformal vector field E is Euler if and only if $E\Phi = (D + 2d_0)\,\Phi$, up to quadratic terms.

Suppose $(M, \nabla, \circ)$ is a semisimple pre-Frob manifold and let us examine the conditions in Theorem 2. For M to be Frob, first we need canonical coordinates. In canonical coordinates, the metric must be diagonal, which can be seen by using the Frobenius algebra property:

$$g_{ij} = (e_i, e_j) = (e_i \circ e_i, e_j) = g(e_i, e_i \circ e_j) = \delta_{ij} g_{ij}. \tag{2.76}$$

Let us call the nonzero diagonal components of the metric η_i. The Christoffel symbols of the Levi-Civita connection are

$$\Gamma^i_{jk} = \frac{1}{2}\sum_r \frac{1}{\eta_i}\delta_{ir}\left(\partial_k\eta_j\delta_{jr} + \partial_j\eta_k\delta_{kr} - \partial_r\eta_j\delta_{jk}\right). \tag{2.77}$$

Next we need to satisfy (2.21) and (2.22). Automatically Γ^i_{jk} is 0 for $i \neq j \neq k \neq i$, so equation (2.21) holds. The non vanishing components are

$$\Gamma^i_{ii} = \frac{1}{2}\frac{\partial_i\eta_i}{\eta_i} \tag{2.78}$$

$$\Gamma^i_{ji} = \frac{1}{2}\frac{\partial_j\eta_i}{\eta_i}, j \neq i \tag{2.79}$$

$$\Gamma^i_{ji} = \frac{1}{2}\frac{\partial_j\eta_i}{\eta_i}, j \neq i \tag{2.80}$$

$$\Gamma^i_{jj} = -\frac{1}{2}\frac{\partial_i\eta_j}{\eta_i}, j \neq i \tag{2.81}$$

In order for (2.22) to hold ($\Gamma^i_{jj} = -\Gamma^i_{ji}$), we need $\partial_j\eta_i = \partial_i\eta_j$, which is true if and only if the exists locally a metric potential η, such that $\eta_i = \partial_i\eta$.

We see that η_i are in fact the connection potentials for the Levi-Civita connection. We can once again define the rotation coefficients as in (2.34):

$$\gamma_{ij} := \frac{1}{2}\frac{\eta_{ij}}{\sqrt{\eta_i\eta_j}}, \tag{2.82}$$

where $\eta_{ij} = \partial_i\partial_j\eta$. The rotation coefficients are now symmetric in i, j. If we specify canonical coordinates and a metric potential η, the Levi-Civita connection is flat if and only if η satisfies the Darboux-Egoroff equations.

In the presence of a conformal Euler field E,

$$E\eta_i = (D - 2d_0)\eta_i, \tag{2.83}$$

which is equivalent to

$$E\eta = (D - d_0)\eta, \text{ up to a constant.} \tag{2.84}$$

2.6 Summary

We provide a summary of semisimple Frob and Frobenius manifolds in the flat coordinate and canonical coordinate descriptions.

Flat picture

Frobenius manifold.

- Flat metric g.
- Potential function Φ (defined up to quadratic terms in flat coordinates) such that

$$g(X \circ Y, Z) = g(X, Y \circ Z) = XYZ\Phi \tag{2.85}$$

  for all flat vector fields X, Y, Z.
- Conformal vector field E with conformal factor D, having the additional property that

$$\mathcal{L}_E \circ = d_0 \circ . \tag{2.86}$$

  E is called the Euler field.
- Homogeneous flat coordinates $t_0, \dots, t_{n-1}$, with $E(t_\alpha) = d_\alpha$.
- Flat identity field $\frac{\partial}{\partial t^0}$.
- $E\Phi = (D + 2d_0)\,\Phi$ plus quadratic terms, so that $\mathcal{L}_E g = Dg$, and $\mathcal{L}_E \circ = d_0 \circ$.

Frob manifold.

- Flat, torsion-free connection ∇.
- Potential vector Φ (defined up to an affine vector) such that

$$X \circ Y = [X, [Y, \Phi]] \tag{2.87}$$

  for all flat vector fields X, Y.

- Affine vector field E with the additional property that

$$\mathcal{L}_E \circ = d_0 \circ . \tag{2.88}$$

  E is called the Euler field.

- Homogeneous flat coordinates $t_0, \ldots, t_{n-1}$, with $E(t_\alpha) = d_\alpha$.
- Flat identity field $\frac{\partial}{\partial t^0}$.
- $[E, \Phi] = d_0\Phi$ up to an affine vector so that $\mathcal{L}_E \circ = d_0 \circ$.

Semisimple picture

Frobenius manifold.

- Canonical coordinates $y^1, \ldots, y^n$ with $e_i \circ e_j = \delta_{ij} e_j$, where $e_i = \frac{\partial}{\partial y^i}$.
- Metric potential η such that g is diagonal with components η_i.
- η satisfies the Darboux-Egoroff equations.
- Euler field $E = d_0\,(y^1 e_1 + \ldots + y^n e_n)$.
- $E\eta = (D - d_0)\eta$ plus a constant (or equivalently $E\eta_i = (D - 2d_0)\eta_i$), so that E is conformal.
- $e\eta$ equal to a constant (or equivalently $e\eta_i = 0$), so that e is flat.

Frob manifold.

- Canonical coordinates $y^1, \ldots, y^n$ with $e_i \circ e_j = \delta_{ij} e_j$, where $e_i = \frac{\partial}{\partial y^i}$.
- Connection coefficients satisfying $\Gamma^i_{ij} = \Gamma^i_{ji} = -\Gamma^i_{jj}$ for $i \neq j$, and $\Gamma^i_{jk} = 0$ for i, j, k distinct. Connection potentials ϕ_i so that $\Gamma^i_{ji} = e_j \left[\frac{1}{2} \log \phi_i\right]$.
- The ϕ_i satisfy the generalized Darboux-Egoroff equations.
- Euler field $E = d_0 \left(y^1 e_1 + \ldots + y^n e_n\right)$.
- $E\phi_i = (D_i - 2d_0)\phi_i$, so that E is affine.
- $e\phi_i = 0$, so that e is flat.

2.7 Structure connections

In the presence of an Euler field E, we can define two new flat connections, called the first and second structure connections. We have already seen a partial definition of the first structure connection. Sometimes these are called *extended* structure connections, but we will avoid that terminology since the word extended will already be used often in this thesis.

Let $\widehat{M} = M \times \mathbb{P}^1$, where $\mathbb{P}^1$ has coordinate λ. Denote by $\widehat{T}$ the pullback of TM to $\widehat{M}$. We will extend the connection ∇^λ on TM defined in (2.5) to a meromorphic connection $\widehat{\nabla}$ on $\widehat{T}$ with poles along $\{0\} \times M$ and $\{\infty\} \times M$, using the Euler field E.

Definition 8. *The first structure connection is defined by*

$$\begin{aligned} \widehat{\nabla}_X Y &= \nabla_X Y + \lambda X \circ Y, \\ d_0 \widehat{\nabla}_{\frac{\partial}{\partial \lambda}} Y &= E \circ Y + \frac{1}{\lambda} \nabla_Y E, \end{aligned} \tag{2.89}$$

where X, Y are λ-independent vector fields.

Proposition 7. *Let M be a pre-Frob manifold, and let E be an affine vector field on M. The first structure connection is flat if and only if M is Frob and E is Euler (that is, $\mathcal{L}_E \circ = d_0 \circ$).*

Proof. Let $\mathcal{E} = E - d_0 \lambda \frac{\partial}{\partial \lambda}$. Then $\mathcal{E}$ along with ∇-flat X generate $T\widehat{M}$. We have already seen that the curvature in the directions tangent to M vanish if and only if M is Frob, since $\widehat{\nabla}$ restricted to M is just ∇^λ. It remains to check the other directions. First we compute that

$$\widehat{\nabla}_{\mathcal{E}} Y = \nabla_E Y + \lambda E \circ Y - \lambda E \circ Y - \nabla_Y E = [E, Y]. \tag{2.90}$$

Assuming Y is ∇-flat, then

$$\widehat{\nabla}_X \widehat{\nabla}_{\mathcal{E}} Y = \lambda \left(X \circ [E, Y] \right), \tag{2.91}$$

where we used that $[E, Y]$ is ∇-flat because E is affine. Also, we have

$$\begin{aligned} \widehat{\nabla}_{\mathcal{E}} \widehat{\nabla}_X Y &= \widehat{\nabla}_{\mathcal{E}} \left(\lambda X \circ Y \right) \\ &= \lambda [E, X \circ Y] - \lambda d_0 X \circ Y. \end{aligned} \tag{2.92}$$

And finally

$$\widehat{\nabla}_{[\mathcal{E}, X]} Y = \widehat{\nabla}_{[E, X]} Y = \lambda \left([E, X] \circ Y \right). \tag{2.93}$$

So the curvature $[\widehat{\nabla}_{\mathcal{E}}, \widehat{\nabla}_X] Y - \widehat{\nabla}_{[\mathcal{E}, X]} Y$ vanishes if and only if

$$[E, X \circ Y] - X \circ [E, Y] - [E, X] \circ Y = d_0 X \circ Y, \tag{2.94}$$

which is the condition for $\mathcal{L}_E \circ = d_0 \circ$. $\square$

Now we will define a second structure connection on $\widehat{T}$. To do this, we will exploit a symmetry of the Darboux-Egoroff equations (2.40), (2.41).

Proposition 8. *Suppose ϕ_i satisfy the Darboux-Egoroff equations, and $E\phi_i = (D_i - 2)\phi_i$ (we have normalized $d_0 = 1$). Let*

$$\check{e}_i = (y^i - \lambda)e_i, \tag{2.95}$$

$$\check{\phi}_i = (y^i - \lambda)^{a_i}\phi_i, \tag{2.96}$$

where the a_i satisfy $a_i - a_j = -(D_i - D_j)$. Then $\check{e}_i, \check{\phi}_i$ also satisfy the Darboux-Egoroff equations.

Proof. We calculate that

$$\check{\gamma}_{ij} = \frac{1}{2}\frac{\check{e}_i\check{\phi}_j}{\sqrt{\check{\phi}_i\check{\phi}_j}} = (y^i - \lambda)^{1-a_i/2}(y^j - \lambda)^{a_j/2}\gamma_{ij}, \text{ for } i \neq j. \tag{2.97}$$

Now it follows that

$$\begin{aligned}\check{e}_k\check{\gamma}_{ij} &= (y^k - \lambda)(y^i - \lambda)^{1-a_i/2}(y^j - \lambda)^{a_j/2}e_k\gamma_{ij} \\ &= (y^k - \lambda)(y^i - \lambda)^{1-a_i/2}(y^j - \lambda)^{a_j/2}\gamma_{ik}\gamma_{kj} \\ &= \check{\gamma}_{ik}\check{\gamma}_{kj},\end{aligned} \tag{2.98}$$

for $i \neq j \neq k \neq i$. Next, we find that

$$\begin{aligned}
\check{e}\check{\gamma}_{ij} &= \sum_k (y^k-\lambda)(y^i-\lambda)^{1-a_i/2}(y^j-\lambda)^{a_j/2} e_k\gamma_{ij} \\
&\quad + (y^i-\lambda)^{1-a_i/2}(y^j-\lambda)^{a_j/2}\left[(1-a_i/2)+a_j/2\right]\gamma_{ij} \\
&= (y^i-\lambda)^{1-a_i/2}(y^j-\lambda)^{a_j/2}\left[-\lambda\sum_k e_k\gamma_{ij} + \sum_k y^k e_k\gamma_{ij} + (1+\frac{a_j-a_i}{2})\gamma_{ij}\right] \\
&= (y^i-\lambda)^{1-a_i/2}(y^j-\lambda)^{a_j/2}\left[-\lambda e\gamma_{ij} + E\gamma_{ij} + \left(1+\frac{a_j-a_i}{2}\right)\gamma_{ij}\right].
\end{aligned} \tag{2.99}$$

The first term in the brackets $e\gamma_{ij}=0$ by assumption. Since $E\gamma_{ij} = \left(\frac{D_j-D_i}{2}-1\right)\gamma_{ij}$, and $a_j - a_i = -(D_j - D_i)$, the other two terms cancel, and so the whole expression on the right vanishes. □

When M has a compatible metric so that it is Frobenius, all the D_i are equal to the conformal factor D. We could take all $a_i = 1$ in (2.96), and the symmetry is induced by the coordinate change $\check{y}^i = \log(y^i - \lambda)$.

These new connection potentials $\check{\phi}$ define a new flat connection $\check{\nabla}$ which by definition has connection coefficients

$$\check{\Gamma}^i_{ji} = \frac{1}{2}\check{e}_j \log\left(\check{\phi}_i\right) = \frac{1}{2}\frac{\check{e}_j\check{\phi}_i}{\check{\phi}_i}. \tag{2.100}$$

Let us write this connection explicitly. For $i \neq j$, we have

$$\begin{aligned}
\check{\nabla}_{\check{e}_i}\check{e}_j &= \frac{1}{2}\frac{\check{e}_j\check{\phi}_i}{\check{\phi}_i} + \frac{1}{2}\frac{\check{e}_i\check{\phi}_j}{\check{\phi}_j} \\
&= (y^i-\lambda)(y^j-\lambda)\left(\frac{1}{2}\frac{\partial_i\phi_j}{\phi_j}e_j + \frac{\partial_j\phi_i}{\phi_i}e_i\right).
\end{aligned} \tag{2.101}$$

Therefore,

$$\check{\nabla}_{e_i} e_j = \frac{1}{2}\frac{\partial_i \phi_j}{\phi_j} e_j + \frac{\partial_j \phi_i}{\phi_i} e_i = \nabla_{e_i} e_j. \tag{2.102}$$

We also have that

$$\begin{aligned} \check{\nabla}_{\check{e}_i} \check{e}_i &= \frac{1}{2}\frac{\check{e}_i \check{\phi}_i}{\check{\phi}_i} \check{e}_i - \frac{1}{2}\sum_{j\neq i} \frac{\check{e}_i \check{\phi}_j}{\check{\phi}_j} \check{e}_j \\ &= \frac{1}{2}(y^i - \lambda)^2 \left[\frac{\partial_i \phi_i}{\phi_i} + \frac{a_i}{y^i - \lambda}\right] e_i - \frac{1}{2}\sum_{j\neq i} (y^i - \lambda)(y^j - \lambda)\frac{\partial_i \phi_j}{\phi_j} e_j. \end{aligned} \tag{2.103}$$

On the other hand,

$$\check{\nabla}_{\check{e}_i} \check{e}_i = (y^i - \lambda)\check{\nabla}_{e_i}\left((y^i - \lambda)e_i\right) = (y^i - \lambda)^2 \left[\check{\nabla}_{e_i} e_i + \frac{1}{y^i - \lambda} e_i\right]. \tag{2.104}$$

So we get that

$$\check{\nabla}_{e_i} e_i = \frac{1}{2}\left[\frac{e_i \phi_i}{\phi_i} - \frac{2 - a_i}{y^i - \lambda}\right] e_i - \frac{1}{2}\sum_{j\neq i} \frac{y^j - \lambda}{y^i - \lambda}\frac{\partial_i \phi_j}{\phi_j} e_j. \tag{2.105}$$

We can write this connection in a more invariant way as follows. Consider ∇E as an operator on TM:

$$\begin{aligned} \nabla E(e_i) &= \nabla_{e_i} E \\ &= \nabla_{e_i}(y^1 e_1 + \ldots + y^n e_n) \\ &= (y^1 \Gamma^1_{1i} + y^i \Gamma^1_{ii}) e_1 + \ldots + (1 + \sum_r y^r \Gamma^i_{ir}) e_i + \ldots + (y^n \Gamma^n_{ni} + y^i \Gamma^n_{ii}) e_n \\ &= \frac{1}{2}\frac{\partial_i \phi_1}{\phi_1}(y^1 - y^i) e_1 + \ldots + \frac{D_i}{2} e_i + \ldots + \frac{1}{2}\frac{\partial_i \phi_n}{\phi_n}(y^n - y^i) e_n, \end{aligned} \tag{2.106}$$

where we have the homogeneity of ϕ to get

$$1 + \sum_r y^r \Gamma^i_{ir} = 1 + \frac{1}{2} \sum_r \frac{y^r \partial_r \phi_i}{\phi_i} = \frac{D_i}{2}. \tag{2.107}$$

Using (2.102), (2.105), and (2.106), we can write

$$\check{\nabla}_{e_i} e_j = \nabla_{e_i} e_j - \left(\nabla E - \frac{D_i}{2} + \frac{2 - a_i}{2} \right) \frac{1}{y^i - \lambda} \delta_{ij} e_i. \tag{2.108}$$

Notice that by the condition for the symmetry, $D_i + a_i = D_j + a_j$ for all i, j, so $(2 - a_i - D_i)/2$ is some constant s. Also, $\delta_{ij} e_i = e_i \circ e_j$.

If we define

$$\mathcal{U}(X) = E \circ X, \tag{2.109}$$

$\check{\nabla}$ can be written as

$$\check{\nabla}_X Y = \nabla_X Y - (s \,\mathrm{id} + \nabla E) (\mathcal{U} - \lambda)^{-1} X \circ Y. \tag{2.110}$$

We define $\check{\nabla}$ in the $\partial/\partial\lambda$ direction by

$$\check{\nabla}_{\partial/\partial\lambda} Y = (s \,\mathrm{id} + \nabla E) (\mathcal{U} - \lambda)^{-1} Y. \tag{2.111}$$

Definition 9. *The second structure connection for a semisimple Frob manifold is defined by*

$$\check{\nabla}_X Y = \nabla_X Y - (s \,\mathrm{id} + \nabla E) (\mathcal{U} - \lambda)^{-1} X \circ Y, \tag{2.112}$$

$$\check{\nabla}_{\partial/\partial\lambda} Y = (s \,\mathrm{id} + \nabla E) (\mathcal{U} - \lambda)^{-1} Y, \tag{2.113}$$

where X, Y are λ-independent vector fields, and $s = (2 - a_i - D_i)/2$ with a_i satisfying the symmetry in Proposition 8.

Note that we can pick any s we like, and this determines a_i.

Proposition 9. *Let M be a semisimple Frob manifold with Euler field E. Then the second structure connection $\check{\nabla}$ is flat.*

Proof. We know that $\check{\nabla}$ is flat in the directions tangent to M, since we constructed it using a symmetry of the Darboux-Egoroff equations. Since $[X, \partial/\partial\lambda] = 0$, it remains to check that

$$\check{\nabla}_{e_i}\check{\nabla}_{\partial/\partial\lambda}e_j = \check{\nabla}_{\partial/\partial\lambda}\check{\nabla}_{e_i}e_j \tag{2.114}$$

for al i, j. Let us collect the formulas for $\check{\nabla}$ in the semisimple basis:

$$\check{\nabla}_{e_i}e_j = \frac{1}{2}\frac{\partial_j\phi_i}{\phi_i}e_i + \frac{1}{2}\frac{\partial_i\phi_j}{\phi_j}e_j, \tag{2.115}$$

$$\check{\nabla}_{e_i}e_i = \frac{1}{2}\left[\frac{e_i\phi_i}{\phi_i} - \frac{2-a_i}{y^i-\lambda}\right]e_i - \frac{1}{2}\sum_{j\neq i}\frac{y^j-\lambda}{y^i-\lambda}\frac{e_i\phi_j}{\phi_j}e_j, \tag{2.116}$$

$$\check{\nabla}_{\partial/\partial\lambda}e_j = \frac{1}{2}\frac{2-a_j}{y^j-\lambda}e_j + \frac{1}{2}\sum_{k\neq j}\frac{y^k-y^j}{y^j-\lambda}\frac{e_j\phi_k}{\phi_k}e_k. \tag{2.117}$$

We have for $i \neq j$:

$$\begin{aligned}\check{\nabla}_{\partial/\partial\lambda}\check{\nabla}_{e_i}(e_j) &= \check{\nabla}_{\partial/\partial\lambda}\left(\frac{1}{2}\frac{\partial_j\phi_i}{\phi_i}e_i + \frac{1}{2}\frac{\partial_i\phi_j}{\phi_j}e_j\right)\\ &= \frac{1}{2}\frac{\partial_j\phi_i}{\phi_i}\left[\frac{1}{2}\frac{2-a_i}{y^i-\lambda}e_i + \frac{1}{2}\sum_{k\neq i}\frac{y^k-y^i}{y^i-\lambda}\frac{\partial_i\phi_k}{\phi_k}e_k\right] + (i\leftrightarrow j),\end{aligned} \tag{2.118}$$

$$
\begin{aligned}
\check{\nabla}_{e_i}\check{\nabla}_{\partial/\partial\lambda}(e_j) = \check{\nabla}_{e_i}&\left(\frac{1}{2}\frac{2-a_j}{y^j-\lambda}e_j + \frac{1}{2}\sum_{k\neq j}\frac{y^k-y^j}{y^j-\lambda}\frac{\partial_j\phi_k}{\phi_k}e_k\right)\\
&= \frac{1}{2}\frac{2-a_j}{y^j-\lambda}\left(\frac{1}{2}\frac{\partial_j\phi_i}{\phi_i}e_i + \frac{1}{2}\frac{\partial_i\phi_j}{\phi_j}e_j\right)\\
&\quad + \frac{1}{2}\sum_{k\neq j}e_i\left(\frac{y^k-y^j}{y^j-\lambda}\frac{\partial_j\phi_k}{\phi_k}\right)e_k\\
&+ \frac{1}{2}\sum_{k\neq j,i}\frac{y^k-y^j}{y^j-\lambda}\frac{\partial_j\phi_k}{\phi_k}\left(\frac{1}{2}\frac{\partial_k\phi_i}{\phi_i}e_i + \frac{1}{2}\frac{\partial_i\phi_k}{\phi_k}e_k\right)\\
&+ \frac{1}{2}\frac{y^i-y^j}{y^j-\lambda}\frac{\partial_j\phi_i}{\phi_i}\left[\frac{1}{2}\left(\frac{\partial_i\phi_i}{\phi_i} - \frac{2-a_i}{y^i-\lambda}\right)e_i - \frac{1}{2}\sum_{k\neq i}\frac{y^k-\lambda}{y^i-\lambda}\frac{\partial_i\phi_k}{\phi_k}e_k\right]. \quad (2.119)
\end{aligned}
$$

In (2.118) the coefficient of e_i is

$$
\frac{1}{4}\frac{2-a_i}{y^i-\lambda}\frac{\partial_j\phi_i}{\phi_i} + \frac{1}{4}\frac{y^i-y^j}{y^j-\lambda}\frac{\partial_i\phi_j\partial_j\phi_i}{\phi_j\phi_i}, \quad (2.120)
$$

whereas in (2.119) we get

$$
\begin{aligned}
&\frac{1}{4}\frac{2-a_i}{y^j-\lambda}\frac{\partial_j\phi_i}{\phi_i} + \frac{1}{2}e_i\left(\frac{y^i-y^j}{y^j-\lambda}\frac{\partial_j\phi_i}{\phi_i}\right)\\
&\quad + \frac{1}{4}\left(\frac{\partial_i\phi_i}{\phi_i} - \frac{2-a_i}{y^i-\lambda}\right)\frac{y^i-y^j}{y^j-\lambda}\frac{\partial_j\phi_i}{\phi_i} + \frac{1}{4}\sum_{k\neq i,j}\frac{y^k-y^j}{y^j-\lambda}\frac{\partial_j\phi_k\partial_k\phi_i}{\phi_k\phi_i}. \quad (2.121)
\end{aligned}
$$

We will use the following identity for the sum $\sum_k$ in (2.121):

$$\frac{1}{4}\sum_{k\neq i,j}\frac{y^k-y^j}{y^j-\lambda}\frac{\partial_j\phi_k\partial_k\phi_i}{\phi_k\phi_i} = \frac{1}{y^j-\lambda}\frac{\sqrt{\phi_j}}{\sqrt{\phi_i}}\sum_{k\neq i,j}\gamma_{jk}\gamma_{ki}(y^k-y^j)$$

$$= \frac{1}{y^j-\lambda}\frac{\sqrt{\phi_j}}{\sqrt{\phi_i}}\sum_{k\neq i,j}\left(y^k e_k\gamma_{ji} - y^j\gamma_{jk}\gamma_{ki}\right)$$

$$= \frac{1}{y^j-\lambda}\frac{\sqrt{\phi_j}}{\sqrt{\phi_i}}\left(E\gamma_{ji} - y^i e_i\gamma_{ji} - y^j e_j\gamma_{ji} + y^j(e_i+e_j)\gamma_{ji}\right)$$

$$= \frac{1}{y^j-\lambda}\frac{\sqrt{\phi_j}}{\sqrt{\phi_i}}\left(\left[\frac{D_i-D_j}{2}-1\right]\gamma_{ji} - y^i e_i\gamma_{ji} + y^j e_i\gamma_{ji}\right)$$

$$= \frac{1}{2}\frac{1}{y^j-\lambda}\left\{\left[\frac{D_i-D_j}{2}-1\right]\frac{\partial_j\phi_i}{\phi_i} + (y^j-y^i)\left(\frac{\partial_i\partial_j\phi_i}{\phi_i} - \frac{1}{2}\frac{\partial_j\phi_i\partial_i\phi_j}{\phi_j\phi_i} - \frac{1}{2}\frac{\partial_j\phi_i\partial_i\phi_i}{\phi_i^2}\right)\right\} \tag{2.122}$$

Using this identity, we look at the coefficient of $\frac{\partial_j\phi_i}{\phi_i}$ in (2.121), which gives

$$\frac{1}{4}\frac{2-a_j}{y^j-\lambda} + \frac{1}{2}\frac{1}{y^j-\lambda} + \frac{1}{2}\left[\frac{D_i-D_j}{2}-1\right]\frac{1}{y^j-\lambda} - \frac{1}{4}\frac{2-a_i}{y^i-\lambda}\left(\frac{y^i-y^j}{y^j-\lambda}\right)$$

$$= \frac{1}{4}\frac{2-a_j}{y^j-\lambda} + \frac{1}{4}\frac{D_i-D_j}{y^j-\lambda} + \frac{1}{4}\frac{2-a_i}{y^i-\lambda}\left(\frac{y^j-y^i}{y^j-\lambda}\right)$$

$$= \frac{1}{4}\frac{2-a_j}{y^j-\lambda} + \frac{1}{4}\frac{D_i-D_j}{y^j-\lambda} + \frac{1}{4}\frac{2-a_i}{y^i-\lambda}\left(1-\frac{y^i-\lambda}{y^j-\lambda}\right)$$

$$= \frac{1}{4}\left(\frac{D_i-D_j+a_i-a_j}{y^j-\lambda} + \frac{2-a_i}{y^i-\lambda}\right). \tag{2.123}$$

Since $D_i - D_j + a_i - a_j = 0$, the above is simply $\frac{1}{4}\frac{2-a_i}{y^i-\lambda}$ which matches the first term in (2.120). The rest of the terms in (2.120) and (2.121) can be compared directly without the use of any identities.

The coefficients of e_j in (2.118) and (2.119) are seen to be equal by directly comparing.

Now we compare coefficients of e_k for $k \neq i \neq j \neq k$. In (2.118), the coefficient of e_k is

$$\frac{1}{4}\frac{y^k - y^i}{y^i - \lambda}\frac{\partial_j\phi_i\partial_i\phi_k}{\phi_i\phi_k} + \frac{1}{4}\frac{y^k - y^j}{y^j - \lambda}\frac{\partial_i\phi_j\partial_j\phi_k}{\phi_j\phi_k} = \\ \frac{1}{4}\left(\frac{y^k - \lambda}{y^i - \lambda} - 1\right)\frac{\partial_j\phi_i\partial_i\phi_k}{\phi_i\phi_k} + \frac{1}{4}\left(\frac{y^k - \lambda}{y^j - \lambda} - 1\right)\frac{\partial_i\phi_j\partial_j\phi_k}{\phi_j\phi_k}. \quad (2.124)$$

The coefficient of e_k in (2.119) is

$$\frac{1}{2}e_i\left(\frac{y^k - y^j}{y^j - \lambda}\frac{\partial_j\phi_k}{\phi_k}\right) + \frac{1}{4}\frac{y^k - y^j}{y^j - \lambda}\frac{\partial_j\phi_k\partial_i\phi_k}{\phi_k^2} - \frac{1}{4}\frac{y^i - y^j}{y^j - \lambda}\frac{y^k - \lambda}{y^i - \lambda}\frac{\partial_j\phi_i\partial_i\phi_k}{\phi_i\phi_k} = \\ \frac{1}{2}\left(\frac{y^k - \lambda}{y^j - \lambda} - 1\right)\left(\frac{\partial_i\partial_j\phi_k}{\phi_k} - \frac{\partial_j\phi_k\partial_i\phi_k}{\phi_k^2}\right) + \frac{1}{4}\left(\frac{y^k - \lambda}{y^j - \lambda} - 1\right)\frac{\partial_j\phi_k\partial_i\phi_k}{\phi_k^2} \\ - \frac{1}{4}\left(\frac{y^i - \lambda}{y^j - \lambda} - 1\right)\frac{y^k - \lambda}{y^i - \lambda}\frac{\partial_j\phi_i\partial_i\phi_k}{\phi_i\phi_k}. \quad (2.125)$$

(2.124) and (2.125) are seen to be equal by the following identity which is equivalent to the generalized Darboux-Egoroff equations (see (2.43)):

$$\frac{\partial_i\partial_j\phi_k}{\phi_k} = \frac{1}{2}\left(\frac{\partial_j\phi_i\partial_i\phi_k}{\phi_i\phi_k} + \frac{\partial_i\phi_j\partial_j\phi_k}{\phi_j\phi_k} + \frac{\partial_j\phi_k\partial_i\phi_k}{\phi_k^2}\right). \quad (2.126)$$

Finally, we must consider the case $i = j$:

$$\check{\nabla}_{\partial/\partial\lambda}\check{\nabla}_{e_i}e_i = \check{\nabla}_{\partial/\partial\lambda}\left[\frac{1}{2}\left(\frac{\partial_i\phi_i}{\phi_i} - \frac{2-a_i}{y^i-\lambda}\right)e_i - \frac{1}{2}\sum_{j\neq i}\frac{y^j-\lambda}{y^i-\lambda}\frac{\partial_i\phi_j}{\phi_j}e_j\right]$$
$$= -\frac{1}{2}\frac{\partial}{\partial\lambda}\left(\frac{2-a_i}{y^i-\lambda}\right)e_i$$
$$+\frac{1}{2}\left(\frac{\partial_i\phi_i}{\phi_i} - \frac{2-a_i}{y^i-\lambda}\right)\left(\frac{1}{2}\frac{2-a_i}{y^i-\lambda}e_i + \frac{1}{2}\sum_{j\neq i}\frac{y^j-y^i}{y^i-\lambda}\frac{\partial_i\phi_j}{\phi_j}e_j\right)$$
$$-\frac{1}{2}\sum_{j\neq i}\frac{\partial}{\partial\lambda}\left(\frac{y^j-\lambda}{y^i-\lambda}\right)\frac{\partial_i\phi_j}{\phi_j}e_j$$
$$-\frac{1}{2}\sum_{j\neq i}\frac{y^j-\lambda}{y^i-\lambda}\frac{\partial_i\phi_j}{\phi_j}\left(\frac{1}{2}\frac{2-a_j}{y^j-\lambda}e_j + \frac{1}{2}\sum_{k\neq j}\frac{y^k-y^j}{y^j-\lambda}\frac{e_j\phi_k}{\phi_k}e_k\right), \quad (2.127)$$

$$\check{\nabla}_{e_i}\check{\nabla}_{\partial/\partial\lambda}e_i = \check{\nabla}_{e_i}\left[\frac{1}{2}\frac{2-a_i}{y^i-\lambda}e_i + \frac{1}{2}\sum_{j\neq i}\frac{y^j-y^i}{y^i-\lambda}\frac{\partial_i\phi_j}{\phi_j}e_j\right]$$
$$= \frac{1}{2}e_i\left(\frac{2-a_i}{y^i-\lambda}\right)e_i$$
$$+\frac{1}{2}\frac{2-a_i}{y^i-\lambda}\left[\frac{1}{2}\left(\frac{\partial_i\phi_i}{\phi_i} - \frac{2-a_i}{y^i-\lambda}\right)e_i - \frac{1}{2}\sum_{j\neq i}\frac{y^j-\lambda}{y^i-\lambda}\frac{\partial_i\phi_j}{\phi_j}e_j\right]$$
$$+\frac{1}{2}\sum_{j\neq i}e_i\left(\frac{y^j-y^i}{y^i-\lambda}\frac{\partial_i\phi_j}{\phi_j}\right)e_j$$
$$+\frac{1}{2}\sum_{j\neq i}\frac{y^j-y^i}{y^i-\lambda}\frac{\partial_i\phi_j}{\phi_j}\left(\frac{1}{2}\frac{\partial_j\phi_i}{\phi_i}e_i + \frac{1}{2}\frac{\partial_i\phi_j}{\phi_j}e_j\right). \quad (2.128)$$

The coefficients of e_i in (2.127) and (2.128) are seen to be equal by directly comparing.

In (2.127), the coefficient of e_j (where $j \neq i$) is

$$\begin{aligned}
&\frac{1}{4}\left(\frac{\partial_i\phi_i}{\phi_i} - \frac{2-a_i}{y^i-\lambda}\right)\frac{y^j-y^i}{y^i-\lambda}\frac{\partial_i\phi_j}{\phi_j} \\
&\qquad - \frac{1}{2}\frac{\partial}{\partial\lambda}\left(\frac{y^j-\lambda}{y^i-\lambda}\right)\frac{\partial_i\phi_j}{\phi_j} - \frac{1}{4}\frac{y^j-\lambda}{y^i-\lambda}\frac{2-a_j}{y^j-\lambda}\frac{\partial_i\phi_j}{\phi_j} \\
&\qquad - \frac{1}{4}\sum_{k\neq i,j}\frac{y^k-\lambda}{y^i-\lambda}\frac{y^j-y^k}{y^k-\lambda}\frac{\partial_i\phi_k}{\phi_k}\frac{\partial_k\phi_j}{\phi_j} \\
&\quad = \frac{1}{4}\left(\frac{\partial_i\phi_i}{\phi_i} - \frac{2-a_i}{y^i-\lambda}\right)\frac{y^j-y^i}{y^i-\lambda}\frac{\partial_i\phi_j}{\phi_j} \\
&\qquad - \frac{1}{2}\frac{\partial}{\partial\lambda}\left(\frac{y^j-\lambda}{y^i-\lambda}\right)\frac{\partial_i\phi_j}{\phi_j} - \frac{1}{4}\frac{2-a_j}{y^i-\lambda}\frac{\partial_i\phi_j}{\phi_j} \\
&\qquad + \frac{1}{4}\sum_{k\neq i,j}\frac{y^k-y^j}{y^i-\lambda}\frac{\partial_i\phi_k}{\phi_k}\frac{\partial_k\phi_j}{\phi_j}.
\end{aligned} \tag{2.129}$$

By a calculation similar to (2.122), we get that the sum $\frac{1}{4}\sum_{k\neq i,j}$ in (2.129) is equal to

$$\frac{1}{2}\frac{1}{y^i-\lambda}\left\{\left[\frac{D_j-D_i}{2}-1\right]\frac{\partial_i\phi_j}{\phi_j} + (y^j-y^i)\left(\frac{\partial_i\partial_i\phi_j}{\phi_j} - \frac{1}{2}\frac{\partial_i\phi_j\partial_i\phi_i}{\phi_i\phi_j} - \frac{1}{2}\frac{\partial_i\phi_j\partial_i\phi_j}{\phi_j^2}\right)\right\}. \tag{2.130}$$

The coefficient of e_j in (2.128) is

$$-\frac{1}{4}\frac{2-a_i}{y^i-\lambda}\frac{y^j-\lambda}{y^i-\lambda}\frac{\partial_i\phi_j}{\phi_j} + \frac{1}{2}e_i\left(\frac{y^j-y^i}{y^i-\lambda}\frac{\partial_i\phi_j}{\phi_j}\right) + \frac{1}{4}\frac{y^j-y^i}{y^i-\lambda}\frac{\partial_i\phi_j\partial_i\phi_j}{\phi_j^2}. \tag{2.131}$$

We need to show that (2.129) and (2.131) are equal. We first look at the coefficients of $\frac{\partial_i\phi_j}{\phi_j}$. In (2.129) (replacing the sum by (2.130)) the coefficient of $\frac{\partial_i\phi_j}{\phi_j}$

is

$$
\begin{aligned}
&-\frac{1}{4}\frac{2-a_i}{y^i-\lambda}\left(\frac{y^j-y^i}{y^i-\lambda}\right)-\frac{1}{2}\frac{\partial}{\partial\lambda}\left(\frac{y^j-\lambda}{y^i-\lambda}\right)-\frac{1}{4}\frac{2-a_j}{y^i-\lambda}+\frac{1}{2}\frac{1}{y^i-\lambda}\left[\frac{D_j-D_i}{2}-1\right]\\
&=-\frac{1}{4}\frac{2-a_i}{y^i-\lambda}\frac{y^j-\lambda}{y^i-\lambda}+\frac{1}{4}\frac{1}{y^i-\lambda}(a_j-a_i+D_j-D_i)-\frac{1}{2}\frac{\partial}{\partial\lambda}\left(\frac{y^j-\lambda}{y^i-\lambda}\right)-\frac{1}{2}\frac{1}{y^i-\lambda}\\
&=-\frac{1}{4}\frac{2-a_i}{y^i-\lambda}\frac{y^j-\lambda}{y^i-\lambda}-\frac{1}{2}\frac{\partial}{\partial\lambda}\left(\frac{y^j-\lambda}{y^i-\lambda}\right)-\frac{1}{2}\frac{1}{y^i-\lambda}, \quad (2.132)
\end{aligned}
$$

where we have used that $a_j - a_i + D_j - D_i = 0$. In (2.131), the coefficient of $\frac{\partial_i \phi_j}{\phi_j}$ is

$$
-\frac{1}{4}\frac{2-a_i}{y^i-\lambda}\frac{y^j-\lambda}{y^i-\lambda}+\frac{1}{2}e_i\left(\frac{y^j-y^i}{y^i-\lambda}\right). \quad (2.133)
$$

The previous two expressions are equal since

$$
-\frac{1}{2}\frac{\partial}{\partial\lambda}\left(\frac{y^j-\lambda}{y^i-\lambda}\right)-\frac{1}{2}\frac{1}{y^i-\lambda}=-\frac{1}{2}\frac{y^j-\lambda}{(y^i-\lambda)^2} \quad (2.134)
$$

while also

$$
\frac{1}{2}e_i\left(\frac{y^j-y^i}{y^i-\lambda}\right)=\frac{1}{2}e_i\left(\frac{y^j-\lambda}{y^i-\lambda}-1\right)=-\frac{1}{2}\frac{y^j-\lambda}{(y^i-\lambda)^2}. \quad (2.135)
$$

All other terms in (2.129) and (2.131) are seen to be equal by directly comparing. □

CHAPTER 3
Extensions of Frob Manifolds

In the previous chapter, we have seen Frob manifolds and morphisms of Frob manifolds. In this chapter, we define a particular kind of morphism called an extension. After giving a general definition, we focus on rank-1 extensions and show that the OWDVV equations [6] arise as the associativity of this extension.

We examine what needs to happen algebraically for rank-1 extensions to be associative. These associativity considerations affect the potential conditions of the extended Frob manifold. We look at this from both the flat and canonical coordinate picture. This interplay between the algebra and the potentiality can be seen very strongly in Theorem 3, which gives the most succinct criteria to specify a Frob extension.

Finally we examine isomonodromic deformations under extensions.

3.1 Definition

Definition 10. *Let M be a Frob manifold. An extension of M is a Frob manifold N that fits into an exact sequence of Frob manifolds, which we denote by*

$$0 \to I \to N \xrightarrow{\pi} M \to 0. \tag{3.1}$$

All the arrows are morphisms of Frob manifolds, as in Definition 3. By an exact sequence, we mean that:

1. *$I \to N \xrightarrow{\pi} M$ is a fibre bundle.*

2. For all p, $0 \to T_pI \to T_pN \to T_{\pi(p)}M \to 0$ is an exact sequence of algebras. We will call the dimension of I the rank of the extension.

Another extension $\tilde{N}$ is equivalent to N if there exists an isomorphism of Frob manifolds $\phi : N \to \tilde{N}$ such that

$$\tilde{\pi} \circ \phi = \pi. \tag{3.2}$$

Such an isomorphism can be viewed as an automorphism of N, or as a change of coordinates.

3.2 Flat coordinates and associativity

Let M be an n-dimensional Frob manifold with local flat coordinates $t^0, \ldots, t^{n-1}$, and local vector potential Φ. We will try to build an extension N of M. For local flat coordinates on N, we use $t^0, \ldots, t^{n-1}$ along with new flat coordinates $u^1, \ldots, u^k$ for the fibre of π. These coordinates define the flat structure on N.

Now we look at the multiplication. Since

$$\pi_* \left(\frac{\partial}{\partial t^\alpha} \circ_p \frac{\partial}{\partial t^\alpha} \right) = \pi_* \left(\frac{\partial}{\partial t^\alpha} \right) \circ_{\pi(p)} \pi_* \left(\frac{\partial}{\partial t^\alpha} \right), \tag{3.3}$$

only the components of $\frac{\partial}{\partial t^\alpha} \circ_N \frac{\partial}{\partial t^\alpha}$ in the kernel of π_* (i.e. the u components) can depend on u. The multiplication table therefore is

$$\begin{aligned} \partial_\alpha \circ \partial_\beta &= \sum_{\gamma=0}^{n-1} \Phi_{\alpha\beta}{}^{\gamma} \partial_\gamma + \sum_{a=1}^{k} \Omega_{\alpha\beta}{}^{a} \partial_a, \\ \partial_\alpha \circ \partial_a &= \sum_{b=1}^{k} \Omega_{\alpha a}{}^{b} \partial_b, \\ \partial_a \circ \partial_b &= \sum_{b=1}^{k} \Omega_{ab}{}^{c} \partial_c, \end{aligned} \tag{3.4}$$

where Φ is the pullback of the vector potential on M, and $\Omega^1, \ldots, \Omega^k$ are extended potential components (which in general depend on u). The total vector potential for N has components

$$\left(\Phi^0, \ldots, \Phi^{n-1}, \Omega^1, \ldots, \Omega^k\right). \tag{3.5}$$

For notational simplicity in (3.4), we have written $\partial_\alpha = \frac{\partial}{\partial t^\alpha}, \partial_a = \frac{\partial}{\partial u^a}$, and denoted partial derivatives of Φ and Ω by subscripts, with Greek letters indexing t and Roman letters indexing u.

The associativity equations for the multiplication (3.4) are

1.
$$\sum_{\delta=0}^{n-1} \Phi_{\alpha\beta}{}^{\delta} \Omega_{\delta\gamma}{}^{a} + \sum_{b=1}^{k} \Omega_{\alpha\beta}{}^{b} \Omega_{b\gamma}{}^{a} = \sum_{\delta=0}^{n-1} \Phi_{\beta\gamma}{}^{\delta} \Omega_{\delta\alpha}{}^{a} + \sum_{b=1}^{k} \Omega_{\beta\gamma}{}^{b} \Omega_{b\alpha}{}^{a}. \tag{3.6}$$

2.
$$\sum_{\gamma=0}^{n-1} \Phi_{\alpha\beta}{}^{\gamma} \Omega_{\gamma a}{}^{b} + \sum_{c=1}^{k} \Omega_{\alpha\beta}{}^{c} \Omega_{ca}{}^{b} = \sum_{c=1}^{k} \Omega_{\beta a}{}^{c} \Omega_{c\alpha}{}^{b}. \tag{3.7}$$

3.
$$\sum_{d=1}^{k} \Omega_{\alpha a}{}^{d} \Omega_{db}{}^{c} = \sum_{d=1}^{k} \Omega_{ab}{}^{d} \Omega_{d\alpha}{}^{c}. \tag{3.8}$$

4.
$$\sum_{e=1}^{k} \Omega_{ab}{}^{e} \Omega_{ec}{}^{d} = \sum_{e=1}^{k} \Omega_{bc}{}^{e} \Omega_{ea}{}^{d}. \tag{3.9}$$

These come from checking the ∂_* components of $(\partial_\alpha \circ \partial_\beta) \circ \partial_\gamma = \partial_\alpha \circ (\partial_\beta \circ \partial_\gamma)$, $(\partial_\alpha \circ \partial_\beta) \circ \partial_a = \partial_\alpha \circ (\partial_\beta \circ \partial_a)$, $(\partial_\alpha \circ \partial_a) \circ \partial_b = \partial_\alpha \circ (\partial_a \circ \partial_b)$, and $(\partial_a \circ \partial_b) \circ \partial_c = \partial_a \circ (\partial_b \circ \partial_c)$. The other components correspond to associativity on M, and we omit these (we assume that M is associative).

Automorphisms of N which commute with π are affine transformations of the form

$$\begin{pmatrix} \tilde{t} \\ \tilde{u} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ A & B \end{pmatrix} \begin{pmatrix} t \\ u \end{pmatrix} + C, \tag{3.10}$$

where we have written the coordinates in column vectors. The associativity equations for the extension are invariant under these transformations.

In order to have an identity field $\partial_0 = \frac{\partial}{\partial t^0}$, as in (2.50), we must impose

$$\Omega_{0a}{}^b = \delta_a{}^b. \tag{3.11}$$

If M has an Euler field E^M, we can extend the Euler field to N (in such a way that $\operatorname{Ad} E$ acts semisimply) by

$$E = E^M + \sum_{a=1}^{k} d_{*i} u^a \partial_a \tag{3.12}$$

(we can replace $d_{*a} u \partial_a$ by a term $s_a \partial_a$ whenever $d_{*a} = 0$). By (2.59), E is an Euler field for N if and only if

$$E\Omega^a = (d_0 + d_{*a})\Omega^a + \text{linear terms}. \tag{3.13}$$

3.3 Rank-1 Extensions

We will now focus on the case where I is 1-dimensional. We specialize the general associativity equations to rank-1 to arrive at the OWDVV equations. Then we study the algebra of rank-1 extensions.

We can divide the algebra of extensions into two types. In the first type, the algebra is that of an extension by a module, and the associativity is related to

Hochschild cohomology. The second type is of more interest to us, and we call it an auxiliary extension.

These algebraic considerations determine how to make the extended multiplication associative. We then look at different criteria for the extended multiplication to be potential.

3.3.1 The open WDVV equations

Locally, we pick flat coordinates $t^0, \dots, t^{n-1}, u$ on N such that π is the projection onto the t^α. From the general case, the multiplication table is

$$\begin{aligned} \partial_\alpha \circ \partial_\beta &= \sum_{\gamma=1}^{n-1} \Phi_{\alpha\beta}{}^{\gamma} \partial_\gamma + \Omega_{\alpha\beta} \partial_*, \\ \partial_i \circ \partial_* &= \Omega_{i*} \partial_*, \\ \partial_* \circ \partial_* &= \Omega_{**} \partial_*. \end{aligned} \tag{3.14}$$

We will sometimes refer to Ω as the extended potential. For notational simplicity in (3.14), we have used $*$ as the index for u, and we have dropped the superscript on Ω.

There are only two associativity conditions to check (compared with four in the case of a general rank extension). They are

1.
$$\sum_{\delta=1}^{n-1} \Phi_{\alpha\beta}{}^{\delta} \Omega_{\delta\gamma} + \Omega_{\alpha\beta}\Omega_{*\gamma} = \sum_{\delta=1}^{n-1} \Phi_{\beta\gamma}{}^{\delta} \Omega_{\delta\alpha} + \Omega_{\beta\gamma}\Omega_{*\alpha}, \tag{3.15}$$

2.
$$\sum_{\gamma} \Phi_{\alpha\beta}{}^{\gamma} \Omega_{\gamma *} + \Omega_{\alpha\beta}\Omega_{**} = \Omega_{\alpha *}\Omega_{*\beta}. \tag{3.16}$$

These are called the open WDVV equations. They come from checking $(\partial_\alpha \circ \partial_\beta) \circ \partial_\gamma = \partial_\alpha \circ (\partial_\beta \circ \partial_\gamma)$ and $(\partial_\alpha \circ \partial_\beta) \circ \partial_* = \partial_\alpha \circ (\partial_\beta \circ \partial_*)$.

Specializing (3.10) to rank-1, the automorphisms of N which commute with π are of the form

$$\tilde{t} = t, \quad \tilde{u} = f(t, u), \tag{3.17}$$

where f is a linear polynomial.

In order to have an identity field ∂_0, we must have $\Omega_{i*} = \delta_{i0}$. We can extend an Euler field E^M on M to an Euler field E on N by

$$E = E^M + d_u u \partial_*. \tag{3.18}$$

For E to be Euler, we must have

$$E\Omega = (d_0 + d_u)\Omega + \text{ linear terms.} \tag{3.19}$$

3.3.2 Algebra

Let A be a n-dimensional commutative algebra, and $e_1, \ldots, e_n$ a basis for A. We denote the multiplication in A by

$$e_i \cdot e_j = \sum_k C_{ij}{}^k e_k, \tag{3.20}$$

where the $C_{ij}{}^k$ are symmetric in i, j and satisfy the associativity condition

$$\sum_r C_{ij}{}^r C_{rk}{}^l = \sum_r C_{kj}{}^r C_{ri}{}^l, \tag{3.21}$$

for all i, j, k, and l.

Consider the possible rank-1 extensions of A. That is, algebras E which fit into the short exact sequence

$$0 \to I \to E \xrightarrow{\pi} A \to 0, \tag{3.22}$$

where I is a 1-dimensional ideal of E. Let e_* be a generator of this ideal. We chose a section $A \to E$ (that is, a map such that the composition $A \to E \xrightarrow{\pi} A$ is the identity), and denote the images of e_i under this section by the same symbols. The most general multiplication for E which fits into the above exact sequence is

$$\begin{aligned} e_i \cdot e_j &= \sum_k C_{ij}{}^k e_k + F_{ij} e_*, \\ e_i \cdot e_* &= G_i e_*, \\ e_* \cdot e_* &= H e_*. \end{aligned} \tag{3.23}$$

Associativity of the multiplication in E imposes constraints on F_{ij}, G_i, and H. Explicitly, these are

$$\sum_r C_{ij}{}^r F_{rk} + F_{ij} G_k = \sum_r C_{ik}{}^r F_{rj} + F_{jk} G_i, \tag{3.24}$$

$$\sum_r C_{ij}{}^r G_r + F_{ij} H = G_i G_j, \tag{3.25}$$

for all i, j, k. These constraints come from looking at the e_* component in $(e_i \cdot e_j) \cdot e_k = e_i \cdot (e_j \cdot e_k)$, and $(e_i \cdot e_j) \cdot e_* = e_i \cdot (e_j \cdot e_*)$. The solutions to these equations are very different depending on whether or not the H term is 0. We will call H the auxiliary term.

Picking a different section amounts to choosing a new basis of the form

$$\tilde{e}_i = e_i - f_i e_* \quad \text{for } i = 1 \ldots n, \quad \tilde{e}_* = c e_*, \tag{3.26}$$

where f_i and c are constants. The structure constants $\tilde{F}_{ij}, \tilde{G}_i, \tilde{H}$ for this new basis are related to the old ones by

$$\tilde{F}_{ij} = \frac{1}{c}\left(F_{ij} + \sum_r C_{ij}{}^r f_r - f_i G_j - G_i f_j + f_i f_j H\right), \tag{3.27}$$

$$\tilde{G}_i = G_i - f_i H, \tag{3.28}$$

$$\tilde{H} = cH. \tag{3.29}$$

We can consider these structure constants as defining another extension $\widetilde{E}$ with basis $\tilde{e}_i, \tilde{e}_*$. Of course, these algebras are isomorphic via $e_i \mapsto \tilde{e}_i, e_* \mapsto \tilde{e}_*$, and this isomorphism commutes with the projection to A. We therefore consider extensions with structure constants related by (3.27), (3.28), (3.29) to be equivalent.

Extension by a module

Let us consider the case where the auxilary term $H = 0$. The second associativity equation (3.25) says that I is an A-module. Equation (3.24) then says that F_{ij} gives a cocyle in the sense of Hochschild cohomology (i.e. the F_{ij} represent an element of $HH^2(A, I)$, see for example [11]). If we pick a new basis of the form

$$\tilde{e}_i = e_i - f_i e_* \quad \text{for } i = 1 \ldots n, \quad \tilde{e}_* = e_*, \tag{3.30}$$

with f_i some constants, the cocyle with respect to this new basis is

$$\widetilde{F}_{ij} = F_{ij} + \sum_r C_{ij}{}^r f_r - G_i f_j - f_i G_j. \tag{3.31}$$

The difference $\widetilde{F}_{ij} - F_{ij}$ is trivial in Hochschild cohomology.

Auxiliary extension

When the auxiliary term H is nonzero, we will call it an auxiliary extension. From the second associativity equation (3.25) we find that

$$F_{ij} = \frac{(G_i G_j - \sum_r C_{ij}{}^r G_r)}{H}. \tag{3.32}$$

If we plug this into the first associativity equation (3.24), we find that it is automatically satisfied. So an extension is completely determined given any G_i and $H \neq 0$. By a choice of basis as in (3.26) with $f_i = G_i/H$ and $c = 1/H$, i.e.

$$e_i - \frac{G_i}{H} e_* \quad \text{for } i = 1 \ldots n, \quad \frac{1}{H} e_*, \tag{3.33}$$

all of these extensions are equivalent to the extension with $F_{ij} = 0, G_i = 0$, and $H = 1$.

Extension of a semisimple algebra

Now let us consider when A is semisimple. We can take the e_i to be the idempotents of A so that

$$e_i \cdot e_j = \delta_{ij} e_i. \tag{3.34}$$

The associativity equations for the extension become

$$\delta_{ij} F_{jk} + F_{ij} G_k = \delta_{jk} F_{ik} + F_{jk} G_i, \tag{3.35}$$

$$\delta_{ij} G_j + F_{ij} H = G_i G_j. \tag{3.36}$$

Extension of a semisimple algebra by a module

Let A be a semisimple algebra. As seen previously, if $H = 0$, then I is an A-module. Since $e_1 + \ldots + e_n$ is the identity in A, we much have $\sum_i G_i = 1$. By the second associativity equation (3.36), we must have $G_i(G_j - \delta_{ij}) = 0$, so that G_i is either 1 or 0, and there can only be 1 non-zero G_i. Without loss of generality, take this $i = 1$. Using the other associativity equation (3.35), we conclude that $F_{ii} = -F_{i1}$ for $i \neq 1$, and $F_{ij} = 0$ for all i, j such that $i \neq j, i \neq 1, j \neq 1$. All these extensions are equivalent to the extension with all $F_{ij} = 0$ by the choice of basis

$$e_i - F_{1i}e_* \quad \text{for } i = 1 \ldots n, \quad e_*. \tag{3.37}$$

Auxiliary extension of a semisimple algebra

If H is nonzero, then the associativity equations (3.35), (3.36) are solved by

$$F_{ij} = \frac{G_i(G_j - \delta_{ij})}{H}. \tag{3.38}$$

By a change of basis with $f_i = \left(\sum_j G_j - 1\right)/(nH)$ (in the notation of equation (3.26)), we can normalize the section so that $\sum_i G_i = 1$.

All of these extensions are isomorphic to the semisimple algebra of dimension $n + 1$ by the choice of basis

$$e_i - \frac{G_i}{H}e_* \quad \text{for } i = 1 \ldots n, \quad \frac{1}{H}e_*. \tag{3.39}$$

The normalization condition $\sum_i G_i = 1$ just says that the identity in A is sent to the identity in E by the section.

3.3.3 Potentiality

Comparing the multiplication tables (3.23) and (3.14), we will examine what the algebraic cases previously considered mean for the potential.

Extension by a module

Let us consider the case when $H = 0$, which corresponds to $\Omega_{**} = 0$. This implies

$$\Omega = A + Bu, \tag{3.40}$$

where A, B are pullbacks of functions on M (i.e. they are only functions of the t variables). Collecting coefficients of powers of u, the associativity equations give the following constraints on A and B:

$$\sum_\delta \Phi_{\alpha\beta}{}^\delta A_{\delta\gamma} + A_{\alpha\beta} B_\gamma = (\alpha \leftrightarrow \gamma), \tag{3.41}$$

$$\sum_\delta \Phi_{\alpha\beta}{}^\delta B_{\delta\gamma} + B_{\alpha\beta} B_\gamma = (\alpha \leftrightarrow \gamma), \tag{3.42}$$

$$\sum_\delta \Phi_{\alpha\beta}{}^\delta B_\delta = B_\alpha B_\beta. \tag{3.43}$$

By differentiating (3.43), we find

$$\Phi_{\alpha\beta}{}^\delta B_{\delta\gamma} = B_{\alpha\gamma} B_\beta + B_\alpha B_{\beta\gamma} - \Phi_{\alpha\beta\gamma}{}^\delta B_\delta. \tag{3.44}$$

If we substitute this into the left hand side of (3.42), we find an expression which is symmetric in α, β, γ, so (3.42) follows from (3.43). Algebraically speaking, at each tangent space, B gives a module structure, and A gives a cocycle in the sense of Hochschild cohomology.

We can consider a change of coordinates that commutes with π and also preserves the module structure:

$$\tilde{t}_\alpha = t_\alpha, \tag{3.45}$$

$$\tilde{u} = u + f(t), \tag{3.46}$$

where $f(t)$ is a linear polynomial. The coordinate vector fields are now

$$\tilde{\partial}_\alpha = \partial_\alpha - f_\alpha \partial_*, \tag{3.47}$$

$$\tilde{\partial}_* = \partial_*, \tag{3.48}$$

where the derivatives f_α are constant. In the new coordinates,

$$\tilde{\Omega}_{\alpha\beta} = \tilde{\partial}_\alpha \circ \tilde{\partial}_\beta = \ A_{\alpha\beta} + \Phi^\gamma_{\alpha\beta} f_\gamma - f_\alpha B_\beta - B_\alpha f_\beta + u B_{\alpha\beta}, \tag{3.49}$$

so

$$\tilde{A}_{\alpha\beta} = A_{\alpha\beta} + \sum_\delta \Phi_{\alpha\beta}{}^\delta f_\delta - B_\alpha f_\beta - f_\alpha B_\beta, \tag{3.50}$$

and

$$\tilde{B}_{\alpha\beta} = B_{\alpha\beta}. \tag{3.51}$$

We see that $\tilde{A}_{\alpha\beta} - A_{\alpha\beta}$ is trivial in Hochschild cohomology. The extended potential in the new coordinates is

$$\tilde{\Omega} = A + \sum_\gamma \Phi^\gamma f_\gamma + uB. \tag{3.52}$$

We summarize what we have just shown in a proposition.

Proposition 10. *Let M be a Frob manifold with potential Φ, and flat coordinates t^α. Let A and B be functions on M satisfying*

$$\sum_\delta \Phi_{\alpha\beta}{}^\delta B_\delta = B_\alpha B_\beta, \tag{3.53}$$

and

$$\sum_\delta \Phi_{\alpha\beta}{}^\delta A_{\delta\gamma} + A_{\alpha\beta} B_\gamma = (\alpha \leftrightarrow \gamma). \tag{3.54}$$

Then there exists a rank-1 extension N of M with extended flat coordinate u and extended potential $\Omega = A + Bu$.

At each point of N, the tangent space has the structure of an extension by a module generated by $\frac{\partial}{\partial u}$ with $A_{\alpha\beta}$ a 2-cocycle in the sense of Hochschild cohomology.

After a change of coordinates

$$\tilde{t}_\alpha = t_\alpha, \tag{3.55}$$

$$\tilde{u} = u + f(t), \tag{3.56}$$

where $f(t)$ is a linear polynomial, the cocycles $\tilde{A}_{\alpha\beta}$ and $A_{\alpha\beta}$ are cohomologous.

Auxiliary extension

Now consider the case when the auxiliary term H, which corresponds to Ω_{**}, is nonzero.

We want there to exist an extended potential Ω such that

$$\begin{aligned} \Omega_{\alpha\beta} &= F_{\alpha\beta}, \\ \Omega_{*\alpha} &= G_\alpha, \\ \Omega_{**} &= H. \end{aligned} \tag{3.57}$$

For such a potential to exist, we need

$$\partial_\gamma \Omega_{\alpha\beta} = \partial_\alpha \Omega_{\gamma\beta}, \tag{3.58}$$

$$\partial_* \Omega_{\alpha\beta} = \partial_\alpha \Omega_{*\beta}, \tag{3.59}$$

$$\partial_* \Omega_{*\alpha} = \partial_\alpha \Omega_{**}. \tag{3.60}$$

Suppose we are given $\theta = \Omega_*$. The last potential condition (3.60) holds automatically. From (3.32), the extended multiplication is associative if

$$\Omega_{\alpha\beta} = \frac{1}{\theta_*}\left(\theta_\alpha\theta_\beta - \Phi_{\alpha\beta}{}^{\gamma}\theta_\gamma\right), \tag{3.61}$$

where for notational simplicity, we are implying summation over repeated upper and lower indices.

Using the associativity condition (3.61), the second potential condition (3.59) is explicitly

$$\theta_{\alpha\beta} = \partial_*\Omega_{\alpha\beta} = \frac{1}{\theta_*}\left(\theta_{\alpha*}\theta_\beta + \theta_\alpha\theta_{\beta*} - \Phi_{\alpha\beta}{}^{\gamma}\theta_{\gamma*}\right) - \frac{\theta_{**}}{(\theta_*)^2}\left(\theta_\alpha\theta_\beta - \Phi_{\alpha\beta}{}^{\gamma}\theta_\gamma\right). \tag{3.62}$$

If this holds, then the first potential condition (3.58) holds as well:

$$\begin{aligned}
\partial_\gamma\Omega_{\alpha\beta} &= \frac{1}{\theta_*}\left(\theta_{\alpha\gamma}\theta_\beta + \theta_\alpha\theta_{\beta\gamma} - \Phi_{\alpha\beta\gamma}{}^{\delta}\theta_\delta - \Phi_{\alpha\beta}{}^{\delta}\theta_{\delta\gamma}\right) - \frac{\theta_{*\gamma}}{(\theta_*)^2}\left(\theta_\alpha\theta_\beta - \Phi_{\alpha\beta}{}^{\delta}\theta_\delta\right) \\
&= \frac{\theta_\beta}{\theta_*}\left[\frac{1}{\theta_*}\left(\theta_{\alpha*}\theta_\gamma + \theta_\alpha\theta_{\gamma*} - \Phi_{\alpha\gamma}{}^{\delta}\theta_{\delta*}\right) - \frac{\theta_{**}}{(\theta_*)^2}\left(\theta_\alpha\theta_\gamma - \Phi_{\alpha\gamma}{}^{\delta}\theta_\delta\right)\right] \\
&\quad + \frac{\theta_\alpha}{\theta_*}\left[\frac{1}{\theta_*}\left(\theta_{\beta*}\theta_\gamma + \theta_\beta\theta_{\gamma*} - \Phi_{\beta\gamma}{}^{\delta}\theta_{\delta*}\right) - \frac{\theta_{**}}{(\theta_*)^2}\left(\theta_\beta\theta_\gamma - \Phi_{\beta\gamma}{}^{\delta}\theta_\delta\right)\right] \\
&\quad - \frac{\Phi_{\alpha\beta}{}^{\delta}}{\theta_*}\left[\frac{1}{\theta_*}\left(\theta_{\delta*}\theta_\gamma + \theta_\delta\theta_{\gamma*} - \Phi_{\delta\gamma}{}^{\epsilon}\theta_{\epsilon*}\right) - \frac{\theta_{**}}{(\theta_*)^2}\left(\theta_\delta\theta_\gamma - \Phi_{\delta\gamma}{}^{\epsilon}\theta_\epsilon\right)\right] \\
&\quad - \frac{\Phi_{\alpha\beta\gamma}{}^{\delta}\theta_\delta}{\theta_*} - \frac{\theta_{*\gamma}}{(\theta_*)^2}\left(\theta_\alpha\theta_\beta - \Phi_{\alpha\beta}{}^{\delta}\theta_\delta\right) \\
&= \frac{1}{(\theta_*)^2}\left[\left(\theta_{\alpha*}\theta_\beta\theta_\gamma + \ldots\right) - \left(\Phi_{\alpha\beta}{}^{\delta}\theta_{\delta*}\theta_\gamma + \ldots\right) + \Phi_{\alpha\beta}{}^{\delta}\Phi_{\delta\gamma}{}^{\epsilon}\theta_{\epsilon*}\right] \\
&\quad - \frac{\theta_{**}}{(\theta_*)^3}\left[2\theta_\alpha\theta_\beta\theta_\gamma - \left(\Phi_{\alpha\beta}{}^{\delta}\theta_\delta\theta_\gamma + \ldots\right) + \Phi_{\alpha\beta}{}^{\delta}\Phi_{\delta\gamma}{}^{\epsilon}\theta_\epsilon\right] \\
&\quad - \frac{\Phi_{\alpha\beta\gamma}{}^{\delta}\theta_\delta}{\theta_*}.
\end{aligned} \tag{3.63}$$

The first equality comes from using (3.62) to replace $\theta_{\alpha\beta}$, then we collect terms. The dots indicate permutations of α, β, γ. From the last expression, we see that $\partial_\gamma \Omega_{\alpha\beta}$ is symmetric in α, β, γ, hence the first potential condition holds.

We have shown the following theorem:

Theorem 3. *Let M be a n-dimensional Frob manifold with flat coordinates t^α and vector potential Φ. Let $\theta = \theta(t_0, \ldots, t_{n-1}, u)$ be a function satisfying $\theta_* \neq 0$ and*

$$\theta_{\alpha\beta} = \partial_* \left[\frac{1}{\theta_*} \left(\theta_\alpha \theta_\beta - \Phi_{\alpha\beta}{}^\gamma \theta_\gamma \right) \right]. \tag{3.64}$$

Then there exists a unique rank-1 extension of M with flat coordinates t^α, u and extended potential Ω with $\Omega_* = \theta$.

Extension of semisimple Frob manifolds

Let y^i be canonical coordinates on M, with $e_i = \frac{\partial}{\partial_i}$, and let u be a flat coordinate for the fibre, with $e_* = \frac{\partial}{\partial_u}$. The multiplication in N is given by

$$\begin{aligned} e_i \circ e_j &= \delta_{ij} e_i + F_{ij} e_*, \\ e_i \circ e_* &= G_i e_*, \\ e_* \circ e_* &= H e_*. \end{aligned} \tag{3.65}$$

Proposition 11. *The pre-Frob manifold with multiplication defined by* (3.65) *is potential if and only if*

$$\sum_r F_{ir}\Gamma^r_{jk} + \partial_i F_{jk} = \sum_r F_{jl}\Gamma^r_{ik} + \partial_j F_{ik}, \tag{3.66}$$

$$\sum_r G_r\Gamma^r_{jk} + \partial_* F_{jk} = \partial_j G_k, \tag{3.67}$$

$$\partial_i G_j = \partial_j G_i, \tag{3.68}$$

$$\partial_* G_j = \partial_j H, \tag{3.69}$$

where Γ *are the connection coefficients in canonical coordinates on* M*.*

Proof. By Theorem 1, we need the λ-linear terms in the curvature of the structure connection to vanish. Computing this, we arrive at

$$e_i \circ (\nabla_j e_k) + \nabla_i(e_j \circ e_k) = (i \leftrightarrow j), \tag{3.70}$$

$$e_* \circ (\nabla_j e_k) + \nabla_*(e_j \circ e_k) = (* \leftrightarrow j), \tag{3.71}$$

$$e_i \circ (\nabla_j e_*) + \nabla_i(e_j \circ e_*) = (i \leftrightarrow j), \tag{3.72}$$

$$e_* \circ (\nabla_j e_*) + \nabla_i(e_* \circ e_*) = (* \leftrightarrow j). \tag{3.73}$$

The e_i components of the first equation are equal since M is potential. There are only e_* components for the last three equations. These equations, along with the e_* component of the first equation give the lemma. □

The last two equations (3.68), (3.69) give the existence of a function $\theta = \theta(y^i, u)$ such that $\partial_i\theta = G_i$, and $\partial_*\theta = H$. The F_{ij} are not in general derivatives.

If we define the 1-forms $\mu_i = \sum_j F_{ij} dy^j$, then the equations (3.66), (3.67) can be restated as

$$\nabla_i \mu_j = \nabla_j \mu_i, \tag{3.74}$$

$$\nabla_* \mu_j = \nabla_j d\theta. \tag{3.75}$$

Extension of semisimple Frob manifolds by a module

When $H = \theta_* = 0$ in (3.65), we have the algebraic structure of an extension by a module. Referring to the results in 3.3.2 on modules over semisimple algebras, we conclude that:

- All the G_i are zero except for, say, $i = 1$. So we have that $\theta = y^1$.
- $F_{ij} = 0$ except for $F_{1i} = F_{i1} = -F_{ii}$.

From equation (3.67) we find that:

$$F_{ij} = E_{ij} - u\Gamma^1_{ij}, \tag{3.76}$$

where the components E_{ij} do not depend on u. Additionally, the E_{ij} must satisfy the cocyle condition (3.66).

Comparing with (3.40), since $y^1 = \theta = \Omega_*$ we see that $\Omega = A + uy^1$. Note that $B = y^i$ are the solutions to (3.43). The differential dy^i is a homomorphism between the algebra T_pM and $\mathbb{C}$.

Auxiliary extension of a semisimple Frob manifold

When $H = \theta_* \neq 0$ in (3.65), we have by (3.38) that

$$F_{ij} = \frac{G_i\left(G_j - \delta_{ij}\right)}{H} = \frac{\theta_i\left(\theta_j - \delta_{ij}\right)}{\theta_*}. \tag{3.77}$$

This ensures our extension is associative. For the extension to be potential, we have the following proposition, which is essentially a restatement of Theorem 3 in canonical coordinates.

Proposition 12. *Let M be a semisimple Frob manifold with canonical coordinates y^i. Let N be the associative pre-Frob extension determined by a function $\theta(y^i, u)$. That is,*

$$H = \theta_*, \quad G_i = \theta_i, \quad F_{ij} = \frac{\theta_i\left(\theta_j - \delta_{ij}\right)}{\theta_*}, \tag{3.78}$$

with multiplication given by (3.65)*. Then N is potential (hence Frob) if and only if*

$$\nabla_i d\theta = \sum_j \partial_* \left(\frac{\theta_i\left(\theta_j - \delta_{ij}\right)}{\theta_*}\right) dy^j. \tag{3.79}$$

More explicitly, this condition is

$$\begin{aligned} \theta_{jk} &= \frac{1}{2}\left(\theta_j \frac{\partial_j \phi_k}{\phi_k} + \theta_k \frac{\partial_k \phi_j}{\phi_j}\right) + \partial_*\left(\frac{\theta_j \theta_k}{\theta_*}\right), \qquad j \neq k, \\ \theta_{jj} &= \frac{1}{2}\left(\theta_j \frac{\partial_j \phi_j}{\phi_j} - \sum_{r \neq j} \theta_r \frac{\partial_j \phi_r}{\phi_r}\right) + \partial_*\left(\frac{\theta_j(\theta_j - 1)}{\theta_*}\right), \end{aligned} \tag{3.80}$$

where ϕ_i are the connection potentials for M.

Proof. The conditions (3.80) are (3.67) written more explicitly by using the properties of the connection coefficients in canonical coordinates (Proposition 2).

Since we have specified $\theta = \Omega_*$, we can use Theorem (3). The condition (3.67) is (3.59) rewritten in canonical coordinates (to see this, start with equation (3.71) and change to flat coordinates). So by the theorem, equation (3.66), which is (3.58) rewritten in canonical coordinates, follows. □

Algebraically, we know that an auxiliary extension of a semisimple algebra is semisimple. It turns out from the potential conditions that we also have canonical coordinates on the extended Frob manifold, i.e. the idempotent fields which are determined algebraically are actually coordinate vector fields.

Proposition 13. *Let M be a semisimple Frob manifold and N an extension with $\theta_* = \Omega_{**} \neq 0$. Then N is semisimple.*

Proof. Let $w^i = y^i, x = \theta(y^i, u)$. These are canonical coordinates since $\theta_i = G_i, \theta_* = H$, and therefore

$$\frac{\partial}{\partial w^i} = e_i - \frac{G_i}{H} e_*, \quad \frac{\partial}{\partial x} = \frac{1}{H} e_*, \tag{3.81}$$

which by (3.39) is a semisimple basis for the extended algebra T_pN at each point p. □

Now that we know an auxiliary extension of a semisimple Frob manifold is semisimple, we examine what the connection potentials are.

Proposition 14. *Let M be a semisimple Frob manifold with canonical coordinates y^i. Let N be the semisimple Frob extension determined by a function θ with $\theta_* \neq 0$, and $w^i = y^i$, $x = \theta$ the canonical coordinates on N. The connection potentials for N are the pullbacks of the connection potentials for M along with the new connection potential for the x direction given by $\phi_x = \theta_*^{-2}$.*

Proof. Using that $\partial_x = \frac{1}{\theta_*}\partial_*$, we compute

$$\nabla_{\partial_x}\partial_x = \left(\partial_x \frac{1}{\theta_*}\right)\partial_* = \left(-\partial_x \log\theta_*\right)\partial_x, \tag{3.82}$$

$$\nabla_{\partial w^i}\partial_x = \left(\frac{\partial}{\partial w^i}\frac{1}{\theta_*}\right)\partial_* = \left(-\frac{\partial}{\partial w^i}\log\theta_*\right)\partial_x, \tag{3.83}$$

which shows that

$$\Gamma^x_{xx} = \partial_x \left[\frac{1}{2} \log \theta_*^{-2} \right], \tag{3.84}$$

$$\Gamma^x_{xi} = -\Gamma^x_{ii} = \frac{\partial}{\partial w^i} \left[\frac{1}{2} \log \theta_*^{-2} \right]. \tag{3.85}$$

and additionally,

$$\Gamma^i_{xx} = -\Gamma^i_{xi} = 0. \tag{3.86}$$

Next we compute

$$\begin{aligned}
\nabla_{\partial w^i} \frac{\partial}{\partial w^i} &= \nabla_{\partial y^i - (\theta_i/\theta_*)\partial_*} \left(\frac{\partial}{\partial y^j} - \frac{\theta_j}{\theta_*} \partial_* \right) \\
&= \sum_k \Gamma^k_{ij} \frac{\partial}{\partial y^k} + \left(\frac{\theta_i}{\theta_*} \partial_* \left[\frac{\theta_j}{\theta_*} \right] \right) \partial_* \\
&= \sum_k \Gamma^k_{ij} \frac{\partial}{\partial w^k} + \left(\sum_k \theta_k \Gamma^k_{ij} + \partial_* \left(\frac{\theta_i \theta_j}{\theta_*} \right) - \theta_{ij} \right) \partial_x.
\end{aligned} \tag{3.87}$$

For $i \neq j$, we have $F_{ij} = \frac{\theta_i \theta_j}{\theta_*}$, and component of ∂_x in the last line is 0 by the potential condition (3.67). This is equivalent to the fact that $\Gamma^x_{ij} = 0$, which is automatically true by (2.21). □

In other words, with $\partial_i = \frac{\partial}{\partial w^i}, \partial_x = \frac{\partial}{\partial x}$, we have

$$\begin{aligned}
\nabla_i \partial_j &= \Gamma^i_{ij} \partial_i + \Gamma^j_{ij} \partial_j, \quad i \neq j, \\
\nabla_i \partial_i &= \sum_k \Gamma^k_{ii} \partial_k + \Gamma^x_{ii} \partial_x, \\
\nabla_i \partial_x &= \nabla_x \partial_i = \Gamma^x_{xi} \partial_x, \\
\nabla_x \partial_x &= \Gamma^x_{xx} \partial_x,
\end{aligned} \tag{3.88}$$

where Γ_{ij}^k are the pullbacks of the connection coefficients in canonical coordinates on M, and

$$\begin{aligned} -\Gamma_{ii}^x = \Gamma_{ix}^x = \Gamma_{xi}^x &= \partial_i \left[\frac{1}{2} \log(\theta_*^{-2}) \right], \\ \Gamma_{xx}^x &= \partial_x \left[\frac{1}{2} \log(\theta_*^{-2}) \right]. \end{aligned} \tag{3.89}$$

In the presence of an Euler field E with d_* the degree of u, $E\theta_* = (d_0 - d_*)\theta_*$, since $\theta_* = \Omega_{**}$. Therefore, since $\phi_x = \theta_*^{-2}$,

$$E\phi_x = (D_x - 2d_0)\phi_x, \tag{3.90}$$

with $D_x = 2d_*$, in the notation of (2.73).

3.3.4 Summary

We summarize extensions of a semisimple Frob manifold M with flat coordinates t^α, vector potential Φ, and canonical coordinates y^i. We let u be the extended flat coordinate, and we have an extended Euler field $E = E^M + d_* u \partial_*$.

Extension by a module

- Extended potential $\Omega = A + uB$ where A, B are functions of t, so that $\Omega_{**} = 0$.
- $B = y^i$ for some i, equivalently $B_\alpha B_\beta = \sum_\gamma {\Phi_{\alpha\beta}}^\gamma \partial_\gamma$.
- A satisfies the cocyle condition

$$\sum_\delta {\Phi_{\alpha\beta}}^\delta A_{\delta\gamma} + A_{\alpha\beta} B_\gamma = \sum_\delta {\Phi_{\gamma\beta}}^\delta A_{\delta\alpha} + A_{\gamma\beta} B_\alpha. \tag{3.91}$$

- $e = \frac{\partial}{\partial t^0}$ is automatically an identity field.
- $E\Omega = (d_0 + d_*)\Omega +$ linear terms, so that E is Euler.

Auxiliary extension

- $\Omega_{**} \neq 0$ so that the auxiliary term in the multiplication is nonzero.

- Ω satisfies open WDVV equations. Alternatively, specify $\theta = \Omega_*$, where θ satisfies the rank-1 potential equation in Theorem (3).
- $e\theta = 1$ or $\Omega_{*i} = \delta_{i0}$, so that $e = \frac{\partial}{\partial t^0}$ is the identity field.
- $E\theta = d_0\theta$ up to a constant, or $E\Omega = (d_0 + d_*)\Omega$ up to linear terms, so that E is Euler.
- The extended Frob manifold N is semisimple with canonical coordinates

$$w^i = y^i, \quad x = \theta. \tag{3.92}$$

- Connection potential for the x direction $\phi_x = \frac{1}{(\theta_*)^2}$. $E\phi_x = (2d_* - 2d_0)\phi_x$.

3.4 Extended Isomonodromy

Let $(M, \nabla, \circ, E)$ be a Frob manifold with Euler field, and $\mathbb{P}^1_\lambda$ the projective line with coordinate λ. Recall from 2.7 the second structure connection on the pullback of TM to $M \times \mathbb{P}^1_\lambda$. It can be written as

$$\check{\nabla} = \nabla + \sum_i A_i(y^1, \ldots, y^n) \frac{d(\lambda - y^i)}{\lambda - y^i}, \tag{3.93}$$

where y^i are the canonical coordinates on M. The A_i are operators $TM \to TM$ given by

$$A_i = AP_i, \tag{3.94}$$

where

$$A = -\left(s\,\mathrm{id} + \nabla E\right), \tag{3.95}$$

with $s = (2 - a_i - D_i)/2$ (a_i satisfies the symmetry in proposition 8), and

$$P_i = e_i \circ \quad , \tag{3.96}$$

which satisfies $\sum_i P_i = \mathrm{id}$.

Let $T \cong \mathbb{C}^n$ be the vector space of flat sections of ∇. A_i can be viewed as an element of $\operatorname{End} T$ which depends on y^i. By restricting $\check{\nabla}$ to $\mathbb{P}^1_\lambda$, we get a $\mathbb{C}^n$ vector bundle over $\mathbb{P}^1_\lambda$ with connection

$$\sum_i A_i(y^1, \ldots, y^n) \frac{d\lambda}{\lambda - y^i}. \tag{3.97}$$

This is a Fuchsian system with poles at y^i. By the flatness of $\check{\nabla}$, varying the position of the poles is isomonodromic.

Taking flat homogeneous coordinate vector fields ∂_α as a basis for T, we can write A, P_i, and A_i as matrices. In components we get

$$(A)^\beta_\alpha = -\left(s + d_\alpha\right) \delta^\beta_\alpha, \tag{3.98}$$

$$(P_i)^\beta_\alpha = \frac{\partial t^\beta}{\partial y^i} \frac{\partial y^i}{\partial t^\alpha}. \tag{3.99}$$

Now we consider a rank-1 auxiliary extension of M

$$0 \to I \to \widetilde{M} \to M \to 0. \tag{3.100}$$

From Proposition 13, $\widetilde{M}$ is semisimple with canonical coordinates $\widetilde{y}^1 = y^1, \ldots, \widetilde{y}^n = y^n$, and $x = \theta(y^i, u)$.

From Proposition 14, the connection potentials on $\widetilde{M}$ are the connection potentials from M, $\phi_1, \ldots, \phi_n$ along with a new connection potential ϕ_x. Then we have a symmetry of the Darboux-Egoroff equations on $\tilde{M}$ (see Proposition 8) with a_i the same as for M, and $a_x = 2(1 - d_* - s)$. This produces a second structure connection with the same s as for M.

The second structure connection for $\widetilde{M}$ can be written in the form

$$\tilde{\widetilde{\nabla}} = \widetilde{\nabla} + \sum_i \widetilde{A}_i(\tilde{y}^1, \ldots, \tilde{y}^n) \frac{d(\lambda - \tilde{y}^i)}{\lambda - \tilde{y}^i} + \tilde{A}_x \frac{d(\lambda - x)}{\lambda - x}. \tag{3.101}$$

Proposition 15. *The matrices for* (3.101) *are given by the following:*

$$\widetilde{A} = \begin{pmatrix} A & 0 \\ 0 & \tilde{a} \end{pmatrix}, \ \textit{where } \tilde{a} = -(s + d_*), \tag{3.102}$$

where s is the constant $(2 - a_i - D_i)/2$ with a_i satisfying the symmetry in proposition 8,

$$\widetilde{P}_i = \begin{pmatrix} P_i & 0 \\ v_i & 0 \end{pmatrix}, \ \textit{with } v_i = -\frac{\theta_i}{\theta_*} \left(\frac{\partial y^i}{\partial t^0}, \ldots, \frac{\partial y^i}{\partial t^{n-1}} \right), \tag{3.103}$$

$$\widetilde{P}_x = \begin{pmatrix} 0 & 0 \\ v_x & 1 \end{pmatrix}, \ \textit{with } v_x = \frac{1}{\theta_*} \left(\frac{\partial \theta}{\partial t^0}, \ldots, \frac{\partial \theta}{\partial t^{n-1}} \right), \tag{3.104}$$

where A and P_i are the matrices associated with M.

Proof. Since $\nabla E(\partial_*) = \nabla_* E = d_* \partial_*$, this gives the last component $\tilde{a}$ in $\widetilde{A}$.

From (3.81) we have that

$$\frac{\partial}{\partial \tilde{y}^i} = \frac{\partial}{\partial y^i} - \frac{\theta_i}{\theta_*} \frac{\partial}{\partial u}, \tag{3.105}$$

and

$$\frac{\partial}{\partial x} = \frac{1}{\theta_*} \frac{\partial}{\partial u}. \tag{3.106}$$

The flat coordinates on $\widetilde{M}$ are $\tilde{t}^\alpha = t^\alpha$ along with the extended flat coordinate u.

Using this, we compute:

$$\frac{\partial \tilde{t}^\beta}{\partial \tilde{y}^i}\frac{\partial \tilde{y}^i}{\partial \tilde{t}^\alpha} = \left(\left[\frac{\partial}{\partial y^i} - \frac{\theta_i}{\theta_*}\frac{\partial}{\partial u}\right] t^\beta\right)\frac{\partial y^i}{\partial t^\alpha} = \frac{\partial t^\beta}{\partial y^i}\frac{\partial y^i}{\partial t^\alpha}, \tag{3.107}$$

$$\frac{\partial u}{\partial \tilde{y}^i}\frac{\partial \tilde{y}^i}{\partial \tilde{t}^\alpha} = \left(\left[\frac{\partial}{\partial y^i} - \frac{\theta_i}{\theta_*}\frac{\partial}{\partial u}\right] u\right)\frac{\partial y^i}{\partial t^\alpha} = -\frac{\theta_i}{\theta_*}\frac{\partial y^i}{\partial t^\alpha}, \tag{3.108}$$

$$\frac{\partial \tilde{t}^\beta}{\partial \tilde{y}^i}\frac{\partial \tilde{y}^i}{\partial u} = 0, \tag{3.109}$$

$$\frac{\partial u}{\partial \tilde{y}^i}\frac{\partial \tilde{y}^i}{\partial u} = 0. \tag{3.110}$$

The vanishing of the last two equations is due to $\frac{\partial \tilde{y}^i}{\partial u} = 0$. Furthermore, we have:

$$\frac{\partial \tilde{t}^\beta}{\partial x}\frac{\partial x}{\partial \tilde{t}^\alpha} = 0, \tag{3.111}$$

$$\frac{\partial \tilde{t}^\beta}{\partial x}\frac{\partial x}{\partial u} = 0. \tag{3.112}$$

since $\frac{\partial \tilde{t}^\beta}{\partial x} = 0$. Lastly, we have:

$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial \tilde{t}^\alpha} = \left(\frac{1}{\theta_*}\frac{\partial}{\partial u}u\right)\frac{\partial \theta}{\partial t^\alpha} = \frac{1}{\theta_*}\frac{\partial \theta}{\partial t^\alpha}, \tag{3.113}$$

$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial u} = \left(\frac{1}{\theta_*}\frac{\partial}{\partial u}u\right)\frac{\partial \theta}{\partial u} = \frac{1}{\theta_*}\theta_* = 1. \tag{3.114}$$

This gives us the components of $\widetilde{P}_i$ and $\widetilde{P}_x$. □

Remark 2. *When M is Frob, we can take $a_i = 1$ for all i in Proposition 8. Then $s = (2 - 1 - D)/2$, where D is the conformal factor for the compatible metric.*

In Chapter 4, we will see an extension of quantum cohomology with $D = 2 - r$ and $d_* = (1 - r)/2$, where r is the dimension of a symplectic manifold. Then we see *that* $\tilde{a}$ *vanishes since*

$$-(s + d_*) = (D-1)/2 - d_* = (1-r)/2 - (1-r)/2 = 0. \tag{3.115}$$

CHAPTER 4
Examples

In this chapter, we present examples of extensions starting from Frobenius manifolds.

We start by giving a case by case classification for extensions of 1- and 2- dimensional Frobenius manifolds, depending on the spectrum of the Euler field E.

We prove the existence of an extension of the Frobenius manifold for the A_n singularity (which can also be viewed as the space of polynomials). This proof relies on Theorem 3 of the previous chapter.

We also give some examples for quantum cohomology. In the case of $\mathbb{P}^2$, and $\mathbb{P}^1 \times \mathbb{P}^1$, these examples are covered in [6]. We conjecture the existence of extensions in the other case, based on the fact that we've computed coefficients which solve the OWDVV equations up to a certain degree without any inconsistencies.

4.0.1 Extensions of 1-dimensional Frobenius Manifolds

Up to conformal transformation, there is one 1-dimensional Frobenius manifold with flat identity field and Euler field. It has Euler field $t_0\partial_0$ and potential

$$\Phi(t_0) = \frac{1}{3}t_0^3. \tag{4.1}$$

We want to classify Frob extensions of this Frobenius manifold having flat identity and Euler field. We consider an extended Euler field

$$E = t_0 \partial_0 + d_* u \partial_*, \quad d_* \neq 0, \tag{4.2}$$

or, when $d_* = 0$,

$$E = t_0 \partial_0 + s \partial_*. \tag{4.3}$$

Using the identity constraint, the extended potential can be written as

$$\Omega(t_0, u) = t_0 u + \omega(u). \tag{4.4}$$

Note that Ω is a component of a vector, and by rescaling $u \mapsto \lambda u + c$, we induce the transformation

$$\Omega(t_0, u) \mapsto \lambda\, \Omega \left(t_0, \frac{u - c}{\lambda} \right). \tag{4.5}$$

We can use this rescaling to fix the s term in the Euler field if we want.

The quasihomogeneity constraint on ω is

$$E\omega = (1 + d_*)\omega + \text{ linear polynomial in } u. \tag{4.6}$$

In the case that $d_* \neq 0$, we can kill a linear polynomial $a + bu$ on the right by adding the linear term $a + \frac{b}{d_*+1} u$ to ω provided that $d_* \neq -1$. When $d_* = -1$, we cannot kill a constant term, so we have $E\omega = c$. When $d_* = 0$, we can always kill the linear polynomial (regardless of what s is).

The associativity equations for the extension hold trivially.

By solving the quasihomogeneity constraint with $E = t_0\partial_0 + d_* u \partial_*$, the possible extended potentials are

$$\Omega(t_0, u) = t_0 u + u^{\frac{1+d_*}{d_*}}, \quad d_* \neq 0, -1, \tag{4.7}$$

$$\Omega(t_0, u) = t_0 u - c \log u, \quad d_* = -1. \tag{4.8}$$

We can fix the parameter c by a transformation $u \mapsto \frac{u}{c}$.

When the Euler field is $E = t_0\partial_0 + s\partial_*$, the quasihomogeneity constraint is solved by

$$\Omega(t_0, u) = t_0 u + e^{\frac{u}{s}}, \quad d_* = 0. \tag{4.9}$$

We can fix the parameter s by a a transformation $u \mapsto \frac{1}{s}u$.

4.0.2 Extensions of 2-dimensional Frobenius Manifolds

Now we will classify extensions of 2-dimensional Frobenius manifolds.

From [3], [8], the classification of 2-dimensional Frobenius manifolds is as follows. For the Euler field

$$E = t_0\partial_0 + (1-r)t_1\partial_1, \quad r \neq 1, \tag{4.10}$$

the potentials are

$$\begin{aligned} \Phi(t_0, t_1) &= \frac{1}{2}t_0^2 t_1 + t_1^k, \quad k = \frac{3-r}{1-r}, \quad r \neq -1, 1, 3, \\ \Phi(t_0, t_1) &= \frac{1}{2}t_0^2 t_1 + t_1^2 \log t_1, \quad r = -1, \\ \Phi(t_0, t_1) &= \frac{1}{2}t_0^2 t_1 + \log t_1, \quad r = 3. \end{aligned} \tag{4.11}$$

When $r = 1$, the Euler field is $E = t_0\partial_0 + 2\partial_1$, and the potential is

$$\Phi(t_0, t_1) = \frac{1}{2}t_0^2 t_1 + e^{t_1}. \tag{4.12}$$

The extension is determined by an extended potential $\Omega(t_0, t_1, u)$. We extend the Euler field by adding either a term $d_* u \partial_*$ when $d_* \neq 0$, or $s\partial_*$ when $d_* = 0$. Note that we can rescale u to fix s so it is not really an extra parameter.

Associativity of the extension has only one constraint

$$\Phi_{111} + \Omega_{11}\Omega_{**} = (\Omega_{1*})^2 . \tag{4.13}$$

The identity constraint $\Omega_{*i} = \delta_{i0}$ means we can write

$$\Omega(t_0, t_1, u) = t_0 u + \omega(t_1, u). \tag{4.14}$$

The quasihomogeneity condition is $E\omega = (d_* + 1)\omega + l$, where l is a linear polynomial. We can try to kill l by adding a linear polynomial to ω. Generically this is possible, however for certain values of r, d_*, we cannot normalize l to 0.

After normalizing $E\omega$, we can find a solution for ω, in terms of an arbitrary function $f(z)$. The associativity constraint then imposes an ODE on f. We list the cases below.

1. $r \neq -d_*$, $r \neq 1$, and $d_* \neq -1, 0$:

   The quasihomogeneity condition can be normalized to

$$E\omega = (1 + d_*)\omega. \tag{4.15}$$

   The solution to the quasihomogeneity condition is

$$\omega = t_1^{\frac{1+d_*}{1-r}} f\left(\frac{u}{t_1^{\frac{d_*}{1-r}}}\right). \tag{4.16}$$

The ODEs for associativity are

$$d_*(1-r)(d_*+r-1)zf'f'' + (r-1)(d_*+1)(d_*+r)ff'' + (1-r)f'^2 + 2(r-3)(r+1) = 0, \quad r \neq -1, 3, \quad (4.17)$$

$$d_*(d_*-2)zf'f'' - (d_*^2-1)ff'' + f'^2 - 8 = 0, \quad r = -1, \quad (4.18)$$

$$d_*(d_*+2)zf'f'' - (d_*+3)(d_*+1)ff'' + f'^2 - 8 = 0, \quad r = 3. \quad (4.19)$$

2. $r = -d_*$, but $r \neq 1$, and $d_* \neq -1, 0$:

We cannot kill a t_1 term, so the quasihomogeneity condition can only be normalized to

$$E\omega = (1-r)\omega + ct_1. \quad (4.20)$$

By rescaling u we can fix c. The solution to the quasihomogeneity condition is

$$\omega = \frac{t_1 \log\left[t_1(1-r)\right] c}{1-r} + t_1 f\left(t_1^{\frac{r}{1-r}} u\right). \quad (4.21)$$

The ODEs for associativity are

$$r(1-r)zf'f'' - c(1-r)^2 f'' + (1-r)f'^2 + 2(r-3)(r+1) = 0, \quad r \neq -1, 3, \quad (4.22)$$

$$zf'f'' + 2cf'' - f'^2 + 8 = 0, \quad r = -1, \quad (4.23)$$

$$3zf'f'' + 2cf'' + f'^2 - 8 = 0, \quad r = 3. \quad (4.24)$$

3. $r = 1$, but $d \neq -1, 0$:

The quasihomogeneity condition can be normalized to

$$E\omega = (1 + d_*)\omega. \tag{4.25}$$

The solution to the quasihomogeneity condition is

$$\omega = e^{\frac{1+d_*}{2}t_1} f\left(\frac{u}{e^{\frac{d_*}{2}t_1}}\right). \tag{4.26}$$

The ODE for associativity is

$$d_*^2 z f' f'' - (1 + d_*)^2 f f'' + f'^2 - 4 = 0. \tag{4.27}$$

4. $r = 1$, $d_* = -1$:

   We cannot kill a constant term, so we can only normalize the quasihomogeneity condition to

$$E\omega = c. \tag{4.28}$$

   By rescaling u we can fix c. The solution to the homogeneity condition is

$$\omega = \frac{c}{2}t_1 + f\left(e^{\frac{t_1}{2}}u\right). \tag{4.29}$$

   The ODE for associativity is

$$z f' f'' + f'^2 - 4 = 0. \tag{4.30}$$

5. $r = 1$, $d_* = 0$:

   The quasihomogeneity condition can be normalized to

$$E\omega = \omega. \tag{4.31}$$

The solution to the homogeneity condition is

$$\omega = e^{\frac{t_1}{2}} f\left(u - \frac{s}{2}t_1\right). \tag{4.32}$$

The ODE for associativity is

$$ff'' - f'^2 + 4 = 0. \tag{4.33}$$

6. $d_* = -1$, but $r \neq 1$:

The quasihomogeneity condition can be normalized to

$$E\omega = c. \tag{4.34}$$

By rescaling u, we can fix c. The solution to the quasihomogeneity condition is

$$\omega = \frac{\log\left[t_1(1-r)\right] c}{1-r} + f\left(t_1^{\frac{1}{1-r}} u\right). \tag{4.35}$$

The ODEs for associativity are

$$(r-2)(r-1)zf'f'' + (1-r)f'^2 + c(1-r)^2 f'' \\ + 2(r-3)(r+1) = 0, \quad r \neq -1, 3, \tag{4.36}$$

$$3zf'f'' + 2cf'' + f'^2 - 8 = 0, \quad r = -1, \tag{4.37}$$

$$zf'f'' + 2cf'' - f'^2 + 8 = 0, \quad r = 3. \tag{4.38}$$

Remark 3. *There is a duality here with the case $r = -d_*$. Namely, the ODE corresponding to $(d_*, r) = (-1, -1)$ is the same as the ODE for $(d_*, r) = (-3, 3)$, and likewise for $(d_*, r) = (-1, 3)$ and $(1, -1)$.*

7. $d_* = 0$, but $r \neq 0, 1$:

The quasihomogeneity condition can be normalized to

$$E\omega = \omega. \tag{4.39}$$

The solution to the quasihomogeneity condition is

$$\omega = t_1^{\frac{1}{1-r}} f\left(u - s\frac{\log t_1}{1-r}\right). \tag{4.40}$$

The ODEs for associativity are

$$s(1-r)^2 f' f'' + r(1-r) f f'' - (1-r) f'^2 - 2(r-3)(r+1) = 0, \quad r \neq -1, 3, \tag{4.41}$$

$$2s f' f'' - f f'' - f'^2 + 8 = 0, \quad r = -1, \tag{4.42}$$

$$2s f' f'' - 3 f f'' + f'^2 - 8 = 0, \quad r = 3. \tag{4.43}$$

8. $d_* = 0$, $r = 0$:

We cannot kill a t_1 term, so the quasihomogeneity condition can only be normalized to

$$E\omega = \omega + c t_1. \tag{4.44}$$

By rescaling u, we can fix c, but in general it would not be possible to fix both c and s.

The solution to the quasihomogeneity condition is

$$\omega = c\left(t_1 \log t_1\right) + t_1 f\left(u - s \log t_1\right). \tag{4.45}$$

The ODE for associativity is

$$sf'f'' + cf'' - f'^2 + 6 = 0. \tag{4.46}$$

4.0.3 Extensions of the space of polynomials (A_n singularity)

Consider the space of polynomials of the form

$$W(x;a) = x^{n+1} + a_{n-1}x^{n-1} + \ldots + a_0. \tag{4.47}$$

This is an n-dimensional space M with coordinates $a_0, \ldots, a_{n-1}$. It can also be thought of as the unfolding of the A_n singularity $C[x]/x^{n+1}$.

M has the structure of a Frobenius manifold, [8], which we will describe next. We identify the tangent space with the Milnor ring

$$T_aM \cong C[x]/W'(x;a), \quad \frac{\partial}{\partial a^i} \mapsto \frac{\partial}{\partial a^i}W(x;a), \tag{4.48}$$

where the prime indicates the derivative with respect to x. This induces an algebra structure on the tangent space. It has an inner product given by the residue pairing

$$g\left(\frac{\partial}{\partial a^i}, \frac{\partial}{\partial a^j}\right) = -(n+1)\,\mathrm{res}_{x=\infty}\left[\frac{\partial W}{\partial a^i}\frac{\partial W}{\partial a^j}/W'\right]. \tag{4.49}$$

The factor $n+1$ is for normalization reasons. This metric turns out to be flat (see [8]), and has flat coordinates t^i, which to leading order are

$$a_i = t_i + \ldots. \tag{4.50}$$

There is also an Euler field which gives degrees $d_i = n+1-i$. From now on, we will work in flat coordinates, so subscripts will indicated derivatives with respect to t^i.

There is an Extension of this Frobenius manifold which arises in a very natural way. By the Euclidean division of polynomials, there exists unique polynomials b_{ij}, r_{ij} in x, such that

$$W_i W_j = b_{ij} W' + r_{ij}, \tag{4.51}$$

where the degree of r_{ij} is less than n. In fact, r_{ij} is the product in the Milnor ring.

For the extended multiplication, we identify ∂_* with W'. We have the multiplication table

$$\begin{aligned} W_i W_j &= b_{ij} W' + r_{ij}, \\ W_i W' &= W_i W', \\ W' W' &= W' W'. \end{aligned} \tag{4.52}$$

This multiplication table looks like an extension if we let x play the role of the extended variable u. The question becomes, does there exist an $\Omega(x,t)$ such that $\Omega_{ij} = b_{ij}$, $\Omega_{*i} = W_i$ and $\Omega_{**} = W'$. From the last two conditions, we see that $\Omega_* = W$. Since we are specifying Ω_*, by Theorem 3, we need to check

$$W_{ij} = \frac{d}{dx}\left[\frac{1}{W'}\left(W_i W_j - r_{ij}\right)\right] = \frac{d}{dx} b_{ij} \tag{4.53}$$

Proposition 16.

$$\frac{d}{dx} b_{ij}(x;t) = W_{ij}(x;t). \tag{4.54}$$

By Theorem 3, this implies the existence of a rank-1 extension of the Frobenius manifold which is the space of polynomials, with $\Omega_* = W$.

Proof. The proof is by a direct calculation using the explicity formulas given in [2]. We have that

$$W_0 = 1, \tag{4.55}$$

$$W_1 = x, \tag{4.56}$$

and for $i > 1$,

$$W_i = \sum \frac{k!}{k_2! \cdots k_n!} \left(\frac{t_2}{n+1}\right)^{k_2} \cdots \left(\frac{t_{n-1}}{n+1}\right)^{k_{n-1}} x^{k_n}, \tag{4.57}$$

where $k = k_2 + \dots k_n$, and the sum is over k_i such that

$$\sum_{j=2}^{n} d_j k_j = i, \tag{4.58}$$

with $d_j = n + 1 - j$ being the degree of t_j for $j < n$ and $d_n = 1$ is the degree of x. Note that for degree reasons, only t_j with $j \geq n + 1 - i$ will appear with nonzero exponent. Similarly,

$$W' = (n+1) \sum \frac{k!}{k_2! \cdots k_n!} \left(\frac{t_2}{n+1}\right)^{k_2} \cdots \left(\frac{t_{n-1}}{n+1}\right)^{k_{n-1}} x^{k_n}, \tag{4.59}$$

where the sum is over k_i such that $\sum_{j=2}^{n} d_j k_j = n$.

Now by multiplying W_i and W_j, and looking at the terms of degree greater or equal to n in x, we can deduce that when $i + j < n$, $b_{ij} = 0$, and when $i + j \geq n$,

$$b_{ij} = \frac{1}{n+1} \sum \frac{k!}{k_2! \cdots k_n!} \left(\frac{t_2}{n+1}\right)^{k_2} \cdots \left(\frac{t_{n-1}}{n+1}\right)^{k_{n-1}} x^{k_n}, \tag{4.60}$$

where the sum is over k_i such that $\sum_{j=2}^{n} d_j k_n = i + j - n$. Note in particular that when $i + j = n$, b_{ij} is the constant $\frac{1}{n+1}$.

On the other hand, when $i+j \leq n$, then $W_{ij} = 0$ since the degree of W is $n+1$, and $d_i + d_j = 2n+2-i-j > n+1$. When $i+j > n$, we take the derivative of W_i with respect to t_j, and compare with the derivative of the formula for b_{ij}, and see that they are equal. □

Now we explicitly list some extended potentials. We also list the polynomial and Frobenius potential for reference.

$n = 2$.

$$\begin{aligned} W(x) &= t_0 + t_1 x + x^3, \\ \Omega(t,u) &= t_0 u + t_1 \frac{u^2}{2} + \frac{u^4}{4} + \frac{t_1^2}{6}, \\ \Phi(t) &= \frac{1}{2} t_0^2 t_1 - \frac{t_1^4}{72}. \end{aligned} \tag{4.61}$$

$n = 3$.

$$\begin{aligned} W(x) &= \left(t_0 + \frac{t_2^2}{8}\right) + t_1 x + t_2 x^2 + x^4, \\ \Omega(t,u) &= \left(t_0 + \frac{t_2^2}{8}\right) u + t_1 \frac{u^2}{2} + t_2 \frac{u^3}{3} + \frac{u^5}{5} + \frac{t_1 t_2}{4}, \\ \Phi(t) &= \frac{1}{2}\left(t_0^2 t_2 + t_0 t_1^2\right) - \frac{t_1^2 t_2^2}{16} + \frac{t_2^5}{960}. \end{aligned} \tag{4.62}$$

$n = 4$.

$$\begin{aligned} W(x) &= \left(t_0 + \frac{t_2 t_3}{5}\right) + \left(t_1 + \frac{t_3^2}{5}\right) x + t_2 x^2 + t_3 x^3 + x^5, \\ \Omega(t,u) &= \left(t_0 + \frac{t_2 t_3}{5}\right) u + \left(t_1 + \frac{t_3^2}{5}\right) \frac{u^2}{2} + t_2 \frac{u^3}{3} + t_3 \frac{u^4}{4} + \frac{u^6}{6} \\ &\quad + \frac{t_1 t_3}{5} + \frac{t_2^2}{10} + \frac{t_3^3}{150}, \\ \Phi(t) &= \frac{t_0^2 t_3}{2} + t_0 t_1 t_2 + \frac{t_1^3}{6} - \frac{t_1^2 t_3^2}{20} - \frac{t_1 t_2^2 t_3}{10} - \frac{t_2^4}{60} + \frac{t_2^2 t_3^3}{150} - \frac{t_3^6}{15000}. \end{aligned} \tag{4.63}$$

$n = 5$.

$$
\begin{aligned}
W(x) &= \left(t_0 + \frac{t_2 t_4}{6} + \frac{t_3^2}{12} + \frac{t_4^3}{108}\right) + \left(t_1 + \frac{t_3 t_4}{3}\right) x + \left(t_2 + \frac{t_4^2}{4}\right) x^2 \\
&\quad + t_3 x^3 + t_4 x^4 + x^6, \\
\Omega(t,u) &= \left(t_0 + \frac{t_2 t_4}{6} + \frac{t_3^2}{12} + \frac{t_4^3}{108}\right) u + \left(t_1 + \frac{t_3 t_4}{3}\right) \frac{u^2}{2} + \left(t_2 + \frac{t_4^2}{4}\right) \frac{u^3}{3} \\
&\quad + t_3 \frac{u^4}{4} + t_4 \frac{u^5}{5} + \frac{u^7}{7} + \frac{t_1 t_4}{6} + \frac{t_2 t_3}{6} + \frac{t_3 t_4^2}{72}, \\
\Phi(t) &= \frac{1}{2}\left(t_0^2 t_4 + t_0 t_2^2\right) + t_0 t_1 t_3 + \frac{t_1^2 t_2}{2} - \frac{t_1^2 t_4^2}{24} - \frac{t_1 t_2 t_3 t_4}{6} - \frac{t_1 t_3^3}{36} \\
&\quad - \frac{t_2^3 t_4}{36} - \frac{t_2^2 t_3^2}{12} + \frac{t_2^2 t_4^3}{216} + \frac{t_2 t_3^2 t_4^2}{72} + \frac{t_3^4 t_4}{216} - \frac{t_3^2 t_4^4}{1728} + \frac{t_4^7}{272160}.
\end{aligned}
\tag{4.64}
$$

$n = 6$.

$$
\begin{aligned}
W(x) &= \left(t_0 + \frac{t_2 t_5}{7} + \frac{t_3 t_4}{7} + \frac{t_4 t_5^2}{49}\right) + \left(t_1 + \frac{2}{7} t_3 t_5 + \frac{t_4^2}{7} + \frac{t_5^3}{49}\right) x \\
&\quad + \left(t_2 + \frac{3}{7} t_4 t_5\right) x^2 + \left(t_3 + \frac{2}{7} t_5^2\right) x^3 + t_4 x^4 + t_5 x^5 + x^7, \\
\Omega(t,u) &= \left(t_0 + \frac{t_2 t_5}{7} + \frac{t_3 t_4}{7} + \frac{t_4 t_5^2}{49}\right) u + \left(t_1 + \frac{2}{7} t_3 t_5 + \frac{t_4^2}{7} + \frac{t_5^3}{49}\right) \frac{u^2}{2} \\
&\quad + \left(t_2 + \frac{3}{7} t_4 t_5\right) \frac{u^3}{3} + \left(t_3 + \frac{2}{7} t_5^2\right) \frac{u^4}{4} + t_4 \frac{u^5}{5} + t_5 \frac{u^6}{6} + \frac{u^8}{8} \\
&\quad + \frac{t_1 t_5}{7} + \frac{t_2 t_4}{7} + \frac{t_3^2}{14} + \frac{t_3 t_5^2}{98} + \frac{t_4^2 t_5}{98} + \frac{t_5^4}{4116}, \\
\Phi(t) &= \frac{t_0^2 t_5}{2} + t_0 t_1 t_4 + t_0 t_2 t_3 \\
&\qquad + \frac{t_1^2 t_3}{2} + \frac{t_1 t_2^2}{2} - \frac{t_1^2 t_5^2}{28} - \frac{t_1 t_2 t_4 t_5}{7} - \frac{t_1 t_3^2 t_5}{14} - \frac{t_1 t_3 t_4^2}{14} \\
&\qquad - \frac{t_2^2 t_3 t_5}{14} - \frac{t_2^2 t_4^2}{14} + \frac{t_2^2 t_5^3}{294} + \frac{t_2 t_3 t_4 t_5^2}{49} - \frac{t_2 t_3^2 t_4}{7} + \frac{t_2 t_4^3 t_5}{147} \\
&\qquad - \frac{t_3^4}{56} + \frac{t_3^3 t_5^2}{294} + \frac{t_3^2 t_4^2 t_5}{49} - \frac{t_3^2 t_5^4}{2744} + \frac{t_3 t_4^4}{196} - \frac{t_3 t_4^2 t_5^3}{686} \\
&\qquad - \frac{t_4^4 t_5^2}{1372} + \frac{t_4^2 t_5^5}{24010} - \frac{t_5^8}{5647152}.
\end{aligned}
\tag{4.65}
$$

4.1 Extensions of Quantum Cohomology

4.1.1 Gromov-Witten invariants

The quantum cohomology of a closed symplectic manifold X is a deformation of the ordinary cohomology ring $H^*(X)$ (with say coefficients in $\mathbb{C}$) [7], [8]. This deformation has the structure of a Frobenius manifold.

Let $n+1$ be the dimension of $H^{2*}(X)$, and let t_i be coordinates for $H^{2*}(X)$. That is,

$$(t_0, \ldots, t_n) = t_0\Delta_0 + \ldots t_n\Delta_n, \tag{4.66}$$

where $\Delta_i \in H^{2p_i}(X)$. The inner product is given by

$$g(\partial_i, \partial_j) = \int_X \Delta_i \wedge \Delta_j, \tag{4.67}$$

where integration is done by selecting representative forms in the cohomology class. These integrals are constant, so t_i are flat coordinates. We pick t_0 to be the coordinate for the identity.

The Frobenius potential is given by

$$\begin{aligned}\Phi = & \text{ classical terms } + \\ & \sum_{d,a}\left[\left(\prod_{\alpha=1}^{r} e^{d_\alpha t_a}\right)\left(\prod_{\beta=r+1}^{n} \frac{t_\beta^{a_\beta}}{a_\beta!}\right) N(d_1, \ldots, d_r; a_{r+1}, \ldots, a_n)\right]. \end{aligned} \tag{4.68}$$

The classical terms arise from the ordinary cohomology ring. We refer to the other terms in the potential as *quantum* terms.

The coefficients $N(d_1, \ldots, d_r; a_{r+1}, \ldots, a_n)$ are the Gromov-Witten invariants of X. Morally, they count the number of holomorphic curves of genus 0 passing through

a_i constraints given by the Poincaré duals of Δ_i, for each i. The image of the curve is $d = (d_1, \ldots, d_r) \in H_2(X, \mathbb{Z})/torsion$ where $d_i = \Delta_i[d]$ for $\Delta_i \in H^2(X)$.

The Gromov-Witten invariants are defined by integration over $\overline{\mathcal{M}}_{0,m}(X, d)$, the Moduli space of stable m-marked maps of genus 0 into X of class $d \in H_2(X, \mathbb{Z})/torsion$, where m is the total number of constraints, $m = \sum_{i=r+1}^{n} a_i$.

These Gromov-Witten invariants are zero unless the codimension given by the constraints is equal to the dimension of this moduli space. That is, unless

$$\sum_{i=r+1}^{n} a_i p_i = \dim_{\mathbb{C}} \overline{\mathcal{M}}_{0,m}(X, d) = c_1(TX)[d] + \dim_{\mathbb{C}} X + m - 3. \tag{4.69}$$

From this dimension constraint, there is an Euler field given by

$$E = \sum_{i:p_i \neq 1} (1 - p_i) t_i \partial_i + \sum_{i:p_i = 1} r_i \partial_i, \tag{4.70}$$

where $c_1(TX) = \sum_i r_i \Delta_i$, which makes the potential Φ quasihomogeneous of degree

$$3 - \dim_{\mathbb{C}} X. \tag{4.71}$$

The associativiy of this Frobenius manifold is a non-trivial result which follows from pulling back a relation in the moduli space $\overline{\mathcal{M}}_{0,4}$ of genus 0 curves with 4 marked points. See [7], [12].

We seek an extended potential of the form

$$\begin{aligned} \Omega = {} & t_0 u + \\ & 2 \sum_{d,a,k} \left[\left(\prod_{\alpha=1}^{r} e^{\frac{1}{2} d_\alpha t_a} \right) \left(\prod_{\beta=r+1}^{n} \frac{\left(\frac{t_\beta}{2}\right)^{a_\beta}}{a_\beta !} \right) \frac{u^k}{k!} M(d_1, \ldots, d_r; a_{r+1}, \ldots, a_n; k) \right]. \end{aligned} \tag{4.72}$$

In certain situations, the coefficients $M(d_1, \ldots, d_r; a_{r+1}, \ldots, a_n; k)$ will be the open Gromov-Witten invariants (up to some factors). Morally, they count maps of open discs with boundary on a Lagrangian L, with a_i interior constrains given by the Poincaré duals of Δ_i and k point-constraints on the boundary.

Although there are technical difficulties, one would like to define these open Gromov-Witten invariants by integration over the moduli space $\overline{\mathcal{M}}_{0,m,k}(X, L, d)$ of stable open discs into X with boundary on L with m interior marked points and k boundary marked points and image $d \in H_2(X, L)$ [10].

The coefficients are zero unless the codimension given by the constraints is equal to the dimension of this moduli space. That is

$$2 \sum_{i=r+1}^{n} a_i p_i + k \dim_{\mathbb{C}} X = \dim_{\mathbb{R}} \overline{\mathcal{M}}_{0,m,k}(X, L, d) = \mu(d) + \dim_{\mathbb{C}} X + 2n + k - 3. \quad (4.73)$$

Here, $\dim_{\mathbb{C}} X$ is the codimension in L of a point, $\mu(d)$ is the Maslov class. If the Lagrangian is real, we can use complex conjugation to double the disk to get a genus-0 curve $\hat{d} \in H_2(X)$, then $\mu(d) = c_1(TX)[\hat{d}]$.

Using this dimensions constraint, and giving u degree $\frac{1}{2}(1 - dim_{\mathbb{C}} X)$, this makes Ω quasi-homogeneous of degree

$$\frac{(3 - \dim_{\mathbb{C}} X)}{2}. \quad (4.74)$$

The potential Φ is sometimes referred to as the closed potential, the extended potential Ω as the open potential, and the corresponding invariants as closed or open Gromov-Witten invariants, respectively.

4.1.2 Small quantum cohomology

There is a variation of the quantum cohomology, called the small quantum cohomology, where we let $t_0 = t_{r+1}, \dots, t_n = 0$, and $q_1 = e^{t_1}, \dots, q_r = e^{t_r}$ (where r is the rank of $H^2(X)$). This ring only depends on terms that are most cubic in the non-divisor variables. These are sometimes called the 3-point invariants.

We can also consider a small extended quantum cohomology by additionally setting $u = 0$. This ring will have also have terms like $q_i^{1/2}$.

4.1.3 Partial reconstruction of the extension

In the quantum cohomology, the first reconstruction theorem by Kontsevich and Manin [7] says that the Gromov-Witten invariants can be computed recursively using the WDVV equations starting from the 3-point invariants whenever $H^*(X)$ is generated by $H^2(X)$.

It appears as though the open WDVV equations can be used to compute the coefficients of Ω when $H^*(X)$ is generated by $H^2(X)$, although in some cases it not enough to only specify the terms in the small extended ring, and it is not always obvious how to recursively solve for certain coefficients in general. We do however, have a partial reconstruction proposition.

We will use the following terminology. The coefficients $M(d_1, \dots, d_r; a_{r+1}, \dots, a_n; k)$ in Ω will be called open coefficients. The sum $a_{r+1} + \dots + a_n$ will be called the number of interior marks. The number k will be called the number of boundary marks. We will say that $d < d'$, or that the degree d is less than d' if $d_i \leq d'_i$ for all i, and at for at least one j, $d_j < d'_j$.

Proposition 17. *Suppose $H^*(X)$ is generated by $H^2(X)$, and suppose Ω of the form (4.72) satisfies the open WDVV equations. An open coefficient with at least 2 interior marks, or at least 1 interior and 1 boundary mark can be computed recursively from the open WDVV equations in terms of coefficients with lower degree, coefficients with fewer marked points, and the coefficients of the closed potential Φ.*

Proof. First, let us consider an open coefficient with at least 2 interior marks:

$$M(d_1, \ldots, d_r; \ldots, a_b + 1, \ldots, a_c + 1, \ldots; k). \tag{4.75}$$

We will call this M for short. Since $H^*(X)$ is generated by divisors, we can write $\Delta_b = \Delta_i \wedge \Delta_j$ with $\Delta_j \in H^2(X)$. Now, consider the following associativity equation:

$$\sum_\alpha \Phi_{ij}{}^\alpha \Omega_{\alpha c} + \Omega_{ij}\Omega_{*c} = \sum_\alpha \Phi_{ic}{}^\alpha \Omega_{\alpha j} + \Omega_{ic}\Omega_{*j}. \tag{4.76}$$

Notice that

$$\Phi_{ij}{}^b = 1 + \text{quantum terms}, \tag{4.77}$$

since $\Delta_i \wedge \Delta_j = \Delta_b$. But for $\alpha \neq b$, $\Phi_{ij}{}^\alpha$ contains only quantum terms.

Now in equation (4.76) we look at the coefficient of

$$\left(\prod_{\alpha=1}^{r} e^{\frac{1}{2}d_\alpha t_a}\right)\left(\prod_{\beta=r+1}^{n} \frac{t_\beta^{a_b}}{a_\beta!}\right)\frac{u^k}{k!}. \tag{4.78}$$

We get

$$M + \ldots = \frac{d_j}{2} M' + \ldots, \tag{4.79}$$

where

$$M' = M(d; \ldots, a_{b'} + 1, \ldots; k), \tag{4.80}$$

with $\Delta_{b'} = \Delta_i \wedge \Delta_k$. The factor of $\frac{d_j}{2}$ come from taking a t_j derivative of $e^{\frac{1}{2}d_j t_j}$. The coefficient M' has one less interior mark. All the other terms have degrees less than d. To see this, notice that all other terms are quadratic in the coefficients (either a product of a closed and open coefficient, or a product of two open coefficients. The sum of the degrees of these quadratic terms must be d (the degree from the closed coefficient will be summed with weight by 2). Since all these coefficients are quantum, they have nonzero degree, thus they must be strictly less than d.

Next, let us consider an open coefficient with at least 1 interior mark and 1 boundary mark:

$$M(d_1, \ldots, d_r; \ldots, a_b + 1, \ldots; k + 1). \tag{4.81}$$

Again we will call this M for short. We can write $\Delta_b = \Delta_i \wedge \Delta_j$ with $\Delta_j \in H^2(X)$. Consider the following associativity equation:

$$\sum_{\alpha} \Phi_{ij}{}^{\alpha} \Omega_{\alpha *} + \Omega_{ij}\Omega_{**} = \Omega_{i*}\Omega_{j*}. \tag{4.82}$$

Once again, we look at the coefficient of (4.78). We get

$$M + \ldots = \ldots, \tag{4.83}$$

where all the other terms have degrees less than d. $\square$

Missing in this partial reconstruction proposition is how to compute open coefficients which are only assumed to have at least 2 boundary marks.

We will now show some specific examples.

4.1.4 $\mathbb{P}^n$

The closed potential is

$$\Phi = \frac{t_0^2 t_n}{2} + \frac{1}{2} t_0 \sum_{i=1}^{n} t_i t_{n-i} + \sum_{d>0,a} e^{dt_1} \frac{t_2^{a_2}}{a_2!} \cdots \frac{t_n^{a_n}}{a_n!} N(d; a_1, \ldots, a_n). \tag{4.84}$$

The open potential is

$$\Omega = t_0 u + 2 \sum_{d>0} e^{\frac{1}{2}dt_1} \frac{\left(\frac{t_2}{2}\right)^{a_2}}{a_2!} \cdots \frac{\left(\frac{t_n}{2}\right)^{a_n}}{a_n!} \frac{u^k}{k!} M(d; a_2, \ldots, a_n; k). \tag{4.85}$$

The Euler field is

$$E = t^0 \partial_0 + (n+1)\partial_1 + (-1)t^2 \partial_2 + \ldots + (1-n)t^n \partial_n + \frac{1-n}{2} u \partial_*, \tag{4.86}$$

which makes Φ and Ω quasihomogeneous of degrees $3-n$, and $(3-n)/2$, respectively.

Small quantum cohomology

The small quantum cohomology ring is the Frobenius manifold at the point $t_2 = \cdots = t_n = 0$, and $q = e^{t_1}$ is arbitrary.

The small quantum cohomology ring of $\mathbb{P}^n$ has the relations

$$\begin{aligned} \partial_i \circ \partial_j &= \partial_{i+j}, \quad i+j < n+1, \\ \partial_1 \circ \partial_n &= q\partial_0 \end{aligned} \tag{4.87}$$

This is a graded algebra, where the degree of ∂_i is $1-i$, the degree of q is $n+1$, and the multiplication has degree 1. The implicit factor of 1 in front of q is the Gromov-Witten invariant which counts the number of lines through 2 points.

This ring is isomorphic to

$$Q = \mathbb{C}[x, q]/\langle x^{n+1} = q \rangle. \tag{4.88}$$

The extended small ring has the relations

$$\begin{aligned}
&\partial_i \circ \partial_j = \partial_{i+j}, \quad i+j < n+1, \\
&\partial_1 \circ \partial_n = q\partial_0 + Aq^{1/2}\partial_*, \\
&\partial_i \circ \partial_* = 0, \\
&\partial_* \circ \partial_* = Bq^{1/2}\partial_*.
\end{aligned} \tag{4.89}$$

Here, $A = \frac{1}{2}M(1;0,\ldots,1;0)$, $B = 2M(1;0,\ldots,0;2)$. Associativity $(\partial_1 \circ \partial_n) \circ \partial_* = \partial_1 \circ (\partial_n \circ \partial_*)$ imposes the constraint $1 + AB = 0$ which, up to rescaling of ∂_*, has the solution

$$M(1;0,\ldots,0;2) = 1, \quad M(1;0,\ldots,1;0) = -1. \tag{4.90}$$

Reconstruction

In addition to the partial reconstruction shown in Proposition 17, we can compute a coefficient M with at least two boundary marks

$$M = M(d;a_2,\ldots,a_n;k+2) \tag{4.91}$$

using the OWDVV equations.

Consider the equation

$$\sum_\alpha {\Phi_{1n}}^\alpha \Omega_{\alpha *} + \Omega_{1n}\Omega_{**} = \Omega_{n*}\Omega_{1*}. \tag{4.92}$$

Let us look at the coefficient of $e^{dt_1/2}t_2^{a_2}\ldots t_n^{a_n}u^k$. From the product

$$\Omega_{1n}\Omega_{**} = (\ldots + M(1;0,\ldots,1;0) + \ldots)\left(\ldots + e^{dt_1/2}t_2^{a_2}\ldots t_n^{a_n}u^k M + \ldots\right), \tag{4.93}$$

we get a term $M(1;0,\dots,1;0)M(d;a_2,\dots,a_n;k+2)$. The term $M(1;0,\dots,1;0)$ is known from the small ring.

So $M = M(d;a_2,\dots,a_n;k+2)$ can be determined from open coefficients of less degree, or coefficients with possibly same degree, but fewer boundary marks. In the special case where $a_2 = \cdots = a_n = 0, k = 0$, we get the relation from the small ring

$$1 + M(1;0,\dots,1;0)M(d;0,\dots,0,\dots;2) = 0. \tag{4.94}$$

4.1.5 $\mathbb{P}^1 \times \mathbb{P}^1$

The closed potential is

$$\Phi = \frac{1}{2}t_0^2 t_3 + t_0 t_1 t_2 + \sum e^{d_1 t_1 + d_2 t_2} \frac{t_3^{a_3}}{a_3!} N(d_1, d_2; a_3). \tag{4.95}$$

The open potential is

$$\Omega = t_0 u + \sum e^{\frac{1}{2}d_1 t_1 + \frac{1}{2}d_2 t_2} \frac{\left(\frac{t_3}{2}\right)^{a_3}}{a_3!} \frac{u^k}{k!} M(d_1, d_2; a_3; k). \tag{4.96}$$

The Euler field is

$$E = t_0\partial_0 + 2\partial_1 + 2\partial_2 - t_3\partial_3 - \frac{1}{2}u\partial_*, \tag{4.97}$$

which makes Φ and Ω quasihomogeneous of degrees 1, and 1/2, respectively.

Small quantum cohomology

The small quantum cohomology ring is the Frobenius manifold at the point $t_3 = 0$ and where $e^{t_1} = q_1, e^{t_2} = q_2$ are arbitrary.

$$\begin{aligned} &\partial_1 \circ \partial_2 = \partial_3, \\ &\partial_3 \circ \partial_1 = q_2 \partial_2, \quad \partial_3 \circ \partial_2 = q_1 \partial_1, \\ &\partial_1 \circ \partial_1 = q_2 \partial_0, \quad \partial_2 \circ \partial_2 = q_1 \partial_0, \\ &\partial_3 \circ \partial_3 = q_1 q_2 \partial_0. \end{aligned} \tag{4.98}$$

The extended small ring has the relations

$$\begin{aligned} &\partial_1 \circ \partial_* = A q_2^{\frac{1}{2}} \partial_*, \quad \partial_2 \circ \partial_* = B q_1^{\frac{1}{2}} \partial_*, \\ &\partial_3 \circ \partial_* = (C q_a + D q_1^{\frac{1}{2}} q_2^{\frac{1}{2}} + E q_2) \partial_*, \\ &\partial_* \circ \partial_* = 0, \end{aligned} \tag{4.99}$$

where A, B, C, D, E are to be determined.

The associativity condition

$$(\partial_1 \circ \partial_1) \circ \partial_* = q_2 \partial_* = \partial_1 \circ (\partial_1 \circ \partial_*) = A^2 q_2 \partial_* \tag{4.100}$$

gives $A^2 = 1$. Similarly, $B^2 = 1$. The other associativity condition $(\partial_1 \circ \partial_2) \circ \partial_* = \partial_3 \circ \partial_* = \partial_1 \circ (\partial_2 \circ \partial_*)$ gives

$$AB = D, \quad C = 0, \quad E = 0. \tag{4.101}$$

The term $M(1,1;0;3)$, which has the fewest boundary marks out of the coefficients with only boundary marks, is not constrained by the OWDVV equations and does not appear here in the small ring.

4.1.6 $G(2,4)$

The closed potential is

$$\Phi = \frac{1}{2}t_0t_2^2 + \frac{1}{2}t_0t_3^2 + t_0t_1t_4 + \frac{1}{2}t_0^2t_5 + \frac{1}{2}t_1^2t_2 + \frac{1}{2}t_1^2t_3 + \\ + \sum_{d>0,a} e^{dt_1}\frac{t_2^{a_2}}{a_2!}\cdots\frac{t_5^{a_5}}{a_5!}N(d;a_1,\ldots,a_5). \quad (4.102)$$

The open potential is

$$\Omega = t_0u + 2\sum_{d>0} e^{\frac{1}{2}dt_1}\frac{\left(\frac{t_2}{2}\right)^{a_2}}{a_2!}\cdots\frac{\left(\frac{t_n}{2}\right)^{a_5}}{a_5!}\frac{u^k}{k!}M(d;a_2,\ldots,a_n;k). \quad (4.103)$$

Small quantum cohomology

The small quantum cohomology ring is the Frobenius manifold at the point $t_2 = \cdots = t_5 = 0$, and $q = e^{t_1}$ is arbitrary.

$$\begin{aligned} \partial_1 \circ \partial_1 &= \partial_2 + \partial_3, \\ \partial_1 \circ \partial_2 &= \partial_4 = \partial_1 \circ \partial_3, \\ \partial_2 \circ \partial_2 &= \partial_5 = \partial_3 \circ \partial_3, \\ \partial_2 \circ \partial_3 &= q\partial_0, \\ \partial_4 \circ \partial_4 &= q\partial_2 + q\partial_3, \\ \partial_4 \circ \partial_5 &= q\partial_4, \\ \partial_5 \circ \partial_5 &= q^2\partial_0. \end{aligned} \quad (4.104)$$

The extended small ring has the relations

$$\begin{aligned}
\partial_* \circ \partial_1 &= 0, \\
\partial_* \circ \partial_2 &= q^{1/2} A \partial_*, \\
\partial_* \circ \partial_3 &= q^{1/2} B \partial_*, \\
\partial_* \circ \partial_4 &= 0, \\
\partial_* \circ \partial_5 &= q C \partial_*, \\
\partial_* \circ \partial_* &= 0,
\end{aligned} \tag{4.105}$$

where A, B, C are to be determined.

The associativity conditions we need to check are the following. First of all,

$$\partial_* \circ (\partial_2 \circ \partial_2) = qC\partial_* = (\partial_* \circ \partial_2) \circ \partial_2 = qA^2 \partial_* \tag{4.106}$$

gives $A^2 = C$. Similarly, replacing 2 with 3, we also get $B^2 = C$. Furthermore

$$(\partial_1 \circ \partial_1) \circ \partial_* = q^{1/2}(A + B)\partial_* = \partial_1 \circ (\partial_1 \circ \partial_*) = 0. \tag{4.107}$$

so $A + B = 0$. Finally

$$(\partial_2 \circ \partial_3) \circ \partial_* = q\partial_* = \partial_2 \circ (\partial_3 \circ \partial_*) = qAB\partial_*, \tag{4.108}$$

so $AB = 1$. The solutions are $A = i, B = -i, C = -1$, or the complex conjugate.

4.2 Tables

We now present tables of open (extended) coefficients we computed using the open WDVV equations.

In the case of $\mathbb{P}^n$, for odd n, the numbers with no boundary marks can be compared with those in the tables given in [4], and they are the same up to sign. However, there are nonzero numbers in our table which according to [4], should be zero.

We also compute for $G(2,4)$, where interestingly the coefficients can be not only negative, but also imaginary.

Except for the case of $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, the existence of an extension is only conjectural, based on the fact that we can find a consistent solution the OWDVV equations up to the degree which we were able to calculate. The cases $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ are blowups of $\mathbb{P}^2$, so they are proved to be solutions of the OWDVV equations by references given in [6].

4.2.1 Extended coefficients for $\mathbb{P}^2$

$M(1;0;2)$	1
$M(1;1;0)$	-1
$M(2;0;5)$	-1
$M(2;1;3)$	1
$M(2;2;1)$	-1
$M(3;0;8)$	8
$M(3;1;6)$	-6
$M(3;2;4)$	4
$M(3;3;2)$	-2
$M(3;4;0)$	0
$M(4;0;11)$	-240
$M(4;1;9)$	144
$M(4;2;7)$	-80
$M(4;3;5)$	40
$M(4;4;3)$	-16
$M(4;5;1)$	0
$M(5;0;14)$	18264
$M(5;1;12)$	-9096
$M(5;2;10)$	4272
$M(5;3;8)$	-1872
$M(5;4;6)$	744
$M(5;5;4)$	-248
$M(5;6;2)$	64
$M(5;7;0)$	-64
$M(6;0;17)$	-2845440
$M(6;1;15)$	1209600
$M(6;2;13)$	-490368
$M(6;3;11)$	188544
$M(6;4;9)$	-67968
$M(6;5;7)$	22400
$M(6;6;5)$	-6400
$M(6;7;3)$	1536
$M(6;8;1)$	-1024

4.2.2 Extended coefficients for $\mathbb{P}^3$

$M(1;0,0;2)$	1
$M(1;0,1;0)$	-1
$M(1;1,0;1)$	0
$M(1;2,0;0)$	-1
$M(2;0,0;4)$	0
$M(2;0,1;2)$	0
$M(2;0,2;0)$	0
$M(2;1,0;3)$	1
$M(2;1,1;1)$	-1
$M(2;2,0;2)$	0
$M(2;2,1;0)$	0
$M(2;3,0;1)$	-3
$M(2;4,0;0)$	0
$M(3;0,0;6)$	-1
$M(3;0,1;4)$	1
$M(3;0,2;2)$	-1
$M(3;0,3;0)$	1
$M(3;1,0;5)$	0
$M(3;1,1;3)$	0
$M(3;1,2;1)$	0
$M(3;2,0;4)$	5
$M(3;2,1;2)$	-3
$M(3;2,2;0)$	1
$M(3;3,0;3)$	0
$M(3;3,1;1)$	0
$M(3;4,0;2)$	-13
$M(3;4,1;0)$	1
$M(3;5,0;1)$	0
$M(3;6,0;0)$	-7
$M(4;0,0;8)$	0
$M(4;0,1;6)$	0
$M(4;0,2;4)$	0
$M(4;0,3;2)$	0
$M(4;0,4;0)$	0
$M(4;1,0;7)$	-12
$M(4;1,1;5)$	8
$M(4;1,2;3)$	-4
$M(4;1,3;1)$	0
$M(4;2,0;6)$	0

$M(4;2,1;4)$	0
$M(4;2,2;2)$	0
$M(4;2,3;0)$	0
$M(4;3,0;5)$	60
$M(4;3,1;3)$	-24
$M(4;3,2;1)$	-4
$M(4;4,0;4)$	0
$M(4;4,1;2)$	0
$M(4;4,2;0)$	0
$M(4;5,0;3)$	-156
$M(4;5,1;1)$	-56
$M(4;6,0;2)$	0
$M(4;6,1;0)$	0
$M(4;7,0;1)$	-660
$M(4;8,0;0)$	0
$M(5;0,0;10)$	45
$M(5;0,1;8)$	-29
$M(5;0,2;6)$	17
$M(5;0,3;4)$	-9
$M(5;0,4;2)$	5
$M(5;0,5;0)$	-5
$M(5;1,0;9)$	0
$M(5;1,1;7)$	0
$M(5;1,2;5)$	0
$M(5;1,3;3)$	0
$M(5;1,4;1)$	0
$M(5;2,0;8)$	-277
$M(5;2,1;6)$	139
$M(5;2,2;4)$	-53
$M(5;2,3;2)$	11
$M(5;2,4;0)$	-5
$M(5;3,0;7)$	0
$M(5;3,1;5)$	0
$M(5;3,2;3)$	0
$M(5;3,3;1)$	0
$M(5;4,0;6)$	1353
$M(5;4,1;4)$	-429
$M(5;4,2;2)$	61
$M(5;4,3;0)$	-73

$M(5;5,0;5)$	0
$M(5;5,1;3)$	0
$M(5;5,2;1)$	0
$M(5;6,0;4)$	-3705
$M(5;6,1;2)$	379
$M(5;6,2;0)$	-1033
$M(5;7,0;3)$	0
$M(5;7,1;1)$	0
$M(5;8,0;2)$	2661
$M(5;8,1;0)$	-14397
$M(5;9,0;1)$	0
$M(5;10,0;0)$	-207325
$M(6;0,1;10)$	0
$M(6;0,2;8)$	0
$M(6;0,3;6)$	0
$M(6;0,4;4)$	0
$M(6;0,5;2)$	0
$M(6;0,6;0)$	0
$M(6;1,0;11)$	1728
$M(6;1,1;9)$	-864
$M(6;1,2;7)$	384
$M(6;1,3;5)$	-144
$M(6;1,4;3)$	48
$M(6;1,5;1)$	-48
$M(6;2,0;10)$	0
$M(6;2,1;8)$	0
$M(6;2,2;6)$	0
$M(6;2,3;4)$	0
$M(6;2,4;2)$	0
$M(6;2,5;0)$	0
$M(6;3,0;9)$	-10368
$M(6;3,1;7)$	4128
$M(6;3,2;5)$	-1280
$M(6;3,3;3)$	304
$M(6;3,4;1)$	-400
$M(6;4,0;8)$	0
$M(6;4,1;6)$	0
$M(6;4,2;4)$	0
$M(6;4,3;2)$	0

$M(6;4,4;0)$	0	$M(7;2,2;8)$	14333	$M(7;10,1;2)$	−9201647
$M(6;5,0;7)$	49344	$M(7;2,3;6)$	−4323	$M(7;10,2;0)$	−23217851
$M(6;5,1;5)$	−13152	$M(7;2,4;4)$	1105	$M(7;11,0;3)$	0
$M(6;5,2;3)$	2560	$M(7;2,5;2)$	−423	$M(7;11,1;1)$	0
$M(6;5,3;1)$	−4496	$M(7;2,6;0)$	85	$M(7;12,0;2)$	−147354705
$M(6;6,0;6)$	0	$M(7;3,0;11)$	0	$M(7;12,1;0)$	−469085619
$M(6;6,1;4)$	0	$M(7;3,1;9)$	0	$M(7;13,0;1)$	0
$M(6;6,2;2)$	0	$M(7;3,2;7)$	0	$M(7;14,0;0)$	−9730160571
$M(6;6,3;0)$	0	$M(7;3,3;5)$	0	$M(8;0,0;16)$	0
$M(6;7,0;5)$	−142464	$M(7;3,4;3)$	0	$M(8;0,1;14)$	0
$M(6;7,1;3)$	24096	$M(7;3,5;1)$	0	$M(8;0,2;12)$	0
$M(6;7,2;1)$	−58240	$M(7;4,0;10)$	−571833	$M(8;0,3;10)$	0
$M(6;8,0;4)$	0	$M(7;4,1;8)$	187693	$M(8;0,4;8)$	0
$M(6;8,1;2)$	0	$M(7;4,2;6)$	−49201	$M(8;0,5;6)$	0
$M(6;8,2;0)$	0	$M(7;4,3;4)$	10437	$M(8;0,6;4)$	0
$M(6;9,0;3)$	256192	$M(7;4,4;2)$	−4089	$M(8;0,7;2)$	0
$M(6;9,1;1)$	−837472	$M(7;4,5;0)$	−1747	$M(8;0,8;0)$	0
$M(6;10,0;2)$	0	$M(7;5,0;9)$	0	$M(8;1,0;15)$	−1177344
$M(6;10,1;0)$	0	$M(7;5,1;7)$	0	$M(8;1,1;13)$	465408
$M(6;11,0;1)$	−13119104	$M(7;5,2;5)$	0	$M(8;1,2;11)$	−170496
$M(6;12,0;0)$	0	$M(7;5,3;3)$	0	$M(8;1,3;9)$	56832
$M(6;0,0;12)$	0	$M(7;5,4;1)$	0	$M(8;1,4;7)$	−16896
$M(7;0,0;14)$	−14589	$M(7;6,0;8)$	2657949	$M(8;1,5;5)$	4608
$M(7;0,1;12)$	6957	$M(7;6,1;6)$	−614271	$M(8;1,6;3)$	−1280
$M(7;0,2;10)$	−3093	$M(7;6,2;4)$	114145	$M(8;1,7;1)$	−1024
$M(7;0,3;8)$	1269	$M(7;6,3;2)$	−47699	$M(8;2,0;14)$	0
$M(7;0,4;6)$	−477	$M(7;6,4;0)$	−53707	$M(8;2,1;12)$	0
$M(7;0,5;4)$	173	$M(7;7,0;7)$	0	$M(8;2,2;10)$	0
$M(7;0,6;2)$	−85	$M(7;7,1;5)$	0	$M(8;2,3;8)$	0
$M(7;0,7;0)$	85	$M(7;7,2;3)$	0	$M(8;2,4;6)$	0
$M(7;1,0;13)$	0	$M(7;7,3;1)$	0	$M(8;2,5;4)$	0
$M(7;1,1;11)$	0	$M(7;8,0;6)$	−8026197	$M(8;2,6;2)$	0
$M(7;1,2;9)$	0	$M(7;8,1;4)$	1354637	$M(8;2,7;0)$	0
$M(7;1,3;7)$	0	$M(7;8,2;2)$	−628685	$M(8;3,0;13)$	7693056
$M(7;1,4;5)$	0	$M(7;8,3;0)$	−1153003	$M(8;3,1;11)$	−2619648
$M(7;1,5;3)$	0	$M(7;9,0;5)$	0	$M(8;3,2;9)$	790272
$M(7;1,6;1)$	0	$M(7;9,1;3)$	0	$M(8;3,3;7)$	−201984
$M(7;2,0;12)$	98073	$M(7;9,2;1)$	0	$M(8;3,4;5)$	44288
$M(7;2,1;10)$	−39855	$M(7;10,0;4)$	17617281	$M(8;3,5;3)$	−9728

$M(8;3,6;1)$	−19200	$M(8;11,1;3)$	−327773440	$M(9;3,0;15)$	0
$M(8;4,0;12)$	0	$M(8;11,2;1)$	−2010495232	$M(9;3,1;13)$	0
$M(8;4,1;10)$	0	$M(8;12,0;4)$	0	$M(9;3,2;11)$	0
$M(8;4,2;8)$	0	$M(8;12,1;2)$	0	$M(9;3,3;9)$	0
$M(8;4,3;6)$	0	$M(8;12,2;0)$	0	$M(9;3,4;7)$	0
$M(8;4,4;4)$	0	$M(8;13,0;3)$	−5660168448	$M(9;3,5;5)$	0
$M(8;4,5;2)$	0	$M(8;13,1;1)$	−41941736448	$M(9;3,6;3)$	0
$M(8;4,6;0)$	0	$M(8;14,0;2)$	0	$M(9;3,7;1)$	0
$M(8;5,0;11)$	−43621632	$M(8;14,1;0)$	0	$M(9;4,0;14)$	796186053
$M(8;5,1;9)$	12146688	$M(8;15,0;1)$	−923081893632	$M(9;4,1;12)$	−232380585
$M(8;5,2;7)$	−2769408	$M(8;16,0;0)$	0	$M(9;4,2;10)$	60291393
$M(8;5,3;5)$	529408	$M(9;0,0;18)$	17756793	$M(9;4,3;8)$	−13422525
$M(8;5,4;3)$	−108544	$M(9;0,1;16)$	−6717465	$M(9;4,4;6)$	2600205
$M(8;5,5;1)$	−316928	$M(9;0,2;14)$	2407365	$M(9;4,5;4)$	−483041
$M(8;6,0;10)$	0	$M(9;0,3;12)$	−812157	$M(9;4,6;2)$	−84119
$M(8;6,1;8)$	0	$M(9;0,4;10)$	256065	$M(9;4,7;0)$	−190933
$M(8;6,2;6)$	0	$M(9;0,5;8)$	−75281	$M(9;5,0;13)$	0
$M(8;6,3;4)$	0	$M(9;0,6;6)$	21165	$M(9;5,1;11)$	0
$M(8;6,4;2)$	0	$M(9;0,7;4)$	−6165	$M(9;5,2;9)$	0
$M(8;6,5;0)$	0	$M(9;0,8;2)$	1993	$M(9;5,3;7)$	0
$M(8;7,0;9)$	198745344	$M(9;0,9;0)$	−1993	$M(9;5,4;5)$	0
$M(8;7,1;7)$	−40704768	$M(9;1,0;17)$	0	$M(9;5,5;3)$	0
$M(8;7,2;5)$	7008000	$M(9;1,1;15)$	0	$M(9;5,6;1)$	0
$M(8;7,3;3)$	−1421056	$M(9;1,2;13)$	0	$M(9;6,0;12)$	−4398941565
$M(8;7,4;1)$	−5509376	$M(9;1,3;11)$	0	$M(9;6,1;10)$	1062186183
$M(8;8,0;8)$	0	$M(9;1,4;9)$	0	$M(9;6,2;8)$	−214704421
$M(8;8,1;6)$	0	$M(9;1,5;7)$	0	$M(9;6,3;6)$	37303455
$M(8;8,2;4)$	0	$M(9;1,6;5)$	0	$M(9;6,4;4)$	−6311805
$M(8;8,3;2)$	0	$M(9;1,7;3)$	0	$M(9;6,5;2)$	−2029881
$M(8;8,4;0)$	0	$M(9;1,8;1)$	0	$M(9;6,6;0)$	−6502789
$M(8;9,0;7)$	−623352576	$M(9;2,0;16)$	−125236233	$M(9;7,0;11)$	0
$M(8;9,1;5)$	99223040	$M(9;2,1;14)$	42113079	$M(9;7,1;9)$	0
$M(8;9,2;3)$	−20619776	$M(9;2,2;12)$	−13095777	$M(9;7,2;7)$	0
$M(8;9,3;1)$	−102065664	$M(9;2,3;10)$	3690831	$M(9;7,3;5)$	0
$M(8;10,0;6)$	0	$M(9;2,4;8)$	−922313	$M(9;7,4;3)$	0
$M(8;10,1;4)$	0	$M(9;2,5;6)$	207511	$M(9;7,5;1)$	0
$M(8;10,2;2)$	0	$M(9;2,6;4)$	−45633	$M(9;8,0;10)$	19716043473
$M(8;10,3;0)$	0	$M(9;2,7;2)$	1039	$M(9;8,1;8)$	−3632745977
$M(8;11,0;5)$	1511345920	$M(9;2,8;0)$	−1993	$M(9;8,2;6)$	578103069

$M(9;8,3;4)$	-92799453
$M(9;8,4;2)$	-42233127
$M(9;8,5;0)$	-173709233
$M(9;9,0;9)$	0
$M(9;9,1;7)$	0
$M(9;9,2;5)$	0
$M(9;9,3;3)$	0
$M(9;9,4;1)$	0
$M(9;10,0;8)$	-63823996593
$M(9;10,1;6)$	9485525655
$M(9;10,2;4)$	-1485706953
$M(9;10,3;2)$	-882676209
$M(9;10,4;0)$	-4337580849
$M(9;11,0;7)$	0
$M(9;11,1;5)$	0
$M(9;11,2;3)$	0
$M(9;11,3;1)$	0
$M(9;12,0;6)$	165343940253
$M(9;12,1;4)$	-25709693961
$M(9;12,2;2)$	-19089752359
$M(9;12,3;0)$	-107385726685
$M(9;13,0;5)$	0
$M(9;13,1;3)$	0
$M(9;13,2;1)$	0
$M(9;14,0;4)$	-478871106213
$M(9;14,1;2)$	-430183080537
$M(9;14,2;0)$	-2699609946829
$M(9;15,0;3)$	0
$M(9;15,1;1)$	0
$M(9;16,0;2)$	-10110796935639
$M(9;16,1;0)$	-69647276547801
$M(9;17,0;1)$	0

4.2.3 Extended coefficients for $\mathbb{P}^4$

$M(1;0,0,0;2)$	1
$M(1;0,0,1;0)$	-1
$M(1;1,1,0;0)$	-1
$M(1;3,0,0;0)$	-1
$M(2;0,2,0;1)$	-1
$M(2;1,0,0;3)$	1
$M(2;1,0,1;1)$	-1
$M(2;2,1,0;1)$	-3
$M(2;4,0,0;1)$	-7
$M(3;0,1,0;4)$	1
$M(3;0,1,1;2)$	-1
$M(3;0,1,2;0)$	1
$M(3;0,4,0;0)$	0
$M(3;1,2,0;2)$	-4
$M(3;1,2,1;0)$	2
$M(3;2,0,0;4)$	5
$M(3;2,0,1;2)$	-3
$M(3;2,0,2;0)$	1
$M(3;2,3,0;0)$	2
$M(3;3,1,0;2)$	-14
$M(3;3,1,1;0)$	2
$M(3;4,2,0;0)$	-12
$M(3;5,0,0;2)$	-44
$M(3;5,0,1;0)$	-6
$M(3;6,1,0;0)$	-138
$M(3;8,0,0;0)$	-984
$M(4;0,0,0;7)$	-1
$M(4;0,0,1;5)$	1
$M(4;0,0,2;3)$	-1
$M(4;0,0,3;1)$	1
$M(4;0,3,0;3)$	-4
$M(4;0,3,1;1)$	-2
$M(4;1,1,0;5)$	9
$M(4;1,1,1;3)$	-5
$M(4;1,1,2;1)$	1
$M(4;1,4,0;1)$	-24
$M(4;2,2,0;3)$	-30
$M(4;2,2,1;1)$	-8
$M(4;3,0,0;5)$	61
$M(4;3,0,1;3)$	-25
$M(4;3,0,2;1)$	-3
$M(4;3,3,0;1)$	-162
$M(4;4,1,0;3)$	-156
$M(4;4,1,1;1)$	-78
$M(4;5,2,0;1)$	-1236
$M(4;6,0,0;3)$	-678
$M(4;6,0,1;1)$	-596
$M(4;7,1,0;1)$	-9166
$M(4;9,0,0;1)$	-67504
$M(5;0,2,0;6)$	18
$M(5;0,2,1;4)$	-8
$M(5;0,2,2;2)$	2
$M(5;0,2,3;0)$	0
$M(5;0,5,0;2)$	-24
$M(5;0,5,1;0)$	0
$M(5;1,0,0;8)$	-19
$M(5;1,0,1;6)$	13
$M(5;1,0,2;4)$	-7
$M(5;1,0,3;2)$	1
$M(5;1,0,4;0)$	5
$M(5;1,3,0;4)$	-64
$M(5;1,3,1;2)$	0
$M(5;1,3,2;0)$	4
$M(5;1,6,0;0)$	-320
$M(5;2,1,0;6)$	160
$M(5;2,1,1;4)$	-60
$M(5;2,1,2;2)$	4
$M(5;2,1,3;0)$	16
$M(5;2,4,0;2)$	-112
$M(5;2,4,1;0)$	-80
$M(5;3,2,0;4)$	-502
$M(5;3,2,1;2)$	-26
$M(5;3,2,2;0)$	22
$M(5;3,5,0;0)$	-3200
$M(5;4,0,0;6)$	1378
$M(5;4,0,1;4)$	-424
$M(5;4,0,2;2)$	26
$M(5;4,0,3;0)$	-8
$M(5;4,3,0;2)$	-960
$M(5;4,3,1;0)$	-1112
$M(5;5,1,0;4)$	-3420
$M(5;5,1,1;2)$	-260
$M(5;5,1,2;0)$	-576
$M(5;5,4,0;0)$	-33864
$M(5;6,2,0;2)$	-8056
$M(5;6,2,1;0)$	-15552
$M(5;7,0,0;4)$	-19410
$M(5;7,0,1;2)$	-3150
$M(5;7,0,2;0)$	-7822
$M(5;7,3,0;0)$	-368144
$M(5;8,1,0;2)$	-71048
$M(5;8,1,1;0)$	-170224
$M(5;9,2,0;0)$	-3850704
$M(5;10,0,0;2)$	-616288
$M(5;10,0,1;0)$	-1720400
$M(5;11,1,0;0)$	-39665920
$M(5;13,0,0;0)$	-410460760

4.2.4 Extended coefficients for $\mathbb{P}^5$

$M(1;0,0,0,0;2)$	1	$M(3;0,2,0,1;1)$	0	$M(3;6,1,0,0;1)$	0
$M(1;0,0,0,1;0)$	-1	$M(3;0,2,2,0;0)$	1	$M(3;6,2,0,0;0)$	-1575
$M(1;0,1,0,0;1)$	0	$M(3;0,3,0,0;2)$	-5	$M(3;7,0,1,0;0)$	-1020
$M(1;0,2,0,0;0)$	-1	$M(3;0,3,0,1;0)$	3	$M(3;8,0,0,0;1)$	0
$M(1;1,0,1,0;0)$	-1	$M(3;0,4,0,0;1)$	0	$M(3;8,1,0,0;0)$	-8443
$M(1;2,0,0,0;1)$	0	$M(3;0,5,0,0;0)$	5	$M(3;10,0,0,0;0)$	-43515
$M(1;2,1,0,0;0)$	-1	$M(3;1,0,1,0;3)$	0	$M(4;0,0,1,0;5)$	1
$M(1;4,0,0,0;0)$	-1	$M(3;1,0,1,1;1)$	0	$M(4;0,0,1,1;3)$	-1
$M(2;0,0,1,0;2)$	0	$M(3;1,0,3,0;0)$	3	$M(4;0,0,1,2;1)$	1
$M(2;0,0,1,1;0)$	0	$M(3;1,1,1,0;2)$	-4	$M(4;0,0,3,0;2)$	0
$M(2;0,1,1,0;1)$	-1	$M(3;1,1,1,1;0)$	2	$M(4;0,1,1,0;4)$	0
$M(2;0,2,1,0;0)$	0	$M(3;1,2,1,0;1)$	0	$M(4;0,1,1,1;2)$	0
$M(2;1,0,0,0;3)$	1	$M(3;1,3,1,0;0)$	0	$M(4;0,1,3,0;1)$	-4
$M(2;1,0,0,1;1)$	-1	$M(3;2,0,0,0;4)$	5	$M(4;0,2,1,0;3)$	-5
$M(2;1,0,2,0;0)$	0	$M(3;2,0,0,1;2)$	-3	$M(4;0,2,1,1;1)$	-1
$M(2;1,1,0,0;2)$	0	$M(3;2,0,0,2;0)$	1	$M(4;0,3,1,0;2)$	0
$M(2;1,1,0,1;0)$	0	$M(3;2,0,2,0;1)$	0	$M(4;0,4,1,0;1)$	-43
$M(2;1,2,0,0;1)$	-3	$M(3;2,1,0,0;3)$	0	$M(4;1,0,0,0;6)$	0
$M(2;1,3,0,0;0)$	0	$M(3;2,1,0,1;1)$	0	$M(4;1,0,0,1;4)$	0
$M(2;2,0,1,0;1)$	-3	$M(3;2,1,2,0;0)$	3	$M(4;1,0,0,2;2)$	0
$M(2;2,1,1,0;0)$	0	$M(3;2,2,0,0;2)$	-15	$M(4;1,0,2,0;3)$	-6
$M(2;3,0,0,0;2)$	0	$M(3;2,2,0,1;0)$	3	$M(4;1,0,2,1;1)$	2
$M(2;3,0,0,1;0)$	0	$M(3;2,3,0,0;1)$	0	$M(4;1,1,0,0;5)$	9
$M(2;3,1,0,0;1)$	-7	$M(3;2,4,0,0;0)$	-35	$M(4;1,1,0,1;3)$	-5
$M(2;3,2,0,0;0)$	0	$M(3;3,0,1,0;2)$	-14	$M(4;1,1,0,2;1)$	1
$M(2;4,0,1,0;0)$	0	$M(3;3,0,1,1;0)$	2	$M(4;1,1,2,0;2)$	0
$M(2;5,0,0,0;1)$	-15	$M(3;3,1,1,0;1)$	0	$M(4;1,2,0,0;4)$	0
$M(2;5,1,0,0;0)$	0	$M(3;3,2,1,0;0)$	-20	$M(4;1,2,0,1;2)$	0
$M(2;7,0,0,0;0)$	0	$M(3;4,0,0,0;3)$	0	$M(4;1,2,2,0;1)$	-30
$M(3;0,0,0,0;5)$	0	$M(3;4,0,0,1;1)$	0	$M(4;1,3,0,0;3)$	-31
$M(3;0,0,0,1;3)$	0	$M(3;4,0,2,0;0)$	-11	$M(4;1,3,0,1;1)$	-15
$M(3;0,0,0,2;1)$	0	$M(3;4,1,0,0;2)$	-45	$M(4;1,4,0,0;2)$	0
$M(3;0,0,2,0;2)$	-1	$M(3;4,1,0,1;0)$	-5	$M(4;1,5,0,0;1)$	-415
$M(3;0,0,2,1;0)$	1	$M(3;4,2,0,0;1)$	0	$M(4;2,0,1,0;4)$	0
$M(3;0,1,0,0;4)$	1	$M(3;4,3,0,0;0)$	-267	$M(4;2,0,1,1;2)$	0
$M(3;0,1,0,1;2)$	-1	$M(3;5,0,1,0;1)$	0	$M(4;2,0,3,0;1)$	-10
$M(3;0,1,0,2;0)$	1	$M(3;5,1,1,0;0)$	-172	$M(4;2,1,1,0;3)$	-31
$M(3;0,1,2,0;1)$	0	$M(3;6,0,0,0;2)$	-123	$M(4;2,1,1,1;1)$	-7
$M(3;0,2,0,0;3)$	0	$M(3;6,0,0,1;0)$	-57	$M(4;2,2,1,0;2)$	0

$M(4;2,3,1,0;1)$	-281
$M(4;3,0,0,0;5)$	61
$M(4;3,0,0,1;3)$	-25
$M(4;3,0,0,2;1)$	-3
$M(4;3,0,2,0;2)$	0
$M(4;3,1,0,0;4)$	0
$M(4;3,1,0,1;2)$	0
$M(4;3,1,2,0;1)$	-184
$M(4;3,2,0,0;3)$	-151
$M(4;3,2,0,1;1)$	-103
$M(4;3,3,0,0;2)$	0
$M(4;3,4,0,0;1)$	-2643
$M(4;4,0,1,0;3)$	-157
$M(4;4,0,1,1;1)$	-77
$M(4;4,1,1,0;2)$	0
$M(4;4,2,1,0;1)$	-1799
$M(4;5,0,0,0;4)$	0
$M(4;5,0,0,1;2)$	0
$M(4;5,0,2,0;1)$	-1314
$M(4;5,1,0,0;3)$	-635
$M(4;5,1,0,1;1)$	-679
$M(4;5,2,0,0;2)$	0
$M(4;5,3,0,0;1)$	-16687
$M(4;6,0,1,0;2)$	0
$M(4;6,1,1,0;1)$	-11461
$M(4;7,0,0,0;3)$	-2283
$M(4;7,0,0,1;1)$	-3935
$M(4;7,1,0,0;2)$	0
$M(4;7,2,0,0;1)$	-104179
$M(4;8,0,1,0;1)$	-69459
$M(4;9,0,0,0;2)$	0
$M(4;9,1,0,0;1)$	-640943
$M(4;11,0,0,0;1)$	-3934115

4.2.5 Extended coefficients for $\mathbb{P}^6$

$M(1;0,0,0,0,0;2)$	1
$M(1;0,0,0,0,1;0)$	-1
$M(1;0,1,1,0,0;0)$	-1
$M(1;1,0,0,1,0;0)$	-1
$M(1;1,2,0,0,0;0)$	-1
$M(1;2,0,1,0,0;0)$	-1
$M(1;3,1,0,0,0;0)$	-1
$M(1;5,0,0,0,0;0)$	-1
$M(2;0,0,2,0,0;1)$	-1
$M(2;0,1,0,1,0;1)$	-1
$M(2;0,3,0,0,0;1)$	-3
$M(2;1,0,0,0,0;3)$	1
$M(2;1,0,0,0,1;1)$	-1
$M(2;1,1,1,0,0;1)$	-3
$M(2;2,0,0,1,0;1)$	-3
$M(2;2,2,0,0,0;1)$	-7
$M(2;3,0,1,0,0;1)$	-7
$M(2;4,1,0,0,0;1)$	-15
$M(2;6,0,0,0,0;1)$	-31
$M(3;0,0,0,3,0;0)$	1
$M(3;0,0,1,1,0;2)$	-1
$M(3;0,0,1,1,1;0)$	1
$M(3;0,0,4,0,0;0)$	0
$M(3;0,1,0,0,0;4)$	1
$M(3;0,1,0,0,1;2)$	-1
$M(3;0,1,0,0,2;0)$	1
$M(3;0,1,2,1,0;0)$	2
$M(3;0,2,0,2,0;0)$	1
$M(3;0,2,1,0,0;2)$	-5
$M(3;0,2,1,0,1;0)$	3
$M(3;0,3,2,0,0;0)$	0
$M(3;0,4,0,1,0;0)$	-5
$M(3;0,6,0,0,0;0)$	-113
$M(3;1,0,1,2,0;0)$	3
$M(3;1,0,2,0,0;2)$	-4
$M(3;1,0,2,0,1;0)$	2
$M(3;1,1,0,1,0;2)$	-4
$M(3;1,1,0,1,1;0)$	2
$M(3;1,1,3,0,0;0)$	-2
$M(3;1,2,1,1,0;0)$	1
$M(3;1,3,0,0,0;2)$	-16
$M(3;1,3,0,0,1;0)$	4
$M(3;1,4,1,0,0;0)$	-63
$M(3;2,0,0,0,0;4)$	5
$M(3;2,0,0,0,1;2)$	-3
$M(3;2,0,0,0,2;0)$	1
$M(3;2,0,2,1,0;0)$	4
$M(3;2,1,0,2,0;0)$	3
$M(3;2,1,1,0,0;2)$	-15

$M(3;2,1,1,0,1;0)$	3
$M(3;2,2,2,0,0;0)$	-46
$M(3;2,3,0,1,0;0)$	-31
$M(3;2,5,0,0,0;0)$	-567
$M(3;3,0,0,1,0;2)$	-14
$M(3;3,0,0,1,1;0)$	2
$M(3;3,0,3,0,0;0)$	-28
$M(3;3,1,1,1,0;0)$	-19
$M(3;3,2,0,0,0;2)$	-46
$M(3;3,2,0,0,1;0)$	-4
$M(3;3,3,1,0,0;0)$	-411
$M(3;4,0,0,2,0;0)$	-11
$M(3;4,0,1,0,0;2)$	-45
$M(3;4,0,1,0,1;0)$	-5
$M(3;4,1,2,0,0;0)$	-304
$M(3;4,2,0,1,0;0)$	-209
$M(3;4,4,0,0,0;0)$	-2941
$M(3;5,0,1,1,0;0)$	-171
$M(3;5,1,0,0,0;2)$	-124
$M(3;5,1,0,0,1;0)$	-56
$M(3;5,2,1,0,0;0)$	-2179
$M(3;6,0,2,0,0;0)$	-1686
$M(3;6,1,0,1,0;0)$	-1131
$M(3;6,3,0,0,0;0)$	-14363
$M(3;7,0,0,0,0;2)$	-314
$M(3;7,0,0,0,1;0)$	-288
$M(3;7,1,1,0,0;0)$	-10671
$M(3;8,0,0,1,0;0)$	-5309
$M(3;8,2,0,0,0;0)$	-67657
$M(3;9,0,1,0,0;0)$	-49527
$M(3;10,1,0,0,0;0)$	-311919
$M(3;12,0,0,0,0;0)$	-1425141
$M(4;0,0,0,2,0;3)$	-1
$M(4;0,0,0,2,1;1)$	1
$M(4;0,0,1,0,0;5)$	1
$M(4;0,0,1,0,1;3)$	-1
$M(4;0,0,1,0,2;1)$	1
$M(4;0,0,3,1,0;1)$	-6
$M(4;0,1,1,2,0;1)$	-3
$M(4;0,1,2,0,0;3)$	-6
$M(4;0,1,2,0,1;1)$	0
$M(4;0,2,0,1,0;3)$	-5
$M(4;0,2,0,1,1;1)$	-1
$M(4;0,2,3,0,0;1)$	-64
$M(4;0,3,1,1,0;1)$	-53
$M(4;0,4,0,0,0;3)$	-27
$M(4;0,4,0,0,1;1)$	-25
$M(4;0,5,1,0,0;1)$	-729
$M(4;1,0,0,3,0;1)$	3

$M(4;1,0,1,1,0;3)$	-6
$M(4;1,0,1,1,1;1)$	2
$M(4;1,0,4,0,0;1)$	-56
$M(4;1,1,0,0,0;5)$	9
$M(4;1,1,0,0,1;3)$	-5
$M(4;1,1,0,0,2;1)$	1
$M(4;1,1,2,1,0;1)$	-36
$M(4;1,2,0,2,0;1)$	-29
$M(4;1,2,1,0,0;3)$	-32
$M(4;1,2,1,0,1;1)$	-14
$M(4;1,3,2,0,0;1)$	-534
$M(4;1,4,0,1,0;1)$	-403
$M(4;1,6,0,0,0;1)$	-5463
$M(4;2,0,1,2,0;1)$	-9
$M(4;2,0,2,0,0;3)$	-32
$M(4;2,0,2,0,1;1)$	-6
$M(4;2,1,0,1,0;3)$	-31
$M(4;2,1,0,1,1;1)$	-7
$M(4;2,1,3,0,0;1)$	-402
$M(4;2,2,1,1,0;1)$	-307
$M(4;2,3,0,0,0;3)$	-141
$M(4;2,3,0,0,1;1)$	-131
$M(4;2,4,1,0,0;1)$	-4151
$M(4;3,0,0,0,0;5)$	61
$M(4;3,0,0,0,1;3)$	-25
$M(4;3,0,0,0,2;1)$	-3
$M(4;3,0,2,1,0;1)$	-206
$M(4;3,1,0,2,0;1)$	-183
$M(4;3,1,1,0,0;3)$	-152
$M(4;3,1,1,0,1;1)$	-102
$M(4;3,2,2,0,0;1)$	-3184
$M(4;3,3,0,1,0;1)$	-2345
$M(4;3,5,0,0,0;1)$	-31457
$M(4;4,0,0,1,0;3)$	-157
$M(4;4,0,0,1,1;1)$	-77
$M(4;4,0,3,0,0;1)$	-2360
$M(4;4,1,1,1,0;1)$	-1881
$M(4;4,2,0,0,0;3)$	-587
$M(4;4,2,0,0,1;1)$	-765
$M(4;4,3,1,0,0;1)$	-24133
$M(4;5,0,0,2,0;1)$	-1313
$M(4;5,0,1,0,0;3)$	-636
$M(4;5,0,1,0,1;1)$	-678
$M(4;5,1,2,0,0;1)$	-18838
$M(4;5,2,0,1,0;1)$	-13623
$M(4;5,4,0,0,0;1)$	-180635
$M(4;6,0,1,1,0;1)$	-11695
$M(4;6,1,0,0,0;3)$	-2117
$M(4;6,1,0,0,1;1)$	-4175

$M(4;6,2,1,0,0;1)$	-138947
$M(4;7,0,2,0,0;1)$	-111656
$M(4;7,1,0,1,0;1)$	-76957
$M(4;7,3,0,0,0;1)$	-1028189
$M(4;8,0,0,0,0;3)$	-6795
$M(4;8,0,0,0,1;1)$	-20929
$M(4;8,1,1,0,0;1)$	-789265
$M(4;9,0,0,1,0;1)$	-413659
$M(4;9,2,0,0,0;1)$	-5801599
$M(4;10,0,1,0,0;1)$	-4382335
$M(4;11,1,0,0,0;1)$	-32479177
$M(4;13,0,0,0,0;1)$	-181335139

4.2.6 Extended coefficients for $\mathbb{P}^1 \times \mathbb{P}^1$

$M(0,1;0;1)$	1
$M(0,2;0;3)$	0
$M(0,2;1;1)$	0
$M(0,3;0;5)$	0
$M(0,3;1;3)$	0
$M(0,3;2;1)$	0
$M(0,4;0;7)$	0
$M(0,4;1;5)$	0
$M(0,4;2;3)$	0
$M(0,4;3;1)$	0
$M(1,0;0;1)$	1
$M(1,1;0;3)$	1
$M(1,1;1;1)$	1
$M(1,2;0;5)$	1
$M(1,2;1;3)$	1
$M(1,2;2;1)$	1
$M(1,3;0;7)$	1
$M(1,3;1;5)$	1
$M(1,3;2;3)$	1
$M(1,3;3;1)$	1
$M(1,4;0;9)$	1
$M(1,4;1;7)$	1
$M(1,4;2;5)$	1
$M(1,4;3;3)$	1
$M(1,4;4;1)$	1
$M(2,0;0;3)$	0
$M(2,0;1;1)$	0
$M(2,1;0;5)$	1
$M(2,1;1;3)$	1
$M(2,1;2;1)$	1
$M(2,2;0;7)$	8
$M(2,2;1;5)$	6
$M(2,2;2;3)$	4
$M(2,2;3;1)$	2
$M(2,3;0;9)$	48
$M(2,3;1;7)$	32
$M(2,3;2;5)$	20
$M(2,3;3;3)$	12
$M(2,3;4;1)$	8

$M(2,4;0;11)$	256
$M(2,4;1;9)$	160
$M(2,4;2;7)$	96
$M(2,4;3;5)$	56
$M(2,4;4;3)$	32
$M(2,4;5;1)$	16
$M(3,0;0;5)$	0
$M(3,0;1;3)$	0
$M(3,0;2;1)$	0
$M(3,1;0;7)$	1
$M(3,1;1;5)$	1
$M(3,1;2;3)$	1
$M(3,1;3;1)$	1
$M(3,2;0;9)$	48
$M(3,2;1;7)$	32
$M(3,2;2;5)$	20
$M(3,2;3;3)$	12
$M(3,2;4;1)$	8
$M(3,3;0;11)$	1086
$M(3,3;1;9)$	606
$M(3,3;2;7)$	318
$M(3,3;3;5)$	158
$M(3,3;4;3)$	78
$M(3,3;5;1)$	46
$M(3,4;0;13)$	18424
$M(3,4;1;11)$	9256
$M(3,4;2;9)$	4432
$M(3,4;3;7)$	2032
$M(3,4;4;5)$	904
$M(3,4;5;3)$	408
$M(3,4;6;1)$	224
$M(4,0;0;7)$	0
$M(4,0;1;5)$	0
$M(4,0;2;3)$	0
$M(4,0;3;1)$	0
$M(4,1;0;9)$	1
$M(4,1;1;7)$	1
$M(4,1;2;5)$	1
$M(4,1;3;3)$	1

$M(4,1;4;1)$	1
$M(4,2;0;11)$	256
$M(4,2;1;9)$	160
$M(4,2;2;7)$	96
$M(4,2;3;5)$	56
$M(4,2;4;3)$	32
$M(4,2;5;1)$	16
$M(4,3;0;13)$	18424
$M(4,3;1;11)$	9256
$M(4,3;2;9)$	4432
$M(4,3;3;7)$	2032
$M(4,3;4;5)$	904
$M(4,3;5;3)$	408
$M(4,3;6;1)$	224
$M(4,4;1;13)$	360896
$M(4,4;2;11)$	152192
$M(4,4;3;9)$	61568
$M(4,4;4;7)$	24064
$M(4,4;5;5)$	9280
$M(4,4;6;3)$	3712
$M(4,4;7;1)$	1536

4.2.7 Extended coefficients for $G(2,4)$

$M(1;0,1,0,0;1)$	i	$M(4;0,4,0,0;3)$	-18	$M(5;0,0,3,0;3)$	0
$M(1;1,0,0,0;1)$	$-i$	$M(4;0,4,0,1;1)$	4	$M(5;0,0,3,1;1)$	0
$M(2;0,0,0,0;3)$	1	$M(4;0,5,1,0;1)$	12	$M(5;0,1,1,0;5)$	$4i$
$M(2;0,0,0,1;1)$	-1	$M(4;0,7,0,0;1)$	74	$M(5;0,1,1,1;3)$	$-2i$
$M(2;0,1,1,0;1)$	-1	$M(4;1,0,0,0;5)$	4	$M(5;0,1,1,2;1)$	0
$M(2;0,3,0,0;1)$	-1	$M(4;1,0,0,1;3)$	-2	$M(5;0,1,4,0;1)$	$-3i$
$M(2;1,0,1,0;1)$	-1	$M(4;1,0,0,2;1)$	0	$M(5;0,2,2,0;3)$	$-14i$
$M(2;1,2,0,0;1)$	-1	$M(4;1,0,3,0;1)$	-4	$M(5;0,2,2,1;1)$	$-2i$
$M(2;2,1,0,0;1)$	-1	$M(4;1,1,1,0;3)$	-4	$M(5;0,3,0,0;5)$	$52i$
$M(2;3,0,0,0;1)$	-1	$M(4;1,1,1,1;1)$	-2	$M(5;0,3,0,1;3)$	$-22i$
$M(3;0,0,1,0;3)$	0	$M(4;1,2,2,0;1)$	-18	$M(5;0,3,0,2;1)$	$4i$
$M(3;0,0,1,1;1)$	0	$M(4;1,3,0,0;3)$	-6	$M(5;0,3,3,0;1)$	$-41i$
$M(3;0,1,2,0;1)$	$-i$	$M(4;1,3,0,1;1)$	-8	$M(5;0,4,1,0;3)$	$-84i$
$M(3;0,2,0,0;3)$	$2i$	$M(4;1,4,1,0;1)$	-36	$M(5;0,4,1,1;1)$	$-12i$
$M(3;0,2,0,1;1)$	$-2i$	$M(4;1,6,0,0;1)$	2	$M(5;0,5,2,0;1)$	$-161i$
$M(3;0,3,1,0;1)$	$-5i$	$M(4;2,0,1,0;3)$	-4	$M(5;0,6,0,0;3)$	$-374i$
$M(3;0,5,0,0;1)$	$-7i$	$M(4;2,0,1,1;1)$	-2	$M(5;0,6,0,1;1)$	$38i$
$M(3;1,0,2,0;1)$	i	$M(4;2,1,2,0;1)$	-18	$M(5;0,7,1,0;1)$	$81i$
$M(3;1,1,0,0;3)$	0	$M(4;2,2,0,0;3)$	-2	$M(5;0,9,0,0;1)$	$1893i$
$M(3;1,1,0,1;1)$	0	$M(4;2,2,0,1;1)$	-12	$M(5;1,0,1,0;5)$	$-4i$
$M(3;1,2,1,0;1)$	$-3i$	$M(4;2,3,1,0;1)$	-76	$M(5;1,0,1,1;3)$	$2i$
$M(3;1,4,0,0;1)$	$-9i$	$M(4;2,5,0,0;1)$	-142	$M(5;1,0,1,2;1)$	0
$M(3;2,0,0,0;3)$	$-2i$	$M(4;3,0,2,0;1)$	-10	$M(5;1,0,4,0;1)$	$3i$
$M(3;2,0,0,1;1)$	$2i$	$M(4;3,1,0,0;3)$	-6	$M(5;1,1,2,0;3)$	0
$M(3;2,1,1,0;1)$	$3i$	$M(4;3,1,0,1;1)$	-8	$M(5;1,1,2,1;1)$	0
$M(3;2,3,0,0;1)$	$-7i$	$M(4;3,2,1,0;1)$	-76	$M(5;1,2,0,0;5)$	$20i$
$M(3;3,0,1,0;1)$	$5i$	$M(4;3,4,0,0;1)$	-294	$M(5;1,2,0,1;3)$	$-10i$
$M(3;3,2,0,0;1)$	$7i$	$M(4;4,0,0,0;3)$	-18	$M(5;1,2,0,2;1)$	$4i$
$M(3;4,1,0,0;1)$	$9i$	$M(4;4,0,0,1;1)$	4	$M(5;1,2,3,0;1)$	$-15i$
$M(3;5,0,0,0;1)$	$7i$	$M(4;4,1,1,0;1)$	-36	$M(5;1,3,1,0;3)$	$-54i$
$M(4;0,0,2,0;3)$	-2	$M(4;4,3,0,0;1)$	-294	$M(5;1,3,1,1;1)$	$-2i$
$M(4;0,0,2,1;1)$	0	$M(4;5,0,1,0;1)$	12	$M(5;1,4,2,0;1)$	$-175i$
$M(4;0,1,0,0;5)$	4	$M(4;5,2,0,0;1)$	-142	$M(5;1,5,0,0;3)$	$-204i$
$M(4;0,1,0,1;3)$	-2	$M(4;6,1,0,0;1)$	2	$M(5;1,5,0,1;1)$	$-100i$
$M(4;0,1,0,2;1)$	0	$M(4;7,0,0,0;1)$	74	$M(5;1,6,1,0;1)$	$-825i$
$M(4;0,1,3,0;1)$	-4	$M(5;0,0,0,0;7)$	0	$M(5;1,8,0,0;1)$	$-645i$
$M(4;0,2,1,0;3)$	-4	$M(5;0,0,0,1;5)$	0	$M(5;2,0,2,0;3)$	$14i$
$M(4;0,2,1,1;1)$	-2	$M(5;0,0,0,2;3)$	0	$M(5;2,0,2,1;1)$	$2i$
$M(4;0,3,2,0;1)$	-10	$M(5;0,0,0,3;1)$	0	$M(5;2,1,0,0;5)$	$-20i$

$M(5;2,1,0,1;3)$	$10i$
$M(5;2,1,0,2;1)$	$-4i$
$M(5;2,1,3,0;1)$	$15i$
$M(5;2,2,1,0;3)$	0
$M(5;2,2,1,1;1)$	0
$M(5;2,3,2,0;1)$	$-65i$
$M(5;2,4,0,0;3)$	$-114i$
$M(5;2,4,0,1;1)$	$-62i$
$M(5;2,5,1,0;1)$	$-967i$
$M(5;2,7,0,0;1)$	$-4155i$
$M(5;3,0,0,0;5)$	$-52i$
$M(5;3,0,0,1;3)$	$22i$
$M(5;3,0,0,2;1)$	$-4i$
$M(5;3,0,3,0;1)$	$41i$
$M(5;3,1,1,0;3)$	$54i$
$M(5;3,1,1,1;1)$	$2i$
$M(5;3,2,2,0;1)$	$65i$
$M(5;3,3,0,0;3)$	0
$M(5;3,3,0,1;1)$	0
$M(5;3,4,1,0;1)$	$-369i$
$M(5;3,6,0,0;1)$	$-4885i$
$M(5;4,0,1,0;3)$	$84i$
$M(5;4,0,1,1;1)$	$12i$
$M(5;4,1,2,0;1)$	$175i$
$M(5;4,2,0,0;3)$	$114i$
$M(5;4,2,0,1;1)$	$62i$
$M(5;4,3,1,0;1)$	$369i$
$M(5;4,5,0,0;1)$	$-1899i$
$M(5;5,0,2,0;1)$	$161i$
$M(5;5,1,0,0;3)$	$204i$
$M(5;5,1,0,1;1)$	$100i$
$M(5;5,2,1,0;1)$	$967i$
$M(5;5,4,0,0;1)$	$1899i$
$M(5;6,0,0,0;3)$	$374i$
$M(5;6,0,0,1;1)$	$-38i$

CHAPTER 5
Summary and Conclusion

In this thesis, we showed that the open WDVV equations arise as the associativity conditions for a rank-1 extension of Frob manifolds. This gives a geometric description of the open WDVV equations in a similar way to how a Frobenius manifold describes the original WDVV equations. The introduction and use of a Frob manifold instead of a Frobenius manifold was necessary due to the lack of natural metric in the motivating examples.

We studied Frob manifolds in detail and showed that, even though they lack a metric, they have many of the same properties as Frobenius manifolds. For example, we gave the notion of semisimple Frob manifolds. We constructed two so-called structure connections for Frob manifolds. We defined extensions in general and showed that there are two kinds of rank-1 extensions, namely an extension by a module and an auxiliary extension. We showed that auxiliary extensions of semisimple Frob manifolds are again semisimple.

We found interesting examples of extensions for two classes of Frobenius manifolds: Quantum Cohomology of $\mathbb{P}^n$ and the universal unfolding of the A_n singularity. The first example example is somewhat conjectural, although we observed a connection to other work done with real enumerative geometry. In addition, we also conjectured the existence of an extension for $G(2,4)$ and we ask whether the coefficients of that extended potential represent enumerative invariants. These coefficients interestingly were complex integers.

We proved the existence of an extension of A_n, and we computed the extended potentials explicitly for some values of n.

Overall, this framework of Frob manifold and extensions has proven useful to describe some interesting phenomenon in different areas of math, which made this a worthwhile object to study. We expect that more examples will be found in the future, and this framework will give some insight into their nature.

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