Symplectic cohomologies and Hodge theories

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Abstract

This thesis is a review of the symplectic cohomology theories of Tseng and Yau [16, 17], which give symplectic analogues of the Dolbeault, Bott-Chern and Aeppli cohomologies for complex manifolds. Basic features of these cohomologies, such as their Hodge theories, finite-dimensionality, duality properties and Poincaré lemmas are reviewed. The symplectic cohomologies are computed explicitly in the case of cotangent bundles. While none of the results of this thesis are new, we have given detailed proofs of a number of facts which are stated without proof in the foundational papers of Tseng and Yau. The thesis concludes with perspectives and open problems.

Résumé

Cette thèse reprend les théories de cohomologie symplectique de Tseng et Yau [16, 17], qui fournissent des analogues symplectiques des cohomologies de Dolbeault, Bott-Chern et Aeppli pour les variétés complexes. Les propriétés de base de ces cohomologies, telles que leurs théories de Hodge, la finitude de leurs dimensions, leurs propriétés de dualité et les lemmes de Poincaré sont étudiés. Les cohomologies symplectiques sont calculées explicitement dans le cas des fibrés cotangents. Bien que cette thèse ne contienne pas de résultats nouveaux, elle contient des preuves détaillées de certains résultats qui sont énoncés sans preuve dans les travaux de Tseng et Yau. Nous concluons thèse avec des perspectives et des questions ouvertes.

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Contents

Introduction

Hodge theory for symplectic manifolds first was discussed by Ehresmann and Libermann [6, 10] and then by Brylinski [4], but their approach was not entirely successful, as we shall see below. Their approach is in parallel with the Hodge theory in Riemannian geometry, which we will shortly review in subsection 1.1. In summary, one defines a symplectic star operator $*_s$ (see subsection 2.1.2) by replacing the Riemannian metric g with the symplectic form ω in the definition of the Hodge star operator $*$. This operator allows us to define the symplectic adjoint of the exterior derivative by the formula

$$d^\Lambda = (-1)^{k+1} *_s d *_s,$$

acting on k-forms in analogy with the adjoint d^* in Riemannian geometry. Defining the symplectic harmonic forms as the differential forms that are both d- and d^Λ-closed, we are looking for the relationship between the de Rham cohomology classes and symplectic harmonic forms. But in contrast with Riemannian geometry, it is not the case that there exists a symplectic harmonic representative for each de Rham cohomology class of a compact symplectic manifold. It was proved by Mathieu [11] and Yan [20] that every de Rham cohomology class contains a symplectic harmonic form if and only if the strong Lefschetz property holds i.e. the maps $H_d^k \to H_d^{2n-k}$ between de Rham cohomology groups given by $[A] \mapsto [\omega^{n-k} \wedge A]$ are surjective for all $0 \leq k \leq n$. Even in this case, symplectic harmonic representatives of de Rham cohomology classes are not unique in contrast with Riemannian geometry. This suggests that this is not an optimal approach and the de Rham cohomology, being a topological invariant, is not an appropriate choice to reflect the symplectic structure properties.

In this thesis, we study a new approach to the Hodge theory on symplectic manifolds recently introduced by Tseng and Yau [16, 17] which is in parallel with complex geometry. The first key step is to define the space $\mathcal{L}^{r,s} = \{\omega^r \wedge B_s : B_s \in \mathcal{P}^s\}$, where $\mathcal{P}^s$ is the space of primitive s-forms, in analogy to the space of type (p, q) forms in complex geometry. This gives a pyramid shape decomposition of differential forms presented in (13), similar to the diamond shape decomposition (6), which familiar from almost complex geometry. Then, the exterior derivative d acting on $\mathcal{L}^{r,s}$ has only two components, so this allows us to define two first order differential operators (∂_+, ∂_-). The pairs (∂_+, ∂_-) and (d, d^Λ) are symplectic counterparts of the pairs $(\partial, \bar{\partial})$ and (d, d^c) in complex geometry. Using these differential operators, we can define some new symplectic cohomologies by the formulas

$$PH^k_{\partial_\pm} = \frac{ker\,\partial_\pm \cap \mathcal{P}^k}{im\,\partial_\pm \cap \mathcal{P}^k},$$

$$PH^k_{d+d^\Lambda} = PH^k_{\partial_+ + \partial_-} = \frac{ker\,(\partial_+ + \partial_-) \cap \mathcal{P}^k}{im\,\partial_+\partial_- \cap \mathcal{P}^k},$$

$$PH^k_{dd^\Lambda} = PH^k_{\partial_+\partial_-} = \frac{ker\,\partial_+\partial_- \cap \mathcal{P}^k}{(im\,\partial_+ + im\,\partial_-) \cap \mathcal{P}^k}.$$

These cohomologies are symplectic analogues of the well-known Dolbeault cohomology, Bott-Chern cohomology [2]

$$H^{p,q}_{\partial+\bar{\partial}} = \frac{ker\,(\partial + \bar{\partial}) \cap \mathcal{A}^{p,q}}{im\,\partial\bar{\partial} \cap \mathcal{A}^{p,q}}, \tag{1}$$

and Aeppli cohomology [1]

$$H^{p,q}_{\partial\bar{\partial}} = \frac{ker\,\partial\bar{\partial} \cap \mathcal{A}^{p,q}}{(im\,\partial + im\,\bar{\partial}) \cap \mathcal{A}^{p,q}}. \tag{2}$$

We can summarize these similarities of symplectic and complex manifolds in the following table:

	Complex manifold	Symplectic manifold
Differential forms	$\mathcal{A}^{p,q}$ Diamond (6)	$\mathcal{L}^{r,s}$ Pyramid (13)
Differential operators	$d = \partial + \bar{\partial}$ $d^c = i(\bar{\partial} - \partial)$ $(\partial, \bar{\partial})$	$d = \partial_+ + L\partial_-$ $d^\Lambda = \frac{1}{H+R+1}\partial_+\Lambda - (H+R)\partial_-$ (∂_+, ∂_-)
Cohomologies	Dolbeault Bott-Chern Aeppli	$PH^k_{\partial_\pm}$ $PH^k_{d+d^\Lambda}$ $PH^k_{dd^\Lambda}$

For a compact manifold, by choosing a compatible Riemannian metric, we define the associated Laplacians for these cohomologies and take advantage of general Hodge theory of self-adjoint elliptic operators on compact manifolds [19] to conclude that all of above cohomologies are finite dimensional. This also implies that the following pairings

$$PH^k_{d+d^\Lambda} \otimes PH^k_{dd^\Lambda} \to \mathbb{R}, \quad PH^k_{\partial_+} \otimes PH^k_{\partial_-} \to \mathbb{R},$$
$$[A] \otimes [A'] \mapsto \int_M A \wedge *_s A',$$

are non-degenerate and therefore we have dualities $PH^k_{d+d^\Lambda} \cong PH^k_{dd^\Lambda}$ and $PH^k_{\partial_+} \cong PH^k_{\partial_-}$.

We then prove Poincaré lemmas for all of above cohomologies and compute them for the general example of cotangent bundles. The results are summarized in the following table:

	Star-shaped open subset of $\mathbb{R}^{2n}$	Cotangent bundle
$PH^k_{d+d^\Lambda}$	$\mathbb{R}$ $\quad k=0$ 0 $\quad 0<k\leq n$	H^k_d $\quad 0\leq k\leq n$
$PH^k_{dd^\Lambda}$	$\mathbb{R}$ $\quad k=1$ 0 $\quad k=0, \quad 2\leq k\leq n$	H^{k-1}_d $\quad 0\leq k<n$ $H^{n-1}_d \oplus H^n_d$ $\quad k=n$
$PH^k_{\partial_+}$	$\mathbb{R}$ $\quad k=0, \quad k=1$ 0 $\quad 2\leq k<n$	$H^{k-1}_d \oplus H^k_d$ $\quad 0\leq k<n$
$PH^k_{\partial_-}$	0 $\quad 0\leq k<n$	0 $\quad 0\leq k<n$

Note that the Euclidean space $\mathbb{R}^{2n}$ with the standard symplectic structure is also the cotangent bundle of $\mathbb{R}^n$ and these two results are the same in this special case.

Our thesis is organized as follows. Section 1 is a short review of classical Hodge theories of Riemannian, complex and Kähler geometries. In section 2, some algebraic and differential operators are introduced and their properties are studied in detail. Different symplectic cohomologies are defined in section 3. We study Hodge theories of these cohomologies and conclude their finite-dimensionality for compact symplectic manifolds. More results for these cohomologies are proved in Section 4 including duality properties, Poincaré lemmas and the fact that under the assumption of compactness and the dd^Λ-lemma, these cohomologies are isomorphic to the de Rham cohomology. The thesis concludes with a brief account of a Mayer-Veitoris-type construction for symplectic cohomologies and a section summarizing the conclusions and outline of open problems.

1 Review of classical Hodge theories

This is just a short review section without details. For further details see the given references for each subsection.

1.1 Compact orientable manifolds

The main reference for this subsection is [18]. Let M be an orientable n-manifold (without boundary). Choose and fix an arbitrary orientation and Riemannian metric g on M. Given a point p in M, we have an isomorphism $f_p : T_p(M) \to T_p^*(M)$ given by $V \mapsto i_V(g_p)$, where $i_\Box$ is the interior product. For $\theta^i, \eta^i \in \Omega^1$ for $i = 1, ..., k$, define

$$< \theta^1 \wedge ... \wedge \theta^k, \eta^1 \wedge ... \wedge \eta^k >_g = det(g_p(f_p^{-1}(\eta_p^i), f_p^{-1}(\theta_p^j))), \tag{3}$$

and extend it linearly to Ω^k. Moreover, we define $< f, g >_g = f.g$ for $f, g \in C^\infty(M) = \Omega^0$ and $< A_{k_1}, A_{k_2} >_g = 0$ for $A_{k_i} \in \Omega^{k_i}$ and $k_1 \neq k_2$. The Hodge star operator $* : \Omega^k \to \Omega^{2n-k}$ is defined by $A \wedge *A' =< A, A' >_g dvol$ for $A, A' \in \Omega^k$, where $dvol$ is the volume element. It satisfies $** = (-1)^{k(n-k)}$.

Moreover, if M is compact, then the formula $< A_k, A'_k >:= \int_M A_k \wedge *A'_k$ is an inner product structure on $\Omega(M)$. For a linear operator T on $\Omega(M)$, if there exists an operator T^* on $\Omega(M)$ satisfying $< TA, A' >=< A, T^*A' >$, then the operator T^* is called the (formal) adjoint of T. Then immediately, we have

$$T^{**} = T, \quad (T + T')^* = T^* + T'^*, \quad (cT)^* = cT^*, \quad (TT')^* = T'^*T^*,$$

for any linear operator T and real constant c. One can show $d^* = (-1)^{n(k+1)+1} * d*$. Define the Hodge Laplacian by $\Delta_d = d^*d + dd^*$ and the space of d-harmonic k-forms by

$$\mathcal{H}_d^k = ker\, \Delta_d \cap \Omega^k = ker\, d \cap ker\, d^* \cap \Omega^k.$$

Also, write $H_d^k = \frac{ker\, d \cap \Omega^k}{im\, d \cap \Omega^k}$ for the k-th real de Rham cohomology class.

Proposition 1.1. *For a compact oriented Riemannian manifold M, the operator Δ_d is elliptic and therefore we have*

a) $dim\, \mathcal{H}_d^k < \infty$

b) The orthogonal decomposition $\Omega^k = \mathcal{H}_d^k \oplus d\Omega^{k-1} \oplus d^*\Omega^{k+1}$

c) $H_d^k \cong \mathcal{H}_d^k$ *i.e. there is a unique d-harmonic form in each de Rham cohomology class.*

d) $* : \mathcal{H}_d^k \to \mathcal{H}_d^{2n-k}$ *is an isomorphism for* $0 \leq k \leq 2n$.

Corollary 1.2. *For a compact orientable manifold* M*, we have dim* $H_d^k < \infty$ *and the following non-degenerate natural pairing and therefore* $H_d^k \cong H_d^{2n-k}$.

$$H_d^k \otimes H_d^{2n-k} \to \mathbb{R}, \qquad [A] \otimes [A'] \mapsto \int_M A \wedge A'. \tag{4}$$

1.2 Almost complex manifolds

The references of this subsection are [9] and [12]. Let M be a $2n$-manifold (without boundary). An almost complex structure J is a linear map $J_p : T_p(M) \to T_p(M)$ with the property $J_p^2 = Id$ for all $p \in M$. Consider the complexified tangent space $T_{\mathbb{C}}(M) = T(M) \otimes \mathbb{C}$ and extend J to a $\mathbb{C}$-linear operator on $T_{\mathbb{C}}(M)$. Write $T'(M)$ and $T''(M)$ for eigenspaces of J corresponding to the eigenvalues i and $-i$, respectively. Then, we have $T_{\mathbb{C}}(M) = T'(M) \oplus T''(M)$ and $\bar{T}'(M) = T''(M)$, where the conjugation is in $T_{\mathbb{C}}(M)$. Define the space of complex-valued k-forms by $\mathcal{A}^k = \bigwedge^k T^*_{\mathbb{C}}(M)$ and the space of type (p, q) forms by the formula

$$\mathcal{A}^{p,q} = \bigwedge^p T'^*(M) \otimes \bigwedge^q T''^*(M) \qquad 0 \leq p, q \leq n. \tag{5}$$

We can arrange these spaces in the following diamond shape order

$$\begin{array}{ccccccc}
 & & & \mathcal{A}^{n,n} & & & \\
 & & \mathcal{A}^{n,n-1} & & \mathcal{A}^{n-1,n} & & \\
 & \mathcal{A}^{n,n-2} & & \mathcal{A}^{n-1,n-1} & & \mathcal{A}^{n-2,n} & \\
 \cdot^{\cdot^{\cdot}} & & & & & & \ddots \\
\mathcal{A}^{n,0} & & & \vdots & & & \mathcal{A}^{0,n} \\
 \ddots & & & & & & \cdot^{\cdot^{\cdot}} \\
 & \mathcal{A}^{2,0} & & \mathcal{A}^{1,1} & & \mathcal{A}^{0,2} & \\
 & & \mathcal{A}^{1,0} & & \mathcal{A}^{0,1} & & \\
 & & & \mathcal{A}^{0,0} & & &
\end{array} \tag{6}$$

The above decomposition and the formula $\bar{T}'(M) = T''(M)$ give

$$\mathcal{A}^k = \bigoplus_{p+q=k} \mathcal{A}^{p,q}, \qquad \bar{\mathcal{A}}^{p,q} = \mathcal{A}^{q,p},$$

An almost complex structure J and a Riemannian metric g are called compatible if we have $g(JV, JW) = g(V, W)$. Likewise, an almost complex structure J and a non-degenerate 2-form ω are called compatible if we have

$$\omega(V, JV) > 0 \qquad V \neq 0,$$
$$\omega(JV, JW) = \omega(V, W).$$

Having an almost complex structure J, there is a 1-1 correspondence between compatible Riemannian metrics and compatible non-degenerate 2-forms related to each other by formulas

$$\omega(V, W) = g(JV, W), \qquad g(V, W) = \omega(V, JW),$$

such a triple (ω, J, g) is called a compatible triple. If we have an almost complex structure J, then there exists a compatible triple (ω, J, g). Also having a non-degenerate 2-form ω, there exists a compatible triple (ω, J, g) (see [12]).

1.3 Compact complex manifolds

The reference of this and next subsections is [7]. An almost complex structure is called integrable if there is an open cover of M with local charts such that the transition maps are holomorphic. A complex structure is an integrable almost complex structure and a complex manifold is a manifold with a complex structure. Any complex manifold is naturally oriented using the standard oreintation on $\mathbb{R}^{2n} = \mathbb{C}^n$ since holomorphic maps are orientation preserving. On a complex manifold, we have $d : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q} \oplus \mathcal{A}^{p,q+1}$ so we can define its components $\partial : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q}$ and $\bar{\partial} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}$ such that $d = \partial + \bar{\partial}$. Also, define the operator $d^c = i(\bar{\partial} - \partial)$. We have

$$\partial^2 = \bar{\partial}^2 = d^{c2} = 0, \qquad \partial\bar{\partial} = -\bar{\partial}\partial, \qquad dd^c = -d^c d = 2i\partial\bar{\partial}.$$

We can define

$$H_d^k(\mathbb{C}) = \frac{ker\, d \cap \mathcal{A}^k}{im\, d \cap \mathcal{A}^k} = H_d^k \otimes \mathbb{C},$$

$$H_{\partial}^{p,q} = \frac{ker\,\partial \cap \mathcal{A}^{p,q}}{im\,\partial \cap \mathcal{A}^{p,q}}, \qquad H_{\bar\partial}^{p,q} = \frac{ker\,\bar\partial \cap \mathcal{A}^{p,q}}{im\,\bar\partial \cap \mathcal{A}^{p,q}}. \tag{7}$$

Note that we have $H_{\bar\partial}^{p,q} = \bar{H}_{\partial}^{q,p}$ and they are called the Dolbeault cohomology groups. We have the $\bar\partial$-Poicaré lemma stating that $H_{\bar\partial}^{p,q} = 0$ for $q > 0$ and M be a polydisk in $\mathbb{R}^{2n} = \mathbb{C}^n$.

Let M be a complex manifold with complex structure J and (real) dimension $2n$. A Hermitian metric on M is a Hermitian inner product on $T'_p(M)$ for all $p \in M$ depending smoothly on p. Choose and fix an arbitrary compatible triple (ω, J, g) on M. Then, there is a Hermitian metric on M defined by $g + i\omega$. Using this Hermitian metric and completely similar to subsection 1.1, we can define a complex Hodge star operator $* : \mathcal{A}^{p,q} \to \mathcal{A}^{n-p,n-q}$ satisfying $** = (-1)^{(p+q)(2n-p-q)} = (-1)^{p+q}$. Acting on Ω^k, it is the same as $*$ operator induced by g.

Moreover, if M is compact, then the formula $< A_k, A'_k >:= \int_M A_k \wedge *A'_k$ makes $\mathcal{A}(M)$ into a Hermitian inner product space such that its real part is the inner product induced by g on $\Omega(M)$. Similarly, define the (formal) adjoint T^* of an operator T on $\mathcal{A}(M)$. Then immediately, we have

$$T^{**} = T, \quad (T + T')^* = T^* + T'^*, \quad (cT)^* = \bar{c}T^*, \quad (TT')^* = T'^*T^*,$$

for any operator T and complex constant c. We have

$$d^* = - * d*, \quad \partial^* = - * \partial *, \quad \bar\partial^* = - * \bar\partial * .$$

Define Laplacians $\Delta_\partial = \partial^*\partial + \partial\partial^*$ and $\Delta_{\bar\partial} = \bar\partial^*\bar\partial + \bar\partial\bar\partial^*$ and the space of $\bar\partial$-harmonic (p,q)-forms by

$$\mathcal{H}_{\bar\partial}^{p,q} = ker\,\Delta_{\bar\partial} \cap \mathcal{A}^{p,q} = ker\,\bar\partial \cap ker\,\bar\partial^* \cap \mathcal{A}^{p,q}. \tag{8}$$

Proposition 1.3. *For a compatible triple (ω, J, g) on a compact complex manifold M, we have*

a) $dim\,\mathcal{H}_{\bar\partial}^{p,q} < \infty$

b) The orthogonal decomposition $\mathcal{A}^{p,q} = \mathcal{H}_{\bar\partial}^{p,q} \oplus \bar\partial\mathcal{A}^{p,q-1} \oplus \bar\partial^*\mathcal{A}^{p,q+1}$

c) $H_{\bar\partial}^{p,q} \cong \mathcal{H}_{\bar\partial}^{p,q}$ i.e. there is a unique $\bar\partial$-harmonic form in each Dolbeault cohomology class.

d) $* : \mathcal{H}_{\bar\partial}^{p,q} \to \mathcal{H}_{\bar\partial}^{n-p,n-q}$ is an isomorphism for $0 \le p,q \le n$.

Corollary 1.4. *For a compact complex manifold M, we have dim $H_{\bar\partial}^{p,q} < \infty$ and the following natural pairing*

$$H_{\bar\partial}^{p,q} \otimes H_{\bar\partial}^{n-p,n-q} \to \mathbb{C}, \qquad [A] \otimes [A'] \mapsto \int_M A \wedge A',$$

is non-degenerate and therefore $H_{\bar\partial}^{p,q} \cong H_{\bar\partial}^{n-p,n-q}$.

A Kähler manifold is a smooth manifold with a compatible triple (ω, J, g) such that J is integrable and ω is d-closed. In other words, Kähler manifolds are both symplectic and complex manifolds in ways that are compatible with each other. Let M be a compact Kähler $2n$-manifold. Consider the operators L and Λ defined in subsection 2.1.1. Then, we have lots of identities involving different operators known as Kähler identities. Some of them are

$$[\partial, \Lambda] = -i\bar\partial^*, \qquad [\bar\partial, \Lambda] = i\partial^*, \qquad \partial\bar\partial^* = -\bar\partial^*\partial,$$
$$\Delta_d = 2\Delta_\partial = 2\Delta_{\bar\partial}, \qquad [\partial, L] = [\bar\partial, L] = [\Delta_d, L] = [\Delta_d, \Lambda] = 0.$$

Using these identities one can prove the following proposition.

Proposition 1.5. *For a compact Kähler manifold M, we have*

a) $H_{\bar\partial}^{p,q} \cong H_{\bar\partial}^{q,p}$ and $H_d^k(\mathbb{C}) \cong \bigoplus_{p+q=k} H_{\bar\partial}^{p,q}$

b) dim H_d^{2k+1} is even, dim $H_{\bar\partial}^{p,p} > 0$ and dim $H_d^{2k} > 0$ for $0 \le k, p \le n$.

c) The maps $L^{n-k} : H_d^k \to H_d^{2n-k}$ and $L^{n-p-q} : H_{\bar\partial}^{p,q} \to H_{\bar\partial}^{n-q,n-p}$ are isomorphisms for $0 \le k, p+q \le n$.

d) The following pairing is non-degenerate for $0 \le k \le n$:

$$H_d^k \otimes H_d^k \to \mathbb{R}, \qquad [A] \otimes [A'] \mapsto \int_M L^{n-k}(A \wedge A').$$

2 Operators on symplectic manifolds

In this section, we define different algebraic and differential operators on symplectic manifolds and study their properties. Each of these operators acts linearly on forms, and thus is a linear map $\Omega(M) \to \Omega(M)$.

From this section onward, (M, ω) is a symplectic manifold (without boundary) with dimension $2n$ i.e. ω is a d-closed non-degenerate 2-form on M. We will use the notation A_k or just simply A for a k-form on M i.e. $A_k \in \Omega^k$. Sometimes we will use other notations as well. Also, unless otherwise specified the main references for this work are either [16] or [17].

2.1 Algebraic operators

Since all the definitions and results in the following two subsections are algebraic, they hold for symplectic vector spaces as well as symplectic manifolds.

2.1.1 Lefschetz decomposition (L, Λ, H and R)

While most of the ideas in this subsection are given in [16] or [17], the proofs given here have been obtained independently. The first three operators are the well-known operators that give a representation of the $sl(2)$ algebra on $\Omega(M)$ and consequently define a Lefschetz decomposition of forms. Since these operators and decomposition are well-studied, we will state some of their elementary properties without proof (properties I-III below).

Definition 2.1. *The* ***Lefschetz operator*** $L : \Omega^k \to \Omega^{k+2}$*, the* ***dual Lefschetz operator*** $\Lambda : \Omega^k \to \Omega^{k-2}$ *and the* ***degree counting operator*** $H : \Omega^k \to \Omega^k$ *are defined by*

$$L(A_k) := \omega \wedge A_k, \tag{9}$$

$$\Lambda(A_k) := \sum_{i,j=1}^{2n} \frac{1}{2}(\omega^{-1})^{ij} i_{\partial_{x^i}} i_{\partial_{x^j}} A_k, \tag{10}$$

$$H(A_k) := (n-k)A_k, \tag{11}$$

where x is a local coordinate, $\omega = \sum_{i,j=1}^{2n} \frac{1}{2}\omega_{ij} dx^i \wedge dx^j$ and $((\omega^{-1})^{ij})$ is the inverse matrix of (ω_{ij}). Also, $i_\square$ stands for the interior product. These operators are extended linearly to $\Omega(M)$. Also, we will set $\Omega^k = \{0\}$ for $k > 2n$ or $k < 0$, for example in $\Lambda : \Omega^1 \to \Omega^{-1}$.

The commutators of these operators are given by

$$[\Lambda, L] = H, \quad [H, \Lambda] = 2\Lambda, \quad [H, L] = -2L, \tag{12}$$

on $\Omega(M)$, so we have a $sl(2)$-representation. Consequently, we have the following three properties:

(I) The maps $L^{n-k} : \Omega^k \to \Omega^{2n-k}$ and $\Lambda^{n-k} : \Omega^{2n-k} \to \Omega^k$ are isomorphisms for $k = 0, ..., n$ (considering L^0 and Λ^0 as the identity map). This implies that $L^r : \Omega^k \to \Omega^{k+2r}$ is injective if $k + r \leq n$ and $\Lambda^r : \Omega^k \to \Omega^{k-2r}$ is injective if $k - r \geq n$.

Definition 2.2. *Define the space of* ***primitive*** *forms as* $\mathcal{P}(M) := ker(\Lambda : \Omega(M) \to \Omega(M))$*. We will use the notation* B_k *or just simply* B *to represent a primitive* k*-form i.e. a member of* $\mathcal{P}^k := \mathcal{P}(M) \cap \Omega^k$*.*

Note that we have $\mathcal{P}^0 = \Omega^0 = C^\infty(M)$ and $\mathcal{P}^1 = \Omega^1$. Also, property (I) implies that $\mathcal{P}^k = \{0\}$ for $k > n$.

(II) The second property gives us an alternative definition of primitive forms:

$$\mathcal{P}^k = ker(L^{n-k+1} : \Omega^k \to \Omega^{2n-k+2}) \qquad k = 0, ..., n.$$

(III) Finally, the last elementary property known as the **Lefschetz decomposition** gives $\Omega^k = \bigoplus_{r \geq 0} L^r \mathcal{P}^{k-2r}$. By equality (2.1.1), if we have $r + k - 2r > n$, then $L^r \mathcal{P}^{k-2r} = \{0\}$. Therefore, we can assume that $max(0, k - n) \leq r \leq \lfloor \frac{k}{2} \rfloor$.

Definition 2.3. *Define the spaces* $\mathcal{L}^{r,s} := L^r \mathcal{P}^s \subseteq \Omega^{2r+s}$ *and the operator* $R : \mathcal{L}^{r,s} \to \mathcal{L}^{r,s}$ *by* $RA := rA$ *for* $A \in \mathcal{L}^{r,s}$*. Extend this operator linearly to a map* $R : \Omega(M) \to \Omega(M)$*.*

Lemma 2.4. *We have the following identities on* $\Omega(M)$*:*

$$\begin{aligned}
[\Lambda, L^r] &= (H + r - 1) r L^{r-1} \qquad r \geq 1, \\
[H, R] &= 0, \\
\Lambda L &= (H + R)(R + 1), \\
L \Lambda &= (H + R + 1) R.
\end{aligned}$$

Proof. The first equality is proved by induction on r. For $r = 1$, it is just $[\Lambda, L] = H$ in (12). Now, assume that it is true for $r - 1$ and we will prove it for r:

$$\begin{aligned}[\Lambda, L^r] &= \Lambda L L^{r-1} - L^r \Lambda = L \Lambda L^{r-1} + H L^{r-1} - L^r \Lambda \\ &= L L^{r-1} \Lambda + L((H + r - 2)(r-1)L^{r-2}) + HL^{r-1} - L^r \Lambda \\ &= (H + 2 + r - 2)(r - 1)LL^{r-2} + HL^{r-1} = (H + r - 1)rL^{r-1}.\end{aligned}$$

For the remaining equalities, it is enough by linearity to verify them only on $\mathcal{L}^{r,s}$. Taking any $B_s \in \mathcal{P}^s$, we have

$$\begin{aligned}HR(L^r B_s) &= (n - 2r - s)rL^r B_s = RH(L^r B_s), \\ \Lambda L(L^r B_s) &= \Lambda L^{r+1} B_s = L^{r+1}\Lambda B_s + (H + r)(r + 1)L^r B_s = (H + R)(R + 1)(L^r B_s),\end{aligned}$$

which prove the second and third formulas. We can prove the last one directly similar to the third one or prove it as follows:

$$L\Lambda = \Lambda L - H = (H + R)(R + 1) - H = (H + R + 1)R. \quad \square$$

By equality (2.1.1), the space $\mathcal{L}^{r,s}$ is nontrivial only if $0 \le r, s$ and $r + s \le n$. We can arrange these spaces in the following pyramid:

$$\begin{array}{ccccccccccc} & & & & & \mathcal{L}^{0,n} & & & & & \\ & & & & \mathcal{L}^{0,n-1} & & \mathcal{L}^{1,n-1} & & & & \\ & & & \mathcal{L}^{0,n-2} & & \mathcal{L}^{1,n-2} & & \mathcal{L}^{2,n-2} & & & \\ & & \cdot^{\cdot^{\cdot}} & & & \vdots & & & \ddots & & \\ & \mathcal{L}^{0,1} & & & & & & & & \mathcal{L}^{n-1,1} & \\ \mathcal{L}^{0,0} & & \mathcal{L}^{1,0} & & & \cdots & & & \mathcal{L}^{n-1,0} & & \mathcal{L}^{n,0} \end{array} \tag{13}$$

The left edge corresponds to the primitive spaces ($r = 0$) and the right edge corresponds to $r + s = n$. The direct sum of k-th vertical line ($0 \le k \le 2n$) gives the Lefschetz decomposition $\Omega^k = \bigoplus_{r=max(0,k-n)}^{\lfloor \frac{k}{2} \rfloor} \mathcal{L}^{r,k-2r}$. On the other hand, all the spaces on the same horizontal line are isomorphic to each other by the following lemma.

Lemma 2.5. *The map $L : \mathcal{L}^{r,s} \to \mathcal{L}^{r+1,s}$ is an isomorphism for $0 \le r, s$ and $r+s < n$ i.e. as long as we don't leave the pyramid. Similarly, the map $\Lambda : \mathcal{L}^{r,s} \to \mathcal{L}^{r-1,s}$ is an isomorphism for $0 < r$, $0 \le s$ and $r+s \le n$ i.e. as long as we don't leave the pyramid.*

Proof. First note that we have $\Lambda(\mathcal{L}^{r,s}) \subseteq \mathcal{L}^{r-1,s}$ by the definition of the spaces $\mathcal{L}^{r,s}$ and the first identity in Lemma 2.4. Also by the definition of the spaces $\mathcal{L}^{r,s}$, the map L is clearly surjective. According to Lemma 2.4, we know that $\Lambda L : \mathcal{L}^{r,s} \to \mathcal{L}^{r,s}$ is simply multiplication by the constant $(n-r-s)(r+1)$. But this constant is positive since we stay in the pyramid and hence $r+s < n$. Consequently, the map ΛL is an isomorphism and L is also injective. Therefore, both such maps L and Λ are isomorphisms. $\square$

For any k-form $A_k \in \Omega^k$, its Lefschetz decomposition gives primitive forms $B_{k-2r} \in \mathcal{P}^{k-2r}$ such that

$$A_k = \sum_{r=max(0,k-n)}^{\lfloor \frac{k}{2} \rfloor} \frac{1}{r!} L^r B_{k-2r}. \tag{14}$$

The nonzero coefficients $\frac{1}{r!}$ added to make some future formulas simpler. By Lemma 2.5, the map $L^r : \mathcal{P}^{k-2r} \to \mathcal{L}^{r,k-2r}$ is an isomorphism for these values of r and therefore these primitive forms are unique and are called the **Lefschetz components** of A_k. They can be computed using the following lemma.

Lemma 2.6. *Let A_k and B_{k-2r} be as above. For some rational coefficients $a_{r,t}$ depending on n, k, r and t, we have*

$$B_{k-2r} = \sum_{t \ge 0} a_{r,t} L^t \Lambda^{r+t} A_k.$$

Proof. We will prove this lemma inductively. Let $r_0 = \lfloor \frac{k}{2} \rfloor$. Applying Λ^{r_0} to the formula (14) and using the first formula in Lemma 2.4 repeatedly, we will find an equation of the form $\Lambda^{r_0} A_k = a B_{k-2r_0}$, where a is a rational number. If $r_0 = 0$, then we have $a = 1$ and if $r_0 > 0$, then the inequality $2r_0 > r_0 \ge k-n$ implies that $k - 2r_0 < n$, so $H > 0$ and therefore a is nonzero. By dividing this equation by a, the lemma is proved for the greatest value of r, i.e. r_0.

Now, let r_1 be $r_0 - 1$ and apply Λ^{r_1} to the formula (14) to find an equation of the form $\Lambda^{r_1} A_k = b B_{k-2r_1} + c L B_{k-2r_0}$, where b and c are rational numbers. By the same argument as above b is nonzero. Consequently, we can replace B_{k-2r_0} from the previous step and solve this equation for B_{k-2r_1} to prove the lemma for r_1, and so on. $\square$

Corollary 2.7. *If we have $\Lambda^t A_k = 0$, then there are at most t nonzero Lefschetz components B_{k-2r} for $r = 0, ..., t-1$.*

Lemma 2.8.

$$\Lambda(A_k) \wedge A_{2n-k+2} = A_k \wedge \Lambda(A_{2n-k+2}),$$

while it is not true in general that $\Lambda(A) \wedge A' = A \wedge \Lambda(A')$.

Proof. Let A' be a nonzero primitive form and $A = \omega$, then $\Lambda(A) \wedge A' = nA' \neq 0 = A \wedge \Lambda(A')$. To prove the equality when the form $\Lambda(A) \wedge A'$ has the maximum degree $2n$, it is enough to check that $(i_{\partial_{x^i}} i_{\partial_{x^j}} A) \wedge A' = A \wedge (i_{\partial_{x^i}} i_{\partial_{x^j}} A')$ for $A = dx^{i_1} \wedge ... \wedge dx^{i_k}$, $A' = dx^{j_1} \wedge ... \wedge dx^{j_{k'}}$ and $k + k' = 2n + 2$, where $x = (x^1, ..., x^{2n})$ is a local coordinate.

Case 1 (Suppose that the indices i and j are not both among i_1, ..., i_k.): Then, we have $i_{\partial_{x^i}} i_{\partial_{x^j}} A = 0$, so the left hand side vanishes. On the other hand, the $2n$-form $A \wedge (i_{\partial_{x^i}} i_{\partial_{x^j}} A')$ is of the form $dx^{i'_1} \wedge ... \wedge dx^{i'_{2n}}$ without both indices i and j appearing, which means there exists a repeated index and the right hand side is also zero.

Case 2 (Suppose that the indices i and j are not both among j_1, ..., $j_{k'}$.): Similar to the previous case.

Case 3 (Both indices i and j are among i_1, ..., i_k and also among j_1, ..., $j_{k'}$.): Without loss of generality, assume that $i_1 = j = j_1$ and $i_2 = i = j_2$. Then the equality follows from the identity

$$(dx^{i_3} \wedge ... \wedge dx^{i_k}) \wedge (dx^j \wedge dx^i \wedge dx^{j_3} \wedge ... \wedge dx^{j_{k'}}) = (dx^j \wedge dx^i \wedge dx^{i_3} \wedge ... \wedge dx^{i_k}) \wedge (dx^{j_3} \wedge ... \wedge dx^{j_{k'}}). \quad \square$$

2.1.2 The symplectic star operator $*_s$

None of the proofs in this subsection are given in [16] or [17]. While these are not new facts and you can find their proofs in many references, the proofs given below appear to be original.

Completely similar to oriented Riemannian manifolds, we can define a bilinear form and star operator on the forms on a symplectic manifold as follows. Given a point p in M, we have an isomorphism $f_p : T_p(M) \to T_p^*(M)$ given by $V \mapsto i_V(\omega_p)$ since the symplectic form ω is nondegenerate. For $\theta^i, \eta^i \in T_p^*(M)$ $(i = 1, ..., k)$, define

$$< \theta^1 \wedge ... \wedge \theta^k, \eta^1 \wedge ... \wedge \eta^k >_\omega = det(< \theta^i, \eta^j >_\omega) = det(\omega_p(f_p^{-1}(\eta^i), f_p^{-1}(\theta^j))), \qquad (15)$$

and extend it linearly to $\bigwedge^k(T_p^*(M))$. For $k = 0$, we use $< f, g >_\omega = f.g$ $(f, g \in C^\infty(M))$. Note that we have $< A_k, A'_k >_\omega = (-1)^k < A'_k, A_k >_\omega$. Finally, we define $< A_k, A'_{k'} >_\omega = 0$ for $k \neq k'$.

Proposition 2.9. *In a local coordinate x, we have*

$$< A, A' >_\omega = \sum_{i_1,...,i_k,j_1,...j_k=1}^{2n} \frac{1}{k!}(\omega^{-1})^{i_1 j_1}...(\omega^{-1})^{i_k j_k} A_{i_1...i_k} A'_{j_1...j_k} \qquad A, A' \in \Omega^k,$$

where $A_{i_1...i_k} = A(\partial_{x^{i_1}}, ..., \partial_{x^{i_k}})$ or equivalently $A = \sum_{i_1,...,i_k=1}^{2n} \frac{1}{k!} A_{i_1...i_k} dx^{i_1} \wedge ... \wedge dx^{i_k}$.

Proof. Since both sides of the equality are pointwise bilinear, it is enough to check it for

$$A = dx^{i_1} \wedge ... \wedge dx^{i_k} = \sum_{\sigma'} \frac{1}{k!} sgn(\sigma') dx^{i_{\sigma'(1)}} \wedge ... \wedge dx^{i_{\sigma'(k)}},$$
$$A' = dx^{j_1} \wedge ... \wedge dx^{j_k} = \sum_{\sigma''} \frac{1}{k!} sgn(\sigma'') dx^{j_{\sigma''(1)}} \wedge ... \wedge dx^{j_{\sigma''(k)}}.$$

Since the isomorphism $f_p^{-1} : T_p^*(M) \to T_p(M)$ sends dx_p^i to $\sum_{j=1}^{2n} (\omega^{-1})^{ij} \partial_{x^j}|_p$, the left hand side is (using formula (15))

$$< A, A' >_\omega = det(< dx^{i_l}, dx^{j_m} >_\omega) = det(\omega(\sum_{b=1}^{2n} (\omega^{-1})^{j_m b} \partial_{x^b}, \sum_{a=1}^{2n} (\omega^{-1})^{i_l a} \partial_{x^a}))$$
$$= det(\sum_{a,b=1}^{2n} (\omega^{-1})^{i_l a} (\omega^{-1})^{j_m b} \omega_{ba}) = det((\omega^{-1})^{i_l j_m}).$$

On the other hand, the right hand side is

$$\sum_{\sigma',\sigma''} \frac{1}{k!} (\omega^{-1})^{i_{\sigma'(1)} j_{\sigma''(1)}} ... (\omega^{-1})^{i_{\sigma'(k)} j_{\sigma''(k)}} sgn(\sigma') sgn(\sigma'')$$
$$= \sum_{\sigma', \sigma = \sigma'' \circ \sigma'^{-1}} \frac{1}{k!} (\omega^{-1})^{i_1 j_{\sigma(1)}} ... (\omega^{-1})^{i_k j_{\sigma(k)}} sgn(\sigma)$$
$$= \sum_{\sigma} (\omega^{-1})^{i_1 j_{\sigma(1)}} ... (\omega^{-1})^{i_k j_{\sigma(k)}} sgn(\sigma) = det((\omega^{-1})^{i_l j_m}). \qquad \square$$

Definition 2.10. *The* ***symplectic star operator*** $*_s : \Omega^k \to \Omega^{2n-k}$ *is defined by*

$$A \wedge *_s A' =< A, A' >_\omega dvol \qquad A, A' \in \Omega^k,$$

where $dvol = \frac{\omega^n}{n!}$ *is the (symplectic) volume element.*

Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds of dimensions $2n_1$ and $2n_2$. We will use the notation L_1, Λ_1 and $*_s^1$ for the above operators on M_1 and similarly for M_2. Then, the product manifold $(M := M_1 \times M_2, \omega := \omega_1 + \omega_2)$ is a symplectic manifold of dimension $2n := 2(n_1 + n_2)$. To prove some facts about the operator $*_s$, we can use mathematical induction on n but to do so we first need to know what is the relationship between $*_s$, $*_s^1$ and $*_s^2$.

Lemma 2.11. *Consider a (k_1+k_2)-form $A_{k_1} \wedge A_{k_2}$ on $M = M_1 \times M_2$ where A_{k_i} is a k_i-form on M_i for $i = 1, 2$. We have*

$$\begin{aligned} *_s(A_{k_1} \wedge A_{k_2}) &= (-1)^{k_1 k_2} *_s^1 A_{k_1} \wedge *_s^2 A_{k_2}, \\ L(A_{k_1} \wedge A_{k_2}) &= L_1(A_{k_1}) \wedge A_{k_2} + A_{k_1} \wedge L_2(A_{k_2}), \\ \Lambda(A_{k_1} \wedge A_{k_2}) &= \Lambda_1(A_{k_1}) \wedge A_{k_2} + A_{k_1} \wedge \Lambda_2(A_{k_2}). \end{aligned}$$

Proof. Note that the second and third formulas are direct consequences of the definitions and $\omega = \omega_1 + \omega_2$. Consider the forms $A_{k_1} = \theta^1 \wedge ... \wedge \theta^{k_1}$, $A'_{k_1} = \theta'^1 \wedge ... \wedge \theta'^{k_1}$, $A_{k_2} = \eta^1 \wedge ... \wedge \eta^{k_2}$ and $A'_{k_2} = \eta'^1 \wedge ... \wedge \eta'^{k_2}$, where θ^i and θ'^i are 1-forms on M_1 and η^i and η'^i are 1-forms on M_2. It is enough to prove the lemma for these particular A_{k_1} and A_{k_2}. By formula (15), we have

$$\begin{aligned} < A'_{k_1} \wedge A'_{k_2}, A_{k_1} \wedge A_{k_2} >_\omega &= det \begin{pmatrix} < \theta'^i, \theta^j >_{\omega_1} & 0 \\ 0 & < \eta'^i, \eta^j >_{\omega_2} \end{pmatrix} \\ &=< A'_{k_1}, A_{k_1} >_{\omega_1} . < A'_{k_2}, A_{k_2} >_{\omega_2} . \end{aligned}$$

So, we can write

$$\begin{aligned} (A'_{k_1} \wedge A'_{k_2}) \wedge *_s(A_{k_1} \wedge A_{k_2}) &=< A'_{k_1} \wedge A'_{k_2}, A_{k_1} \wedge A_{k_2} >_\omega dvol \\ &=< A'_{k_1}, A_{k_1} >_{\omega_1} . < A'_{k_2}, A_{k_2} >_{\omega_2} dvol_1 \wedge dvol_2 \\ &= (A'_{k_1} \wedge *_s^1 A_{k_1}) \wedge (A'_{k_2} \wedge *_s^2 A_{k_2}) \\ &= (-1)^{k_1 k_2} (A'_{k_1} \wedge A'_{k_2}) \wedge (*_s^1 A_{k_1} \wedge *_s^2 A_{k_2}), \end{aligned}$$

which proves the lemma. □

Proposition 2.12. *The operator $*_s$ acts like the reflection by the middle vertical line on the pyramid (13). More precisely, we have $*_s : \mathcal{L}^{r,s} \to \mathcal{L}^{n-r-s,s}$ given by*

$$*_s \frac{1}{r!} L^r B_s = \frac{(-1)^{\frac{s(s+1)}{2}}}{(n-r-s)!} L^{n-r-s} B_s, \tag{16}$$

where $B_s \in \mathcal{P}^s$ and $r+s \leq n$.

Proof. I obtained the idea of this proof from [9]. Proof is by induction on n. Note that all the operators L, Λ and $*_s$ act pointwise, so it is enough to prove this proposition at a fixed point $p \in M$.

The basis case ($n=1$): Using a Darboux coordinate (x,y) such that $\omega = dx \wedge dy$ and easy computations, one can check that we have $*_s 1 = \omega = L^1(1)$, $*_s L^1(1) = *_s \omega = 1$, $*_s dx = -dx$ and $*_s dy = -dy$. By pointwise linearity, checking these four equalities is sufficient to prove the basis case.

The inductive case: Using a cubical Darboux coordinate around $p \in M$, without loss of generality, we can assume that (M_1, ω_1) and (M_2, ω_2) are respectively open submanifolds of $\mathbb{R}^2$ and $\mathbb{R}^{2(n-1)}$ with the standard symplectic forms (i.e. $\omega_2 = dx^2 \wedge dy^2 + ... + dx^n \wedge dy^n$ and $\omega_1 = dx^1 \wedge dy^1$) and (M, ω) is the product symplectic manifold. A general s-form B_s at $p = (p_1, p_2) \in M_1 \times M_2$ is of the form

$$B_s = A_s + dx^1 \wedge B_{s-1} + dy^1 \wedge B'_{s-1} + \omega_1 \wedge B_{s-2},$$

where A_s, B_{s-1}, B'_{s-1} and B_{s-2} are forms at $p_2 \in M_2$. Now, if B_s is primitive, we must have (using third formula in Lemma 2.11)

$$0 = \Lambda B_s = \Lambda_2 A_s + dx^1 \wedge \Lambda_2 B_{s-1} + dy^1 \wedge \Lambda_2 B'_{s-1} + B_{s-2} + \omega_1 \wedge \Lambda B_{s-2},$$

and therefore B_{s-1}, B'_{s-1} and B_{s-2} should be primitive and $0 = \Lambda_2 A_s + B_{s-2}$. Then, Corollary 2.7 and Lemma 2.4 imply that $A_s = B'_s - \frac{1}{n-s+1} L_2 B_{s-2}$ for some primitive form B'_s at p_2. Combining these with the second formula in Lemma 2.11, we find

$$\begin{aligned} L^r B_s = {} & r\omega_1 \wedge L_2^{r-1} B'_s + L_2^r B'_s - \frac{1}{n-s+1} L_2^{r+1} B_{s-2} \\ & + dx^1 \wedge L_2^r B_{s-1} + dy^1 \wedge L_2^r B'_{s-1} + \frac{n-r-s+1}{n-s+1} \omega_1 \wedge L_2^r B_{s-2}. \end{aligned}$$

Using this equation, the fist formula in Lemma 2.11 and the induction hypothesis, we can compute

$$\begin{aligned}(-1)^{\frac{s(s+1)}{2}}(n-r-s)!*_s\frac{1}{r!}L^rB_s = {} & L_2^{n-r-s}B'_s+(n-r-s)\omega_1\wedge L_2^{n-r-s-1}B'_s \\ & +\frac{r+1}{n-s+1}\omega_1\wedge L_2^{n-r-s}B_{s-2}+dx^1\wedge L_2^{n-r-s}B_{s-1} \\ & +dy^1\wedge L_2^{n-r-s}B'_{s-1}-\frac{1}{n-s-1}L_2^{n-r-s+1}B_{s-2},\end{aligned}$$

whose right hand side is the same as the right hand side of the previous equation after replacing r by $n-r-s$. $\square$

Corollary 2.13. *The maps $*_s:\Omega^k\to\Omega^{2n-k}$ and $*_s:\mathcal{L}^{r,s}\to\mathcal{L}^{n-r-s,s}$ are isomorphisms and the operators L and Λ are symplectic adjoints of each other:*

$$*_s *_s = 1,$$

$$\Lambda = *_sL*_s, \qquad L = *_s\Lambda *_s .$$

Proof. Proof of the first formula is just using the formula (16) twice and the fact that $*_s$ is pointwise linear. The first and second formulas give the third one. To prove the second one, it is enough to check that both hand sides act similarly on $\frac{1}{r!}L^rB_s$ (using the formula (16) and Lemma 2.4):

$$\begin{aligned}*_sL*_s(\frac{1}{r!}L^rB_s) &= *_s\frac{(-1)^{\frac{s(s+1)}{2}}}{(n-r-s)!}L^{n-s-r+1}B_s=\frac{n-s-r+1}{(r-1)!}L^{r-1}B_s \\ &= \frac{H+r-1}{(r-1)!}L^{r-1}B_s=\Lambda(\frac{1}{r!}L^rB_s).\end{aligned}$$ $\square$

Let (ω,J,g) be a compatible triple on M that always exists for any symplectic manifold (see [12]). Define $\mathcal{J}:\Omega^k\to\Omega^k$ by

$$(\mathcal{J}A_k)(V_1,...,V_k):=A_k(JV_1,...,JV_k), \tag{17}$$

then we have $\mathcal{J}^2=(-1)^k=\mathcal{J}^{-2}$ on k-forms and $\mathcal{J}\omega=\omega$. Note that any symplectic manifold is naturally oriented using the orientation induced by ω^n. Using notations in subsection 1.1, we have:

Proposition 2.14. *For a compatible triple (ω, J, g), the following identities hold*

$$[\mathcal{J}, L] = [\mathcal{J}, \Lambda] = 0,$$

$$< A_k, A'_k >_g = < A_k, \mathcal{J} A'_k >_\omega,$$

$$* = *_s \mathcal{J},$$

$$*\frac{1}{r!} L^r B_s = \frac{(-1)^{\frac{s(s+1)}{2}}}{(n-r-s)!} L^{n-r-s} \mathcal{J}(B_s),$$

$$[\mathcal{J}, *] = [\mathcal{J}, *_s] = 0,$$

$$** = (-1)^k,$$

$$\Lambda = (-1)^k * L * .$$

Proof. From the definition, we have $\mathcal{J}(A \wedge A') = \mathcal{J}(A) \wedge \mathcal{J}(A')$. Together with the equality $\mathcal{J}(\omega) = \omega$, we can conclude that $\mathcal{J} L = L \mathcal{J}$. In a Darboux coordinate system such that $\omega = \sum_{i=1}^n dx^i \wedge dy^i$, we have $J(\partial_{x^i}) = \partial_{y^i}$, $J(\partial_{y^i}) = -\partial_{x^i}$ and $\Lambda = \sum_{i=1}^n i_{\partial_{y^i}} i_{\partial_{x^i}}$. These formulas prove $\mathcal{J}\Lambda = \Lambda \mathcal{J}$.

Using the definitions of bilinear forms $<,>_g$ and $<,>_\omega$ on k-forms, the definition of $\mathcal{J}$ and compatibility conditions of (ω, J, g), we can easily see that $< A_k, A'_k >_g = < A_k, \mathcal{J} A'_k >_\omega$. Then, we can write

$$A_k \wedge *A'_k = < A_k, A'_k >_g dvol = < A_k, \mathcal{J} A'_k >_\omega dvol = A_k \wedge *_s \mathcal{J} A'_k,$$

which proves $* = *_s \mathcal{J}$.

Combining $* = *_s \mathcal{J}$, the formula (16) and commutativity of $\mathcal{J}$ with operators L and Λ proves the fourth one. Note that $\mathcal{J}$ preserves primitivity by commutativity with Λ.

Computing $\mathcal{J} *_s$ and $* = *_s \mathcal{J}$ on $\mathcal{L}^{r,s}$ using the formula (16) and the previous formula, we can see that $\mathcal{J} *_s = *_s \mathcal{J}$ and therefore the operators $*$ and $*_s$ commute with $\mathcal{J}$.

For the last two equations, we have:

$$** = *_s \mathcal{J} *_s \mathcal{J} = *_s *_s \mathcal{J}^2 = (-1)^k,$$

$$(-1)^k * L * = (-1)^k *_s \mathcal{J} L *_s \mathcal{J} = (-1)^k *_s L *_s \mathcal{J}^2 = *_s L *_s = \Lambda,$$

□

We can extend the operator $\mathcal{J}$ to act on the complex-valued forms $\mathcal{A}(M)$ by the formula $\mathcal{J} = \sum_{p,q} i^{p-q}\Pi^{p,q}$, where $i = \sqrt{-1}$ and $\Pi^{p,q} : \mathcal{A}(M) \to \mathcal{A}^{p,q}$ is the type (p,q) projection map. To see the relationship between these two definitions, recall that J acts on $T'(M)$ as multiplication by i and on $T''(M)$ as multiplication by $-i = i^{-1}$. Then, the formula (17) gives to the above formula formula.

2.2 Differential operators

2.2.1 d and d^Λ

Definition 2.15. *Define the operator $d^\Lambda : \Omega^k \to \Omega^{k-1}$ by $d^\Lambda := d\Lambda - \Lambda d$, where d is the exterior derivative. Therefore, the operator dd^Λ is a map $\Omega^k \to \Omega^k$.*

Lemma 2.16. $[d, L] = 0, \qquad [d, \Lambda] = d^\Lambda, \qquad [d, H] = d, \qquad dd^\Lambda = -d\Lambda d = -d^\Lambda d.$

Proof. The second assertion is just the definition of d^Λ. The proof of others are as follows:

$$dLA_k = d(\omega \wedge A_k) = d(\omega) \wedge A_k + (-1)^2 \omega \wedge dA_k = LdA_k,$$

$$dHA_k - HdA_k = (n-k)dA_k - (n-(k+1))dA_k = dA_k,$$

$$dd^\Lambda = dd\Lambda - d\Lambda d = -d\Lambda d = -(d\Lambda d - \Lambda dd) = -d^\Lambda d.$$

□

Lemma 2.17. *For any $B_k \in \mathcal{P}^k$, there exist primitive forms $B^0_{k+1} \in \mathcal{P}^{k+1}$ and $B^1_{k-1} \in \mathcal{P}^{k-1}$ such that*

$$\begin{aligned} dB_k &= B^0_{k+1} + LB^1_{k-1}, \\ d^\Lambda B_k &= -HB^1_{k-1}. \end{aligned}$$

In particular, B_k is d^Λ-closed if it is d-closed.

Proof. First, we want to show that the Lefschetz decomposition of dB_k has at most two nonzero components. If $k > n$, then $B_k = 0 = dB_k$. Otherwise, since B_k is primitive and $k \leq n$, then $L^{n-k+1}B_k = 0$. Consider the Lefschetz decomposition $dB_k = \sum_{r\geq 0} L^r B^r_{k+1-2r}$. Using $[d, L] = 0$, we have $0 = dL^{n-k+1}B_k = \sum_{r\geq 0} L^{n-k+1+r} B^r_{k+1-2r}$. This implies that $L^{n-k+1+r}B^r_{k+1-2r} = 0$ for all $r \geq 0$. By (I) in the subsection 2.1.1, for $k' + r' \leq n$, the operator

$L^{r'} : \Omega^{k'} \to \Omega^{k'+2r'}$ is injective. If $r \geq 2$, then $(k+1-2r)+(n-k+1+r) = n+2-r \leq n$ and therefore $B^r_{k+1-2r} = 0$. Consequently, we have $dB_k = B^0_{k+1} + LB^1_{k-1}$.

Next, we have

$$d^\Lambda B_k = d\Lambda B_k - \Lambda d B_k = 0 - \Lambda B^0_{k+1} - \Lambda L B^1_{k-1} = -HB^1_{k-1}. \quad \square$$

Proposition 2.18. *We have $d^\Lambda = (-1)^{k+1} *_s d *_s$ on k-forms i.e. the operator d^Λ is the symplectic adjoint of d.*

Proof. To prove the proposition, we will check that both operators act similarly on $\frac{1}{r!}L^r B_s$.

$$\begin{aligned} d\Lambda(\frac{1}{r!}L^r B_s) &= \frac{1}{r!}d(H+r-1)rL^{r-1}B_s = \frac{1}{(r-1)!}(H+r)L^{r-1}dB_s \\ &= \frac{1}{(r-1)!}(H+r)L^{r-1}B^0_{s+1} + \frac{1}{(r-1)!}(H+r)L^r B^1_{s-1}, \end{aligned}$$

$$\begin{aligned} \Lambda d(\frac{1}{r!}L^r B_s) &= \frac{1}{r!}\Lambda L^r B^0_{s+1} + \frac{1}{r!}\Lambda L^{r+1} B^1_{s-1} \\ &= \frac{1}{(r-1)!}(H+r-1)L^{r-1}B^0_{s+1} + \frac{r+1}{r!}(H+r)L^r B^1_{s-1}, \\ d^\Lambda(\frac{1}{r!}L^r B_s) &= \frac{1}{(r-1)!}L^{r-1}B^0_{s+1} - \frac{n-r-s+1}{r!}L^r B^1_{s-1}, \\ (-1)^{2r+s+1} *_s d *_s (\frac{1}{r!}L^r B_s) &= \frac{(-1)^{\frac{(s+1)(s+2)}{2}}}{(n-r-s)!} *_s L^{n-r-s}B^0_{s+1} - \frac{(-1)^{\frac{(s-1)s}{2}}}{(n-r-s)!} *_s L^{n-r-s+1}B^1_{s-1} \\ &= \frac{1}{(r-1)!}L^{r-1}B^0_{s+1} - \frac{n-r-s+1}{r!}L^r B^1_{s-1}. \end{aligned} \quad \square$$

Corollary 2.19. *We have the following identities:*

$$d^\Lambda d^\Lambda = 0, \qquad d = (-1)^{k+1} *_s d^\Lambda *_s, \qquad dd^\Lambda = *_s dd^\Lambda *_s,$$

$$[d^\Lambda, L] = d, \qquad [d^\Lambda, \Lambda] = 0, \qquad [d^\Lambda, H] = -d^\Lambda,$$

$$[dd^\Lambda, L] = [dd^\Lambda, \Lambda] = [dd^\Lambda, H] = 0.$$

Proof. On k-forms, we have

$$\begin{aligned} d^\Lambda d^\Lambda &= (-1)^{k-1+1} *_s d *_s d^\Lambda = - *_s d *_s *_s d *_s = - *_s dd *_s = 0, \\ (-1)^{k+1} *_s d^\Lambda *_s &= (-1)^{k+1} *_s ((-1)^{2n-k+1} *_s d *_s) *_s = d, \\ dd^\Lambda &= (-1)^{k-1+1} *_s d^\Lambda *_s d^\Lambda = - *_s d^\Lambda *_s *_s d *_s = *_s dd^\Lambda *_s . \end{aligned}$$

To prove the equalities in the second line, we multiply the corresponding identities in Lemma 2.16 from left and right by $*_s$. Finally, we use the equality $[a,[b,c]] = a[b,c] + [a,c]b$ for commutators to prove the identities in the third line. For example acting on k-forms, we can write

$$\begin{aligned} d &= (-1)^{k+1} *_s (d\Lambda - \Lambda d)*_s = (-1)^{k+3} *_s d *_s *_s \Lambda *_s - *_s \Lambda *_s (-1)^{k+1} *_s d *_s \\ &= d^\Lambda L - L d^\Lambda = [d^\Lambda, L], \\ [dd^\Lambda, L] &= d[d^\Lambda, L] + [d, L]d^\Lambda = dd + 0 = 0. \end{aligned}$$

□

Corollary 2.20. *Let A be a differential form, then we have*

a) A is d-closed (d-exact) $\iff *_s A$ is d^Λ-closed (d^Λ-exact) and vice versa.

b) A is dd^Λ-closed (dd^Λ-exact) $\iff *_s A$ is dd^Λ-closed (dd^Λ-exact).

Proof. The computation $d^\Lambda *_s A = \pm *_s d *_s *_s A = \pm *_s dA$ proves part a). For vice versa cases just note that $*_s *_s = 1$. For part b), write $dd^\Lambda *_s A = *_s dd^\Lambda *_s *_s A = *_s dd^\Lambda A$. □

Definition 2.21. *We say a k-form A_k is $d + d^\Lambda$-exact if there are $(k-1)$ and $(k+1)$-forms A_{k-1} and A_{k+1} satisfying $A_k = dA_{k-1} + d^\Lambda A_{k+1}$.*

Proposition 2.22. *Let $A_k = \sum_r \frac{1}{r!} L^r B_{k-2r}$ be the Lefschetz decomposition of a k-form A_k. We have:*

a) A_k is both d- and d^Λ-closed $\iff B_{k-2r}$ is d-closed for all r.

b) A_k is dd^Λ-closed (dd^Λ-exact) $\iff B_{k-2r}$ is dd^Λ-closed (dd^Λ-exact) for all r.

c) A_k is $(d + d^\Lambda)$-exact $\iff B_{k-2r}$ is $(d + d^\Lambda)$-exact for all r.

Proof. Part a) $\Leftarrow$: Assume that $dB_{k-2r} = 0$ for all r. Then, the following equalities

$$\begin{aligned} dA_k &= \sum_r \frac{1}{r!} L^r dB_{k-2r}, \\ d^\Lambda A_k &= \sum_r \frac{1}{r!} L^r (d^\Lambda + rd) B_{k-2r}, \end{aligned}$$

and the identity $d^\Lambda B_{k-2r} = 0$ from Lemma 2.17 imply that $dA_k = d^\Lambda A_k = 0$.

Part a) $\Rightarrow$: Assume that $dA_k = d^\Lambda A_k = 0$. By Lemma 2.6, we can compute

$$dB_{k-2r} = \sum_t a_{r,t} L^r d\Lambda^{r+t} A_k = \sum_t a_{r,t} L^r \Lambda^{r+t} (d + (r+t) d^\Lambda) A_k,$$

and therefore $dB_{k-2r} = 0$. The proof of the part b) is similar and trivial since the operator dd^Λ commutes with both L and Λ operators.

Part c) $\Leftarrow$: Assume that we have $B = dA + d^\Lambda A'$. Then, we can write

$$L^r B = dL^r A + d^\Lambda L^r A' - r dL^{r-1} A' = d(L^r A - rL^{r-1}A') + d^\Lambda (L^r A'),$$

even when the form B is not primitive.

Part c) $\Rightarrow$: Assume that we have $A_k = dA + d^\Lambda A'$. Then, we can write

$$\begin{aligned} B_{k-2r} = \sum_t a_{r,t} L^r \Lambda^{r+t} A_k = d \sum_t a_{r,t} L^r \Lambda^{r+t} A - d^\Lambda \sum_t a_{r,t}(r+t) L^r \Lambda^{r+t-1} A \\ + d \sum_t a_{r,t}(r+t) r L^{r-1} \Lambda^{r+t-1} A + d^\Lambda \sum_t a_{r,t} L^r \Lambda^{r+t} A' - d \sum_t a_{r,t} r L^{r-1} \Lambda^{r+t} A'. \end{aligned}$$ $\square$

Corollary 2.23. *Let A be a k-form for $0 \le k \le n$ and $A' = L^{n-k}A$, then we have*

a) A is d-closed (d-exact) $\Longrightarrow$ A' is d-closed (d-exact).

b) A is dd^Λ-closed (dd^Λ-exact) $\Longleftrightarrow$ A' is dd^Λ-closed (dd^Λ-exact).

c) A is both d- and d^Λ-closed $\Longleftrightarrow$ A' is both d- and d^Λ-closed.

d) A is $(d + d^\Lambda)$-exact $\Longleftrightarrow$ A' is $(d + d^\Lambda)$-exact.

Proof. For part a), just note that $[d, L] = 0$. To prove other parts, assume that we have the Lefschetz decompositions $A = \sum_r \frac{1}{r!} L^r B_{k-2r}$ and $A' = \sum_{r'} \frac{1}{r'!} L^{r'} B'_{(2n-k)-2r'}$. Since the map $L^{n-k} : \Omega^k \to \Omega^{2n-k}$ is an isomorphism, we should have $\frac{1}{r!} B_{k-2r} = \frac{1}{r'!} B'_{(2n-k)-2r'}$ for $r' = n - k + r$. Now Proposition 2.22 proves parts b, c and d. $\square$

Lemma 2.24. *Let V be a symplectic vector field on M i.e. the 1-form $\nu := i_V \omega$ is d-closed. The Lie derivative $\mathcal{L}_V$ on differential forms satisfies*

$$\mathcal{L}_V L = L\,\mathcal{L}, \qquad \mathcal{L}_V B = -d^\Lambda(\nu \wedge B) - \nu \wedge d^\Lambda B,$$

for a primitive form B.

Proof. We use the Cartan formula $\mathcal{L}_V = i_V d + d\, i_V$ to prove the lemma. This formula shows that $\mathcal{L}_V \omega = 0$. We have the following commutator formulas

$$\mathcal{L}_V LA = \mathcal{L}_V(\omega \wedge A) = (\mathcal{L}_V \omega) \wedge A + \omega \wedge (\mathcal{L}_V A) = L\,\mathcal{L}_V A,$$

$$i_V \Lambda = \Lambda\, i_V, \qquad i_V LA = i_V(\omega) \wedge A + \omega \wedge i_V A = \nu \wedge A + L i_V A.$$

The last two formulas imply that

$$-\Lambda(\nu \wedge B) = \Lambda L i_V B - i_V \Lambda L B = H i_V B - i_V H B = i_V B,$$

for a primitive form B. If we have $dB = B' + LB''$, then we can compute

$$\begin{aligned}
i_V dB &= i_V B' + \nu \wedge B'' - L\Lambda(\nu \wedge B'') = -\Lambda(\nu \wedge B') - \Lambda L(\nu \wedge B'') + (H+1)(\nu \wedge B'') \\
&= -\Lambda(\nu \wedge dB) + \nu \wedge (HB'') = \Lambda d(\nu \wedge B) - \nu \wedge (d^\Lambda B) \\
&= -d^\Lambda(\nu \wedge B) - d i_V B - \nu \wedge (d^\Lambda B).
\end{aligned}$$

□

Corollary 2.25. *Let V be a Hamiltonian vector field on M i.e. there exists $h \in C^\infty(M)$ satisfying $i_V \omega = dh$. The Lie derivative $\mathcal{L}_V$ on differential forms satisfies*

a) A_k is both d- and d^Λ-closed $\Longrightarrow \mathcal{L}_V A_k$ is dd^Λ-exact.

b) A_k is dd^Λ-closed $\Longrightarrow \mathcal{L}_V A_k$ is $(d + d^\Lambda)$-exact.

Proof. Let $A_k = \sum_r \frac{1}{r!} L^r B_{k-2r}$ be the Lefschetz decomposition of A_k.

Part a): Assume that A_k is both d- and d^Λ-closed, then B_{k-2r} is d and d^Λ-closed for all r. Using the previous lemma, we have

$$\begin{aligned}
\mathcal{L}_V B_{k-2r} &= -d^\Lambda((dh) \wedge B_{k-2r}) - (dh) \wedge d^\Lambda B_{k-2r} = -d^\Lambda d(hB_{k-2r}) - 0 = dd^\Lambda(hB_{k-2r}) \\
&\Longrightarrow \mathcal{L}_V A_k = \sum_r \frac{1}{r!} L^r \mathcal{L}_V B_{k-2r} = \sum_r \frac{1}{r!} L^r dd^\Lambda(hB_{k-2r}) = dd^\Lambda(hA_k).
\end{aligned}$$

Part b): Assume that A_k is dd^Λ-closed, then B_{k-2r} is dd^Λ-closed for all r. Using the previous lemma, we have

$$\mathcal{L}_V B_{k-2r} = d(-h \wedge d^\Lambda B_{k-2r}) + d^\Lambda(-(dh) \wedge B_{k-2r}),$$

hence $\mathcal{L}_V B_{k-2r}$ is $(d + d^\Lambda)$-exact for all r and therefore $\mathcal{L}_V A_k = \sum_r \frac{1}{r!} L^r \mathcal{L}_V B_{k-2r}$ is also $(d+d^\Lambda)$-exact by the proof of Proposition 2.22 (note that $\mathcal{L}_V B_{k-2r}$ can be non-primitive). □

Lemma 2.26. *If M is compact, then we have*

$$\begin{aligned}
\int_M (d^\Lambda A_k) \wedge A' &= (-1)^k \int_M A_k \wedge (d^\Lambda A'), \\
\int_M (dd^\Lambda A_k) \wedge A'' &= -\int_M A_k \wedge (dd^\Lambda A'').
\end{aligned}$$

Proof. Note that the Stokes' theorem implies that $\int_M (dA_k)\wedge A' = (-1)^{k+1}\int_M A_k \wedge (dA')$. Using this fact and Lemma 2.8, we can write:

$$\int_M (d\Lambda A_k)\wedge A' = (-1)^{k-2+1}\int_M (\Lambda A_k)\wedge (dA') = (-1)^{k+1}\int_M A_k\wedge(\Lambda dA'),$$
$$\int_M (\Lambda dA_k)\wedge A' = \int_M (dA_k)\wedge(\Lambda A') = (-1)^{k+1}\int_M A_k\wedge(d\Lambda A').$$

Subtracting these two formulas gives the first identity. For the second identity, write

$$\int_M (dd^\Lambda A_k)\wedge A'' = (-1)^k(-1)^{(k-1)+1}\int_M A_k\wedge(d^\Lambda dA'') = -\int_M A_k\wedge(dd^\Lambda A''). \quad \square$$

Note that the previous lemma serves as a version of Stokes' theorem for operators d^Λ and dd^Λ.

Definition 2.27. *For a compatible triple (ω, J, g), the differential operator $d^c : \Omega^k \to \Omega^{k+1}$ is defined by $d^c := \mathcal{J}^{-1} d\mathcal{J}$.*

Lemma 2.28. *For a compatible triple (ω, J, g), we have $d^\Lambda = -* d^c *$.*

Proof. Acting on k-forms, we have

$$d^\Lambda = (-1)^{k+1} *_s d *_s = (-1)^{k+1} * \mathcal{J}^{-1} d * \mathcal{J}^{-1} = (-1)^{k+1} * \mathcal{J}^{-1} d\mathcal{J} * \mathcal{J}^{-2} = - * d^c * . \quad \square$$

Lemma 2.29. *For a compatible triple (ω, J, g) on a compact manifold, whenever acting on k-forms, we have*

$$L^* = (-1)^k * L * = \Lambda, \qquad H^* = H, \qquad d^* = - * d *,$$

$$d^{\Lambda *} = * d^\Lambda *, \qquad (dd^\Lambda)^* = (-1)^{k+1} * dd^\Lambda *, \qquad d^{c*} = - * d^c * = d^\Lambda.$$

Proof. The equality $H^* = H$ is obvious.

$$\begin{aligned} < LA_{k-2}, A'_k > &= \int_M (LA_{k-2})\wedge(*A'_k) = \int_M A_{k-2}\wedge(L * A'_k) \\ &= \int_M (-1)^{2n-k+2} A_{k-2}\wedge(* * L * A'_k) =< A_{k-2}, (-1)^k * L * A'_k >, \end{aligned}$$

$$\begin{aligned} < dA_{k-1}, A'_k > &= \int_M (dA_{k-1})\wedge(*A'_k) = \int_M (-1)^k A_{k-1}\wedge(d * A'_k) \\ &= \int_M (-1)^k(-1)^{2n-k+1} A_{k-1}\wedge(* * d * A'_k) =< A_{k-1}, - * d * A'_k >, \end{aligned}$$

$$< d^\Lambda A_{k+1}, A'_k > = \int_M (d^\Lambda A_{k+1}) \wedge (*A'_k) = \int_M (-1)^{k+1} A_{k+1} \wedge (d^\Lambda * A'_k)$$

$$= \int_M (-1)^{k+1}(-1)^{2n-k-1} A_{k-1} \wedge (* * d^\Lambda * A'_k) =< A_{k-1}, *d^\Lambda * A'_k >,$$

$$(dd^\Lambda)^* = d^{\Lambda *} d^* = - * d^\Lambda * * d* = -(-1)^{2n-k+1} * d^\Lambda d* = (-1)^{k+1} * dd^\Lambda *,$$

$$< d^c A_{k-1}, A'_k > = \int_M (\mathcal{J}^{-1} d\mathcal{J} A_{k-1}) \wedge (*A'_k) = \int_M d\mathcal{J} A_{k-1} \wedge (\mathcal{J} * A'_k)$$

$$= \int_M (-1)^k \mathcal{J} A_{k-1} \wedge (d\mathcal{J} * A'_k) = \int_M (-1)^k A_{k-1} \wedge (\mathcal{J}^{-1} d\mathcal{J} * A'_k)$$

$$= \int_M (-1)^k (-1)^{2n-k+1} A_{k-1} \wedge (* * d^c * A'_k) =< A_{k-1}, - * d^c * A'_k > .$$

For the last one, note that a nonzero $2n$-form is of type (n, n), so it is invariant under the action of $\mathcal{J}$ and $\mathcal{J}(A \wedge A') = \mathcal{J}(A) \wedge \mathcal{J}(A')$. □

Corollary 2.30. $\mathcal{J} d = -d^{\Lambda *}\mathcal{J}, \quad \mathcal{J} d^\Lambda = d^* \mathcal{J}, \quad \mathcal{J} d^* = -d^\Lambda \mathcal{J}, \quad \mathcal{J} d^{\Lambda *} = d\mathcal{J}.$

Proof. The last equality is just $d^{\Lambda *} = d^c = \mathcal{J}^{-1} d\mathcal{J}$. For the others, we have

$$-d^{\Lambda *}\mathcal{J} = -\mathcal{J}^{-1} d\mathcal{J}^2 = (-1)^{k+1}\mathcal{J}^{-2}\mathcal{J} d = \mathcal{J} d,$$

$$\mathcal{J} d^\Lambda = (-1)^{k+1}\mathcal{J} *_s d*_s = - * d *_s \mathcal{J}^2 = - * d * \mathcal{J} = d^* \mathcal{J},$$

$$-d^\Lambda \mathcal{J} = -\mathcal{J}^{-1} d^* \mathcal{J}^2 = (-1)^{k-1}\mathcal{J}^{-2}\mathcal{J} d^* = \mathcal{J} d^*.$$

□

Lemma 2.31. *For a compatible triple (ω, J, g) on a compact manifold, we have*

$$[d^*, L] = -d^{\Lambda *}, \qquad [d^*, \Lambda] = 0, \qquad [d^*, H] = -d^*,$$

$$[d^{\Lambda *}, L] = 0, \qquad [d^{\Lambda *}, \Lambda] = -d^*, \qquad [d^{\Lambda *}, H] = d^{\Lambda *},$$

$$[(dd^\Lambda)^*, L] = [(dd^\Lambda)^*, \Lambda] = [(dd^\Lambda)^*, H] = 0.$$

Proof. We obtain these by taking adjoint of previous commutation relations and the formula $[a, b]^* = -[a^*, b^*]$. □

Remark. *For a Kähler manifold, the operator d^c is given by the formula $i(\bar{\partial} - \partial)$. To see this, let A be a type (p, q)-form, then we have*

$$d^c A = \mathcal{J}^{-1} d\mathcal{J} A = \mathcal{J}^{-1}(\partial + \bar{\partial})\mathcal{J} A = i^{p-q}\mathcal{J}^{-1}\partial A + i^{p-q}\mathcal{J}^{-1}\bar{\partial} A$$

$$= i^{p-q} i^{q-(p+1)}\partial A + i^{p-q} i^{(q+1)-p}\bar{\partial} A = i(\bar{\partial} - \partial) A.$$

2.2.2 ∂_+ and ∂_-

Using Lemmas 2.16 and 2.17, we have the map $d : \mathcal{L}^{r,s} \to \mathcal{L}^{r,s+1} \oplus \mathcal{L}^{r+1,s-1}$.

Definition 2.32. *Define the differential operators $\partial_+ : \mathcal{L}^{r,s} \to \mathcal{L}^{r,s+1}$ and $\partial_- : \mathcal{L}^{r,s} \to \mathcal{L}^{r,s-1}$ by the identity $d = \partial_+ + L\partial_-$.*

Lemma 2.33.

$$\begin{aligned}[]
[\partial_+, L] = L([\partial_-, L]) &= 0,\\
L(\partial_+\partial_-) &= -L(\partial_-\partial_+),\\
\partial_+^2 = \partial_-^2 &= 0.
\end{aligned}$$

Proof. Since we have $[\partial_+, L] : \mathcal{L}^{r,s} \to \mathcal{L}^{r+1,s+1}$ and $L([\partial_-, L]) : \mathcal{L}^{r,s} \to \mathcal{L}^{r+2,s-1}$, the following computation

$$0 = [d, L] = [\partial_+ + L\partial_-] = [\partial_+, L] + L([\partial_-, L]),$$

implies that both terms should be zero. Similarly, the computation

$$0 = d^2 = \partial_+^2 + (\partial_+ L\partial_- + L\partial_-\partial_+) + L\partial_- L\partial_- = \partial_+^2 + (L\partial_+\partial_- + L\partial_-\partial_+) + L^2\partial_-^2,$$

implies $L(\partial_+\partial_-) = -L(\partial_-\partial_+)$ and $\partial_+^2 = L^2\partial_-^2 = 0$. Note that acting on $\mathcal{L}^{r,s}$ with $s < 2$, we have $\partial_-^2 = 0$ because we leave the pyramid and outside of the pyramid we have $\mathcal{L}^{\square,\square} = \{0\}$. On the other hand, when the operator $L^2\partial_-^2 = 0$ acts on $\mathcal{L}^{r,s}$ with $s \geq 2$, we will never leave the pyramid so the operator L^2 is an isomorphism by Lemma 2.5 and again we have $\partial_-^2 = 0$. □

Remark. *We have $[\partial_-, L] = 0$ and $\partial_+\partial_- = -\partial_-\partial_+$, if we don't leave the pyramid i.e. acting on $\mathcal{L}^{r,s}$ with $r + s < n$.*

Proposition 2.34. *We have $d^\Lambda : \mathcal{L}^{r,s} \to \mathcal{L}^{r-1,s+1} \oplus \mathcal{L}^{r,s-1}$ and $dd^\Lambda : \mathcal{L}^{r,s} \to \mathcal{L}^{r,s}$ satisfying the following formulas*

$$(-1)^{k+1} *_s \partial_+ *_s = \frac{1}{H+R+1}\partial_+\Lambda, \qquad (-1)^{k+1} *_s L\partial_- *_s = -(H+R)\partial_-,$$

$$d^\Lambda = \frac{1}{H+R+1}\partial_+\Lambda - (H+R)\partial_-,$$

$$dd^\Lambda = -(H+2R+1)\partial_+\partial_-,$$

$$\partial_+ = \frac{1}{H+2R+1}((H+R+1)d + Ld^\Lambda),$$
$$\partial_- = \frac{-1}{(H+2R+1)(H+R)}((H+R+1)d^\Lambda - d\Lambda).$$

Proof. Using Lemma 2.17, we have $dB_s = B^0_{s+1} + LB^1_{s-1}$ and

$$\begin{aligned}(-1)^{k+1} *_s \partial_+ *_s (\frac{1}{r!}L^r B_s) &= (-1)^{2r+s+1}(-1)^{\frac{s(s+1)}{2}} *_s \frac{1}{(n-r-s)!}L^{n-r-s}B^0_{s+1} \\ &= \frac{1}{(r-1)!}L^{r-1}B^0_{s+1},\end{aligned}$$
$$\frac{1}{H+R+1}\partial_+\Lambda(\frac{1}{r!}L^r B_s) = \frac{1}{H+R+1}\partial_+\frac{H+r-1}{(r-1)!}L^{r-1}B_s = \frac{1}{(r-1)!}L^{r-1}B^0_{s+1}.$$

Similarly, we have

$$\begin{aligned}(-1)^{k+1} *_s L\partial_- *_s (\frac{1}{r!}L^r B_s) &= (-1)^{2r+s+1}(-1)^{\frac{s(s+1)}{2}} *_s \frac{1}{(n-r-s)!}L^{n-r-s+1}B^1_{s-1} \\ &= -\frac{n-r-s+1}{r!}L^r B^1_{s-1} = -(H+R)\partial_-(\frac{1}{r!}L^r B_s).\end{aligned}$$

Adding these two, the formula for d^Λ is proved. Consequently, we have

$$\begin{aligned}dd^\Lambda &= (\partial_+ + L\partial_-)(\frac{1}{H+R+1}\partial_+\Lambda - (H+R)\partial_-) = -\partial_+(H+R)\partial_- + L\partial_-\frac{1}{H+R+1}\partial_+\Lambda \\ &= -(H+R+1)\partial_+\partial_- - \frac{1}{H+R+1}\partial_+\partial_- L\Lambda = -(H+2R+1)\partial_+\partial_-.\end{aligned}$$

Using the previous formulas for d and d^Λ and acting on $\mathcal{L}^{r,s}$, we have

$$\begin{aligned}(H+R+1)d + Ld^\Lambda &= ((n-r-s)\partial_+ + (n-r-s+1)L\partial_-) \\ &\quad + (\frac{1}{n-r-s+1}\partial_+ L\Lambda - (n-r-s+1)L\partial_-) = (n-s)\partial_+ = (H+2R+1)\partial_+, \\ (H+R+1)d^\Lambda - d\Lambda &= (\partial_+\Lambda - (n-r-s+1)(n-r-s+2)\partial_-) - (\partial_+\Lambda + \partial_- L\Lambda) \\ &= -(n-r-s+1)(n-s+2)\partial_- = -(H+R)(H+2R+1)\partial_-.\end{aligned}$$

Finally, note that we always have $H+R+1>0$ and $H+2R+1>0$. Moreover, $(H+R)$ is positive after applying ∂_-, d^Λ or $d\Lambda$. $\square$

Corollary 2.35. *The operators $\partial_+ : \mathcal{P}^k \to \mathcal{P}^{k+1}$ and $\partial_- : \mathcal{P}^k \to \mathcal{P}^{k-1}$ on primitive forms satisfy the following formulas*

$$\partial_+ = d + LH^{-1}d^\Lambda = (1 - LH^{-1}\Lambda)d,$$
$$\partial_- = -H^{-1}d^\Lambda = H^{-1}\Lambda d,$$
$$\partial_+\partial_- = -(H+1)^{-1}dd^\Lambda = (H+1)^{-1}d\Lambda d.$$

Proof. Formulas for $\partial_\pm$ are just formulas in Lemma 2.17. The last formula follows from $dd^\Lambda = -(H+2R+1)\partial_+\partial_-$ in the above proposition after inserting $R = 0$. Note that we always have $H+1 > 0$ on primitive forms. Moreover, H is positive after applying d^Λ on primitive forms. $\square$

Lemma 2.36. *For a compatible triple (ω, J, g) on a compact manifold, we have*

$$\mathcal{J}\partial_+\mathcal{J}^{-1} = \partial_-^*(H+R), \qquad \mathcal{J}\partial_+^*\mathcal{J}^{-1} = -(H+R)\partial_-.$$

Proof. Taking adjoint of formulas for d and d^Λ, we have

$$\begin{aligned} d^* &= \partial_+^* + \partial_-^*\Lambda, \\ d^{\Lambda *} &= L\partial_+^*\frac{1}{H+R+1} - \partial_-^*(H+R). \end{aligned}$$

Using these and various results from previous subsections, we have

$$\begin{aligned} (H+2R+1)\mathcal{J}\partial_+\mathcal{J}^{-1} &= (-1)^{k+1}\mathcal{J}((H+R+1)*_s d^\Lambda *_s + L *_s d*_s)\mathcal{J}^{-1} \\ &= -((H+R+1)*d^\Lambda * + L*d*) = Ld^* - (H+R+1)d^{\Lambda *} \\ &= L\partial_-^*\Lambda + (H+R+1)\partial_-^*(H+R) = L\Lambda\partial_-^* + (H+R+1)^2\partial_-^* \\ &= (H+2R+1)(H+R+1)\partial_-^* = (H+2R+1)\partial_-^*(H+R) \end{aligned}$$

Since we have $H+2R+1 > 0$, the proof is complete. Similarly, we can write

$$\begin{aligned} -(H+2R+1)\mathcal{J}^{-1}(H+R)\partial_-\mathcal{J} &= (-1)^{k+1}\mathcal{J}^{-1}((H+R+1)*_s d *_s - *_s d^\Lambda *_s \Lambda)\mathcal{J} \\ &= -((H+R+1)*d* - *d^\Lambda*\Lambda) = (H+R+1)d^* + d^{\Lambda *}\Lambda \\ &= (H+R+1)\partial_+^* + \frac{1}{H+R+1}L\Lambda\partial_+^* = (H+2R+1)\partial_+^*. \end{aligned}$$

$\square$

3 Symplectic cohomologies and Hodge theories

In this section, we will define different symplectic cohomologies. All these cohomologies are invariant under symplectomorphisms in the following sense. Let $f : (M_1, \omega_1) \to (M_2, \omega_2)$ be a symplectomorphism i.e. a diffeomorphism satisfying $f^*\omega_2 = \omega_1$ or $f^*L_2 = L_1 f^*$. Then, the pullback map $f^* : \Omega(M_2) \to \Omega(M_1)$ commutes with all defined operators on differential forms in Section 2. Consequently, it induces well-defined linear isomorphisms on the level of cohomologies.

3.1 d^Λ cohomology

Definition 3.1. *Having $d^\Lambda d^\Lambda = 0$, we can define the d^Λ cohomology by*

$$H^k_{d^\Lambda} := \frac{ker\, d^\Lambda \cap \Omega^k}{d^\Lambda \Omega^{k+1}}, \tag{18}$$

for $0 \leq k \leq 2n$.

Definition 3.2. *For a compatible triple (ω, J, g) on a compact manifold M, we define the self-adjoint operator $\Delta_{d^\Lambda} : \Omega^k \to \Omega^k$ called Laplacian associated with the d^Λ cohomology by*

$$\Delta_{d^\Lambda} := d^{\Lambda*} d^\Lambda + d^\Lambda d^{\Lambda*}, \tag{19}$$

and the space of d^Λ-harmonic k-forms by $\mathcal{H}^k_{d^\Lambda} := ker\, \Delta_{d^\Lambda} \cap \Omega^k$.

We use the similar notations $\Delta_d = d^*d + dd^*$ and $\mathcal{H}^k_d := ker\, \Delta_d \cap \Omega^k$ for de Rham cohomology.

Lemma 3.3. $*_s \Delta_d = \Delta_{d^\Lambda} *_s, \quad \mathcal{J}\Delta_d = \Delta_{d^\Lambda}\mathcal{J}, \quad *\Delta_d = \Delta_d *, \quad *\Delta_{d^\Lambda} = \Delta_{d^\Lambda} *.$

Proof. Corollary 2.30 implies the second equality. First and second equalities give the others using formula $* = *_s \mathcal{J}$. For the first equality, write

$$\begin{aligned}\Delta_{d^\Lambda} *_s = *d^\Lambda * d^\Lambda *_s + d^\Lambda * d^\Lambda * *_s &= (-1)^{(k+1)+1}(-1)^{(2n-k)+1} *_s *d * d \\ &+ (-1)^{(2n-k+1)+1}(-1)^{k+1} *_s d * d* = - *_s *d * d - *_s d * d* = *_s \Delta_d.\end{aligned}$$

□

Lemma 3.4. *Let (ω, J, g) denotes a compatible triple on a compact manifold M. A form A is d^Λ-harmonic if and only if $d^\Lambda A = d^{\Lambda*} A = 0$.*

Proof. The implication $\Leftarrow$ is an immediate consequence of the above definition. For the other implication under the assumption $\Delta_{d^\Lambda} A = 0$, we have

$$0 =< \Delta_{d^\Lambda} A, A >=< d^{\Lambda *} d^\Lambda A, A > + < d^\Lambda d^{\Lambda *} A, A >$$
$$=< d^\Lambda A, d^\Lambda A > + < d^{\Lambda *} A, d^{\Lambda *} A >= ||d^\Lambda A||^2 + ||d^{\Lambda *} A||^2. \qquad \square$$

Proposition 3.5. *For a compatible triple (ω, J, g) on a compact manifold M, we have*

a) dim $\mathcal{H}^k_{d^\Lambda} < \infty$

b) The orthogonal decomposition $\Omega^k = \mathcal{H}^k_{d^\Lambda} \oplus d^\Lambda \Omega^{k+1} \oplus d^{\Lambda *} \Omega^{k-1}$

c) $H^k_{d^\Lambda} \cong \mathcal{H}^k_{d^\Lambda}$ *i.e. there is a unique d^Λ-harmonic form in each d^Λ cohomology class.*

Proof. To prove first two parts, it is enough to show that the self-adjoint operator Δ_{d^Λ} is elliptic [19]. To show this, we will prove that the operator Δ_{d^Λ} has the same symbol as the Hodge Laplacian $\Delta_d = d^* d + dd^*$. Choose a local unitary coframe $\{\theta^1, ..., \theta^n, \bar{\theta}^1, ..., \bar{\theta}^n\}$ of the cotangent bundle such that the metric is $g = \sum_i (\theta^i \otimes \bar{\theta}^i + \bar{\theta}^i \otimes \theta^i)$. A general (p, q)-form $A_{p,q}$ can be written in the form of $A_{p,q} = \sum_{\#I=p,\#I'=q} A_{I,I'} \theta^I \wedge \bar{\theta}^{I'}$ and therefore we have

$$dA_{p,q} = \sum_{i,\#I=p,\#I'=q} \partial_i A_{I,I'} \theta^i \wedge \theta^I \wedge \bar{\theta}^{I'} + \sum_{i',\#I=p,\#I'=q} \bar{\partial}_{i'} A_{I,I'} \bar{\theta}^{i'} \wedge \theta^I \wedge \bar{\theta}^{I'} + \sum_{\#I=p,\#I'=q} A_{I,I'} d(\theta^I \wedge \bar{\theta}^{I'}).$$

In the calculation of the symbol, only the highest-order derivatives of functions $A_{I,I'}$ matter and therefore the only effective terms in the above equation are the first two terms. Consequently, the operator d is equivalent to the operator $\partial + \bar{\partial}$ in terms of symbols. We thus write $d \simeq \partial + \bar{\partial}$. This implies that as long as we are computing the symbol, we can use all the Kähler identities. So, we can compute

$$\begin{aligned}
\Delta_{d^\Lambda} &= d^{\Lambda *} d^\Lambda + d^\Lambda d^{\Lambda *} = d^c d^{c*} + d^{c*} d^c \simeq i(\bar{\partial} - \partial)(-i)(\bar{\partial}^* - \partial^*) + (-i)(\bar{\partial}^* - \partial^*) i(\bar{\partial} - \partial) \\
&= \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} + \partial\partial^* + \partial^*\partial - \partial\bar{\partial}^* - \bar{\partial}\partial^* - \partial^*\bar{\partial} - \bar{\partial}^*\partial \\
&\simeq \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} + \partial\partial^* + \partial^*\partial + \bar{\partial}^*\partial + \partial^*\bar{\partial} + \bar{\partial}\partial^* + \partial\bar{\partial}^* \\
&= (\partial^* + \bar{\partial}^*)(\partial + \bar{\partial}) + (\partial + \bar{\partial})(\partial^* + \bar{\partial}^*) = d^* d + dd^* = \Delta_d.
\end{aligned}$$

In the above computation, we have used $d^\Lambda = d^{c*}$, $d^c \simeq i(\bar{\partial} - \partial)$, $\bar{\partial}^* \partial \simeq -\partial \bar{\partial}^*$ and their adjoints.

Finally, to show part c) we should prove that $ker\, d^\Lambda \cap \Omega^k = \mathcal{H}^k_{d^\Lambda} \oplus d^\Lambda \Omega^{k+1}$ or equivalently $ker\, d^\Lambda \cap d^{\Lambda *}\Omega^{k-1} = \{0\}$ by the decomposition in part b). Assume that the form $A = d^{\Lambda *} A_{k-1}$ is d^Λ-closed, then we have

$$0 =< d^\Lambda A, A_{k-1} >=< A, d^{\Lambda *} A_{k-1} >= ||A||^2. \quad \square$$

Corollary 3.6. *For a compact symplectic manifold, dim* $H^k_{d^\Lambda} < \infty$*.*

3.2 $d + d^\Lambda$ cohomology

Definition 3.7. *Define the* $d + d^\Lambda$ *cohomology by*

$$H^k_{d+d^\Lambda} := \frac{ker\,(d + d^\Lambda) \cap \Omega^k}{im\, dd^\Lambda \cap \Omega^k} = \frac{ker\, d \cap ker\, d^\Lambda \cap \Omega^k}{dd^\Lambda \Omega^k}, \tag{20}$$

for $0 \le k \le 2n$*.*

Definition 3.8. *For a compatible triple* (ω, J, g) *on a compact manifold* M*, we define the self-adjoint operator* $\Delta_{d+d^\Lambda} : \Omega^k \to \Omega^k$ *called Laplacian associated with the* $d+d^\Lambda$ *cohomology by the formula*

$$\Delta_{d+d^\Lambda} := dd^\Lambda (dd^\Lambda)^* + \lambda(d^* d + d^{\Lambda *} d^\Lambda), \tag{21}$$

where λ *is a positive real number. Also, define the space of* $(d + d^\Lambda)$*-harmonic* k*-forms by* $\mathcal{H}^k_{d+d^\Lambda} := ker\, \Delta_{d+d^\Lambda} \cap \Omega^k$*.*

Lemma 3.9. $$[\Delta_{d+d^\Lambda}, L] = [\Delta_{d+d^\Lambda}, \Lambda] = [\Delta_{d+d^\Lambda}, H] = 0.$$

Proof. It is trivial the the Laplacian Δ_{d+d^Λ} commutes with H since it preserves the degree. For the other equality, write

$$[\Delta_{d+d^\Lambda}, L] = 0 + \lambda[d^* d, L] + \lambda[d^{\Lambda *} d^\Lambda, L] = \lambda[d^*, L]d + \lambda d^{\Lambda *}[d^\Lambda, L] = -\lambda d^{\Lambda *} d + \lambda d^{\Lambda *} d = 0,$$
$$[\Delta_{d+d^\Lambda}, \Lambda] = 0 + \lambda[d^* d, \Lambda] + \lambda[d^{\Lambda *} d^\Lambda, \Lambda] = \lambda d^*[d, \Lambda] + \lambda[d^{\Lambda *}, \Lambda]d^\Lambda = \lambda d^* d^\Lambda - \lambda d^* d^\Lambda = 0. \quad \square$$

Lemma 3.10. *For a compatible triple* (ω, J, g) *on a compact manifold* M*, define the following operator*

$$D_{d+d^\Lambda} = \Delta_{d+d^\Lambda} + (dd^\Lambda)^* dd^\Lambda + d^* d^\Lambda d^{\Lambda *} d + d^{\Lambda *} dd^* d^\Lambda.$$

For a differential form A*, we have*

$$\Delta_{d+d^\Lambda} A = 0 \iff dA = d^\Lambda A = (dd^\Lambda)^* A = 0 \iff D_{d+d^\Lambda} A = 0.$$

Proof. Assuming $dA = d^\Lambda A = (dd^\Lambda)^* A = 0$, we have $\Delta_{d+d^\Lambda} A = 0 = D_{d+d^\Lambda} A$ directly from the definitions of operators. Now, suppose that $\Delta_{d+d^\Lambda} A = 0$, we have

$$0 =< \Delta_{d+d^\Lambda} A, A >= ||(dd^\Lambda)^* A||^2 + \lambda ||dA||^2 + \lambda ||d^\Lambda A||^2,$$

and therefore $dA = d^\Lambda A = (dd^\Lambda)^* A = 0$. Similarly, assuming $D_{d+d^\Lambda} A = 0$ implies that

$$0 =< D_{d+d^\Lambda} A, A >= ||(dd^\Lambda)^* A||^2 + \lambda ||dA||^2 + \lambda ||d^\Lambda A||^2 + ||dd^\Lambda A||^2 + ||d^{\Lambda *} dA||^2 + ||d^* d^\Lambda A||^2,$$

and again this implies $dA = d^\Lambda A = (dd^\Lambda)^* A = 0$. □

Proposition 3.11. *For a compatible triple (ω, J, g) on a compact manifold M, we have*

a) dim $\mathcal{H}^k_{d+d^\Lambda} < \infty$

b) The orthogonal decomposition $\Omega^k = \mathcal{H}^k_{d+d^\Lambda} \oplus dd^\Lambda \Omega^k \oplus (d^* \Omega^{k+1} + d^{\Lambda *} \Omega^{k-1})$

c) $H^k_{d+d^\Lambda} \cong \mathcal{H}^k_{d+d^\Lambda}$ i.e. there is a unique (d+d^Λ)-harmonic form in each d+d^Λ cohomology class.

Proof. The operator Δ_{d+d^Λ} is not elliptic but the self-adjoint operator D_{d+d^Λ} in the previous lemma is elliptic as it is proved below. Since we have $ker\, \Delta_{d+d^\Lambda} = ker\, D_{d+d^\Lambda}$ by the previous lemma, this proves first two parts. To compute the symbol of D_{d+d^Λ}, we can use all the Kähler identities by the same reasoning as in the proof of Proposition 3.5. Also, note that only fourth-order terms in D_{d+d^Λ} matter when we are computing the symbol. So, we have

$$\begin{aligned} D_{d+d^\Lambda} &\simeq dd^\Lambda d^{\Lambda *} d^* + d^{\Lambda *} d^* dd^\Lambda + d^* d^\Lambda d^{\Lambda *} d + d^{\Lambda *} dd^* d^\Lambda \\ &\simeq -dd^\Lambda d^* d^{\Lambda *} - d^* d^{\Lambda *} dd^\Lambda - d^* d^\Lambda dd^{\Lambda *} - dd^{\Lambda *} d^* d^\Lambda \\ &\simeq dd^* d^\Lambda d^{\Lambda *} + d^* dd^{\Lambda *} d^\Lambda + d^* dd^\Lambda d^{\Lambda *} + dd^* d^{\Lambda *} d^\Lambda = \Delta_d \Delta_{d^\Lambda} \simeq \Delta_d^2. \end{aligned}$$

In the above computation, we have used $d^\Lambda = d^{c*}$, $dd^\Lambda = -d^\Lambda d$, $dd^c \simeq -d^c d$ and their adjoints. We also used $\Delta_{d^\Lambda} \simeq \Delta_d$ proved in Proposition 3.5.

Finally, to show c) we should prove that $ker\, d \cap ker\, d^\Lambda \cap \Omega^k = \mathcal{H}^k_{d+d^\Lambda} \oplus dd^\Lambda \Omega^k$ or

$$ker\, d \cap ker\, d^\Lambda \cap (d^* \Omega^{k+1} + d^{\Lambda *} \Omega^{k-1}) = \{0\},$$

by the decomposition in b). Assume that the form $A = d^* A_{k+1} + d^{\Lambda *} A_{k-1}$ is both d- and d^Λ-closed, then we have

$$0 =< dA, A_{k+1} > + < d^\Lambda A, A_{k-1} >=< A, d^* A_{k+1} > + < A, d^{\Lambda *} A_{k-1} >= ||A||^2. \quad \square$$

Corollary 3.12. *For a compact symplectic manifold, dim* $H^k_{d+d^\Lambda} < \infty$.

Proposition 3.13. *We have dim* $H^{2k}_{d+d^\Lambda} > 0$ *for* $0 \leq k \leq n$. *In fact, we have* $[\omega^k] \neq 0$.

Proof. Having $d\omega = 0$ imply that $d\omega^k = 0$ and

$$d^\Lambda \omega^k = d\Lambda L^k 1 - 0 = dL^k \Lambda 1 + d((H + k - 1)kL^{k-1}1) = 0 + (n - k + 1)kd\omega^{k-1} = 0.$$

Note that $\omega^k = L^k 1$ and the constant function 1 is not dd^Λ-exact, hence ω^k is not dd^Λ-exact for $0 \leq k \leq n$ by Proposition 2.22. □

Proposition 3.14. *Let* $f_t : M \to M$ *be a Hamiltonian isotopy of* M *i.e. a smooth family of diffeomorphisms* f_t *generated by a family of Hamiltonian vector fields* V_t *and* $f_0 = Id$. *Then, the* $d + d^\Lambda$ *cohomology class is invariant under this isotopy i.e. the isomorphism* $f_t^* : H^k_{d+d^\Lambda} \xrightarrow{\cong} H^k_{d+d^\Lambda}$ *is actually the identity map for all* t.

Proof. Let the form A be both d- and d^Λ-closed i.e. $[A] \in H^k_{d+d^\Lambda}$. By Corollary 2.25 part a), we have

$$\frac{d}{dt} f_t^* A = f_t^* \mathcal{L}_{V_t} A = f_t^* dd^\Lambda A_t' = dd^\Lambda (f_t^* A_t'),$$

which proves that the cohomology class $[f_t^* A]$ is independent of time t. But we know that f_0 and f_0^* are the identity maps. □

Remark 3.15. *Let* B *be a* ∂_+*-exact primitive form, i.e. there exists a differential form* A *satisfying* $B = \partial_+ A$. *Then the primitive component* B' *of* A *also satisfies* $B = \partial_+ B'$. *Therefore for primitive forms, being* ∂_+*-exact in terms of differential forms and primitive forms are equivalent. The case is the same for* ∂_-*-exactness and* $\partial_+\partial_-$*-exactness.*

Definition 3.16. *Having Lemma 2.22 and Definition 3.7 in mind, define the primitive* $d+d^\Lambda$ *cohomology by*

$$PH^k_{d+d^\Lambda} := \frac{ker\, d \cap \mathcal{P}^k}{im\, dd^\Lambda \cap \mathcal{P}^k} = \frac{ker\, d \cap \mathcal{P}^k}{dd^\Lambda \mathcal{P}^k} = \frac{ker\, \partial_+ \cap ker\, \partial_- \cap \mathcal{P}^k}{\partial_+ \partial_- \mathcal{P}^k} = \frac{ker\, (\partial_+ + \partial_-) \cap \mathcal{P}^k}{\partial_+ \partial_- \mathcal{P}^k}, \tag{22}$$

for $0 \leq k \leq n$. *Because of the last term, we also use the notation* $PH^k_{\partial_+ + \partial_-}$ *for these cohomologies.*

Note that we have $im\, dd^\Lambda \cap \mathcal{P}^k = dd^\Lambda \mathcal{P}^k = \partial_+\partial_- \mathcal{P}^k$ because of Remark 3.15 and the fact that dd^Λ and $\partial_+\partial_-$ are equal upto a nonzero constant.

The Lefschetz decomposition of differential forms and Proposition 2.22 give the Lefschetz decomposition at the level of $d + d^\Lambda$ cohomology:

$$H^k_{d+d^\Lambda} = \bigoplus_r L^r P H^{k-2r}_{d+d^\Lambda}. \tag{23}$$

Having primitive cohomology $PH^k_{d+d^\Lambda}$, we can compute cohomology $H^k_{d+d^\Lambda}$ by this formula.

Definition 3.17. *For a compatible triple (ω, J, g) on a compact manifold M, we define the primitive Laplacian $\Delta^p_{d+d^\Lambda} : \mathcal{P}^k \to \mathcal{P}^k$ by*

$$\Delta^p_{d+d^\Lambda} := dd^\Lambda (dd^\Lambda)^* + \lambda d^* d, \tag{24}$$

where λ is a positive real number. Also, define the space of primitive $(d + d^\Lambda)$-harmonic k-forms by $P\mathcal{H}^k_{d+d^\Lambda} := ker\, \Delta^p_{d+d^\Lambda} \cap \mathcal{P}^k = ker\, d \cap ker\, (dd^\Lambda)^* \cap \mathcal{P}^k$.

Note that we have $ker\, \Delta^p_{d+d^\Lambda} = ker\, \Delta_{d+d^\Lambda} = ker\, D_{d+d^\Lambda}$ on primitive forms since d-closed primitive forms are also d^Λ-closed. By ellipticity of the self-adjoint opreator D_{d+d^Λ} defined in Lemma 3.10, we have a primitive version of Proposition 3.11.

3.3 dd^Λ cohomology

Definition 3.18. *Define the dd^Λ cohomology by*

$$H^k_{dd^\Lambda} := \frac{ker\, dd^\Lambda \cap \Omega^k}{(im\, d + im\, d^\Lambda) \cap \Omega^k} = \frac{ker\, dd^\Lambda \cap \Omega^k}{d\Omega^{k-1} + d^\Lambda \Omega^{k+1}}, \tag{25}$$

for $0 \le k \le 2n$.

Definition 3.19. *For a compatible triple (ω, J, g) on a compact manifold M, we define the self-adjoint operator $\Delta_{dd^\Lambda} : \Omega^k \to \Omega^k$ called Laplacian associated with the dd^Λ cohomology by*

$$\Delta_{dd^\Lambda} := (dd^\Lambda)^* dd^\Lambda + \lambda(dd^* + d^\Lambda d^{\Lambda *}), \tag{26}$$

where λ is a positive real number. Also, define the space of dd^Λ-harmonic k-forms by the formula $\mathcal{H}^k_{dd^\Lambda} := ker\, \Delta_{dd^\Lambda} \cap \Omega^k$.

Lemma 3.20. *For a compatible triple (ω, J, g) on a compact manifold M, we have*

$$*\Delta_{d+d^\Lambda} = \Delta_{dd^\Lambda} *, \quad \mathcal{J}\Delta_{d+d^\Lambda} = \Delta_{dd^\Lambda}\mathcal{J}, \quad *_s\Delta_{d+d^\Lambda} = \Delta_{d+d^\Lambda} *_s, \quad *_s\Delta_{dd^\Lambda} = \Delta_{dd^\Lambda} *_s,$$
$$[\Delta_{dd^\Lambda}, L] = [\Delta_{dd^\Lambda}, \Lambda] = [\Delta_{dd^\Lambda}, H] = 0.$$

Proof. The proof of the second line is exactly similar to the proof of Lemma 3.9. Corollary 2.30 implies the second equality in the first line. First and second equalities give the next two equalities using formula $* = *_s\mathcal{J}$. For the first equality, write

$$\begin{aligned} *\Delta_{d+d^\Lambda} &= (-1)^{k+1} * dd^\Lambda * dd^\Lambda * + \lambda(-(-1)^{2n-k} d * d + (-1)^{2n-k} d^\Lambda * d^\Lambda) \\ &= (-1)^{k+1} * dd^\Lambda * dd^\Lambda * + \lambda(-(-1)^{k} d * d + (-1)^{k} d^\Lambda * d^\Lambda) = \Delta_{dd^\Lambda} * . \end{aligned}$$
□

Lemma 3.21. *For a compatible triple (ω, J, g) on a compact manifold M, consider the following operator*

$$D_{dd^\Lambda} = \Delta_{dd^\Lambda} + dd^\Lambda(dd^\Lambda)^* + dd^{\Lambda *} d^\Lambda d^* + d^\Lambda d^* dd^{\Lambda *}.$$

For a differential form A, we have $\Delta_{dd^\Lambda} A = 0$ if and only if $d^*A = d^{\Lambda *}A = dd^\Lambda A = 0$ if and only if $D_{dd^\Lambda} A = 0$.

Proof. Exactly similar to the proof of Lemma 3.10. □

Proposition 3.22. *For a compatible triple (ω, J, g) on a compact manifold M, we have*

a) $\dim \mathcal{H}^k_{dd^\Lambda} < \infty$

b) The orthogonal decomposition $\Omega^k = \mathcal{H}^k_{dd^\Lambda} \oplus (d\Omega^{k-1} + d^\Lambda\Omega^{k+1}) \oplus (dd^\Lambda)^*\Omega^k$

c) $H^k_{dd^\Lambda} \cong \mathcal{H}^k_{dd^\Lambda}$ i.e. there is a unique dd^Λ-harmonic form in each dd^Λ cohomology class.

Proof. Similar to the proof of Proposition 3.11, we can prove that the self-adjoint operator D_{dd^Λ} is elliptic. □

Corollary 3.23. *For a compact symplectic manifold, $\dim H^k_{dd^\Lambda} < \infty$.*

Proposition 3.24. *For a compact symplectic manifold, the dimensions of H^{2k}_d, $H^{2k}_{d^\Lambda}$, $H^{2k}_{d+d^\Lambda}$ and $H^{2k}_{dd^\Lambda}$ are all positive for $0 \le k \le n$. In fact, the class $[\omega^k]$ is nontrivial for all of these cohomologies.*

Proof. By Proposition 3.13, ω^k is d-, d^Λ- and dd^Λ-closed. Assume that $0 \le k \le n$ and $\omega^k = dA + d^\Lambda A'$. Then, we have

$$\begin{aligned} 0 \neq n! \int_M dvol = \int_M \omega^n &= \int_M (dA) \wedge \omega^{n-k} + \int_M (d^\Lambda A') \wedge \omega^{n-k} \\ &= (-1)^{(2k-1)+1} \int_M A \wedge d\omega^{n-k} + (-1)^{2k+1} \int_M A' \wedge d^\Lambda \omega^{n-k} = 0 \end{aligned}$$

This contradiction shows that ω^k is not $(d + d^\Lambda)$- exact and therefore it also can't be d- or d^Λ- or dd^Λ-exact. □

Proposition 3.25. *Let $f_t : M \to M$ be a Hamiltonian isotopy of M. Then, the dd^Λ cohomology class is invariant under this isotopy i.e. the isomorphism $f_t^* : H^k_{dd^\Lambda} \xrightarrow{\cong} H^k_{dd^\Lambda}$ is actually the identity map for all t.*

Proof. Exactly similar to the proof of Proposition 3.14. □

Definition 3.26. *Having Lemma 2.22 and Definition 3.18 in mind, define the primitive dd^Λ cohomology by*

$$PH^k_{dd^\Lambda} := \frac{ker\, dd^\Lambda \cap \mathcal{P}^k}{(im\, d + im\, d^\Lambda) \cap \mathcal{P}^k} = \frac{ker\, \partial_+ \partial_- \cap \mathcal{P}^k}{\partial_+ \mathcal{P}^{k-1} + \partial_- \mathcal{P}^{k+1}}, \tag{27}$$

for $0 \le k \le n$. Because of the last term, we also use the notation $PH^k_{\partial_+\partial_-}$ for these cohomologies.

Let us prove the equality $(im\, d + im\, d^\Lambda) \cap \mathcal{P}^k = \partial_+ \mathcal{P}^{k-1} + \partial_- \mathcal{P}^{k+1}$ in this definition. Assume that the form $B''' = dA + d^\Lambda A'$ is primitive and that the primitive component of the differential form A is B. Also, suppose that $B' + LB''$ gives the first two terms in the Lefschetz decomposition of A'. Then by Proposition 2.34 and Lemma 2.4, the primitive components of dA and $d^\Lambda A'$ are $\partial_+ B$ and

$$\frac{1}{H+R+1} \partial_+ \Lambda L B'' - (H+R) \partial_- B' = \partial_+ B'' - \partial_- (H+1) B',$$

respectively. Consequently, we have $B''' = \partial_+(B + B'') + \partial_-(-(H+1)B')$. Conversely, assume that we have $B = \partial_+ B' + \partial_- B''$. Then by Corollary 2.35, we have

$$B = (d + LH^{-1} d^\Lambda) B' - H^{-1} d^\Lambda B'' = d(\frac{H}{H+1} B') + d^\Lambda (L(H+1)^{-1} B' - (H+1)^{-1} B'').$$

The Lefschetz decomposition of differential forms and Proposition 2.22 give the Lefschetz decomposition at the level of dd^Λ cohomology:

$$H^k_{dd^\Lambda} = \bigoplus_r L^r PH^{k-2r}_{dd^\Lambda}. \tag{28}$$

Having primitive cohomologys $PH^k_{dd^\Lambda}$, we can compute cohomology $H^k_{dd^\Lambda}$ by this formula.

Definition 3.27. *For a compatible triple (ω, J, g) on a compact manifold M, we define the primitive Laplacian $\Delta^p_{dd^\Lambda} : \mathcal{P}^k \to \mathcal{P}^k$ by*

$$\Delta^p_{dd^\Lambda} := (dd^\Lambda)^* dd^\Lambda + \lambda d^\Lambda d^{\Lambda *}, \tag{29}$$

where λ is a positive real number. Also, define the space of primitive dd^Λ-harmonic k-forms by $P\mathcal{H}^k_{dd^\Lambda} := ker\, \Delta^p_{dd^\Lambda} \cap \mathcal{P}^k = ker\, d^{\Lambda *} \cap ker\, dd^\Lambda \cap \mathcal{P}^k$.

Note that we have $ker\, \Delta^p_{dd^\Lambda} = ker\, \Delta_{dd^\Lambda} = ker\, D_{dd^\Lambda}$ on primitive forms because $d^{\Lambda *}$-closed primitive forms are also d^*-closed by the formula $[d^{\Lambda *}, \Lambda] = -d^*$. By ellipticity of the self-adjoint opreator D_{dd^Λ} defined in Lemma 3.21, we have a primitive version of Proposition 3.22.

3.4 $\partial_\pm$ cohomologies

Definition 3.28. *Having $\partial_+^2 = \partial_-^2 = 0$, we can define the $\partial_\pm$ and primitive $\partial_\pm$ cohomologies by*

$$H^{r,s}_{\partial_+} := \frac{ker\, \partial_+ \cap \mathcal{L}^{r,s}}{\partial_+ \mathcal{L}^{r,s-1}}, \qquad PH^k_{\partial_+} := \frac{ker\, \partial_+ \cap \mathcal{P}^k}{\partial_+ \mathcal{P}^{k-1}} = H^{0,k}_{\partial_+}, \tag{30}$$

$$H^{r,s}_{\partial_-} := \frac{ker\, \partial_- \cap \mathcal{L}^{r,s}}{\partial_- \mathcal{L}^{r,s+1}}, \qquad PH^k_{\partial_-} := \frac{ker\, \partial_- \cap \mathcal{P}^k}{\partial_- \mathcal{P}^{k+1}} = H^{0,k}_{\partial_-}, \tag{31}$$

for $0 \le r+s < n$ and $0 \le k < n$.

Note that we have $H^{r,s}_{\partial_\pm} \cong H^{0,s}_{\partial_\pm} = PH^s_{\partial_\pm}$ for $0 \le r+s < n$ by commutativity of $\partial_\pm$ with L and isomorphism $L^r : \mathcal{P}^s \xrightarrow{\cong} \mathcal{L}^{r,s}$. Therefore, we only study the primitive cohomologies $PH^k_{\partial_\pm}$.

Proposition 3.29. *The following differential complex*

$$\begin{array}{ccccccccc} 0 & \xrightarrow{\partial_+} & \mathcal{P}^0 & \xrightarrow{\partial_+} & \dots & \xrightarrow{\partial_+} & \mathcal{P}^{n-1} & \xrightarrow{\partial_+} & \mathcal{P}^n \\ & & & & & & & & \Big\downarrow{\scriptstyle \partial_+\partial_-} \\ 0 & \xleftarrow[\partial_-]{} & \mathcal{P}^0 & \xleftarrow[\partial_-]{} & \dots & \xleftarrow[\partial_-]{} & \mathcal{P}^{n-1} & \xleftarrow[\partial_-]{} & \mathcal{P}^n \end{array} \tag{32}$$

is elliptic.

Proof. First note that this is a differential complex since $\partial_\pm^2 = (\partial_+\partial_-)\partial_+ = \partial_-(\partial_+\partial_-) = 0$. To prove that it is elliptic we should show that the associated symbol complex is exact everywhere. Let $p \in M$, $\xi \in T_p^*M - \{0\}$ and $B_k \in \mathcal{P}_p^k$, where by $\mathcal{P}_p^k$ we mean the primitive subspace of the k-th exterior power $\bigwedge^k T_p^*M$, in other words B_k is a primitive k-form at point p. Let the operator $T : \mathcal{P}^k \to \mathcal{P}^{k+d}$ be one of operators ∂_+, ∂_- or $\partial_+\partial_-$. Recall that the symbol of T is the linear map $\sigma_T(\xi) : \mathcal{P}_p^k \to \mathcal{P}_p^{k+d}$ defined by

$$\sigma_T(\xi)(B_k) := T(f^2 B)(p),$$

where we have $B \in \mathcal{P}^k$ and $f \in C^\infty(M)$ such that $B_p = B_k$, $f(p) = 0$ and $df(p) = \xi$. We know that the symbol $\sigma_d(\xi)$ of the exterior derivative operator d is simply left exterior multiplication by ξ [18]. Using Corollary 2.35, we can easily compute the following symbols

$$\begin{aligned} \sigma_{\partial_+}(\xi)(B_k) &= (1 - LH^{-1}\Lambda)(\xi \wedge B_k), \\ \sigma_{\partial_-}(\xi)(B_k) &= H^{-1}\Lambda(\xi \wedge B_k), \\ \sigma_{\partial_+\partial_-}(\xi)(B_k) &= (H+1)^{-1}(\xi \wedge \Lambda(\xi \wedge B_k)). \end{aligned}$$

Take a basis $\{e_1, ..., e_{2n}\}$ of T_p^*M such that $\omega_p = e_1 \wedge e_2 + ... + e_{2n-1} \wedge e_{2n}$ and $\xi = e_1$. Using the notation $\omega_1 := e_1 \wedge e_2$ and $\omega_2 := e_3 \wedge e_4 + ... + e_{2n-1} \wedge e_{2n}$, we showed in the proof of Proposition 2.12 that we have

$$B_k = e_1 \wedge B_{k-1} + e_2 \wedge B'_{k-1} + (\omega_1 - \frac{1}{n-k+1}\omega_2) \wedge B_{k-2} + B'_k,$$

for some primitive forms $B_{k-1}, B'_{k-1} \in \mathcal{P}_p^{k-1}$, $B_{k-2} \in \mathcal{P}_p^{k-2}$ and $B'_k \in \mathcal{P}_p^k$ at point p involving only $e_3, ..., e_{2n}$. In other words, we have

$$\mathcal{P}_p^k = span\{e_1 \wedge B_{k-1},\ e_2 \wedge B'_{k-1},\ (\omega_1 - (H+1)^{-1}\omega_2) \wedge B_{k-2},\ B'_k\},$$

where $B_{k-1}, B'_{k-1}, B_{k-2}$ and B'_k range over all such forms. Using $\xi = e_1$, we have

$$\sigma_{\partial_+}(\xi)(e_1 \wedge B_{k-1}) = \sigma_{\partial_-}(\xi)(e_1 \wedge B_{k-1}) = \sigma_{\partial_+\partial_-}(\xi)(e_1 \wedge B_{k-1}) = 0,$$
$$\sigma_{\partial_+}(\xi)(e_2 \wedge B'_{k-1}) = (\omega_1 - (H+2)^{-1}\omega) \wedge B'_{k-1} = \frac{H+1}{H+2}((\omega_1 - (H+1)^{-1}\omega_2) \wedge B'_{k-1}),$$
$$\sigma_{\partial_-}(\xi)(e_2 \wedge B'_{k-1}) = H^{-1}B'_{k-1}, \qquad \sigma_{\partial_+\partial_-}(\xi)(e_2 \wedge B'_{k-1}) = (H+1)^{-1}(e_1 \wedge B'_{k-1}),$$
$$\xi \wedge (\omega_1 - (H+1)^{-1}\omega_2) \wedge B_{k-2} = -(H+2)^{-1}(\omega_2 \wedge e_1 \wedge B_{k-2}) = -LH^{-1}(e_1 \wedge B_{k-2}),$$

$$\sigma_{\partial_+}(\xi)((\omega_1 - (H+1)^{-1}\omega_2) \wedge B_{k-2}) = \sigma_{\partial_+\partial_-}(\xi)((\omega_1 - (H+1)^{-1}\omega_2) \wedge B_{k-2}) = 0,$$
$$\sigma_{\partial_-}(\xi)((\omega_1 - (H+1)^{-1}\omega_2) \wedge B_{k-2}) = -H^{-1}(e_1 \wedge B_{k-2}),$$
$$\sigma_{\partial_-}(\xi)(B'_k) = \sigma_{\partial_+\partial_-}(\xi)(B'_k) = 0, \qquad \sigma_{\partial_+}(\xi)(B'_k) = e_1 \wedge B'_k.$$

Consequently, we have computed the kernels and images of symbols of different operators appearing in the differential complex:

$$ker\, \sigma_{\partial_+}(\xi) = span\{e_1 \wedge B_{k-1},\, (\omega_1 - (H+1)^{-1}\omega_2) \wedge B_{k-2}\},$$
$$ker\, \sigma_{\partial_-}(\xi) = span\{e_1 \wedge B_{k-1},\, B'_k\},$$
$$ker\, \sigma_{\partial_+\partial_-}(\xi) = span\{e_1 \wedge B_{k-1},\, (\omega_1 - (H+1)^{-1}\omega_2) \wedge B_{k-2},\, B'_k\},$$
$$im\, \sigma_{\partial_+}(\xi) = span\{(\omega_1 - (H+1)^{-1}\omega_2) \wedge B'_{k-1},\, e_1 \wedge B'_k\},$$
$$im\, \sigma_{\partial_-}(\xi) = span\{B'_{k-1},\, e_1 \wedge B_{k-2}\}, \qquad im\, \sigma_{\partial_+\partial_-}(\xi) = span\{e_1 \wedge B'_{k-1}\},$$

which show the exactness of the symbol complex (note that we have $B'_n = 0$ because if there exists a nonzero primitive k-form at p not involving e_1 and e_2, then we should have $0 \leq k \leq n-2$). □

Remark. *Note that two cohomologies at position $\mathcal{P}^n$ in the above differential complex are previously defined primitive cohomologies $PH^n_{dd^\Lambda}$ and $PH^n_{d+d^\Lambda}$.*

Definition 3.30. *For a compatible triple (ω, J, g) on a compact manifold M, we define the self-adjoint operator $\Delta_{\partial_\pm} : \mathcal{P}^k \to \mathcal{P}^k$ called Laplacian associated with the cohomologies $PH_{\partial_\pm}$ by the formula*

$$\Delta_{\partial_\pm} := \partial_\pm(\partial_\pm)^* + (\partial_\pm)^*\partial_\pm. \tag{33}$$

Also, define the space of primitive $\partial_\pm$-harmonic k-forms by the formula

$$P\mathcal{H}^k_{\partial_\pm} := ker\,\Delta_{\partial_\pm} \cap \mathcal{P}^k = ker\,\partial_\pm \cap ker\,\partial^*_\pm \cap \mathcal{P}^k, \tag{34}$$

for $0 \leq k < n$.

Let p be a given point in M and $\xi \in T^*_pM$ be nonzero. Using Proposition 3.29 and the following lemma, the symbol $\sigma_{\Delta_{\partial_\pm}}(\xi) : \mathcal{P}^k_p \to \mathcal{P}^k_p$ is an isomorphism for $0 \leq k < n$, so the self-adjoint operators $\Delta_{\partial_\pm}$ are elliptic and we have Proposition 3.32.

Lemma 3.31. *Let U, V and W be finite dimensional inner product spaces. Assume that we have the exact sequence $U \xrightarrow{S} V \xrightarrow{T} W$ of linear maps. Let $T^* : W \to V$ be the adjoint of T i.e $< Tv, w >=< v, T^*w >$ for all $v \in V$ and $w \in W$ and similarly $S^* : V \to U$ be the adjoint of S. Then, the linear map $(T^*T + SS^*) : V \to V$ is an isomorphism [18].*

Proof. Since the space V is finite dimensional, it is sufficient to prove that the above map is injective. Let $v \in V$ be nonzero. If we have $Tv = 0$, then exactness of the above sequence implies that there exists $u \in U$ such that $v = Su$. So, we have

$$< u, S^*v >=< Su, v >=< v, v >\neq 0,$$

which shows that $S^*v \neq 0$. Consequently, at least one of Tv and S^*v are nonzero and therefore

$$< (T^*T + SS^*)v, v >=< Tv, Tv > + < S^*v, S^*v >\neq 0,$$

which completes the proof. □

Proposition 3.32. *For a compatible triple (ω, J, g) on a compact manifold M, we have*

a) $dim\,P\mathcal{H}^k_{\partial_\pm} < \infty$

b) The orthogonal decomposition $\mathcal{P}^k = P\mathcal{H}^k_{\partial_\pm} \oplus \partial_\pm\mathcal{P}^{k\mp1} \oplus \partial^*_\pm\mathcal{P}^{k\pm1}$

c) $PH^k_{\partial_\pm} \cong P\mathcal{H}^k_{\partial_\pm}$ i.e. there is a unique primitive $\partial_\pm$-harmonic form in each primitive $\partial_\pm$ cohomology class.

Corollary 3.33. *For a compact symplectic manifold, $dim\,PH^k_{\partial_\pm} < \infty$ for $0 \leq k < n$.*

4 Further properties of symplectic cohomologies

4.1 Dualities

Proposition 4.1. *For any symplectic manifold M, we have the following isomorphisms:*

$$*_s : H^k_d \to H^{2n-k}_{d^\Lambda}, \qquad *_s : H^k_{d+d^\Lambda} \to H^{2n-k}_{d+d^\Lambda}, \qquad *_s : H^k_{dd^\Lambda} \to H^{2n-k}_{dd^\Lambda} \qquad 0 \le k \le 2n,$$
$$L^{n-k} : H^k_{d+d^\Lambda} \to H^{2n-k}_{d+d^\Lambda}, \qquad L^{n-k} : H^k_{dd^\Lambda} \to H^{2n-k}_{dd^\Lambda} \qquad 0 \le k \le n.$$

Proof. Since these maps at the level of differential forms are isomorphisms, Corollaries 2.20 and 2.23 prove this proposition. $\square$

Proposition 4.2. *For a compatible triple (ω, J, g) on a compact manifold M, we have the following isomorphisms:*

$$*_s : \mathcal{H}^k_d \to \mathcal{H}^{2n-k}_{d^\Lambda}, \qquad *_s : \mathcal{H}^k_{d+d^\Lambda} \to \mathcal{H}^{2n-k}_{d+d^\Lambda}, \qquad *_s : \mathcal{H}^k_{dd^\Lambda} \to \mathcal{H}^{2n-k}_{dd^\Lambda} \qquad 0 \le k \le 2n,$$
$$L^{n-k} : \mathcal{H}^k_{d+d^\Lambda} \to \mathcal{H}^{2n-k}_{d+d^\Lambda}, \qquad L^{n-k} : \mathcal{H}^k_{dd^\Lambda} \to \mathcal{H}^{2n-k}_{dd^\Lambda} \qquad 0 \le k \le n,$$
$$* : \mathcal{H}^k_d \to \mathcal{H}^{2n-k}_d, \qquad * : \mathcal{H}^k_{d^\Lambda} \to \mathcal{H}^{2n-k}_{d^\Lambda}, \qquad * : \mathcal{H}^k_{d+d^\Lambda} \to \mathcal{H}^{2n-k}_{dd^\Lambda} \qquad 0 \le k \le 2n,$$
$$\mathcal{J} : \mathcal{H}^k_d \to \mathcal{H}^k_{d^\Lambda}, \qquad \mathcal{J} : \mathcal{H}^k_{d+d^\Lambda} \to \mathcal{H}^k_{dd^\Lambda} \qquad 0 \le k \le 2n,$$
$$\mathcal{J} : P\mathcal{H}^k_{d+d^\Lambda} \to P\mathcal{H}^k_{dd^\Lambda} \qquad 0 \le k \le n,$$
$$\mathcal{J} : P\mathcal{H}^k_{\partial_+} \to P\mathcal{H}^k_{\partial_-} \qquad 0 \le k < n.$$

Proof. Since these maps at the level of differential forms are isomorphisms, Lemmas 3.3, 3.20, 3.9 and the equality $\mathcal{J}\Delta^p_{d+d^\Lambda} = \Delta^p_{dd^\Lambda}\mathcal{J}$ prove first five lines of isomorphisms. For the last line, note that the commutation relation $\mathcal{J}\Lambda = \Lambda\mathcal{J}$ implies the isomorphism $\mathcal{J} : \Omega^k \to \Omega^k$ reduces to the isomorphism $\mathcal{J} : \mathcal{P}^k \to \mathcal{P}^k$. It remains to check that for a primitive k-form B, we have $B \in P\mathcal{H}^k_{\partial_\pm}$ if and only if $\mathcal{J}B \in P\mathcal{H}^k_{\partial_\mp}$. But using Lemma 2.36, we can write

$$\partial_+(\mathcal{J}B) = (-1)^k(n-k)\mathcal{J}^{-1}\partial_-^* B, \qquad \partial_+^*(\mathcal{J}B) = -(-1)^k(n-k+1)\mathcal{J}^{-1}\partial_- B,$$
$$\partial_-(\mathcal{J}B) = \frac{-1}{n-k+1}\mathcal{J}\partial_+^* B, \qquad \partial_-^*(\mathcal{J}B) = \frac{1}{n-k}\mathcal{J}\partial_+ B.$$
$\square$

Proposition 4.3. *For a compact symplectic manifold M, we have the following non-degenerate* *natural pairings*

$$H_d^k \otimes H_d^{2n-k} \to \mathbb{R}, \qquad H_{d^\Lambda}^k \otimes H_{d^\Lambda}^{2n-k} \to \mathbb{R}, \qquad H_{d+d^\Lambda}^k \otimes H_{dd^\Lambda}^{2n-k} \to \mathbb{R},$$
$$[A] \otimes [A'] \mapsto \int_M A \wedge A',$$

for $0 \leq k \leq 2n$. *Consequently, we have* $H_{d+d^\Lambda}^k \cong H_{dd^\Lambda}^{2n-k}$.

Proof. Proof of first two pairings are easier and similar to the last one which is proved below. To prove well-definedness, assume that we have

$$C = A + dd^\Lambda D, \qquad C' = A' + dD' + d^\Lambda D'', \qquad dA = d^\Lambda A = dd^\Lambda A' = 0.$$

Then using the Stokes' theorem and Lemma 2.26, we have

$$\begin{aligned}\int_M C \wedge C' &= \int_M A \wedge A' + \int_M A \wedge dD' + \int_M A \wedge d^\Lambda D'' + \int_M (dd^\Lambda D) \wedge A' \\ &+ \int_M (dd^\Lambda D) \wedge dD' + \int_M (dd^\Lambda D) \wedge d^\Lambda D'' = \int_M A \wedge A' + (-1)^{k+1} \int_M (dA) \wedge D' \\ &+ (-1)^k \int_M (d^\Lambda A) \wedge D'' - \int_M D \wedge dd^\Lambda A' + \int_M D \wedge d^\Lambda dd D' - \int_M D \wedge dd^\Lambda d^\Lambda D'' \\ &= \int_M A \wedge A'.\end{aligned}$$

To prove non-degeneracy, choose any compatible triple (ω, J, g) and assume that we have $A \in \mathcal{H}_{d+d^\Lambda}^k$ and $A \neq 0$. Then, it is the case that $*A \in \mathcal{H}_{dd^\Lambda}^{2n-k}$, $[*A] \in H_{dd^\Lambda}^{2n-k}$ and

$$[A] \otimes [*A] \mapsto \int_M A \wedge *A =< A, A > \neq 0. \quad \square$$

Proposition 4.4. *For a compact symplectic manifold* M*, we have the following non-degenerate natural pairings*

$$H_d^k \otimes H_{d^\Lambda}^k \to \mathbb{R}, \quad H_{d+d^\Lambda}^k \otimes H_{dd^\Lambda}^k \to \mathbb{R}, \quad PH_{d+d^\Lambda}^k \otimes PH_{dd^\Lambda}^k \to \mathbb{R}, \quad PH_{\partial_+}^k \otimes PH_{\partial_-}^k \to \mathbb{R},$$
$$[A] \otimes [A'] \mapsto \int_M A \wedge *_s A',$$

for $0 \leq k \leq 2n$, $0 \leq k \leq n$ *and* $0 \leq k < n$, *respectively. Consequently, we have isomorphisms* $H_{d+d^\Lambda}^k \cong H_{dd^\Lambda}^k$, $PH_{d+d^\Lambda}^k \cong PH_{dd^\Lambda}^k$ *and* $PH_{\partial_+}^k \cong PH_{\partial_-}^k$.

Proof. Well-definedness of first three pairings is implied by Corollary 2.20, Stokes' theorem and Lemma 2.26 similar to the previous proposition. To prove well-definedness of the last isomorphism, note that for primitive forms B and B' we have

$$\int_M B \wedge *_s B' = \frac{(-1)^{\frac{k(k+1)}{2}}}{(n-k)!} \int_M L^{n-k}(B \wedge B'),$$

by Proposition (2.12). For $0 \leq k < n$ and primitive forms B_k and B_{k+1}, we have

$$\begin{aligned} dL^{n-k-1}(B_k \wedge B_{k+1}) &= L^{n-k-1}((dB_k) \wedge B_{k+1}) + (-1)^k L^{n-k-1}(B_k \wedge dB_{k+1}) \\ &= L^{n-k-1}((\partial_+ B_k) \wedge B_{k+1}) + (\partial_- B_k) \wedge (L^{n-k} B_{k+1}) \\ &+ (-1)^k B_k \wedge L^{n-k-1}(\partial_+ B_{k+1}) + (-1)^k L^{n-k}(B_k \wedge \partial_- B_{k+1}) \\ &= L^{n-k-1}((\partial_+ B_k) \wedge B_{k+1}) + (-1)^k L^{n-k}(B_k \wedge \partial_- B_{k+1}). \end{aligned}$$

Then, the Stokes' theorem implies that

$$\int_M L^{n-k-1}((\partial_+ B_k) \wedge B_{k+1}) = (-1)^{k+1} \int_M L^{n-k}(B_k \wedge \partial_- B_{k+1}).$$

We can use this formula to prove well-definedness: Assume that we have

$$C = B + \partial_+ D, \qquad C' = B' + \partial_- D', \qquad \partial_+ B = \partial_- B' = 0,$$

where all differential forms are primitive. We can write

$$\begin{aligned} \int_M L^{n-k}(C \wedge C') &= \int_M L^{n-k}(B \wedge B') + \int_M L^{n-k}(B \wedge \partial_- D') + \int_M L^{n-k}((\partial_+ D) \wedge B') \\ &+ \int_M L^{n-k}((\partial_+ D) \wedge \partial_- D') = \int_M L^{n-k}(B \wedge B') + (-1)^{k+1} \int_M L^{n-k-1}((\partial_+ B) \wedge D') \\ &+ (-1)^k \int_M L^{n-k+1}(D \wedge \partial_- B') + (-1)^k \int_M L^{n-k+1}(D \wedge \partial_-^2 D') = \int_M L^{n-k}(B \wedge B'). \end{aligned}$$

Proof of non-degeneracy is completely similar for all parings and we do it only for the last pairing. Choose any compatible triple (ω, J, g) and assume that we have $B \in P\mathcal{H}^k_{\partial_+}$ and $B \neq 0$. Then, it is the case that $\mathcal{J}B \in P\mathcal{H}^k_{\partial_-}$, $[\mathcal{J}B] \in PH^k_{\partial_-}$ and

$$[B] \otimes [\mathcal{J}B] \mapsto \int_M B \wedge *_s \mathcal{J}B = \int_M B \wedge *B =< B, B > \neq 0. \quad \square$$

4.2 Poincaré lemmas

All over this subsection, we assume that M is a star-shaped open subset of $\mathbb{R}^{2n}$ with the standard symplectic form $\omega = \sum_{i=0}^{n} dx^i \wedge dx^{i+n}$. Hence, M is connected and d-closed 0-forms are constant functions. For the 1-form $\alpha = \sum_{i=0}^{n} x^i dx^{i+n}$, we have $d\alpha = \omega$. Then, the form α is primitive and $\partial_+\alpha = 0$ and $\partial_-\alpha = 1$. Our goal is to compute different symplectic cohomologies for this special case. Note that M is not compact and some of dualities don't hold for M.

Proposition 4.5. *We have $H^k_{d^\Lambda} = \{0\}$ for $0 \leq k < 2n$ and $\dim H^{2n}_{d^\Lambda} = 1$ with the generator $[\omega^n]$.*

Proof. This is just applying Proposition 4.1 to the standard Poincaré lemma. $\square$

Lemma 4.6. *If B is a d-closed primitive k-form for $0 < k \leq n$, then there exists a primitive $k-1$-form B' satisfying $B = dB'$. Moreover, we should have $\partial_- B' = d^\Lambda B' = 0$ and $B = \partial_+ B'$.*

Proof. First note that any primitive form B' such that dB' is also primitive should be both ∂_--closed and d^Λ-closed. By the Poincaré lemma for the de Rham cohomology, we know that there exists a $k-1$-form A with the property $B = dA$. For $k = 1$, since any 0-form is primitive, there is nothing to prove. For $2 \leq k \leq n$ to see that A can be chosen to be primitive, we should do the proof of the Poincaré lemma given in [14], again.

Consider the vector field $V = \sum_{i=1}^{n} x^i \partial_{x^i}$ on M. We have

$$\mathcal{L}_V(f dx^I) = (Vf)dx^I + f(d(i_V dx^I)) = (Vf + kf)dx^I,$$

where $\#I = k$. Without loss of generality, assume that M is star-shaped around the origin. Define the linear map $T : \Omega^k \to \Omega^k$ for $0 < k \leq n$ by

$$T(f dx^I) := (\int_0^1 t^{k-1} f(tx)dt)dx^I,$$

where $\#I = k$. We have $T\mathcal{L}_V = Id$ and $Td = dT$ by the following computations:

$$T\mathcal{L}_V(f dx^I) = (\int_0^1 t^{k-1}(Vf + kf)(tx)dt)dx^I = (\int_0^1 (\frac{d}{dt} t^k f(tx))dt)dx^I = f dx^I,$$
$$Td(f dx^I) = \sum_{i=1}^{n} (\int_0^1 t^k \partial_{x^i} f(tx)dt)dx^i \wedge dx^I = (d(\int_0^1 t^{k-1} f(tx)dt)) \wedge dx^I = dT(f dx^I).$$

For $2 \leq k \leq n$, take $B' := T(i_V B)$. Then, we have

$$dB' = dT(i_V B) = Td(i_V B) = T\mathcal{L}_V(B) = B.$$

Since we have $\Lambda\, i_V = i_V \Lambda$, the form $i_V B$ is primitive. It remains to prove that T maps primitive forms to primitive forms. Note that a differential form is primitive if and only if it can be written in the form of $\sum_{I,I'} f_{I,I'}(dx^I + dx^{I'})$ such that for any I and I' with $f_{I,I'} \neq 0$, the form $dx^I + dx^{I'}$ is primitive. This is an easy consequence of the definition of primitivity and we will illustrate this by an example. Let $M = \mathbb{R}^6$ and the form

$$f_0 dx^1 \wedge dx^2 + f_1 dx^1 \wedge dx^4 + f_2 dx^2 \wedge dx^5 + f_3 dx^3 \wedge dx^6,$$

be primitive. Then, we must have $f_1 + f_2 + f_3 = 0$ and therefore this form can be written as

$$\frac{f_0}{2}(dx^1 \wedge dx^2 + dx^1 \wedge dx^2) + f_1(dx^1 \wedge dx^4 + dx^6 \wedge dx^3) + f_2(dx^2 \wedge dx^5 \wedge +dx^6 \wedge dx^3),$$

where all three parentheses are primitive. Now, since we have

$$T(\sum_{I,I'} f_{I,I'}(dx^I + dx^{I'})) = \sum_{I,I'}(\int_0^1 t^{k-1} f_{I,I'}(tx)dt)(dx^I + dx^{I'}),$$

T maps primitive forms to primitive forms. $\square$

Proposition 4.7. *We have* $PH^k_{d+d^\Lambda} = \{0\}$ *for* $0 < k \leq n$ *and* $\dim PH^0_{d+d^\Lambda} = 1$ *with the generator* $[1]$.

Proof. ($k = 0$): Any constant function is primitive, d-closed and not dd^Λ-exact. On the other hand, any d-closed 1-form is a constant function.

($0 < k \leq n$): Let B be a d-closed primitive k-form for $0 < k \leq n$. By Lemma 4.6, there is a primitive $(k-1)$-form B' satisfying $B = dB'$. Since B' is d^Λ-closed by Proposition 4.5, there is a k-form A with the property $B' = d^\Lambda A$ and hence $B = dd^\Lambda A$. $\square$

Corollary 4.8. *We have* $H^{2k+1}_{d+d^\Lambda} = \{0\}$ *for* $0 \leq k < n$ *and* $\dim H^{2k}_{d+d^\Lambda} = 1$ *for* $0 \leq k \leq n$ *with the generator* $[\omega^k]$.

Proof. This is just applying Proposition 4.7 to the formula (23). Note that this result is consistent with Proposition 3.13. $\square$

Proposition 4.9. *We have $PH^k_{dd^\Lambda} = \{0\}$ for $k = 0$ or $2 \le k \le n$ and dim $PH^1_{dd^\Lambda} = 1$ with the generator $[\alpha]$.*

Proof. ($k = 0$): By Proposition 4.5, any 0-form is d^Λ- and hence $(d + d^\Lambda)$-exact.

($k = 1$): Define a linear map $T : PH^1_{dd^\Lambda} \to \mathbb{R}$ by $T([B]) = d^\Lambda B$. We will show that T is well-defined, injective and nonzero, so that it is an isomorphism. First note that $dd^\Lambda B = 0$ implies that $d^\Lambda B$ is a constant function. To prove well-definedness, write

$$d^\Lambda(B + dA_0 + d^\Lambda A_2) = d^\Lambda B - dd^\Lambda A_0 + 0 = d^\Lambda B.$$

The equality $T([B]) = d^\Lambda B = 0$ and Proposition 4.5 ($1 < 2n$) implies that B is d^Λ-exact or $[B] = 0$ and T is injective. Finally, note that $dd^\Lambda\alpha = -d^\Lambda\omega = 0$ and

$$T([\alpha]) = d^\Lambda\alpha = d\Lambda\alpha - \Lambda d\alpha = -\Lambda\omega = -n \neq 0.$$

($2 \le k \le n$): Let B be a dd^Λ-closed primitive k-form. Since $d^\Lambda B$ is primitive and d-closed, Proposition 4.7 ($0 < k - 1$) implies that there is a $(k-1)$-form A satisfying $d^\Lambda B = dd^\Lambda A$. Hence, we have $d^\Lambda(B + dA) = 0$. Using Proposition 4.5 ($k < 2n$), the form $B + dA$ is d^Λ-exact or B is $(d + d^\Lambda)$-exact. □

Corollary 4.10. *We have $H^{2k}_{dd^\Lambda} = \{0\}$ for $0 \le k \le n$ and dim $H^{2k+1}_{dd^\Lambda} = 1$ for $0 \le k < n$ with the generator $[\omega^k \wedge \alpha]$.*

Proof. This is just applying Proposition 4.9 to the formula (28). □

Proposition 4.11. *We have $PH^k_{\partial_+} = \{0\}$ for $2 \le k < n$ and dim $PH^0_{\partial_+}$ = dim $PH^1_{\partial_+} = 1$ with generators $[1]$ and $[\alpha]$, respectively.*

Proof. ($k = 0$): Since for 0-forms the equality $d = \partial_+$ holds, we have $PH^0_{\partial_+} = H^0_d$ for any symplectic manifold.

($k = 1$): Define a linear map $T : PH^1_{\partial_+} \to \mathbb{R}$ by $T([B]) = \partial_- B$. We will show that T is well-define, injective and nonzero, hence it is an isomorphism. First note that $\partial_+ B = 0$ implies that $0 = d^2 B = L\partial_+\partial_- B$ or $0 = \partial_+\partial_- B = d\partial_- B$ and therefore $\partial_- B$ is a constant function. To prove well-definedness, write

$$\partial_-(B + \partial_+ B_0) = \partial_- B - \partial_+\partial_- B_0 = \partial_- B.$$

The equality $T([B]) = \partial_- B = 0$ implies that $dB = 0$. Then by the standard Poincaré lemma, there is a 0-form B_0 with the property $B = dB_0 = \partial_+ B_0$ or $[B] = 0$ and T is injective. Finally, we have $\partial_+ \alpha = 0$ and $T([\alpha]) = \partial_- \alpha = 1 \neq 0$.

($2 \leq k < n$): By the same argument as in the previous case, we should have $d\partial_- B = 0$ and therefore $\partial_- B = dA$ for some $k-2$-form A by the standard Poincaré lemma ($0 < k-1$). This implies that $d(B - LA) = 0$ and consequently $B - LA = dA'$ for some $k-1$-form A'. If the primitive component of A' is B', then we have $B = \partial_+ B'$. □

Proposition 4.12. *We have $PH^k_{\partial_-} = \{0\}$ for $0 \leq k < n$.*

Proof. ($k = 0$): Since all 0- and 1-forms are primitive and acting on primitive forms, operators ∂_- and d^Λ are equivalent, we have $PH^0_{\partial_-} = H^0_{d^\Lambda}$ for any symplectic manifold.

($0 < k < n$): Let B be a ∂_--closed primitive k-form or $dB = \partial_+ B$. Then, the primitive form dB is d-closed and using Proposition 4.7 ($0 < k+1 \leq n$), there exists a primitive $k+1$-form B' satisfying $dB = dd^\Lambda B'$. Applying the same proposition again, there exists a primitive k-form B'' satisfying $B - d^\Lambda B' = dd^\Lambda B''$ or

$$B = -H\partial_- B' - (H+1)\partial_+\partial_- B'' = \partial_-(-(n-k)B' + (n-k+1)\partial_+ B''). \quad \square$$

4.3 dd^Λ-lemma and comparison with H^k_d

Definition 4.13. *Using Corollary 2.23 part a), the induced maps $L^{n-k} : H^k_d \to H^{2n-k}_d$ on the level of de Rham cohomology are well-defined for $0 \leq k \leq n$. The symplectic manifold M satisfies the* ***strong*** *or* ***hard Lefschetz property*** *if these maps are surjective for all $0 \leq k \leq n$. Note that if M is compact, then being a surjection is the same as being an isomorphism for the above maps by Poincaré duality, i.e. $dim\, H^k_d = dim\, H^{2n-k}_d$.*

Proposition 4.14. *The strong Lefschetz property holds if and only if every de Rham cohomology class contains a form that is both d- and d^Λ-closed.*

Proof. This theorem is proved by Mathieu [11] and Yan [20] but the following proof is obtained from [5].

($\Leftarrow$): Given $[A] \in H^{2n-k}_d$ for some $0 \leq k \leq n$, we should find $[A'] \in H^k_d$ such that $L^{n-k}[A'] = [A]$. By our assumption, we can assume that $A \in \Omega^{2n-k}$ is d- and d^Λ-closed.

Since the map $L^{n-k} : \Omega^k \to \Omega^{2n-k}$ is an isomorphism, we can take $A' \in \Omega^k$ such that $L^{n-k}A' = A$. By Corollary 2.23 part c), the form A' is d-closed and hence a representative for a de Rham cohomology class which is mapped to $[A]$ by L^{n-k}.

($\Rightarrow$): Note that it is enough to prove the result only for cohomology classes in H_d^k for $0 \leq k \leq n$ by the following reasoning. Given $[A] \in H_d^{2n-k}$ for $0 \leq k \leq n$, by having the strong Lefschetz property, there is $[A'] \in H_d^k$ mapped to $[A]$ by L^{n-k}. If we know that we can take A' being both d- and d^Λ-closed, then $L^{n-k}A'$ is also both d- and d^Λ-closed representative of $[A]$ by Corollary 2.23 part c). The proof for H_d^k, where $0 \leq k \leq n$, is by induction on k.

Basis step ($k = 0, 1$): All 0-forms are d^Λ-closed. Also, all d-closed 1-forms are d^Λ-closed because of the formula $d^\Lambda = d\Lambda - \Lambda d$. Therefore, any member of any cohomology class in H_d^0 and H_d^1 is both d- and d^Λ-closed.

Inductive step ($(k-2) \Rightarrow k$, $2 \leq k \leq n$): Given $[A] \in H_d^k$, we have $[L^{n-k+1}A] \in H_d^{2n-k+2}$. Having the strong Lefschetz property, there there is $[A'] \in H_d^{k-2}$ mapped to $[L^{n-k+1}A]$ by L^{n-k+2}. In other words, there is $C \in \Omega^{2n-k+1}$ such that $L^{n-k+1}A = L^{n-k+2}A' + dC$. Since the map $L^{n-k+1} : \Omega^{k-1} \to \Omega^{2n-k+1}$ is an isomorphism, there exists $C' \in \Omega^{k-1}$ mapped to C by L^{n-k+1}. Define the k-form $B := A - LA' - dC'$ with the properties $L^{n-k+1}B = dC - dC = 0$ and $dB = 0$. Consequently, the form B is primitive and being d-closed it is also d^Λ-closed. Using induction hypothesis for $[A'] \in H_d^{k-2}$, we have $A' = A'' + dD$, where A'' is both d- and d^Λ-closed. This implies that $dLA'' = d^\Lambda LA'' = 0$ using $[d^\Lambda, L] = d$. Finally combining all these together, we can write $A = (LA'' + B) + d(LD + C')$, where $(LA'' + B)$ is both d- and d^Λ-closed. This completes the proof. $\square$

Note that the previous theorem only talks about the existence of such representatives and not uniqueness. In fact as you can see in the above proof (see the basis step), such representatives are not unique in general.

Proposition 4.15. *The induced map $L^{n-k} : H_{d+d^\Lambda}^k \to H_d^{2n-k}$ is surjective for $0 \leq k \leq n$ if and only if the strong Lefschetz property holds.*

Proof. First note that this map is well-defined by Corollary 2.23. The implication $\Rightarrow$ is obvious. For the other implication, take $[A] \in H_d^{2n-k}$ for $0 \leq k \leq n$. By Proposition 4.14, we can assume that A is both d- and d^Λ-closed. Since the map $L^{n-k} : \Omega^k \to \Omega^{2n-k}$ is an isomorphism, there exists a k-form A' mapped to A by L^{n-k}. By Corollary 2.23, the form A' is a representative for a $H^k_{d+d^\Lambda}$ cohomology class. $\square$

Definition 4.16. *The symplectic manifold M satisfies the dd^Λ**-lemma** if we have*

$$im\, d \cap ker\, d^\Lambda = im\, d^\Lambda \cap ker\, d = im\, dd^\Lambda.$$

In other words, for a d- and d^Λ-closed form, being d-exact or d^Λ-exact or dd^Λ-exact are equivalent to each other.

Proposition 4.17. *The induced map $L^{n-k} : H^k_{d+d^\Lambda} \to H^{2n-k}_d$ is injective for $0 \leq k \leq n$ if and only if the dd^Λ-lemma holds.*

Proof. ($\Rightarrow$): Let A be a d- and d^Λ-closed form. First note that dd^Λ-exactness implies both d- and d^Λ-exactness. By Corollary 2.20, it is enough to prove that d-exactness implies dd^Λ-exactness. Assume that $A \in \Omega^k$ is d-exact and $0 \leq k \leq n$. By Corollary 2.23, the form $L^{n-k}A$ is also d-exact. Injectivity of $L^{n-k} : H^k_{d+d^\Lambda} \to H^{2n-k}_d$ implies that A should be dd^Λ-exact. Now, assume that $A \in \Omega^{2n-k}$ is d-exact and $0 \leq k \leq n$. Since the map $L^{n-k} : \Omega^k \to \Omega^{2n-k}$ is an isomorphism, there exists a k-form A' mapped to A by L^{n-k}. By Corollary 2.23, the form A' should be both d- and d^Λ-closed. Injectivity of $L^{n-k} : H^k_{d+d^\Lambda} \to H^{2n-k}_d$ implies that A' should be dd^Λ-exact. Finally Corollary 2.23, gives dd^Λ-exactness of A.

($\Leftarrow$): Assume that A be a d- and d^Λ-closed k-form ($0 \leq k \leq n$) such that $L^{n-k}A$ is d-exact. Corollary 2.23 implies that $L^{n-k}A$ is both d- and d^Λ-closed and dd^Λ-lemma gives dd^Λ-exactness of $L^{n-k}A$. Finally by Corollary 2.23, A is dd^Λ-exact. $\square$

Proposition 4.18. *For a compact symplectic manifold, the dd^Λ-lemma is equivalent to the strong Lefschetz property.*

Proof. This theorem is proved by Merkulov [13] and Guillemin [8]. The proof is more involved than the previous theorem, so we will not give the proof here. The interested reader can find the proof in the above references or in [5]. $\square$

Corollary 4.19. *For a compact symplectic manifold satisfying dd^Λ-lemma, the induced map $L^{n-k} : H^k_{d+d^\Lambda} \to H^{2n-k}_d$ is an isomorphism for $0 \leq k \leq n$.*

Note that compact Kähler manifolds satisfy both the strong Lefschetz property and dd^{Λ}-lemma, but that is not true for all Kähler manifolds. For example, consider the standard Euclidean Kähler manifold $\mathbb{R}^{2n}$. The map $L^{n-k} : H^k_{d+d^{\Lambda}} \to H^{2n-k}_d$ is surjective but not injective by Corollary 4.8. Hence, the Euclidean space satisfies the strong Lefschetz property but not dd^{Λ}-lemma.

Proposition 4.20. *We have $PH^0_{\partial_+} = H^0_d$ and $PH^0_{\partial_-} = H^0_{d^{\Lambda}}$. Moreover, if M is compact, then we have also $PH^1_{\partial_+} = H^1_d$ and $PH^1_{\partial_-} = H^1_{d^{\Lambda}}$.*

Proof. Equalities for 0 are trivial and justified in proofs of Propositions 4.11 and 4.12. Suppose that M is compact and B_1 is a 1-form. Since M is compact, it has finitely many connected components that are compact. It is enough to prove equalities for each component and hence without loss of generality assume that M is connected.

($PH^1_{\partial_+} = H^1_d$): The only non-obvious fact that should be checked is that if B_1 is ∂_+-closed, then it is also d-closed. By the proof of Proposition 4.11, $\partial_- B_1$ is a constant function c. If $c \neq 0$, then we have $d(\frac{B_1}{c}) = \omega$ but $[\omega] \in H^2_d$ is nonzero by compactness. Consequently, we have $c = 0$ and $dB_1 = 0$.

($PH^1_{\partial_-} = H^1_{d^{\Lambda}}$): The only non-obvious fact that should be checked is that if B_1 is d^{Λ}-exact, then it is also ∂_--exact. Suppose that there is a 2-form $A_2 = B_2 + LB_0$ such that

$$B_2 = d^{\Lambda} A_2 = d^{\Lambda} B_2 + L d^{\Lambda} B_0 + dB_0 = -\partial_-((n-1)B_2) + dB_0$$

It remains to prove that dB_0 is ∂_--exact. Without loss of generality, we can assume that $\int_M B_0 = 0$ by adding a constant function and without changing dB_0. Because for any d-closed 0-form i.e. a constant function c we have

$$\int_M c \wedge *_s B_0 = c \int_M B_0 = 0,$$

Proposition 4.4 implies that B_0 is d^{Λ}-exact i.e. $B_0 = d^{\Lambda} B_1'$. Then, the 1-form dB_0 should be ∂_--exact:

$$dB_0 = dd^{\Lambda} B_1' = -(H+1)\partial_+\partial_- B_1' = \partial_-(n\partial_+ B_1'). \quad \square$$

Lemma 4.21. *Let M be a symplectic manifold satisfying the dd^{Λ}-lemma and $B_k \in \mathcal{P}^k$ be d-closed and ∂_+-exact. Then, B_k is $\partial_+\partial_-$-exactness.*

Proof. First, note that B_k is both d- and d^Λ-closed. Assume that we have $B_k = \partial_+ B_{k-1}$. We will show that B_k is d-exact. Then using the dd^Λ-lemma, it should be dd^Λ-exact and hence $\partial_+\partial_-$-exact. We can write $dB_{k-1} = B_k + LB_{k-2}$, where we have

$$B_{k-2} = \partial_- B_{k-1} = -H^{-1} d^\Lambda B_{k-1} = d^\Lambda\left(\frac{-1}{n-k+2} B_{k-1}\right),$$
$$LdB_{k-2} = dB_k + dLB_{k-2} = d^2 B_{k-1} = 0 \Rightarrow dB_{k-2} = 0.$$

Applying the dd^Λ-lemma to B_{k-2}, there is a form A_{k-3} satisfying $B_{k-2} = dA_{k-3}$ and therefore $B_k = d(B_{k-1} - LA_{k-3})$. $\square$

Definition 4.22. *Define*

$$H_d^{r,s} := \frac{ker\, d \cap \mathcal{L}^{r,s}}{im\, d \cap \mathcal{L}^{r,s}}, \qquad PH_d^k := H_d^{0,k}.$$

Note that we have $dim\, H_d^{r,s} \leq dim\, H_d^{2r+s}$.

Proposition 4.23. *For a compact symplectic manifold satisfying dd^Λ-lemma, we have*

$$PH^k_{d+d^\Lambda} \cong PH^k_{dd^\Lambda} \cong PH^k_d \qquad 0 \leq k \leq n,$$
$$PH^k_{\partial_+} \cong PH^k_{\partial_-} \cong PH^k_d \qquad 0 \leq k < n.$$

Proof. We already know that $PH^k_{d+d^\Lambda} \cong PH^k_{dd^\Lambda}$ and $PH^k_{\partial_+} \cong PH^k_{\partial_-}$. By Proposition 4.18 and Corollary 4.19, the maps $L^{n-k} : PH^k_d \to H_d^{n-k,k}$ and $L^{n-k} : PH^k_{d+d^\Lambda} \to H_d^{n-k,k}$ are isomorphisms for $0 \leq k \leq n$. Now suppose that $0 \leq k < n$ and we show that $PH^k_{\partial_-} \cong H_d^{n-k,k}$ to complete the proof. Acting on $\mathcal{L}^{n-k,k}$, we have $\partial_+ = 0$ and $d = L\partial_-$. Therefore, the following commutative diagram

$$\begin{tikzcd}
0 \arrow[d, "Id"] & \mathcal{P}^0 \arrow[l, "\partial_-"'] \arrow[d, "L^n"] & \dots \arrow[l, "\partial_-"'] & \mathcal{P}^{n-1} \arrow[l, "\partial_-"'] \arrow[d, "L"] & \mathcal{P}^n \arrow[l, "\partial_-"'] \arrow[d, "L^0=Id"] \\
0 & \mathcal{L}^{n,0} \arrow[l, "d"'] & \dots \arrow[l, "d"'] & \mathcal{L}^{1,n-1} \arrow[l, "d"'] & \mathcal{L}^{0,n} \arrow[l, "d"']
\end{tikzcd}$$

gives $PH^k_{\partial_-} \cong \frac{ker\, d \cap \mathcal{L}^{n-k,k}}{d\mathcal{L}^{n-k-1,k+1}}$ for any symplectic manifold. It remains to prove that under the assumption of dd^Λ-lemma we have $im\, d \cap \mathcal{L}^{n-k,k} \subseteq d\mathcal{L}^{n-k-1,k+1}$. Let $dA \in \mathcal{L}^{n-k,k}$ and $A = \sum_{i=0}^{j} L^{n-k-1+i} B_{k+1-2i}$ be the Lefschetz decomposition of A. We use induction on j. For $j = 0$, we have $A \in \mathcal{L}^{n-k-1,k+1}$ and we are done. For $j > 0$, we will find another form A'

such that $dA' = dA$ and $A' = \sum_{i=0}^{j-1} L^{n-k-1+i} B'_{k+1-2i}$ is the Lefschetz decomposition of A'. Because we have $dA \in \mathcal{L}^{n-k,k}$ and $j > 0$, we should have $\partial_- B_{k+1-2j} = 0$. Applying Lemma 4.21 to $\partial_+ B_{k+1-2j}$, we have $\partial_+ B_{k+1-2j} = \partial_+ \partial_- B'$ and therefore

$$d(L^{n-k-1+j} B_{k+1-2j}) = L^{n-k-1+j} \partial_+ \partial_- B' = -L^{n-k-1+j-1} L \partial_- \partial_+ B' = d(-L^{n-k-1+j-1} \partial_+ B').$$

It is enough to take

$$A' = A - L^{n-k-1+j} B_{k+1-2j} - L^{n-k-1+j-1} \partial_+ B' = \sum_{i=0}^{j-1} L^{n-k-1+i} B_{k+1-2i} - L^{n-k-1+j-1} \partial_+ B'. \quad \square$$

Corollary 4.24. *For the complex projective space $\mathbb{CP}^n$, we have*

$$PH^0_{d+d^\Lambda} \cong PH^0_{dd^\Lambda} \cong PH^0_{\partial_+} \cong PH^0_{\partial_-} \cong PH^0_d \cong \mathbb{R},$$

and the higher degree primitive cohomologies are trivial.

Proof. Being a compact Kähler manifold, the previous proposition is applicable. We know that $H^{2k}_d \cong \mathbb{R}$ for $0 \leq k \leq n$ with the generator $[\omega^k]$ and other de Rham cohomology groups are trivial. Using $dim\, PH^k_d \leq dim\, H^k_d$, the only possible nontrivial PH^k_d are even degrees. All 0-forms are primitive so we have $PH^0_d = H^0_d \cong \mathbb{R}$. To prove $PH^{2k}_d = \{0\}$ for $k > 0$, we must show that if the form $B = c\,\omega^k + dA$ is primitive, then it is d-exact. Let $A = \sum_r L^r B_{2k-1-2r}$ be the Lefschetz decomposition of A. We have

$$0 = \Lambda B = (n+k-1)kL^{k-1}c + \sum_r (n-2k+3r-1)rL^{r-1}(\partial_+ B_{2k-1-2r} + \partial_- B_{2k+1-2r}),$$

and therefore $\partial_- B_1 = -c$ and $\partial_- B_{2k+1-2r} = -\partial_+ B_{2k-1-2r}$ for $0 < r < k$. Replacing these formulas in the equation $B = c\,\omega^k + dA$, we see that $B = \partial_+ B_{2k-1}$. Using Lemma 4.21, B is $\partial_+\partial_-$-exact and consequently d-exact. $\square$

4.4 Cotangent bundles

All over this subsection, assume that N is a smooth manifold with dimension n and the $2n$-manifold $M := T^*(N)$ is the cotangent bundle of N considered as a symplectic manifold with the canonical symplectic form described below. For more details about this symplectic structure see [12]. The reference for this subsection is [15].

Let $(x^1, ..., x^n)$ be a local coordinate system for N and $(x^1, ..., x^n, y^1, ..., y^n)$ be the associated local coordinates for M. There is a canonical 1-form on M defined by $\alpha := \sum_{i=1}^n y^i dx^i$. The canonical symplectic structure on M is defined by $\omega := -d\alpha = \sum_{i=1}^n dx^i \wedge dy^i$. Note that we have $\partial_+\alpha = 0$ and $\partial_-\alpha = -1$. Let $\pi : M \to N$ be the projection map and β_k be a k-form on N. Then, the forms $\pi^*\beta_k$ and $\alpha \wedge \pi^*\beta_k$ are both obviously primitive on M since they don't involve dy^is. We simply write β_k instead of $\pi^*\beta_k$ and stick to our previous notation of writing B_k and A_k for a general primitive k-form and differential k-form on M.

Lemma 4.25. *For any k-form β on N, we have*

$$d^\Lambda(\alpha \wedge \beta) = H\beta, \qquad \partial_+(\alpha \wedge \beta) = -\alpha \wedge d\beta \qquad 0 \leq k \leq n.$$

$$\partial_-(\alpha \wedge \beta) = -\beta \qquad 0 \leq k < n.$$

Proof.

$$\partial_+(\alpha \wedge \beta) + L\partial_-(\alpha \wedge \beta) = d(\alpha \wedge \beta) = (d\alpha) \wedge \beta - \alpha \wedge d\beta = -\alpha \wedge d\beta - L\beta,$$

$$d^\Lambda(\alpha \wedge \beta) = -\Lambda d(\alpha \wedge \beta) = \Lambda(\alpha \wedge d\beta) + \Lambda L\beta = \Lambda L\beta = H\beta. \qquad \square$$

Our goal is to compute different symplectic cohomologies of M. Note that M is not compact and some of previous results like dualities don't hold for M. Formulas (23) and (28) give cohomologies $H^k_{d+d^\Lambda}(M)$ and $H^k_{dd^\Lambda}(M)$.

Proposition 4.26. *We have the following isomorphisms:*

a) $H^k_d(M) \cong H^k_d(N)$ and $H^k_{d^\Lambda}(M) \cong H^{2n-k}_d(N)$ for $0 \leq k \leq 2n$.

b) $PH^k_{d+d^\Lambda}(M) \cong H^k_d(N)$ for $0 \leq k \leq n$.

c) $PH^k_{dd^\Lambda}(M) \cong H^{k-1}_d(N)$ for $0 \leq k < n$ and $PH^n_{dd^\Lambda}(M) \cong H^{n-1}_d(N) \oplus H^n_d(N)$.

d) $PH^k_{\partial_+}(M) \cong H^{k-1}_d(N) \oplus H^k_d(N)$ for $0 \leq k < n$.

e) $PH^k_{\partial_-}(M) = \{0\}$ for $0 \leq k < n$.

Proof. (Part a): Since M is a deformation retract of N and the de Rham cohomology is a topological invariant, we have first isomorphism ($\pi^* : H^k_d(N) \xrightarrow{\cong} H^k_d(M)$ for $0 \leq k \leq 2n$). Proposition 4.1 gives the other isomorphism ($*_s\pi^* : H^{2n-k}_d(N) \xrightarrow{\cong} H^k_{d^\Lambda}(M)$ for $0 \leq k \leq 2n$).

(Part e): We will prove it by strong induction on k. Proposition 4.20 and part a) give the result for $k = 0$. Assume that we have $0 < k < n$ and $PH^{k'}_{\partial_-}(M) = \{0\}$ for all $k' < k$. Now, we can repeat the proof of $PH^k_{\partial_-} \cong H^{n-k,k}_d$ in Proposition 4.23 but instead of using dd^Λ-lemma, we use induction hypothesis to find a primitive form B' satisfying $B_{k+1-2j} = \partial_- B'$. Since $dim\, H^{n-k,k}_d \leq dim\, H^{2n-k}_d = 0$, the induction is complete.

(Part b): We will prove that the map $\pi^* : H^k_d(N) \to PH^k_{d+d^\Lambda}(M)$ is a well-defined isomorphism for $0 \leq k \leq n$. (Well-definedness): Given a d-closed k-form β_k on N, it is both primitive and d-closed on M. If we have $\beta_k = d\beta_{k-1}$, then $\partial_-\beta_{k-1} = 0$ and part e) $(k-1 < n)$ implies that $\beta_{k-1} = \partial_- B_k$ and therefore

$$\beta_k = d\beta_{k-1} = \partial_+\partial_- B_k.$$

(Injectivity): Let β_k be d-closed on N and dd^Λ-exact on M, then it is trivially d-exact on M. Then by part a), the form β_k should be d-exact also on N.

(Surjectivity): Let B be a d-closed primitive k-form on M. We have $B - \beta_k = dA$ for some d-closed k-form β_k on N and $k-1$-form on M by part a). Let $A = \sum_{r=0}^{j} L^r B_{k-1-2r}$ be the Lefschetz decomposition of A. Because we have $dA \in \mathcal{P}^k$, we should have $\partial_- B_{k-1-2j} = 0$. By part e)$(k-1-2j < n)$, there exists $B' \in \mathcal{P}^{k-2j}$ such that $B_{k-1-2j} = \partial_- B'$. We use induction on j to prove that dA is $\partial_+\partial_-$-exact which completes the proof of surjectivity. For $j = 0$, we have $dA = \partial_+\partial_- B'$. For $j > 0$, take

$$A' = A - L^j B_{k-1-2j} - L^{j-1}\partial_+ B' = \sum_{i=0}^{j-1} L^r B_{k-1-2r} - L^{j-1}\partial_+ B',$$

similar to the proof of part e). Induction hypothesis implies that $dA' = dA$ is $\partial_+\partial_-$-exact.

(Part c) for $0 \leq k < n$): Proposition 4.20 and part a) give the result for $k = 0$. We will prove that the map $T_1 : H^{k-1}_d(N) \to PH^k_{dd^\Lambda}(M)$ given by $[\beta] \mapsto [\alpha \wedge \beta]$ is a well-defined isomorphism for $0 < k < n$. (Well-definedness of T_1 for $0 < k \leq n$): The form $\alpha \wedge \beta$ is primitive and satisfies

$$dd^\Lambda(\alpha \wedge \beta) = dH\beta = (n-k+1)d\beta = 0.$$

If we have $\beta = d\beta'$, then the form $\alpha \wedge \beta = \partial_+(-\alpha \wedge \beta')$ is $d + d^\Lambda$-exact.

(Injectivitiy of T_1 for $0 < k \leq n$): Assume that $\alpha \wedge \beta = \partial_+ B + \partial_- B'$. Then, we have

$$\beta = -\partial_-(\alpha \wedge \beta) = -\partial_-\partial_+ B = \partial_+\partial_- B,$$

and part b) implies that β is d-exact on N.

(Surjectivity of T_1 for $0 < k < n$): Let B be a $\partial_+\partial_-$-closed primitive k-form on M. Then, the primitive form $\partial_- B$ is d-closed. Using part b), there exist a d-closed $k-1$-form β on N and a primitive $k-1$-form B' on M satisfying

$$\partial_+\partial_- B' = \partial_- B - \beta = \partial_- B + \partial_-(\alpha \wedge \beta).$$

Then, the primitive form $B + \alpha \wedge \beta + \partial_+ B'$ is ∂_--closed. Using part e) and $k < n$, it is ∂_--exact and $T_1([-\beta]) = [B]$.

(Part c) for $k = n$): We will prove that the map $T_2 : H_d^{n-1}(N) \oplus H_d^n(N) \to PH^n_{dd^\Lambda}(M)$ given by $([\beta], [\beta']) \mapsto [\alpha \wedge \beta + \beta']$ is a well-defined isomorphism. Combining proofs of Well-definedness for part b) and T_1 proves that T_2 is well-defined.

(Injectivitiy of T_2): Assume that $\alpha \wedge \beta + \beta' = \partial_+ B$. We must show that both β and β' are d-exact on N. Using

$$\partial_+\partial_- B = -\partial_-\partial_+ B = -\partial_-(\alpha \wedge \beta) = \beta,$$

part a) implies that $\beta = d\beta''$ and therefore $\partial_+(\alpha \wedge \beta'') = -\alpha \wedge \beta$. Taking $B' := B + \alpha \wedge \beta''$, we have $\beta' = \partial_+ B'$ and

$$0 = d\beta' = -L\partial_+\partial_- B' \Longrightarrow \partial_+\partial_- B' = 0.$$

Part c) for $k = n-1$ implies that $B' = \alpha \wedge \beta''' + \partial_+ B'' + \partial_- B'''$ and therefore

$$\beta' = \partial_+ B' = -\alpha \wedge d\beta''' + \partial_+^2 B'' + \partial_+\partial_- B''' = \partial_+\partial_- B'''.$$

Finally, the form β' is d-exact on N by part a).

(Surjectivity of T_2): Let B be a $\partial_+\partial_-$-closed primitive n-form on M. The argument in the proof of surjectivity of T_1 is valid until we obtain the ∂_--closed form $B + \alpha \wedge \beta + \partial_+ B'$. This form is ∂_+- and d-closed since primitive n-forms are always ∂_+-closed. Using part b), we have

$$B + \alpha \wedge \beta + \partial_+ B' = \beta' + \partial_+\partial_- B'' \Longrightarrow T_2([-\beta], [\beta']) = [B].$$

(Part d): Proposition 4.20 and part a) give the result for $k = 0$. We will prove that the map $T_3 : H_d^{k-1}(N) \oplus H_d^k(N) \to PH^k_{\partial_+}(M)$ given by $([\beta], [\beta']) \mapsto [\alpha \wedge \beta + \beta']$ is a well-defined isomorphism for $0 < k < n$. Combining proofs of Well-definedness for part b) and T_1 proves that T_3 is well-defined. Proof of injectivity is the same as injectivity of T_2.

(Surjectivity of T_3): Let B be a ∂_+-closed primitive k-form on M for $0 < k < n$. Using $k < n$, we have $\partial_+\partial_- B = -\partial_-\partial_+ B = 0$. The argument in the proof of surjectivity of T_1 is valid until we obtain the ∂_--closed form $B + \alpha \wedge \beta + \partial_+ B'$. This form is ∂_+- and d-closed because $\partial_+ B = 0$. The remain of proof is the same as surjectivity of T_2. $\square$

Note that if U is a star-shaped open subset of $\mathbb{R}^n$, then $M = U \times \mathbb{R}^n$ as a symplectic manifold can be considered both as the cotangent bundle of U and a star-shaped open subset of $\mathbb{R}^{2n}$. Then, we can compute its symplectic cohomologies both by Poincaré lemmas and the previous proposition and the results are the same.

4.5 Mayer-Vietoris sequence

In this subsection, we justify that there exists a Mayer-Vietoris sequence corresponding to the differential complex (32) and by an example we show its effectiveness for computation of symplectic cohomologies. This is an original work and it is in parallel with [3].

Proposition 4.27. *Let $\{U, V\}$ be an open cover of the symplectic manifold M. Then, the following Mayer-Vietoris sequence*

$$\begin{array}{c} 0 \longrightarrow \mathcal{P}^k(M) \xrightarrow{r} \mathcal{P}^k(U) \oplus \mathcal{P}^k(V) \xrightarrow{\delta} \mathcal{P}^k(U \cap V) \longrightarrow 0 \\ B \mapsto (B|_U, B|_V) \quad (B, B') \mapsto B'|_{U\cap V} - B|_{U\cap V} \end{array}$$

is exact for $0 \leq k \leq n$.

Proof. First note that an open subset U of a symplectic manifold (M, ω) is also symplectic manifold by restricting ω to U. Being primitive is a pointwise condition so it is preserved under restrictions and the above maps are well-defined.

(Exactness at $\mathcal{P}^k(M)$): If B is a primitive form on M satisfying $r(B) = 0$, then it is 0 on both U and V and therefore $B = 0$ by $M = U \cup V$.

(Exactness at $\mathcal{P}^k(U) \oplus \mathcal{P}^k(V)$): For a primitive form B on M, we have

$$\delta r(B) = B|_{U\cap V} - B|_{U\cap V} = 0.$$

Let B and B' be primitive forms on U and V, respectively such that $\delta(B,B')=0$. Then, the differential form

$$B'' = \begin{cases} B & \text{on } U \\ B' & \text{on } V \end{cases}$$

is well-defined. It is primitive by the same reason mentioned above and $r(B'')=(B,B')$.

(Exactness at $\mathcal{P}^k(U\cap V)$): Let B be a primitive form on $U\cap V$ and $\{\rho_U,\rho_V\}$ be a partition of unity subordinate to $\{U,V\}$. Then, the following differential forms

$$B' = \begin{cases} -\rho_V B & \text{on } U\cap V \\ 0 & \text{on } U - supp(\rho_V) \end{cases} \qquad B'' = \begin{cases} \rho_U B & \text{on } U\cap V \\ 0 & \text{on } V - supp(\rho_U) \end{cases}$$

are well-defined on U and V, respectively. They are primitive because 0 is primitive and multiplication by a function doesn't change primitivity. Finally, we have

$$\delta(B',B'') = \rho_U B - (-\rho_V B) = B. \quad \square$$

Using this short exact sequence, the differential complex (32) and the snake lemma, we find a long exact sequence of primitive symplectic cohomologies. As a nontrivial example, we use this result to compute primitive symplectic cohomologies of the cotangent bundle of the 2-torus T^2 and compare the results with Proposition 4.26. Let U' and V' be some ϵ-neighborhoods of the upper and lower half torus. Note that U' are V' are diffeomorphic to the cylinder $S^1\times\mathbb{R}$ and $U'\cap V'$ is diffeomorphic to the disjoint union of two cylinders. Taking $U=T^*(U')$ and $V=T^*(V')$, We have $\{U,V\}$ is an open cover of $M=T^*(T^2)$. Using a separate Mayer-Vietoris sequence or Proposition 4.26, we have

$$PH^0_{\partial_+} \cong PH^2_{dd^\Lambda} \cong \mathbb{R}, \qquad PH^1_{\partial_+} \cong \mathbb{R}^2, \qquad PH^2_{d+d^\Lambda} \cong PH^1_{\partial_-} \cong PH^0_{\partial_-} \cong 0,$$

for the cotangent bundle of the cylinder. Then, we have the long exact sequence in the next page, where the difference maps δ_1 and δ_3 are given by $(c_1,c_2)\mapsto(c_2-c_1,c_2-c_1)$ and have rank 1 and the difference maps δ_2 is given by $(c_1,c_2,c_3,c_4)\mapsto(c_2-c_1,c_2-c_1,c_4-c_3,c_4-c_3)$ and have rank 2. We can compute the symplectic cohomologies of M as follows

$$PH^0_{\partial_+} \cong im\, r_1 \cong ker\,\delta_1 \cong \mathbb{R}, \qquad PH^2_{d+d^\Lambda} \cong im\, f_3 \cong \mathbb{R}, \qquad PH^1_{\partial_-} \cong PH^0_{\partial_-} \cong 0,$$
$$PH^1_{\partial_+} \cong im\, f_1 \oplus ker\,\delta_2 \cong \mathbb{R}\oplus\mathbb{R}^2 \cong \mathbb{R}^3, \qquad PH^2_{dd^\Lambda} \cong im\, f_2 \oplus ker\,\delta_3 \cong \mathbb{R}^2\oplus\mathbb{R} \cong \mathbb{R}^3,$$

which are the same as the results from Proposition 4.26.

$$
\begin{tikzcd}
 & & M = T^*(T^2) & U \sqcup V & U \cap V & \\
PH^0_{\partial_+} & 0 \arrow[r] & ? \arrow[r, "r_1"] & \mathbb{R} \oplus \mathbb{R} \arrow[r, "\delta_1"] & \mathbb{R} \oplus \mathbb{R} \arrow[dll, "f_1"] & \\
PH^1_{\partial_+} & & ? \arrow[r, "r_1"] & \mathbb{R}^2 \oplus \mathbb{R}^2 \arrow[r, "\delta_2"] & \mathbb{R}^2 \oplus \mathbb{R}^2 \arrow[dll, "f_2"] & \\
PH^2_{dd^\Lambda} & & ? \arrow[r, "r_3"] & \mathbb{R} \oplus \mathbb{R} \arrow[r, "\delta_3"] & \mathbb{R} \oplus \mathbb{R} \arrow[dll, "f_3"] & \\
PH^2_{d+d^\Lambda} & & ? \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] & \\
PH^1_{\partial_-} & & ? \arrow[r] & 0 \arrow[r] & 0 \arrow[dll] & \\
PH^0_{\partial_-} & & ? \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}
$$

The final remark is that if a symplectic manifold M has finite open cover $\{U_1, ..., U_m\}$ such that the symplectic cohomologies of any intersection $U_{i_1} \cap ... \cap U_{i_j}$ are finite-dimensional, then we can conclude that $PH^k_{\partial_\pm}$ for $0 \leq k < n$, $PH^n_{d+d^\Lambda}$ and $PH^n_{dd^\Lambda}$ of the manifold M are also finite-dimensional using the Mayer-Vietoris sequence and induction on m. For example for $PH^n_{d+d^\Lambda}$, we have

$$... \to PH^{n-1}_{\partial_+}(U \cap V) \xrightarrow{f} PH^n_{d+d^\Lambda}(U \cup V) \xrightarrow{r} PH^n_{d+d^\Lambda}(U) \oplus PH^n_{d+d^\Lambda}(V) \to ... \, ,$$

and therefore

$$PH^n_{d+d^\Lambda}(U \cup V) \cong ker\, r \oplus im\, r \cong im\, f \oplus im\, r.$$

This implies that if $PH^{n-1}_{\partial_+}(U \cap V)$, $PH^n_{d+d^\Lambda}(U)$ and $PH^n_{d+d^\Lambda}(V)$ are finite-dimensional, then $PH^n_{d+d^\Lambda}(U \cup V)$ is also finite-dimensional.

Conclusion

As a conclusion, I will give the general picture behind all the results given in this thesis. First, let us summarize the purely algebraic aspects in this theory. Having a compatible triple (ω, J, g) on a manifold M (or a vector space), we have the two different decompositions (6) and (13) of forms and the following isomorphisms. For the diamond, the conjugation and the operators $*$ and L^{n-p-q} give isomorphisms around the middle vertical line, centre point and the middle horizontal line, respectively. Note that the map $\mathcal{J} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q}$ is just multiplication by a constant, so it is a trivial isomorphism which is not interesting. As a result, the operators $*$ and $*_s = *\mathcal{J}^{-1}$ are equivalent on $\mathcal{A}^{p,q}$. So, the only different nontrivial isomorphisms on the diamond are the above three types of isomorphisms. On the other hand for the pyramid, the operators $*$, $*_s$ and L^{n-2r-s} give different isomorphisms around the middle vertical line. We also have nontrivial isomorphisms $\mathcal{J} : \mathcal{L}^{r,s} \to \mathcal{L}^{r,s}$ and isomorphisms of spaces in each horizontal line given by L^r (Lemma 2.5).

Moving on to the case of complex manifolds, for a general almost complex structure J, there is no obvious relation between d and the diamond decomposition. Under the geometric assumption of integrability, the operator d has only two components acting on $\mathcal{A}^{p,q}$. This gives first order differential operators $(\partial, \bar{\partial})$ and complex cohomologies of Dolbeault, Bott-Chern and Aeppli. On a complex manifold, conjugation is the only natural isomorphism among the above isomorphisms for the diamond, in the sense that it doesn't depend on a choice of a compatible triple, so it gives the dualities $H^{p,q}_{\bar{\partial}} = \bar{H}^{q,p}_{\partial}$, $H^{p,q}_{\partial+\bar{\partial}} = \bar{H}^{q,p}_{\partial+\bar{\partial}}$ and $H^{p,q}_{\partial\bar{\partial}} = \bar{H}^{q,p}_{\partial\bar{\partial}}$ even for non-compact complex manifolds. To obtain the finite dimensionality of complex cohomologies and transfer more dualities to the level of cohomologies, we need to add the assumption of compactness to be able to use the general Hodge theory of elliptic operators on a compact manifold (see Corollary 1.4). The new duality for compact complex manifold comes from the isomorphism $*$ on the diamond and we still don't have any duality coming from L^{n-p-q} because there is no obvious relation between the the algebraic operator L and differential operators in general. Adding one more assumption that ω is d-closed or $[d, L] = 0$, i.e. the manifold is Kähler, implies this missing duality and many more dualities (see Proposition 1.5).

Next, for symplectic manifolds, we see that given a general non-degenerate 2-form ω, there is no obvious relation between d and the pyramid decomposition. Under the geometric assumption of d-closedness, the operator d has only two components acting on $\mathcal{L}^{r,s}$. This gives first order differential operators (∂_+, ∂_-) and new symplectic cohomologies. On a symplectic manifold, the isomorphisms among the above isomorphisms which are natural for the pyramid, in the sense that they don't depend on a choice of a compatible triple, are $*_s$ and L^r. Therefore, we have the isomorphism $H^{r,s}_{\partial_\pm} \cong PH^s_{\partial_\pm}$, Proposition 4.1 and formulas (23) and (28) even in the case of non-compact symplectic manifolds. As a result, we can compute all symplectic cohomologies only knowing the de Rham and primitive cohomologies. To conclude the finite dimensionality of symplectic cohomologies and transfer more dualities to the level of cohomologies, we need to add the assumption of compactness to use the elliptic theory on a compact manifold (see Propositions 4.3 and 4.4). Assuming furthermore that the dd^Λ-lemma holds, we find even more isomorphisms like $L^{n-k} : H^k_d \to H^{2n-k}_d$ and $L^{n-k} : H^k_{d+d^\Lambda} \to H^{2n-k}_d$ and all primitive symplectic cohomologies become isomorphic to the primitive de Rham cohomology (see Propositions 4.23).

We finish this thesis with an outline of open problems. Since the symplectic cohomologies have only been introduced recently, there are many questions to be answered about them. Among these open problems, maybe the most important and natural questions are the following ones:

- What is the relationship between the Lie algebra cohomology of a Lie group and the symplectic cohomologies of the orbits of the co-adjoint action? Answering this problem, one could compute the symplectic cohomologies for a homogeneous symplectic manifold.

- What is the behavior of the symplectic cohomologies under symplectic reductions?

- Is there a Mayer-Vietoris construction for mixed symplectic cohomologies H_{d+d^Λ} and H_{dd^Λ} that are not coming from a differential complex?

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