The Sequential Spectral Method for Integro-Differential Equations

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Abstract

Using the Galerkin method to solve nonlinear integro-differential equations of elliptic or parabolic type one needs to solve the resulting nonlinear systems of algebraic or ordinary differential equations. To solve these equations with Newtons method or a variant thereof can be very difficult and one needs a good initial guess for the methods to converge. Also there might be multiple solutions and it is virtually impossible to track all of them. In addition it is hard to study the parameter dependence of solutions. We developed a remedy for these problems by developing the sequential spectral method which avoids solving a nonlinear system altogether. In the sequential spectral method a scalar nonlinear algebraic or ordinary differential equation is solved at the initial stage and then the solution of the original problem is obtained through iterations, we never have to solve a nonlinear system at any stage of the method. The sequential spectral method converges linearly for steady state problems and superlinearly in the case of evolution. With the sequential spectral method we can obtain solutions to any desired accuracy with much less effort than with the Galerkin method. We can also increase the spectral degree of accuracy while the method is running. In addition one can easily detect the existence of multiple solutions by observing only a single equation and one can track those solutions. The behavior of the solution and the dependence on parameters can be estimated and one can also determine the blow up time for the corresponding parameter values by studying only a single equation. We further show that the sequential spectral method can be applied to a system of nonlinear elliptic partial differential equations.

Résumé

En utilisant la méthode de Galerkin pour résoudre des équations intégro-différentielle non-linéaires de type elliptique ou parabolique, on a besoin de résoudre le système non-linéaire d'équations algébriques ou d'équations différentielles ordinaires résultant. Résoudre ces équations avec la méthode de Newton ou une variante peut se révéler très difficile et une bonne conjecture initiale est nécessaire pour que les méthodes convergent. Il peut y avoir également de multiples solutions et en pratique il est impossible de toutes les repérer. De plus il est difficile d'étudier la dépendance des paramètres des solutions. Nous avons developpé un remède à ces problèmes en développant la méthode spectrale séquentielle qui permet d'éviter de résoudre entièrement un système non-linéaire. Dans la méthode spectrale séquentielle, nous résolvons une équation scalaire algébrique non-lináire ou une équation différentielle ordinaire à l'étape initiale et ensuite la solution du problème original est obtenue par itérations, nous n'avons jamais besoin de résoudre un systeme non-linéaire, à aucune étape de la méthode. La méthode spectrale séquentielle converge linéairement pour des problèmes stationaires et superlinéairement pour des problèmes d'évolution. Avec la méthode spectrale séquentielle, nous pouvons obtenir des solutions aussi précises que nous le souhaitons en beaucoup moins d'effort qu'avec la méthode de Galerkin. Nous pouvons aussi augmenter le degré spectral de précision pendant que la méthode s'exécute. De plus il est facile de détecter l'existence de solutions multiples en observant seulement une seule équation et on peut donc retracer ces solutions. Le comportement de la solution et la dépendance des paramétres peuvent être estimés et on peut également déterminer le temps où la solution devient singulire pour les valeurs correspondantes des paramètres en étudiant seulement une équation. En outre nous montrons que la méthode spectrale séquentielle peut être appliquée à un système d'équations différentielles partielles elliptiques non-linéaires.

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Chapter 1

Introduction

1.1 Introduction

Many problems arising from combustion theory, fluid dynamics, quantum mechanics, microwave theory, elasticity and structural mechanics can be described by a mathematical model consisting of ordinary differential equations, partial differential equations or integro-differential equations. These problems can be linear in special cases, but in general they are nonlinear. For linear problems, the Laplace and Fourier transform are often used to find analytical solutions. It is however not possible to find analytical solutions for all linear problems. In addition most of the real life problems are nonlinear and the search for analytical solutions is a difficult task. Therefore, numerical methods have been developed to solve these problems numerically. There are different types of numerical methods, for example the finite difference method, the finite element method and spectral methods. Each numerical method has its own advantages and limitations so it is difficult to compare different numerical methods. However, many numerical techniques are related. The Galerkin formulation, which is being used in many subject areas, provides the connection. Within the Galerkin frame-work we can generate finite difference, finite element and spectral methods. The idea of the sequential

spectral method developed in this thesis also stems from the Galerkin method so we describe the Galerkin methods in Section 1.2 for the purpose of comparison and clarity. The sequential spectral method for elliptic and parabolic partial differential equations or integro-differential equations is developed in Chapter 2.

1.2 The Galerkin Method

The origin of the Galerkin method is generally associated with a paper published in 1915 by Galerkin, a Russian mechanical engineer, on the elastic equilibrium of rods and thin plates [Gal15]. The use of the Galerkin methods increased rapidly during the 1950's. The Galerkin method has been used to solve many problems in structural mechanics, dynamics, heat flow, hydrodynamic stability, magnetohydrodynamics, heat and mass transfer, acoustics, microwave theory, neutron transport, etc. Problems governed by ordinary differential equations, partial differential equations, integral equations and integro-differential equations have been investigated via Galerkin formulations. Steady, unsteady and eigenvalue problems have proved to be equally amenable to the Galerkin treatment. Essentially, any problem for which governing equations can be written down is a candidate for the Galerkin method. The Galerkin method is a member of the larger class of methods known as the methods of weighted residuals [LP99].

Let Ω be a spatial domain with boundary $\partial\Omega$. We consider an initial boundary value problem

$$\begin{aligned} \mathcal{L}u &= 0, && \mathbf{x} \in \Omega,\ t > 0, \\ \mathcal{B}u &= 0, && \mathbf{x} \in \partial\Omega,\ t > 0, \\ u(\mathbf{x}, 0) &= u_0(\mathbf{x}), && \mathbf{x} \in \Omega, \end{aligned} \tag{1.1}$$

where $\mathcal{L}$ is a differential operator in space and time and $\mathcal{B}$ is a linear boundary operator. We assume that some conditions are satisfied to ensure existence,

uniqueness and certain regularity of the solution of (1.1). The Galerkin method is to find an approximate solution

$$\tilde{u}(\mathbf{x},t) = \sum_{j=1}^{N} a_j(t)\ \phi_j(\mathbf{x}), \tag{1.2}$$

where $\phi_j(\mathbf{x})$ are known basis functions and they are members of a complete set of functions. The functions $\phi_j(\mathbf{x})$ are called trial functions and the coefficients $a_j(t)$ are to be determined. Substituting the approximate solution (1.2) into the original problem (1.1), we get in general a nonzero residual $R(\tilde{u})$, given by

$$R(\tilde{u}) := \mathcal{L}\tilde{u} \neq 0. \tag{1.3}$$

The Galerkin method requires the inner product of the residual and $\phi_k(\mathbf{x})$ to be zero for all k, that is,

$$(R(\tilde{u}), \phi_k) = 0, \quad k = 1, 2, 3, \ldots, N. \tag{1.4}$$

The coefficients $a_j(t)$ are determined by solving the system of N ordinary differential equations given in (1.4). In the steady case, the coefficients are just constants and we have to solve a system of N algebraic equations. To obtain a solution by the Galerkin method most of the effort is required to solve the system of equations (1.4). The accuracy increases by increasing N [Fle84] but the solution of equations (1.4) then becomes more and more difficult.

In this thesis, we are mostly concerned with nonlinear initial boundary value problems of the form

$$\begin{aligned} \mathcal{L}u &\equiv \alpha\tfrac{\partial u}{\partial t} - \triangle u - F(u) = 0, && \mathbf{x} \in \Omega,\ t > 0, \\ \mathcal{B}u &\equiv 0, && \mathbf{x} \in \partial\Omega,\ t > 0, \\ u(\mathbf{x},0) &= u_0(\mathbf{x}), && \mathbf{x} \in \Omega, \end{aligned} \tag{1.5}$$

where $\alpha \geq 0$. Substituting the approximate solution (1.2) into problem (1.5), the system of equations (1.4) can be written in matrix form as

$$\mathbf{M}\dot{\mathbf{a}} + \mathbf{B}\,\mathbf{a} + \mathbf{c} = \mathbf{0}, \tag{1.6}$$

where $\dot{\mathbf{a}} = [\frac{da_1}{dt}, \frac{da_2}{dt}, \ldots, \frac{da_N}{dt}]$. The elements of $\mathbf{M}$, $\mathbf{B}$ and $\mathbf{c}$ are respectively

$$\begin{aligned} & m_{jk} = \alpha(\phi_j, \phi_k), \quad b_{jk} = -(\triangle\phi_j, \phi_k) \\ \text{and} \quad & c_k = (F(\sum_{j=1}^{N} a_j\phi_j), \phi_k), \quad j,k = 1,2,\cdots,N. \end{aligned} \tag{1.7}$$

If orthogonal trial functions $\phi_j(\mathbf{x})$ are used, then $\mathbf{M}$ is a diagonal matrix. Since orthogonal functions are linearly independent, the resulting equations (1.4) will also be independent if the problem is linear. The use of orthogonal functions $\phi_j(\mathbf{x})$ also avoids a matrix factorization and subsequent matrix multiplication. However, it maintains the high accuracy of using global trial functions. The choice of orthogonal trial functions leads to *spectral methods*. If the trial functions are chosen to be polynomials defined in small domains, called elements, then the Galerkin method leads to the *finite element methods*.

An important feature of the traditional Galerkin methods, which has contributed to its widespread use, is the ability to achieve high accuracy with few terms in the approximate solution, provided the trial functions are chosen to take advantage of a priori knowledge of the expected solution. Often an eigenfunction expansion of a related (and presumably simpler) problem is used. Mikhlin [Mik64] proved that if a unique solution of the elliptic boundary value problem exists, then a Galerkin solution will converge to it, under appropriate conditions. Temam [Tem73] established such a result for a nonlinear elliptic boundary value problem. Convergence of the Galerkin method for nonlinear initial boundary value problems is discussed by Finlayson in [Fin72].

The idea of the Galerkin method is simple and can be applied to solve many different problems but the most difficult task is to solve the system of equations (1.4). For nonlinear parabolic partial differential or integro-differential equations, the Galerkin method reduces the problem to one of solving a nonlinear system of ordinary differential equations, which is not easy to solve in general. For nonlinear elliptic boundary value problems, the coefficients a_j are constants and are determined by solving the system of N nonlinear algebraic equations given by (1.4). For such systems solutions are usually sought using iterative schemes such as the classical Newton method or variations thereof [BP81]. For such methods, the ability to determine a solution depends crucially on providing the solver with an initial guess close to the solution. The difficulty in determining such an initial guess invariably grows in proportion to the dimension of the system. Another problem that is encountered with such classical root-finding algorithms is their inability to consistently predict the multiplicity of solutions. Consider, for example, the two-dimensional problem

$$\begin{aligned} f(x,y) &= 0, \\ g(x,y) &= 0. \end{aligned}$$

Both f and g are completely arbitrary functions each of which has zero contour lines. The solutions that are sought are those points which are common to the zero contours of both f and g. In order to find all common points, the full zero contours of both functions must, in some sense, be mapped out. These contours will, in general, consist of several disjoint closed or open curves. It is difficult to know whether all disjoint pieces of the zero contours have been mapped out. For a general nonlinear system, information about the number of solutions is therefore difficult to obtain. Now assume, for argument's sake, that one does know with some degree of certainty that there are p solutions to the nonlinear system. In addition, assume that p "good" initial guesses are available. Will

Newton's method, for example, produce these p solutions? In [TAK96], it was shown numerically that the Galerkin method (using Newton's method for the simultaneous equations) could not converge to a known, "supercritical" solution of a certain elliptic problem. An example of a nonlinear integral equation where the method fails to reveal all the solutions was presented by Katina in [Mic97].

1.3 Proposed Work

From the previous section we recognize the difficulty of solving the system of equations obtained by the Galerkin method. To alleviate the situation, Tam *et al.* introduced the sequential eigenfunction expansion method and applied it to solve a semi-linear elliptic partial differential equation [TAK96]. The method sought a solution in terms of a series expansion of the form $\sum_{j=1}^{\infty} a_j \phi_j$, where $\{\phi_j(\mathbf{x})\}$ are the eigenfunctions corresponding to the operator, the domain and the boundary conditions. In contrast to Galerkin methods, the coefficients are determined sequentially by solving a single equation at each step and then through iteration achieve convergence to (1.4). The sequential nature of the computations makes the size of N somewhat immaterial, as only a single equation is solved at any stage of the procedure and hence the aforementioned difficulties are avoided.

The sequential eigenfunction expansion method, which we call the sequential spectral method (SSM), has been applied to nonlinear integral equations [Mic97] and to parabolic partial differential equations [Alr00]. Independently a related frequency decomposition and subspace correction algorithm for an abstract parabolic evolution equation has been proposed and analyzed in [Gan97]. The present work investigates the applicability and robustness of the sequential spectral method for nonlinear integro-differential equations of elliptic and parabolic type and gives new convergence results for this method. The sequential spectral method is developed in Chapter 2. We apply the sequential spectral method to

nonlinear Fredholm integro-differential equations of elliptic type in Chapter 3 and of parabolic type in Chapter 4. We then extend the idea of Tam *et al.* [TAK96] for elliptic partial differential equations to systems of elliptic partial differential equations in Chapter 5. In all cases, the convergence of the sequential spectral method is analyzed and problems from real applications are solved. The numerical results are presented and are compared with the solutions by the Galerkin method where possible.

The sequential spectral method has mainly the following advantages over the Galerkin method:

1. Using the Galerkin method we need to solve a system of nonlinear algebraic or ordinary differential equations, whereas with the sequential spectral method only a single algebraic or ordinary differential equation needs to be solved at any step.

2. Using the Galerkin method we need to truncate the infinite expansion at a fixed number N, where N has to be determined in advance. Using the sequential spectral method we can continue to add new components a_j, without affecting the previous process and looking at the values of the coefficients obtained, we can get an idea about the right choice of N.

3. With the sequential spectral method we can track multiple solutions which are not easy to find by the Galerkin method.

Further advantages of the sequential spectral method will be seen in the next chapters when we apply the method to solve different problems.

1.4 Definitions and Notations

Here are some of the notations and definitions used throughout the thesis. Bold lower case letters, like $\mathbf{x}$, are used for vectors and bold upper case letters, like

$\mathbf{A}$, are used for matrices. The inner product of the functions f and g is denoted by (f, g) . If f and g are continuous functions in their domain Ω, then $(f, g) = \int_\Omega f(\mathbf{x})g(\mathbf{x})d\mathbf{x}$. Since we are working with the eigenfunctions in this thesis, it is natural to work in the L_2 norm. The L_2 norm for a function f is defined by $||f(\mathbf{x})||_2 = \sqrt{(f, f)}$. The L_2 norm for a vector $\mathbf{x}$ in $\mathbb{R}^n$ is defined by $||\mathbf{x}||_2 = \sqrt{\sum_{j=1}^n |x_j|^2}$. In some places, we use the L_1 norm of vectors and matrices, defined by $||\mathbf{x}||_1 = \sum_{j=1}^n |x_j|$ and $||\mathbf{A}||_1 = \max_j \sum_{i=1}^n |a_{ij}|$, respectively. We denote the spectral radius of a matrix $\mathbf{A}$ by $\rho(\mathbf{A})$, defined by $\rho(\mathbf{A}) = \max_j |\lambda_j|$, where λ_j are the eigenvalues of matrix $\mathbf{A}$.

Definition 1.1 *A function $f : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ is* ***Lipschitz*** *if there exists a constant $L > 0$, called the Lipschitz constant, such that*

$$|f(\mathbf{x}) - f(\mathbf{y})| \leq L \, ||\mathbf{x} - \mathbf{y}||, \quad \text{for all } \mathbf{x}, \mathbf{y} \in \Omega.$$

The completeness of functions can be defined in many ways. In particular the concept of complete functions and an orthonormal basis are equivalent. Here we define a complete set as follows

Definition 1.2 *A set of orthogonal functions $\{\phi_j\}$, $j = 1, 2, 3, \ldots$ is called a* ***complete set*** *if $(f, \phi_j) = 0$, for $j = 1, 2, 3, \ldots$, implies $f \equiv 0$, that is, if there is no function in space which is orthogonal to every function ϕ_j.*

Definition 1.3 *A sequence $\{x_n\}$ of real numbers is said to be a* ***contractive sequence*** *if there exists a constant $C \in (0, 1)$, such that*

$$|x_{n+1} - x_n| \leq C|x_n - x_{n-1}|, \quad \text{for all } n \in N.$$

The definition 1.3 can be extended for the vectors and functions with appropriate norms.

1.5 Theorems

Throughout the thesis, we frequently use the results given by the following theorems.

Theorem 1.1 (Mean Value Theorem in $\mathbb{R}^n$) *Let Ω be an open convex set in $\mathbb{R}^n$ and $f : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a differentiable function on Ω. Then for any $\mathbf{x}$, $\mathbf{y} \in \Omega$ there exists an $\alpha \in (0,1)$ such that*

$$f(\mathbf{x}) - f(\mathbf{y}) = \nabla f(\mathbf{d}) \cdot (\mathbf{x} - \mathbf{y}),$$

where $\mathbf{d} = (1-\alpha)\mathbf{x} + \alpha\mathbf{y} \in \Omega$ and $\nabla f = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n})$.

Proof We define a curve $g : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$g(t) = f(\mathbf{x} + t\mathbf{u}), \quad \text{for } 0 \leq t \leq ||\mathbf{y} - \mathbf{x}||,$$

where $\mathbf{u}$ is a unit vector $\mathbf{u} = \frac{\mathbf{y}-\mathbf{x}}{||\mathbf{y}-\mathbf{x}||}$. The function $g(t)$ is differentiable and by applying the Mean Value Theorem of a single variable to $g(t)$ we get the result. The details of the proof can be found in [Kos99] (pages 568-569). ■

Theorem 1.2 (Mean Value Theorem for Integrals) *Let $f : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function in a closed, connected and bounded domain Ω and $g : \Omega \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be an integrable function on Ω and $g(\mathbf{x}) \geq 0$. Then there exists $\mathbf{p} = (p_1, p_2, \ldots, p_n) \in \Omega$ such that*

$$\int_\Omega f(\mathbf{x})g(\mathbf{x})d\mathbf{x} = f(\mathbf{p}) \int_\Omega g(\mathbf{x})d\mathbf{x}.$$

Proof Since $f(\mathbf{x})$ is continuous on the closed, bounded and connected domain Ω, by the Extreme Value Theorem, there exist m and M, real constants, such

that $m \leq f(\mathbf{x}) \leq M$ for all $\mathbf{x} \in \Omega$. And since $g(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \Omega$, we have $mg(\mathbf{x}) \leq f(\mathbf{x})g(\mathbf{x}) \leq Mg(\mathbf{x})$. Integrating over the domain we have

$$m \int_\Omega g(\mathbf{x})d\mathbf{x} \leq \int_\Omega f(\mathbf{x})g(\mathbf{x})d\mathbf{x} \leq M \int_\Omega g(\mathbf{x})d\mathbf{x}.$$

Now if $\int_\Omega g(\mathbf{x})d\mathbf{x} = 0$, then $\int_\Omega f(\mathbf{x})g(\mathbf{x})d\mathbf{x} = 0$ and the desired equality is true. If $\int_\Omega g(\mathbf{x})d\mathbf{x} \neq 0$, then we can divide by it to obtain

$$m \leq \frac{\int_\Omega f(\mathbf{x})g(\mathbf{x})d\mathbf{x}}{\int_\Omega g(\mathbf{x})d\mathbf{x}} \leq M.$$

By the Intermediate Value Theorem, there exists $\mathbf{p} \in \Omega$, such that $f(\mathbf{p}) = \frac{\int_\Omega f(\mathbf{x})g(\mathbf{x})d\mathbf{x}}{\int_\Omega g(\mathbf{x})d\mathbf{x}}$, which proves the result. ■

The proof for the one dimensional case of this theorem can be found in many books, for example [Kos99] but for the general case, it is often given as an exercise in these books.

Theorem 1.3 *Let* $\{\phi_j(\mathbf{x})\}$, $j = 1, 2, 3, \ldots$ *be a complete set of orthonormal functions. Then every square integrable function* $f(\mathbf{x})$ *can be expanded in a series using the orthonormal system of functions* $\{\phi_j(\mathbf{x})\}$,

$$f(\mathbf{x}) = \sum_{j=1}^{\infty} a_j \phi_j(\mathbf{x}), \quad \text{where } a_j = (f, \phi_j), \tag{1.8}$$

which converges in the mean to $f(\mathbf{x})$. *In addition, Parseval's identity* $||f||_2^2 = \sum_{j=1}^{\infty} |a_j|^2$ *holds.*

Proof For any $\alpha_1, \alpha_2, \ldots, \alpha_n$ consider $S_n = \sum_{j=1}^{n} \alpha_j \phi_j$. Then

$$\begin{aligned} ||f - S_n||_2^2 &= (f, f) - 2(f, S_n) + (S_n, S_n) \\ &= ||f||_2^2 - 2\sum_{j=1}^{n} \alpha_j (f, \phi_j) + \sum_{j=1}^{n} |\alpha_j|^2. \end{aligned}$$

Adding and subtracting $\sum_{j=1}^{n} |(f, \phi_j)|^2$ to complete the square, we obtain

$$||f - S_n||_2^2 = ||f||_2^2 + \sum_{j=1}^{n} |(f, \phi_j) - \alpha_j|^2 - \sum_{j=1}^{n} |(f, \phi_j)|^2.$$

The minimum will be achieved by choosing $\alpha_j = (f, \phi_j) = a_j$ (say) and the above relation becomes

$$||f - S_n||_2^2 = ||f||_2^2 - \sum_{j=1}^{n} |a_j|^2. \tag{1.9}$$

Because the set $\{\phi_j(\mathbf{x})\}$ is complete and hence closed so

$$\lim_{n \to \infty} ||f - S_n||_2^2 = 0, \tag{1.10}$$

which proves the required relation (1.8). Parseval's identity follows from relations (1.9) and (1.10). The details of the proof can be found in [Kan97] (pages 148-150) or [Det62] (pages 39-42). ■

Theorem 1.4 *The set of eigenfunctions $\{\phi_j\}$ of Laplace's equation with Dirichlet, Neumann or Robin boundary conditions forms a complete orthonormal system.*

Proof The general proof is quite lengthy and can be found in detail in [DN66] (pages 245-249). Here we briefly describe the idea of the proof for Dirichlet boundary

conditions. In order to prove completeness of the eigenfunctions, it is sufficient to show that the set of eigenfunctions of Laplace's equation spans a vector space with convergence in the mean. Consider $r_n = f - \sum_{j=1}^{n} a_j \phi_j$, where a_j are the Fourier coefficients given by $a_j = (f, \phi_j)$. Then according to relation (1.9)

$$||r_n||_2^2 = ||f||_2^2 - \sum_{j=1}^{n} |a_j|^2.$$

Also $(r_n, \phi_j) = 0$, for $j = 1, 2, \ldots, n$. By the variational properties of the eigenvalues and eigenfunctions, the Rayleigh quotient of r_n is at least as great as the $(n+1)$st eigenvalue, that is,

$$\frac{E(r_n)}{||r_n||_2^2} \geq \lambda_{n+1}, \text{ which implies } ||r_n||_2^2 \leq \frac{E(r_n)}{\lambda_{n+1}}, \tag{1.11}$$

where $E(r_n) = \int_\Omega (\triangle r_n)^2 d\mathbf{x}$ is the energy integral. The bilinear term is given by $E(u, v) = \int_\Omega \nabla u \nabla v d\mathbf{x}$. For the Laplace equation, we have
$E(f, \phi_j) = (\nabla f, \nabla \phi_j) = -(f, \triangle \phi_j) = \lambda_j a_j$. Hence

$$E(r_n) = E(f - \sum_{j=1}^{n} a_j \phi_j) = E(f) - \sum_{j=1}^{n} \lambda_j a_j^2 \leq E(f).$$

From relation (1.11) we conclude $||r_n||_2^2 \leq \dfrac{E(f)}{\lambda_{n+1}}$. Since $\lambda_{n+1} \to \infty$ as $n \to \infty$, we can make $||r_n||_2^2$ as small as we please by choosing n sufficiently large and this proves that any function can be written as a convergent series of the eigenfunctions and hence the eigenfunctions form a complete orthonormal system. ■

Theorem 1.5 *Let* $\mathbf{A}$ *be an* $n \times n$ *matrix with eigenvalues,* λ_j, $j = 1, 2, \ldots, n$. *Then* $\lim_{k \to \infty} \mathbf{A}^k = \mathbf{0}$ *if and only if the spectral radius of* $\mathbf{A}$, $\rho(\mathbf{A}) = \max_{1 \leq j \leq n} |\lambda_j|$, *is less than 1.*

Proof For the given matrix $\mathbf{A}$, there exists a nonsingular $n \times n$ matrix $\mathbf{P}$ which reduces the matrix $\mathbf{A}$ to its Jordan canonical form denoted by $\mathbf{S}$, that is,

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{S} = \begin{pmatrix} \mathbf{J}_1 & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{J}_2 & \cdots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{J}_r \end{pmatrix}, \tag{1.12}$$

where each of the $n_l \times n_l$ submatrices $\mathbf{J}_l$ are the Jordan blocks. Since each submatrix $\mathbf{J}_l$ is upper triangular, so is $\mathbf{S}$ and

$$\mathbf{S}^k = \begin{pmatrix} \mathbf{J}_1^k & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{J}_2^k & \cdots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{J}_r^k \end{pmatrix}. \tag{1.13}$$

The Jordan blocks $\mathbf{J}_l$ are upper triangular matrices with the eigenvalues on the diagonal and 1 on the super diagonal and zero everywhere else and $\lim_{k\to\infty} \mathbf{J}_l^k = 0$ if and only if $|\lambda_j| < 1$ for all j. Now from (1.12), we have $\mathbf{A} = \mathbf{P}\mathbf{S}\mathbf{P}^{-1}$, so $\mathbf{A}^k = \mathbf{P}\mathbf{S}^k\mathbf{P}^{-1}$. Thus $\mathbf{A}^k \to \mathbf{0}$ if and only if $\rho(\mathbf{A}) < 1$. The details of the proof can be found in [Mey00] (pages 617-618). ■

Chapter 2

The Sequential Spectral Method

2.1 Introduction

In this chapter, we consider the nonlinear initial boundary value problem

$$\begin{aligned} \mathcal{L}u &\equiv \alpha\frac{\partial u}{\partial t} - \triangle u - F(u) = 0, && \mathbf{x} \in \Omega,\ t > 0, \\ \mathcal{B}u &\equiv r(\mathbf{x})\frac{\partial u}{\partial n} + s(\mathbf{x})u = 0, && \mathbf{x} \in \partial\Omega,\ t > 0, \\ u(\mathbf{x}, 0) &= u_0(\mathbf{x}), && \mathbf{x} \in \Omega, \end{aligned} \tag{2.1}$$

where $\alpha \geq 0$ and $\frac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial\Omega$, the boundary of Ω. The boundary functions $r(\mathbf{x})$ and $s(\mathbf{x})$ are nonnegative with either $r(\mathbf{x}) \equiv 0$, $s(\mathbf{x}) > 0$ (Dirichlet condition) or $r(\mathbf{x}) > 0$, $s(\mathbf{x}) \geq 0$ (Neumann or Robin condition). The function $F(u)$ is a nonlinear function of u and may contain spatial integrals of some nonlinear function of u. For $\alpha = 0$, the problem becomes an elliptic boundary value problem and there is no initial condition needed.

In general, the solution of the above problem is obtained numerically using the finite difference or finite element method. Spectral methods are used if the domain Ω is simple enough so that the set of eigenfunctions associated with the Laplacian, the domain, and the homogeneous boundary conditions can be ob-

tained easily. However, newer spectral methods can deal with more complicated boundary conditions [LT00]. In the spectral method, the solution is sought as an infinite expansion in eigenfunctions with time dependent coefficients. The coefficients are then determined by solving an infinite system of first order nonlinear ordinary differential equations obtained by using some closure conditions, coupled with initial conditions derived from $u(\mathbf{x}, 0) = u_0(\mathbf{x})$. An obvious advantage of spectral methods is that the solution is amenable for further analysis which is not possible by purely numerical methods. Many applications and advantages of spectral methods can be found in [KPY98], [CCZ87] and [LT00]. To proceed numerically, the infinite system has to be truncated to a finite N-dimensional system. In the Galerkin method, the closure condition is that the residual has zero projection on the first N coordinates of the space spanned by the eigenfunctions. This gives rise to solving an N-dimensional system of nonlinear ordinary differential equations (algebraic equations when $\alpha = 0$). While there exist now quite robust algorithms in available software, such as Auto [DK], to carry out integration, the nonlinearity can make the integration very difficult, if not impossible, for N sufficiently large. Also in the case of $\alpha = 0$ the solution of the system of nonlinear algebraic equations is not easy to find as discussed in Section 1.2. For this reason, we develop an alternate method, in which the size of N does not play an important role. We recognize that the difficulty with large N stems from the requirement of the Galerkin method that the residual $R(\tilde{u})$, where $\tilde{u}$ is the approximate solution, should have zero projection on the first N coordinates of the eigenfunctions space simultaneously. To alleviate the situation, we relax the requirement of simultaneity, and compensate for it with iteration. We calculate the coefficients of the eigenfunctions sequentially, using the condition of zero projection of $R(\tilde{u})$ sequentially, and then through iteration achieve the condition of zero projection of $R(\tilde{u})$ on the N eigenfunction coordinates. The sequential nature of this algorithm makes the size of N somewhat immaterial, as only a

single equation has to be solved at any stage of the procedure. The sequential spectral method is developed in Section 2.2. The convergence of the sequential spectral method for different concrete cases is analyzed in Chapters 3, 4 and 5.

2.2 The Sequential Spectral Method (SSM)

We develop the sequential spectral method (SSM) for problem (2.1). Let $\{\lambda_j\}$ and $\{\phi_j(\mathbf{x})\}$ be the eigenvalues and the orthonormal eigenfunctions of the Laplacian,

$$\triangle\phi_j = -\lambda_j\phi_j, \tag{2.2}$$

where the eigenfunctions $\{\phi_j(\mathbf{x})\}$ satisfy homogeneous boundary conditions. We expand the initial condition in the eigenfunctions,

$$u(\mathbf{x},0) = u_0(\mathbf{x}) = \sum_{j=1}^{\infty} c_j\phi_j(\mathbf{x}).$$

Since our primary concern is to construct a solution, we suppose that the existence of a solution of problem (2.1) has been established. Of course, if we are able to construct a solution, we have also proved its existence. To avoid the proliferation of subscripts and superscripts for the approximating functions, we use $\tilde{u}$ as a generic symbol to denote an approximation to u in our development. After the idea is explained, the formal treatment will be presented with all the subscripts and superscripts in place. Also we use u instead of $u(\mathbf{x},t)$, but keep in mind that wherever u and $\tilde{u}$ appears, these are functions of space and time variables. For the steady case there will be no time dependence.

We first set $\tilde{u} = a_1(t)\phi_1(\mathbf{x})$ and compute the residual

$$R(\tilde{u}) = \alpha\frac{da_1}{dt}\phi_1 + \lambda_1 a_1\phi_1 - F(a_1\phi_1). \tag{2.3}$$

Now according to the Galerkin procedure we set the projection of $R(\tilde{u})$ onto the first eigenfunction ϕ_1 to zero. Using the notation of the inner product of two functions, the condition $(R(\tilde{u}), \phi_1) = 0$ leads to

$$\alpha \frac{da_1}{dt} + \lambda_1 a_1 - (F(a_1\phi_1), \phi_1) = 0. \tag{2.4}$$

This is a first order ordinary differential equation for $a_1(t)$, which together with the initial condition $a_1(0) = c_1$ determines $a_1(t)$. For $\alpha = 0$ it would be a scalar equation for a_1 and there is no initial condition needed. Equation (2.4) may have more than one solution, in which case, each value of the coefficient a_1 can generate an expansion for u and the problem (2.1) could have multiple solutions. To proceed, we focus on one solution. With $a_1(t)$ so determined, we next take $\tilde{u} = a_1(t)\phi_1(\mathbf{x}) + a_2(t)\phi_2(\mathbf{x})$ and compute again the residual,

$$R(\tilde{u}) = \alpha \left(\frac{da_1}{dt}\phi_1 + \frac{da_2}{dt}\phi_2 \right) + \lambda_1 a_1 \phi_1 + \lambda_2 a_2 \phi_2 - F(a_1\phi_1 + a_2\phi_2)$$

Now, we require $(R(\tilde{u}), \phi_2) = 0$ and using that the eigenfunctions $\{\phi_j(\mathbf{x})\}$ are orthonormal, we get

$$\alpha \frac{da_2}{dt} + \lambda_2 a_2 - (F(a_1\phi_1 + a_2\phi_2), \phi_2) = 0, \tag{2.5}$$

which, together with the condition $a_2(0) = c_2$, determines the function $a_2(t)$.

Proceeding in this manner, we generate a sequence of coefficients $\{a_j(t)\}$, and consider formally $\tilde{u} = \sum_{j=1}^{\infty} a_j(t)\phi_j(\mathbf{x})$. We have the residual

$$R(\tilde{u}) = \sum_{j=1}^{\infty} (\alpha \frac{da_j}{dt}\phi_j + \lambda_j a_j \phi_j) - F(\sum_{j=1}^{\infty} a_j \phi_j). \tag{2.6}$$

Now using the completeness of the eigenfunctions (Theorem 1.4), we expand

$$F(\sum_{j=1}^{\infty} a_j \phi_j) = \sum_{k=1}^{\infty} b_k(t)\phi_k(\mathbf{x}), \tag{2.7}$$

where the coefficients $b_k(t)$ (b_k will be constant if $\alpha = 0$) are given by

$$b_k(t) = (F(\sum_{j=1}^{\infty} a_j \phi_j), \phi_k). \tag{2.8}$$

Equation (2.6) then takes the form

$$R(\tilde{u}) = \sum_{j=1}^{\infty} \left(\alpha \frac{da_j}{dt} + \lambda_j a_j - b_j \right) \phi_j. \tag{2.9}$$

Now for $\tilde{u}$ to be a solution, we would need $(R(\tilde{u}), \phi_j) = 0$ for all j, which would imply

$$\alpha \frac{da_j}{dt} + \lambda_j a_j - b_j = 0, \qquad \text{for } j = 1, 2, \ldots . \tag{2.10}$$

where $a_j(0) = c_j$. Unfortunately our $a_j(t)$ just computed will almost certainly not satisfy (2.10) and we need to introduce an iteration scheme to achieve the required equality in (2.10). Let $u^{(n)}$ denote the nth approximation of u, given by

$$u^{(n)} = \sum_{j=1}^{\infty} a_j^{(n)}(t)\phi_j(\mathbf{x}) \tag{2.11}$$

and $b_j^{(n)}(t)$ be the coefficients in the expansion

$$F(u^{(n)}) = \sum_{j=1}^{\infty} b_j^{(n)}(t)\phi_j(\mathbf{x}). \tag{2.12}$$

We construct a new sequence $\{a_j^{(n+1)}(t)\}$ from the solution of

$$\alpha \frac{da_j^{(n+1)}}{dt} + \lambda_j a_j^{(n+1)}(t) = b_j^{(n)}(t), \quad a_j^{(n+1)}(0) = c_j, \tag{2.13}$$

where the coefficients $b_j^{(n)}(t)$ are given by

$$b_j^{(n)}(t) = (F(u^{(n)}), \phi_j). \tag{2.14}$$

Once the $\{a_j^{(n+1)}(t)\}$ are obtained, we construct a new approximate solution

$$u^{(n+1)} = \sum_{j=1}^{\infty} a_j^{(n+1)}(t)\phi_j(\mathbf{x}). \tag{2.15}$$

The decoupling procedure described in (2.3) -(2.5) provides the initial guess $\{a_j^{(0)}(t)\}$ for the above iteration scheme. Depending on prior knowledge of the solution, any other initial guess could also be chosen. If we multiply equation (2.13) by $e^{\frac{\lambda_j}{\alpha}t}$ and integrate from 0 to t, we obtain

$$a_j^{(n+1)}(t) = e^{-\frac{\lambda_j}{\alpha}t}[c_j + \frac{1}{\alpha}\int_0^t b_j^{(n)}(\tau)e^{\frac{\lambda_j}{\alpha}\tau}d\tau]. \tag{2.16}$$

If the iteration scheme (2.16) converges, then $R(u^{(\infty)}) = 0$ and hence a solution of problem (2.1) is obtained. The numerical results in Chapter 4 demonstrate that the method captures the essence of the problem, and $a_j^{(0)}(t)$ is within a few percent of $a_j^{(\infty)}(t)$. Convergence to a tolerance of 10^{-6} is achieved in only a few iterations.

For $\alpha = 0$, the coefficients $\{a_j\}$ will be just constants and obtained sequentially by solving the corresponding algebraic equations. The conditions given in (2.10) become

$$\lambda_j a_j - b_j = 0, \qquad \text{for} \quad j = 1, 2, \ldots . \tag{2.17}$$

In order to achieve this equality, we introduce again an iteration. In the steady case we assume that all the eigenvalues $\{\lambda_j\}$ are positive. Let $u^{(n)}$ denote the nth approximation of u, given by

$$u^{(n)} = \sum_{j=1}^{\infty} a_j^{(n)} \phi_j(\mathbf{x}) \tag{2.18}$$

and $b_j^{(n)}$ be the coefficients in the expansion

$$F(u^{(n)}) = \sum_{j=1}^{\infty} b_j^{(n)} \phi_j(\mathbf{x}). \tag{2.19}$$

We construct a new sequence $\{a_j^{(n+1)}\}$ by setting

$$a_j^{(n+1)} = \frac{1}{\lambda_j} b_j^{(n)} = \frac{1}{\lambda_j} (F(u^{(n)}), \phi_j), \tag{2.20}$$

from which we construct a new approximation

$$u^{(n+1)} = \sum_{j=1}^{\infty} a_j^{(n+1)} \phi_j(\mathbf{x}). \tag{2.21}$$

Similar to the parabolic case the decoupling procedure of finding the coefficients $\{a_j\}$ sequentially provides the initial guess $\{a_j^{(0)}\}$ for the above iteration scheme. Depending on a priori knowledge of the solution, any other initial guess can also be chosen. If the iteration scheme (2.18)-(2.21) converges, then $R(u^{(\infty)}) = 0$ and hence a steady state solution of the problem (2.1) is obtained. Note that the iteration (2.20) can only be defined under the assumption that all eigenvalues are nonzero. However, in the case of Neumann boundary conditions, problem (2.2) can have a zero eigenvalue. So, in general, for Neumann boundary conditions, solutions can not be obtained by the sequential spectral method, but there are certain forms of $F(u)$, which allow us to define the iteration scheme. These

are discussed in Section 2.2.1. The sequential spectral method is applied to an elliptic integro-differential equation in Chapter 3. We analyze the convergence of the iteration scheme (2.18) - (2.21) and apply the method to a problem from an application.

Note that in practice, we truncate the infinite expansion and use a finite number of eigenfunctions but here we have presented the sequential spectral method for an infinite number of coefficients. The number of coefficients needed is dependent on the problem at hand and the required accuracy of the solution. In the next chapters, we apply the sequential spectral method to particular types of problems, analyze the convergence of the method and solve examples from applications. The numerical results in the next chapters demonstrate that the method captures the essence of the problem and the initial guess is within a few percent of the exact solution in all applications treated in this thesis.

2.2.1 The Sequential Spectral Method with Zero Eigenvalue or Non-homogeneous Boundary Conditions

The iteration scheme (2.18)-(2.21) has been defined under the assumption that the eigenvalues λ_j are nonzero. If there exists a zero eigenvalue of the eigenvalue problem (2.2), $\lambda_1 = 0$, then the iteration (2.20) can not be defined for $j = 1$. Hence, in general, the iteration scheme can not be defined if the eigenvalue problem (2.2) has a zero eigenvalue. However, if $\lambda_1 = 0$ and the nonlinear function $F(u)$ is a combination of a linear function ζu, and a nonlinear function $F_1(u)$, that is ,

$$F(u) = \zeta u + F_1(u). \tag{2.22}$$

Then we can define the iteration scheme as

$$a_j^{(n+1)} = \frac{1}{(\lambda_j - \zeta)} (F_1(u^{(n)}), \phi_j), \tag{2.23}$$

provided $\zeta \neq \lambda_j$, $j = 1, 2, 3, \ldots$.

In Chapter 5, the algorithm is applied to a system of nonlinear elliptic partial differential equations and to illustrate the idea, an example having a zero eigenvalue is solved by defining the iteration as in (2.23).

We developed the sequential spectral method for homogeneous boundary conditions. However in some problems with non-homogeneous boundary conditions, the boundary conditions could be made homogeneous by certain transformation of variables. Those problems can in turn be solved by the sequential spectral method. We solve a problem in Chapter 5, where the original problem has non-homogeneous boundary conditions.

Chapter 3

Nonlinear Elliptic Integro-Differential Equations

3.1 Introduction

In this chapter, we apply the sequential spectral method to the nonlinear elliptic boundary value problem

$$\begin{aligned} \mathcal{L}u &\equiv \triangle u + F(u) = 0\ , \qquad \mathbf{x} \in \Omega, \\ \mathcal{B}u &\equiv r(\mathbf{x})\frac{\partial u}{\partial n} + s(\mathbf{x})u = 0, \quad \mathbf{x} \in \partial\Omega, \end{aligned} \tag{3.1}$$

where $\frac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial\Omega$, the boundary of Ω. The boundary functions $r(\mathbf{x})$ and $s(\mathbf{x})$ are nonnegative with either $r(\mathbf{x}) \equiv 0$, $s(\mathbf{x}) > 0$ (Dirichlet condition) or $r(\mathbf{x}) > 0$, $s(\mathbf{x}) > 0$ (Robin condition). In accordance to the discussion in Section 2.2.1, we have eliminated the case of Neumann boundary conditions due to the possible existence of a zero eigenvalue in that case. The function $F(u)$ is a nonlinear function of u and also contains spatial integrals of some nonlinear functions of u. The existence of solution of problem (3.1) depends on the nature of $F(u)$. We describe here two particular forms of $F(u)$, for which

we solve examples later in the thesis.

In the first case

$$F(u) = \frac{\delta\, f(u)}{(\int_\Omega f(u)d\mathbf{x})^p}, \tag{3.2}$$

where f is Lipschitz continuous and positive, $p \geq 0$ and $\delta > 0$. Such nonlocal problems arise, for example, in the study of phenomena associated with the occurrence of shear bands in metals being deformed under high strain rates [Bur94], [ONNN94], [BT96], in modeling the phenomena of Ohmic heating [Lac95a], [Lac95b], in the investigation of the fully turbulent behavior of a real flow, using invariant measures for the Euler equation [CLMP92], and in the theory of gravitational equilibrium of polytropic stars [KN91]. In the case of homogeneous Dirichlet boundary conditions, the following facts follow from the classical theory of partial differential equations [BL97]:

1. Any solution of the boundary value problem (3.1), where $F(u)$ is given by (3.2), is positive for $\mathbf{x} \in \Omega$ with outer normal derivative $\frac{\partial u}{\partial n} \leq 0$ for $\mathbf{x} \in \partial\Omega$.

2. For $\Omega = \{x : |x| < 1\}$, any solution of the boundary value problem (3.1)-(3.2) is radially symmetric and radially decreasing.

Second, we consider nonlinear functions of the form

$$F(u) = \sigma\left(f_1(u) + b\int_\Omega f_2(u)d\mathbf{x}\right), \tag{3.3}$$

where b and σ are positive constants. The nonlinear functions $f_1(u)$ and $f_2(u)$ are continuous on Ω, locally Lipschitz with respect to u and convex, $f_1(0) > 0$, $f_2(0) > 0$ and f_2 is increasing in u. Such a problem arises in the thermal explosion process of a compressible reactive gas, see [BB82] and [Pao92]. For $f_1(u) = f_2(u) = e^{\gamma u}$, $\gamma > 0$, Pao has proved that there exists a critical value of σ, denoted by σ^*, such that for $\sigma < \sigma^*$, the boundary value problem (3.1) has

a positive solution [Pao92]. The convergence of the sequential spectral method for a general nonlinear function $F(u)$ is analyzed in Section 3.2 under a Lipschitz condition on F. In Section 3.3 a nonlinear elliptic integro-differential equation arising in the model of shear banding having $F(u)$ of the form given by (3.2) is solved and the numerical results are compared with the Galerkin method. The advantages of the sequential spectral method over the Galerkin method become apparent in this first application.

Our convergence analysis is based on the following result about contractive sequences.

Theorem 3.1 *Every contractive sequence of real numbers is a Cauchy sequence, and hence, convergent.*

Proof We follow the proof in [Kos99](page 105). Let the sequence $\{x_n\}$ be a contractive sequence. Then, by definition, there exists a constant $C \in (0,1)$, such that

$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n|, \quad \text{for all } n \in N.$$

Using the above inequality repeatedly, we have

$$|x_{n+2} - x_{n+1}| \leq C|x_{n+1} - x_n| \leq C^2|x_n - x_{n-1}| \leq \cdots \leq C^n|x_2 - x_1|.$$

For $m > n$ we obtain

$$\begin{aligned}
|x_m - x_n| &\leq |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\
&\leq (C^{m-2} + C^{m-3} + \cdots + C^{n-1})|x_2 - x_1| \\
&= C^{n-1}(C^{m-n-1} + C^{m-n-2} + \cdots + 1)|x_2 - x_1| \\
&= C^{n-1}(\tfrac{1-C^{m-n}}{1-C})|x_2 - x_1| \\
&< C^{n-1}(\tfrac{1}{1-C})|x_2 - x_1|.
\end{aligned}$$

Since $(\frac{1}{1-C})|x_2 - x_1|$ is a constant and $\lim_{n\to\infty} C^{n-1} = 0$, because $0 < C < 1$, the sequence $\{x_n\}$ is a Cauchy sequence. Since a Cauchy sequence is convergent, the sequence $\{x_n\}$ is convergent. ■

3.2 Convergence Analysis

We analyze now the convergence of the iteration scheme (2.18) - (2.21). Let $u^{(n)}(\mathbf{x})$ denote the nth approximation of $u(\mathbf{x})$, given by

$$u^{(n)}(\mathbf{x}) = \sum_{j=1}^{\infty} a_j^{(n)} \phi_j(\mathbf{x}), \tag{3.4}$$

and $b_j^{(n)}$ be the coefficients in the expansion

$$F(u^{(n)}(\mathbf{x})) = \sum_{j=1}^{\infty} b_j^{(n)} \phi_j(\mathbf{x}). \tag{3.5}$$

We construct a new sequence $\{a_j^{(n+1)}\}$ by setting

$$a_j^{(n+1)} = \frac{1}{\lambda_j} b_j^{(n)} = \frac{1}{\lambda_j} (F(u^{(n)}), \phi_j), \tag{3.6}$$

from which we construct a new approximate

$$u^{(n+1)}(\mathbf{x}) = \sum_{j=1}^{\infty} a_j^{(n+1)} \phi_j(\mathbf{x}). \tag{3.7}$$

Here $\{\lambda_j\}$ are the eigenvalues and $\{\phi_j(\mathbf{x})\}$ are the orthonormal eigenfunctions of the corresponding eigenvalue problem,

$$\triangle \phi = -\lambda \phi, \quad \mathcal{B}\phi = 0. \tag{3.8}$$

To investigate the convergence of the above iteration scheme, we consider the difference of corresponding coefficients a_j between two iteration steps,

$$a_j^{(n+1)} - a_j^{(n)} = \frac{1}{\lambda_j} \int_\Omega [F(u^{(n)}) - F(u^{(n-1)})]\phi_j d\mathbf{x}. \tag{3.9}$$

In order to prove convergence, we will prove that the sequence $\{a_j^{(n)}\}$ is contractive, that is, $|a_j^{(n+1)} - a_j^{(n)}| \leq C|a_j^{(n)} - a_j^{(n-1)}|$ with a constant $0 < C < 1$ and then convergence follows by Theorem 3.1. In our convergence analysis, we do not write the spatial dependence of the functions involved explicitly to avoid long expressions but we mention the dependence of every function separately when it first appears. Suppose $F(u^{(n)})$ is continuous in $\Omega \cup \partial\Omega$ and differentiable in Ω for each nonnegative integral value of n, then by the Mean Value Theorem (Theorem 1.1),

$$F(u^{(n)}) - F(u^{(n-1)}) = \frac{\partial F}{\partial u}(\xi^{(n)})(u^{(n)} - u^{(n-1)})$$

for some $\xi^{(n)}(\mathbf{x})$ between $u^{(n)}(\mathbf{x})$ and $u^{(n-1)}(\mathbf{x})$. Inserting this result into equation (3.9) we obtain

$$a_j^{(n+1)} - a_j^{(n)} = \frac{1}{\lambda_j} \int_\Omega \frac{\partial F}{\partial u}(\xi^{(n)})(u^{(n)} - u^{(n-1)})\phi_j d\mathbf{x}. \tag{3.10}$$

Now if $(u^{(n)} - u^{(n-1)})\phi_j$ does not change sign in Ω then by applying the Mean Value Theorem for Integrals (Theorem 1.2), we can write

$$a_j^{(n+1)} - a_j^{(n)} = \frac{1}{\lambda_j} \frac{\partial F}{\partial u}(\eta_j^{(n)}) \int_\Omega (u^{(n)} - u^{(n-1)})\phi_j d\mathbf{x}, \tag{3.11}$$

for $\eta_j^{(n)} = \xi^{(n)}(\mathbf{p})$, for some $\mathbf{p} \in \Omega$. The right hand side of this last expression can now be related back to the coefficients a_j,

$$\begin{aligned}\int_\Omega (u^{(n)} - u^{(n-1)})\phi_j d\mathbf{x} &= \int_\Omega \left(\sum_{i=1}^{\infty} (a_i^{(n)} - a_i^{(n-1)})\phi_i \right) \phi_j d\mathbf{x} \\ &= a_j^{(n)} - a_j^{(n-1)}.\end{aligned} \tag{3.12}$$

Using this result in equation (3.11) we get

$$a_j^{(n+1)} - a_j^{(n)} = \frac{1}{\lambda_j} \frac{\partial F}{\partial u}(\eta_j^{(n)})(a_j^{(n)} - a_j^{(n-1)}).$$

Since the eigenvalues λ_j are nonnegative and increasing, we obtain

$$|a_j^{(n+1)} - a_j^{(n)}| \leq \frac{1}{\lambda_1} |\frac{\partial F}{\partial u}(\eta_j^{(n)})| \, |a_j^{(n)} - a_j^{(n-1)}|. \tag{3.13}$$

Therefore the mapping (3.9) is contractive if

$$\max |\frac{\partial F}{\partial u}(\eta_j^{(n)})| \leq \lambda_1, \quad \forall \; n \text{ and } j. \tag{3.14}$$

Hence if the derivatives of F are uniformly bounded by λ_1, the method converges, if $(u^{(n)} - u^{(n-1)})\phi_j$ does not change sign in Ω. Now if $(u^{(n)} - u^{(n-1)})\phi_j$ changes sign in Ω, then we can not apply the Mean Value Theorem for Integrals to equation (3.10) and thus can not get the relation (3.13) for each of the coefficients a_j separately. Instead, we prove that the sequence $\{u^{(n)}\}$ is contractive. From equation (3.9), we have

$$a_j^{(n+1)} - a_j^{(n)} = \frac{1}{\lambda_j} \int_\Omega \left[F(u^{(n)}) - F(u^{(n-1)}) \right] \phi_j d\mathbf{x}, \tag{3.15}$$

Let

$$g^{(n)}(\mathbf{x}) := F(u^{(n)}) - F(u^{(n-1)}). \tag{3.16}$$

Since the set of eigenfunctions $\{\phi_j\}$ is a complete set of orthonormal functions (Theorem 1.4), we can expand $g^{(n)}(\mathbf{x})$ in a Fourier series in $\{\phi_j\}$ (Theorem 1.3),

$$g^{(n)}(\mathbf{x}) = \sum_{j=1}^{\infty} g_j^{(n)} \phi_j(\mathbf{x}), \tag{3.17}$$

where the Fourier coefficients $g_j^{(n)}$ are given by

$$g_j^{(n)} = (g^{(n)}, \phi_j) = ([F(u^{(n)}) - F(u^{(n-1)})], \phi_j). \tag{3.18}$$

Relation (3.15) then becomes

$$a_j^{(n+1)} - a_j^{(n)} = \frac{1}{\lambda_j} g_j^{(n)}. \tag{3.19}$$

Multiplying both sides by ϕ_j, $j = 1, 2, 3, \ldots$ and summing over j we obtain

$$\sum_{j=1}^{\infty} \left(a_j^{(n+1)} - a_j^{(n)} \right) \phi_j = \sum_{j=1}^{\infty} \frac{g_j^{(n)}}{\lambda_j} \phi_j. \tag{3.20}$$

We now define a new function

$$h^{(n)}(\mathbf{x}) := \sum_{j=1}^{\infty} \frac{g_j^{(n)}}{\lambda_j} \phi_j(\mathbf{x}), \tag{3.21}$$

where $\frac{g_j^{(n)}}{\lambda_j}$ are the Fourier coefficients of $h^{(n)}(\mathbf{x})$. Using the definitions of $u^{(n)}$ and $u^{(n+1)}$ from relations (3.4) and (3.7) in the left hand side of equation (3.20), we

obtain

$$u^{(n+1)} - u^{(n)} = h^{(n)}. \tag{3.22}$$

Taking the L_2 norm squared on both sides we have

$$||u^{(n+1)} - u^{(n)}||_2^2 = ||h^{(n)}||_2^2. \tag{3.23}$$

Since Parseval's identity holds for the complete set of orthonormal eigenfunctions $\{\phi_j\}$ (Theorem 1.3), we obtain

$$||u^{(n+1)} - u^{(n)}||_2^2 = \sum_{j=1}^{\infty} |\frac{g_j^{(n)}}{\lambda_j}|^2 \leq \frac{1}{\lambda_1^2} \sum_{j=1}^{\infty} |g_j^{(n)}|^2, \tag{3.24}$$

because the eigenvalues $\{\lambda_j\}$ are positive and increasing. Since $g_j^{(n)}$ are the Fourier coefficients of the function $g^{(n)}(\mathbf{x})$, using the Parseval's identity again we obtain

$$||u^{(n+1)} - u^{(n)}||_2^2 \leq \frac{1}{\lambda_1^2} ||g^{(n)}||_2^2. \tag{3.25}$$

From relation (3.16), we have

$$||g^{(n)}||_2^2 = ||F(u^{(n)}) - F(u^{(n-1)})||_2^2. \tag{3.26}$$

Assuming $F(u)$ to be globally Lipschitz in L_2 with Lipschitz constant L_F, we obtain

$$||g^{(n)}||_2^2 \leq L_F^2 ||u^{(n)} - u^{(n-1)}||_2^2. \tag{3.27}$$

Inserting this result into relation (3.25), we obtain

$$||u^{(n+1)} - u^{(n)}||_2^2 \leq \frac{L_F^2}{\lambda_1^2}||u^{(n)} - u^{(n-1)}||_2^2. \tag{3.28}$$

The above inequality is for the L_2 norm squared, but this was done only to facilitate the presentation. If we follow the procedure of (3.22) - (3.28) without the square, or simply taking the square root of (3.28) we obtain

$$||u^{(n+1)} - u^{(n)}||_2 \leq \frac{L_F}{\lambda_1}||u^{(n)} - u^{(n-1)}||_2. \tag{3.29}$$

Therefore the sequence $\{u^{(n)}\}$ is contractive if $L_F < \lambda_1$ and the sequential spectral method converges. Thus we have proved

Theorem 3.2 *Let $F(u)$ be globally Lipschitz with Lipschitz constant L_F,*

$$||F(u) - F(v)||_2 \leq L_F||u - v||_2, \quad \forall\ u, v \in L_2,$$

and let λ_1 be the smallest positive eigenvalue of the eigenvalue problem (3.8). If

$$L_F < \lambda_1, \tag{3.30}$$

then the sequential spectral method converges.

Hence under condition (3.30), we have $R(u^{(n)}) \to 0$ for $n \to \infty$, implying that $u^{(\infty)}$ is a solution of equation (3.1). The convergence rate and the a priori error bound is given by the following theorem.

Theorem 3.3 (Linear Convergence) *Let $F(u)$ satisfy the hypotheses of Theorem 3.2. Then the sequential spectral method (3.4)-(3.7) converges to a solution $u(\mathbf{x})$ of problem (3.1) at the rate*

$$||u - u^{(n)}||_2 \leq (\frac{L_F}{\lambda_1})^n||u - u^{(0)}||_2. \tag{3.31}$$

In addition, we have the a priori error estimate

$$||u - u^{(n)}||_2 \leq \frac{(\frac{L_F}{\lambda_1})^n}{1 - \frac{L_F}{\lambda_1}} ||u^{(1)} - u^{(0)}||_2. \tag{3.32}$$

Proof Let $u^{(n)}(\mathbf{x})$ be the nth iterate of the approximate solution, $u^{(n)}(\mathbf{x}) = \sum_{j=1}^{\infty} a_j^{(n)} \phi_j(\mathbf{x})$. Let $u(\mathbf{x})$ be a solution of the problem (3.1), then $u(\mathbf{x}) = \sum_{j=1}^{\infty} a_j \phi_j(\mathbf{x})$. Since $u(\mathbf{x})$ is a solution of equation (3.1), we have

$$\sum_{j=1}^{\infty} \lambda_j a_j \phi_j(\mathbf{x}) = F(u), \quad \text{which gives} \quad a_j = \frac{1}{\lambda_j}(F(u), \phi_j).$$

Using the value of $a_j^{(n)}$ from equation (3.6), we consider

$$a_j - a_j^{(n)} = \frac{1}{\lambda_j} \int_{\Omega} [F(u) - F(u^{(n)})] \phi_j d\mathbf{x}. \tag{3.33}$$

For finding the error we follow the derivations in (3.16) - (3.29) and obtain

$$||u - u^{(n)}||_2 \leq \frac{L_F}{\lambda_1} ||u - u^{(n-1)}||_2. \tag{3.34}$$

Applying this inequality inductively, we get

$$||u - u^{(n)}||_2 \leq \left(\frac{L_F}{\lambda_1}\right)^2 ||u - u^{(n-2)}||_2 \leq \cdots \leq \left(\frac{L_F}{\lambda_1}\right)^n ||u - u^{(0)}||_2. \tag{3.35}$$

Since $\frac{L_F}{\lambda_1} < 1$,

$$\lim_{n \to \infty} ||u - u^{(n)}||_2 \leq \lim_{n \to \infty} \left(\frac{L_F}{\lambda_1}\right)^n ||u - u^{(0)}||_2 = 0, \tag{3.36}$$

and $\{u^{(n)}\}$ converges to u linearly.

Now we prove the second error bound. By inequality (3.29), we have

$$||u^{(n+1)} - u^{(n)}||_2 \leq \frac{L_F}{\lambda_1}||u^{(n)} - u^{(n-1)}||_2 \leq \cdots \leq \left(\frac{L_F}{\lambda_1}\right)^n ||u^{(1)} - u^{(0)}||_2. \quad (3.37)$$

For $m > n$, we have

$$\begin{aligned}
||u^{(m)} - u^{(n)}||_2 &\leq ||u^{(m)} - u^{(m-1)}||_2 + ||u^{(m-1)} - u^{(m-2)}||_2 + \cdots + ||u^{(n+1)} - u^{(n)}||_2 \\
&\leq \left(\frac{L_F}{\lambda_1}\right)^{m-1} ||u^{(1)} - u^{(0)}||_2 + \cdots + \left(\frac{L_F}{\lambda_1}\right)^n ||u^{(1)} - u^{(0)}||_2 \\
&= \left(\frac{L_F}{\lambda_1}\right)^n \left[\left(\frac{L_F}{\lambda_1}\right)^{m-n-1} + \left(\frac{L_F}{\lambda_1}\right)^{m-n-2} + \cdots + \frac{L_F}{\lambda_1} + 1\right] ||u^{(1)} - u^{(0)}||_2
\end{aligned}$$

Now $\lim_{m\to\infty} u^{(m)} = u$, so

$$\begin{aligned}
||u - u^{(n)}||_2 = \lim_{m\to\infty} ||u^{(m)} - u^{(n)}||_2 &\leq \left(\frac{L_F}{\lambda_1}\right)^n ||u^{(1)} - u^{(0)}||_2 \sum_{j=0}^{\infty} \left(\frac{L_F}{\lambda_1}\right)^j \\
&= \frac{(\frac{L_F}{\lambda_1})^n}{1 - \frac{L_F}{\lambda_1}} ||u^{(1)} - u^{(0)}||_2,
\end{aligned}$$

which gives the a priori error bound. ■

Note that the relation (3.31) also gives an error bound in terms of the exact solution. Since the exact solution is usually not known, the a priori bound (3.32) can be useful to estimate the error. Note that the condition (3.14), which is obtained under the assumption that $\{u^{(n)} - u^{(n-1)}\}\phi_j$ does not change sign, is in accordance with condition (3.30), because for a continuous differentiable function, the maximum value of the derivative serves as a Lipschitz constant.

In Theorem 3.2, we assume $F(u)$ to be globally Lipschitz in L_2. From relation (3.31), we see that all iterates stay in a ball $B(u, r)$, centered at the solution u, with radius $r = ||u - u^{(0)}||_2$. So we can relax the global Lipschitz condition, $F(u)$

only needs to be Lipschitz in the ball $B(u,r)$.

Corollary 3.1 *Let $F(u)$ be locally Lipschitz in a ball $B(u,r)$, centered at the solution u, with radius $r = ||u - u^{(0)}||_2$, where $u^{(0)}$ is the initial guess, with Lipschitz constant L_F, and let λ_1 be the smallest positive eigenvalue of the eigenvalue problem (3.8). If $L_F < \lambda_1$, then the sequential spectral method converges at the rate given in (3.31)*

We emphasize that condition (3.30) which allows us to prove convergence, is a sufficient condition but it is not necessary. The iteration may converge even if condition (3.30) is violated. In order to check condition (3.30) we need to find the Lipschitz constant L_F, but in practice we just run the algorithm and let the numerical results demonstrate whether convergence is achieved or not.

3.2.1 Convergence Analysis for Neumann Conditions

We defined the sequential spectral method (3.4) - (3.7) for homogeneous Dirichlet or Robin conditions to ensure that the eigenvalues of problem (3.8) are positive. In the case of Neumann conditions, the problem (3.8) can have a zero eigenvalue and the equation (3.6) cannot be defined. According to Section 2.2.1, if $F(u) = \zeta u + F_1(u)$ then we define the iteration scheme (3.4) - (3.7) by replacing equation (3.6) with

$$a_j^{(n+1)} = \frac{1}{(\lambda_j - \zeta)} (F_1(u^{(n)}), \phi_j), \tag{3.38}$$

provided $\zeta \neq \lambda_j$, $j = 1, 2, 3, \ldots$. If $F_1(u)$ is Lipschitz in a ball which contains all iterates of $u^{(n)}$ with Lipschitz constant L_{F_1}, then the convergence analysis of the sequential spectral method follows in the same way and we obtain that if $L_{F_1} < |\zeta|$, then the sequence $\{u^{(n)}\}$ is contractive.

3.3 Numerical Results

To illustrate the sequential spectral method, we choose an integro-differential elliptic boundary value problem coming from the formation of shear bands in materials [BLT99]. The formation of shear bands in metals has important implications to a variety of technological processes. These bands are observed in very thin zones and generally regarded as a precursor to material failure. Shear band formation is caused by the heat generated in regions with highest strain rate. With insufficient time for diffusion of this heat, a localized thermal softness of the metal occurs which enhances plastic flow in a thin zone. This localization of plastic strain into an adiabatic shear band during rapid plastic shear shares some interesting similarities with the ignition problem in combustion for chemically reactive systems.

Consider loading a thin walled tube of metal of length d in torsion with the ends held at constant temperature T_0, the initial temperature of the tube. One end of the tube is fixed and the other end is twisted at a constant rate $v = v_0$. Let $\tau(\mathbf{x}, t)$ denote the shear stress and $S(\mathbf{x}, t)$ the shear strain. If the plastic strain rate is given by Arrhenius law

$$\frac{\partial S}{\partial t} = \mu \, e^{\frac{-\triangle H(\tau)}{KE}}, \tag{3.39}$$

where E is the absolute temperature, $\triangle H(\tau)$ is the activation enthalpy and K is the Boltzmann constant, then the mathematical model for the shearing process can be written as a reaction diffusion equation which describes the energy balance

coupled with a compatibility equation,

$$\begin{aligned} \frac{\partial E}{\partial t} - \kappa \frac{\partial^2 E}{\partial x^2} &= \tau \mu^p \, e^{\frac{-\triangle H(\tau)}{KE}}, \\ \frac{\partial v}{\partial x} &= \mu \, e^{\frac{-\triangle H(\tau)}{KE}}, \\ E(0,t) &= E(d,t) = T_0, \; E(x,0) = T_0, \\ v(0,t) &= 0, v(d,t) = v_0. \end{aligned} \tag{3.40}$$

By integrating the compatibility equation and nondimensionalising, we obtain the non-local parabolic problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} &= \frac{\delta \, e^u}{(\int_{-1}^{1} e^u dx)^p}, \\ u(-1,t) &= u(1,t) = 0, \\ u(x,0) &= u_0(x) \geq 0, \end{aligned} \tag{3.41}$$

where $\delta > 0$ and $p \geq 0$. A detailed derivation can be found in [BLT99] or [BT96]. The nonlocal initial boundary value problem (3.41) appears to be very much like the classical ignition model for rigid reactive materials assuming one step Arrhenius chemistry and in fact reduces to the ignition model for $p = 0$. The model developed by Burns in [Bur94] has $p = 1$. Burns model has been studied in detail by Bebernes and Talaga in [BT96] and they have verified that a unique bounded solution exists for all $\delta > 0$. This implies that no shear banding occurs in Burns model. If, however, the model is given with $0 \leq p < 1$, then there exists a critical value δ^*, such that, for all $\delta > \delta^*$, the solutions blow up in finite time. This does predict shear banding as observed in the experiments of Merchand and Duffy [MD88].

The associated steady state problem is

$$-\frac{\partial^2 u}{\partial x^2} = \frac{\delta \, e^u}{(\int_{-1}^{1} e^u dx)^p}, \quad u(-1) = u(1) = 0. \tag{3.42}$$

From the classical theory of partial differential equations any solution of the

boundary value problem (3.42) is radially symmetric and radially decreasing [BL97]. So the problem (3.42) can be written as

$$-\frac{\partial^2 u}{\partial x^2} = \frac{\delta\, e^u}{(2\int_0^1 e^u dx)^p}, \quad u'(0) = 0, \quad u(1) = 0. \tag{3.43}$$

Bebernes and Lacey in [BL97] have proved

Theorem 3.4

1. *If $p \geq 1$, then the boundary value problem (3.43) has a unique solution for all $\delta > 0$.*

2. *If $0 \leq p < 1$, then there exists $\delta^* > 0$ such that the boundary value problem (3.43) has*

   (a) two solutions for $\delta < \delta^*$

   (b) one solution for $\delta = \delta^*$ and

   (c) no solution for $\delta > \delta^*$.

We examine solutions of the problem (3.43) for different values of p by applying the sequential spectral method and the Galerkin method and compare the numerical results. We will denote the truncated solution obtained by the Galerkin method by u^g. From the eigenvalue problem

$$\frac{d^2\phi}{dx^2} = -\lambda\phi, \qquad \frac{d\phi}{dx}(0) = 0, \quad \phi(1) = 0,$$

we obtain the eigenvalues $\lambda_j = [\frac{(2j-1)\pi}{2}]^2$ and orthonormal eigenfunctions $\phi_j = \sqrt{2}\cos(\frac{(2j-1)\pi x}{2})$, $j = 1, 2, 3, \ldots$.

For $p = 1$, we obtained, with the sequential spectral method (SSM), a unique solution for all values of $\delta > 0$. Tables 3.1 and 3.2 compare the values of the coefficients a_j from the SSM and g_j from the Galerkin method for $\delta = 1$ and

	Galerkin	SSM	
j	g_j	$a_j^{(0)}$	$a_j^{(6)}$
1	0.189173752	0.189561292	0.189173752
2	-0.005704984	-0.005705579	-0.005704984
3	0.001228008	0.001227847	0.001228008
4	-0.000469654	-0.000446864	-0.000469654
5	0.000210188	0.000210130	0.000210188
6	-0.000115091	-0.000115055	-0.000115091
7	0.000069716	0.000069691	0.000069716

Table 3.1: Comparison of the coefficients obtained by the Galerkin method and the sequential spectral method for $p = 1$ and $\delta = 1$.

	Galerkin	SSM	
j	g_j	$a_j^{(0)}$	$a_j^{(8)}$
1	0.610427863	0.612456619	0.610427863
2	-0.009285960	-0.009276831	-0.009285960
3	0.002380523	0.002379187	0.002380523
4	-0.000878048	-0.000877113	-0.000878048
5	0.000415311	0.000414748	0.000415311
6	-0.000228054	-0.000227697	-0.000228054
7	0.000138411	0.000138124	0.000138411

Table 3.2: Comparison of the coefficients obtained by the Galerkin method and the sequential spectral method for $p = 1$ and $\delta = 3$.

$\delta = 3$. The coefficients $a_j^{(0)}$ are found by using bisection method and the Galerkin coefficients g_j are obtained by using the nonlinear equation solver in Maple with standard tolerance of 10^{-10}. Figures 3.1 and 3.2 show the Galerkin solution $u^g = \sum_{j=1}^{7} g_j \phi_j(x)$ and the initial solution $u^{(0)} = \sum_{j=1}^{7} a_j^{(0)} \phi_j(x)$ at the start of first iteration in the sequential spectral method and the error $|u^g - u^{(0)}|$ between these solutions. We see from the figures and tables that the $a_j^{(0)}$ are very close to the Galerkin coefficients g_j and converge to g_j in only a few iterations.

We see from Tables 3.1 and 3.2 that the coefficients a_j become smaller when j increases, and the coefficient of first eigenfunction is important. This is one of the great features of the spectral methods and gives spectral accuracy. While using the sequential spectral method, we can use this decaying property of the

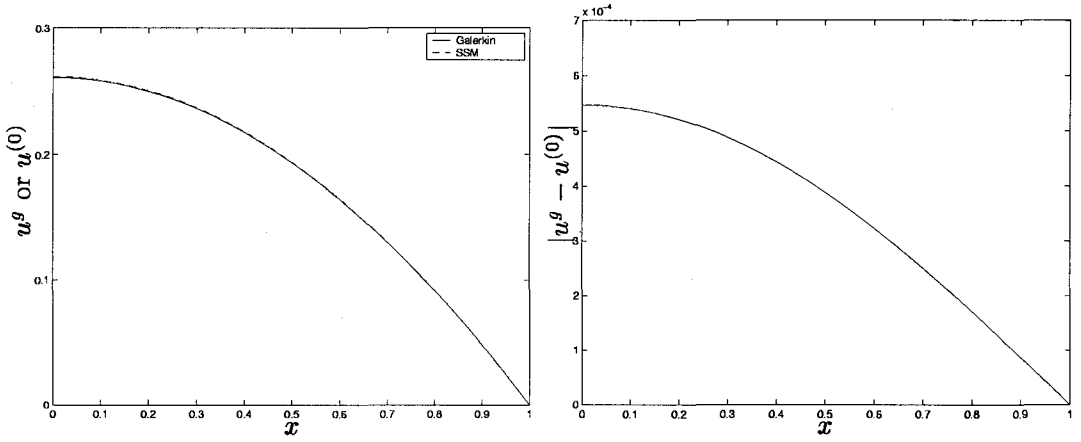

Figure 3.1: Comparison of u^g and $u^{(0)}$ and the error $|u^g - u^{(0)}|$ for $p = 1$ and $\delta = 1$.

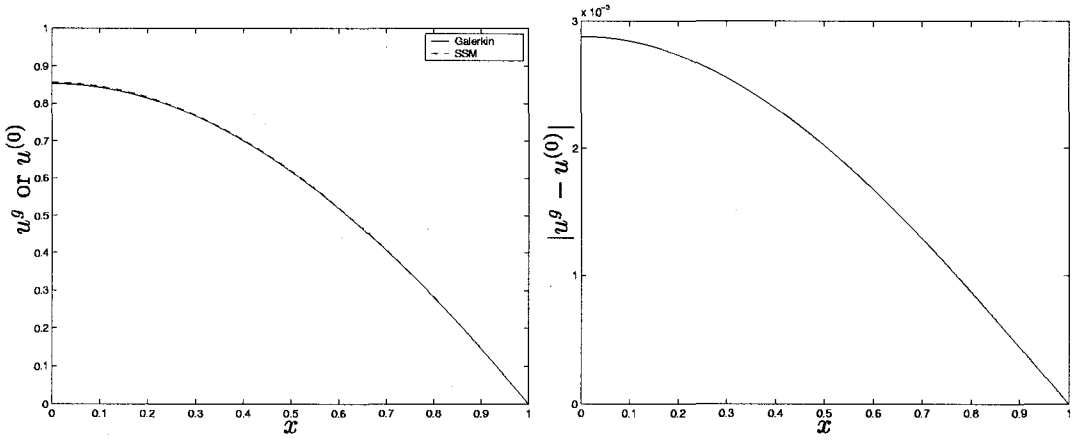

Figure 3.2: Comparison of u^g and $u^{(0)}$ and the error $|u^g - u^{(0)}|$ for $p = 1$ and $\delta = 3$.

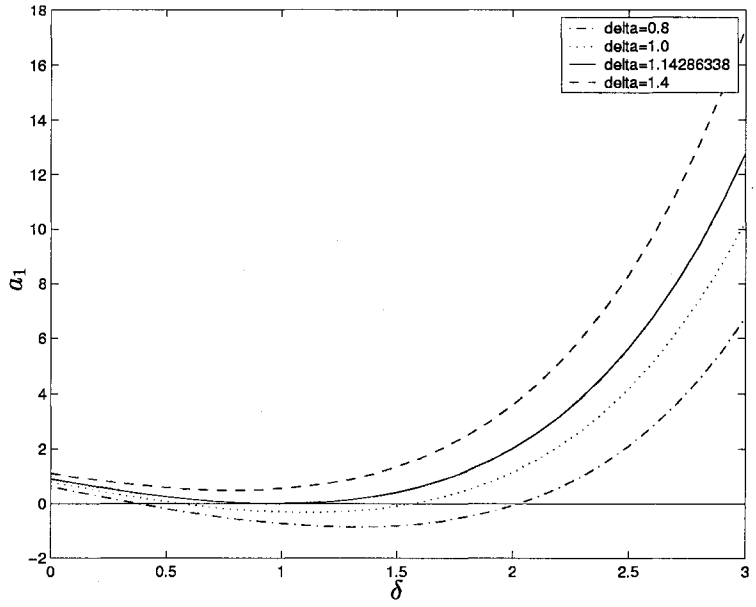

Figure 3.3: Solutions for $a_1^{(0)}$ of equation (3.44) for $p = 1/6$ and for different values of δ, where $\delta^* \approx 1.14286338$.

coefficients to determine N, the number of terms in the expansion to achieve required accuracy.

For $0 \leq p < 1$, we found two solutions for $\delta < \delta^*$. By the Galerkin method the existence of multiple solutions can not be found easily. While in the sequential spectral method, we first solved the equation

$$-\frac{\pi^2}{4}a_1 + \frac{\delta\sqrt{2}\int_0^1 e^{\sqrt{2}\,a_1\cos\frac{\pi x}{2}}\cos\frac{\pi x}{2}dx}{\left(2\int_0^1 e^{\sqrt{2}\,a_1\cos\frac{\pi x}{2}}dx\right)^p} = 0, \tag{3.44}$$

to find the value of a_1. Two values of a_1 were found for $0 \leq p < 1$ and $\delta < \delta^*$. Each value of a_1 can generate an expansion and we immediately see that multiple solutions are possible. By plotting the graphs of equation (3.44) for different values of δ, we observe that equation (3.44) has two values of a_1 for $\delta < \delta^*$ and the two solutions come closer and closer together as δ approaches δ^* from the left and merge to one solution at $\delta = \delta^*$. Figure 3.3 shows the graphs of equation (3.44) for $p = 1/6$ and different values of δ.

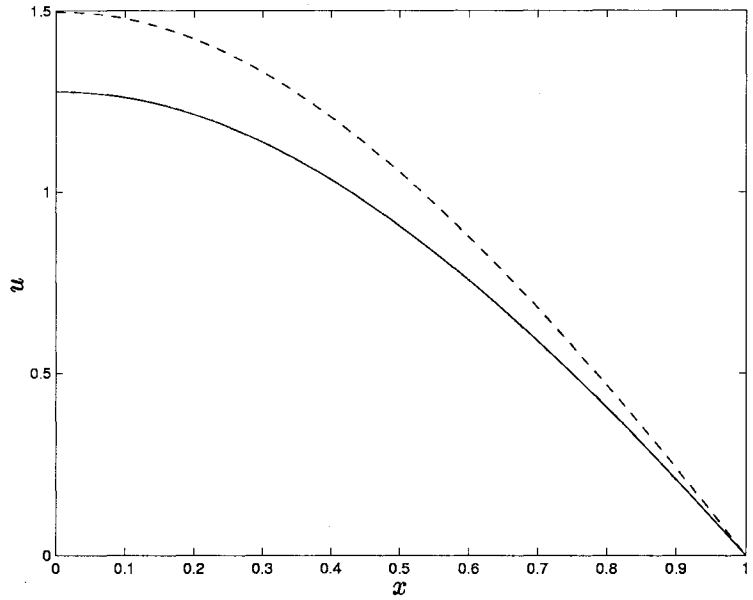

Figure 3.4: Two solutions $u = \sum_{j=1}^{7} a_j \phi_j(x)$ of problem (3.43) for $p = 1/6$ and $\delta = 1.14 < \delta^*$.

We computed the two solutions of the problem (3.43) for $p = 1/6$ and $\delta = 1.14$ by the Galerkin method and the sequential spectral method. The solutions are shown in Figure 3.4 and a comparison of the coefficients a_j and g_j is given in Tables 3.3 and 3.4. Here again the initial coefficients $a_j^{(0)}$ are not far from the Galerkin coefficients g_j but it takes a bit more iterations to converge. Figures 3.5 and 3.6 show the solutions u^g, $u^{(0)}$ and the error $|u^g - u^{(0)}|$ for the two solutions.

In using the Galerkin method, we need to truncate the infinite expansion at a fixed number N, where N has to be determined in advance. Using the sequential spectral method we can continue to add new components a_j without affecting the previous process and looking at the values of the coefficients obtained, we can get an idea about the right choice of N. The time of computations depends upon the nature of the nonlinearity in the problem and the number N. In the sequential spectral method, we always have to solve a single scalar equation regardless of the value of N. However, the time of computations depends much on N in the Galerkin method. For solving a system of N equations a good choice of the initial

	Galerkin	SSM	
j	g_j	$a_j^{(0)}$	$a_j^{(70)}$
1	0.904836965	0.918163001	0.904836965
2	-0.004699530	-0.004422257	-0.004699530
3	0.002457131	0.002460107	0.002457131
4	-0.000902004	-0.000900857	-0.000902004
5	0.000430335	0.000429725	0.000430335
6	-0.000237206	-0.000236814	-0.000237206
7	0.000144341	0.000143966	0.000144341

Table 3.3: Comparison of the coefficients of the first solution obtained by the Galerkin method and the sequential spectral method for $p = 1/6$ and $\delta = 1.14$.

	Galerkin	SSM	
j	g_j	$a_j^{(0)}$	$a_j^{(63)}$
1	1.056103439	1.054778681	1.056103439
2	-0.000228582	-0.000357669	-0.000228582
3	0.002430340	0.002439498	0.002430340
4	-0.000865839	-0.000868569	-0.000865839
5	0.000416076	0.000416994	0.000416076
6	-0.000229851	-0.000230235	-0.000229851
7	0.000140093	0.000140128	0.000140093

Table 3.4: Comparison of the coefficients of the second solution obtained by the Galerkin method and the sequential spectral method for $p = 1/6$ and $\delta = 1.14$.

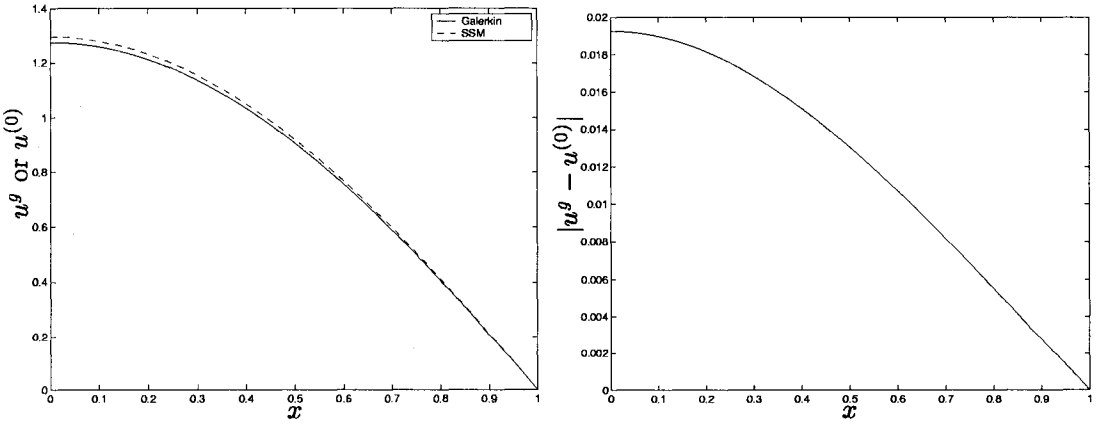

Figure 3.5: Comparison of u^g and $u^{(0)}$ and the error $|u^g - u^{(0)}|$ for the first solution for $p = 1/6$ and $\delta = 1.14$.

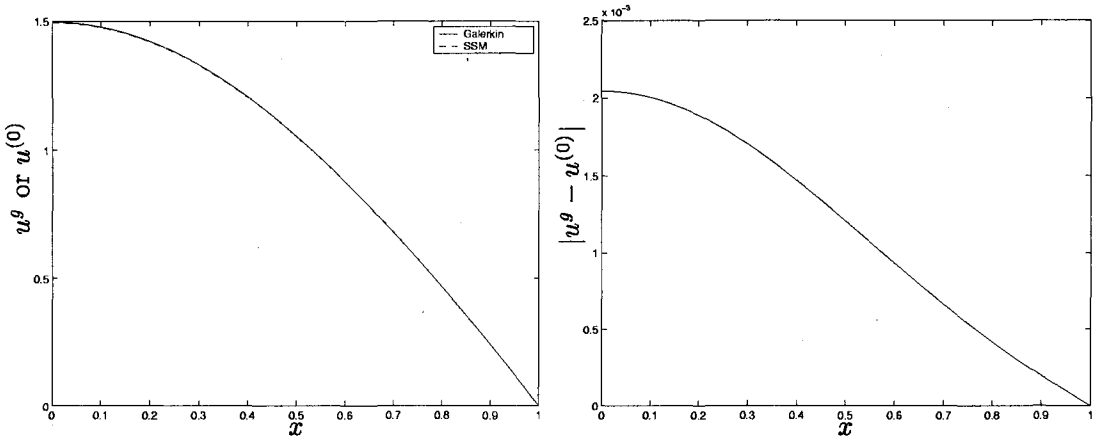

Figure 3.6: Comparison of u^g and $u^{(0)}$ and the error $|u^g - u^{(0)}|$ for the second solution for $p = 1/6$ and $\delta = 1.14$.

guess is crucial. Using $a_j^{(0)}$ from our SSM as the initial guesses for g_j, the time of computations decreases but still it is about ten times the time taken by SSM for $N = 7$. The ratio of computational time of the Galerkin and the sequential spectral method increases as we increase N. Figure 3.7 compares the logarithm of the time taken by the Galerkin method and the sequential spectral method for $p = 1$, $\delta = 1$ and different values of N. Here computational time is the time taken by using the nonlinear equation solver in Maple for both the Galerkin and the sequential spectral methods. Using the method of least squares, the approximate slopes for the Galerkin and the sequential spectral methods are 6.086 and 3.381 respectively. This means that the approximate computational complexity for the Galerkin method is $t_g \approx N^6$ and for the SSM is $t_s \approx N^3$.

If $0 \leq p < 1$, then by Theorem 3.4, there exists a value δ^* such that the boundary value problem (3.43) has no solution for $\delta > \delta^*$. We tried to estimate the critical value δ^* by examining equation (3.44) numerically. The estimates for δ^* for different values of p are given in Table 3.5. The dependence of δ^* on p is also shown in Figure 3.8. These estimates are virtually impossible to find by the Galerkin method. This is a big advantage of the sequential spectral method: the behavior of the solution can be estimated by examining a single scalar equation

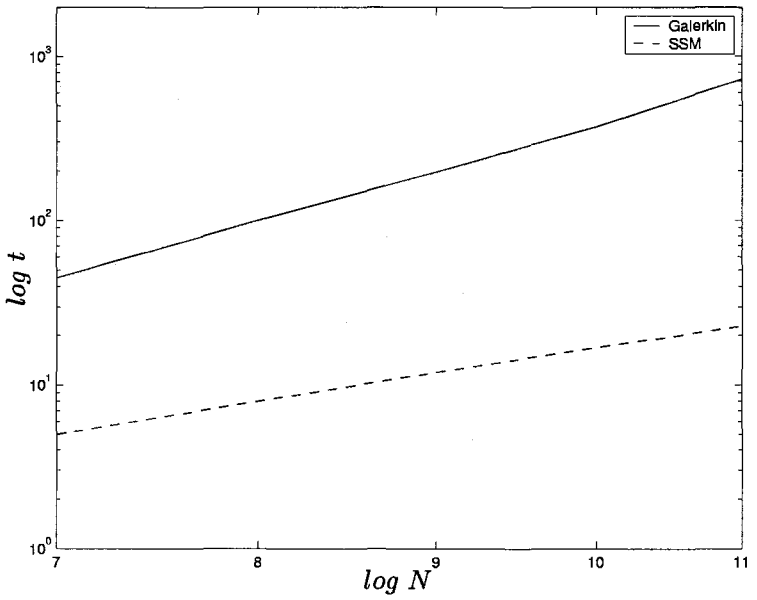

Figure 3.7: Comparison of the computational time for the Galerkin method and SSM for $p = 1$, $\delta = 1$ and different values of N. Approximate slope for the Galerkin method is 6.086 and for SSM is 3.381.

instead of solving a system of N equations in the Galerkin method.

p	δ^*	p	δ^*
0.00	[0.8773,0.8774]	0.60	[2.7456,2.7457]
0.04	[0.9327,0.9328]	0.70	[3.6693,3.6694]
0.08	[0.9930,0.9931]	0.75	[4.3724,3.2725]
0.10	[1.0251,1.0252]	0.80	[5.3903,5.3904]
0.20	[1.2087,1.2088]	0.90	[10.232,10.233]
0.25	[1.3179,1.3180]	0.92	[12.596,12.597]
0.30	[1.4417,1.4418]	0.94	[16.501,16.502]
0.40	[1.7454,1.7455]	0.96	[24.238,24.239]
0.50	[2.1567,2.1568]	0.98	[47.229,47.230]

Table 3.5: Range for the values of δ^* for different values of p

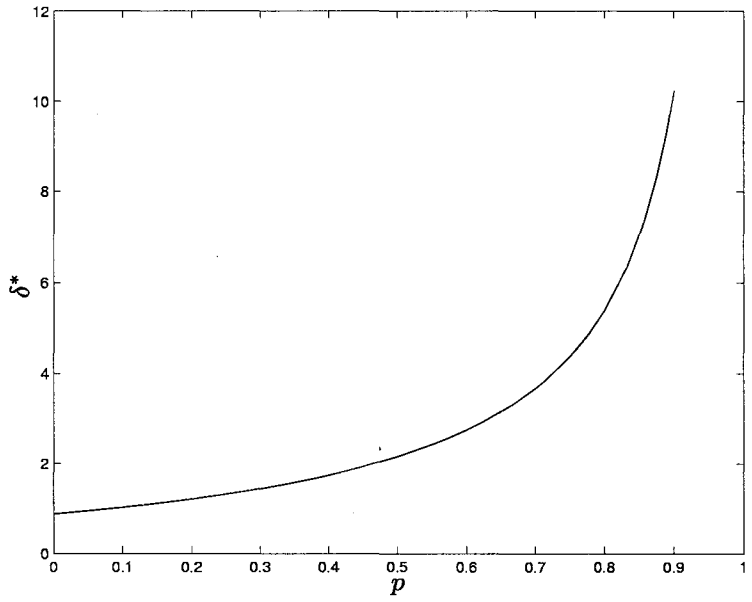

Figure 3.8: Dependence of δ^* on p.

Chapter 4

Nonlinear Parabolic Integro-Differential Equations

4.1 Introduction

In this chapter, we consider the nonlinear Fredholm integro-parabolic equation

$$\begin{aligned} \mathcal{L}u &\equiv \frac{\partial u}{\partial t} - \triangle u - F(u) = 0, && \mathbf{x} \in \Omega,\ t > 0, \\ \mathcal{B}u &\equiv r(\mathbf{x})\frac{\partial u}{\partial n} + s(\mathbf{x})u = 0, && \mathbf{x} \in \partial\Omega,\ t > 0, \\ u(\mathbf{x}, 0) &= u_0(\mathbf{x}), && \mathbf{x} \in \Omega, \end{aligned} \tag{4.1}$$

where $\frac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial\Omega$, the boundary of Ω. The boundary functions $r(\mathbf{x})$ and $s(\mathbf{x})$ are nonnegative with either $r(\mathbf{x}) \equiv 0$, $s(\mathbf{x}) > 0$ (Dirichlet condition) or $r(\mathbf{x}) > 0$, $s(\mathbf{x}) \geq 0$ (Neumann or Robin condition). The nonlinear function $F(u)$ also contains spatial integral of some nonlinear function of u. In this thesis we are concerned with $F(u)$ of the form

$$F(u) = f_1(u) + \int_\Omega f_2(u)d\mathbf{x}, \tag{4.2}$$

where f_1 and f_2 are nonlinear functions of u and are continuous on $\Omega \times [0, \infty)$, locally Lipschitz with respect to u and convex, $f_1(0) > 0$, $f_2(0) > 0$ and f_2 is increasing in u. In the case of homogeneous Dirichlet conditions, existence of solutions has been proved in [BE82] and is summarized in the following theorem

Theorem 4.1 *If* $r(\mathbf{x}) \equiv 0$, $u_0 \in L^2(\Omega)$ *and* $\sup_{\mathbf{x}\in\Omega} u_0(\mathbf{x}) < \infty$, *then problem (4.1) has a unique classical solution on* $\Omega \times [0, T)$, *where either* $T = \infty$ *or* $T < \infty$ *and* $\lim_{t\to T^-} \sup_{\mathbf{x}\in\Omega} u(\mathbf{x}, t) = \infty$.

Theorem 4.2 *If* $r(\mathbf{x}) \equiv 0$ *and if* $u_0(\mathbf{x}) \equiv 0$ *for* $\mathbf{x} \in \Omega$, *then the solution* $u(\mathbf{x}, t)$ *of the problem (4.1) is nonnegative and nondecreasing as a function of* t *on* $\Omega \times [0, T)$, *provided the derivatives* f_1' *and* f_2' *are Lipschitz continuous.*

We apply the sequential spectral method developed in Chapter 2 to problem (4.1), where now $\alpha = 1$. We analyze the convergence of the iteration scheme in Section 4.2 and find a surprising new convergence result in the evolution case: the convergence rate is faster than any linear rate if $\alpha \neq 0$. In Section 4.3 we carry out numerical computations on a nonlinear parabolic integro-differential equation arising from combustion theory and compare the results obtained by the sequential spectral method and the Galerkin method.

4.2 Convergence Analysis

Now we analyze the convergence of the sequential spectral method (2.11)-(2.16) applied to the problem (4.1). Let $u^{(n)}(\mathbf{x}, t)$ denote the nth approximation of $u(\mathbf{x}, t)$, given by

$$u^{(n)}(\mathbf{x}, t) = \sum_{j=1}^{\infty} a_j^{(n)}(t)\phi_j(\mathbf{x}) \tag{4.3}$$

and let $b_j^{(n)}(t)$ be the coefficients in the expansion

$$F(u^{(n)}(\mathbf{x},t)) = \sum_{j=1}^{\infty} b_j^{(n)}(t)\phi_j(\mathbf{x}). \tag{4.4}$$

We construct a new sequence $\{a_j^{(n+1)}(t)\}$ from the solution of

$$\frac{da_j^{(n+1)}(t)}{dt} + \lambda_j a_j^{(n+1)}(t) = b_j^{(n)}(t), \quad a_j^{(n+1)}(0) = c_j \tag{4.5}$$

or equivalently by setting

$$a_j^{(n+1)}(t) = e^{-\lambda_j t}[c_j + \int_0^t b_j^{(n)}(\tau)e^{\lambda_j \tau}d\tau], \tag{4.6}$$

where the coefficients $b_j^{(n)}(t)$ and c_j are given by

$$b_j^{(n)}(t) = (F(u^{(n)}), \phi_j) \quad \text{and} \quad c_j = (u_0, \phi_j), \tag{4.7}$$

from which we construct a new approximate

$$u^{(n+1)}(\mathbf{x},t) = \sum_{j=1}^{\infty} a_j^{(n+1)}(t)\phi_j(\mathbf{x}). \tag{4.8}$$

In the following convergence analysis we do not write the dependence of the functions on $\mathbf{x}$ and t explicitly. To investigate the convergence of the sequential spectral method we consider the difference of coefficients a_j at two iteration steps,

$$a_j^{(n+1)}(t) - a_j^{(n)}(t) = e^{-\lambda_j t}\int_0^t e^{\lambda_j \tau}[b_j^{(n)} - b_j^{(n-1)}]d\tau. \tag{4.9}$$

Inserting the value of $b_j^{(n)}(\tau)$ from (4.7), we have

$$a_j^{(n+1)}(t) - a_j^{(n)}(t) = e^{-\lambda_j t} \int_0^t e^{\lambda_j \tau} \int_\Omega [F(u^{(n)}) - F(u^{(n-1)})]\phi_j d\mathbf{x} d\tau. \tag{4.10}$$

Let

$$S_j^{(n)}(\tau) = \int_\Omega [F(u^{(n)}) - F(u^{(n-1)})]\phi_j d\mathbf{x}. \tag{4.11}$$

Then equation (4.10) becomes

$$a_j^{(n+1)}(t) - a_j^{(n)}(t) = e^{-\lambda_j t} \int_0^t e^{\lambda_j \tau} S_j^{(n)}(\tau) d\tau. \tag{4.12}$$

Suppose $F(u^{(n)})$ is continuous and differentiable in its domain for each nonnegative integer n, then by the Mean Value Theorem (Theorem 1.1), we have

$$F(u^{(n)}) - F(u^{(n-1)}) = \frac{\partial F}{\partial u}(\xi^{(n)})(u^{(n)} - u^{(n-1)}),$$

for some $\xi^{(n)}(\mathbf{x}, t)$ between $u^{(n)}(\mathbf{x}, t)$ and $u^{(n-1)}(\mathbf{x}, t)$, and so $S_j^{(n)}(\tau)$ can be written as

$$S_j^{(n)}(\tau) = \int_\Omega \frac{\partial F}{\partial u}(\xi^{(n)})(u^{(n)} - u^{(n-1)})\phi_j d\mathbf{x}, \tag{4.13}$$

Now if $(u^{(n)} - u^{(n-1)})\phi_j$ does not change sign in Ω, then applying the Mean Value Theorem for Integrals (Theorem 1.2), we find

$$S_j^{(n)}(\tau) = \frac{\partial F}{\partial u}(\eta_j^{(n)}) \int_\Omega (u^{(n)} - u^{(n-1)})\phi_j d\mathbf{x}, \tag{4.14}$$

for $\eta_j^{(n)}(\tau) = \xi^{(n)}(\mathbf{p}, \tau)$, for some $\mathbf{p} \in \Omega$. Now using equation (4.3) and the

orthonormality of the eigenfunctions $\{\phi_j\}$, we have

$$\begin{aligned}\int_\Omega (u^{(n)} - u^{(n-1)})\phi_j d\mathbf{x} &= \int_\Omega \left(\sum_{i=1}^{\infty}(a_i^{(n)} - a_i^{(n-1)})\phi_i\right)\phi_j d\mathbf{x} \\ &= a_j^{(n)}(\tau) - a_j^{(n-1)}(\tau).\end{aligned} \tag{4.15}$$

Using this result in equation (4.14), we get the value of $S_j^{(n)}(\tau)$ in terms of the coefficients $a_j(\tau)$ as

$$S_j^{(n)}(\tau) = \frac{\partial F}{\partial u}(\eta_j^{(n)})\left(a_j^{(n)}(\tau) - a_j^{(n-1)}(\tau)\right).$$

Hence by equation (4.12) we have

$$a_j^{(n+1)}(t) - a_j^{(n)}(t) = e^{-\lambda_j t}\int_0^t e^{\lambda_j \tau}\frac{\partial F}{\partial u}(\eta_j^{(n)})\left(a_j^{(n)}(\tau) - a_j^{(n-1)}(\tau)\right)d\tau. \tag{4.16}$$

Therefore

$$|a_j^{(n+1)}(t) - a_j^{(n)}(t)| \leq e^{-\lambda_j t}\int_0^t e^{\lambda_j \tau}\,|\frac{\partial F}{\partial u}(\eta_j^{(n)})|\,|a_j^{(n)}(\tau) - a_j^{(n-1)}(\tau)|d\tau. \tag{4.17}$$

Let t be in the interval $[0,T]$ and assume that $\frac{\partial F}{\partial u}$ is uniformly bounded. Let $\max\limits_{0\leq t\leq T}|\frac{\partial F}{\partial u}(\eta_j^{(n)})| = M_j$ for all n. Then we have

$$|a_j^{(n+1)}(t) - a_j^{(n)}(t)| \leq M_j e^{-\lambda_j t}\int_0^t e^{\lambda_j \tau_1}\,|a_j^{(n)}(\tau_1) - a_j^{(n-1)}(\tau_1)|d\tau_1. \tag{4.18}$$

Now using induction on n we have

$$\begin{aligned}
|a_j^{(n+1)}(t) - a_j^{(n)}(t)| &\leq M_j^2 e^{-\lambda_j t} \int_0^t \int_0^{\tau_1} e^{\lambda_j \tau_2} \, |a_j^{(n-1)} - a_j^{(n-2)}| \, d\tau_2 d\tau_1 \\
&\leq M_j^3 e^{-\lambda_j t} \int_0^t \int_0^{\tau_1} \int_0^{\tau_2} e^{\lambda_j \tau_3} \, |a_j^{(n-2)} - a_j^{(n-3)}| \, d\tau_3 d\tau_2 d\tau_1 \\
&\vdots \\
&\leq M_j^n e^{-\lambda_j t} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} e^{\lambda_j \tau_n} \, |a_j^{(1)} - a_j^{(0)}| \, d\tau_n \ldots d\tau_2 d\tau_1.
\end{aligned} \tag{4.19}$$

Let $E_j = \max_{0 \leq t \leq T} |a_j^{(1)}(t) - a_j^{(0)}(t)|$ and $\lambda_j > 0$ for all j then we have

$$\begin{aligned}
|a_j^{(n+1)}(t) - a_j^{(n)}(t)| &\leq M_j^n E_j e^{-\lambda_j t} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} e^{\lambda_j \tau_n} d\tau_n \ldots d\tau_2 d\tau_1 \\
&= M_j^n E_j e^{-\lambda_j t} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-2}} \left(\frac{e^{\lambda_j \tau_{n-1}} - 1}{\lambda_j} \right) d\tau_{n-1} \ldots d\tau_2 d\tau_1 \\
&= M_j^n E_j e^{-\lambda_j t} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-3}} \left(\frac{e^{\lambda_j \tau_{n-2}} - 1}{\lambda_j^2} - \frac{\tau_{n-2}}{\lambda_j} \right) d\tau_{n-2} \ldots d\tau_2 d\tau_1 \\
&\vdots \\
&= M_j^n E_j e^{-\lambda_j t} \left(\frac{e^{\lambda_j t} - 1}{\lambda_j^n} - \frac{t}{\lambda_j^{n-1}} - \frac{t^2}{2! \, \lambda_j^{n-2}} - \cdots - \frac{t^{n-1}}{(n-1)! \, \lambda_j} \right) \\
&= \frac{M_j^n E_j}{\lambda_j^n} \left[1 - \left(1 + \frac{\lambda_j t}{1!} + \frac{(\lambda_j t)^2}{2!} + \cdots + \frac{(\lambda_j t)^{n-1}}{(n-1)!} \right) e^{-\lambda_j t} \right] \\
&= \frac{M_j^n E_j}{\lambda_j^n} \left(1 - \sum_{k=0}^{n-1} \frac{(\lambda_j t)^k}{k!} e^{-\lambda_j t} \right).
\end{aligned} \tag{4.20}$$

Since $\sum_{k=0}^{n-1} \frac{(\lambda_j t)^k}{k!} \to e^{\lambda_j t}$ as $n \to \infty$, the right hand side of (4.20) will be zero as $n \to \infty$ provided $\frac{\partial F}{\partial u}(\eta_j^{(n)})$ is bounded for all values of n, j and $t \in [0, T]$ and the iteration scheme converges. In fact, boundedness of the derivative is only required in the ball which contains iterates of $u^{(n)}$. The inequality (4.20) can be obtained provided there is no zero eigenvalue. If $\lambda_1 = 0$, then for $j = 1$ the relation (4.20)

becomes

$$\begin{aligned} |a_1^{(n+1)}(t) - a_1^{(n)}(t)| &\leq M_1^n E_1 \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} d\tau_n \ldots d\tau_2 d\tau_1. \\ &\leq M_1^n E_1 \frac{t^n}{n!}, \end{aligned} \tag{4.21}$$

which also proves convergence for $a_1^{(n)}(t)$, but the rate of convergence is different.

The bound obtained in (4.20) is the sharpest bound we could get. In order to obtain a simpler bound which shows more clearly the super linear convergence rate, we consider equation (4.18) again, that is,

$$|a_j^{(n+1)}(t) - a_j^{(n)}(t)| \leq M_j \int_0^t e^{-\lambda_j(t-\tau_1)} \, |a_j^{(n)}(\tau_1) - a_j^{(n-1)}(\tau_1)| d\tau_1. \tag{4.22}$$

Since $e^{-\lambda_j(t-\tau_1)} \leq 1$ for $\tau_1 \in [0,t]$, we obtain

$$|a_j^{(n+1)}(t) - a_j^{(n)}(t)| \leq M_j \int_0^t |a_j^{(n)}(\tau_1) - a_j^{(n-1)}(\tau_1)| d\tau_1. \tag{4.23}$$

Now using induction on n we obtain as for the case of $\lambda_1 = 0$

$$\begin{aligned} |a_j^{(n+1)}(t) - a_j^{(n)}(t)| &\leq M_j^2 \int_0^t \int_0^{\tau_1} |a_j^{(n-1)} - a_j^{(n-2)}| \, d\tau_2 d\tau_1 \\ &\leq M_j^3 \int_0^t \int_0^{\tau_1} \int_0^{\tau_2} |a_j^{(n-2)} - a_j^{(n-3)}| \, d\tau_3 d\tau_2 d\tau_1 \\ &\vdots \\ &\leq M_j^n \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} |a_j^{(1)} - a_j^{(0)}| \, d\tau_n \ldots d\tau_2 d\tau_1 \\ &\leq M_j^n E_j \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} d\tau_n \ldots d\tau_2 d\tau_1 \\ &= M_j^n E_j \frac{t^n}{n!}, \end{aligned} \tag{4.24}$$

where $E_j = \max_{0 \leq t \leq T} |a_j^{(1)}(t) - a_j^{(0)}(t)|$. This proves that the sequence $\{a_j^{(n)}(t)\}$ converges, provided $\frac{\partial F}{\partial u}(\eta_j^{(n)})$ is bounded for all values of n, j and $t \in [0,T]$, again

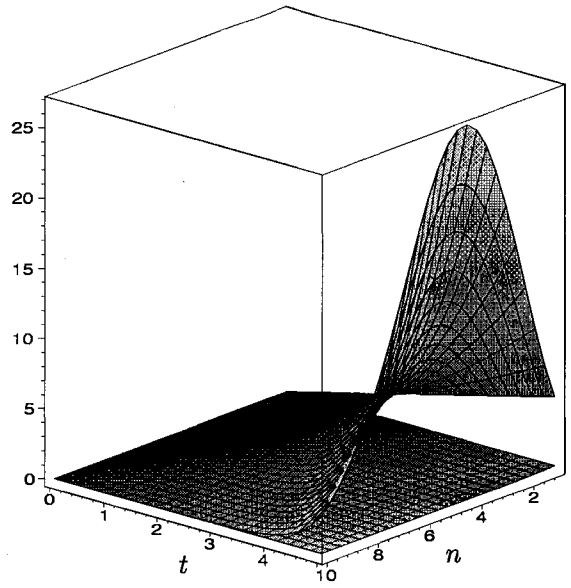

Figure 4.1: Comparison for the bounds (4.20) and (4.24), the lower one corresponds to $(\frac{1}{\lambda_1})^n[1 - \sum_{k=0}^{n-1} \frac{(\lambda_1 t)^k}{k!} e^{-\lambda_1 t}]$ and the upper one is $\frac{t^n}{n!}$.

boundedness is required in a ball which contains all iterates of $u^{(n)}$, and the iteration scheme (4.3) - (4.8) converges. We compare the two bounds given in (4.20) and (4.24), by omitting the common terms $M_j^{(n)}$ and E_j. Figure 4.1 shows a comparison of $(\frac{1}{\lambda_1})^n[1 - \sum_{k=0}^{n-1} \frac{(\lambda_1 t)^k}{k!} e^{-\lambda_1 t}]$ and $\frac{t^n}{n!}$. The first bound depends on the eigenvalues but the latter does not and one can clearly see in the Figure 4.1 that the first bound is much sharper. Here we have taken $\lambda_1 = \pi^2$, which is the smallest eigenvalue for the numerical example solved in Section 4.3.

Now if $(u^{(n)} - u^{(n-1)})\phi_j$ changes sign in Ω then we can not apply the Mean Value Theorem for Integrals to equation (4.13) and would thus not be able to prove convergence of each of the coefficients $a_j^{(n)}(t)$ seperately. But as in Chapter 3 we can analyze the convergence of the sequence $\{u^{(n)}\}$ rather than for the coefficients separately. We consider equation (4.10), that is,

$$a_j^{(n+1)}(t) - a_j^{(n)}(t) = \int_0^t e^{-\lambda_j(t-\tau)} \int_\Omega [F(u^{(n)}) - F(u^{(n-1)})]\phi_j d\mathbf{x} d\tau. \qquad (4.25)$$

Let

$$g^{(n)}(\mathbf{x},t) := F(u^{(n)}) - F(u^{(n-1)}). \tag{4.26}$$

Using the completeness of the set of orthonormal eigenfunctions $\{\phi_j\}$ we expand $g^{(n)}(\mathbf{x},t)$ as a Fourier series in $\{\phi_j(\mathbf{x})\}$ (Theorem 1.3),

$$g^{(n)}(\mathbf{x},t) = \sum_{j=1}^{\infty} g_j^{(n)}(t)\phi_j(\mathbf{x}), \tag{4.27}$$

where the Fourier coefficients $g_j^{(n)}(t)$ are given by

$$g_j^{(n)}(t) = (g^{(n)}, \phi_j) = ([F(u^{(n)}) - F(u^{(n-1)})], \phi_j). \tag{4.28}$$

Then relation (4.25) becomes

$$a_j^{(n+1)}(t) - a_j^{(n)}(t) = \int_0^t e^{-\lambda_j(t-\tau)} g_j^{(n)}(\tau)d\tau. \tag{4.29}$$

Multiplying both sides by ϕ_j, $j = 1, 2, 3, \ldots$ and summing over j we obtain

$$\sum_{j=1}^{\infty}\{a_j^{(n+1)}(t) - a_j^{(n)}(t)\}\phi_j(\mathbf{x}) = \sum_{j=1}^{\infty}\int_0^t e^{-\lambda_j(t-\tau)} g_j^{(n)}(\tau)d\tau\ \phi_j(\mathbf{x}). \tag{4.30}$$

We now define the new function

$$h^{(n)}(\mathbf{x},t) := \sum_{j=1}^{\infty}\int_0^t e^{-\lambda_j(t-\tau)} g_j^{(n)}(\tau)d\tau\ \phi_j(\mathbf{x}), \tag{4.31}$$

where $\int_0^t e^{-\lambda_j(t-\tau)} g_j^{(n)}(\tau)d\tau$ are the Fourier coefficients of $h^{(n)}(\mathbf{x},t)$. Using the definitions of $u^{(n)}$ and $u^{(n+1)}$ from relations (4.3) and (4.8), in the left hand side

of equation (4.30), we obtain

$$u^{(n+1)} - u^{(n)} = h^{(n)}. \tag{4.32}$$

Taking the L_2 norm squared with respect to the spatial variables on both sides we have

$$||u^{(n+1)} - u^{(n)}||_2^2 = ||h^{(n)}||_2^2. \tag{4.33}$$

Now Parseval's identity holds for the complete set of orthonormal eigenfunctions $\{\phi_j\}$ (Theorem 1.3). Thus we obtain

$$\begin{aligned} ||u^{(n+1)} - u^{(n)}||_2^2 &= \sum_{j=1}^{\infty} |\int_0^t e^{-\lambda_j(t-\tau)} g_j^{(n)}(\tau) d\tau|^2 \\ &\leq \sum_{j=1}^{\infty} \left(\int_0^t e^{-\lambda_j(t-\tau)} |g_j^{(n)}(\tau)| d\tau \right)^2 \\ &\leq \sum_{j=1}^{\infty} \left(\int_0^t |g_j^{(n)}(\tau)| d\tau \right)^2, \end{aligned} \tag{4.34}$$

because $e^{-\lambda_j(t-\tau)} \leq 1$ for $0 \leq \tau \leq t$. Using Cauchy Schwarz we have

$$\left(\int_0^t |g_j^{(n)}(\tau)| d\tau \right)^2 \leq \left(\int_0^t |g_j^{(n)}(\tau)|^2 d\tau \right) \left(\int_0^t 1 d\tau \right) = t \int_0^t |g_j^{(n)}(\tau)|^2 d\tau \tag{4.35}$$

and since $t \in [0, T]$, the relation (4.34) becomes

$$\begin{aligned} ||u^{(n+1)} - u^{(n)}||_2^2 &\leq T \sum_{j=1}^{\infty} \int_0^t |g_j^{(n)}(\tau)|^2 d\tau \\ &= T \int_0^t \sum_{j=1}^{\infty} |g_j^{(n)}(\tau)|^2 d\tau, \end{aligned} \tag{4.36}$$

because the series $\sum_{j=1}^{\infty} |g_j^{(n)}(\tau)|^2$ converges uniformly. Since $g_j^{(n)}(t)$ are the

Fourier coefficients of the function $g^{(n)}(\mathbf{x}, t)$, using Parseval's identity again, we arrive at

$$\begin{aligned} ||u^{(n+1)} - u^{(n)}||_2^2 &\leq T \int_0^t ||g^{(n)}||_2^2 d\tau \\ &= T \int_0^t ||F(u^{(n)}) - F(u^{(n-1)})||_2^2 d\tau. \end{aligned} \tag{4.37}$$

Assuming that $F(u)$ is globally Lipschitz in L_2 for every $t \in [0, T]$ with Lipschitz constant L_F, we obtain

$$||u^{(n+1)} - u^{(n)}||_2^2 \leq T L_F^2 \int_0^t ||u^{(n)} - u^{(n-1)}||_2^2 d\tau, \tag{4.38}$$

and by induction on n we get

$$\begin{aligned} ||u^{(n+1)} - u^{(n)}||_2^2 &\leq T^2 L_F^4 \int_0^t \int_0^{\tau_1} ||u^{(n-1)} - u^{(n-2)}||_2^2 d\tau_2 d\tau_1 \\ &\vdots \\ &\leq T^n L_F^{2n} \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} ||u^{(1)} - u^{(0)}||_2^2 d\tau_n \ldots d\tau_2 d\tau_1 \\ &\leq T^n L_F^{2n} E_0 \int_0^t \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} d\tau_n \ldots d\tau_2 d\tau_1 \\ &\leq T^n L_F^{2n} E_0 \frac{t^n}{n!}, \end{aligned} \tag{4.39}$$

where $E_0 = \max_{0 \leq t \leq T} ||u^{(1)} - u^{(0)}||_2^2$ is the initial error. The above inequality holds for every $t \in [0, T]$, therefore we have

$$\max_{0 \leq t \leq T} ||u^{(n+1)} - u^{(n)}||_2^2 \leq \frac{(T^2 L_F^2)^n}{n!} \max_{0 \leq t \leq T} ||u^{(1)} - u^{(0)}||_2^2, \tag{4.40}$$

which proves that $\{u^{(n)}\}$ is a Cauchy sequence and hence $\{u^{(n)}\}$ converges to a limit. Thus we have proved

Theorem 4.3 *Let $F(u)$ be globally Lipschitz in L_2 with Lipschitz constant L_F,*

$$||F(u) - F(v)||_2 \leq L_F||u - v||_2, \quad \forall\ u, v \in L_2.$$

Then the sequential spectral method (4.3) - (4.8) converges.

Note that we have obtained convergence also in the case when $\{u^{(n)} - u^{(n-1)}\}\phi_j$ does not change sign provided $\frac{\partial F}{\partial u}(\eta_j^{(n)})$ is bounded for all n, j and $t \in [0, T]$ which means that L_F is finite because for a continuous differentiable function the maximum value of the derivative serves as a Lipschitz constant. Hence the conditions in that typical case are in accordance with the conditions for the general case. The following theorem gives the convergence rate of the sequential spectral method for evolution problems of the type (4.1).

Theorem 4.4 (Superlinear Convergence) *Let $F(u)$ satisfy the hypotheses of Theorem 4.3. Then the sequence $\{u^{(n)}(\mathbf{x}, t)\}$ defined by the sequential spectral method (4.3) - (4.8) converges to a solution $u(\mathbf{x}, t)$ of problem (4.1) at the rate*

$$\max_{0\leq t\leq T} ||u - u^{(n)}||_2^2 \leq \frac{(T^2 L_F^2)^n}{n!} \max_{0\leq t\leq T} ||u - u^{(0)}||_2^2. \tag{4.41}$$

Proof Let $u(\mathbf{x}, t)$ be a solution of the problem (4.1), $u(\mathbf{x}, t) = \sum_{j=1}^{\infty} a_j(t)\phi_j(\mathbf{x})$, and let $u^{(n)}(\mathbf{x}, t)$ be the nth iterate of the approximate solution, $u^{(n)}(\mathbf{x}, t) = \sum_{j=1}^{\infty} a_j^{(n)}(t)\phi_j(\mathbf{x})$. Then the difference in the Fourier coefficients satisfies

$$a_j(t) - a_j^{(n)}(t) = \int_0^t e^{-\lambda_j(t-\tau)} \int_\Omega [F(u) - F(u^{(n)})]\phi_j d\mathbf{x} d\tau. \tag{4.42}$$

Following the steps from (4.26) - (4.40), we obtain

$$\max_{0\leq t\leq T} ||u - u^{(n)}||_2^2 \leq \frac{(T^2 L_F^2)^n}{n!} \max_{0\leq t\leq T} ||u - u^{(0)}||_2^2. \tag{4.43}$$

Since $\lim_{n\to\infty} \frac{(T^2L_F^2)^n}{n!} = 0$, the sequence $\{u^{(n)}\}$ converges to u at the rate $\frac{(T^2L_F^2)^n}{n!}$ which shows superlinear convergence of $\{u^{(n)}\}$, since $\frac{(T^2L_F^2)^n}{n!}$ goes to zero faster than any fixed γ^n. ■

The results obtained above are similar to the results for the Picard-Lindelöf iterations for initial value problems (see [Pic93] and [Lin94]).

In Theorem 4.3, we assume $F(u)$ to be globally Lipschitz in L_2. From relation (4.41), we see that all iterates stay in a ball $B(u,r)$, centered at the solution u, with radius $r^2 = \max_{n\in\mathbb{N}}(\frac{(T^2L_F^2)^n}{n!}) \max_{0\le t\le T} ||u-u^{(0)}||_2^2$. So we can relax the global Lipschitz condition, $F(u)$ only needs to be Lipschitz in the ball $B(u,r)$.

Corollary 4.1 *Let $F(u)$ be locally Lipschitz in L_2 in a ball $B(u,r)$,centered at the solution u, with $r^2 = \max_{n\in\mathbb{N}}(\frac{(T^2L_F^2)^n}{n!}) \max_{0\le t\le T} ||u-u^{(0)}||_2^2$, where $u^{(0)}$ is the initial guess, with Lipschitz constant L_F. Then the sequential spectral method (4.3) - (4.8) converges at the rate given in (4.41).*

We emphasize that the conditions in Theorem 4.3 are again sufficient conditions for the iteration to converge and they are not necessary. In order to check this sufficient condition we have to check that $F(u)$ is Lipschitz, but in practice we just run the algorithm and let the numerical results demonstrate whether convergence is achieved or not. The numerical results in Section 4.3 demonstrate that the method captures the essence of the problem, and $a_j^{(0)}(t)$ is within a few percent of $a_j^{(\infty)}(t)$. Convergence to a tolerance 10^{-6} is achieved in only a few iterations.

4.3 Numerical Results

Using the sequential spectral method, we analyze an integro-parabolic boundary value problem arising from combustion theory of thermal explosion [Pao92]. The

governing equations are

$$\begin{aligned} \frac{\partial u}{\partial t} - D\triangle u &= \sigma\left(e^{\gamma u} + b\int_\Omega e^{\gamma u(\mathbf{y},t)}d\mathbf{y}\right), && \mathbf{x}\in\Omega,\ t>0, \\ r(\mathbf{x})\frac{\partial u}{\partial n} + s(\mathbf{x})u &= 0, && \mathbf{x}\in\partial\Omega,\ t>0, \\ u(\mathbf{x},0) &= u_0(\mathbf{x}), && \mathbf{x}\in\Omega, \end{aligned} \tag{4.44}$$

where u is the temperature distribution of a gas, D, σ and γ are positive constants, b is a nonnegative gas constant and $\frac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial\Omega$. Here σ is the Frank-Kamenetski parameter. The last term on the right hand side is due to the compressibility of the gas.

When $b = 0$ and $u_0 = 0$, the existence of a solution is proved in [Bel87]. Bellout also proved that solutions blow-up in finite time for certain values of σ. For $b > 0$, Pao discussed the case of Neumann boundary conditions in [Pao80] and showed that the solution $u(\mathbf{x},t)$ of the problem (4.44) blows-up in finite time for any $\sigma > 0$. The case of homogeneous Dirichlet boundary conditions is discussed in [BB82] and [BE82] and it is proved that there exists a critical value of σ, denoted by σ^*, such that for $\sigma < \sigma^*$, the problem (4.44) has a positive solution which converges to a steady state solution and for $\sigma > \sigma^*$ no positive steady state solution can exist and there exists a finite time T^* such that a unique solution $u(\mathbf{x},t)$ exists in $\overline{\Omega}\times[0,T^*)$ and $\lim_{t\to T^*}\{\max_{\mathbf{x}\in\overline{\Omega}} u(\mathbf{x},t)\} = \infty$. Pao proved the same results for the more general case of b and u_0 being nonnegative, not necessarily zero, in [Pao92].

We will do numerical computations for the one dimensional case with homogeneous Dirichlet boundary conditions

$$\begin{aligned} \tfrac{\partial u}{\partial t} - Du_{xx} &= \sigma[e^{\gamma u} + b\int_0^1 e^{\gamma u(y,t)}dy], && x\in(0,1),\ t>0, \\ u(0,t) &= u(1,t) = 0, && t>0, \\ u(x,0) &= 0, && x\in(0,1). \end{aligned} \tag{4.45}$$

Existence of a nonnegative and nondecreasing solution of the problem (4.45) is obtained by Theorem 4.2. We will solve the problem for the values $\sigma = 1/2$, $b = D = \gamma = 1$. The orthonormal eigenfunctions $\{\phi_j\}$ and the corresponding eigenvalues $\{\lambda_j\}$ satisfy

$$\frac{\partial^2 \phi_j}{\partial x^2} = -\lambda_j \phi_j, \quad \text{where } \phi_j(0) = \phi_j(1) = 0$$

and hence the eigenfunctions and the eigenvalues are

$$\phi_j = \sqrt{2}\sin(j\pi x), \quad \lambda_j = j^2\pi^2, \quad j = 1, 2, 3, \ldots .$$

To use the sequential spectral method (SSM), we expand the initial condition $0 = u(x,0) = \sum_{j=1}^{\infty} c_j \phi_j(x)$, which gives $c_j = 0$ for all values of j. Using the sequential spectral method, we found that the even Fourier coefficients satisfy $|a_{2j}^{(0)}(t)| < 10^{-20}$ for $j = 1, 2, 3, 4$, $t > 0$ and hence decided to investigate this further:

Lemma 4.1 *Let $u^{(n)}(\mathbf{x}, t) = \sum_{j=1}^{\infty} a_j^{(n)}(t)\phi_j(\mathbf{x})$ be the nth approximation of solution of problem (4.45), obtained by the SSM, then the even Fourier coefficients $a_{2j}^{(n)}(t) = 0$, $\forall\, j > 0, t > 0, n \in \mathbb{N}$.*

Proof The proof is technical and is given in the Appendix A. ■

Since by the above lemma the even Fourier coefficients are zero, we drop them from the expansion. Therefore, using the Galerkin method to solve the above problem, the procedure reduces the problem of solving a system of $2N$ nonlinear ordinary differential equations to the solution of a system of N nonlinear ordinary differential equations. Note that this kind of observation can not be obtained directly by using the Galerkin method.

For $N = 10$, Tables 4.1 to 4.5 show a comparison between the coefficients

	Galerkin	SSM	
t	$g_1(t)$	$a_1^{(0)}(t)$	$a_1^{(6)}(t)$
0.05	0.036309792	0.036295110	0.036309792
0.10	0.059549494	0.059514302	0.059549494
0.50	0.099915189	0.099861831	0.099915189
1.00	0.101112835	0.101050321	0.101112835
1.50	0.101116171	0.101057200	0.101116171

Table 4.1: Comparison of the coefficients $a_1(t)$ and $g_1(t)$.

$g_j(t)$ and $a_j(t)$ obtained by using the Galerkin method and the sequential spectral method for different values of t. We have used an adaptive Runge-Kutta method of order 4 with error control of order 5 to solve the ordinary differential equations. The results show that the coefficients $a_j^{(0)}(t)$ are very close to the Galerkin coefficients $g_j(t)$ and in only 6 iterations $a_j(t)$ converges to the Galerkin coefficients to a tolerance of 10^{-6}. The solution to the problem is given by $u(x,t) = \sum_{j=1}^{5} g_{2j-1}(t)\phi_{2j-1}(x)$. Figures 4.2 to 4.6 also show that there is a very small difference in the initial coefficients $a_j^{(0)}(t)$ and $g_j(t)$ already, for $j = 1, 3, 5, 7, 9$. The steady state for $a_j^{(0)}(t)$ is reached at $t = 1$ except for $a_1^{(0)}(t)$. The errors $|g_j(t) - a_j^{(0)}(t)|$ are very small and the maximum values of the errors for $j = 1, 3, 5, 7, 9$ are given in the captions of Figures 4.2 to 4.6.

We note from the above results that the initial coefficients $a_j^{(0)}(t)$, for $j = 1, 3, 5, \ldots$ is a decreasing sequence and also $a_j^{(6)}(t)$ is decreasing for the given problem. Figure 4.7 shows $a_1(t)$ and $a_3(t)$, at the initial step and then at the final step, which indicates that the coefficient of the first eigenfunction is the most important. This is because the eigenfunctions are intrinsic to the operator, the domain, and the boundary conditions and leads to the spectacular performance of spectral methods in general.

	Galerkin	SSM	
t	$g_3(t)$	$a_3^{(0)}(t)$	$a_3^{(6)}(t)$
0.05	0.003403936	0.003403852	0.003403936
0.10	0.003484777	0.003484172	0.003484777
0.50	0.003555878	0.003557170	0.003555878
1.00	0.003557947	0.003557664	0.003557947

Table 4.2: Comparison of the coefficients $a_3(t)$ and $g_3(t)$.

	Galerkin	SSM	
t	$g_5(t)$	$a_5^{(0)}(t)$	$a_5^{(6)}(t)$
0.05	7.43001×10^{-4}	7.43029×10^{-4}	7.43001×10^{-4}
0.10	7.51272×10^{-4}	7.51463×10^{-4}	7.51272×10^{-4}
0.50	7.66064×10^{-4}	7.66340×10^{-4}	7.66064×10^{-4}
1.00	7.66504×10^{-4}	7.66482×10^{-4}	7.66504×10^{-4}

Table 4.3: Comparison of the coefficients $a_5(t)$ and $g_5(t)$.

	Galerkin	SSM	
t	$g_7(t)$	$a_7^{(0)}(t)$	$a_7^{(6)}(t)$
0.05	2.70664×10^{-4}	2.70804×10^{-4}	2.70664×10^{-4}
0.10	2.73640×10^{-4}	2.73290×10^{-4}	2.73640×10^{-4}
0.50	2.78979×10^{-4}	2.79383×10^{-4}	2.78979×10^{-4}
1.00	2.79140×10^{-4}	2.78768×10^{-4}	2.79140×10^{-4}

Table 4.4: Comparison of the coefficients $a_7(t)$ and $g_7(t)$.

	Galerkin	SSM	
t	$g_9(t)$	$a_9^{(0)}(t)$	$a_9^{(6)}(t)$
0.05	1.27363×10^{-4}	1.27451×10^{-4}	1.27363×10^{-4}
0.10	1.28719×10^{-4}	1.29053×10^{-4}	1.28719×10^{-4}
0.50	1.31221×10^{-4}	1.31048×10^{-4}	1.31221×10^{-4}
1.00	1.31296×10^{-4}	1.31279×10^{-4}	1.31296×10^{-4}

Table 4.5: Comparison of the coefficients $a_9(t)$ and $g_9(t)$.

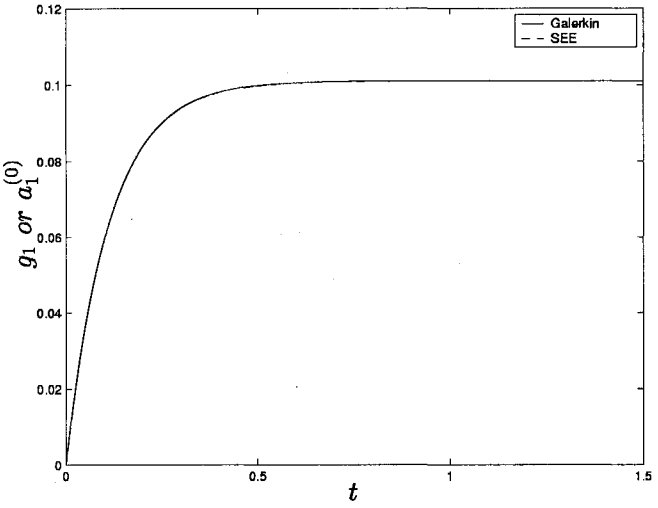

Figure 4.2: Comparison of initial coefficients $a_1^{(0)}(t)$ and the Galerkin coefficient $g_1(t)$. The maximum difference is $\max_t |g_1(t) - a_1^{(0)}(t)| = 6.8567 \times 10^{-5}$.

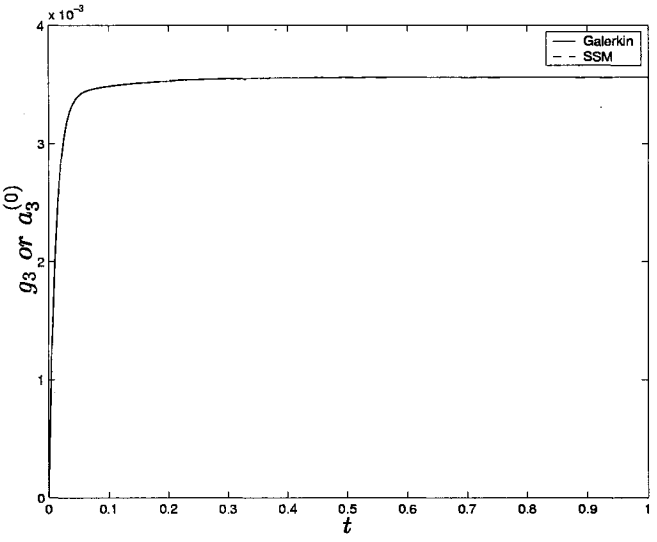

Figure 4.3: Comparison of initial coefficient $a_3^{(0)}(t)$ and the Galerkin coefficient $g_3(t)$. The maximum difference is $\max_t |g_3(t) - a_3^{(0)}(t)| = 4.3968 \times 10^{-6}$.

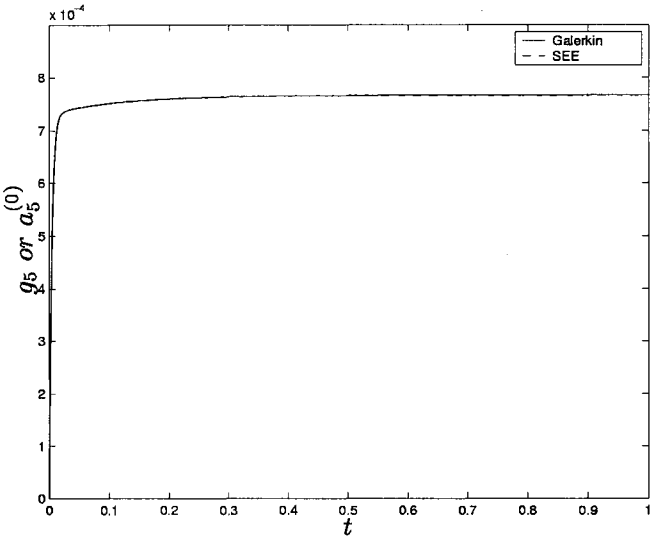

Figure 4.4: Comparison of initial coefficient $a_5^{(0)}(t)$ and the Galerkin coefficient $g_5(t)$. The maximum difference is $\max_t |g_5(t) - a_5^{(0)}(t)| = 1.0251 \times 10^{-6}$.

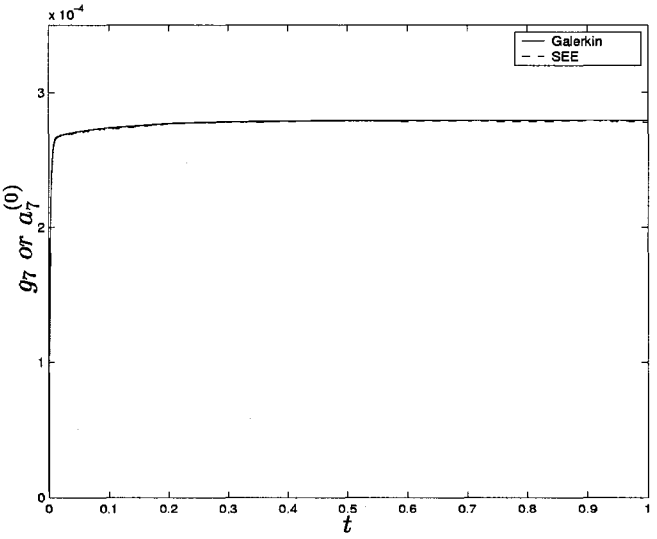

Figure 4.5: Comparison of initial coefficient $a_7^{(0)}(t)$ and the Galerkin coefficient $g_7(t)$. The maximum difference is $\max_t |g_7(t) - a_7^{(0)}(t)| = 1.1768 \times 10^{-6}$.

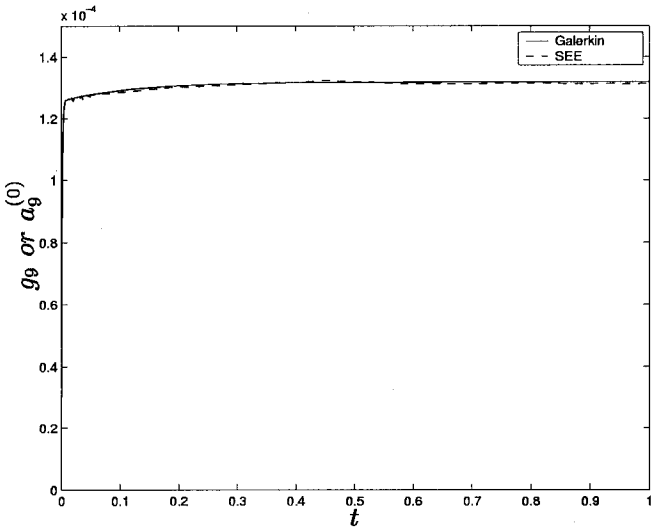

Figure 4.6: Comparison of initial coefficient $a_9^{(0)}(t)$ and the Galerkin coefficient $g_9(t)$, here $\max_t |g_9(t) - a_9^{(0)}(t)| = 2.1151 \times 10^{-6}$.

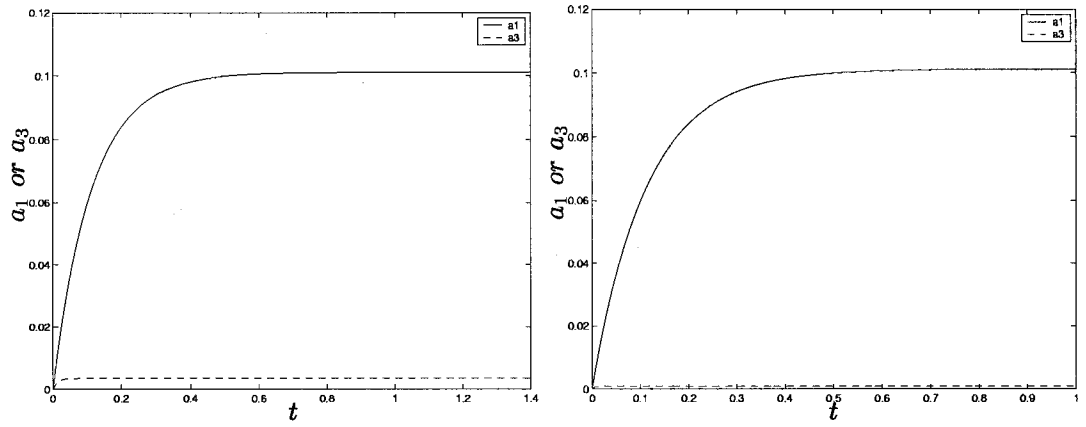

Figure 4.7: Comparison of the values of $a_1^{(0)}(t)$ and $a_3^{(0)}(t)$ on the left and on the right comparison of the values of $a_1^{(6)}(t)$ and $a_3^{(6)}(t)$. The first eigenfunction is dominant in the solution

Using the sequential spectral method, we have to solve a single ordinary differential equation at each step, which is easier than solving N simultaneous ordinary differential equations, obtained by using the Galerkin method, especially when N is large and implicit methods are used. For $N = 10$, the computational time for the Galerkin method is about eight times the time required by the sequential spectral method. The ratio of the computational time grows as N becomes large because the Galerkin method takes much more time for large N, but in our method we always have to solve one single ordinary differential equation regardless of the value of N.

According to the results by Pao in [Pao92] the problem (4.45) has a positive solution for $\sigma < \sigma^*$, which converges to a steady state solution. We applied the iteration scheme from Chapter 3 to find a steady state solution of the problem (4.45) for the values $\sigma = 1/2$, $b = D = \gamma = 1$. Using the sequential spectral method, we again found that $|a_j^{(0)}| < 10^{-30}$ for j even in steady case also, so a_j are important for only odd values of j. This information reduces the problem of solving a system of $2N$ algebraic equations to a system of N equations in the Galerkin method. Table 4.6 shows the numerical results obtained using the Galerkin and the sequential spectral method for $N = 10$. The result shows that the initial coefficients $a_j^{(0)}$ are not very far from the Galerkin coefficients g_j and in only 6 iterations a_j converge to the Galerkin solution. We see that the solution for the problem (4.45) converges to this steady state solution. Figure 4.8 also shows that there is a very small difference between the solution obtained from the initial coefficients $a_j^{(0)}$ and the Galerkin coefficients g_j, $|u^g - u^{(0)}| \leq 10^{-4}$ for all values of $x \in \Omega$.

Equation (4.44) has been studied in detail in [Pao92] to establish existence of a solution and Pao has derived a bound for the critical value σ^* given by

$$\sigma^* < \frac{\lambda_0}{\gamma e(1 + b|\Omega|/\Phi)} \tag{4.46}$$

	Galerkin	SSM	
j	g_j	$a_j^{(0)}$	$a_j^{(6)}$
1	0.101116171	0.1010572002	0.101116171
3	0.003557947	0.0035576649	0.003557947
5	0.000766504	0.0007664819	0.000766504
7	0.000279139	0.0002791332	0.000279139
9	0.000131298	0.0001312960	0.000131298

Table 4.6: Comparison of the coefficients obtained by the Galerkin and the sequential spectral method.

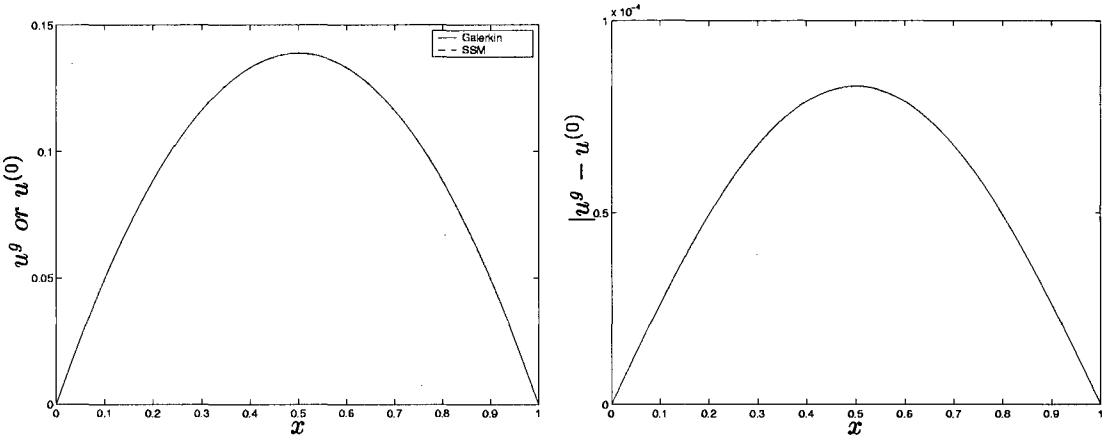

Figure 4.8: Comparison of $u^g = \sum_{j=1}^{10} g_j \phi_j$ and $u^{(0)} = \sum_{j=1}^{10} a_j^{(0)} \phi_j$ and the error $|u^g - u^{(0)}|$.

b	$\gamma = 1/4$	$\gamma = 1/2$	$\gamma = 1$	$\gamma = 2$	$\gamma = 3$
1/4	[11.548,11.549]	[5.774,5.775]	[2.887,2.888]	[1.443,1.444]	[0.962,0.963]
1/2	[9.811,9.812]	[4.905,4.906]	[2.452,2.453]	[1.226,1.227]	[0.817,0.818]
1	[7.544,7.545]	[3.772,3.773]	[1.886,1.887]	[0.943,0.944]	[0.628,0.629]
2	[5.162,5.163]	[2.581,2.582]	[1.290,1.291]	[0.645,0.646]	[0.430,0.431]
3	[3.924,3.925]	[1.962,1.963]	[0.981,0.982]	[0.490,0.491]	[0.327,0.328]

Table 4.7: Range for values of $\tilde{\sigma} = \sigma^*/D$

where λ_0 is the principal eigenvalue of the eigenvalue problem

$$D\triangle\phi + \lambda\phi = 0, \quad \mathcal{B}\phi = 0 \ \text{ on } \partial\Omega,$$

and $\phi(\mathbf{x})$ is the eigenfunction corresponding to λ_0 and it is normalized such that $|\Omega|^{-1}\int_\Omega \phi dx = 1$. Φ is defined by $\Phi = \max_{\mathbf{x}\in\overline{\Omega}}\phi(\mathbf{x})$. We consider the ordinary differential equation for finding the coefficient $a_1^{(0)}(t)$ to estimate the critical value σ^*. Table 4.7 shows ranges for σ^* for certain values of D, γ and b. These results show that σ^* is directly proportional to D and inversely proportional to γ. Figure 4.9 shows the dependence of σ^* on b obtained by Pao and by using the first equation in the SSM, the estimates are very close for small values of b. The functional dependence of σ^* obtained in this way agrees with the functional dependence obtained by Pao. This indicates that only by looking at a single equation for $a_1^{(0)}(t)$, we can get an estimate of the critical value of σ^* for the problem, which is not easy to obtain by using the Galerkin method.

It has been proved in [Pao92] and [BE82], that for $\sigma > \sigma^*$, the solution of the problem (4.45) will blow up in finite time, $t = T^*$. We tried to estimate the dependence of T^* on σ, by looking again at the single ordinary differential equation obtained for solving $a_1^{(0)}(t)$. For $D = b = \gamma = 1$, the values of T^* for different values of σ are shown in Table 4.8. The dependence of T^* on σ is also shown in Figure 4.10.

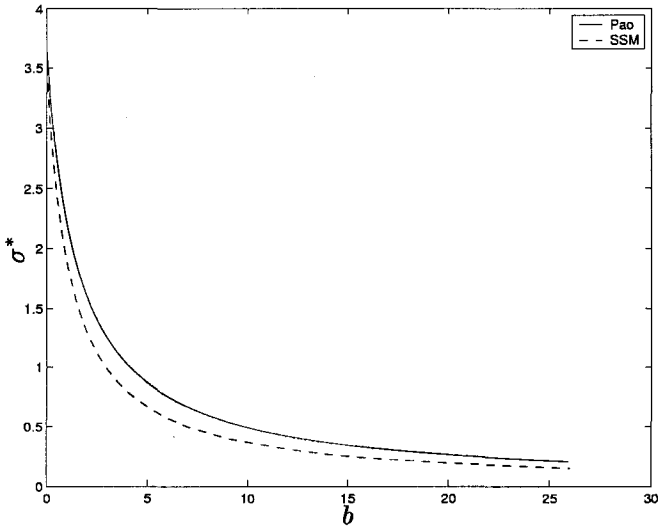

Figure 4.9: The functional dependence of the critical value σ^* on b obtained analytically by Pao and by the SSM.

σ	T^*
1.887	20.2213
1.89	9.28358
1.90	4.75030
2.00	1.47360
2.50	0.49790
5.00	0.13790
10.0	0.05790

Table 4.8: The values of T^* for $D = b = \gamma = 1$.

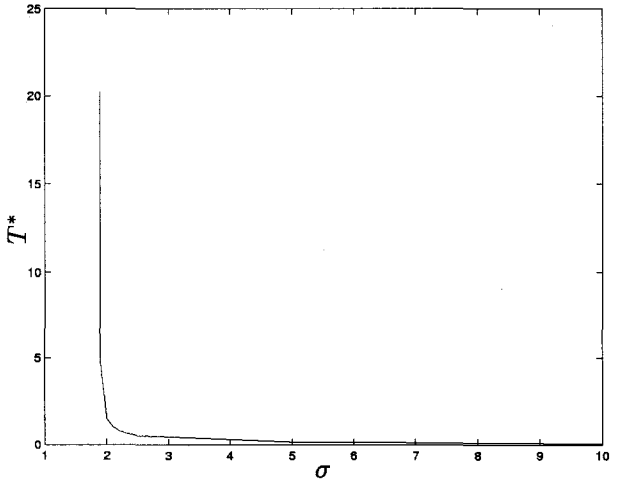

Figure 4.10: The values of critical time T^* for different values of σ.

Chapter 5

Systems of Nonlinear Elliptic Partial Differential Equations

5.1 Introduction

We introduced the sequential spectral method in Chapter 2 and applied the algorithm to nonlinear elliptic and parabolic integro-differential equations in Chapters 3 and 4. The idea was first introduced by Tam *et al.* in [TAK96] to solve a semilinear elliptic partial differential equation arising from microwave heating. In this chapter, we extend the idea to a system of nonlinear elliptic partial differential equations. To facilitate the presentation, we consider a system of only two equations. The method can however be applied to a system of N equations in the same manner. We develop the procedure in Section 5.2. In Section 5.3, we analyze the convergence of the method in the case of identical and different boundary conditions for u and v. The numerical results are presented in Section 5.4 and a comparison with the Galerkin method is performed.

5.2 The Expansion Procedure

We consider the system of nonlinear elliptic partial differential equations in a domain Ω,

$$\begin{aligned} \triangle u + F(u,v) &= 0, \\ \triangle v + G(u,v) &= 0, \end{aligned} \tag{5.1}$$

subject to homogeneous Dirichlet or Robin boundary conditions

$$\begin{aligned} \mathcal{A}u \equiv r_1(\mathbf{x})\frac{\partial u}{\partial n} + s_1(\mathbf{x})u = 0, \quad \mathbf{x} \in \partial\Omega, \\ \mathcal{B}v \equiv r_2(\mathbf{x})\frac{\partial v}{\partial n} + s_2(\mathbf{x})v = 0, \quad \mathbf{x} \in \partial\Omega, \end{aligned} \tag{5.2}$$

where $\frac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial\Omega$, the boundary of Ω. The boundary functions $r_1(\mathbf{x})$ and $r_2(\mathbf{x})$ are nonnegative and $s_1(\mathbf{x})$ and $s_2(\mathbf{x})$ are positive. The functions F and G are nonlinear functions of u and v. Note that we have not included the case of Neumann boundary conditions. In fact, in the case of Neumann conditions, we can have a zero eigenvalue, which will not allow us to define the iteration scheme (5.17) or (5.18). Therefore, we are constructing the procedure under the assumption that the eigenvalues are positive. However, as discussed in Section 2.2.1, there are certain special forms of the functions F and G which can allow us to apply the method in the case of a zero eigenvalue also and we discuss those in Section 5.3.1. We develop the algorithm for the general case, that is, when u and v satisfy different boundary conditions. If the boundary conditions are identical, $\mathcal{A} = \mathcal{B}$, then the algebra is simpler.

Let $\{\lambda_j\}$ and $\{\phi_j(\mathbf{x})\}$ be the eigenvalues and the orthonormal eigenfunctions for the eigenvalue problem

$$\triangle\phi = -\lambda\phi, \quad \mathcal{A}\phi = 0, \quad \mathbf{x} \in \partial\Omega \tag{5.3}$$

and let $\{\mu_j\}$ and $\{\psi_j(\mathbf{x})\}$ be the eigenvalues and the orthonormal eigenfunctions for the eigenvalue problem

$$\triangle\psi = -\mu\psi, \quad \mathcal{B}\psi = 0, \quad \mathbf{x} \in \partial\Omega. \tag{5.4}$$

In the case of identical boundary conditions, $\lambda_j = \mu_j$ and $\phi_j(\mathbf{x}) = \psi_j(\mathbf{x})$.
We denote

$$\begin{aligned} Q(u) &:= \triangle u + F(u,v), \\ R(v) &:= \triangle v + G(u,v). \end{aligned} \tag{5.5}$$

As in the case of integro-differential equations, we use $\tilde{u}$ and $\tilde{v}$ as generic symbols for the approximations of u and v. To begin, we take $\tilde{u} = a_1\phi_1(\mathbf{x})$ and $\tilde{v} = c_1\psi_1(\mathbf{x})$. Then the residuals become

$$\begin{aligned} Q(\tilde{u}) &= -\lambda_1 a_1\phi_1 + F(a_1\phi_1, c_1\psi_1), \\ R(\tilde{v}) &= -\mu_1 c_1\psi_1 + G(a_1\phi_1, c_1\psi_1). \end{aligned} \tag{5.6}$$

Using the notation of the inner product of two functions, and following the Galerkin idea, we set $(Q(\tilde{u}), \phi_1) = 0$ and $(R(\tilde{v}), \psi_1) = 0$. These conditions give

$$\begin{aligned} -\lambda_1 a_1 + (F(a_1\phi_1, c_1\psi_1), \phi_1) &= 0, \\ -\mu_1 c_1 + (G(a_1\phi_1, c_1\psi_1), \psi_1) &= 0. \end{aligned} \tag{5.7}$$

Solving this pair of algebraic equations, we find the values of a_1 and c_1. This system may have more than one solution, in which case, each solution will give rise to an expansion for u and v. To proceed, we focus on one solution. After finding the values of a_1 and c_1, we next take $\tilde{u} = a_1\phi_1(\mathbf{x}) + a_2\phi_2(\mathbf{x})$ and $\tilde{v} = c_1\psi_1(\mathbf{x}) + c_2\psi_2(\mathbf{x})$. Now we set the projection of $Q(\tilde{u})$ onto ϕ_2 and the projection of $R(\tilde{v})$ onto ψ_2 to be zero, that is, $(Q(\tilde{u}), \phi_2) = 0$ and $(R(\tilde{v}), \psi_2) = 0$. These

conditions and the orthonormality of $\{\phi_i(\mathbf{x})\}$ and $\{\psi_i(\mathbf{x})\}$ lead to

$$\begin{aligned} -\lambda_2 a_2 + (F(a_1\phi_1 + a_2\phi_2, c_1\psi_1 + c_2\psi_2), \phi_2) &= 0, \\ -\mu_2 c_2 + (G(a_1\phi_1 + a_2\phi_2, c_1\psi_1 + c_2\psi_2), \psi_2) &= 0. \end{aligned} \tag{5.8}$$

We find the values of a_2 and c_2 by solving this system. In the case of multiple solutions, we have to take one at a time. Proceeding in the same manner, we generate a sequence of coefficients $\{a_j\}$ and $\{c_j\}$ and consider formally

$$\tilde{u} = \sum_{j=1}^{\infty} a_j\phi_j(\mathbf{x}) \quad \text{and} \quad \tilde{v} = \sum_{j=1}^{\infty} c_j\psi_j(\mathbf{x}).$$

Then the residuals are given by

$$\begin{aligned} Q(\tilde{u}) &= \sum_{j=1}^{\infty} -\lambda_j a_j\phi_j + F(\sum_{j=1}^{\infty} a_j\phi_j, \sum_{j=1}^{\infty} c_j\psi_j), \\ R(\tilde{v}) &= \sum_{j=1}^{\infty} -\mu_j c_j\psi_j + G(\sum_{j=1}^{\infty} a_j\phi_j, \sum_{j=1}^{\infty} c_j\psi_j). \end{aligned} \tag{5.9}$$

Now using the completeness of the eigenfunctions $\{\phi_j\}$ and $\{\psi_j\}$ we can expand

$$F(\sum_{j=1}^{\infty} a_j\phi_j, \sum_{j=1}^{\infty} c_j\psi_j) = \sum_{k=1}^{\infty} b_k\phi_k(\mathbf{x}) \tag{5.10}$$

and

$$G(\sum_{j=1}^{\infty} a_j\phi_j, \sum_{j=1}^{\infty} c_j\psi_j) = \sum_{k=1}^{\infty} d_k\psi_k(\mathbf{x}), \tag{5.11}$$

where the coefficients b_k and d_k are given by

$$\begin{aligned} b_k &= (F(\sum_{j=1}^{\infty} a_j\phi_j, \sum_{j=1}^{\infty} c_j\psi_j), \phi_k), \\ d_k &= (G(\sum_{j=1}^{\infty} a_j\phi_j, \sum_{j=1}^{\infty} c_j\psi_j), \psi_k). \end{aligned} \tag{5.12}$$

Then (5.9) becomes

$$\begin{aligned} Q(\tilde{u}) &= \sum_{j=1}^{\infty}(-\lambda_j a_j + b_j)\phi_j, \\ R(\tilde{v}) &= \sum_{j=1}^{\infty}(-\mu_j c_j + d_j)\psi_j. \end{aligned} \tag{5.13}$$

For $\tilde{u}$ and $\tilde{v}$ to be a solution, we need $(Q(\tilde{u}), \phi_j) = 0$ and $(R(\tilde{v}), \psi_j) = 0$, for all j, as in the Galerkin method. This means that we need

$$b_j - \lambda_j a_j = 0 \quad \text{and} \quad d_j - \mu_j c_j = 0, \quad \text{for all } j. \tag{5.14}$$

If the above conditions hold then (5.13) implies that $Q(\tilde{u}) = 0$ and $R(\tilde{v}) = 0$, and so $\tilde{u}$ and $\tilde{v}$ will be a true solution. But for the a_j, b_j, c_j and d_j just computed, (5.14) will not hold in general and we need to introduce an iteration scheme to achieve the required equality (5.14). Let $u^{(n)}$ and $v^{(n)}$ denote the nth approximations of u and v, given by

$$\begin{aligned} u^{(n)}(\mathbf{x}) &= \sum_{j=1}^{\infty} a_j^{(n)}\phi_j(\mathbf{x}), \\ v^{(n)}(\mathbf{x}) &= \sum_{j=1}^{\infty} c_j^{(n)}\psi_j(\mathbf{x}). \end{aligned} \tag{5.15}$$

and $b_j^{(n)}$, $d_j^{(n)}$ be the coefficients in the expansion

$$\begin{aligned} F(u^{(n)}, v^{(n)}) &= \sum_{j=1}^{\infty} b_j^{(n)} \phi_j(\mathbf{x}), \\ G(u^{(n)}, v^{(n)}) &= \sum_{j=1}^{\infty} d_j^{(n)} \psi_j(\mathbf{x}). \end{aligned} \tag{5.16}$$

We construct new sequences $\{a_j^{(n+1)}\}$ and $\{c_j^{(n+1)}\}$ by defining

$$a_j^{(n+1)} = \frac{1}{\lambda_j} b_j^{(n)} = \frac{1}{\lambda_j} (F(u^{(n)}, v^{(n)}), \phi_j) \tag{5.17}$$

and

$$c_j^{(n+1)} = \frac{1}{\mu_j} d_j^{(n)} = \frac{1}{\mu_j} (G(u^{(n)}, v^{(n)}), \psi_j). \tag{5.18}$$

Using these new values of the coefficients, we form the new iterates

$$u^{(n+1)}(\mathbf{x}) = \sum_{j=1}^{\infty} a_j^{(n+1)} \phi_j(\mathbf{x}) \quad \text{and} \quad v^{(n+1)}(\mathbf{x}) = \sum_{j=1}^{\infty} c_j^{(n+1)} \psi_j(\mathbf{x}). \tag{5.19}$$

We denote the values of $\{a_j\}$ and $\{c_j\}$ obtained by the decoupling procedure by $\{a_j^{(0)}\}$ and $\{c_j^{(0)}\}$ to have the initial guess $u^{(0)}$ and $v^{(0)}$ for the iteration scheme. In fact, depending on prior knowledge of the solution, any other initial guess could also be chosen. If the iteration scheme (5.15) - (5.19) converges, then $Q(u^{(\infty)}) = 0$ and $R(v^{(\infty)}) = 0$ and hence a solution of the boundary value problem (5.1)-(5.2) will be obtained. The numerical work in Section 5.4 indicates that the method captures the essence of the problem and $a_j^{(0)}$ and $c_j^{(0)}$ are within a few percent of $a_j^{(\infty)}$ and $c_j^{(\infty)}$, and we get convergence to a tolerance of 10^{-6} in only a few iterations.

5.3 Convergence Analysis

In this section we analyze the convergence of the sequential spectral method (5.15)-(5.19). We need the following result for the convergence of sequences of vectors.

Theorem 5.1 *Let* $\{\mathbf{x}^{(n)}\}$ *be a nonnegative sequence satisfying* $\mathbf{x}^{(n)} \leq \mathbf{A}\mathbf{x}^{(n-1)}$, *where* $\mathbf{A}$ *is any* $n \times n$ *matrix. If the spectral radius* $\rho(\mathbf{A})$ *is less than 1, then the sequence* $\{\mathbf{x}^{(n)}\}$ *converges to* $\mathbf{0}$ *for any initial guess* $\mathbf{x}^{(0)}$.

Proof From $\mathbf{x}^{(n)} \leq \mathbf{A}\mathbf{x}^{(n-1)}$ we have by induction

$$\mathbf{x}^{(n)} \leq \mathbf{A}\mathbf{x}^{(n-1)} \leq \mathbf{A}^2\mathbf{x}^{(n-2)} \leq \cdots \leq \mathbf{A}^n\mathbf{x}^{(0)}.$$

Taking the limit, we have

$$\lim_{n\to\infty} \mathbf{x}^{(n)} \leq \lim_{n\to\infty} \mathbf{A}^n\mathbf{x}^{(0)}. \tag{5.20}$$

By Theorem 1.5, $\lim_{n\to\infty} \mathbf{A}^n = 0$ if $\rho(\mathbf{A}) < 1$, therefore $\lim_{n\to\infty} \mathbf{x}^{(n)} = 0$, because $\{\mathbf{x}^{(n)}\}$ is a nonnegative sequence. Hence the sequence $\{\mathbf{x}^{(n)}\}$ converges to zero. ■

In order to investigate convergence of the sequential spectral method, we consider the difference of the corresponding coefficients a_j and c_j between two iteration steps,

$$a_j^{(n+1)} - a_j^{(n)} = \frac{1}{\lambda_j}\int_\Omega [F(u^{(n)}, v^{(n)}) - F(u^{(n-1)}, v^{(n-1)})]\phi_j d\mathbf{x} \tag{5.21}$$

and

$$c_j^{(n+1)} - c_j^{(n)} = \frac{1}{\mu_j}\int_\Omega [G(u^{(n)}, v^{(n)}) - G(u^{(n-1)}, v^{(n-1)})]\psi_j d\mathbf{x}. \tag{5.22}$$

By applying the Mean Value Theorem to equations (5.21) and (5.22) and then applying the Mean Value Theorem for Integrals under the assumptions that $(u^{(n)} - u^{(n-1)})\phi_j$, $(u^{(n)} - u^{(n-1)})\psi_j$, $(v^{(n)} - v^{(n-1)})\phi_j$ and $(v^{(n)} - v^{(n-1)})\psi_j$ do not change sign, we can prove convergence of the iteration scheme as we have done in Chapter 3. We observed in Chapter 3 that the conditions of convergence in this particular case are in accordance with the conditions obtained in the general case. Therefore here we describe only the general case. Of course, the results will imply convergence for any particular case. Let

$$\begin{aligned} g^{(n)}(\mathbf{x}) :=& F(u^{(n)}) - F(u^{(n-1)}), \\ p^{(n)}(\mathbf{x}) :=& G(u^{(n)}) - G(u^{(n-1)}). \end{aligned} \tag{5.23}$$

Since each of the set of eigenfunctions $\{\phi_j\}$ and $\{\psi_j\}$ forms a complete set of orthonormal functions (Theorem 1.4), we can expand $g^{(n)}$ and $p^{(n)}$ in a Fourier series in $\{\phi_j\}$ and $\{\psi_j\}$ (Theorem 1.3) respectively,

$$\begin{aligned} g^{(n)}(\mathbf{x}) &= \sum_{j=1}^{\infty} g_j^{(n)} \phi_j(\mathbf{x}), \\ p^{(n)}(\mathbf{x}) &= \sum_{j=1}^{\infty} p_j^{(n)} \psi_j(\mathbf{x}), \end{aligned} \tag{5.24}$$

where the Fourier coefficients $g_j^{(n)}$ and $p_j^{(n)}$ are given by

$$\begin{aligned} g_j^{(n)} &= (g^{(n)}, \phi_j) = ([F(u^{(n)}, v^{(n)}) - F(u^{(n-1)}, v^{(n-1)})], \phi_j), \\ p_j^{(n)} &= (p^{(n)}, \psi_j) = ([G(u^{(n)}, v^{(n)}) - G(u^{(n-1)}, v^{(n-1)})], \psi_j). \end{aligned} \tag{5.25}$$

Using the definitions in (5.23), the relations (5.21) and (5.22) become

$$\begin{aligned} a_j^{(n+1)} - a_j^{(n)} &= \frac{1}{\lambda_j} g_j^{(n)}, \\ c_j^{(n+1)} - c_j^{(n)} &= \frac{1}{\mu_j} p_j^{(n)}. \end{aligned} \tag{5.26}$$

Multiplying both sides of the first equation with ϕ_j, $j = 1, 2, 3, \ldots$ and of the second equation with ψ_j, $j = 1, 2, 3, \ldots$ and summing over j we obtain

$$\begin{aligned} \sum_{j=1}^{\infty} \left(a_j^{(n+1)} - a_j^{(n)} \right) \phi_j &= \sum_{j=1}^{\infty} \frac{g_j^{(n)}}{\lambda_j} \phi_j, \\ \sum_{j=1}^{\infty} \left(c_j^{(n+1)} - c_j^{(n)} \right) \psi_j &= \sum_{j=1}^{\infty} \frac{p_j^{(n)}}{\mu_j} \psi_j. \end{aligned} \tag{5.27}$$

We now define the new functions $h^{(n)}(\mathbf{x})$ and $q^{(n)}(\mathbf{x})$,

$$\begin{aligned} h^{(n)}(\mathbf{x}) &:= \sum_{j=1}^{\infty} \frac{g_j^{(n)}}{\lambda_j} \phi_j(\mathbf{x}), \\ q^{(n)}(\mathbf{x}) &:= \sum_{j=1}^{\infty} \frac{p_j^{(n)}}{\mu_j} \psi_j(\mathbf{x}), \end{aligned} \tag{5.28}$$

where $\frac{g_j^{(n)}}{\lambda_j}$ and $\frac{p_j^{(n)}}{\mu_j}$ are the Fourier coefficients of $h^{(n)}(\mathbf{x})$ and $q^{(n)}(\mathbf{x})$ respectively. Using the definitions of $u^{(n)}$, $u^{(n+1)}$, $v^{(n)}$ and $v^{(n+1)}$ from relations (5.15) and (5.19), in the left hand side of equation (5.27), we obtain

$$\begin{aligned} u^{(n+1)} - u^{(n)} &= h^{(n)}, \\ v^{(n+1)} - v^{(n)} &= q^{(n)}. \end{aligned} \tag{5.29}$$

Taking the L_2 norm squared on both sides we have

$$\begin{aligned} ||u^{(n+1)} - u^{(n)}||_2^2 &= ||h^{(n)}||_2^2, \\ ||v^{(n+1)} - v^{(n)}||_2^2 &= ||q^{(n)}||_2^2. \end{aligned} \tag{5.30}$$

Now we consider only the first equation in the above relation, that is, $||u^{(n+1)} - u^{(n)}||_2^2 = ||h^{(n)}||_2^2$. Parseval's identity holds for the complete set of

orthonormal eigenfunctions $\{\phi_j\}$ (Theorem 1.3), therefore we obtain

$$||u^{(n+1)} - u^{(n)}||_2^2 = \sum_{j=1}^{\infty} |\frac{g_j^{(n)}}{\lambda_j}|^2 \leq \frac{1}{\lambda_1^2} \sum_{j=1}^{\infty} |g_j^{(n)}|^2, \tag{5.31}$$

because the eigenvalues $\{\lambda_j\}$ are positive and increasing. Since $g_j^{(n)}$ are the Fourier coefficients of the function $g^{(n)}(\mathbf{x})$, using Parseval's identity again we obtain

$$||u^{(n+1)} - u^{(n)}||_2^2 \leq \frac{1}{\lambda_1^2} ||g^{(n)}||_2^2,$$

or taking the square root we get

$$||u^{(n+1)} - u^{(n)}||_2 \leq \frac{1}{\lambda_1} ||g^{(n)}||_2. \tag{5.32}$$

From relation (5.23), we have

$$||g^{(n)}||_2 = ||F(u^{(n)}, v^{(n)}) - F(u^{(n-1)}, v^{(n-1)})||_2. \tag{5.33}$$

Assuming $F(u, v)$ to be globally Lipschitz in both u, v, with Lipschitz constant L_u and L_v respectively,

$$\begin{aligned} ||F(u^{(n)}, v^{(n)}) - F(u^{(n-1)}, v^{(n)})||_2 &\leq L_u ||u^{(n)} - u^{(n-1)}||_2, \\ ||F(u^{(n)}, v^{(n)}) - F(u^{(n)}, v^{(n-1)})||_2 &\leq L_v ||v^{(n)} - v^{(n-1)}||_2. \end{aligned} \tag{5.34}$$

Using the triangle inequality and the above assumption, with $L_F = \max(L_u, L_v)$, we have

$$||F(u^{(n)}, v^{(n)}) - F(u^{(n-1)}, v^{(n-1)})||_2 \leq L_F(||u^{(n)} - u^{(n-1)}||_2 + ||v^{(n)} - v^{(n-1)}||_2). \tag{5.35}$$

Using this assumption in (5.33), we obtain

$$||g^{(n)}||_2 \leq L_F(||u^{(n)} - u^{(n-1)}||_2 + ||v^{(n)} - v^{(n-1)}||_2). \tag{5.36}$$

Inserting this result into (5.32), we have

$$||u^{(n+1)} - u^{(n)}||_2 \leq \frac{L_F}{\lambda_1}\left(||u^{(n)} - u^{(n-1)}||_2 + ||v^{(n)} - v^{(n-1)}||_2\right). \tag{5.37}$$

Performing operations similar to (5.31)-(5.37) the second relation in (5.30) gives

$$||v^{(n+1)} - v^{(n)}||_2 \leq \frac{L_G}{\mu_1}\left(||u^{(n)} - u^{(n-1)}||_2 + ||v^{(n)} - v^{(n-1)}||_2\right), \tag{5.38}$$

where $G(u, v)$ is assumed to be globally Lipschitz with Lipschitz constant L_G. Combining relations (5.37) and (5.38), we have

$$\begin{pmatrix} ||u^{(n+1)} - u^{(n)}||_2 \\ ||v^{(n+1)} - v^{(n)}||_2 \end{pmatrix} \leq \begin{pmatrix} \frac{L_F}{\lambda_1} & \frac{L_F}{\lambda_1} \\ \frac{L_G}{\mu_1} & \frac{L_G}{\mu_1} \end{pmatrix} \begin{pmatrix} ||u^{(n)} - u^{(n-1)}||_2 \\ ||v^{(n)} - v^{(n-1)}||_2 \end{pmatrix}. \tag{5.39}$$

Hence we have an inequality over one iteration step of the form

$$\mathbf{x}^{(n+1)} \leq \mathbf{A}\mathbf{x}^{(n)}, \tag{5.40}$$

where

$$\mathbf{x}^{(n)} = \begin{pmatrix} ||u^{(n)} - u^{(n-1)}||_2 \\ ||v^{(n)} - v^{(n-1)}||_2 \end{pmatrix} \quad \text{and} \quad \mathbf{A} = \begin{pmatrix} \frac{L_F}{\lambda_1} & \frac{L_F}{\lambda_1} \\ \frac{L_G}{\mu_1} & \frac{L_G}{\mu_1} \end{pmatrix}. \tag{5.41}$$

By Theorem 5.1, if the spectral radius of $\mathbf{A}$ is less than 1, then the sequence $\mathbf{x}^{(n)}$ converges to zero, which means that the iteration scheme (5.15)-(5.19) converges. Now the eigenvalues of the matrix $\mathbf{A}$ are $\lambda_{\mathbf{A}} \in \{0, \frac{L_F}{\lambda_1} + \frac{L_G}{\mu_1}\}$. So the spectral radius is $\rho(\mathbf{A}) = \frac{L_F}{\lambda_1} + \frac{L_G}{\mu_1}$, because all quantities are positive. Thus we have

proved

Theorem 5.2 *Let $F(u,v)$ and $G(u,v)$ be globally Lipschitz in u and v, with Lipschitz constants L_F and L_G,*

$$\begin{aligned} ||F(u_1,v_1) - F(u_2,v_2)||_2 &\leq L_F\left(||u_1-u_2||_2 + ||v_1-v_2||_2\right), \\ ||G(u_1,v_1) - G(u_2,v_2)||_2 &\leq L_G\left(||u_1-u_2||_2 + ||v_1-v_2||_2\right), \end{aligned} \quad \forall\ u_1,u_2,v_1,v_2 \in L_2. \tag{5.42}$$

Let λ_1 and μ_1 be the smallest positive eigenvalues of the eigenvalue problems (5.3) and (5.4) respectively. If

$$\left(\frac{L_F}{\lambda_1} + \frac{L_G}{\mu_1}\right) < 1, \tag{5.43}$$

then the sequential spectral method (5.15)-(5.19) converges.

Hence under the conditions of Theorem 5.2, $u^{(\infty)}$ and $v^{(\infty)}$ are a solution of problem (5.1). The following theorem gives the rate of convergence and the a priori error estimate.

Theorem 5.3 (Linear Convergence) *Let $F(u,v)$, $G(u,v)$ satisfy (5.42) with (5.43). Then the sequences $\{u^{(n)}\}$ and $\{v^{(n)}\}$ defined by the sequential spectral method (5.15) - (5.19) converge to a solution u,v of problem (5.1)-(5.2) at the rate*

$$\left(||u-u^{(n)}||_2 + ||v-v^{(n)}||_2\right) \leq \left(\frac{L_F}{\lambda_1} + \frac{L_G}{\mu_1}\right)^n \left(||u-u^{(0)}||_2 + ||v-v^{(0)}||_2\right) \tag{5.44}$$

and the a priori error estimate is given by

$$\left(||u-u^{(n)}||_2 + ||v-v^{(n)}||_2\right) \leq \frac{(\frac{L_F}{\lambda_1} + \frac{L_G}{\mu_1})^n}{1-(\frac{L_F}{\lambda_1} + \frac{L_G}{\mu_1})} \left(||u^{(1)}-u^{(0)}||_2 + ||v^{(1)}-v^{(0)}||_2\right). \tag{5.45}$$

Proof Let $u^{(n)}(\mathbf{x})$ and $v^{(n)}(\mathbf{x})$ be the nth iterate of the approximate solution, given by $u^{(n)}(\mathbf{x}) = \sum_{j=1}^{\infty} a_j^{(n)}\phi_j(\mathbf{x})$ and $v^{(n)}(\mathbf{x}) = \sum_{j=1}^{\infty} c_j^{(n)}\psi_j(\mathbf{x})$. Let $u(\mathbf{x})$ and $v(\mathbf{x})$ be a solution of the problem (5.1)-(5.2), $u(\mathbf{x}) = \sum_{j=1}^{\infty} a_j\phi_j(\mathbf{x})$ and $v(\mathbf{x}) = \sum_{j=1}^{\infty} c_j\psi_j(\mathbf{x})$. Since u, v is a solution of problem (5.1)-(5.2), we have $a_j = \frac{1}{\lambda_j}(F(u,v),\phi_j)$ and $c_j = \frac{1}{\mu_j}(G(u,v),\psi_j)$. Using the values of $a_j^{(n)}$ and $c_j^{(n)}$ from (5.17) and (5.18), we consider

$$\begin{aligned} a_j - a_j^{(n)} &= \frac{1}{\lambda_j}\int_\Omega [F(u,v) - F(u^{(n)},v^{(n)})]\phi_j d\mathbf{x}, \\ c_j - c_j^{(n)} &= \frac{1}{\mu_j}\int_\Omega [G(u,v) - G(u^{(n)},v^{(n)})]\psi_j d\mathbf{x}. \end{aligned} \tag{5.46}$$

Following the steps from (5.23) to (5.39), we obtain

$$\begin{pmatrix} ||u-u^{(n)}||_2 \\ ||v-v^{(n)}||_2 \end{pmatrix} \leq \begin{pmatrix} \frac{L_F}{\lambda_1} & \frac{L_F}{\lambda_1} \\ \frac{L_G}{\mu_1} & \frac{L_G}{\mu_1} \end{pmatrix} \begin{pmatrix} ||u-u^{(n-1)}||_2 \\ ||v-v^{(n-1)}||_2 \end{pmatrix}. \tag{5.47}$$

and defining

$$\mathbf{e}_{(n)} := \begin{pmatrix} ||u-u^{(n)}||_2 \\ ||v-v^{(n)}||_2 \end{pmatrix} \quad \text{and} \quad \mathbf{A} := \begin{pmatrix} \frac{L_F}{\lambda_1} & \frac{L_F}{\lambda_1} \\ \frac{L_G}{\mu_1} & \frac{L_G}{\mu_1} \end{pmatrix}, \tag{5.48}$$

the relation (5.47) becomes $\mathbf{e}_{(n)} \leq \mathbf{A}\mathbf{e}_{(n-1)}$. Taking the L_1 norm on both sides, we have $||\mathbf{e}_{(n)}||_1 \leq ||\mathbf{A}\mathbf{e}_{(n-1)}||_1 \leq ||\mathbf{A}||_1||\mathbf{e}_{(n-1)}||_1$. Applying this inequality inductively, we obtain

$$||\mathbf{e}_{(n)}||_1 \leq ||\mathbf{A}||_1||\mathbf{e}_{(n-1)}||_1 \leq ||\mathbf{A}||_1^2||\mathbf{e}_{(n-2)}||_1 \leq \cdots \leq ||\mathbf{A}||_1^n||\mathbf{e}_{(0)}||_1,$$

which by the definition of the L_1 norm gives

$$\left(||u-u^{(n)}||_2 + ||v-v^{(n)}||_2\right) \leq \left(\frac{L_F}{\lambda_1} + \frac{L_G}{\mu_1}\right)^n \left(||u-u^{(0)}||_2 + ||v-v^{(0)}||_2\right), \tag{5.49}$$

because the columns of $\mathbf{A}$ are identical and positive. Since $(\frac{L_F}{\lambda_1} + \frac{L_G}{\mu_1}) < 1$, we have $\lim_{n\to\infty} \left(\frac{L_F}{\lambda_1} + \frac{L_G}{\mu_1}\right)^n = 0$ and therefore

$$\lim_{n\to\infty} \left(||u - u^{(n)}||_2 + ||v - v^{(n)}||_2\right) = 0. \tag{5.50}$$

Hence $\{u^{(n)}\}$, $\{v^{(n)}\}$ converges to a solution u, v and the rate of convergence is given by (5.49).

Now we will find the a priori estimate for the error. We define

$$\mathbf{e}_{(n)}^{(n+1)} := \begin{pmatrix} ||u^{(n+1)} - u^{(n)}||_2 \\ ||v^{(n+1)} - v^{(n)}||_2 \end{pmatrix}, \tag{5.51}$$

and by using the definition of $\mathbf{A}$ from (5.48), the relation (5.39) can be written as $\mathbf{e}_{(n)}^{(n+1)} \leq \mathbf{A}\mathbf{e}_{(n-1)}^{(n)}$. Taking the L_1 norm on both sides, we obtain $||\mathbf{e}_{(n)}^{(n+1)}||_1 \leq ||\mathbf{A}||_1 ||\mathbf{e}_{(n-1)}^{(n)}||_1$. Applying the inequality inductively, we obtain

$$||\mathbf{e}_{(n)}^{(n+1)}||_1 \leq ||\mathbf{A}||_1 ||\mathbf{e}_{(n-1)}^{(n)}||_1 \leq ||\mathbf{A}||_1^2 ||\mathbf{e}_{(n-2)}^{(n-1)}||_1 \leq \cdots \leq ||\mathbf{A}||_1^n ||\mathbf{e}_{(0)}^{(1)}||_1.$$

For $m > n$, we have

$$\begin{aligned} ||\mathbf{e}_{(n)}^{(m)}||_1 &\leq ||\mathbf{e}_{(m-1)}^{(m)}||_1 + ||\mathbf{e}_{(m-2)}^{(m-1)}||_1 + \cdots + ||\mathbf{e}_{(n)}^{(n+1)}||_1 \\ &\leq ||\mathbf{A}||_1^{m-1} ||\mathbf{e}_{(0)}^{(1)}||_1 + ||\mathbf{A}||_1^{m-2} ||\mathbf{e}_{(0)}^{(1)}||_1 + \cdots + ||\mathbf{A}||_1^n ||\mathbf{e}_{(0)}^{(1)}||_1 \\ &= ||\mathbf{A}||_1^n \left(||\mathbf{A}||_1^{m-n-1} + ||\mathbf{A}||_1^{m-n-2} + \cdots + ||\mathbf{A}||_1 + 1\right) ||\mathbf{e}_{(0)}^{(1)}||_1. \end{aligned} \tag{5.52}$$

Since the sequences $\{u^{(n)}\}$ and $\{v^{(n)}\}$ converge to u and v, we have $\lim_{m\to\infty} ||\mathbf{e}_{(n)}^{(m)}||_1 = ||\mathbf{e}_{(n)}||_1$. Hence letting $m \to \infty$ in the above inequality, we obtain

$$||\mathbf{e}_{(n)}||_1 = ||\mathbf{A}||_1^n \sum_{j=1}^{\infty} ||\mathbf{A}||_1^j ||\mathbf{e}_{(0)}^{(1)}||_1 = \frac{||\mathbf{A}||_1^n}{1 - ||\mathbf{A}||_1} ||\mathbf{e}_{(0)}^{(1)}||_1. \tag{5.53}$$

Using the definition of the L_1 norm we get the a priori estimate for the error

$$\left(||u-u^{(n)}||_2+||v-v^{(n)}||_2\right) \leq \frac{(\frac{L_F}{\lambda_1}+\frac{L_G}{\mu_1})^n}{1-(\frac{L_F}{\lambda_1}+\frac{L_G}{\mu_1})}\left(||u^{(1)}-u^{(0)}||_2+||v^{(1)}-v^{(0)}||_2\right).$$

■

In Theorem 5.2, we assume $F(u,v)$, $G(u,v)$ to be globally Lipschitz in both u and v. From the relation (5.44), we found that all the iterates are in the ball $B((u,v),r)$, centered at the solution (u,v), with radius $r = ||u-u^{(0)}||_2 + ||v-v^{(0)}||_2$. So we need $F(u,v)$ and $G(u,v)$ to be Lipschitz only in the ball $B((u,v),r)$.

Corollary 5.1 *Let $F(u,v)$ and $G(u,v)$ be locally Lipschitz in a ball $B((u,v),r)$, centered at the solution (u,v), with radius $r = ||u-u^{(0)}||_2 + ||v-v^{(0)}||_2$, where $u^{(0)}$, $v^{(0)}$ are the initial guesses, with Lipschitz constants L_F and L_G. Let λ_1 and μ_1 be the smallest positive eigenvalues of the eigenvalue problems (5.3) and (5.4) respectively. If $\left(\frac{L_F}{\lambda_1}+\frac{L_G}{\mu_1}\right) < 1$, then the sequential spectral method (5.15)-(5.19) converges at the rate given in (5.44).*

We emphasize that the condition (5.43) which allows us to prove convergence is a sufficient condition and it is not necessary. The iteration may converge even if the condition (5.43) is violated. The above discussion is for general boundary boundary conditions $\mathcal{A}$ and $\mathcal{B}$. In the particular case of $\mathcal{A} = \mathcal{B}$, that is, when u and v satisfy identical boundary conditions, then the eigenvalues and eigenfunctions of the problems (5.3) and (5.4) are the same. The iteration scheme and convergence analysis follows in the same manner and we obtain that if $L_F + L_G < \lambda_1$, then the sequential spectral method converges.

5.3.1 Convergence Analysis for Neumann Conditions

We developed the sequential spectral method (5.15)-(5.19) under the assumption that neither λ_1 nor μ_1 is zero. This is because (5.17) or (5.18) cannot be defined

for $j = 1$, if $\lambda_1 = 0$ or $\mu_1 = 0$. Hence, in general, the iteration scheme can not be defined if the eigenvalue problem (5.3) or (5.4) has a zero eigenvalue. However, there are certain forms of $F(u, v)$ and $G(u, v)$ which can allow us to define the iteration scheme in that case as well and they are discussed in Section 2.2.1. If $\lambda_1 = 0$ and the nonlinear function $F_1(u, v)$ is a combination of a linear term $\zeta_1 u$ and a nonlinear term $F_1(u, v)$, that is,

$$F(u, v) = \zeta_1 u + F_1(u, v), \tag{5.54}$$

then we can define the iteration scheme

$$a_j^{(n+1)} = \frac{1}{(\lambda_j - \zeta_1)}(F_1(u^{(n)}, v^{(n)}), \phi_j), \tag{5.55}$$

provided $\zeta_1 \neq \lambda_j$, $j = 1, 2, 3, \ldots$. Similarly, if $\mu_1 = 0$ and $G(u, v)$ has the form

$$G(u, v) = \zeta_2 v + G_1(u, v), \tag{5.56}$$

then we can define the iteration scheme

$$c_j^{(n+1)} = \frac{1}{(\mu_j - \zeta_2)}(G_1(u^{(n)}, v^{(n)}), \psi_j), \tag{5.57}$$

provided $\zeta_2 \neq \mu_j$, $j = 1, 2, 3, \ldots$. If both λ_1 and μ_1 are zero, then the equations (5.17) and (5.18) in the iteration scheme (5.15)- (5.19) will be replaced by equations (5.55) and (5.57) respectively. The convergence of the iteration scheme can be analyzed similarly. It is found that a sufficient condition for convergence is that $\frac{L_F}{|\zeta_1|} + \frac{L_G}{|\zeta_2|} < 1$. To illustrate this idea, we have solved a system of elliptic partial differential equations with $\mu_1 = 0$ where $G(u, v)$ is of the form given by (5.56). The numerical results are presented in the next section and a comparison with the solution obtained by the Galerkin method is given. Convergence is

obtained again in only a few iterations.

5.4 Numerical Results

In this section we solve a system of elliptic partial differential equations arising from a chemical reaction. Suppose the reaction $A \rightarrow B$ takes place in two steps, the first is a reversible binding to an enzyme, $A + E \rightleftharpoons C$, and the second is the dissociation of the complex so formed into the product with the release of the enzyme $C \rightarrow B + E$. The reaction is to take place within a region Ω in which all the species can freely diffuse but whose boundary $\partial\Omega$ is permeable only to A and B. The original system consist of four equations but under certain conditions and by nondimensionalising the quantities we obtain system of two equations. A detailed derivation can be found in [Ari75]. The steady state equations governing the concentrations can be written as

$$\begin{aligned} \triangle u &= C_1^2[u(1-v) - (\kappa - \chi)v], && \mathbf{x} \in \Omega, \\ \triangle v &= C_2^2[-u(1-v) + \kappa v], && \mathbf{x} \in \Omega, \\ \beta\frac{\partial u}{\partial n} + u &= 1, \quad \frac{\partial v}{\partial n} = 0, && \mathbf{x} \in \partial\Omega, \end{aligned} \tag{5.58}$$

where $\frac{\partial}{\partial n}$ is the outward normal derivative on $\partial\Omega$. The parameters C_1 and C_2 are positive constants and their values depend on the diffusion coefficients. The constants κ and χ corresponds to the rate constants. The above system has been studied in [Ari75] and [Ari72]. We solve the problem for the slab $0 \leq x \leq 1$. In order to have homogeneous boundary conditions for u, we put $u = w + 1$. So we will solve the problem

$$\begin{aligned} \triangle w &= C_1^2[(w+1)(1-v) - (\kappa - \chi)v], && x \in (0,1), \\ \triangle v &= C_2^2[-(w+1)(1-v) + \kappa v], && x \in (0,1), \\ \beta\frac{\partial w}{\partial n} + w &= 0, \quad \frac{\partial v}{\partial n} = 0, && x = 0, 1. \end{aligned} \tag{5.59}$$

The orthonormal eigenfunctions $\phi_j(x)$, $\psi_j(x)$ and the corresponding eigenvalues λ_j, μ_j satisfy the equations

$$\triangle\phi_j = -\lambda_j\phi_j, \quad \text{where} \quad \beta\,\frac{\partial\phi_j}{\partial n} + \phi_j = 0 \quad \text{for } x = 0, 1.$$

$$\triangle\psi_j = -\mu_j\psi_j, \quad \text{where} \quad \frac{\partial\psi_j}{\partial n} = 0 \qquad \text{for} \quad x = 0, 1.$$

Solving these equations, we get

$$\begin{aligned}
\phi_j &= \sqrt{\tfrac{2}{1+\beta^2 j^2\pi^2}}\,[\sin(j\pi x) - \beta j\pi\,\cos(j\pi x)], \quad \lambda_j = j^2\pi^2, \quad j = 1, 2, 3, \ldots . \\
\psi_j &= \sqrt{2}\,\cos\,((j-1)\pi x), \qquad\qquad\qquad\quad \mu_j = (j-1)^2\,\pi^2, \qquad j = 1, 2, 3, \ldots .
\end{aligned}$$

Numerically we truncate the expansion at $N = 6$. Using the Galerkin method we obtain the system of equations

$$\begin{aligned}
(\lambda_j + C_1^2)g_j + C_1^2(1 - \sum_{k=1}^{6} g_k\phi_k \sum_{k=1}^{6} e_k\psi_k - (1+\kappa-\chi)\sum_{k=1}^{6} e_k\psi_k, \phi_j) = 0, \\
(\mu_j + (1+\kappa)C_2^2)e_j - C_2^2(1 - \sum_{k=1}^{6} g_k\phi_k \sum_{k=1}^{6} e_k\psi_k + \sum_{k=1}^{6} g_k\phi_k, \psi_j) = 0,
\end{aligned} \tag{5.60}$$

for $j = 1, 2, 3, \ldots, 6$. To solve these 12 nonlinear equations, we use a fixed point iteration of the form

$$\begin{aligned}
g_j^{(l+1)} &= \frac{C_1^2}{(\lambda_j + C_1^2)}\,(-1 + \sum_{k=1}^{6} g_k^{(l)}\phi_k \sum_{k=1}^{6} e_k^{(l)}\psi_k + (1+\kappa-\chi)\sum_{k=1}^{6} e_k^{(l)}\psi_k, \phi_j) \\
e_j^{(l+1)} &= \frac{C_2^2}{\mu_j + (1+\kappa)C_2^2}\,(1 - \sum_{k=1}^{6} g_k^{(l)}\phi_k \sum_{k=1}^{6} e_k^{(l)}\psi_k + \sum_{k=1}^{6} g_k^{(l)}\phi_k, \psi_j),
\end{aligned} \tag{5.61}$$

for $j = 1, 2, \ldots, 6$. The computations are performed for the parameter values

	Galerkin	SSM	
j	g_j	$a_j^{(0)}$	$a_j^{(8)}$
1	-3.324326×10^{-3}	-3.302877×10^{-3}	-3.324326×10^{-3}
2	5.569387×10^{-7}	7.730235×10^{-7}	5.569387×10^{-7}
3	-4.444987×10^{-5}	-4.427130×10^{-5}	-4.444987×10^{-5}
4	3.720939×10^{-8}	4.121142×10^{-8}	3.720939×10^{-8}
5	-5.801788×10^{-6}	-5.804807×10^{-6}	-5.801788×10^{-6}
6	7.692260×10^{-9}	9.428359×10^{-9}	7.692260×10^{-9}

Table 5.1: Comparison of the coefficients a_j and g_j for $\beta = 1$.

$C_1 = C_2 = \kappa = 1$, $\chi = 0.25$. A similar fixed point method is used to solve the system of two equations obtained at each step in the sequential spectral method. Tables 5.1 and 5.2 show a comparison between the coefficients g_j and a_j for w and the coefficients e_j and c_j for v obtained by using the Galerkin method and the sequential spectral method. The results show that the initial coefficients $a_j^{(0)}$ and $c_j^{(0)}$ are very close to the Galerkin coefficients g_j and e_j and converge to the Galerkin coefficients to a tolerance $tol = 10^{-8}$ in only 8 iterations. The truncated solution to the problem (5.58) is given by $u(x) = 1 + \sum_{j=1}^{6} g_j \phi_j(x)$ and $v(x) = \sum_{j=1}^{6} e_j \psi_j(x)$. Figures (5.1) and (5.2) also show that there is a very small difference in the solutions $u^{(0)}$, u^g and $v^{(0)}$, v^g.

We note from the above results that $a_j^{(0)}$ and $c_j^{(0)}$, for $j = 1, 3, 5, \ldots$ are decreasing sequences and also for $j = 2, 4, 6, \ldots$. The same behavior is observed for $a_j^{(8)}$ and $c_j^{(8)}$ for the given problem. So the coefficients of the first eigenfunctions are more important. This is because the eigenfunctions are intrinsic to the operator, the domain and the boundary conditions.

	Galerkin	SSM	
j	e_j	$c_j^{(0)}$	$c_j^{(8)}$
1	3.533922×10^{-1}	3.533947×10^{-1}	3.533922×10^{-1}
2	1.336607×10^{-4}	1.326501×10^{-4}	1.336607×10^{-4}
3	5.110983×10^{-6}	5.069182×10^{-6}	5.110983×10^{-6}
4	2.424781×10^{-7}	2.415259×10^{-7}	2.424781×10^{-7}
5	2.750798×10^{-7}	2.727424×10^{-7}	2.750798×10^{-7}
6	1.149816×10^{-8}	1.166550×10^{-8}	1.149816×10^{-8}

Table 5.2: Comparison of the coefficients c_j and e_j for $\beta = 1$.

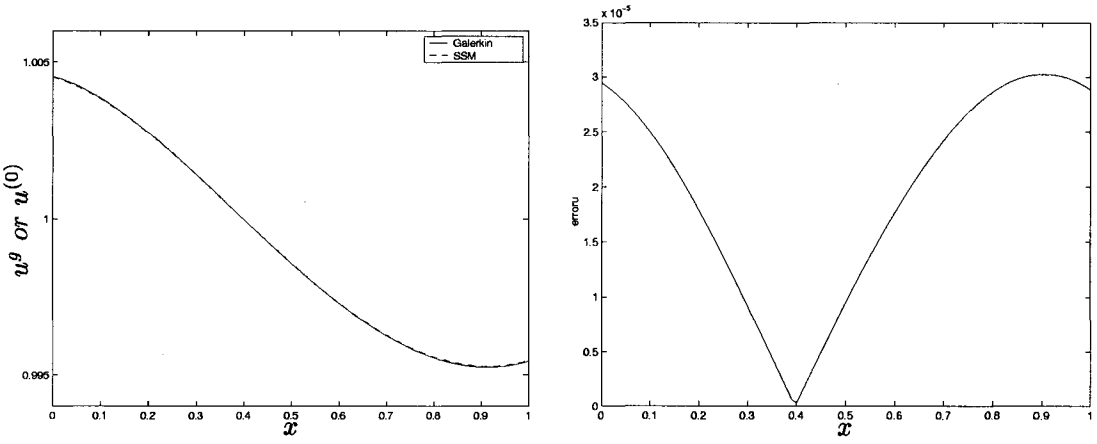

Figure 5.1: Comparison of $u^g = 1 + \sum_{j=1}^{N} g_j \phi_j$ and $u^{(0)} = 1 + \sum_{j=1}^{N} a_j^{(0)} \phi_j$ for $N = 6$, $\beta = 1$ and the error $|u^g - u^{(0)}|$.

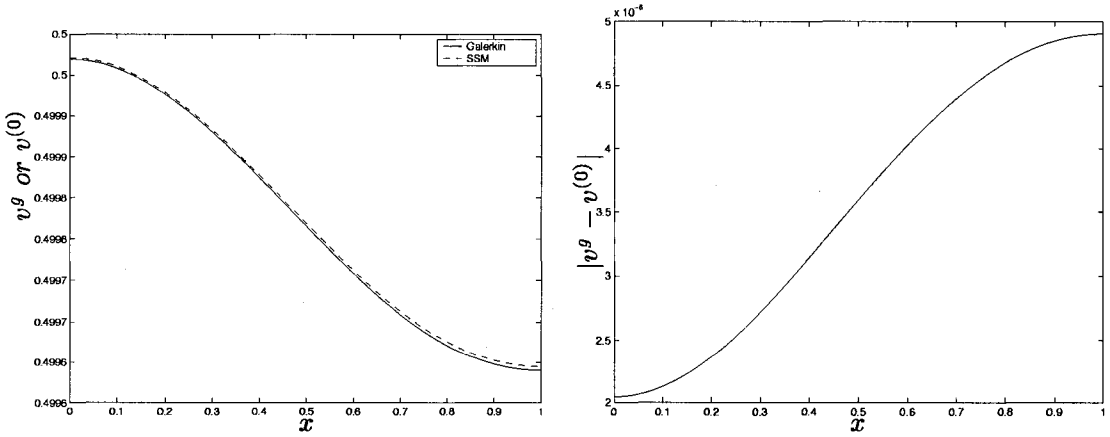

Figure 5.2: Comparison of $v^g = \sum_{j=1}^{N} e_j \psi_j$ and $v^{(0)} = \sum_{j=1}^{N} c_j^{(0)} \psi_j$ for $N = 6$, $\beta = 1$ and the error $|v^g - v^{(0)}|$.

Chapter 6

Conclusion

6.1 Main Research Results

In this thesis, we developed the sequential spectral method for nonlinear partial differential equations and integro-differential equations of elliptic and parabolic type. We proved that the sequential spectral method converges linearly for the elliptic case and superlinearly for the parabolic case. We found many advantages of the sequential spectral method over the classical Galerkin methods:

(i) In the sequential spectral method, one needs to solve a single algebraic (in the elliptic case) or ordinary differential equation (in the parabolic case) at each step instead of a system of equations in the Galerkin method.

(ii) In the sequential spectral method, we can continue to add components until we get the required accuracy and this does not affect the previous process.

(iii) The existence of multiple solutions can be detected by solving a single equation (see Figure 3.3).

(iv) The time of computation is much lower than for the Galerkin method. For a certain example in Section 3.3, the approximate computational complexity

for the Galerkin method is $t_g \approx N^6$ and for the SSM is $t_s \approx N^3$ (see Figure 3.7).

(v) The dependence of the solution on parameters can be analyzed by studying a single equation, and also estimates for critical values of the parameters can be obtained (see Tables 3.5 and 4.7, Figures 3.8 and 4.9).

(vi) The blow up time can be estimated for corresponding parametric values (see Table 4.8 and Figure 4.10).

The sequential spectral method is a new method, introduced by Tam *et. al* in [TAK96] for a semi-linear elliptic partial differential equation. Al Refai has developed the method for parabolic partial differential equations and systems in [Alr00] but the conditions for convergence obtained were not transparent and difficult to verify. We have obtained convergence conditions depending on the Lipschitz constant of the nonlinear function $F(u)$ and proved that the convergence is linear in the elliptic case (Theorem 3.3) and superlinear in the parabolic case (Theorem 4.4), and we also found a priori estimates for the error. We have also extended the idea to a system of elliptic partial differential equations and proved linear convergence of the sequential spectral method and an a priori estimate for error (Theorem 5.3). Since the sequential spectral method is a new method, many issues remain to be investigated.

6.2 Future Research Directions

Observing the advantages of the sequential spectral method, one naturally wants to apply the method to a variety of problems described by partial differential equations, integro-differential equations and integral equations from applications, especially problems which were hard to solve up to now. We developed the method for the case of $F(u)$, that is, the nonlinear term depends only on u.

Al Refai has applied the sequential spectral method to the Kuramoto-Sivashinsky equation, which has a nonlinear term of the form $F(u, u_x)$ and proved convergence for that equation only [Alr00]. There is no general theory for equations with nonlinearity $F(u, u_x)$. We have started to work on a parabolic integro-differential equation with a nonlinear term of the form $F(u, u_t)$. We have developed the iteration scheme and applied it to a model problem and obtained very promising numerical results but we have not yet proved convergence. This will be one of our research projects in the near future. We hope that the idea can be applied to more general systems containing higher derivatives of u with respect to the spatial variables. Also hyperbolic equations seem to be a good candidate for this method, we have especially nonlinear wave equations in mind. Al Refai has applied the method to a wave type equation and got numerical results but there is no convergence analysis yet [Alr00]. Katina has developed the sequential spectral method for nonlinear integral equations [Mic97] and we have also obtained first results to extend this idea to systems of integral equations. Finally the method should be extended to systems of integro-differential equations.

A complete different approach would be to study a discretized version of this sequential spectral method. This would allow us to use the method on arbitrary domains with general boundary conditions. The interesting question would be if it pays off to compute some of the lower eigenfunctions numerically and then to apply the sequential spectral method to resolve the nonlinearity, compared to a Finite Element approach. For consistent discretizations we would expect that similar convergence for the sequential spectral method could be obtained as far the continuous case shown in this thesis.

Appendix A

Proof of Lemma 4.1

We now prove Lemma 4.1. To this end, we need

Lemma A.1 *For every even m, odd k_l and $n_l \in \mathbb{N}$, and $N \in \mathbb{N}$, we have*

$$\int_0^1 \prod_{l=1}^{N} \sin^{n_l}(k_l \pi x) \sin(m\pi x) dx = 0. \tag{A.1}$$

Proof From the trigonometric identities, we know that

$$\sin(k\pi(1-x)) = \begin{cases} -\sin(k\pi x), & \text{if } k \text{ is even,} \\ \sin(k\pi x), & \text{if } k \text{ is odd.} \end{cases} \tag{A.2}$$

We first split the integral in half,

$$\begin{aligned} \int_0^1 \prod_{l=1}^{N} \sin^{n_l}(k_l \pi x) \sin(m\pi x) dx = & \int_0^{1/2} \prod_{l=1}^{N} \sin^{n_l}(k_l \pi x) \sin(m\pi x) dx \\ & + \int_{1/2}^1 \prod_{l=1}^{N} \sin^{n_l}(k_l \pi x) \sin(m\pi x) dx. \end{aligned} \tag{A.3}$$

Then putting $x = 1 - u$ in the first integral on the right hand side of equation (A.3) and using (A.2), the first integral becomes

$$\begin{aligned} \int_0^{1/2} \prod_{l=1}^{N} \sin^{n_l}(k_l \pi x) \sin(m\pi x) dx &= \int_1^{1/2} \prod_{l=1}^{N} \sin^{n_l}(k_l \pi (1-u)) \sin(m\pi(1-u))(-du) \\ &= -\int_{1/2}^{1} \prod_{l=1}^{N} \sin^{n_l}(k_l \pi u) \sin(m\pi u) du \end{aligned} \tag{A.4}$$

and hence with (A.3)

$$\int_0^1 \prod_{l=1}^{N} \sin^{n_l}(k_l \pi x) \sin(m\pi x) dx = 0,$$

for every even m, odd k_l and $n_l \in \mathbb{N}$, and $N \in \mathbb{N}$. ■

Lemma A.2 *For every $j > 0$,*

$$\int_0^1 e^{\sum_{k=1}^{j} a_{2k-1}(t) \sin(2k-1)\pi x} \sin(2j\pi x) dx = 0. \tag{A.5}$$

Proof Using the series expansion of exponential functions, we have,

$$\begin{aligned} &\int_0^1 e^{\sum_{k=1}^{j} a_{2k-1}(t) \sin(2k-1)\pi x} \sin(2j\pi x) dx = \\ &\qquad \int_0^1 \sum_{n=1}^{\infty} \frac{[\sum_{k=1}^{j} a_{2k-1}(t) \sin(2k-1)\pi x]^n}{n!} \sin(2j\pi x) dx. \end{aligned} \tag{A.6}$$

By Lemma A.1, all the integrals on the right hand side will be zero and hence

$$\int_0^1 e^{\sum_{k=1}^{j} a_{2k-1}(t) \sin(2k-1)\pi x} \sin(2j\pi x) dx = 0, \quad \forall\ j > 0. \tag{A.7}$$

■

Lemma A.3 *Let* $u^{(0)}(\mathbf{x},t)=\sum_{j=1}^{\infty}a_j^{(0)}(t)\phi_j(\mathbf{x})$ *be the initial approximation for the solution of problem 4.45, obtained by the SSM. Then the even Fourier coefficients* $a_{2l}^{(0)}(t)=0,\ \forall\ l>0, t>0.$

Proof To find the initial guess $u^{(0)}(\mathbf{x},t)=\sum_{j=1}^{\infty}a_j^{(0)}(t)\phi_j(\mathbf{x})$, we need to solve the nonlinear ordinary differential equations for the Fourier coefficients $a_j^{(0)}(t)$,

$$\frac{da_j^{(0)}}{dt}+D\lambda_j a_j^{(0)}(t)=\sigma\sqrt{2}\int_0^1 e^{\gamma\sqrt{2}[\sum_{k=1}^{j}a_k^{(0)}(t)\sin(k\pi x)]}\sin(j\pi x)dx, \tag{A.8}$$

with initial condition $a_j^{(0)}(0)=0$, sequentially for $j=1,2,3,....$ Here D, σ and γ are positive constants. In order to prove that the even coefficients $a_{2l}^{(0)}(t)=0,\ \forall\ l>0, t>0$, we prove that $a_{2l}^{(0)}(t)=0$ is the unique solution of the corresponding equation. First we prove, by induction, that $a_{2l}^{(0)}(t)=0$ is a solution of the corresponding ordinary differential equation.

For $l=1$, we find $a_2^{(0)}(t)$ by solving

$$\frac{da_2^{(0)}}{dt}+D\lambda_2 a_2^{(0)}=\sigma\sqrt{2}\int_0^1 e^{\gamma\sqrt{2}[a_1^{(0)}\sin(\pi x)+a_2^{(0)}\sin(2\pi x)]}\sin(2\pi x)dx, \tag{A.9}$$

together with initial condition $a_2^{(0)}(0)=0$. We see that $a_2^{(0)}(t)=0$ is a solution of equation (A.9), if

$$\int_0^1 e^{\gamma\sqrt{2}a_1^{(0)}(t)\sin(\pi x)}\sin(2\pi x)dx=0,$$

which is true by Lemma A.2. So the result is true for $l=1$.

Now suppose that the result is true for $l=m$, that is, $a_{2m}^{(0)}(t)=0$ is a solution of

$$\frac{da_{2m}^{(0)}}{dt}+D\lambda_{2m}a_{2m}^{(0)}=\sigma\sqrt{2}\int_0^1 e^{\gamma\sqrt{2}[\sum_{k=1}^{m}a_{2k-1}^{(0)}\sin((2k-1)\pi x)+a_{2m}^{(0)}\sin(2m\pi x)]}\sin(2m\pi x)dx, \tag{A.10}$$

where $a_{2m}^{(0)}(0) = 0$. To prove that the result is true for $l = m+1$, we consider

$$\frac{da_{2(m+1)}^{(0)}}{dt} + D\lambda_{2(m+1)}a_{2(m+1)}^{(0)} = \sigma\sqrt{2}\int_0^1 e^{\gamma\sqrt{2}[\sum_{k=1}^{m+1} a_{2k-1}^{(0)}\sin((2k-1)\pi x)+a_{2(m+1)}^{(0)}\sin(2(m+1)\pi x)]}\sin(2(m+1)\pi x)dx, \tag{A.11}$$

where $a_{2(m+1)}^{(0)}(0) = 0$. By Lemma A.2, we have

$$\int_0^1 e^{\sum_{k=1}^{m+1} a_{2k-1}^{(0)}\sin(2k-1)\pi x}\sin(2(m+1)\pi x)dx = 0,$$

which implies that $a_{2(m+1)}^{(0)}(t) = 0$ satisfies equation (A.11) for all $t > 0$. Hence the result is true for $l = m+1$ and the induction is complete.

In order to prove uniqueness, we use the Lipschitz property as for the Picard-Lindelöf iterations for initial value problems (see [Pic93] and [Lin94]). We have to solve the ordinary differential equations (A.8) for the coefficients $a_j^{(0)}(t)$,

$$\frac{da_j^{(0)}}{dt} + D\lambda_j a_j^{(0)}(t) = \sigma\sqrt{2}\int_0^1 e^{\gamma\sqrt{2}\sum_{k=1}^{j} a_k^{(0)}(t)\sin(k\pi x)}\sin(j\pi x)dx, \tag{A.12}$$

with the initial condition $a_j^{(0)}(0) = 0$, sequentially for $j = 1, 2, 3, \ldots$.

Let

$$f(a_j^{(0)}(t)) := \sigma\sqrt{2}\int_0^1 e^{\gamma\sqrt{2}[\sum_{k=1}^{j-1} a_k^{(0)}(t)\sin(k\pi x)+a_j^{(0)}(t)\sin(j\pi x)]}\sin(j\pi x)dx. \tag{A.13}$$

To see that this function is Lipschitz, we compute

$$\left|f(a_j^{(0)}) - f(\tilde{a}_j^{(0)})\right| = \sigma\sqrt{2}\left|\int_0^1 e^{\gamma\sqrt{2}\sum_{k=1}^{j-1} a_k^{(0)}\sin(k\pi x)}\left[e^{\gamma\sqrt{2}a_j^{(0)}\sin(j\pi x)} - e^{\gamma\sqrt{2}\tilde{a}_j^{(0)}\sin(j\pi x)}\right]\sin(j\pi x)dx\right|. \tag{A.14}$$

By the Mean Value Theorem, we have

$$|e^{\gamma\sqrt{2}a_j^{(0)}\sin(j\pi x)} - e^{\gamma\sqrt{2}\tilde{a}_j^{(0)}\sin(j\pi x)}| = e^{\xi(x)}\gamma\sqrt{2}|a_j^{(0)} - \tilde{a}_j^{(0)}| \cdot |\sin(j\pi x)|, \qquad \text{(A.15)}$$

for some $\xi(x)$ between $\gamma\sqrt{2}a_j^{(0)}\sin(j\pi x)$ and $\gamma\sqrt{2}\tilde{a}_j^{(0)}\sin(j\pi x)$. Using this result in relation (A.14), we obtain

$$|f(a_j^{(0)}) - f(\tilde{a}_j^{(0)})| \leq 2\sigma\gamma \int_0^1 e^{\gamma\sqrt{2}\sum_{k=1}^{j-1} a_k^{(0)}\sin(k\pi x)+\xi(x)}|a_j^{(0)} - \tilde{a}_j^{(0)}| \cdot |\sin(j\pi x)|^2 dx. \qquad \text{(A.16)}$$

Therefore $f(a_j^{(0)})$ is Lipschitz ,

$$|f(a_j^{(0)}) - f(\tilde{a}_j^{(0)})| \leq L_f|a_j^{(0)} - \tilde{a}_j^{(0)}|, \qquad \text{(A.17)}$$

with Lipschitz constant,

$$L_f = 2\sigma\gamma \int_0^1 e^{\gamma\sqrt{2}\sum_{k=1}^{j-1} a_k^{(0)}\sin(k\pi x)+\xi(x)}|\sin(j\pi x)|^2 dx. \qquad \text{(A.18)}$$

Using the definition of $f(a_j^{(0)})$ in equation (A.12), we have

$$\frac{da_j^{(0)}}{dt} = -D\lambda_j a_j^{(0)} + f(a_j^{(0)}). \qquad \text{(A.19)}$$

Let

$$g(a_j^{(0)}(t)) := -D\lambda_j a_j^{(0)}(t) + f(a_j^{(0)}(t)), \qquad \text{(A.20)}$$

then $g(a_j^{(0)})$ is Lipschitz,

$$\begin{aligned} |g(a_j^{(0)}) - g(\tilde{a}_j^{(0)})| &= | - D\lambda_j a_j^{(0)} + f(a_j^{(0)}) + D\lambda_j \tilde{a}_j^{(0)} - f(\tilde{a}_j^{(0)})| \\ &\leq (D\lambda_j + L_f)|a_j^{(0)} - \tilde{a}_j^{(0)}| \\ &= L_g |a_j^{(0)} - \tilde{a}_j^{(0)}|, \end{aligned} \tag{A.21}$$

with Lipschitz constant,

$$L_g = D\lambda_j + L_f, \tag{A.22}$$

because D and λ_j are positive. Hence there exists a unique solution of equations (A.12) and thus the solution $a_{2l}^{(0)}(t) = 0$ we found is unique. ■

We are now able to prove Lemma 4.1:

Proof From Lemma A.3 we found that $a_{2l}^{(0)}(t) = 0,\ \forall\ l > 0, t > 0$ in the initial guess $u^{(0)}(\mathbf{x}, t)$. For the nth approximation by the SSM, we need to solve linear ordinary differential equations for the Fourier coefficients $a_j^{(n)}(t)$,

$$\frac{da_j^{(n)}}{dt} + D\lambda_j a_j^{(n)}(t) = \sigma\sqrt{2} \int_0^1 e^{\gamma\sqrt{2}[\sum_{k=1}^{j} a_k^{(n-1)}(t)\sin(k\pi x)]} \sin(j\pi x) dx, \tag{A.23}$$

with initial condition $a_j^{(n)}(0) = 0$, sequentially for $j = 1, 2, 3, \ldots$. Here D, σ and γ are positive constants. For $n = 1$, we have

$$\frac{da_j^{(1)}}{dt} + D\lambda_j a_j^{(1)}(t) = \sigma\sqrt{2} \int_0^1 e^{\gamma\sqrt{2}[\sum_{k=1}^{j} a_{2k-1}^{(0)}(t)\sin((2k-1)\pi x)]} \sin(j\pi x) dx, \tag{A.24}$$

with initial condition $a_j^{(1)}(0) = 0,\ j = 1, 2, 3, \ldots$. Now for the even coefficients $j = 2l$, we have by Lemma A.2

$$\frac{da_{2l}^{(1)}}{dt} + D\lambda_{2l} a_{2l}^{(1)} = 0, \quad a_{2l}^{(1)}(0) = 0, \tag{A.25}$$

which has solution $a_{2l}^{(1)}(t) = 0$, for $l = 1, 2, 3, \ldots$. Similarly by induction and the results of Lemma A.2, all the approximations are zero if the initial approximation is zero.Therefore the even Fourier coefficients $a_{2l}^{(n)}(t) = 0$, $\forall\, l > 0, t > 0$. ■

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