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ABSTRACT

The main purpose of the thesis is to provide a comprehensive study of the stability theory of difference equations using the second method of Liapunov. This study not only unifies the rather fragmentary results of the present status of the theory, but also provides a number of new results. The work consists of three major parts:

a) A series of theorems is established for each of eight different types of stability and asymptotic stability for the difference equations under consideration. Among these, the concept of l_p-stability is introduced and investigated extensively. Also, a new approach to studying which stability properties are preserved under small perturbations is introduced and thoroughly studied. In general, most of the results fall into two categories. The first involves the existence of a certain class of real scalar functions possessing particular properties whose existence imply the type of stability being studied. The second deals with the converse problem of determining conditions of stability and conditions on the difference equation which will guarantee the existence of such functions.

b) These results are further extended to obtain a series of theorems on various forms of stability and asymptotic stability in the whole. In addition, a number of results on the boundedness and uniform boundedness of all solutions of the difference equation are presented.

c) The preceding theory is then applied to a rather wide class of difference equations of the form

$$X(n+m) + a_1X(n+m-1) + \dots + a_mX(n) - F(n,X(n),\dots X(n+m-1)) = 0.$$

A technique is developed which can be used to determine the conditions to be imposed on the coefficients a_1 and on the function F in order to insure stability and asymptotic stability for the solutions of the equation. By way of illustration, the method is applied in detail to the case $m = 4$.

McGILL UNIVERSITY
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THE STABILITY THEORY OF DIFFERENCE EQUATIONS USING LIAPUNOV'S DIRECT METHOD

by

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INTRODUCTION

One of the major breakthroughs in the study of differential equations occurred in 1893 with the publication of Liapunov's now famous paper "Probleme general de la stabilite du mouvement", in which he introduced his Direct Method for studying the behavior of the solutions of differential equations. The method is, in reality, rather indirect in that the behavior of the solutions is inferred from the existence of certain real scalar functions with various particular properties, instead of from a direct knowledge of the solutions. In essence, this class of scalar functions, commonly called Liapunov functions in much of the literature on the subject, represents a generalization of the total energy of a physical system. It is this possibility of determining the behavior of the solutions implicitly that makes Liapunov's method so eminently useful, especially for non-linear differential equations where one cannot usually solve the equations explicitly.

While the full implications of Liapunov's approach were not fully appreciated for forty years, and in fact, his work faded into obscurity during this time, it was finally "rediscovered" about 35 years ago and the subsequent use of it has made it the principal mathematical tool for dealing with problems involving linear and non-linear stability questions of all types, particularly in the theory of control systems.

It was during this same latter period that the study of difference equations was given new impetus by the realization that such equations had certain extremely useful applications. With the development of high speed computing machines, many differential equations which previously were considered insoluble, in the practical sense, were converted to approximate difference equations, which the computers could easily handle. Also, many problems arose in the field of control theory in engineering which were expressible in terms of difference equations. Notable among these is the field of sampled data systems, in which a process is examined at periodic intervals to test various aspects of it. The mathematical formulation of the problem is essentially a difference equation and the behavior of the process, interpreted as the solution of the difference equation, is to be determined.

As a result of such developments, not only was an intensive study of difference equations warranted, but also an approach to yield knowledge of the behavior of the solutions was needed. Such an approach turns out to be an application of Liapunov's method to discrete variable systems. While only a few such investigations have yet been carried out, these few indicate that the method which was so fruitful for differential equations possesses an analogue for difference equations which is equally powerful. Krasovskii (7) transferred a number of results on the stability of differential equations to difference equations, though

Hahn (3) was the first to apply the direct method systematically. The latter work, however, was mainly concerned with the linear difference equation

$$X(n+1) = A(n)X(n)$$

and, in particular, with the constant case in which stability criteria were obtained in terms of the size of the eigenvalues of the matrix A. Halanay (4) extended some of this work and also gave the first results in the converse direction; that is, to determine under what conditions the proper Liapunov functions exist. Kalman and Bertram (6) gave rather far-reaching extensions of the stability theory for difference equations, especially for the general equation

$$X(n+1) = f(n,X(n)). \qquad (*)$$

Finally, a new book by Hahn (Stability of Motion, Academic Press, New York, 1968) was brought to the author's attention after this thesis was completed. A number of the results which appear in the book were obtained independently in the present work. These are Theorems 1,13,17,35,37,38.

In a somewhat different direction, stability criteria for a particular class of difference equations were established by Puri and Drake (11) for equations of the second and third order.

In the present work, both of these aspects, the theoretical and the practical, are further investigated and the works previously mentioned are extended considerably. In Chapter 2 of Part I, a series of theorems is obtained for each of eight different types of stability. In particular, the concept of l_p-stability for difference equations is introduced for the first time and the development of this theory parallels Strauss' (12) study of L^p-stability for differential equations. Moreover, a new approach to studying stability properties being preserved under small perturbations is introduced and treated extensively. In addition, the concept of instability for difference equations is also presented.

Essentially, for each of the types of stability under investigation, the behavior of the solutions of the difference equation is shown to be guaranteed by the existence of certain real scalar functions possessing particular properties. On the other hand, we also consider the converse problem of determining under what conditions of stability for the equilibrium is the existence of such scalar function assured. This latter problem is approached in a number of ways, either by strengthening the type of stability assumed or by restricting the function f(n,X) in the difference equation (*).

In Chapter 3, the concepts of boundedness of solutions of the difference equation and various types of stability in the whole are studied and a series of theorems is obtained for each. Moreover, throughout the work, those results which have been obtained by previous researchers are indicated in the appropriate places. As a consequence, Part I represents a comprehensive study of the application of Liapunov's Direct Method to difference equations. Similar extensive surveys for using this approach for differential equations have been given in a paper by Antosiewicz (1) and in a book by Hahn (2).

Incidentally, the term Liapunov function has been considerably overworked in that different sets of properties are ascribed to such functions by different authors throughout the literature. As a consequence, it was felt that itemization of the particular properties required of the scalar functions for each individual theorem is preferable and this convention will be adhered to throughout the present work.

In Part II, the methods of Puri and Drake are generalized to a scheme for treating difference equations of any order which fall into a rather wide class. The basis of this method is to obtain conditions on the coefficients and on the given function in the non-linear equation

$$X(n+m) + a_1 X(n+m-1) + \dots + a_m X(n) + F(n, X(n), \dots X(n+m-1)) = 0$$

which will guarantee both stability and asymptotic stability for the equilibrium. The extension of their approach is done in such a way as to result in significant simplifications in the calculations which are required to obtain the appropriate conditions of the a_i and on F. This approach is illustrated in Chapter 5, where the fourth order equation is treated in detail to determine conditions for stability and asymptotic stability. Moreover, certain of the theorems obtained by Puri and Drake yield only stability, not the asymptotic stability claimed by the authors. These theorems have been repaired in Chapter 6 and the corrections have been accomplished using the modified technique from Chapter 4.

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PART II

STABILITY CRITERIA

FOR A CERTAIN CLASS OF DIFFERENCE EQUATIONS

PART I

SOME RESULTS ON THE STABILITY THEORY OF DIFFERENCE EQUATIONS

Chapter 1: Basic Definitions and Concepts

Let S be a divergent strictly monotonic increasing sequence $\{s_i\}$ of real numbers:

$$s_i < s_{i+1} \qquad \lim_{i \to \infty} s_i = \infty .$$

The general difference equation is then given by

$$X(s_{i+1}) = f(s_i, s_0, x_0)$$

subject to the initial condition

$$X(s_0) = x_0,$$

where $X(s_i)$ and f may be vectors and f is a function of the indicated arguments.

However, we will concern ourselves with a somewhat simpler case; namely, when

$$s_{i+1} - s_i = a,$$

a constant, for all values of i. We thus will assume that there is a constant difference between the elements of the sequence S. Further, there is obviously no loss of generality in taking this constant difference a = 1 and, accordingly, we may then choose for the sequence S simply the set I of all nonnegative integers.

In view of the above remarks, the difference equation we shall study is

$$X(n+1) = f(n, X(n)), \qquad (*)$$

where

$$X(n) = \begin{pmatrix} X_1(n) \\ X_2(n) \\ \vdots \\ X_t(n) \end{pmatrix}$$

$$f(n,X(n)) = \begin{pmatrix} f_1(n,X(n)) \\ f_2(n,X(n)) \\ \vdots \\ f_t(n,X(n)) \end{pmatrix} .$$

Here f is a function assuming values in E^t, an arbitrary t-dimensional vector space and defined on

$$D^t_{n_0 R} = \left\{ (n,X) \in I \times E^t : \ n \geq n_0 \geq 0, \ \ 0 \leq \|X\| \leq R \right\} .$$

Here $\|X\|$ denotes any t-dimensional norm of the vector X. The difference equation (*) will be subject to the initial condition

$$X(n_0) = x_0 .$$

Finally, in all of the following, we assume that

$$f(n,0) = 0$$

for all $n \geq n_0$; that is, f is identically zero whenever X(n) is identically zero. An equivalent way of stating this is that $X(n) = 0$ is the trivial solution of equation (*).

The difference equation problem stated above, consisting of equation (*) and the initial condition, will always have a solution and this solution will be unique for all $n \geq n_0$. This may easily be seen since, for given $x_0 = X(n_0)$, $X(n_0+1)$ is uniquely determined by

$$X(n_0+1) = f(n_0,X(n_0)) = f(n_0,x_0) .$$

Similarly, $X(n_o+2)$ is uniquely determined by

$$X(n_o+2) = f(n_o+1, X(n_o+1)) ,$$

and so on, inductively, for every value of $n \geqslant n_o$.

The unique solution of the above difference equation problem, which is equal to x_o for $n = n_o$, is denoted by

$$X(n) = F(n, n_o, x_o),$$

and is such that

$$x_o = X(n_o) = F(n_o, n_o, x_o).$$

Furthermore, we note that if for a particular point (m,Y) in $D^t_{n_o R}$,

$$\| f(m,Y) \| > R,$$

then obviously $Y(m+1)$ is also larger in norm than R. Consequently, $Y(m+2)$ is not defined by the difference equation (*), since $f(m+1,Y(m+1))$ is not defined. As a result of these remarks, unless otherwise mentioned, we will concern ourselves solely in the sequel with those solutions which are defined for all $n \geqslant n_o$. Equivalently, the only solutions considered are those which start in $D^t_{n_o R}$ and remain in it for all $n \geqslant n_o$.

We now define the various types of possible behavior of the solutions of the difference equation problems which will be of interest to us in the sequel.

<u>Definition 1:</u> The equilibrium (or trivial solution) $X = 0$ of the difference equation (*) is said to be <u>stable</u> if, for any $\epsilon > 0$ and any $n_o \in I$, there exists a $\delta(\epsilon, n_o) > 0$ such that $\|x_o\| < \delta$ implies that

$$\| F(n,n_0,x_0) \| < \epsilon$$

for all $n \geq n_0$.

Definition 2: The equilibrium X = 0 of the difference equation (*) is said to be uniformly stable if, for any $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that $n_0 \in I$ and $\| x_0 \| < \delta$ imply

$$\| F(n,n_0,x_0) \| < \epsilon$$

for all $n \geq n_0$.

Definition 3: The equilibrium X = 0 of the difference equation (*) is said to be quasi-asymptotically stable if for any $n_0 \in I$, there exists a $\delta(n_0) > 0$ such that $\| x_0 \| < \delta$ implies

$$F(n,n_0,x_0) \to 0$$

as $n \to \infty$.

Definition 4: The equilibrium X = 0 of the difference equation (*) is said to be asymptotically stable if it is both stable and quasi-asymptotically stable.

Definition 5: The equilibrium X = 0 of the difference equation (*) is said to be quasi-equiasymptotically stable if for any $n_0 \in I$, there exists a $\delta(n_0) > 0$ such that $\| x_0 \| < \delta$ implies

$$F(n,n_0,x_0) \to 0$$

uniformly on $\| x_0 \| < \delta$ as $n \to \infty$.

Definition 6: The equilibrium X = 0 of the difference equation (*) is said to be equiasymptotically stable if it is both stable and quasi-equiasymptotically stable.

Definition 7: The equilibrium X = 0 of the difference equation (*) is said to be quasi-uniformly-asymptotically stable if there exists a $\delta > 0$ such that $n_o \varepsilon I$, $\|x_o\| < \delta$ imply

$$F(n,n_o,x_o) \rightarrow 0$$

uniformly on $n_o \varepsilon I$, $\|x_o\| < \delta$ as $n \rightarrow \infty$.

Definition 8: The equilibrium X = 0 of the difference equation (*) is said to be uniformly-asymptotically stable if it is both stable and quasi-uniformly-asymptotically stable.

Definition 9: The equilibrium X = 0 of the difference equation (*) is said to be exponentially stable if there exists a $B > 1$ and, given any $\epsilon > 0$, a $\delta(\epsilon) > 0$, such that $n_o \varepsilon I$, $\|x_o\| < \delta$ imply

$$\|F(n,n_o,x_o)\| < \epsilon B^{-(n-n_o)}$$

for all $n \geqslant n_o$.

Definition 10: The equilibrium X = 0 of the difference equation (*) is said to be l_p-stable if it is stable and if for all $n_o \varepsilon I$, there exists a $\delta(n_o) > 0$ such that $\|x_o\| < \delta$ implies

$$\sum_{k=n_o}^{\infty} \|F(k,n_o,x_o)\|^p < \infty$$

for some $p > 0$.

Definition 11: The equilibrium X = 0 of the difference equation (*) is said to be unstable if for every $\epsilon > 0$ and for every $n_o \varepsilon I$, there exists some x_o with $\|x_o\| < \epsilon$ such that

$$\|F(n_1,n_o,x_o)\| \geqslant \epsilon$$

for some $n_1 \geqslant n_0$.

<u>Definition 12:</u> The equilibrium $X = 0$ of the difference equation (*) is said to be <u>exponentially unstable</u> if there exist $B > 1$ and $C > 0$ such that for all $r > 0$, there exist x_0 with $\|x_0\| < r$, and for all $n_1 \in I$, there exist $n_0 \geqslant n_1$ such that

$$\|F(n, n_0, x_0)\| \geqslant C\|x_0\| B^{(n-n_0)}$$

for all $n \geqslant n_0$. If the above relation holds for all x_0 with $\|x_0\| < r$, for some $r > 0$, then the equilibrium is said to be <u>completely exponentially unstable.</u>

The investigation of the various types of stability for the trivial solution $X = 0$ of the difference equation (*) will be carried out by using a certain class of real scalar functions, $V(n,X)$, defined on

$$D^t_{n_0',R'} = \left\{ (n,X) \in I \times E^t : n \geqslant n_0' \geqslant 0, \ \|X\| \leqslant R' \right\}$$

and such that $V(n,0) = 0$ for all $n \geqslant n_0'$. In addition, a number of further properties will be required of these functions in various instances. The function $V(n,X)$ is said to be Lipschitzian if for two points (n,X_1) and (n,X_2) in its domain of definition,

$$|V(n,X_1) - V(n,X_2)| \leq B\|X_1 - X_2\|,$$

where B is a positive constant. If this property holds only locally on $D^t_{n_0',R'}$, then we say that the function $V(n,X)$ is locally Lipschitzian on this set.

The function $V(n,X)$ is said to be positive definite on $D^t_{n_o,R'}$ if, given any r, $0 < r < R'$, there exists a real number $b(r) > 0$ such that $V(n,X) \geqslant b$ for all $n \geqslant n_o'$ and all X with $r \leq \|X\| < R'$ and if, for X = 0, V(n,0) = 0 for all $n \geqslant n_o'$. The concept of positive definiteness can also be expressed in terms of the class M_o of all real-valued monotone increasing functions a(r), defined and positive for $r > 0$ and satisfying the condition a(0) = 0. The function V(n,X) is then positive definite if there exists a function a(r) of class M_o such that

$$V(n,X) \geqslant a(\|X\|)$$

for all $n \geqslant n_o'$. This equivalent formulation will often prove more useful.

The function V(n,X) is said to be negative definite if -V(n,X) is positive definite. Finally, V(n,X) is said to be positive semi-definite if $V(n,X) \geqslant 0$ for all $n \geqslant n_o'$; that is, V(n,X) may assume the value zero for some X other than X = 0, and a similar definition holds for V(n,X) being negative semi-definite.

The non-negative scalar function V(n,X) is said to be decrescent (or, equivalently, admits of an infinitesimal upper bound) if there exists a function a(r) of class M_o such that

$$V(n,X) \leq a(\|X\|)$$

for all $n \geqslant n_o'$.

The non-negative scalar function V(n,X) is said to be radially unbounded if for each $a > 0$, there is a $b > 0$ such that $V(n,X) > a$ whenever $\|X\| > b$ and $n \geqslant n_o'$.

Corresponding to the function V(n,X), we define the total difference

$$\Delta V(n,X) = V(n+1,f(n,X)) - V(n,X).$$

For convenience, we shall occasionally write this as

$$\Delta V(n,X) = V(n+1,X(n+1)) - V(n,X).$$

$\Delta V(n,X)$ is obviously a measure of the growth or decay of the function V(n,X) with regard to increasing n along the discrete trajectories represented by the solutions of the difference equation (*). It should be noted that, in general, this can be calculated without direct knowledge of the actual solutions. Moreover, since we are considering only those solutions which start in $D^t_{n_o'R'}$ and remain there for all $n \geqslant n_o'$, $\Delta V(n,X)$ is well-defined for all n and all X.

In the sequel, we consider the set

$$D^t_{n_o R} \cap D^t_{n_o' R'} ,$$

the intersection of the domains of definition of the functions f(n,X) in the difference equation (*) and the functions V(n,X). For simplicity, this intersection will be denoted by $D_{n_o R}$.

Chapter 2: Stability Theory of Difference Equations

2.1 Stability of the Equilibrium

The basic theorem on the stability of the trivial solution of the difference equation

$$X(n+1) = f(n,X(n)) \qquad (*)$$

has been given by Hahn (3) in his Theorem 1.

<u>Theorem (Hahn):</u> If there exists a real scalar function $V(n,X)$ for which, on D_{n_oR} ,

a) $V(n,X)$ is positive definite

b) $\Delta V(n,X)$ is negative semi-definite,

then the equilibrium $X = 0$ of the difference equation (*) is stable.

The following theorem is a partial converse of this result. Its proof depends on the fact that all solutions start at some initial time n_o. Moreover, we consider the set D which consists of those points in D_{n_oR} which are specifically determined by the given difference equation. To illustrate this, consider the scalar equation $X(n+1) = \frac{1}{2}X(n)$. The point $(n_o+1, \frac{3}{4}R)$ is not a point of D since it is not the image under the equation of any point in D_{n_oR}.

<u>Theorem 1:</u> If the equilibrium $X = 0$ of the difference equation (*) is stable on D_{n_oR} , then there exists a real scalar function $V(n,X)$ for which, on D,

a) $V(n,X)$ is positive definite

b) $\Delta V(n,X)$ is negative semi-definite.

<u>Proof:</u> For convenience, we introduce the following notation. We write (n,X) to represent any parameter point in D and N as the independent variable. Thus, $F(N,n,X)$ represents that solution of the difference equation evaluated at time N which passes through the point (n,X). In order to consider values of N for which $n_o \leq N < n$, it is necessary to interpret X as $X = F(n,n_o,x_o)$, for any appropriate x_o from which a solution emanates which passes through (n,X). This x_o need not be unique.

We now consider the scalar function

$$V(n,X) = \|F(n_o,n,X)\| .$$

Since the equilibrium is stable, for all $\epsilon > 0$, there exists a $\delta > 0$ such that $\|F(n_o,n,X)\| < \delta$ implies $\|X\| < \epsilon$. Correspondingly, it follows that for $\|X\| \geq \epsilon$, $V(n,X) \geq \delta > 0$, so that $V(n,X)$ is positive definite.

Moreover,

$$\Delta V(n,X) = \|F(n_o,n+1,X(n+1))\| - \|F(n_o,n,X(n))\| = 0,$$

since $(n,X(n))$ and $(n+1,X(n+1))$ are two successive points along the same trajectory. As a consequence, it follows that $\Delta V(n,X)$ is negative semi-definite and the proof is complete.

2.2 Uniform Stability of the Equilibrium

The following theorem, originally given by Kalman and Bertram (6), extends Hahn's result to sufficient conditions for uniform stability, in their Theorem 1.1.4.

<u>Theorem (Kalman and Bertram):</u> If there exists a real scalar function V(n,X) for which, on $D_{n_o R}$,

a) V(n,X) is positive definite

b) V(n,X) is decrescent

c) Δ V(n,X) is negative semi-definite

then the equilibrium X = 0 of the difference equation (*) is uniformly stable.

Halanay (4) has supplied a converse theorem to the above result, in his Theorem 3.

<u>Theorem (Halanay):</u> If the equilibrium X = 0 of the difference equation (*) is uniformly stable, then there exists a real scalar function V(n,X) such that

$$a(\|X\|) \leq V(n,X) \leq b(\|X\|),$$

for some a(r) and b(r) of class M_o, and such that

$$\Delta V(n,X) \leq 0$$

for all $n \geq n_o$.

These two theorems taken together supply necessary and sufficient conditions for uniform stability. The following theorem gives an alternate set of necessary and sufficient conditions for uniform stability.

Theorem 2: The equilibrium $X = 0$ of the difference equation (*) is uniformly stable if and only if a continuous function $a(r)$ of class M_0 exists such that

$$\| F(n,n_0,x_0) \| \leq a(\| x_0 \|)$$

for every x_0 satisfying $\| x_0 \| \leq \delta$ for some $\delta \leq R$.

Proof: This result, for the case of differential equations, has been given by Hahn (3) in his Theorem 17.1. However, the proof that he gives depends solely on properties of the real number system and functions of real variables and hence carries over unchanged to the present case in which we are considering functions with arguments assuming only discrete values.

This theorem can be used to derive a consequence dealing with the uniform stability of the equilibrium of difference equations in which the function $f(n,X)$ is periodic in n.

Theorem 3: If $f(n,X)$ is periodic in n on $D_{n_0 R}$ and if the equilibrium $X = 0$ of the difference equation

$$X(n+1) = f(n,X(n))$$

is stable, then it is uniformly stable.

Proof: Let the period of $f(n,X)$ with respect to n be m. Then

$$\begin{aligned} F(n_o+m+1,n_o+m,x_o) &= f(n_o+m,F(n_o+m,n_o+m,x_o)) \\ &= f(n_o,x_o) \\ &= F(n_o+1,n_o,x_o). \end{aligned}$$

In a similar way, it follows that

$$F(n+m,n_o+m,x_o) = F(n,n_o,x_o)$$

for all $n \geqslant n_o$. As a result, the proof of the analogous result for differential equations, as given by Hahn (3) in his Theorem 17.2, is equally valid here since only properties of real numbers and real valued functions are used to construct a comparison function satisfying the conditions of Theorem 2.

We note that if $f(n,X)$ is independent of n, that is, if the equation is autonomous, then it is trivially periodic. Consequently, the above theorem generalizes the corresponding result given by Kalman and Bertram (6) for this particular case.

2.3 Asymptotic Stability of the Equilibrium

We now turn to an examination of conditions under which the solutions to the difference equation converge to zero. To begin, we cite a second theorem due to Hahn (3), his Theorem 2, which gives sufficient conditions for asymptotic stability of the equilibrium.

<u>Theorem (Hahn):</u> If there exists a real scalar function $V(n,X)$ for which, on D_{n_oR},

a) $V(n,X)$ is positive definite

b) $V(n,X)$ is decrescent

c) $\Delta V(n,X)$ is negative definite

then the equilibrium $X = 0$ of the difference equation (*) is asymptotically stable.

The following theorem demonstrates that the condition that the function $V(n,X)$ be decrescent in Hahn's result is unnecessary. The author originally used a longer and more complicated proof and wishes to thank Professor R. Datko for suggesting the present more elegant proof.

<u>Theorem 4:</u> If there exists a real scalar function $V(n,X)$ for which, on D_{n_oR},

a) $V(n,X)$ is positive definite

b) $\Delta V(n,X)$ is negative definite

then the equilibrium $X = 0$ of the difference equation (*) is asymptotically stable.

<u>Proof:</u> From the hypotheses, there exist functions $a(r)$ and $b(r)$ of class M_0 such that

$$V(n,X) \geq a(\|X\|) \qquad \Delta V(n,X) \leq -b(\|X\|).$$

Moreover,

$$V(n_0+1, X(n_0+1)) = \Delta V(n_0, X(n_0)) + V(n_0, X(n_0)),$$

$$V(n_0+2, X(n_0+2)) = \Delta V(n_0, x_0) + \Delta V(n_0+1, X(n_0+1)) + V(n_0, x_0),$$

and so, by induction,

$$V(n_0+k, F(n_0+k, n_0, x_0)) = \sum_{j=0}^{k-1} \Delta V(n_0+j, F(n_0+j, n_0, x_0)) + V(n_0, x_0)$$

$$\leq \sum_{j=0}^{k-1} [-b(\|F(n_0+j, n_0, x_0)\|)] + V(n_0, x_0)$$

Taking the limit as $k \to \infty$, and using the fact that $V(n,X)$ is non-negative, we find that

$$\lim_{k\to\infty} \sum_{j=0}^{k-1} b(\|F(n_0+j, n_0, x_0)\|) \leq V(n_0, x_0)$$

which implies that

$$b(\| F(n_0+k, n_0, x_0) \|) \to 0$$

as $k \to \infty$, and therefore, since $b(r)$ is monotonically increasing,

$$F(n_0+k, n_0, x_0) \to 0$$

as $k \to \infty$; i.e., the equilibrium is asymptotically stable.

The following result considerably sharpens the above theorem.

<u>Theorem 5:</u> If there exists a real scalar function $V(n,X)$ for which, on D_{n_oR},

a) $V(n,X)$ is positive definite

b) $\Delta V(n,X)$ is negative definite

then, given any r, $0 < r < R$, there exists a $\delta(n_o,r) > 0$ such that for any x_o with $\|x_o\| < \delta$ and any ϵ, $0 < \epsilon < r < R$, there exists an integer $\nu_o(n_o,r,\epsilon) > 0$ and an integer $n_1(n_o,x_o)$ in the interval $[n_o, n_o+\nu_o]$ such that

$$\| F(n_1,n_o,x_o) \| < \epsilon.$$

<u>Proof:</u> The assumptions on the existence of the function $V(n,X)$ on D_{n_oR} guarantee that the equilibrium $X = 0$ is stable. Hence, given any r, $0 < r < R$, and any n_o in I, there exists a $\delta(n_o,r) > 0$ such that for any x_o with $\|x_o\| < \delta$, we have

$$\| F(n,n_o,x_o) \| < r$$

for $n \geq n_o$.

Now, given any ϵ, $0 < \epsilon < r$, there exist constants $a(\epsilon)$ and $b(\epsilon)$, both positive, such that

$$V(n,X) \geq a \qquad \Delta V(n,X) \leq -b$$

for $n \geq n_o$ and $0 < \epsilon \leq \|X\| < r$. Define

$$q(n_o,r) = \sup \{ V(n_o,X) : \|X\| < \delta \}$$

$$\nu_o(n_o,r,\epsilon) = [\![q/b]\!] + 1,$$

where $[\![q/b]\!]$ represents the greatest integer in q/b.

Now, given any x_o with $\|x_o\| < \delta$, either

$$\| x_o \| \geq \epsilon \qquad \text{or} \qquad \| x_o \| < \epsilon.$$

In the former case, for some $n \geq n_o$,

$$\| F(n,n_o,x_o) \| \geq \epsilon .$$

If $\epsilon \leq \| F(n,n_o,x_o) \| < r$ throughout the interval $[n_o, n_o + \nu_o]$, then

$$\Delta V(n,F(n,n_o,x_o)) \leq -b$$

on this interval. Moreover, we have that

$$\begin{aligned} V(n_o+1,F(n_o+1,n_o,x_o)) &= \Delta V(n_o,F(n_o,n_o,x_o)) + V(n_o,x_o) \\ &\leq -b + V(n_o,x_o). \end{aligned}$$

Continuing in this manner, we determine that

$$\begin{aligned} V(n_o+\nu_o,F(n_o+\nu_o,n_o,x_o)) &= \sum_{k=0}^{\nu_o-1} \Delta V(n_o+k,F(n_o+k,n_o,x_o)) \\ &\qquad + V(n_o,x_o) \\ &\leq V(n_o,x_o) - \nu_o b. \end{aligned}$$

As a consequence,

$$\begin{aligned} a &\leq V(n_o+\nu_o, F(n_o+\nu_o,n_o,x_o)) \\ &\leq V(n_o, F(n_o,n_o,x_o)) - \nu_o b \\ &= V(n_o,x_o) - b [\![q/b]\!] - b \\ &\leq V(n_o,x_o) + (b - q) - b \\ &= V(n_o,x_o) - q \\ &\leq 0, \end{aligned}$$

since $q \geq V(n_o,X)$, which contradicts the assumption that $a > 0$. Therefore, there exists an integer $n_1(n_o,x_o)$ in the interval $[n_o, n_o+\nu_o]$ such that

$$\| F(n_1,n_o,x_o) \| < \epsilon .$$

In the second case, where $\| x_o \| < \epsilon$, we simply let $n_1 = n_o$, so that

$$\| F(n_1,n_o,x_o) \| = \| x_o \| < \epsilon .$$

As an extension of this theorem, we have the following corollary.

Corollary 5.1: Assume the same hypotheses on $V(n,X)$ hold as in Theorem 5, then given any $n_o \in I$ and any r, $0 < r < R$, there exists a $\delta(n_o,r) > 0$, such that for any x_o with $\|x_o\| < \delta$ and any sequence $\{\epsilon_k\}$, $0 < \epsilon_k < r$, such that $\{\epsilon_k\}$ converges monotonically to zero, there exists a non-decreasing sequence $\{n_k\}$ of integers, $n_{k+1} \geq n_k \geq n_o$ such that

$$\| F(n_k,n_o,x_o) \| < \epsilon_k$$

for all k.

Proof: Given any r, $0 < r < R$, and $n_o \in I$, let $\delta(n_o,r)$ and $q(n_o,r) = \sup V(n_o,X) > 0$ be defined as in the proof of Theorem 5. Given the sequence $\{\epsilon_k\}$ converging to zero, $0 < \epsilon_k < r$, there exist sequences $\{a_k\}$ and $\{b_k\}$ of positive elements such that

$$V(n,X) \geq a_k \qquad \Delta V(n,X) \leq -b_k$$

for all $n \geq n_o$ and all X with $\epsilon_k \leq \|X\| < r$. Now define

$$\nu_{k-1}(n_o,r,\epsilon_k) = [\![q/b_k]\!] + 1.$$

From the proof of the theorem, there exists some integer m_k in the interval $[n_o, n_o + \nu_{k-1}]$ such that

$$\| F(m_k,n_o,x_o) \| < \epsilon_k.$$

Now let n_k be the smallest such integer in the interval for which this holds. This procedure determines a sequence $\{n_k\}$, $k = 1,2,\ldots$ with $n_{k+1} \geq n_k \geq n_o$ and such that

$$\| F(n_j, n_o, x_o) \| < \epsilon_j$$

for all j.

The proof of the following result follows directly from that of Theorem 5.

<u>Theorem 6:</u> If there exists a real scalar function $V(n,X)$ for which, on $D_{n_o R}$,

a) $V(n,X)$ is positive definite

b) $V(n,X)$ is bounded

c) $\Delta V(n,X)$ is negative definite,

then, given any $n_o \varepsilon I$ and any r, $0 < r < R$, there exists a $\delta(n_o, r) > 0$ such that for any x_o with $\| x_o \| < \delta$ and any ϵ, $0 < \epsilon < r$, there exists an integer $\nu(r, \epsilon) > 0$ which is independent of n_o, and an integer $n_1(n_o, x_o)$ in the interval $[n_o, n_o + \nu]$ such that

$$\| F(n_1, n_o, x_o) \| < \epsilon .$$

<u>Proof:</u> Let Q be any upper bound for $V(n,X)$ on $D_{n_o R}$. Then the proof of this theorem is the same as the proof of Theorem 5 with q replaced by Q.

We now consider some results on asymptotic stability in the converse direction. The problem can be approached in several ways, either by imposing conditions on the class of functions $f(n,X)$ or by assuming a more stringent form of asymptotic stability. The second possibility will be dealt with in the later sections. We begin the study of the first approach by stating the following Lemma due to Massera (10).

<u>Lemma (Massera):</u> Given any real scalar function $g(r)$ defined and positive on every compact interval $J \subset [0,\infty)$ such that $g(r) \to 0$ as $r \to \infty$; and given any real scalar function $h(r)$, defined and continuous, positive and non-decreasing on $[0,\infty)$, then there exists, for any integer $k > 0$, a positive real scalar function $G(r)$, of class C^k, and increasing together with its first k derivatives on $[0,\infty)$ and with $G^{(i)}(0) = 0$, $i = 0,1,\ldots k$, such that, for any real scalar function $g^*(r)$ on $[0,\infty)$,

$$0 \leq g^*(r) \leq cg(r),$$

for some constant $c > 0$, the integrals

$$\int_0^\infty G^{(i)}(\ g^*(r)\ )\ h(r)\ dr \qquad\qquad 0 \leq i \leq k$$

converge uniformly in g^*.

The analogue of this lemma for difference equations would guarantee the existence of the same scalar function $G(r)$ and the uniform convergence of

$$\sum_{j=0}^{\infty} G^{(i)}(\ g^*(j)\ )\ h(j) \qquad\qquad 0 \leq i \leq k.$$

However, the convergence of these sums in the discrete case follows immediately from the convergence of the corresponding integrals, as given in the Lemma, by the integral test for the convergence of a series. Hence, it follows that the Lemma is valid for the discrete cases we are considering.

We shall make use of this Lemma in the following converse theorem, as well as in other theorems in later sections, where it serves in the construction of a real scalar function $V(n,X)$ under the hypothesis that the equilibrium is asymptotically stable, if the given function $f(n,X)$ is restricted to be linear; i.e.,

$$f(n,X) = A(n)X(n).$$

<u>Theorem 7:</u> If the equilibrium $X = 0$ of the linear difference equation

$$X(n+1) = A(n)X(n)$$

is asymptotically stable, then there exists a real scalar function $V(n,X)$ for which, on D_{n_oR},

a) $V(n,X)$ is positive definite

b) $\Delta V(n,X)$ is negative definite.

<u>Proof:</u> Denote by $Z(n)$ the fundamental matrix solution of the linear difference equation which satisfies the initial condition

$$Z(0) = I,$$

the identity matrix. The general solution of the equation is then given by

$$F(n,n_o,x_o) = Z(n)Z^{-1}(n_o)x_o.$$

Thus, for $n \geq n_o$, replacing x_o, n_o and n respectively by

X, n, and N, we find

$$\| X \| = \| Z(n)Z^{-1}(N)F(N,n,X) \|$$
$$\leq \| Z(n)Z^{-1}(N)\| \, \|F(N,n,X)\| \quad .$$

Now let

$$g(n) = \| Z(n)Z^{-1}(N) \| \quad .$$

For fixed X, g(n) goes to zero as n goes to infinity since the equilibrium is asymptotically stable.

Given $\epsilon > 0$, there exists a constant $a(\epsilon) > 0$ such that

$$\| F(N,n,X) \| \geq a$$

for all $n \geq N$ and $\|X\| \geq \epsilon$. Furthermore, for each X with $\| X \| \geq q$, for any $q > 0$,

$$\| F(N,n,X) \| \to \infty$$

uniformly as $n \to \infty$.

We now define

$$V(n,X) = \sum_{k=n}^{\infty} G\left[\| Z(k)Z^{-1}(N)\| \, \|F(N,n,X)\|\right]$$
$$+ \sum_{k=N}^{\infty} G\left[\| Z(k)Z^{-1}(N)\| \, \|F(N,n,X)\|\right] ,$$

using the discrete form of Massera's Lemma. This function is positive definite since

$$V(n,X) \geq G\left[\| Z(N)Z^{-1}(N)\| \, \| F(N,n,X) \|\right]$$
$$= G(\|F(N,n,X)\|) ,$$

which is zero only for X = 0, for any n.

Moreover,

$$\Delta V(n,X) = \sum_{k=n+1}^{\infty} G\left[\| Z(k)Z^{-1}(N)\| \, \| F(N,n+1,X(n+1))\|\right]$$
$$- \sum_{k=n}^{\infty} G\left[\| Z(k)Z^{-1}(N)\| \, \| F(N,n,X(n))\|\right]$$

$$= \lim_{k\to\infty} G\left[\| Z(k)Z^{-1}(N)\| \, \|F(N,n+1,X(n+1))\|\right]$$

$$- G\left[\| Z(N)Z^{-1}(N)\| \, \|F(N,n,X(n))\|\right]$$

$$= - G\left[\| Z(n)Z^{-1}(N)\| \, \|F(N,n,X)\|\right]$$

$$\leq - G(\|X\|),$$

since

$$\| X\| \leq \|Z(n)Z^{-1}(N)\| \, \|F(N,n,X)\| .$$

This proves that $\Delta V(n,X)$ is negative definite and hence completes the proof of the theorem.

The function $V(n,X)$ constructed in the proof of this theorem possesses the following interesting and useful property.

Corollary 7.1: Given the function $V(n,X)$ constructed in Theorem 7, then given any r, $0<r<R$, and any $\nu \geqslant N$, there exists a $\mu(\nu,r)\geqslant\nu$ such that for any integer m in the interval $[N,\nu]$ and any Y with $\|Y\|\leq r$, the conditions $n\geqslant\mu$ and $V(n,X) \leq V(m,Y)$ both imply that $\|X\|\leq r$.

Proof: For all r, $0<r<R$, and for all $\nu\geqslant N$, choose an integer $\mu(\nu,r)\geqslant\nu$ so large that for m in the interval $[N,\nu]$ and Y with $\|Y\|\leq r$, we have that $n\geqslant\mu$ and

$$2\sum_{k=N}^{\infty} G\left[\| Z(k)Z^{-1}(N)\| \, \|F(N,m,Y)\|\right]$$

$$\geqslant \sum_{k=N}^{\infty} G\left[\| Z(k)Z^{-1}(N)\| \, \|F(N,n,X)\|\right]$$

imply that $\|X\| \leq r$. This is possible since the solutions tend to zero. Hence, if m is in the interval $[N, \nu]$ and Y is such that $\|Y\| \leq r$, then $n \geq \mu$ and

$$V(n,X) \leq V(m,Y)$$

both imply that

$$\sum_{k=N}^{\infty} G\left[\|Z(k)Z^{-1}(N)\| \; \|F(N,n,X)\|\right] \leq V(n,X)$$
$$\leq V(m,Y)$$
$$\leq 2\sum_{k=N}^{\infty} G\left[\|Z(k)Z^{-1}(N)\| \; \|F(N,m,Y)\|\right]$$

which implies that $\|X\| \leq r$.

We shall now consider an alternative set of sufficient conditions for asymptotic stability of the equilibrium. Although the hypotheses are stronger than those in Theorem 4, they may be easier to apply.

Theorem 8: If there exists a real scalar function V(n,X) for which, on $D_{n_0 R}$,

a) V(n,X) is positive definite

b) $\Delta V(n,X) \leq -W(V(n,X))$, for some scalar function W(r) of class M_0

then the equilibrium X = 0 of the difference equation (*) is asymptotically stable.

Proof: Since V(n,X) is positive definite, there exists a function a(r) of class M_0 such that

$$V(n,X) \geq a(\|X\|).$$

Therefore

$$W[V(n,X)] \geqslant W[a(\|X\|)] ,$$

so that

$$\Delta V(n,X) \leqslant - W[V(n,X)] \leqslant - W[a(\|X\|)] = - b(\|X\|),$$

where the composite function $b = W \circ a$ is also of class M_0. Thus, $\Delta V(n,X)$ is negative definite and the equilibrium is asymptotically stable by Theorem 4.

We note that if $V(n,X)$ is also decrescent, then it is possible to reverse the implication that $\Delta V \leqslant -W(V)$ yields $\Delta V \leqslant -b(\|X\|)$.

2.4 Equiasymptotic Stability of the Equilibrium

We next consider a series of theorems dealing with a more restrictive form of asymptotic stability; namely, equiasymptotic stability.

Theorem 9: If there exists a real scalar function $V(n,X)$ for which, on D_{n_0R},

a) $V(n,X)$ is positive definite

b) $\Delta V(n,X)$ is negative definite

c) given any r, $0 < r < R$, and any integer $\nu \geqslant n_0$, there exists an $\epsilon(r)$, $0 < \epsilon < r$, and an integer $\mu(\nu, r) \geqslant \nu$ such that, for some integer m in the interval $[n_0, \nu]$ and for some Y with $\|Y\| < \epsilon$ the conditions $n \geqslant \mu$ and $V(n,X) \leqslant V(m,Y)$ together imply that $\|X\| < r$,

then the equilibrium $X = 0$ of the difference equation (*) is equiasymptotically stable.

Proof: Using Hahn's theorem, it follows that the equilibrium is stable.

Given any r, $0 < r < R$, let $\epsilon(r)$ be the constant corresponding to r, according to the hypothesis c), $0 < \epsilon < r$. By Theorem 5, given r and any $n_o \in I$, there exists a $\delta(n_o, r) > 0$ such that for any x_o with $\|x_o\| < \delta$ and any ϵ, $0 < \epsilon < r$, there exist integers $\nu(n_o, \epsilon) > 0$ and m in the interval $[n_o, n_o + \nu]$ such that

$$\|F(m, n_o, x_o)\| < \epsilon.$$

Moreover, given $n_o + \nu$, let $\mu(n_o + \nu, r) \geq n_o + \nu$ be the integer corresponding to $n_o + \nu$ according to the hypothesis.

Now, it follows that

$$V(n, F(n, n_o, x_o)) \leq V(m, F(m, n_o, x_o))$$

for all $n \geq m$, and this certainly holds for $n \geq \mu$. Consequently,

$$\|F(n, n_o, x_o)\| < r,$$

independently of x_o, for all $\|x_o\| < \delta$, which proves the equiasymptotic stability.

The following theorem gives an alternative set of conditions which also imply the equiasymptotic stability of the equilibrium.

Theorem 10: If there exist two real scalar functions $U(n,X)$ and $V(n,X)$ for which, on D_{n_oR},

a) $U(n,X)$ is positive definite

b) $V(n,X)$ is positive definite

c) $V(n,X)$ is decrescent

d) for any positive r_1, $r_2 < R$, the quantity

$$\Delta V(n,X) + U(n,X) \to 0$$

uniformly as $n \to \infty$ for all X such that

$r_1 \leq \|X\| \leq r_2$,

then the equilibrium $X = 0$ of the difference equation (*) is equiasymptotically stable.

Proof: Let $\{c_k\}$ be any non-increasing sequence which converges to zero,

$$0 < c_k < R, \qquad k = 1,2,\ldots$$

Define

$$\mu_k(c_k) = \inf \{ V(n,X) : n \geq n_o, \ \|X\| = c_k \}$$
$$\lambda_k(c_k) = \sup \{ V(n,X) : n \geq n_o, \ \|X\| < c_k \}.$$

Note that $0 \leq \lambda_k < \infty$, while $\mu_k > 0$. Further, if $V(n,X) = \mu_k$, then $n \geq n_o$ and $b_k \leq \|X\| \leq c_k$, for some constants $b_k(c_k) > 0$. In addition, there exist constants $a_k(c_k) > 0$ and $\rho_k(c_k) > 0$ such that

$$V(n,X) < \mu_k \qquad \text{for } n \geq n_o, \ \|X\| < a_k;$$
$$U(n,X) \geq 2\rho_k \qquad \text{for } n \geq n_o, \ a_k \leq \|X\| < R;$$

that is, $a_k < b_k \leq c_k$, for all k.

Furthermore, there exists a divergent sequence $\{\nu_k\}$ of integers, $\nu_k(c_k) > n_o$, such that

$$\Delta V(n,X) + U(n,X) < \rho_k$$

for $n \geqslant \nu_k$, $a_k \leqslant \|X\| \leqslant c_1$; that is,

$$\Delta V(n,X) < -\rho_k$$

for $n \geqslant \nu_k$, $a_k \leqslant \|X\| \leqslant c_1$. Without any loss of generality, we assume that

$$\nu_{k+2} > \nu_{k+1} + \lambda_k / \rho_{k+1}$$

for all k.

We now make use of the fact that $F(n,n_o,x_o)$ depends continuously on the initial value x_o. Thus, for all $n_o' \geqslant n_o$, there exists a $\delta(c_1,n_o') = \delta^*(\nu_1,n_o') > 0$ such that for any x_o with $\|x_o\| < \delta$, it follows that

$$\|F(n,n_o',x_o)\| < a_1$$

on the interval $[n_o', n_o'+\nu_1]$. Therefore, for $n \geqslant n_o'+\nu_1$,

$$\|F(n,n_o',x_o)\| < c_1,$$

for if not, there would be some $n = n' \geqslant n_o'+\nu_1$ such that

$$\|F(n',n_o',x_o)\| \geqslant c_1.$$

This in turn would imply that

$$V(n',F(n',n_o',x_o)) \geqslant \mu_1 .$$

However, from the construction,

$$V(n_o'+\nu_1, F(n_o'+\nu_1, n_o',x_o)) < \mu_1 .$$

As a consequence,

$$\mu_1 \leq V(n',F(n',n_o',x_o)) \leq V(n_o'+\nu_1,F(n_o'+\nu_1,n_o',x_o)) < \mu_1$$

which is a contradiction. Therefore, we must have

$$\|F(n,n_o',x_o)\| < c_1$$

for all $n \geqslant n_o'$ and for all x_o with $\|x_o\| < \delta$, and accordingly, the equilibrium is stable.

Now, either

$$\| F(n_o' + \nu_2, n_o', x_o) \| < a_2$$

or it is not. In the former case, we conclude that

$$\| F(n, n_o', x_o) \| < c_2$$

for all $n \geqslant n_o' + \nu_2$ by the above chain of reasoning. On the other hand, suppose that

$$\| F(n_o' + \nu_2, n_o', x_o) \| \geqslant a_2;$$

then there would exist an integer n_2 in I_2, where

$$I_2 = [\, n_o' + \nu_2 \, , \; n_o' + \nu_2 + [\![\lambda_1 / \rho_2]\!] \,],$$

such that

$$\| F(n_2, n_o', x_o) \| < a_2;$$

for if not, we would have the condition

$$a_2 \leq \| F(n, n_o', x_o) \| \leq c_1$$

holding throughout I_2. This would imply that

$$\Delta V(n, F(n, n_o', x_o)) < -\rho_2$$

holds throughout I_2 and therefore

$$\begin{aligned} V(n_o' + \nu_2 + [\![\lambda_1/\rho_2]\!], \; & F(n_o' + \nu_2 + [\![\lambda_1/\rho_2]\!], n_o', x_o)) \\ & \leq V(n_o' + \nu_2, \; F(n_o' + \nu_2, n_o', x_o)) - \lambda_1 \\ & \leq 0, \end{aligned}$$

which is a contradiction. Hence, by an argument similar to the one employed above,

$$\| F(n, n_o', x_o) \| < c_2$$

for all $n \geqslant n_2$ and all x_o with $\| x_o \| < \delta$. Moreover, this inequality holds for all $n \geqslant n_o' + \nu_2 + [\![\lambda_1/\rho_2]\!]$; that is, it holds for all $n \geqslant n_o' + \nu_3$. Continuing in this manner, we can show that

$$\| F(n,n_o',x_o) \| < c_k$$

for all $n \geqslant n_o' + \nu_{k+1}$ and for all x_o with $\|x_o\| < \delta$, for all k, which concludes the proof.

<u>Theorem 11:</u> If all of the hypotheses of Theorem 10 are satisfied and if, in addition, the function f(n,X) is Lipschitzian on $D_{n_o R}$ for some constant $K > 0$, then the equilibrium X = 0 of the difference equation (*) is uniform-asymptotically stable.

<u>Proof:</u> The proof of this theorem is essentially the same as that for Theorem 10, except that the new assumption on f(n,X) allows the introduction of a decaying exponential bound of the form

$$\| F(n,n_o,x_o) \| \leq \|x_o\| \quad K^{n-n_o}$$

on the solution $F(n,n_o,x_o)$.

<u>Theorem 12:</u> If the equilibrium X = 0 of the difference equation (*) is uniformly stable and if there exists a real scalar function V(n,X) for which, on $D_{n_o R}$,

a) V(n,X) is positive definite

b) Δ V(n,X) is negative definite

then the equilibrium is equiasymptotically stable.

<u>Proof:</u> The equilibrium is obviously asymptotically stable, by Theorem 4, so that all that need be done is to show that all solutions tend to zero uniformly;

that is, we must show that the equilibrium is quasi-equiasymptotically stable.

Choose $r < R$ such that for any ϵ, $0 < \epsilon \leq r$, there exists a $\delta(\epsilon) > 0$ such that n_0 in I, $\| x_0 \| < \delta$ imply

$$\| F(n,n_0,x_0) \| < \epsilon$$

for $n \geqslant n_0$. Let $\delta_0 = \delta(r)$ be the particular δ corresponding to $\epsilon = r$. By Theorem 5, given n_0 in I and any x_0 with $\| x_0 \| < \delta_0$, there is an integer $\nu(n_0,r,\epsilon) > 0$ and an integer n' in the interval $[n_0,n_0+\nu]$ such that

$$\| F(n',n_0,x_0) \| < \delta(\epsilon).$$

This implies

$$\| F(n,n_0,x_0) \| < \epsilon$$

for all $n \geqslant n'$, and hence, a fortiori, this inequality holds for all $n \geqslant n_0 + \nu \geqslant n'$ and for any x_0 with $\| x_0 \| < \delta$ which concludes the proof.

2.5 Uniform-Asymptotic Stability of the Equilibrium

The results which follow deal with uniform-asymptotic stability, an even more restrictive form of asymptotic stability than the equiasymptotic stability considered in the previous section. The first theorem in this direction has been given by Kalman and Bertram (6) in their Theorem 1.1.

<u>Theorem (Kalman and Bertram):</u> If there exists a real scalar function $V(n,X)$ for which, on D_{n_0R},

a) $V(n,X)$ is positive definite

b) $V(n,X)$ is decrescent

c) $\Delta V(n,X)$ is negative definite

then the equilibrium $X = 0$ of the difference equation (*) is uniformly-asymptotically stable.

We now present a converse to this theorem.

<u>Theorem 13:</u> If the equilibrium $X = 0$ of the difference equation (*) is uniformly-asymptotically stable, then there exists a real scalar function $V(n,X)$ which satisfies on D_{n_0r}, for some r, $0 < r \leq R$, the following conditions:

a) $V(n,X)$ is positive definite

b) $V(n,X)$ is decrescent

c) $V(n,X)$ is locally Lipschitzian

d) $\Delta V(n,X)$ is negative definite.

<u>Proof:</u> Choose r^*, $0 < r^* < R$, so that for all ϵ, $0 < \epsilon \leq r^*$, there exists a $\delta(\epsilon) > 0$ such that for $n \in I$ and X with $\|X\| < \delta$,

$$\|F(n+k,n,X)\| < \epsilon$$

for $k \geq 0$. By the hypothesis that the equilibrium is uniformly-asymptotically stable, it follows that there exists a $\delta_0 > 0$ and, for all $\eta > 0$, there exists an integer $\nu(\eta) \geq n_0$, such that, for $n \in I$ and $\|X\| < \delta_0$,

$$\| F(n+k,n,X) \| < \eta$$

for $k \geq \nu$. Let

$$r = \min(\delta_0 , \delta(r^*))$$

and consider the region $D_{n_o r} \subset D_{n_o R}$ defined by

$$\{ X : \|X\| < r \}.$$

Now, given any non-increasing sequence $\{c_j\}$, $0 < c_j < r$, there exists an increasing divergent sequence $\{n_j\}$, $n_j(c_j) > 0$, such that $(n,X) \in D_{n_o r}$ implies that

$$\| F(n+k,n,X) \| < c_j$$

for all $k \geq n_j$.

Let g(k) be a real scalar function, positive and non-increasing for $k > 0$, such that $g(k) \to 0$ as $k \to \infty$ and, for all $(n,X) \in D_{n_o r}$,

$$\| F(n+k,n,X) \| \leq g(k)$$

on the interval $[0,n_j]$ and let

$$g(n_{j+1}) = c_j$$

for all j. As a result,

$$g(n_{j+1}) \leq g(k) \leq g(n_j)$$

for all k in the interval $[n_j, n_{j+1}]$, which implies

$$\| F(n+k,n,X) \| < c_j \leq g(k)$$

on the interval $[n_j, n_{j+1}]$. This in turn implies

$$\| F(n+k,n,X) \| \leq g(k)$$

for all $k \geq 0$.

Now let G(k) be the function associated with g(k), as given in the discrete form of Massera's Lemma, where we take $h(k) = 1$. Consider the scalar function

$$V(n,X) = \sum_{k=0}^{\infty} G(\|F(n+k,n,X)\|).$$

This function is well-defined on $D_{n_0 r}$ and by the Lemma, $G(s)$ is continuously differentiable, which implies that $V(n,X)$ is also continuously differentiable with respect to X. Also, by the Lemma,

$$\sum_{k=0}^{\infty} G'(\|F(n+k,n,X)\|)$$

converges uniformly, and hence is bounded on $D_{n_0 r}$. As a consequence, the matrix Φ of partial derivatives of $V(n,X)$ with respect to the components of X is also bounded. Thus, applying a generalized form of the mean value theorem to $V(n,X)$, we obtain

$$\begin{aligned} |V(n,X_1) - V(n,X_2)| &= \|\Phi(n,X^*)\| \, \|X_1 - X_2\| \\ &\leq M \|X_1 - X_2\| , \end{aligned}$$

where X^* is some value of X between X_1 and X_2 for each n. The above inequality demonstrates that $V(n,X)$ is locally Lipschitzian.

Moreover, choosing $X_2 = 0$, we see that

$$\begin{aligned} |V(n,X_1) - V(n,0)| &= |V(n,X_1)| \\ &\leq M \|X_1\| , \end{aligned}$$

for each X with $\|X\| < r$; that is,

$$|V(n,X)| \leq M \|X\|,$$

which implies that $V(n,X)$ converges to zero with X, independently of n, which means that $V(n,X)$ is decrescent.

Furthermore,

$$V(n,X) = \sum_{k=0}^{\infty} G[\|F(n+k,n,X)\|]$$

$$\geq G\,[\|F(n,n,X)\|]$$
$$= G(\|X\|),$$

so that V(n,X) is positive definite.

Finally, we must investigate the total difference for V(n,X). This is given by

$$\Delta V(n,X) = \sum_{k=0}^{\infty}\Big[G(\|F(n+k+1,n+1,X(n+1))\|) - G(\|F(n+k,n,X(n))\|)\Big] .$$

This series, however, telescopes and leaves only the first term corresponding to k = 0 and the limiting term. Hence,

$$\Delta V(n,X) = \lim_{k\to\infty} G(\|F(n+k+1,n+1,X(n+1))\|) - G(\|F(n,n,X(n))\|)$$
$$= -\,G(\|X\|),$$

since, by the uniform-asymptotic stability of the equilibrium,

$$\|F(n+k,n,X)\| \to 0$$

as $k \to \infty$ and G(0) = 0. Thus, Δ V(n,X) is negative definite and the theorem is proved.

The previous Theorem 13 has been proved by Halanay (4) in a much more restrictive form. His Theorem 4 is as follows.

<u>Theorem (Halanay):</u> If there exists a function m(r) of class M_0 such that the function f(n,X) satisfies the condition

$$\| f(n,X) \| \geq m(\|X\|),$$

and if the equilibrium $X = 0$ of the difference equation (*) is uniformly-asymptotically stable, then there exists a real scalar function $V(n,X)$ for which, on D_{n_oR},

a) $V(n,X)$ is positive definite

b) $V(n,X)$ is decrescent

c) $\Delta V(n,X)$ is negative definite.

We now present another criterion for the uniform-asymptotic stability of the equilibrium.

<u>Theorem 14:</u> If the equilibrium $X = 0$ of the difference equation (*) is uniformly stable and if there exists a real scalar function $V(n,X)$ for which, on D_{n_oR},

a) $V(n,X)$ is positive definite

b) $V(n,X)$ is bounded

c) $\Delta V(n,X)$ is negative definite

then the equilibrium is uniformly-asymptotically stable.

<u>Proof:</u> Choose $r < R$ such that for any ϵ, $0 < \epsilon \leq r$, there exists a $\delta(\epsilon) > 0$ such that $n_o \varepsilon I$, $\| x_o \| < \delta$ imply that

$$\| F(n,n_o,x_o) \| < \epsilon$$

for all $n \geq n_o$. Let $\delta_o = \delta(r)$ be the particular δ corresponding to $\epsilon = r$. By Theorem 6, given any $n_o \varepsilon I$ and any x_o with $\| x_o \| < \delta_o$, there exists an integer $\nu(r,\epsilon) > 0$, which is independent of n_o, and an integer

$n'(n_o,x_o)$ in the interval $[n_o, n_o+\nu]$ such that

$$\| F(n',n_o,x_o)\| < \delta(\epsilon).$$

Consequently,

$$\| F(n,n_o,x_o)\| < \epsilon$$

for all $n \geq n'$ and hence, a fortiori, this inequality holds for all $n \geq n_o + \nu \geq n'$ and for all x_o with $\|x_o\| < \delta_o$, which concludes the proof.

Before continuing, we digress to develop some additional theory regarding solutions of difference equations which will prove useful in the sequel. The difference equation under consideration is still

$$X(n+1) = f(n,X(n)),$$

but we now require that the function $f(n,X)$ satisfy a Lipschitz condition in D_{n_oR} with constant K with respect to the second coordinate; i.e.,

$$\| f(n,X_1) - f(n,X_2)\| \leq K \| X_1 - X_2 \| .$$

A function $\phi(n)$ is called an ϵ-approximate solution of the difference equation under consideration if

$$\| \phi(n+1) - f(n,\phi(n)) \| < \epsilon$$

for all $n \geq n_o$. The existence of such approximate solutions follows directly from their existence for differential equations.

We now consider two such approximate solutions, $\phi_1(n)$ and $\phi_2(n)$, to the difference equation under consideration which differ from the actual solution by at most ϵ_1 and ϵ_2, respectively. That is,

$$\| \phi_1(n+1) - f(n,\phi_1(n)) \| < \epsilon_1$$
$$\| \phi_2(n+1) - f(n,\phi_2(n)) \| < \epsilon_2 .$$

Further, we assume that at some initial value of n, n_o, these two approximate solutions differ from one another by at most some amount δ. Thus,

$$\| \phi_1(n_o) - \phi_2(n_o) \| \leq \delta .$$

As a result, we have

$$\| \phi_1(n+1) - \phi_2(n+1) - [f(n,\phi_1(n)) - f(n,\phi_2(n))] \| < \epsilon_1 + \epsilon_2 = \epsilon .$$

Hence,

$$\| \phi_1(n+1) - \phi_2(n+1) \| < \epsilon + \| f(n,\phi_1(n)) - f(n,\phi_2(n)) \| \leq \epsilon + K \| \phi_1(n) - \phi_2(n) \| .$$

In particular,

$$\| \phi_1(n_o+1) - \phi_2(n_o+1) \| < \epsilon + K \| \phi_1(n_o) - \phi_2(n_o) \| \leq \epsilon + K\delta .$$

Furthermore,

$$\| \phi_1(n_o+2) - \phi_2(n_o+2) \| < \epsilon + K \| \phi_1(n_o+1) - \phi_2(n_o+1) \| < \epsilon + K\epsilon + K^2\delta .$$

Proceeding inductively, it follows that

$$\| \phi_1(n_o+j) - \phi_2(n_o+j) \| < \epsilon + \epsilon K + \epsilon K^2 + \dots + \epsilon K^{j-1} + \delta K^j = \epsilon \frac{1-K^j}{1-K} + K^j \delta .$$

If, in the above, we now take the solution $F(n,n_o,0)$ to be ϕ_1, so that the corresponding $\epsilon_1 = 0$, and choose

$F(n,n_0,x_0)$ as ϕ_2, where $\|x_0\|<\delta$ and $\epsilon_2 = 0$ also, then the preceding estimate yields the following inequality.

$$\|F(n,n_0,x_0) - F(n,n_0,0)\| = \|F(n,n_0,x_0)\| \leq K^{n-n_0}\delta ,$$

since $\epsilon = \epsilon_1 + \epsilon_2 = 0$. This estimate will be employed in the proof of the next theorem which gives further conditions for uniform-asymptotic stability of the equilibrium.

<u>Theorem 15:</u> If $f(n,X)$ is Lipschitzian with constant K on D_{n_0R} and if the equilibrium $X = 0$ of the difference equation (*) satisfies the following condition:

> given $\delta_0>0$ and any $\epsilon>0$, there exists an integer $N(\epsilon)>0$ such that $\|x_0\|<\delta_0$, $n_0 \in I$ imply that
>
> $$\|F(n,n_0,x_0)\| < \epsilon$$
>
> for all $n \geq n_0+N$,

then the equilibrium is uniformly-asymptotically stable.

<u>Proof:</u> By definition, we need only show that the equilibrium is uniformly stable. Given ϵ, put $\delta(\epsilon) = \epsilon$, for $K<1$, and for $K\geq 1$, let $\delta(\epsilon) = \epsilon K^{-N}$, where N is the integer whose existence is specified in the hypothesis for given ϵ. Using the estimate for the solution of the difference equation obtained above, we find

$$\|F(n,n_0,x_0)\| \leq \delta K^{n-n_0}$$

$$= \begin{cases} \epsilon K^{n-n_o} & \leq \quad \epsilon \qquad (K<1) \\ \epsilon K^{-N}K^{n-n_o} & \leq \quad \epsilon \qquad (K \geqslant 1) \end{cases}$$

for all n in the interval $[n_o, n_o+N]$.

Furthermore, the hypothesis guarantees that for any $n \geqslant n_o+N$, and for the given ϵ,

$$\| F(n,n_o,x_o) \| < \epsilon.$$

Thus, this inequality holds for all $n \geqslant n_o$, and hence we conclude that the equilibrium is indeed uniformly stable.

The following theorem has been given by Hahn(2) for the case of differential equations. However, as was the case with Theorem 2, the proof given by Hahn depends solely on properties of the real number system and of functions of real variables. Accordingly, no modifications are required to deal with functions whose arguments are discrete and the result is stated without proof.

<u>Theorem 16:</u> The equilibrium $X = 0$ of the difference equation (*) is quasi-uniformly-asymptotically stable if and only if there exists a continuous monotonically decreasing function $\sigma(r)$, defined for all $r \geqslant 0$, satisfying the following conditions:

a) $\lim_{r \to \infty} \sigma(r) = 0$

b) $\|F(n,n_o,x_o)\| \leq \sigma(n-n_o)$

for any x_o with $\|x_o\| < \delta$, for some $\delta < R$.

We may combine the two results, Theorem 2 and Theorem 16, to obtain a single necessary and sufficient criterion for uniform-asymptotic stability of the equilibrium.

<u>Theorem 17:</u> The equilibrium $X = 0$ of the difference equation (*) is uniformly-asymptotically stable if and only if there exist two real functions $\rho(r)$ and $\sigma(r)$, such that the following conditions are fulfilled:

a) $\rho(r)$ is defined, continuous, and monotonically increasing for $0 < r < R$ and $\rho(0) = 0$

b) $\sigma(r)$ is defined, continuous, and monotonically decreasing for all $r \geq 0$ and

$$\lim_{r\to\infty} \sigma(r) = 0$$

c) $$\| F(n,n_0,x_0)\| \leq \sigma(n-n_0)\, \rho(\|x_0\|),$$

for any x_0 with $\|x_0\| < \delta$, for some $\delta \leq R$.

In Theorem 3, we saw that if $f(n,X)$ is periodic in n, stability implied uniform stability. We now present a similar result when the equilibrium is known to be asymptotically stable.

<u>Theorem 18:</u> If $f(n,X)$ is periodic in n, and if the equilibrium $X = 0$ of the difference equation (*) is asymptotically stable, then it is also uniformly-asymptotically stable.

Proof: The comparison function, which Hahn (2) constructs in his Theorem 17.5, satisfies the continuous form of Theorem 17. Moreover, this function fulfills the same requirements of Theorem 17 for discrete variables.

As in Theorem 3, the special case dealing with $f(n,X)$ being independent of n has been done by Kalman and Bertram (6) in their Theorem 1.1.1.

Finally, we consider a property which follows from some of the conditions imposed on $V(n,X)$.

Theorem 19: If there exists a real scalar function $V(n,X)$ for which, on $D_{n_o R}$,

a) $V(n,X)$ is positive definite

b) $\Delta V(n,X)$ is negative definite

then, given any r' and r'', $0 < r'' < r' < R$, there exist numbers $q(r',r'') > 0$ and $c(r'') > 0$ such that the function

$$W(n,X) = q^n V(n,X)$$

satisfies the condition

$$\Delta W(n,X) \leq -c$$

for all $n \geqslant n_o$ and $r'' \leq \|X\| < r'$.

Proof: For $0 < r'' \leqslant \|X\| < r' < R$, there exist positive constants a and b such that $V(n,X) \geqslant a$ and $\Delta V(n,X) \leq -b$. Hence, for the function $W(n,X) = q^n V(n,X)$, for any q, $0 < q < 1$, it follows that

$$\begin{aligned}\Delta W(n,X) &= q^n\left[qV(n+1,X(n+1)) - V(n,X(n))\right]\\ &= q^n\left[q\Delta V(n,X) + (q-1)V(n,X)\right]\\ &\le q^n\left[-qb + (q-1)a\right]\\ &= -c.\end{aligned}$$

2.6 Exponential Stability of the Equilibrium

We now consider an even more restrictive form of asymptotic stability; namely, exponential stability, where the solutions of the difference equation must decay exponentially with increasing n. The first theorem of this section deals with sufficient conditions for this type of decay in terms of the existence of a scalar function V(n,X).

<u>Theorem 20:</u> If there exists a real scalar function V(n,X) for which, on D_{n_oR},

a) $$a_1\|X\|^p \le V(n,X) \le a_2\|X\|^p$$

for some positive constants a_1 and a_2 and some $p > 0$

b) $$\Delta V(n,X) \le -a_3\|X\|^p$$

for some positive constant a_3

c) $$a_3/a_2 < 1$$

then the equilibrium X = 0 of the difference equation (*) is exponentially stable.

<u>Proof:</u> From the hypothesis, it follows that

$$\Delta V(n,X) \leq -a_3 \|X\|^p$$
$$\leq (-a_3/a_2)\, V(n,X)$$
$$= -a_4 \; V(n,X),$$

where, by assumption, $0 < a_4 < 1$. Hence,

$$\Delta V(n,X) = V(n+1,X(n+1)) - V(n,X(n)) \leq -a_4 \; V(n,X(n));$$

that is,

$$V(n+1,X(n+1)) \leq (1-a_4)\; V(n,X(n)),$$

so that

$$V(n+2,X(n+2)) \leq (1-a_4)\; V(n+1,X(n+1))$$
$$\leq (1-a_4)^2\; V(n,X(n)).$$

Proceeding inductively, we obtain

$$V(n+k,X(n+k)) \leq (1-a_4)^k\; V(n,X).$$

Thus, for $n = n_o+k$, this becomes

$$V(n,X(n)) \leq (1-a_4)^{n-n_o}\; V(n_o,X(n_o)).$$

Hence,

$$a_1 \;\|X\|^p \leq V(n,X(n)) \leq (1-a_4)^{n-n_o}\; V(n_o,x_o)$$
$$\leq a_2(1-a_4)^{n-n_o} \|x_o\|^p.$$

As a result,

$$\|F(n,n_o,x_o)\|^p \leq (a_2/a_1)(1-a_4)^{n-n_o} \;\|x_o\|^p,$$

so that

$$\|F(n,n_o,x_o)\| \leq B\cdot C^{n-n_o} \|x_o\|;$$

i.e., the equilibrium is exponentially stable.

It might be noted that this result is somewhat more general than the analogous one given by Krasovskii (8) for the case of differential equations where he assumed $p = 2$.

We have seen previously that exponential stability implies all of the other forms of stability considered so far. We consider now a partial converse of one of these implications when the given function $f(n,X)$ is linear.

<u>Theorem 21:</u> If the equilibrium $X = 0$ of the linear difference equation

$$X(n+1) = A(n)X(n)$$

is uniformly asymptotically stable, then it is also exponentially stable.

<u>Proof:</u> Since the difference equation under consideration is linear, its general solution is given by

$$F(n,n_o,x_o) = Z(n)Z^{-1}(n_o)x_o,$$

where $Z(n)$ again denotes the fundamental matrix solution for which

$$Z(0) = I.$$

Let the norm of $Z(n)Z^{-1}(n_o)$ be denoted by $b(n,n_o)$, so that

$$\| F(n,n_o,x_o) \| = \| Z(n)Z^{-1}(n_o)x_o \| \leq b(n,n_o) \| x_o \| .$$

Comparing this estimate with the hypotheses of Theorem 17, we see that we can take

$$\rho(r) = r$$

and

$$b(n,n_o) \leq \sigma(n-n_o);$$

that is,

$$\| F(n,n_o,x_o) \| \leq \| x_o \| \sigma(n-n_o).$$

Now, for $n \geqslant n_1 \geqslant n_o$, it follows that

$$Z(n)Z^{-1}(n_o) = Z(n)Z^{-1}(n_1)Z(n_1)Z^{-1}(n_o),$$

and as a consequence,

$$\begin{aligned} b(n,n_o) &= \| Z(n)Z^{-1}(n_o) \| \\ &\leq \| Z(n)Z^{-1}(n_1) \| \, \| Z(n_1)Z^{-1}(n_o) \| \\ &= b(n,n_1)\, b(n_1,n_o). \end{aligned}$$

Now consider $n = n_o + k\nu$, for some $\nu > 0$. Thus

$$b(n_o + \nu,\, n_o) \leqslant \sigma(\nu).$$

Further,

$$b(n_o + k\nu, n_o) \leqslant \left[\sigma(\nu)\right]^k,$$

as seen from the following induction argument. The inequality has already been established for $k = 1$, so that we assume

$$b(n_o + k\nu, n_o) \leqslant \left[\sigma(\nu)\right]^k$$

and consider the (k+1)st term.

$$\begin{aligned} b\left[n_o+(k+1)\nu, n_o\right] &\leqslant b\left[n_o+(k+1)\nu, n_o+k\nu\right] b(n_o+k\nu, n_o) \\ &\leqslant \sigma\left[(n_o+k\nu+\nu) - (n_o+k\nu)\right]\left[\sigma(\nu)\right]^k \\ &= \sigma(\nu)\left[\sigma(\nu)\right]^k \\ &= \left[\sigma(\nu)\right]^{k+1}. \end{aligned}$$

Since $\sigma(n-n_o)$ goes monotonically to zero by assumption, there exists a ν sufficiently large so that for $n = n_o + \nu$, or $n - n_o = \nu$,

$$\sigma(\nu) < \tfrac{1}{2}.$$

Thus,

$$b(n_o+k\nu, n_o) \leqslant \left[\sigma(\nu)\right]^k < (\tfrac{1}{2})^k.$$

But, since $n = n_o + k\nu$, we have

$$k = (n-n_o)/\nu ,$$

so that

$$b(n,n_o) < (2)^{-(n-n_o)/\nu} \quad .$$

It therefore follows that

$$\| F(n,n_o,x_o) \| \leq b(n,n_o) \|x_o\|$$

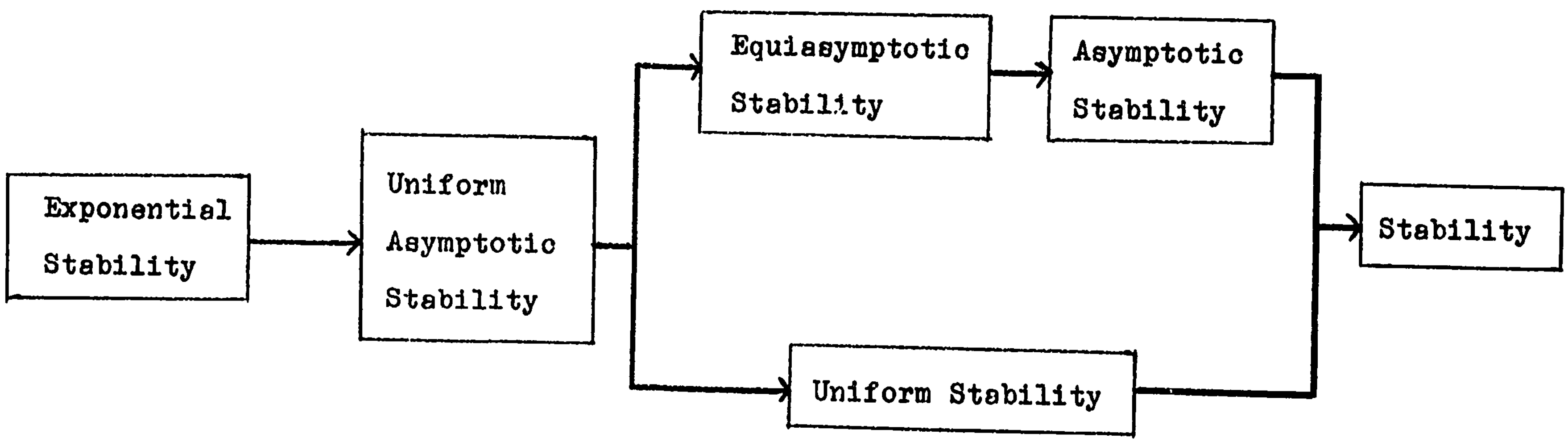
$$< (2)^{-(n-n_o)/\nu} \|x_o\|$$

and the equilibrium is exponentially stable.

2.7 l_p-Stability of the Equilibrium

Thus far, the types of stability considered may be grouped into a series of chains in which each successive concept of stability implies all that precede it. Thus, the most restrictive definition, that of exponential stability, implies uniform-asymptotic stability, which in turn implies equiasymptotic stability. Continuing, equiasymptotic stability implies asymptotic stability, which then implies stability itself. In a separate direction, the chain branches so that uniform-asymptotic stability implies uniform stability, which in turn gives stability. The various implications are illustrated in the accompanying diagram.

We now consider another type of stability which does not quite fit into either of these chains of successive types of stability; namely, l_p-stability. From its definition, given in Definition 10, we require that the


Exponential Stability
Uniform Asymptotic Stability
Equiasymptotic Stability
Asymptotic Stability
Uniform Stability
Stability

p-th powers of the solution of the difference equation

$$X(n+1) = f(n,X(n))$$

are summable for some $p > 0$. That is,

$$\sum_{k=n_0}^{\infty} \| F(k,n_0,x_0) \|^p < \infty .$$

The convergence of such a sum for positive p implies automatically that

$$\| F(n,n_0,x_0) \|^p \to 0$$

as $n \to \infty$, so that $\| F(n,n_0,x_0) \|$ itself goes to zero with increasing n. Thus, l_p-stability obviously implies asymptotic stability. The following theorem treats the relationship between l_p-stability and exponential stability, the other end of the chain.

<u>Theorem 22:</u> If the equilibrium $X = 0$ of the difference equation (*) is exponentially stable, then it is also l_p-stable.

<u>Proof:</u> If the equilibrium is exponentially stable, the solution to the difference equation satisfies an estimate of the form

$$\| F(n,n_0,x_0) \| \leq B \| x_0 \| c^{n-n_0},$$

for some c, $0 < c < 1$, and some $B > 0$. Therefore

$$\| F(n,n_0,x_0) \|^p \leq B^p \| x_0 \|^p c^{p(n-n_0)}$$

and hence

$$\sum_{k=n_0}^{\infty} \| F(k,n_0,x_0) \|^p \leq B^p \| x_0 \|^p \sum_{k=n_0}^{\infty} (c^p)^{k-n_0}$$

$$= \frac{B^p \|x_0\|^p}{c^{n_0 p}} \sum_{k=n_0}^{\infty} (c^p)^k$$

$$\leq \frac{B^p \|x_0\|^p}{c^{n_0 p}} \sum_{k=0}^{\infty} (c^p)^k$$

$$= \frac{B^p \|x_0\|^p}{c^{n_0 p}} \cdot \frac{1}{1-c^p}$$

$$< \infty$$

for any c, $0 < c < 1$.

On the other hand, the following rather trivial example indicates that asymptotic stability, and even equiasymptotic stability, does not necessarily imply l_p-stability.

Example: Consider the scalar difference equation

$$X(n+1) = \frac{\log(n+2)}{\log(n+3)} X(n),$$

which may be written as $X(n+1) = A(n)X(n)$, so that the equation is linear. The solution of the linear difference equation is given by

$$F(n,n_o,x_o) = Z(n)Z^{-1}(n_o)x_o,$$

which for this particular equation becomes

$$F(n,n_o,x_o) = \frac{1}{\log(n+2)} \log(n_o+2)\, x_o.$$

It is obvious that as n goes to infinity, the solution tends to zero, so that the equilibrium is indeed asymptotically stable. In fact, since the difference equation is linear, the equilibrium must be equiasymptotically stable.

However,

$$\sum_{k=n_o}^{\infty} \|F(k,n_o,x_o)\|^p = [\log(n_o+2)]^p \|x_o\|^p \sum_{k=n_o}^{\infty} \left[\frac{1}{\log(k+2)}\right]^p$$

and the latter series is known to diverge for all $p \geq 0$. Hence, the equilibrium for this difference equation is not l_p-stable for any p.

We now present a sufficient criterion for l_p-stability in the form of the existence of a scalar function $V(n,X)$.

<u>Theorem 23:</u> If there exists a real scalar function $V(n,X)$ for which, on D_{n_oR},

a) $V(n,X)$ is positive definite

b) $\Delta V(n,X) \leq -c\|X\|^p$, for some $p \geq 0$ and some $c > 0$

then the equilibrium $X = 0$ of the difference equation (*) is l_p-stable.

<u>Proof:</u> The existence of the function $V(n,X)$ with a negative definite total difference implies that the equilibrium is stable. Thus, given any $n_o \geq 1$, a δ_o can be chosen such that $\|x_o\| < \delta_o$ implies that

$$\|F(n,n_o,x_o)\| < M$$

for some $M > 0$ and for all $n \geq n_o$.

Now, define, for $n \geq n_1$, for any $n_1 > n_o$,

$$G(n) = V(n,F(n,n_o,x_o)) + c\sum_{k=n_1-1}^{n-1} \|F(k,n_o,x_o)\|^p.$$

We note that

$$G(n_1) = V(n_1,F(n_1,n_o,x_o)) + c\|F(n_1-1,n_o,x_o)\|^p$$
$$= V(n_1,x_1) + c\|F(n_1-1,n_o,x_o)\|^p ,$$

where we have written $x_1 = F(n_1,n_o,x_o)$. Furthermore,

$$\Delta G(n) = \Delta V(n,F(n,n_o,x_o)) + c\|F(n,n_o,x_o)\|^p$$
$$\leq -c\|F(n,n_o,x_o)\|^p + c\|F(n,n_o,x_o)\|^p ,$$

so that

$$\Delta G(n) \leq 0.$$

As a consequence, it follows that

$$G(n) \leq G(n_1)$$

for all $n \geq n_1$. That is,

$$V(n,F(n,n_o,x_o)) + c\sum_{k=n_1-1}^{n-1}\|F(k,n_o,x_o)\|^p$$
$$\leq V(n_1,x_1) + c\|F(n_1-1,n_o,x_o)\|^p ,$$

or equivalently,

$$0 \leq V(n,F(n,n_o,x_o)) \leq V(n_1,x_1) - c\sum_{k=n_1}^{n-1}\|F(k,n_o,x_o)\|^p ,$$

so that

$$c\sum_{k=n_1}^{n-1}\|F(k,n_o,x_o)\|^p \leq V(n_1,x_1) \leq V(n_o,x_o)$$

for all $n \geq n_1$. This implies that

$$\sum_{k=n_1}^{n-1}\|F(k,n_o,x_o)\|^p \leq (1/c)V(n_o,x_o).$$

Taking the limit as $n \to \infty$, we find that

$$\sum_{k=n_1}^{\infty}\|F(k,n_o,x_o)\|^p \leq (1/c)V(n_o,x_o),$$

which implies

$$\sum_{k=n_o}^{\infty}\|F(k,n_o,x_o)\|^p < \infty ,$$

and thus the equilibrium is l_p-stable.

We now turn to possible converses to the previous theorem which will guarantee the existence of scalar functions $V(n,X)$. The first such result, for the case of a linear difference equation, requires nothing more than the asumption that the equilibrium is l_p-stable. However, for the case of arbitrary $f(n,X)$, additional asumptions will be necessary.

Theorem 24: If the equilibrium $X = 0$ of the linear difference equation

$$X(n+1) = A(n)X(n)$$

is l_p-stable, then there exists a real scalar function $V(n,X)$ for which, on D_{n_oR},

a) $V(n,X)$ is positive definite

b) $V(n,X)$ is locally Lipschitzian

c) $\Delta V(n,X) \leq -\|X\|^p$.

Proof: Since the difference equation is linear, the general solution is given by

$$F(n,n_o,x_o) = Z(n)Z^{-1}(n_o)x_o.$$

Further, since l_p-stability implies stability, there exists an $M > 0$ such that

$$\| Z(n) \| \leq M$$

for all $n \geq n_o$.

We now define

$$s(n) = \| Z(n) \|$$

and

$$\phi(n,X) = \| Z^{-1}(n)X \| ,$$

so that

$$\phi(n_o,x_o) = \| Z^{-1}(n_o)x_o \| ,$$

and therefore, it follows that

$$\| F(n,n_o,x_o) \| \leq s(n)\phi(n_o,x_o).$$

As a consequence,

$$\| x_o \| = \| F(n_o,n_o,x_o) \| \leq s(n_o)\phi(n_o,x_o).$$

Now define

$$V(n,X) = \sum_{k=n}^{\infty} [s(k)\phi(n,X)]^p + \sum_{k=0}^{\infty} [s(k)\phi(n,X)]^p$$

for all points (n,X) in D_{n_oR}. Both of these series converge since the equilibrium is l_p-stable. The proof that this function satisfies the theorem depends on the following three properties for this choice of $V(n,X)$.

Property 1: There exist two positive constants c_1 and c_2 such that

$$c_1\|X\|^p \leq V(n,X) \leq c_2\| Z^{-1}(n)\|^p\|X\|^p$$

for all points (n,X) in D_{n_oR}.

We have

$$\begin{aligned} V(n,X) &\leq 2\sum_{k=0}^{\infty} [s(k)\phi(n,X)]^p \\ &= 2 [\phi(n,X)]^p \sum_{k=0}^{\infty} [s(k)]^p \\ &\leq 2 \| Z^{-1}(n)\|^p \|X\|^p \sum_{k=0}^{\infty} [s(k)]^p \\ &= c_2 \| Z^{-1}(n)\|^p \|X\|^p. \end{aligned}$$

Furthermore,

$$\begin{aligned} V(n,X) &\geq \sum_{k=0}^{\infty} [s(k)\ \phi(n,X)]^p \\ &= \sum_{k=0}^{\infty} [s(k)\ \| Z^{-1}(n)X\|]^p \\ &\geq \sum_{k=0}^{\infty} [s(k)\,\|X\|]^p/[s(n)]^p \end{aligned}$$

$$= \sum_{k=0}^{\infty} [s(k)]^p \, \|X\|^p \,/\, \|Z(n)\|^p$$
$$\geqslant (1/M)^p \, \|X\|^p \sum_{k=0}^{\infty} [s(k)]^p$$
$$= c_1 \|X\|^p.$$

Moreover, it is evident that $V(n,0) = 0$, for all $n \geqslant n_o$.

Property 2: $\Delta V(n,X) \leq -\|X\|^p.$

We have

$$\phi(n,X) = \|Z^{-1}(n)X\| = \|Z(0)Z^{-1}(n)X\|$$
$$= \|F(0,n,X)\|.$$

Hence,

$$\phi(n,F(n,n_o,x_o)) = \|F(0,n,F(n,n_o,x_o))\|$$
$$= \|Z^{-1}(n)Z(n)Z^{-1}(n_o)x_o\|$$
$$= \|Z^{-1}(n_o)x_o\|$$
$$= \phi(n_o,x_o).$$

Now

$$V(n,F(n,n_o,x_o)) = \sum_{k=n}^{\infty} [s(k)\phi(n_o,x_o)]^p + \sum_{k=0}^{\infty} [s(k)\phi(n_o,x_o)]^p,$$

where the second term is independent of n for all (n_o,x_o) in $D_{n_o R}$. Therefore,

$$\Delta V(n,F(n,n_o,x_o)) = V(n+1,F(n+1,n_o,x_o)) - V(n,F(n,n_o,x_o))$$
$$= -[s(n)\ \phi(n_o,x_o)]^p$$
$$= -[\|Z(n)\| \ \|Z^{-1}(n_o)x_o\|]^p.$$

As a result,

$$\Delta V(n_o,F(n_o,n_o,x_o)) = -[\|Z(n_o)\| \ \|Z^{-1}(n_o)x_o\|]^p$$
$$\leq -\|x_o\|^p.$$

However, this inequality is valid for every (n_o,x_o) in

D_{n_0R}, so that we may conclude that

$$\Delta V(n,X) \leq -\|X\|^p$$

for every (n,X) in D_{n_0R}.

Property 3: There exists a positive constant c_3 such that

$$|V(n,X_1) - V(n,X_2)| \leq c_3 \left[\|X_1\|^{p-1} + \|X_2\|^{p-1}\right]\|Z^{-1}(n)\|^p \|X_1-X_2\|.$$

We have

$$\begin{aligned} |V(n,X_1) - V(n,X_2)| &= \Big\| \sum_{k=n}^{\infty}\left\{[s(k)\phi(n,X_1)]^p - [s(k)\phi(n,X_2)]^p\right\} \\ &\quad + \sum_{k=0}^{\infty}\left\{[s(k)\phi(n,X_1)]^p - [s(k)\phi(n,X_2)]^p\right\}\Big\| \\ &\leq 2 \Big\| \sum_{k=0}^{\infty} [s(k)]^p\left\{[\phi(n,X_1)]^p - [\phi(n,X_2)]^p\right\}\Big\| \\ &\leq 2 \sum_{k=0}^{\infty} [s(k)]^p \Big|\|Z^{-1}(n)X_1\|^p - \|Z^{-1}(n)X_2\|^p\Big| . \end{aligned}$$

Now, by the mean value theorem, given any two real numbers r_1 and r_2,

$$\begin{aligned} |r_1{}^p - r_2{}^p| &= |p\, r^{p-1} (r_1-r_2)| \\ &\leq p(r_1{}^{p-1} + r_2{}^{p-1})\, |r_1-r_2| , \end{aligned}$$

where $r_1 \leq r \leq r_2$. Applying this result to the above, we obtain

$$\begin{aligned} &\Big|\|Z^{-1}(n)X_1\|^p - \|Z^{-1}(n)X_2\|^p\Big| \\ &\quad\leq p\left[\|Z^{-1}(n)X_1\|^{p-1} + \|Z^{-1}(n)X_2\|^{p-1}\right] \cdot \\ &\qquad\cdot \Big|\|Z^{-1}(n)X_1\| - \|Z^{-1}(n)X_2\|\Big| \end{aligned}$$

$$\leq p \|Z^{-1}(n)\|^{p-1} \left[\|X_1\|^{p-1} + \|X_2\|^{p-1}\right] \cdot$$

$$\cdot \|Z^{-1}(n)X_1 - Z^{-1}(n)X_2\|$$

$$\leq p \|Z^{-1}(n)\|^{p} \left[\|X_1\|^{p-1} + \|X_2\|^{p-1}\right] \|X_1 - X_2\| .$$

Hence,

$$|V(n,X_1) - V(n,X_2)|$$

$$\leq 2\sum_{k=0}^{\infty}[s(k)]^p \, p \|Z^{-1}(n)\|^p \left[\|X_1\|^{p-1} + \|X_2\|^{p-1}\right] \cdot$$

$$\cdot \|X_1 - X_2\|$$

$$= c_3 \|Z^{-1}(n)\|^p \left[\|X_1\|^{p-1} + \|X_2\|^{p-1}\right] \|X_1 - X_2\| .$$

Thus, Property 1 shows that $V(n,X)$ is positive definite, while Property 2 demonstrates the condition on $\Delta V(n,X)$ and Property 3 proves that $V(n,X)$ is locally Lipschitzian.

We now consider the possibility of proving the existence of a scalar function $V(n,X)$ for the arbitrary difference equation (*). In order to obtain any results, it is necessary to impose more stringent conditions than merely l_p-stability for the equilibrium, as was done in Theorem 24.

<u>Theorem 25:</u> Suppose that the equilibrium $X = 0$ of the difference equation (*) is l_p-stable and further, suppose that for each fixed n, $\frac{d}{dX} F(k,n,X)$ exists for any point in D_{n_oR}; moreover, suppose that

$$s = \sup_{(k,X)\in D_{n_oR}} \left\{ \sum_{k=0}^{\infty} \|F(k+n,n,X)\|^{p-1} \|\Phi(k+n,n,X_1^*,\ldots X_t^*)\| \right\}$$

is bounded, where Φ is the matrix of partial derivatives

$$\Phi(k+n,n,X_1^*,\dots X_t^*) = \left(\frac{\partial F_i(k+n,n,X_i^*)}{\partial X^j}\right),$$

X^j are the components of the vector X and $(n,X_1^*),\dots(n,X_t^*)$ are points in D_{n_oR}. Then there exists a real scalar function $V(n,X)$ for which, on D_{n_oR},

a) $V(n,X)$ is positive definite

b) $V(n,X)$ is locally Lipschitzian

c) $\Delta V(n,X) \leq -\|X\|^p$.

<u>Proof:</u> We define the function

$$V(n,X) = \sum_{k=n}^{\infty} \|F(k,n,X)\|^p$$

and demonstrate that it possesses all of the required properties for the function described in the statement of the theorem. This function can be rewritten as

$$V(n,X) = \sum_{k=0}^{\infty} \|F(k+n,n,X)\|^p .$$

Furthermore,

$$V(n,0) = \sum_{k=0}^{\infty} \|F(k+n,n,0)\|^p = 0,$$

since $F(k,n,0) = 0$ for all k. Moreover,

$$V(n,X) \geqslant \|F(n,n,X)\|^p = \|X\|^p,$$

which proves that $V(n,X)$ is positive definite.

We now apply the estimate derived in the proof of Property 3 in the previous theorem to obtain

$$|V(n,X_1) - V(n,X_2)|$$

$$\leq \sum_{k=0}^{\infty} \left| \|F(k+n,n,X_1)\|^p - \|F(k+n,n,X_2)\|^p \right|$$

$$\leq \sum_{k=0}^{\infty} p\left[\|F(k+n,n,X_1)\|^{p-1} + \|F(k+n,n,X_2)\|^{p-1}\right] \cdot$$

$$\cdot \left| \|F(k+n,n,X_1)\| - \|F(k+n,n,X_2)\| \right| .$$

Imposing the requirement that the solution $F(k,n,X)$ be a differentiable function of X for each fixed n, then on the closed interval $[(n,X_1), (n,X_2)]$ in D_{n_oR}, we may apply a generalized form of the mean value theorem to obtain

$$F(k+n,n,X_1) - F(k+n,n,X_2) = \Phi(k+n,n,X_1^*,..X_t^*)\ (X_1-X_2),$$

where Φ and the X_i^* are as defined in the statement of the theorem, for some points (n,X_1^*), ... (n,X_t^*) in the closed interval. Hence, the above estimate for the difference between $V(n,X)$ evaluated for two different values of X becomes

$$|V(n,X_1) - V(n,X_2)|$$

$$\leq \sum_{k=0}^{\infty} p\left[\|F(k+n,n,X_1)\|^{p-1} + \|F(k+n,n,X_2)\|^{p-1}\right] \cdot$$

$$\cdot \|\Phi(k+n,n,X_1^*,...X_t^*)\| \, \|X_1-X_2\|,$$

$$\leq 2ps\|X_1 - X_2\| ;$$

that is, $V(n,X)$ is locally Lipschitzian.

Finally, we consider the total difference for $V(n,X)$. To begin, we note that

$$F(k,n,F(n,n_o,x_o)) = F(k,n_o,x_o)$$

since the solution to a difference equation through any given point is unique and since, at the point (n_o,x_o),

$$F(k,n_0,F(n_0,n_0,x_0)) = F(k,n_0,x_0).$$

Hence,

$$V(n,F(n,n_0,x_0)) = \sum_{k=n}^{\infty} \|F(k,n_0,x_0)\|^p,$$

so that

$$\begin{aligned} \Delta V(n,X) &= \sum_{k=n+1}^{\infty} \|F(k,n_0,x_0)\|^p - \sum_{k=n}^{\infty} \|F(k,n_0,x_0)\|^p \\ &= -\|F(n,n_0,x_0)\|^p \\ &\leq 0. \end{aligned}$$

In fact, $\Delta V(n,X)$ is equal to zero if and only if $F(n,n_0,x_0)$ is equal to zero, which occurs if and only if $x_0 = 0$. Thus we conclude that $\Delta V(n,X)$ is negative definite and hence this function $V(n,X)$ satisfies all of the requirements of the theorem.

We now consider the linear difference equation

$$X(n+1) = A(n)X(n)$$

and the associated equation

$$X(n+1) = A(n)X(n) + g(n,X(n)),$$

where the term $g(n,X(n))$ may be considered as a perturbation of the linear system. If the equilibrium $X = 0$ of the linear equation possesses some form of stability and if the perturbation is small in some sense, then it is reasonable to expect that the equilibrium of the perturbed equation should share the same stability property. The following theorem provides proof of this expectation for the case of l_p-stability.

<u>Theorem 26:</u> If the equilibrium $X = 0$ of the linear diffe-

rence equation

$$X(n+1) = A(n)X(n)$$

is l_p-stable, then the equilibrium of the associated perturbed equation

$$X(n+1) = A(n)X(n) + g(n,X(n))$$

is also l_p-stable provided the perturbation $g(n,X(n))$ satisfies the conditions:

a) $g(n,0) = 0$ for all $n \geqslant n_o$

b) $\| A(n)X(n) + g(n,X(n))\| \leq \|A(n)X(n)\|$

c) $\dfrac{\|Z^{-1}(n)\|^p \|g(n,X(n))\|}{\|X(n)\|} \longrightarrow 0$ uniformly as $\|X\| \to 0$.

<u>Proof:</u> If $X(n) = 0$ is l_p-stable for the linear equation, then, by Theorem 24, there exists a scalar function $V(n,X)$ satisfying the conditions on $D_{n_o R}$,

1. $c_1\|X\|^p \leq V(n,X) \leq c_2 \|Z^{-1}(n)\|^p \|X\|^p$
2. $\Delta V(n,X) \leq -\|X\|^p$
3. $|V(n,X_1) - V(n,X_2)| \leq c_3 \left[\|X_1\|^{p-1} + \|X_2\|^{p-1}\right] \cdot \|Z^{-1}(n)\|^p \|X_1-X_2\|$.

This function will be used to prove the l_p-stability of the perturbed equation. The total difference of this function for the perturbed equation is given by

$$\begin{aligned}
\Delta V_p(n,X) &= V(n+1,AX + g) - V(n,X) \\
&= V(n+1,AX + g) - V(n+1,AX) + V(n+1,AX) - V(n,X) \\
&\leq |V(n+1,AX+g) - V(n+1,AX)| + V(n+1,AX) - V(n,X) \\
&\leq c_3 \left[\| AX+g\|^{p-1} + \| AX\|^{p-1}\right] \| Z^{-1}(n)\|^p \|g\| \\
&\quad + \Delta V_L(n,X)
\end{aligned}$$

$$\leq c_3 \left[2\|AX\|^{p-1} \right] \|Z^{-1}(n)\|^p \|g\| - \|X\|^p,$$

using condition b) on $AX + g$ and the definition of total difference of $V(n,X)$ for the linear difference equation. Thus,

$$\Delta V_p(n,X) \leq 2c_3\|A\|^{p-1}\|X\|^{p-1} \| Z^{-1}(n)\|^p \|g\| - \|X\|^p$$

$$= -\|X\|^p \left[1 - \frac{2c_3 \| A\|^{p-1} \|Z^{-1}(n)\|^p \|g\|}{\|X\|} \right]$$

$$\leq - c\|X\|^p,$$

for some positive $c < 1$ for $\|X\|$ sufficiently small since by assumption c),

$$\frac{\|Z^{-1}(n)\|^p \|g\|}{\|X\|} \to 0$$

with $\|X\|$. Hence,

$$\Delta V_p(n,X) \leq - \|X\|^p$$

and the equilibrium of the perturbed equation is also l_p-stable by an application of Theorem 23.

2.8 Stability Under Perturbations

We now consider in greater depth the situation where a given difference equation is altered by the addition of a "small" perturbing term to $f(n,X)$. In particular, we consider the difference equation (*)

$$X(n+1) = f(n,X(n))$$

and the associated perturbed difference equation

$$Y(n+1) = f(n,Y(n)) + g(n,Y(n)).$$

As before, we impose the condition

$$f(n,0) = 0$$

for all $n \geqslant n_0$. However, there are two possible approaches to follow with regard to $g(n,Y)$. The first is where the perturbing function satisfies a condition of the form

$$g(n,0) = 0$$

for all $n \geqslant n_0$ also. In this case, the trivial solution $X = 0$ is a common solution to both the perturbed and the unperturbed difference equations. The second possibility arises when the above condition on $g(n,Y)$ does not hold, but when it is known that the perturbation is "small" in some sense.

As an example of the first possibility, we consider the following theorem originally given by Hahn (3) and dealing with the case where the equilibrium of the unperturbed equation is exponentially stable.

Theorem (Hahn): If the equilibrium $X = 0$ of the linear difference equation

$$X(n+1) = A(n)X(n)$$

is exponentially stable, then the equilibrium of the perturbed equation

$$Y(n+1) = A(n)Y(n) + g(n,Y(n))$$

is also exponentially stable, provided that

a) $g(n,0) = 0$ for all $n \geqslant n_0$

b) $\|g(n,Y)\| \leq a\,\|Y\|$,

for some sufficiently small constant a.

We now turn to an examination of maintaining some type of stability property when the perturbation is small, but not necessarily zero for $Y = 0$. We begin by introducing one definition of stability under such small perturbations.

<u>Definition 13:</u> The equilibrium $X = 0$ of the unperturbed difference equation (*) is said to be <u>totally stable</u> if, for every $\epsilon > 0$, there exist two positive constants $\delta_1(\epsilon)$ and $\delta_2(\epsilon)$ such that

$$\|x_0\| < \delta_1 \qquad \|g(n,Y(n))\| < \delta_2$$

for all (n,Y) in D_{n_0R} imply that

$$\|F^*(n,n_0,x_0)\| < \epsilon$$

for all $n \geq n_0$ for every solution $F^*(n,n_0,x_0)$ of the perturbed difference equation

$$Y(n+1) = f(n,Y(n)) + g(n,Y(n)).$$

The following theorem for total stability of the equilibrium is a consequence of this definition.

<u>Theorem 27:</u> If there exists a real scalar function $V(n,X)$ for which, on D_{n_0R},

a) $V(n,X)$ is positive definite
b) $V(n,X)$ is decrescent
c) $V(n,X)$ is locally Lipschitzian
d) $\Delta V(n,X)$ is negative definite for the unperturbed

difference equation (*)

then the equilibrium $X = 0$ of the difference equation (*) is totally stable.

<u>Proof:</u> By the conditions in the hypothesis on $V(n,X)$, there exist functions $a(r)$ and $b(r)$ of class M_0 such that

$$a(\|X\|) \leq V(n,X) \leq b(\|X\|)$$

and a function $c(r)$, also of class M_0, such that for the unperturbed difference equation,

$$\Delta V(n,X) \leq -c(\|X\|).$$

Now, given ϵ, $0 < \epsilon < R$, choose a constant q, $0 < q < a(\epsilon)$. Then there exists a constant $r(q) > 0$ such that

$$V(n,X^*) = q$$

implies $r < \|X^*\| < \epsilon$ for some X^* with $\|X^*\| > 0$. Furthermore,

$$\Delta V(n,X^*) \leq -c(\|X^*\|) \leq -c(r),$$

since $r \leq \|X^*\|$.

We now consider the total difference of $V(n,X)$ for the perturbed difference equation. This becomes

$$\begin{aligned}\Delta V_P(n,X(n)) &= V(n+1,f(n,X)+g(n,X)) - V(n,X(n)) \\ &= V(n+1,f(n,X)+g(n,X)) - V(n+1,f(n,X)) \\ &\quad + V(n+1,f(n,X)) - V(n,X(n)) \\ &\leq |V(n+1,f(n,X)+g(n,X)) - V(n+1,f(n,X))| \\ &\quad + \Delta V_L(n,X)\end{aligned}$$

$$\leq K\|f(n,X)+g(n,X) - f(n,X)\| + \Delta V_L(n,X)$$

$$= K\|g(n,X)\| + \Delta V_L(n,X)$$

$$\leq K\delta_2 + \Delta V_L(n,X)$$

$$\leq 0$$

for δ_2 sufficiently small, using the fact that V(n,X) is locally Lipschitzian with constant K.

We now choose $\delta_1 = \delta_1(\epsilon)$ such that $\delta_1 < \epsilon$ and

$$V(n_0, x_0) < q$$

for $\|x_0\| < \delta_1$. Then, for all $n \geq n_0$,

$$\|F^*(n, n_0, x_0)\| < \epsilon ,$$

for if not, there would exist an $n_1 \geq n_0$ such that

$$V(n_1, X) > q,$$

since $q < a(\epsilon) = \epsilon$. But

$$V(n_0, x_0) < q$$

and $\Delta V_P(n,X)$ is negative definite, which implies that V(n,X) is monotonically decreasing. Thus we indeed have

$$\|F^*(n, n_0, x_0)\| < \epsilon$$

for $\|x_0\| < \delta_1$ and $\|g(n, X(n))\| < \delta_2$. Hence, the equilibrium is totally stable.

If we compare this result with Theorem 13, we obtain the following relation.

<u>Theorem 28:</u> If the equilibrium X = 0 of the difference equation (*) is uniformly asymptotically stable, then it is also totally stable.

We next consider a slightly different approach to this entire problem. In particular, we will investigate how the solutions of the perturbed equation behave with respect to the solutions of the unperturbed difference equation. Thus, for example, we will consider such possibilities as whether the perturbed solutions will remain close to, or even approach, the unperturbed solution.

Definition 14: The solutions of the perturbed difference equation

$$Y(n+1) = f(n,Y(n)) + g(n,Y(n))$$

are said to be stable with respect to the unperturbed difference equation (*) if, for all $\epsilon > 0$ and all $n_0 \in I$, there exists a $\delta(\epsilon) > 0$ such that $\|x_0^* - x_0\| < \delta$ implies

$$\|F^*(n,n_0,x_0^*) - F(n,n_0,x_0)\| < \epsilon$$

for all $n \geq n_0$, for every solution $F^*(n,n_0,x_0^*)$ of the perturbed difference equation.

Definition 15: The solutions of the perturbed difference equation

$$Y(n+1) = f(n,Y(n)) + g(n,Y(n))$$

are said to be asymptotically stable with respect to the unperturbed difference equation (*) if they are stable with respect to it and if, for all $n_0 \in I$, there exists a $\delta_0(n_0) > 0$ such that $\|x_0^* - x_0\| < \delta_0$ implies that

$$\|F^*(n,n_0,x_0^*) - F(n,n_0,x_0)\| \to 0$$

as $n \to \infty$ for every solution $F^*(n,n_0,x_0^*)$ of the perturbed difference equation.

The last definition is equivalent to the statement that all solutions of the perturbed difference equation which start sufficiently near to the unperturbed solution eventually approach it. Moreover, we note that both of these definitions are independent of the behavior of the solutions of the unperturbed equation. In fact, the following simple examples show that these solutions may be stable, asymptotically stable, or even unstable.

Example 1: Consider the unperturbed difference equation

$$X(n+1) = x_o$$

whose stable solution is $F(n,n_o,x_o) = x_o$, and the perturbed equation

$$Y(n+1) = x_o + y_o,$$

for some sufficiently small y_o. The perturbed solution is given by

$$F^*(n,n_o,x_o^*) = x_o^* = x_o + y_o.$$

Consequently,

$$F^*(n,n_o,x_o^*) - F(n,n_o,x_o) = y_o$$

and Definition 14 holds with $\delta = \epsilon$, for any $\epsilon > 0$.

Example 2: Consider the unperturbed difference equation

$$X(n+1) = aX(n)$$

with $|a| < 1$, whose asymptotically stable solution is

$$F(n,n_o,x_o) = a^{n-n_o}x_o.$$

In addition, consider the perturbed equation

$$Y(n+1) = (a+b)Y(n),$$

with b sufficiently small; in particular, take any b in the open interval (0,1-a). The associated solution is then given by

$$F^*(n,n_o,x_o^*) = (a+b)^{n-n_o}x_o^*$$

and therefore,

$$\|F^*(n,n_o,x_o^*) - F(n,n_o,x_o)\| = \|(a+b)^{n-n_o}x_o^* - a^{n-n_o}x_o\|,$$

which approaches zero as $n \to \infty$, for any x_o^*.

<u>Example 3:</u> Consider the unperturbed difference equation

$$X(n+1) = X(n) + x_o/n_o,$$

whose unstable solution is given by

$$F(n,n_o,2x_o) = x_o + nx_o/n_o.$$

In addition, consider

$$Y(n+1) = Y(n) + x_o/n_o + g(n),$$

where g(n) is any sequence for which $\sum^{\infty} g(k) = 0.$

The corresponding solution is then given by

$$F^*(n,n_o,x_o^*) = x_o + nx_o/n_o + \sum_{k=n_o}^{n-1} g(k),$$

and, by the choice of g(n), it is obvious that the difference between the two solutions approaches zero as $n \to \infty$.

We now present several theorems which supply sufficient conditions for these types of behavior to hold in terms of the existence of real scalar functions U(n,X), which are similar to those used in the previous results.

Theorem 29: If there exists a real scalar function $U(n,X)$ for which, on $D_{n_0 R}$,

a) $U(n,X)$ is positive definite

b) $\Delta U(n,Y(n)-X(n))$ is negative semi-definite,

then the solutions of the perturbed difference equation

$$Y(n+1) = f(n,Y) + g(n,Y)$$

are stable with respect to the unperturbed difference equation (*), provided that

$$\| F^*(n,n_0,x_0^*) - F(n,n_0,x_0)\| \leq R$$

for all $n \geq n_0$.

Proof: Since $U(n,X)$ is positive definite, there is a function $a(r)$ of class M_0 such that

$$U(n,X) \geq a(\|X\|).$$

Now, given any ϵ, choose x_0^* sufficiently close to x_0 so that

$$\|x_0^*-x_0\| < \epsilon \qquad U(n_0,x_0^* - x_0) < a(\epsilon).$$

It then follows that

$$\|F^*(n,n_0,x_0^*) - F(n,n_0,x_0)\| < \epsilon$$

for all $n \geq n_0$; for, if not, there would be some $n_1 > n_0$ such that

$$\| F^*(n_1,n_0,x_0^*) - F(n_1,n_0,x_0)\| \geq \epsilon \quad .$$

This, however, would imply that

$$U(n_1,F^*(n_1,n_0,x_0^*) - F(n_1,n_0,x_0))$$

$$\geq a(\|F^*(n_1,n_0,x_0^*) - F(n_1,n_0,x_0)\|)$$

$$\geq a(\epsilon)$$

$$> U(n_o, x_o^* - x_o)$$

$$\geqslant U(n_1, F^*(n_1, n_o, x_o^*) - F(n_1, n_o, x_o)),$$

which is a contradiction.

<u>Theorem 30:</u> If there exists a real scalar function $U(n,X)$ for which, on D_{n_oR},

a) $U(n,X)$ is positive definite

b) $\Delta U(n, Y(n)-X(n))$ is negative definite,

then the solutions of the perturbed difference equation

$$Y(n+1) = f(n,Y) + g(n,Y)$$

are asymptotically stable with respect to the unperturbed difference equation (*), provided that

$$\|F^*(n,n_o,x_o^*) - F(n,n_o,x_o)\| \leq R$$

for all $n \geqslant n_o$.

<u>Proof:</u> The proof of this theorem follows directly from that for Theorem 4, taking into account the type of modifications which appear in the proof of Theorem 29.

It should be noted that both of these results hinge on the requirement that

$$\|F^*(n,n_o,x_o^*) - F(n,n_o,x_o)\| \leq R$$

for all n. The following theorem gives one fairly simple set of conditions on the functions $f(n,X)$ and $g(n,Y)$ which will guarantee that this condition holds.

__Theorem 31__: If the function $f(n,X)$ satisfies a Lipschitz condition with respect to the variable X with constant $L<1$ and if the function $g(n,Y)$ satisfies

$$\|g(n,Y)\| \leq a \|Y\| ,$$

for some sufficiently small positive constant a, then

$$\|F^*(n,n_o,x_o^*) - F(n,n_o,x_o)\| \leq R$$

for all $n \geqslant n_o$, provided that x_o^* is chosen sufficiently close to x_o.

__Proof:__ For simplicity, we will denote

$$f(n_o+j) = f(n_o+j,F^*(n_o+j,n_o,x_o^*)).$$

It then follows, after a somewhat involved inductive argument, that

$$\|F^*(n_o+k,n_o,x_o^*) - F(n_o+k,n_o,x_o)\|$$

$$\leq L^k\|x_o^*-x_o\| + \|x_o^*\| [a^k + a^{k-1}L + a^{k-2}L^2 + \ldots + aL^{k-1}]$$

$$+ \|f(n_o)\| [a^{k-1} + a^{k-2}L + \ldots + aL^{k-2}] + \ldots +$$

$$+ \|f(n_o+k-3)\| [a^2 + aL] + a \|f(n_o+k-2)\|$$

$$\leq L\|x_o^*-x_o\| + R [(a^k + a^{k-1}L + \ldots + aL^{k-1}) + (a^{k-1} + a^{k-2}L + \ldots + aL^{k-2}) + \ldots + (a^2 + aL) + a]$$

$$= L\|x_o^*-x_o\| + R [(a + a^2 + \ldots + a^k) + L(a + a^2 + \ldots + a^{k-1}) + \ldots + L^{k-1}(a)]$$

$$= L\|x_o^*-x_o\| + Ra/(1-a) [(1-a^k) + L(1-a^{k-1}) + \ldots + L^{k-1}(1-a)]$$

$$\leq L\|x_o^*-x_o\| + Ra/(1-a) [1 + L + \ldots + L^{k-1}]$$

$$\leq L\|x_o^*-x_o\| + aR/(1-a)(1-L).$$

This quantity, however, can be made smaller than R by choosing x_0^* sufficiently close to x_0 and by taking a sufficiently small, since $L<1$.

By way of example, we now present one of the usual type of results on preserving stability under perturbations which is now merely an immediate application of Theorems 30 and 31. Essentially, this is Hahn's result just mentioned.

<u>Theorem 32:</u> If the linear equation

$$X(n+1) = A(n)X(n)$$

is asymptotically stable with $\|A(n)\| \leq b < 1$ for all n, then the solutions of the perturbed difference equation

$$Y(n+1) = A(n)Y(n) + g(n,Y(n))$$

where

$$\|g(n,Y)\| \leq a \|Y\|$$

for some sufficiently small positive constant a, are also asymptotically stable.

It is fairly apparent at this point that the notions introduced in this section can easily be extended to encompass as well the various refinements of the stability properties which have already been studied. For example, if definitions analogous to Definitions 14 and 15 are introduced for the solutions of the perturbed difference equation being either l_p-stable or exponentially stable with respect to the unperturbed equation (*), the following results can easily be demonstrated.

<u>Theorem 33:</u> If there exists a real scalar function $U(n,X)$ for which, on D_{n_oR},

a) $U(n,X)$ is positive definite

b) $\Delta U(n,Y(n)-X(n)) \leq - c\|Y(n)-X(n)\|^p$

for some $p \geq 0$ and some $c > 0$,

then the solutions of the perturbed difference equation

$$Y(n+1) = f(n,Y(n)) + g(n,Y(n))$$

are l_p-stable with respect to the unperturbed difference equation (*), provided that

$$\|F^*(n,n_o,x_o^*) - F(n,n_o,x_o)\| \leq R$$

for all $n \geq n_o$.

<u>Theorem 34:</u> If there exists a real scalar function $U(n,X)$ for which, on D_{n_oR},

a) $a_1\|X\|^p \leq U(n,X) \leq a_2\|X\|^p$

for some positive constants a_1 and a_2 and for some positive p

b) $\Delta U(n,Y(n)-X(n)) \leq - a_3 \|Y(n)-X(n)\|^p$

for some positive constant a_3, where $a_3/a_2 < 1$,

then the solutions of the perturbed difference equation

$$Y(n+1) = f(n,Y(n)) + g(n,Y(n))$$

are exponentially stable with respect to the unperturbed difference equation (*), provided that

$$\|F^*(n,n_o,x_o^*) - F(n,n_o,x_o)\| \leq R$$

for all $n \geq n_o$.

Finally, it should be noted that an entirely analogous theory can be developed for perturbations of differential equations.

2.9 Instability of the Equilibrium

We next consider sufficient conditions to guarantee that the equilibrium is unstable. Without such criteria, the inability to determine an appropriate Liapunov-type function to deduce stability or asymptotic stability of any kind would be totally inconclusive. On the other hand, the determination of a function satisfying the conditions in the theorems below resolves the situation immediately. The first result is the discrete analogue of Liapunov's Second Theorem on instability.

<u>Theorem 35:</u> If there exists a real scalar function $V(n,X)$ for which, on D_{n_oR},

a) $V(n,X)$ is bounded

b) $\Delta V(n,X) \;=\; a\,V(n,X) + W(n,X)$,

   where a is a positive constant and $W(n,X)$ is a semi-definite function defined on D_{n_oR}

c) if $W(n,X)$ is not identically zero, then in each subdomain $D_{n_1r} \subset D_{n_oR}$, there exist points (n,X) for which $V(n,X)$ and $W(n,X)$ have the same sign for all $n \geq n_1$, then the equilibrium $X = 0$ of the difference equation (*) is unstable.

<u>Proof:</u> Suppose that $D_{n_1 r}$ is any subdomain of $D_{n_o R}$ in which $V(n,X)$ and $W(n,X)$ are both positive at some points. Let (n_1,x_1) be one such point and consider the solution $F(n,n_1,x_1)$. We have

$$\begin{aligned}\Delta\left[(a+1)^{-n} V(n,F(n,n_1,x_1))\right] &= (a+1)^{-(n+1)} V(n+1,F(n+1,n_1,x_1)) - (a+1)^{-n}V(n,F(n,n_1,x \\ &= (a+1)^{-(n+1)}\left[V(n+1,F(n+1,n_1,x_1)) - (a+1)V(n,F(n,n_1,x_1)\right. \\ &= (a+1)^{-(n+1)}\left[\Delta V(n,F(n,n_1,x_1)) - aV(n,F(n,n_1,x_1))\right].\end{aligned}$$

Consequently, along this particular trajectory,

$$\begin{aligned}W(n,F(n,n_1,x_1)) &= \Delta V(n,F(n,n_1,x_1)) - aV(n,F(n,n_1,x_1)) \\ &= (a+1)^{+(n+1)}\Delta\left[(a+1)^{-n}V(n,F(n,n_1,x_1))\right]\end{aligned}$$

and since $W(n,X)$ is positive,

$$\Delta\left[(a+1)^{-n} V(n,F(n,n_1,x_1))\right] \geqslant 0.$$

Thus, $(a+1)^{-n} V(n,X)$ increases along this particular trajectory and hence

$$(a+1)^{-n} V(n,F(n,n_1,x_1)) \geqslant (a+1)^{-n_1} V(n_1,x_1)$$

and therefore

$$V(n,F(n,n_1,x_1)) \geqslant (a+1)^{n-n_1} V(n_1,x_1),$$

which becomes arbitrarily large as $n \to \infty$. However, we assumed that $V(n,X)$ is bounded on $D_{n_o R}$, and so the solution must leave $D_{n_o R}$ and it must do so across the boundary $\|X\| = R$; that is, the equilibrium is unstable.

The second result is the discrete analogue of Cetaev's theorem on instability. It was originally given by Hahn (3).

<u>Theorem (Hahn):</u> If there exists a real scalar function $V(n,X)$ for which

a) in every $D_{n_1 r} \subset D_{n_0 R}$, for r arbitrarily small, there exist X such that $V(n,X) < 0$, for all $n \geq n_1$,

b) $V(n,X)$ is bounded from below in some subdomain $D \subset D_{n_0 R}$ in which $V(n,X) < 0$,

c) in this particular subdomain D,

$$\Delta V(n,X) \leq -a(|V|) < 0$$

for some function a(r) of class M_0,

then the equilibrium $X = 0$ of the difference equation (*) is unstable.

2.10 Some Stability Theorems given by Hurt

We conclude this chapter by citing some results obtained by Hurt (5) in a paper brought to the author's attention after the research work for this thesis was completed. In this paper, a number of theorems are presented dealing with stability theory for difference equations using a somewhat different approach than the one employed here. The principal concepts and results are given below.

A point X^* in E^t is said to be a positive limit point of $X(n)$ if there exists a strictly monotonic divergent sequence n_k of integers such that $X(n_k) \to X^*$ as $k \to \infty$. The union of all the positive limit points of $X(n)$ is the positive limit set.

<u>Theorem (Hurt):</u> If there exists a real scalar function $V(n,X)$ for which, on some set G in E^t and for all $n \geqslant n_o$,

a) $V(n,X)$ is bounded below

b) $V(n,X)$ is continuous as a function of X

c) $\Delta V(n,X) \leq - W(X) < 0$,

for some continuous function $W(X)$,

then every solution which starts in G and remains in G for all n approaches the set

$$A^* = \{X : W(X) = 0\} \cup \{\infty\} = A \cup \{\infty\},$$

where $\{\infty\}$ represents the vector at infinity.

We note that if $V(n,X)$ is positive definite and $\Delta V(n,X)$ is negative semi-definite, then this theorem reduces to Hahn's theorem on stability. If, in addition, the function $\Delta V(n,X)$ is negative definite, or equivalently, if the function $W(X)$ is of class M_o, then we obtain Theorem 4 on asymptotic stability. In this case, we have $A = \{0\}$ and all solutions will thus approach the origin as $n \to \infty$.

A function $X^*(n)$ is said to be a solution of the autonomous difference equation

$$X(n+1) = f(X(n))$$

on $(-\infty, \infty)$ if for any n_0 in $(-\infty, \infty)$,

$$F(n-n_0, n_0, X^*(n_0)) = X^*(n).$$

A set B is said to be an invariant set for the autonomous difference equation if x_0 in B implies that there is a solution $X^*(n)$ for the equation on $(-\infty, \infty)$ such that $X^*(n)$ is in B for all n and further that $X^*(0) = x_0$.

Theorem (Hurt): If there exists a real scalar function V(X) for the autonomous difference equation

$$X(n+1) = f(X(n))$$

for which, on some set G in E^t,

a) V(X) is bounded below

b) V(X) is continuous

c) $\Delta V(X) \leq 0$,

then every solution which starts in G and remains in G for all n is either unbounded or approaches some invariant set contained in

$$A = \{X : \Delta V(X) = 0\}$$

as $n \to \infty$.

The difference equation (*) is said to be asymptotically autonomous if it is possible to write it as

$$X(n+1) = g(X) + h(n,X),$$

where $h(n,X) \to 0$ as $n \to \infty$ uniformly for all X in any compact set.

<u>Theorem (Hurt):</u> If a solution X(n) of the difference equation (*) approaches a closed bounded set A as $n \to \infty$ and if X(n) is also a solution of the asymptotically autonomous difference equation

$$X(n+1) = g(X) + h(n,X),$$

then it approaches the largest invariant set for the autonomous difference equation

$$X(n+1) = g(X)$$

contained in A as $n \to \infty$.

Chapter 3: Boundedness of Solutions and Stability in the Whole

3.1 Boundedness of Solutions

Thus far, we have been concerned with the difference equation

$$X(n+1) = f(n,X(n)), \qquad (*)$$

where the function $f(n,X)$ has been assumed bounded and has been defined only for the semi-bounded domain

$$D_{n_oR} = \left\{ (n,X) \in I \times E^t : n \geq n_o \geq 0, \ \| X \| \leq R \right\} .$$

In the present chapter, we will consider the case where the norm of the vector X will be allowed to be possibly unbounded. The difference equation to be studied will be

$$X(n+1) = g(n,X(n)), \qquad (**)$$

where the function $g(n,X)$ considered is defined throughout $I \times E^t$, for all $n \geq n_o$, and where $g(n,X)$ assumes values throughout E^t, though otherwise, the equation (**) will possess all of the properties previously indicated for the equation (*). Moreover, the complement of D_{n_oR} in $I \times E^t$ will be denoted by D'_{n_oR} ; that is,

$$D'_{n_oR} = \left\{ (n,X) \in I \times E^t : n \geq n_o \geq 0, \ \| X \| \geq R > 0 \right\}.$$

In addition, we will consider the region

$$D_{n_o\infty} = \left\{ (n,X) \in I \times E^t ; \ n \geq n_o \geq 0 \right\}.$$

<u>Definition 16:</u> A solution to the difference equation (**) is said to be <u>bounded</u> if, given any $n_o \geq 0$ and any $r_o > 0$, there exists a $B(n_o,r_o) > 0$ such that for any x_o with $\|x_o\| < r_o$,

$$\| F(n,n_o,x_o) \| < B$$

for all $n \geq n_o$.

<u>Definition 17:</u> A solution to the difference equation (**) is said to be <u>uniformly bounded</u> if, given any $r_o > 0$, there exists a $B(r_o) > 0$ such that for any $n_o \geq 0$, and any x_o with $\| x_o \| < r_o$,

$$\| F(n,n_o,x_o) \| < B$$

for all $n \geq n_o$.

<u>Definition 18:</u> A solution to the difference equation (**) is said to be <u>ultimately bounded</u> if, given any $n_o \varepsilon I$ and any r_o and r_1, $r_o > r_1 > 0$, there exists a $B(r_1) > 0$ and an integer $\nu(r_o,r_1) > 0$ such that for any x_o with $\| x_o \| < r_o$,

$$\| F(n,n_o,x_o) \| < B$$

for all $n \geq n_o + \nu$.

<u>Definition 19:</u> A solution to the difference equation (**) is said to be <u>uniformly ultimately bounded</u> if, given any r_o and r_1, $r_o > r_1 > 0$, there exists a $B(r_1) > 0$ and an integer $\nu(r_o,r_1) > 0$ such that for any $n_o \varepsilon I$ and any x_o with $\| x_o \| < r_o$,

$$\| F(n,n_o,x_o) \| < B$$

for all $n \geq n_o + \nu$.

We now develop criteria for the various types of boundedness enumerated above in terms of the existence of certain real scalar functions with particular properties which are defined on $D'_{n_o R}$ and which are quite similar to the functions used in the previous chapter.

<u>Theorem 36:</u> If there exists a real scalar function $W(n,X)$ for which, on $D'_{n_o R}$,

a) $W(n,X)$ is positive definite

b) $W(n,X) \to \infty$ uniformly as $X \to \infty$

c) $\Delta W(n,X)$ is negative semi-definite

then every solution of the difference equation (**) is bounded.

<u>Proof:</u> Choose any $n_o \geqslant 0$ and any $r_o > R$ and define

$$w(n_o,r_o) = \sup \left\{ W(n_o,X) \;:\; R < \|X\| < r_o \right\}.$$

Further, take $B(n_o,r_o) > R$ such that $W(n,X) > w$ for $n \geqslant 0$ and X such that $\|X\| \geqslant B$. As a consequence, if $\|x_o\| < r_o$, then

$$\| F(n,n_o,x_o) \| < B$$

for all $n \geqslant n_o$; for if not, there would be some $n' > n_o$ such that

$$\| F(n',n_o,x_o) \| \geqslant B.$$

Thus,

$$w < W(n',F(n',n_o,x_o)) \leqslant W(n_o,x_o) \leqslant w,$$

since $W(n,X)$ has a negative semi-definite total difference so that this is a contradiction. Hence, $\| F(n,n_o,x_o) \|$ is bounded by B for all $n \geqslant n_o$.

Theorem 37: If there exists a real scalar function $W(n,X)$ for which, on D'_{n_oR},

a) $W(n,X)$ is positive definite

b) $W(n,X)$ is bounded on the set $I \times (S \cap N')$, where S is any open sphere containing the set

$$N = \{ X : \|X\| \leq R \}$$

c) $W(n,X) \to \infty$, uniformly as $X \to \infty$

d) $\Delta W(n,X)$ is negative semi-definite

then every solution to the difference equation (**) is uniformly bounded.

Proof: Given any $r_o > R$, there exists a sphere of radius S, $r_o \leq S$, containing the set N defined in the statement of the theorem and such that $W(n,X)$ is bounded on $I \times (S \cap N')$. Define

$$w(r_o) = \sup \{ W(n,X) : n \geq 0, \quad R \leq \|X\| < r_o \}.$$

We observe that since $W(n,X)$ is bounded, $w < \infty$. Let $B(r_o) > R$ be such that $W(n,X) > w$ for $n \geq 0$ and X such that $\|X\| \geq B$. As a consequence, $\|x_o\| < r_o$ and $n_o \geq 0$ imply

$$\| F(n,n_o,x_o) \| < B$$

for all $n \geq n_o$; for if not, then there would be some $n' > n_o$ for which

$$\| F(n',n_o,x_o) \| \geq B.$$

Hence,

$$w < W(n',F(n',n_o,x_o)) \leq W(n_o,x_o) \leq w,$$

since $\Delta W(n,X)$ is negative semi-definite, which is a contradiction. Thus, $\|F(n,n_o,x_o)\|$ is uniformly bounded by B.

The hypotheses of the previous theorem can be expressed in alternate terms which occasionally may be easier to apply. The conditions are stated in the following equivalent theorem.

<u>Theorem 37A:</u> If there exists a real scalar function $W(n,X)$ for which, on D'_{n_oR},

a) $W(n,X) \leq a(\|X\|)$, for some positive increasing function $a(r)$

b) $W(n,X) \geq b(\|X\|)$, for some non-negative increasing function $b(r)$

c) $\Delta W(n,X)$ is non-positive,

then every solution of the difference equation (**) is uniformly bounded.

We now present a criterion for uniform ultimate boundedness of all solutions to the difference equation under consideration which is a direct analogue of Theorem 36. As with Theorem 36, there is also an equivalent formulation which will be stated after the proof of Theorem 38.

<u>Theorem 38:</u> If there exists a real scalar function $W(n,X)$ for which, on D'_{n_oR},

a) $W(n,X)$ is positive definite

b) $W(n,X)$ is bounded on the set $I \times (S \cap N')$, where S is any open sphere containing the set

$$N = \{ X : \| X \| \leq R \}$$

c) $W(n,X) \to \infty$ uniformly as $X \to \infty$

d) $\Delta W(n,X)$ is negative definite

then every solution to the difference equation (**) is uniformly ultimately bounded.

<u>Proof:</u> Given any r_o and r_1, $r_o > r_1 > R$, we define

$$w(r_o) = \sup \{ W(n,X) : n \geq 0, \quad R \leq \| X \| < r_o \}.$$

Let $a(r_1) > 0$ be a constant such that

$$\Delta W(n,X) \leq -a$$

for $n \geq 0$ and any X with $\| X \| \geq r_1$. Further, let

$$\nu (r_o, r_1) = [\![w/a]\!] + 1.$$

Now, given any $n_o \geq 0$ and any x_o with $\| x_o \| < r_o$, then either $\| x_o \| < r_1$ or $r_1 \leq \| x_o \| < r_o$. In the former case, there exists a $B(r_1) \geq R$ such that

$$\| F(n, n_o, x_o) \| < B$$

for all $n \geq n_o$, by the argument in the previous proofs.

In the latter case, where $r_1 \leq \| x_o \| < r_o$, there exists an integer n' between n_o and $n_o + \nu$ such that

$$\| F(n', n_o, x_o) \| < r_1;$$

for if not, then

$$\| F(n, n_o, x_o) \| \geq r_1$$

for all n in the interval $[n_o, n_o + \nu]$; hence,

$$W(n_o + \nu, F(n_o + \nu, n_o, x_o)) \leq W(n_o, x_o) - a\nu$$

$$
\begin{aligned}
&= W(n_o,x_o) - a([\![w/a]\!] + 1) \\
&\leq W(n_o,x_o) - a(w/a - 1) - a \\
&= W(n_o,x_o) - w \\
&\leq 0,
\end{aligned}
$$

which is impossible by the definition of W. Hence, in either case,

$$\| F(n,n_o,x_o) \| < B$$

for all $n \geq n'$ and therefore, this inequality holds for all $n \geq n_o + \nu$.

The alternative formulation of this theorem, which was mentioned previously, is given by the following statement.

Theorem 38A: If there exists a real scalar function $W(n,X)$ for which, on D'_{n_oR},

a) $W(n,X) \leq a(\|X\|)$, for some positive increasing function $a(r)$

b) $W(n,X) \geq b(\|X\|)$, for some non-negative increasing function $b(r)$

c) $\Delta W(n,X) \leq -c(\|X\|)$, for some positive continuous function $c(r)$

then every solution of the difference equation (**) is uniformly ultimately bounded.

3.2 Stability in the Whole

We now consider the possibility of stability for the equilibrium of the difference equation under study when the initial values are allowed to become arbitrarily large in norm. We begin by citing the pertinent definitions.

Definition 20: The equilibrium $X = 0$ of the difference equation (**) is said to be asymptotically stable in the whole if it is stable and if (n_0,x_0) in $I \times E^k$ implies

$$F(n,n_0,x_0) \to 0$$

as $n \to \infty$.

Definition 21: The equilibrium $X = 0$ of the difference equation (**) is said to be equiasymptotically stable in the whole if it is stable and if

$$F(n,n_0,x_0) \to 0$$

uniformly in x_0 for $\|x_0\| \leq r$, where r is fixed but arbitrarily large, as $n \to \infty$.

Definition 22: The equilibrium $X = 0$ of the difference equation (**) is said to be uniformly-asymptotically stable in the whole if every solution of the difference equation is uniformly bounded and if, given any positive numbers r_0 and r_1, there exists an integer $\nu(r_0,r_1) > 0$ such that, given $n_0 \geqslant 0$, and any x_0 with $\|x_0\| < r_0$, we have

$$\| F(n,n_0,x_0) \| < r_1$$

for all $n \geq n_o + \nu$.

<u>Definition 23:</u> The equilibrium X = 0 of the difference equation (**) is said to be <u>l_p-stable in the whole</u> if it is stable and if the series

$$\sum_{k=n_o}^{\infty} \|F(k,n_o,x_o)\|^p < \infty$$

for every (n_o,x_o) in $D_{n_o\infty}$.

Finally, we consider a series of theorems which yield conditions under which we may conclude that the various types of stability and asymptotic stability hold in the whole. The following two results were obtained originally by Kalman and Bertram (6) in their Theorems 1.1.2 and 1.1.

<u>Theorem (Kalman and Bertram):</u> If there exists a real scalar function V(n,X) for which, on $D_{n_o\infty}$,

a) V(n,X) is positive definite

b) V(n,X) is radially unbounded

c) Δ V(n,X) is negative definite

then the equilibrium X = 0 of the difference equation (**) is equiasymptotically stable in the whole.

<u>Theorem (Kalman and Bertram):</u> If there exists a real scalar function V(n,X) for which, on $D_{n_o\infty}$,

a) V(n,X) is positive definite

b) V(n,X) is decrescent

c) $V(n,X)$ is radially unbounded

d) $\Delta V(n,X)$ is negative definite

then the equilibrium $X = 0$ of the difference equation (**) is uniformly-asymptotically stable in the whole.

We now consider the case where the function $g(n,X)$ is periodic in n.

Theorem 39: If the function $g(n,X(n))$ is periodic in n and if the equilibrium $X = 0$ of the difference equation (**) is stable in the whole, then it is also uniformly stable in the whole.

Proof: The comparison function used to prove Theorem 3 satisfies all of the requirements for the function described in Theorem 2, where the bound, δ, on the values of $\|x_0\|$ may be allowed to become arbitrarily large. Hence, the stability is uniform in the whole.

Theorem 40: If the function $g(n,X(n))$ is periodic in n and if the equilibrium $X = 0$ of the difference equation (**) is asymptotically stable in the whole, then it is uniformly asymptotically stable in the whole.

Proof: The comparison function used to prove Theorem 18 satisfies all of the requirements for the function described in Theorem 16, where the bound on the values of $\|x_o\|$, δ, may be allowed to become arbitrarily large. As a consequence, the asymptotic stability is indeed uniform asymptotic stability in the whole.

Kalman and Bertram (6) have also supplied the following criterion for uniform asymptotic stability in the whole in the autonomous case, in their Theorem 1.2.

Theorem (Kalman and Bertram): If the function $g(n,X(n))$ is independent of n and if there exists a real scalar function $V(n,X)$ for which, on $D_{n_o\infty}$,

a) $V(n,X)$ is positive definite

b) $V(n,X)$ is continuous as a function of X

c) $V(n,X)$ is radially unbounded

d) $\Delta V(n,X)$ is negative definite

then the equilibrium $X = 0$ of the difference equation

$$X(n+1) = g(X(n))$$

is uniformly asymptotically stable in the whole.

The following theorem extends the concept of l_p-stability to l_p-stability in the whole.

Theorem 41: If there exists a real scalar function $V(n,X)$ for which, on $D_{n_o\infty}$,

a) $V(n,X)$ is positive definite

b) $V(n,X) \to \infty$ as $\|X\| \to \infty$ uniformly on the set

$n_o \leq n \leq N$, for every integer $N \geqslant n_o$

c) $\Delta V(n,X) \leq -c\|X\|^p$, for some constant $c > 0$ and some $p \geqslant 0$

then the equilibrium $X = 0$ of the difference equation (**) is l_p-stable in the whole.

<u>Proof:</u> The proof of this theorem follows directly from that given for Theorem 23 since, from condition b) on $V(n,X)$, it follows that $(n,F(n,n_o,x_o))$ is in $D_{n_o\infty}$ for all $n \geqslant n_o$ and every (n_o,x_o) in $D_{n_o\infty}$.

PART II

STABILITY CRITERIA

FOR A CERTAIN CLASS OF DIFFERENCE EQUATIONS

Chapter 4: The Difference Equation of Order m

We will now apply part of the preceding theory on the stability and asymptotic stability of the equilibrium to the solutions of a certain class of difference equations. In particular, we will concern ourselves with the following difference equation of order m

$$X(n+m) + a_1X(n+m-1) + \dots + a_mX(n) - F(n,X(n), \dots X(n+m-1)) = 0, \tag{4.1}$$

where the a_i, $i = 1,2,\dots m$, are real constants and F is a real scalar function of the indicated arguments satisfying the condition

$$F(n,0,\dots 0) = 0$$

for all n greater than or equal to some n_o in I. When the function F is identically equal to zero, we shall speak of the difference equation as being homogeneous; otherwise, if the function F is present, we will say that the equation is non-homogeneous.

The present chapter is concerned with the general m^{th} order case for which we develop a method for determinig the conditions under which the equilibrium of the difference equation (4.1) is stable or asymptotically stable. In Chapter 5, we will treat the particular case m = 4 in complete detail. The cases $m = 2$ and $m = 3$ have already been considered by Puri and Drake (11). Several comments on and extensions of their work appear in Chapter 6.

That the class of equations represented by equation (4.1) actually encompasses a very wide set of difference equations can be seen from the fact that an arbitrary nonlinear difference equation of the form

$$\begin{aligned} X(n+m) = {} & f_1\left[n, X(n), \ldots X(n+m-1)\right] X(n+m-1) + \ldots \\ & \ldots + f_m\left[n, X(n)\right] X(n) \\ & + f\left[n, X(n), \ldots X(n+m-1)\right], \end{aligned} \tag{4.2}$$

where $f(n,0,\ldots 0) = 0$ for all $n \geqslant n_o$, can be expressed in the form given in equation (4.1). This is possible by writing equation (4.2) as

$$\begin{aligned} & X(n+m) + a_1 X(n+m-1) + \ldots + a_m X(n) + \\ & \quad + \left\{ f_1\left[n, X(n), \ldots X(n+m-1)\right] - a_1 \right\} X(n+m-1) + \ldots \\ & \quad + \ldots + \left\{ f_m\left[n, X(n)\right] - a_m \right\} X(n) + \\ & \quad + f\left[n, X(n), \ldots X(n+m-1)\right] \\ & = X(n+m) + a_1 X(n+m-1) + \ldots + a_m X(n) - F\left[n, X(n), \ldots X(n+m-1)\right] \\ & = 0, \end{aligned}$$

where the a_i are any real constants.

In order to discuss the possible stability and asymptotic stability of the trivial solution of the difference equations in the class under consideration, we will determine what conditions must be imposed on the coefficients a_i and on the function F which will guarantee such stability or asymptotic stability.

Before proceeding with this discussion, we note that it is possible, and indeed far more convenient, to rewrite equation (4.1) as the matrix equation

$$X(n+1) = A\,X(n) - bF, \tag{4.3}$$

where

$$X(n) = \begin{pmatrix} X_1(n) \\ X_2(n) \\ \vdots \\ X_m(n) \end{pmatrix} = \begin{pmatrix} X(n) \\ X(n+1) \\ \vdots \\ X(n+m-1) \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 0 & 1 & \cdots & & 0 \\ \vdots & \vdots & & & & & \vdots \\ 0 & 0 & \cdots & & & 0 & 1 \\ -a_m & -a_{m-1} & \cdots & & & -a_2 & -a_1 \end{pmatrix}$$

and

$$b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

We first consider the homogeneous case, where $F = 0$ in equation (4.3). In order to obtain a useful scalar function $V(n,X)$ with which to study the stability and asymptotic stability of the trivial solution of equation (4.3), it is expedient to transform the variable vector $X(n)$ into a new vector quantity

$$Y(n) = Q\,X(n), \tag{4.4}$$

where Q is a real $m \times m$ matrix which is to be determined. If the transformation Q is applied to the matrix equation (4.3), we obtain

$$\begin{aligned} Y(n+1) &= Q\,X(n+1) \\ &= QA\,X(n) \\ &= QAQ^{-1}\,Y(n) \\ &= R\,Y(n), \end{aligned} \tag{4.5}$$

where we have put

$$R = QAQ^{-1}, \tag{4.6}$$

which is a similarity transformation between the matrix A and some matrix R, where Q is assumed non-singular.

Anticipating an expression which will arise in the sequel, it is convenient to impose the following condition on the matrix R.

$$R^T R = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \ddots & 1 & \vdots \\ 0 & 0 & 0 & \cdots 0 & r^2 \end{pmatrix}, \tag{4.7}$$

where R^T denotes the transpose of the matrix R. From this condition, it is possible to determine an explicit form for R. Writing

$$R = \begin{pmatrix} r_{11} & r_{12} & \cdots\cdots & r_{1,m-1} & R_1 \\ r_{21} & r_{22} & \cdots\cdots & r_{2,m-1} & R_2 \\ \vdots & \vdots & & \vdots & \vdots \\ r_{m,1} & r_{m,2} & \cdots\cdots & r_{m,m-1} & R_m \end{pmatrix},$$

condition (4.7) becomes

$$\begin{pmatrix} r_{11} & r_{21} & \cdots & r_{m-1,1} & r_{m,1} \\ r_{12} & r_{22} & \cdots & & r_{m,2} \\ \vdots & \vdots & & & \vdots \\ R_1 & R_2 & \cdots & & R_m \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1,m-1} & R_1 \\ r_{21} & \cdots & & r_{2,m-1} & R_2 \\ \vdots & & & & \vdots \\ r_{m1} & \cdots & & r_{m,m-1} & R_m \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots 1 & 0 \\ 0 & 0 & 0 & \cdots 0 & r^2 \end{pmatrix} .$$

This matrix product represents a system of m^2 equations. However, of these, the equations corresponding to the subdiagonal terms in the matrix on the right will be identical with the equations corresponding to the symmetric superdiagonal terms. That is, $(m-1)m/2$ equations will be repeated and hence there are only

$$m^2 - \frac{m(m-1)}{2} = \frac{m^2 + m}{2}$$

independent relations, one of which is

$$r^2 = R_1{}^2 + R_2{}^2 + \ldots + R_m{}^2. \tag{4.8}$$

As a consequence, we have $(m^2+m)/2 - 1$ expressions with which to determine the $m(m-1)$ unknown r_{ij} in terms of the m R_i's. Thus,

$$m(m-1) - \frac{m^2+m}{2} + 1 = \frac{(m-2)(m-1)}{2} \tag{4.9}$$

of the r_{ij} may be chosen arbitrarily. In particular, we choose all of the terms below the first subdiagonal as zero; i.e., $r_{31} = r_{41} = \cdots = r_{m1} = r_{42} = \ldots = r_{m2} = \ldots = r_{m,m-2} = 0$.

As a result,

$$R = \begin{pmatrix} r_{11} & r_{12} & \cdots\cdots\cdots\cdots & R_1 \\ r_{21} & r_{22} & \cdots\cdots\cdots\cdots & R_2 \\ 0 & r_{32} & \cdots\cdots\cdots\cdots & R_3 \\ 0 & 0 & \cdots\cdots\cdots\cdots & R_4 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots\cdots r_{m,m-1} & R_m \end{pmatrix}.$$

If equation (4.7) is now expanded in terms of this choice of R, a form for R is obtained, but not a unique one. For any given column of this matrix R, there is a choice of signs for the elements as indicated below in equation (4.10), but the choice for any given column is independent of the choice for any other column. Thus,

$$R = \begin{pmatrix} \pm\frac{R_2}{\rho_2} & \pm\frac{R_1R_3}{\rho_2\rho_3} & \pm\frac{R_1R_4}{\rho_3\rho_4} & \cdots\cdots & \pm\frac{R_1R_m}{\rho_{m-1}\rho_m} & R_1 \\ \mp\frac{R_1}{\rho_2} & \pm\frac{R_2R_3}{\rho_2\rho_3} & \pm\frac{R_2R_4}{\rho_3\rho_4} & & \vdots & \vdots \\ 0 & \mp\frac{\rho_2}{\rho_3} & \pm\frac{R_3R_4}{\rho_3\rho_4} & & \vdots & \vdots \\ 0 & 0 & \mp\frac{\rho_3}{\rho_4} & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \pm\frac{R_{m-1}R_m}{\rho_{m-1}\rho_m} & R_{m-1} \\ 0 & 0 & 0 & \cdots\cdots & \mp\frac{\rho_{m-1}}{\rho_m} & R_m \end{pmatrix}, \tag{4.10}$$

where

$$\begin{aligned} \rho_2^2 &= R_1^2 + R_2^2 \\ \rho_3^2 &= R_1^2 + R_2^2 + R_3^2 \\ &\vdots \\ \rho_m^2 &= R_1^2 + R_2^2 + \dots + R_m^2 = r^2. \end{aligned} \tag{4.11}$$

That the condition imposed on R to yield equation (4.10) is not as arbitrary as would appear may be seen from the following example for m = 5. From condition (4.9), we see that six zero positions are permitted in the R matrix. We assume that they are chosen as shown:

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} & r_{14} & R_1 \\ r_{21} & r_{22} & r_{23} & r_{24} & R_2 \\ 0 & 0 & r_{33} & r_{34} & R_3 \\ 0 & 0 & r_{43} & r_{44} & R_4 \\ 0 & 0 & r_{53} & r_{54} & R_5 \end{pmatrix}.$$

We now apply condition (4.7) and obtain, among others, the following five equations,

a) $r_{11}^2 + r_{21}^2 = 1$

b) $r_{11}R_1 + r_{21}R_2 = 0$

c) $r_{12}^2 + r_{22}^2 = 1$

d) $r_{12}R_1 + r_{22}R_2 = 0$

e) $r_{11}r_{12} + r_{21}r_{22} = 0.$

Using a) and b), it follows that

$$r_{11} = \frac{\pm R_2}{\rho_2}$$

$$r_{21} = \frac{\mp R_1}{\rho_2},$$

where ρ_2 is defined in equations (4.11). Further, from c) and d), we obtain

$$r_{22} = \frac{\pm R_1}{\rho_2}$$

$$r_{12} = \frac{\mp R_2}{\rho_2}.$$

However, when we attempt to introduce these four expressions, with any appropriate combination of signs, into relation e), we are led to a contradiction. For example, choosing

$$r_{11} = \frac{R_2}{\rho_2}, \; r_{21} = \frac{-R_1}{\rho_2}, \; r_{22} = \frac{R_1}{\rho_2}, \; r_{12} = \frac{-R_2}{\rho_2},$$

equation e) becomes

$$\begin{aligned} r_{11}r_{12} + r_{21}r_{22} &= \frac{R_2}{\rho_2}\left[-\frac{R_2}{\rho_2}\right] + \left[\frac{-R_1}{\rho_2}\right]\frac{R_1}{\rho_2} \\ &= -\left[R_2^2/\rho_2^2 + R_1^2/\rho_2^2\right] \\ &= -1 \neq 0. \end{aligned}$$

Since the matrices A and R are similar, they have the same characteristic polynomials. Equating similar powers of λ in

$$\det(A - \lambda I) = \det(R - \lambda I),$$

we obtain a set of m equations expressing the a_i in terms of the R_i and the ρ_i. As a further consequence of the similarity of A and R, it follows that

$$\det(A) = \det(R),$$

where

$$\det(R) = \sqrt{\det(R^T R)} = \sqrt{r^2}$$

from condition (4.7). By specifying the signs of the elements of R as given by

$$R = \begin{pmatrix} \frac{R_2}{\rho_2} & \frac{R_1 R_3}{\rho_2 \rho_3} & \frac{R_1 R_4}{\rho_3 \rho_4} & \cdots\cdots & \frac{R_1 R_m}{\rho_{m-1}\rho_m} & R_1 \\ -\frac{R_1}{\rho_2} & \frac{R_2 R_3}{\rho_2 \rho_3} & \frac{R_2 R_4}{\rho_3 \rho_4} & \cdots\cdots & \frac{R_2 R_m}{\rho_{m-1}\rho_m} & R_2 \\ 0 & -\frac{\rho_2}{\rho_3} & \frac{R_3 R_4}{\rho_3 \rho_4} & & & \vdots \\ \vdots & & & & & \vdots \\ 0 & \cdots & \cdots & 0 & -\frac{\rho_{m-1}}{\rho_m} & R_m \end{pmatrix}, \qquad (4.12)$$

we will have that

$$\det(R) = r.$$

Furthermore, from the given form of A, it follows that

$$\det(A) = (-1)^m a_m \det(I_m)$$

$$= (-1)^m a_m ,$$

where I_m represents the m x m identity matrix. As a consequence, we may conclude that

$$r = (-1)^m a_m ,$$

or equivalently,

$$a_m = (-1)^m r. \qquad (4.13)$$

Thus, this will be one of the m conditions (the one, in particular, corresponding to the zeroth terms) obtained by equating the coefficients of the powers of λ in the characteristic polynomials of A and R, since the constant term in the characteristic polynomial of a matrix is simply the determinant of that matrix. Unfortunately, there appears to be no general expression for these conditions.

At this point, it would seem natural to calculate the matrix Q from the relationship $QA = RQ$. However, this will lead to m^2 equations which are exceedingly difficult to solve for $m > 3$. Accordingly, it is instead advisable to carry out the details of the next few steps in the present development to see what freedom exists to impose conditions on Q which will simplify it. In particular, we seek to find which permissible conditions will yield the greatest simplification for the later detailed calculations.

To determine the stability and asymptotic stability of the equilibrium of either the homogeneous or the nonhomogeneous difference equation, we shall introduce as a possible choice for the scalar function $V(n,X)$

$$V(n,X) = Y^T(n)Y(n). \tag{4.14}$$

This is equivalent to

$$V(n,X) = X^T(n)Q^TQX(n),$$

which is a sum of squares and hence is positive definite, whatever the particular form of Q.

Next, we investigate the total difference of $V(n,X)$ for the homogeneous equation. This is given by

$$\begin{aligned}\Delta V(n,X) &= V(n+1,X(n+1)) - V(n,X(n)) \\ &= Y^T(n+1)Y(n+1) - Y^T(n)Y(n) \\ &= Y^T(n)R^TRY(n) - Y^T(n)Y(n) \\ &= Y^T(n)\left[R^TR - I\right]Y(n) \\ &= (r^2-1)\,Y_m{}^2 \\ &\leq 0\end{aligned}$$

for $r^2-1 < 0$, using equation (4.7). That is, we must have

$$r^2 = a_m{}^2 < 1. \tag{4.15}$$

Thus, the reason for imposing condition (4.7) on R becomes apparent. Furthermore, we note that one of the principal conditions under which the equilibrium X = 0 of the homogeneous system is stable is that equation (4.15) holds for a_m.

Moreover, it follows that

$$\begin{aligned} V(n,X) &= X^T(n)Q^TQX(n) \\ &= (QX)^T(QX) \\ &= \langle QX,QX \rangle \\ &= \|QX\|^2 \\ &\leq \|Q\|^2 \|X\|^2, \end{aligned}$$

where $\langle x,y \rangle$ represents the inner product of the vectors x and y. As a consequence, we note that the function V(n,X) is decrescent, though in view of Theorem 4, this condition is not required for asymptotic stability.

We now consider the non-homogeneous difference equation

$$X(n+1) = AX(n) - bF.$$

As before, we apply the transformation Q to obtain

$$\begin{aligned} Y(n+1) &= QX(n+1) \\ &= QAX(n) - QbF \\ &= RY(n) - QbF. \end{aligned}$$

We again choose

$$V(n,X) = Y^T(n)Y(n) = X^T(n)Q^TQX(n),$$

which is positive definite, as noted previously. The total difference of V(n,X) for the non-homogeneous equation is given by

$$\Delta V(n,X) = \left[Y^T R^T - b^T Q^T F\right]\left[RY - QbF\right] - Y^T Y$$
$$= Y^T R^T R Y - \left[Y^T R^T Q b + b^T Q^T R Y\right] F + + b^T Q^T Q b F^2 - Y^T Y.$$

Hence,

$$\Delta V(n,X) = Y^T\left[R^T R - I\right]Y - 2\, Y^T R^T Q b F + b^T Q^T Q b F^2, \qquad (4.16)$$

using the fact that

$$Y^T R^T Q b = b^T Q^T R Y,$$

since both expressions are scalars.

In order to simplify condition (4.16), we shall first simplify the term Qb. Now

$$Q = \begin{pmatrix} q_{11} & q_{12} & \cdots & q_{1m} \\ q_{21} & & & \vdots \\ \vdots & & & \vdots \\ q_{m1} & \cdots & \cdots & q_{mm} \end{pmatrix}$$

and

$$Qb = \begin{pmatrix} q_{1m} \\ q_{2m} \\ \vdots \\ q_{mm} \end{pmatrix}.$$

Accordingly, if we could choose $q_{1m} = q_{2m} = \cdots = q_{m-1,m} = 0$ and $q_{mm} = 1$, then

$$Qb = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} = b. \qquad (4.17)$$

That this choice is indeed possible is a consequence of the following consideration. When expanding $QA = RQ$, as noted before, we obtain m^2 relations between the $m^2 + 2m$ quantities, q_{ij}, a_i and R_i, $i,j = 1,2,...m$. In addition, the equations obtained by equating the characteristic polynomials of A and R yield m further relations between these quantities. Hence, we have $m^2 + m$ equations relating the $m^2 + 2m$ quantities q_{ij}, a_i, R_i, so that m of the q_{ij} can be chosen arbitrarily and so condition (4.17) is justified.

Ideally, now, we should be able to determine Q explicitly by expanding $QA = RQ$, with

$$Q = \begin{pmatrix} q_{11} & q_{12} \cdots\cdots & q_{1,m-1} & 0 \\ \vdots & & & \vdots \\ \vdots & & . & \vdots \\ \vdots & & & 0 \\ q_{m1} & \cdots\cdots\cdots\cdots & q_{m,m-1} & 1 \end{pmatrix} \qquad (4.18)$$

and then simplifying the m^2 equations which result by introducing the expressions for the a_i in terms of the R_i, as given by the relations obtained by equating the characteristic polynomials of A and R. Unfortunately, the computation is still quite involved and there appears to be no general form which the Q matrix assumes for any choice of m. Of course, for any particular value of m, the calculation can be performed, as is done for the case $m = 4$ in the following chapter.

If we now consider again the total difference of $V(n,X)$ in the non-homogeneous case, as given by equation

(4.16), we see that essentially all that remain to be calculated are the two quantities b^TQ^TQb and Y^TR^TQb. Since, by our choice for Q, Qb = b, we immediately see that the former term is simply 1. As for the latter term, it involves the product of Y^TR^T and Qb. However, since Qb = b, only the last element of the vector RY will contribute to the product, so that

$$Y^TR^TQb = -\frac{\rho_{m-1}}{\rho_m} Y_{m-1} + R_mY_m. \tag{4.19}$$

Hence, equation (4.16) reduces to

$$\Delta V(n,X) = (r^2-1)Y_m{}^2 - 2F\left[-\frac{\rho_{m-1}}{\rho_m}Y_{m-1} + R_mY_m\right] + F^2. \tag{4.20}$$

Moreover, once Q is known, we may calculate Y_{m-1} and Y_m, using the fact that Y = QX. Finally, after $\Delta V(n,X)$ is expressed in terms of the original X_i, all that is necessary to insure asymptotic stability of the equilibrium X = 0 of the non-homogeneous difference equation is to impose the condition of negative definiteness on $\Delta V(n,X)$. This is accomplished by determining those conditions on the a_i which will guarantee the negative definiteness of $\Delta V(n,X)$. The precise method of doing this is illustrated in the next chapter.

Chapter 5: The Difference Equation of Fourth Order

We now apply the results and techniques of the preceding chapter on the m-th order difference equation to the particular case where m = 4. Thus, the difference equation under consideration is

$$X(n+4) + a_1X(n+3) + a_2X(n+2) + a_3X(n+1) + a_4X(n) + F\left[n, X(n), \ldots X(n+3)\right] = 0, \tag{5.1}$$

where a_1, a_2, a_3 and a_4 are real constants and F is a real scalar function of the indicated arguments satisfying

$$F(n,0,0,0,0) = 0$$

for all $n \geqslant n_0 \geqslant 0$. Equivalently, equation (5.1) can be written as the matrix equation

$$X(n+1) = AX(n) - bF, \tag{5.2}$$

where

$$X(n) = \begin{pmatrix} X_1(n) \\ X_2(n) \\ X_3(n) \\ X_4(n) \end{pmatrix} = \begin{pmatrix} X(n) \\ X(n+1) \\ X(n+2) \\ X(n+3) \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_4 & -a_3 & -a_2 & -a_1 \end{pmatrix},$$

and

$$b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Treating the homogeneous case

$$X(n+1) = AX(n)$$

first, we again introduce the transformation Q to obtain

$$Y(n) = QX(n) \tag{5.3}$$

and

$$Y(n+1) = RY(n), \tag{5.4}$$

where R is defined by

$$RQ = QA. \tag{5.5}$$

We now impose the condition that R satisfy

$$R^T R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & r^2 \end{pmatrix}. \tag{5.6}$$

For the case m = 4, condition (4.9) of the previous chapter guarantees that we may set

$$\frac{(m-1)(m-2)}{2} = 3$$

of the elements of R to zero, and following the general pattern established in that chapter, we take

$$r_{31} = r_{41} = r_{42} = 0,$$

so that

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} & R_1 \\ r_{21} & r_{22} & r_{23} & R_2 \\ 0 & r_{32} & r_{33} & R_3 \\ 0 & 0 & r_{43} & R_4 \end{pmatrix}. \tag{5.7}$$

As a result, equation (5.6) becomes

$$\begin{pmatrix} r_{11} & r_{21} & 0 & 0 \\ r_{12} & r_{22} & r_{32} & 0 \\ r_{13} & r_{23} & r_{33} & r_{43} \\ R_1 & R_2 & R_3 & R_4 \end{pmatrix} \begin{pmatrix} r_{11} & r_{12} & r_{13} & R_1 \\ r_{21} & r_{22} & r_{23} & R_2 \\ 0 & r_{32} & r_{33} & R_3 \\ 0 & 0 & r_{43} & R_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & r^2 \end{pmatrix},$$

which leads to the following set of equations:

a) $r_{11}^2 + r_{21}^2 = 1$

b) $r_{11}r_{12} + r_{21}r_{22} = 0$

c) $r_{11}r_{13} + r_{21}r_{23} = 0$

d) $r_{11}R_1 + r_{21}R_2 = 0$

e) $r_{12}^2 + r_{22}^2 + r_{32}^2 = 1$

f) $r_{12}r_{13} + r_{22}r_{23} + r_{32}r_{33} = 0$

g) $r_{12}R_1 + r_{22}R_2 + r_{32}R_3 = 0$

h) $r_{13}^2 + r_{23}^2 + r_{33}^2 + r_{43}^2 = 1$

i) $r_{13}R_1 + r_{23}R_2 + r_{33}R_3 + r_{43}R_4 = 0$

j) $R_1^2 + R_2^2 + R_3^2 + R_4^2 = r^2.$

To solve these equations, we proceed as follows. From d) and a), we obtain

$$r_{11} = \pm \frac{R_2}{\sigma},$$

where

$$\sigma^2 = R_1^2 + R_2^2. \tag{5.8}$$

Accordingly,

$$r_{21} = \mp \frac{R_1}{\sigma}.$$

In a similar manner (as shown in Appendix A), the other elements of R can be determined in terms of R_1, R_2, R_3 and R_4 up to their signs, to yield

$$R = \begin{pmatrix} \pm\frac{R_2}{\sigma} & \pm\frac{R_1R_3}{\sigma\rho} & \pm\frac{R_1R_4}{r\rho} & R_1 \\ \mp\frac{R_1}{\sigma} & \pm\frac{R_2R_3}{\sigma\rho} & \pm\frac{R_2R_4}{r\rho} & R_2 \\ 0 & \mp\frac{\sigma}{\rho} & \pm\frac{R_3R_4}{r\rho} & R_3 \\ 0 & 0 & \mp\frac{\rho}{r} & R_4 \end{pmatrix}, \quad (5.9)$$

where

$$\rho^2 = R_1^{\,2} + R_2^{\,2} + R_3^{\,2}. \quad (5.10)$$

Since the matrices A and R are similar, we may equate their characteristic polynomials and thus obtain

$$\begin{aligned} \det(A - \lambda I) &= \det(R - \lambda I) \\ &= \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 \\ &= \lambda^4 - \lambda^3\left[+R_4 + \frac{R_3R_4}{r\rho} + \frac{R_2R_3}{\sigma\rho} + \frac{R_2}{\sigma}\right] \\ &\quad + \lambda^2\left[\frac{R_3 r}{\rho} + \frac{R_2R_3R_4}{r\sigma\rho} + \frac{R_2R_4}{\sigma} + \frac{R_2R_4}{r\sigma} + \frac{R_2R_3R_4}{\sigma\rho} + \frac{R_3}{\rho}\right] \\ &\quad - \lambda\left[+\frac{R_2 r}{\sigma} + \frac{R_2R_3 r}{\sigma\rho} + \frac{R_3R_4}{\rho} + \frac{R_4}{r}\right] \\ &\quad + r, \end{aligned}$$

on expanding $\det(R - \lambda I)$, collecting the coefficients of like powers of λ, and simplifying the terms using equations (5.8), (5.10), and relation j) from the expansion of equation (5.6). Hence,

$$a_1 = -R_4 - \frac{R_3R_4}{r\rho} - \frac{R_2R_3}{\sigma\rho} - \frac{R_2}{\sigma} \quad (5.11a)$$

$$a_2 = \frac{rR_3}{\rho} + \frac{R_2R_3R_4}{r\sigma\rho} + \frac{R_2R_4}{\sigma} + \frac{R_2R_4}{r\sigma} + \frac{R_2R_3R_4}{\sigma\rho} + \frac{R_3}{\rho} \quad (5.11b)$$

$$a_3 = \frac{-rR_2}{\sigma} - \frac{R_2R_3 r}{\sigma\rho} - \frac{R_3R_4}{\rho} - \frac{R_4}{r} \quad (5.11c)$$

$$a_4 = +r. \tag{5.11d}$$

For purposes of reference in the sequel, we shall refer to the above four relations as equations (5.11).

In the process of performing this calculation, as was noted previously, a particular choice is made of the signs of the components of R, so that we finally have a unique expression for R given by

$$R = \begin{pmatrix} \frac{R_2}{\sigma} & \frac{R_1R_3}{\sigma\rho} & \frac{R_1R_4}{r\rho} & R_1 \\ -\frac{R_1}{\sigma} & \frac{R_2R_3}{\sigma\rho} & \frac{R_2R_4}{r\rho} & R_2 \\ 0 & -\frac{\sigma}{\rho} & \frac{R_3R_4}{r\rho} & R_3 \\ 0 & 0 & -\frac{\rho}{r} & R_4 \end{pmatrix}. \tag{5.12}$$

We now turn to an explicit calculation of the matrix Q from the relationship QA = RQ. From the discussion in the previous chapter, we know that we may choose

$$q_{14} = q_{24} = q_{34} = 0,$$

$$q_{44} = 1,$$

so that Q has the general form

$$Q = \begin{pmatrix} q_{11} & q_{12} & q_{13} & 0 \\ q_{21} & q_{22} & q_{23} & 0 \\ q_{31} & q_{32} & q_{33} & 0 \\ q_{41} & q_{42} & q_{43} & 1 \end{pmatrix}.$$

Substituting this form for the matrix Q into the relation QA = RQ leads to the equation

$$\begin{pmatrix} q_{11} & q_{12} & q_{13} & 0 \\ q_{21} & q_{22} & q_{23} & 0 \\ q_{31} & q_{32} & q_{33} & 0 \\ q_{41} & q_{42} & q_{43} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -a_4 & -a_3 & -a_2 & -a_1 \end{pmatrix} =$$

$$\begin{pmatrix} \frac{R_2}{\sigma} & \frac{R_1 R_3}{\sigma\rho} & \frac{R_1 R_4}{r\rho} & R_1 \\ -\frac{R_1}{\sigma} & \frac{R_2 R_3}{\sigma\rho} & \frac{R_2 R_4}{r\rho} & R_2 \\ 0 & -\frac{\sigma}{\rho} & \frac{R_3 R_4}{r\rho} & R_3 \\ 0 & 0 & -\frac{\rho}{r} & R_4 \end{pmatrix} \begin{pmatrix} q_{11} & q_{12} & q_{13} & 0 \\ q_{21} & q_{22} & q_{23} & 0 \\ q_{31} & q_{32} & q_{33} & 0 \\ q_{41} & q_{42} & q_{43} & 1 \end{pmatrix}.$$

Multiplying this out, we obtain sixteen equations. Among them, we find such simple expressions as

$$q_{13} = R_1$$
$$q_{23} = R_2$$
$$q_{33} = R_3$$
$$q_{43} - a_1 = R_4.$$

Substituting from equations (5.11), the last relation yields

$$q_{43} = -\frac{R_3 R_4}{r\rho} - \frac{R_2 R_3}{\sigma\rho} - \frac{R_2}{\sigma} .$$

The other elements of Q are not so easily determined, as the remaining twelve equations are far more complicated. The actual computations are carried out in Appendix B. The resulting expressions for the q_{ij} there obtained are simplified by introducing the values for the a_i in terms of the R_i, as given by equations (5.11). Thus, we finally obtain for Q,

$$Q = \begin{pmatrix} 0 & 0 & R_1 & 0 \\ 0 & -\sigma & R_2 & 0 \\ +\rho & -\frac{R_2R_3}{\sigma} - \frac{R_2\rho}{\sigma} & R_3 & 0 \\ -\frac{R_4}{r} & \frac{R_2R_4}{\sigma r} + \frac{R_2R_3R_4}{r\sigma\rho} + \frac{R_3}{\rho} & -\frac{R_2}{\sigma} - \frac{R_3R_4}{r\rho} - \frac{R_2R_3}{\sigma\rho} & 1 \end{pmatrix}. \quad (5.13)$$

With these preliminary results, we may now consider the scalar function $V(n,X) = Y^T(n)Y(n)$. As mentioned previously, this will always be positive definite, so all that remains is to consider the sign of its total difference $\Delta V(n,X)$ for the difference equation under study. From equation (4.15) of the previous chapter, with $m = 4$, we have for the homogeneous case

$$\Delta V(n,X) = (r^2 - 1)\, Y_4^{\,2},$$

which will be non-positive for $r^2 = a_4^{\,2} < 1$. Hence we have

<u>Theorem 42:</u> If, for real constants a_1, a_2, a_3, a_4, equations (5.11) can be solved for real numbers R_1, R_2, R_3, R_4 which satisfy the condition

$$R_1^{\,2} + R_2^{\,2} + R_3^{\,2} + R_4^{\,2} = r^2 < 1,$$

then the equilibrium $X = 0$ of the homogeneous difference equation of fourth order

$$X(n+4) + a_1X(n+3) + a_2X(n+2) + a_3X(n+1) + a_4X(n) = 0$$

is stable.

We now attempt to determine those conditions on the a_i which will guarantee that $\Delta V(n,X)$ is negative definite and which therefore suffice for asymptotic stability for the homogeneous difference equation. That is, subject to these conditions, $\Delta V(n,X)$ will be zero when, and only when, $X_1 = X_2 = X_3 = X_4 = 0$. Thus we consider

$$\Delta V(n,X) = (r^2-1)\,Y_4^{\,2} = 0.$$

We note that this does not automatically imply that $\Delta V(n,X)$ is negative definite in the Y coordinates. Using equation (5.3) to determine Y_4, this becomes

$$(r^2-1)\left\{-\frac{R_4X_1}{r} + \left[\frac{R_2R_4}{r\sigma} + \frac{R_2R_3R_4}{r\sigma\rho} + \frac{R_3}{\rho}\right]X_2 - \left[\frac{R_2}{\sigma} + \frac{R_3R_4}{r\rho} + \frac{R_2R_3}{\sigma\rho}\right]X_3 + X_4\right\}^2 = 0.$$

This expression is obviously zero if each of the X_i is zero. Therefore, we must consider under what conditions

$$-\frac{R_4X_1}{r} + \left[\frac{R_2R_4}{r\sigma} + \frac{R_2R_3R_4}{r\sigma\rho} + \frac{R_3}{\rho}\right]X_2 - \left[\frac{R_2}{\sigma} + \frac{R_3R_4}{r\rho} + \frac{R_2R_3}{\sigma\rho}\right]X_3 + X_4 \tag{5.14}$$

is equal to zero. This is equivalent to

$$X(n+3) = \frac{R_4}{r}X(n) - \left[\frac{R_2R_4}{r\sigma} + \frac{R_2R_3R_4}{r\sigma\rho} + \frac{R_3}{\rho}\right]X(n+1) + \left[\frac{R_2}{\sigma} + \frac{R_3R_4}{r\rho} + \frac{R_2R_3}{\sigma\rho}\right]X(n+2). \tag{5.15}$$

If, in relation (5.15), we replace n by n+1, we also have

$$X(n+4) = \frac{R_4}{r}X(n+1) - \left[\frac{R_2R_4}{r\sigma} + \frac{R_2R_3R_4}{r\sigma\rho} + \frac{R_3}{\rho}\right]X(n+2) + \left[\frac{R_2}{\sigma} + \frac{R_3R_4}{r\rho} + \frac{R_2R_3}{\sigma\rho}\right]X(n+3). \tag{5.16}$$

We now substitute for X(n+3) and X(n+4) from equations (5.15) and (5.16) into the homogeneous difference equation and obtain, after simplifying the resulting expression using equations (5.11),

$$R_3X(n+2) - X(n+1)\left[\frac{R_2R_3}{\sigma} + \frac{R_2\rho}{\sigma}\right] + \rho X(n) = 0. \qquad (5.17)$$

We first investigate the case in which $R_3 = 0$. Then equation (5.17) reduces to

$$-\frac{R_2\rho}{\sigma} X(n+1) + \rho X(n) = 0,$$

or equivalently, if $\rho \neq 0$,

$$R_2X(n+1) = \sigma X(n).$$

Now, if $R_2 = 0$ also, then $\sigma X(n) = 0$. However, if we assume that σ is non-zero, then it follows that

$$X(n) = 0$$

for all $n \geqslant n_o$. On the other hand, if $R_2 \neq 0$, we obtain

$$X(n+1) = \frac{\sigma}{R_2} X(n).$$

Substituting this expression into the homogeneous difference equation, we find

$$X(n)\left[\frac{\sigma^4}{R_2{}^4} + a_1\frac{\sigma^3}{R_2{}^3} + a_2\frac{\sigma^2}{R_2{}^2} + a_3\frac{\sigma}{R_2} + a_4\right] = 0.$$

The term in the brackets is precisely the characteristic polynomial of the matrix A evaluated for

$$\lambda = \sigma / R_2.$$

Hence, for asymptotic stability in the present case, we must have the expression in the brackets non-zero; i.e., we must assume that σ/R_2 is not an eigenvalue of the matrix A. As a consequence, we conclude that

$$X(n) = 0$$

for all $n \geqslant n_c$.

We now consider equation (5.17) in the case where $R_3 \neq 0$. It then follows that

$$X(n+2) = X(n+1)\left[\frac{R_2\rho}{R_3\sigma} + \frac{R_2}{\sigma}\right] - \frac{\rho}{R_3} X(n).$$

As was done previously, this equation can be used to express both $X(n+3)$ and $X(n+4)$ in terms of simply $X(n)$ and $X(n+1)$. After substituting these three relationships into the homogeneous difference equation, we find that it reduces to

$$aX(n+1) + bX(n) = 0, \tag{5.18}$$

where

$$a = \frac{R_2^3\rho^3}{R_3^3\sigma^3} - \frac{R_2\rho}{R_3\sigma} - 2\frac{R_1^2R_2}{R_3^2\sigma} + \frac{R_1^2R_4\rho}{R_3\sigma^2} + \frac{R_1^2R_2R_3}{\sigma^3\rho} - \frac{R_2^2R_4}{R_3^2} - \frac{R_1^2R_3R_4}{\rho}$$

$$b = \frac{R_1^2}{R_3^2} + \frac{R_2R_4\sigma}{R_3^2} - \frac{R_2^2\rho}{R_3^3}.$$

We now consider separately the cases where a is zero and non-zero. In the first instance, where $a = 0$, equation (5.18) reduces to

$$bX(n) = 0,$$

so it is necessary to show that $b \neq 0$. If $b = 0$, then it can be written in the following two forms.

$$0 = \sigma^2 - R_2^2 + \sigma R_2 R_4 - \frac{R_2^2 \rho}{R_3}$$

$$0 = R_2^2 (1 + \rho / R_3) - \sigma R_2 R_4 - \sigma^2 .$$

The first is a quadratic equation in σ, so that

$$\sigma = \frac{-R_2 R_4 \pm R_2 (R_4^2 + 4 + 4\rho / R_3)^{\frac{1}{2}}}{2} .$$

Equivalently,

$$\frac{\sigma}{R_2} = \frac{-R_4 \pm (R_4^2 + 4 + 4\rho / R_3)^{\frac{1}{2}}}{2} .$$

The second expression for b = 0 is quadratic in R_2, so that

$$R_2 = \frac{+\sigma R_4 \pm \sigma (R_4^2 + 4 + 4\rho / R_3)^{\frac{1}{2}}}{2} ,$$

or equivalently,

$$\frac{R_2}{\sigma} = \frac{+R_4 \pm (R_4^2 + 4 + 4\rho / R_3)^{\frac{1}{2}}}{2} .$$

As a consequence,

$$\frac{\sigma}{R_2} - \frac{R_2}{\sigma} = \frac{\sigma^2 - R_2^2}{\sigma R_2} = \frac{R_1^2}{\sigma R_2} = -R_4 .$$

That is,

$$R_1^2 = -\sigma R_2 R_4 .$$

Thus, if b = 0, it follows that

$$0 = R_1^2 + \sigma R_2 R_4 - \frac{R_2^2 \rho}{R_3} = -\frac{R_2^2 \rho}{R_3} .$$

Hence, $R_2 = 0$ also. But if $R_2 = 0$, then $b = R_1^2 = 0$, which is impossible since we assume $R_1^2 + R_2^2 = \sigma^2 > 0$.

Secondly, we consider the case where $a \neq 0$. Then

$$X(n+1) = -b/a\ X(n).$$

Substituting this into the homogeneous equation, we find

$$X(n)\left[(b/a)^4 - a_1(b/a)^3 + a_2(b/a)^2 - a_3(b/a) + a_4\right] = 0.$$

The term in the brackets is the characteristic polynomial of the matrix A evaluated for $\lambda = -b/a$. Thus, if we assume that $-b/a$ is not an eigenvalue of A, then

$$X(n) = 0$$

for all $n \geqslant n_o$.

Collecting the above results, we have the following theorem.

<u>Theorem 43:</u> If

a) for real constants a_1, a_2, a_3, a_4, equations (5.11) can be solved for real numbers R_1, R_2, R_3 and R_4 which satisfy the conditions

$$R_1^2 + R_2^2 + R_3^2 + R_4^2 = r^2 < 1$$

$$R_1^2 + R_2^2 = \sigma^2 > 0,$$

b) $R_3 = 0$, $R_2 \neq 0$ implies $+\sigma/R_2$ is not an eigenvalue of A,

c) for real constants a and b defined by equation (5.18) for $R_3 \neq 0$, $a \neq 0$ implies $-b/a$ is not an eigenvalue of A,

then the equilibrium $X = 0$ of the homogeneous difference equation of fourth order

$$X(n+4) + a_1X(n+3) + a_2X(n+2) + a_3X(n+1) + a_4X(n) = 0$$

is asymptotically stable.

We now turn our attention to a consideration of the non-homogeneous difference equation (5.1). From the discussion of the previous chapter, we see that we need only examine the behavior of the total difference $\Delta V(n,X)$, which is given by

$$\Delta V(n,X) = (r^2-1)\, Y_4^2 - 2(-\rho/r\, Y_3 + R_4 Y_4)F + F^2, \qquad (5.19)$$

for the case m = 4. Further, using the expression for Q given in equation (5.13) and the matrix equation Y = QX, we find

$$Y_3 = \rho X_1 - (R_2/\sigma)(R_3+\rho)X_2 + R_3X_3$$

$$Y_4 = -\frac{R_4}{r} X_1 + \left[\frac{R_2R_4}{\sigma r} + \frac{R_2R_3R_4}{r\sigma\rho} + \frac{R_3}{\rho}\right]X_2 + \left[-\frac{R_2}{\sigma} - \frac{R_3R_4}{r\rho} - \frac{R_2R_3}{\sigma\rho}\right]X_3 + X_4.$$

Hence, equation (5.19) reduces to

$$\begin{aligned}\Delta V(n,X) = \; & (r^2-1)\left[-\frac{R_4}{r} X_1 + \left(\frac{R_2R_4}{r\sigma} + \frac{R_2R_3R_4}{r\sigma\rho} + \frac{R_3}{\rho}\right) X_2 \right.\\ & \left. + \left(-\frac{R_2}{\sigma} - \frac{R_3R_4}{r\rho} - \frac{R_2R_3}{\sigma\rho}\right) X_3 + X_4\right]^2 \\ & -2\left[-rX_1 + \left(\frac{rR_2}{\sigma} + \frac{rR_2R_3}{\sigma\rho} + \frac{R_3R_4}{\rho}\right)X_2 \right.\\ & \left. + \left(-\frac{R_2R_4}{\sigma} - \frac{R_2R_3R_4}{\sigma\rho} - \frac{rR_3}{\rho}\right) X_3 + R_4X_4\right] F \\ & + F^2.\end{aligned}$$

We may rewrite this as

$$\Delta V(n,X) = K - 2\left[-\frac{\rho^2}{r} X_1 + \left(\frac{R_2\rho^2}{r\sigma} + \frac{R_2R_3\rho}{r\sigma}\right) X_2 - \frac{R_3\rho}{r} X_3\right] F + \left[1 - \frac{R_4^2}{r^2-1}\right] F^2, \qquad (5.20)$$

where

$$K = (r^2-1)\left[-\frac{R_4}{r}X_1 + \left(\frac{R_2R_4}{r\sigma} + \frac{R_2R_3R_4}{r\sigma\rho} + \frac{R_3}{\rho}\right)X_2 + \left(-\frac{R_2}{\sigma} - \frac{R_3R_4}{r\rho} - \frac{R_2R_3}{\sigma\rho}\right)X_3 + X_4 - \frac{R_4}{r^2-1}F\right]^2. \tag{5.21}$$

Since, by hypothesis, $r^2-1 < 0$, we see immediately that $K \leq 0$. Hence, $\Delta V(n,X)$ as given by equation (5.20) will be non-positive for

$$-2F\left[-\frac{\rho^2}{r}X_1 + \left(\frac{\rho^2R_2}{r\sigma} + \frac{R_2R_3\rho}{r\sigma}\right)X_2 - \frac{R_3\rho}{r}X_3\right] + F^2\left[\frac{\rho^2-1}{r^2-1}\right] \leq 0,$$

or equivalently,

$$0 < \frac{F(n,X_1,X_2,X_3,X_4)}{\left[-\frac{\rho^2}{r}X_1 + \left(\frac{R_2\rho^2}{r\sigma} + \frac{R_2R_3\rho}{r\sigma}\right)X_2 - \frac{R_3\rho}{r}X_3\right]} \leq \frac{2(r^2-1)}{\rho^2-1}. \tag{5.22}$$

Again, collecting the above results, we have the following theorem.

<u>Theorem 44:</u> If,

a) for real constants a_1, a_2, a_3, and a_4, equations (5.11) can be solved for real numbers R_1, R_2, R_3 and R_4 which satisfy the conditions

$$R_1^2 + R_2^2 + R_3^2 + R_4^2 = r^2 < 1$$

$$R_1^2 + R_2^2 = \sigma^2 > 0,$$

b) there exists a real function $F(n,U_1,U_2,U_3,U_4)$ such that $F(n,U_1,U_2,U_3,U_4) = 0$ if and only if $U_1 = U_2 = U_3 = U_4 = 0$ and which satisfies the condition

$$0 < \frac{F(n,U_1,U_2,U_3,U_4)}{\left[-\frac{\rho^2}{r}U_1 + \left(\frac{R_2\rho^2}{r\sigma} + \frac{R_2R_3\rho}{r\sigma}\right)U_2 - \frac{\rho R_3}{r}U_3\right]} \leq \frac{2(r^2-1)}{\rho^2-1},$$

then the equilibrium $X = 0$ of the non-homogeneous difference equation of fourth order

$$X(n+4) + a_1X(n+3) + a_2X(n+2) + a_3X(n+1) + a_4X(n) + F(n,X(n),\ldots,X(n+3)) = 0$$

is stable.

Finally, we will now investigate under what conditions the equilibrium of the non-homogeneous equation (5.1) is asymptotically stable. From the discussion for the general case in the previous chapter, we already know that the scalar function

$$V(n,X) = Y^T(n)Y(n)$$

is decrescent, so that we need only determine under what conditions its total difference, as given by equation (5.20), is negative definite for the non-homogeneous equation. To do this, we set $\Delta V(n,X) = 0$ and obtain $K = 0$ and

$$F^2\left[\frac{\rho^2-1}{r^2-1}\right] = 2F\left[-\frac{\rho^2}{r}X_1 + \left(\frac{R_2\rho^2}{r\sigma} + \frac{R_2R_3\rho}{r\sigma}\right)X_2 - \frac{R_3\rho}{r}X_3\right], \qquad (5.23)$$

both of which must hold simultaneously. Equation (5.23) is valid if $F = 0$, but we assume that F is zero if and only if each of the $X(n) = 0$, for all n, which in turn would guarantee that $K = 0$ also. Thus, the case which needs investigation is where equation (5.23) is divided by F to reduce to

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$$F\left[\frac{\rho^2-1}{r^2-1}\right] = 2\left[-\frac{\rho^2}{r}X_1 + \left(\frac{R_2\rho^2}{r\sigma} + \frac{R_2R_3\rho}{r\sigma}\right)X_2 - \frac{R_3\rho}{r}X_3\right] \qquad (5.24)$$

and where K = 0 yields

$$\frac{R_4F}{r^2-1} = -\frac{R_4}{r}X_1 + \left(\frac{R_2R_4}{r\sigma} + \frac{R_2R_3R_4}{r\sigma\rho} + \frac{R_3}{\rho}\right)X_2 + \left(-\frac{R_2}{\sigma} - \frac{R_3R_4}{r\rho} - \frac{R_2R_3}{\sigma\rho}\right)X_3 + X_4. \qquad (5.25)$$

If equation (5.25) is multiplied by R_4 and the result is added to equation (5.24), we obtain

$$F = -(r+\rho^2/r)X_1 + \left(\frac{R_2\rho^2}{r\sigma} + \frac{R_2R_3}{r\sigma} + \frac{R_2r}{\sigma} + \frac{R_2R_3r}{\sigma\rho} + \frac{R_3R_4}{\rho}\right)X_2 + \left(-\frac{R_3\rho^2}{r\rho} - \frac{R_2R_4}{\sigma} - \frac{R_3r}{\rho} - \frac{R_2R_3R_4}{\sigma\rho}\right)X_3 + R_4X_4. \qquad (5.26)$$

We now substitute this value for F into equation (5.1) to obtain

$$\begin{aligned} X(n+4) &+ X(n+3)\left[-\frac{R_3R_4}{r\rho} - \frac{R_2R_3}{\sigma\rho} - \frac{R_2}{\sigma}\right] \\ &+ X(n+2)\left[+\frac{R_3\rho}{r} + \frac{R_2R_3R_4}{r\sigma\rho} + \frac{R_2R_4}{r\sigma} + \frac{R_3}{\rho}\right] \\ &+ X(n+1)\left[\frac{R_2\rho^2}{r\sigma} - \frac{R_2R_3}{r\sigma} - \frac{R_4}{r}\right] - X(n)\frac{\rho^2}{r} = 0. \end{aligned} \qquad (5.27)$$

For simplicity, we shall write this as

$$X(n+4) + b_1X(n+3) + b_2X(n+2) + b_3X(n+1) + b_4X(n) = 0. \qquad (5.28)$$

Equations (5.24) and (5.26) yield two separate expressions for the function F. If we equate these two formulations, we find that

$$X_4 = X_1\left[\frac{R_4}{r}\left(\frac{\rho^2+1}{\rho^2-1}\right)\right] + X_2\left[\left(\frac{R_2R_4}{r\sigma} + \frac{R_2R_3R_4}{r\sigma\rho}\right)\left(\frac{\rho^2+1}{\rho^2-1}\right) - \frac{R_3}{\rho}\right]$$

$$+ X_3 \left[- \frac{R_3 R_4 (\rho^2+1)}{r \rho (\rho^2-1)} + \frac{R_2}{\sigma} + \frac{R_2 R_3}{\sigma \rho} \right].$$

If we now introduce this expression into equation (5.27), we can eliminate X(n+4) and X(n+3), leaving

$$X(n+2) \left[\frac{2R_2 R_4 \sigma}{r(\rho^2-1)} + \frac{2R_3{}^2 R_4{}^2 (\rho^2+1)}{r^2 (\rho^2-1)^2} + \frac{R_3 \rho}{r} - \frac{R_2 R_3 R_4}{r \sigma (\rho+1)} \right]$$
$$+ X(n+1) \left[\frac{2R_4}{r(\rho^2-1)} - \frac{2R_2 R_3 R_4{}^2 \rho (\rho^2+1)}{r^2 \sigma (\rho^2-1)^2} - \frac{2R_2 R_3{}^2 R_4{}^2 (\rho+1)}{r^2 \sigma (\rho^2-1)^2} + \right.$$
$$\left. + \frac{2R_3{}^2 R_4}{r(\rho^2-1)} + \frac{R_2 \rho^2}{r \sigma} - \frac{R_2 R_3}{r \sigma} \right] + X(n) \left[\frac{-2R_3 R_4{}^2 \rho (\rho^2+1)}{r^2 (\rho^2-1)^2} - \frac{\rho^2}{r} \right]$$

$$= c_3 X(n+2) + c_2 X(n+1) + c_1 X(n) = 0. \qquad (5.29)$$

Now, if $c_3 = 0$, then equation (5.29) reduces to

$$c_1 X(n) + c_2 X(n+1) = 0.$$

If, further, $c_2 = 0$, then

$$c_1 X(n) = 0.$$

Thus, if we assume that $c_1 \neq 0$ in this case, it follows that

$$X(n) = 0$$

for all $n \geqslant n_o$.

If, on the other hand, $c_3 = 0$ and $c_2 \neq 0$, then we may solve for X(n+1) as

$$X(n+1) = -c_1/c_2 \, X(n).$$

Introducing this into equation (5.28), we have

$$X(n) \left[(-c_1/c_2)^4 + b_1 (-c_1/c_2)^3 + b_2 (-c_1/c_2)^2 \right.$$
$$\left. + b_3 (-c_1/c_2) + b_4 \right] = 0. \qquad (5.30)$$

Hence, if we assume that the term in the brackets is non-zero, then we must again have that

$$X(n) = 0$$

for all $n \geqslant n_o \geqslant 0$.

We now consider the case when $c_3 \neq 0$, so that we may solve for X(n+2) in equation (5.29).

$$X(n+2) = -c_1/c_3\ X(n) - c_2/c_3\ X(n+1).$$

Using this expression to eliminate X(n+3) and X(n+4) and substituting them into equation (5.28), we obtain

$$X(n+1)\Big[(-c_2/c_3)^2 + c_1c_2/c_3{}^2 + b_1(-c_2/c_3)^2 - b_1(c_1/c_3) - b_2(c_2/c_3) + b_3\Big]$$
$$+ X(n)\Big[-c_1c_2{}^2/c_3{}^3 + b_1(c_1c_2/c_3{}^2) - b_2(c_1/c_3) + b_4\Big]$$
$$= d_1X(n+1) + d_2X(n) = 0. \qquad (5.31)$$

Now, if $d_1 = 0$, then $d_2X(n) = 0$ also. Hence, if we impose the assumption that $d_2 \neq 0$, we obtain

$$X(n) = 0$$

for all $n \geqslant n_o \geqslant 0$. For the other alternative,

$$X(n+1) = -d_2/d_1\ X(n).$$

Substituting this expression into equation (5.28), we find

$$X(n)\Big[(-d_2/d_1)^4 + b_1(-d_2/d_1)^3 + b_2(-d_2/d_1)^2 + b_3(-d_2/d_1) + b_4\Big] = 0. \qquad (5.32)$$

Thus, as in equation (5.30), we must assume that the term in the brackets is non-zero to insure that

$$X(n) = 0$$

for all $n \geqslant n_o \geqslant 0$.

Summarizing the above results, we obtain the following theorem.

<u>Theorem 45:</u> If,

a) for real constants a_1, a_2, a_3 and a_4, equations (5.11) can be solved for real numbers R_1, R_2, R_3 and R_4 which satisfy the conditions

$$R_1^2 + R_2^2 + R_3^2 + R_4^2 = r^2 < 1$$

$$R_1^2 + R_2^2 = \sigma^2 > 0,$$

b) there exists a real function $F(n,U_1,U_2,U_3,U_4)$ such that $F(n,U_1,U_2,U_3,U_4) = 0$ if and only if $U_1 = U_2 = U_3 = U_4 = 0$ and which further satisfies the condition

$$0 < \frac{F(n,U_1,U_2,U_3,U_4)}{\left[\frac{-\rho^2}{r} U_1 + \left(\frac{R_2\rho^2}{r\sigma} + \frac{R_2 R_3 \rho}{r\sigma}\right) U_2 - \frac{\rho R_3}{r} U_3\right]} \leq \frac{2(r^2-1)}{\rho^2-1},$$

c) for real constants b_1, b_2, b_3 and b_4 as defined by equation (5.28), c_1, c_2 and c_3 as defined by equation (5.29), and d_1 and d_2 as defined by equation (5.31),

1) $c_3 = 0$, $c_2 = 0$ imply that $c_1 \neq 0$

2) $c_3 = 0$, $c_2 \neq 0$ imply that

$$(-c_1/c_2)^4 + b_1(-c_1/c_2)^3 + b_2(-c_1/c_2)^2 + b_3(-c_1/c_2) + b_4 \neq 0,$$

3) $c_3 \neq 0$, $d_1 = 0$ imply that $d_2 \neq 0$

4) $c_3 \neq 0$, $d_1 \neq 0$ imply that

$$(-d_2/d_1)^4 + b_1(-d_2/d_1)^3 + b_2(-d_2/d_1)^2 + b_3(-d_2/d_1) + b_4 \neq 0,$$

then the equilibrium of the non-homogeneous difference equation of fourth order

$$X(n+4) + a_1X(n+3) + a_2X(n+2) + a_3X(n+1) + a_4X(n) + F(n,X(n),\ldots X(n+3)) = 0$$

is asymptotically stable.

We shall now apply the above results to several examples of specific non-homogeneous difference equations of the fourth order.

<u>Example 1:</u> Consider the difference equation

$$X(n+4) + \tfrac{1}{2}(1-\sqrt{2})\, X(n+2) - (1/\sqrt{2})\, X(n) + \frac{c}{1+h^2(n)}\left[-(1/\sqrt{2})X(n) - \tfrac{1}{2}X(n+2)\right] = 0,$$

where h(n) is any real scalar function of n. Then equations (5.11) may be solved to yield

$$R_1 = R_3 = \tfrac{1}{2}, \qquad R_2 = R_4 = 0.$$

As a result, $\sigma^2 = \frac{1}{4}$ and $\rho^2 = r^2 = \frac{1}{2}$. Therefore, Theorem 42 insures stability in the homogeneous case (c = 0).

Moreover, using the notation of Theorem 43,

$$a = 0 \qquad\qquad b = 1.$$

Thus the equilibrium of the homogeneous case (c = 0) is also asymptotically stable.

Furthermore, the non-homogeneous term satisfies the estimate

$$\frac{\frac{c}{1+h^2(n)}\left[-(1/\sqrt{2})X(n) - \tfrac{1}{2}X(n+2)\right]}{-(1/\sqrt{2})X(n) - \tfrac{1}{2}X(n+2)} \leq \frac{c}{1+h^2(n)}$$

$$\leq \frac{2(r^2-1)}{\rho^2-1}$$

$$= 2,$$

whenever $c \leq 2$ for all $n \geq n_0 \geq 0$. Hence, by Theorem 44, the equilibrium of the non-homogeneous equation with $c \leq 2$ is stable.

Finally, using the notation of Theorem 45,

$$b_1 = 0 \quad b_2 = \tfrac{1}{2}(1+\sqrt{2}) \quad b_3 = 0 \quad b_4 = 1/\sqrt{2}$$

$$c_1 = -1/\sqrt{2} \quad c_2 = 0 \quad c_3 = \tfrac{1}{2}$$

$$d_1 = 0 \quad d_2 = 1+\sqrt{2}\,.$$

These values correspond to the case given in the Theorem for $c_3 \neq 0$, but $d_1 = 0$. The only additional condition for asymptotic stability is that d_2 be non-zero, which is fulfilled.

<u>Example 2:</u> Consider the difference equation

$$X(n+4) + (1/\sqrt{2})\,X(n+3) - \tfrac{1}{2}X(n+1) - (1/\sqrt{2})\,X(n)$$

$$- \frac{c}{1+h^2(n)} \sin\left[(1/\sqrt{2})X(n) - \tfrac{1}{2}X(n+1)\right] = 0,$$

where $h(n)$ is any real scalar function of n. As in Example 1, equations (5.11) are solvable to yield

$$R_1 = R_2 = \tfrac{1}{2} \qquad R_3 = R_4 = 0$$

and, as a result,

$$\sigma^2 = \rho^2 = r^2 = \tfrac{1}{2}.$$

Thus, by Theorem 42, we conclude that the equilibrium of the homogeneous equation ($c = 0$) is stable.

To investigate asymptotic stability of the homogeneous equation, we must examine the situation when $R_3 = 0$. In that case,

$$\sigma/R_2 = \sqrt{2}\,,$$

and hence

$$(\sigma/R_2)^4 + a_1(\sigma/R_2)^3 + a_2(\sigma/R_2)^2 + a_3(\sigma/R_2) + a_4$$

$$= (\sqrt{2})^4 + (1/\sqrt{2})(\sqrt{2})^3 - \tfrac{1}{2}(\sqrt{2}) - (1/\sqrt{2})$$
$$= 6 - \sqrt{2} \neq 0.$$

Therefore, the equilibrium is asymptotically stable for $c = 0$.

If $c \neq 0$, then the non-homogeneous term satsifies

$$-\frac{c}{1 + h^2(n)} \left[\frac{\sin\left[(1/\sqrt{2})\, X(n) - \tfrac{1}{2}X(n+1)\right]}{-(1/\sqrt{2})\, X(n) + \tfrac{1}{2}X(n+1)} \right] \leq \frac{c}{1 + h^2(n)} \cdot 1$$
$$\leq 2\,\frac{r^2 - 1}{\rho^2 - 1}$$
$$= 2,$$

whenever $c \leq 2$ for all $n \geq n_0 \geq 0$. Thus, this gives the condition for stability of the equilibrium in the non-homogeneous case.

Finally, using the notation of Theorem 45, we find

$$b_1 = -1/\sqrt{2} \quad b_2 = 0 \quad b_3 = \tfrac{1}{2} \quad b_4 = -1/\sqrt{2}$$
$$c_1 = -1/\sqrt{2} \quad c_2 = \tfrac{1}{2} \quad c_3 = 0.$$

Thus, we must investigate the case $c_3 = 0$ and $c_2 \neq 0$. Here

$$c_1/c_2 = -\sqrt{2}$$

and

$$(\sqrt{2})^4 - (1/\sqrt{2})(\sqrt{2})^3 + \tfrac{1}{2}(\sqrt{2}) - 1/\sqrt{2}$$
$$= 4 - 2 \neq 0,$$

and so, by Theorem 45, the equilibrium is asymptotically stable for all $n \geq n_0 \geq 0$ in the non-homogeneous case whenever $c \leq 2$.

Chapter 6: Some Remarks on the Work of Puri and Drake

At this point, several remarks on the work done by Puri and Drake (11) are in order. Essentially, the theorems that they state in their paper purport to be sufficient conditions for asymptotic stability of the equilibria of the non-homogeneous difference equations of second and third orders. However, the conditions they obtain actually yield no more than the usual stability. The principal reason for this is that the scalar functions $V(n,X)$ that they consider are required to possess negative semi-definite total differences. That is, they impose no conditions on $\Delta V(n,X)$ to insure that it is negative definite. It is the purpose of the present chapter to determine the supplementary conditions on the scalar functions $V(n,X)$ for the cases $m = 2$ and $m = 3$ which will guarantee asymptotic stability of the equilibria.

Moreover, it turns out that utilization of the approach outlined in Chapter 4 leads to a far simpler format for the conditions and so will be adopted here instead of the formulation used by Puri and Drake.

We first consider the case $m = 2$. For reference, the forms for the various quantities which appear when working through the entire scheme will be stated. We have

$$R = \begin{pmatrix} R_2/r & R_1 \\ -R_1/r & R_2 \end{pmatrix},$$

so that

$$a_1 = -(R_2/r + R_2)$$
$$a_2 = r \tag{6.1}$$

and

$$Q = \begin{pmatrix} R_1 & 0 \\ -\frac{R_2}{r} & 1 \end{pmatrix}.$$

As a result,

$$Y_1 = R_1 X_1$$
$$Y_2 = -(R_2/r)X_1 + X_2.$$

As mentioned above, it is necessary only to study under what conditions the total difference of $V(n,X) = Y^T Y$ is negative definite for the difference equation

$$X(n+2) + a_1 X(n+1) + a_2 X(n) + F(n,X(n),X(n+1)) = 0.$$

We have

$$\Delta V(n,X) = (r^2-1)Y_2^2 - 2\left[(-R_1/r)Y_1 + R_2 Y_2\right]F + F^2.$$

Substituting the values for the Y_i, we find

$$\Delta V(n,X) = (r^2-1)\left[(-R_2/r)X_1 + X_2\right]^2 - 2\left[-rX_1 + R_2 X_2\right]F + F^2$$
$$= (r^2-1)\left[(-R_2/r)X_1 + X_2 - (R_2/r^2-1)F\right]^2$$
$$+ 2(R_1{}^2/r)FX_1 + (R_1{}^2-1)/(r^2-1)\ F^2.$$

In order that this expression be non-positive, we impose the condition

$$2(R_1{}^2/r)X_1 F + (R_1{}^2-1)/(r^2-1)\ F^2 \leq 0,$$

or equivalently,

$$-2(R_1{}^2/r)(r^2-1)/(R_1{}^2-1) \leq \frac{F}{X_1} \leq 0.$$

Then, in order for $\Delta V(n,X)$ to be zero, it is necessary that

$$\frac{R_2}{r^2-1} F = -\frac{R_2}{r} X_1 + X_2 \tag{6.2}$$

and

$$\frac{R_1^2 - 1}{r^2 - 1} F = -2\frac{R_1^2}{r} X_1. \tag{6.3}$$

If equation (6.2) is now multiplied by R_2 and the result added to equation (6.3), we obtain

$$F = (-r - \frac{R_1^2}{r}) X_1 + R_2X_2. \tag{6.4}$$

Also, by equating the two values for the function F given in equations (6.2) and (6.4), we obtain

$$X_2 = -\frac{R_2}{r}\frac{(R_1^2 + 1)}{(R_1^2 - 1)} X_1. \tag{6.5}$$

Equation (6.5) gives an expression for X(n+2) in terms of X(n+1) simply by replacing n by n+1. As a consequence, we can substitute in the original difference equation for X(n+1), X(n+2) and F from equations (6.4) and (6.5) and thus obtain

$$\left[\frac{R_2^2(R_1^2+1)^2}{r^2(R_1^2-1)^2} + (+\frac{1}{r} + 1)\frac{R_2^2(R_1^2 + 1)}{r(R_1^2 - 1)} + r - \frac{R_2^2(R_1^2 + 1)}{r(R_1^2 - 1)} - (r + \frac{R_1^2}{r})\right] X(n) = 0.$$

If this expression is now expanded and simplified, we finally find that

$$\left[\frac{2R_1^2R_2^2(R_1^2 + 1)}{r^2(R_1^2 - 1)^2} - \frac{R_1^2}{r}\right] X(n) = 0.$$

Hence, if we specify that

$$\left[\frac{2R_2^{\,2}(R_1^{\,2}+1)}{r(R_1^{\,2}-1)^2} - 1\right]R_1^{\,2} \neq 0,$$

then it follows that

$$X(n) = 0$$

for all $n \geqslant n_0 \geqslant 0$. This leads to the following result.

Theorem 45: If

a) for real constants a_1 and a_2, equations (6.1) can be solved for real constants R_1 and R_2 which satisfy the conditions

$$R_1^{\,2} + R_2^{\,2} = r^2 < 1$$

$$\left[\frac{2R_2^{\,2}(R_1^{\,2}+1)}{r(R_1^{\,2}-1)^2} - 1\right]R_1^{\,2} \neq 0,$$

b) there exists a real function $F(n,U_1,U_2)$ such that $F(n,U_1,U_2) = 0$ if and only if $U_1 = U_2 = 0$ and which further satisfies the condition

$$\frac{-2R_1^{\,2}(r^2-1)}{r(R_1^{\,2}-1)} \leqslant \frac{F(n,U_1,U_2)}{U_1} < 0,$$

then the equilibrium $X = 0$ of the non-homogeneous difference equation of second order

$$X(n+2) + a_1X(n+1) + a_2X(n) + F(n,X(n),X(n+1)) = 0$$

is asymptotically stable.

It is worth remarking here that in the homogeneous case, for m = 2, Puri and Drake have actually shown that the total difference of their scalar function V(n,X) is indeed negative definite.

We now turn to an investigation of the asymptotic stability of the equilibrium of the difference equation of third order. As before, we list the relevant quantities prior to the study. They are

$$R = \begin{pmatrix} \frac{R_2}{\rho} & \frac{R_1 R_3}{r\rho} & R_1 \\ -\frac{R_1}{\rho} & \frac{R_2 R_3}{r\rho} & R_2 \\ 0 & -\frac{\rho}{r} & R_3 \end{pmatrix},$$

$$\begin{aligned} a_1 &= -\frac{R_2 R_3}{r\rho} - \frac{R_2}{\rho} - R_3 \\ a_2 &= \frac{R_2 R_3}{\rho} + \frac{rR_2}{\rho} + \frac{R_3}{r} \\ a_3 &= -r, \end{aligned} \qquad (6.7)$$

$$Q = \begin{pmatrix} 0 & R_1 & 0 \\ -\rho & R_2 & 0 \\ \frac{R_3}{r} & -\frac{R_2}{\rho} - \frac{R_2 R_3}{r\rho} & 1 \end{pmatrix}.$$

As a consequence, we find that

$$\begin{aligned} Y_1 &= R_1 X_1 \\ Y_2 &= -\rho X_1 + R_2 X_2 \\ Y_3 &= \frac{R_3}{r} X_1 + \left[-\frac{R_2}{\rho} - \frac{R_2 R_3}{r\rho}\right] X_2 + X_3. \end{aligned}$$

The total difference for the scalar function $V(n,X) = Y^T Y$ in this case is given by

$$\Delta V(n,X) = (r^2-1)Y_3^{\,2} - 2\left[(-\rho/r)Y_2 + R_3Y_3\right]F + F^2$$

$$= (r^2-1)\left[\frac{R_3}{r}X_1 + \left(-\frac{R_2}{\rho} - \frac{R_2R_3}{r\rho}\right)X_2 + X_3\right]^2$$

$$-2\left[rX_1 + \left(\frac{-R_2R_3}{\rho} - \frac{R_2r}{\rho}\right)X_2 + R_3X_3\right]F + F^2.$$

This may be rewritten as

$$\Delta V(n,X) = (r^2-1)\left[\frac{R_3}{r}X_1 + \left(-\frac{R_2}{\rho} - \frac{R_2R_3}{r\rho}\right)X_2 + X_3 - \frac{R_3}{r^2-1}F\right]^2$$

$$-2\left[\frac{\rho^2}{r}X_1 - \frac{R_2\rho}{r}X_2\right]F + \frac{\rho^2-1}{r^2-1}F^2. \tag{6.8}$$

In order to have this negative semi-definite, we must impose the condition

$$-2\left[\frac{\rho^2}{r}X_1 - \frac{R_2\rho}{r}X_2\right]F + \frac{\rho^2-1}{r^2-1}F^2 \leqslant 0,$$

or equivalently,

$$\frac{-2(r^2-1)}{r(\rho^2-1)} \leqslant \frac{F}{-\rho^2X_1 + R_2\rho X_2} < 0.$$

We now examine under what conditions $\Delta V(n,X)$, as given by equation (6.8), is zero. We must have

$$\frac{R_3}{r^2-1}F = \frac{R_3}{r}X_1 + \left(-\frac{R_2}{\rho} - \frac{R_2R_3}{r\rho}\right)X_2 + X_3 \tag{6.9}$$

$$\frac{\rho^2-1}{r^2-1}F = 2\frac{\rho^2}{r}X_1 - 2\frac{R_2\rho}{r}X_2. \tag{6.10}$$

If equation (6.9) is multiplied by R_3 and the result added to equation (6.10), we obtain

$$F = \left(\frac{\rho^2}{r} + r\right)X_1 + \left(\frac{-R_2R_3}{\rho} - \frac{R_2\rho}{r} - \frac{R_2r}{\rho}\right)X_2 + R_3X_3. \tag{6.11}$$

Moreover, equating the two values of F obtained in equations (6.9) and (6.11), we find

$$\left(-\frac{R_3\rho^2}{r} - \frac{R_3}{r}\right)X_1 + \left(-R_2\rho + \frac{R_2R_3\rho}{r} + \frac{R_2}{\rho} + \frac{R_2R_3}{r\rho}\right)X_2 + (\rho^2-1)X_3 = 0 \quad (6.1$$

The difference equation under consideration is

$$X(n+3) + a_1X(n+2) + a_2X(n+1) + a_3X(n) + F(n,X(n),X(n+1),X(n+2)) = 0.$$

With the expressions for the a_i given in equations (6.7) and for F from equation (6.11), this equation becomes

$$X(n+3) + \left(-\frac{R_2}{\rho} - \frac{R_2R_3}{r\rho}\right)X(n+2) + \left(\frac{R_3}{r} - \frac{R_2\rho}{r}\right)X(n+1) + \frac{\rho^2}{r}X(n)$$

$$= X(n+3) + b_1X(n+2) + b_2X(n+1) + b_3X(n) = 0 \qquad (6.13)$$

Equation (6.12) can now be solved for X(n+2), and correspondingly for X(n+3) by replacing n by n+1, in terms of X(n) and X(n+1). Introducing these values into equation (6.13) and simplifying, we obtain

$$\left[-\frac{2R_2R_3^2\rho}{r^2(\rho^2-1)^2} - \frac{2R_2R_3^{\,2}}{r^2\rho(\rho^2-1)^2} + \frac{\rho^2}{r}\right]X(n)$$

$$+\left[\frac{2R_3\,\rho^2}{(\rho^2-1)r} + \frac{2R_2^{\,2}R_3^{\,2}(\rho^2+1)}{r^2\rho^2(\rho^2-1)^2} - \frac{2R_2^{\,2}R_3}{r\rho^2(\rho^2-1)} - \frac{R_2\rho}{r}\right]X(n+1)$$

$$= c_1X(n+1) + c_2X(n) = 0. \qquad (6.14)$$

Now, if $c_1 = 0$, then we must have $c_2 \neq 0$ to insure that

$$X(n) = 0$$

for all $n \geq n_o \geq 0$. On the other hand, if $c_1 \neq 0$, then

$$X(n+1) = -c_2/c_1\; X(n),$$

so that equation (6.13) becomes

$$\left[(-c_2/c_1)^3 + b_1(-c_2/c_1)^2 + b_2(-c_2/c_1) + b_3\right]X(n) = 0.$$

Thus, if the term in the brackets in non-zero, then

$$X(n) = 0$$

for all $n \geqslant n_o \geqslant 0$.

Summarizing the above results, we obtain the following theorem.

<u>Theorem 46:</u> If

a) for real constants a_1, a_2 and a_3, equations (6.7) can be solved for real numbers R_1, R_2 and R_3 which further satisfy the conditions

$$R_1{}^2 + R_2{}^2 + R_3{}^3 = r^2 < 1$$

$$R_1{}^2 + R_2{}^2 = \rho^2 > 0,$$

b) there exists a real function $F(n,U_1,U_2,U_3)$ such that $F(n,U_1,U_2,U_3) = 0$ if and only if $U_1 = U_2 = U_3 = 0$ and which further satisfies the condition

$$\frac{-2(r^2-1)}{r(\rho^2-1)} \leq \frac{F(n,U_1,U_2,U_3)}{-\rho^2 U_1 + R_2 \rho U_2} < 0,$$

c) given real constants b_1, b_2 and b_3, as defined by equation (6.13), and c_1 and c_2, as defined by equation (6.14),

1) $c_1 = 0$ implies $c_2 \neq 0$

2) $c_1 \neq 0$ implies

$$(-c_2/c_1)^3 + b_1(-c_2/c_1)^2 + b_2(-c_2/c_1) + b_3 \neq 0,$$

then the equilibrium $X = 0$ of the non-homogeneous difference equation of third order

$$X(n+3) + a_1X(n+2) + a_2X(n+1) + a_3X(n) + F(n,X(n),\ldots X(n+2)) = 0$$

is asymptotically stable.

Finally, we consider the homogeneous case for m = 3. For this difference equation, the total difference of V(n,X) is given by

$$\Delta V(n,X) = (r^2-1)\left[\frac{R_3}{r} X_1 - \left(\frac{R_2}{\rho} + \frac{R_2R_3}{r\rho}\right)X_2 + X_3\right]^2.$$

If this expression is zero, then

$$X(n+2) = \frac{-R_3}{r} X(n) + \left(\frac{R_2R_3}{r\rho} + \frac{R_2}{\rho}\right) X(n+1). \qquad (6.15)$$

Substituting this into the difference equation and simplifying the result, we have

$$-\frac{\rho^2}{r} X(n) + \frac{R_2\rho}{r} X(n+1) = 0. \qquad (6.16)$$

This immediately reduces to

$$R_2X(n+1) = \rho\, X(n).$$

Now, if $R_2 = 0$, it follows that

$$X(n) = 0$$

for all $n \geqslant n_0$, since $\rho \neq 0$. On the other hand, if $R_2 \neq 0$, then we have

$$X(n+1) = [\rho /R_2]X(n).$$

Substituting this into the original difference equation of third order, we obtain

$$R_1{}^2(\rho - R_2R_3)\, X(n) = 0.$$

Thus, if we assume that the term preceding X(n) is non-zero, then we conclude again that

$$X(n) = 0$$

for all $n \geqslant n_o$.

Summarizing the above results, we obtain the following theorem.

<u>Theorem 47:</u> If

a) for real constants a_1, a_2 and a_3, equations (6.7) can be solved for real numbers R_1, R_2 and R_3 which satisfy the conditions

$$R_1^2 + R_2^2 + R_3^2 = r^2 < 1$$

$$R_1^2 + R_2^2 = \rho^2 > 0,$$

b) $R_2 \neq 0$ implies that $R_1^2(\rho - R_2R_3) \neq 0$,

then the equilibrium $X = 0$ of the homogeneous difference equation of third order

$$X(n+3) + a_1X(n+2) + a_2X(n+1) + a_3X(n) = 0$$

is asymptotically stable.

Finally, to conclude this section, we consider how "good" the results in Part II are. In particular, we will compare some of the preceding results for homogeneous difference equations with the standard conditions for asymptotic stability for such homogeneous equations; namely, the Schur-Cohn criterion. For the case $m = 2$, the Schur-Cohn conditions are

$$|a_2| < 1$$

$$|a_1| < |1 + a_2| .$$

The corresponding conditions based on the work in this chapter are obtained by actually solving equations (6.1) for R_1 and R_2. We already have the condition

$$r^2 = a_2^2 < 1.$$

The first part of equations (6.1) yields

$$R_2 = -a_1/(1+1/r)$$

$$= \frac{-a_1 a_2}{1+a_2} .$$

As a consequence,

$$R_1^2 = r^2 - R_2^2 = a_2^2 \left[1 - \left(\frac{a_1}{1+a_2} \right)^2 \right]$$

and therefore,

$$R_1 = \left(\frac{a_2}{1+a_2} \right) \left[(1+a_2)^2 - a_1^2 \right]^{1/2},$$

where it is necessary to impose the condition

$$(1+a_2)^2 > a_1^2,$$

so that R_1 is indeed a real number. Thus, we note that the conditions obtained here are identical with the Schur-Cohn conditions, as was pointed out by Puri and Drake.

However, for the case $n = 3$, Puri and Drake merely observed that their conditions were similar in form to those given by the Schur-Cohn criterion. The latter are

$$|a_3| < 1$$

$$|a_3^2 - 1| > |a_1 a_3 - a_2|$$

$$1 + a_1 + a_2 + a_3 > 0$$

$$1 - a_1 + a_2 - a_3 > 0.$$

As above, we obtain our conditions by solving equations

(6.7). To begin, we find that

$$R_3 = a_3(a_2 - a_1a_3)/(a_3^2 - 1),$$

where we have already imposed

$$a_3^2 = r^2 < 1.$$

Consequently, we have

$$\rho^2 = r^2 - R_3^2 = a_3^2\left[(a_3^2 - 1)^2 - (a_2 - a_1a_3)^2\right]/(a_3^2-1)^2$$

and therefore,

$$\rho = a_3\left[(a_3^2 - 1)^2 - (a_2 - a_1a_3)^2\right]^{1/2}/(a_3^2 - 1),$$

where we must assume

$$(a_3^2 - 1)^2 > (a_2 - a_1a_3)^2.$$

Thus, we note, the first two conditions here are identical with the first two Schur-Cohn conditions. Moreover, it is possible to solve for R_2 by substituting for R_3, r and ρ into equations (6.7), and based on this, R_1 can be determined. As before, the evaluation of R_1 results in a square root and it is necessary to impose the additional condition

$$\left[(a_3^2 - 1) - (a_2 - a_1a_3)\right]^2 > (a_2a_3 - a_1)^2.$$

We then find, after a somewhat detailed calculation, that the last two conditions in the Schur-Cohn criterion imply the above inequality. Thus, for the case m = 3, the conditions stated in Theorem 47 are at least as good as those given by the Schur-Cohn conditions.

However, when a similar analysis is applied to the case m = 4, the complexity of the terms becomes so great that no definite conclusion can be formed. All that can be said is that the first two conditions obtained based on the development in this chapter coincide with the first two Schur-Cohn conditions, while the remaining conditions are similar in form to the remaining Schur-Cohn conditions. It is felt by the author, though, that the conclusion obtained above for m = 3 will probably also hold for the case m = 4.

Appendix A: Calculation of the Matrix R

In Chapter 5, condition (5.6) was imposed on the matrix R. The expansion of this condition led to a series of ten relations. It was shown there that equations d) and a) together yield

$$r_{11} = \pm \frac{R_2}{\sigma} \quad \text{and} \quad r_{21} = \mp \frac{R_1}{\sigma} .$$

From relations b) and g), it follows that

$$\begin{aligned} r_{22} &= (R_2/R_1) r_{12} \\ r_{32} &= -(\sigma^2/R_1 R_3) r_{12} . \end{aligned}$$

Substituting these expressions into equation e), there results

$$r_{12}^2 \, (\sigma^2/R_1^2)(\rho^2/R_3^2) = 1,$$

and hence we obtain

$$r_{12} = \pm \frac{R_1 R_3}{\sigma \rho} ,$$

which implies that

$$r_{22} = \pm \frac{R_2 R_3}{\sigma \rho} \quad \text{and} \quad r_{32} = \mp \frac{\sigma}{\rho} .$$

In a totally similar manner, relations c), f) and i) yield expressions for r_{23}, r_{33} and r_{43}, respectively, in terms of r_{13}. Substituting these results into equation h), we find

$$r_{13}^2 (r^2 \rho^2 / R_1^2 R_4^2) = 1,$$

which gives that

$$r_{13} = \pm \frac{R_1 R_4}{r \rho} .$$

This in turn implies that

$$r_{23} = \pm \frac{R_2 R_4}{r\rho}$$

$$r_{33} = \pm \frac{R_3 R_4}{r\rho}$$

$$r_{43} = \mp \frac{\rho}{r} ,$$

which results in the form for the matrix R given in equation (5.9).

Appendix B; Calculation of the Matrix Q

When the relation QA = RQ is expanded, the following set of equations is obtained:

a) $0 = (-R_2/\sigma)q_{11} + (R_1R_3/\sigma\rho)q_{21} + (R_1R_4/r\rho)q_{31} + R_1q_{41}$

b) $q_{11} = (-R_2/\sigma)q_{12} + (R_1R_3/\sigma\rho)q_{22} + (R_1R_4/r\rho)q_{32} + R_1q_{42}$

c) $q_{12} = (-R_2/\sigma)q_{13} + (R_1R_3/\sigma\rho)q_{23} + (R_1R_4/r\rho)q_{33} + R_1q_{43}$

d) $q_{13} = R_1$

e) $0 = (R_1/\sigma)q_{11} + (R_2R_3/\sigma\rho)q_{21} + (R_2R_4/r\rho)q_{31} + R_2q_{41}$

f) $q_{21} = (R_1/\sigma)q_{12} + (R_2R_3/\sigma\rho)q_{22} + (R_2R_4/r\rho)q_{32} + R_2q_{42}$

g) $q_{22} = (R_1/\sigma)q_{13} + (R_2R_3/\sigma\rho)q_{23} + (R_2R_4/r\rho)q_{33} + R_2q_{43}$

h) $q_{23} = R_2$

i) $0 = (-\sigma/\rho)q_{21} + (R_3R_4/r\rho)q_{31} + R_3q_{41}$

j) $q_{31} = (-\sigma/\rho)q_{22} + (R_3R_4/r\rho)q_{32} + R_3q_{42}$

k) $q_{32} = (-\sigma/\rho)q_{23} + (R_3R_4/r\rho)q_{33} + R_3q_{43}$

l) $q_{33} = R_3$

m) $-a_4 = (-\rho/r)q_{31} + R_4q_{41}$

n) $q_{41} - a_3 = (-\rho/r)q_{32} + R_4q_{42}$

o) $q_{42} - a_2 = (-\rho/r)q_{33} + R_4q_{43}$

p) $q_{43} - a_1 = R_4.$

As was noted previously, we immediately have, from relation p),

$$q_{43} = -(R_3R_4/r\rho) - (R_2R_3/\sigma\rho) - R_2/\sigma .$$

Substituting this expression for q_{43} and the ones from equations d), h) and l) into relation o), we find upon using equation (5.11b) to express a_2,

$$q_{42} = (R_2R_4/r\sigma)(1 + R_3/\rho) + R_3/\rho .$$

Similarly, substituting q_{23}, q_{33} and q_{43} into relation k) we obtain

$$q_{32} = (-R_2/\rho)(\sigma + R_3^2/\sigma) + R_2R_3/\sigma$$

$$= (R_2/\sigma)(-R_3 - \rho).$$

Substitution of these same quantities into relation g) yields

$$q_{22} = (-R_1^2/\sigma) + (-R_2^2/\sigma) = -\sigma ,$$

and similarly, from equation c),

$$q_{12} = 0.$$

The derived expressions for q_{32} and q_{42} are now substituted into relation n), along with the value for a_3 given in equation (5.11c) to give

$$q_{41} = (R_2/r\sigma)(R_4^2+\rho^2) - (R_2R_3/r\sigma)(\rho + R_4^2/\rho)$$

$$+ rR_2R_3/\sigma\rho - rR_2/\sigma - R_4/r$$

$$= -R_4/r,$$

upon simplification. Furthermore, equation m) yields

$$(\rho/r)q_{31} = a_4 + R_4q_{41} = +r + R_4^2/r = +\rho^2/r,$$

which implies that

$$q_{31} = +\rho .$$

Once q_{31} and q_{41} are known, relation i) gives

$$(\sigma/\rho)q_{21} = (R_3R_4/r\rho)(\rho) - R_3(R_4/r) = 0,$$

and hence,

$$q_{21} = 0.$$

Finally, equation a) is used to determine

$$q_{11} = 0.$$

Thus far, all ten of the unknown q_{ij} have been determined from only ten of the sixteen equations a) - p). However, the remaining six relations are satisfied for the values for the q_{ij} obtained, as can be seen from direct substitution and subsequent simplification.

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