An introduction to functions of bounded variation, sets of finite perimeter and some applications to geometric variational problems

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Abstract

In this thesis, we explore how the theory of functions of bounded variation (BV) establishes an appropriate and versatile framework in the study of geometric variational problems. We begin with a presentation of some fundamental results on BV functions that will allow us to link them to Radon measures. In the special case of characteristic functions with bounded variation, we present structural results on sets of finite perimeter, including a generalization of the Gauss-Green Theorem. This machinery will allow us to assign a notion of perimeter to any set of finite Lebesgue measure, hence allowing non-smooth competitors to be considered in minimization problems involving the surface area. We will then address Plateau's problem and the first variation of the area functional. Finally, we will present the ideas of Steiner symmetrization to provide a proof of the Isoperimetric inequality.

Abrégé

Dans cette thèse, nous explorons comment la théorie des fonctions de variation bornée (BV) établit un cadre approprié et polyvalent dans l'étude des problèmes variationnels géométriques. Nous commençons par une présentation de quelques résultats fondamentaux sur les fonctions BV qui nous permettront de les relier à des mesures de Radon. Dans le cas particulier des fonctions caractéristiques à variation bornée, nous présentons des résultats structuraux sur des ensembles de périmètre fini, incluant une généralisation du théorème de Gauss-Green. Cette machinerie nous permettra d'attribuer une notion de périmètre à tout ensemble de measures de Lebesgue finie, permettant ainsi de considérer des concurrents non lisses dans les problèmes de minimisation impliquant la surface. Nous aborderons ensuite le problème de Plateau et la première variation de la fonctionnelle d'aire. Pour terminer, nous présenterons les idées de la symétrisation de Steiner pour fournir une preuve de l'inégalité isopérimétrique.

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Contents

Notation

- For $x \in \mathbb{R}^n$, denote the Euclidean norm on $\mathbb{R}^n$ by $|x| = \sqrt{\sum_{i=1}^n x_i^2}$.
- $C_c^1(\Omega; \mathbb{R}^n) = \{\phi = (\phi_1, \dots, \phi_n) \,|\, \phi_i \in C_c^1(\Omega) \text{ for all } i \in \{1, \dots, n\}\}$.
- For $\phi \in C_c^1(\Omega; \mathbb{R}^n)$, denote $||\phi||_\infty = \sup_{x \in \Omega} |\phi(x)|$.
- B_r is the open ball of radius $r > 0$ centred at the origin. We write $B_r(x)$ for the open ball of radius $r > 0$ centred at $x \in \mathbb{R}^n$.
- $S^{n-1} = \{x \in \mathbb{R}^n \,|\, |x| = 1\}$.
- If $E \subseteq \mathbb{R}^n$, then χ_E is the characteristic function of E, i.e. $\chi_E(x) = 1$ if and only if $x \in E$, and $\chi_E(x) = 0$ if and only if $x \notin E$.
- We denote e_i for $i \in \{1, \dots, n\}$ to be the unit vector in the direction of i-th coordinate axis.
- If $V \subseteq \Omega$ such that $\overline{V}$ is compact in Ω, then we write $V \subseteq\subseteq \Omega$.
- For $\{E_h\}_{h=1}^\infty$ a sequence of subsets of $\mathbb{R}^n$ and $E \subset \mathbb{R}^n$, we write $E_h \to E$ in $L^1(\mathbb{R}^n)$ to mean $\chi_{E_h} \to \chi_E$ in $L^1(\mathbb{R}^n)$.
- If μ is a measure on $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is μ-measurable, then we denote $\nu = \mu \llcorner f$ to be the measure defined by $\nu(B) = \int_B f \, d\mu$, for every μ-measurable set B.
- If μ is a measure on $\mathbb{R}^n$ and $E \subset \mathbb{R}^n$, then we denote $\nu = \mu \llcorner E$ to be the measure defined by $\nu(B) = \mu(B \cap E)$, for every μ-measurable set B.
- If $G : \mathbb{R}^n \to \mathbb{R}^n$, then denote DG to be the Jacobian matrix of G.
- If M is an $n \times n$ matrix, then denote $|M|$ to be the determinant of M.

Introduction

Originally posed by Lagrange in the 1760s, Plateau's problem aims to find the surface with the least area amongst all surfaces with a prescribed boundary curve. The problem is named after the physicist Joseph Plateau, who experimented with soap films; by dipping a closed wire into soapy water, the resulting film always took a shape that minimized the area. This suggested the existence of minimal surfaces. Translating Plateau's experiment into a formal proof has been an elusive problem for mathematicians for over a century, with major advancements made only in the mid-1900s [6]. The classical approach in $\mathbb{R}^3$ is to treat 2-dimensional surfaces as the mappings of smooth functions over the closed unit ball in $\mathbb{R}^2$, i.e. $f : \overline{B_1} \subset \mathbb{R}^2 \to \mathbb{R}^3$ such that $f(\overline{B_1})$ is homeomorphic to $\overline{B_1}$. Through this approach, J.Douglas and T. Rado were able to present the first complete proof of the existence of minimal surfaces for all Jordan curves, smooth functions $\phi : \partial B_1 \subset \mathbb{R}^2 \to \mathbb{R}^3$ such that $\phi(\partial B_1)$ is homeomorphic to ∂B_1. Unfortunately, there was no generalization to higher dimensions as their method relied on conformal mappings, hence they were intrinsically limited to functions with domains in $\mathbb{R}^2$ [4]. Moreover, there were other drawbacks to treating surfaces as mappings, such as the lack of compactness properties in the natural mapping topology. A compactness result would later prove to be essential in showing the existence of minimal surfaces in higher dimensions, in what would be known as the Direct-Method [8].

To generalize to higher dimensions and to obtain a compactness result, mathematicians started working in more abstract spaces, often employing the tools of geometric measure theory. Several different formulations of the surface were proposed during this time, including Federer's rectifiable currents, Almgren's varifold and Reifenerg's clever use of Cech homology [4]. In the perspective of De Giorgi, $(n-1)$-dimensional surfaces are the boundaries of n-dimensional subsets for which the characteristic function has bounded variation (BV). In the setting of BV, these characteristic functions induce finite Radon measures, which not only capture information about the area but also possess compactness properties. De Giorgi named these subsets "sets of finite perimeter". He showed that many of the desired properties of smooth surfaces may be generalized to sets of finite perimeter, while still allowing the possibility of singularities and other complex geometry.

For example, one such generalization of smooth sets comes in the Gauss-Green Theorem. In the classical statement, the integral of the divergence of a vector-field over a smooth set is equal to the flux. As an alternative to the topological boundary in the computation of the flux, De Giorgi suggested a measure-theoretic notion of the boundary that he dubbed the reduced boundary. With this, he showed that the Gauss-Green Theorem may be extended to sets of finite perimeter without any additional assumption on the regularity of the boundary. In his Structure Theorem, he argued that the reduced boundary of a set of finite perimeter is a C^1 hyper-surface up to a null-measure set. Moreover, De Giorgi showed that if a set is a minimizer to the Plateau's problem, then its reduced boundary must be an analytic hyper-surface. This was the first major regularity result to Plateau's problem in higher dimensions [4].

In this thesis, we will follow De Giorgi's approach. We begin by introducing the foundation of BV functions; its connection to Radon measures and general approximation methods including the lower-semi continuity property and compactness. In Chapter 2, we shift our attention to sets of finite perimeter and discuss the alternative notions of the boundary along with De Giorgi's regularity result. Once we have established the setting we will be working in, the remaining chapters will be dedicated to studying minimization problems, with a particular focus on the problem of Plateau and the Isoperimetric problem.

1 Functions of Bounded Variation

1.1 The Structure Theorem of BV

In the first chapter, we will introduce the basic properties of BV functions and establish the primary tools of the space. We begin with one of the defining characteristics of BV functions; a link between the distributional derivatives and Radon measures. From this point and onwards, we will always assume $\Omega \subseteq \mathbb{R}^n$ is open.

Definition 1.1.1. *A function $f \in L^1(\Omega)$ has bounded variation in Ω, if*

$$||Df||(\Omega) = \sup\left\{ \int_\Omega f \ \mathrm{div}(\phi)\, dx \,\middle|\, \phi \in C_c^1(\Omega;\mathbb{R}^n),\ ||\phi||_\infty \leq 1 \right\} < \infty. \tag{1.1.1}$$

We call $||Df||(\Omega)$ the total variation of f in Ω, and we denote the space of functions with bounded variation in Ω by $BV(\Omega)$.

If we equip $BV(\Omega)$ with the following norm

$$||f||_{BV(\Omega)} = ||f||_{L^1(\Omega)} + ||Df||(\Omega),$$

then $BV(\Omega)$ forms a Banach space. We will prove this claim in the next subsection once we have established the necessary tools.

Definition 1.1.2. *A function $f \in L^1_{loc}(\Omega)$ has local bounded variation in Ω, if for all $V \subseteq\subseteq \Omega$,*

$$||Df||(V) = \sup\left\{ \int_V f \mathrm{div}(\phi)\, dx \,\middle|\, \phi \in C_c^1(V;\mathbb{R}^n),\ ||\phi||_\infty \leq 1 \right\} < \infty. \tag{1.1.2}$$

Likewise, we denote the set of functions with local bounded variation in Ω by $BV_{loc}(\Omega)$.

We are primarily interested in characteristic functions with bounded variation, so we give this special class of $BV(\Omega)$ an appropriate name.

Definition 1.1.3. *For $E \subseteq \mathbb{R}^n$, we say E has finite perimeter in Ω, if $\chi_E \in BV(\Omega)$. If $\chi_E \in BV_{loc}(\Omega)$, then we say E has local finite perimeter in Ω. We write $E \in BV(\Omega)$*

and $E \in BV_{loc}(\Omega)$ respectively. Furthermore, we will denote $||\partial E||(\Omega) = ||D\chi_E||(\Omega)$ as in (1.1.1), and refer to it as the perimeter of E in Ω.

We wish to link these functions with Radon measures. We begin by recalling the Riesz Representation Theorem.

Theorem 1.1.1. *(Riesz Representation Theorem.) [[3], Theorem 1.38] Let $L : C_c(\mathbb{R}^n;\mathbb{R}^n) \to \mathbb{R}$ be a linear functional satisfying*

$$\sup\left\{ L(\phi) \,\middle|\, \phi \in C_c(V;\mathbb{R}^n),\ ||\phi||_\infty \le 1 \right\} < \infty, \tag{1.1.3}$$

for all compact sets $V \subseteq \mathbb{R}^n$. Then, there exists a Radon measure μ on $\mathbb{R}^n$ and a μ-measurable function $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ such that

1. *$|\sigma(\cdot)| = 1$ μ-a.e.*
2. *$L(\phi) = \int_{\mathbb{R}^n} \phi \cdot \sigma \, d\mu$, for all $\phi \in C_c(\mathbb{R}^n;\mathbb{R}^n)$.*

Observe the set in which the supremum is taken over in (1.1.3) looks similar to the set criterion of (1.1.2) for local bounded variation. If we can find an appropriate linear operator L for $\int_\Omega f \operatorname{div}(\phi)\, dx$ for all $\phi \in C_c^1(\Omega;\mathbb{R}^n)$, then a direct application of the Riesz Representation Theorem should allow us to connect f to an appropriate measure.

Theorem 1.1.2. *(Structure Theorem for Functions of Local Bounded Variation.)[[3], Theorem 5.1] Let $f \in BV_{loc}(\Omega)$. There exists a Radon measure μ on Ω and a $\mu-$measurable function $\sigma : \Omega \to \mathbb{R}^n$ such that*

1. *$|\sigma(\cdot)| = 1\, \mu - a.e,$*
2. *for all $\phi \in C_c^1(\Omega;\mathbb{R}^n)$,*

$$\int_\Omega f \operatorname{div}(\phi)\, dx = -\int_\Omega \sigma \cdot \phi \, d\mu. \tag{1.1.4}$$

Proof. Let $L : C_c^1(\Omega;\mathbb{R}^n) \to \mathbb{R}$ be defined as

$$L(\phi) = -\int_\Omega f \operatorname{div}(\phi)\, dx.$$

We wish to extend L to $C_c(\Omega;\mathbb{R}^n)$ before applying the Riesz Representation Theorem. To do this we define for all $k \in \mathbb{N}$, the open sets

$$\Omega_k = \left\{ x \in \Omega \,\middle|\, d(x, \partial\Omega) > \frac{1}{k} \right\} \cap B_k.$$

Clearly, $\Omega_k \subseteq \Omega_{k+1}$ and $\Omega = \cup_{k=1}^{\infty} \Omega_k$. Now let

$$C_k = \sup \left\{ \int_{\Omega} f \operatorname{div}(\phi)\, dx \,\middle|\, \phi \in C_c^1(\Omega_k;\mathbb{R}^n),\ ||\phi||_\infty \leq 1 \right\}.$$

Since $f \in BV_{loc}(\Omega)$ and $\Omega_k \subseteq\subseteq \Omega$, we see that $C_k < \infty$. This implies $L|_{C_c^1(\Omega_k;\mathbb{R}^n)}$ is continuous. Next, we define the sublinear functional $\rho_k : C_c(\Omega_k;\mathbb{R}^n) \to \mathbb{R}$ by

$$\rho_k(\phi) = C_k ||\phi||_\infty.$$

Observe, for all $\phi \in C_c^1(\Omega_k;\mathbb{R}^n)$,

$$L\left(\frac{\phi}{||\phi||_\infty}\right) = \frac{L(\phi)}{||\phi||_\infty} \leq C_k.$$

That is, $L|_{C_c^1(\Omega_k;\mathbb{R}^n)} \leq \rho_k|_{C_c^1(\Omega_k;\mathbb{R}^n)}$. By the Hahn-Banach Theorem, there exists a continuous linear functional $L_k : C_c(\Omega_k;\mathbb{R}^n) \to \mathbb{R}$ such that $L_k|_{C_c^1(\Omega_k;\mathbb{R}^n)} = L|_{C_c^1(\Omega_k;\mathbb{R}^n)}$, and for all $\phi \in C_c(\Omega_k;\mathbb{R}^n)$,

$$L_k(\phi) \leq C_k ||\phi||_\infty. \tag{1.1.5}$$

By density of $C_c^1(\Omega_k;\mathbb{R}^n)$ in $C_c(\Omega_k;\mathbb{R}^n)$, L_k is a unique extension of L on $C_c(\Omega_k;\mathbb{R}^n)$. Now notice if $l > k$, then $\Omega_k \subseteq \Omega_l$. Which in turn implies if $\phi \in C_c^1(\Omega_k;\mathbb{R}^n)$, then $L_l(\phi) = L(\phi)$. Therefore, L_l is also an extension of L on $C_c(\Omega_k;\mathbb{R}^n)$. The uniqueness of the extension of L on $C_c(\Omega_k;\mathbb{R}^n)$ would imply $L_l|_{C_c(\Omega_k;\mathbb{R}^n)} = L_k$. We now define the linear functional $F : C_c(\Omega;\mathbb{R}^n) \to \mathbb{R}$ by

$$F(\phi) = \lim_{k\to\infty} L_k(\phi).$$

We note if $\phi \in C_c(\Omega;\mathbb{R}^n)$, there exists k large so that $supp(\phi) \subseteq \Omega_k$. Thus, for all $l > k$,

$L_l(\phi) = L_k(\phi)$. So, F is well-defined. Likewise, if $\phi \in C_c^1(\Omega; \mathbb{R}^n)$, then $F(\phi) = L(\phi)$ by the same argument. Lastly, we note if $V \subseteq\subseteq \Omega$, there exists k large so that $V \subseteq \Omega_k$. By (1.1.5),

$$\sup\{F(\phi)\,|\,\phi \in C_c(V; \mathbb{R}^n),\, ||\phi||_\infty \leq 1\} \leq \sup\{L_k(\phi)\,|\,\phi \in C_c(\Omega_k; \mathbb{R}^n),\, ||\phi||_\infty \leq 1\} \leq C_k.$$

By the Riesz Representation Theorem, there exists a Radon measure μ on Ω and a μ-measurable function σ satisfying the desired properties. $\square$

We will denote the measure μ that arises from the Structure Theorem as $||Df||$ for $f \in BV_{loc}(\Omega)$. In the special case $f = \chi_E$ for some $E \subseteq \mathbb{R}^n$, we will denote the associated measure by $||\partial E||$ and $\nu_E = -\sigma$. In addition to (1.1.4), there is another property of $||Df||$ we may deduce as a consequence of the Riesz Representation Theorem. If $V \subseteq\subseteq \Omega$ is open, then

$$||Df||(V) = \sup\left\{\int_\Omega f \operatorname{div}(\phi)\, dx \,\middle|\, \phi \in C_c^1(V; \mathbb{R}^n),\, ||\phi||_\infty \leq 1\right\}. \tag{1.1.6}$$

Remark. *If $f \in BV(\Omega)$, then $||Df||$ is a finite measure on Ω and $||Df||(\Omega)$ equates to the total variation of f in Ω as in (1.1.1).*

The Structure Theorem coincides with some well-known results. Let us look at some examples.

Example 1.1.1. *Let $\Omega = \mathbb{R}^2$ and $E = B_1$. If we denote ν as the outward unit normal vector along ∂B_1, then by the Divergence Theorem, for all $\phi \in C_c^1(\mathbb{R}^2; \mathbb{R}^2)$*

$$\int_{B_1} \operatorname{div}(\phi)\, dx = \int_{\partial B_1} \nu \cdot \phi \, d\mathcal{H}^1. \tag{1.1.7}$$

If $||\phi||_\infty \leq 1$, we see by the Cauchy-Schwarz inequality (1.1.7) is bounded by $\mathcal{H}^1(\partial B_1)$. Therefore, $\chi_{B_1} \in BV(\mathbb{R}^2)$. From the Structure Theorem we may deduce $\nu_{B_1} = -\nu$ along ∂B_1 and $||\partial B_1|| = \mathcal{H}^1 \llcorner \partial B_1$. This tells us for a given function $\phi \in C_c^1(\mathbb{R}^2; \mathbb{R}^2)$, the behaviour of ϕ along ∂B_1 dictates the duality formula of the perimeter of (1.1.1). We claim $\mathcal{H}^1(\partial B_1) = ||\partial B_1||(\mathbb{R}^2)$. It only remains to show $||\partial B_1||(\mathbb{R}^2) \geq \mathcal{H}^1(\partial B_1)$. Let

$\eta \in C_c^1(\mathbb{R})$ such that $\eta \equiv 1$ on $[0,1]$. We know $\nu(x) = x$ for all $x \in \partial B_1$. So, if we take $N(x) = \eta(|x|)x$, then $N \in C_c^1(\mathbb{R}^2;\mathbb{R}^2)$ and N is a continuous extension of ν. If $g \in C_c^1(\mathbb{R}^2)$, then $gN \in C_c^1(\mathbb{R}^2;\mathbb{R}^2)$ and (1.1.7) becomes

$$\int_{B_1} \operatorname{div}(gN)\, dx = \int_{\partial B_1} g\, d\mathcal{H}^1.$$

Consequently,

$$||\partial B_1||(\mathbb{R}^2) \geq \sup\left\{ \int_{\partial B_1} g\, d\mathcal{H}^1 \,\middle|\, g \in C_c^1(\mathbb{R}^2),\, |g| \leq 1 \right\} = \mathcal{H}^1(\partial B_1).$$

We conclude $||\partial B_1||(\mathbb{R}^2) = \mathcal{H}^1(\partial B_1)$. Moreover, for any open $V \subseteq \mathbb{R}^2$, utilizing (1.1.6) and the same prior argument, we may deduce $||\partial E||(V) = \mathcal{H}^1(\partial B_1 \cap V)$. That is, $||\partial E||(V)$ equates to the arc-length of B_1 in V. It follows $||\partial B_1||$ captures information about the perimeter of B_1.

In general, if $E \subset \mathbb{R}^n$ is open and bounded with smooth boundaries, by applying the same argument we see that $||\partial E||(\mathbb{R}^n) = \mathcal{H}^{n-1}(\partial E)$. More precisely, $||\partial E|| = \mathcal{H}^{n-1} \llcorner \partial E$.

Theorem 1.1.3. *If $f \in W^{1,1}(\Omega)$, then $f \in BV(\Omega)$ and*

$$||Df||(\Omega) = \int_\Omega |\nabla f|\, dx. \tag{1.1.8}$$

Moreover, from the Structure Theorem, $\sigma = \frac{\nabla f}{|\nabla f|}$ and $||Df|| = \mathcal{L}^n \llcorner |\nabla f|$.

Proof. Let $\phi \in C_c^1(\Omega;\mathbb{R}^n)$ with $||\phi||_\infty \leq 1$. By the Cauchy-Schwarz inequality,

$$\begin{aligned} \left| \int_\Omega f \operatorname{div}(\phi)\, dx \right| &= \left| \int_\Omega \nabla f \cdot \phi\, dx \right| \\ &\leq \int_\Omega |\nabla f \cdot \phi|\, dx \\ &\leq \int_\Omega |\nabla f|\, dx. \end{aligned}$$

That is, $\int_\Omega |\nabla f|\, dx$ uniformly bounds $\int_\Omega f \operatorname{div}(\phi)\, dx$ for all $\phi \in C_c^1(\Omega;\mathbb{R}^n)$ with $||\phi||_\infty \leq 1$. It follows that $f \in BV(\Omega)$ and $||Df||(\Omega) \leq \int_\Omega |\nabla f|\, dx$. For the reverse inequality, we define the normalized gradient $g : \mathbb{R}^n \to \mathbb{R}^n$ by

$$g = \begin{cases} -\frac{\nabla f}{|\nabla f|} & \text{if } \nabla f \neq 0 \\ 0 & \text{if } \nabla f = 0. \end{cases}$$

Next, we construct an increasing sequence of nested open sets by

$$\Omega_k = \left\{ x \in \Omega \,\middle|\, dist(x, \partial\Omega) > \frac{1}{k} \right\} \cap B_k.$$

Clearly, $\Omega = \cup_{k=1}^{\infty} \Omega_k$. If $g_k = g\chi_{\Omega_k}$ and $\epsilon > 0$, we notice upon convoluting each component of g_k by the usual mollifier ρ_ϵ, as in

$$\rho_\epsilon \star g_k = (\rho_\epsilon \star (g_k)_1, \ldots, \rho_\epsilon \star (g_k)_n),$$

then for ϵ sufficiently small, $\rho_\epsilon \star g_k \in C_c^1(\Omega; \mathbb{R}^n)$. Furthermore, we may extract a sequence $\{\epsilon_l\}_{l \in \mathbb{N}}$ such that $\rho_{\epsilon_l} \star g_k \xrightarrow{\epsilon_l \to 0} g_k$ point-wise $\mathcal{L}^n$-a.e. Next, we see by Jensen's inequality, for all $x \in \Omega$,

$$|\rho_{\epsilon_l} \star g_k(x)|^2 = \left(\int_{\mathbb{R}^n} \rho_{\epsilon_l}(x-y) g_k(y)\, dy \right)^2 \leq \int_{\mathbb{R}^n} \rho_{\epsilon_l}(x-y) |g_k(y)|^2\, dy \leq 1. \tag{1.1.9}$$

Thus, $||\rho_{\epsilon_l} \star g_k||_\infty \leq 1$. Moreover, $|\nabla f \cdot (\rho_{\epsilon_l} \star g_k)| \leq |\nabla f| \chi_\Omega$. By the Dominated Convergence Theorem,

$$\lim_{\epsilon_l \to 0} \int_\Omega f \operatorname{div}(\rho_{\epsilon_l} \star g_k)\, dx = \lim_{\epsilon_l \to 0} -\int_\Omega \nabla f \cdot (\rho_{\epsilon_l} \star g_k)\, dx = -\int_\Omega \nabla f \cdot g_k\, dx = \int_{\Omega_k} |\nabla f|\, dx.$$

Therefore, for all $\epsilon' > 0$ and $k \in \mathbb{N}$, there exists $\phi_k \in C_c^1(\Omega; \mathbb{R}^n)$ with $||\phi_k||_\infty \leq 1$ and

$$\int_\Omega f \operatorname{div}(\phi_k)\, dx \geq \int_{\Omega_k} |\nabla f|\, dx - \epsilon'.$$

We now take the lim sup across both sides with respect to $k \to \infty$ to get

$$\limsup_{k \to \infty} \int_\Omega f \operatorname{div}(\phi_k)\, dx \geq \limsup_{k \to \infty} \int_{\Omega_k} |\nabla f|\, dx - \epsilon' = \int_\Omega |\nabla f|\, dx - \epsilon'.$$

Sending ϵ' to 0 and noting $||Df||(\Omega) \geq \limsup_{k\to\infty} \int_\Omega f \text{div}(\phi_k)\, dx$, we see that

$$||Df||(\Omega) \geq \int_\Omega |\nabla f|\, dx.$$

We conclude $||Df||(\Omega) = \int_\Omega |\nabla f|\, dx$. $\square$

Theorem 1.1.3 implies if $f \in W^{1,1}(\Omega)$, then $||f||_{BV(\Omega)} = ||f||_{W^{1,1}(\Omega)}$. It follows as an immediate consequence that $W^{1,1}(\Omega)$ may be embedded into $BV(\Omega)$ as a closed subspace. In fact, this subset relation is strict. Example 1.1.1 shows this. If $E \subseteq \Omega$ is open and bounded with smooth boundaries, then $\chi_E \in BV(\Omega)$. But, $\chi_E \notin W^{1,1}(\Omega)$. Therefore, $C_c^\infty(\Omega)$ while dense in $W^{1,1}(\Omega)$, is not dense in $BV(\Omega)$.

1.2 Approximation and Compactness

The results of this subsection will be our primary tools in studying BV functions. We begin with one of the most versatile properties and our main method of bounding the total variation.

Theorem 1.2.1. *(Lower-Semicontinuity.)[[3],Theorem 5.2] Let $\{f_k\}_{k=1}^\infty \subseteq BV(\Omega)$ and $f_k \to f$ in $L^1_{loc}(\Omega)$, then*

$$||Df||(\Omega) \leq \liminf_{k\to\infty} ||Df_k||(\Omega). \tag{1.2.1}$$

Proof. Let $\phi \in C_c^1(\Omega;\mathbb{R}^n)$ with $||\phi||_\infty \leq 1$. Denote K $= supp(\phi)$, then $f_k \to f$ in $L^1(K)$. So,

$$\int_\Omega f\ \text{div}(\phi)dx = \lim_{k\to\infty} \int_\Omega f_k\ \text{div}(\phi)dx = \liminf_{k\to\infty} \int_\Omega f_k\ \text{div}(\phi)dx \leq \liminf_{k\to\infty} ||Df_k||(\Omega).$$

That is, $\liminf_{k\to\infty} ||Df_k||(\Omega)$ uniformly bounds $\int_\Omega f \text{div}(\phi)\, dx$ for all $\phi \in C_c^1(\Omega;\mathbb{R}^n)$ with $||\phi||_\infty \leq 1$. Hence, (1.2.1) follows. $\square$

Remark. *Theorem 1.2.1 also implies $||Df||(V) \leq \liminf_{k\to\infty} ||Df_k||(V)$, for all open $V \subseteq\subseteq \Omega$. However, it does not imply $f \in BV(\Omega)$, nor $f \in BV_{loc}(\Omega)$, as the right-hand side of (1.2.1) may be infinite.*

As we will later see in Chapter 3, Theorem 1.2.1 allows us to verify candidate functions as solutions to minimization problems involving the total variation. Indeed, if given a sequence of BV functions such that the total variation converges to the infimum across all competitors, and if the limit exists, then by lower-semi continuity the total variation of the limit must be bounded from above by the corresponding infimum. For a simpler application, we will utilize the lower-semi continuity property of BV to show $BV(\Omega)$ is Banach.

Theorem 1.2.2. *[[6], Remark 1.12] $BV(\Omega)$ is Banach.*

Proof. Recall, if $f \in BV(\Omega)$, then

$$||f||_{BV(\Omega)} = ||f||_{L^1(\Omega)} + ||Df||(\Omega).$$

Let $\{f_k\}_{k=1}^{\infty} \subseteq BV(\Omega)$ be a Cauchy sequence under $||\cdot||_{BV(\Omega)}$, then there exists $M > 0$ such that $||f_k||_{BV(\Omega)} < M$ for all $k \in \mathbb{N}$. Notice, $\{f_k\}_{k=1}^{\infty}$ is also Cauchy in $L^1(\Omega)$, so by completeness of $L^1(\Omega)$, there exists $f \in L^1(\Omega)$ such that $f_k \to f$ in $L^1(\Omega)$. By Theorem 1.2.1,

$$||Df||(\Omega) \leq \liminf_{k\to\infty} ||Df_k||(\Omega) < M.$$

We conclude $f \in BV(\Omega)$. It only remains to show $||D(f_k - f)||(\Omega) \to 0$. Let $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $k, j > N$, $||f_j - f_k||_{BV(\Omega)} < \epsilon$. In particular, $||D(f_j - f_k)||(\Omega) < \epsilon$. We observe that for a fixed k, $||(f_j - f_k) - (f - f_k)||_{L^1(\Omega)} = ||f_j - f||_{L^1(\Omega)}$. Thus, $f_j - f_k \xrightarrow{j\to\infty} f - f_k$ in $L^1(\Omega)$. If $k > N$, then by Theorem 1.2.1,

$$||D(f_k - f)||(\Omega) \leq \liminf_{j\to\infty} ||D(f_j - f_k)||(\Omega) < \epsilon.$$

It follows $||D(f_k - f)||(\Omega) \to 0$. $\square$

Despite our prior comment on the lack of density of smooth functions in $BV(\Omega)$ under $||\cdot||_{BV(\Omega)}$, we may still wish to approximate BV functions through smooth functions. Given that we know the total variation of smooth functions to be the L^1-norm of the gradient by Theorem 1.1.3, and we know we can approximate the L^1-norm component of the BV norm using smooth functions, it seems we are very close. We present a weaker form of approximation, but sufficient for our applications.

Theorem 1.2.3. *(Smooth Approximation.)[[3], Theorem 5.3] For $f \in BV(\Omega)$, there exists $\{f_k\}_{k=1}^{\infty} \subseteq BV(\Omega) \cap C^{\infty}(\Omega)$ such that $f_k \to f$ in $L^1(\Omega)$ and $||Df_k||(\Omega) \to ||Df||(\Omega)$.*

Proof. Let $\epsilon > 0$ and $m, k \in \mathbb{N}$. We construct a double indexed increasing nested sequence of open sets by

$$\Omega_k^m = \left\{ x \in \Omega \,\middle|\, \operatorname{dist}(x, \partial\Omega) > \frac{1}{m+k} \right\} \cap B_{m+k}.$$

Clearly, $\Omega = \cup_{k=1}^{\infty} \Omega_k^m$ for all $m \in \mathbb{N}$. Similarly, we also have $\Omega = \cup_{m=1}^{\infty} \Omega_1^m$. We may use the latter to obtain

$$||Df||(\Omega) = \lim_{m \to \infty} ||Df||(\Omega_1^m).$$

Since $f \in BV(\Omega)$, $||Df||(\Omega) < \infty$, so there exists $m \in \mathbb{N}$ so that

$$||Df||(\Omega \setminus \Omega_1^m) < \epsilon. \tag{1.2.2}$$

For such m, let $\Omega_0^m = \emptyset$. For $k \geq 1$, define

$$A_k = \Omega_{k+1}^m \setminus \overline{\Omega_{k-1}^m}. \tag{1.2.3}$$

The set $\{A_k\}_{k=1}^{\infty}$ forms an open cover on Ω, hence there exists a partition of unity $\{\psi_k\}_{k=1}^{\infty}$ subordinate to the open cover $\{A_k\}_{k=1}^{\infty}$, such that

(P1) $\psi_k \in C_c^{\infty}(\Omega)$, $0 \leq \psi_k \leq 1$ for all $k \in \mathbb{N}$,

(P2) $supp(\psi_k) \subset A_k$ for all $k \in \mathbb{N}$,

(P3) $\sum_{k=1}^{\infty} \psi_k(x) = 1$ for all $x \in \Omega$.

We now let ρ_ϵ denote the usual mollifier for $\epsilon > 0$, then for each k there exists $\epsilon_k > 0$ such that

$*1.$ $supp(\rho_{\epsilon_k} \star f\psi_k) \subset A_k$,

$*2.$ $||\rho_{\epsilon_k} \star f\psi_k - f\psi_k||_{L^1(\Omega)} < \frac{\epsilon}{2^k}$,

$*3.$ $||\rho_{\epsilon_k} \star f\nabla\psi_k - f\nabla\psi_k||_{L^1(\Omega)} < \frac{\epsilon}{2^k}$.

Define

$$f_\epsilon = \sum_{k=1}^{\infty} \rho_{\epsilon_k} \star f\psi_k. \tag{1.2.4}$$

We observe that for a fixed $x \in \Omega$, there exists $k \in \mathbb{N}$ such that $x \in A_k$. Since, A_k is open, there exists $r > 0$ such that $B_r(x) \subseteq A_k$. By $*1$, $B_r(x) \cap supp(\rho_{\epsilon_j} \star f\psi_j) = \emptyset$ for all $j > k+2$ and $j < k-2$. Thus, for all $y \in B_r(x)$, $f_\epsilon(y)$ is a finite sum of smooth functions. We deduce $f_\epsilon \in C^\infty(\Omega)$. To see that $f_\epsilon \in L^1(\Omega)$, we observe

$$\int_\Omega |f_\epsilon|\, dx = \int_\Omega \left|\sum_{k=1}^{\infty} \rho_{\epsilon_k} \star f\psi_k\right| dx \leq \int_\Omega \sum_{k=1}^{\infty} |\rho_{\epsilon_k} \star f\psi_k - f\psi_k| + |f\psi_k|\, dx.$$

We apply the Monotone Convergence Theorem on $\sum_{k=1}^{\infty} |\rho_{\epsilon_k \star f\psi_k} - f\psi_k|$ to get

$$\int_\Omega |f_\epsilon|\, dx \leq \sum_{k=1}^{\infty} \int_\Omega |\rho_\epsilon \star f\psi_k - f\psi_k|\, dx + \int_\Omega \sum_{k=1}^{\infty} |f\psi_k|\, dx. \tag{1.2.5}$$

Then, by Property $*2$ and P3 we see

$$\int_\Omega |f_\epsilon|\, dx \leq \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} + \int_\Omega \sum_{k=1}^{\infty} |f|\psi_k\, dx \leq \epsilon + \int_\Omega |f|\, dx.$$

It follows that $f_\epsilon \in L^1(\Omega)$. Consequently, $f_\epsilon \in BV(\Omega)$. For $L^1(\Omega)$ convergence of f_ϵ to f, we apply the Monotone Convergence Theorem as in (1.2.5), along with $*2$ to see that

$$\int_\Omega |f - f_\epsilon|\, dx \leq \int_\Omega \sum_{k=1}^{\infty} |f\psi_k - \rho_{\epsilon_k} \star f\psi_k|\, dx \tag{1.2.6}$$

$$= \sum_{k=1}^{\infty} \int_\Omega |f\psi_k - \rho_{\epsilon_l} \star f\psi_k|\, dx \tag{1.2.7}$$

$$\leq \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} \tag{1.2.8}$$

$$= \epsilon. \tag{1.2.9}$$

Thus, $f_\epsilon \xrightarrow{\epsilon\to 0} f$ in $L^1(\Omega)$. By Theorem 1.2.1,

$$||Df||(\Omega) \leq \liminf_{\epsilon\to 0} ||Df_\epsilon||(\Omega).$$

It remains to show $\limsup_{\epsilon\to 0} ||Df_\epsilon||(\Omega) \leq ||Df||(\Omega)$. Let $\epsilon > 0$ and f_ϵ be defined as before. Let $\phi \in C_c^1(\Omega;\mathbb{R}^n)$ with $||\phi||_\infty \leq 1$, then

$$\int_\Omega f_\epsilon \mathrm{div}(\phi)dx = \int_\Omega \left(\sum_{k=1}^\infty \rho_{\epsilon_k} \star f\psi_k\right) \mathrm{div}(\phi)dx. \tag{1.2.10}$$

Since ϕ has compact support, $supp(\phi) \subseteq \cup_{j=1}^m A_{k_j}$ for some finite subset $\{A_{k_j}\}_{j=1}^m \subset \{A_k\}_{k=1}^\infty$. We note that each A_{k_j} intersects only finitely many A_k's, so there are only finitely many non-zeros functions in $\{\rho_{\epsilon_k} \star f\psi_k\}_{k=1}^\infty$ on $supp(\phi)$. This implies Equation (1.2.10) is actually a finite sum. We may interchange the summation and integral, giving us

$$\int_\Omega f_\epsilon \mathrm{div}(\phi)dx = \sum_{k=1}^\infty \int_\Omega \left(\rho_{\epsilon_k} \star f\psi_k\right) \mathrm{div}(\phi)\, dx. \tag{1.2.11}$$

We wish to relate the left hand side of (1.2.11) to an expression involving the integrand of f and the divergence of a compactly supported function. We note that for each $i \in \{1,\ldots,n\}$, for all $k \in \mathbb{N}$,

$$\int_\Omega \left(\rho_{\epsilon_k} \star f\psi_k\right) \partial_i\phi_i \, dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \rho_{\epsilon_k}(x-y)\, f(y)\psi_k(y)\, \partial_i\phi_i(x)\, dy\, dx. \tag{1.2.12}$$

Given that ρ_{ϵ_k} and $\partial_i\phi_i$ are continuous with compact support, we see that $\rho_{\epsilon_k}\partial_i\phi_i \in L^\infty(\mathbb{R}^n)$. By Young's Inequality,

$$||(\rho_{\epsilon_k} \star f\psi_k)\partial_i\phi_i||_{L^1(\mathbb{R}^n)} \leq ||\rho_{\epsilon_k}\partial_i\phi_i||_{L^\infty(\mathbb{R}^n)}||f\psi_k||_{L^1(\mathbb{R}^n)} < \infty.$$

By Fubini's Theorem, we may interchange the order of integration of (1.2.12), giving us

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \rho_{\epsilon_k}(x-y)\, f(y)\psi_k(y)\, \partial_i\phi_i(x)\, dy\, dx &= \int_{\mathbb{R}^n} f(y)\psi_k(y) \int_{\mathbb{R}^n} \rho_{\epsilon_k}(y-x)\partial_i\phi_i(x)\, dx\, dy \\ &= \int_\Omega f\psi_k(\rho_{\epsilon_k} \star \partial_i\phi_i)\, dy \\ &= \int_\Omega f\psi_k\, \partial_i(\rho_{\epsilon_k} \star \phi_i)\, dy.\end{aligned}$$

Next, we note upon rearrangement that

$$\psi_k \mathrm{div}(\rho_{\epsilon_k} \star \phi) = \mathrm{div}(\psi_k(\rho_{\epsilon_k} \star \phi)) - \nabla\psi_k \cdot (\rho_{\epsilon_k} \star \phi). \tag{1.2.13}$$

We further integrate the latter term against f along with an application of Fubini's Theorem as before to obtain

$$\begin{aligned}\int_\Omega f\, \nabla\psi_k \cdot (\rho_{\epsilon_k} \star \phi) dx &= \int_\Omega f(x)\, \nabla\psi_k(x) \cdot \left(\int_{\mathbb{R}^n} \rho_{\epsilon_k}(x-y)\phi(y)\, dy\right) dx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)\nabla\psi_k(x) \cdot \rho_{\epsilon_k}(x-y)\phi(y)\, dy\, dx \\ &= \int_{\mathbb{R}^n} \phi(y) \cdot \int_{\mathbb{R}^n} f(x)\nabla\psi_k(x)\rho_{\epsilon_k}(x-y)\, dx\, dy \\ &= \int_\Omega \phi \cdot (\rho_{\epsilon_k} \star (f\nabla\psi_k))\, dx. \end{aligned} \tag{1.2.14}$$

If we combine (1.2.12), (1.2.13) and (1.2.14), then (1.2.11) becomes

$$\begin{aligned}\sum_{k=1}^{\infty}\int_\Omega (\rho_{\epsilon_k} \star f\psi_k)\, \mathrm{div}(\phi)\, dx &= \sum_{k=1}^{\infty}\int_\Omega f\psi_k\, \mathrm{div}(\rho_{\epsilon_k} \star \phi)\, dx \\ &= \sum_{k=1}^{\infty}\int_\Omega f\, \mathrm{div}(\psi_k(\rho_{\epsilon_k} \star \phi))\, dx \\ &\quad - \sum_{k=1}^{\infty}\int_\Omega f\, \nabla\psi_k \cdot (\rho_{\epsilon_k} \star \phi) dx \\ &= \sum_{k=1}^{\infty}\int_\Omega f\, \mathrm{div}(\psi_k(\rho_{\epsilon_k} \star \phi))\, dx \\ &\quad - \sum_{k=1}^{\infty}\int_\Omega \phi \cdot (\rho_{\epsilon_k} \star (f\nabla\psi_k) - f\nabla\psi_k)\, dx \\ &=: I_\epsilon^1 + I_\epsilon^2.\end{aligned}$$

By construction of ϵ_k in $*3$,

$$
\begin{aligned}
|I_\epsilon^2| &\le \sum_{k=1}^{\infty} \int_\Omega |\phi \cdot (\rho_{\epsilon_k} \star (f\,\nabla\psi_k) - f\,\nabla\psi_k)|\,dx \\
&\le \sum_{k=1}^{\infty} \int_\Omega |(\rho_{\epsilon_k} \star (f\,\nabla\psi_k) - f\,\nabla\psi_k)|\,dx \\
&\le \epsilon.
\end{aligned}
$$

By the same application of Jensen's Inequality in (1.1.9), we see that $||\psi_k(\rho_{\epsilon_k} \star \phi)||_\infty \le 1$. Next, we claim each point $x \in \Omega$ is contained in no more than three sets of $\{A_k\}_{k=1}^{\infty}$. To see this, we note $x \in A_k$ if and only if $1/(m+k+1) \le d(x, \partial\Omega) \le 1/(m+k-1)$. Now if $l \ge 2$ or $l \le -2$, then

$$
\left(\frac{1}{m+k+l+1}, \frac{1}{m+k+l-1}\right) \cap \left(\frac{1}{m+k+1}, \frac{1}{m+k-1}\right) = \emptyset.
$$

Therefore, $x \notin A_{k+l}$ for all $l \le -2$ or $l \ge 2$. This implies we can subdivide $\{A_k\}_{k=2}^{\infty}$ into three subsequences of pairwise disjoint sets, $\{A_k^1\}_{k=2}^{\infty}$, $\{A_k^2\}_{k=2}^{\infty}$ and $\{A_k^3\}_{k=2}^{\infty}$. Since each subsequence of $\{A_k^i\}_{k=2}^{\infty}$ for $i \in \{1,2,3\}$, is contained in $\Omega \setminus \Omega_1$, by (1.2.2) we see that

$$
\begin{aligned}
|I_\epsilon^1| &= \left| \int_\Omega f\,\mathrm{div}(\psi_1(\rho_{\epsilon_1} \star \phi))\,dx + \sum_{k=2}^{\infty} \int_\Omega f\,\mathrm{div}(\psi_k(\rho_{\epsilon_k} \star \phi))\,dx \right| \\
&\le ||Df||(\Omega) + \sum_{k=2}^{\infty} ||Df||(A_k^1) + ||Df||(A_k^2) + ||Df||(A_k^3) \\
&\le ||Df||(\Omega) + 3||Df||(\Omega \setminus \Omega_1) \\
&\le ||Df||(\Omega) + 3\epsilon.
\end{aligned}
$$

Putting I_ϵ^1 and I_ϵ^2 together, we see for all $\phi \in C_c^1(\Omega;\mathbb{R}^n)$ with $||\phi||_\infty \le 1$,

$$
\int_\Omega f_\epsilon\,\mathrm{div}(\phi)\,dx \le ||Df||(\Omega) + 4\epsilon.
$$

Taking the supremum across all $\phi \in C_c^1(\Omega;\mathbb{R}^n)$ with $||\phi||_\infty \le 1$ yields

$$
||Df_\epsilon||(\Omega) \le ||Df||(\Omega) + 4\epsilon.
$$

Thus,

$$\limsup_{\epsilon \to 0} ||Df_\epsilon||(\Omega) \leq ||Df||(\Omega),$$

as desired. $\square$

Notice, if we replace Ω in Theorem 1.2.3 with $V \subseteq\subseteq \Omega$ open, then $f_\epsilon \xrightarrow{\epsilon \to 0} f$ in $L^1(V)$, so lower-semicontinuity still holds. Furthermore, we may show $\limsup_{\epsilon \to 0} ||Df_\epsilon||(V) \leq ||Df||(V)$ by applying the same argument as in the prior proof. In other words, we obtain the following:

Corollary 1.2.1. *If $\{f_k\}_{k=1}^\infty$ is the sequence stated in Theorem 1.2.3, then for all $V \subseteq\subseteq \Omega$ open,*

$$\lim_{k \to \infty} ||Df_k||(V) = ||Df||(V). \tag{1.2.15}$$

If we recall the Relich-Kondrachov Theorem, we know $W^{1,1}(\Omega)$ may be compactly embedded into $L^1(\Omega)$. Theorem 1.2.3 will allow us to to obtain a generalization of the Relich-Kondrachov Theorem for $BV(\Omega)$.

Theorem 1.2.4. *(Compactness.) [[3],Theorem 5.5] Let $\Omega \subset \mathbb{R}^n$ be open and bounded with Lipschitz boundary $\partial\Omega$. Assume $\{f_k\}_{k=1}^\infty \subseteq BV(\Omega)$ is a bounded sequence under $||\cdot||_{BV}$, then there exists a $f \in BV(\Omega)$ and a subsequence $\{f_{k_j}\}_{j=1}^\infty$ such that $f_{k_j} \xrightarrow{j \to \infty} f$ in $L^1(\Omega)$.*

Proof. By the assumption of $||\cdot||_{BV(\Omega)}$ boundedness, there exists $M > 0$ such that for all $k \in \mathbb{N}$,

$$||f_k||_{BV(\Omega)} = ||f_k||_{L^1(\Omega)} + ||Df_k||(\Omega) < M.$$

By Theorem 1.2.3, we may choose a smooth function g_k such that $||g_k - f_k||_{L^1(\Omega)} < 1/k$, and $||Dg_k||(\Omega)$ is sufficiently close to $||Df_k||(\Omega)$ so that $||Dg_k||(\Omega)$ is also less than M. Since $g_k \in W^{1,1}(\Omega)$, it follows from Theorem 1.1.3 that $||Dg_k||(\Omega) = ||\nabla g_k||_{L^1(\Omega)}$. Therefore, for all $k \in \mathbb{N}$

$$||g_k||_{BV(\Omega)} = ||g_k||_{W^{1,1}(\Omega)} = ||g_k||_{L^1(\Omega)} + ||\nabla g_k||_{L^1(\Omega)} < 2M + 1.$$

By the Rellich-Kondrachov Theorem, there exists $f \in L^1(\Omega)$ and a subsequence $\{g_{k_j}\}_{j=1}^{\infty}$ such that $g_{k_j} \to f$ in $L^1(\Omega)$. We see by construction of g_{k_j}

$$\begin{aligned}
\lim_{j\to\infty} ||f_{k_j} - f||_{L^1(\Omega)} &\leq \lim_{j\to\infty} ||f_{k_j} - g_{k_j}||_{L^1(\Omega)} + \lim_{j\to\infty} ||g_{k_j} - f||_{L^1(\Omega)} \\
&\leq \lim_{j\to\infty} \frac{1}{k_j} \\
&= 0
\end{aligned}$$

Thus, $f_{k_j} \xrightarrow{j\to\infty} f$ in $L^1(\Omega)$. It remains to show $f \in BV(\Omega)$, but this is immediate by Theorem 1.2.1 as

$$||Df||(\Omega) \leq \liminf_{k\to\infty} ||Df_{k_j}||(\Omega) < M,$$

which completes the proof. □

By combining a diagonal argument with Theorem 1.2.4 we may obtain a compactness result for $BV_{loc}(\mathbb{R}^n)$.

Corollary 1.2.2. *Let $\{f_k\}_{k=1}^{\infty} \subseteq BV_{loc}(\mathbb{R}^n)$ such that for all compact $V \subset \mathbb{R}^n$, $\{f_k\}_{k=1}^{\infty}$ forms a bounded sequence in $||\cdot||_{BV(V)}$. There exists a $f \in BV_{loc}(\mathbb{R}^n)$ and a subsequence $\{f_{k_j}\}_{j=1}^{\infty}$ such that $f_{k_j} \xrightarrow{j\to\infty} f$ in $L^1_{loc}(\mathbb{R}^n)$.*

As we have alluded to before, the compactness result of BV is essential in proving existence of solutions to minimization problems via the Direct-Method. It allows one to validate the existence of a limit from a minimizing sequence of competitors, while the lower-semi continuity property allows one to show the limit is indeed a minimizer. We will see the Direct-Method in its complete form in Chapter 3.

1.3 Isoperimetric Inequalities

We have seen $W^{1,1}(\Omega)$ is dense in $BV(\Omega)$ in the weak sense of Theorem 1.2.3. We will utilize this smooth approximation by $W^{1,1}(\Omega)$ to derive Sobolev-like inequalities for BV. We will then show these Sobolev-like inequalities implies the Isoperimetric inequality.

The following two theorems we will state but not prove, as their proofs are either a straightforward application of the Gagliardo-Nirenberg inequality or identical to the proof for $W^{1,1}_{loc}(\mathbb{R}^n)$.

Theorem 1.3.1. *[[3], Theorem 5.10 (i)] There exists $C_1 > 0$ such that for all $f \in BV(\mathbb{R}^n)$,*

$$||f||_{L^{1^*}(\mathbb{R}^n)} \leq C_1||Df||(\mathbb{R}^n), \tag{1.3.1}$$

where $1^* = \frac{n}{n-1}$.

Theorem 1.3.2. *(Poincare's Inequality for BV.) [[3], Theorem 5.10 (ii)] Denote $(f)_{x,r}$ by*

$$(f)_{x,r} = \frac{1}{\mathcal{L}^n(B_r(x))}\int_{B_r(x)} f(y)\,dy.$$

There exists a constant $C_2 > 0$ such that for all $f \in BV_{loc}(\mathbb{R}^n)$ and for all $r > 0$,

$$||f - (f)_{x,r}||_{L^{1^*}(B_r(x))} \leq C_2||Df||(B_r(x)).$$

Theorem 1.3.1 and 1.3.2 allows us to relate the total variation of a function to its L^{1^*}-norm. In the special case of characteristic functions, these inequalities become statements about the volume of a set and its perimeter. This relation between volume and perimeter is commonly known as the Isoperimetric inequality. In the following theorem, we will present a weak variant of the Isoperimetric inequality, then in Chapter 4 we will refine the bound, and discuss the relation to the Isoperimetric problem.

Theorem 1.3.3. *(Isoperimetric inequality.)[[3], Theorem 5.11] There exists $C > 0$, such that if $E \in BV(\mathbb{R}^n)$, then*

$$(\mathcal{L}^n(E))^{\frac{1}{1^*}} \leq C||\partial E||(\mathbb{R}^n). \tag{1.3.2}$$

Furthermore, there exists $C_0 > 0$ such that for all $x \in \mathbb{R}^n$ and for all $r > 0$,

$$\min\{\mathcal{L}^n((B_r(x) \cap E), \mathcal{L}^n(B_r(x) \setminus E)\}^{\frac{1}{1^*}} \leq 2C_0||\partial E||(B_r(x)). \tag{1.3.3}$$

(1.3.3) is known as the local Isoperimetric inequality.

Proof. (1.3.2) is immediate by taking $f = \chi_E$ into Theorem 1.3.1. To get (1.3.3), we first observe

$$(\chi_E)_{x,r} = \frac{\mathcal{L}^n(B_r(x) \cap E)}{\mathcal{L}^n(B_r(x))},$$

then

$$\begin{aligned}
\int_{B_r(x)} |\chi_E - (\chi_E)_{x,r}|^{1^*}\, dx &= \int_{B_r(x)\cap E} \left| \frac{\mathcal{L}^n(B_r(x)) - \mathcal{L}^n(B_r(x) \cap E)}{\mathcal{L}^n(B_r(x))} \right|^{1^*} dx \\
&\quad + \int_{B_r(x)\setminus E} \left| \frac{\mathcal{L}^n(B_r(x) \cap E)}{\mathcal{L}^n(B_r(x))} \right|^{1^*} dx \\
&= \mathcal{L}^n(B_r(x) \cap E) \left| \frac{\mathcal{L}^n(B_r(x) \setminus E)}{\mathcal{L}^n(B_r(x))} \right|^{1^*} \\
&\quad + \mathcal{L}^n(B_r(x) \setminus E) \left| \frac{\mathcal{L}^n(B_r(x) \cap E)}{\mathcal{L}^n(B_r(x))} \right|^{1^*}.
\end{aligned}$$

Now assuming without loss of generality that $\mathcal{L}^n(B_r(x) \cap E) \leq \mathcal{L}^n(B_r(x) \setminus E)$, we have

$$\mathcal{L}^n(B_r(x)) = \mathcal{L}^n(B_r(x) \setminus E) + \mathcal{L}^n(B_r(x) \cap E) \leq 2\mathcal{L}^n(B_r(x) \setminus E).$$

Thus,

$$\begin{aligned}
\left(\int_{B_r(x)} |\chi_E - (\chi_E)_{x,r}|^{1^*}\, dx \right)^{\frac{1}{1^*}} &\geq \mathcal{L}^n(B_r(x) \cap E)^{\frac{1}{1^*}} \left| \frac{\mathcal{L}^n(B_r(x) \setminus E)}{\mathcal{L}^n(B_r(x))} \right| \\
&\geq \frac{1}{2} \mathcal{L}^n(B_r(x) \cap E)^{\frac{1}{1^*}}.
\end{aligned}$$

Likewise, if $\mathcal{L}^n(B_r(x) \setminus E) \leq \mathcal{L}^n(B_r(x) \cap E)$, we get

$$\left(\int_{B_r(x)} |\chi_E - (\chi_E)_{x,r}|^{1^*}\, dx \right)^{\frac{1}{1^*}} \geq \frac{1}{2} \mathcal{L}^n(B_r(x) \setminus E)^{\frac{1}{1^*}}.$$

In general,

$$\left(\int_{B_r(x)} |\chi_E - (\chi_E)_{x,r}|^{1^*}\, dx \right)^{\frac{1}{1^*}} \geq \frac{1}{2} \min\{\mathcal{L}^n(B_r(x) \cap E), \mathcal{L}^n(B_r(x) \setminus E)\}^{\frac{1}{1^*}}. \tag{1.3.4}$$

By Theorem 1.3.2, there exists $C_2 > 0$ such that for all $r > 0$,

$$C_2 ||Df||(B_r(x)) \geq \left(\int_{B_r(x)} |\chi_E - (\chi_E)_{x,r}|^{1^*}\, dx \right)^{\frac{1}{1^*}}.$$

Then, combined with (1.3.4) we see

$$\min\{\mathcal{L}^n(B_r(x) \cap E), \mathcal{L}^n(B_r(x) \setminus E)\}^{1^*} \leq 2C_2||\partial E||(B_r(x)).$$

$\square$

1.4 Coarea Formula

In this subsection we give a short presentation of the coarea formula and its applications. Given that we will not utilize the coarea formula directly in the later chapters, we will omit its proof to the appendix.

Assume $\Omega \subseteq \mathbb{R}^n$ is open and $f : \Omega \to \mathbb{R}$. For $t \in \mathbb{R}$, we define the level set of f by

$$E_t = \{x \in \Omega \,|\, f(x) > t\}.$$

As we will see, the coarea formula will allow us to relate the level set of a function to its total variation.

Theorem 1.4.1. *(Coarea Formula.)[[3], Theorem 5.9] If $f \in BV(\Omega)$, then for $\mathcal{L}^1$-a.e $t \in \mathbb{R}$, E_t has finite perimeter. Moreover,*

$$||Df||(\Omega) = \int_{-\infty}^{\infty} ||\partial E_t||(\Omega)\, dt. \tag{1.4.1}$$

Conversely, if $f \in L^1(\Omega)$ and

$$\int_{-\infty}^{\infty} ||\partial E_t||(\Omega)\, dt < \infty,$$

then $f \in BV(\Omega)$.

If one recalls a result from probability, then (1.4.1) may look familiar; the expectation of a positive random variable X may be obtained by integrating over the probability of the level sets of X. The coarea formula states exactly that. Let us look at a simple example.

Example 1.4.1. *Let $\Omega = \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be a continuous and compactly supported function. By continuity, we observe for each $t \in \mathbb{R}$, E_t is open, hence may be expressed as the union of countably many disjoint open intervals. Let us fix a $t \in \mathbb{R}$, and for simplicity let us assume $E_t = (a, b)$. We wish to evaluate the perimeter of E_t. Notice, if $\phi \in C_c^1(\mathbb{R})$, then by the Fundamental Theorem of Calculus,*

$$\int_{E_t} \phi' \, dx = \int_a^b \phi' \, dx = \phi(b) - \phi(a). \tag{1.4.2}$$

Now if $|\phi| \leq 1$, then (1.4.2) is bounded by 2, and we conclude $||\partial E_t||(\mathbb{R}) \leq 2$. On the other hand, it is easy to construct $\phi \in C_c^1(\mathbb{R})$ such that $|\phi| \leq 1$, $\phi(b) = 1$ and $\phi(a) = -1$. See Figure 1 for an example. For such ϕ, $\int_{E_t} \phi' \, dx = 2$. It follows $||\partial E_t||(\mathbb{R}) = 2$, which is precisely the number of boundary points of (a, b). Since f is continuous, we also have $\partial E_t = \{x \in \mathbb{R} \,|\, f(x) = t\}$. If we repeat this argument for finitely many open intervals, as f has compact support, we would get

$$||\partial E_t||(\mathbb{R}) = \mathcal{H}^0(\partial E_t) = \mathcal{H}^0(\{x \in \mathbb{R} \,|\, f(x) = t\}).$$

More precisely, $||\partial E_t||(\mathbb{R})$ counts the number of times $f(x) = t$. The coarea formula then translates to

$$||Df||(\mathbb{R}) = \int_{-\infty}^{\infty} ||\partial E_t||(\mathbb{R}) \, dt = \int_{-\infty}^{\infty} \mathcal{H}^0(\{x \in \mathbb{R} \,|\, f(x) = t\}) \, dt.$$

In the general case, let $n \geq 2$ and $f : \mathbb{R}^n \to \mathbb{R}$ a smooth compactly supported function. By the Morse-Sard Theorem, for $\mathcal{L}^1$-a.e $t \in \mathbb{R}$, E_t is open with smooth boundaries. We see by the same argument with the the Divergence Theorem as in Example 1.1.1, $||\partial E_t||(\mathbb{R}^n) = \mathcal{H}^{n-1}(\partial E_t)$. The coarea formula then implies

$$||Df||(\mathbb{R}^n) = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\partial E_t) \, dt = \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\{x \in \mathbb{R} \,|\, f(x) = t\}) \, dt.$$

We will explore more on the implications of when $||\partial E||(\mathbb{R}^n) = \mathcal{H}^{n-1}(\partial E)$ for E a set of finite perimeter in the next section.

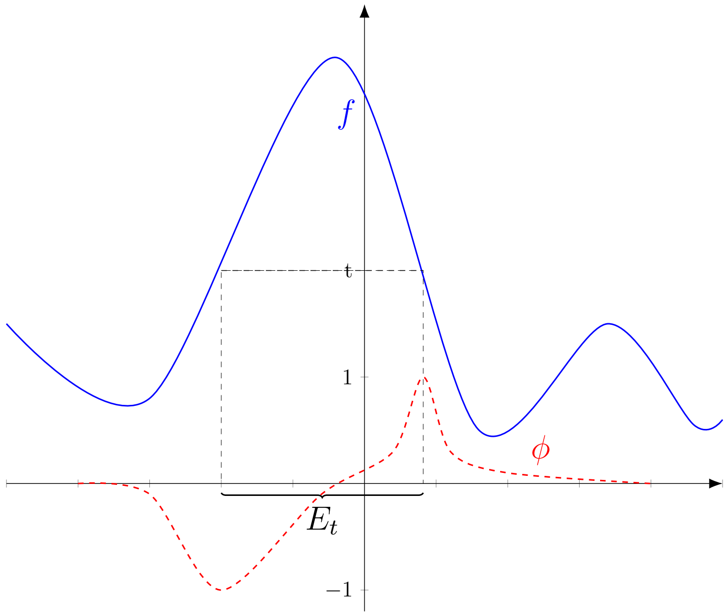

Figure 1: The level set of f at t and $\phi \in C_c^1(\mathbb{R})$ such that $\phi = \pm 1$ on ∂E_t.

The coarea formula, as we saw allows one to decompose the total variation of $f \in BV(\Omega)$ into a formulation involving only the perimeter of its level sets. Now if f is smooth, these level sets are open with smooth boundaries. Naturally, one may ask, through an approximating sequence of smooth function, if it is possible to extract a sequence of smooth sets converging to a set $E \in BV(\Omega)$ from the level sets of smooth functions?

Theorem 1.4.2. *[[7], Theorem 13.8] If $E \in BV(\mathbb{R}^n)$, then there exists a sequence of smooth sets $\{E_h\}_{h=1}^{\infty}$ such that $E_h \to E$ in $L^1(\mathbb{R}^n)$ and $||\partial E_h||(\mathbb{R}^n) \to ||\partial E||(\mathbb{R}^n)$.*

We will provide only a sketch of the proof, the complete details may be found in [6] and [7].

Sketch of Proof of Theorem 1.4.2. We first assume E is bounded. As we wish to bound the approximating sequence of smooth functions, instead of taking the sequence that arises from Theorem 1.2.3, we take a sequence of mollifiers on χ_E. Let $f_\epsilon = \rho_\epsilon \star \chi_E$. Clearly, $f_\epsilon \xrightarrow{\epsilon \to 0} \chi_E$ in $L^1(\mathbb{R}^n)$. By noting that for all $\phi \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$ such that $||\phi||_\infty \leq 1$, we see that

$$\int_{\mathbb{R}^n} f_\epsilon \text{div}(\phi)\, dx = \int_E \text{div}(\rho_\epsilon \star \phi)\, dx.$$

By Jensen's inequality, we know $||\rho_\epsilon \star \phi||_\infty \leq 1$, thus $||Df_\epsilon||(\mathbb{R}^n) \leq ||\partial E||(\mathbb{R}^n)$. Combined with Theorem 1.2.1, we obtain

$$||\partial E||(\mathbb{R}^n) \leq \liminf_{\epsilon \to 0} ||Df_\epsilon||(\mathbb{R}^n) \leq ||\partial E||(\mathbb{R}^n).$$

Therefore, $\liminf_{\epsilon\to 0} ||Df_\epsilon||(\mathbb{R}^n) = ||\partial E||(\mathbb{R}^n)$, we may assume up to a subsequence $\{\epsilon_h\}_{h=1}^\infty$, $||Df_{\epsilon_h}||(\mathbb{R}^n) \xrightarrow{\epsilon_h \to 0} ||\partial E||(\mathbb{R}^n)$. Let us denote the level sets of f_{ϵ_h} by $E_t^h = \{x \in \mathbb{R}^n \,|\, f_{\epsilon_h}(x) > t\}$. Utilizing the coarea formula and the fact that $0 \leq f_{\epsilon_h} \leq 1$, we get

$$||\partial E||(\mathbb{R}^n) = \lim_{\epsilon_h \to 0} ||Df_{\epsilon_h}||(\mathbb{R}^n) = \liminf_{\epsilon_h \to 0} \int_0^1 ||\partial E_t^h||(\mathbb{R}^n)\, dt.$$

By Fatou's Lemma,

$$||\partial E||(\mathbb{R}^n) = \liminf_{\epsilon_h \to 0} \int_0^1 ||\partial E_t^h||(\mathbb{R}^n)\, dt \geq \int_0^1 \liminf_{\epsilon_h \to 0} ||\partial E_t^h||(\mathbb{R}^n)\, dt. \tag{1.4.3}$$

We may show that for all $t \in (0,1)$, $E_t^h \xrightarrow{\epsilon_h \to 0} \chi_E$ in $L^1(\mathbb{R}^n)$ by observing that if $x \in E_t^h \backslash E$, then $f_{\epsilon_h}(x) - \chi_E(x) > t$. Likewise, if $x \in E \setminus E_t^h$, then $\chi_E(x) - f_{\epsilon_h}(x) > 1 - t$. By noting that

$$\int_{\mathbb{R}^n} |f_{\epsilon_h} - \chi_E|\, dx = \int_{E \backslash E_t^h} |f_{\epsilon_h} - \chi_E|\, dx + \int_{E \backslash E_t^h} |f_{\epsilon_h} - \chi_E|\, dx,$$

it is easy to show that

$$\mathcal{L}^n(E \Delta E_t^h) \leq \frac{1}{\min\{t, 1-t\}} \int_{\mathbb{R}^n} |f_{\epsilon_h} - \chi_E|\, dx.$$

Since, $f_{\epsilon_h} \xrightarrow{\epsilon_h \to 0} \chi_E$ in $L^1(\mathbb{R}^n)$, we see that $E_t^h \xrightarrow{\epsilon_h \to 0} E$ in $L^1(\mathbb{R}^n)$ for all $t \in (0,1)$. By Theorem 1.2.1, $||\partial E||(\mathbb{R}^n) \leq \liminf_{\epsilon_h \to 0} ||\partial E_t^h||(\mathbb{R}^n)$ for all $t \in (0,1)$. Plugging this into (1.4.3) yields

$$||\partial E||(\mathbb{R}^n) \geq \int_0^1 \liminf_{\epsilon_h \to 0} ||\partial E_t^h||(\mathbb{R}^n)\, dt \geq \int_0^1 ||\partial E||(\mathbb{R}^n)\, dt = ||\partial E||(\mathbb{R}^n).$$

That is, $\liminf_{\epsilon_h \to 0} ||\partial E_t^h||(\mathbb{R}^n) = ||\partial E||(\mathbb{R}^n)$ for $\mathcal{L}^1$-a.e $t \in (0,1)$. Hence, by taking a further subsequence, we may assume $||\partial E_t^h||(\mathbb{R}^n) \xrightarrow{\epsilon_h \to 0} ||\partial E||(\mathbb{R}^n)$ for $\mathcal{L}^1$-a.e $t \in (0,1)$. Moreover, by the Morse-Sard Theorem, these level sets are smooth.

In the case $E \in BV(\mathbb{R}^n)$ not necessarily bounded, one may show that $B_r \cap E \xrightarrow{r\to\infty} E$ in $L^1(\mathbb{R}^n)$ and $||\partial B_r \cap E||(\mathbb{R}^n) \xrightarrow{r\to\infty} ||\partial E||(\mathbb{R}^n)$. Since we may approximate each $B_r \cap E$ with a smooth and bounded set, by a diagonal argument, we may construct a sequence of smooth bounded sets $\{E_h\}_{h=1}^{\infty}$ such that $E_h \to E$ in $L^1(\mathbb{R}^n)$ and $||\partial E_h||(\mathbb{R}^n) \to ||\partial E||(\mathbb{R}^n)$. □

We remark, the original formulation of sets of finite perimeter by Caccioppoli, before it was refined by De Giorgi, was through the approximation by polyhedral sets [4]. Through a further diagonalization argument onto Theorem 1.4.2, where we approximate the smooth boundaries by a sequence of affine functions, we may recover Caccioppoli's polyhedral approximation. This approximation technique will be prove to be very convenient in solving the Isoperimetric problem in Chapter 4.

Theorem 1.4.3. *[[7], Remark 13.13] If $E \in BV(\mathbb{R}^n)$, then there exists a sequence $\{E_h\}_{h=1}^{\infty}$ of open bounded sets with polyhedral boundary such that $E_h \to E$ in $L^1(\mathbb{R}^n)$ and $||\partial E_h||(\mathbb{R}^n) \to ||\partial E||(\mathbb{R}^n)$.*

1.5 Single Variable Case

An important aspect of the theory of BV is the restriction of multi-variable functions onto 1-dimensional lines. To be precise, if $f \in BV(\mathbb{R}^n)$, $x \in \mathbb{R}$, and $\nu \in S^{n-1}$, then we define the restriction $f_x^\nu : \mathbb{R} \to \mathbb{R}$ by $f_x^\nu(t) = f(x + t\nu)$. Effectively, this reduces a problem involving several variables down into a problem of only a single variable. In the particular case of BV functions of a single variable, there exist an alternate formulation of the variation that provides a more geometric interpretation of this function class. In this subsection, we will look at this alternate definition and discuss the implications it has on our duality definition of (1.1.1).

Definition 1.5.1. *If $f \in L^1(\mathbb{R})$, we define the the total variation function of f by*

$$T_f(x) = \sup \left\{ \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \;\middle|\; \{x_j\}_{j=0}^{n},\, n \in \mathbb{N},\, x_0 < \cdots < x_n \le x \right\}. \qquad (1.5.1)$$

We say f has bounded total variation if $\lim_{x\to\infty} T_f(x) < \infty$.

Let us start with some basic properties of functions with bounded total variation, all of which we will state without proof.

Lemma 1.5.1. *If f has bounded total variation, then $f \in L^\infty(\mathbb{R})$.*

We note the converse is not true. Take for example

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \in (0,1) \\ 0 & \text{if } x \in \mathbb{R} \setminus (0,1). \end{cases}$$

Clearly, $f \in L^\infty(\mathbb{R})$. However, we notice on $(0,1)$, f oscillates indefinitely between 1 and -1. In fact, for all $x > 0$ and $m \in \mathbb{N}$, it is easy to construct a finite sequence $\{x_j\}_{j=0}^m \subset (0,x]$, in which $\sin(1/x_j)$ alternates between -1 and 1. For such a sequence

$$\sum_{j=1}^m |\sin(1/x_j) - \sin(1/x_{j-1})| = 2m.$$

It follows $T_f(x) = \infty$, so f cannot have bounded total variation.

On the other hand, bounded functions that are also monotonic do have bounded total variation. In fact, $\lim_{x\to\infty} T_f(x) = \sup_{y\in\mathbb{R}} f(y) - \inf_{y\in\mathbb{R}} f(y)$.

Lemma 1.5.2. *If $f \in L^1(\mathbb{R})$ is monotone and bounded, then f has bounded total variation.*

One of the characterizing features of functions with bounded total variation is the Jordan decomposition. We state the result in the following:

Lemma 1.5.3. *[[5], Lemma 3.26] If f has bounded total variation, then $T_f + f$ and $T_f - f$ are non-decreasing. Moreover, $f = \frac{1}{2}(T_f + f) - \frac{1}{2}(T_f - f)$. That is, f may be decomposed into the difference of non-decreasing functions.*

We shall relate functions with bounded total variation to a unique signed Radon measure. In turn, we will obtain an integration by parts formula similar to (1.1.4).

Theorem 1.5.1. *If $f : \mathbb{R} \to \mathbb{R}$ has bounded total variation, then there exists a finite signed measure μ such that for all $\phi \in C_c^1(\mathbb{R})$,*

$$\int_{\mathbb{R}} f\,\phi'\,dx = -\int_{\mathbb{R}} \phi\,d\mu. \tag{1.5.2}$$

Furthermore, for all open intervals $(a,b) \subseteq \mathbb{R}$, μ satisfies

$$\mu((a,b)) = f(b^-) - f(a^+). \tag{1.5.3}$$

Proof. We first prove the result for f non-decreasing and bounded. Define $\lambda : C_c^1(\mathbb{R}) \to \mathbb{R}$ by

$$\lambda(\phi) = -\int_{\mathbb{R}} f\,\phi'\,dx.$$

Clearly λ is linear. Let $\phi \in C_c^1(\mathbb{R})$ and define for $h > 0$,

$$\phi_h(x) = \frac{\phi(x+h) - \phi(x)}{h}.$$

By definition $\phi_h \xrightarrow{h\to 0} \phi'$ point-wise. By the Mean Value Theorem, there exists $z \in (x, x+h)$ such that $\phi_h(x) = \phi'(z)$. Since $\phi \in C_c^1(\mathbb{R})$, there exists $M > 0$ so that $M > \phi'(\cdot)$, and there exists compact set $K \subset \mathbb{R}$ such that $supp(\phi') \subseteq K$. Hence, $\phi_h \leq M\chi_K$ for all $h > 0$. By the Dominated Convergence Theorem,

$$\lambda(\phi) = -\int_{\mathbb{R}} f(x)\,\phi'(x)\,dx = -\lim_{h\to 0}\int_{\mathbb{R}} f(x)\,\frac{\phi(x+h)-\phi(x)}{h}\,dx.$$

By a change of variables, we may write

$$\begin{aligned}\int_{\mathbb{R}} f(x)\,\frac{\phi(x+h)-\phi(x)}{h}\,dx &= \frac{1}{h}\left(\int_{\mathbb{R}} f(x-h)\,\phi(x)\,dx - \int_{\mathbb{R}} f(x)\,\phi(x)\,dx\right)\\ &= \int_{\mathbb{R}} \frac{f(x-h)-f(x)}{h}\,\phi(x)\,dx.\end{aligned}$$

Then,

$$\lambda(\phi) = \lim_{h\to 0}\int_{\mathbb{R}} \frac{f(x)-f(x-h)}{h}\,\phi(x)\,dx.$$

If $\phi \geq 0$, since f is non-decreasing, it follows $\lambda(\phi) \geq 0$. Thus, λ is a positive linear functional. Now suppose $|\phi| \leq 1$ and $supp(\phi) \subset (a,b)$ for some $a < b$, then

$$\begin{aligned}
\lambda(\phi) &\leq \lim_{h\to 0} \int_{\mathbb{R}} \frac{f(x) - f(x-h)}{h} \chi_{(a,b)}\, dx \\
&= \lim_{h\to 0} \frac{1}{h} \left(\int_a^b f(x)\, dx - \int_{a-h}^{b-h} f(x)\, dx \right) \\
&= \lim_{h\to 0} \frac{1}{h} \left(\int_{b-h}^b f(x)\, dx - \int_{a-h}^{a} f(x)\, dx \right) \\
&= f(b^-) - f(a^-) \\
&\leq 2||f||_{L^\infty}
\end{aligned}$$

Hence, λ is a bounded positive linear functional on $C^1_c(\mathbb{R})$. By the Hahn-Banach Theorem, there exists a positive linear functional $\overline{\lambda} : C_c(\mathbb{R}) \to \mathbb{R}$, such that $\overline{\lambda}|_{C^1_c(\mathbb{R})} = \lambda$. Since $C^1_c(\mathbb{R})$ is dense in $C_c(\mathbb{R})$, this extension is unique. By the Riesz-Representation Theorem, there exists a unique Radon measure μ such that for all $\phi \in C_c(\mathbb{R})$,

$$\overline{\lambda}(\phi) = \int_{\mathbb{R}} \phi\, d\mu.$$

In particular, for all $\phi \in C^1_c(\mathbb{R})$,

$$\int_{\mathbb{R}} f\, \phi'\, dx = -\overline{\lambda}(\phi) = -\int_{\mathbb{R}} \phi\, d\mu,$$

as desired.

To prove (1.5.3) for a non-decreasing and bounded function f, we claim that (1.5.2) also holds for Lipschitz functions ϕ with compact support. We know by Radamacher's Theorem, if ϕ is Lipschitz, then ϕ' exists $\mathcal{L}^1-$a.e, so (1.5.2) is well-defined. Now let ρ_ϵ be the usual mollifier, then $\rho_\epsilon \star \phi \in C^1_c(\mathbb{R})$ and $\rho_\epsilon \star \phi \xrightarrow{\epsilon\to 0} \phi$ uniformly. In addition, $(\rho_\epsilon \star \phi)' \xrightarrow{\epsilon\to 0} \phi'$ in $L^1(\mathbb{R})$. Therefore, for all Lipschitiz function ϕ,

$$\int_{\mathbb{R}} f\, \phi'\, dx = \lim_{\epsilon\to 0} \int_{\mathbb{R}} f\, (\rho_\epsilon \star \phi)'\, dx = -\lim_{\epsilon\to 0} \int_{\mathbb{R}} \rho_\epsilon \star \phi\, d\mu = -\int_{\mathbb{R}} \phi\, d\mu.$$

Now let $a < b$ and $\epsilon < (b-a)/2$, define a Lipschitz function ϕ_ϵ by

$$\phi_\epsilon(x) = \begin{cases} 0 & \text{if } x \le a \\ \frac{x-a}{\epsilon} & \text{if } x \in (a, a+\epsilon) \\ 1 & \text{if } x \in [a+\epsilon, b-\epsilon] \\ \frac{b-x}{\epsilon} & \text{if } x \in (b-\epsilon, b) \\ 0 & \text{if } x \ge b. \end{cases}$$

As $\epsilon \to 0$, ϕ_ϵ increases to $\chi_{(a,b)}$. By the Monotone Convergence Theorem,

$$\begin{aligned} \mu((a,b)) &= \int_{\mathbb{R}} \chi_{(a,b)}\, d\mu \\ &= \lim_{\epsilon\to 0} \int_{\mathbb{R}} \phi_\epsilon \, d\mu \\ &= -\lim_{\epsilon\to 0} \int_{\mathbb{R}} f\, \phi_\epsilon' \, d\mu \\ &= -\lim_{\epsilon\to 0} \left(\frac{1}{\epsilon} \int_a^{a+\epsilon} f(x)\, dx - \frac{1}{\epsilon} \int_{b-\epsilon}^{b} f(x)\, dx \right) \\ &= f(b^-) - f(a^+), \end{aligned}$$

which proves (1.5.3). Since f is bounded, $|f(b^-) - f(a^+)| \le 2||f||_{L^\infty}$ for all $a < b$. Then, as $a \to -\infty$ and $b \to \infty$, we get $\mu(\mathbb{R}) \le 2||f||_{L^\infty}$. Therefore, if f is non-decreasing and bounded, μ is a finite Radon measure.

To extend this result to all function f with bounded total variation, we know by Lemma 1.5.3, f may be expressed as the difference of non-decreasing functions f_1 and f_2. That is, $f = f_1 - f_2$. In addition, Lemma 1.5.1 and 1.5.3 tells us f_1 and f_2 must also be bounded. From our prior work, there exists finite Radon measures μ_1 and μ_2 corresponding to f_1 and f_2 respectively, satisfying (1.5.2). We now define the signed measure $\mu = \mu_1 - \mu_2$. Clearly, μ is finite. It follows, if $\phi \in C_c^1(\mathbb{R})$, then

$$-\int_{\mathbb{R}} f\, \phi'\, dx = -\int_{\mathbb{R}} (f_1 - f_2)\, \phi'\, dx = \int_{\mathbb{R}} \phi\, d\mu_1 - \int_{\mathbb{R}} \phi\, d\mu_2 = \int_{\mathbb{R}} \phi\, d\mu.$$

Moreover, for all $a < b$

$$
\begin{aligned}
\mu((a,b)) &= \mu_1((a,b)) - \mu_2((a,b)) \\
&= f_1(b^-) - f_1(a^+) - f_2(b^-) + f_2(a^+) \\
&= f(b^-) - f(a^+),
\end{aligned}
$$

which concludes the proof. $\square$

Theorem 1.5.2. *Conversely, if $f : \mathbb{R} \to \mathbb{R}$ and there exists a finite signed measure μ satisfying (1.5.2), then there exists a function $g : \mathbb{R} \to \mathbb{R}$ with bounded total variation such that $f = g$ $\mathcal{L}^1$-a.e.*

Proof. By the Jordan Decomposition Theorem, there exists unique Radon measures μ^+ and μ^- such that $\mu = \mu^+ - \mu^-$. Since μ is finite, it follows $\mu^\pm$ are also finite. We now define the functions $g^\pm : \mathbb{R} \to \mathbb{R}$ by

$$g^+(x) = \mu^+((-\infty, x]) \quad \text{and} \quad g^-(x) = \mu^-((-\infty, x]).$$

Clearly, $g^\pm$ are monotone and bounded. By Lemma 1.5.2, $g^\pm$ are functions with bounded total variation. If we define $g(x) = g^+(x) - g^-(x)$, then g also has bounded total variation and g has the property

$$g(x) = \mu((-\infty, x]).$$

By Theorem 1.5.1, there exists a finite signed measure μ_g such that for all $\phi \in C_c^1(\mathbb{R})$,

$$\int_{\mathbb{R}} g\,\phi'\,dx = -\int_{\mathbb{R}} \phi\,d\mu_g,$$

and for any interval (a, b),

$$\mu_g((a,b)) = g(b^-) - g(a^+).$$

Notice that g is right continuous by continuity from above of measures, so $g(a^+) = g(a)$. If $\{x_n\}_{n=1}^\infty \subset \mathbb{R}$ is an increasing sequence such that $x_n \to b$, then by continuity from below of measures, we see that

$$g(b^-) = \lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} \mu((-\infty, x_n]) = \mu((-\infty, b)).$$

Therefore, for all $a < b$

$$\mu_g((a,b)) = g(b^-) - g(a^+) = \mu((-\infty, b)) - \mu((-\infty, a]) = \mu((a,b)).$$

We deduce that $\mu(B) = \mu_g(B)$ for all Borel sets B. This implies for all $\phi \in C_c^1(\mathbb{R})$,

$$\int_{\mathbb{R}} (f-g)\,\phi'\,dx = -\int_{\mathbb{R}} \phi\,d\mu + \int_{\mathbb{R}} \phi\,d\mu_g = 0.$$

We conclude $f = g$ $\mathcal{L}^1$-a.e. □

We have shown there is an intimate link between signed Radon measures and functions with bounded total variation. Let us now link this alternate notion of variation to our definition of $BV(\mathbb{R})$ from (1.1.1). Let $f : \mathbb{R} \to \mathbb{R}$ be a function of bounded total variation and let μ be its associated signed measure from Theorem 1.5.1. As in the prior proof, we may write $\mu = \mu^+ - \mu^-$, where $\mu^\pm$ are finite Radon measures. Recall, the total variation of a signed measure μ is defined to be $|\mu| = \mu^+ + \mu^-$. We notice if $\phi \in C_c^1(\mathbb{R})$ and $|\phi| \leq 1$, then

$$\begin{aligned}
\left|\int_{\mathbb{R}} f\phi'\,dx\right| &= \left|\int_{\mathbb{R}} \phi\,d\mu\right| \\
&= \left|\int_{\mathbb{R}} \phi\,d\mu^+ - \int_{\mathbb{R}} \phi\,d\mu^-\right| \\
&\leq \int_{\mathbb{R}} |\phi|\,d\mu^+ + \int_{\mathbb{R}} |\phi|\,d\mu^- \\
&\leq \mu^+(\mathbb{R}) + \mu^-(\mathbb{R}) \\
&= |\mu|(\mathbb{R}).
\end{aligned}$$

However, $|\mu|(\mathbb{R}) < \infty$, so $\int_{\mathbb{R}} f\phi'\,dx$ is uniformly bounded for all $\phi \in C_c^1(\mathbb{R})$ with $|\phi| \leq 1$. It follows $f \in BV(\mathbb{R})$ and $||Df||(\mathbb{R}) \leq |\mu|(\mathbb{R})$.

Conversely, if $f \in BV(\mathbb{R})$, by Theorem 1.1.2, there exists a finite Radon measure $||Df||$ and $\sigma : \mathbb{R} \to \mathbb{R}$ with $|\sigma| = 1$ $||Df||$-a.e, such that for all $\phi \in C_c^1(\mathbb{R})$,

$$\int_{\mathbb{R}} f\phi' \, dx = -\int_{\mathbb{R}} \phi \, \sigma \, d||Df||.$$

If we define $\mu^+ = ||Df|| \llcorner \sigma^+$ and $\mu^- = ||Df|| \llcorner \sigma^-$, then $\mu^\pm$ are finite Radon measures. Hence, $\mu = \mu^+ - \mu^-$ is a finite signed measure satisfying (1.5.2). Moreover,

$$||Df||(\mathbb{R}) = \int_{\mathbb{R}} |\sigma| \, d||Df|| = \int_{\mathbb{R}} \sigma^+ \, d||Df|| + \int_{\mathbb{R}} \sigma^- \, d||Df|| = \mu^+(\mathbb{R}) + \mu^-(\mathbb{R}) = |\mu|(\mathbb{R}).$$

By Theorem (1.5.2), there exists $g : \mathbb{R} \to \mathbb{R}$ with bounded total variation such that $f = g$ $\mathcal{L}^1$-a.e. We summarize the equivalency in the following theorem.

Theorem 1.5.3. *$f \in BV(\mathbb{R})$ if and only if there exists $g : \mathbb{R} \to \mathbb{R}$ such that $f = g$ $\mathcal{L}^1$-a.e. Moreover, $||Df||(\mathbb{R}) = |\mu|(\mathbb{R})$, for μ the associated signed measure of g.*

We end the study of the single variable case with a short analysis on the geometry of $BV(\mathbb{R})$. In light of Theorem 1.5.3, we may assume $f \in BV(\mathbb{R})$ and f has bounded total variation. The name bounded variation suggest functions of this class have minimal fluctuation. This can be seen in the definition of (1.5.1). Indeed, $T_f(x)$ quantifies the frequency and size of oscillations before the point x. Then as $x \to \infty$, the bounded total variation of f is in some sense the total oscillation of f. Therefore, functions in $BV(\mathbb{R})$ may characterized by those whose graph have finite oscillatory behaviour. In addition, Lemma 1.5.3 tells us these functions are also continuous $\mathcal{L}^1$-a.e.

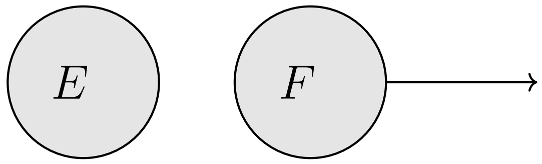

Figure 2: The BV perimeter of a set is invariant under attachment of an infinite tail but the $(n-1)$ Hausdorff measure becomes infinite.

2 Sets of finite perimeter

With the machinery we have developed thus far, we are ready to shift our attention to the special case of characteristic functions with bounded variation and begin our study of sets and perimeter. In Example 1.1.1 and 1.4.1, we saw that under sufficient regularity conditions on the boundary, the perimeter of a set E is equal to $\mathcal{H}^{n-1}(\partial E)$. Moreover, we were able to deduce

$$||\partial E|| = \mathcal{H}^{n-1} \llcorner \partial E. \tag{2.0.1}$$

In other words, all information on the perimeter of a smooth set E is embedded into its boundary. We would like to generalize these results to non-smooth sets.

As we will soon see, there are many instances in which the topological boundary is too crude for such task, as its Hausdorff measure may lead to an overestimation of the perimeter. Take for example $\Omega = \mathbb{R}^2$ and F to be the closed unit ball B_1 with an infinite tail attached. See Figure 2. Recalling the definition of the total variation in (1.1.1), we notice that functions that are equivalent up to $\mathcal{L}^2$-measure 0 output the same total variation. Therefore, F is equivalent to the unit ball in $BV(\mathbb{R}^2)$ and $||\partial F|| = ||\partial B_1||$. But notice by (2.0.1), this implies $||\partial F||$ is computed using only $\mathcal{H}^1$ on ∂B_1. Indeed, F has finite perimeter whereas $\mathcal{H}^1(\partial F) = \infty$. This suggest $||\partial F||$ omits $\partial F \setminus \partial B$ in its computation of the perimeter. Therefore, if we wish to obtain a similar expression to (2.0.1), we will need a new notion of the boundary. In this section, we introduce the reduced boundary and the measure-theoretic boundary. We will show these new boundaries are the analogs to the topological boundaries for smooth sets, which in turn will not only allow us to derive an expression similar to (2.0.1), but also the generalized Gauss-Green Theorem.

2.1 BV Boundaries

2.1.1 The Reduced Boundary

Definition 2.1.1. *Let $E \in BV_{loc}(\mathbb{R}^n)$ and ν_E be the associated function to $||\partial E||$ of Theorem 1.1.2. We define the reduced boundary $\partial^* E$ to be the set of $x \in \mathbb{R}^n$ such that*

1. $||\partial E||(B_r(x)) > 0$ *for all* $r > 0$,
2. $\lim_{r\to 0} \frac{1}{||\partial E||(B_r(x))} \int_{B_r(x)} \nu_E \, d||\partial E|| = \nu_E(x)$,
3. $|\nu_E(x)| = 1$.

We call $\nu_E(x)$ the measure-theoretic normal on x.

Before proceeding with the ongoing theory, there are some simple geometric implications that we may deduce from the given definition. We notice Property (1) of Definition 2.1.1 accounts for the addition of any $\mathcal{L}^n$-measure 0 appendages and removes them from the reduced boundary. To see this, let us look at the previous example. Let $F = \overline{B_1} \cup T$, where $T = \{(x_1, 0) \,|\, x_1 \geq 1\}$ as in Figure 2. We observe for all $x \in T \setminus \overline{B_1}$, there exists $r > 0$ such that $B_r(x) \cap \overline{B_1} = \emptyset$. Now if $\phi \in C_c^1(B_r(x); \mathbb{R}^n)$, then

$$\int_F \operatorname{div}(\phi)\, dx = \int_T \operatorname{div}(\phi)\, dx = 0.$$

Thus, $||\partial E||(B_r(x)) = 0$ and $x \notin \partial^* E$. This suggests that sets that are equivalent up to $\mathcal{L}^n$-measure 0 should have the same reduced boundary, and under the pretext of $\mathcal{H}^{n-1}(\partial^* E)$, the same perimeter.

We claim the perimeter of a set is computed using only the reduced boundary. Let us recall Property (1) of Theorem 1.1.2; for $||\partial E||$-a.e $x \in \mathbb{R}^n$, $|\nu_E|(x) = 1$. We also know by the Lebesgue Differentiation Theorem, for $||\partial E||$-a.e $x \in \mathbb{R}^n$,

$$\lim_{r\to 0} \frac{1}{||\partial E||(B_r(x))} \int_{B_r(x)} \nu_E \, d||\partial E|| = \nu_E(x). \tag{2.1.1}$$

Therefore, Properties (2) and (3) hold $||\partial E||$-a.e. But for equality in (2.1.1), it must be that x satisfies Property (1). This implies the perimeter is at most restricted to $\partial^* E$ as

$$||\partial E||(\mathbb{R}^n \setminus \partial^* E) = 0. \tag{2.1.2}$$

We claim $||\partial E|| = \mathcal{H}^{n-1} \llcorner \partial^* E$. Before we can prove this claim, we will need some additional results pertaining to $\partial^* E$. The first are several useful local inequalities of $\partial^* E$.

Lemma 2.1.1. *[[3], Lemma 5.3] If $E \in BV_{loc}(\mathbb{R}^n)$, there exists constants $A_1, \dots, A_5 > 0$, such that for all $x \in \partial^* E$,*

1. $\liminf_{r \to 0} \frac{\mathcal{L}^n(B_r(x) \cap E)}{r^n} \geq A_1$,
2. $\liminf_{r \to 0} \frac{\mathcal{L}^n(B_r(x) \setminus E)}{r^n} \geq A_2$,
3. $\liminf_{r \to 0} \frac{||\partial E||(B_r(x))}{r^{n-1}} \geq A_3$,
4. $\limsup_{r \to 0} \frac{||\partial E||(B_r(x))}{r^{n-1}} \leq A_4$,
5. $\limsup_{r \to 0} \frac{||\partial (E \cap B_r(x))||(\mathbb{R}^n)}{r^{n-1}} \leq A_5$.

Note, (1) and (2) tells us $\partial^* E \subseteq \partial E$. This makes sense as $\partial^* E$ is a more restrictive notion of the boundary. In contrast, (3), (4), and (5) suggest $||\partial E||$ is comparable to $\mathcal{H}^{n-1}$ locally.

Our primary method of studying the reduced boundary will be through blow-ups and hyperplanes.

Definition 2.1.2. *Let $E \in BV_{loc}(\mathbb{R}^n)$. For each $x \in \partial^* E$, define the hyperplane*

$$H(x) = \{y \in \mathbb{R}^n \,|\, \nu_E(x) \cdot (y - x) = 0\},$$

and the half-spaces

$$H^+(x) = \{y \in \mathbb{R}^n \,|\, \nu_E(x) \cdot (y - x) \geq 0\},$$

$$H^-(x) = \{y \in \mathbb{R}^n \,|\, \nu_E(x) \cdot (y - x) \leq 0\}.$$

For $r > 0$, define the blow-up of E at x by a factor of $\frac{1}{r}$ to be

$$E_r = \{y \in \mathbb{R}^n;\, r(y - x) + x \in E\}.$$

Theorem 2.1.1. *[[3], Theorem 5.13] If $E \in BV_{loc}(\mathbb{R}^n)$ and $x \in \partial^* E$, then $\chi_{E_r} \xrightarrow{r \to 0} \chi_{H^-(x)}$ in $L^1_{loc}(\mathbb{R}^n)$.*

Proof. If $x \in \partial^* E$, then upon a change of coordinates, we may without loss of generality assume $x = 0$ and $\nu_E(x) = (0, \ldots, 1) = e_n$. Then,

$$H(0) = \{y \in \mathbb{R}^n \,|\, y_n = 0\},$$
$$H^+(0) = \{y \in \mathbb{R}^n \,|y_n \geq 0\},$$
$$H^-(0) = \{y \in \mathbb{R}^n \,|y_n \leq 0\},$$

and for $r > 0$,

$$E_r = \{y \in \mathbb{R}^n; ry \in E\}.$$

The proof will be broken down into 3 main steps. Step (1) will be to show χ_{E_r} converges to a characteristic function in $L^1_{loc}(\mathbb{R}^n)$. Step (2) will be to show that this characteristic function is a half-space. And finally step 3, we show this half-space is $H^-(0)$.

Fix an $L > 0$ and define the following sets for $r > 0$,

$$D_r = E_r \cap B_L.$$

We will also define a function $g_r : \mathbb{R}^n \to \mathbb{R}^n$ by

$$g_r(z) = \frac{z}{r}.$$

If $\phi \in C^1_c(\mathbb{R}^n; \mathbb{R}^n)$ with $||\phi||_\infty \leq 1$, by applying the change of variables $x = z/r$ and noting

$$\operatorname{div}(\phi \circ g_r) = \frac{1}{r}\operatorname{div}(\phi) \circ g_r,$$

we see that

$$\begin{aligned}
\int_{D_r} \operatorname{div}(\phi)\, dx &= \frac{1}{r^n} \int_{E \cap B_{rL}} \operatorname{div}(\phi) \circ g_r \, dz && (2.1.3) \\
&= \frac{1}{r^{n-1}} \int_{E \cap B_{rL}} \operatorname{div}(\phi \circ g_r)\, dz \\
&= \frac{1}{r^{n-1}} \int_{E \cap B_{rL}} \nu_{E \cap B_{rL}} \cdot (\phi \circ g_r)\, d||\partial(E \cap B_{rL})||. \\
&\leq \frac{||\partial(E \cap B_{rL})||(\mathbb{R}^n)}{r^{n-1}} && (2.1.4)
\end{aligned}$$

By Property (5) of Lemma 2.1.1, there exists $A_5 > 0$ and $0 < r_0 \leq 1$ such that if $0 < r \leq r_0$, then (2.1.4) is bounded by A_5. Hence,

$$\int_{D_r} \operatorname{div}(\phi)\, dx \leq \frac{||\partial(E \cap B_{rL})||(\mathbb{R}^n)}{r^{n-1}} \leq A_5.$$

It follows for all $r \leq r_0$,

$$||\partial D_r||(\mathbb{R}^n) \leq A_5,$$

hence $D_r \in BV(\mathbb{R}^n)$. Clearly,

$$||\chi_{D_r}||_{L^1(\mathbb{R}^n)} = \mathcal{L}^n(D_r) \leq \mathcal{L}^n(B_L).$$

There exists $C > 0$ such that for all $r \leq r_0$,

$$||\chi_{D_r}||_{BV(\mathbb{R}^n)} = ||\partial D_r||(\mathbb{R}^n) + ||\chi_{D_r}||_{L^1(\mathbb{R}^n)} \leq C.$$

Thus, we may bound $|| \cdot ||_{BV(B_L)}$ of χ_{E_r} for all $L > 0$. By Corollary 1.2.2, given a sequence $\{r_k\}_{k=1}^{\infty} \subset \mathbb{R}$ such that $r_k \to 0$, there exists $f \in BV_{loc}(\mathbb{R}^n)$ and a subsequence $\{s_j\}_{j=1}^{\infty} \subseteq \{r_k\}_{k=1}^{\infty}$ such that $\chi_{E_{s_j}} \xrightarrow{j \to \infty} f$ in $L^1_{loc}(\mathbb{R}^n)$. In addition, by extracting a further subsequence, we may assume $\chi_{E_{s_j}}$ converges to f point-wise $\mathcal{L}^n$-a.e. Now note that $\chi_{E_{s_j}}$ is either 0 or 1, so if f is the point-wise limit, f must also only take on the values of either 0 or 1 $\mathcal{L}^n$-a.e. Therefore, $f = \chi_F$ $\mathcal{L}^n$-a.e for some $F \in BV_{loc}(\mathbb{R}^n)$. It remains to show that F is a half-space.

For notational convenience, we will write $E_j = E_{s_j}$ and $\nu_j = \nu_{E_{s_j}}$. Let $r > 0$, for all $\phi \in C_c^1(B_r; \mathbb{R}^n)$, we see by applying the change of variables $z = x/s_j$ as in (2.1.3), along with Theorem 1.1.2,

$$\int_{\mathbb{R}^n} \phi \cdot \nu_j \, ||\partial E_j|| = \int_{E_j} \operatorname{div}(\phi) \, dx = \frac{1}{s_j^{n-1}} \int_E \operatorname{div}(\phi \cdot g_{s_j}) \, dx = \int_{\mathbb{R}^n} (\phi \circ g_{s_j}) \cdot \nu_E \, d||\partial E||. \tag{2.1.5}$$

We note that if $\phi \in C_c^1(B_r; \mathbb{R}^n)$, then $\phi \circ g_{s_j} \in C_c^1(B_{rs_j}; \mathbb{R}^n)$. So, upon taking the supremum across both sides we obtain

$$||\partial E_j||(B_r) = \frac{||\partial E||(B_{rs_j})}{s_j^{n-1}}. \tag{2.1.6}$$

If $\{\xi_m\}_{m=1}^{\infty} \subseteq C_c^1(B_r)$ such ξ_m forms an increasing sequence and $\xi_m \xrightarrow{m\to\infty} \chi_{B_r}$ point-wise, then plugging $\phi_m = e_n \xi_m$ into (2.1.5) yields

$$\begin{aligned} \int_{\mathbb{R}^n} (\xi_m)(e_n \cdot \nu_j) \, d||\partial E_j|| &= \int_{\mathbb{R}^n} \phi_m \cdot \nu_j \, d||\partial E_j|| \\ &= \frac{1}{s_j^{n-1}} \int_{\mathbb{R}^n} (\phi_m \circ g_{s_j}) \cdot \nu_E \, d||\partial E|| \\ &= \frac{1}{s_j^{n-1}} \int_{\mathbb{R}^n} (\xi_m \circ g_{s_j})(\nu_E \cdot e_n) \, d||\partial E||. \end{aligned}$$

By the Monotone Convergence Theorem, as $m \to \infty$,

$$\int_{B_r} \nu_j \cdot e_n \, d||\partial E_j|| = \frac{1}{s_j^{n-1}} \int_{B_{rs_j}} \nu_E \cdot e_n \, d||\partial E||. \tag{2.1.7}$$

Since $0 \in \partial^* E$ and $\nu_E(0) = e_n$, by Property (2) of the reduced boundary it follows

$$\lim_{r\to 0} \frac{1}{||\partial E||(B_r)} \int_{B_r} |\nu_E - e_n|^2 \, d||\partial E|| = \lim_{r\to 0} \frac{2}{||\partial E||(B_r)} \int_{B_r} 1 - \nu_E \cdot e_n \, d||\partial E|| = 0 \tag{2.1.8}$$

Now combining (2.1.6) and (2.1.7) we get

$$\begin{aligned}\lim_{j\to\infty}\frac{1}{||\partial E_j||(B_r)}\int_{B_r}|e_n-\nu_j|^2\,d||\partial E_j|| &= \lim_{j\to\infty}\frac{2s_j^{n-1}}{||\partial E||(B_{rs_j})}\int_{B_r}|e_n-\nu_j|^2\,d||\partial E_j|| \\ &= \lim_{j\to\infty}\frac{2s_j^{n-1}}{||\partial E||(B_{rs_j})}\int_{B_r}1-\nu_j\cdot e_n\,d||\partial E_j|| \\ &= \lim_{j\to\infty}\frac{2}{||\partial E||(B_{rs_j})}\int_{B_{rs_j}}1-\nu_E\cdot e_n\,d||\partial E|| \\ &= 0.\end{aligned}$$

Therefore, for all $r>0$

$$\lim_{j\to\infty}\int_{B_r}|e_n-\nu_j|^2\,d||\partial E_j|| = 2\lim_{j\to\infty}\int_{B_r}1-\nu_j\cdot e_n\,d||\partial E_j|| = 0. \tag{2.1.9}$$

Since, $\chi_{E_j}\to\chi_F$ in $L^1_{loc}(\mathbb{R}^n)$, we have for all $\phi\in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$,

$$\lim_{j\to\infty}\int_{\mathbb{R}^n}\phi\cdot\nu_j\,d||\partial E_j|| = \lim_{j\to\infty}\int_{E_j}\operatorname{div}(\phi)\,dx = \int_F\operatorname{div}(\phi)\,dx = \int_{\mathbb{R}^n}\phi\cdot\nu_F\,d||\partial F|| \tag{2.1.10}$$

Now if $\xi\in C^1_c(\mathbb{R}^n)$ and $\phi=e_n\xi$, then (2.1.10) implies

$$\int_{\mathbb{R}^n}\xi e_n\cdot\nu_F\,d||\partial F|| = \lim_{j\to\infty}\int_{\mathbb{R}^n}\xi e_n\cdot\nu_j\,d||\partial E_j||. \tag{2.1.11}$$

If $r>0$ such that $supp(\xi)\subset B_r$, then (2.1.9) implies

$$\lim_{j\to\infty}\int_{B_r}\xi-\xi e_n\cdot\nu_j\,d||\partial E_j|| = 0.$$

So (2.1.11) becomes

$$\int_{\mathbb{R}^n}\xi e_n\cdot\nu_F\,d||\partial F|| = \lim_{j\to\infty}\int_{\mathbb{R}^n}\xi e_n\cdot\nu_j\,d||\partial E_j|| = \lim_{j\to\infty}\int_{\mathbb{R}^n}\xi\,d||\partial E_j||. \tag{2.1.12}$$

Let $r>0$, if we take $\xi\in C^1_c(\mathbb{R}^n)$ such that $\xi\geq 0$, $\xi=1$ on B_r and $\xi=0$ on $\mathbb{R}^n\setminus B_{r+h}$, for some $h > 0$, then by Theorem 1.2.1 and (2.1.12),

$$||\partial F||(B_r) \leq \liminf_{j\to\infty} ||\partial E_j||(B_r) \leq \liminf_{j\to\infty} \int_{B_{r+h}} \xi \, d||\partial E_j|| = \int_{B_{r+h}} \xi e_n \cdot \nu_F \, d||\partial F||$$

As $h \to 0$, we get

$$||\partial F||(B_r) \leq \int_{B_r} e_n \cdot \nu_F \, d||\partial F||. \tag{2.1.13}$$

By the Cauchy-Schwarz inequality, $e_n \cdot \nu_f \leq 1$. So, (2.1.13) is bounded above and below by $||\partial F||(B_r)$ for all $r > 0$. But, equality of the Cauchy-Schwarz inequality holds if and only if $\nu_F = e_n$ $||\partial F||$-a.e. Therefore, $\nu_F = e_n$ $||\partial F||$-a.e.

It remains to show that F is $H^-(0)$. Given that $\nu_F = e_n$, for all $\phi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$,

$$\int_F \text{div}(\phi)\, dx = \int_{\mathbb{R}^n} \phi \cdot e_n \, d||\partial F||. \tag{2.1.14}$$

Let $\epsilon > 0$, then using the usual mollifier ρ_ϵ, we define $f_\epsilon = \rho_\epsilon \star \chi_F$. For all $\phi \in C_c^1(\mathbb{R}^n)$,

$$\begin{aligned}
\int_{\mathbb{R}^n} f_\epsilon \text{div}(\phi)\, dx &= \int_F \text{div}(\rho_\epsilon \star \phi)\, dx \\
&= \int_{\mathbb{R}^n} (\rho_\epsilon \star \phi) \cdot e_n \, d||\partial F|| \\
&= \int_{\mathbb{R}^n} \rho_\epsilon \star (\phi \cdot e_n)\, d||\partial F||.
\end{aligned}$$

Since $f_\epsilon \in C^\infty(\mathbb{R}^n) \cap W^{1,1}_{loc}(\mathbb{R}^n)$, we also have

$$\int_{\mathbb{R}^n} f_\epsilon \text{div}(\phi)\, dx = -\int_{\mathbb{R}^n} \nabla f_\epsilon \cdot \phi \, dx.$$

Therefore,

$$-\int_{\mathbb{R}^n} \nabla f_\epsilon \cdot \phi \, dx = \int_{\mathbb{R}^n} \rho_\epsilon \star (\phi \cdot e_n)\, d||\partial F||. \tag{2.1.15}$$

Observe for all $g \in C_c^1(\mathbb{R}^n)$, if we define $\phi = e_i g$ for $i \in \{1, \ldots, n-1\}$, then (2.1.15) implies

$$\int_{\mathbb{R}^n} \frac{\partial f_\epsilon}{\partial z_i} g \, dx = 0.$$

Thus, for all $i \in \{1, \ldots, n-1\}$,

$$\frac{\partial f_\epsilon}{\partial z_i} = 0. \tag{2.1.16}$$

Now suppose there exists $x \in \mathbb{R}^n$ such that $\frac{\partial f_\epsilon}{\partial z_n}(x) > 0$, then by continuity there exists a neighbourhood U of x such that $\frac{\partial f_\epsilon}{\partial z_n}\Big|_U > 0$. We may choose a positive $g \in C^1_c(\mathbb{R}^n)$ such that $supp(g) \subset U$. Upon setting $\phi = e_n g$, we see that

$$-\int_U \frac{\partial f_\epsilon}{\partial z_n} g \, dx = \int_{\mathbb{R}^n} \rho_\epsilon \star g \, d||\partial F|| > 0.$$

But this contradicts the negative integrand of $-\frac{\partial f_\epsilon}{\partial z_n} g|_U$. Therefore,

$$\frac{\partial f_\epsilon}{\partial z_n} \leq 0. \tag{2.1.17}$$

We will use (2.1.16) and (2.1.17) to recover the precise set F. Since $f_\epsilon \xrightarrow{\epsilon \to 0} \chi_F$ in $L^1_{loc}(\mathbb{R}^n)$, we may extract a sequence $\{\epsilon_j\}_{j=1}^\infty$ such that $f_{\epsilon_j} \xrightarrow{\epsilon_j \to 0} \chi_F$ point-wise $\mathcal{L}^n$-a.e. Let $x \in F$ such that the limit converges. By (2.1.16), we see that for all $i \in \{1, \ldots, n-1\}$ and $r \in \mathbb{R}$, $f_{\epsilon_j}(x + re_i) = f_{\epsilon_j}(x)$. By the point-wise convergence of the sequence; $f_{\epsilon_j}(x) \xrightarrow{\epsilon_j \to 0} 1$ and $f_{\epsilon_j}(x+re_i) \xrightarrow{\epsilon_j \to 0} \chi_F(x+re_i)$, we deduce $\chi_F(x+re_i) = 1$ for all $r \in \mathbb{R}$ and $i \in \{1, \ldots, n-1\}$. That is, up to a $\mathcal{L}^n$-null set, F is invariant under translation along the i-th coordinate axis for all $i \in \{1, \ldots, n-1\}$.

On the other hand, if $x \in F$ such that $\lim_{\epsilon_j \to 0} f_{\epsilon_j}(x) = \chi_F(x)$, (2.1.17) tells us for all $r \leq 0$, $f_{\epsilon_j}(x) \leq f_{\epsilon_j}(x + re_n)$. Upon taking the limit we see that

$$1 = \chi_F(x) = \lim_{\epsilon_j \to 0} f_{\epsilon_j}(x) \leq \lim_{\epsilon_j \to 0} f_{\epsilon_j}(x + re_n) = \chi_F(x + re_n) \leq 1.$$

Thus, $\chi_F(x + re_n) = 1$ for all $r \leq 0$. That is, F is invariant under downward translations along the n-th axis. We conclude for some $\gamma \in \mathbb{R}$,

$$F = \{x \in \mathbb{R}^n | \, x_n \leq \gamma\}.$$

It only remains to show $\gamma = 0$. Suppose for contradiction $\gamma > 0$, then we see that

$B_\gamma \cap F = B_\gamma$. Let $\omega_n = \mathcal{L}^n(B_1)$, then given $\chi_{E_j} \to \chi_F$ in $L^1_{loc}(\mathbb{R}^n)$, we have

$$\begin{aligned}\omega_n \gamma^n = \mathcal{L}^n(B_\gamma \cap F) &= \lim_{j\to\infty} \mathcal{L}^n(B_\gamma \cap E_j) \\ &= \lim_{j\to\infty} \frac{\mathcal{L}^n(B_{\gamma s_j} \cap E)}{s_j^n}.\end{aligned} \tag{2.1.18}$$

Implying

$$\begin{aligned}\lim_{j\to\infty} \frac{\mathcal{L}^n(B_{\gamma s_j} \setminus E)}{(\gamma s_j)^n} &= \lim_{j\to\infty} \frac{\mathcal{L}^n(B_{\gamma s_j}) - \mathcal{L}^n(B_{\gamma s_j} \cap E)}{(\gamma s_j)^n} \\ &= \omega_n - \lim_{j\to\infty} \frac{\mathcal{L}^n(B_{\gamma s_j} \cap E)}{(\gamma s_j)^n} \\ &= 0,\end{aligned}$$

But this contradicts Property (2) of Theorem 2.1.1. It must be that $\gamma \leq 0$. Suppose $\gamma < 0$, then $B_{|\gamma|} \cap F = \emptyset$. Thus,

$$\begin{aligned}0 = \mathcal{L}^n(B_{|\gamma|} \cap F) &= \lim_{j\to\infty} \mathcal{L}^n(B_{|\gamma|} \cap E_j) \\ &= \lim_{j\to\infty} \frac{\mathcal{L}^n(B_{|\gamma| s_j} \cap E)}{s_j^n} \\ &= \lim_{j\to\infty} \frac{\mathcal{L}^n(B_{|\gamma| s_j} \cap E)}{s_j^n}.\end{aligned}$$

But this contradicts Property (1) of Theorem 2.1.1. Therefore, it must be that $\gamma = 0$. This concludes the proof. $\square$

If we recall for $t \in [0,1]$, the set of points with density t of E is defined by

$$E^{(t)} = \left\{ x \in \mathbb{R}^n \,\middle|\, \lim_{r\to 0} \frac{\mathcal{L}^n(E \cap B_r(x))}{\omega_n r^n} = t \right\},$$

where $\omega_n = \mathcal{L}^n(B_1)$. Utilizing Theorem 2.1.1, we claim $\partial^* E \subseteq E^{(1/2)}$. To see this, let $x \in \partial^* E$, upon translation of the set E, we may assume without loss of generality that

$x = 0$. By the change of variables $z = y/r$ and Theorem 2.1.1, we see that

$$\begin{aligned}
\lim_{r\to 0} \frac{\mathcal{L}^n(E\cap B_r)}{\omega_n r^n} &= \lim_{r\to 0} \frac{1}{\omega_n r^n} \int_{B_r} \chi_E(y)\,dy \\
&= \lim_{r\to 0} \frac{1}{\omega_n r^n} \int_{B_1} r^n \chi_E(rz)\,dz \\
&= \lim_{r\to 0} \frac{1}{\omega_n} \int_{B_1} \chi_{E_r}(z)\,dz \\
&= \frac{1}{\omega_n} \mathcal{L}^n(H^-(0)\cap B_1) \\
&= \frac{1}{2}.
\end{aligned}$$

Thus, $0 \in E^{(1/2)}$, completing the proof of our claim. Moreover, Theorem 2.1.1 tell us that points on the reduced boundary are locally flat and their outward unit normals corresponds to their measure-theoretic normal ν_E. In light of this local flatness of the reduced boundary, we deduce the following properties:

Theorem 2.1.2. *[[3], Theorem 5.14] Let $E \in BV_{loc}(\mathbb{R}^n)$. If $x \in \partial^* E$, then*

1. $\lim_{r\to 0} \frac{\mathcal{L}^n(B_r(x))\cap E\cap H^+(x))}{r^n} = 0$,
2. $\lim_{r\to 0} \frac{\mathcal{L}^n((B_r(x)\setminus E)\cap H^+(x))}{r^n} = 0$,
3. $\lim_{r\to 0} \frac{||\partial E||(B_r(x))}{\omega_{n-1} r^{n-1}} = 1$.

Notice that Property (3) of Theorem 2.1.2 is a refinement of properties (3) and (4) of Lemma 2.1.1. We shall use this local comparison between the two measures to derive an absolute-continuity like result between $\mathcal{H}^{n-1}$ and $||\partial E||$ on $\partial^* E$.

Lemma 2.1.2. *[[3], Lemma 5.4] Let $E \in BV_{loc}(\mathbb{R}^n)$. There exists a constant $C > 0$ such that if $A \subseteq \partial^* E$, then*

$$\mathcal{H}^{n-1}(A) \le C||\partial E||(A).$$

Proof. Let ϵ, $\delta > 0$. Since $||\partial E||$ is a Radon measure, there exists an open set U containing A such that

$$||\partial E||(U) \le ||\partial E||(A) + \epsilon.$$

By Lemma 2.1.1, there exists $A_3 > 0$ such that for all $x \in \partial^* E$,

$$\liminf_{r \to 0} \frac{||\partial E||(B_r(x))}{r^{n-1}} > A_3.$$

Since every point in A is an interior point of U, we may define the following open cover of A by

$$\mathcal{F} = \left\{ B_r(x) \,\middle|\, x \in A, B_r(x) \subseteq U,\, r < \frac{\delta}{10},\; \frac{||\partial E||(B_r(x))}{r^{n-1}} \geq A_3 \right\},$$

By the Vitali Covering Lemma, there exists a countable collection of disjoint balls $\{B_{r_i}(x_i)\}_{i=1}^{\infty} \subseteq \mathcal{F}$ such that

$$A \subseteq \bigcup_{i=1}^{\infty} B_{5r_i}(x_i).$$

Notice by construction, $r_i^{n-1} \leq \frac{||\partial E||(B_r(x_i))}{A_3}$ for all $i \in \mathbb{N}$. It follows that

$$\begin{aligned}
\mathcal{H}^{n-1}_{\delta}(A) &\leq \sum_{i=1}^{\infty} \alpha(n-1) \left(\frac{diam(B_{5r_i}(x_i))}{2} \right)^{n-1} \\
&= \sum_{i=1}^{\infty} \alpha(n-1)(5r_i)^{n-1} \\
&\leq \frac{\alpha(n-1)5^{n-1}}{A_3} \sum_{i=1}^{\infty} ||\partial E||(B_{r_i}(x_i)) \\
&\leq \frac{\alpha(n-1)5^{n-1}}{A_3} ||\partial E||(U) \\
&\leq \frac{\alpha(n-1)5^{n-1}}{A_3} (||\partial E||(A) + \epsilon).
\end{aligned}$$

Set $C = \frac{\alpha(n-1)5^{n-1}}{A_3}$, then as $\epsilon \to 0$ we obtain

$$\mathcal{H}^{n-1}_{\delta}(A) \leq C||\partial E||(A).$$

Sending δ to 0 gives $\mathcal{H}^{n-1}(A) \leq C||\partial E||(A)$ as desired. $\square$

We are ready to prove De Giorgi's regularity result on $\partial^* E$. We claim $\partial^* E$ is C^1 in a measure-theoretic sense.

Theorem 2.1.3. *(Structure theorem for sets of finite perimeter.)[[3], Theorem 5.15] If $E \in BV_{loc}(\mathbb{R}^n)$, then*

1. $\partial^* E = \bigcup_{k=1}^{\infty} K_k \cup N$, *where* N *is a* $||\partial E||$*-null set and* K_k *are compact subsets of* C^1 *hypersurfaces* S_k*,*

2. $\nu_E|_{S_k}$ *is normal for each* S_k*,*

3. $||\partial E|| = \mathcal{H}^{n-1} \llcorner \partial^* E$.

Proof. Let $r > 0$, define $\psi_r, \phi_r : \partial^* E \to \mathbb{R}$ by

$$\psi_r(x) = \frac{\mathcal{L}^n(B_r(x) \cap E \cap H^+(x))}{r^n}$$

and

$$\phi_r(x) = \frac{\mathcal{L}^n((B_r(x) \setminus E) \cap H^+(x))}{r^n}.$$

By Theorem 2.1.2, for each $x \in \partial^* E$, $\phi_r(x)$, $\psi_r(x) \to 0$ point-wise as $r \to 0$. By Egoroff's Theorem, there exists a sequence of disjoint $||\partial E||$-measurable sets $\{F_i\}_{i=1}^{\infty} \subseteq \partial^* E$ such that

$$||\partial E||(\partial^* E \setminus \cup_{i=1}^{\infty} F_i) = 0,$$

and ψ_r, ϕ_r converge uniformly to 0 on each F_i. Next, we apply Lusin's Theorem to each i for a collection of disjoint compact sets $\{G_j^i\}_{j=1}^{\infty} \subset F_i$ such that $\nu_E|_{G_j^i}$ is continuous, and

$$||\partial E||(F_i \setminus \cup_{j=1}^{\infty} G_j^i) = 0.$$

Up to reindexing, we may denote $\{G_j^i\}_{i,j=1}^{\infty}$ as $\{K_k\}_{k=1}^{\infty}$. Let $N = \partial^* E \setminus \cup_{k=1}^{\infty} K_k$, then

$$||\partial E||(N) = ||\partial E||(\partial^* E \setminus \cup_{i=1}^{\infty} F_i) + ||\partial E||(\cup_{i=1}^{\infty} F_i \setminus \cup_{k=1}^{\infty} K_k) = 0.$$

Clearly,

$$\partial^* E = \cup_{k=1}^{\infty} K_k \cup N,$$

and by construction, ϕ_r and ψ_r converge uniformly to 0 on each K_k, and $\nu_E|_{K_k}$ is continuous. The objective for the remainder of the proof is to apply the Whitney's Extension

Theorem to show the existence of hypersurfaces S_k containing K_k. We define for $\delta > 0$,

$$\rho_k(\delta) = \sup\left\{ \frac{\nu_E(x)\cdot(y-x)|}{|y-x|} \,\middle|\, 0 < |x-y| \le \delta,\ x, y \in K_k \right\}.$$

We want to show $\rho_k(\delta) \xrightarrow{\delta\to 0} 0$. Fix $k \in \mathbb{N}$, and let $0 < \epsilon < 1$. Denote $\omega_n = \mathcal{L}^1(B_1)$. By the uniform convergence of ψ_r and ϕ_r, if r is sufficiently small, for all $x \in K_k$,

$$\psi_r(x) = \frac{\mathcal{L}^n(B_r(x)\cap E \cap H^+(x))}{r^n} < \frac{\epsilon^n \omega_n}{2^{n+2}}, \tag{2.1.19}$$

and

$$\phi_r(x) = \frac{\mathcal{L}^n((B_r(x)\setminus E)\cap H^-(x))}{r^n} < \frac{\epsilon^n \omega_n}{2^{n+2}}. \tag{2.1.20}$$

Equation (2.1.20) implies

$$\begin{aligned}
\frac{\mathcal{L}^n(B_r(x)\cap E\cap H^-(x))}{r^n} &= \frac{\mathcal{L}^n(B_r(x)\cap H^-(x)) - \mathcal{L}^n((B_r(x)\setminus E)\cap H^-(x))}{r^n} \\
&\ge \frac{\omega_n}{2} - \frac{\mathcal{L}^n((B(x,\delta)\setminus E)\cap H^-(x))}{\delta^n} \\
&\ge \frac{\omega_n}{2} - \frac{\epsilon^n\omega_n}{2^{n+2}}.
\end{aligned} \tag{2.1.21}$$

In particular, there exists δ_0 such that for all $r < 2\delta_0$, for all $x \in K_K$, both (2.1.19) and (2.1.21) hold.

We claim that for $r < \delta_0$, $|\nu_E(x)\cdot(y-x)| < \epsilon|y-x|$ for all $x, y \in K_k$. Suppose for the purpose of contradiction that there exists $x, y \in K_k$ such that $\nu_E(x)\cdot(y-x) \ge \epsilon|y-x|$, and $0 < |x-y| \le r$. Let $r_0 = |x-y|$ and $z \in B_{\epsilon r_0}(y)$. There exists $w \in \mathbb{R}^n$ such that $z = y + w$ and $|w| < \epsilon r_0$. By the Cauchy-Schwarz Inequality, $|\nu_E(x)\cdot w| \le |w|$. Therefore,

$$\nu_E(x)\cdot(z-x) = \nu_E(x)\cdot(y-x) + \nu_E(x)\cdot w \ge \epsilon r_0 - |w| \ge 0.$$

Since this holds for all $z \in B_{\epsilon r_0}(y)$, it follows $B_{\epsilon r_0}(y) \subseteq H^+(x)$. In addition,

$$|x-z| \le |x-y| + |y-z| < 2r_0,$$

so $z \in B_{2r_0}(x)$. In summary, $B_{\epsilon r_0}(y) \subseteq H^+(x) \cap B_{2r_0}(x)$. Now observe by (2.1.19),

$$\begin{aligned}\mathcal{L}^n(B_{2r_0}(x) \cap E \cap H^+(x)) &< \frac{\epsilon^n \omega_n}{2^{n+2}}(2r_0)^n \\ &= \frac{\epsilon^m \omega_n}{4} r_0^n. \end{aligned} \tag{2.1.22}$$

Likewise, (2.1.21) implies

$$\begin{aligned}\mathcal{L}^n(E \cap B_{\epsilon r_0}(y)) &\geq \mathcal{L}^n(B_{\epsilon r_0}(y) \cap E \cap H^-(y)) \\ &\geq \frac{\epsilon^n \omega_n r_0^n}{2}\left(1 - \frac{\epsilon^n}{2^{n+1}}\right) \\ &> \frac{\epsilon^n \omega_n r_0^n}{4}. \end{aligned} \tag{2.1.23}$$

Putting (2.1.22) and (2.1.23) together, we get

$$\mathcal{L}^n(E \cap B_{\epsilon r_0}(y)) > \mathcal{L}^n(E \cap B_{2r_0}(x)) \cap H^+(x)).$$

However, we first showed $B_{\epsilon r_0}(y) \subseteq H^+(x) \cap B_{2r_0}(x)$, so this contradicts the monotonicity of $\mathcal{L}^n$. The same argument shows there does not exist $x, y \in K_k$ such that $|x-y| < r$, and $\nu_E(x) \cdot (y-x) < -\epsilon|x-y|$. Therefore, for all $k \in \mathbb{N}$, there exists δ_0^k such that if $r < 2\delta_0^k$, then $\rho_k(r) < \epsilon$. By the Whitney's Extension Theorem, there exists $g_k \in C^1(\mathbb{R}^n)$ for each $k \in \mathbb{N}$, such that

$$g_k = 0 \qquad \text{and} \qquad \nabla g_k = \nu_E \text{ on } K_k. \tag{2.1.24}$$

Let

$$S_k = \left\{ x \in \mathbb{R}^n \mid g_k(x) = 0,\, |\nabla g_k(x)| > \frac{1}{2} \right\}.$$

Since $|\nu_E(x)| = 1$ for all $x \in K_k$, (2.1.24) tells us K_k is contained in S_k. By the Implicit Function Theorem, S_k is a C^1 hyper-surface.

We have thus proved Properties (1) and (2), it remains to show Property (3). Let $A \subseteq \mathbb{R}^n$. By (2.1.2), we may assume $A \subseteq \partial^* E$. We may further assume that $A \subseteq \cup_{k=1}^{\infty} K_k$ as $||\partial E||(A \cap N) = 0$. Let $A_k = A \cap K_k$, then $A = \cup_{k=1}^{\infty} A_k$. We have $A_k \subseteq S_k$ for all

$k \in \mathbb{N}$. Define

$$\nu_k = \mathcal{H}^{n-1} \llcorner S_k.$$

By Lemma 2.1.2, $\nu_k << ||\partial E||$. Since S_k is C^1, for all $x \in S_k$, we have

$$\lim_{r \to 0} \frac{\nu(B_r(x))}{\omega_{n-1} r^{n-1}} = 1. \tag{2.1.25}$$

By Property (3) of Theorem 2.1.2, for all $x \in A_k$,

$$\frac{||\partial E||(B_r(x))}{\omega_{n-1} r^{n-1}} = 1.$$

Combining this with (2.1.25), we get

$$D_{||\partial E||}\nu := \lim_{r \to 0} \frac{\nu(B_r(x))}{||\partial E||(B_r(x))} = 1.$$

By the Radon-Nikodym Theorem,

$$\begin{aligned} ||\partial E||(A_k) &= \int_{A_k} 1 \, d||\partial E|| \\ &= \int_{A_k} D_{||\partial E||}\nu \, d||\partial E|| \\ &= \nu(A_k) \\ &= \mathcal{H}^{n-1}(A_k). \end{aligned}$$

Therefore,

$$||\partial E||(A) = \sum_{k=1}^{\infty} ||\partial E||(A_k) = \sum_{k=1}^{\infty} \mathcal{H}^{n-1}(A_k) = \mathcal{H}^{n-1}(A).$$

We conclude $||\partial E|| = \mathcal{H}^{n-1} \llcorner \partial^* E$. □

We have just characterized the measure $||\partial E||$ for sets of local finite perimeter and derived a direct method of computing the perimeter that aligns with our intuition. In light of the recent regularity result, we end this subsection with the generalized Gauss-Green Theorem.

Theorem 2.1.4. *(Gauss-Green Theorem.)[[3], Theorem 5.16] Let $E \in BV_{loc}(\mathbb{R}^n)$, then* *for all* $\phi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$,

$$\int_E \operatorname{div}(\phi)\,dx = \int_{\partial^* E} \phi \cdot \nu_E \, d\mathcal{H}^{n-1}.$$

Proof. Let $\phi \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$, then by Theorem 1.1.2 and Property (3) of Theorem 2.1.3,

$$\int_E \operatorname{div}(\phi)\,dx = \int_{\mathbb{R}^n} \phi \cdot \nu_E \, d||\partial E|| = \int_{\partial^* E} \phi \cdot \nu_E \, d\mathcal{H}^{n-1},$$

as desired. □

2.1.2 The Measure-Theoretic Boundary

We have seen the reduced boundary characterizes the Radon measure of a set of local finite perimeter. We have also seen the geometry of points along the reduced boundary are quite limited to being locally flat. In this subsection, we introduce a more robust notion of the boundary, one that possess all of the perimeter computing properties of the reduced boundary, but allows for more complex local geometry.

Definition 2.1.3. *Let* $E \in BV_{loc}(\mathbb{R}^n)$*. For* $x \in \mathbb{R}^n$*, we say* $x \in \partial_* E$*, the measure-theoretic boundary of* E*, if*

$$\limsup_{r \to 0} \frac{\mathcal{L}^n(B_r(x) \cap E)}{r^n} > 0,$$

and

$$\limsup_{r \to 0} \frac{\mathcal{L}^n(B_r(x) \setminus E)}{r^n} > 0.$$

It is easy to see that $\partial_* E$ is a further generalization of $\partial^* E$. To see this, we recall a consequence of Theorem 2.1.1; $\partial^* E \subseteq E^{(1/2)}$. Next, we notice if $x \in \mathbb{R}^n \setminus (E^0 \cup E^1)$, then $\lim_{r\to 0} \frac{\mathcal{L}^n(B_r(x)\cap E)}{\omega_n r^n} \neq 0$ or 1. So, $x \in \partial_* E$. Therefore, we have $\partial^* E \subseteq E^{(1/2)} \subseteq \mathbb{R}^n \setminus (E^0 \cup E^1) \subseteq \partial_* E$. We claim $\partial^* E$ and $\partial_* E$ are equivalent in a measure-theoretic sense.

Theorem 2.1.5 ([3], Lemma 5.5)**.** *If* $E \in BV_{loc}(\mathbb{R}^n)$*, then* $\partial^* E \subseteq \partial_* E$ *and* $\mathcal{H}^{n-1}(\partial_* E \setminus \partial^* E) = 0$.

Proof. We only need to show $\mathcal{H}^{n-1}(\partial_* E \backslash \partial^* E) = 0$. Fix a $x \in \partial_* E$ and define $f : \mathbb{R}^+ \to \mathbb{R}$ by

$$f(r) = \frac{\mathcal{L}^n(B_r(x) \cap E)}{\omega_n r^n}.$$

By continuity of $\mathcal{L}^n$, f is continous. Since $x \in \partial_* E$, there exists a sequence $\{r_j\}_{j=1}^{\infty}$ and $0 < \alpha_1 < \alpha_2 < 1$ such that

$$\alpha_1 \leq f(r_j) \leq \alpha_2.$$

Hence,

$$\min\{\mathcal{L}^n(B_r(x) \cap E), \mathcal{L}^n(B_r(x) \setminus E)\} \geq \min\{\alpha_1, 1 - \alpha_2\} r_j^n \omega_n.$$

By the relative Isoperimetric inequality of Theorem 1.3.3, there exists constant $C_0 > 0$ such that

$$\begin{aligned}
||\partial E||(B_{r_j}(x_j))) &\geq \frac{\min\{\mathcal{L}^n(B_r(x) \cap E), \mathcal{L}^n(B_r(x) \setminus E)\}^{1-1/n}}{2C_0} \\
&\geq \frac{(\min\{\alpha_1, 1 - \alpha_2\} r_j^n \omega_n)^{1-1/n}}{2C_2} \\
&= \frac{(\min\{\alpha_1, 1 - \alpha_2\} \omega_n)^{1-1/n} r_j^{n-1}}{2C_2}.
\end{aligned}$$

Therefore,

$$\limsup_{r \to 0} \frac{||\partial E||(B_r(x))}{r^{n-1}} > 0. \tag{2.1.26}$$

Now define

$$L_k = \left\{ x \in \partial_* E \setminus \partial^* E \;\middle|\; \limsup_{r \to 0} \frac{||\partial E||(B_r(x))}{r^{n-1}} > \frac{1}{2^k} \right\}.$$

Equation (2.1.26) implies $\partial_* E \setminus \partial^* E = \cup_{k=1}^{\infty} L_k$, so it suffices to show $\mathcal{H}^{n-1}(L_k) = 0$ for all $k \in \mathbb{N}$. We will apply a covering argument as in the proof of Lemma 2.1.2. Let $\epsilon, \delta > 0$. We know that $||\partial E|| = \mathcal{H}^{n-1} \llcorner \partial^* E$, so $||\partial E||(L_k) = 0$ for all $k \in \mathbb{N}$. Given that $||\partial E||$ is a Radon measure, there exists an open set A_k^ϵ containing L_k such that

$$||\partial E||(A_k^\epsilon) \leq ||\partial E||(L_k) + \frac{\epsilon}{2^k} = \frac{\epsilon}{2^k}. \tag{2.1.27}$$

Next, define

$$\mathcal{F}_k = \left\{ B_r(x) \middle| x \in L_k,\ B_r(x) \subset A_k^\epsilon, 0 < r < \frac{\delta}{10}, ||\partial E||(B_r(x)) > \frac{r^{n-1}}{2^k} \right\}.$$

$\mathcal{F}_k$ clearly covers L_k. By the Vitali-Covering Lemma, there exists a countable collection of disjoint balls $\{B_{r_i}(x_i)\}_{i=1}^\infty$ such that

$$L_k \subseteq \bigcup_{i=1}^\infty B_{5r_i}(x_i).$$

Notice by construction $diam(B_{5r_i}(x_i)) < \delta$, hence

$$\begin{aligned}
\mathcal{H}_\delta^{n-1}(L_k) &\leq \sum_{i=1}^\infty \alpha(n-1)(5r_i)^{n-1} \\
&\leq \alpha(n-1)5^{n-1}2^k \sum_{i=1}^\infty ||\partial E||(B_{r_i}(x_i)) \\
&\leq \alpha(n-1)5^{n-1}2^k ||\partial E||(A_k^\epsilon).
\end{aligned}$$

However, (2.1.27) implies $2^k||\partial E||(A_k^\epsilon) \leq \epsilon$, so we get

$$\mathcal{H}_\delta^{n-1}(L_k) \leq \alpha(n-1)5^{n-1}\epsilon.$$

As $\epsilon \to 0$, we get $\mathcal{H}_\delta^{n-1}(L_k) = 0$. Thus, $\mathcal{H}^{n-1}(L_k) = 0$. □

Remark. *The measure-theoretic boundary is related to the topological boundary by the following; $\partial E_* \subseteq \partial E$. To see this, we note if $x \in \partial_* E \setminus \partial E$, then there exists $r > 0$ such that $B_r(x) \subseteq \mathbb{R}^n \setminus E$ or $B_r(x) \subseteq E$. In both scenarios, either* $\limsup_{r\to 0} \frac{\mathcal{L}^n(B_r(x)\cap E)}{r^n} = 0$, *or* $\limsup_{r\to 0} \frac{\mathcal{L}^n(B_r(x)\setminus E)}{r^n} = 0$*, contradicting $x \in \partial_* E$.*

In summary, we have three different notions of the boundary for sets of finite perimeter; the topological boundary, the reduced boundary, and the measure-theoretic boundary. We showed $\partial^* E \subseteq \partial_* E \subseteq \partial E$. The topological boundary is the coarsest boundary and often leads to overestimation of the perimeter. The reduced boundary captures information about the perimeter. However, it is only composed of points that upon blow-up resembles that of a half-space. On the other hand, points on the measure-theoretic bound-

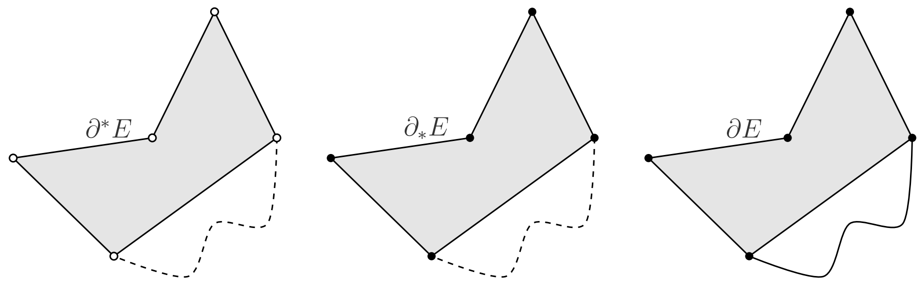

Figure 3: On the left, the reduced boundary comprises of only the edges of the polyhedron. In the middle, the measure-theoretic boundary comprises of all edges and all vertices. On the right, the topological boundary comprises everything.

ary are less limited with their blow-ups, allowing for cusps and other more complicated geometry. See Figure 3 for an example. Nonetheless, the reduced boundary and the measure-theoretic boundary are interchangeable in the study of the perimeter.

3 Plateau Type Problems and The First Variation of the Area Formula

There are numerous geometric minimization problems which are accessible through the setting of BV. Such problems may include boundary value constraints, fixed mass constraints, or prescribed mean curvature. Although each problem relies on a different set of analytical techniques to derive an explicit solution, there is a universal approach to proving the existence of a solution. We call this the Direct-Method. The idea of the Direct-Method is rather simple; the goal is to establish a compactness result in order to obtain a solution as the limit of a minimizing sequence of competitors. Through an application of the lower-semicontinuity property of BV functions, we show this limit is in fact in our space. In this section, we will demonstrate the Direct-Method by proving the existence of solutions to a Plateau-type problem. In addition, we will derive a first derivative test result as a means of verifying solutions of minimization problems pertaining to the perimeter.

3.1 Plateau-Type Problem and the Direct-Method

If we wish to solve a perimeter-induced minimization problem, then we must have a notion of minimality. We define the following;

Definition 3.1.1. *Let $A \subset \mathbb{R}^n$ be open and bounded. Let $E \in BV_{loc}(\mathbb{R}^n)$. We say E is a minimal in A, if for every $F \in BV_{loc}(\mathbb{R}^n)$ such that $F \setminus A = E \setminus A$,*

$$||\partial E||(A) \leq ||\partial F||(A).$$

The classical Plateau's problem consists of finding a surface with minimal surface area amongst all surfaces with a prescribed boundary curve. We present a simpler, more topological based problem in the same spirit of that of Plateau. Fix $L \in BV_{loc}(\mathbb{R}^n)$ and $A \subset \mathbb{R}^n$ open and bounded. We wish to find a set with the least perimeter amongst all sets that coincides with L outside of A. More precisely, let

$$\Theta = \{F \in BV_{loc}(\mathbb{R}^n) \mid F \setminus A = L \setminus A\}. \tag{3.1.1}$$

We say $E^* \in BV_{loc}(\mathbb{R}^n)$ is a solution to the Plateau-type problem in A, if $E^* \in \Theta$ and

$$||\partial E^*||(A) = \inf\{||\partial F||(A) \mid F \in \Theta\}. \tag{3.1.2}$$

In short, we say E^* solves

$$\gamma(L, A) = \inf\{||\partial F||(A) \mid F \in \Theta\}. \tag{3.1.3}$$

In some sense, the criterion of $L \setminus A$ acts as a boundary constraint for this problem; we are only interested in sets with boundary $\partial A \cap L$ in A.

Theorem 3.1.1. *(Existence of Minimizers.)[[6], Theorem 1.20] For all open and bounded $A \subset \mathbb{R}^n$ and $L \in BV_{loc}(\mathbb{R}^n)$, there exists a solution to (3.1.3). That is, there exists a set $E \in BV_{loc}(\mathbb{R}^n)$, such that E is minimal in A, and $E \setminus A = L \setminus A$.*

Proof. Let Θ be defined as in (3.1.1). Given that A is bounded, we may choose $R > 0$ so that A is compactly contained in B_R. Recalling (1.1.6), for all $F \in \Theta$, we know the measure $||\partial F||$ captures the perimeter of F on all open sets. Hence, we may write

$$||\partial F||(B_R) = ||\partial F||(A) + ||\partial F||(B_R \setminus A).$$

But since $F \setminus A = L \setminus A$, it follows that $\partial^* F = \partial^* L$ outside of A. By Proprty (3) of Theorem 2.1.3,

$$||\partial F||(B_R \setminus A) = \mathcal{H}^{n-1}(\partial^* F \cap (B_R \setminus A)) = \mathcal{H}^{n-1}(\partial^* L \cap (B_R \setminus A)) = ||\partial L||(B_R \setminus A).$$

Therefore,

$$||\partial F||(B_R) = ||\partial F||(A) + ||\partial L||(B_R \setminus A). \tag{3.1.4}$$

It suffices to find a minimal set in B_R. As in (3.1.3), let

$$\gamma(L, B_R) = \inf\{||\partial F||(B_R) \,|\, F \in \Theta\}.$$

By definition of the infimum, there exists $\{E_k\}_{k=1}^{\infty} \subseteq \Theta$ such that $||\partial E_k||(B_R) \to \gamma(L, B_R)$. Since, $\{||\partial E_k||(B_R)\}_{k=1}^{\infty}$ is a converging sequence, there exists $M > 0$ so that $||\partial E_k||(B_R) \leq M$ for all $k \in \mathbb{N}$. Therefore,

$$||\chi_{E_k}||_{BV(B_R)} = ||\chi_{E_k}||_{L^1(B_R)} + ||\partial E_k||(B_R) \leq \mathcal{L}^n(B_R) + M.$$

By Theorem 1.2.4, there exists $f \in BV(B_R)$, and a subsequence $\{E_{s_j}\}_{j=1}^{\infty} \subseteq \{E_k\}_{k=1}^{\infty}$ such that $\chi_{E_{s_j}} \xrightarrow{j\to\infty} f$ in $L^1(B_R)$. Given that $\chi_{E_{s_j}}$ takes only the values of 0 or 1, and up to a subsequence $\chi_{E_{s_j}} \xrightarrow{j\to\infty} f$ point-wise $\mathcal{L}^n-$a.e on B_R, we may assume $f = \chi_E$ for some $E \subseteq B_R$. By Theorem 1.2.1,

$$||\partial E||(B_R) \leq \liminf_{j\to\infty} ||\partial E_{s_j}||(B_R) = \gamma(L, B_R). \tag{3.1.5}$$

We notice for all $x \in B_R \setminus A$, $\chi_{E_{s_j}}(x) = \chi_L(x)$. Thus, for $\mathcal{L}^n$-a.e $x \in B_R \setminus A$, $\chi_E(x) = \chi_L(x)$. Now recall by definition of (1.1.2), sets that are identical $\mathcal{L}^n$-a.e are equivalent in $BV_{loc}(\mathbb{R}^n)$, so we may assume $E \setminus A = L \setminus A$. Furthermore, by the same decomposition as in (3.1.4), we see $||\partial E||(B_R) < \infty$ for all $R > 0$. Therefore, $E \in \Theta$. Moreover, (3.1.5) now implies $||\partial E||(B_R) = \gamma(L, B_R)$. We conclude E is a minimal in A. $\square$

3.2 First Variation of the Area

We have shown that minimal sets exist, but how do we verify if a given set is minimal? Following the usual methodology of Euler-Lagrange equations, one idea is to obtain a necessary criticality condition on minimal sets by looking at small local perturbations. We see that if E is minimal in A, then we can construct a series of new sets E_t^* for $t \in \mathbb{R}$, by continuously deforming E in A. See Figure 4. By restricting these deformations to the interior of A, the acquired sets would agree with E outside of A, i.e. $E_t^* \setminus A = E \setminus A$.

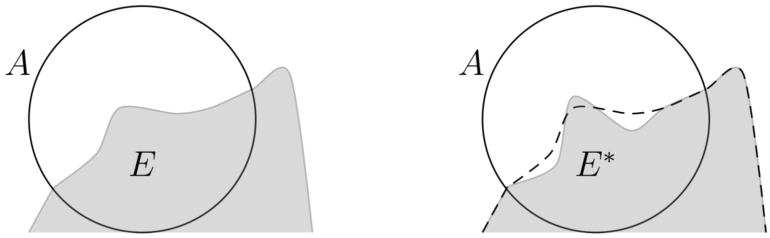

Figure 4: By restricting the deformation of E to the interior of A, we may obtain a new set E^*, such that $E^* \setminus A = E \setminus A$.

Since E is minimal in A, we see that for all $t \in \mathbb{R}$, $||\partial E||(A) \leq ||\partial E_t^*||(A)$. If we assume at $t = 0$, $E_0^* = E$, and upon further assumption that the perimeter function $t \mapsto ||\partial E_t^*||(A)$ is differentiable, then the minimality of E would lead us to expect

$$\frac{d}{dt}||\partial E_t^*||(A)\bigg|_{t=0} = 0, \tag{3.2.1}$$

and

$$\frac{d^2}{dt^2}||\partial E_t^*||(A)\bigg|_{t=0} > 0. \tag{3.2.2}$$

If E satisfies (3.2.1) and (3.2.2) for all continuous deformations, then Fermat's Theorem on stationary points would allows us to conclude that E is indeed a minimal set. The goal now is to derive an explicit formula for the left-hand side of (3.2.1), more commonly known as the first variation of the area.

A natural method for defining these deformations on a set E is through a diffeomorphism $G : \mathbb{R}^n \to \mathbb{R}^n$. We will denote DG to be the Jacobian matrix of G and $|DG|$ to be the corresponding determinant. The following lemma will allow us to compute the perimeter of a set under a diffeomorphism transformation in terms of the perimeter of the original set. We present the general result for BV functions.

Lemma 3.2.1. *[[6], Lemma 10.1] Let $f \in BV_{loc}(\mathbb{R}^n)$ and $G : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. Define $f^* = f \circ G^{-1}$. For all $A \subset \mathbb{R}^n$ open and bounded, we define $A^* = G(A)$. If σ is the resulting function associated to f from Theorem 1.1.2, then*

$$||Df^*||(A^*) = \int_A |H\sigma| \, d||Df||, \tag{3.2.3}$$

where $H = |DG|(DG)^{-1}$.

Proof. If $\phi \in C_c^1(A;\mathbb{R}^n)$, denote $\phi^* = \phi \circ G^{-1}$. We first claim

$$||Df^*||(A^*) = \sup\left\{ \int_{A^*} f^* \mathrm{div}(\phi^*)\, dx \,\middle|\, \phi \in C_c^1(A;\mathbb{R}^n),\ ||\phi||_\infty \leq 1 \right\}. \tag{3.2.4}$$

That is, taking the supremum across $\phi \in C_c^1(A;\mathbb{R}^n)$ with $||\phi||_\infty \leq 1$ is equivalent to taking the supremum across $\eta \in C_c^1(A^*;\mathbb{R}^n)$ with $||\eta||_\infty \leq 1$. To see this, let $\phi \in C_c^1(A;\mathbb{R}^n)$. Clearly, if $||\phi||_\infty \leq 1$, then $||\phi^*||_\infty \leq 1$. By definition, $supp(\phi) = \overline{\{x \in \mathbb{R}^n \,|\, \phi(x) \neq 0\}} = \overline{\phi^{-1}(\mathbb{R}^n \setminus \{0\})} \subseteq\subseteq A$. Likewise, $supp(\phi^*) = \overline{(\phi^*)^{-1}(\mathbb{R}^n \setminus \{0\})} = \overline{G(\phi^{-1}(\mathbb{R}^n \setminus \{0\})}$. By continuity, $\overline{G(\phi^{-1}(\mathbb{R}^n \setminus \{0\})} = G(\overline{\phi^{-1}(\mathbb{R}^n \setminus \{0\}}) \subseteq\subseteq A^*$. Thus, $\phi^* \in C_c^1(A^*;\mathbb{R}^n)$. Similarly, if $\eta \in C_c^1(A^*;\mathbb{R}^n)$, then by the same argument, we see that $\eta \circ F \in C_c^1(A;\mathbb{R}^n)$. This concludes the proof of the claim.

Next, we notice that if for all $\phi \in C_c^1(A;\mathbb{R}^n)$, f satisfies

$$\int_{A^*} f^* \mathrm{div}(\phi^*)\, dx = -\int_A H\sigma \cdot \phi \, d||Df||, \tag{3.2.5}$$

where $H = |DG|(DG)^{-1}$, then by (3.2.4), taking the supremum across both sides with respect to $\phi \in C_c^1(A;\mathbb{R}^n)$ and $||\phi||_\infty \leq 1$ yields

$$||Df^*||(A^*) = \sup\left\{ -\int_A H\sigma \cdot \phi \, d||Df|| \,\middle|\, \phi \in C_c^1(A;\mathbb{R}^n),\ ||\phi||_\infty \leq 1 \right\}. \tag{3.2.6}$$

By applying the same argument as in the proof of Theorem 1.1.3, the right-hand side of (3.2.6) can be shown to be equal to $\int_A |H\sigma| \, d||Df||$. Thus,

$$||Df^*||(A^*) = \int_A |H\sigma| \, d||Df||.$$

Therefore, it suffices to show (3.2.5) holds for all $f \in BV_{loc}(\mathbb{R}^n)$.

We first show (3.2.5) holds for $f \in BV_{loc}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$. Let $\phi \in C_c^1(A;\mathbb{R}^n)$ and $g = G^{-1}$, then

$$\int_{A^*} f^* \,\mathrm{div}(\phi^*)\, dx = -\int_{A^*} \nabla f^* \cdot \phi^* \, dx.$$

By the Chain-rule, $\nabla f^* = (Dg\nabla f) \circ g$, so

$$\int_{A^*} f^* \mathrm{div}(\phi^*)\, dx = -\int_{A^*} ((Dg\nabla f) \circ g) \cdot (\phi \circ g)\, dx.$$

We apply the change of variables $y = G^{-1}(x) = g(x)$ to get

$$\int_{A^*} ((Dg\nabla f) \circ g) \cdot (\phi \circ g)\, dx = \int_A ((Dg \circ G)\nabla f) \cdot \phi\, |DG|\, dy. \tag{3.2.7}$$

By the Inverse Function Theorem, $Dg \circ G = (DG)^{-1}$, so (3.2.7) becomes

$$\int_A ((Dg \circ G)\nabla f) \cdot \phi\, |DG|\, dy = \int_A (DG)^{-1}\nabla f \cdot \phi\, |DG|\, dy = \int_A H\nabla f \cdot \phi\, dy.$$

It follows for all $\phi \in C_c^1(A;\mathbb{R}^n)$,

$$\int_{A^*} f^* \mathrm{div}(\phi^*)\, dx = -\int_A H\nabla f \cdot \phi\, dx.$$

Moreover, $f \in BV_{loc}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$, so $f \in W^{1,1}_{loc}(\mathbb{R}^n)$. Recalling Theorem 1.1.3, we know that $||Df|| = \mathcal{L}^n \llcorner |\nabla f|$ and $\sigma = \frac{\nabla f}{|\nabla f|}$. Therefore,

$$\int_{A^*} f^* \mathrm{div}(\phi^*)\, dx = -\int_A H\frac{\nabla f}{|\nabla f|} \cdot \phi |\nabla f|\, dx = \int_A H\sigma \cdot \phi\, d||Df||,$$

and (3.2.5) holds for $BV_{loc}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$.

For the general case, let $f \in BV_{loc}(\mathbb{R}^n)$. By Theorem 1.2.3, there exists $\{f_k\}_{k=1}^\infty \subseteq BV(A) \cap C^\infty(A)$, such that $f_k \to f$ in $L^1(A)$. We see by the change of variables $y = g(x)$,

$$\lim_{k\to\infty} \int_{A^*} |f^* - f_k^*|\, dx = \lim_{k\to\infty} \int_A |f - f_k||Dg|\, dy \le \max_{y \in \overline{A}} |Dg(y)| \lim_{k\to\infty} \int_A |f - f_k|\, dy = 0.$$

So, $f_k^* \to f^*$ in $L^1(A^*)$. It follows for all $\phi^* \in C_c^1(A^*;\mathbb{R}^n)$,

$$\int_{A^*} f^* \mathrm{div}(\phi^*)\, dx = \lim_{k\to\infty} \int_{A^*} f_k^* \mathrm{div}(\phi^*)\, dx. \tag{3.2.8}$$

We now note that for all $\phi \in C_c^1(A;\mathbb{R}^n)$, if H^T is the transpose of H, then $H^T\phi \in$

$C^1_c(A;\mathbb{R}^n)$, so by Theorem 1.1.2,

$$\int_A f\mathrm{div}(H^T\phi)\,dx = -\int_A \sigma\cdot H^T\phi\,d||Df||.$$

Thus,

$$\begin{aligned}\lim_{k\to\infty}\left|\int_A H\nabla f_k\cdot\phi\,dx - \int_A H\sigma\cdot\phi\,d||Df||\right| &= \lim_{k\to\infty}\left|\int_A \nabla f_k\cdot H^T\phi\,dx - \int_A \sigma\cdot H^T\phi\,d||Df||\right|\\ &= \lim_{k\to\infty}\left|-\int_A f_k\mathrm{div}(H^T\phi)\,dx + \int_A f\mathrm{div}(H^T\phi)\,dx\right|\\ &= \lim_{k\to\infty}\left|\int_A (f-f_k)\mathrm{div}(H^T\phi)\,dx\right|\\ &= 0.\end{aligned}$$

That is,

$$\lim_{k\to\infty}\int_A H\nabla f_k\cdot\phi\,dx = \int_A H\sigma\cdot\phi\,d||Df||. \tag{3.2.9}$$

Combining (3.2.8), and (3.2.9), we see for all $\phi\in C^1_c(A;\mathbb{R}^n)$,

$$\int_{A^*} f^*\mathrm{div}(\phi^*)\,dx = \lim_{k\to\infty}\int_{A^*} f_k^*\mathrm{div}(\phi^*)\,dx = -\lim_{k\to\infty}\int_A H\nabla f_k\cdot\phi\,dx = -\int_A H\sigma\cdot\phi\,d||Df||,$$

as desired. □

Before we apply Lemma 3.2.1 to sets of local finite perimeter, we first recall some results from prior points in the thesis. If $E\in BV_{loc}(\mathbb{R}^n)$, then ν_E is the measure-theoretic normal of E along ∂^*E, as well as the associated function to E from Theorem 1.1.2. Moreover, by Theorem 2.1.3, we know $||\partial E|| = \mathcal{H}^{n-1}\llcorner\partial^*E$. If we take $f=\chi_E$ for $E\in BV_{loc}(\mathbb{R}^n)$, then upon noting that $(\chi_E)^* = \chi_E\circ G^{-1} = \chi_{E^*}$, where $E^*=G(E)$, Lemma 3.2.1 may be written as

$$||\partial E^*||(A^*) = \int_A |H\nu_E|\,d||\partial E|| = \int_{A\cap\partial^*E}|H\nu_E|\,d\mathcal{H}^{n-1}. \tag{3.2.10}$$

Notice that this is only for a single diffeomorphism. To obtain a continuous series of diffeomorphisms, we will need the following:

Definition 3.2.1. *Given an open and bounded sets $A \subset \mathbb{R}^n$, we say $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ forms a local variation in A if*

1. *F is smooth,*
2. *$F_t(\cdot) = F(\cdot, t)$ is a diffeomorphism for all $t \in \mathbb{R}$,*
3. *$F_0(\cdot) = I$, the identity function,*
4. *$\{x \in \mathbb{R}^n \,|\, F_t(x) \neq x\} \subseteq\subseteq A$ for all $t \in \mathbb{R}$.*

If we set $E_t^* = F_t(A)$, then by the smoothness of F, we see $\{E_t^*\}_{t\in\mathbb{R}}$ forms our desired continuous deformations of E. Moreover, Property (4), ensures the deformations are restricted to the interior of A. Hence, $E_t^* \setminus A = E \setminus A$ and $F_t(A) = A$ for all $t \in \mathbb{R}$.

It is easy to see that if $T \in C_c^\infty(A; \mathbb{R}^n)$, then $F_t(x) = x + tT(x)$ forms a local variation in A. For our applications, it suffices to assume all local variations in A are of this form. The choice of variation is convenient because we wish to compute the derivative at $t = 0$ as seen in (3.2.1), so we are only interested in the behaviour of the local variation around $t = 0$. If F is a local variation in A, then by the Taylor expansion of F with respect to t, for some $\epsilon > 0$,

$$F(x,t) = x + t\frac{dF(x,t)}{dt}\bigg|_{t=0} + O(t^2), \tag{3.2.11}$$

for all $|t| < \epsilon$. Setting $T = \frac{dF(x,t)}{dt}\Big|_{t=0}$ yields the generalization.

The following Lemma will be useful in expressing (3.2.10) in a more differentiable form when $DF_t = Id + tT$ for $T \in C_c^\infty(A; \mathbb{R}^n)$, and Id the identity matrix.

Lemma 3.2.2. *[[7], Lemma 17.4] Let M be an $n \times n$-matrix. Denote Id to be the identity matrix, and $M^2 = M \circ M$. The second order Taylor expansion close to the identity states*

1. *$(Id + tM)^{-1} = Id - tM + t^2M^2 + O(t^3)$,*
2. *$|Id + tM| = 1 + tTrace(M) + \frac{t^2}{2}(Trace(M)^2 - Trace(M^2)) + O(t^3)$.*

We are now ready to derive the first variation of the area.

Theorem 3.2.1. *(The First Variation of the Area.)[[7], Theorem 17.5] Let $A \subseteq \mathbb{R}^n$ be open and bounded. For $E \in BV_{loc}(\mathbb{R}^n)$ and $T \in C_c^\infty(A;\mathbb{R}^n)$, define the local variation in A by $F_t(x) = x + tT(x)$. If $E_t^* = F_t(E)$, then*

$$||\partial E_t^*||(A) = ||\partial E||(A) + t\int_{\partial^* E \cap A} \mathrm{div}(T) - \nu_E \cdot (DT\nu_E)\, d\mathcal{H}^{n-1} + O(t^2).$$

Moreover,

$$\left.\frac{d||\partial E_t^*||(A)}{dt}\right|_{t=0} = \int_{\partial^* E} \mathrm{div}(T) - \nu_E \cdot (DT\nu_E)\, d\mathcal{H}^{n-1}. \tag{3.2.12}$$

Proof. By Lemma 3.2.2,

$$(DF_t)^{-1} = (Id - tDT)^{-1} = Id - tDT + O(t^2),$$

and

$$det(Id + tDT) = 1 + tTrace(DT) + O(t^2) = 1 + t\mathrm{div}(T) + O(t^2).$$

Therefore,

$$\begin{aligned}|(DF_t)^{-1}\nu_E|^2 &= \nu_E \cdot \nu_E - 2t\nu_E \cdot (DT\nu_E) + t^2(DT\nu_E)\cdot(DT\nu_E) + O(t^2)\\ &= 1 - 2t\nu_E \cdot (DT\nu_E) + O(t^2)\\ &= (1 - t\nu_E \cdot (DT\nu_E) + O(t^2))^2.\end{aligned}$$

It follows that

$$\begin{aligned}|H_t\nu_E| &= det(Id + tDT)(DF_t)^{-1} = (1 + t\mathrm{div}(T) + O(t^2))(1 - t\nu_E \cdot (DT\nu_E) + O(t^2))\\ &= 1 + t\mathrm{div}(T) - t\nu_E \cdot (DT\nu_E) + O(t^2).\end{aligned}$$

Plugging this into Lemma 3.2.1 yields

$$\begin{aligned}
||\partial E_t^*||(A) &= \int_A |H_t \nu_E| \, d||\partial E|| \\
&= \int_{\partial^* E \cap A} 1 + t\mathrm{div}(T) - t\nu_E \cdot (DT\nu_E) + O(t^2) \, d\mathcal{H}^{n-1} \\
&= ||\partial E||(A) + t \int_{\partial^* E \cap A} \mathrm{div}(T) - \nu_E \cdot (DT\nu_E) \, d\mathcal{H}^{n-1} + O(t^2),
\end{aligned}$$

as desired. □

Equation (3.2.12) provides the necessary condition of (3.2.1) for minimal sets. We define the following:

Definition 3.2.2. *Let $A \subset \mathbb{R}^n$ be open and bounded. We say $E \in BV_{loc}(\mathbb{R}^n)$ is stationary for perimeter in A, if for all $T \in C_c^\infty(A;\mathbb{R}^n)$,*

$$\int_{\partial^* E \cap A} \mathrm{div}(T) - \nu_E \cdot (DT\nu_E) \, d\mathcal{H}^{n-1} = 0. \tag{3.2.13}$$

It is clear that if E is minimal in A, then E must be stationary for perimeter in A. However, as in the case of real-valued functions, having a zero derivative does not imply a local extremum, so the converse is not true. To see this, consider $E = \{(x,y) \in \mathbb{R}^2 \,|\, xy > 0\}$. It is easy to see that $\partial^* E = \{(x,0)\,|x \neq 0\} \cup \{(0,y)\,|y \neq 0\}$. If $A = B_1$, and $T = (T_1, T_2) \in C_c^\infty(B_1;\mathbb{R}^2)$, then

$$\begin{aligned}
\int_{\partial^* E \cap B_1} \mathrm{div}(T) - \nu_E \cdot (DT\nu_E) d\mathcal{H}^{n-1} = &\int_{\{(x,0)|x\neq 0\}\cap B_1} \mathrm{div}(T) - \nu_E \cdot (DT\nu_E) \, d\mathcal{H}^{n-1} \\
&+ \int_{\{(0,y)|y\neq 0\}\cap B_1} \mathrm{div}(T) - \nu_E \cdot (DT\nu_E) \, d\mathcal{H}^{n-1}.
\end{aligned}$$

Now observe that

$$\{(x,0)\,|\, x \neq 0\} \cap B_1 = \{(x,0)\,|\, x \in (-1,0)\} \cup \{(x,0)\,|\, x \in (0,1)\} =: X^- \cup X^+.$$

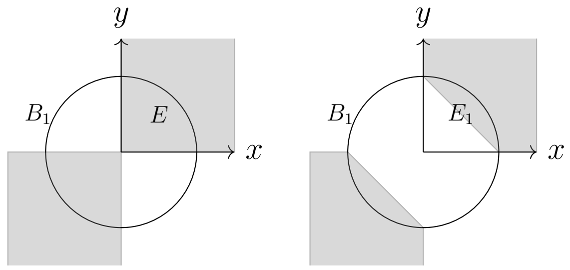

Figure 5: E is stationary for perimeter in B_1, but not a minimizer in B_1.

We note $\nu_E = (0,1)$ on X^-, and $\nu_E = (0,-1)$ on X^+. Thus,

$$\begin{aligned}
\int_{\{(x,0)|x\neq 0\}\cap B_1} \operatorname{div}(T) - \nu_E \cdot (DT\nu_E)\, d\mathcal{H}^{n-1} &= \int_{X^-} \operatorname{div}(T) - (0,1)\cdot\left(\frac{\partial T_1}{\partial y}, \frac{\partial T_2}{\partial y}\right) d\mathcal{H}^{n-1} \\
&\quad + \int_{X^+} \operatorname{div}(T) - (0,-1)\cdot\left(-\frac{\partial T_1}{\partial y}, -\frac{\partial T_2}{\partial y}\right) d\mathcal{H}^{n-1} \\
&= \int_{X^-} \frac{\partial T_1}{\partial x}\, d\mathcal{H}^{n-1} + \int_{X^+} \frac{\partial T_1}{\partial x}\, d\mathcal{H}^{n-1} \\
&= \int_{-1}^{0} \frac{\partial T_1}{\partial x}(x,0)\, dx + \int_{0}^{1} \frac{\partial T_1}{\partial x}(x,0)\, dx \\
&= T_1((0,0)) - T_1((-1,0)) + T_1((1,0)) - T_1((0,0)) \\
&= 0.
\end{aligned}$$

By a similar decomposition, we see that

$$\int_{\{(0,y)|y\neq 0\}\cap B_1} \operatorname{div}(T) - \nu_E \cdot (DT\nu_E)\, d\mathcal{H}^{n-1} = 0,$$

for all $T \in C_c^\infty(B_1;\mathbb{R}^n)$. Therefore, E is stationary for perimeter in B_1. However, E is not minimal in B_1. As illustrated in Figure 5, there exists a set E_1 such that $E_1 \setminus B_1 = E \setminus B_1$, and $||\partial E_1||(B_1) < ||\partial E||(B_1)$. To differentiate between sets that are stationary but not minimal, from those that are truly minimal, one relies on a stability result on stationary sets. This is commonly known as the second variation of the area. This topic is beyond the scope of this thesis, we instead refer to Chapter 17.6 from [7] for the complete theory.

4 Euclidean Isoperimetric Problem

In this section, we study a different geometric variation problem from that of Plateau. Instead of boundary constraints, we now impose a mass constraint. This problem is famously known as the Euclidean Isoperimetric problem, and it is one of the oldest mathematical problems to be studied. The original problem aims to find a closed curve in $\mathbb{R}^2$ which minimizes the perimeter and also encloses a specified area. The Direct-Method provides a means of proving the existence of a minimizing curve. However, unlike most minimization problems, the unique solution to the Isoperimetric problem has been known for centuries. The ancient Greeks knew that the circle was the minimizing curve, but their proof was flawed due to the exclusion of irregular curves. A complete rigorous proof was only obtained in the 1800s by Steiner, who proposed a symmetrization algorithm that turned any closed curve into a circle of the same area. This algorithm is the key to solving the Isoperimetric problem not just in $\mathbb{R}^2$, but all dimensions greater than 2.

We start by posing the Isoperimetric problem in the setting of $BV(\mathbb{R}^n)$. As we have seen in the previous chapters, by working with sets of finite perimeter, we are not limited to sets with smooth boundaries. Let $m > 0$ and denote

$$\Lambda_m = \{E \in BV(\mathbb{R}^n) \,|\, \mathcal{L}^n(E) = m\}. \tag{4.0.1}$$

We wish to find a solution to the following minimization problem,

$$\gamma_m = \inf\{||\partial E||(\mathbb{R}^n) \,|\, E \in \Lambda_m\}. \tag{4.0.2}$$

That is, we wish to find a set $E^* \in \Lambda_m$ such that $||\partial E^*||(\mathbb{R}^n) = \gamma_m$. If we recall the Isoperimetric inequality of Theorem 1.3.3, there exists $C > 0$ such that for all $E \in \Lambda_m$,

$$m^{(n-1)/n} \leq C||\partial E||(\mathbb{R}^n).$$

More precisely, the Isoperimetric inequality tells us the perimeter of a set is bounded from below by the volume of the set. Therefore, if E is a solution to (4.0.2), then $||\partial E||(\mathbb{R}^n)$ must be atleast $\frac{m^{(n-1)/n}}{C}$. This implies if we know the precise value of this constant C and if we can find a set with perimeter $\frac{m^{(n-1)/n}}{C}$, then we have solved the Isoperimetric problem.

Theorem 4.0.1. *[[7], Theorem 14.1] Denote $\omega_n = \mathcal{L}^n(B_1)$. If $E \in BV(\mathbb{R}^n)$, then*

$$||\partial E||(\mathbb{R}^n) \geq n\omega_n^{1/n}\mathcal{L}^n(E)^{(n-1)/n}. \tag{4.0.3}$$

We notice that the ball of volume $\mathcal{L}^n(E)$ has perimeter $n\omega_n^{1/n}(\mathcal{L}^n(E))^{(n-1)/n}$. Therefore, if we show the ball is the minimizer of $\gamma_{\mathcal{L}^n(E)}$, then this solves both the Isoperimetric problem and Theorem 4.0.1.

We will need the ideas of Steiner symmetrization in order to prove Theorem 4.0.1, thus we postpone the proof of Theorem 4.0.1 to the end of this section. Before proceeding with the precise construction of Steiner symmetrization, we begin with the general idea as presented in [2]. For our demonstration, we will assume $n = 2$, and we will reduce our sample pool of (4.0.1) to only convex sets. That is, we only consider sets in

$$\Lambda_m^C = \{E \in BV(\mathbb{R}^2)\,|E \text{ is convex and } \mathcal{L}^2(E) = m\}.$$

As before, we wish to solve the following minimization problem

$$\inf\{||\partial E||(\mathbb{R}^n)\,|E \in \Lambda_m^C\}. \tag{4.0.4}$$

Now if $E \in \Lambda_m^C$, there exists convex functions $\psi_1, \psi_2 : [a, b] \to \mathbb{R}$, for some $a < b$, such that

$$E = \{(x, y)\,|\,a \leq x \leq b,\ \psi_1(x) \leq y \leq \psi_2(x)\}.$$

It is clear that,

$$m = \mathcal{L}^2(E) = \int_a^b \psi_2 - \psi_1\,dx.$$

In addition, by Theorem 2.1.3,

$$||\partial E||(\mathbb{R}^n) = \mathcal{H}^1(\partial E) = \psi_2(a) - \psi_1(a) + \psi_2(b) - \psi_1(b) + \int_a^b \sqrt{1 + (\psi_1')^2} + \sqrt{1 + (\psi_2')^2}\, dx. \tag{4.0.5}$$

We define the Steiner symmetrization of E to be

$$E^s = \left\{ (x, y) \,\middle|\, a \leq x \leq b,\ |y| \leq \frac{\psi_1(x) - \psi_2(x)}{2} \right\}.$$

We see that the Steiner symmetrization of a set E results in a new set E^s that is symmetric with respect to the x-axis. Clearly,

$$\mathcal{L}^2(E^s) = 2 \int_a^b \frac{\psi_1(x) - \psi_2(x)}{2}\, dx = \mathcal{L}^2(E) = m,$$

so E^s preserves the area of E. It is easy to see that E^s is also convex, so $E^s \in \Lambda_m^C$. Likewise, we compute the perimeter of E^s to be

$$||\partial E^s||(\mathbb{R}^n) = \mathcal{H}^1(\partial E^s) = \psi_2(a) - \psi_1(a) + \psi_2(b) - \psi_1(b) + 2 \int_a^b \sqrt{1 + \left(\frac{\psi_1' - \psi_2'}{2} \right)^2}\, dx. \tag{4.0.6}$$

Notice by the convexity of $f(z) = \sqrt{1 + z^2}$, for all $t \in [0, 1]$ and $z_1, z_2 \in \mathbb{R}$,

$$f(tz_1 + (1 - t)z_2) \leq tf(z_1) + (1 - t)f(z_2) \tag{4.0.7}$$

Choosing $t = 1/2$, $z_1 = \psi_1'$, and $z_2 = -\psi_2'$, yields

$$\int_a^b \sqrt{1 + \left(\frac{\psi_1' - \psi_2'}{2} \right)^2}\, dx \leq \frac{1}{2} \int_a^b \sqrt{1 + (\psi_1')^2}\, dx + \frac{1}{2} \int_a^b \sqrt{1 + (\psi_2')^2}\, dx.$$

Plugging this into (4.0.5) and (4.0.6), we see that

$$||\partial E^s||(\mathbb{R}^2) \leq ||\partial E||(\mathbb{R}^2). \tag{4.0.8}$$

Now, observe equality in (4.0.7) holds if and only if $z_1 = z_2$, or more particularly when

$\psi_1' = -\psi_2'$. Therefore, $||\partial E^s||(\mathbb{R}^2) < ||\partial E||(\mathbb{R}^2)$, unless $\psi_1 = -\psi_2 + c$, for some $c \in \mathbb{R}$. That is, equality holds when E already possesses a horizontal axis of symmetry. In such a case, Steiner symmetrization translates the set so that the axis of symmetry is precisely the x-axis. Thus, equality holds if and only if $E = E^s + c$ for some $c \in \mathbb{R}$.

Now notice, we may apply the same symmetrization argument across any hyperplane $y = ax$, for $a \in \mathbb{R}$. We claim if E^* is a solution to (4.0.4), then it must follow that for every hyperplane $\mathcal{P}$ through the origin, E^* must possess an axis of symmetry that is parallel to $\mathcal{P}$. If not, then we may apply the Steiner symmetrization to obtain a new set with an even smaller perimeter. We notice the only convex set that satisfies this symmetry criterion is the circle with area m.

It is now clear that the objective for the remainder of this section is the following; obtain a general form of (4.0.8) for all sets of finite perimeter, derive a convexity result for the case of equality of (4.0.8), then apply the symmetrization argument above to conclude the ball minimizes the perimeter. We now define Steiner symmetrization for arbitrary sets in $\mathbb{R}^n$.

Definition 4.0.1. *Let $E \subseteq \mathbb{R}^n$ and $z \in \mathbb{R}^{n-1}$, we define the vertical slice of E at z by*

$$E_z = \{t \in \mathbb{R} \,|\, (z,t) \in E\}.$$

Definition 4.0.2. *If $E \subseteq \mathbb{R}^n$, we define the Steiner symmetrization of E as*

$$E^s = \left\{ (z,t) \in \mathbb{R}^{n-1} \times \mathbb{R} \,\middle|\, |t| < \frac{\mathcal{L}^1(E_z)}{2} \right\}.$$

In essence, Steiner symmetrization converts each vertical slice of E into an interval of the same length, then situates the interval so that the midpoint lies on the hyperplane $\{x \in \mathbb{R}^n \,|\, x_n = 0\}$. See Figure 6. It is easy to see by Fubini's Theorem, $\mathcal{L}^n(E^s) = \mathcal{L}^n(E)$. So Steiner symmetrization preserves the volume of sets. As we will now see, the perimeter is decreased under Steiner symmetrization.

Theorem 4.0.2. *(Steiner's inequality.) [[7], Theorem 14.4] If $E \in BV(\mathbb{R}^n)$ and $\mathcal{L}^n(E) <$*

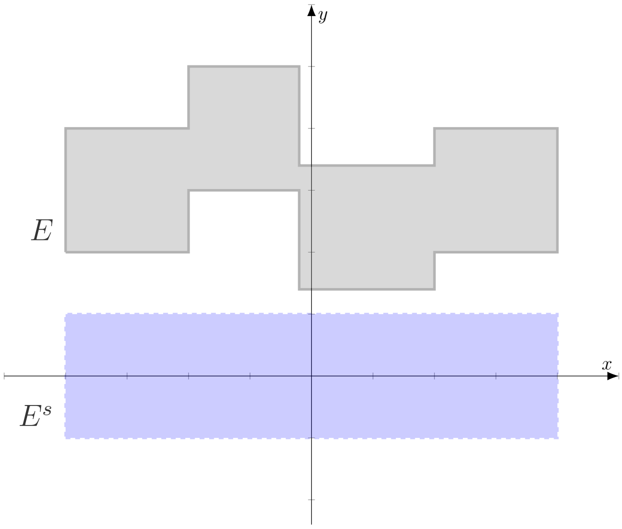

Figure 6: Steiner symmetrization preserves the volume but decreases the perimeter of a set.

∞, *then*

$$||\partial E^s||(\mathbb{R}^n) \leq ||\partial E||(\mathbb{R}^n). \tag{4.0.9}$$

Moreoever,

1. *If equality holds in (4.0.9), then for* $\mathcal{L}^{n-1}$- *a.e* $z \in \mathbb{R}^{n-1}$, E_z *is equivalent to an interval.*

2. *If* E *is equivalent to a convex set, then equality of (4.0.9) holds if and only if there exists* $c \in \mathbb{R}$ *so that* $E^s = E + ce_n$.

Although Property (1) will be crucial to proving the convexity of the solution to the Isoperimetric problem, we will omit its proof as it requires several geometric results that we have not covered in Chapter 2 on sets of finite perimeter in 1-dimension. The proof of Property (2) follows by a very similar convexity argument as in our demonstration of Steiner symmetrization in $\mathbb{R}^2$. The complete details of both proofs may be found in [7].

Proof of Theorem 4.0.2. We will first prove the result for a set E with polyhedral boundaries, then through an application of Theorem 1.4.3, we obtain the result for all sets of finite perimeter.

We start by assuming E is open and bounded with polyhedral boundaries. Let us denote the faces of E by $\{\partial_i E\}_{i=1}^N$. It is clear that $\partial E = \cup_{i=1}^N \partial_i E$. By rotating

E we may assume the outward normal vector ν_E on $\partial_i E$ for all $i \in \{1, \dots, N\}$, is never orthogonal to e_n. Hence, by the Implicit Function Theorem, for each $i \in \{1, \dots, N\}$, there exists $U_i \subset \mathbb{R}^{n-1}$ and an affine function $u_i : U_i \to \mathbb{R}$, such that the graph $\Gamma(u_i, U_i) = \{(z, u_i(z)) \,|\, z \in U_i\} = \partial_i E$. It follows,

$$\partial E = \cup_{i=1}^N \Gamma(u_i, U_i). \tag{4.0.10}$$

We define the projection of E onto $\mathbb{R}^{n-1}$ to be

$$G = \{z \in \mathbb{R}^{n-1} \,|\, \mathcal{L}^1(E_z) > 0\}.$$

Through a series of finite intersections between G and $\{U_i\}_{i=1}^N$, we may obtain a partition $\{G_h\}_{h=1}^M$ such that $G = \cup_{h=1}^M G_h$. Moreover, by taking the restriction of u_i on G_h for all $i \in \{1, \dots, N\}$ such that $G_h \cap U_i \neq \emptyset$, we obtain a finite collection of affine functions $u_k^h, v_k^h : G_h \to \mathbb{R}$, for $1 \leq k \leq N(h)$, satisfying

1. $u_k^h \geq v_k^h$, for all $1 \leq k \leq N(h)$.
2. If $1 \leq j < k \leq N(h)$, then $u_k^h > v_j^h$, and $v_k^h > u_j^h$.
3. If $i \in \{1, \dots, N\}$ such that $G_h \cap U_i \neq \emptyset$, then for some $k \in \{1, \dots, N(h)\}$, $u_k^h = u_i|_{G_h}$ if $(\nu_E)_n|_{U_i} > 0$, or $v_k^h = u_i|_{G_h}$ if $(\nu_E)_n|_{U_i} < 0$.

In other words, $\{u_k^h, v_k^h\}_{k=1}^{N(h)}$, forms the faces of E over G_h. By (4.0.10), we get

$$\partial E = \bigcup_{h=1}^{M} \bigcup_{k=1}^{N(h)} \Gamma(u_k^h, G_h) \cup \Gamma(v_k^h, G_H), \tag{4.0.11}$$

and since E is a polyhedral set,

$$E = \bigcup_{h=1}^{M} \left\{(z, t) \in G_h \times \mathbb{R} : \, t \in \cup_{k=1}^{N(h)} (v_k^h(z), u_k^h(z))\right\}. \tag{4.0.12}$$

We now define $m : \mathbb{R}^{n-1} \to \mathbb{R}$, by $m(z) = \mathcal{L}^1(E_z)$. Clearly, if $z \notin G$, then $m(z) = 0$. We see from (4.0.12), if $z \in G_h$ for some $h \in \{1, \dots, M\}$, then $m(z) = \sum_{k=1}^{N(h)} u_k^h(z) - v_k^h(z)$.

Therefore, m is a continuous piecewise affine function on $\mathbb{R}^{n-1}$. Recall, $E^s = \{(z,t) \in G \times \mathbb{R} \,|\, |t| < m(z)/2\}$, so $\partial E^s = \Gamma(m, G)$. That is, the boundary of E^s is the graph of a piecewise affine function. We deduce that E^s is also an open bounded set with polyhedral boundaries. By Theorem 2.1.3, $||\partial E^s|| = \mathcal{H}^{n-1} \llcorner \partial E^s$, so by the surface area formula,

$$\begin{aligned}
||\partial E^s||(\mathbb{R}^n) &= \mathcal{H}^{n-1}(\partial E^s) \\
&= 2\int_G \sqrt{1 + \left|\frac{\nabla m(z)}{2}\right|^2}\, dz \\
&= \sum_{h=1}^{M} \int_{G_h} \sqrt{4 + |\nabla m(z)|^2}\, dz. \qquad (4.0.13)
\end{aligned}$$

Likewise, $||\partial E|| = \mathcal{H}^{n-1} \llcorner \partial E$, so combined with (4.0.11), we see that

$$\begin{aligned}
||\partial E||(\mathbb{R}^n) &= \mathcal{H}^{n-1}(\partial E) \\
&= \sum_{h=1}^{M} \sum_{k=1}^{N(h)} \mathcal{H}^{n-1}(\Gamma(u_k^h, G_h)) + \mathcal{H}^{n-1}(\Gamma(v_k^h, G_H)) \\
&= \sum_{h=1}^{M} \sum_{k=1}^{N(h)} \int_{G_h} \sqrt{1 + |\nabla v_k^h(z)|^2} + \sqrt{1 + |\nabla u_k^h(z)|^2}\, dz. \qquad (4.0.14)
\end{aligned}$$

By convexity of $x \mapsto \sqrt{1+x^2}$, we see for all $t \in [0,1]$,

$$\sqrt{1 + |t\nabla v_k^h + (1-t)\nabla u_k^h|^2)} \le t\sqrt{1 + |\nabla v_k^h|^2} + (1-t)\sqrt{1 + |\nabla u_k^h|^2}. \qquad (4.0.15)$$

Taking $t = 1/2$, we apply (4.0.15) to (4.0.14) to get

$$\begin{aligned}
||\partial E||(\mathbb{R}^n) &= \sum_{h=1}^{M} \sum_{k=1}^{N(h)} \int_{G_h} \sqrt{1 + |\nabla v_k^h(z)|^2} + \sqrt{1 + |\nabla u_k^h(z)|^2}\, dz \\
&\ge \sum_{h=1}^{M} 2 \sum_{k=1}^{N(h)} \int_{G_h} \sqrt{1 + \left|\frac{\nabla u_k^h(z) - \nabla v_k^h(z)}{2}\right|^2}\, dz \\
&= \sum_{h=1}^{M} 2N(h) \left\{ \int_{G_h} \frac{1}{N(h)} \sum_{k=1}^{N(h)} \sqrt{1 + \left|\frac{\nabla u_k^h(z) - \nabla v_k^h(z)}{2}\right|^2}\, dz \right\}. \qquad (4.0.16)
\end{aligned}$$

Noting that

$$\sqrt{1+\left|\sum_{k=1}^{N(h)}\frac{\nabla u_k^h-\nabla v_k^h}{2N(h)}\right|^2}=\frac{1}{N(h)}\sqrt{(N(h))^2+\left|\sum_{k=1}^{N(h)}\frac{\nabla u_k^h-\nabla v_k^h}{2}\right|^2}$$
$$\leq \frac{1}{N(h)}\sum_{k=1}^{N(h)}\sqrt{1+\left|\frac{\nabla u_k^h-\nabla v_k^h}{2}\right|^2},$$

and $\nabla m(z)=\sum_{k=1}^{N(h)}\nabla u_h^k(z)-\nabla v_h^k(z)$, we see that (4.0.16) becomes

$$\begin{aligned}
||\partial E||(\mathbb{R}^n) &\geq \sum_{h=1}^{M} 2N(h)\int_{G_h}\sqrt{1+\left|\sum_{k=1}^{N(h)}\frac{\nabla u_k^h(z)-\nabla v_k^h(z)}{2N(h)}\right|^2}\,dz.\\
&=\sum_{h=1}^{M}\int_{G_h}\sqrt{4(N(h))^2+\left|\sum_{k=1}^{N(h)}\nabla u_k^h(z)-\nabla v_k^h(z)\right|^2}\,dz\\
&=\sum_{h=1}^{M}\int_{G_h}\sqrt{4(N(h))^2+|\nabla m(z)|^2}\,dz. \qquad (4.0.17)
\end{aligned}$$

Combining (4.0.13) with (4.0.17), and noting that $N(h)\geq 1$ for all $h\in\{1\ldots,M\}$, we conclude

$$\begin{aligned}
||\partial E||(\mathbb{R}^n) &\geq \sum_{h=1}^{M}\int_{G_h}\sqrt{4N(h)^2+|\nabla m(z)|^2}\,dz\\
&\geq \sum_{h=1}^{M}\int_{G_h}\sqrt{4+|\nabla m(z)|^2}\,dz\\
&=||\partial E^s||(\mathbb{R}^n). \qquad (4.0.18)
\end{aligned}$$

Therefore, (4.0.9) holds for all bounded polyhedral sets.

We are now ready to prove the general case. Assume $E\in BV(\mathbb{R}^n)$ and $\mathcal{L}^n(E)<\infty$. By Theorem 1.4.3, there exists a sequence $\{E_h\}_{h=1}^{\infty}$ of bounded open sets with polyhedral boundaries such that $E_h\to E$ in $L^1(\mathbb{R}^n)$ and $||\partial E_h||(\mathbb{R}^n)\to||\partial E||(\mathbb{R}^n)$. Define for all $h\in\mathbb{N}$, $m_h:\mathbb{R}^{n-1}\to\mathbb{R}$, by $m_h(z)=\mathcal{L}^1((E_h)_z)$. In addition, we define

$$D_h=\{z\in\mathbb{R}^{n-1}\,|\,E_z\text{ is not an interval}\}.$$

From (4.0.18), we see that for all $h \in \mathbb{N}$,

$$||\partial E_h^s||(\mathbb{R}^n) \leq ||\partial E_h||(\mathbb{R}^n). \tag{4.0.19}$$

We want to show $E_h^s \xrightarrow{h\to\infty} E^s$ in $L^1(\mathbb{R}^n)$. We first notice that for all $z \in \mathbb{R}^{n-1}$, $t \in (E_h \Delta E)_z$ if and only if $(z,t) \in E_h \Delta E$. Now $(z,t) \in E_h \setminus E$ if and only if $t \in (E_h)_z \setminus E_z$. Similarly, $(z,t) \in E \setminus E_h$ if and only if $t \in E_z \setminus (E_h)_z$. Hence, we conclude for all $z \in \mathbb{R}^{n-1}$, $(E_h \Delta E)_z = (E_h)_z \Delta E_z$. By Fubini's Theorem,

$$\begin{aligned}
\mathcal{L}^n(E_h \Delta E) &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \chi_{E_h \Delta E}(z,t)\, dt\, dz \\
&= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \chi_{(E_h \Delta E)_z}(t)\, dt dz \\
&= \int_{\mathbb{R}^{n-1}} \mathcal{L}^1((E_h \Delta E)_z)\, dz \\
&= \int_{\mathbb{R}^{n-1}} \mathcal{L}^1((E_h)_z \Delta E_z)\, dz.
\end{aligned}$$

Now recall, for all $z \in \mathbb{R}^{n-1}$, $t \in (E^s)_z$ if and only if $|t| < \mathcal{L}^1(E_z)/2 = m(z)/2$, and $t \in (E_h^s)_z$ if and only if $|t| < \mathcal{L}^1((E_h)_z)/2 = m_h(z)/2$. It follows, if $m_h(z) > m(z)$, then

$$(E_h^s)_z \Delta (E^s)_z = (E_h^s)_z \setminus (E^s)_z = \left\{ t \in \mathbb{R} \, : \, \frac{m(z)}{2} \leq |t| < \frac{m_h(z)}{2} \right\},$$

and if $m(z) > m_h(z)$, then

$$(E_h^s)_z \Delta (E^s)_z = (E^s)_z \setminus (E_h^s)_z = \left\{ t \in \mathbb{R} \, : \, \frac{m_h(z)}{2} \leq |t| < \frac{m(z)}{2} \right\}.$$

In the case of equality $m_h(z) = m(z)$, then $(E_h^s)_z = (E^s)_z$ and $(E_h^s)_z \Delta (E^s)_z = \emptyset$. In all cases, we compute

$$\mathcal{L}^1((E_h^s)_z \Delta (E^s)_z) = |m_h(z) - m(z)| \tag{4.0.20}$$

We may relate $(E_h^s)_z \Delta (E^s)_z$ to $(E_h)_z \Delta E_z$ through the following

$$\begin{aligned} |m(z) - m_h(z)| &= |\mathcal{L}^1(E_z) - \mathcal{L}^1((E_h)_z)| \\ &\leq |\mathcal{L}^1(E_z) - \mathcal{L}^1(E_z \cap (E_h)_z)| + |\mathcal{L}^1(E_z \cap (E_h)_z) - \mathcal{L}^1((E_h)_z)| \\ &= \mathcal{L}^1((E_h)_z \Delta E_z). \end{aligned} \tag{4.0.21}$$

By Fubini's Theorem, (4.0.20), and (4.0.21), we see that

$$\begin{aligned} \mathcal{L}^n(E^s \Delta (E_h^s)_z) &= \int_{\mathbb{R}^{n-1}} \mathcal{L}^1((E_h^s)_z \Delta (E^s)_z)\, dz \\ &= \int_{\mathbb{R}^{n-1}} |m(z) - m_h(z)|\, dz \\ &\leq \int_{\mathbb{R}^{n-1}} \mathcal{L}^1(E_z \Delta (E_h)_z)\, dz \\ &= \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \chi_{E \Delta E_h}(z,t)\, dt\, dz \\ &= \mathcal{L}^n(E_h \Delta E). \end{aligned}$$

Given that $E_h \to E$ in $L^1(\mathbb{R}^n)$, it must follow that $E_h^s \xrightarrow{h\to\infty} E^s$ in $L^1(\mathbb{R}^n)$ as well. Therefore, by Theorem 1.2.1 and (4.0.19),

$$||\partial E^s||(\mathbb{R}^n) \leq \liminf_{h\to\infty} ||\partial E_h^s||(\mathbb{R}^n) \leq \liminf_{h\to\infty} ||\partial E_h||(\mathbb{R}^n) = ||\partial E||(\mathbb{R}^n),$$

thereby proving (4.0.9). □

The following lemma will provide a stronger topological implication of when equality in (4.0.9) holds. This will be essential for arguing that minimizers of the Isoperimetric problems are convex.

Lemma 4.0.1. *[[7], Lemma 14.6] Let $E \in BV(\mathbb{R}^n)$ and $E^{(1)}$ the Lebesgue set of E, i.e.*

$$E^{(1)} = \left\{ x \in \mathbb{R}^n \, : \, \lim_{r\to\infty} \frac{|E \cap B_r(x)|}{\omega_n r^n} = 1 \right\}.$$

If for almost every $z \in \mathbb{R}^{n-1}$, E_z is equivalent to an interval, then $E^{(1)}$ has the property that for all $z \in \mathbb{R}^{n-1}$, $(E^{(1)})_z$ is an interval.

Before we proceed with the proof of Theorem 4.0.1, we remark Steiner symmetrization may be applied to any hyperplane through the origin. To be precise, if $\nu \in S^{n-1}$, we denote the hyperplane $\nu^{\perp} = \{x \in \mathbb{R}^n \,|\, x \cdot \nu = 0\}$. And for $x \in \nu^{\perp}$, we define the vertical slice at x with respect to $\nu^{\perp}$ to be

$$E_x^{\nu} = \{t \in \mathbb{R} \,|x + t\nu \in E\}.$$

We define the Steiner symmetrization with respect to $\nu^{\perp}$ by

$$E_{\nu}^{s} = \left\{ x + t\nu \,\middle|\, x \in \nu^{\perp},\ |t| < \frac{\mathcal{L}^1(E_x^{\nu})}{2} \right\}.$$

We see by a change of coordinates, Theorem 4.0.2 and Lemma 4.0.1 still hold for E_{ν}^{s}, for all $\nu \in S^{n-1}$.

Proof of Theorem 4.0.1. As stated before, it suffices to show that the ball of a fixed volume is the minimizer across all sets of the same volume. We will first show this for bounded sets. Let $m > 0$ and $R > 0$, so that $m < \omega_n R^n$. We define

$$\Lambda_R = \{F \in BV(\mathbb{R}^n) \,|\, F \subseteq B_R \text{ and } \mathcal{L}^n(F) = m\}.$$

We note, by choosing R so that $m < \omega_n R^n$, we guarantee Λ_R is non-empty and also not a singleton set. We wish to solve the following minimization problem

$$\gamma_m^R = \inf\{||\partial F||(\mathbb{R}^n) \,|F \in \Lambda_R\}.$$

By the Direct-Method as in the proof of Theorem 3.1.1, there exists a set $E \in \Lambda_R$ such that $||\partial E||(\mathbb{R}^n) = \gamma_m^R$. By the Lebesgue Differentiation Theorem, we know $\mathcal{L}^n(E^{(1)} \Delta E) = 0$, so we may assume without loss of generality that $E^{(1)} = E$. Now, let $\nu \in S^{n-1}$ and $\nu^{\perp}$ be the hyperplane through the origin with normal vector ν. If E_{ν}^{s} is the Steiner Symmetrization of E with respect to $\nu^{\perp}$, then by Theorem 4.0.2,

$$||\partial E_{\nu}^{s}||(\mathbb{R}^n) \leq ||\partial E||(\mathbb{R}^n).$$

We note if $x \in \nu^\perp$ and $t_0 = \sup\{t \geq 0 \,|\, x + t\nu \in B_R\}$, then $\frac{\mathcal{L}^1(E_x^\nu)}{2} \leq t_0$ and $|x + t_0\nu| \leq R$. We see that if $x + t\nu \in E_\nu^s$ for $|t| < \frac{\mathcal{L}^1(E_x^\nu)}{2}$, then

$$|x + t\nu|^2 = |x|^2 + |t|^2 < |x|^2 + \left|\frac{\mathcal{L}^1(E_x^\nu)}{2}\right|^2 \leq |x|^2 + |t_0|^2 \leq R.$$

That is, $E_\nu^s \subseteq B_R$. Now combined with the fact that $\mathcal{L}^n(E_\nu^s) = \mathcal{L}^n(E) = m$, we see $E_\nu^s \in \Lambda_R$. But E is the minimizer, so $||\partial E||(\mathbb{R}^n) \leq ||\partial E_\nu^s||(\mathbb{R}^n)$. It follows

$$||\partial E||(\mathbb{R}^n) = ||\partial E_\nu^s||(\mathbb{R}^n),$$

for all $\nu \in S^{n-1}$. By Theorem 4.0.1, for all $\nu \in S^{n-1}$, E_x^ν is an interval for all $x \in \nu^\perp$. We claim E is convex. To see this, let $x, y \in E$ and $\nu = \frac{y-x}{|y-x|}$. There exists $x_0 \in \nu^\perp$ and $t_x \in E_{x_0}^\nu$ such that $x = x_0 + t_x\nu$. Now observe

$$y = x_0 + (t_x + |y-x|)\frac{y-x}{|y-x|} = x_0 + (t_x + |y-x|)\nu.$$

So, $t_x + |y-x| \in E_{x_0}^\nu$. But $E_{x_0}^\nu$ is an interval, hence for all $t_x \leq t \leq t_x + |y-x|$, $x_0 + t\nu \in E$. In other words, $tx + (1-t)y \in E$ for all $t \in [0,1]$. We deduce E is convex. We have by Property (2) of Theorem 4.0.2, for all $\nu \in S^{n-1}$, there exists $c_\nu \in \mathbb{R}$ such that

$$E = c_\nu\nu + E_\nu^s.$$

To show E is a ball, although not necessarily centred at the origin, we construct a new set F by

$$F = -(c_{e_1}e_1 + \cdots + c_{e_n}e_n) + E. \tag{4.0.22}$$

Since F is a translation of E, F is convex, and for all $\nu \in S^{n-1}$, $F_\nu^s = x + E_\nu^s$, for some $x \in \nu^\perp$. It is easy to see that the perimeter operator is invariant under translation, so

$$||\partial F||(\mathbb{R}^n) = ||\partial E||(\mathbb{R}^n) = ||\partial E_\nu^s||(\mathbb{R}^n) = ||\partial F_\nu^s||(\mathbb{R}^n).$$

By Property (2) of Theorem 4.0.2, for all $\nu \in S^{n-1}$, there exists $d_\nu \in \mathbb{R}$ so that

$$F = d_\nu \nu + F_\nu^s. \tag{4.0.23}$$

If we choose $\nu = e_1$, then upon noting $E_{e_1}^s = E - c_{e_1} e_1$ and (4.0.22), we get

$$\begin{aligned} d_{e_1} e_1 + F_{e_1}^s &= -(c_{e_1} e_1 + \cdots + c_{e_n} e_n) + E \\ &= -(c_{e_2} e_2 + \cdots + c_{e_n} e_n) + E_{e_1}^s. \end{aligned}$$

Through rearrangement, we see

$$E_{e_1}^s = d_{e_1} e_1 + c_{e_2} e_2 + \cdots + c_{e_n} e_n + F_{e_1}^s.$$

However, by construction $E_{e_1}^s$ and $F_{e_1}^s$ are symmetric with respect to $e_1^\perp$. So any translation in the direction of e_1 would contradict this symmetry. Thus, $d_1 = 0$. By the same reasoning we see that $d_i = 0$ for all $i \in \{1, \ldots, n\}$. In summary, we have shown $F = F_{e_i}^s$, for all $i \in \{1, \ldots, n\}$. In other words, F is invariant under reflection with respect to the coordinate hyperplanes. Equivalently, F is invariant under the mapping $x \mapsto -x$. We claim this implies $d_\nu = 0$ for all $\nu \in S^{n-1}$. The invariance of the antipodal mapping implies $F = -F$. Consequently, for all $\nu \in S^{n-1}$, $-F_\nu^s = F_\nu^s$, so by (4.0.23)

$$d_\nu \nu + F_\nu^s = F = -F = -d_\nu \nu - F_\nu^s = -d_\nu \nu + F_\nu^s.$$

But this implies $d_\nu = -d_\nu$, so it must follow that $d_\nu = 0$. Thus, $F = F_\nu^s$ for all $\nu \in S^{n-1}$. We have shown that F is symmetric with respect to all hyperplanes through the origin, and we know F is convex. Therefore, F must be a ball of volume m and by (4.0.22), E must be a ball of volume m. We conclude the ball of volume m is the minimizer across all bounded sets of volume m. It is now clear that if $E \in BV(\mathbb{R}^n)$ is bounded, and noting that the ball of volume $\mathcal{L}^n(E)$ has perimeter $n\omega_n^{1/n}(\mathcal{L}^n(E))^{(n-1)/n}$, we see

$$||\partial E||(\mathbb{R}^n) \geq n\omega_n^{1/n}(\mathcal{L}^n(E))^{(n-1)/n}. \tag{4.0.24}$$

For the general case where $E \in BV(\mathbb{R}^n)$, by Theorem (1.4.3), there exists $\{E_h\}_{h=1}^{\infty} \subset BV(\mathbb{R}^n)$ such that E_h is bounded for all $h \in \mathbb{N}$, $E_h \to E$ in $L^1(\mathbb{R}^n)$ and $||\partial E_h||(\mathbb{R}^n) \to ||\partial E||(\mathbb{R}^n)$. By (4.0.24), we get

$$||\partial E||(\mathbb{R}^n) = \lim_{h\to\infty} ||\partial E_h||(\mathbb{R}^n) \geq \lim_{h\to\infty} n\omega_n^{1/n}(\mathcal{L}^n(E_h))^{(n-1)/n} = n\omega_n^{1/n}(\mathcal{L}^n(E))^{(n-1)/n},$$

which concludes the proof. $\square$

Conclusion

In summary, we have shown that sets of finite perimeter provide a simple but effective framework for studying geometric variational problems. The structural nature between BV functions and Radon measures allows for seamless transitions between measure-theoretic notions and geometric ones. We have seen that the associated Radon measure of sets of finite perimeter possesses perimeter computing capabilities. At the same time, the compactness property gives accessibility to the Direct-Method, which almost trivializes the existence of solutions to minimization problems.

What we have presented is only the tip of what De Giorgi's approach has to offer. In Chapter 2, we established a regularity result on the reduced boundary. One may take this further by studying the regularity of minimal sets. As it turns out, minimal sets have analytic reduced boundary. Another key aspect of De Giorgi's theory is the study of minimal cones; sets that are "tangent" to the boundary of a minimal set. By the method of blow-ups, one may extract a converging sequence in measure to a cone C, defined by $C = \{tx \,|\, t > 0,\ x \in A\}$ for some $A \subset \mathbb{R}^n$. Moreover, if the blow-ups are minimal, then C is also minimal. There is a direct relation between minimal cones and regularity of minimal sets. Effectively, the problem of singularities along the topological boundary of a minimal set may be reduced to the problem of the existence of minimal cones in $\mathbb{R}^n$ with singularities [6]. De Giorgi, Almgren and Simon showed that there does not exist minimal cones with singularities in $\mathbb{R}^n$ for $n \leq 7$, thus proving regularity of minimal sets up to dimension-7[4]. The example of Simon's cone provides the counter argument in $\mathbb{R}^8$. Henceforth, there may exist minimal sets with singularities in dimensions greater than 7.

Appendix A Some Additional Theorems

Theorem A.0.1. *(Hahn Banach.)[[1], Theorem 1.1] Let E be a vector space over $\mathbb{R}$, and $p : E \to \mathbb{R}$ satisfying*

1. *$p(\lambda x) = \lambda p(x)$, for all $x \in E$ and $\lambda > 0$,*

2. *$p(x + y) \le p(x) + p(y)$, for all $x, y \in E$.*

If $G \subset E$ a linear subspace, and $g : G \to \mathbb{R}$ a linear functional such that $g(x) \le p(x)$, for all $x \in G$, then there exists an extension $f : E \to \mathbb{R}$, such that $f|_G = g$, and $f(x) \le p(x)$, for all $x \in E$.

Theorem A.0.2. *(Whitney's Extension Theorem.)[[3], Theorem 6.10] Let $C \subset \mathbb{R}^n$ be a closed subset. Suppose $f : C \to \mathbb{R}$ and $d : C \to \mathbb{R}^n$ are continuous functions. For each compact set V, define*

$$\rho_V(\delta) = \sup\left\{ \frac{|f(x) - f(y) - d(x)\cdot(y - x)|}{|x - y|} \,\middle|\, 0 < |x - y| \le \delta,\ x, y \in V \right\}.$$

If for all compact $V \subset \mathbb{R}^n$, $\rho_V \xrightarrow{\delta \to 0} 0$, then there exists a function $g : \mathbb{R}^n \to \mathbb{R}$ such that

1. *g is C^1,*

2. *$g = f$, $\nabla g = d$ on C.*

Theorem A.0.3. *(Relich Kondrachov Theorem.)[[3], Theorem 4.11] Assume $\Omega \subset \mathbb{R}^n$ is open and bounded with Lipschitz boundary $\partial\Omega$. If $1 < p < n$ and $\{f_k\}_{k=1}^{\infty} \subset W^{1,p}(\Omega)$ satisfying $\sup_{k\in\mathbb{N}} ||f_k||_{W^{1,p}(\Omega)} < \infty$. Then, there exists a subsequence $\{f_{k_j}\}_{j=1}^{\infty}$ and $f \in W^{1,p}(\Omega)$ such that $f_{k_j} \xrightarrow{j\to\infty} f$ in $L^q(\Omega)$ for all $1 \le q \le np/(n - p)$.*

Theorem A.0.4. *(Morse-Sard Theorem.)[[7], Lemma 13.15] If $f \in C^\infty(\mathbb{R}^n)$, then for $\mathcal{L}^1$-a.e $t \in \mathbb{R}$, $\{x \in \mathbb{R}^n \,|\, f(x) = t\}$ is a smooth hyper-surface in $\mathbb{R}^n$.*

Theorem A.0.5. *(Vitali-Covering Lemma.)[[3], Theorem 1.24] Let $\mathcal{F} = \{B_{r_i}(x_i)\}_{i\in\mathcal{I}}$ be a collection of balls such that $\sup\{r_i \,|\, i \in \mathcal{I}\} < \infty$. Then there exists a countable subset $\{B_{r_j}(x_j)\}_{j\in\mathbb{N}} \subseteq \mathcal{F}$ such that*

$$\mathcal{F} \subseteq \bigcup_{j\in\mathbb{N}} B_{5r_j}(x_j).$$

Theorem A.0.6. *(Jordan Decomposition Theorem.)[[5], Theorem 3.4] If μ is a signed measure, there exists positive measure μ^+ and μ^- such that $\mu = \mu^+ - \mu^-$.*

Theorem A.0.7. *(Lebesgue Differentiation Theorem.)[[5], Theorem 3.21] If $f \in L^1_{loc}(\mathbb{R}^n)$ with respect to measure μ, then for μ-a.e $x \in \mathbb{R}^n$,*

$$\lim_{r\to 0} \frac{1}{\mu(B_r)} \int_{B_r} f \, d\mu = f(x).$$

Theorem A.0.8. *(Lusin's Theorem.)[[3], Theorem 1.14] Let μ be a Borel regular measure on $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ be μ-measurable. If $A \subseteq \mathbb{R}^n$ such that $\mu(A) < \infty$, for a fix $\epsilon > 0$, there exists $K \subseteq A$ such that $\mu(A \setminus K) < \epsilon$ and $f|_K$ is continuous.*

Appendix B Proof of the Coarea formula

Proof of Theorem 1.4.1. We first show that if $f \in L^1(\Omega)$ and $\phi \in C^1_c(\Omega; \mathbb{R}^n)$, then

$$\int_\Omega f \operatorname{div}(\phi) \, dx = \int_{-\infty}^{\infty} \int_{E_t} \operatorname{div}(\phi) \, dx \, dt.$$

First assume $f \geq 0$ and $x \in \Omega$, then if $f(x)$ is finite, we observe that for all $t \in \mathbb{R}$, $\chi_{E_t}(x) = 1$ if and only if $f(x) > t$. Equivalently, we may write

$$f(x) = \int_0^{f(x)} 1 \, dt = \int_0^{\infty} \chi_{E_t}(x) \, dt. \tag{B.0.1}$$

Since f is integrable, f must be finite $\mathcal{L}^n$-a.e, so (B.0.1) holds for $\mathcal{L}^n$-a.e $x \in \Omega$. By Fubini's Theorem,

$$\begin{aligned} \int_\Omega f \operatorname{div}(\phi) \, dx &= \int_\Omega \left(\int_0^\infty \chi_{E_t}(x) \, dt \right) \operatorname{div}(\phi) \, dx \\ &= \int_\Omega \left(\int_0^\infty \chi_{E_t} \operatorname{div}(\phi) \, dt \right) \, dx \\ &= \int_0^\infty \int_{E_t} \operatorname{div}(\phi) \, dx \, dt. \end{aligned}$$

Now notice that if $t < 0$ and $f \geq 0$, we get $E_t = \Omega$. This implies

$$\int_{E_t} \operatorname{div}(\phi)\, dx = \int_\Omega \operatorname{div}(\phi)\, dx = 0.$$

Thus,

$$\int_\Omega f \operatorname{div}(\phi)\, dx = \int_{-\infty}^{\infty} \int_{E_t} \operatorname{div}(\phi)\, dx\, dt.$$

Likewise, if $f \leq 0$, for a fix $x \in \Omega$, if $f(x)$ is finite, then $\chi_{E_t}(x) - 1 = -1$ if and only if $t \geq f(x)$. We may then write for $\mathcal{L}^n$-a.e $x \in \Omega$,

$$f(x) = \int_{-\infty}^{0} (\chi_{E_t}(x) - 1)\, dt.$$

By Fubini's Theorem,

$$\begin{aligned}
\int_\Omega f \operatorname{div}(\phi)\, dx &= \int_\Omega \left(\int_{-\infty}^{0} (\chi_{E_t}(x) - 1)\, dt \right) \operatorname{div}(\phi)\, dx \\
&= \int_{-\infty}^{0} \int_\Omega (\chi_{E_t} \operatorname{div}(\phi) - \operatorname{div}(\phi))\, dx\, dt \\
&= \int_{-\infty}^{0} \int_{E_t} \operatorname{div}(\phi)\, dx\, dt.
\end{aligned}$$

Since $f \leq 0$, we notice if $t > 0$, then $E_t = \emptyset$. Combining this with the above we get

$$\int_\Omega f \operatorname{div}(\phi)\, dx = \int_{-\infty}^{\infty} \int_{E_t} \operatorname{div}(\phi)\, dx\, dt.$$

For an arbitrary $f \in L^1(\Omega)$, we may write $f = f^+ - f^-$. We will denote $E_t^+ = \{x \in \Omega \,|\, f^+(x) > t\}$ and $E_t^- = \{x \in \Omega \,|\, -f^-(x) > t\}$. If $t > 0$, we have $E_t = E_t^+$ and $E_t^- = \emptyset$. If $t < 0$, we have $E_t = E_t^-$ and $E_t^+ = \Omega$. Putting all this together we see

$$\begin{aligned}
\int_\Omega f \operatorname{div}(\phi)\, dx &= \int_\Omega (f^+ - f^-) \operatorname{div}(\phi)\, dx \\
&= \int_{-\infty}^{\infty} \left(\int_{E_t^+} \operatorname{div}(\phi) dx + \int_{E_t^-} \operatorname{div}(\phi)\, dx \right) dt \\
&= \int_{-\infty}^{\infty} \int_{E_t} \operatorname{div}(\phi)\, dx\, dt. \qquad \text{(B.0.2)}
\end{aligned}$$

Now if $||\phi||_\infty \leq 1$, then (B.0.2) implies

$$\int_\Omega f\,\mathrm{div}(\phi)\,dx \leq \int_{-\infty}^{\infty} ||\partial E_t||(\Omega)\,dt.$$

That is, $\int_\Omega f \mathrm{div}(\phi)\,dx$ is uniformly bounded by $\int_{-\infty}^{\infty} ||\partial E_t||(\Omega)\,dt$ for all $\phi \in C_c^1(\Omega;\mathbb{R}^n)$ with $||\phi||_\infty \leq 1$. Therefore,

$$||Df||(\Omega) \leq \int_{-\infty}^{\infty} ||\partial E_t||(\Omega)\,dt. \tag{B.0.3}$$

If the right-hand side of (B.0.3) is finite, it follows that $f \in BV(\Omega)$.

To establish equality in (B.0.3), we will first show we have equality when $f \in BV(\Omega) \cap C^\infty(\Omega)$. We start off by defining $m : \mathbb{R} \to \mathbb{R}$ by

$$m(t) = \int_{\Omega \setminus E_t} |\nabla f|\,dx = \int_{\{f \leq t\}} |\nabla f|\,dx.$$

If $t_1 < t_2$, then $\{f \leq t_1\} \subseteq \{f \leq t_2\}$, so $m(t_1) \leq m(t_2)$. In other words, m is non-decreasing and its derivative m' exists $\mathcal{L}^1$-a.e. Next, we notice for all $[a, b] \subset \mathbb{R}$,

$$\int_a^b m'(t)\,dt \leq m(b) - m(a) = \int_{\{a<f\leq b\}} |\nabla f|\,dx \leq \int_\Omega |\nabla f|\,dx.$$

Given that $m' \geq 0$ $\mathcal{L}^1$-a.e, by taking the limit as $a \to -\infty$ and $b \to \infty$, we see that

$$\int_{-\infty}^{\infty} m'(t)\,dt \leq \int_\Omega |\nabla f|\,dx. \tag{B.0.4}$$

Now fix $t \in \mathbb{R}$, and define $\eta_r : \mathbb{R} \to \mathbb{R}$ for $r > 0$ by

$$\eta_r(s) = \begin{cases} 0 & \text{if } s \leq t \\ \frac{s-t}{r} & \text{if } t \leq s \leq t + r \\ 1 & \text{if } s \geq t + r. \end{cases}$$

Then,

$$\eta_r'(s) = \begin{cases} \frac{1}{r} & \text{if } t \leq s \leq t+r \\ 0 & \text{otherwise.} \end{cases}$$

We have that $\eta_r \circ f \in W^{1,1}(\Omega)$, hence we see that for all $\phi \in C_c^1(\Omega; \mathbb{R}^n)$,

$$\begin{aligned} -\int_\Omega \eta_r(f(x))\,\mathrm{div}(\phi)\,dx &= \int_\Omega \eta_r'(f(x))\,\nabla f \cdot \phi\,dx \\ &= \frac{1}{r}\int_{E_t \setminus E_{t+r}} \nabla f \cdot \phi\,dx. \end{aligned} \tag{B.0.5}$$

Now observe that for all $t \in \mathbb{R}$ such that $m'(t)$ exists, (B.0.5) gives

$$\begin{aligned} \frac{m(t+r) - m(t)}{r} &= \frac{1}{r}\left(\int_{\Omega\setminus E_{t+r}} |\nabla f|\,dx - \int_{\Omega \setminus E_t} |\nabla f|\,dx\right) \\ &= \frac{1}{r}\int_{E_t\setminus E_{t+r}} |\nabla f|\,dx \\ &\geq \frac{1}{r}\int_{E_t\setminus E_{t+r}} \nabla f \cdot \phi\,dx \\ &= -\int_\Omega \eta_r(f(x))\,\mathrm{div}(\phi)\,dx. \end{aligned}$$

As $r \to 0$, we see that $\{\eta_r\}_{r>0}$ forms an increasing sequence that converges point-wise to $\chi_{(t,\infty)}$. By the Monotone Convergence Theorem,

$$\begin{aligned} m'(t) &\geq -\lim_{r\to 0}\int_\Omega \eta_r(f(x))\,\mathrm{div}(\phi)\,dx \\ &= -\int_\Omega \chi_{(t,\infty)}(f(x))\mathrm{div}(\phi)\,dx \\ &= -\int_{E_t} \mathrm{div}(\phi)\,dx \\ &= \int_{E_t} \mathrm{div}(-\phi)\,dx. \end{aligned}$$

But notice that

$$\sup\left\{\int_{E_t} \mathrm{div}(-\phi)\,dx \,\middle|\, \phi \in C_c^1(\Omega;\mathbb{R}^n),\ ||\phi||_\infty \leq 1\right\} = ||\partial E_t||(\Omega).$$

Thus,

$$m'(t) \geq ||\partial E_t||(\Omega).$$

By (B.0.4), we get

$$\int_{-\infty}^{\infty} ||\partial E_t||(\Omega)\,dt \leq \int_{-\infty}^{\infty} m'(t)\,dt \leq \int_{\Omega} |\nabla f|\,dx = ||Df||(\Omega).$$

Together with $(B.0.3)$, we see that for all $f \in BV(\Omega) \cap C^{\infty}(\Omega)$,

$$||Df||(\Omega) = \int_{-\infty}^{\infty} ||\partial E_t||(\Omega)\,dt. \tag{B.0.6}$$

To prove the general case, let $f \in BV(\Omega)$. By Theorem 1.2.3, there exists $\{f_k\}_{k=1}^{\infty} \subseteq BV(\Omega) \cap C^{\infty}(\Omega)$ such that $f_k \to f$ in $L^1(\Omega)$ and $||Df_k||(\Omega) \to ||Df||(\Omega)$. We will denote the level sets of f_k by $E_t^k = \{x \in \Omega \,|\, f_k(x) > t\}$. Fix $x \in \Omega$ and $k \in \mathbb{N}$, without loss of generality we assume $f_k(x) \leq f(x)$. We see that

$$|\chi_{E_t^k}(x) - \chi_{E_t}(x)| = \begin{cases} 0 & \text{if } t < f_k(x) \\ 1 & \text{if } f_k(x) \leq t \leq f(x) \\ 0 & \text{if } t > f(x). \end{cases}$$

It follows,

$$\int_{-\infty}^{\infty} |\chi_{E_t^k}(x) - \chi_{E_t}(x)|\,dt = \int_{f_k(x)}^{f(x)} 1\,dx = |f(x) - f_k(x)|.$$

By Fubini's Theorem,

$$\int_{\Omega} |f(x) - f_k(x)|\,dx = \int_{\Omega}\int_{-\infty}^{\infty} |\chi_{E_t^k} - \chi_{E_t}|\,dt\,dx = \int_{-\infty}^{\infty}\int_{\Omega} |\chi_{E_t^k} - \chi_{E_t}|\,dx\,dt.$$

But, $f_k \to f$ in $L^1(\Omega)$, so by Fatou's Lemma,

$$\int_{-\infty}^{\infty} \liminf_{k\to\infty} \int_{\Omega} |\chi_{E_t^k} - \chi_{E_t}|\,dx\,dt \leq \liminf_{k\to\infty} \int_{\Omega} |f(x) - f_k(x)|\,dx. = 0$$

Since the integrand is always greater or equal to 0, we get that for $\mathcal{L}^1$-a.e $t \in \mathbb{R}$,

$$\liminf_{k\to\infty} \int_\Omega |\chi_{E_t^k} - \chi_{E_t}|\, dx = 0.$$

We may extract a subsequence $\{k_j\}_{j=1}^\infty$ such that $\chi_{E_t^{k_j}} \xrightarrow{j\to\infty} \chi_{E_t}$ in $L^1(\Omega)$ for $\mathcal{L}^1$-a.e $t \in \mathbb{R}$. By Theorem 1.2.1,

$$||\partial E_t||(\Omega) \leq \liminf_{j\to\infty} ||\partial E_t^{k_j}||(\Omega).$$

Finally, by applying Fatou's Lemma once again, we get the desired inequality

$$\begin{aligned}
\int_{-\infty}^{\infty} ||\partial E_t||(\Omega)\, dt &\leq \int_{-\infty}^{\infty} \liminf_{j\to\infty} ||\partial E_t^{k_j}||(\Omega)\, dt \\
&\leq \liminf_{j\to\infty} \int_{-\infty}^{\infty} ||\partial E_t^{k_j}||(\Omega), dt \\
&= \liminf_{j\to\infty} ||Df_{k_j}||(\Omega) \\
&= ||Df||(\Omega).
\end{aligned}$$

Putting this together with (B.0.3), we see that if $f \in BV(\Omega)$, then

$$||Df||(\Omega) = \int_{-\infty}^{\infty} ||\partial E_t||(\Omega)\, dt.$$

Hence, $||\partial E_t||(\Omega) < \infty$ for $\mathcal{L}^1$-a.e $t \in \mathbb{R}$. $\square$

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