Modular Transformation of the $T\bar{T}$-deformed Conformal Field Theory

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Abstract

This thesis explores the consequences of modular transformations on a $T\bar{T}$-deformed two-dimensional conformal field theory defined on a torus geometry, where $T\bar{T}$ denotes the negative determinant of the stress-energy tensor of the theory. The investigation resulted in the observation that the coupling of the $T\bar{T}$-deformation transforms as a modular form under a modular transformation of the theory on the torus. This can be derived from either the modular invariance of the partition function and the energy spectrum of the theory, or from the variation of the partition function with the coupling of the $T\bar{T}$-deformation. The significance of this deformation lies in its holographic interpretation as the boundary theory dual to a theory of quantum gravity defined on a radially-cutoff bulk. We therefore begin with a review of the formalism of two-dimensional conformal field theories and then continue with a discussion of the energy spectrum and the partition function of the $T\bar{T}$-deformed theory. Next, we sketch out the AdS/CFT correspondence, before describing the radially-cutoff AdS/$T\bar{T}$-deformed CFT correspondence. We end with a few suggestions for further investigations of the $T\bar{T}$-deformed theory.

Abrégé

Cette thèse explore les conséquences des transformations modulaires sur une théorie de champs conforme à deux dimensions définie sur le tore et déformée par le déterminant négatif du tenseur énergie-impulsion de la théorie, ou $T\bar{T}$. Cette investigation nous conduit a l'observation que le couplage de la déformation $T\bar{T}$ se transforme comme une forme modulaire sous une transformation modulaire de la théorie sur le tore. Ceci peut être obtenu soit par l'invariance modulaire de la fonction de partition et le spectre d'énergie de la théorie, soit à partir de la variation de la fonction de partition avec le couplage de la déformation $T\bar{T}$. L'importance de cette déformation réside dans le fait qu'ou l'interpréte holographiquement comme la théorie sur le bord duale à une théorie de la gravité quantique définie dans un volume coupé radialement. Nous commençons donc par une revue du formalisme des théories de champs conformes bidimensionnelles et poursuivons avec une discussion du spectre d'énergie et de la fonction de partition de la théorie déformée par $T\bar{T}$. Nous esquissons ensuite la correspondance AdS/CFT, avant de décrire la correspondence AdS coupé radialement / CFT déformée par $T\bar{T}$. Nous terminons avec quelques suggestions pour approfondir la théorie déformée par $T\bar{T}$.

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Chapter 1

Introduction

1.1 Motivation for the Thesis

Unification has been a driving force in theoretical physics over the last two centuries. Einstein's theories of relativity was born out of the the inconsistency of Maxwell's theory of electromagnetism with Newton's theories of mechanics and gravity. In a similar vein, Dirac's quantum field theory unified the special theory of relativity with quantum mechanics and extended the predictive power of physics down to subatomic scales. A successful and experimentally viable description of physics at the Planck scale therefore *ought to* require a unification of the mathematical structures of the general theory of relativity and quantum field theory. A tantalizing clue to the structure of this theory of quantum gravity was discovered around 1997 by Maldacena [1] and subsequently elaborated by Witten [2] (see also [3]). The discovery is a dictionary of correspondences between the *asymptotically* anti-de-sitter spacetime (AdS) of a theory of quantum gravity and the conformal field theory (CFT) on the boundary of the spacetime. The $\text{AdS}_d/\text{CFT}_{d-1}$ correspondence, where d is the number of dimensions of the spacetime, is a window into the structure of a theory of quantum gravity as seen through the lens of the corresponding conformal field theory.

A conformal field theory is the simplest kind of quantum field theory. Whereas quantum field theories are invariant under transformations which define the Poincaré group, conformal field theories are invariant under transformations which define the *much larger* conformal group. More explicitly, conformal field theories are invariant under a transformation of length scale and therefore cannot be parameterized by a mass scale. Maxwell's theory of electromagnetism and the classical Yang-Mills theory are examples of conformal field theories, while their quantized counterparts are not conformally invariant.

Given that the AdS/CFT correspondence is a *holographic* dictionary between physical observables in the bulk and the boundary, it is natural to wonder if *a finite perturbation* of the conformal field theory on the boundary corresponds, in holographic terms, to *some perturbation* of the asymptotically anti-de sitter spacetime in the bulk. An example of such a *deformed correspondence* has been explicitly demonstrated by McGough, Mezeí and Verlinde [4]: the *finite* deformation, *in infinitesimal steps*, of a two-dimensional conformal field theory on the boundary by the $T\bar{T}$-operator corresponds, in holographic terms, to a radially cut-off anti-de sitter spacetime in the bulk. Here, $T\bar{T}$ *denotes*[1] the negative determinant of the stress-energy tensor of the *'unperturbed'* theory *at any deformation 'time-step'*, which is in fact $T\bar{T}$ *only* at the ultraviolet fixed point, but is otherwise $T\bar{T} - \Theta^2$. Furthermore, the *finite* strength or coupling of the $T\bar{T}$-deformation of the conformal field theory scales with the inverse-square of the radius of the cut-off anti-de sitter spacetime.

In this thesis, we attempt to add to the $T\bar{T}$-deformed holographic dictionary of the AdS/CFT correspondence. We do so by exploring the characteristics of the $T\bar{T}$-deformed conformal field theory in order to identify results on the boundary which can be holographically interpreted in the bulk. We find that the deformation coupling of the $T\bar{T}$-deformed conformal field theory transforms on a torus as a modular form of weight 4. This result is fleshed out in the final section of Chapter 3.

[1]we follow the notational convention in the literature

Although the observable universe is not a three-dimensional anti-de sitter spacetime, an exploration of two-dimensional conformal field theories, their holographic three-dimensional counterparts and deformations of such theories offers physical insights into the much richer yet mathematically intractable physical theory of quantum gravity which ought to describe the interior of black holes and the first few seconds of the Big Bang.

1.2 Outline of the Thesis

In this thesis, we derive the modular transformation of the deformation coupling of the $T\bar{T}$-deformed conformal field theory. In order to achieve this goal, we use the modular invariance of the energy spectrum and the partition function of the deformed theory and apply a series of S-transformations. We then confirm our result using the variation of the partition function with the deformation coupling. The T-transformation is rather trivial, so we next postulate and then explicitly confirm the general modular transformation using the modular group multiplication law.

Chapter 2 is a review of two-dimensional conformal field theories. The chapter begins with a discussion of the conformal group and the conformal algebra, then continues with a discussion of a plethora of topics such as correlation functions of quasi-primary fields, conserved currents and the energy-momentum tensor, radial quantization and the operator product expansion, before ending with a discussion of the modular invariance of the partition function on a torus. Chapter 3 defines the $T\bar{T}$-deformation as the simplest integrable operator and explains the solvability of $T\bar{T}$-deformed theories, then continues with a discussion of the energy spectrum of the primary states and the variation of the partition function with the deformation coupling, before ending with the derivation of the modular transformation of the deformation coupling. Chapter 4 introduces the AdS/CFT correspondence and then delves into a detailed discussion of the radially-cutoff AdS/$T\bar{T}$-deformed CFT correspondence, ending with a demonstration of the correspondence for the thermodynamics of the bulk and the boundary. Chapter 5 outlines several future directions of research that could extend the work of this thesis.

Chapter 2

Two-dimensional Conformal Field Theories

The defining characteristic of a field theory, be it classical or quantum, is the invariance of physical observables of the theory under the space-time translations and the Lorentz transformations of the Poincaré group. In this chapter, we define conformal field theories as the set of field theories, be it classical or quantum, with physical observables which are invariant under transformations of the conformal group, which is itself a *superset* of the Poincaré group. This chapter is an exploration of the consequences on a field theory of the symmetries of the conformal group which are not simply space-time translations and Lorentz transformations. More specifically, this chapter *builds*, in section 2.2, the quantum Hilbert space of a conformal field theory by promoting physical fields to quantum operators. To that end, we *delineate*, in subsection 2.1.2, the physical fields of a conformal field theory in terms of their transformations under the conformal group. Therefore, we start in section 2.1.1 by *characterizing* the algebra of the generators of a conformal group in two dimensions. For completeness, we also include a discussion of the conserved currents of a two-dimensional conformal field theory in subsection 2.1.3. Finally, section 2.3 is a foray into the constraints (due to *additional* (modular) symmetries) on conformal field theories on a torus $\mathbb{T}^2$. This section paves the way to a discussion, in section 3.4, of the transformation of the deformation coupling of the $T\bar{T}$-deformed conformal field theory under a modular transformation on a torus.

There is an extensive literature of two-dimensional conformal field theories. Introductory textbooks on the subject include those by Blumenhagen and Plauschinn [5] and di Francesco, Mathieu and Sénéchal [6]. The former is a pedagogical review of the subject while the latter is an extensive survey of the popular topics in the field. The lecture notes by Qualls [7] cover two-dimensional conformal field theories while those by Rychkov [8] cover theories in higher dimensions. Additional excellent surveys of the field include those by Simmons-Duffin [9], Schellekens [10] and Ginsparg [11].

As a final note, we confine ourselves in this chapter to a discussion of conformal field theories on flat spacetimes, as the field theory on the boundary of a bulk-boundary correspondence is defined on a flat spacetime. Equally importantly, we relegate ourselves to a study of conformal field theories on Euclidean spacetimes, so that the results we derive are only a Wick rotation away from the equivalent results on the corresponding Lorentzian spacetimes. Finally, sections 2.1 and 2.2 focus exclusively[1] on conformal field theories defined on a *(genus-0) Riemann surface* S^2, while section 2.3 discusses the *additional* (modular) symmetries which constrain two-dimensional conformal field theories on a *(genus-1) torus* $\mathbb{T}^2$.

[1] although we digress in subsection 2.1.1.1 to discuss the types of conformal transformations on a $(d > 2)$-dimensional conformal compactification of Minkowski spacetime $\mathbb{R}^{d-1,1}$

2.1 Conformal Transformations in Classical Theories

Conformal transformations are (invertible) coordinate transformations under which the metric of a spacetime transforms by a *local* scale factor. In other words, a conformal transformation is defined as a transformation on the flat metric η such that

$$\eta_{\mu\nu} \to \Lambda(x)\eta_{\mu\nu}, \tag{2.1}$$

where $\Lambda(x)$ is some arbitrary *local* function. Evidently, the coordinate transformations for which $\Lambda(x) = 1$ form the Poincaré (sub)group.

In this section, we discuss the nature of conformally invariant classical field theories. We first enumerate the types of conformal symmetries on two- and higher-dimensional spacetimes. Next, we demonstrate that correlation functions of (quasi-primary) fields of a two-dimensional conformal field theory are fixed by conformal symmetry. We end with a discussion of the conserved charges of a two-dimensional conformal field theory.

2.1.1 Types of conformal transformations

The nature of conformal transformations on a flat spacetime depends on the number of dimensions of the spacetime. Quite remarkably, the number of conformal transformations is infinite on a two-dimensional spacetime, but is finite for all other dimensions. We can explicitly derive all the conformal transformations for any dimension by solving for the infinitesimal conformal transformations.

An infinitesimal transformation

$$x^\mu \to x^\mu + \epsilon^\mu(x) \tag{2.2}$$

of the coordinates x of a flat spacetime induces the following transformation on the metric η of the spacetime:

$$\eta_{\mu\nu} \to \frac{\partial x'^{\rho}}{\partial x^{\mu}} \frac{\partial x'^{\sigma}}{\partial x^{\nu}} \eta_{\rho\sigma}$$
$$\eta_{\mu\nu} \to (\delta^{\rho}_{\mu} + \partial_{\mu}\epsilon^{\rho})(\delta^{\sigma}_{\nu} + \partial_{\nu}\epsilon^{\sigma})\eta_{\rho\sigma}$$
$$\eta_{\mu\nu} \to \eta_{\mu\nu} + (\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu}).$$

Therefore, the infinitesimal transformation (2.2) qualifies as a conformal transformation if the metric η satisfies the equation

$$\Lambda(x)\eta_{\mu\nu} = \eta_{\mu\nu} + (\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu}), \tag{2.3}$$

which can be simplified after contraction with the metric η to give

$$\Lambda(x) = 1 + \frac{2}{d}\left(\partial \cdot \epsilon\right). \tag{2.4}$$

Comparing equations (2.3) and (2.4), we find that the infinitesimal transformation (2.2) qualifies as a conformal transformation if

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = \frac{2}{d}\left(\partial \cdot \epsilon\right)\eta_{\mu\nu}. \tag{2.5}$$

The solution of the above equation would allow us to determine the complete set of infinitesimal conformal coordinate transformations on a spacetime of any dimension. For example, in a one-dimensional spacetime, any smooth transformation of the coordinate is a conformal transformation as equation (2.5) does not constrain the transformation (2.2).

2.1.1.1 in a three- or higher-dimensional spacetime

The conformal transformations on a d-dimensional flat spacetime, where $d \geq 3$, consist of one dilatation, d translations, $\frac{d(d-1)}{2}$ rotations and d special conformal transformations.

To see this explicitly, we make use of the equations

$$\Box(\partial \cdot \epsilon) = 0 \tag{2.6}$$

and

$$\partial_\mu \partial_\nu \epsilon_\rho = \frac{1}{d}(-\eta_{\mu\nu}\partial_\rho + \eta_{\rho\mu}\partial_\nu + \eta_{\nu\rho}\partial_\mu)(\partial \cdot \epsilon), \tag{2.7}$$

which can be derived rather straightforwardly from equation (2.5). The former can be obtained by simply contracting (2.5) with $\partial^\mu \partial^\nu$. The latter can be obtained by first acting on (2.5) with ∂_ρ to obtain

$$\partial_\rho \partial_\mu \epsilon_\nu + \partial_\rho \partial_\nu \epsilon_\mu = \frac{2}{d}\, \eta_{\mu\nu}\, \partial_\rho\, (\partial \cdot \epsilon) \tag{2.8}$$

and then judiciously combining (2.8) with the following equations (which are cyclic permutations of the indices of (2.8)):

$$\begin{aligned}
\partial_\nu \partial_\rho \epsilon_\mu + \partial_\mu \partial_\rho \epsilon_\nu &= \frac{2}{d}\, \eta_{\rho\mu}\, \partial_\nu\, (\partial \cdot \epsilon) \\
\partial_\mu \partial_\nu \epsilon_\rho + \partial_\nu \partial_\mu \epsilon_\rho &= \frac{2}{d}\, \eta_{\nu\rho}\, \partial_\mu\, (\partial \cdot \epsilon).
\end{aligned}$$

Equation (2.6) implies that ϵ is at most quadratic in x, so that

$$\epsilon_\mu(x) = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho, \tag{2.9}$$

where $c_{\mu\nu\rho}$ must be symmetric in indices ν and ρ. We now physically interpret each term in equation (2.9) *independently* - this is possible as the constraints for conformal invariance are independent of the position x.

- The term $\epsilon_\mu = a_\mu$ corresponds to an infinitesimal translation, which at the finite level is also a translation.

- The term $\epsilon_\mu = b_{\mu\nu} x^\nu$ corresponds to

  - an infinitesimal scaling $x^\mu \to x^\mu + \alpha x^\mu$, with $\alpha = \frac{1}{d}\, b^\lambda{}_\lambda$, which at the finite level is the scaling $x^\mu \to \lambda x^\mu$, for some scale factor λ, and
  - an infinitesimal rotation $x^\mu \to (\delta^\mu{}_\nu + m^\mu{}_\nu) x^\nu$, with $m^\mu{}_\nu = \frac{1}{2}\,(b^\mu{}_\nu - b^\nu{}_\mu)$ which at the finite level is the Lorentz transformation $x^\mu \to \Lambda^\mu{}_\nu x^\nu$, for some Lorentz matrix $\Lambda^\mu{}_\nu$.

  This can be seen by inserting $\epsilon_\mu = b_{\mu\nu} x^\nu$ into equation (2.5) to obtain

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d}\, b^\lambda{}_\lambda\, \eta_{\mu\nu},$$

  which shows that

  - the symmetric part $\frac{1}{2}\,(b_{\mu\nu} + b_{\nu\mu})$ of $b_{\mu\nu}$ is constrained to be $\alpha\eta_{\mu\nu} = \frac{1}{d}\, b^\lambda{}_\lambda\, \eta_{\mu\nu}$ and corresponds, at the finite level, to a scaling transformation, and
  - the antisymmetric part $m_{\mu\nu} = \frac{1}{2}\,(b_{\mu\nu} - b_{\nu\mu})$ of $b_{\mu\nu}$ is not constrained, and corresponds, at the finite level, to a Lorentz transformation.

- The term $\epsilon_\mu = c_{\mu\nu\rho} x^\nu x^\rho$ corresponds to an infinitesimal special conformal transformation $x^\mu \to x^\mu + 2(x\cdot b) x^\mu - x^2 b^\mu$, with $b_\mu = \frac{1}{d}\, c^\lambda{}_{\lambda\mu}$, which at the finite level is an inversion followed by a translation and another inversion as described by $\frac{x^\mu}{x^2} \to \frac{x^\mu}{x^2} - b^\mu$. This can be seen by inserting $\epsilon_\mu = c_{\mu\nu\rho} x^\nu x^\rho$ into equation (2.7) to obtain

$$c_{\mu\nu\rho} = \eta_{\mu\nu} b_\rho + \eta_{\rho\mu} b_\nu - \eta_{\nu\rho} b_\mu, \qquad b_\mu = \frac{1}{d}\, c^\lambda{}_{\lambda\mu},$$

which can be used to derive the infinitesimal transformation straightforwardly.

2.1.1.2 in two dimensions

The conformal transformations on a two-dimensional flat spacetime form an infinite group. To see this explicitly, we flesh out the indices in equation (2.5) to find that

$$\partial_0 \epsilon_0 = \partial_1 \epsilon_1, \qquad \partial_0 \epsilon_1 = -\partial_1 \epsilon_0, \tag{2.10}$$

which *happen to be* the Cauchy-Riemann equations. As a holomorphic function and an anti-holomorphic function on the complex plane each satisfy the Cauchy-Riemann equations *independently*, we can rewrite the Cauchy-Riemann equations (2.10) as the *equivalent* relations

$$\partial \bar{\epsilon}(z, \bar{z}) = 0, \qquad \bar{\partial} \epsilon(z, \bar{z}) = 0,$$

using the set of variables

$$\begin{aligned} z &= x^0 + i x^1, \qquad & \epsilon &= \epsilon^0 + i \epsilon^1, \qquad & \partial &\equiv \partial_z = \frac{1}{2}\left(\partial_0 - i \partial_1\right), \\ \bar{z} &= x^0 - i x^1, \qquad & \bar{\epsilon} &= \epsilon^0 - i \epsilon^1, \qquad & \bar{\partial} &\equiv \partial_{\bar{z}} = \frac{1}{2}\left(\partial_0 + i \partial_1\right), \end{aligned}$$

where ϵ and $\bar{\epsilon}$ are respectively holomorphic and anti-holomorphic functions of the complex variable z. As the functions $\epsilon(z, \bar{z})$ and $\bar{\epsilon}(z, \bar{z})$ are *arbitrary*, the *infinitesimal* conformal transformations $z \to z + \epsilon(z)$ and $\bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z})$ on a two-dimensional spacetime form an infinite set. By extension, the finite transformations $z \to f(z)$ and $\bar{z} \to \bar{f}(\bar{z})$ which correspond respectively to the infinitesimal transformations $z \to z + \epsilon(z)$ and $\bar{z} \to \bar{z} + \bar{\epsilon}(\bar{z})$ also form an infinite set.

The finite number of *global* conformal transformations of the coordinates on a two-dimensional spacetime are transformations which take the spacetime point z to the spacetime point $\dfrac{az+b}{cz+d}$, where matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ form the Mobius group[2] $PSL(2,\mathbb{C}) \equiv SL(2,\mathbb{C})/\mathbb{Z}_2$. To see this explicitly, we physically interpret (the generators of) the *local* algebra for the *global* conformal transformations:

- The generators $\{\ell_n, \bar{\ell}_n\}$ of the *classical* conformal algebra form two commuting copies of the Witt algebra (but with $z^* = \bar{z}$):

$$\begin{aligned} [\ell_m, \ell_n] &= (m-n)\ell_{m+n}, \\ [\ell_m, \bar{\ell}_n] &= 0, \\ [\bar{\ell}_m, \bar{\ell}_n] &= (m-n)\,\bar{\ell}_{m+n}. \end{aligned} \tag{2.11}$$

We can demonstrate this with the choice

$$\begin{aligned} \ell_n &= -z^{n+1}\partial, \\ \bar{\ell}_n &= -\bar{z}^{n+1}\bar{\partial}, \end{aligned}$$

[2] $SL(2,\mathbb{C})/\mathbb{Z}_2$ is the group of matrices of complex entries with unit determinant such that each matrix and its negative signify the same (conformal) transformation.

for the generators of the Witt algebra, which corresponds to the local conformal transformations

$$z \to z + \sum_{n\in\mathbb{Z}} \epsilon_n(-z^{n+1}),$$
$$\bar{z} \to \bar{z} + \sum_{n\in\mathbb{Z}} \bar{\epsilon}_n(-\bar{z}^{n+1}).$$

These local conformal transformations use the Laurent mode-expansions $\epsilon(z) = \sum_{n\in\mathbb{Z}} \epsilon_n(-z^{n+1})$ and $\bar{\epsilon} = \sum_{n\in\mathbb{Z}} \bar{\epsilon}_n(-\bar{z}^{n+1})$ about $z = 0$, which is a valid expansion as the functions $\epsilon(z)$ and $\bar{\epsilon}(z)$ are meromorphic on the complex plane.

- The subalgebra of the generators $\{\ell_{-1}, \ell_0, \ell_1\} \bigcup \{\bar{\ell}_{-1}, \bar{\ell}_0, \bar{\ell}_1\}$ of the *local* Witt algebra corresponds to the *global* conformal group. For example, on the Riemann sphere, the generators $\{\ell_{-1}, \ell_0, \ell_1\} \bigcup \{\bar{\ell}_{-1}, \bar{\ell}_0, \bar{\ell}_1\}$ generate the following transformations:
  - The generators ℓ_{-1} and $\bar{\ell}_{-1}$ are momentum operators and therefore generate translations $z \to z + b$ and $\bar{z} \to \bar{z} + \bar{b}$.
  - The generators ℓ_0 and $\bar{\ell}_0$, in polar coordinates $z = re^{i\phi}$, are the linear combinations

$$\ell_0 + \bar{\ell}_0 = -r\partial_r,$$
$$i(\ell_0 - \bar{\ell}_0) = -\partial_\phi,$$

    which generate dilatations and rotations respectively.
  - The generators ℓ_1 and $\bar{\ell}_1$ generate special conformal transformations $-\frac{1}{z} \to -\frac{1}{z} - c$ and $-\frac{1}{\bar{z}} \to -\frac{1}{\bar{z}} - c$.

In other words, the generators $\{\ell_{-1}, \ell_0, \ell_1\} \bigcup \{\bar{\ell}_{-1}, \bar{\ell}_0, \bar{\ell}_1\}$, which form a (closed) subalgebra of the Witt algebra, generate the global conformal transformations on the Riemann sphere. The other generators are not defined globally on the Riemann sphere: the generators ℓ_n and $\bar{\ell}_n$ for $n < -1$ are singular at $z \to 0$, and the generators ℓ_n and $\bar{\ell}_n$ for $n > 1$ are singular at $z \to \infty$. The former is rather obvious, while the latter can be observed with the change of coordinates to $w = -1/z$.

2.1.2 Correlation functions of quasi-primary fields

We introduce the following vocabulary:

- *chiral field*: a field that depends only on z
- *anti-chiral field*: a field that depends only on $\bar{z}$
- *quasi-primary field*: a field that transforms, under some *global conformal* transformation $f \in PSL(2, \mathbb{C})$, as

$$\phi(z, \bar{z}) \to \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z), \bar{f}(\bar{z})) \tag{2.12}$$

  - *conformal dimensions*: the (real-valued) weights $(h, \bar{h})$ of the quasi-primary field $\phi(z, \bar{z})$
  - *(scaling) dimension*: the dimension $\Delta = h + \bar{h}$ of the quasi-primary field $\phi(z, \bar{z})$
  - *conformal spin*: the spin $s = h - \bar{h}$ of the quasi-primary field $\phi(z, \bar{z})$

- *primary field*: a *quasi-primary* field that transforms, under some *finite conformal* transformation $z \to f(z)$, as

$$\phi(z, \bar{z}) \to \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \phi(f(z), \bar{f}(\bar{z}))$$

- *secondary field*: a field, such as the derivative of a primary field, that is not quasi-primary

The invariance of a correlation function of a *quasi-primary field* of a two-dimensional conformal field theory under the symmetries of the theory fixes the function up to a structure constant. For example, the three-point function of the *chiral quasi-primary field* ϕ for spacetime points z_1, z_2 and z_3 is, by conformal symmetry, given by

$$\langle \phi_1(z_1)\phi_2(z_2)\phi_3(z_3)\rangle = \frac{C_{123}}{(z_1 - z_2)^{h_1+h_2-h_3}(z_2 - z_3)^{h_2+h_3-h_1}(z_1 - z_3)^{h_1+h_3-h_2}}$$

for the structure constants C_{123}, while the two-point function of the *quasi-primary field* is

$$\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\rangle = \frac{d_{12}\delta_{h_1,h_2}\delta_{\bar{h}_1,\bar{h}_2}}{(z_1 - z_2)^{h_1+h_2}(\bar{z}_1 - \bar{z}_2)^{\bar{h}_1+\bar{h}_1}}$$

for the structure constant d_{12}.

To help readers see this explicitly, we offer a template of the proof: we show explicitly that the two-point function of the *chiral quasi-primary field* ϕ for spacetime points z_1 and z_2 is, by conformal symmetry, given by

$$\langle \phi_1(z_1)\phi_2(z_2)\rangle = \frac{d_{12}\delta_{h_1,h_2}}{(z_1 - z_2)^{h_1+h_2}} \tag{2.13}$$

for the structure constant d_{12}. To that end, we apply each *global conformal* transformation on the 'unknown' function $\langle \phi_1(z_1)\phi_2(z_2)\rangle$ in turn:

- Invariance under spacetime translation is consistent with the form $\langle \phi_1(z_1)\phi_2(z_2)\rangle = g(z_1 - z_2)$ of the correlation function for some function g.

- Invariance under scale transformations $z_1 \to \lambda z_1$ and $z_2 \to \lambda z_2$ is consistent with the form $\langle \phi_1(z_1)\phi_2(z_2)\rangle = \lambda^{h_1+h_2}\langle \phi_1(\lambda z_1)\phi_2(\lambda z_2)\rangle$ of the correlation function, so that the correlation function is of the form

$$\langle\phi_1(z_1)\phi_2(z_2)\rangle = \frac{d_{12}(h_1,h_2)}{(z_1-z_2)^{h_1+h_2}}$$

for some function $d_{12}(h_1,h_2)$.

- Invariance under the transformation $z \to -1/z$ is consistent with the form

$$\langle\phi_1(z_1)\phi_2(z_2)\rangle = \frac{\langle\phi_1\left(-1/z_1\right)\phi_2\left(-1/z_2\right)\rangle}{z_1^{2h_1}z_2^{2h_2}} = \frac{\langle\phi_1(z_1)\phi_2(z_2)\rangle}{z_1^{h_1-h_2}z_2^{h_2-h_1}}$$

  of the correlation function, so that we have $d_{12}(h_1,h_2) = d_{12}\delta_{h_1,h_2}$ for some constant d_{12}.

2.1.3 Conserved currents and the energy-momentum tensor

The *invariance* of the observables of a conformal field theory under the *continuous* coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$ is consistent with the existence of the *conserved current* $j_\mu = T_{\mu\nu}\epsilon^\nu$, where $T_{\mu\nu}$ is a symmetric energy-momentum tensor. This energy-momentum tensor is *conserved* and *traceless*:

- Under the *translation* $x^\mu \to x^\mu + \epsilon^\mu(x)$ (where $\epsilon^\mu(x)$ is an *arbitrary constant*), we find that

$$\partial^\mu j_\mu = \epsilon^\nu\partial^\mu T_{\mu\nu},$$

  so that the conservation of the current j_μ implies the conservation of the energy-momentum tensor $T_{\mu\nu}$.

- Under the *transformation* $x^\mu \to x^\mu + \epsilon^\mu(x)$ (where $\epsilon^\mu(x)$ is an *arbitrary function*), we find that

$$\partial^\mu j_\mu = \epsilon^\nu \left(\partial_\mu T_{\mu\nu}\right) + T_{\mu\nu}\left(\partial^\mu \epsilon^\nu\right) = \frac{1}{2} T_{\mu\nu}(\partial^\mu \epsilon^\nu + \partial^\nu \epsilon^\mu) = \frac{1}{d} T_{\mu\nu}(\partial \cdot \epsilon)\eta^{\mu\nu} = \frac{1}{2}\left(\Lambda(x) - 1\right) T^\mu_\mu,$$

where we used equations (2.4) and (2.5). Therefore, the conservation of the current j_μ implies the tracelessness of the energy-momentum tensor $T_{\mu\nu}$.

More specifically, the energy-momentum tensor of a *two-dimensional* conformal field theory on the *complex plane* is given by

$$\begin{pmatrix} T(z) & 0 \\ 0 & \bar{T}(\bar{z}) \end{pmatrix}.$$

To see this explicitly, we use the transformation rule for a tensor to find the components in the $\{z, \bar{z}\}$ basis:

$$\begin{aligned}
T_{zz} &= \frac{\partial x^\mu}{\partial z}\frac{\partial x^\nu}{\partial z} T_{\mu\nu} = \frac{1}{4}(T_{00} - 2iT_{10} - T_{11}) \\
T_{\bar{z}\bar{z}} &= \frac{\partial x^\mu}{\partial \bar{z}}\frac{\partial x^\nu}{\partial \bar{z}} T_{\mu\nu} = \frac{1}{4}(T_{00} + 2iT_{10} - T_{11}) \\
T_{z\bar{z}} &= \frac{\partial x^\mu}{\partial z}\frac{\partial x^\nu}{\partial \bar{z}} T_{\mu\nu} = \frac{1}{4}(T_{00} + T_{11}) \\
T_{\bar{z}z} &= \frac{\partial x^\mu}{\partial \bar{z}}\frac{\partial x^\nu}{\partial z} T_{\mu\nu} = \frac{1}{4}(T_{00} + T_{11}).
\end{aligned}$$

Due to the tracelessness of the energy-momentum tensor, we have that $T_{z\bar{z}} = 0 = T_{\bar{z}z}$. Therefore, we find that

$$T_{zz} = \frac{1}{2}(T_{00} - iT_{10}), \qquad T_{\bar{z}\bar{z}} = \frac{1}{2}(T_{00} + iT_{10}),$$

so that

$$\begin{aligned}\bar{\partial}T_{zz} &= \frac{1}{4}(\partial_0 + i\partial_1)(T_{00} - iT_{10}) \\ &= \frac{1}{4}(\partial_0 T_{00} + \partial_1 T_{10} + i\partial_1 T_{00} - i\partial_0 T_{10}) \\ &= \frac{1}{4}(\partial^\mu T_{\mu 0} - i\partial^\mu T_{\mu 1}) \\ &= 0,\end{aligned}$$

and similarly for $\partial T_{\bar{z}\bar{z}}$.

2.2 Conformal Transformations in Quantum Theories

In this section, we discuss a quantization procedure for two-dimensional conformal field theories. This procedure includes a compactification of the spatial dimension of the flat spacetime (on which the theory is defined) to a circle of fixed radius and is therefore named radial quantization. We then discuss the operator product expansions for primary fields and the energy-momentum tensor, which leads naturally to a discussion of the Virasoro algebra. Finally, we introduce the state-operator correspondence and construct the quantum Hilbert space of two-dimensional conformal field theories.

2.2.1 Radial quantization

A classical two-dimensional conformal field theory is quantized by promoting the modes in the Fourier expansion of primary fields to operators. However, the correlation functions of a two-dimensional *quantum* conformal field theory defined on the flat spacetime are infrared-divergent. We therefore introduce the following caveats in the quantization procedure:

- The correlation functions of a two-dimensional *quantum* conformal field theory defined on the space compactification of the flat spacetime are finite in the infrared limit. Therefore, we perform the quantization procedure not on the flat spacetime, but on its space compactification. As conformal field theories are scale invariant, the size of the compactification is not physically significant; therefore, we follow the usual convention and set the radius of the infinite cylinder to 2π.

- Computations for a two-dimensional quantum conformal field theory defined on the cylinder are rather cumbersome. Therefore, we perform computations for the conformal field theory on the complex plane obtained by the conformal map $z = e^w$

of the cylinder, where the complex variable $w = x^0 + ix^1$ parametrizes the surface of the cylinder. This would mean that

- fixed time slices on the cylinder are mapped to circles of fixed radius on the plane, increasing from the origin to infinity, and
- fixed spatial slices on the cylinder are mapped to lines on the plane, each of which extends from the origin to infinity.

Furthermore, we note that

- time translations $x^0 \to x^0 + a$ generated by the Hamiltonian operator $H = L_0 + \bar{L}_0$ on the cylinder are mapped to dilatations $z \to e^a z$ generated by the dilatation operator $D = L_0 + \bar{L}_0$ on the plane, and
- spatial translations $x^1 \to x^1 + b$ generated by the momentum operator $P = i(L_0 - \bar{L}_0)$ on the cylinder are mapped to spatial rotations $z \to e^{ib} z$ generated by the angular momentum $J = i(L_0 - \bar{L}_0)$ on the plane.

The radial quantization procedure is therefore the promotion of the modes of the Fourier expansion of the primary fields on the cylinder to operators, or equivalently the promotion of the modes

$$\phi_{n,\bar{n}} = \frac{1}{2\pi i} \oint dz\, z^{n+h-1} \frac{1}{2\pi i} \oint d\bar{z}\, \bar{z}^{\bar{n}+\bar{h}-1} \phi(z, \bar{z})$$

of the Laurent expansion

$$\phi(z, \bar{z}) = \sum_{n,\bar{n} \in \mathbb{Z}} z^{-n-h} \bar{z}^{-\bar{n}-\bar{h}} \phi_{n,\bar{n}} \tag{2.14}$$

of the primary field $\phi(z, \bar{z})$ (with conformal weight $(h, \bar{h})$) on the plane to operators.

2.2.2 Operator product expansion

A product of operators (in a correlation function) of a classical field theory maps to a number of products of operators in the corresponding quantum field theory. We opt to study the correlation functions of time-ordered products of operators of two-dimensional quantum conformal field theories defined on the surface of the cylinder, which map to the correlation functions of radially-ordered products

$$R\left[A(z_1)B(z_2)\right] = \begin{cases} A(z_1)B(_2) \text{ for } |z_1| > |z_2| \\ B(z_2)A(z_1) \text{ for } |z_2| > |z_1| \end{cases}$$

for operators $A(z_1)$ and $B(z_2)$ defined on the plane. For notational convenience, we omit the symbol R which denotes the radial ordering of operator products.

2.2.2.1 for primary fields

The operator product expansion of the primary field $\phi(w,\bar{w})$ with the energy-momentum tensor $T(z)$ offers a definition of primary quantum fields and is given by

$$T(z)\phi(w,\bar{w}) = \frac{h}{(z-w)^2}\phi(w,\bar{w}) + \frac{1}{(z-w)}\partial_w\phi(w,\bar{w}) + \ldots, \tag{2.15}$$

$$\bar{T}(\bar{z})\phi(w,\bar{w}) = \frac{h}{(\bar{z}-w)^2}\phi(w,\bar{w}) + \frac{1}{(\bar{z}-w)}\partial_w\phi(w,\bar{w}) + \ldots, \tag{2.16}$$

where the ellipsis are regular terms. To see this explicitly, we compare the transformation rule of a *generic* field under an infinitesimal conformal transformation (using the conserved charge of the theory) with the transformation rule of a *primary* field under an infinitesimal coordinate transformation:

- The conserved current $j_\mu = T_{\mu\nu}\epsilon^\nu$ of a two-dimensional conformal field theory has the conserved charge

$$Q = \int_{\text{fixed } x^0} dx^1 j_0$$

on the surface of the infinite cylinder and the *corresponding* conserved charge

$$Q = \frac{1}{2\pi i}\oint_C \left[dz\, T(z)\, \epsilon(z) + d\bar{z}\, \bar{T}(\bar{z})\, \bar{\epsilon}(\bar{z})\right] \tag{2.17}$$

on the plane, where the contour C is a counter-clockwise circle. As a conserved charge generates symmetry transformations of an operator ϕ with the commutator $\delta\phi = [Q, \phi]$, we find that the transformation of a *generic* field under an infinitesimal conformal transformation is

$$\delta\phi(w,\bar{w}) = \frac{1}{2\pi i}\oint_C dz\, [T(z)\epsilon(z), \phi(w,\bar{w})] + \frac{1}{2\pi i}\oint_C d\bar{z}\, \left[\bar{T}(\bar{z})\bar{\epsilon}(\bar{z}), \phi(w,\bar{w})\right],$$

where w is a point not on the infinite cylinder, but on the plane. As the contour C in equation (2.17) is conserved in time and can have an arbitrary radius, we can write

$$\begin{aligned}
\delta\phi(w,\bar{w}) = \quad & \frac{1}{2\pi i}\oint_{|z|>|w|} dz\, \epsilon(z)\, T(z)\, \phi(w,\bar{w}) - \frac{1}{2\pi i}\oint_{|w|<|z|} dz\, \epsilon(z)\, \phi(w,\bar{w})\, T(z) \\
& + \frac{1}{2\pi i}\oint_{|\bar{z}|>|\bar{w}|} d\bar{z}\, \bar{\epsilon}(\bar{z})\, \bar{T}(\bar{z})\, \phi(w,\bar{w}) - \frac{1}{2\pi i}\oint_{|\bar{w}|<|\bar{z}|} d\bar{z}\, \bar{\epsilon}(\bar{z})\, \phi(w,\bar{w})\, \bar{T}(\bar{z}) \\
= \quad & \frac{1}{2\pi i}\left(\oint_{|z|>|w|} - \oint_{|w|<|z|}\right) dz\, \epsilon(z)\, [T(z)\phi(w,\bar{w})] \\
& + \frac{1}{2\pi i}\left(\oint_{|\bar{z}|>|\bar{w}|} - \oint_{|\bar{w}|<|\bar{z}|}\right) d\bar{z}\, \bar{\epsilon}(\bar{z})\, \left[\bar{T}(\bar{z})\phi(w,\bar{w})\right].
\end{aligned}$$

Therefore, we find that

$$\delta\phi(w,\bar{w}) = \frac{1}{2\pi i}\oint_{C(w)} dz\,\epsilon(z)\left[T(z)\phi(w,\bar{w})\right] + \frac{1}{2\pi i}\oint_{C(\bar{w})} d\bar{z}\,\bar{\epsilon}(\bar{z})\left[\bar{T}(\bar{z})\phi(w,\bar{w})\right].$$

- Under the infinitesimal conformal transformation $f(w) = w + \epsilon(w)$, a primary field $\phi(w,\bar{w})$ on the plane transforms from (2.12) as

$$\begin{aligned}\phi(w,\bar{w}) &\to (1+\partial\epsilon)^{h}\,\left(1+\bar{\partial}\bar{\epsilon}\right)^{\bar{h}}\,\phi\left(w+\epsilon,\bar{w}+\bar{\epsilon}\right)\\ &= \left(1+h\,\partial\epsilon+\bar{h}\,\bar{\partial}\bar{\epsilon}\right)\,\left[\phi(w,\bar{w})+\epsilon\,\partial\phi(w,\bar{w})+\bar{\epsilon}\,\bar{\partial}\phi(w,\bar{w})\right]\\ &= \phi(w,\bar{w})+\epsilon\,\partial\phi(w,\bar{w})+\bar{\epsilon}\,\bar{\partial}\phi(w,\bar{w})+h\,\phi(w,\bar{w})\,\partial\epsilon+\bar{h}\,\phi(w,\bar{w})\,\bar{\partial}\bar{\epsilon}.\end{aligned}$$

Therefore, we find that

$$\delta\phi(w,\bar{w}) = \left(\epsilon\,\partial+\bar{\epsilon}\,\bar{\partial}+h\,\partial\epsilon+\bar{h}\,\bar{\partial}\bar{\epsilon}\right)\,\phi(w,\bar{w}),$$

which we can rewrite using

$$\begin{aligned}\phi(w,\bar{w})\,\partial_w\epsilon(w) &= \frac{1}{2\pi i}\oint_{C(w)} dz\,\frac{\epsilon(z)}{(z-w)^2}\,\phi(w,\bar{w}),\\ \epsilon(w)\,\partial_w\phi(w,\bar{w}) &= \frac{1}{2\pi i}\oint_{C(w)} dz\,\frac{\epsilon(z)}{(z-w)}\,\partial_w\phi(w,\bar{w}),\end{aligned}$$

and the corresponding anti-holomorphic equations.

2.2.2.2 for the energy-momentum tensor

The operator product expansion for the energy-momentum tensor is of the form

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \dots, \tag{2.18}$$

$$\bar{T}(z)\bar{T}(w) = \frac{\bar{c}/2}{(z-w)^4} + \frac{2\bar{T}(w)}{(z-w)^2} + \frac{\partial_w \bar{T}(w)}{(z-w)} + \dots, \tag{2.19}$$

where c and $\bar{c}$ are respectively the *holomorphic* and *anti-holomorphic* central charges of the theory, and the ellipses denote regular terms. The central charges are the *additional Noether charges* which appear upon the *quantization* of a *classical* conformal field theory. More explicitly, the Virasoro symmetry algebra of the infinite number of generators $\{L_m\}$, c and $\bar{c}$ of a quantum conformal field theory is given by

$$\begin{aligned}
&[L_m, L_n] = (m-n)\, L_{m+n} + \frac{c}{12}\,(m^3 - m)\,\delta_{m+n,0}, \\
&[L_m, \bar{L}_n] = 0, \\
&[\bar{L}_m, \bar{L}_n] = (m-n)\, \bar{L}_{m+n} + \frac{\bar{c}}{12}\,(m^3 - m)\,\delta_{m+n,0},
\end{aligned} \tag{2.20}$$

and is therefore the *central* extension of the Witt symmetry algebra (2.11) of the infinite number of generators $\{\ell_m\}$ of a classical conformal field theory.

To prove the operator product expansion[3] (2.18), we note the following points:

- The singular terms are of the form $\mathcal{O}_n(z-w)^{-n}$, with the scaling dimension $\Delta_n = 4-n$ for the operator $\mathcal{O}_n$: every term in the TT-operator-product expansion scales as $\Delta = 4$, as the energy-momentum tensor itself has scaling dimension $\Delta = 2$.

- n can be at most 4, as operators with negative scaling dimension cannot occur in a unitary conformal field theory.

[3] the proof of the operator product expansion (2.19) follows the same template

- The operator product is radially ordered so that $T(z)T(w) = T(w)T(z)$ and the operator product expansion must be invariant under $z \leftrightarrow w$. Therefore, odd values of n are not allowed. However, the term with $n = 1$ is allowed because of the derivative of T in the numerator.

To prove the Virasoro algebra (2.20), we note the following points:

- $T(z)$ is a quasi-primary field of conformal dimension $(2, 0)$: the expectation value of the operator product expansion (2.18) is

$$\langle T(z)T(w)\rangle = \frac{c/2}{(z-w)^4},$$

  where the constant value of the one-point function vanishes due to scale invariance; this matches the correlation function (2.13) of a quasi-primary field.

- As such, the Laurent expansion (2.14) of the energy-momentum tensor is

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \implies L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1}\, T(z),$$

  where the modes L_n are the generators of the symmetry algebra of the quantum conformal field theory:

$$\begin{aligned} Q_n &= \oint \frac{dz}{2\pi i}\, T(z)\, (-\epsilon_n z^{n+1}) \\ &= -\epsilon_n \sum_{m\in\mathbb{Z}} \oint \frac{dz}{2\pi i}\, L_m\, z^{n-m-1} \\ &= -\epsilon_n \sum_{m\in\mathbb{Z}} L_m\, \delta_{mn} \\ &= -\epsilon_n\, L_n. \end{aligned}$$

- Using the operator product expansion (2.18), we can show then that the modes L_n satisfy the Virasoro algebra:

$$
\begin{aligned}
[L_m, L_n] &= \frac{1}{2\pi i}\oint dz\, z^{m+1}\frac{1}{2\pi i}\oint dw\, w^{n+1}[T(z), T(w)] \\
&= \oint_{C(0)} \frac{dw}{2\pi i}\, w^{n+1} \oint_{C(w)} \frac{dz}{2\pi i}\, z^{m+1} \left[\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} + \dots\right] \\
&= \oint_{C(0)} \frac{dw}{2\pi i}\, w^{n+1} \bigg[\frac{c}{2\cdot 3!}\,(m+1)m(m-1)\, w^{m-2} \\
&\qquad\qquad + 2\,(m+1)\, w^m\, T(w) + w^{m+1}\,\partial_w T(w)\bigg] \\
&= \oint_{C(0)} \frac{dw}{2\pi i}\bigg[\frac{c}{12}\,(m^3-m)\, w^{n+m-1} \\
&\qquad\qquad + 2\,(m+1)\, w^{n+m+1}\, T(w) + w^{n+m+2}\,\partial_w T(w)\bigg] \\
&= \frac{c}{12}\,(m^3-m)\,\delta_{m+n,0} + 2\,(m+1)\, L_{m+n} \\
&\qquad + \oint_{C(0)} \frac{dw}{2\pi i}\left[\partial_w\left(w^{n+m+2}\, T(w)\right) - \partial_w\left(w^{n+m+2}\right) T(w)\right] \\
&= \frac{c}{12}\,(m^3-m)\,\delta_{m+n,0} + 2\,(m+1)\, L_{m+n} \\
&\qquad + 0 - (n+m+2)\oint_{C(0)} \frac{dw}{2\pi i}\, w^{n+m+1}\, T(w) \\
&= \frac{c}{12}\,(m^3-m)\,\delta_{m+n,0} + 2\,(m+1)\, L_{m+n} + 0 - (m+n+2)\, L_{m+n} \\
&= (m-n)\, L_{m+n} + \frac{c}{12}\,(m^3-m)\,\delta_{m+n,0}.
\end{aligned}
$$

To prove that the Virasoro algebra is the central extension of the Witt algebra, we note the following points:

- The central extension of the Witt algebra of generators $\{\ell_m\}$ is the symmetry algebra

$$[L_m, L_n] = (m-n)\, L_{m+n} + cp(m,n),$$
$$[L_m, \bar{L}_n] = 0,$$
$$[\bar{L}_m, \bar{L}_n] = (m-n)\, \bar{L}_{m+n} + \bar{c}p(m,n),$$

  with generators $\{L_m\}$, independent holomorphic central charge c and anti-holomorphic central charge $\bar{c}$, and the unknown function $cp(m,n)$ which commutes with every element of the extended algebra.

- The commutator and therefore $p(m,n)$ is anti-symmetric.

- Using $\hat{L}_0 = L_0 + \dfrac{cp(1,-1)}{2}$ and $\hat{L}_n = L_n + \dfrac{cp(n,0)}{n}$ for $n \neq 0$, we obtain the commutators

$$[\hat{L}_n, \hat{L}_0] = nL_n + cp(n,0) = n\hat{L}_n$$
$$[\hat{L}_1, \hat{L}_{-1}] = 2L_0 + cp(1,-1) = 2\hat{L}_0.$$

  for $p(n,0) = p(1,-1) = 0$. We switch to these commutators and rename $\hat{L}_n$ as L_n.

- Using the Jacobi identity

$$0 = [[L_m, L_n], L_0] + [[L_0, L_m], L_n] + [[L_n, L_0], L_m],$$

we find that

$$\begin{aligned}
0 &= [(m-n)L_{m+n} + cp(m,n), L_0] + [-mL_m + cp(0,m), L_n] \\
&\quad + [nL_n + cp(n,0), L_m] \\
0 &= (m-n)[(m+n)L_{m+n} + cp(m+n,0)] - m[(m-n)L_{m+n} + cp(m,n)] \\
&\quad + n[(n-m)L_{m+n} + cp(n,m)] \\
0 &= (m-n)p(m+n,0) - mp(m,n) + np(n,m) \\
0 &= (m+n)p(n,m).
\end{aligned}$$

Therefore, $p(m,n) = 0$ if $m \neq -n$, and we find that $p(n,-n) \neq 0$ for $|n| \geq 2$.

- Using the Jacobi identity

$$0 = [[L_{1-n}, L_n], L_{-1}] + [[L_n, L_{-1}], L_{1-n}] + [[L_{-1}, L_{1-n}], L_n],$$

we find that

$$0 = (1-2n)p(1,-1) + (n+1)p(n-1,1-n) + (n-2)p(-n,n),$$

so that

$$\begin{aligned}
p(n,-n) &= \left(\frac{n+1}{n-2}\right) p(n-1,1-n) \\
&= \left(\frac{n+1}{n-2}\right)\left(\frac{n}{n-3}\right) p(n-2,2-n) \\
&= \cdots \\
&= \frac{(n+1)n(n-1)\cdots 5\cdot 4}{(n-2)(n-3)\cdots 2\cdot 1}\, p(2,-2) \\
&= \frac{(n+1)n(n-1)}{6}\, p(2,-2).
\end{aligned}$$

- We use the normalization $p(2,-2) = 1/2$ by convention.

It is crucial to point out that, as with the Witt algebra, only the subalgebra $\{L_{-1}, L_0, L_1\}$ of the Virasoro algebra generates the global conformal group $PSL(2,\mathbb{C})$, as the central extension does not affect the Virasoro generators: $p(m,n) = 0$ for $n, m = -1, 0, 1$.

As a sidenote, we mention the following points:

- The energy-momentum tensor transforms under a *finite* conformal transformation as

$$T(z) \to \left(\frac{\partial f}{\partial z}\right)^2 T\left(f(z)\right) + \frac{c}{12} S\left(f(z), z\right), \tag{2.21}$$

where $S(w,z)$ is the Schwarzian derivative defined by

$$S(w,z) = \frac{1}{(\partial_z w)^2}\left(\left(\partial_z w\right)\left(\partial_z^3 w\right) - \frac{3}{2}\left(\partial_z^2 w\right)^2\right).$$

- A primary field ϕ_n satisfies

$$[L_m, \phi_n] = \left((h-1)m - n\right)\phi_{m+n}, \tag{2.22}$$

$$[\bar{L}_m, \phi_n] = \left((\bar{h}-1)m - n\right)\phi_{m+n}, \tag{2.23}$$

where $\{L_m, \bar{L}_m\}$ are the generators of the Virasoro algebra (2.20). These commutation relations can be proved using the operator product expansions (2.15) and (2.16). The proof is similar in spirit to the proof of the Virasoro algebra using the operator product expansion (2.18) for energy-momentum tensors.

2.2.3 Hilbert Space

The Hilbert space of a two-dimensional conformal field theory with central charges c and $\bar{c}$ is rather simply the sum $\oplus_{h,\bar{h}} V(h,c) \otimes \bar{V}(\bar{h},\bar{c})$ of Verma modules $V(h,c) \otimes \bar{V}(\bar{h},\bar{c})$ of the *primary states* $\phi_{h,\bar{h}}|0\rangle$ of the theory. More explicitly, each Verma module is a *lowest-weight* representation of (the generators of) the Virasoro algebra and *consists of* a primary (ground) state and its *associated* descendant (excited) states. We unpack this series of statements in the following:

- The state-operator correspondence defines the primary states of a two-dimensional conformal field theory:
  - The in-vacuum $|0\rangle$ is defined as

$$L_n|0\rangle = 0 = \bar{L}_n|0\rangle \qquad \text{for} \qquad n > -2,$$

and the out-vacuum $\langle 0|$ is defined as

$$\langle 0|L_n = 0 = \langle 0|\bar{L}_n \qquad \text{for} \qquad n < 2.$$

This is because the central charges in the Virasoro algebra prevents us from having the full symmetry: if, for example, we required $L_2|0\rangle = 0 = L_{-2}|0\rangle$ and $L_0|0\rangle = 0$, then there would be a contradiction because

$$||L_{-2}|0\rangle||^2 = \langle 0|L_2L_{-2}|0\rangle = \langle 0|L_{-2}L_2|0\rangle + 4\langle 0|L_0|0\rangle + \frac{c}{2} = \frac{c}{2} \neq 0.$$

- A *primary in-state* $|\phi\rangle$ is defined from the mode $\phi_{-h,-\bar{h}}$ of the *corresponding* field $\phi(z,\bar{z})$ with weights $(h,\bar{h})$ as

  $$|\phi\rangle = \phi_{-h,-\bar{h}}|0\rangle.$$

  To see this explicitly, we note that the in-state $|\phi\rangle$ can be obtained by applying the *corresponding* operator $\phi(x,t)$ on the vacuum $|0\rangle$ at past infinity $t \to -\infty$, or equivalently, by applying the *corresponding* operator $\phi(z,\bar{z})$ on the vacuum $|0\rangle$ at the origin $z,\bar{z} \to 0$:

  $$|\phi\rangle = \lim_{t\to-\infty} \phi(x,t)|0\rangle \iff |\phi\rangle = \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})|0\rangle. \tag{2.24}$$

  However, the mode expansion (2.14) is consistent with the definition (2.24) only if the modes $\phi_{n,\bar{n}}$ for $n > -h$ or $\bar{n} > -\bar{h}$ are annihilated by the vacuum $|0\rangle$:

  $$\phi_{n,\bar{n}}|0\rangle = 0 \text{ for } n > -h, \bar{n} > -\bar{h}.$$

- A *primary out-state* $\langle\phi|$ is the hermitian conjugate of the *corresponding* in-state $|\phi\rangle$:

  $$\langle\phi| = |\phi\rangle^\dagger.$$

  As the Wick rotation $x^0 = it \to -x^0$ from Minkowski space to Euclidean space implies the Hermitian conjugation $z^\dagger = 1/\bar{z}$, the hermitian conjugate of a field is

  $$\phi(z,\bar{z})^\dagger = \bar{z}^{-2h} z^{-2\bar{h}} \phi\left(\frac{1}{\bar{z}}, \frac{1}{z}\right),$$

which can be Laurent expanded as

$$\phi(z,\bar{z})^\dagger = \bar{z}^{-2h} z^{-2\bar{h}} \sum_{n,\bar{n}\in\mathbb{Z}} \left(\frac{1}{\bar{z}}\right)^{-n-h} \left(\frac{1}{z}\right)^{-\bar{n}-\bar{h}} \phi_{n,\bar{n}} = \sum_{n,\bar{n}\in\mathbb{Z}} \bar{z}^{n-h} z^{\bar{n}-\bar{h}} \phi_{n,\bar{n}}.$$

Comparing with the mode expansion

$$\phi(z,\bar{z})^\dagger = \sum_{n,\bar{n}\in\mathbb{Z}} z^{-n-h} \bar{z}^{-\bar{n}-\bar{h}} \phi^\dagger_{n,\bar{n}},$$

we find that $\phi^\dagger_{n,\bar{n}} = \phi_{-n,-\bar{n}}$. Therefore, an out-state is defined as

$$\langle\phi| = \lim_{z,\bar{z}\to 0} \langle 0|\phi(z,\bar{z})^\dagger = \lim_{w,\bar{w}\to\infty} w^{2h}\bar{w}^{2\bar{h}} \langle 0|\phi(w,\bar{w}) = \langle 0|\phi_{h,\bar{h}}$$

with $w = z^{-1}$, and

$$\langle 0|\phi_{n,\bar{n}} = 0 \text{ for } n < h, \bar{n} < \bar{h}.$$

- The discrete spectrum of the Hamiltonian $L_0 + \bar{L}_0$ is bounded *from below* by the *scaling dimension* $h + \bar{h}$ of the primary in-state $\phi_{-h,-\bar{h}}|0\rangle$. To prove this, we use the commutation relations (2.22) and (2.23), and obtain

$$L_0\phi_{-h,-\bar{h}}|0\rangle = \phi_{-h,-\bar{h}}L_0|0\rangle + h\phi_{-h,-\bar{h}}|0\rangle = h\phi_{-h,-\bar{h}}|0\rangle,$$
$$\bar{L}_0\phi_{-h,-\bar{h}}|0\rangle = \phi_{-h,-\bar{h}}\bar{L}_0|0\rangle + \bar{h}\phi_{-h,-\bar{h}}|0\rangle = \bar{h}\phi_{-h,-\bar{h}}|0\rangle,$$

  where h and $\bar{h}$ are the holomorphic conformal weight and the anti-holomorphic conformal weight respectively of the primary in-state $\phi_{-h,-\bar{h}}|0\rangle$.

- The (lowering) operators L_n with $n > 0$ decrease the conformal dimension of L_0 by n, while the (raising) operators L_{-n} with $n > 0$ increase the conformal dimension by

n. This can be seen using the Virasoro algebra (2.20):

$$L_0 L_n \phi_{h,\bar{h}}|0\rangle = (L_n L_0 - nL_n)\phi_{h,\bar{h}}|0\rangle = (h-n)L_n\phi_{h,\bar{h}}|0\rangle.$$

- The (lowering) operators L_n with $n > 0$, not surprisingly, annihilate primary states. This can be seen using the commutation relations (2.22) and (2.23):

$$L_n|\phi\rangle = L_n\phi_{-h}|0\rangle = \phi_{-h}L_n|0\rangle + (h(n+1)-n)\phi_{-h+n}|0\rangle.$$

- *Each* primary state $\phi_{h,\bar{h}}|0\rangle$ of the conformal field theory can be used to build *one* lowest weight representation of (the generators of) the Virasoro algebra - *a* Verma module (which closes under the action of the generators of the Virasoro algebra) - where the lowest weight is the eigenvalue of the $L_0 + \bar{L}_0$ operator when it acts on the primary state $\phi_{h,\bar{h}}|0\rangle$. The lowering operators L_n with $n > 0$ annihilate the primary (ground) state, while the raising operators L_{-n} with $n > 0$ create descendant (excited) states from the primary state. Therefore, the basis of the Verma module $V(h,c)$ is the *conformal family*

$$\{L_{k_{-1}} \dots L_{k_{-n}}\phi_h|0\rangle : k_i \geq 1\},$$

each with eigenvalue $h + k_1 + \dots + k_n$, and similarly for the Verma module $\bar{V}(\bar{h},\bar{c})$. If the primary state is the vacuum $|0\rangle$, then the condition $k_i \geq 1$ is replaced by $k_i \geq 2$ as $L_{-1}|0\rangle = 0$.

2.3 Conformal Field Theory on a Torus

Thus far, we explored the consequences of conformal symmetry on a field theory defined on a *genus-zero surface - the Riemann sphere*. We now explore two-dimensional conformal field theories defined on a *genus-one surface - the two-torus*.

In this section, we identify the modular group as a symmetry group of a two-dimensional conformal field theory on a torus, and determine the generators of the group. We also write down several equivalent forms of the partition function of the theory on the torus.

2.3.1 Modular transformations on a torus

A two-dimensional conformal field theory on a two-torus is equivalent to the corresponding conformal field theory on a Riemann sphere S^2 which is *compactified* as

$$z \sim z + m\alpha_1 + n\alpha_2, \qquad z \in \mathbb{C}, \qquad m, n \in \mathbb{Z}, \qquad \alpha_1, \alpha_2 \in \mathbb{C}, \tag{2.25}$$

where α_1 and α_2 are the circumferences of the torus. As illustrated in Figure 2.1, the pair (α_1, α_2) of complex numbers spans a lattice whose smallest cell is the *fundamental* domain of the torus. The torus is then obtained by identifying opposite edges of the fundamental domain.

The shape of the torus is described by the parameter

$$\tau = \frac{\alpha_2}{\alpha_1} = \tau_1 + i\tau_2, \tag{2.26}$$

which is called the *modular parameter* as the conformal field theory is invariant under the action of the *modular group* $PSL(2, \mathbb{Z}) \equiv SL(2, \mathbb{Z})/\mathbf{Z}_2$ on the parameter τ. The *modular group* $PSL(2, \mathbb{Z}) \equiv SL(2, \mathbb{Z})/\mathbf{Z}_2$ is the group of *Mobius* transformations $PGL(2, C)$ of the

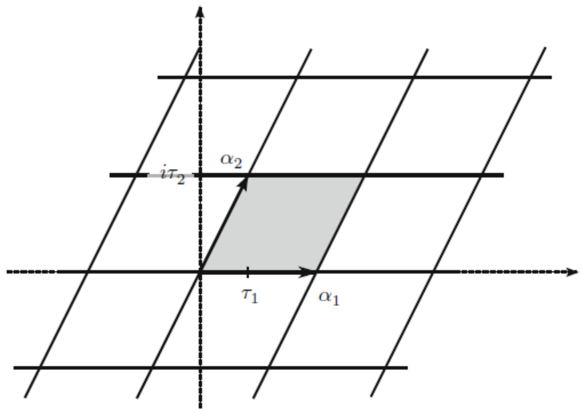

FIGURE 2.1: Lattice of a torus generated by (α_1, α_2), conveniently chosen as $(1, \tau)$. The shaded region indicates the fundamental domain of the torus, and the torus itself is obtained by identifying opposite edges thereof.

upper half of the complex plane of the form

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}, \qquad \{a, b, c, d\} \equiv \{-a, -b, -c, -d\}, \qquad ad - bc \neq 0, \qquad a, b, c, d \in \mathbb{C}, \tag{2.27}$$

but with $ad - bc = 1$ and $a, b, c, d \in \mathbb{Z}$. This group is generated by the S and T transformations, defined as

$$T : \tau \to \tau + 1 \implies T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tag{2.28}$$

$$S : \tau \to -\frac{1}{\tau} \implies S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.29}$$

Any element of the modular group can be written as a succession of applications of these two generators. For example, the identity element of the modular group is given by

$$S^2 = (ST)^3 = 1. \tag{2.30}$$

2.3.2 Partition function on a torus

The partition function of a two-dimensional conformal field theory defined on a torus generated by the lattice $(1, \tau)$ is the modular invariant trace

$$Z(\tau_1, \tau_2) = \text{Tr } \left(e^{-2\pi\tau_2 H + 2\pi i\tau_1 P}\right) \tag{2.31}$$

over the Hilbert space of the theory, where the momentum operator $P = (L_0)_{\text{cyl}} - (\bar{L}_0)_{\text{cyl}}$ generates translations in the spatial direction τ_1 and the Hamiltonian operator $H = (L_0)_{\text{cyl}} + (\bar{L}_0)_{\text{cyl}}$ generates translations in the temporal direction τ_2.

The partition function of the theory on the *corresponding* complex plane is

$$Z(\tau, \bar{\tau}) = \text{Tr } \left(q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}\right), \tag{2.32}$$

where $q = e^{2\pi i\tau}$, and L_0 and $\bar{L}_0$ are operators on the plane. To see this explicitly, we use $\left(L_0\right)_{\text{cyl}} = L_0 - \frac{c}{24}$ and $\left(\bar{L}_0\right)_{\text{cyl}} = \bar{L}_0 - \frac{c}{24}$, which follow from the mode expansion of the transformation rule for the energy-momentum tensor from the cylinder to the plane:

$$T_{\text{cyl}}(w) = z^2 T_{\text{plane}}(z) - \frac{c}{24}, \tag{2.33}$$

$$\bar{T}_{\text{cyl}}(w) = z^2 \bar{T}_{\text{plane}}(z) - \frac{c}{24}. \tag{2.34}$$

These equations, in turn, can be derived from the transformation rule (2.21):

$$T_{\text{cyl}}(w) = \left(\frac{\partial z(w)}{\partial w}\right)^2 T_{\text{plane}}(z(w)) + \frac{c}{12} S(z(w), w) = z^2 T_{\text{plane}}(z) - \frac{c}{24}, \tag{2.35}$$

$$\bar{T}_{\text{cyl}}(w) = \left(\frac{\partial z(w)}{\partial w}\right)^2 \bar{T}_{\text{plane}}(z(w)) + \frac{c}{12} S(z(w), w) = z^2 \bar{T}_{\text{plane}}(z) - \frac{c}{24}. \tag{2.36}$$

As a final remark, the partition function (2.31) for a two-dimensional conformal field theory on the torus is also the partition function for the theory on the surface of an infinite cylinder at finite temperature, as the torus is related to the infinite cylinder by a compactification of the temporal direction. Therefore, the component τ_2 is the inverse temperature $\beta/2\pi$, and the component τ_1 is the angular potential $K/2\pi$. Then, we find that the partition function on the torus is also given by

$$Z(\beta, K) = \text{Tr}\,\left(e^{-\beta H + iKP}\right). \tag{2.37}$$

Chapter 3

$T\bar{T}$-deformation of a Conformal Field Theory

Recently, Smirnov and Zamolodchikov discovered a class of *exactly solvable irrelevant* deformations of two-dimensional quantum field theories, the simplest of which is the $T\bar{T}$-deformation [12]. The $T\bar{T}$-deformed theory takes the form

$$S_{\text{QFT}}^{\mu+\delta\mu} = S_{\text{QFT}}^{\mu} + \delta\mu \int d^2x \left(T\bar{T}\right)_{\mu},$$

where the coupling $\delta\mu$ characterizes an *infinitesimal* perturbation and $\left(T\bar{T}\right)_{\mu}$ *denotes* the negative determinant of the stress-energy tensor of the '*unperturbed*' theory[1]:

$$\begin{aligned} T\bar{T} - \Theta^2 &= T_{zz}T_{\bar{z}\bar{z}} - T_{z\bar{z}}^2 \\ &= \frac{1}{4}\left[\,(T_{00} - 2iT_{01} - T_{11})\,(T_{00} + 2iT_{01} - T_{11}) - (T_{00} + T_{11})^2\,\right] \\ &= T_{01}^2 - T_{00}T_{11} \\ &= -\text{det}\,T. \end{aligned}$$

For the purposes of this thesis, $S_{\text{QFT}}^{\mu=0}$ describes a two-dimensional *conformal* field theory.

[1]a single finite deformation by the $T\bar{T}$ operator and a finite deformation composed of infinitesimal $T\bar{T}$-steps generate *different* theories!

As μ has dimensions of length squared, the $T\bar{T}$-deformation is irrelevant in the infrared, and conversely relevant in the ultraviolet. Although the $T\bar{T}$-deformation is (infrared-)irrelevant, it does not spoil the existence of the ultraviolet fixed point. Rather, the deformation is exactly solvable, in the sense that, even if the original two-dimensional conformal field theory itself has no extra symmetries other than Virasoro symmetry, the deformed theory possesses an infinite set of conserved charges and allows for exact computation of interesting physical quantities such as scattering phases, energy levels, and the thermodynamic equation of state [12, 13, 14].

This chapter proves, in section 3.4, that the strength of the $T\bar{T}$-deformation transforms as a modular form of weight 4 using the energy spectrum and the partition function of the deformed conformal field theory. To that end, we derive the energy spectrum of the theory in section 3.2 and the partition function in section 3.4. The proof of the energy spectrum uses the Zamolodchikov equation which follows from the notion of $T\bar{T}$ as an integrable deformation - section 3.1 therefore introduces $T\bar{T}$ as the simplest of an infinite number of integrable operators. The proof of the relation for the evolution of the partition function follows from the equivalence of $T\bar{T}$ to perturbation by a random, locally correlated metric - section 3.3 therefore explains the solvability of $T\bar{T}$-deformed theories as arising from their stochastic nature.

3.1 Integrability of the $T\bar{T}$-deformation

The infinitesimal $T\bar{T}$-deformation of an integrable quantum field theory generates an integrable quantum field theory. In other words, the $T\bar{T}$-composite operator is integrable. To prove this statement, we show in subsection 3.1.2 that *any integrable quantum field theory can be integrably deformed by any one of an infinite set of operators, each of which is in one-to-one correspondence with the conserved current densities of the integrable quantum field theory* - the $T\bar{T}$ deformation is the simplest of these integrable operators. Therefore, we start in subsection 3.1.1 by *defining* an integrable quantum field theory.

In order to set the notation for this section, we let Σ be the space of two-dimensional quantum field theories, and we let $\Sigma^{\text{int}} \subset \Sigma$ be the space of two-dimensional *integrable* quantum field theories. Then, the tangent space $T\Sigma$ is *clearly* the span of *scalar local* operators *modulo derivatives* and the tangent space $T\Sigma^{(\text{int})}$ is *clearly* is the span of *integrable* operators. Furthermore, we note that the quantum field theories in this section are defined on the Euclidean plane $\mathbb{R}^2$ and we denote points on this plane as $z = (\mathrm{z}, \bar{\mathrm{z}})$, where $\mathrm{z} = x + iy$ and $\bar{\mathrm{z}} = x - iy$.

3.1.1 Integrable quantum field theories

Integrable quantum field theories are quantum field theories with an *infinite* set of *local charges* (or *local integrals of motion*) P_s for positive spin $s > 0$ and $\bar{P}_s$ for negative spin $s > 0$. Each local charge is associated to a *local current density* (T_{s+1}, Θ_{s-1}) for positive spin $s > 0$ and $(\bar{\Theta}_{s-1}, \bar{T}_{s+1})$ for negative spin $s > 0$. Each of these current densities most definitely *satisfies* the *continuity equation*. These facts are illustrated below in Table 3.1. The positive-spin currents and negative-spin currents are related by $(T_{-s+1}, \Theta_{-s-1}) = (\bar{\Theta}_{s-1}, \bar{T}_{s+1})$.

As P_s and $\bar{P}_s$ are conserved charges, the integral over the contour C in any of the local charges P_s or $\bar{P}_s$ is invariant under deformations of the contour if the contour does not

	positive spin, $s > 0$	negative spin, $s > 0$
Local charges	$P_s = \int_C [T_{s+1}(z)d\mathrm{z} + \Theta_{s-1}(z)d\bar{\mathrm{z}}]$	$\bar{P}_s = \int_C [\bar{T}_{s+1}(z)d\bar{\mathrm{z}} + \bar{\Theta}_{s-1}(z)dz]$
Local currents	$(\mathrm{T}_{s+1}, \Theta_{s-1})$	$(\bar{\Theta}_{s-1}, \bar{T}_{s+1})$
Continuity equation	$\partial_{\bar{\mathrm{z}}} T_{s+1}(z) = \partial_{\mathrm{z}} \Theta_{s-1}(z)$	$\partial_{\mathrm{z}} \bar{T}_{s+1}(z) = \partial_{\bar{\mathrm{z}}} \bar{\Theta}_{s-1}(z)$

TABLE 3.1: Local conserved charges and local conserved current densities of integrable quantum field theories

cross over any of the insertion points $z_1, \ldots, z_n$:

$$\oint_C \langle [T_{s+1}(z)d\mathrm{z} + \Theta_{s-1}(z)d\bar{\mathrm{z}}] O_1(z_1)\ldots O_n(z_n)\rangle = 0. \tag{3.1}$$

The charges form a commuting set, that is,

$$[P_s, P_\sigma] = [P_s, \bar{P}_\sigma] = [\bar{P}_s, \bar{P}_\sigma] = 0.$$

This implies that

$$[P_\sigma, T_{s+1}(z)] = \partial_{\mathrm{z}} A_{\sigma,s}(z), \qquad [P_\sigma, \Theta_{s-1}(z)] = \partial_{\bar{\mathrm{z}}} A_{\sigma,s}(z)$$
$$[P_\sigma, \bar{T}_{s+1}(z)] = \partial_{\bar{\mathrm{z}}} B_{\sigma,s}(z), \qquad [P_\sigma, \bar{\Theta}_{s-1}(z)] = \partial_{\mathrm{z}} B_{\sigma,s}(z)$$

and similarly for the commutators of $\bar{P}_\sigma$ with the currents. The commutators can be expressed as the contour integral

$$[P_\sigma, O(z)] = \frac{1}{2\pi i}\oint_{C_z} [T_{\sigma+1}(w)d\mathrm{w} + \Theta_{\sigma-1}(w)d\bar{\mathrm{w}}]O(z),$$

where C_z is a contour encircling z.

3.1.2 Infinite number of integrable deformations

The goal of this section is to show that *the infinite set of operators*

$$X_s(z') = \lim_{z\to z'}[T_{s+1}(z)\bar{T}_{s+1}(z') - \Theta_{s-1}(z)\bar{\Theta}_{s-1}(z')]$$

form the set of integrable deformations of an integrable quantum field theory, i.e. $X_s(z') \in T\Sigma^{(int)}$ [12]. The $T\bar{T}$-operator is the simplest of these integrable deformations.

We first prove that the operators $X_s(z')$ span the tangent space $T\Sigma$. As the tangent space $T\Sigma$ is the span of *scalar local* operators *modulo derivatives*, we need to prove that the operators $X_s(z')$ are *scalar local* operators *modulo derivatives*. In other words, we need to prove that $\partial_{\mathrm{z}} X_s(z')$ and $\partial_{\bar{\mathrm{z}}} X_s(z')$ are derivative operators spanned by Span$\{\partial_{\mathrm{z}} O_a(z), \partial_{\bar{\mathrm{z}}} O_a(z)\}$. This is rather easy to prove. Using the continuity equations

$$\partial_{\bar{\mathrm{z}}} T_{s+1}(z) = \partial_{\mathrm{z}}\Theta_{s-1}(z), \qquad \partial_{\mathrm{z}}\bar{T}_{s+1}(z) = \partial_{\bar{\mathrm{z}}}\bar{\Theta}_{s-1}(z)$$

from Table 3.1, we find that

$$\begin{aligned}\partial_{\mathrm{z}} X_s(z') &= \lim_{z\to z'} \partial_{\mathrm{z}}[T_{s+1}(z)\bar{T}_{s+1}(z') - \Theta_{s-1}(z)\bar{\Theta}_{s-1}(z')] \\ &= \lim_{z\to z'} \left[(\partial_{\bar{\mathrm{z}}} + \partial_{\bar{\mathrm{z}}'})\Theta_{s-1}(z)\bar{T}_{s+1}(z') - (\partial_{\bar{\mathrm{z}}} + \partial_{\bar{\mathrm{z}}'})\Theta_{s-1}(z)\bar{\Theta}_{s-1}(z')\right]\end{aligned}$$

and

$$\begin{aligned}\partial_{\bar{\mathrm{z}}} X_s(z') &= \lim_{z\to z'} \partial_{\bar{\mathrm{z}}}\left[T_{s+1}(z)\bar{T}_{s+1}(z') - \Theta_{s-1}(z)\bar{\Theta}_{s-1}(z')\right] \\ &= \lim_{z\to z'} \left[(\partial_{\mathrm{z}} + \partial_{\mathrm{z}'})\Theta_{s-1}(z)\bar{T}_{s+1}(z') - (\partial_{\bar{\mathrm{z}}} + \partial_{\bar{\mathrm{z}}'})\Theta_{s-1}(z)\bar{\Theta}_{s-1}(z')\right].\end{aligned}$$

As the differential operators $(\partial_{\mathrm{z}}+\partial_{\mathrm{z}'})$ and $(\partial_{\bar{\mathrm{z}}}+\partial_{\bar{\mathrm{z}}'})$ each annihilate the coefficient functions $C^c_{ab}(z-z')$ in the operator product expansion

$$O_a(z)O_b(z') = \sum_c C^c_{ab}(z-z')O_c(z'),$$

we find that $\partial_{\mathrm{z}}X_s(z')$ and $\partial_{\bar{\mathrm{z}}}X_s(z')$ are derivative operators (which are singular at $z = z'$!) spanned by Span$\{\partial_{\mathrm{z}}O_a(z), \partial_{\bar{\mathrm{z}}}O_a(z)\}$.

We now show that the operators $X_s(z')$ span the tangent space $T\Sigma^{(\mathrm{int})}$. In other words, we show that, given an integrable quantum field theory with a set of local integrals of motion $\{P_\sigma\}$ (for spins $\{\sigma\}$) and action $\mathcal{A}_0$, the infinitesimal perturbation

$$\mathcal{A}_0 + \delta\mathcal{A}_s, \qquad \delta\mathcal{A}_s = \delta g_s \int d^2w X_s(w)$$

preserves the set of integrals of motion. More specifically, we show that the relation (3.1) holds not only in the unperturbed theory, but also in its infinitesimal perturbation.

We can break down the relation (3.1) for the perturbed theory into the sum of terms

$$\begin{aligned}
&\langle \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n) \oint_C [T_{\sigma+1}(z)d\mathrm{z} + \Theta_{\sigma-1}(z)d\bar{\mathrm{z}}]\rangle_{\mathcal{A}_0+\delta\mathcal{A}_s} \\
&\qquad = \langle \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n) \oint_C [T_{\sigma+1}(z)d\mathrm{z} + \Theta_{\sigma-1}(z)d\bar{\mathrm{z}}]\rangle_{\mathcal{A}_0} \\
&\qquad\quad - \delta g_s \int d^2w \langle \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n) \oint_C [T_{\sigma+1}(z)d\mathrm{z} + \Theta_{\sigma-1}(z)d\bar{\mathrm{z}}] X_s(w)\rangle_{\mathcal{A}_0},
\end{aligned}$$

where we neglected to write down terms with $\delta\mathcal{O}_k(z_k)$ (as they are not relevant to the proof). The first term in the above equation vanishes (as the unperturbed theory is integrable). The integral over w in the second term can be split as

$$-\int d^2w = -\int_{D(C)} d^2w - \int_{D'(C)} d^2w, \tag{3.2}$$

where $D(C)$ is the region of the plane bounded by the contour C which encloses the point w, and $D'(C)$ is the complement of $D(C)$. The integral over $D'(C)$ vanishes as $D'(C)$ does not enclose the point w. As the integral over $D(C)$ encloses the point w, the integral yields the commutator $[P_\sigma, X_s(w)]$, yet to be integrated in w over $D(C)$. It turns out that the commutator $[P_\sigma, X_s(w)] = 0 \bmod \partial\mathcal{F}$:

$$\begin{aligned}
[P_\sigma, X_s(w)] &= \lim_{w\to w'}[P_\sigma, T_{s+1}(w)\bar{T}_{s+1}(w') - \Theta_{s-1}(w)\bar{\Theta}_{s-1}(w')] \\
&= \lim_{w\to w'}\Big(T_{s+1}(w)[P_\sigma, \bar{T}_{s+1}(w')] + [P_\sigma, T_{s+1}(w)]\bar{T}_{s+1}(w') \\
&\qquad - \Theta_{s-1}(w)[P_\sigma, \bar{\Theta}_{s-1}(w')] - [P_\sigma, \Theta_{s-1}(w)]\bar{\Theta}_{s-1}(w')\Big) \\
&= \lim_{w\to w'}\Big(\partial_{\bar{\mathrm{w}}'}B_{\sigma,s}(w')T_{s+1}(w) + \partial_{\mathrm{w}}A_{\sigma,s}(w)\bar{T}_{s+1}(w') \\
&\qquad - \partial_{\mathrm{w}'}B_{\sigma,s}(w')\Theta_{s-1}(w) - \partial_{\bar{\mathrm{w}}}A_{\sigma,s}(w)\bar{\Theta}_{s-1}(w')\Big).
\end{aligned}$$

Therefore, we can write the commutator $[P_\sigma, X_s(w)]$ as

$$[P_\sigma, X_s(w)] = \frac{1}{4\pi i}\left(\partial_{\bar{\mathrm{w}}}\hat{T}_{\sigma+1,s}(w) + \partial_{\mathrm{w}}\hat{\Theta}_{\sigma-1,s}(w)\right)$$

where $\hat{T}_{\sigma+1,s}$ and $\hat{\Theta}_{\sigma-1,s}$ are some local fields of spins $\sigma+1$ and $\sigma-1$, respectively. We then find by Stokes' theorem that the integral over w evaluates to

$$-\langle\mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\oint_C[\hat{T}_{\sigma+1,s}(w)d\mathrm{w} + \hat{\Theta}_{\sigma-1,s}(w)d\bar{\mathrm{w}}]\rangle_{\mathcal{A}_0},$$

and that

$$\begin{aligned}
&\langle\mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\oint_C[T_{\sigma+1}(z)d\mathrm{z} + \Theta_{\sigma-1}(z)d\bar{\mathrm{z}}]\rangle_{\mathcal{A}_0+\delta\mathcal{A}_s} \\
&\quad = -\delta g_s\langle\mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\oint_C[\hat{T}_{\sigma+1,s}(z)d\mathrm{z} + \hat{\Theta}_{\sigma-1,s}(z)d\bar{\mathrm{z}}]\rangle_{\mathcal{A}_0}.
\end{aligned}$$

Therefore, we find that the integrals of motion $\{P_\sigma\}$, with currents $(T_{\sigma+1}+\delta g_s\,\bar{T}_{\sigma+1,s}, \Theta_{\sigma-1}+\delta g_s\,\bar{\Theta}_{\sigma-1,s})$ are conserved in the perturbed theory.

3.2 Energy Spectrum of the Primary States

The energy spectrum of a $T\bar{T}$-deformed conformal field theory with conformal dimension $(\Delta_n, \bar{\Delta}_n)$ on a cylinder with circumference L [12, 13, 14] is[2]

$$E_n(\mu, L)L = \frac{2\pi}{\tilde{\mu}}\left(1 - \sqrt{1 - 2\tilde{\mu}M_n + \tilde{\mu}^2 J_n^2}\right), \qquad \tilde{\mu} \equiv \frac{\pi\mu}{L^2}, \tag{3.3}$$

where $M_n = \Delta_n + \bar{\Delta}_n - \frac{c}{12}$ and $J_n = \Delta_n - \bar{\Delta}_n$.

3.2.1 Derivation of the spectrum

The derivation of the spectrum is rather simple and uses the Zamolodchikov equation (3.5).[3] The deformed conformal field theory on a spatial cylinder with circumference L has an energy spectrum E_n and a momentum spectrum P_n of the form

$$E_n = \frac{\mathcal{E}_n(\mu/L^2)}{L} \qquad \text{and} \qquad P_n = \frac{2\pi J_n}{L}, \qquad J_n \in \mathbb{Z},$$

such that, in the limit of the original conformal field theory, we have

$$E_n = \frac{2\pi M_n}{L} \qquad \text{and} \qquad P_n = \frac{2\pi J_n}{L}, \qquad J_n \in \mathbb{Z}. \tag{3.4}$$

The energy and momentum eigenstates are stationary and translation invariant, so we can use the Zamolodchikov relation (3.5). Letting $|n\rangle$ denote an energy and momentum

[2]The right-hand side becomes imaginary above some critical conformal dimension (for fixed $\tilde{\mu} > 0$) or above some critical value of $\tilde{\mu}$ (for fixed $\Delta_n + \bar{\Delta}_n > \frac{c}{12}$. This behavior is called the 'shock singularity' in [12] and indicates the presence of a UV cutoff.

[3]We can also derive the ground state energy of the deformed conformal field theory from its two-particle S-matrix [14].

eigenstate in the conformal field theory, we have

$$\begin{aligned}\langle n|T\bar{T} - \Theta\bar{\Theta}|n\rangle &= \langle n|T|n\rangle\langle n|\bar{T}|n\rangle - \langle n|\Theta|n\rangle\langle n|\Theta|n\rangle \\ &= -\frac{1}{4}\left(\langle n|T_{\tau\tau}|n\rangle\langle n|T_{xx}|n\rangle - \langle n|T\tau x|n\rangle\langle n|T_{\tau x}|n\rangle\right).\end{aligned}$$

The stress tensor components have physical meaning as the energy density, pressure and momentum density:

$$\langle n|T_{\tau\tau}|n\rangle = \frac{E_n}{L}, \qquad \langle n|T_{xx}|n\rangle = \frac{\partial E_n}{\partial L}, \qquad \langle n|T_{\tau x}|n\rangle = \langle n|i(T-\bar{T})|n\rangle = \frac{iP_n}{L}.$$

Using

$$H_{\text{int}} = \int d\theta\, \mathcal{L}_{\text{int}} = \mu \int d\theta\, (T\bar{T} - \Theta\bar{\Theta}),$$

where θ is the angular coordinate on the cylinder, we find that

$$\langle n|T\bar{T} - \Theta\bar{\Theta}|n\rangle = \frac{1}{L}\frac{\partial H_{\text{int}}}{\partial \mu} = \frac{1}{L}\frac{\partial E_n}{\partial \mu}.$$

Therefore, the Zamolodchikov equation in the energy and momentum eigenstate $|n\rangle$ becomes

$$0 = 4\frac{\partial E_n}{\partial \mu} + E_n\frac{\partial E_n}{\partial L} + \frac{P_n^2}{L}.$$

Using equation (3.4) as the initial conditions, we find that the energy spectrum (3.3) solves the above differential equation.

3.2.2 A small digression: the Zamolodchikov equation

The Zamolodchikov equation [20], for any *translation-invariant* and *stationary* state in any two-dimensional relativistic quantum field theory with a local stress-energy tensor, is the remarkable factorization property

$$\langle T\bar{T}\rangle = \langle T\rangle\langle\bar{T}\rangle - \langle\Theta\rangle^2. \tag{3.5}$$

Here $\Theta = T_{z\bar{z}} = \frac{1}{4}T^\alpha_\alpha$ denotes the trace of the stress tensor.

The derivation of this formula is quite straightforward. We first take two opposite limits of the expectation value of the operator product $T(z)\bar{T}(w) - \Theta(z)\Theta(w)$ to obtain

$$\begin{aligned}&\lim_{w\to z}\langle T(z)\bar{T}(w)\rangle - \langle\Theta(z)\Theta(w)\rangle = \langle T\bar{T}\rangle - \langle\Theta^2\rangle \qquad \text{and}\\ &\lim_{w\to\infty}\langle T(z)\bar{T}(w)\rangle - \langle\Theta(z)\Theta(w)\rangle = \langle T\rangle\langle\bar{T}\rangle - \langle\Theta\rangle^2,\end{aligned}$$

where the first equality is trivial and the second equality follows from the cluster decomposition theorem of local quantum field theory. The key insight that relates the two limits is that the gradient of $\langle T(z)\bar{T}(w)\rangle - \langle\Theta(z)\Theta(w)\rangle$ with respect to z and w identically vanishes:

$$\begin{aligned}&\langle\partial_z T(z)\bar{T}(w)\rangle - \langle\partial_z\Theta(z)\Theta(w)\rangle = \cdots = 0,\\ &\langle\partial_{\bar{z}} T(z)\bar{T}(w)\rangle - \langle\partial_{\bar{z}}\Theta(z)\Theta(w)\rangle = -\langle\Theta(z)\partial_w\bar{T}(w)\rangle + \langle\bar{T}\partial_w\Theta(z)(w)\rangle = 0,\end{aligned}$$

using

- the conservation laws

$$\partial_{\bar{z}}T_s = \partial_z\Theta_s, \qquad \partial_z\bar{T}_s = \partial_{\bar{z}}\bar{\Theta}_s,$$

- and the fact that in a translation invariant state, the two point functions in $\langle T(z)\bar{T}(w)\rangle - \langle \Theta(z)\Theta(w)\rangle$ depend only on the coordinate difference $z - w$.

Therefore, the function $\langle T(z)\bar{T}(w)\rangle - \langle \Theta(z)\Theta(w)\rangle$ is a constant and $\langle T\bar{T}\rangle - \langle \Theta^2\rangle = \langle T\rangle\langle \bar{T}\rangle - \langle \Theta\rangle^2$.

The Zamolodchikov equation suggests that the $T\bar{T}$-operator has exact scale dimension 4 for translationally-invariant states (that is, for states at zero momentum). In other words, $T\bar{T}$ behaves as a local scaling operator with scale dimension 4 up to total derivative terms.[4] Therefore,

- the Zamolodchikov equation describes the behavior of more general states to leading order in a derivative expansion, and
- the total derivative terms may require extra assumptions about the ultraviolet behavior of the quantum field theory, which may not obviously hold for the $T\bar{T}$-deformed theory.

[4] The absence of anomalous dimensions makes it possible that the energy spectrum is independent of ultraviolet cutoff.

3.3 Solvablity of $T\bar{T}$-deformed Theories

In this section, we show that the effective action for the $T\bar{T}$-deformation is a boundary term, which then explains the property that the $T\bar{T}$ deformation is 'solvable', even when the undeformed theory is not integrable.

The $T\bar{T}$-deformation $Z_{\mathcal{D}}^{(\delta\mu)}$ to the generating functional $Z_{\mathcal{D}}^{(\mu)}$ on a compact (simply connected) manifold $\mathcal{D}$ (which may be endowed with a flat metric) is weighted by the exponential factor

$$\begin{aligned}
&\exp\left(16\,\delta\mu\int_{\mathcal{D}} d^2x\ \left(T^{(\mu)}\bar{T}^{(\mu)} - \Theta^{(\mu)2}\right)\right)\\
&\quad= \exp\left(-16\,\delta\mu\int_{\mathcal{D}} d^2x\ \left(\det T^{(\mu)}\right)\right)\\
&\quad= \exp\left(-8\,\delta\mu\int_{\mathcal{D}} d^2x\ \epsilon_{ik}\epsilon_{jl}T^{(\mu)}_{ij}T^{(\mu)}_{kl}\right)\\
&\quad\propto \int[dh]\ \exp\left(-\frac{1}{32\delta\mu}\int_{\mathcal{D}} d^2x\ \epsilon_{ik}\epsilon_{jl}h^{(\mu)}_{ij}h^{(\mu)}_{kl} + \int_{\mathcal{D}} d^2x\ h^{(\mu)}_{ij}\epsilon_{ik}\epsilon_{jl}\epsilon_{km}\epsilon_{ln}T^{(\mu)}_{mn}\right)\\
&\quad= \int[dh]\ \exp\left(-\frac{1}{32\delta\mu}\int_{\mathcal{D}} d^2x\ \epsilon_{ik}\epsilon_{jl}h^{(\mu)}_{ij}h^{(\mu)}_{kl} + \int_{\mathcal{D}} d^2x\ h^{(\mu)}_{ij}T^{(\mu)}_{ij}\right), \qquad (3.6)
\end{aligned}$$

where, in the penultimate step, we decoupled the perturbation to the action by applying the Hubbard-Stratonovich transformation on the gaussian integral over $\mathcal{D}$ to obtain a new gaussian integral over a *symmetric*[5] tensor field $h^{(\mu)}_{ij}$, and in the final step, we used $\epsilon_{ik}\epsilon_{jl}\epsilon_{km}\epsilon_{ln} = \delta_{im}\delta_{jn}$ to simplify the integral. As the stress-energy tensor $T^{(\mu)}_{ij}$ is the conserved current associated to the *infinitesimal* diffeomorphism $g^{(\mu)}_{ij} = \delta_{ij} + h^{(\mu)}_{ij}$ of the Euclidean flat spacetime, we note the following points:

- We can get by *without* a specfication of the contour of integration over the space of *perturbations* $h^{(\mu)}_{ij}$ of the metric as we only use the configuration of the stress-energy tensor $T^{(\mu)}_{ij}$ that lies at the extremum of the action functional in this proof.

[5] $h^{(\mu)}_{ij}$ is symmetric as $T^{(\mu)}_{ij}$ is symmetric

- The perturbation $h_{ij}^{(\mu)}$ has the scalar-vector decomposition

$$h_{ij}^{(\mu)} = \partial_i \alpha_j^{(\mu)} + \partial_j \alpha_i^{(\mu)} - \delta_{ij}\Phi^{(\mu)}, \tag{3.7}$$

where $\partial_i \alpha_j^{(\mu)} + \partial_j \alpha_i^{(\mu)}$ corresponds to the infinitesimal diffeomorphism $x_i \to x_i + \alpha_i^{(\mu)}(x)$ of the euclidean metric, and $e^{\Phi^{(\mu)}}$ is an infinitesimal change in the conformal factor.

Having determined the form of $h_{ij}^{(\mu)}$, we now simplify the exponents in the gaussian integral (3.6) separately. The first exponent simplifies to

$$\begin{aligned}
\int_{\mathcal{D}} d^2x\ \epsilon_{ik}\epsilon_{jl} h_{ij}^{(\mu)} h_{kl}^{(\mu)} &= \int_{\mathcal{D}} d^2x\ \epsilon_{ik}\epsilon_{jl} \Big(\partial_i \alpha_j^{(\mu)} + \partial_j \alpha_i^{(\mu)} + \delta_{ij}\Phi^{(\mu)}\Big)\Big(\partial_k \alpha_l^{(\mu)} + \partial_l \alpha_k^{(\mu)} + \delta_{kl}\Phi^{(\mu)}\Big) \\
&= \int_{\mathcal{D}} d^2x\ \epsilon_{ik}\epsilon_{jl} \Big(4(\partial_i \alpha_j^{(\mu)})(\partial_k \alpha_l^{(\mu)}) + 4(\partial_i \alpha_j^{(\mu)})(\delta_{kl}\Phi^{(\mu)}) + (\delta_{ij}\Phi^{(\mu)})(\delta_{kl}\Phi^{(\mu)})\Big) \\
&= 4\int_{\mathcal{D}} d^2x\ \left(\partial_i \left[\epsilon_{ik}\epsilon_{jl}\alpha_j^{(\mu)}\partial_k \alpha_l^{(\mu)}\right] + \Phi^{(\mu)}\partial_k \alpha_k^{(\mu)} + \left(\Phi^{(\mu)}\right)^2\right) \\
&= 4\int_{\partial\mathcal{D}} dn_i\ \left(\epsilon_{ik}\epsilon_{jl}\alpha_j^{(\mu)}\partial_k \alpha_l^{(\mu)}\right) + 4\int_{\mathcal{D}} d^2x\ \left(\Phi^{(\mu)}\partial_k \alpha_k^{(\mu)} + \left(\Phi^{(\mu)}\right)^2\right),
\end{aligned}$$

where dn_i is the outward pointing normal line element. As only the symmetric combination $\partial_i \alpha_j^{(\mu)} + \partial_j \alpha_i^{(\mu)}$ enters in $h_{ij}^{(\mu)}$, we may restrict $\alpha_i^{(\mu)}$ to be irrotational - if we decompose in general $\alpha_i^{(\mu)} = \partial_i \phi^{(\mu)} + \epsilon_{ik}\partial_k \psi^{(\mu)}$, then $\partial_k\partial_k \psi^{(\mu)} = 0$. Then, we find that

$$\begin{aligned}
\int_{\mathcal{D}} d^2x\ \epsilon_{ik}\epsilon_{jl} h_{ij}^{(\mu)} h_{kl}^{(\mu)} &= 4\int_{\partial\mathcal{D}} dn_i\ \left(\epsilon_{ik}\epsilon_{jl}\alpha_j^{(\mu)}\partial_k \alpha_l^{(\mu)}\right) + 4\int_{\mathcal{D}} d^2x\ \left(\Phi^{(\mu)}\partial_k \alpha_k^{(\mu)} + \left(\Phi^{(\mu)}\right)^2\right) \\
&= 4\int_{\partial\mathcal{D}} dn_i\ \left(\epsilon_{ik}\epsilon_{jl}\alpha_j^{(\mu)}\partial_k \alpha_l^{(\mu)}\right) + 4\int_{\mathcal{D}} d^2x\ \left(\Phi^{(\mu)}\partial_k\partial_k \phi^{(\mu)} + \left(\Phi^{(\mu)}\right)^2\right) \\
&= 4\int_{\partial\mathcal{D}} dn_i\ \left(\epsilon_{ik}\epsilon_{jl}\alpha_j^{(\mu)}\partial_k \alpha_l^{(\mu)}\right) \\
&\quad + 4\int_{\mathcal{D}} d^2x\ \left(\partial_k\left[\Phi^{(\mu)}\partial_k \phi^{(\mu)} - \phi^{(\mu)}\partial_k^{(\mu)}\Phi^{(\mu)}\right] + \phi^{(\mu)}\partial_k\partial_k \Phi^{(\mu)} + \left(\Phi^{(\mu)}\right)^2\right) \\
&= 4\int_{\partial\mathcal{D}} dn_i\ \left(\epsilon_{ik}\epsilon_{jl}\alpha_j^{(\mu)}\partial_k \alpha_l^{(\mu)} + \Phi^{(\mu)}\partial_k \phi^{(\mu)} - \phi^{(\mu)}\partial_k^{(\mu)}\Phi^{(\mu)}\right) + 4\int_{\mathcal{D}} d^2x\ \left(\Phi^{(\mu)}\right)^2,
\end{aligned} \tag{3.8}$$

where we dropped the third term as $\partial_k \partial_k \Phi = 0$. Moreover, the second exponent simplifies to

$$\begin{aligned}
\int_{\mathcal{D}} d^2x \, h_{ij}^{(\mu)} T_{ij}^{(\mu)} &= \int_{\mathcal{D}} d^2x \, T_{ij}^{(\mu)} \left(\partial_i \alpha_j^{(\mu)} + \partial_j \alpha_i^{(\mu)} + \delta_{ij} \Phi^{(\mu)} \right) \\
&= \int_{\mathcal{D}} d^2x \, \left[2\partial_i \left(T_{ij}^{(\mu)} \alpha_j^{(\mu)} \right) + \delta_{ij} T_{ij}^{(\mu)} \Phi^{(\mu)} \right] \\
&= \int_{\partial\mathcal{D}} dn_i \, T_{ij}^{(\mu)} \alpha_j^{(\mu)} + \int_{\mathcal{D}} d^2x \, \delta_{ij} T_{ij}^{(\mu)} \Phi^{(\mu)}. \qquad (3.9)
\end{aligned}$$

We now note that, because h couples to a conserved quantity, *the integration is over flat metrics only*. This means that, the last term in (3.7) may be absorbed into a shift in α, and we can drop the terms in $\Phi^{(\mu)}$ in (3.8) and (3.9). Therefore, the exponential factor (3.6) becomes the boundary term

$$\begin{aligned}
&\exp \left[\left(\frac{1}{8\,\delta\mu} \right) \int_{\partial\mathcal{D}} dn_i \left(\epsilon_{ik} \epsilon_{jl} \left(\alpha_j^{(\mu)} \partial_k \alpha_l^{(\mu)} \right) - \alpha_j^{(\mu)} T_{ij}^{(\mu)} \right) \right] \\
&\quad = \exp \left[\left(\frac{1}{8\,\delta\mu} \right) \int_{\partial\mathcal{D}} ds_k \left(\epsilon_{jl} \left(\alpha_j^{(\mu)} \partial_k \alpha_l^{(\mu)} \right) - \alpha_j^{(\mu)} T_{ij}^{(\mu)} \epsilon_{ik} \right) \right], \qquad (3.10)
\end{aligned}$$

where ds_k is the tangential line element.

Therefore, adding the infinitesimal $T\bar{T}$-deformation is equivalent to coupling the theory to a random, locally correlated metric. However, it is sufficient to restrict to flat metrics, which correspond to infinitesimal diffeomorphisms $x_i \to x_i + \alpha_i(x)$, and the resulting quadratic action for α_i is then a total derivative. This simple fact at the root of the solvability of the deformation.

3.4 Partition Function on a Torus

In this section, we show that the partition function $Z^{(\mu)}$ on a flat torus satisfies the differential equation [21]

$$\frac{\partial Z^{(\mu)}}{\partial \mu} = \left(\frac{\partial}{\partial L_0} \frac{\partial}{\partial L_1'} - \frac{\partial}{\partial L_1} \frac{\partial}{\partial L_0'} \right) Z^{(\mu)}, \tag{3.11}$$

where the points $(0, 0)$, (L_1, L_2), (L_1', L_2') and $(L_1 + L_1', L_2 + L_2')$ define the vertices of the fundamental domain of the torus.

We begin with the exponential weight (3.10) of the $T\bar{T}$-deformation $Z_{\mathcal{D}}^{(\delta\mu)}$ to the generating functional $Z_{\mathcal{D}}^{(\mu)}$ on a compact (simply connected) manifold $\mathcal{D}$ (which may be endowed with a flat metric). The first integral in this factor evaluates to

$$\begin{aligned}
\int_{\partial\mathcal{D}} ds_k \, \epsilon_{jl} \left(\alpha_j^{(\mu)} \partial_k \alpha_l^{(\mu)} \right) &= \int_{\partial\mathcal{D}} ds_0 \, \left[\epsilon_{jl} \left(\alpha_j^{+(\mu)} \partial_0 \alpha_l^{+(\mu)} - \alpha_j^{-(\mu)} \partial_0 \alpha_l^{-(\mu)} \right) \right] \\
&+ \int_{\partial\mathcal{D}} ds_1 \, \left[\epsilon_{jl} \left(\alpha_j^{+(\mu)} \partial_1 \alpha_l^{+(\mu)} - \alpha_j^{-(\mu)} \partial_1 \alpha_l^{-(\mu)} \right) \right] \\
&= \int_{\partial\mathcal{D}} ds_0 \, \Big[\left(\alpha_0^{+(\mu)} \partial_0 \alpha_1^{+(\mu)} - \alpha_0^{-(\mu)} \partial_0 \alpha_1^{-(\mu)} \right) \\
&\qquad - \left(\alpha_1^{+(\mu)} \partial_0 \alpha_0^{+(\mu)} - \alpha_1^{-(\mu)} \partial_0 \alpha_0^{-(\mu)} \right) \Big] \\
&+ \int_{\partial\mathcal{D}} ds_1 \, \Big[\left(\alpha_0^{+(\mu)} \partial_1 \alpha_1^{+(\mu)} - \alpha_0^{-(\mu)} \partial_1 \alpha_1^{-(\mu)} \right) \\
&\qquad - \left(\alpha_1^{+(\mu)} \partial_1 \alpha_0^{+(\mu)} - \alpha_1^{-(\mu)} \partial_1 \alpha_0^{-(\mu)} \right) \Big],
\end{aligned} \tag{3.12}$$

where $\pm$ are the values of α on the opposite boundaries. The first term in (3.12) evaluates to

$$
\begin{aligned}
\int_{\partial\mathcal{D}} ds_0 \, \Big[\Big(\alpha_0^{+(\mu)} \partial_0 \alpha_1^{+(\mu)} - \alpha_0^{-(\mu)} \partial_0 \alpha_1^{-(\mu)} \Big) - \Big(\alpha_1^{+(\mu)} \partial_0 \alpha_0^{+(\mu)} - \alpha_1^{-(\mu)} \partial_0 \alpha_0^{-(\mu)} \Big) \Big] \\
= \Big[\alpha_0^{(\mu)} \Big]_1 \Big(\int_{\partial\mathcal{D}} ds_0 \; \partial_0 \alpha_1^{(\mu)} \Big) - \Big[\alpha_1^{(\mu)} \Big]_1 \Big(\int_{\partial\mathcal{D}} ds_0 \; \partial_0 \alpha_0^{(\mu)} \Big) \\
= \Big[\alpha_0^{(\mu)} \Big]_1 \Big[\alpha_1^{(\mu)} \Big]_0 - \Big[\alpha_1^{(\mu)} \Big]_1 \Big[\alpha_0^{(\mu)} \Big]_0 ,
\end{aligned}
$$

where, in the first step, we used continuity of partial derivatives to write $\partial_i \alpha_j^{+(\mu)} = \partial_i \alpha_j^{-(\mu)}$ and defined $[\alpha_i^{(\mu)}]_j$ to be the difference in the values $\alpha_i^{+(\mu)}$ and $\alpha_i^{-(\mu)}$ at opposite edges parallel to s_j.

Therefore, we find that only the discontinuities in α contribute to the effective action, and that they are constant along each edge. This means that, as expected, only *constant* flat metrics h_{ij} contribute. Therefore, we can write (3.6) as

$$
\frac{A}{\delta\mu} \int \prod_{i,j=0}^{1} dh_{ij} \; \exp\left(-\frac{A}{16\,\delta\mu} (h_{00}h_{11} - h_{01}^2) + A h_{ij} T_{ij} \right),
$$

where we have incorporated some inessential factors into the measure, and A is the total area of the torus. The second term in the exponent is equivalent to shifting the vertices of the parallelogram according to $L \to L + \delta L$, where

$$
\delta L_0 = \frac{1}{2} h_{00} L_0 + \frac{1}{2} h_{01} L_1, \qquad \delta L_1 = \frac{1}{2} h_{10} L_0 + \frac{1}{2} h_{11} L_1,
$$

and similarly for L'. Thus, the $T\bar{T}$-deformation to the partition function is

$$
\begin{aligned}
Z^{(\mu+\delta\mu)}(L, L') &= \frac{A}{\delta\mu} \int \prod_{i,j=0}^{1} dh_{ij} \; \exp\left(-\frac{A}{16\,\delta\mu} (h_{00}h_{11} - h_{01}^2) + \log \; Z^{(\mu)}(L + \delta L, L' + \delta L') \right) \\
&= \frac{A}{\delta\mu} \int \prod_{i,j=0}^{1} dh_{ij} \; Z^{(\mu)}(L + \delta L, L' + \delta L') \exp\left(-\frac{A}{16\,\delta\mu} (h_{00}h_{11} - h_{01}^2) \right)
\end{aligned}
$$

where we Taylor expand the right hand side in powers of h_{ij}. The first non-zero term is that proportional to

$$(\partial_{h_{00}}\partial_{h_{11}} - \partial_{h_{01}}^2)Z^{(\mu)}(L + \delta L, L' + \delta L'),$$

which gives a result proportional to

$$\begin{aligned}\frac{\delta\mu}{A}&\left[\left(L_0\frac{\partial}{\partial L_0} + L_0'\frac{\partial}{\partial L_0'}\right)\left(L_1\frac{\partial}{\partial L_1} + L_1'\frac{\partial}{\partial L_1'}\right) - \left(L_1\frac{\partial}{\partial L_0} + L_1'\frac{\partial}{\partial L_0'} + L_0\frac{\partial}{\partial L_1} + L_0'\frac{\partial}{\partial L_1'}\right)^2\right]\\ &= \frac{\delta\mu}{A}(L_0L_1' - L_1L_0')\left(\frac{\partial}{\partial L_0}\frac{\partial}{\partial L_1'} - \frac{\partial}{\partial L_1}\frac{\partial}{\partial L_0'}\right).\end{aligned}$$

Recognizing the factor $L_0L_1' - L_1L_0'$ as the area A, we then find our main result (3.11) for the torus.

3.5 Modular Transformation of the $T\bar{T}$ Coupling

In this section, we determine the transformation of the coupling μ under the modular transformation of the quantum field theory (4.4) on a torus. We find that *the coupling μ transforms as a modular form of weight 4*. This is the central result of this thesis.

The partition function of a *thermal* conformal field theory *on a cylinder* is periodic in imaginary time: $it = t_E \sim t_E + \beta$, where β is the inverse temperature. This means that the conformal field theory lives on a two-torus with modular parameter $\tau = \dfrac{\theta + i\beta}{L}$, where θ is an angular potential and τ transforms under the modular group $PSL(2,\mathbb{Z}) \equiv SL(2,\mathbb{Z})/\mathbf{Z}_2$.

The modular group $PSL(2,\mathbb{Z}) \equiv SL(2,\mathbb{Z})/\mathbf{Z}_2$ is generated by the two transformations

$$S : z \mapsto -1/z,$$
$$T : z \mapsto z + 1,$$

so that every element in the modular group $PSL(2,\mathbb{Z}) \equiv SL(2,\mathbb{Z})/\mathbf{Z}_2$ can be represented (in a non-unique way) by the composition of powers of S and T. The generators S and T obey the relations $S^2 = I$ and $(ST)^3 = I$. These are a complete set of relations, so the modular group Γ has the presentation[6]:

$$\Gamma \cong \langle S, T \mid S^2 = I, (ST)^3 = I \rangle.$$

We would like to determine the transformation of the coupling μ under the S-transformation $\tau \to -1/\tau$ and the T-transformation $\tau \to \tau + 1$ of the two-torus.

[6]One method of defining a group is by a presentation. One specifies a set S of generators so that every element of the group can be written as a product of powers of some of these generators, and a set R of relations among those generators. We then say that G has presentation $\langle S \mid R \rangle$.

The S-transformation of the two-torus reduces to

$$\tau \to -1/\tau \implies \frac{\theta + i\beta}{L} \to -\frac{L}{\theta + i\beta} \implies \theta + i\beta \to -\left(\frac{L^2\theta}{\theta^2 + \beta^2}\right) + i\left(\frac{L^2\beta}{\theta^2 + \beta^2}\right)$$

Therefore, we would like to determine the transformation of the coupling μ under the *simultaneous* transformations $\theta \to -\left(\frac{L^2\theta}{\theta^2 + \beta^2}\right)$ and $\beta \to \left(\frac{L^2\beta}{\theta^2 + \beta^2}\right)$, that is, under the *simultaneous* scaling $\theta \to -\lambda\theta$ and $\beta \to \lambda\beta$ for the dimensionless number $\lambda = \frac{L^2}{\theta^2 + \beta^2}$. To that end, we use the energy spectrum (3.3) of the deformed theory. We then check our answer using the differential equation (3.11) for the partition function.

Proof using the energy spectrum:

We use the following facts:

1. As the S-transformation $\tau \to -1/\tau$ is the interchange $(\beta, L) \to (L, -\beta)$, we find that a series of S-transformations is the interchange

$$(\beta, L) \to (L, -\beta) \to (-\beta, -L) \to (-L, \beta) \to (\beta, L).$$

2. Under a scaling $L \to \lambda L$ of the deformed energy spectrum (3.3) by some dimensionless number λ, we have $\mu \to \lambda^2\mu$.

3. The scaling $L \to \lambda L$ is due to the scaling $\phi \to \lambda\phi$, where $\phi = \theta + i\beta$, and therefore accompanies the scaling $\theta \to \lambda\theta$.

We then find that

$$\begin{aligned}
Z(\beta, L, \mu, \theta) &= Z(L, -\beta, \mu, \theta) && \text{(by property 1)} \\
&= Z(L, -\lambda\beta, \lambda^2\mu, \lambda\theta) && \text{(by property 2)} \\
&= Z\left(L, -\frac{L^2\beta}{\theta^2+\beta^2}, \frac{L^4}{(\theta^2+\beta^2)^2}\mu, \frac{L^2\theta}{\theta^2+\beta^2}\right) && \left(\text{by the choice } \lambda = \frac{L^2}{\theta^2+\beta^2}\right) \\
&= Z\left(L, \frac{L^2\beta}{\theta^2+\beta^2}, \frac{L^4}{(\theta^2+\beta^2)^2}\mu, -\frac{L^2\theta}{\theta^2+\beta^2}\right) && \text{(by property 3)} \\
&= Z\left(\frac{L^2\beta}{\theta^2+\beta^2}, L, \frac{L^4}{(\theta^2+\beta^2)^2}\mu, -\frac{L^2\theta}{\theta^2+\beta^2}\right) && \text{(as the torus is Euclidean).}
\end{aligned}$$

Therefore, the S-transformation induces the transformation $\mu \to \frac{L^4}{(\theta^2+\beta^2)^2}\mu$ of the coupling μ, that is, the transformation $\mu \to \frac{\mu}{|\tau|^4}$.

Proof using the evolution of the partition function:

We first note that the scaling $L \to \lambda L$ accompanies the scalings $L_0 + iL_1 \to \lambda\left(L_0 + iL_1\right)$ and $L_0' + iL_1' \to \lambda\left(L_0' + iL_1'\right)$, and that the partial derivative in (3.11) scales as

$$\frac{\partial}{\partial L_0}\frac{\partial}{\partial L_1'} - \frac{\partial}{\partial L_1}\frac{\partial}{\partial L_0'} \to \frac{1}{\lambda^2}\left(\frac{\partial}{\partial L_0}\frac{\partial}{\partial L_1'} - \frac{\partial}{\partial L_1}\frac{\partial}{\partial L_0'}.\right)$$

The latter follows straightforwardly from an application of the chain rule. For example, the first term in the derivative transforms as

$$\begin{aligned}
\frac{\partial}{\partial L_0^{\text{after}}}\frac{\partial}{\partial L_1'^{\text{after}}} &= \frac{\partial}{\partial L_0^{\text{after}}}\left(\frac{\partial L_1'^{\text{before}}}{\partial L_1'^{\text{after}}}\frac{\partial}{\partial L_1'^{\text{before}}}\right) \\
&= \frac{1}{\lambda}\frac{\partial}{\partial L_0^{\text{after}}}\frac{\partial}{\partial L_1'^{\text{before}}} \\
&= \frac{1}{\lambda}\left(\frac{\partial L_0^{\text{before}}}{\partial L_0^{\text{after}}}\frac{\partial}{\partial L_0^{\text{before}}}\right)\frac{\partial}{\partial L_1'^{\text{before}}} \\
&= \frac{1}{\lambda^2}\frac{\partial}{\partial L_0^{\text{before}}}\frac{\partial}{\partial L_1'^{\text{before}}}.
\end{aligned}$$

Therefore, we find that the coupling μ scales as $\mu \to \lambda^2\mu$, or as $\mu \to \dfrac{\mu}{|\tau|^4}$. This confirms the effect on the coupling of the S-transformation.

We now complete the remainder of the proof. The S-transformation and the T-transformation together generate the following transformation for the modular parameter τ:

$$\tau \to \frac{a\tau + b}{c\tau + d} = \gamma\tau.$$

We *postulate* that the S-transformation and the T-transformation together generate the following transformation for the coupling μ:

$$\mu \to \frac{\mu}{|c\tau + d|^4}. \tag{3.13}$$

Let's *confirm* our *postulated* modular transformation for the coupling μ by *checking* that the group multiplication law for $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\mathbb{Z}_2$ holds. For matrices

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{and} \qquad \gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

we have that

$$\gamma'\gamma = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a'a + b'c & a'b + b'd \\ c'a + d'c & c'b + d'd \end{pmatrix},$$

which means that, under the action of $\gamma'\gamma$, the modular parameter τ and the coupling μ *must* transform as

$$\tau \to \frac{(a'a + b'c)\tau + (a'b + b'd)}{(c'a + d'c)\tau + (c'b + d'd)}, \qquad \mu \to \frac{\mu}{|(c'a + d'c)\tau + (c'b + d'd)|^4}.$$

Let's check this explicitly:

$$\tau \to \frac{a\tau + b}{c\tau + d} \to \frac{a'\left(\frac{a\tau+b}{c\tau+d}\right) + b'}{c'\left(\frac{a\tau+b}{c\tau+d}\right) + d'} = \frac{(a'a + b'c)\tau + (a'b + b'd)}{(c'a + d'c)\tau + (c'b + d'd)}$$

$$\mu \to \frac{\mu}{|c\tau + d|^4} \to \frac{\mu/(c\tau + d)^4}{\left|c'\left(\frac{a\tau+b}{c\tau+d}\right) + d'\right|^4} = \frac{\mu}{|(c'a + d'c)\tau + (c'b + d'd)|^4}.$$

This *confirms* our *postulated* modular transformation (3.13) for the coupling μ.

Chapter 4

Cutoff-AdS/deformed-CFT correspondence

This chapter introduces, in section 4.2, the gravitational spacetime which is dual to the $T\bar{T}$-deformed conformal field theory. The deformed correspondence is a generalization of the AdS/CFT correspondence under a flow along the $T\bar{T}$-deformation in the space of couplings which perturb the boundary conformal field theory. Subsection 4.2.1, in particular, narrows the discussion of the deformed holographic dictionary to its thermodynamics and outlines a proof that each pair of the thermodynamic properties of the deformed bulk and boundary match one another. We precede this discussion with an introduction to the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence in section 4.1.

As a final note, this chapter includes a discussion of BTZ black holes in the context of the AdS/CFT correspondence. So, we review below the characteristics of BTZ black holes:

- the metric of a BTZ black hole with mass M and angular momentum J (with left- and right inverse temperature $\beta_\pm$) can be written as

$$
\begin{aligned}
ds^2 &= -f^2(r)dt^2 + f^{-2}(r)dr^2 + r^2(d\theta - \omega(r)dt)^2, \\
f^2(r) &= r^2 - 8GM + \frac{16G^2J^2}{r^2}, \qquad \omega(r) = \frac{4GJ}{r^2}, \\
M &= \frac{r_+^2 + r_-^2}{8G}, \qquad J = \frac{r_+ r_-}{4G}, \qquad \beta_\pm = \frac{2\pi}{r_+ \mp r_-}
\end{aligned}
$$

- the *uncharged* and *non-rotating* Einstein geometries with a local AdS_3 metric have the following *mass* spectrum:

$M = -1/8G :$	global AdS_3
$-1/8G < M < 0 :$	AdS_3 with a conical defect
$M = 0 :$	Poincaré-AdS_3
$M > 0 :$	massive BTZ black hole

4.1 AdS versus CFT

Anti-de Sitter spacetime is the maximally symmetric solution of the Einstein equations with negative cosmological constant. In global coordinates, the metric of three-dimensional anti-de Sitter spacetime is

$$ds^2 = \ell^2 \left(-\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\phi^2\right).$$

Under the conformal compactification $d\sigma = \dfrac{d\rho}{\cosh \rho}$, the spacetime transforms into a solid cylinder, where the radial coordinate σ ranges from 0 to $\pi/2$, and the coordinates (t, ϕ) along the surface of the cylinder each describes a sphere S^2. The Poincaré patch covers a wedge of the Euclidean solid cylinder and is described by

$$ds^2 = \frac{\ell^2}{z^2}(dz^2 + dx^2),$$

where x is a coordinate on $\mathbb{R}^3$ and the boundary is at $z = 0$. The following discussion of the AdS/CFT correspondence will describe AdS as the Poincaré patch.

AdS/CFT is a dictionary of correspondences between a theory of quantum gravity on an asymptotically anti-de Sitter spacetime and a conformal field theory on the boundary of the spacetime. As examples, we note the following:

- The thermal partition functions $Z_{\text{CFT}}(\beta) = \text{Tr } e^{-\beta H_{\text{CFT}}}$ and $Z_{\text{grav}}(\beta) = e^{-S_E[g]} + \ldots$, where the ellipsis denotes quantum corrections, must equal one other.

- The GKPW dictionary [2, 3] states that

$$Z_{\text{grav}}[\phi_0^i(x); \partial M] = \left\langle \exp\left(-\sum_i \int d^2x \, \phi_0^i(x) O^i(x)\right) \right\rangle_{\text{CFT on } \partial M}, \tag{4.1}$$

  where

- the boundary value of a bulk field, such as the metric, in $\{\phi_0^i(x)\}$ acts as a source for the corresponding boundary operator, the stress tensor, in $\{O^i(x)\}$,

- the boundary conditions on bulk scalars are

$$\phi^i(z,x) = \lim_{z\to 0}\left(z^{d-\Delta}\phi_0^i(x) + \dots\right), \tag{4.2}$$

  where the ellipsis denotes subleading terms, and

- the mass m of a bulk scalar is related to the scaling dimension Δ of the corresponding boundary operator by

$$m^2 = \Delta(d-\Delta). \tag{4.3}$$

- The correlation functions $\langle O_1(x_1)\cdots O_n(x_n)\rangle_{\text{CFT}}$ and $\lim_{\rho\to\infty}\langle\phi_1(\rho,x_1)c\dots\phi_n(\rho,x_n)\rangle_{\text{grav}}$ computed from the GKPW dictionary must equal one another.

4.2 Radially-cutoff AdS versus $T\bar{T}$-deformed CFT

The AdS/CFT correspondence states that a (massive) anti-de sitter black hole spacetime is dual to a (thermal) conformal field theory on a cylinder. We can deform the bulk anti-de sitter spacetime and the dual conformal field theory on the boundary and still retain the bulk/boundary correspondence. For example [4], if we slice off the asymptotic region of the bulk to leave the compact subregion described by

$$ds^2_{\text{AdS}} = \frac{dr^2}{r^2} + r^2 g_{\alpha\beta} dx^\alpha dx^\beta, \qquad r < r_c,$$

then the dual quantum field theory is simply the original conformal field theory deformed by the composite operator $T\bar{T}$ so that

$$S_{\text{QFT}} = S_{\text{CFT}} + \mu \int d^2x \, T\bar{T}, \tag{4.4}$$

where μ is related to r_c via

$$\mu = \frac{16\pi G}{r_c^2} = \frac{24\pi}{c}\frac{1}{r_c^2}.$$

As we set $\ell_{\text{AdS}} = 1$, the Brown-Henneaux relation [18] used in the above equation is $c = 3/2G$. We therefore find that the coupling μ acts as a geometric cutoff that removes the asymptotic region of the anti-de sitter spacetime, and thereby places the quantum field theory on a Dirichlet wall at a finite radial distance $r = r_c$ from the center of the bulk.

We note that the deformation preserves Lorentz invariance because $T\bar{T} = \frac{1}{8}T^{\alpha\beta}T_{\alpha\beta} - \frac{1}{16}(T^\alpha_\alpha)^2$. Moreover, by finite μ, we mean that there is a one-parameter family of theories defined by $dS^{(\mu)}_{\text{QFT}}/d\mu = \int d^2x \, (T\bar{T})_\mu$, where the μ subscript of $T\bar{T}$ emphasizes that in this equation we have to use the stress tensor of $S^{(\mu)}_{\text{QFT}}$.

The deformation affects the standard holographic dictionary between physical quantities of the conformal field theory and corresponding quantities in the anti-de sitter spacetime. This can be quantitatively checked for certain examples of the special subclass of quantities that can be created or measured by the stress tensor, or equivalently, by deformations of the metric. Examples of such quantities are signal propagation speed, finite size effects, thermodynamic properties (such as the equation of state, energy spectrum, pressure, temperature, heat capacity, etc.), and the Euclidean partition function $Z_{\text{QFT}}(g_{\alpha\beta}, \mu)$ in a general background metric $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$.

We note that, at large central charge c (which is the classical limit of small gravitational constant G), we can identify

$$Z_{\text{QFT}}(g_{\alpha\beta}, \mu) = \exp\left(-\frac{1}{16\pi G} S_{\text{cl}}(r_c^2 g_{\alpha\beta})\right),$$

where $S_{\text{cl}}(r_c^2 g_{\alpha\beta})$ is the action of three-dimensional classical pure gravity restricted to the region $r < r_c$, with Dirichlet boundary conditions $ds^2|_{r=r_c} = r_c^2 g_{\alpha\beta}dx^\alpha dx^\beta$ on the metric and $\phi_i|_{r=r_c} = 0$ on all bulk fields ϕ_i.

4.3 Thermodynamics of the Deformed Correspondence

The energy spectrum (3.3) of the $T\bar{T}$-deformed conformal field theory is dual to the quasi-local energy of the radially-cutoff BTZ black hole, with Dirichlet boundary conditions

$$ds^2|_{r=r_c} = r_c^2 dx^+ dx^-,$$

where r_c is the radius of the black hole. The quasi-local energy for this spacetime was computed in [19] (without any reliance on or reference to the holographic dictionary) by integrating the Brown-York stress-energy tensor over the boundary surface. The computation returns

$$E = \frac{r_c}{4G}\left[1 - \sqrt{1 - \frac{8GM}{r_c^2} + \frac{16G^2J^2}{r_c^4}}\right]$$

for the quasi-local energy in the bulk with mass M and angular momentum J, which matches the energy spectrum (3.3) of the $T\bar{T}$-deformed conformal field theory, again provided we identify

$$\mu = \frac{24\pi}{c}\frac{1}{r_c^2}.$$

The agreement between the energy spectra in the bulk and the boundary extends to a precise holographic correspondence between all thermodynamic quantities, such as the equation of state, pressure, temperature, heat capacity, etc. As the equations for the energy spectrum remain valid for finite values of the coupling, these provide a tool to probe for new physical insights which manifest themselves deep inside the bulk anti-de sitter spacetime. For example, the shock singularity of the deformed conformal field theory (above which (3.3) becomes imaginary) maps to the singular characteristics (such as a diverging temperature and pressure) of the radially-cutoff BTZ black hole as the radius approaches the horizon. An investigation of this physical transition would reveal insights about the physics of black hole horizons.

In the following subsection, we summarize the derivation of the total quasi-local gravitational energy of the black hole as a function of the radius r_c, and show that it agrees with the energy spectrum (3.3) of the $T\bar{T}$-deformed conformal field theory.

4.3.1 Energy spectrum on the deformed bulk and boundary

We use the following points to guide our computation of the quasi-local energy of a rotating BTZ black hole (4.6) with mass M and angular momentum J inside a Dirichlet wall $B = \{r = r_c\}$ defined by the boundary condition

$$ds^2|_B = g_{\alpha\beta}dx^\alpha dx^\beta = -N^2 dt^2 + e^{2\varphi}(d\theta - \omega dt)^2. \tag{4.5}$$

- The gravitational action for a pure three-dimensional spacetime with negative cosmological constant $\Lambda = -1$ and with a timelike boundary B is

  $$S = \frac{1}{16\pi G}\int d^3x\,\sqrt{-g_3}(R+2) - \frac{1}{8\pi G}\int_B d^2x\,\sqrt{-g}(K+1), \tag{4.6}$$

  where K denotes the extrinsic curvature of the boundary, and the boundary cosmological constant is tuned so that it cancels the leading volume divergence of the bulk action. The classical saddle points of this action describe locally pure three-dimensional anti-de sitter spacetimes, which include rotating BTZ black holes.

- The gravitational energy of any saddle point black hole space-time is defined in terms of the variation of the action $S[g]$, where $S[g]$ is a functional which depends on the boundary metric. In the parametrization (4.5), this variation takes the general form [19]

$$\delta S = \int_B d^2x \pi^{\alpha\beta} \delta g_{\alpha\beta} = \int_B d^2x \sqrt{-g} \left(-\epsilon \delta N - j \delta \omega + p \delta \varphi \right).$$

The quantities ϵ, j and p can respectively be interpreted as gravitational energy density, momentum density, and pressure, as measured on the boundary B. The total energy is thus obtained by integrating the energy density over the spatial section of B

$$E = \oint d\theta e^{\varphi} \epsilon.$$

Using the above points and following [19], we find that

$$E = \frac{r_c}{4G} \left[1 - \sqrt{1 - \frac{8GM}{r_c^2} + \frac{16G^2 J^2}{r_c^4}} \right]. \tag{4.7}$$

If we multiply (4.7) by the circumference of the circle $2\pi r_c$, we obtain a formula for the dimensionless quantity $\mathcal{E} = 2\pi r_c E$, which perfectly matches with the result (3.3) for the energy spectrum of the boundary conformal field theory, provided we identify

$$M = M_n = \Delta_n + \bar{\Delta}_n - \frac{c}{12}, \qquad J = J_n = \Delta_n - \bar{\Delta}_n, \qquad \tilde{\mu} = \frac{4G}{r_c^2} = \frac{6}{c} \frac{1}{r_c^2}.$$

The first two identifications are completely standard in AdS/CFT, and the third identification relates the deformation parameter $\tilde{\mu}$ the cutoff radius r_c.

Chapter 5

Conclusion

5.1 Summary

In this thesis, we explored properties of the $T\bar{T}$-deformed conformal field theory to identify results on the boundary which can be interpreted in the bulk. In particular, we found that the $T\bar{T}$-coupling transforms as a modular form of weight 4 on a torus. In order to explain this result and describe its significance, we started in chapter 2 with an introduction to two-dimensional conformal field theories, continued in chapter 3 with an exploration of $T\bar{T}$-deformed conformal field theories, and ended in chapter 4 with a discussion of the radially-cutoff AdS/$T\bar{T}$-deformed CFT correspondence. In the following, we summarize the contents of each individual chapter.

In chapter 2, we started with a derivation of the conformal group and the conformal algebra in an arbitrary number of dimensions. In particular, we showed that two-dimensional conformal field theories consist of an infinite number of symmetries and are therefore highly constrained. We then provided a template to compute the correlation functions of quasi-primary operators using conformal symmetry and demonstrated that the energy-momentum tensor is not only conserved, but also traceless. Next, we introduced the radial quantization procedure for conformal field theories and the operator product expansion for the energy-momentum tensor and primary fields, which led naturally to a discussion of the Virasoro algebra of the quantized theory. We then built the Hilbert space of two-dimensional conformal field theories using the state-operator correspondence and the highest weight representations of the Virasoro algebra. We ended the chapter with an introduction to the modular transformations on a torus and the modular invariance of the partition function of a conformal field theory on a torus.

In chapter 3, we introduced the the $T\bar{T}$-deformation of a conformal field theory as an irrelevant yet solvable theory. We defined an integrable quantum field theory and then proved that the $T\bar{T}$-deformation is the simplest of an infinite set of integrable deformations of an integrable quantum field theory. Next, we proved an expression for the energy spectrum of the deformed theory on a cylinder using the Zamolodchikov equation. We then reviewed a proof of a differential equation which describes the evolution of the deformed partition function as a function of the $T\bar{T}$ coupling. We ended the chapter with the proof that the $T\bar{T}$ coupling transforms on a torus as a modular form of weight 4 - we used both the energy spectrum and the evolution of the partition function to compute this result.

In chapter 5, we started with an introduction to the statement of the AdS/CFT correspondence. We then delved into a discussion of the correspondence of the $T\bar{T}$-deformed conformal field theory on the boundary with the radially-cutoff BTZ black hole spacetime in the bulk. We demonstrated this correspondence by outlining a proof that the energy spectrum on the boundary and the bulk agree with each other, and demonstrating that the thermodynamics on the bulk and boundary agree with each other.

5.2 Outlook

There are several ways to extend the work done in this thesis. For example, it would be interesting to prove that the coupling of the $T\bar{T}$-deformation transforms as a modular form of weight 4 using the dual BTZ path integral. To this end, we can use the papers [22, 23] by Witten and Maloney. In particular, [23] computes the partition function of quantum gravity in asymptotically AdS_3 spacetime with a torus conformal structure on the boundary as a sum over all the geometries of the $SL(2,\mathbb{Z})/\mathbb{Z}_2$ family of black holes [24, 25], which includes the thermal anti-de Sitter spacetime, the Euclidean BTZ black hole and geometries with contractible cycles $cT + dX$, where c and d are relatively prime integers. In this regard, we note that the $T\bar{T}$-deformed thermal conformal field theory on a cylinder is dual *specifically* to the radially-cutoff Euclidean BTZ black hole.

We can also compute the Casimir energy of a free massive scalar field in the radially-cut-off global-AdS_3 geometry and check that it matches the energy spectrum of the deformed conformal field theory. The direct computation of the Casimir energy in the bulk is rather cumbersome. However, we can use the heat-kernel approach [26] to circumvent the computation of the eigenvalues of the equation of motion in the bulk, and directly compute the one-loop correction to the vacuum energy in the bulk using the heat equation with appropriate boundary conditions in the bulk.

It would also be interesting to compute the deficit angle for a conical defect radially-cutoff AdS_3 geometry. The deficit angle for the vanilla conical defect AdS_3 geometry was first computed by [27] and [28].

Lastly, we can attempt to generalize the computation of the gravity dual of the Ising model [29] to include deformations by the composite $T\bar{T}$-operator.

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