Quasi-Local Energy in Minkowski, Anti de-Sitter, and Kerr Spacetimes.

Bartosz Syroka

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Abstract

The paper of Chen, Wang, and Yau, titled "The Minkowski Formula and the Quasi-Local Mass" [4], shows the application of Minkowski curvature formulas to quasi-local mass and energy in spacetimes, deriving several bounds on these quantities and using these to prove rigidity theorems. We first provide detailed explanations of the proofs, and then follow the methods that they use in the cases of Minkowski, Anti de-Sitter and Schwarzschild spacetimes to show that the main result on quasi-local energy can also be used for the Kerr spacetime. The necessary background is included in the first chapters and the appendix, covering spacetime geometry and curvature equations.

Résumé

L'article de Chen, Wang, et Yau, titué "The Minkowski Formula and the Quasi-Local Mass" [4], démontre l'application des formules de courbure de Minkowski à la masse et l'énergie quasi-locales dans des espace-temps, dérivant plusieurs bonds sur ces quantités et les employant afin prouver des théorèmes de rigidité. Premièrement, nous pourvoyons des explications détaillées de ces preuves, et puis nous suivons les méthodes qu'ils emploient dans les cas des espace-temps de Minkowski, Anti de-Sitter et Schwarzschild pour démontrer que le théorème principal sur l'énergie quasi-locale peut être utilisé en l'espace-temps de Kerr. Le contexte nécessaire est inclus parmi les premiers chapitres et l'appendice, couvrant la géometrie des espace-temps et les equations de courbure.

Contents

1. Introduction

In a recent paper by Chen, Wang, and Yau [4], the authors found estimates on the Wang-Yau quasi-local mass in Minkowski and Anti de-Sitter spacetimes using a Minkowski curvature formula, which in turn was possible due to the existence of conformal Killing-Yano (CKY) tensors in these geometries. We describe here the necessary background comprising curvature equations for submanifolds of Lorentz manifolds and CKY 2-forms present in 4-dimensional spacetimes. Also included are some examples of connection and curvature calculations using the structure equations, to showcase the relevant methods for computations.

The motivating problem is that the concepts of mass and energy in general relativity are not well-defined locally, as one cannot make use of an energy density. If one can make use of some asymptotic symmetry, then this can still be salvaged. Viewing systems possessing such symmetry from spatial infinity, we obtain the ADM energy-momentum, whereas viewing them from null infinity yields the Bondi energy-momentum [23]. This is limited in scope, as the system need not be isolated and thus viewed from infinity where this symmetry exists. As a result, various definitions of so-called quasi-local mass were proposed, with natural physical requirements such as nonnegativity in the general case, vanishing in the case of a flat spacetime, and gauge invariance, among others [28] .

We will consider mainly the Wang-Yau quasi-local energy. To describe the energy of a region of spacetime, it will be sufficient to consider quantities on a spacelike 2-surface which bounds it. In particular, we will explain how to construct this in a gauge-invariant (that is, a frame invariant) way, starting with the extrinsic curvature of the boundary surface in the ambient physical spacetime. Once this is done, the estimates which Chen, Wang, and Yau have obtained will give way to rigidity theorems for 3-dimensional Riemannian manifolds through isometric embeddings of the boundary surface into Minkowski or Anti de-Sitter spacetimes, and using the quasi-local energy as a geometric tool available therein.

To facilitate the discussion of these results, we will present the necessary prerequisites from general relativity. In particular, we will need to transition from the Riemannian to the Lorentzian setting, thus introducing the distinction between spacelike, timelike, and null vector fields and submanifolds of spacetime. This allows for the aforementioned viewing of isolated systems from null or spatial infinity by following the appropriate geodesic curves. We also make a brief mention of the Einstein field equations, which sit at the heart of the theory, and comment on some properties of vacuum solutions. The exact static and stationary solutions of Schwarzschild and Kerr, respectively, are then explored in more depth.

We follow this up with a discussion of various curvature equations which will be vital in the computations required for later proofs. A necessary component of this examination is the hierarchy of Killing tensors, where we will find a particular importance in the conformal Killing-Yano tensors. Such tensors of rank 2 exist for certain spherically symmetric spacetimes that we will consider. They constitute so-called "hidden symmetries" of the spacetime, which may allow for separability of certain equations, which we discuss in section 3.3. It is critical for section 4.4 that the Kerr spacetime admits such conformal Killing-Yano tensors.

The equations themselves are found by taking the familiar Gauss curvature formulas and extending the results to a Lorentzian setting, with embedded submanifolds of arbitrary codimension. Our focus will be on codimension 2 surfaces lying in 4-dimensional spacetimes, for which we derive the Gauss and Codazzi equations in section 3.5. In this, we largely shadow the work in [19], while making necessary adjustments for our specific case. These will be used frequently in the proofs of the main results.

Throughout chapter 4, we will describe the many proofs in the main paper [4], explaining many of the details which were not included explicitly. In particular, the exact formula which uses the conformal Killing-Yano 2-forms to present quasi-local energy in a more tractable way requires a long chain of computations which we attempt to present. Moreover, to show how some of the tensor calculations we are assuming or quoting could be verified directly, we include an appendix with some basic calculations and outlines of (co)frames so that the reader may see the methods underlying some of the statements.

Finally, the authors showed that the Minkowski curvature formula used in the discussion of quasi-local mass with respect to Minkowski and Anti de-Sitter spacetimes is also valid in Schwarzschild. We will introduce the conformal Killing-Yano 2-form for the Kerr spacetime, and follow the proof of Chen, Wang, and Yau to show that the formula continues to hold in the rotating Kerr solution. The critical fact is to see that the exact expression of this 2-form comes is not used explicitly, where the proof instead requires us to use the divergence of the tensor instead. The conformal Killing-Yano 2-form for the Kerr spacetime can be shown to have the same divergence as that for the Minkowski and Anti de-Sitter spacetimes [15], whence the theorem will follow. The exact expression is of course used in deriving certain bounds for the quasi-local energy, which we do not pursue in this case.

2. General Relativity Background

2.1. Basic definitions

The context in which we are working will largely be Lorentzian geometry, and so we define the basic notions required beyond the Riemannian setting. Throughout chapter 2, we are using material found in [5], [20], [19], and [22].

Definition 2.1.1. Let V be an n-dimensional vector space. A Lorentz inner product on V is a symmetric, nondenerate, bilinear form g with 1 negative and $n-1$ positive eigenvalues.

We will often write $\langle \cdot, \cdot \rangle$ in place of g to represent the scalar product. Now we present the meaning of the negative eigenvalue in the Lorentz metric:

Definition 2.1.2. A vector v in a Lorentz vector space V is called

1. spacelike if $\langle v, v \rangle > 0$
2. null if $\langle v, v \rangle = 0$
3. timelike $\langle v, v \rangle < 0$

We see that the negative eigenvalue is meant to correspond to vectors describing time, whereas null vectors represent light. Moreover, the set of all null vectors in a Lorentz vector space V is called its nullcone, and it separates the timelike vectors into two regions joined at the origin. After a suitable choice of orientation, we will be able to refer to timelike vectors as either future- or past-directed, depending on the region in which they lie. Using the above, we may single out vector subspaces of V of particular importance:

Definition 2.1.3. Let V be a Lorentz vector space with a scalar product g. A subspace W of V is called

1. spacelike if $g|_W$ is positive definite
2. null if $g|_W$ is degenerate
3. timelike if $g|_W$ defines a Lorentz metric on W

Having defined the above in the case of vector spaces, we are able to smoothly transition to the case of spacetime manifolds.

Definition 2.1.4. A spacetime N is a connected and time-oriented 4-dimensional Lorentz manifold, by which we mean that each tangent space of N is a Lorentz vector space, and that there exists a continuous and globally defined nonspacelike vector field on N.

A simple example of a spacetime is the Minkowski space $\mathbb{R}^{3,1}$, which is the space $\mathbb{R}^4$ equipped with the Lorentz metric:

$$ds^2 = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$$

where we denote, as expected, the spatial coordinates by x^i and the time coordinate by t, with ds^2 the arc length element. The time orientation is given by the constant vector field ∂_t. This spacetime is modelled after the flat Euclidean space.

We may present another example, called Anti de-Sitter space (AdS), which is modelled after hyperbolic space. Topologically, AdS is $S^1 \times \mathbb{R}^3$, endowed with the metric

$$ds^2 = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2 dS^2$$

where we denote the sphere metric by dS^2.

Finally, we define the corresponding special submanifolds of a spacetime N:

Definition 2.1.5. A submanifold M of a spacetime, or more generally a Lorentz manifold, N, is called

1. spacelike if T_pM is spacelike for all p in M
2. null if T_pM is null for all p in M
3. timelike if T_pM is timelike for all p in M

In order to be as self-contained as possible, we pause here to review distributions and Killing vector fields, before continuing on to stationary and static spacetimes.

Definition 2.1.6. A distribution $\mathscr{D}$ of dimension k on a smooth manifold M is a smooth subbundle of rank k of the tangent bundle TM. At each point p in M, it assigns a subspace $\mathscr{D}_p \subset T_pM$.

In particular, we need the following:

Definition 2.1.7. A distribution $\mathscr{D}$ is called integrable if for every p in M, there exists a submanifold P of M whose tangent space agrees with the distribution at this point: $T_pP = \mathscr{D}_p$.

By virtue of the Frobenius theorem, the above condition is equivalent to other useful statements. A distribution $\mathscr{D}$ is integrable if and only if it is involutive, which means that it is closed under the Lie bracket. Precisely, if X, Y are vector fields in $\mathscr{D}$, then so is $[X, Y]$. Another equivalent statement is the complete integrability of $\mathscr{D}$. We will call a chart (U, ϕ) on M flat for $\mathscr{D}$ if $\phi(U)$ is a cube in $\mathbb{R}^n$, and if at point of U, $\mathscr{D}$ is spanned by the coordinate vector fields $\{\partial_1, \dots, \partial_k\}$. Then we call the distribution $\mathscr{D}$ completely integrable if each point of M admits a neighbourhood with a flat chart for $\mathscr{D}$.

Moving onto the topic of Killing vector fields, we recall that the motivation here is to consider infinitesimal isometries of M. The flows ϕ^t of these vector fields are isometries of M for the times t where they are defined. Put another way, we may define them using the assumption that $\mathscr{L}_X g = 0$, where we denote the Lie derivative by $\mathscr{L}$. This equation leads us to the equivalent formulation below:

Definition 2.1.8. A Killing vector field X on a pseudo-Riemannian manifold (M, g) satisfies the Killing equation:

$$g(\nabla_V X, W) + g(\nabla_W X, V) = 0$$

for all vector fields V, W. Equivalently, we may write this in components as

$$\nabla_{(a} X_{b)} = \nabla_a X_b + \nabla_b X_a = 0$$

Given the context of general relativity, we need to introduce also the concept of an observer field. An observer in a spacetime is a material particle parametrized by so-called "proper time". What is important for us is the following definition:

Definition 2.1.9. An observer field X on (M, g) is a future-pointing, timelike unit vector field, whose integral curves are called X-observers.

1. We call X stationary if there exists a smooth function $f > 0$ on M such that fX is a Killing vector field, in which case we say that M is stationary relative to X.

2. If X is hypersurface-orthogonal, that is, if $X^\perp$ is integrable, then X is static, and we say that M is static relative to X.

We call X hypersurface-orthogonal whenever $X^\perp$, its orthogonal complement in TM, is an integrable distribution. This is because the integral manifolds to such a vector field are hypersurfaces normal to X.

Given a stationary field X, the flow ϕ^t of fX is an isometry for each t which carries each X-observer to itself. This means that the local universe is not changing for X-observers. On the other hand, should the observer field X be static, we know that the integral manifolds of $X^\perp$ are 3-dimensional spacelike manifolds which are invariant under the flow of X. This provides a strong separation of time and space.

Before moving onto the structure equations, let us briefly introduce the Einstein field equations. These are a set of nonlinear partial differential equations which involve the curvature tensor R of the spacetime metric g, and relate these to the energy-momentum tensor T which represents the matter fields. We denote the cosmological constant by Λ. Then the equations state that:

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$$

The desired solutions of these equations are spacetimes with a Lorentz metric g_{ab}, and their complexity makes exact solutions quite difficult to come by. We will, however, study a few such solutions in the next section.

The Minkowski spacetime is of course one possible solution, taking the cosmological constant $\Lambda = 0$ and assuming a vacuum $T = 0$. Allowing a nonzero cosmological constant, we may arrive at constant scalar curvature solutions of the equations. When $R > 0$, we obtain the so-called de-Sitter spacetime, with topology $\mathbb{R}^1 \times S^3$. On the other hand, if $R < 0$ then we recover the Anti de-Sitter spacetime discussed in a previous example. We note that vacuum solutions of the above, where $T = 0$, must be Ricci flat. To see this, raise an index and take the trace over the left hand side of the above equation. This will yield for us that $R = 0$. Substituting this back into the field equations, we get $R_{ab} = 0$.

Although not directly pertinent to our discussion, it is interesting to point out that the above system of PDEs admits a well-posed Cauchy problem. In particular, we may prescribe initial data on a spacelike hypersurface S, and ask whether there is a development into a spacetime M satisfying the field equations, such that the metric of M restricts to the metric on S, and the hypersuface S is a Cauchy surface for M (i.e. every non-spacelike curve intersects it exactly once). The formulation of the initial data on S involves a nonlinear elliptic system, called the constraint equations, whereas the development itself requires one to solve a nonlinear hyperbolic problem involving the reduced Einstein equations.

It is also worthwhile to note that the Einstein field equations are self-interacting, which means that they remain nonlinear in the absence of other fields, since the gravitational field defines the spacetime over which it propagates. Furthermore, they are unique only up to diffeomorphism, so in the solving of the Cauchy problem one must impose certain gauge conditions on the covariant derivatives of the metric and so remove the additional degrees of freedom. Last of all, the metric itself defines the spacetime structure, so we do not even know the domain of dependence of the initial surface S to begin with. Despite all this, given sufficiently smooth initial data satisfying the empty space constraint equations on a spacelike hypersurface, there will exist a maximal development of the empty space Einstein field equations.

2.2. Schwarzschild and Kerr solutions

We conclude this chapter with an excursion into the well-known Schwarzschild and Kerr solutions of Einstein's field equations. These are both vacuum solutions, and so Ricci-flat. The Schwarzschild solution represents the spacetime surrounding a static black hole, whereas the Kerr solution allows for the black hole to be rotating. Where the Schwarzschild solution is recovered as a limit of the Kerr spacetime, they are both special cases of the Kerr-Newman solution, which generalizes Kerr by allowing an electric charge.

The Schwarzschild spacetime has the metric:

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta + \sin^2\theta \, d\phi^2)$$

for $r > 2m$. This is a static spacetime, as the vector field ∂_t is a timelike Killing vector which is a gradient. It is spherically symmetric, by which we mean that it is invariant under the $SO(3)$ action on the spacelike two-spheres $\{t, r = \text{constant}\}$. Furthermore, it is an asymptotically flat solution, in the sense that the metric is of the form $g = g_0 + O(1/r)$ as r tends to infinity, for g_0 the metric of the Minkowski spacetime. The m variable in the metric represents the mass of the black hole as measured from infinity. We note that the Schwarzschild spacetime is unique in the sense that, given any spherically symmetric vacuum solution of the field equations, it must be locally isometric to the Schwarzschild solution.

The Kerr spacetime, on the other hand, has the metric of the form:

$$ds^2 = \rho^2\left(\frac{dr^2}{\Delta} + d\theta^2\right) + (r^2 + a^2)\sin^2\theta \, d\phi^2 - dt^2 + \frac{2mr}{\rho^2}(a\sin^2\theta \, d\phi - dt)^2$$

in the Boyer-Lindquist coordinates. The newly introduced variables stand for:

$$\rho^2(r,\theta) = r^2 + a^2\cos^2\theta$$

$$\Delta(r) = r^2 - 2mr + a^2$$

and m, ma represent the mass and angular momentum as measured from infinity, respectively. To see the connection between Kerr and Schwarzschild explicitly, we need only make the following substitutions in the Schwarzschild case:

$$g_{tt} = -1 + 2m/r \longrightarrow -1 + 2mr/\rho^2$$

$$g_{rr} = (1 - 2m/r)^{-1} \longrightarrow \rho^2/\Delta$$

$$g_{\theta\theta} = r^2 \longrightarrow \rho^2$$

$$g_{\phi\phi} = r^2\sin^2\theta \longrightarrow (r^2 + a^2 + 2mra^2\sin^2\theta/\rho^2)\sin^2\theta$$

and with the addition of the mixed term $g_{t\phi} = g_{\phi t} = -2mra\sin^2\theta/\rho^2$.

The Kerr family is parametrized by the parameters a and m. In the limit as a tends to 0, we recover the Schwarzschild spacetime. Otherwise, we are left with three distict cases: (1) if $0 < a^2 < m^2$, then we obtain the slowly rotating Kerr spacetime, (2) if $a^2 = m^2$, then we obtain the extreme Kerr spacetime, and (3) if $m^2 < a^2$, then we have the rapidly rotating Kerr spacetime. These lead to different horizons of the spacetime, denoted by H and defined as the vanishing sets of $\Delta = r^2 - 2mr + a^2$. Apparent in the metric is also another type of singularity, called the ring singularity Σ, and defined by $\rho^2 = r^2 + a^2\cos\theta = 0$.

The roots of Δ have to be examined for each of the above scenarios. In the Schwarzschild case $a = 0$, we see that the roots are 0 and $2m$. For slow Kerr, we have two distinct positive roots $r_\pm = m \pm (m^2 - a^2)^{\frac{1}{2}} < 2m$. In the extreme case, we obtain a double root $r = m$. Last of all, for fast Kerr, Δ has no real roots. We are interested in seeing the maximal domains for the Kerr metric, and it can be shown that the Boyer-Lindquist form has to be considered on the connected components of $\mathbb{R} \times S^2 \setminus (H \cup \Sigma)$, cut using the horizons $H : \Delta = 0$. We call these components the Boyer-Lindquist blocks.

Definition 2.2.1. The Boyer-Lindquist blocks, denoted I, II, III, are the open subsets of $\mathbb{R}^2 \times S^2 \setminus \Sigma$ described by:

1. For slow Kerr, there are two horizons r_+ and r_-;
   $I : r > r_+$, $II : r_- < r < r_+$, $III : r < r_-$.

2. For extreme Kerr, there is a single horizon $r = m$;
   $I : r > m$, $III : r < m$.

3. For fast Kerr, there are no horizons, so we may view the spacetime;
   $I = III = \mathbb{R}^2 \times S^2 \setminus \Sigma$.

A noteworthy result is that the maximally extended Kerr spacetimes are composed precisely of the Boyer-Lindquist blocks glued together along the horizons. This is interesting in particular when we consider the causal structure of the blocks. Note that we may call a spacetime region chronological if it contains no closed timelike curves, and causal if it contains no closed non-spacelike curves. Blocks I and

II are both causal. Furthermore, since ∂_t is a Killing vector field on I which is future-directed and timelike, we have a stationary observer field $X = \partial_t/|\partial_t|$ on the open set $\{g_{tt} < 0\}$. The integral curves of this vector field are called Kerr stationary observers. However, ∂_t fails to be hypersuface-orthogonal on any open set in a Kerr spacetime, and thus Kerr observers are not static. However, block III is vicious, meaning that for any two points p and q in the block, there exists a timelike future-directed curve in block III from p to q. The consequence is a violation of causality in the interior block III of the Kerr spacetime. Furthermore, all maximal models of the Kerr spacetime contain this block.

We may end the section by summarizing a few meaningful properties of the Kerr family of solutions. First of all, the Kerr spacetime is axially symmetric, as a consequence of the rotation. Second, it is stationary for $r > 2m$, so if we imagine a star of such a radius being the source of gravity, it would not be expanding or collapsing. Third of all, it is Ricci flat, which means that the spacetime is a vacuum solution: the star is the only source of gravity. Last of all, the Kerr solution is asymptotically flat, which implies that gravity becomes very weak when far from the source.

3. Curvature Equations

3.1. The structure equations

In order to perform the necessary calculations for deriving the connection and curvature associated to the metric tensor, we will be employing the first and second structure equations. This section is largely based on the work in [20] and [22]. Let us begin by introducing connection and curvature forms. We choose an orthonormal frame $\{E_1, \dots, E_n\}$ of a pseudo-Riemannian manifold (where the metric g has p positive and q negative eigenvalues, with $p + q = n$), which we may do at least locally. Associated to this is the orthonormal coframe $\{\theta^1, \dots, \theta^n\}$ defined by the relation

$$\theta^i(E_j) = \delta^i_j$$

where δ^i_j is the Kronecker delta.

Definition 3.1.1. Given a pseudo-Riemannian manifold (M, g) with Levi-Civita connection ∇, we define the connection 1-forms ω^i_j acting on vector fields X by:

$$\omega^i_j(X) = \theta^i(\nabla_X E_j)$$

Theorem 3.1 (Cartan's first structure equation)**.** *If ω^i_j are the connection 1-forms and $\{\theta^i\}$ the orthonormal coframe as above, we have the relation:*

$$d\theta^i = -\sum_m \omega^i_m \wedge \theta^m$$

Proof. We make use of the equation

$$d\theta(X, Y) = X\theta(Y) - Y\theta(X) - \theta([X, Y])$$

where θ is a 1-form and X, Y are vector fields. It follows that:

$$\begin{aligned} d\theta^i(E_a, E_b) &= E_a\theta^i(E_b) - E_b\theta^i(E_a) - \theta^i([E_a, E_b]) \\ &= -\theta^i([E_a, E_b]) \end{aligned}$$

The right hand side gives us:

$$\begin{aligned} \Big(-\sum_m \omega^i_m \wedge \theta^m \Big)(E_a, E_b) &= -\sum_m \omega^i_m(E_a)\theta^m(E_b) + \sum_m \omega^i_m(E_b)\theta^m(E_a) \\ &= -\omega^i_b(E_a) + \omega^i_a(E_b) \\ &= -\theta^i(\nabla_{E_a} E_b) + \theta^i(\nabla_{E_b} E_a) \\ &= -\theta^i([E_a, E_b]) \end{aligned}$$

■

If we denote $\omega = (\omega^i_j)$ the matrix of connection 1-forms, then the first structure equation may be written as

$$d\theta = -\omega \wedge \theta$$

The second equation will relate the connection 1-forms to the curvature forms. We define the latter:

Definition 3.1.2. If X and Y are vector fields on the pseudo-Riemannian manifold M, then we define the curvature forms Ω^i_j by:

$$R(X,Y)E_j = \sum_i \Omega^i_j(X,Y)E_i$$

where Ω^i_j are 2-forms.

Realize $(R(X,Y)E_j)$ as a vector and let $E = (E_1, \dots, E_n)$. Then the above definition states that

$$(R(X,Y)E_j) = E(\Omega^i_j(X,Y))$$

with $(\Omega^i_j(X,Y))$ acting as matrix multiplication on the right on E.

Theorem 3.2 (Cartan's second structure equation)**.** *Given the connection* 1*-forms* ω^i_j *and curvature* 2*-forms* Ω^i_j*, the following relation holds:*

$$\Omega^i_j = d\omega^i_j + \sum_m \omega^i_m \wedge \omega^m_j$$

Proof. Once again, we will prove this by evaluating both sides of the equation on the frame field vectors.

$$R(E_a, E_b)E_j = \nabla_{E_a}(\nabla_{E_b}E_j) - \nabla_{E_b}(\nabla_{E_a}E_j) - \nabla_{[E_a,E_b]}E_j$$

The connection forms can be used to rewrite the terms $\nabla_{E_b}E_j = \sum_i \omega^i_j(E_b)E_i$. We evaluate:

$$\begin{aligned}\nabla_{E_a}(\nabla_{E_b}E_j) &= \sum E_a\omega^i_j(E_b)E_i + \sum \omega^m_j(E_b)\nabla_{E_a}E_m \\ &= \sum E_a\omega^i_j(E_b)E_i + \sum \omega^m_j(E_b)\omega^i_m(E_a)E_i\end{aligned}$$

Using the above, we may write the curvature terms using connection 1-forms:

$$\begin{aligned}R(E_a, E_b)E_j = &\sum \big(E_a\omega^i_j(E_b) - E_b\omega^i_j(E_a)\big)E_i \\ &+ \sum \big(\omega^m_j(E_b)\omega^i_m(E_b) - \omega^m_j(E_a)\omega^i_m(E_b)\big)E_i \\ &- \sum \omega^i_j\big([E_a, E_b]\big)E_i\end{aligned}$$

Notice that the middle line is exactly the evaluation of $\omega \wedge \omega$, while the first and last lines compose the evaluation of $d\omega$. This proves the second structure equation. ■

Once again, we may use matrix notation to write the structure equation in short form as:

$$\Omega = d\omega + \omega \wedge \omega$$

3.2. Gauss Codazzi equations for arbitrary codimension

Here we extend some basic curvature formulas to the context of pseudo-Riemannian submanifolds of arbitrary codimension. The case of 2-dimensional spacelike surfaces sitting in a 4-dimensional spacetime is the most relevant for the later sections. However, for the sake of generality, we take a codimension k submanifold M embedded in a pseudo-Riemannian manifold $(\overline{M}, \overline{g})$ of dimension n, with the induced metric denoted by g. Further, we denote the curvature tensors by R and $\overline{R}$, respectively. We quickly recall some basics, following the work of Lee in [19], as well as [11] and [16] for some foundations.

Definition 3.2.1. The second fundamental form h of a hypersurface $(M, g) \hookrightarrow (\overline{M}, \overline{g})$ is a symmetric mapping into the normal bundle of M in $\overline{M}$.

$$h : TM \times TM \to NM$$
$$(X, Y) \mapsto (\overline{\nabla}_X Y)^{\perp}$$

In higher codimensions, there are multiple independent normal vectors to M. If we write them as $\{\nu_1, \ldots \nu_k\}$ for codimension k, then by h we mean the summation of the second fundamental forms in each of the normal directions:

$$h(X, Y) = \sum h_i(X, Y) = \sum \langle (\overline{\nabla}_X Y)^{\perp}, \nu_i \rangle \nu_i$$

where by $\langle \cdot, \cdot \rangle$ we are denoting $\overline{g}$.

It can be shown that the second fundamental form measures the discrepancy between the connections on the ambient and the embedded manifolds; namely, we have the relation

$$\overline{\nabla}_X Y - \nabla_X Y = h(X, Y)$$

Therefore, it will also relate the curvature tensors of the two manifolds, described by the Gauss equation. If X, Y, Z, W are sections of TM, then we arrive at:

$$\overline{R}(X, Y, Z, W) - R(X, Y, Z, W) = \langle h(X, Z), h(Y, W) \rangle - \langle h(X, W), h(Y, Z) \rangle$$

where this formula holds in the general case that we have been discussing. (This fact follows from checking that the usual proofs carry over to the general case without much effort.)

However, we need not stop there. The vector field W does not have to be chosen in TM, as the proof of the Gauss equation still holds more generally. Of particular interest to us will be the case where W is one of the normal directions ν_i. In this case, we arrive at the equation:

$$\overline{R}(X, Y, Z, \nu_i) = \langle h(X, Z), (\overline{\nabla}_Y \nu_i)^{\perp} \rangle - \langle h(Y, Z), (\overline{\nabla}_X \nu_i)^{\perp} \rangle$$

We may immediately drop the orthogonal projection, since h already takes values in NM. We arrive at:

$$\overline{R}(X, Y, Z, \nu_i) = \left\langle \sum h_j(X, Z), \overline{\nabla}_Y \nu_i \right\rangle - \left\langle \sum h_j(Y, Z), \overline{\nabla}_X \nu_i \right\rangle$$

Note that for some scalar function f, we will have

$$X\langle f\nu_j, \nu_i\rangle = \langle Xf\nu_j, \nu_i\rangle + f\langle \overline{\nabla}_X \nu_j, \nu_i\rangle + f\langle \nu_j, \overline{\nabla}_X \nu_i\rangle$$

which implies for us that we can write

$$\begin{aligned}\overline{R}(X, Y, Z, \nu_i) = &\left\langle \sum \big(\nabla_X h_j(Y, Z) - \nabla_Y h_j(X, Z)\big)\nu_j, \nu_j \right\rangle \\ &+ \sum h_j(Y, Z)\langle \overline{\nabla}_X \nu_j, \nu_i\rangle - \sum h_j(X, Z)\langle \overline{\nabla}_Y \nu_j, \nu_i\rangle\end{aligned}$$

where we are using h_j to denote both the vector and the coefficient, which should be clear from context. We state the specific case we need in the following theorem:

Theorem 3.3. *Consider a codimension 1 or 2 submanifold M of an n-dimensional pseudo-Riemannian manifold $\bar{M}$, with the normal directions to M denoted by ν_1, ν_2. It follows from the Gauss equation that we have the following relations:*
In codimension 1:

$$\overline{R}(X, Y, Z, \nu_1) = \nabla_X h_1(Y, Z) - \nabla_Y h_1(X, Z)$$

whereas in codimension 2:

$$\begin{aligned}\overline{R}(X, Y, Z, \nu_1) =& \nabla_X h_1(Y, Z) - \nabla_Y h_1(X, Z) \\ &+ h_2(Y, Z)\langle \nu_1, \overline{\nabla}_X \nu_2\rangle - h_2(X, Z)\langle \nu_1, \overline{\nabla}_Y \nu_2\rangle\end{aligned}$$

We will use exactly this last equation in the proofs of many results in chapter 4.

3.3. Killing tensors

In the paper of [26], the authors generalize the above Gauss and Codazzi curvature equations with Minkowski formulas in spacetime, using the existence of conformal Killing-Yano (CKY) tensors. Before specifying the result we will need later, we explain some of the hierarchy of Killing tensors [8], [9], [10]. Since we will only be concerned with rank 2 CKY tensors, we present the setup with this in mind.

Note that in what follows, we will denote the tensor $\nabla_{\partial_a} T_b$ by $T_{b;a}$. That is, we let the subscript ; denote the covariant derivative applied to the tensor T. Moreover, the symmetric part of a tensor, for example T_{ab}, is denoted using open brackets, $T_{(ab)}$, while its antisymmetric part is denoted using closed brackets, $T_{[ab]}$.

Definition 3.3.1. A Killing tensor of rank p on a spacetime (M, g) is a symmetric tensor which satisfies the Killing equation:

$$T_{(a_1 \ldots a_p;b)} = 0$$

Notably, this is a symmetric generalization of a Killing vector. Let us provide some physical motivation for the examination of such objects [8], [9]. The geodesics on a spacetime manifold can be studies using the Hamilton-Jacobi equation:

$$H(q, \partial_q S(q)) = E$$

for E the energy, and S Hamilton's characteristic function. Precisely, H in this case is $\|\nabla S\|^2$ in the metric of the cotangent bundle. An important question to ask is when this equation is separable. To this end, the existence of Killing vectors and Killing tensors satisfying some commutativity conditions is integral to the definition of so-called separability structures. These structures are composed of charts which allow for an additive separation of variables of the Hamilton-Jacobi equation.

Furthermore, let us consider the Klein-Gordon equation

$$\Box\phi = m^2\phi$$

where we have the wave operator $\Box = g^{ab}\nabla_a\nabla_b$. If the manifold in question is Einstein, that is a manifold whose Ricci tensor is a scalar multiple of the metric, then the Klein-Gordon equation is separable if and only if the Hamilton-Jacobi equation is separable.

On the other hand, a Killing-Yano tensor is an antisymmetric version of the Killing tensor, in the following sense:

Definition 3.3.2. A Killing-Yano tensor of rank p on a spacetime (M, g) is an antisymmetric tensor which satisfies the Killing-Yano equation:

$$T_{a_1 \ldots a_{p-1}(a_p;b)} = 0$$

Since T is antisymmetric, this shows us that the covariant derivative on T is also antisymmetric.

One may motivate the Killing-Yano tensor geometrically as follows: Let γ be a geodesic curve on a manifold, and let T be a Killing-Yano tensor of rank two. Denote also by T the associated endomorphism obtained by raising the index. Then T gives rise to a vector field X that is parallel translated along γ, where $X = T\dot{\gamma}$. We may check that the necessary condition holds:

$$\begin{aligned}
\dot{\gamma}^a\nabla_a X^b &= \dot{\gamma}^a\nabla_a(T^b_c\dot{\gamma}^c) \\
&= \dot{\gamma}^a(\nabla_a T^b_c)\dot{\gamma}^c + T^b_c(\dot{\gamma}\nabla_a\dot{\gamma}^c) \\
&= \frac{1}{2}\dot{\gamma}^a\dot{\gamma}^c(\nabla_a T^b_c + \nabla_c T^b_a) \\
&= 0
\end{aligned}$$

We see that the vanishing occurs due to the appearance of the geodesic condition on γ, and the Killing-Yano equation $T_{b(c;a)} = 0$.

Moreover, Killing-Yano tensors act as "square roots" of Killing tensors in the following sense: If we have two Killing-Yano tensors f, g of rank p, then the product

$$K^{ab} = f^{(a}{}_{c_2 \ldots c_p} g^{b)c_2 \ldots c_p}$$

is a Killing tensor of rank 2. In certain spacetimes, the Killing and Killing-Yano tensors allow for the separability of the Dirac equation [9].

We may generalize both of these to their conformal versions via:

Definition 3.3.3.

1. A conformal Killing tensor of rank p on a spacetime (M, g) is a symmetric tensor which satisfies the modified equation:

$$T_{(a_1 \dots a_p;b)} = g_{b(a_1}\overline{T}_{a_2 \dots a_p)}$$

for some symmetric tensor $\overline{T}$ of rank $p-1$

2. A conformal Killing-Yano tensor of rank p on a spacetime (M, g) is an antisymmetric tensor which satisfies the modified equation (∇ the Levi-Civita connection of g):

$$\nabla_{(a_1}T_{a_2)a_3 \dots a_{p+1}} = g_{a_1 a_2}\overline{T}_{a_3 \dots a_{p+1}} - (p-1)g_{[a_3(a_1}\overline{T}_{a_2) \dots a_{p+1}]}$$

where by taking a trace we find that

$$\overline{T}_{a_2 \dots a_p} = \frac{1}{n-p+1}\nabla^{a_1}T_{a_1 a_2 \dots a_p}$$

for n the dimension of M.

If ω is a conformal Killing-Yano p-form on a manifold with metric g, then $\Omega^{p+1}\omega$ is a conformal Killing-Yano p-form with respect to the conformally scaled metric $\Omega^2 g$.

Of particular importance are two subsets of the conformal Killing-Yano forms: the Killing-Yano forms we had discussed above, and closed conformal Killing-Yano forms (with respect to exterior differentiation). Let us rewrite the above definitions in terms of differential forms for clarity. We will denote by $\iota(\cdot)$ the interior product. Then we may define a p-form ω to be conformal Killing-Yano if and only if, for any smooth vector field X, its covariant derivative satisfies:

$$\nabla_X \omega = \iota(X)\kappa + X \wedge \xi$$

where κ is a $(p+1)$-form, and ξ is a $(p-1)$-form. Explicitly, we may find that these forms are:

$$\kappa = \frac{1}{p+1}\nabla \wedge \omega$$
$$\xi = = \frac{1}{n-p+1}\iota(\nabla)\omega$$

where n is the dimension of the spacetime we are working with. To be clear:

$$\iota(X)\omega = X^a \omega_{aa_2 \cdots a_p}$$
$$\iota(\nabla)\omega = \nabla^a \omega_{aa_2 \cdots a_p}$$

The defining equation with the κ and ξ specified becomes:

$$\nabla_X \omega = \frac{1}{p+1}\iota(X)(\nabla \wedge \omega) + \frac{1}{n-p+1}X \wedge \iota(\nabla)\omega$$

With this notation, we may characterize the special subsets of conformal Killing-Yano tensors quite simply. The Killing-Yano forms are exactly the forms α which satisfy

$$\nabla_X \alpha = \iota(X)\kappa$$

whereas the closed conformal Killing-Yano forms β satisfy

$$\nabla_X \beta = X \wedge \xi$$

Moreover, we may also note the behaviour of the conformal Killing-Yano tensors under the Hodge star $*$. The defining equation above becomes:

$$\nabla_X(*\omega) = \frac{1}{p_* + 1}\iota(X)(\nabla \wedge *\omega) + \frac{1}{n - p_* + 1} X \wedge \iota(\nabla) * \omega$$

for $p_* = n - p$. This shows that the Hodge star of a conformal Killing-Yano form is another conformal Killing-Yano form. Furthermore, it acts as a mapping between Killing-Yano forms and closed conformal Killing-Yano forms (so that $*$ maps KY forms to closed CKY forms, and conversely, closed CKY forms to KY forms).

We should note that, for our desired context, we will focus on conformal Killing-Yano 2-forms on a 4-dimensional spacetime. In this case, we have a far simpler identification for such tensors. Let us call Q the form, and $(N, \langle\cdot,\cdot\rangle)$ the spacetime manifold. Denoting by ∇ its Levi-Civita connection, and by ξ the divergence of Q with respect to ∇, the aforementioned equations reduce to:

$$\nabla_X Q(Y,Z) + \nabla_Y Q(X,Z) = \frac{1}{3}(2\langle X,Y\rangle\langle \xi, Z\rangle - \langle X,Z\rangle\langle \xi,Y\rangle - \langle Y,Z\rangle\langle \xi,X\rangle$$

for all fields X, Y, Z, where the similarity to the basic Killing equation is more easily appreciated [4].

3.4. Killing-Yano tensors in the Kerr spacetime

We present a basic application of the aforementioned ideas to the Kerr spacetime. The cited results can be found throughout [4], [9], and [15]. It is a well-known result that the Kerr solution admits a closed conformal Killing-Yano tensor, namely:

$$Q = a\cos\theta\sin\theta\; d\theta \wedge ((r^2 + a^2)\; d\phi - a\; dt) + r\; dr \wedge (a\sin^2\theta\; d\phi - dt)$$

written in the Boyer-Lindquist coordinate system discussed in section 2.2. This tensor Q has a well-known divergence of $3\partial_t$ [15]. It is also interesting to note that in the limit as $a \to 0$, we recover the conformal Killing-Yano 2-form for the Schwarzschild spacetime, $r\; dr \wedge dt$. This is, in fact, a conformal Killing-Yano 2-form present in any spherically symmetric spacetime endowed with a metric of the type:

$$-f(r)^2\; dt^2 + \frac{dr^2}{f(r)^2} + r^2\; dS^2$$

where we again denote by dS the line element of the sphere metric. The tensor retains the divergence of $3\partial_t$ in such spacetimes.

Q will map to a Killing-Yano tensor $P = *Q$ under the Hodge star. This yields

$$P = *Q = r \sin\theta \, d\theta \wedge ((r^2 + a^2) \, d\phi - a \, dt) + a \cos\theta \, dr \wedge (dt - a \sin^2\theta \, d\phi)$$

which is divergence-free, $P^{ab}_{;a} = 0$. Now recall that Killing-Yano tensors act as "square roots" of Killing tensors, in the sense that we may write a new Killing tensor $k_{ab} = P_{ac} P^c_b$. Moreover, the coordinate vector fields ∂_t and ∂_ϕ are Killing vector fields with respect to the Boyer-Lindquist coordinates. It can then be shown that the four tensors - $\partial_t, \partial_\phi, k$, and the metric g - are linearly independent and commute with respect to the so-called Nijenhuis-Schouten bracket (a generalization of the Lie bracket). This further implies that the integrals of motion associated with these tensors are all independent and in involution. Consequently, this means that the geodesic motion in the Kerr spacetime is completely integrable.

3.5. Minkowski curvature formulas in spacetime

The primary method used to prove the results of the paper is a kind of Minkowski curvature formula in the setting of a spacelike surface in spacetime. This work can be found in great detail in the thesis of Y-K. Wang [25] and paper of Wang, Wang and Yau [26]. We recall some classical results first, and then proceed to introduce the necessary preliminaries for the formula. Recall that in the case of a 2-dimensional surface $\Sigma \subset \mathbb{R}^3$, with outward unit normal ν, the Minkowski formula states that

$$\int_\Sigma H \, d\sigma = \int_\Sigma K(X \cdot \nu) \, d\sigma$$

where we denote by σ the induced metric on the surface Σ, by H the mean curvature vector which is defined as the trace of the second fundamental form associated to ν on Σ, and finally by K the Gauss curvature of the surface Σ which can be viewed as the determinant of said second fundamental form.

A more general version of the Minkowski formula holds as well, again for a hypersurface embedded in $\mathbb{R}^n$. Recall that the principal curvatures of Σ are the eigenvalues of its second fundamental form. Now denote by σ_k the k-th elementary symmetric function of the principal curvatures, and by μ the induced volume measure on Σ, to avoid misunderstanding. For $X : \Sigma \to \mathbb{R}^n$ the embedding, and ν the outer unit normal as before, the formula reads:

$$(n-k) \int_\Sigma \sigma_{k-1} \, d\mu = k \int_\Sigma \sigma_k \langle X, \nu \rangle \, d\mu$$

Now let Σ be a 2-dimensional surface in N, with normal directions $\nu_1 = e_3, \nu_2 = e_4$, where $\{e_i\}$ compose a frame of the tangent bundle of N. We write the connection 1-form of the normal bundle of Σ as

$$\alpha_{e_3}(X) = \langle \overline{\nabla}_X e_3, e_4 \rangle$$

We may finally quote the special case of the Minkowski formula:

Theorem 3.4. *Let Σ be a 2-dimensional spacelike surface in a spacetime manifold $(N, \bar{g})$. We denote by σ the induced metric on Σ. Let e_3, e_4 be a frame of the normal bundle, h_3, h_4 be the corresponding second fundamental forms, H_0 the mean* *curvature vector field, R the Riemann curvature tensor of $\bar{g}$, and α_{e_3} the connection 1-form in the normal bundle. Then*

$$\begin{aligned}-\int_\Sigma \langle J_0, \partial_t\rangle \, d\sigma = \int_\Sigma & 2(\det h_3 - \det h_4)Q_{34} + (R^{ab}_{a3}Q_{b4} - R^{ab}_{a4}Q_{b3}) \\ & + (R^{ab}_{43} - (d\alpha_{e_3})^{ab})Q_{ab} \, d\sigma\end{aligned}$$

where by J_0 we denote the reflection of H_0 through the light cone in the normal bundle of Σ. Explicitly,

$$J_0 = \langle H_0, e_4\rangle e_3 - \langle H_0, e_3\rangle e_4$$

This is in particular the formula given in [26] with the choice of $(r, s) = (2, 0)$, [4]. The mean curvature is vital to calculating the quasi-local mass and energy terms in spacetimes.

4. Quasi-Local Mass Estimates

4.1. Quasi-local mass

In this chapter, we describe the results of the paper by Chen, Wang, and Yau [4], and explain the proofs they have given. As we will be dealing with isometric embeddings of surfaces from some physical spacetime into a chosen reference spacetime, let us make the choice at the outset to denote with a prime $'$ the quantities present in the source space, and without it the quantities living in the target. Therefore, as at the end of chapter 3, we take a 2-dimensional surface Σ in a physical spacetime N with a frame $\{e_3', e_4'\}$ of its normal bundle, and let h_3', h_4' be the second fundamental forms of Σ in the normal directions e_3', e_4', respectively. Furthermore, we write D for the ambient connection, whereas we use simply ∇ for the connection of the induced metric σ on Σ. With this in mind, recall the definition of the connection 1-form of the normal bundle to the surface:

$$\alpha_{e_3'}(X) = \langle D_X e_3', e_4' \rangle$$

where for convenience we sometimes denote by $\langle \cdot, \cdot \rangle$ the metric of the ambient spacetime N. We take an isometric embedding X of Σ into, at first, Minkowski spacetime $\mathbb{R}^{3,1}$, although we will consider cases of Anti de-Sitter, Schwarzschild, and Kerr spacetimes shortly. We choose also a frame $\{e_3, e_4\}$ of the normal bundle of the embedded surface $X(\Sigma)$, with h_3, h_4 being the associated second fundamental forms, and α_{e_3} the connection 1-form of its normal bundle.

We begin with the critical idea of the paper, which combines Killing-Yano tensors with Minkowski curvature formulas. The evaluation of quasi-local mass or energy of a region of spacetime comes down to considering the differences of the mean curvature along a spacelike 2-dimensional surface which forms a boundary of a spacelike region (a time-slice), measured with respect to the ambient spacetime in which the surface lives, against the extrinsic curvature it attains in the reference spacetime through some embedding. It can be shown that there is a gauge-invariant method for producing this reference extrinsic curvature, meaning that the choice of the frame will not remain an ambiguity [27], [28], [29].

Theorem 4.1. *Given a spacelike 2-dimensional surface Σ in a physical spacetime N, and a frame $\{e_3', e_4'\}$ of its normal bundle in N, let X be an isometric embedding of Σ into Minkowski space $\mathbb{R}^{3,1}$. Suppose that there is a frame of the normal bundle of $X(\Sigma)$ in $\mathbb{R}^{3,1}$ such that*

$$\alpha_{e_3'} = \alpha_{e_3} = \zeta$$

Then

$$\int -\langle\partial_t, e_4\rangle(trh_3 - trh_3') + \langle\partial_t, e_3\rangle(trh_4 - trh_4')d\sigma$$
$$= \int (2\det h_3 - 2\det h_4 - trh_3trh_3' + h_3\cdot h_3' + trh_4trh_4' - h_4\cdot h_4')Q_{34}$$
$$+ (R^{ab}_{a4}Q_{b3} - R^{ab}_{a3}Q_{b4}) - Q_{bc}\sigma^{cd}[(h_3)_{da}h_4'^{ab} - (h_4)_{da}h_3'^{ab}]\, d\sigma$$

for R the curvature tensor of N. We have denoted by ζ the common connection one form. Moreover, we are using the conformal Killing-Yano tensor inside the target spacetime, which for $\mathbb{R}^{3,1}$ is $r\ dr\wedge dt$, with divergence $3\partial_t$.

Proof. The idea of the proof is to consider two divergence quantities and, using Stokes' theorem, derive the above relations. The first one concerns the frame in the physical spacetime before any embedding, whereas the second is for the embedded surface.

$$\nabla_a((trh_3'\sigma^{ab} - h_3'^{ab})Q_{b4} - (trh_4'\sigma^{ab} - h_4'^{ab})Q_{b3})$$
$$\nabla_a((trh_3\sigma^{ab} - h_3^{ab})Q_{b4} - (trh_4\sigma^{ab} - h_4^{ab})Q_{b3})$$

where the connection ∇ is the induced connection on the surface.

We first do the necessary calculations for the first quantity. We use the second equation of Theorem 3.3. Raising two indices and setting $X = \partial_a = Z, Y = \partial_b, \nu_1 = e_3', \nu_2 = e_4'$, and $\zeta_a = (\alpha_{e_3'})_a$ we derive

$$\begin{aligned} R^{ab}_{a3} &= \nabla^a(h_3')^b_a - \nabla^b(h_3')^a_a + (h_4')^b_a\zeta^a - (h_4')^a_a\zeta^b \\ &= \nabla_a(h_3')^{ab} - \nabla_a\sigma^{ab}trh_3' + (h_4')^{cb}\sigma_{ca}\zeta^a - trh_4'\zeta^b \\ &= \nabla_a((h_3')^{ab} - \sigma^{ab}trh_3') + (h_4')^{ab}\zeta_a - trh_4'\zeta^b \end{aligned}$$

We can rearrange the above, and repeat the calculation with $\nu_2 = e_4'$ to write:

$$\nabla_a(\sigma^{ab}trh_3' - (h_3')^{ab}) = -R^{ab}_{a3} + (h_4')^{ab}\zeta_a - trh_4'\zeta^b$$
$$\nabla_a(\sigma^{ab}trh_4' - (h_4')^{ab}) = -R^{ab}_{a4} + (h_3')^{ab}\zeta_a - trh_3'\zeta^b$$

Now we turn our attention to differentiating the Killing-Yano 2-form.

$$\nabla_aQ_{b3} = (\nabla_aQ)_{b3} + Q(\nabla_a\partial_b, e_3) + Q(\partial_b, \nabla_ae_3)$$

Recall that the differentiation is done using the connection of σ, that is, the induced connection on the surface Σ. However, the fact that we have an isometric embedding into $\mathbb{R}^{3,1}$ allows us to use the fact that the discrepancy between the ambient connection D and the induced connection ∇ is described using the second fundamental forms. Let us work term by term. The first is simply differentiation of a function, so the two connections agree:

$$(\nabla_aQ)_{b3} = (D_aQ)_{b3}$$

The second term can be rewritten as follows:

$$\begin{aligned}&Q(D_a\partial_b - (h_3)_{ab} + (h_4)_{ab}, e_3)\\&= -Q((h_3)_{ab}, e_3) + Q((h_4)_{ab}, e_3)\\&= (h_4)_{ab}Q_{43}\end{aligned}$$

Note that we can choose for $Q(D_a\partial_b, e_3)$ to be pointwise equal to zero, so this part vanishes. Moreover, the tensor h_3 is in the direction of e_3, which by the antisymmetry of Q also forces this part to vanish. Thus we arrive at the above. It should be noted that the positive on h_4 is coming from the e_4 direction being timelike, and thus under the Lorentzian metric having a negative sign. This can be observed from the equations following Definition 3.2.1. Finally, the third term becomes:

$$\begin{aligned}&Q(\partial_b, D_ae_3)\\&= Q(\partial_b, \Sigma_c\langle D_ae_3, \partial_c\rangle\partial_c + \langle D_ae_3, e_4\rangle e_4)\\&= Q_{bc}(h_3)_a^c - \zeta_aQ_{b4}\\&= Q_{bc}\sigma^{cd}(h_3)_{ad} - \zeta_aQ_{b4}\end{aligned}$$

Using the fact that we may use the ambient connection and subtract off the second fundamental forms, but these are defined over $T\Sigma$, there is no discrepancy. The rest follows by summing up all the components, and separating appropriately.

Although the calculations above were done in the case of Q_{b3}, we may read off the results for Q_{b4} by following the same steps. Therefore, we may put everything together:

$$\begin{aligned}&\nabla_a((\sigma^{ab}trh_3' - (h_3')^{ab})Q_{b4} - (\sigma^{ab}trh_4' - (h_4')^{ab})Q_{b3})\\&= (R_{a4}^{ab} - (h_3')^{ab}\zeta_a + trh_3'\zeta^b)Q_{b3} - (R_{a3}^{ab} - (h_4')^{ab}\zeta_a + trh_4'\zeta^b)Q_{b4}\\&+ (\sigma^{ab}trh_3' - (h_3')^{ab})((D_aQ)_{b4} - (h_3)_{ab}Q_{34} + Q_{bc}\sigma^{cd}(h_4)_{da})\\&- (\sigma^{ab}trh_4' - (h_4')^{ab})((D_aQ)_{b3} - (h_4)_{ab}Q_{34} + Q_{bc}\sigma^{cd}(h_3)_{da})\end{aligned}$$

We can use the definition of conformal Killing-Yano 2-forms to rewrite some of the above terms. Since the divergence of Q is $3\partial_t$, we get:

$$(D_aQ)_{b4} + (D_bQ)_{a4} = \frac{1}{3}(2\langle\partial_a, \partial_b\rangle\langle 3\partial_t, e_4\rangle)$$

where all other terms vanish because $\langle e_4, \partial_b\rangle = \langle e_4, \partial_a\rangle = 0$. Notice further that we are summing over both the indices a and b. Thus:

$$\begin{aligned}&(\sigma^{ab}trh_3' - (h_3')^{ab})(D_aQ)_{b4}\\&= \frac{1}{2}(\sigma^{ab}trh_3' - (h_3')^{ab})((D_aQ)_{b4} + (D_bQ)_{a4})\\&= (\sigma^{ab}trh_3' - (h_3')^{ab})\langle\partial_a, \partial_b\rangle\langle\partial_t, e_4\rangle\\&= (2trh_3' - trh_3')\langle\partial_t, e_4\rangle\\&= \langle\partial_t, trh_3'e_4\rangle\end{aligned}$$

We will treat the terms separately, then put them together.

$$
\begin{aligned}
&\nabla_a((\sigma^{ab} trh_3' - (h_3')^{ab})Q_{b4} - (\sigma^{ab} trh_4' - (h_4')^{ab})Q_{b3}) \\
&\quad = (R^{ab}_{a4} - (h_3')^{ab}\zeta_a + trh_3'\zeta^b)Q_{b3} - (R^{ab}_{a3} - (h_4')^{ab}\zeta_a + trh_4'\zeta^b)Q_{b4} \\
&\quad + (\sigma^{ab} trh_3' - (h_3')^{ab})\nabla_a Q_{b4} \\
&\quad - (\sigma^{ab} trh_4' - (h_4')^{ab})\nabla_a Q_{b3}
\end{aligned}
$$

We now deal with the latter two terms. Expanding the derivative of Q as we have above, then grouping terms, we obtain:

$$
\begin{aligned}
&trh_3'\langle \partial_t, e_4\rangle - trh_4'\langle \partial_t, e_3\rangle \\
&+ (\sigma^{ab} trh_4' - (h_4')^{ab})((h_4)_{ab}Q_{34} - Q_{bc}\sigma^{cd}(h_3)_{da} + \zeta_a Q_{b4}) \\
&- (\sigma^{ab} trh_3' - (h_3')^{ab})((h_3)_{ab}Q_{34} - Q_{bc}\sigma^{cd}(h_4)_{da} + \zeta_a Q_{b3})
\end{aligned}
$$

We split this up further into manageable pieces. The $Q_{bc}\sigma^{cd}$ terms are:

$$
Q_{bc}\sigma^{cd}[(h_3)_{da}(h_4')^{ab} - (h_4)_{da}(h_3')^{ab}]
$$

The Q_{34} terms give:

$$
\begin{aligned}
&Q_{34}[(h_3')^{ab}(h_3)_{ab} - trh_3'\sigma^{ab}(h_3)_{ab} - (h_4')^{ab}(h_4)_{ab} - trh_4'\sigma^{ab}(h_4)_{ab}] \\
&= Q_{34}[h_3' \cdot h_3 - trh_3' trh_3 - h_4' \cdot h_4 + trh_4' trh_4]
\end{aligned}
$$

where we have used the notation $h_3 \cdot h_3'$ to denote the product $(h_3)^{ab}(h_3)_{ab}$, as well as the fact that the metric σ satisfies $\sigma^{ab} = \delta^{ab}$, the Kronecker delta. Next, the ζ_a terms give us:

$$
\begin{aligned}
&(\sigma^{ab} trh_4' - (h_4')^{ab})\zeta_a Q_{b4} - (\sigma^{ab} trh_3' - (h_3')^{ab})\zeta_a Q_{b3} \\
&= (trh_4'\zeta^b - (h_4')^{ab}\zeta_a)Q_{b4} - (trh_3'\zeta^b - (h_3')^{ab}\zeta_a)Q_{b3}
\end{aligned}
$$

which will cancel with the other terms from the first computation. Finally, the left over terms we have give rise to:

$$
\begin{aligned}
&trh_3'\sigma^{ab}Q_{bc}\sigma^{cd}(h_4)_{da} - trh_4'\sigma^{ab}Q_{bc}\sigma^{cd}(h_3)_{da} \\
&= trh_3' Q_{bc}(h_4)^{cb} - trh_4' Q_{bc}(h_3)^{cb} \\
&= Q_{ab}(trh_3' h_4^{ab} - trh_4' h_3^{ab}) \\
&= 0
\end{aligned}
$$

Let us write down the final divergence quantity we have arrived at:

$$
\begin{aligned}
&\nabla_a((\sigma^{ab} trh_3' - (h_3')^{ab})Q_{b4} - (\sigma^{ab} trh_4' - (h_4')^{ab})Q_{b3}) \\
&\quad = R^{ab}_{a4}Q_{b3} - R^{ab}_{a3}Q_{b4} + trh_3'\langle \partial_t, e_4\rangle - trh_4'\langle \partial_t, e_3\rangle \\
&\quad + Q_{bc}\sigma^{cd}[(h_3)_{da}(h_4')^{ab} - (h_4)_{da}(h_3')^{ab}] \\
&\quad + Q_{34}[h_3' \cdot h_3 - trh_3' trh_3 - h_4' \cdot h_4 + trh_4' trh_4]
\end{aligned}
$$

where we note that the ζ_a terms cancelled with some of the terms in the first line.

Now that we have considered every term arising from this divergence term, it remains for us to examine what happens for the other divergence quantity with no prime $'$ terms. All the equations we have above continue to hold, although the final expressions have a slightly simpler form.

Indeed, first let us replace every h_3' with h_3, and every h_4' with h_4. For the Q_{34} terms, let us use the fact that the second fundamental forms are symmetric 2×2 matrices, say of the forms:

$$\begin{bmatrix} a & c \\ c & b \end{bmatrix}, \begin{bmatrix} e & g \\ g & f \end{bmatrix}$$

$$\begin{aligned}
&Q_{34}[h_3 \cdot h_3 - (trh_3)^2 - h_4 \cdot h_4 + (trh_4)^2] \\
&= Q_{34}[a^2 + 2c^2 + b^2 - (a+b)^2 - e^2 - f^2 - 2g^2 + (e+f)^2] \\
&= Q_{34}[2c^2 - 2ab - 2g^2 + 2ef] \\
&= Q_{34}[2 \det h_4 - 2 \det h_3]
\end{aligned}$$

The term $trh_3'\langle \partial_t, e_4 \rangle$ changes to $trh_3\langle \partial_t, e_4 \rangle$, and likewise for 3 and 4 interchanged. Next, the Gauss equations whence we obtained the curvature terms in the ambient spacetime do not give any curvature terms after the embedding, because the curvature tensor of the flat Minkowski spacetime is zero. Before we move onto putting everything together, we make more sense of one of the expressions after the embedding. Let us calculate the differential of the connection one form of the normal bundle.

$$\begin{aligned}
&d\zeta(\partial_a, \partial_b) \\
&= \partial_a \zeta(\partial_b) - \partial_b \zeta(\partial_a) - \zeta([\partial_a, \partial_b]) \\
&= \partial_a \langle D_b e_3, e_4 \rangle - \partial_b \langle D_a e_3, e_4 \rangle \\
&= \langle (D_a D_b - D_b D_a) e_3, e_4 \rangle + \langle D_b e_3, D_a e_4 \rangle - \langle D_a e_3, D_b e_4 \rangle
\end{aligned}$$

The first term is a curvature $R(\partial_a, \partial_b, e_3, e_4)$, which for a codimension 2 surface will vanish by the equations of chapter 3, since:

$$R(\partial_a, \partial_b, e_3, e_4) = \langle (D_a e_3)^\perp, (D_b e_4)^\perp \rangle - \langle (D_b e_3)^\perp, (D_a e_4)^\perp \rangle$$

where we note that $D_a e_3$ is orthogonal to e_3 by $\partial_a \langle e_3, e_3 \rangle = 2\langle D_a e_3, e_3 \rangle = 0$, and so similarly $D_b e_4$ is orthogonal to e_4. This means that the inner products are some components multiplying $\langle e_3, e_4 \rangle = 0$. For what remains, the above arguments show us that any of the terms in the normal directions will vanish, and so we may split up the derivatives into components along the tangent directions ∂_a, ∂_b, and so obtain:

$$\begin{aligned}
&\langle \langle D_b e_3, \partial_a \rangle \partial_a + \langle D_b e_3, \partial_b \rangle \partial_b, \langle D_a e_4, \partial_a \rangle \partial_a + \langle D_a e_4, \partial_b \rangle \partial_b \rangle \\
&- \langle \langle D_a e_3, \partial_a \rangle \partial_a + \langle D_a e_3, \partial_b \rangle \partial_b, \langle D_b e_4, \partial_a \rangle \partial_a + \langle D_b e_4, \partial_b \rangle \partial_b \rangle \\
&= (h_3)_{ab}(h_4)_{aa} + (h_3)_{bb}(h_4)_{ab} - (h_3)_{aa}(h_4)_{ab} - (h_3)_{ab}(h_4)_{bb}
\end{aligned}$$

These computations give us the Ricci equation, which means that after the embedding, we obtain the term:

$$Q_{bc}\sigma^{cd}[(h_3)_{da}(h_4)^{ab} - (h_4)_{da}(h_3)^{ab}] = Q_{bc}(d\zeta)^{bc} = 0$$

the vanishing due to $Q_{bc} = 0$ in our case, as there are no timelike vectors among the coordinate vector fields ∂_b. To finish the proof, we need only to take one divergence quantity, subtract it from the other, and then integrate over the surface Σ. The divergence quantities under integration vanish by Stokes' theorem, and so we obtain the desired integral equalities.

■

Let us return to the connection with the Wang-Yau quasi-local mass. We are given, a priori, the frame $\{e_3', e_4'\}$ of the normal bundle to Σ inside N, and an isometric embedding X of the surface into Minkowski spacetime $\mathbb{R}^{3,1}$. To construct a "canonical" gauge in the reference space, we further require a choice of a timelike unit vector, T_0, in $\mathbb{R}^{3,1}$. We take the image $X(\Sigma)$ and project it onto the orthogonal complement of T_0, which for convenience we denote by $T_0^{\perp}$. The projection now lies in this 3-dimensional "time-slice" of Minkowski space, and so admits a unique normal unit vector field e_3, defined globally by parallel translation along T_0. Now we must choose the second normal vector field e_4 to Σ. We take it to be that vector field which is orthogonal to e_3, and uniquely determined by the condition that

$$\langle H', e_4' \rangle = \langle H, e_4 \rangle$$

Namely, that the inner product of the mean curvature vector field with the timelike normal e_4 is the same whether we measure it in the source, or in the target spacetime. This means that the expansion of the surface along the timelike vector fields is the same before and after and embedding [28].

The connection with the quasi-local mass is that, if we have a surface Σ in spacetime N, and an observer (X, T_0) satisfying $\alpha_{e_3} = \alpha_{e_3'}$, then

$$\begin{aligned} E(\Sigma, X, T_0) &= \int -\langle \partial_t, e_4 \rangle (trh_3 - trh_3') + \langle \partial_t, e_3 \rangle (trh_4 - trh_4') d\sigma \\ &= \frac{1}{8\pi} \int (2 \det h_3 - 2 \det h_4 - trh_3 trh_3' + h_3 \cdot h_3' + trh_4 trh_4' - h_4 \cdot h_4') Q_{34} \\ &+ (R^{ab}_{a4} Q_{b3} - R^{ab}_{a3} Q_{b4}) - Q_{bc}\sigma^{cd}[(h_3)_{da} h_4'^{ab} - (h_4)_{da} h_3'^{ab}] \, d\sigma \end{aligned}$$

and here we immediately see the Minkowski formula above allowing us to rewrite the difference of the mean curvatures.

4.2. Estimates in Minkowski space

The goal of this section is to provide applications of the quasi-local mass as a geometric gadget that can be used, for example, to prove rigidity theorems for 3-dimensional manifolds fitting a certain framework of hypetheses. Such results are reliant on a variety of bounds on the quasi-local mass and energy with reference to the Minkowski spacetime. We begin with the choice of isometric embedding X and timelike unit vector $T_0 = \partial_t$. The Liu-Yau quasi-local mass of the surface, $m_{LY}(\Sigma)$, is then equal to the quasi-local energy $E(\Sigma, X, \partial_t)$. Under the assumption that $\alpha_H = 0$, that is, $\langle D_X H', e_4' \rangle = 0$ for all choices of fields X, the Liu-Yau quasi-local mass is a critical point of the Wang-Yau quasi-local energy [4].

Theorem 4.2. *If $\alpha_H = 0$, then the Liu-Yau mass is*

$$m_{LY}(\Sigma) = \frac{1}{8\pi}\int (2\det h_3 - trh_3 trh_3' + h_3 \cdot h_3')(X \cdot e_3) - R^{ab}_{a3}(X \cdot e_b)\, d\sigma$$

Proof. We embed the surface into $(\partial_t)^\perp$, and so take $e_4 = \partial_t$. Therefore, the mean curvature in the embedding must have no component in the ∂_t direction.

$$\langle H, e_4 \rangle = 0$$

Recall that when constructing these frames, we make the assumption that

$$\langle H', e_4' \rangle = \langle H, e_4 \rangle$$

and so we obtain $e_4' = J/|H|$. Next, recall that h_3 and h_4 are in the directions e_3 and e_4, respectively, and thus so are their traces. By a slight abuse of notation, we again denote by h_i both the vector in the direction e_i as well as the scalar value, which should be clear in the following context:

$$\begin{aligned}
\alpha_H(X) &= \langle D_X H, e_4 \rangle \\
&= \langle D_X(trh_3 + trh_4), e_4 \rangle \\
&= trh_3 \langle D_X e_3, e_4 \rangle + trh_4 \langle D_X e_4, e_4 \rangle \\
&= trh_3 \langle D_X e_3, e_4 \rangle \\
&= 0
\end{aligned}$$

By the equality $\langle H', e_4' \rangle = \langle H, e_4 \rangle$, we obtain the same result with e_3', e_4', whence it follows that

$$\alpha_{e_3} = \alpha_{e_3'} = 0$$

Next, note that since the isometric embedding is into the orthogonal complement of ∂_t, we actually obtain $h_4 = 0$. Now, let us apply Theorem 4.1, which yields:

$$m_{LY}(\Sigma) = \frac{1}{8\pi}\int (2\det h_3 - trh_3 trh_3' + h_3 \cdot h_3')Q_{34} + R^{ab}_{a4}Q_{b3} - R^{ab}_{a3}Q_{b4}\, d\sigma$$

We compute the necessary Q components, using the form $Q = r\ dr \wedge dt$, with $e_4 = \partial_t$.

$$Q_{ab} = 0,\ Q_{a3} = 0,\ Q_{34} = (X \cdot e_3),\ Q_{b4} = (X \cdot e_4)$$

using $X \cdot e_i$ to denote the inner product in this direction. Using these values, we recover the desired expression.

■

This allows us to prove the following bound on the Liu-Yau mass:

Theorem 4.3. *Given Σ a topological sphere in spacetime N, and choice of frame $e_3' = -H/|H|$, $e_4' = J/|H|$. If, in addition, $\alpha_{e_3'} = 0$, $R^{ab}_{a4} = 0$, the second fundamental form in direction e_3', h_3', is positive definite, and the Gauss curvature of σ is positive, then:*

$$m_{LY}(\Sigma) \leq \frac{1}{8\pi}\int 2R^+_{1212}(X \cdot e_3) - R^{ab}_{a3}(X \cdot e_b)\, d\sigma$$

with $R^+_{1212} = \max\{R_{1212}, 0\}$.

Proof. The proof uses the Gauss and Codazzi equations to estimate the integrand. The Gauss equation describes the Gauss curvature K of the surface Σ, an invariant under the isometric embedding from N into Minkowski space. It states:

$$K = \det h_3' - \det h_4' + R_{1212}$$

which we may read off of the general Gauss equation in section 3.2, using the fact that $K = R_{1221}$ in the curvature tensor induced on Σ (whereas the R above is the curvature of the ambient spacetime N). Next, the Codazzi equations are the codimension 2 case of Theorem 3.3:

$$\begin{aligned}
\nabla_a(h_3')_{bc} - \nabla_b(h_3')_{ac} &= R_{abc3} + (\alpha_{e_3'})_b(h_4')_{ac} - (\alpha_{e_3'})_a(h_4')_{bc} \\
\nabla_a(h_4')_{bc} - \nabla_b(h_4')_{ac} &= R_{abc4} + (\alpha_{e_3'})_b(h_3')_{ac} - (\alpha_{e_3'})_a(h_3')_{bc}
\end{aligned}$$

We have assumed that $\alpha e_3' = 0$, so most of the above terms vanish. Moreover, we may show that $trh_4' = 0$, using the fact that our choice of frame assumes $\langle H, e_4 \rangle = \langle H', e_4' \rangle$. Recall from the previous theorem that in the case of the Liu-Yau mass m_{LY}, we are embedding into the orthogonal complement $(\partial_t)^\perp$, yielding $h_4 = 0$. As we had done before, we may derive:

$$\begin{aligned}
\langle H, e_4 \rangle &= \langle (trh_3')e_3 + (trh_4)e_4, e_4 \rangle \\
&= 0 \\
&= \langle H', e_4' \rangle \\
&= \langle (trh_3')e_3' + (trh_4')e_4', e_4' \rangle \\
&= -trh_4'
\end{aligned}$$

Let us rearrange the indices in the Codazzi equations, and take a trace in the following way:

$$\nabla^c(h_4')_{bc} = \nabla^c(h_4')_{bc} + \nabla_b trh_4' = 0$$

where adding the trh_4' term of course does not change anything, and the $R^{ab}_{a4} = 0$ assumption removes the curvature term. It then follows that $h_4' = 0$, by symmetry and being traceless. Therefore, we only need to consider the second fundamental forms in the e_3, e_3' directions.

We diagonalize and write them as 2×2 matrices:

$$h_3 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \; h_3' = \begin{pmatrix} c & e \\ e & d \end{pmatrix}$$

with which the Gauss equations for before and after the embedding become, respectively:

$$\begin{aligned}
K &= cd - e^2 + R_{1212} \\
K &= ab
\end{aligned}$$

with the curvature of the Minkowski spacetime being zero, and thus yielding no term to match R_{1212}.

Now, to begin estimating m_{LY}, we need a bound on the terms $2\det h_3 - tr h_3 tr h_3' + h_3 \cdot h_3'$ appearing in the integral in Theorem 4.2. Using the matrices for h_3 and h_3', this quantity is

$$2ab - bc - ad$$

We will argue first assuming $R_{1212} \leq 0$, then with $R_{1212} > 0$.

For the first case, we may read off from the Gauss equations that $K \leq cd$. So in particular:

$$\begin{aligned} bc + ad &= bc + \frac{abcd}{bc} \\ &\geq bc + \frac{K^2}{bc} \end{aligned}$$

Note that for any positive real numbers, we have

$$\frac{a}{b} + \frac{b}{a} \geq 2$$

so for our case we have

$$\begin{aligned} \frac{bc}{K} + \frac{K}{bc} &\geq 2 \\ bc + \frac{K^2}{bc} &\geq 2K \end{aligned}$$

So altogether, we attain:

$$2ab - bc - ad \leq 2K - 2K = 0$$

For the case of $R_{1212} > 0$, the Gauss equations tell us that

$$cd = K - R_{1212} + e^2 > K - R_{1212}$$

We may choose a constant $C > 1$ such that

$$(K - R_{1212})C^2 = K = ab$$

Dividing through, we can get the estimate

$$K - R_{1212} = \frac{K}{C^2} \leq cd$$

With this, let us again estimate $bc + ad$:

$$\begin{aligned} bc + ad &= \frac{abcd}{ad} + ad \\ &\geq \frac{K^2}{C^2 ad} + ad \\ &= \frac{K}{C}\left(\frac{K}{Cad} + \frac{Cad}{K}\right) \\ &\geq \frac{2K}{C} \end{aligned}$$

Using this result, we get:

$$
\begin{aligned}
2ad - bc - ad &\leq 2K - \frac{2K}{C} \\
&\leq 2K - \frac{2K}{C^2} \\
&= 2R_{1212}
\end{aligned}
$$

Setting $R^{+}_{1212} = \max\{R_{1212}, 0\}$, we arrive at the desired inequality. ■

Finally, we may use the previous bound to obtain a rigidity result, which makes use of the quasi-local mass to yield geometric properties of a 3-dimensional manifold. This method will come into play again in the case of embeddings into Anti de-Sitter space. We note that among the hypotheses of the following theorem lies the dominant energy condition. We do not, however, belabour this point, as the methods of proof remain clear.

Theorem 4.4. *Given Σ a surface in spacetime N satisfying the dominant energy condition, and bounding a spacelike hypersurface M, suppose that we have: frame $e_3' = -H/|H|, e_4' = J/|H|$, with $\alpha_{e_3'} = 0$, $R^{ab}_{a4} = 0, \nabla_b R^{ab}_{a3} = 0$, and $R_{1212} \leq 0$ on Σ. Assume that the second fundamental form in direction of e_3, h_3 is positive definite, and Gauss curvature of Σ is positive. Then the domain of dependence of M is isometric to an open set of $\mathbb{R}^{3,1}$.*

Proof. The proof uses by Theorem 4.3, integration by parts, and realizing that the vector field

$$(X \cdot e_a)e_a + (X \cdot e_b)e_b$$

is the gradient quantity

$$\frac{1}{2}\nabla|X|^2$$

Then we may apply the hypotheses of the theorem directly to the Liu-Yau mass inequality:

$$
\begin{aligned}
m_{LY}(\Sigma) &\leq \frac{1}{8\pi}\int 2R^{+}_{1212}(X \cdot e_3) - R^{ab}_{a3}(X \cdot e_b)\, d\sigma \\
&\leq \frac{1}{8\pi}\int \frac{1}{2}|X|^2 \nabla_b R^{ab}_{a3}\, d\sigma \\
&= 0
\end{aligned}
$$

Now, the mass term satisfies positivity and rigidity properties that then force the isometry of the domain of dependence to an open set of Minkowski space. However, we do not discuss these properties in any further detail.

■

The next upper bound is a necessary lemma before the final rigidity theorem we shall state for the Minkowski reference case.

Theorem 4.5. *If Σ is a convex surface in a time-symmetric hypersurface $(M, \bar{g})$ with positive Gauss curvature, then its Brown-York quasi-local mass satisfies the* *upper bound:*

$$m_{BY}(\Sigma) = \frac{1}{8\pi}\int (2\det h_3 - trh_3 trh_3' + h_3 \cdot h_3')(X\cdot e_3) - R_{a3}^{ab}(X\cdot e_b)\, d\sigma$$
$$\leq \frac{1}{8\pi}\int 2\bar{R}^{+}_{1212}(X\cdot e_3)\, d\sigma + \frac{1}{16\pi}\int |X|^2 \nabla_b \bar{R}_{a3}^{ab}\, d\sigma$$

Proof. To prove this, we begin with a static spacetime N with metric $g = -dt^2 + \bar{g}$, realizing our surface Σ as lying in the time slice. Now, notice that the curvature R of the spacetime is related to the curvature $\bar{R}$ of the original manifold by the relations:

$$R_{ijkl} = \bar{R}_{ijkl}$$
$$R_{ijk0} = 0$$

meaning that the curvature tensor is zero in the timelike direction. Now, the upper bound may be calculated using the same ideas as in Theorem 4.3. The first term $2R^{+}_{1212}$ is clear from the result of said theorem. The second term follows by applying the methods used in Theorem 4.4. ■

We arrive at the rigidity theorem using Brown-York quasi-local mass.

Theorem 4.6. *Let $(M, \bar{g})$ be a 3-dimensional manifold with boundary Σ, with scalar curvature of $\bar{g}$ nonnegative, and Σ a convex 2-dimensional sphere with positive Gauss curvature. If $\nabla_b \bar{R}_{a3}^{ab} = 0, \bar{R}_{1212} \leq 0$ on Σ, then $\bar{g}$ is the flat metric.*

Proof. Notice that under the given assumptions, Theorem 4.5 gives us the bound

$$m_{BY} \leq 0$$

The fact that $\bar{g}$ must be the flat metric then follows from results on the Brown-York mass' positivity and rigidity, which we again do not delve into. ■

Before moving onto other reference spacetimes, we would like to describe a result on the asymptotic behaviour of the quasi-local mass. In particular, that the Brown-York-Liu-Yau quasi-local mass limits to the ADM mass in a precise sense [4]. For this purpose, we must first define asymptotically flat manifolds.

Definition 4.2.1. A 3-dimensional manifold $(M, \bar{g})$ is called asymptotically flat of order τ if, outside a compact set, M is diffeomorphic to $\mathbb{R}^3 \setminus \{|x| \leq r_0\}$ for some $r_0 > 0$, and under the diffeomorphism we have:

$$\bar{g}_{ij} - \delta_{ij} = O(|x|^{-\tau});\ \partial\bar{g}_{ij} = O(|x|^{-\tau-1});\ \partial^2\bar{g}_{ij} = O(|x|^{-\tau-2})$$

for some $\tau > 1/2$. We have denoted by ∂ the partial differentiation on $\mathbb{R}^3$.

Theorem 4.7. *If we have an asymptotically flat manifold of order $\tau > 1/2$ with Σ_r the coordinate spheres of asymptotically flat coordinates, then*

$$\lim_{r\to\infty}\int_{\Sigma_r} H - H' + (\overline{Ric} - \frac{1}{2}\bar{R}\bar{g})(X, e_3)\, d\sigma_r = 0$$

Proof.

$$\begin{aligned}\int H - H' \, d\sigma &= 8\pi m_{LY}(\Sigma) \\ &= \int (2\det h_3 - trh_3' trh_3 + h_3' \cdot h_3) - \overline{Ric}(e_3, e_a)X \cdot e_a \, d\sigma \\ &= \int \det(h_3 - h_3')(X \cdot e_3) + (\det h_3 - \det h_3')(X \cdot e_3) \\ &\quad - \overline{Ric}(e_3, e_a)X \cdot e_a \, d\sigma\end{aligned}$$

where the above equalities can be checked by again using the fact that the second fundamental forms are 2×2 symmetric matrices, and computing directly. Now, notice that the determinant of the second fundamental form gives us the Gauss curvature. In this case, for a 2-dimensional surface within a 3-dimensional space, this is simply the quantity $\bar{R}_{1212}$. But we may rewrite this using:

$$\begin{aligned}\bar{R} &= \overline{Ric}_{11} + \overline{Ric}_{22} + \overline{Ric}_{33} \\ &= \bar{R}_{1212} + \bar{R}_{1313} + \bar{R}_{2121} + \bar{R}_{2323} + \bar{R}_{3131} + \bar{R}_{3232} \\ &= 2\bar{R}_{1212} + 2\overline{Ric}_{33}\end{aligned}$$

Using this in our calculations, we obtain:

$$\begin{aligned}&\int \det(h_3 - h_3')(X \cdot e_3) + (\det h_3 - \det h_3')(X \cdot e_3) - \overline{Ric}(e_3, e_a)X \cdot e_a \, d\sigma \\ &= \int \det(h_3 - h_3')(X \cdot e_3) + (\frac{\bar{R}}{2} - \overline{Ric}(e_3, e_3))(X \cdot e_3) - \overline{Ric}(e_3, e_a)(X \cdot e_a) \, d\sigma\end{aligned}$$

Notice that the Ricci terms can be combined into:

$$\overline{Ric}_{33}(X \cdot e_3) + \overline{Ric}_{3a}(X \cdot e_a) = \overline{Ric}(X, e_3)$$

and therefore the integral above becomes

$$\int \det(h_3 - h_3')(X \cdot e_3) + (\frac{\bar{R}}{2}\bar{g} - \overline{Ric})(X, e_3) \, d\sigma$$

Finally, notice that $\det(h_3 - h_3') = O(r^{-2\tau-2})$, and $X \cdot e_3 = O(r)$, so that as $r \to \infty$, we arrive at the desired limit expression. ■

4.3. Estimates in Anti de-Sitter space

The case of embedding our surface Σ isometrically into Anti de-Sitter space will be treated analogously to the Minkowski case, so that the processes involved will seem much the same. We begin with the mirror result at the basis of the paper.

Theorem 4.8. *Let Σ be a 2-dimensional surface in a physical spacetime N, and take a frame $\{e_3', e_4'\}$ of its normal bundle. Let X be an isometric embedding of*

Σ into the Anti de-Sitter spacetime. Suppose that there is a frame $\{e_3, e_4\}$ of the normal bundle of $X(\Sigma)$ so that $\alpha_{e_3'} = \alpha_{e_3}$. Then we again have the formula:

$$\begin{aligned}
&\int -\langle \partial_t, e_4 \rangle (trh_3 - trh_3') + \langle \partial_t, e_3 \rangle (trh_4 - trh_4') d\sigma \\
&= \int (2 \det h_3 - 2 \det h_4 - trh_3 trh_3' + h_3 \cdot h_3' + trh_4 trh_4' - h_4 \cdot h_4') Q_{34} \\
&\quad + (R^{ab}_{a4} Q_{b3} - R^{ab}_{a3} Q_{b4}) - Q_{bc} \sigma^{cd} [(h_3)_{da} h'^{ab}_4 - (h_4)_{da} h'^{ab}_3] \, d\sigma
\end{aligned}$$

for R the curvature tensor of N.

Proof. Notice that the Anti de-Sitter spacetime once again has the spherical symmetry discussed in section 3.4, and thus admits the same Killing-Yano 2-form as in the Minkowski case, with the same divergence of $3\partial_t$. Therefore, the same proof yields for us the desired integral expression. The only significant difference is that the curvature tensor of the AdS spacetime does not vanish. However, this is a complete manifold of constant curvature, that is, a space form, and it is a well known fact that the curvature tensor of such a manifold satisfies the equality:

$$R(X, Y)Z = \mathrm{Sec}(g)(g(Y, Z)X - g(X, Z)Y)$$

and with $Z = e_3$ or $Z = e_4$, we see that the relevant curvature terms vanish. ■

We again have a connection to quasi-local mass for the above formula, as taking the Killing vector field $T_0 = \partial_t$ and embedding X with $\alpha'_{e_3} = \alpha_{e_3}$ yields for us

$$\begin{aligned}
E(\Sigma, X, T_0) = \frac{1}{8\pi} \int &(2 \det h_3 - 2 \det h_4 - trh_3 trh_3' + h_3 \cdot h_3' + trh_4 trh_4' - h_4 \cdot h_4') Q_{34} \\
&+ (R^{ab}_{a4} Q_{b3} - R^{ab}_{a3} Q_{b4}) - Q_{bc} \sigma^{cd} [(h_3)_{da} h'^{ab}_4 - (h_4)_{da} h'^{ab}_3] \, d\sigma
\end{aligned}$$

Mimicking the setup of the Minkowski case, we take $\alpha_H = 0$, and deduce the following:

Theorem 4.9. *If $\alpha_H = 0$ and X is an isometric embedding of Σ into the $t = 0$ slice of the Anti de-Sitter spacetime, then we have*

$$E(\Sigma, X, \partial_t) = \frac{1}{8\pi} \int (2 \det h_3 - trh_3 trh_3' + h_3 \cdot h_3')(r\partial_r \cdot e_3) - R^{ab}_{a3}(r\partial_r \cdot e_b) \, d\sigma$$

Proof. Recall that with an embedding into a time-slice, we have $h_4 = 0$. Moreover, $Q_{ab} = Q_{a3} = 0$, and $Q_{a4} = r\partial_r \cdot e_a$. The result then follows. ■

We next obtain a rigidity result in the spirit of Theorem 4.6 above, where we use the embedding into the Anti de-Sitter spacetime to bring forth the quasi-local mass unavailable in the original setting. Once again, we see that this introduces for us a useful tool in the analysis of the given manifold.

Theorem 4.10. *Let $(M, \bar{g})$ be a 3-dimensional manifold with boundary Σ which is convex and has Gauss curvature (of the induced σ) bounded from below by -1. Suppose that the scalar curvature $\bar{R}(\bar{g}) \geq -6$. Let $\bar{R}_{ijkl}$, the curvature tensor of $\bar{g}$, satisfy $\nabla_b \bar{R}^{ab}_{a3} = 0$ and $\bar{R}_{1212} \leq -1$ on the boundary. Then $\bar{g}$ is the hyperbolic metric.*

Proof. Let us choose an isometric embedding X of Σ into hyperbolic space, with the constraint $r\partial_r \cdot e_3 > 0$. The hyperbolic metric

$$\frac{dr^2}{1+r^2} + r^2\, dS^2$$

can be viewed as the $t = 0$ time-slice of the AdS metric

$$-dt^2 + \frac{dr^2}{1+r^2} + r^2\, dS^2$$

Moreover, the vector field $r\partial_r \cdot e_b$ can be viewed as the gradient $\nabla_b(r^2/2 + r^4/4)$ with respect to the connection on hyperbolic space. Last of all, notice that $R = \bar{R}$ in the case that $g = -dt^2 + \bar{g}$. Therefore, we obtain:

$$\int R^{ab}_{a3}(r\partial_r \cdot e_b)\, d\sigma = -\int \nabla_b R^{ab}_{a3}\big(\frac{r^2}{2} + \frac{r^4}{4}\big)\, d\sigma = 0$$

Consequently, the energy quantity becomes

$$E(\Sigma, X, \partial_t) = \frac{1}{8\pi}\int 2\det h_3 - trh_3 trh_3' + h_3 \cdot h_3')(r\partial_r \cdot e_3)\, d\sigma$$

As we had in Theorem 4.3, we must now estimate the quantity inside the integral. We employ the Gauss equations, which in the case of an embedding into hyperbolic space give us

$$\begin{aligned} K &= \det h_3' + R_{1212} \\ K &= \det h_3 - 1 \end{aligned}$$

with -1 being the sectional curvature of hyperbolic space. Use the notation of Theorem 4.3 to rewrite these using the components of the second fundamental forms:

$$\begin{aligned} K &= cd - e^2 + R_{1212} \\ K &= ab - 1 \end{aligned}$$

Finally, notice that with our assumption of $R_{1212} \leq -1$, these equations reduce to the case of $R_{1212} \leq 0$ of Theorem 4.3, and so the same argument yields

$$2\det h_3 - trh_3 trh_3' + h_3 \cdot h_3' = 2ab - bc - ad \leq 0$$

Once more, although we do not discuss the details, theorems concerning positivity and rigidity of the energy force the metric $\bar{g}$ to be the standard hyperbolic one. ∎

At the end of this section, we once again wish to study the asymptotic behaviour of the quasi-local mass, now with reference to AdS, which corresponds to the hyperbolic metric. In a way parallel to the flat case, we come to the following definition:

Definition 4.3.1. A 3-dimensional manifold $(M, \bar{g})$ is called asymptotically hyperbolic of order τ if, outside a compact set, M is diffeomorphic to $\mathbb{H}^3 \setminus \{|x| \leq r_0\}$ for some $r_0 > 0$, and under the diffeomorphism we have:

$$|\bar{g} - g_0| = O(|x|^{-\tau});\ |\partial(\bar{g} - g_0)| = O(|x|^{-\tau-1});\ |\partial^2(\bar{g} - g_0)| = O(|x|^{-\tau-2})$$

for some $\tau > 3/2$. We have denoted by ∂ the partial differentiation on hyperbolic space $\mathbb{H}^3$, and its metric by g_0, with respect to which we measure the norm.

We also arrive at a similar theorem, taking $V = \sqrt{r^2+1}$ to be the static potential.

Theorem 4.11. *If we have an asymptotically hyperbolic manifold of order $\tau > 3/2$, and Σ_r denote the coordinate spheres of the asymptotically hyperbolic coordinates, then we have*

$$\lim_{r\to\infty} \int V(H_0 - H) + (\overline{Ric} - \frac{1}{2}(\bar{R}+2)\bar{g})(X, e_3)\, d\sigma = 0$$

We forgo the proof given that it follows the same calculation as in the Minkowski case, though with the change of $\bar{R}$ to $\bar{R}+2$ to account for the hyperbolic metric's curvature.

4.4. Schwarzschild and Kerr results

In this final section, we note that the authors have showed the critical theorem holds for embeddings into the Schwarzschild spacetime with minimal adjustments. We will show, further, that this theorem will hold analogously in the case of the Kerr solution, largely because the dependence on the explicit form of the conformal Killing-Yano 2-form within the result is in fact focused upon its divergence, which remains as $3\partial_t$ as we allow for rotation $a > 0$. First, for the static solution $a = 0$ we retain:

Theorem 4.12. *Let Σ be a 2-dimensional surface in a physical spacetime N, and take a frame $\{e_3', e_4'\}$ of its normal bundle. Let X be an isometric embedding of Σ into the Schwarzschild spacetime. Suppose that there is a frame $\{e_3, e_4\}$ of the normal bundle of $X(\Sigma)$ so that $\alpha_{e_3'} = \alpha_{e_3}$. Then we have the formula:*

$$\begin{aligned}
&\int -\langle \partial_t, e_4\rangle(trh_3 - trh_3') + \langle \partial_t, e_3\rangle(trh_4 - trh_4')d\sigma \\
&= \int (2\det h_3 - 2\det h_4 - trh_3 trh_3' + h_3\cdot h_3' + trh_4 trh_4' - h_4 \cdot h_4')Q_{34} \\
&\quad + (R_{a4}^{ab} - \mathscr{S}_{a4}^{ab})Q_{b3} - (R_{a3}^{ab} - \mathscr{S}_{a3}^{ab})Q_{b4} - Q_{bc}\sigma^{cd}[(h_3)_{da}h_4'^{ab} - (h_4)_{da}h_3'^{ab}]\, d\sigma
\end{aligned}$$

for R the curvature tensor of N, and $\mathscr{S}$ the curvature tensor of the Schwarzschild spacetime.

Arguing in the more general case of Kerr will suffice, and so we may move onto the following generalization:

Theorem 4.13. *Let Σ be a 2-dimensional surface in a physical spacetime N, and take a frame $\{e_3', e_4'\}$ of its normal bundle. Let X be an isometric embedding of Σ into the Kerr spacetime. Suppose that there is a frame $\{e_3, e_4\}$ of the normal bundle of $X(\Sigma)$ so that $\alpha_{e_3'} = \alpha_{e_3}$. Then we have the formula:*

$$\begin{aligned}
&\int -\langle \partial_t, e_4\rangle(trh_3 - trh_3') + \langle \partial_t, e_3\rangle(trh_4 - trh_4')d\sigma \\
&= \int (2\det h_3 - 2\det h_4 - trh_3 trh_3' + h_3\cdot h_3' + trh_4 trh_4' - h_4 \cdot h_4')Q_{34} - Q_{ab}(d\zeta)^{ab} \\
&\quad + (R_{a4}^{ab} - \mathscr{K}_{a4}^{ab})Q_{b3} - (R_{a3}^{ab} - \mathscr{K}_{a3}^{ab})Q_{b4} - Q_{bc}\sigma^{cd}[(h_3)_{da}h_4'^{ab} - (h_4)_{da}h_3'^{ab}]\, d\sigma
\end{aligned}$$

for R the curvature tensor of N, and $\mathscr{K}$ the curvature tensor of the Kerr spacetime.

Proof. The proof follows the same calculations for the divergence quantities as in Theorem 4.1. Going through the individual components, we note that first of all, the Gauss equations give the same terms regardless of the target spacetime into which we embed the surface. However, unlike the cases of Minkowski and Anti de-Sitter spacetimes, the curvature terms after the embedding do not vanish, hence the appearance of $\mathscr{K}$ above. The other terms come from differentiating the conformal Killing-Yano 2-form of the Kerr spacetime using the induced connection ∇ on Σ. Recall that the form is:

$$Q = a \cos\theta \sin\theta \; d\theta \wedge ((r^2 + a^2) \; d\phi - a \; dt) + r \; dr \wedge (a \sin^2\theta \; d\phi - dt)$$

with divergence $3\partial_t$ [15].

Now, we first related the induced connection ∇ to the ambient connection D, making no use of the specific form of Q, and so these terms remain the same. Further, we used the definition of the conformal Killing-Yano tensors of rank 2 in 4-dimensional spacetimes to obtain the terms of form $trh_3'\langle\partial_t, e_4\rangle$. Notice that within these calculations, we made explicit use of only the divergence of the tensor Q, which is the same in the Kerr solution. Therefore, the same conclusions follow here.

The latter parts of the computations make only one explicit use of Q, which is to allow for the vanishing of $Q_{ab}(d\zeta)^{ab}$. In the Kerr spacetime, this term remains. Thus the simplifications made to arrive at the final formula hold in this general case, as stated in the theorem. Letting a vanish in the metric and tensor, we recover the result of Theorem 4.12 for the Schwarzschild spacetime, as then $Q_{ab}(d\zeta)^{ab} = 0$.

■

5. Appendix

5.1. Curvature calculations in spacetimes

We wish to include several examples of how the connection and curvature forms can be used along with the structural equations described in chapter 3 in order to calculate the tensors in question. It is most illustrative to begin with the case of Euclidean space, after which we continue with the static Schwarzschild black hole. The Kerr calculations are not presented explicitly, but follow an analogous process, starting with a standard frame and coframe described at the end.

We follow the presentation of Sternberg [22] for $\mathbb{R}^2$ in polar coordinates. The metric we consider is given by

$$ds^2 = dr^2 + \phi(r)^2 \, d\psi^2$$

for which $\{dr, \phi(r)\, d\psi\}$ is a coframe field. Using compact notation, we write that

$$\theta = \begin{pmatrix} dr \\ \phi \, d\psi \end{pmatrix}, \; d\theta = \begin{pmatrix} 0 \\ \phi' \, dr \wedge \psi \end{pmatrix}$$

By the first structure equation, $d\theta + \omega \wedge \theta = 0$, we find that

$$\omega = \begin{pmatrix} 0 & -\phi' \, d\psi \\ \phi' \, d\psi & 0 \end{pmatrix}$$

The curvature form then follows from the second structure equation, $d\omega + \omega \wedge \omega = \Omega$. However, $\omega \wedge \omega = 0$, so this reduces to $\Omega = d\omega$.

$$\Omega = \begin{pmatrix} 0 & -\phi'' \, dr \wedge d\psi \\ \phi'' \, dr \wedge d\psi & 0 \end{pmatrix}$$

We identify the components in terms of the coframe field:

$$\theta = \begin{pmatrix} \theta^1 \\ \theta^2 \end{pmatrix}, \; \Omega = \begin{pmatrix} 0 & -\frac{\phi''}{\phi}\theta^1 \wedge \theta^2 \\ \frac{\phi''}{\phi}\theta^1 \wedge \theta^2 & 0 \end{pmatrix}$$

This now allows us to easily compute these terms on the frame field $\{E_1, E_2\}$ corresponding to the coframe field $\{\theta_1, \theta_2\}$.

$$\Omega(E_1, E_2) = \begin{pmatrix} 0 & -\frac{\phi''}{\phi} \\ \frac{\phi''}{\phi} & 0 \end{pmatrix}$$

Matrix multiplication on the right gives us the curvature tensor evaluation:

$$(E_1, E_2)\begin{pmatrix} 0 & -\frac{\phi''}{\phi} \\ \frac{\phi''}{\phi} & 0 \end{pmatrix} = \frac{\phi''}{\phi}(E_2, -E_1)$$

Which tells us that $R(E_1, E_2, E_2, E_1) = -\phi''/\phi$.

For the example of Schwarzschild calculations, we again follow [22]. The metric on $\mathbb{R}^2 \times S^2$ is given by

$$ds^2 = -h\, dt^2 + \frac{1}{h}\, dr^2 + r^2(d\theta^2 + S^2\, d\phi^2)$$

using the shorthand:

$$h = 1 - \frac{2M}{r},\ S = \sin\theta,\ C = \cos\theta$$

On the Schwarzschild exterior, that is, where $r > 2M$, we take the orthonormal coframe

$$\theta = \begin{pmatrix} \sqrt{h}\, dt \\ \frac{1}{\sqrt{h}}\, dr \\ r\, d\theta \\ rS\, d\phi \end{pmatrix}$$

calling its components $\theta^i, i = 0, 1, 2, 3$. We again employ the equation $d\theta + \omega \wedge \theta = 0$. We arrive at the expressions:

$$\begin{aligned} d\theta^0 &= -\frac{M}{r^2\sqrt{h}}\theta^0 \wedge \theta^1 \\ d\theta^1 &= 0 \\ d\theta^2 &= \frac{\sqrt{h}}{r}\theta^1 \wedge \theta^2 \\ d\theta^3 &= \frac{\sqrt{h}}{r}\theta^1 \wedge \theta^3 + \frac{C}{rS}\theta^2 \wedge \theta^3 \end{aligned}$$

Applying the structural equation for each of the components of $d\theta$ is sufficient to solve for the components of ω:

$$\omega = \begin{pmatrix} 0 & \frac{M}{r^2}\, dt & 0 & 0 \\ \frac{M}{r^2}\, dt & 0 & -\sqrt{h}\, d\theta & -S\sqrt{h}\, d\phi \\ 0 & \sqrt{h}\, d\theta & 0 & -C\, d\phi \\ 0 & S\sqrt{h}\, d\phi & C\, d\phi & 0 \end{pmatrix}$$

To find the curvature 2-form, we need to apply the second structural equation as before, $d\omega + \omega \wedge \omega = \Omega$. Since we know the components of the left hand side, this is purely computational. It yields:

$$\Omega = \begin{pmatrix} 0 & 2\theta^0 \wedge \theta^1 & -\theta^0 \wedge \theta^2 & -\theta^0 \wedge \theta^3 \\ 2\theta^0 \wedge \theta^1 & 0 & -\theta^1 \wedge \theta^2 & -\theta^1 \wedge \theta^3 \\ -\theta^0 \wedge \theta^2 & \theta^1 \wedge \theta^2 & 0 & 2\theta^2 \wedge \theta^3 \\ -\theta^0 \wedge \theta^3 & \theta^1 \wedge \theta^3 & -2\theta^2 \wedge \theta^3 & 0 \end{pmatrix}$$

Last of all, we describe the setup for the Kerr spacetime, using both Sternberg [22] and O'Neill [20]. Following the exposition of section 2.2, define on the region $I \cup II \cup III$ the Boyer-Lindquist frame field:

$$
\begin{aligned}
E_0 &= \frac{1}{\rho\sqrt{\varepsilon\Delta}}((r^2+a^2)\partial_t + a\partial_\phi) \\
E_1 &= \frac{\sqrt{\varepsilon\Delta}}{\rho}\partial_r \\
E_2 &= \frac{1}{\rho}\partial_\theta \\
E_3 &= \frac{1}{S\rho}(\partial_\phi + aS^2\partial_t)
\end{aligned}
$$

where we use the shorthard:

$$\rho^2 = r^2 + a^2C^2,\ \Delta = r^2 - 2Mr + a^2$$

and ε is a function which is equal to 1 on $I \cup III$, and to -1 on II. Now, the above induced the coframe:

$$
\begin{aligned}
\omega^0 &= \frac{\sqrt{\varepsilon\Delta}}{\rho}(dt - aS^2\, d\phi) \\
\omega^1 &= \frac{\rho}{\sqrt{\varepsilon\Delta}}\, dr \\
\omega^2 &= \rho\, d\theta \\
\omega^3 &= \frac{S}{\rho}((r^2+a^2)\, d\phi - a\, dt)
\end{aligned}
$$

What follows is then many calculations, using the two structural equations in the same way as in the above examples, to yield for us both the connection 1-form components for the connection in the Kerr metric, and then for the curvature. It can be shown directly in this way that the Kerr solution is Ricci-flat, as is expected of a vacuum solution of Einstein's field equations.

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