CONSEQUENCES OF A MATRIX MECHANICS & A RADIATING HARMONIC OSCILLATOR WITHOUT THE QUANTUM POSTULATE

A Thesis Submitted for the M.Sc. Degree

CONSEQUENCES OF A MATRIX MECHANICS AND A RADIATING HARMONIC OSCILLATOR WITHOUT THE QUANTUM POSTULATE.

By

A. Press

April, 1927

CONSEQUENCES OF A MATRIX MECHANICS AND A RADIATING HARMONIC OSCILLATOR WITHOUT THE QUANTUM POSTULATE.

By

A. Press

Summary

In order to avoid postulating quantum conditions from the start, as done in the Heisenberg-Born-Jordan-Dirac Matrix Mechanics, the writer conceived the idea of giving concrete visualization to the elements of the pq-qp matrix by relating them to a classical formula leading to an action. (See paper published in the Phil. Mag. for Sept., 1924). This required a slightly modified way of defining the unit Matrix.

If we consider a portion of a radiating gas capable of sending out a series of wave-lengths according to an observed law, then we may attribute to each periodicity a sine element of the p and q matrices. Such aggregates when expressed in the form of an action matrix pq-qp, constitute a standard diagonal matrix. In the text it has been shown that there is an analogy also between $h/2\pi i$ of matrix mechanics and the X-expression developed on the basis of the present matrix system.

An important distinction has also manifested itself. On account of the special character assigned to the

p and q elements in the modified matrices employed, the squares of the p(mn) elements rather than of the q(mn) elements are associated with the frequencies $\nu(mn)$. It must also be said that the X-function does not require the i-term to appear in the denominator. Nor is it necessary, for matrix processes, to assume that the value of all the diagonal elements in the Unit Matrix shall be unity or even equal to each other.

The requirement for the i-factor in the numerator has naturally led to the minus sign appearing when doubly differentiating a matrix, as occurs when considering the problem of a harmonic oscillator. Nevertheless, it must be borne in mind that the harmonic oscillator functions, heretofore used, have not been strictly radiating systems. This is especially true since the Hamiltonian H-function has been considered as constant and independent of the time. The radiating harmonic oscillator of the text, on the other hand, implies that only the mean H over the cyclic period is constant. Thus during a quarter of a cycle absorption of energy is allowed for, and mathematically expressed, whereas during the next quarter of a cycle, radiation of energy is presumed to take place. Besides, for a radiating oscillator the momentum is not wholly in phase with $\dot{q}$. Thus the observed frequencies are interpreted to indicate an upper level n=+1 and a lower level n=-1 with respect to a mean level of energy H_o. Strangely enough the ratio of upper (or lower) level energy H_n to mean level energy H_o is is of the order of 1/3 as against the one-half obtained from the Heisenberg-Born mechanics for E_o. Considerable freedom however is left open for specifying the "orbital frequency" of the generators.

The evidence goes to show that the postulates of the Quantum Mechanics do not necessarily involve such bold assumptions as appear to be the case at first sight. Part of their strange character seems to be due to the use of non-radiating harmonic oscillators, whereas those of the radiating type are now made mathematically available.

It is significant that it should be possible to show from purely classical considerations that the non-radiating harmonic oscillator is only then capable of becoming of radiating type when a discrete amount of energy (1/3 of the non-radiating content) is continuously being absorbed and re-radiated. This appears to suggest that the radiating harmonic oscillator of the text is analagous in its properties to that of an ordinary organ pipe.

....

The equation of the non-radiating harmonic oscillator is

$$\underline{H} = \tfrac{1}{2}\underline{p}^2 + 2\pi\, \nu_0{}^2 \underline{q}^2 \qquad \ldots\ldots (1)$$

where $\underline{p}$ is the generalized momentum and $\underline{q}$ is the generalized displacement. Consider, then, matrices of elements such as those of the Heisenherg-Born type:-

$$q = \left(q_{mn}\, \epsilon^{2\pi i\, \nu(mn)t} \right) ; \quad p = \left(p_{mn}\, \epsilon^{2\pi i\, \nu(mn)t} \right) \ldots (2)^{(*)}$$

It is possible to get a correspondence between the matrices p and q and the true momenta and displacements $\underline{p}$ and $\underline{q}$ if we consider the following resultant matrix:

$$d = pq - qp = U \qquad \ldots\ldots\ldots\ldots (3)$$

Thus let it be assumed that the process of differentiation with respect to the time can be performed in the following way:

$$\left(\frac{d}{dt}\right)d = \dot{d} = \left(\frac{d}{dt}\right)p.q + p.\left(\frac{d}{dt}\right)q - \left(\frac{d}{dt}\right)q.p - q.\left(\frac{d}{dt}\right)p$$

or

$$\dot{d} = \dot{p}q + p\dot{q} - \dot{q}p - q\dot{p} \qquad \ldots\ldots (4)$$

Let it be further assumed that we can have a matrix function H analagous to $\underline{H}$ above such that we can have corresponding canonical equations, viz.,

$$\frac{\partial H}{\partial p} = \dot{q} \; ; \; -\frac{\partial H}{\partial q} = \dot{p} \qquad \ldots\ldots (5)$$

It follows at once on substitution in (4) that

(*) A different type of matrix will be further in question.

$$\dot{d} = -\frac{\partial H}{\partial q}.q + p.\frac{\partial H}{\partial p} - \frac{\partial H}{\partial p}.p + q.\frac{\partial H}{\partial q}$$

$$\dot{d} = \left\{q.\frac{\partial H}{\partial q} - \frac{\partial H}{\partial q}.q\right\} + \left\{p.\frac{\partial H}{\partial p} - \frac{\partial H}{\partial p}.p\right\} \quad . . (6)$$

It can be shown quite generally in connection with matrices that adopting Dirac's Poisson-bracket notation (*)

$$-\left[\frac{\partial H}{\partial q}.q\right] = -q\frac{\partial H}{\partial q} + \frac{\partial H}{\partial q}q = \left[q.\frac{\partial H}{\partial q}\right] = \frac{\partial^2 H}{\partial p \partial q}$$

whereas

$$\left[\frac{\partial H}{\partial p}.p\right] = p.\frac{\partial H}{\partial p} - \frac{\partial H}{\partial p}.p = \frac{\partial^2 H}{\partial q \partial p} \quad . . . . . . (7)$$

There can therefore be a correspondence between the canonical equations of Hamilton and the canonical equations of Heisenberg for matrices generally, provided that the matrices are such that

$$\dot{d} = 0 \quad . . . . . (8)$$

This means that the matrices must be of the type that

$$pq - qp = \text{constant, (independent of t)} \quad . . . (9)$$

The above two conditions can be met, first by interpreting the multiplication of matrices so that the time functions should not appear. This implies that a special meaning needs to be given to "multiplication" for by (4) it is presumed that the p's and q's are in fact functions of the time. Secondly, in order to meet the condition (9) a diagonal matrix condition is necessary. This can also be satisfied by properly interpreting the elements of the resultant multiplication matrix.

(*) See Appendix

Let us therefore form matrices and then develop their products. Thus let, for example,

$$q = \begin{pmatrix} q_{11}q_{12}q_{13} \\ q_{21}q_{22}q_{23} \\ q_{31}q_{32}q_{33} \end{pmatrix} \quad \ldots(10) \qquad\qquad p = \begin{pmatrix} p_{11}p_{12}p_{13} \\ p_{21}p_{22}p_{23} \\ q_{31}q_{32}q_{33} \end{pmatrix} \ldots(11)$$

$$pq = \begin{pmatrix} (p_{11}q_{11}+p_{12}q_{21}+p_{13}q_{31}) & (p_{11}q_{12}+p_{12}q_{22}+p_{13}q_{32}) & (p_{11}q_{13}+p_{12}q_{23}+p_{13}q_{33}) \\ (p_{21}q_{11}+p_{22}q_{21}+p_{23}q_{31}) & (p_{21}q_{12}+p_{22}q_{22}+p_{23}q_{32}) & (p_{21}q_{13}+p_{22}q_{23}+p_{23}q_{33}) \\ (p_{31}q_{11}+p_{32}q_{21}+p_{33}q_{31}) & (p_{31}q_{12}+p_{32}q_{22}+p_{33}q_{32}) & (p_{31}q_{13}+p_{32}q_{23}+p_{33}q_{33}) \end{pmatrix}$$

$$\ldots\ldots(12)$$

$$qp = \begin{pmatrix} (q_{11}p_{11}+q_{12}p_{21}+q_{13}p_{31}) & (q_{11}p_{12}+q_{12}p_{22}+q_{13}p_{32}) & (q_{11}p_{13}+q_{12}p_{23}+q_{13}p_{33}) \\ (q_{21}p_{11}+q_{22}p_{21}+q_{23}p_{31}) & (q_{21}p_{12}+q_{22}p_{22}+q_{23}p_{32}) & (q_{21}p_{13}+q_{22}p_{23}+q_{23}p_{33}) \\ (q_{31}p_{11}+q_{32}p_{21}+q_{33}p_{31}) & (q_{31}p_{12}+q_{32}p_{22}+q_{33}p_{32}) & (q_{31}p_{13}+q_{32}p_{23}+q_{33}p_{33}) \end{pmatrix}$$

$$\ldots\ldots(13)$$

It will be noticed that in pq as well as in qp that the diagonal elements are either of the form

$$\sum p_{mn}q_{nm} \quad \text{or} \quad \sum q_{mn}p_{nm} \qquad \ldots\ldots(14)$$

The condition then needs to be imposed by definition that

$$p_{mn}q_{rs} = 0 \quad \text{for} \quad m \neq s \ ; \ n \neq r \quad \ldots\ldots(15)$$

The significance of this condition will be brought out later and will amount to ignoring term elements producing expressions in which the frequency of a p_{mn} element differs from the frequency of a q_{rs} element.

Forming now the subtraction of matrices (12) and (13) it is at once apparent that only diagonal elements need to be considered and we have, subject to (15) that

$$pq-qp =$$

$$\begin{pmatrix} (p_{11}q_{11}-q_{11}p_{11})+(p_{12}q_{21}-q_{12}p_{21})+(p_{13}q_{31}-q_{13}p_{31}) & 0 & 0 \\ 0 & (p_{21}q_{12}-q_{21}p_{12})+(p_{22}q_{22}-q_{22}p_{22})+(p_{23}q_{32}-q_{23}p_{32}) & 0 \\ 0 & 0 & (p_{31}q_{13}-q_{31}p_{13})+(p_{32}q_{23}-q_{32}p_{23})+(p_{33}q_{33}-q_{33}p_{33}) \end{pmatrix} \quad ----(16)$$

The general term of any diagonal element is therefore seen to be of the form of

$$\sum (p_{mn}q_{nm} - q_{mn}p_{nm}) \qquad .....(17)$$

The multiplication of matrices has then to be so limited by definition that an expression such as (17) has no longer to involve the time function per se. It will then result that the canonical equations of classical mechanics can be translated in invariant form into the domain of general matrix mechanics. The first important thing, then, in a rationalization of matrix mechanics, is to give a physical basis for the interpretation of (17).

By a theorem in classical generalized mechanics already deduced by the writer, (see Phil. Mag., Sept. 1924) it was proved that, if we have a component of generalized force defined by

$$\underset{\cdot}{E}(mn) = E_{nm} \sin \omega t + E_{mn} \cos \omega t$$

and a consequent generalized displacement) . . . . .(18)

$$\underset{\cdot}{D}(mn) = D_{nm} \sin \omega t + D_{mn} \cos \omega t$$

the rate of doing work depended on the expression

$$\underset{\cdot}{E}(mn) \cdot \frac{d}{dt} \underset{\cdot}{D}(mn) = \omega \Big[\left\{ E_{mn}D_{nm} - D_{mn}E_{nm} \right\} \cos^2 \omega t + E_{nm}D_{mn} \cos 2 \omega t + \tfrac{1}{2} \left(E_{nm}D_{nm} - D_{mn}E_{mn} \right) \sin 2 \omega t \Big] \quad \ldots\ldots(19)$$

For a real activity (Force x Velocity of Displacement) therefore, it is required that

$$E_{mn} D_{nm} - D_{mn} E_{nm} \neq 0 \quad \ldots\ldots(20)$$

The 2 ωt terms can contribute nothing to the real average activity. A similar expression will be developed for the action $\underset{\cdot}{p}$ (mn) .$\underset{\cdot}{q}$ (nm) and will be used for the interpretation of (17). One thing is certain from (20) we can never have an expression

$$\frac{D_{mn}}{E_{mn}} = k = \frac{D_{nm}}{E_{nm}} \quad \ldots\ldots(21)$$

where k has an ordinary real scalar value. Expression (20) would under those circumstances reduce to zero, which is contrary to hypothesis. In other words $\underset{\cdot}{E}$(mn) and $\underset{\cdot}{D}$(mn) must, for real activity, be out of phase to some degree at least and we should write instead

$$\underset{\cdot}{D} = \underset{\cdot}{k}\, \underset{\cdot}{E} \quad \ldots\ldots(22)$$

where $\underset{\cdot}{k}$ for complex operands is also complex. (*)

(*) By $\underset{\cdot}{k}$ being complex is meant that a component k $\underset{\cdot}{E}$ of $\underset{\cdot}{D}$ is in time phase with $\underset{\cdot}{E}$ but the component of magnitude k_2 E of $\underset{\cdot}{D}$

(next page)

That is let

$$\underset{\cdot}{k} = k_1 - k_2 j \qquad (**) \; \ldots\ldots (23)$$

then with

$$\underset{\cdot}{D}(mn) = (D_{nm} + D_{mn} j) .\sin \omega t \qquad \ldots\ldots (24)$$

$$\left.\begin{aligned} (k_1 - k_2 j)\underset{\cdot}{E}(mn) &= (D_{nm} + D_{mn} j) .\sin \omega t \\ \underset{\cdot}{E}(mn) &= (E_{nm} + E_{mn} j) .\sin \omega t \end{aligned}\right\} \qquad \ldots\ldots (25)$$

We have by definition for the work done

$$dW = \underset{\cdot}{E}.\, d\underset{\cdot}{D} = \underset{\cdot}{E}\, dt.\, \frac{d\underset{\cdot}{D}}{dt} = \dot{\underset{\cdot}{D}}.d\,\underset{\cdot}{P} = \frac{d\underset{\cdot}{P}}{dt} .d\underset{\cdot}{D} \qquad ..(26)$$

This follows because for a generalized Momentum $\underset{\cdot}{P}$ the following obtains

$$\frac{d\underset{\cdot}{P}}{dt} = \underset{\cdot}{E} \; ; \; \underset{\cdot}{P} = \int \underset{\cdot}{E}.dt \qquad \ldots\ldots (27)$$

Yet it is to be borne in mind that with

$$\frac{d}{dt} = \omega j \qquad \ldots\ldots (28)$$

$$\underset{\cdot}{E} = \frac{\underset{\cdot}{D}}{\underset{\cdot}{k}} = \frac{d\underset{\cdot}{P}}{dt} = \omega j\, \underset{\cdot}{P} \qquad \ldots\ldots (29)$$

showing that the generalized momentum $\underset{\cdot}{P}$ cannot be in time phase with $\dot{\underset{\cdot}{D}}$. To deduce the latter we have

---

(*)(continued)is in time quadrature with respect to $\underset{\cdot}{E}$. The Heaviside-Perry method of complexes, or Resistance Operators, was first extensively treated in Perry's "Calculus for Engineers"(see pp. 236 et seq.). The method is much more powerful than that of Steinmetz since the latter has to do with the effective values of the variables, whereas the Heaviside operational method deals with instaneous values throughout - an all important difference.

(**) See author's "Harmonic Algebra" - Univ. of Calif. Publications. Sept. 30,1919. Also Heaviside's Elec. Mag. Theory. Vol.11,p.228.

$$\dot{P} = \frac{\dot{D}}{k\omega j} = -\frac{1}{\omega^2}\cdot\frac{1}{\dot{k}}\cdot\frac{d\dot{D}}{dt} = \frac{-(k_1 + k_2\, j)}{\omega^2\, k^2}\cdot\frac{d\dot{D}}{dt}\ (*) \quad ..(30)$$

where we define for convenience that

$$k^2 = k_1^{\ 2} + k_2^{\ 2} \quad .....(31)$$

A relation corresponding to (19) can be obtained for $\dot{p}\dot{q}$ by noting first that

$$\frac{d\dot{D}(mn)}{dt} = \omega(D_{nm}\cos\omega t - D_{mn}\sin\omega t) \quad ...(32)$$

$$\dot{E}(mn) = E_{nm}\sin\omega t + E_{mn}\cos\omega t \quad .....(33)$$

It is the Multiplication of the right-hand expressions that leads to (19). For (32), (33) we can therefore substitute

$$\left.\begin{aligned} \dot{p}(mn) &= (\omega p'_{nm})\cos\omega t - (\omega p'_{mn})\sin\omega t \\ \dot{q}(mn) &= q_{nm}\sin\omega t + q_{mn}\cos\omega t \end{aligned}\right\} \quad ...(34)$$

These should lead to a form similar to (19) by using the substitution $\omega p'$ for D_{nm} and q_{nm} for E_{nm} etc. However, if instead we write

$$\dot{p}(mn) = p_{nm}\cos\omega t - p_{mn}\sin\omega t \quad ...(35)$$

the result will be

(*) This important result will be employed in developing the differential equation of a Radiating Harmonic Oscillator.

$$\underset{\cdot}{p}(mn) . \underset{\cdot}{q}(mn) = \Big[(q_{mn}p_{nm} - p_{mn}q_{nm}) \cos^2 \omega t + q_{nm}p_{mn} \cos 2\omega t + \tfrac{1}{2} (q_{nm}p_{nm} - p_{mn}q_{mn}) \sin 2\omega t \Big] \quad \ldots\ldots (36)$$

The mean action would thus depend on the expression

$$p_{mn}q_{nm} - q_{mn}p_{nm} = \beta \neq 0 \quad \ldots\ldots (37)$$

which will be plus or minus depending on whether radiation of energy or absorption exists. Expression (37) indicates in what manner the matrices for p and q are to be built up, - and moreover indicates in what manner matrix multiplication is to be understood and more especially with regard to d or U of (13).

Given that to each generator of a radiating system S there are to be allocated a displacement coordinate $\underset{\cdot}{q}$ (mn) and a momentum coordinate $\underset{\cdot}{p}$ (mn), with reference to a unit or standard aggregation of generators (time t = zero) acting as reference, then the two expressions are to be written in the form

$$\underset{\cdot}{q}(mn) = q_{nm} \sin 2\pi\nu (mn) t + q_{mn} \cos 2\pi\nu(mn)t \quad \ldots\ldots (38)$$

$$\underset{\cdot}{p}(mn) = -p_{mn} \sin 2\pi\nu (mn)t + p_{nm} \cos 2\pi\nu(mn)t \quad \ldots\ldots (39)$$

For simplicity we can write

$$\underset{\cdot}{q}(mn) = (q_{nm}+q_{mn}j) \sin 2\pi\nu(mn)t = q_{nm}s_{mn}+q_{mn}c_{mn} = q(nm)+q(mn) \quad \ldots\ldots (40)$$

and for the momentum function

$$\dot{p}(mn) = (-p_{mn}+p_{nm}j) \sin 2\pi v(mn)t = -p_{mn}c_{mn}+p_{nm}c_{mn}$$

$$= -p(mn) + p(nm) \qquad \ldots\ldots (41)$$

Writing out the matrix expressions for the S-system we then have for example

$$p= \begin{pmatrix} p(11) & p(12) & p(13) \\ p(21) & p(22) & p(23) \\ p(31) & p(32) & p(33) \end{pmatrix} \quad ; \quad q= \begin{pmatrix} q(11) & q(12) & q(13) \\ q(21) & q(22) & q(23) \\ q(31) & q(32) & q(33) \end{pmatrix} \qquad \ldots\ldots (42)$$

Proper regard must, however, be paid to the fact whereas any q(mn) in q of (42) corresponds to the cosine function such that

$$q(mn) = q_{mn} \cos 2\pi v(mn)t$$

whereas the p(mn) of p corresponds to the sine function so that

$$p(mn) = -p_{mn} \sin 2\pi v(mn)t$$

In a similar way the following values hold

$$q(nm) = q_{nm} \sin 2\pi v(mn)t \quad ; \quad p(nm) = p_{nm} \cos 2\pi v(mn)t.$$

For convenience we can set that

$$v(mn) = -v(nm) \qquad \ldots\ldots (43)$$

We can then write

$$\left.\begin{aligned} q(mn) &= q_{mn} \cos 2\pi\, v(mn)t \\ q(nm) &= q_{nm} \sin 2\pi v\, (mn)t \end{aligned}\right\} \qquad \ldots\ldots (44)$$

$$p(mn) = p_{mn} \sin 2\pi\nu(nm)t$$
$$p(nm) = p_{nm} \cos 2\pi\nu(nm)t \quad \ldots\ldots(45)$$

It is then the multiplication of two matrices of the form (10) and (11) rather than (42) that will give the resultant d according to (44) and (45) viz:

$$pq-qp=\begin{pmatrix} (p_{11}q_{11}-q_{11}p_{11})+(p_{12}q_{21}-q_{12}p_{21})+(p_{13}q_{31}-q_{13}p_{13}), & 0, & 0 \\ 0, & (p_{21}q_{12}-q_{21}p_{12})+(p_{22}q_{22}-q_{22}p_{22})+(p_{23}q_{32}-q_{23}p_{23}), & 0 \\ 0, & 0, & (p_{31}q_{13}-q_{31}p_{13})+(p_{32}q_{23}-q_{32}p_{23})+(p_{33}q_{33}-q_{33}p_{33}) \end{pmatrix} \quad \ldots\ldots(46)$$

A simple form of the bracket values represented by $p_{mn}\ q_{nm}$ $-q_{mn}\ p_{nm}$ in (46) will now be in order.

It has already been pointed out in (30) that the generalized momentum $\underset{\cdot}{P}$ can be expressed in terms of the generalized displacement $\underset{\cdot}{D}$ and of $\dot{\underset{\cdot}{D}}$. We have, in fact, that

$$\underset{\cdot}{P} = \frac{\dot{\underset{\cdot}{D}}}{k\omega \underset{\cdot}{j}} = \frac{k_1+k_2j}{\omega jk^2}\,(D_1+D_2j)\,\sin\omega t \quad \ldots\ldots(47)$$

It follows, therefore, that

$$k^2\omega \underset{\cdot}{P} = (k_2-k_1j)(D_1+D_2j)\,\sin\omega t=(k_2D_1+k_1D_2)s+(D_2k_2-k_1D_1)c \quad (*) \quad \ldots\ldots(48)$$

(*) Here s is employed for $\sin\omega t$ and c for $\cos\omega t$. When dealing with the double periodicity terms, as in the product $\underset{\cdot}{P}\ \underset{\cdot}{D}$ of (49) the operators must be translated into c's, and appropriate s's introduced for the terms involving the sine functions of the time.

Thus multiplying through with $\dot{D}$ to obtain the action we have

$$k^2\omega\dot{P}\,\dot{D}=\left[(k_2\,D_1+k_1D_2)s + (D_2k_2-D_1k_1)c\right](D_1s + D_2c)$$
$$=\left\{D_1\,(k_2D_1+k_1D_2)\sin^2\omega t + D_2(D_2k_2-D_1k_1)\cos^2\omega t\right\}$$
$$+\left\{D_1(D_2k_2-D_1k_1) + D_2(k_2D_1+k_1D_2)\right\}\sin\omega t.\cos\omega t \quad \ldots\ldots(49)$$

It thus appears that the average value of the action depends on the expression

$$\frac{1}{2\pi}\oint \dot{P}\,\dot{D}\,dt = \left[\dot{P}\,\dot{D}\right]_{av} = \left\{\tfrac{1}{2}\frac{(D_1{}^2 + D_2{}^2)}{\omega^2{}_{mn}}.\ \frac{2\pi k_2}{k^2}\right\}\frac{1}{2\pi} \quad \ldots\ldots(50)$$

Translated into the notation of (36) therefore, we have by (38) that

$$\left|\dot{p}(mn).\dot{q}(mn)\right|av.= q_{mn}p_{nm}-p_{mn}q_{nm}=\left[\frac{2\pi k_2}{k^2}.\tfrac{1}{2}.\frac{(q^2_{mn}+q_{nm}{}^2)}{\omega^2_{mn}}\right]\frac{1}{2\pi}$$

$$=\left[\frac{\pi k_2}{2k^2}\ \frac{|\dot{q}(mn)|^2}{\omega^2_{mn}}\right]\frac{1}{2\pi} \quad \ldots\ldots(51)$$

The latter brings out an analogy with the postulate of the quantum theory, for we can write (instead of h/i),

$$X_{mn} = \frac{\pi k_2}{\omega^2_{mn}k^2}\quad(q_{mn}{}^2 + q_{nm}{}^2) \quad (*) \quad \ldots\ldots(52)$$

where X_{mn} would be a constant for an observation steady state. In any event (51) and (52) do not contain the time. There is then justification in regarding U of (3) as a "unit" matrix.

(*) A later improved form will be shown to involve the momenta rather than the coordinates divided by the frequency.

The following rule can, therefore, be enunciated. In forming a matrix product of elements the ordinary multiplication rule of algebraic matrices is understood, but in addition, it is implied that average or mean time values be inserted in the resultant. As to differentiation with respect to time, it means that the differential of the resultant matrix is the same as the differential with regard to the individual matrices comprising the operand originally, and then taking average time values.

It is at once apparent from (26) that the amount of work done, as by radiation, can be put into two forms:-

$$dW = \dot{D}.\ dP \quad ; \quad dW = \dot{P}.\ dD,$$

indicating that W must be a function of P, $\dot{P}$, D, $\dot{D}$. If then by partial differentiation it is understood that

$$dW = \frac{\partial W}{\partial D}\, dD + \frac{\partial W}{\partial P}\, dP \qquad \ldots\ldots(53)$$

then this will lead to a solution

$$W = f\,(P,D) \qquad \ldots\ldots(54)$$

provided that no $\dot{P}$'s or $\dot{D}$'s are presumed to appear in the last equation. If they are to appear at all they must do so by substition only in the partial derivatives

$$\frac{\partial W}{\partial D} = \dot{P}$$

$$\frac{\partial W}{\partial P} = \dot{D} \qquad \ldots\ldots(55)$$

The latter equations expressing radiation conditions, as a system, are more symmetrical even that the analagous Hamiltonian equations.

Transposing to the notation (5) we have for matrices

$$\frac{\partial W}{\partial q} = \dot{p} \quad ; \quad \frac{\partial W}{\partial p} = \dot{q} \qquad \ldots\ldots(56)$$

with the reservation as known from (30) that p is not in phase with $\dot{q}$. In fact, from (34) we have

$$\underset{\bullet}{q}(mn) = (q_{nm} + jq_{mn}) \sin \omega t \; ; \quad \underset{\bullet}{q}(mn) = \omega j(q_{nm} + q_{mn} j) \sin \omega t \qquad \ldots\ldots(57)$$

On the other hand, from (30) we also note that

$$\underset{\bullet}{p}(mn) = - \frac{k_1 + k_2 j}{\omega^2 \; k^2} \cdot \frac{d\underset{\bullet}{q}}{dt} \qquad \ldots\ldots(58)$$

Thus

$$\underset{\bullet}{p}(mn) = - \frac{1}{\omega^2 k^2} \left\{ \frac{k_2}{\omega} \cdot \frac{d}{dt} + k_1 \right\} \underset{\bullet}{\dot{q}} \qquad \ldots\ldots(59)$$

$$p(mn) = - \frac{k_1}{\omega^2 k^2} \underset{\bullet}{\dot{q}} - \frac{k_2}{\omega^3 k^2} \underset{\bullet}{\ddot{q}} \qquad \ldots\ldots(60)$$

To develope the differential equation for the Radiating Harmonic Oscillator the Hamiltonian canonical equations are

$$\frac{\partial H}{\partial \underline{p}} = \dot{\underline{q}} \quad ; \quad - \frac{\partial H}{\partial \underline{q}} = \dot{\underline{p}} \qquad \ldots\ldots(61)$$

It was the above equations that were made the basis of treatment for the Matrix Mechanics. We have, however, seen by (30) that as a condition for real activity (or action) we must have that

$$\frac{-\omega^2 k^2}{k_1 - k_2 j} \cdot \underline{p} = \dot{\underline{q}} \qquad \text{.....(62)}$$

This gives a clue for the POSSIBLE form of $\underline{H}$. Thus by combining (60) and (62) we have

$$\frac{\partial \underline{H}}{\partial \underline{p}} = -\omega^2 (k_1 - k_2 j)\underline{p} = -\omega^2 k_1 p + k_2 \omega \frac{dp}{dt}$$

$$= -\omega^2 k_1 p + k_2 \omega \dot{p} \qquad \text{.....(63)}$$

On integrating the last equation it follows

$$\underline{H} = -\frac{\omega^2 k_1}{2} \underline{p}^2 + k_2 \omega \dot{\underline{p}}\underline{p} + f(\underline{q}) \qquad \text{.....(64)}$$

Applying now the second equation of (61) it is seen that

$$\frac{-\partial \underline{H}}{\partial \underline{q}} = \dot{\underline{p}} = \frac{\partial}{\partial \underline{q}} f(\underline{q}) \qquad \text{.....(65)}$$

whence integrating, the form of $\underline{H}$ must be

$$\underline{H} = -\frac{\omega^2 k_1}{2} \underline{p}^2 + k_2 \omega \underline{p}\, \dot{\underline{p}} - \dot{\underline{p}}\, \underline{q} \qquad \text{.....(66)}$$

IT IS THE LATTER EQUATION AND NOT (1) that applies to the problem of radiating systems.

Taking now the case of equation (30) we have, dropping subscripts,

$$\underset{\bullet}{p} = \frac{\underset{\bullet}{q}}{\underset{\bullet}{k}\,\omega j} \; ; \quad \underset{\bullet}{q} = (k_1 - k_2 j)\,\omega j\,\underset{\bullet}{p} \qquad \ldots\ldots(67)$$

If then for convenience we let

$$\underset{\bullet}{p} = P \sin \omega t' \qquad \ldots\ldots(68)$$

$$\underset{\bullet}{q} = \omega P(k_1 c + k_2 s) \qquad (*) \quad \ldots\ldots(69)$$

This means that

$$\dot{p} = \omega j\ P \sin \omega t' = \omega P c \qquad \ldots\ldots(70)$$

$$\dot{p} q = \omega^2 P^2\ (k_1 c^2 + k_2 sc) \qquad \ldots\ldots(71)$$

Turning now to the next to the last term in (66) we likewise have

$$p\ \dot{p} = Ps\,\omega\,Pc = \omega P^2 sc$$

so that on multiplying with $k_2\omega$ we have

$$k_2\,\omega\,p\,\dot{p} = k_2\,\omega^2 P^2 sc \qquad \ldots\ldots(72)$$

(*) From (68) and (69) we note that for the absolute value $|\underset{\bullet}{p}|^2 = P^2$ whereas $|\underset{\bullet}{q}|^2 = \omega^2 P^2 k^2$. In other words $|\underset{\bullet}{q}|^2 = 4\pi^2 \nu^2 k^2 |\underset{\bullet}{p}|^2$ This is the result of (77)

The remaining term gives

$$p^2 = P^2 s^2$$

$$\frac{\omega^2 k_1}{2} p^2 = \omega^2 P^2 \frac{k_1}{2} s^2 \quad \ldots\ldots(73)$$

Now adding all the terms together it follows that

$$- H = \omega^2 P^2 \left\{ \frac{k_1}{2} s^2 + k_1 c^2 \right\}$$

$$= \frac{3}{4} \omega^2 P^2 k_1 \left\{ 1 + \frac{1}{3} \cos 2\omega t^1 \right\} \quad \ldots\ldots(74)$$

The latter equation it is seen indicates a constant component for the total energy H which is given by

$$- H_0 = \frac{3}{4} \omega^2 P^2 k_1 \quad \ldots\ldots(75)$$

This type of term means NO RADIATION. Such radiation as does appear must come from the variable remainder represented by

$$-H_v = \frac{1}{4} \omega^2 P^2 k_1 \cdot \cos 2\omega t^1 \quad \ldots\ldots(76)$$

The amplitude of the radiation is seen to fluctuate about a mean level H_0 of (75) with equal ranges (energy levels) plus and minus. This accords with the Heisenberg-Born Matrix Condition $n = \pm 1$. In (76) such amplitude has preferably been expressed in terms of the square of the momentum and the frequency.

To transform (76) as well as condition (52) use can be made of (69) which gives

$$|\dot{q}|^2 = \omega^2 P^2 k^2 = \omega^2 k^2 |\dot{p}|^2 \qquad \ldots\ldots(77)$$

yet it is better to refer to the momenta amplitudes rather than the coordinate ones in order to emphasize the analogies with the quantum theory. We then have that

$$X = \frac{\pi}{2} k_2 P^2 = \frac{\pi}{2} k_2 |\dot{p}|^2 \qquad \ldots\ldots(78)$$

which is of the order of an energy (or quantum).

The ratio of the two energies above is given by

$$\frac{H_v}{H_o} = \frac{1}{3} \qquad \ldots\ldots(79)$$

It is significant that H_o corresponds to the constant aggregate energy of the non-radiating harmonic oscillator heretofore employed, whereas H_v represents the amplitude of the fluctuating absorption and radiating component. Nothing is indicated about the "orbital periodicity" with which the potential and kinetic energies interchange in the H_o system. Whether there is a relationship of this latter with the half frequency component of the Nernst-Lindemann formula has yet to be determined, at least is it suggestive. In any case one thing is certain, equation (79) has been arrived at on purely classical lines and it shows as a consequence of applying Hamilton's canonical equations that an unexcited harmonic oscillator when caused to radiate by virtue of an impressed field of force only then becomes radiating when it can absorb and re-radiate a definite, discrete quantum of energy equal to 1/3 of its normal unexcited content. This corresponds exactly with one of the conditions of the Planck-Bohr developments.

Indeed an harmonic oscillator of the organ pipe type does not give an appreciable increase of volume with increased blowing pressure. A point is soon reached when the dominant frequency takes a discrete jump in conformity with Bohr requirements.

. . . .

Appendix

Employing the bracket notation of Dirac let

$$pq - qp = [q,p] + U$$

with

$$-[q,p] = [p,q]$$

To interpret $[q,p^2]$ we note that

$$[q,p^2] = q^2p - qp^2 = p(qp)-(qp)p = p(pq-qp)+(pq-qp)p$$

$$= p\,U + U\,p = 2p.U$$

In a similar manner it can be shown that

$$[q^2,p] = 2\,q\,U$$

indicating quite generally

$$[q,p^n] = n\,p^{n-1}.\,U = \frac{d}{dp}\,p^n.$$

$$[q^n,\,p] = \frac{d}{dq}\cdot q^n.$$

In fact as a simple extension for functions of q and p expressible as a series we should have dropping the unity matrix that

$$[\mathcal{F}(p,q),\,p] = \frac{\partial \mathcal{F}}{\partial q} \quad ; \quad [q,\,F\,(p,q)] = \frac{\partial F}{\partial p} \quad (*)$$

(*) Kind acknowledgements are made to Dr. A.S. Eve, Director of the Physics Department of McGill University, to Professor A.H.S. Gillson, and to Dr. J.S. Foster, for their interest and criticisms.