Subgroup Theorems in Relatively Hyperbolic Groups and Small-Cancellation Theory

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To my lovely wife and my dear parents, I love you all.

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ABSTRACT

In the first part, we study amalgams of relatively hyperbolic groups and also the relatively quasiconvex subgroups of such amalgams. We prove relative hyperbolicity for a group that splits as a finite graph of relatively hyperbolic groups with parabolic edge groups; this generalizes a result proved independently by Dahmani, Osin and Alibegovic. More generally, we prove a combination theorem for a group that splits as a finite graph of relatively hyperbolic groups with total, almost malnormal and relative quasiconvex edge groups. Moreover, we provide a criterion for detecting quasiconvexity of subgroups in relatively hyperbolic groups that split as above. As an application, we show local relative quasiconvexity of any f.g. group that is hyperbolic relative to Noetherian subgroups and has a small-hierarchy. Studying free subgroups of relatively hyperbolic groups, we reprove the existence of a malnormal, relatively quasiconvex, rank 2 free subgroup F in a non-elementary relatively hyperbolic group G. Using this result and with the aid of a variation on a result of Arzhantseva, we show that if G is also torsion-free then "generically" any subgroup of F is aparabolic, malnormal in G and quasiconvex relative to $\mathbb{P}$ and therefore hyperbolically embedded. As an application, generalizing a result of I. Kapovich, we prove that for any f.g., non-elementary, torsion-free group G that is hyperbolic relative to $\mathbb{P}$, there exists a group G^* containing G such that G^* is hyperbolic relative to $\mathbb{P}$ and G is not relatively quasiconvex in G^*.

In the second part, we investigate the existence of $F_2 \times F_2$ in the non-metric small-cancellation groups. We show that a C(6)-T(3) small-cancellation group cannot contain a subgroup isomorphic to $F_2 \times F_2$. The analogous result is also proven in the C(3)-T(6) case.

ABRÉGÉ

Dans la première partie, nous étudions les amalgames de groupes relativement hyperboliques et également les sous-groupes relativement quasiconvexes de ces amalgames. Nous prouvons l'hyperbolicité relative pour un groupe qui se sépare comme un graphe fini de groupes relativement hyperboliques avec des groupes d'arêtes paraboliques, ce qui généralise un résultat prouvé indépendamment par Dahmani, Osin et Alibegović. Nous l'étendons au cas où les groupes d'arêtes sont totalaux, malnormal et relativement quasiconvexes. En outre, nous fournissons un critère de détection de quasiconvexité relative des sous-groupes dans les groupes hyperboliques qui divisent. Comme application, nous montrons la quasiconvexité locale relative d'un groupe qui est relativement hyperbolique à certains sous-groupes noethériens et qui a une petite hiérarchie. Nous étudions également les sous-groupes libres de groupes relativement hyperboliques, et reprouvons l'existence d'un sous-groupe libre, malnormal, relativement quasiconvexe F_2 dans un groupe non-élémentaire relativement hyperbolique G. En combinant ce résultat avec une variation sur un théorème de Arzhantseva, nous montrons que si G est aussi sans-torsion, "génériquement" tout sous-groupe de F_2 est aparabolique, malnormal dans G et quasiconvexe par rapport à $\mathbb{P}$. Comme application, nous montrons que pour tout groupe G non-élémentaire, sans-torsion, qui est hyperbolique par rapport à $\mathbb{P}$, il existe un groupe G^* contenant G tel que G^* est hyperbolique par rapport à $\mathbb{P}$ et G n'est pas quasiconvexe dans G^*. Dans la deuxième partie, nous étudions l'existence de sous-groupe $F_2 \times F_2$ dans des groupes à petite simplification. Nous montrons que les groupes C(6) ne peuvent pas contenir un sous-groupe isomorphe à $F_2 \times F_2$. Le résultat analogue est également prouvé dans le dossier C(3)-T(6) affaire.

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CHAPTER 1
Introduction

1.1 Background and Results

This thesis concerns problems in *relatively hyperbolic groups* and *small-cancellation groups*. These classes of groups are two of the main objects of study in geometric group theory.

1.1.1 Relatively hyperbolic groups

Hyperbolic groups were introduced by Gromov in his seminal work [Gro87] in the 1980s. A geodesic metric space X is *δ-hyperbolic* if for any geodesic triangle $\triangle$ in X, each side of $\triangle$ lies in the δ-neighbourhood of the other two sides. A group G is *hyperbolic* if its Cayley graph is δ-hyperbolic for some $\delta \geq 0$. This class of groups generalizes the fundamental group of compact Riemannian manifolds with strictly negative sectional curvature. For example, finitely generated free groups, and the fundamental groups of surfaces with negative Euler characteristic are hyperbolic. As a generalization of hyperbolic groups, Gromov introduced relatively hyperbolic groups [Gro87]. According to his definition, G is relatively hyperbolic if it admits a discrete isometric action on a complete, locally compact, hyperbolic geodesic space such that the quotient space is quasi-isometric to a union of some copies of $[0, \infty]$ joined at 0. Later various approaches to relative hyperbolicity were developed [Far98], [Bow99b], [Osi06c], [Hru10], [DS05], [Yam04], and later Hruska [Hru10] showed that these definitions are equivalent for finitely generated groups. Following Bowditch's

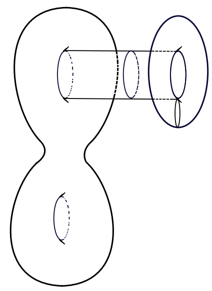

Figure 1–1: X is constructed by connecting a surface of genus 2, S and a torus T via a cylinder.

approach [Bow99b], a group G is *hyperbolic relative to a finite family of subgroups* $\mathbb{P} = \{P_1, \ldots, P_n\}$ if G acts on a connected, "fine" and hyperbolic graph K such that the stabilizers of edges are finite and infinite valence vertex stabilizers are conjugates of elements of $\mathbb{P}$. A conjugate of an element of $\mathbb{P}$ is called a *maximal parabolic*. A motivating example for this definition is $\pi_1(X) = \pi_1 S *_{\mathbb{Z}} \pi_1 T$ where S is a surface of genus 2 and T is a torus such that X is constructed by connecting S and T using a cylinder as illustrated in Figure 1–1. In this case $\pi_1 X$ is hyperbolic relative to $\pi_1 T$. Indeed we can construct a fine, hyperbolic graph K by "conning-off" (see Definition 6) all $\pi_1 X$-translates of the flat in $\widetilde{X}$ whose stabilizer is $\pi_1 T$.

The class of relatively hyperbolic groups contains many interesting groups: word hyperbolic groups, fundamental groups of finite volume hyperbolic manifolds, limit groups [Dah03], geometrically finite convergence groups [Yam04], groups acting freely on $\mathbb{R}^n$-trees [Gui04], CAT(0)-groups with isolated flats [HK05] and many other examples. Relatively hyperbolic groups benefit from many interesting properties, for instance if elements of $\mathbb{P}$ have finite asymptotic dimension (respectively satisfies

Baum-Connes conjecture) then the group G has finite asymptotic dimension (satisfies Baum-Connes conjecture) and therefore G satisfies the analytic Novikov conjecture [Osi05], [FO12], [Yu98].

One of the natural inductive approaches in studying a group G, is splitting it as the an amalgamated free products or HNN-extension and then investigating whether G has certain properties that the vertex groups has. In this direction, the following fundamental question arises which was asked by Swarup [Swa96](see also question 1.13 in [Bes04]).

Problem 1.1.1. *Let G split as a finite graph of groups with relatively hyperbolic vertex groups. Under what conditions is G relatively hyperbolic?*

Bestvina and Feighn proved a combination theorem by answering Problem 1.1.1 for hyperbolic groups when the edge groups are quasiconvex and malnormal [BF92]. We prove the following result:

Theorem 3.1.2 (H.B. and D. Wise)**.** *Let G split as a finite graph of groups. Suppose each vertex group is relatively hyperbolic and each edge group is parabolic in its vertex groups. Then G is hyperbolic relative to $\mathbb{Q} = \{Q_1, \ldots, Q_j\}$ where each Q_i is the stabilizer of a "parabolic tree".*

For the definition of a parabolic tree, see Definition 7. Theorem 3.1.2 has been proven in a special case when the edge groups are "Maximal" on at least one side in [Dah03], [Ali05], [Osi06a] and [Gau07] with different approaches. Also Mj and Reeves gave a generalization of the Bestvina-Feighn combination theorem that follows Farb's approach but uses a generalized partial electrocution [MR08].

Theorem 3.1.2, can be illustrated in the following example: Let $G = G_1 *_C G_2$ where each $G_i = \pi_1 M_i$ with M_i a cusped hyperbolic manifold with a single boundary torus T_i. Here C is an arbitrary common subgroup of $\pi_1 T_1$ and $\pi_1 T_2$. Then G is hyperbolic relative to the parabolic subgroup $\pi_1 T_1 *_C \pi_1 T_2$. Note that in this example the edge groups are not maximal and $\pi_1 T_1 *_C \pi_1 T_2$ is the stabilizer of a parabolic tree which maps injectively to the Bass-Serre tree of the splitting.

Introduced also by Gromov [Gro87], *relatively quasiconvex* subgroups of relatively hyperbolic groups play a central role in the theory of relatively hyperbolic groups. These groups are the most natural subgroups of relatively hyperbolic groups in the sense that their intrinsic geometry is preserved under the embedding into the relatively hyperbolic group. A subspace Y of a geodesic metric space X is quasiconvex if there exists an $\varepsilon > 0$ such that all of the geodesics in X with endpoints in Y lie in ε-neighborhood of Y. A subgroup H of a group G generated by S is quasiconvex if the 0-cells corresponding to H form a quasiconvex subspace of the Cayley graph $\Gamma(G, S)$. Roughly speaking, a subgroup H of a relatively hyperbolic group G is relatively quasiconvex if the 0-cells corresponding to H form a quasiconvex subspace of the Cayley graph constructed by the generating set consisting of the disjoint union $S \sqcup \bigsqcup_i P_i$ where P_i s are the maximal parabolic subgroups of G. As examples, any parabolic subgroup of a relatively hyperbolic group is relatively quasiconvex, moreover any virtually cyclic subgroup of a relatively hyperbolic group is relatively quasiconvex. Relatively quasiconvex subgroups have important properties; for instance, any relatively quasiconvex subgroup is relatively hyperbolic and the intersection of two of these subgroups is again relatively quasiconvex. Relative quasiconvexity was formulated by Dahmani [Dah03] and later by Osin in [Osi06c] and Hruska investigated several equivalent definitions of relatively quasiconvex subgroups [Hru10]. Martinez-Pedroza and Wise [MPW11] introduced a definition of relative quasiconvexity in the context of fine hyperbolic graphs and proved that their definition is equivalent to Osin's definition.

Although, Theorem 3.1.2 covers a large class of groups, we expect to generalize the theorem to the case that the edge groups are not necessarily parabolic, similar to Bestvina and Feighn's result for hyperbolic groups. Indeed the result holds when the edge groups are "total", malnormal and relatively quasiconvex which is shown in the following, using Theorem 3.1.2 :

Theorem 3.2.2 (H.B. and D. Wise)**.**

1. *Let G_i be hyperbolic relative to $\mathbb{P}_i$ for $i = 1, 2$. Let $C_i \leq G_i$ be almost malnormal, total and relatively quasiconvex. Let ${C_1}' \leq C_1$. Then $G = G_1 *_{{C_1}'=C_2} G_2$ is hyperbolic relative to $\mathbb{P} = \mathbb{P}_1 \cup \mathbb{P}_2 - \{P_2 \in \mathbb{P}_2 \ : \ P_2^g \subseteq C_2,\ \text{for some } g \in G_2\}$.*
2. *Let G_1 be hyperbolic relative to $\mathbb{P}$. Let $\{C_1, C_2\}$ be almost malnormal and assume each C_i is total and relatively quasiconvex. Let ${C_1}' \leq C_1$. Then $G = G_1 *_{{C_1}'={C_2}^t}$ is hyperbolic relative to $\mathbb{P} = \mathbb{P} - \{P_2 \in \mathbb{P}_2 \ : \ P_2^g \subseteq C_2,\ \text{for some } g \in G_2\}$.*

Note that a special case of this result were proved in [Dah03], [Ali05], [Osi06a] and [Gau07]. In all those articles the edge groups C_is are maximal parabolic subgroups.

By induction and using Theorem 3.2.2, we obtain the following corollary for a finite graph of groups:

Corollary 3.2.4. *Let G split as a finite graph of groups. Suppose*

(a) Each G_ν is hyperbolic relative to $\mathbb{P}_\nu$;

(b) Each G_e is total and relatively quasiconvex in G_ν;

(c) Each G_e is almost malnormal in G_ν.

Then G is hyperbolic relative to $\bigcup_\nu \mathbb{P}_\nu - \{repeats\}$.

Another interesting application of the combination theorems proven in this thesis, is the following result for 3-manifolds:

Corollary 5.2.4. *Let M be a compact irreducible 3-manifold. Moreover, let $M_1, \ldots, M_r$ denote the graph manifolds obtained by removing each (open) hyperbolic piece in the geometric decomposition of M. Then $\pi_1 M$ is hyperbolic relative to $\{\pi_1 M_1, \ldots, \pi_1 M_r\}$.*

This result was previously proved by Drutu-Sapir using work of Kapovich-Leeb and relative hyperbolicity of the asymptotically tree graded groups [KL95, DS05].

We devote two chapters to studying the relation between relative quasiconvexity of a subgroup H of a relatively hyperbolic group G that splits and relative quasiconvexity of the intersection of H with the vertex groups. Indeed, we are interested in investigating under what conditions a subgroup of G is relatively quasiconvex if its intersections with the vertex groups are relatively quasiconvex. Let G split as a graph of groups with relatively hyperbolic vertex groups and let H be a finitely generated subgroup of G. If there are finitely many H-orbits of vertices v in the Bass-Serre tree T with H_v nontrivial, and each such H_v relatively quasiconvex in G_v then H is "tamely generated". Note that the definition of tamely generated subgroup is more general than the statement mentioned here, see Definition 10. The following result implies that in quasiconvex, malnormal splitting a tamely generated subgroup is relatively quasiconvex.

Theorem 5.1.4 (H.B. and D. Wise)**.** *Let G be finitely generated and hyperbolic relative to $\mathbb{P}$ such that G splits as a finite graph of groups. Suppose*

(a) Each G_e is total in G;

(b) Each G_e is relatively quasiconvex in G;

(c) Each G_e is almost malnormal in G.

Let $H \leq G$ be tamely generated. Then H is relatively quasiconvex in G.

A relatively hyperbolic group G is *locally relatively quasiconvex* if every finitely generated subgroup of G is relatively quasiconvex. For a group G, being locally relatively quasiconvex is an strong property as it implies relative hyperbolicity of all finitely generated subgroups of G. Theorem 5.1.4 yields the following interesting application:

Corollary 5.1.6. *Let G be finitely generated and hyperbolic relative to $\mathbb{P}$. Suppose G splits as a finite graph of groups. Assume:*

(a) Each G_ν is locally relatively quasiconvex;

(b) Each G_e is Noetherian, total and relatively quasiconvex in G;

(c) Each G_e is almost malnormal in G.

Then G is locally relatively quasiconvex relative to $\mathbb{P}$.

Dahmani [Dah03] proved that "limit groups"; also called fully residually free groups; are locally relatively quasiconvex. A group is *small* if it does not contain a subgroup isomorphic to F_2. If a finitely generated group G is hyperbolic relative to

"Noetherian" subgroups and G can be built by a series of Amalgamated free products and HNN-extensions along small subgroups then G is locally relatively quasiconvex. More precisely, we have the following generalization of Dahmani's result:

Theorem 4.3.5 (H.B. and D. Wise)**.** *Let G be finitely generated and hyperbolic relative to $\mathbb{P}$ where each element of $\mathbb{P}$ is Noetherian. Suppose G has a small-hierarchy. Then G is locally relatively quasiconvex.*

Tits [Tit72], proved his important and famous "Tits Alternative" for linear groups which states that a finitely generated linear group over a field is either virtually solvable or contains a non-cyclic free subgroup. The Tits alternative was proved for several classes of groups, for instance Gromov [Gro87] showed that hyperbolic groups satisfy the Tits alternative. This result was generalized to relatively hyperbolic groups in [Tuk96]. Later Kapovich [Kap99] showed that any non-elementary subgroup of a hyperbolic group contains a malnormal and quasiconvex free subgroup. The following result was proved in [MOY11]. We reprove this using the fine graph approach which yields a natural proof.

Theorem 6.3.3 ([MOY11] and, H.B.)**.** *Let G be torsion-free, non-elementary and hyperbolic relative to $\mathbb{P}$. Let g and $\bar{g}$ be hyperbolic elements of G such that $\langle g, \bar{g} \rangle$ is not cyclic. Then there exists k such that for any $n \geq k$ the subgroup $F = \langle g^n, \bar{g}^n \rangle$ is free of rank 2, aparabolic and quasiconvex in G relative to $\mathbb{P}$.*

Using a result of Arzhantseva [Arz00], we obtain the following theorem. This strengthens Theorem 6.3.3.

Theorem 6.3.6 (H.B.)**.** *Let G be torsion-free, non-elementary and hyperbolic relative to $\mathbb{P}$. Then there exists a rank 2 free subgroup F of G such that "generically"* *any finitely generated subgroup of F is aparabolic, malnormal in G and quasiconvex relative to $\mathbb{P}$, and therefore hyperbolically embedded relative to $\mathbb{P}$.*

In a relatively hyperbolic group, any non-parabolic, infinite cyclic subgroup is relatively quasiconvex. Thus for a non-parabolic, infinite cyclic group, relative quasiconvexity is an absolute property and does not depend on the embedding in an ambient hyperbolic group. This motivates us to give the definition of "absolutely relatively quasiconvex" subgroups. Let G be hyperbolic relative to $\mathbb{P}$. We say that G is *absolutely relatively quasiconvex* if for any group G^* that is hyperbolic relative to $\mathbb{P}$, containing G as a non-parabolic subgroup, G is relatively quasiconvex in G^*. I. Kapovich [Kap99] defined absolute quasiconvexity for hyperbolic groups and he showed that any absolutely quasiconvex, torsion-free hyperbolic group is infinite cyclic. The following result generalizes his result:

Theorem 6.4.2 (H.B.)**.** *Let G be a finitely generated, torsion-free group that is non-elementary and hyperbolic relative to $\mathbb{P}$. There exists a group G^* that is hyperbolic relative to $\mathbb{P}$ such that G is a subgroup of G^* and G is not quasiconvex in G^* relative to $\mathbb{P}$.*

1.1.2 Small-cancellation theory

Small-cancellation theory studies small-cancellation complexes and the groups acting on them. These complexes are those whose two cells have "small overlap" which each other. Small-cancellation theory has proven to be a powerful theory in studying groups, especially in construction of groups with some given properties. Some ideas underlying this theory go back to the work of Max Dehn in the 1910s and later it was generalized by various people in [Tar49], [Gre60], [LS77], [Ol′91], [Gro03], [MW02], [Osi10], and many others. Recently in his outstanding work, Wise [Wis] generalized small-cancellation theory to CAT(0) cube complexes which yielded a proof for the "Virtually Haken conjecture" [AGM12]. Here, we follow the geometric language given in [MW02].

A 2-complex X satisfies the C(p)-T(q) small-cancellation condition if each reduced disc diagram D mapping to X has the property that its internal 2-cells have at least p neighbouring 2-cells and that internal 0-cells of D have at least q adjacent 2-cells (or have degree 2). In a certain sense, the C(p)-T(q) condition represents a combinatorial comparison condition with a simply-connected surface tiled by p-gons with q of them meeting around each vertex. Note that C(p)-T(q) condition sometimes is called the *non-metric small-cancellation condition.* A group G is C(p)-T(q) if it is the fundamental group of a C(p)-T(q) complex. We are interested in the cases when $\frac{1}{p}+\frac{1}{q}\leq\frac{1}{2}$, in which the corresponding surface tiling corresponds to a regular tiling of the Euclidean or Hyperbolic plane.

If a finite 2-complex X satisfies C(p)-T(q) with $\frac{1}{p}+\frac{1}{q}<\frac{1}{2}$, then the group $\pi_1 X$ is word-hyperbolic and indeed in this case C(p)-T(q) is one of the followings: C(7)-T(3), C(5)-T(4), C(4)-T(5), or C(3)-T(7). However, $\pi_1 X$ only necessarily manifests features of nonpositive curvature when $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$ (i.e. C(6)-T(3), C(4)-T(4), & C(3)-T(6)). It is well-known that a word-hyperbolic group cannot contain a $\mathbb{Z}\times\mathbb{Z}$ subgroup, and in this setting, the $\mathbb{Z}\times\mathbb{Z}$ subgroup in $\pi_1 X$ leads to a combinatorial flat plane in $\widetilde{X}$. More generally, failure of word-hyperbolicity corresponds to failure of the existence of a linear isoperimetric function which corresponds to the existence of a combinatorial flat plane in $\widetilde{X}$ as shown by Ivanov and Schupp [IS98]. However the degree to which $\pi_1 X$ fails to be word-hyperbolic has not yet been studied deeply. One sense in which $\pi_1 X$ can "strongly fail" to be hyperbolic is if there is a profusion of $\mathbb{Z}\times\mathbb{Z}$ subgroups, or indeed, if these subgroups richly "interact" with each other, as in $F_2 \times F_2$. This can certainly occur when X is C(4)-T(4), as indeed $F_2 \times F_2 \cong \pi_1 X$ when $X = B \times B$ where B is a bouquet of 2 circles. However, this is not true in the C(6)-T(3) and C(3)-T(6) cases and we have:

Theorem 9.3.1 (H.B. and D. Wise). *A $C(6)$-$T(3)$ group cannot contain $F_2 \times F_2$. The same statement holds for a $C(3)$-$T(6)$ group.*

To prove Theorem 9.3.1, we define objects called "bitorus", see Definition 31 and we show that if $Y \to X$ is a locally convex map where Y is a connected and compact C(6)-T(3) (or C(3)-T(6)) and X is a hexagonal (respectively a triangular) bitorus then either $\pi_1 Y \cong 1$ or $\pi_1 Y \cong \mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z} \subseteq \pi_1 Y$. Combining this result with the fact that $F_2 \times F_2 \cong (F_2 \times \mathbb{Z}) *_{F_2} (F_2 \times \mathbb{Z})$, we prove the theorem.

In fact, Theorem 9.3.1 could be seen as evidence that the following conjecture is true:

Conjecture 1.1.2. *Let G be a $C(6)$-$T(3)$ or $C(3)$-$T(6)$, finitely presented group. Then G is hyperbolic if and only if G does not contain a subgroup isomorphic to $\mathbb{Z} \times \mathbb{Z}$.*

1.2 Outline

Chapter 2 contains the necessary background for relatively hyperbolic groups and their relatively quasiconvex subgroups. We recall Bowditch's definition of relative hyperbolicity in Section 2.1. We also provide some properties and examples. The definition of relatively quasiconvex subgroups is reviewed in Section 2.2.

In Chapter 3, we prove some combination theorems for relatively hyperbolic groups. In Section 3.1, we define "parabolic trees" and we prove a combination theorem for relatively hyperbolic groups along parabolic subgroups. Several corollaries will then be discussed. A result of Yang for extended peripheral structure is recalled in Section 3.2. We use this together with our results to prove a combination theorem for relatively hyperbolic groups along total, malnormal and relatively quasiconvex subgroups.

Chapter 4 contains results about relatively quasiconvex subgroups. In Section 4.1, we define subgroups called "tamely generated", and show that tamely generated subgroups in parabolic splittings are relatively quasiconvex. After providing a relative quasiconvexity criterion, we prove some local relative quasiconvexity results. Section 4.2, explores groups with a small-hierarchy and provides their local relative quasiconvexity.

Chapter 5 contains theorems concerning relative quasiconvexity of a subgroup of a relatively hyperbolic group that splits. In Section 5.1, using results in Chapter 4, we investigate the conditions ensuring relative quasiconvexity for a tamely generated subgroups of a relatively hyperbolic group that splits. Some applications of these theorems are given in Section 5.2.

In Chapter 6, we study free subgroups of relatively hyperbolic groups. In Section 6.1, we give a criterion for a path in a δ-hyperbolic space to be a quasigeodesic. Our argument extracts the hyperbolic case of a relative hyperbolic argument given in [HWb]. Section 6.2 contains a variation on a result of Arzhantseva [Arz00] which implies that given finitely many infinite index subgroups of a free group F, "generically" subgroups of F avoid conjugates of these infinite index subgroups. Using this, we provide a generic statement for free subgroups of relatively hyperbolic groups in Section 6.3. Indeed, we first show the existence of an aparabolic, malnormal, relatively quasiconvex subgroup in a torsion free, non-elementary relatively hyperbolic groups. As an application, we prove a non-quasiconvex embedding theorem for relatively hyperbolic group.

A brief review of the basic notation of small-cancellation theory is provided in Chapter 7. We will recall the basic terminology and the background in Section 7.1. We then review the combinatorial Gauss-Bonnet theorem and Greendlinger's lemma in Section 7.2.

Chapter 8 is devoted to the study of locally convex maps in C(6)-T(3) and C(3)-T(6) small-cancellation complexes. In this chapter, after defining the "thickening" of a subcomplex of a C(6)-T(3) and C(3)-T(6) complex, we show that the thickening of a locally convex subcomplex is again locally convex.

The main object of interest in Chapter 9 is a "bitorus" which is a compact and connected 2-complex homeomorphic to $B \times S^1$ where B is a finite connected leafless graph and $\chi(B) = -1$. We characterize compact, connected complexes mapping to a bitorus via a locally convex map. We then prove the objective of Chapter 8: A C(6)-T(3) group cannot contain $F_2 \times F_2$ as a subgroup. The analogous result is also proven in the C(3)-T(6) case.

Mandatory originality statement: This manuscript is based on the author's work in [Big12], [BW12] and [BW13], two of which are joint with Daniel Wise. It also contains results that have not been submitted yet: the results for C(3)-T(6) complexes in Chapters 8 and 9. However, the materials have been rewritten and reorganized. In Chapter 2, the definition of relatively hyperbolic groups is based on Bowditch's approach in [Bow99a]. Moreover, we use the definition of relative quasiconvexity proposed by Martinez-Pedroza and Wise in [MPW11]. In Chapter 7, which contains the basic terminology and background for small-cancellation theory, we follow the geometric language given in [MW02]. However, a more classical reference is [LS77]. All of the results in this thesis are the work of the author (including joint work with Daniel Wise), unless noted otherwise.

CHAPTER 2
Relatively Hyperbolic Groups and Relatively Quasiconvex Subgroups

This chapter contains the definition and properties of relatively hyperbolic groups and their quasiconvex subgroups. We also provide some properties and examples.

2.1 Relatively Hyperbolic Groups

2.1.1 Basic Terminology and Background

The class of relatively hyperbolic groups was introduced by Gromov [Gro87] as a generalization of the class of fundamental groups of complete finite-volume manifolds of pinched negative sectional curvature. Various approaches to relative hyperbolicity were developed by [Far98], [Bow99b], [Osi06c], [Hru10], [DS05], [Yam04] and Hruska [Hru10] showed that these definitions are equivalent for finitely generated groups.

Definition 1 (δ-Hyperbolic space)**.** A map φ defined on a metric space (X, d) is called *ϵ-thin* if $\varphi(x) = \varphi(y)$ implies $d(x, y) \leq \epsilon$. Let X be a geodesic metric space and let $\triangle[x_1, x_2, x_3]$ be a geodesic triangle in X. Let T be a tripod with three extremal vertices y_1, y_2 and y_3 so that $d(y_i, y_j) = d(x_i, x_j)$, see Figure 2–1. The triangle $\triangle$ is called *ϵ-thin* if the map $\varphi_\triangle : \triangle \to T$ which sends x_i to y_i and which is an isometry on the sides of $\triangle$, is an ϵ-thin map. A geodesic metric space X is called *δ-hyperbolic*, for $\delta > 0$, if any geodesic triangle in X is δ-thin.

Definition 2 (Fine)**.** A *circuit* in a graph is an embedded cycle. A graph Γ is *fine* if each edge of Γ lies in finitely many circuits of length n for each n.

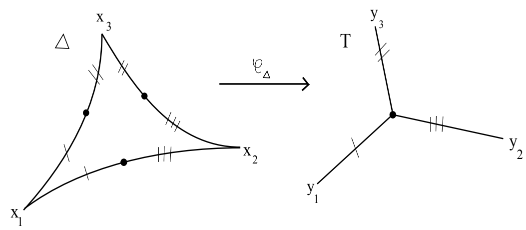

Figure 2–1: $\varphi_{\triangle}$ is a map from the geodesic triangle to the tripod.

2.1.2 Definition of Relatively Hyperbolic Groups

We employ the following definition of relatively hyperbolic groups that was formulated by Bowditch [Bow99b]:

Definition 3 (Relatively Hyperbolic Group)**.** A group G is *hyperbolic relative to a finite collection of subgroups* $\mathbb{P} = \{P_1, \ldots, P_n\}$ if G acts on a connected graph Γ (without inversions) with the following properties:

1. Γ is hyperbolic and fine;
2. Γ is cocompact, i.e. $G \backslash \Gamma$ is compact;
3. The stabilizer of each edge of Γ is finite;
4. Each element of $\mathbb{P}$ equals the stabilizer of a vertex of Γ, and each infinite vertex stabilizer is conjugate to a unique element of $\mathbb{P}$.

Remark 2.1.1. We refer to a connected, fine, hyperbolic graph Γ equipped with such an action as a $(G; \mathbb{P})$*-graph*. Subgroups of G that are conjugate into subgroups in $\mathbb{P}$ are *parabolic*.

2.1.3 Properties and Examples

The following example was discussed in [Far98]:

Let $G = \langle a, b \rangle$ be the fundamental group of a punctured torus. Let the cyclic subgroup $H = \langle aba^{-1}b^{-1} \rangle$ be the cusped subgroup of G. Let Γ be the Cayley graph of G. Using Γ, we form *the Coned-off Cayley graph* $\widehat{\Gamma}$ *of* G as follows: For each left coset gH of H in G, add a vertex $v(gH)$ to Γ and add an edge $e(gh)$ of length $\frac{1}{2}$ from each element gh of gH to the vertex $v(gH)$. The group G acts on $\widehat{\Gamma}$. One can see that $\widehat{\Gamma}$ with the graph metric is a $(G;\{H\})$-graph and therefore G is hyperbolic relative to $\{H\}$.

The class of relatively hyperbolic groups contains word hyperbolic groups, fundamental groups of finite volume hyperbolic manifolds, limit groups [Dah03], geometrically finite convergence groups [Yam04], groups acting freely on $\mathbb{R}^n$-trees [Gui04], CAT(0)-groups with isolated flats [HK05] and many other examples.

Definition 4 (Almost Malnormal)**.** A subgroup H is *malnormal* in G if $H \cap H^g = \{1\}$ for $g \notin H$, and similarly H is *almost malnormal* if this intersection $H \cap H^g$ is always finite. Likewise, a collection of subgroups $\{H_i\}$ is *almost malnormal* if $H_i^g \cap H_j^h$ is finite unless $i = j$ and $gh^{-1} \in H_i$.

The following well-known property was proven in [Bow99b], one can see also [MPW11, Lem 2.2].

Lemma 2.1.2 (Almost Malnormal)**.** *Let* G *be hyperbolic relative to* $\mathbb{P}$ *then* $\{P^g \mid P \in \mathbb{P}, g \in G\}$ *is an almost malnormal collection of subgroups.*

The following associative property was proved in [DS05]:

Lemma 2.1.3. *If* G *is finitely generated and hyperbolic relative to* $\mathbb{P} = \{P_1, \ldots, P_n\}$ *and each* P_i *is hyperbolic relative to* $\mathbb{H}_i = \{H_{i1}, \ldots, H_{im_i}\}$*, then* G *is hyperbolic relative to* $\bigcup_{1 \le i \le n} \mathbb{H}_i$*.*

2.2 Relatively Quasiconvex Subgroups

The notion of relatively quasiconvex subgroup was formulated by Dahmani [Dah03] and by Osin [Osi06c], and Hruska investigated several equivalent definitions of relatively quasiconvex subgroups [Hru10]. Martinez-Pedroza and Wise introduced a definition of relative quasiconvexity in the context of fine hyperbolic graphs and showed this definition is equivalent to Osin's definition [MPW11].

2.2.1 Definition of Relatively Quasiconvex Subgroups

We will use the following definition of Martinez-Pedroza and Wise [MPW11]:

Definition 5 (Relatively Quasiconvex subgroup)**.** Let G be hyperbolic relative to $\mathbb{P}$. A subgroup H of G is *quasiconvex relative to* $\mathbb{P}$ if for some (and hence any) $(G;\mathbb{P})$-graph K, there is a nonempty connected and quasi-isometrically embedded, H-cocompact subgraph L of K.

Note that in this thesis, we sometimes refer to L as a *quasiconvex H-cocompact subgraph* of K.

Definition 6 (Coned-off Cayley Graph)**.** Let S be a set of generators of a group G. The *Cayley graph* $\Gamma(G,S)$ is an oriented labelled 1-complex with vertex set G and edge set $G \times S$. An edge (g,s) has initial vertex g and terminal vertex gs and label s.

Let G be a group and $\mathbb{P}$ a finite collection of subgroups of G. A set $S \subseteq G$ is a *relative generating set* for the pair $(G,\mathbb{P})$ if $G = \langle S \cup \bigcup_{P\in\mathbb{P}} P\rangle$

Let S be a finite relative generating set for $(G,\mathbb{P})$. The *coned-off Cayley graph* $\widehat{\Gamma}(G,\mathbb{P},S)$ is the graph constructed from $\Gamma(G,S)$ as follows: For each left coset gP with $g \in G$ and $P \in \mathbb{P}$, add a new vertex $\bar{v}(gP)$ to $\Gamma(G,S)$, and add a 1-cell from

$\bar{v}(gP)$ to each element of gP. These new vertices of $\widehat{\Gamma}(G, \mathbb{P}, S)$ that are not in Γ are called *cone-vertices*. Each 1-cell of $\widehat{\Gamma}(G, \mathbb{P}, S)$ between an element of G and a cone vertex is a *cone-edge*. Note that $\widehat{\Gamma}(G, \mathbb{P}, S)$ is connected since S is a relative generating set for $(G, \mathbb{P})$. We will also consider the (ordinary) Cayley graph with respect to the generating set consisting of the disjoint union $S \sqcup \bigsqcup_i P_i$ which we denote by $\Gamma(G, S \cup \mathbb{P})$.

The following is the definition of relative quasiconvexity proposed by Osin [Osi06c]:

Let G be a group generated by a finite set S and $\mathbb{P}$ be a finite collection of subgroups of G. A subgroup H of G is called relatively quasiconvex with respect to $\mathbb{P}$ if there exists a constant $\sigma > 0$ such that the following condition holds. Let f, g be two elements of H, and p an arbitrary geodesic path from f to g in $\Gamma(G, S \cup \mathbb{P})$. Then for any vertex $v \in p$ there exists a vertex $w \in H$ such that $dist_S(v, w) \leq \sigma$.

This definition is equivalent to Definition 5 by [MPW11].

2.2.2 Properties and Examples

Remark 2.2.1. It is immediate from the Definition 5 that in a relatively hyperbolic group, any parabolic subgroup is relatively quasiconvex, and any relatively quasiconvex subgroup is also relatively hyperbolic. In particular, the relatively quasiconvex subgroup H is hyperbolic relative to the collection $\mathbb{P}_H$ consisting of representatives of H-stabilizers of vertices of $L \subseteq K$. Note that a conjugate of a relatively quasiconvex subgroup is also relatively quasiconvex.

And the intersection of two relatively quasiconvex subgroups is relatively quasiconvex. Specifically, this last statement was proven when G is finitely generated in [MP09], and when G is countable in [Hru10].

Relative quasiconvexity has the following transitive property proven by Hruska for countable relatively hyperbolic groups in [Hru10]:

Lemma 2.2.2. *Let G be hyperbolic relative to $\mathbb{P}_G$. Suppose that B is relatively quasiconvex in G, and note that B is then hyperbolic relative to $\mathbb{P}_B$ as in Remark 2.2.1. Then $A \leq B$ is quasiconvex relative to $\mathbb{P}_B$ if and only if A is quasiconvex relative to $\mathbb{P}_G$.*

Proof. Let K be a $(G; \mathbb{P}_G)$-graph. As B is quasiconvex relative to $\mathbb{P}_G$, there is a B-cocompact and quasiconvex subgraph $L \subset K$. Note that L is a $(B; \mathbb{P}_B)$-graph. Let $A \leq B$.

If A is quasiconvex in B relative to $\mathbb{P}_B$, there is an A-cocompact quasiconvex subgraph $M \subset L$. Since the composition $L_A \to L_B \to K$ is a quasi-isometric embedding, A is quasiconvex relative to $\mathbb{P}_G$. Conversely, if A is quasiconvex in G relative to $\mathbb{P}_G$, then there is an A-cocompact quasiconvex subgraph $M \subset K$. Let $L' = L \cup BM$ and note that L' is B-cocompact and hence also quasiconvex, and thus L' also serves as a fine hyperbolic graph for B. Now $M \subset L'$ is quasiconvex since $M \subset L$ is quasiconvex so A is relatively quasiconvex in B. □

Remark 2.2.3. One consequence of Theorem 3.1.2 and its various Corollaries, is that when G splits as a graph of relatively hyperbolic groups with parabolic subgroups, then each of the vertex groups is quasiconvex relative to the peripheral structure of G. (For Theorem 3.1.2 this is $\mathbb{Q}$, and for Corollary 3.1.4 this is $\mathbb{P} - \{\text{repeats}\}$.)

Indeed, K_v is a G_v-cocompact quasiconvex subgraph in the fine graph K constructed in the proof.

CHAPTER 3
Combination of Relatively Hyperbolic Groups

In this chapter, we first define "parabolic trees" which will be the parabolic subgroups of combination of relatively hyperbolic groups. We then prove a combination theorem for relatively hyperbolic groups along parabolic subgroups. Some corollaries will be provided. Moreover, we will prove a combination theorem for relatively hyperbolic groups along malnormal, relatively quasiconvex subgroups.

3.1 Combination along Parabolic Subgroups

This section contains combination theorem for relatively hyperbolic groups along parabolic subgroups. We also provide several corollaries.

Technical Remark 3.1.1. *Given a finite collection of parabolic subgroups* $\{A_1, \ldots, A_r\}$, *we choose* $\mathbb{P}$ *so that there is a prescribed choice of parabolic subgroup* $P_i \in \mathbb{P}$ *so that* A_i *is* "declared" *to be conjugate into* P_i. *This is automatic for an infinite parabolic subgroup* A *but for finite subgroups there could be ambiguity. One way to resolve this is to revise the choice of* $\mathbb{P}$ *as follows: For any finite collection of parabolic subgroups* $\{A_1, \ldots, A_r\}$ *in* G, *we moreover assume each* A_i *is conjugate to a subgroup of* $\mathbb{P}$ *and we assume that no two (finite) subgroups in* $\mathbb{P}$ *are conjugate. We note that finite subgroups can be freely added to or omitted from the peripheral structure of* G *(see e.g. [MPW11]).*

Definition 7 (Parabolic tree)**.** Let G split as a finite graph of groups where each vertex group G_ν is hyperbolic relative to $\mathbb{P}_\nu$, and where each edge group G_e embeds as a parabolic subgroup of its two vertex groups. Let T be the Bass-Serre tree. Define the *parabolic forest* F by:

1. A *vertex* in F is a pair (u, P) where $u \in T^0$ and P is a G_u-conjugate of an element of $\mathbb{P}_u$.
2. An *edge* in F is a pair (e, G_e) where e is an edge of T and G_e is its stabilizer.
3. The edge (e, G_e) is *attached* to $(\iota(e), \iota(P_e))$ and $(\tau(e), \tau(P_e))$ where $\iota(e)$ and $\tau(e)$ are the initial and terminal vertex of e and $\iota(P_e)$ is the $G_{\iota(e)}$-conjugate of an element of $\mathbb{P}$ that is declared to contain G_e. Likewise for $(\tau(e), \tau(P_e))$. We arranged for this unique determination in Technical Remark 3.1.1.

Each component of F is a *parabolic tree* and the map $F \to T$ is injective on the set of edges, and in particular each parabolic tree embeds in T. Let $S_1, \ldots, S_j$ be representatives of the finitely many orbits of parabolic trees under the G action on F. Let $Q_i = stab(S_i)$, for each i.

Theorem 3.1.2 (Combining Relatively Hyperbolic Groups Along Parabolics)**.** *Let G split as a finite graph Γ of groups. Suppose each vertex group is relatively hyperbolic and each edge group is parabolic in its vertex groups. Then G is hyperbolic relative to $\mathbb{Q} = \{Q_1, \ldots, Q_j\}$.*

Proof. For $u \in \Gamma^0$, let G_u be hyperbolic relative to $\mathbb{P}_u$ and let K_u be a $(G_u; \mathbb{P}_u)$-graph. For each $P \in \mathbb{P}_u$, following the Technical Remark 3.1.1, we choose a specific vertex of K_u whose stabilizer equals P. Note that, in general there could be more than one possible choice when $|P| < \infty$, but by Technical Remark 3.1.1 we have a unique choice. Translating determines a "choice" of vertex for conjugates.

We now construct a $(G;\mathbb{Q})$-graph $\bar{K}$. Let K be the tree of spaces whose underlying tree is the Bass-Serre tree T with the following properties:

1. Vertex spaces of K are copies of appropriate elements in $\{K_u : u \in \Gamma^0\}$. Specifically, K_ν is a copy of K_u where u is the image of ν under $T \to \Gamma$.
2. Each edge space K_e is an ordinary edge, denoted as an ordered pair (e, G_e) that is attached to the vertices in $K_{\iota(e)}$ and $K_{\tau(e)}$ that were chosen to contain G_e.

Note that each G_ν acts on K_ν and there is a G-equivariant map $K \to T$. Let $\bar{K}$ be the quotient of K obtained by contracting each edge space. Observe that G acts on $\bar{K}$ and there is a G-equivariant map $K \to \bar{K}$. Moreover the preimage of each open edge of $\bar{K}$ is a single open edge of K.

We now show that $\bar{K}$ is a $(G;\mathbb{Q})$-graph. Since any embedded cycle lies in some vertex space, the graph $\bar{K}$ is fine and hyperbolic. There are finitely many orbits of vertices in K and therefore finitely many orbits of vertices in $\bar{K}$. Likewise, there are finitely many orbits of edges in $\bar{K}$. The stabilizer of an (open) edge of $\bar{K}$ equals the stabilizer of the corresponding (open) edge in K, and is thus finite. By construction, there is a G-equivariant embedding $F \hookrightarrow K$ where F is the parabolic forest associated to G and T. Finally, the preimage in K of a vertex of $\bar{K}$ is precisely a parabolic tree and thus the stabilizer of a vertex of $\bar{K}$ is a conjugate of some Q_j. An example of this construction is illustrated in Figure 3–1. □

3.1.1 Corollaries

We now examine some conclusions that arise when the parabolic trees are small. An extreme case arises when the edge groups are isolated from each other as follows:

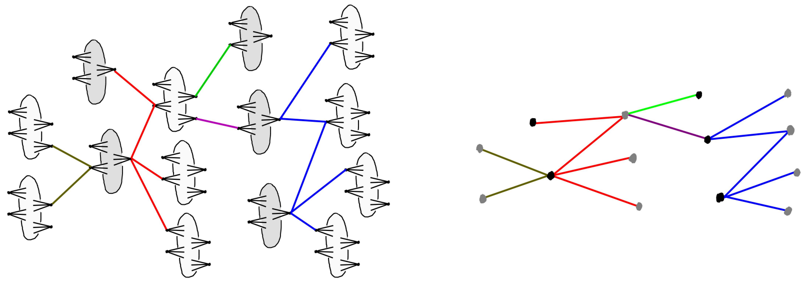

Figure 3–1: A fine graph K_G for $G = A *_C B$ is built from copies of fine graphs K_A and K_B for A and B by gluing new edges together along vertices stabilized by C. The parabolic trees of T are images of trees formed from the new edges in K_G. We obtain a fine hyperbolic graph $\bar{K}_G$ with finite edge stabilizers as a quotient $K_G \to \bar{K}_G$.

Corollary 3.1.3. *Let G split as a finite directed graph of groups where each vertex group G_ν is hyperbolic relative to $\mathbb{P}_\nu$. Suppose that:*

1. *Each edge group is parabolic in its vertex groups.*
2. *Each outgoing infinite edge group $G_{\vec{e}}$ is maximal parabolic in its initial vertex group G_ν and for each other incoming and outgoing infinite edge group $G_{\overleftarrow{e}}$ or $G_{\vec{d}}$ or $G_{\overleftarrow{d}}$, none of its conjugates lie in $G_{\vec{e}}$.*

Then G is hyperbolic relative to $\mathbb{P} = \bigcup_\nu \mathbb{P}_\nu - \{outgoing\ edge\ groups\}$.

Proof. We can arrange for finitely stabilized edges of F to be attached to distinct chosen vertices when they correspond to distinct edges of T. Thus, parabolic trees are singletons and/or i-pods consisting of edges that all terminate at the same vertex $\{(\nu, P^g)\}$ where $P \in \mathbb{P}_\nu$ and $g \in G_\nu$. Recall that an *i-pod* is a tree consisting of i edges glued to a central vertex. $\square$

Corollary 3.1.4. *Let G split as a finite graph of groups. Suppose each vertex group G_ν is hyperbolic relative to $\mathbb{P}_\nu$. For each G_ν assume that the collection $\{G_e :$ e is attached to $\nu\}$ is a collection of maximal parabolic subgroups of G_ν. Then G is* *hyperbolic relative to* $\mathbb{P} = \bigcup_\nu \mathbb{P}_\nu - \{repeats\}$. *Specifically, we remove an element of* $\bigcup_\nu \mathbb{P}_\nu$ *if it is conjugate to another one.*

The first two of the following cases were treated by Dahmani, Alibegović, and Osin [Ali05, Dah03, Osi06a]:

Corollary 3.1.5. *1. Let* G_1 *and* G_2 *be hyperbolic relative to* $\mathbb{P}_1$ *and* $\mathbb{P}_2$. *Let* $G = G_1 *_{P_1=P_2'} G_2$ *where each* $P_i \in \mathbb{P}_i$ *and* P_1 *is identified with the subgroup* ${P_2}'$ *of* P_2. *Then* G *is hyperbolic relative* $\mathbb{P}_1 \cup \mathbb{P}_2 - \{P_1\}$.

2. Let G_1 *be hyperbolic relative to* $\mathbb{P}$. *Let* $P_1 \in \mathbb{P}$ *be isomorphic to a subgroup* ${P_2}'$ *of a maximal parabolic subgroup* P_2 *not conjugate to* P_1. *Let* $G = G_1 *_{{P_1}^t={P_2}'}$ *where* ${P_1}^t = t^{-1}P_1 t$. *Then* G *is hyperbolic relative to* $\mathbb{P} - \{P_1\}$.

3. Let G_1 *be hyperbolic relative to* $\mathbb{P}$. *Let* $P \in \mathbb{P}$ *be isomorphic to* $P' \leq P$. *Let* $G = G_1 *_{P^t=P'}$. *Then* G *is hyperbolic relative to* $\mathbb{P} \cup \langle P, t\rangle - \{P\}$.

Remark 3.1.6. Note that in this Corollary and some similar results when we say $P_i \in \mathbb{P}_i$, we mean if $P_i^g \in \mathbb{P}_i$ then replace P_i^g by P_i in $\mathbb{P}_i$.

Proof. **(1)**: In this case, the parabolic trees are either singletons stabilized by a conjugate of an element of $\mathbb{P}_1 \cup \mathbb{P}_2 - \{P_1\}$, or parabolic trees are i-pods stabilized by conjugates of P_2.

(2): The proof is similar.

(3): All parabolic trees are singletons except for those that are translates of a copy of the Bass-Serre tree for $P*_{P^t=P'}$. Following the proof of Theorem 3.1.2, let $\nu \in \bar{K}$, if the preimage of ν in K is not attached to an edge space, then G_ν is conjugate to an element of $\mathbb{P} - \{P\}$, otherwise G_ν is conjugate to $\langle P, t\rangle$.

□

Example 3.1.7. We encourage the reader to consider the case of Theorem 3.1.2 and Corollaries 3.1.4 and 3.1.5, in the scenario where G splits as a graph of free groups with cyclic edge groups. A very simple case is: Let $G = \langle a, b, t \mid (W^n)^t = W^m \rangle$ where $W \in \langle a, b \rangle$ and $m, n \geq 1$. Then G is hyperbolic relative to $\langle W, t \rangle$.

3.2 Combination along Total, Malnormal and Quasiconvex Subgroups

In this section, we first recall the notion of extended peripheral structure defined by Yang [Yan11] and we state his criterion for hyperbolicity of extended peripheral structure. Using his result, we then prove a combination theorem for relatively hyperbolic groups along total, malnormal and quasiconvex subgroups. This generalizes results in Chapter 3.

3.2.1 Extended Peripheral Structure and Hyperbolicity

Gersten [Ger96] and then Bowditch [Bow99b] showed that a hyperbolic group G is hyperbolic relative to an almost malnormal quasiconvex subgroup. Generalizing work of Martinez-Pedroza [Mar08], Yang introduced and characterized a class of parabolically extended structures for countable relatively hyperbolic groups [Yan11]. We use his results to generalize our previous results. The following was defined in [Yan11] for countable groups.

Definition 8 (Extended Peripheral Structure)**.** A *peripheral structure* consists of a finite collection $\mathbb{P}$ of subgroups of a group G. Each element $P \in \mathbb{P}$ is a *peripheral subgroup* of G. The peripheral structure $\mathbb{E} = \{E_j\}_{j \in J}$ *extends* $\mathbb{P} = \{P_i\}_{i \in I}$ if for each $i \in I$, there exists $j \in J$ such that $P_i \subseteq E_j$. For $E \in \mathbb{E}$, we let $\mathbb{P}_E = \{P_i : P_i \subseteq E, P_i \in \mathbb{P}, i \in I\}$.

We will use the following result of Yang [Yan11].

Theorem 3.2.1 (Hyperbolicity of Extended Peripheral Structure)**.** *Let G be hyperbolic relative to $\mathbb{P}$ and let the peripheral structure $\mathbb{E}$ extend $\mathbb{P}$. Then G is hyperbolic relative to $\mathbb{E}$ if and only if the following hold:*

1. *$\mathbb{E}$ is almost malnormal;*
2. *Each $E \in \mathbb{E}$ is quasiconvex in G relative to $\mathbb{P}$.*

3.2.2 Combination along Total, Malnormal and Quasiconvex Subgroups

Definition 9 (Total)**.** Let G be hyperbolic relative to $\mathbb{P}$. The subgroup H of G is *total relative to* $\mathbb{P}$ if: either $H \cap P^g = P^g$ or $H \cap P^g$ is finite for each $P \in \mathbb{P}$ and $g \in G$.

We now generalize Corollary 3.1.5 to handle the case where edge groups are quasiconvex and not merely parabolic.

Theorem 3.2.2 (Combination along Malnormal and Quasiconvex Subgroups)**.**

1. *Let G_i be hyperbolic relative to $\mathbb{P}_i$ for $i = 1, 2$. Let $C_i \leq G_i$ be almost malnormal, total and relatively quasiconvex. Let ${C_1}' \leq C_1$. Then $G = G_1 *_{{C_1}'=C_2} G_2$ is hyperbolic relative to $\mathbb{P} = \mathbb{P}_1 \cup \mathbb{P}_2 - \{P_2 \in \mathbb{P}_2 \ : \ P_2^g \subseteq C_2, \text{ for some } g \in G_2\}$.*
2. *Let G_1 be hyperbolic relative to $\mathbb{P}$. Let $\{C_1, C_2\}$ be almost malnormal and assume each C_i is total and relatively quasiconvex. Let ${C_1}' \leq C_1$. Then $G = G_1 *_{{C_1}'={C_2}^t}$ is hyperbolic relative to $\mathbb{P} = \mathbb{P} - \{P_2 \in \mathbb{P}_2 \ : \ P_2^g \subseteq C_2, \text{ for some } g \in G_2\}$.*

Proof. **(1)**: For each i, let

$$\mathbb{E}_i = \mathbb{P}_i - \{P \in \mathbb{P}_i \colon P^g \leq C_i, \text{ for some } g \in G_i\} \cup \{C_i\}$$

Without loss of generality, we can assume that $\mathbb{E}_i$ extends $\mathbb{P}_i$, since we can replace an element of $\mathbb{P}_i$ by its conjugate. We now show that G_i is hyperbolic relative to $\mathbb{E}_i$ by verifying the two conditions of Theorem 6.3.4: $\mathbb{E}_i$ is malnormal in G_i, since $\mathbb{P}_i$ is almost malnormal and C_i is total and almost malnormal. Each element of $\mathbb{E}_i$ is relatively quasiconvex, since C_i is relatively quasiconvex by hypothesis and each element of $\mathbb{P}_i$ is relatively quasiconvex by Remark 2.2.1.

We now regard each G_i as hyperbolic relative to $\mathbb{E}_i$. Therefore since the edge group $C_2 = {C_1}'$ is maximal on one side, by Corollary 3.1.5, G is hyperbolic relative to $\mathbb{E} = \mathbb{E}_1 \cup \mathbb{E}_2 - \{C_2\}$.

We now apply Lemma 6.4.1 to show that G is hyperbolic relative to $\mathbb{P}$. We showed that G is hyperbolic relative to $\mathbb{E}$. But each element of $\mathbb{E}$ is hyperbolic relative to $\mathbb{P}$ that it contains. Thus by Lemma 6.4.1, we obtain the result.

(2): The proof is analogous to the proof of (1). $\square$

The following can be obtained by induction using Theorem 3.2.2 or can be proven directly using the same mode of proof.

Corollary 3.2.3. *Let G split as a finite graph of groups. Suppose*

(a) Each G_ν is hyperbolic relative to $\mathbb{P}_\nu$;

(b) Each G_e is total and relatively quasiconvex in G_ν;

(c) $\{G_e : e \text{ is attached to } \nu\}$ is almost malnormal in G_ν for each vertex ν.

Then G is hyperbolic relative to $\bigcup_\nu \mathbb{P}_\nu - \{\text{repeats}\}$.

Corollary 3.2.4. *Let G split as a finite graph of groups. Suppose*

(a) Each G_ν is hyperbolic relative to $\mathbb{P}_\nu$;

(b) Each G_e is total and relatively quasiconvex in G_ν;

(c) Each G_e is almost malnormal in G_ν.

Then G is hyperbolic relative to $\bigcup_\nu \mathbb{P}_\nu - \{repeats\}$.

Proof. Let Γ be the graph of groups. We prove the result by induction on the number of edges of the graph of groups Γ. The base case where Γ has no edge is contained in the hypothesis. Suppose that Γ has at least one edge e (regarded as an open edge). If e is nonseparating, then $G = A*_{C^t=D}$ where A is the graph of groups over $\Gamma - e$, and C, D are the two images of G_e. Condition (c) ensures that $\{C, D\}$ is almost malnormal in A, and by induction, A is hyperbolic relative to $\bigcup_\nu \mathbb{P}_\nu - \{\text{repeats}\}$ where $\nu \in \Gamma - e$, and thus G is hyperbolic relative to $\bigcup_\nu \mathbb{P}_\nu - \{\text{repeats}\}$ Corollary 3.2.3. A similar argument concludes the separating case. $\square$

CHAPTER 4
Small-hierarchies and Local Relative Quasiconvexity

In this chapter, we define a tamely generated subgroup and then we prove relative quasiconvexity of a tamely generated subgroup in parabolic splitting of a relatively hyperbolic group. We also give a criterion for locally relative quasiconvexity. Moreover, we define small-hierarchy and we show that any finitely generated group with small-hierarchy that is hyperbolic relative to Noetherian subgroups is locally relatively quasiconvex.

4.1 Quasiconvexity of a Subgroup in Parabolic Splitting

4.1.1 Tamely Generated Subgroup

Definition 10 (Tamely generated)**.** Let G split as a graph of groups with relatively hyperbolic vertex groups. A subgroup H is *tamely generated* if the induced graph of groups Γ_H has a π_1-isomorphic subgraph of groups Γ'_H that is a finite graph of groups each of whose vertex groups is relatively quasiconvex in the corresponding vertex group of G.

Note that H is tamely generated when H is finitely generated and there are finitely many H-orbits of vertices v in T with H_v nontrivial, and each such H_v is relatively quasiconvex in G_v. However the above condition is not necessary. For instance, let $G = F_2 \times \mathbb{Z}_2$, and consider a splitting where Γ is a bouquet of two circles, and each vertex and edge group is isomorphic to $\mathbb{Z}_2$. Then every finitely generated subgroup H of $F_2 \times \mathbb{Z}_2$ is tamely generated, but no subgroup containing

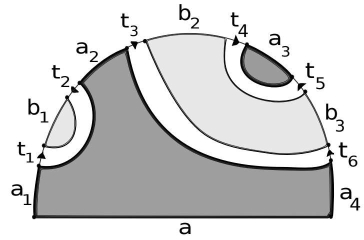

Figure 4–1: D is a disc diagram whose boundary path is $a_1t_1b_1{t_2}^{-1}a_2t_3b_2t_4a_3t_5^{-1}b_3t_6^{-1}a_4$

$\mathbb{Z}_2$ satisfies the condition that there are finitely many H-orbits of vertices ω with H_ω nontrivial.

4.1.2 Quasiconvexity of a Subgroup in Parabolic Splitting

Lemma 4.1.1. *Let G be a finitely generated group that split as a finite graph of groups Γ. If each edge group is finitely generated then each vertex group is finitely generated*

Proof. Let $G = \langle g_1, \ldots, g_n \rangle$. We regard G as π_1 of a 2-complex corresponding to Γ. We show that each vertex group G_v equals $\langle \{G_e\}_{e \text{ attached to } v} \cup \{g \in G_v : g \text{ in normal form of some } g_i\} \rangle$. Let $a \in G_v$ and consider an expression of a as a product of normal forms of the $g_i^{\pm 1}$. Then a equals some product $a_1{t_1}^{\epsilon_1}b_1{t_2}^{\epsilon_2}a_2 \cdots a_n{t_m}^{\epsilon_m}b_k$. There is a disc diagram D whose boundary path is $a^{-1}a_1{t_1}^{\epsilon_1}b_1{t_2}^{\epsilon_2}a_2 \cdots a_n{t_m}^{\epsilon_m}b_k$. See Figure 4–1. The region of D that lies along a shows that a equals the product of elements in edge groups adjacent to G_v, together with elements of G_v that lie in the normal forms of $g_1, \ldots, g_n$. □

Theorem 4.1.2 (Quasiconvexity of a Subgroup in Parabolic Splitting)**.** *Let G split as a finite graph Γ of relatively hyperbolic groups such that each edge group is parabolic in its vertex groups. (Note that G is hyperbolic relative to $\mathbb{Q} = \{Q_1, \ldots, Q_j\}$ by Theorem 3.1.2.) Let $H \leq G$ be tamely generated. Then H is quasiconvex relative to $\mathbb{Q}$. Moreover if each H_v in the Bass-Serre tree T is finitely generated then H is finitely generated.*

Proof. Since there are finitely many orbits of vertices whose stabilizers are finitely generated, H is finitely generated. For each $u \in \Gamma^0$, let G_u be hyperbolic relative to $\mathbb{P}_u$ and let K_u be a $(G_u; \mathbb{P}_u)$-graph. Let K be the $(G; \mathbb{Q})$-graph constructed in the proof of Theorem 3.1.2 and let $\bar{K}$ be its quotient. We will construct an H-cocompact quasiconvex, connected subgraph $\bar{L}$ of $\bar{K}$.

Let T_H be the minimal H-invariant subgraph of T. Recall that each edge of T (and hence T_H) corresponds to an edge of K. Let F_H denote the subgraph of K that is the union of all edges correspond to edges of T_H. Let $\{\nu_1, \ldots, \nu_n\}$ be a representatives of H-orbits of vertices of T_H. For each i, let $L_i \hookrightarrow K_{\nu_i}$ be a $(H \cap G_{\nu_i}^{g_i})$-cocompact quasiconvex subgraph such that L_i contains $F_H \cap K_{\nu_i}$. (There are finitely many $(H \cap G_{\nu_i}^{g_i})$-orbits of such endpoints of edges in K_{ν_i}.) Let $L = F_H \cup \bigcup_{i=1}^{n} HL_i$ and let $\bar{L}$ be the image of L under $K \to \bar{K}$. Observe that L is quasiconvex in K since K is a "tree union" and each such L_i of L is quasiconvex in K_{ν_i}. And likewise, $\bar{L}$ is quasiconvex in $\bar{K}$.

□

Corollary 4.1.3 (Characterizing Quasiconvexity in Maximal Parabolic Splitting)**.** *Let G split as a finite graph of countable groups. For each ν, let G_ν be hyperbolic relative to $\mathbb{P}_\nu$ and let the collection $\{G_e : e$ is attached to $\nu\}$ be a collection of maximal parabolic subgroups of G_ν. (Note that G is hyperbolic relative to $\mathbb{P} = \bigcup_\nu \mathbb{P}_\nu - \{repeats\}$ by Corollary 3.1.4.) Let T be the Bass-Serre tree and let H be a subgroup of G. The following are equivalent:*

1. *H is tamely generated and each H_v in the Bass-Serre tree T is finitely generated*
2. *H is finitely generated and quasiconvex relative to $\mathbb{P}$.*

Proof. **(1 $\Rightarrow$ 2)**: Follows from Theorem 3.1.2 and Theorem 4.1.2.

(2 $\Rightarrow$ 1): Since H is finitely generated, the minimal H-subtree T_H is H-cocompact, and so H splits as a finite graph of groups Γ_H. Since H is quasiconvex relative to $\mathbb{P}$, it is hyperbolic relative to intersections with conjugates of $\mathbb{P}$. In particular, the infinite edge groups in the induced splitting of H are maximal parabolic, and are thus finitely generated since the maximal parabolic subgroups of a finitely generated relatively hyperbolic group are finitely generated. [Osi06c]. Each vertex group of Γ_H is finitely generated by Lemma 4.1.1.

By Remark 2.2.3, each vertex group of G is quasiconvex relative to $\mathbb{P}$, and hence each G_ν is relatively quasiconvex by Remark 2.2.1 since it is a conjugate of a vertex group. Thus $H_\nu = H \cap G_\nu$ is quasiconvex relative to $\mathbb{P}$ by Remark 2.2.1. Finally, H_ν is quasiconvex in G_ν by Lemma 2.2.2. □

4.2 A Criterion for Local Relative Quasiconvexity

Definition 11 (Locally Relatively Quasiconvex)**.** A relatively hyperbolic group G is *locally relatively quasiconvex* if each finitely generated subgroup of G is relatively quasiconvex.

The focus of this section is the following criterion for showing that the combination of locally relatively quasiconvex groups is again locally relatively quasiconvex.

Recall that N is *Noetherian* if each subgroup of N is finitely generated. We now give a criterion for local quasiconvexity of a group that splits along parabolic subgroups.

Theorem 4.2.1 (A Criterion for Local Relative Quasiconvexity)**.**

1. *Let G_1 and G_2 be locally relatively quasiconvex relative to $\mathbb{P}_1$ and $\mathbb{P}_2$. Let $G = G_1 *_{P_1 = P_2'} G_2$ where each $P_i \in \mathbb{P}_i$ and P_1 is identified with the subgroup ${P_2}'$ of P_2. Suppose P_1 is Noetherian. Then G is locally quasiconvex relative to $\mathbb{P}_1 \cup \mathbb{P}_2 - \{P_1\}$.*
2. *Let G_1 be a locally relatively quasiconvex relative to $\mathbb{P}$. Let $P_1 \in \mathbb{P}$ be isomorphic to a subgroup ${P_2}'$ of a maximal parabolic subgroup P_2 not conjugate to P_1. Let $G = G_1 *_{{P_1}^t = {P_2}'}$. Suppose P_1 is Noetherian. Then G is locally quasiconvex relative to $\mathbb{P} - \{P_1\}$.*
3. *Let G_1 be a locally quasiconvex relative to $\mathbb{P}$. Let P be a maximal parabolic subgroup of G_1, isomorphic to $P' \leq P$. Let $G = G_1 *_{P^t = P'}$ and suppose P is Noetherian. Then G is also locally quasiconvex relative to $\mathbb{P} \cup \langle P, t \rangle - \{P\}$.*

Proof. (1): By Corollary 3.1.5, G is hyperbolic relative to $\mathbb{P} = \mathbb{P}_1 \cup \mathbb{P}_2 - \{P_1\}$. Let H be a finitely generated subgroup of G. We show that H is quasiconvex relative to

$\mathbb{P}$. Let T be the Bass-Serre tree of G. Since H is finitely generated, the minimal H-subtree T_H is H-cocompact, and so H splits as a finite graph of groups Γ_H. Moreover, the edge groups of this splitting are finitely generated, since the edge groups of G are Noetherian by hypothesis. Thus each vertex group of Γ_H is finitely generated by Lemma 4.1.1. Since G_1 and G_2 are locally relatively quasiconvex, each vertex group of T_H is relatively quasiconvex in its "image vertex group" under the map $T_H \to T$. Now by Theorem 4.1.2, H is quasiconvex relative to $\mathbb{P}$. The proof of (2) and (3) are similar. □

Corollary 4.2.2. *Let G split as a finite graph of groups. Suppose*

a) Each G_ν is locally relatively quasiconvex;

b) Each G_e is Noetherian and maximal parabolic in its vertex groups;

c) $\{G_e : e \text{ is attached to } \nu\}$ is almost malnormal in G_ν, for any vertex ν.

Then G is locally relatively quasiconvex relative to $\mathbb{P}$, see Corollary 3.1.4.

4.3 Small-hierarchies and Local Relative Quasiconvexity

The main result in this section is a consequence of Theorem 4.2.1. We first prove that when a relatively hyperbolic group G splits then relative quasiconvexity of vertex groups is equivalent to relative quasiconvexity of the edge groups.

We recall the following observation of Bowditch (see [MPW11, Lem 2.7 and 2.9]).

Lemma 4.3.1 (G-attachment)**.** *Let G act on a graph K. Let $p, q \in K^0$ and e be a new edge whose endpoints are p and q. The G-*attachment *of e is the new graph $K' = K \cup Ge$ which consists of the union of K and copies ge of e attached at gp and gq for any $g \in G$. Note that K' is G-cocompact/fine/hyperbolic if K is.*

Lemma 4.3.2 (Quasiconvex Edges $\Longleftrightarrow$ Quasiconvex Vertices)**.** *Let G be hyperbolic relative to $\mathbb{P}$. Suppose G splits as a finite graph of groups whose vertex groups and edge groups are finitely generated. Then the edge groups are quasiconvex relative to $\mathbb{P}$ if and only if the vertex groups are quasiconvex relative to $\mathbb{P}$.*

Proof. If the vertex groups are quasiconvex relative to $\mathbb{P}$ then so are the edge groups, since relative quasiconvexity is preserved by intersection (see [Hru10, MP09]) in the finitely generated group G. Assume the edge groups are quasiconvex relative to $\mathbb{P}$. Let K be a $(G; \mathbb{P})$ graph and let T be the Bass-Serre tree for G. Let $f : K \to T$ be a G-equivariant map that sends vertices to vertices and edges to geodesics. Subdivide K and T, so that each edge is the union of two length $\frac{1}{2}$ *halfedges*. Let ν be a vertex in T. It suffices to find a G_ν-cocompact quasiconvex subgraph L of K.

Let $\{e_1, \ldots, e_m\}$ be representatives of the G_ν-orbits of halfedges attached to ν. Let ω_i be the other vertex of e_i for $1 \leq i \leq m$. Since each $G_{\omega_i} = G_{e_i}$ is finitely generated by hypothesis, we can perform finitely many G_{ω_i}-attachments of arcs so that the preimage of ω_i is connected for each i. This leads to finitely many G-attachments to K to obtain a new fine hyperbolic graph K'. By mapping the newly attached edges to their associated vertices in T, we thus obtain a G-equivariant map $f' : K' \to T$ such that $M_i' = f'^{-1}(\omega_i)$ is connected and G_{ω_i}-cocompact for each i.

Consider $L' = f'^{-1}(N_{\frac{1}{2}}(\nu))$ where $N_{\frac{1}{2}}(\nu)$ is the closed $\frac{1}{2}$-neighborhood of ν. To see that L' is connected, consider a path σ in K' between distinct components of L'. Moreover choose σ so that its image in T is minimal among all such choices. Then σ must leave and enter L' through the same $g_\nu M_i'$ which is connected by construction.

We now show that L' is quasiconvex. Consider a geodesic γ that intersects L' exactly at its endpoints. As before the endpoints of γ lie in the same $g_\nu M_i'$. Since $g_\nu M_i'$ is κ_i-quasiconvex for some κ_i, we see that γ lies in κ-neighborhood of $g_\nu M_i'$ and hence in the κ-neighborhood of L'. □

Definition 12 (Small-Hierarchy)**.** A group is *small* if it has no rank 2 free subgroup. Any small group has a *length 0 small-hierarchy*. G has a *length n small-hierarchy* if $G \cong A *_C B$ or $G \cong A*_{C^t=C'}$, where A and B have length $(n-1)$ small-hierarchies, and C is small and finitely generated.

We say G has a *small-hierarchy* if it has a length n small-hierarchy for some n.

We can define $\mathcal{F}$-hierarchy by replacing "small" by a class of groups $\mathcal{F}$ closed under subgroups and isomorphisms. For instance, when $\mathcal{F}$ is the class of finite groups, the class of groups with an $\mathcal{F}$-hierarchy is precisely the class of virtually free groups. Other notable classes are the groups with a *Noetherian hierarchy*.

We will employ the following Theorem of Yang [Yan11] in which relative quasiconvexity has been characterized with respect to extensions:

Theorem 4.3.3 (Quasiconvexity in Extended Peripheral Structure)**.** *Let G be hyperbolic relative to $\mathbb{P}$ and relative to $\mathbb{E}$. Suppose that $\mathbb{E}$ extends $\mathbb{P}$. Then*

1. *If $H \leq G$ is quasiconvex relative to $\mathbb{P}$, then H is quasiconvex relative to $\mathbb{E}$.*
2. *Conversely, if $H \leq G$ is quasiconvex relative to $\mathbb{E}$, then H is quasiconvex relative to $\mathbb{P}$ if and only if $H \cap E^g$ is quasiconvex relative to $\mathbb{P}$ for all $g \in G$ and $E \in \mathbb{E}$.*

Remark 4.3.4. The *Tits alternative* for relatively hyperbolic groups states that every finitely generated subgroup is either: elementary, parabolic, or contains a subgroup isomorphic to F_2. The Tits alternative is proven for countable relatively hyperbolic groups in [Gro87, Thm 8.2.F]. A proof is given for convergence groups in [Tuk96]. It is shown in [Osi06c] that every cyclic subgroup H of a finitely generated relatively hyperbolic group G is relatively quasiconvex.

Theorem 4.3.5. *Let G be finitely generated and hyperbolic relative to $\mathbb{P}$ where each element of $\mathbb{P}$ is Noetherian. Suppose G has a small-hierarchy. Then G is locally relatively quasiconvex.*

Proof. The proof is by induction on the length of the hierarchy. Since edge groups are finitely generated, the Tits alternative shows that there are three cases according to whether the edge group is finite, virtually cyclic, or infinite parabolic, and we note that the edge group is relatively quasiconvex in each case. These three cases are each divided into two subcases according to whether $G = A *_{C_1} B$ or $G = A*_{C_1^t=C_2}$.

Since C_1 and G are finitely generated the vertex groups are finitely generated by Lemma 4.1.1. Thus, since C_1 is relatively quasiconvex the vertex groups are relatively quasiconvex by Lemma 4.3.2.

When C_1 is finite the conclusion follows in each subcase from Theorem 4.2.1.

When C_1 is virtually cyclic but not parabolic, then C_1 lies in a unique maximal virtually cyclic subgroup Z that is almost malnormal and relatively quasiconvex by [Osi06b]. Thus G is hyperbolic relative to $\mathbb{P}' = \mathbb{P} \cup \{Z\}$ by Theorem 6.3.4.

Observe that C_1 is maximal infinite cyclic on at least one side, since otherwise there would be a nontrivial splitting of Z as an Amalgamated free product over C_1. We equip the (relatively quasiconvex) vertex groups with their induced peripheral structures. Note that C_1 is maximal parabolic on at least one side and so G is locally relatively quasiconvex relative to $\mathbb{P}'$ by Theorem 4.2.1. Finally, by Theorem 4.3.3, any subgroup H is quasiconvex relative to the original peripheral structure $\mathbb{P}$ since intersections between H and conjugates of Z are quasiconvex relative to $\mathbb{P}$.

When C_1 is infinite parabolic, we will first produce a new splitting before verifying local relative quasiconvexity.

When $G = A *_{C_1} B$. Let D_a, D_b be the maximal parabolic subgroups of A, B containing C_1, and refine the splitting to:

$$A *_{D_a} (D_a *_{C_1} D_b) *_{D_b} B$$

The two outer splittings are along a parabolic that is maximal on the outside vertex group. The inner vertex group $D_a *_{C_1} D_b$ is a single parabolic subgroup of G. Indeed, as C_1 is infinite, $D_a \supset C_1 \subset D_b$ must all lie in the same parabolic subgroup of G. It is obvious that $D_a *_{C_1} D_b$ is locally relatively quasiconvex with respect to its induced peripheral structure since it is itself parabolic in G. Consequently $(D_a *_{C_1} D_b) *_{D_b} B$ is locally relatively quasiconvex by Theorem 4.2.1, therefore $G = A *_{D_a} \big((D_a *_{C_1} D_b) *_{D_b} B\big)$ is locally relatively quasiconvex by Theorem 4.2.1.

When $G \cong A*_{{C_1}^t = C_2}$, let M_i be the maximal parabolic subgroup of G containing C_i. There are two subsubcases:

$\big[t \in M_1\big]$ Then $C_2 \leq M_1$ and

we revise the splitting to $G \cong A *_{D_1} M_1$ where $D_1 = M_1 \cap A$. And in this splitting the edge group is maximal parabolic at $D_1 \subset A$, and M_1 is parabolic.

$\big[t \notin M_1\big]$

Let D_i denote the maximal parabolic subgroup of A containing C_i. Observe that $\{D_1, D_2\}$ is almost malnormal since $D_i = M_i \cap A$. We revise the HNN extension to the following:

$$\left(D_1^t *_{C_1^t = C_2} A\right) *_{D_1{}^t = D_1}$$

where the conjugated copies of D_1 in the HNN extension embed in the first and second factor of the Amalgamated Free Product.

In both cases, the local relative quasiconvexity of G now holds by Theorem 4.2.1 as before. □

We obtain the the following corollary which was proved first by Dahmani [Dah03].

Corollary 4.3.6. *Every limit group is locally relatively quasiconvex.*

CHAPTER 5
Relative Quasiconvexity in Graphs of Groups

This chapter contains theorems concerning relative quasiconvexity of a subgroup of a relatively hyperbolic group that splits. As an application, we generalize local relative quasiconvexity results in the previous chapter.

5.1 Quasiconvexity Criterion for Relatively Hyperbolic Groups that Split

Lemma 5.1.1 (Total Edges $\Longleftrightarrow$ Total Vertices)**.** *Let G be hyperbolic relative to $\mathbb{P}$. Let G act on a tree T. For each $P \in \mathbb{P}$ let T_P be a minimal P-subtree. Assume that no T_P has a finite edge stabilizer in the P-action. Then edge groups of T are total in G iff vertex groups are total in G.*

Proof. Since the intersection of two total subgroups is total, if the vertex groups are total then the edge groups are also total. We now assume that the edge groups are total. Let G_ν be a vertex group and $P \in \mathbb{P}$ such that $P^g \cap G_\nu$ is infinite for some $g \in G$. If $|P^g \cap G_e| = \infty$ for some edge e attached to ν, then $P \subseteq G_e$, thus $P \subseteq G_e \subseteq G_\nu$. Now suppose that $|P^g \cap G_e| < \infty$ for each e attached to ν. If $P^g \nsubseteq G_\nu$ then the action of P^g on gT violates our hypothesis. $\square$

Remark 5.1.2. Suppose G is finitely generated and G is hyperbolic relative to $\mathbb{P}$. Let $P \in \mathbb{P}$ such that $P = A *_C B$ [$P = A*_{C=C'^t}$] where C is a finite group. Since P is hyperbolic relative to $\{A, B\}$ [$\{A\}$], by Lemma 6.4.1, G is hyperbolic relative to $\mathbb{P}' = \mathbb{P} - \{P\} \cup \{A, B\}$ [$\mathbb{P}' = \mathbb{P} - \{P\} \cup \{A\}$].

We now describe a more general criterion for relative quasiconvexity which is proven by combining Corollary 4.1.3 with Theorem 4.3.3.

Theorem 5.1.3. *Let G be finitely generated and hyperbolic relative to $\mathbb{P}$. Suppose G splits as a finite graph of groups. Suppose*

(a) Each G_e is total in G;

(b) Each G_e is relatively quasiconvex in G;

(c) $\{G_e : e$ is attached to $\nu\}$ is almost malnormal in G_ν for each vertex ν.

Let $H \leq G$ be tamely generated subgroup of G. Then H is relatively quasiconvex in G.

Proof. Technical Point: By splitting certain elements of $\mathbb{P}$ to obtain $\mathbb{P}'$ as in Remark 5.1.2, we can assume that G is hyperbolic relative to $\mathbb{P}'$ and each G_ν is hyperbolic relative to the conjugates of elements of $\mathbb{P}'$ that it contains.

Indeed for any $P \in \mathbb{P}$, if the action of P on a minimal subtree T_P of the Bass-Serre tree T, yields a finite graph Γ of groups some of whose edge groups are finite, then following Remark 5.1.2, we can replace $\mathbb{P}$ by the groups that complement these finite edge groups, (i.e. the fundamental groups of the subgraphs obtained by deleting these edges from Γ.) Therefore G is hyperbolic relative to $\mathbb{P}'$.

No $P \in \mathbb{P}'$ has a nontrivial induced splitting as a graph of groups with a finite edge group. The edge groups are total relative to $\mathbb{P}'$ since they are total relative to $\mathbb{P}$. Therefore by Lemma 5.1.1 the vertex groups are total in G relative to $\mathbb{P}'$. By Lemma 4.3.2, each vertex group G_ν is relatively quasiconvex in G relative to $\mathbb{P}$, therefore by Theorem 4.3.3 each G_ν is quasiconvex in G relative to $\mathbb{P}'$. Thus G_ν has an induced relatively hyperbolic structure $\mathbb{P}'_\nu$ as in Remark 2.2.1. By totality of G_ν, we can assume each element of $\mathbb{P}'_\nu$ is a conjugate of an element of $\mathbb{P}'$. And as usual we may omit the finite subgroups in $\mathbb{P}'_\nu$.

Step 1: We now extend the peripheral structure of each G_ν from $\mathbb{P}'_\nu$ to $\mathbb{E}_\nu$ where

$$\mathbb{E}_\nu = \{G_e : e \text{ is attached to } \nu\} \cup \{P \in \mathbb{P}'_\nu : P^g \not\leq G_e \text{ for any } g \in G_\nu\}$$

Almost malnormality of $\mathbb{E}_\nu$ follows from Condition (c) and the totality of the edge groups in their vertex groups which follows by the totality of the edge groups in G, also relative quasiconvexity of the new elements G_e is Condition (b). Thus G_ν is hyperbolic relative to $\mathbb{E}_\nu$ by Theorem 4.3.3.

Step 2: For each $\tilde{\nu}$ in the Bass-Serre tree, its H-stabilizer $H_{\tilde{\nu}}$ lies in $G_{\tilde{\nu}}$ which we identify (by a conjugacy isomorphism) with the chosen vertex stabilizer G_ν in the graph of group decomposition. Then $H_{\tilde{\nu}}$ is quasiconvex in G_ν relative to $\mathbb{E}_\nu$ for each ν by Theorem 4.3.3, since $\mathbb{E}_\nu$ extends $\mathbb{P}'_\nu$ and each $H_{\tilde{\nu}}$ is quasiconvex in G_ν relative to $\mathbb{P}'_\nu$. Therefore H is quasiconvex relative to $\bigcup \mathbb{E}_\nu$ by Corollary 4.1.3.

Step 3: H is quasiconvex relative to $\mathbb{P}' = \bigcup \mathbb{P}'_\nu$. Since $\bigcup \mathbb{E}_\nu$ extends $\mathbb{P} = \bigcup \mathbb{P}'_\nu$, by Theorem 4.3.3, it suffices to show that $H \cap K^g$ is quasiconvex relative to $\mathbb{P}'$ for all $K \in \bigcup \mathbb{E}_\nu$ and $g \in G$. There are two cases:

Case 1: $K \in \mathbb{P}'_\nu$ for some ν. Now $H \cap K^g$ is a parabolic subgroup of G relative to $\mathbb{P}'$ and is thus quasiconvex relative to $\mathbb{P}'$.

Case 2: $K = G_e$ for some e attached to some ν. The group K is relatively quasiconvex in G_ν, therefore by Remark 2.2.1, K^g is also relatively quasiconvex but in $G_{g\nu}$. Now since $K^g \cap H = K^g \cap H_{g\nu}$ and K^g and $H_{g\nu}$ are both relatively quasiconvex in $G_{g\nu}$, the group $K^g \cap H$ is relatively quasiconvex in $G_{g\nu}$. Since by Lemma 4.3.2,

$G_{g\nu}$ is quasiconvex relative to $\mathbb{P}'$, Lemma 2.2.2 implies that $K^g \cap H$ is quasiconvex relative to $\mathbb{P}'$.

Now H is quasiconvex relative to $\mathbb{P}$ by Theorem 4.3.3, since $\mathbb{P}$ extends $\mathbb{P}'$. $\square$

The following result strengthens Theorem 5.1.3, by relaxing Condition (c).

Theorem 5.1.4 (Quasiconvexity Criterion for Relatively Hyperbolic Groups that Split)**.** *Let G be finitely generated and hyperbolic relative to $\mathbb{P}$ such that G splits as a finite graph of groups. Suppose*

(a) Each G_e is total in G;

(b) Each G_e is relatively quasiconvex in G;

(c) Each G_e is almost malnormal in G.

Let $H \leq G$ be tamely generated. Then H is relatively quasiconvex in G.

Remark 5.1.5. By Lemma 2.2.2 and Remark 2.2.3, Condition (b) is equivalent to requiring that each G_e is quasiconvex in G_ν. Also we can replace Condition (a) by requiring G_e to be total in G_ν.

Proof. We prove the result by induction on the number of edges of the graph of groups Γ. The base case where Γ has no edge is contained in the hypothesis. Suppose that Γ has at least one edge e (regarded as an open edge). If e is nonseparating, then $G = A*_{C^t=D}$ where A is the graph of groups over $\Gamma - e$, and C, D are the two images of G_e. Condition (c) ensures that $\{C, D\}$ is almost malnormal in A, and by induction, the various nontrivial intersections $H \cap A^g$ are relatively quasiconvex in A^g, and thus H is relatively quasiconvex in G by Theorem 5.1.3. A similar argument concludes the separating case. $\square$

Corollary 5.1.6. *Let G be finitely generated and hyperbolic relative to $\mathbb{P}$. Suppose G splits as a finite graph of groups. Assume:*

(a) Each G_ν is locally relatively quasiconvex;

(b) Each G_e is Noetherian, total and relatively quasiconvex in G;

(c) Each G_e is almost malnormal in G.

Then G is locally relatively quasiconvex relative to $\mathbb{P}$.

5.2 Some Applications

Theorem 5.2.1. *Let G be hyperbolic relative to $\mathbb{P}$. Suppose G splits as a graph Γ of groups with relatively quasiconvex edge groups. Suppose Γ is bipartite with $\Gamma^0 = V \sqcup U$ and each edge joins vertices of V and U. Suppose each G_v is maximal parabolic for $v \in V$, and for each $P \in \mathbb{P}$ there is at most one v with P conjugate to G_v. Let $H \leq G$ be tamely generated. Then H is quasiconvex relative to $\mathbb{P}$.*

The scenario of Theorem 5.2.1 arises when M is a compact aspherical 3-manifold, from its JSJ decomposition. The manifold M decomposes as a bipartite graph Γ of spaces with $\Gamma^0 = U \sqcup V$. The submanifold M_v is hyperbolic for each $v \in V$, and M_u is a graph manifold for each $u \in U$. The edges of Γ correspond to the "transitional tori" between these hyperbolic and complementary graph manifold parts. Some of the graph manifolds are complex but others are simpler Seifert fibered spaces; in the simplest cases, thickened tori between adjacent hyperbolic parts or I-bundles over Klein bottles where a hyperbolic part terminates. Hence $\pi_1 M$ decomposes accordingly as a graph Γ of groups, and $\pi_1 M$ is hyperbolic relative to $\{\pi_1 M_u : u \in U\}$ by Theorem 3.1.2 or indeed, Corollary 3.1.3.

Proof. Let K_o be a fine hyperbolic graph for G. Each vertex group is quasiconvex in G by Lemma 4.3.2, and so for each $u \in U$ let K_u be a G_u-quasiconvex subgraph, and in this way we obtain finite hyperbolic G_u-graphs, and for $v \in V$, we let K_v be a singleton. We apply the Construction in the proof of Theorem 3.1.2 to obtain a fine hyperbolic G-graph K and quotient $\bar{K}$. Note that the parabolic trees are i-pods. We form the H-cocompact quasiconvex subgraph L by combining H_ω-cocompact quasiconvex subgraphs K_ω as in the proof of Theorem 4.1.2. $\square$

Theorem 5.2.2. *Let G be finitely generated and hyperbolic relative to $\mathbb{P}$. Suppose G splits as graph Γ of groups with relatively quasiconvex edge groups. Suppose Γ is bipartite with $\Gamma^0 = V \sqcup U$ and each edge joins vertices of V and U. Suppose each G_v is almost malnormal and total in G for $v \in V$.*

Let $H \leq G$ be tamely generated. Then H is quasiconvex relative to $\mathbb{P}$.

Theorem 5.2.2 covers the case where edge groups are almost malnormal on both sides since we can subdivide to put barycenters of edges in V.

Another special case where Theorem 5.2.2 applies is where $G = G_1 *_{{C_1}'=C_2} G_2$ is hyperbolic relative to $\mathbb{P}$, and $C_2 \leq G_2$ is total and relatively quasiconvex in G and almost malnormal in G_2.

Proof. Following the Technical Point in the proof of Theorem 5.1.3, by splitting certain elements of $\mathbb{P}$ to obtain $\mathbb{P}'$ as in Remark 5.1.2, we can assume that G is hyperbolic relative to $\mathbb{P}'$ where each $P' \in \mathbb{P}'$ is elliptic with respect to the action of G on the Bass-Serre tree T. Since $\mathbb{P}$ extends $\mathbb{P}'$ and each $G_v \cap P^g$ is conjugate to an element of $\mathbb{P}'$, we see that each G_v is quasiconvex in G relative to $\mathbb{P}'$ by Theorem 4.3.3, and moreover, since elements of $\mathbb{P}'$ are vertex groups of elements of $\mathbb{P}$, each G_v is total relative to $\mathbb{P}'$. Therefore each G_v is hyperbolic relative to a collection $\mathbb{P}'_v$ of conjugates of elements of $\mathbb{P}'$.

We argue by induction on the number of edges of Γ. If Γ has no edge the result is contained in the hypothesis. Suppose Γ has at least one edge e. If e is separating and $\Gamma = \Gamma_1 \sqcup e \sqcup \Gamma_2$ where e attaches $v \in \Gamma_1^0$ to $u \in \Gamma_2^0$ then $G = G_1 *_{G_e} G_2$ where $G_i = \pi_1(\Gamma_i)$. Each G_e is the intersection of vertex groups and hence quasiconvex relative to $\mathbb{P}'$. By Lemma 4.3.2, the groups G_1 and G_2 are quasiconvex in G relative to $\mathbb{P}'$. Thus G_i is hyperbolic relative to $\mathbb{P}'_i$ by Remark 2.2.1.

Observe that T contains subtrees T_1 and T_2 that are the Bass-Serre trees of Γ_1 and Γ_2, and $T - G\tilde{e} = \{gT_1 \cup gT_2 \ : \ g \in G\}$. The Bass-Serre tree $\bar{T}$ of $G_1 *_{G_e} G_2$ is the quotient of T obtained by identifying each gT_i to a vertex.

Since H is relatively finitely generated, there is a finite graph of groups Γ_H for H, and a map $\Gamma_H \to \Gamma$. Removing the edges mapping to e from Γ_H, we obtain a collection of finitely many graphs of groups - some over Γ_1 and some over Γ_2. Each component of Γ_H corresponds to the stabilizer of some gT_i and is denoted by H_{gT_i}, and since that component is a finite graph with relatively quasiconvex vertex stabilizers, we see that each H_{gT_i} is relatively quasiconvex in G_i relative to $\mathbb{P}'_i$ by induction on the number of edges of Γ_H.

We extend the peripheral structure $\mathbb{P}'_1$ of G_1 to $\mathbb{E}_1 = \{G_1\}$. Note that now each H_{gT_1} is quasiconvex in G_1 relative to $\mathbb{E}_1$ by Theorem 4.3.3. Let

$$\mathbb{E} = \mathbb{E}_1 \cup \mathbb{P}'_2 - \{P \in \mathbb{P}'_2 \colon P^g \leq G_e, \text{ for some } g \in G_2\}.$$

Observe that $\mathbb{E}$ extends $\mathbb{P}'$. Since G_v is total and quasiconvex in G relative to $\mathbb{P}'$ and $\mathbb{E}$ extends $\mathbb{P}'$, the group G_1 is total and quasiconvex in G relative to $\mathbb{E}$ by Theorem 4.3.3. Therefore G is hyperbolic relative to $\mathbb{E}$ by Theorem 6.3.4.

Since G_1 is maximal parabolic in G, by Theorem 5.2.1 H is quasiconvex in G relative to $\mathbb{E}$. The graph Γ_H shows that H is generated by finitely many hyperbolic elements and vertex stabilizers $H_{g\bar{T}_i}$ and each $H_{g\bar{T}_i} = H_{gT_i}$ which we explained above is relatively quasiconvex in G_i.

We now show that H is quasiconvex relative to $\mathbb{P}'$ and therefore relative to $\mathbb{P}$ by Theorem 4.3.3. Since $\mathbb{E}$ extends $\mathbb{P}'$, by Theorem 4.3.3, it suffices to show that $H \cap E^g$ is quasiconvex relative to $\mathbb{P}'$ for all $E \in \mathbb{E}$ and $g \in G$. There are two cases:

Case 1: $E \in \mathbb{P}'_2$. Now $H \cap E^g$ is a parabolic subgroup of G relative to $\mathbb{P}'$ and is thus quasiconvex relative to $\mathbb{P}'$.

Case 2: $E = G_1$. Then $H \cap E^g$ is quasiconvex relative to $\mathbb{P}'_1$ since $(H \cap E^g) = H_{gT_1}$ is quasiconvex in G_1^g relative to $\mathbb{E}_1^g = \{G_1^g\}$. Since $E^g = G_1^g$ is quasiconvex relative to $\mathbb{P}'$, Lemma 2.2.2 implies that $H \cap E^g$ is quasiconvex relative to $\mathbb{P}'$.

Now assume that e is nonseparating. Let $u \in U$ and $v \in V$ be the endpoints of e. Then $G = G_1 *_{C^t = D}$ where G_1 is the graph of groups over $\Gamma - e$, and C and D are the images of G_e in G_v and G_u respectively. We first reduce the peripheral structure of G from $\mathbb{P}$ to $\mathbb{P}'$, and we then extend from $\mathbb{P}'$ to $\mathbb{E}$ with:

$$\mathbb{E} = \{G_v\} \cup \mathbb{P}' - \{P \in \mathbb{P}' \colon P^g \leq G_v, \text{ for some } g \in G\}.$$

G is hyperbolic relative to $\mathbb{E}$ by Theorem 6.3.4 as G_v is almost malnormal, total, and quasiconvex relative to $\mathbb{P}$. The argument follows by induction and Theorem 5.2.1 as in the separating case. $\square$

Theorem 5.1.4 suggests the following criterion for relative quasiconvexity:

Conjecture 5.2.3. *Let G be hyperbolic relative to $\mathbb{P}$. Suppose G splits as a finite graph of groups with finitely generated relatively quasiconvex edge groups. Suppose $H \leq G$ is tamely generated such that each H_v is finitely generated for each v in the Bass-Serre tree. Then H is relatively quasiconvex in G.*

When the edge groups are separable in G, there is a finite index subgroup G' whose splitting has relatively malnormal edge groups (see e.g. [HW09, HWa]). Consequently, if moreover, the edge groups of G are total, then the induced splitting of G' satisfies the criterion of Theorem 5.1.4, and we see that Conjecture 5.2.3 holds in this case. In particular, Conjecture 5.2.3 holds when G is virtually special and hyperbolic relative to virtually abelian subgroups, provided that edge groups are also total. We suspect the totalness assumption can be dropped totally.

Consider a hyperbolic 3-manifold M virtually having a malnormal quasiconvex hierarchy (conjecturally all closed M). Theorem 5.1.4 suggests an alternate approach to the tameness theorem, which could be reproven by verifying:

If the intersection of a finitely generated H with a malnormal quasiconvex edge group is infinitely generated then H is a virtual fiber.

The following consequence of Corollary 3.1.3 is a natural consequence of the viewpoint developed here.

Corollary 5.2.4. *Let M be a compact irreducible 3-manifold. And let $M_1, \ldots, M_r$ denote the graph manifolds obtained by removing each (open) hyperbolic piece in the geometric decomposition of M. Then $\pi_1 M$ is hyperbolic relative to $\{\pi_1 M_1, \ldots, \pi_1 M_r\}$.*

The relative hyperbolicity of $\pi_1(M)$ was previously proved by Drutu-Sapir using work of Kapovich-Leeb. This previous proof is deep as it uses the structure of the asymptotic cone due to Kapovich-Leeb together with the technical proof of Drutu-Sapir that asymptotically tree graded groups are relatively hyperbolic [KL95, DS05].

CHAPTER 6
Free Subgroups and A Non-quasiconvex Embedding of Relatively Hyperbolic Groups

In this chapter, we first give a criterion for a path in a δ-hyperbolic space to be a quasigeodesic. Our argument extracts the hyperbolic case of a relative hyperbolic argument given in [HWb]. Then we prove a variant result of Arzhantseva [Arz00] which implies that given finitely many infinite index subgroups of a free group F, "generically" subgroups of F avoid conjugates of these infinite index subgroups. Using this, we provide a generic statement for free subgroups of relatively hyperbolic groups. As an application, we prove a non-quasiconvex embedding theorem for relatively hyperbolic groups which generalizes a result of Kapovich in [Kap99].

6.1 Quasigeodesics in hyperbolic spaces

6.1.1 A Quasigeodesic Criterion in Hyperbolic Spaces

Definition 13 (Quasigeodesic)**.** Let $[a, b]$ be a real interval and let $\lambda > 0$ and $\epsilon \geq 0$. A (λ, ϵ)-*quasigeodesic* in a metric space (X, d) is a function $\alpha : [a, b] \to X$ for all $s, t \in [a, b]$ such that

$$d(s, t) \leq \lambda d(\alpha(s), \alpha(t)) + \epsilon.$$

Recall that two paths α and β in a metric space X, ϵ-fellow travel for a distance $\geq c$ if the length of the part of α lying in $N_\epsilon(\beta)$ is $\geq c$.

Lemma 6.1.1. *Let X be a δ-hyperbolic space. Let τ_1, σ_2, τ_2, σ_3, τ_3,... be a sequence of concatenatable geodesics. Let γ_k be a geodesic with the same endpoints of concatenation $\tau_1\sigma_2\tau_2\sigma_3\ldots\sigma_k\tau_k$. Suppose that there exists c with the following properties:*

1. *$|\tau_i| > 4c$ for $2 \leq i \leq k-1$;*
2. *τ_i, τ_{i+1} can not 3δ-fellow travel for a distance $\geq c$;*
3. *σ_i, τ_i and σ_{i+1}, τ_i can not 2δ-fellow travel for a distance $\geq c$.*

Then

1. *$|\gamma_{k+1}| \geq |\gamma_k| + |\sigma_{k+1}| + |\tau_{k+1}| - (6c + 2\delta)$;*
2. *The terminal subpaths of γ_k and τ_k, δ-fellow travel for a distance of at least $|\tau_k| - 2c$.*

Proof. We prove the result by induction on k. When $k = 1$, it is obvious. For each $k \geq 1$, let ω_k be a geodesic with the same endpoints as $\tau_1\sigma_2\tau_2\sigma_3\ldots\sigma_k$. Consider two δ-thin geodesic triangles $\gamma_k\sigma_{k+1}\omega_{k+1}$ and $\omega_{k+1}\tau_{k+1}\gamma_{k+1}$. Now consider the map to tripods with δ-diameter fibers, and draw the geodesic triangles with sides $\leq \delta$. There are two cases according to the position of the points of these triangles on ω_{k+1}. The first case is illustrated in Figure 6–1-(II) and the second case in Figure 6–2.

The notion $[i, j]$ is for the geodesic between points denoted by numbers i and j in the following figures and $|i, j|$ is the length of $[i, j]$.

First note that in both cases $4 \in [2, 5]$, otherwise if $4 \in [1, 2]$ by considering the point 13 which is the comparison point of 2 on σ_{k+1} in Figure 6–1-(I), since the assumption (3) holds for σ_{k+1}, τ_k, we have $|3, 5| < c$. Also by induction the terminal subpaths of γ_k and τ_k, δ-fellow travel for a distance of at least $|\tau_k| - 2c$, therefore

$|3,5| \geq |\tau_k| - 2c$. Combining these two facts, $|\tau_k| < 3c$ which is contradiction with assumption (1).

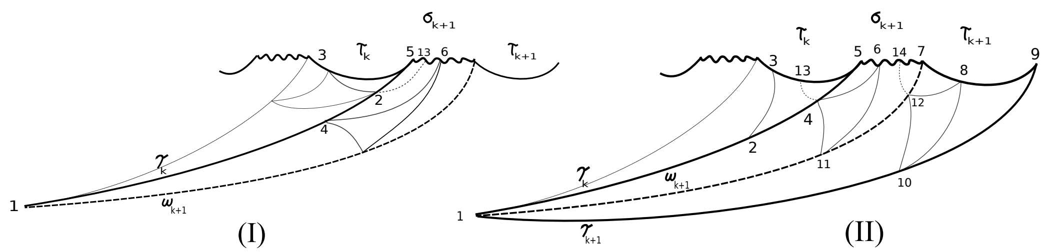

Figure 6–1:

We consider the first case. Let the point 13 be the comparison point of 4 on τ_k and the point 14 be the comparison point of 12 on σ_{k+1} and draw the geodesics $[4,13]$ and $[12,14]$.

Since by assumption (3), τ_k and σ_{k+1} 2δ-fellow travel for a distance less than c, we have $|13,5| = |4,5| = |5,6| < c$. By similar argument $|14,7| = |7,12| = |7,8| < c$. Now

$$\begin{aligned} |\omega_{k+1}| &= |1,11| + |11,12| + |12,7| = |1,4| + |11,12| + |14,7| \\ &\geq |1,4| + (|6,14| - 2\delta) + |14,7| + (|4,5| - c) + (|5,6| - c) \\ &= |1,5| + |5,7| - 2\delta - 2c = |\gamma_k| + |\sigma_{k+1}| - 2\delta - 2c \end{aligned}$$

$$
\begin{aligned}
|\gamma_{k+1}| &= |1,10| + |10,9| = |1,12| + |8,9| = |1,11| + |11,12| + |8,9| \\
&\geq |1,11| + |11,12| + |8,9| + (|12,7| - c) + (|7,8| - c) \\
&= |1,7| + |7,9| - 2c = |\omega_{k+1}| + |\tau_{k+1}| - 2c \\
&\geq |\gamma_k| + |\sigma_{k+1}| + |\tau_{k+1}| - 4c - 2\delta \geq |\gamma_k| + |\sigma_{k+1}| + |\tau_{k+1}| - 6c - 2\delta
\end{aligned}
$$

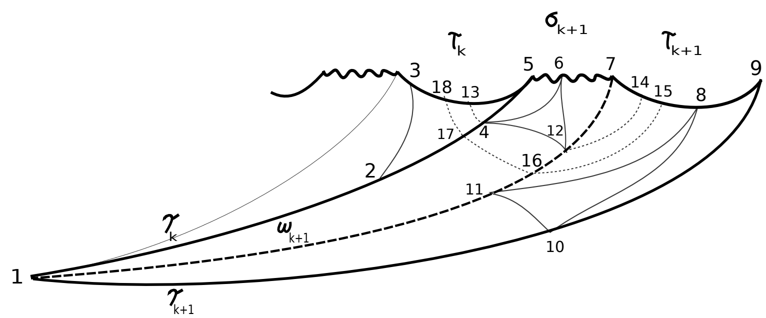

Figure 6–2:

We now consider the second case. Note that the point 13 is the comparison point of 4 on τ_k and 14 is the comparison point of 12 on τ_{k+1}. Assumption (3), implies that $|13,5| = |4,5| = |5,6| < c$, by similar argument $|6,7| = |7,12| = |7,14| < c$.

First we prove $|7,8|$ is bounded by showing that $|7,8| < 2c$. Assume that $|14,8| \geq c$, then we consider the point $15 \in [14,8]$ such that $|14,15| = c$. Consider comparison points $16, 17$ and 18 of 15 and geodesics between them. Note that $18 \in [3,13]$, since by induction $|3,5| > 2c$. Now since $|13,18| = |14,8| = c$ and by assumption (2), τ_k and τ_{k+1} can not 3δ-fellow travel for a distance $\geq c$, we reach a contradiction. Therefore $|14,8| < c$ which implies $|7,8| < 2c$. Now we have:

$$
\begin{aligned}
|\omega_{k+1}| &= |1,12| + |12,7| = |1,4| + |6,7| \\
&\geq |1,4| + |6,7| + (|4,5| - c) + (|5,6| - c) \\
&= |1,5| + |5,7| - 2c = |\gamma_k| + |\sigma_{k+1}| - 2c
\end{aligned}
$$

$$
\begin{aligned}
|\gamma_{k+1}| &= |1,10| + |10,9| = |1,11| + |8,9| \\
&\geq |1,11| + |8,9| + (|11,12| - c) + (|14,8| - c) + (|12,7| - c) + (|7,14| - c) \\
&= |1,7| + |1,9| - 4c = |\omega_{k+1}| + |\tau_{k+1}| - 4c \\
&\geq |\gamma_k| + |\sigma_{k+1}| + |\tau_{k+1}| - 6c \geq |\gamma_k| + |\sigma_{k+1}| + |\tau_{k+1}| - 6c - 2\delta
\end{aligned}
$$

□

The following Lemma provides a criterion for a path in a δ-hyperbolic space to be a quasigeodesic.

Proposition 6.1.2 (Quasigeodesic Criterion)**.** *Let X be a δ-hyperbolic space. Let*

$$\gamma = \sigma_1\tau_1\sigma_2\tau_2\ldots\sigma_n\tau_n\sigma_{n+1}$$

be a piecewise geodesic path. Suppose that:

1. $|\tau_i| > 2(6c + 2\delta)$ *for* $i \geq 1$*;*
2. τ_i, τ_{i+1} *can not* 3δ*-fellow travel for a distance* $\geq c$*;*
3. σ_i, τ_i *and* σ_{i+1}, τ_i *can not* 2δ*-fellow travel for a distance* $\geq c$.

Then γ is quasigeodesic. Indeed it is $(2,0)$-quasigeodesic.

Proof. Let γ' be a subpath of γ and let λ be a geodesic with the same endpoints as γ'. We show that $|\gamma'| \leq 2|\lambda|$. First note that any subpath γ' of γ can be expressed in the following form,

$$\gamma' = \overline{\tau_{i+1}}\sigma_{i+2}\tau_{i+2}\sigma_{i+3}\tau_{i+3}\ldots\tau_{i+k-1}\sigma_{i+k}\overline{\tau_{i+k}}$$

where $\overline{\tau_{i+1}}$ and $\overline{\tau_{i+k}}$ are respectively subpathes of τ_{i+1} and τ_{i+k}. Note that $\overline{\tau_{i+1}}$, $\overline{\tau_{i+k}}$, σ_{i+2} and σ_{i+k} can be trivial paths. Without loss of generality, we can shift the indices to make computation easier so that

$$\gamma' = \overline{\tau_1}\sigma_2\tau_2\sigma_3\tau_3\ldots\tau_{k-1}\sigma_k\overline{\tau_k}$$

Let λ be a geodesic with the same endpoints as γ'. Since γ' satisfies in hypotheses of Lemma 6.1.1, letting γ_k in the lemma equal to λ and using the hypothesis that $|\tau_i| > 2(6c+2\delta)$ for $2 \leq i \leq k-1$, we have:

$$\begin{aligned}
|\lambda| = |\gamma_k| \quad &\geq \quad \sum_{i=2}^{k-1}|\tau_i| + \sum_{i=2}^{k}|\sigma_i| + |\overline{\tau_1}| + |\overline{\tau_k}| - (k-1)(6c+2\delta) \\
&\geq \quad \sum_{i=2}^{k-1}[|\tau_i| - (6c+2\delta)] + \sum_{i=2}^{k}|\sigma_i| + (6c+2\delta) + |\overline{\tau_1}| + |\overline{\tau_k}| \\
&\geq \quad \frac{1}{2}\sum_{i=2}^{k-1}|\tau_i| + \frac{1}{2}\sum_{i=2}^{k}|\sigma_i| + \frac{1}{2}\left(|\overline{\tau_1}| + |\overline{\tau_k}|\right) = \frac{1}{2}|\gamma'|
\end{aligned}$$

□

6.2 A Generic Property for Subgroups of Free Groups

Definition 14 ([Arz00])**.** Let F_n be a nonabelian free group. Let $\mathsf{N}(n, m, t)$ denote the number of m-tuples $(r_1, \ldots, r_m)$ of cyclically reduced words in F_n such that

$|r_i| \leq t$ for each i. Moreover, let $\mathsf{N}_{\mathcal{P}}(n, m, t)$ be the number of such m-tuples such that $\langle r_1, \ldots, r_m \rangle$ has property $\mathcal{P}$. We say *generically any subgroup of F_n has property $\mathcal{P}$* if

$$\lim_{t \to \infty} \frac{\mathsf{N}_{\mathcal{P}}(n, m, t)}{\mathsf{N}(n, m, t)} = 1$$

The following Proposition is a variant of Arzhantseva's result in [Arz00, Thm 1] and it follows by the same proof. We show the normalizer of a subgroup H of G by $\mathcal{N}(H)$.

Proposition 6.2.1. *Let F_n be a nonabelian free group. Let $H_1, \ldots, H_s$ be finitely generated infinite index subgroups of F_n. Generically, the group generated by randomly chosen words $r_1, \ldots, r_m$ in F_n has the property that $\langle r_1 \ldots, r_m \rangle \cap H_i^f = 1$ for each i and any $f \in F_n$.*

Proof. Let $F_n = \langle x_1, \ldots, x_n \rangle$. Let Γ_i be the labelled directed graph corresponding to the core of the cover associated to H_i. Γ_i is indeed the Stallings reduced folded graph. Let $(r_1, \ldots, r_m)$ be an m-tuple of cyclically reduced words generated by $x_i^{\pm 1}$, $1 \leq i \leq n$ of length $|r_i| \leq t$. By the argument in the proof of [Arz00, Thm 1], we have the following facts:

(**i**) The proportion of all m-tuples $(r_1, \ldots, r_m)$ such that $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_m \rangle$ is not $C'(\frac{1}{6})$ decreases exponentially when $t \to \infty$. So we assume that $\langle r_1, \ldots, r_m \rangle$ satisfies the small-cancellation condition $C'(\frac{1}{6})$.

(**ii**) We can assume that no cyclic shift of any r_i contains a subword of length $\geq \frac{|r_i|}{2}$ which is the label of a path in some Γ_i, $1 \leq i \leq s$. Since the proportion of m-tuples $(r_1, \ldots, r_m)$ such that for at least one k, a subword of length $\geq \frac{|r_k|}{2}$ of r_k, is a label of a reduced path in some Γ_i, decreases exponentially when $t \to \infty$.

Now assuming (i)-(ii), we prove the result. It is enough to show that $\langle r_1, \ldots, r_m\rangle^f \cap H_i = 1$ for any $f \in F_n$ and each i. Assume that $\langle r_1, \ldots, r_m\rangle^f \cap H_i \neq 1$ for some $f \in F_n$ and some i. Then $\mathcal{N}(\langle r_1, \ldots, r_m\rangle^f) \cap H_i \neq 1$. Let $v \in \mathcal{N}(\langle r_1, \ldots, r_m\rangle^f) \cap H_i$. There is a nonempty reduced path in Γ_i whose label is v. Also, $\bar{v} = v\mathcal{N}(\langle r_1, \ldots, r_m\rangle^f)$ is trivial element in the group $G = \frac{F_n}{\mathcal{N}(\langle r_1, \ldots, r_m\rangle^f)}$. Since $G = \frac{F_n}{\mathcal{N}(\langle r_1, \ldots, r_m\rangle^f)}$ satisfies $C'(\frac{1}{6})$, by Theorem 7.2.2, there is some r_j with a subword v_j such that $|v_j| > \frac{|r_j|}{2}$ and, $r_j \cap v$ contains v_j. Indeed, there is a disc diagram D mapping to the standard 2-complex corresponding to G whose boundary map has the label $\bar{v}$ and by Theorem 7.2.2, D contains a i-shell where $i \leq 3$. However, this contradicts (ii). $\square$

Generically a k-tuple in a free group generates a rank k free group which was proved by different people, [Jit02], [AO96], [BMN$^+$10, Thm 3.6, 4.3]. Indeed, we have:

Theorem 6.2.2 ([Jit02], [BMN$^+$10])**.** *Exponentially generically, a k-tuple of elements of F_n generates a malnormal, free subgroup of rank k.*

Combining Proposition 6.2.1, with Theorem , we obtain the following result:

Corollary 6.2.3. *Let F_n be a nonabelian free group. Let $H_1, \ldots, H_s$ be finitely generated infinite index subgroups of F_n. Exponentially generically, the group generated by randomly chosen words $r_1, \ldots, r_m$ in F_n, is the free group of rank m, malnormal in F_n and $\langle r_1 \ldots, r_m\rangle \cap H_i^f = 1$ for each i and any $f \in F_n$.*

6.3 Free Subgroups of Relatively Hyperbolic Groups

This section contains two main theorems. The first one give a generic statement about existence of aparabolic, malnormal and quasiconvex subgroup of a relatively hyperbolic group and the second one provides a non-quasiconvex embedding of relatively hyperbolic groups.

6.3.1 Malnormal and Relatively Quasiconvex, Free Subgroups of Relatively Hyperbolic Groups

Recall that, in a δ-hyperbolic space X, for any hyperbolic element $g \in G$, we call any geodesic with two disjoint endpoints in ∂X which is g-invariant, *axis* of g.

Lemma 6.3.1 (Parabolic or Hyperbolic)**.** *Let G be hyperbolic relative to $\mathbb{P}$ and let K be a $(G;\mathbb{P})$-graph. Then for any infinite order $g \in G$, either g is parabolic or g^m has a quasiconvex axis in K for some m.*

Proof. Let d be the graph metric for K. By [Bow08, lem 2.2, 3.4], either g^m has a quasiconvex axis in K for some m, or some (therefore any) $\langle g \rangle$-orbit is bounded. Now we show that if some $\langle g \rangle$-orbit is bounded then g is parabolic. Let $x \in K^0$ and assume that $gx \neq x$. Let γ be a geodesic between x and gx and let e be an edge in γ. Now consider geodesic triangles $\triangle_n$ where for each n the vertices of $\triangle_n$ are x, gx and $g^n x$. Since the set $\{g^n x \mid n \geq 1\}$ is bounded and K is fine and for each n the triangle $\triangle_n$ contains e, the cardinality of the set $\{\triangle_n \mid n \geq 1\}$ is finite. Therefore there are $i > j$ such that $g^i x = g^j x$ which implies $g^{i-j} x = x$. Therefore $gx = x$ otherwise if stab(x)=P then $g^{i-j} \in P \cap P^{g^{i-j}}$ but g has infinite order which is contradiction since by Lemma 2.1.2, the intersection of two parabolic subgroups is finite.. □

Let H be a subgroup of a group G. The commensurator subgroup of H is $Comm(H) = \{g \mid [H : H \cap H^g] < \infty\}$. The following Lemma was proved by Kapovich in [Kap99] in the special case when G is hyperbolic.

Lemma 6.3.2 (Commensurator)**.** *Let G be torsion-free and hyperbolic relative to $\mathbb{P}$. Let F be a rank ≤ 2 free subgroup of G which is relatively quasiconvex and non-parabolic. Then $Comm(F) = F$.*

Proof. Since F is relatively quasiconvex in G, by [HW09, Thm 1.6], F has finite index in $\text{Comm}(F)$. Moreover, since G is torsion-free and F is a free group, $\text{Comm}(F)$ is torsion-free that has a free subgroup of finite index. By the theorem of J.Stallings [Sta68], this implies that $\text{Comm}(F)$ is itself a free group. Now suppose that $\text{Comm}(F) \neq F$, that is the index of F in $\text{Comm}(F)$ is greater than 1. By the Theorem II of [Kur60, ch 17], this implies that the rank of $\text{Comm}(F)$ is strictly less than the rank of F. However the rank of F is ≤ 2 and $F \leq \text{Comm}(F)$ which gives us a contradiction. $\square$

Definition 15 (Aparabolic)**.** Let G be hyperbolic relative to $\mathbb{P}$. A subgroup H is called *aparabolic* if $H \cap P^g = 1$ for any $P \in \mathbb{P}$ and $g \in G$.

Definition 16 (Quasi-isometry)**.** Let (X, d_X) and (Y, d_Y) be metric spaces. A map $\varphi : X \to Y$ is *quasi-isometry* if for every $x_1,\, x_2 \in X$,

$$\frac{1}{\kappa} d_X(x_1, x_2) - \epsilon \leq d_Y\big(\varphi(x_1), \varphi(x_2)\big) \leq \kappa d_X(x_1, x_2) + \epsilon,$$

for some constants $\kappa \geq 1$ and $\epsilon \geq 0$. We also call φ a (κ, ϵ)-quasi-isometry.

The following was proved in [MOY11]. Here, we give an independent and simple proof using Proposition 6.1.2 and Bowditch's fine graph approach to relatively hyperbolic groups.

Theorem 6.3.3. *Let G be torsion-free, non-elementary and hyperbolic relative to $\mathbb{P}$. Let g and $\bar{g}$ be hyperbolic elements of G such that $\langle g, \bar{g}\rangle$ is not cyclic. Then there* *exists k such that for any $n \geq k$ the subgroup $F = \langle g^n, \bar{g}^n \rangle$ is free of rank 2, aparabolic and quasiconvex in G relative to $\mathbb{P}$.*

Proof. Let K be a $(G;\mathbb{P})$-graph. We use $[x, y]$ to denote a geodesic in K from the vertex x to the vertex y, and $|[x, y]|$ denotes the length of $[x, y]$. The open ϵ-neighborhood of x is denoted by $N_\epsilon(x)$. Note that since $\langle g, \bar{g} \rangle$ is not cyclic, $\langle g^r, \bar{g}^s \rangle$ is not cyclic for any r, $s \geq 1$. Indeed if $\langle g^r, \bar{g}^s \rangle$ is cyclic for some r, s then $g^i = \bar{g}^j$, for some i, $j \geq 1$, and so g commutes with $\bar{g}^j$. This implies that $\langle \bar{g}^j \rangle \subseteq \langle \bar{g} \rangle \cap \langle \bar{g} \rangle^g$, hence $[\langle \bar{g} \rangle \ : \ \langle \bar{g} \rangle \cap \langle \bar{g} \rangle^g] < \infty$. Therefore $g \in \mathrm{Comm}(\langle \bar{g} \rangle)$ which equals $\langle \bar{g} \rangle$ by Lemma 6.3.2. This contradicts that $\langle g, \bar{g} \rangle$ is not cyclic.

For sufficiently large n, we will construct a quasiconvex tree $L \subseteq K$ upon which $\langle g^n, \bar{g}^n \rangle$ acts freely and cocompactly. By Lemma 6.3.1, $h = g^{n_1}$ has a quasiconvex axis Υ for some n_1 and $\bar{h} = \bar{g}^{n_2}$ has a quasiconvex axis $\bar{\Upsilon}$ for some n_2. Choose x_h and $x_{\bar{h}}$ to be vertices in Υ and $\bar{\Upsilon}$ such that

$$d(x_h, x_{\bar{h}}) = \min\{d(x, y) \mid x \text{ is a vertex in } \Upsilon \text{ and } y \text{ is a vertex in } \bar{\Upsilon}\}.$$

Either $d(x_h, x_{\bar{h}}) = 0$ or $d(x_h, x_{\bar{h}}) > 0$. We give the proof in the second case and the first case is similar. Let $\sigma = [x_h, x_{\bar{h}}]$ be a geodesic. Choose $\mu > 0$ such that both $\Upsilon \subseteq K$ and $\bar{\Upsilon} \subseteq K$ are μ-quasiconvex, i.e. if a geodesic α in K has endpoints on Υ then $\alpha \subseteq N_\mu(\Upsilon)$. Moreover, let Υ and $\bar{\Upsilon}$ be (κ, ϵ)-quasiconvex where $\kappa \geq 1$.

Let $\tau = [x_h, h^m x_h]$ and $\bar{\tau} = [x_{\bar{h}}, \bar{h}^m x_{\bar{h}}]$. Choose m large enough that $\tau \nsubseteq N_{3\delta+4\mu+R}(\bar{\tau})$ and $\tau \nsubseteq N_{3\delta+4\mu+R}(h\bar{\tau})$. Note that such m exists as otherwise Υ and $\bar{\Upsilon}$ are coarsely the same and so $\langle h, \bar{h} \rangle = \langle g^{n_1}, \bar{g}^{n_2} \rangle$ would be cyclic which is impossible, as shown. Also as Υ and $\bar{\Upsilon}$ are quasiconvex we can choose m large enough so that

$\tau \nsubseteq N_{2\delta+4\mu}(\sigma)$ and $\bar{\tau} \nsubseteq N_{2\delta+4\mu}(\sigma)$. Let $c = \max\{|\tau|, |\bar{\tau}|\}$ where τ and $\bar{\tau}$ satisfy all previously mentioned conditions. Without loss of generality, we can assume m is large enough that $|\tau| > 2(6c + 2\delta) + 1$ and $|\bar{\tau}| > 2(6c + 2\delta) + 1$.

Let $F = \langle h^m, \bar{h}^m \rangle$ and let L be the graph consisting of the union of all F-translates of the connected graph $\Upsilon \cup \bar{\Upsilon} \cup \sigma$ (see Figure 6–3-(I)). Let γ be the fundamental domain for the actin of $\langle h^m \rangle$ on Υ and $\bar{\gamma}$ be the fundamental domain for the actin of $\langle \bar{h}^m \rangle$ on $\bar{\Upsilon}$. Let J be the graph obtained from $\gamma \sqcup \sigma \sqcup \bar{\gamma}$ by identifying the endpoints of γ with the initial vertex of σ and identifying the endpoints of $\bar{\gamma}$ with the terminal vertex of σ (see Figure 6–3-(II)). Note that $\pi_1 J \cong F$. Let $\tilde{J}$ be the universal cover of J and consider $\pi_1 J$-equivariant map $\varphi : \tilde{J} \to L$ where the group $\pi_1 J$ maps to $F = \langle h^m, \bar{h}^m \rangle$. We show φ is a quasi-isometry which implies that L is a quasiconvex subgraph of K and by Definition 5, this implies $F \leq G$ is relatively quasiconvex subgroup of G. We also prove that φ is injective which shows that L is a tree, therefore F is free.

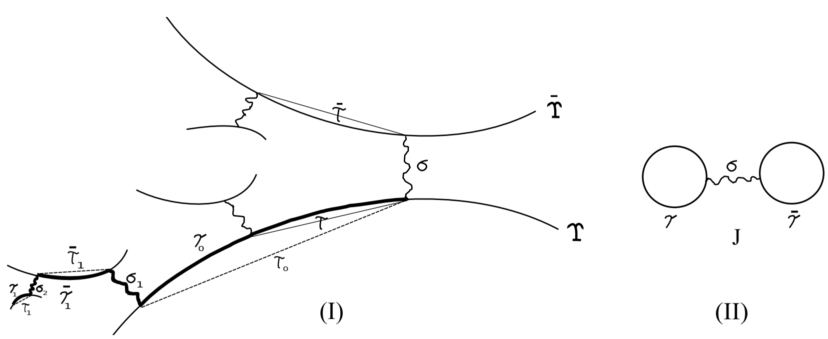

Figure 6–3: The dark path is $\psi = \gamma_0 \sigma_1 \bar{\gamma}_1 \sigma_2 \gamma_1$. The path τ_i is a geodesic between endpoints of γ_i and $\bar{\tau}_i$ is a geodesic between endpoints of $\bar{\gamma}_i$.

We show φ is $\big(2(\kappa+\epsilon),0\big)$-quasi-isometry. Let ξ be a path in $\tilde{J}$ and let $\varphi(\xi)=\psi$ such that

$$\psi=\gamma_0\sigma_1\bar{\gamma}_1\sigma_2\gamma_1\sigma_3\bar{\gamma}_2\sigma_4\gamma_2\cdots\sigma_r\gamma_r$$

where each σ_i is an F-translate of σ, and each γ_i and each $\bar{\gamma}_i$ is a subpath of an F-translate of Υ and $\bar{\Upsilon}$ respectively (see Figure 6–3-(I)). Note that there are similar cases to consider where ψ starts/ends with a proper path of σ_i and where ψ ends with $\bar{\gamma}_r$ instead of γ_r.

Replace each γ_i and each $\bar{\gamma}_i$ respectively by geodesics τ_i and $\bar{\tau}_i$ with the same endpoints to get

$$\psi'=\tau_0\sigma_1\bar{\tau}_1\sigma_2\tau_1\sigma_3\bar{\tau}_2\sigma_4\tau_2\ldots\sigma_r\tau_r$$

Now ψ' satisfies the hypotheses of Proposition 6.1.2: Firstly, $|\tau_i|$, $|\bar{\tau}_i|>2(6c+2\delta)$ for each i except perhaps for $i=0$ and $i=r$. Secondly, τ_i and $\bar{\tau}_i$ (and similarly $\bar{\tau}_i$ and τ_{i+1}) do not 3δ-fellow travel for distance $\geq c$. Indeed $\tau_i\subseteq N_\mu(\gamma_i)$ and $\gamma_i\subseteq N_\mu(f\tau)$ where $f\tau$ is an f-translate of τ whose initial vertex is the same as γ_i. Therefore $\tau_i\subseteq N_{2\mu}(f\tau)$ and since $\tau\nsubseteq N_{3\delta+4\mu}(\bar{\tau})$, we see that τ_i and $\bar{\tau}_i$ do not 3δ-fellow travel for distance $\geq c$. A similar argument shows that ψ' satisfies the third property of Proposition 6.1.2. Therefore ψ' is a $(2,0)$-quasigeodesic. We now show that $|\psi|\leq(\kappa+\epsilon)|\psi'|$ and since ψ' is a $(2,0)$-quasigeodesic, this implies that φ is $\big(2(\kappa+\epsilon),0\big)$-quasi-isometry.

First note that since Υ and $\bar{\Upsilon}$ are (κ,ϵ)-quasiconvex, $|\gamma_i|\leq\kappa|\tau_i|+\epsilon$ and $|\bar{\gamma}_i|\leq\kappa|\bar{\tau}_i|+\epsilon$, combining this with the assumption $|\tau_i|$ and $|\bar{\tau}_i|\geq 1$, we see that $|\gamma_i|\leq(\kappa+\epsilon)|\tau_i|$ and $|\bar{\gamma}_i|\leq(\kappa+\epsilon)|\bar{\tau}_i|$ (note that $\kappa+\epsilon\geq 1$). Therefore

$$\begin{aligned}
|\psi| &= |\gamma_0| + |\sigma_1| + |\bar{\gamma}_1| + |\sigma_2| + |\gamma_1| + |\sigma_3| + |\bar{\gamma}_2| + |\sigma_4| + |\gamma_2| + \cdots + |\sigma_r| + |\gamma_r| \\
&\leq (\kappa+\epsilon)\big(|\tau_0| + |\sigma_1| + |\bar{\tau}_1| + |\sigma_2| + |\tau_1| + |\sigma_3| + |\bar{\tau}_2| + |\sigma_4| + |\tau_2| + \cdots + |\sigma_r| + |\tau_r|\big) \\
&= (\kappa+\epsilon)|\psi'|
\end{aligned}$$

Thus φ is quasi-isometry, therefore L is a quasiconvex subgraph of K and by Definition 5, this implies $F \leq G$ is relatively quasiconvex subgroup of G.

We now show that φ is an injective map which implies L is a tree and F is free. Suppose $\varphi(x_1) = \varphi(x_2)$ where x_1 and x_2 are points on $\tilde{J}$. Since φ is $\big(2(\kappa+\epsilon), 0\big)$-quasi-isometry

$$d_{\tilde{J}}(x_1, x_2) \leq 2(\kappa+\epsilon) d_K\big(\varphi(x_1), (x_2)\big) = 0.$$

Therefore $x_1 = x_2$ and φ is injective.

Since φ is an injective F-equivariant map and F acts freely on $\tilde{J}$, F acts freely on L. This implies that $F \cap P^g = 1$ for any $g \in G$ and any $P \in \mathbb{P}$ and thus F is aperiodic. $\square$

Let G be hyperbolic relative to $\mathbb{P}$. Recall that $Q \leq G$ is called *hyperbolically embedded relative to* $\mathbb{P}$ if G is hyperbolic relative to $\mathbb{P} \cup \{Q\}$.

The following was proved in [Osi06b].

Theorem 6.3.4. *Let G be hyperbolic relative to $\mathbb{P}$ and let $Q \leq G$ be a subgroup. Then G is hyperbolic relative to $\mathbb{P} \cup \{Q\}$ if and only if $\mathbb{P} \cup \{Q\}$ is almost malnormal and Q is quasiconvex in G relative to $\mathbb{P}$*

Lemma 6.3.5. *Let G be torsion-free, non-elementary and hyperbolic relative to $\mathbb{P}$. Let $F \leq G$ be a free and relatively quasiconvex subgroup that contains no non-trivial parabolic element. Then generically any finitely generated subgroup of F is aparabolic, malnormal in G and quasiconvex relative to $\mathbb{P}$, and therefore hyperbolically embedded relative to $\mathbb{P}$.*

Proof. We show that generically any finitely generated subgroup M of F is malnormal in G and quasiconvex relative to $\mathbb{P}$. Clearly $M \cap P^g = 1$ for any $g \in G$ and any $P \in \mathbb{P}$, thus M is aparabolic. Therefore by Theorem 6.3.4, M will be hyperbolically embedded in G relative to $\mathbb{P}$. Since G is torsion-free and F has rank 2, by Lemma 6.3.2, $\mathrm{Comm}(F) = F$. Since the intersection of F with any parabolic subgroup is trivial, by [HW09, thm 1.6], F has finite width i.e. there is a finite set of elements $\{g_1, \ldots, g_s\}$ in $G - F$ such that if $F \cap F^g$ is infinite then $g \in g_i F$ for some i. Let $H_i = F \cap F^{g_i}$. Each H_i is an infinite group which has infinite index in F. By Corollary 6.2.3, generically, the subgroup $M = \langle r_1, \ldots, r_m \rangle$ generated by randomly chosen words $r_1, \ldots, r_m$ in F is free group of rank m, malnormal in F and $M \cap H_i^f = 1$ for each i and any $f \in F$. We show that M is relatively quasiconvex and malnormal in G. Since M is quasiconvex in F and F is relatively quasiconvex in G, by Remark 2.2.2, M is relatively quasiconvex in G.

We now show that M is malnormal in G. Let $g \in G$ such that $M \cap M^g \neq 1$. If $[F : F \cap F^g] < \infty$ then $g \in \mathrm{Comm}(F) = F$, thus $g \in M$ since $M \leq F$ is malnormal. So we suppose that $[F : F \cap F^g] = \infty$, in which case $g = g_i f$ for some $f \in F$ and some i. Now

$$1 \neq M \cap M^g = M \cap M^{g_i f}$$

therefore

$$1 \neq M^{f^{-1}} \cap M^{g_i} = (M^{f^{-1}} \cap F) \cap M^{g_i} = M^{f^{-1}} \cap (M^{g_i} \cap F) \subseteq M^{f^{-1}} \cap (F \cap F^{g_i}) = M^{f^{-1}} \cap H_i$$

which contradicts that $M \cap H_i^f = 1$ for any $f \in F$. Thus M is malnormal in G □

The existence of hyperbolically embedded free subgroup in the following theorem, has been shown in [MOY11].

Theorem 6.3.6. *Let G be a torsion-free, non-elementary and hyperbolic relative to $\mathbb{P}$. Then there exists a rank 2 free subgroup F of G such that generically any finitely generated subgroup of F is aparabolic, malnormal in G and quasiconvex relative to $\mathbb{P}$, and therefore hyperbolically embedded relative to $\mathbb{P}$. In particular, G contains a hyperbolically embedded rank 2 free subgroup.*

Proof. By [Osi06b, Cor 4.5], G contains a hyperbolic element g. By 6.3.2 the commensurator subgroup of $\langle g \rangle$ is $\langle g \rangle$. Since G is not elementary and not parabolic, there is a hyperbolic element h in $G - \langle g \rangle$. Now the proof follows by Theorem 6.3.3 and Lemma 6.3.5. □

6.4 A Non-quasiconvex Embedding of Relatively Hyperbolic Groups

The following was proved in [DS05].

Lemma 6.4.1. *If G is finitely generated and hyperbolic relative to $\mathbb{P} = \{P_1, \ldots, P_n\}$ and each P_i is hyperbolic relative to $\mathbb{H}_i = \{H_{i1}, \ldots, H_{im_i}\}$, then G is hyperbolic relative to $\bigcup_{1 \leq i \leq n} \mathbb{H}_i$.*

The following theorem generalizes Kapovich's result in [Kap99].

Theorem 6.4.2. *Let G be a finitely generated torsion-free group that is non-elementary and hyperbolic relative to $\mathbb{P}$. There exists a group G^* that is hyperbolic relative to $\mathbb{P}$ such that G is a subgroup of G^* and G is not quasiconvex in G^* relative to $\mathbb{P}$.*

Proof. By Theorem 6.3.6, there is a rank 2 free subgroup $F = \langle a, b \rangle$ in G such that G is hyperbolic relative to $\mathbb{P} \cup \{F\}$. Let $\phi : F \to F$ such that $\phi(a) = abab^2 \cdots ab^{100}$ and $\phi(b) = baba^2 \cdots ba^{100}$. Then the group

$$F^* = \langle a, b, t \mid tat^{-1} = \phi(a), tbt^{-1} = \phi(b) \rangle.$$

is $C'(\frac{1}{6})$ and therefore hyperbolic. Moreover $F \leq F^*$ is exponentially distorted and thus F is not quasiconvex in F^*. Indeed $d_F(1, t^n a t^{-n}) \geq 100^n$, where d_F is the word metric for F.

Consider the group

$$G^* = \langle G, t \mid tat^{-1} = \phi(a), tbt^{-1} = \phi(b) \rangle.$$

We show that G^* is hyperbolic relative to $\mathbb{P}$ and G is not relatively quasiconvex in G^*. By Corollary 3.1.5, since G is hyperbolic relative to $\mathbb{P} \cup \{F\}$, the group G^* is hyperbolic relative to $\mathbb{P} \cup \{\langle F, t \rangle\}$. Since $F^* = \langle F, t \rangle$ is hyperbolic, G^* is hyperbolic relative to $\mathbb{P}$, by Lemma 6.4.1.

Now suppose that G is relatively quasiconvex in G^*.

$$G^* = G*_{F^t = \varphi(F)} \cong G *_F F^*.$$

F is relatively quasiconvex in G and G is relatively quasiconvex in G^*. Therefore F is relatively quasiconvex in G^*. Therefore by Lemma 4.3.2, F^* is relatively quasiconvex in G^*. So

$$F \leq F^* \leq G^*.$$

and both F and F^* are relatively quasiconvex in G^*, therefore F is relatively quasiconvex in F^*. Since F is aparabolic, and F is relatively quasiconvex in F^*, we see that F is quasiconvex in F^* which is contradiction. $\square$

CHAPTER 7
Small-cancellation Theory

Small-cancellation theory studies small-cancellation complexes and the groups acting on them. These complexes are the one whose two cells have "small overlap" which each other. Small-cancellation theory has proven to be a powerful theory in studying groups, specially in construction of groups with some given properties. Some ideas underlying this theory go back to the work of Max Dehn in the 1910s and later it was generalized by various people in [Tar49], [Gre60], [LS77], [Ol′91], [Gro03], [MW02], [Osi10], and many others. Recently in his outstanding work, Wise generalized small-cancellation theory to CAT(0) cube complexes [Wis]. This chapter contains a brief review of the basic notions of small-cancellation theory. We follow the geometric language given in [MW02], and more details and examples can be found there. A more classical reference is [LS77]. We also provide some examples.

Convention 7.0.3. *All the maps in this thesis will be combinatorial which will be defined in the following.*

7.1 Basic Terminology and Background

In this section, we review the definitions of small-cancellation theory.

Definition 17 (Combinatorial maps and complexes)**.** A map $Y \to X$ between CW complexes is *combinatorial* if its restriction to each open cell of Y is a homeomorphism onto an open cell of X. A CW complex X is *combinatorial* if the attaching map of each open cell of X is combinatorial for a suitable subdivision.

Definition 18 (Disc diagram)**.** A *disc diagram* D is a compact contractible 2-complex with a fixed embedding in the plane. A *boundary cycle* P of D is a closed path in ∂D which travels entirely around D (in a manner respecting the planar embedding of D).

A *disc diagram in* X is a map $D \to X$. It is a well-known fact, due to van Kampen, that whenever $P \to X$ is a nullhomotopic closed path, there is a disc diagram $D \to X$ such that $P \to X$ factors as $P \to \partial D \to X$.

Let R_1 and R_2 be 2-cells that meet along a 1-cell e in the disc diagram $D \to X$. We say R_1 and R_2 are a *cancellable pair* if the boundary paths of R_1 and R_2 starting at e map to the same closed path in X. $D \to X$ is *reduced* if it has no cancellable pair of 2-cells.

Definition 19 (Piece)**.** Let X be a combinatorial 2-complex. Intuitively, a piece of X is a path which is contained in the boundaries of the 2-cells of X in at least two distinct ways. More precisely, a nontrivial path $P \to X$ is a *piece* of X if there are 2-cells R_1 and R_2 such that $P \to X$ factors as $P \to R_1 \to X$ and as $P \to R_2 \to X$ but there does not exist a homeomorphism $\partial R_1 \to \partial R_2$ such that there is a commutative diagram:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \partial R_2 \arrow[d] \\ \partial R_1 \arrow[r] \arrow[ur] & X \end{tikzcd}$$

Excluding commutative diagrams of this form ensures that P occurs in ∂R_1 and ∂R_2 in essentially distinct ways.

Note that in the combinatorial group theory setting a piece has the following explanation:

Let $F(S)$ be the free group generated by a finite set S. Let R be a set of words in $S \cup S^{-1}$. Let $G = \langle S \mid R \rangle$. We assume the elements of R are nontrivial and cyclically reduced. Denote by R^* the set of all cyclic conjugates of elements of $R \cup R^{-1}$. A piece is a common prefix of two distinct elements of R^*. In other word a piece is a subword that could cancel in the product $r.s$ where r and $s \in R^*$.

Definition 20 (C(p)-T(q) complex)**.** An *arc* in a disc diagram is a path whose internal vertices have valence 2 and whose initial and terminal vertices have valence $\geq$ 3. The arc is *internal* if its interior lies in the interior of D, and it is a *boundary arc* if it lies entirely in ∂D. A 2-complex X satisfies the C(p) condition if the boundary path of each 2-cell in each reduced disc diagram D either contains a nontrivial boundary arc, or is the concatenation of at least p nontrivial internal arcs. X is a *T(q) complex* if in any reduced disc diagram $D \to X$, any 0-cell v satisfies valenc$(v) = 2$ or $\geqslant q$. A 2-complex which satisfies both C(p) and T(q) is a C(p)-T(q) *complex.* A group G is C(p)-T(q) if it is the fundamental group of a C(p)-T(q) 2-complex.

For a fixed positive real number α, the complex X is a C'(α) complex provided that for each 2-cell $R \to X$, and each piece $P \to X$ which factors as $P \to R \to X$, we have $|P| < \alpha|\partial D|$. A group G is C'(α) if it is the fundamental group of a C'(α) 2-complex.

A *triangular* T(6) *complex* is a T(6) complex such that in any disc diagram, all the internal 2-cells are triangles.

Definition 21 (Ladder)**.** A *ladder* L is a disc diagram which is the union of a sequence of closed 1-cells and 2-cells $c_1, \ldots, c_n$, such that for $1 < j < n$, there are exactly two components in $L - c_j$, and exactly one component in $L - c_1$ and $L - c_n$. Finally, any c_i which is a 1-cell is not contained in any other closed c_j.

Remark 7.1.1. If X is a T(q) complex with $q > 5$, then every piece in X has length 1, observed originally by Pride, (see [MW02, Lem 3.5]).

Any C(6) complex is a C(6)-T(3) complex, therefore we do not mention the term T(6), for these complexes.

Remark 7.1.2. Note that triangular T(6) complexes are the 2-dimensional analogous of systolic complexes defined by T. Januszkiewicz and J. Swiatkowski [JŚ06] and independently by F. Haglund [Hag03].

Definition 22 (i-shell, spur, fan)**.** Let D be a diagram. An *i-shell* of D is a 2-cell $R \hookrightarrow D$ whose boundary cycle ∂R is the concatenation $P_0 P_1 \cdots P_i$ where $P_0 \to D$ is a boundary arc, the interior of $P_1 \cdots P_i$ maps to the interior of D, and $P_j \to D$ is a nontrivial interior arc of D for all $j > 0$. The path P_0 is the *outer path* of the i-shell. Note that $P_0 = \partial R \cap \partial D$. In Figure 7–1-A, the 2-cell R is a 3-shell and in Figure 7–1-B, it is a 2-shell.)

A *fan* is a sequence of consecutive 2-shells such that two neighbouring 2-shells intersect in a piece which is not outer path of any of them. We say F *is a fan with two 2-shells* if F has two 2-shells. Figure 7–1-B illustrates an example of a fan F with two 2-shells R_1 and R_2. Note that *the outer path of* F is the path containing the outer paths of 2-shells of F and the complement of the outer path in the boundary of F is the inner path of F.

A 1-cell e in ∂D that is incident with a valence 1 0-cell v is a *spur*.

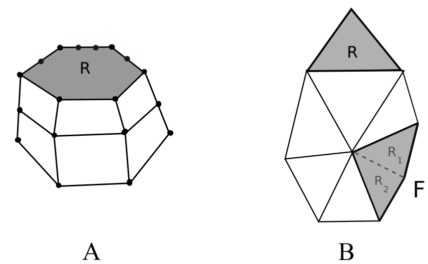

Figure 7–1: In Figure B, R is a 1-shell and F is a fan with two 2-shells R_1 and R_2.

Definition 23 (Missing i-shell, Missing fan with two 2-shell)**.** Consider the commutative diagram on the left

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Y \arrow[d] \\ R \arrow[r] & X \end{tikzcd} \qquad\qquad \begin{tikzcd} P \arrow[r] \arrow[d] & Y \arrow[d] \\ R \arrow[r] \arrow[ur] & X \end{tikzcd}$$

where Y is a 2-complex, R is a closed 2-cell, and $P \to X$ is a path which factors through both Y and R. Let $\partial R = PS$ where S is the concatenation of i pieces. We say that R is a *missing i-shell* for Y if the map $P \to Y$ does not extend to a map $R \to Y$ so that the diagram on the right commutes. We call P the *outer path* of R, see Figure 8–1.

Now consider the previous commuting diagram on the left, where Y is a 2-complex and $R = F$ is a fan with two 2-shells R_1 and R_2. Let e be the common piece (indeed this is an edge) with end 0-cells v and w. Let $\partial F = PS$ where S is inner path which is the concatenation of 2 pieces joining along the vertex v. We say that F is a *missing fan with two 2-shells* for Y if the map $P \to Y$ does not extend to a map $F \to Y$ so that the diagram on the right commutes, Figure 8–3 illustrates an example. e is called *the common piece of the missing fan* and v is called *the tip vertex of the missing fan*.

7.2 Greendlinger's Lemma, Hexagonal and Triangular Torus

In this section, after recalling the Greendlinger's Lemma, a crucial theorem in small-cancellation theory, we define hexagonal and triangular torus.

A combinatorial 2-complex X is an angled 2-complex if an angle is assigned to each corner of each 2-cell. Since the edges in links of 0-cells of X are in one-to-one correspondence with the corners of 2-cells of X, this is equivalent to an angle assignment to each edge of $link(v)$ for each $v \in X^0$. The curvature $\kappa(C)$ of an n-sided 2-cell C of X is defined by

$$\kappa(C) = \left(\sum_{c \in \text{Corners(C)}} \measuredangle(c) \right) - (n-2)\pi.$$

The curvature $\kappa(v)$ of a 0-cell $v \in X^0$ is defined by

$$\kappa(v) = 2\pi - \pi\chi(link(v)) - \sum_{c \in \text{Corners(v)}} \measuredangle(c).$$

The following result is the fundamental tool of small-cancellation theory [MW02, Thm 4.6]

Theorem 7.2.1 (Combinatorial Gauss-Bonnet Theorem)**.** *Let X be an angled 2-complex then*

$$\sum_{C \in 2-cells(X)} \kappa(C) + \sum_{v \in 0-cells(X)} \kappa(v) = 2\pi \cdot \chi(X) \tag{7.1}$$

The following result is "Greendlinger's Lemma" for C(6) complexes [MW02, Thm 9.4] (see [LS77, Thm V.4.5] for a classical version of this statement).

Theorem 7.2.2 (Greendlinger's Lemma for C(6)). *If D is a $C(6)$-$T(3)$ disc diagram, then one of the following holds:*

1. *D contains at least three spurs and/or i-shells with $i \leq 3$;*
2. *D is a ladder, and hence has a spur or 1-shell at each end;*
3. *D consists of a single 0-cell or a single 2-cell.*

Definition 24 (Avoiding and meeting 0-cells). Let $F \leq D$ be a fan with two 2-shells or a 1-shell, let Q be its outer path, and let v be a 0-cell in ∂D. We say F meets v if the outer path Q contains v in its interior. Otherwise, we say that F avoids v.

Any nontrivial disc diagram contains at least two 1-shells and/or fans with two 2-shells. Indeed in [MW02, Thm 9.5], it was proved:

Theorem 7.2.3. *Let D be a C(3)-$T(6)$ disc diagram that is nontrivial i.e. not a* 0*-cell and not a* 2*-cell. Let v be a* 0*-cell in ∂D, then D contains a spur, a* 1*-shell, or a pointed fan with two* 2*-shells which avoids v.*

Definition 25 (Hexagonal torus and Honeycomb). A *honeycomb* in a C(6) complex X is a hexagonal tiling of $\mathbb{Z}^2$ with some valence 2 vertices added. A *hexagonal torus* is a C(6) complex homeomorphic to a torus. A very simple hexagonal torus is indicated in Figure 7–2. Of course, any hexagonal torus is the quotient of a honeycomb by a free cocompact action of $\mathbb{Z} \times \mathbb{Z}$.

Definition 26 (Triangular torus). A *triangular flat plane F* is a C(3)-T(6) complex homeomorphic to the Euclidean plane such that each closed 2-cell intersects exactly

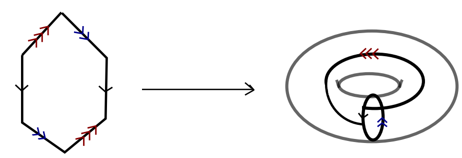

Figure 7–2: Identifying opposite sides of a hexagon yields a hexagonal torus.

three neighbouring closed 2-cells. A *triangular torus* is a C(3)-T(6) complex homeomorphic to a torus. Note that similar to hexagonal torus, any triangular torus is the quotient of a triangular flat plane by a free cocompact action of $\mathbb{Z} \times \mathbb{Z}$.

CHAPTER 8
Locally Convex Maps

In this chapter, we define locally convex and strongly locally convex maps for C(6) and C(3)-T(6) complexes and we show that the "thickening" of a strongly locally convex subcomplex is also strongly locally convex. We will follow the following convention for the remainder of the thesis:

Convention 8.0.4. *We will assume that no two* 2*-cells of* X *have the same attaching map.*

Definition 27 (Thickening)**.** Let X be a C(6) or C(3)-T(6) complex and $\widetilde{Y} \subseteq \widetilde{X}$ be a subcomplex. The *thickening* $\mathsf{N}(\widetilde{Y})$ *of* $\widetilde{Y}$ is the subcomplex

$$\mathsf{N}(\widetilde{Y}) = \widetilde{Y} \cup \{\bar{R} \mid R \text{ is a 2-cell and } \bar{R} \cap \widetilde{Y} \neq \varnothing\}.$$

We use the notation $\mathsf{N}^0(\widetilde{Y}) = \widetilde{Y}$ and $\mathsf{N}^{i+1}(\widetilde{Y}) = \mathsf{N}(\mathsf{N}^i(\widetilde{Y}))$. Note that $\mathsf{N}(\widetilde{Y}) = \mathsf{N}^1(\widetilde{Y})$ might not contain an open neighborhood of $\widetilde{Y}$.

Remark 8.0.5. If $\widetilde{X}$ is connected and has no isolated 1-cell and $\widetilde{Y} \neq \varnothing$ then $\widetilde{X} = \cup_{k\geq 0}\mathsf{N}^k(\widetilde{Y})$. Indeed for any path P whose initial vertex is on a cell α in $\widetilde{X}$ and whose terminal vertex lies on $\widetilde{Y}$, we see that $\alpha \subset \mathsf{N}^{|P|}(\widetilde{Y})$.

8.1 Locally Convex Maps in C(6)

8.1.1 Definition of Locally Convex Maps in C(6)

Definition 28 (Locally convex map)**.** A combinatorial map between C(6) complexes is an *immersion* if it is locally injective. An immersion $\phi : Y \to X$ between C(6) complexes is *locally convex* if it does not have a missing i-shell where $1 \leq i \leq 3$. Figure 8–1 illustrates an example in which the inclusion of the shaded subcomplex Y is not a locally convex map, indeed it has a missing 3-shell R.

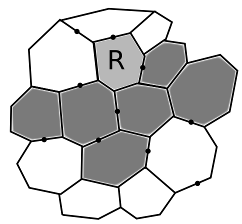

Figure 8–1: R is a 3-shell in the complement of the dark shaded complex and is missing along its outer path in the shaded complex. So the inclusion map from the dark complex to the whole complex is not locally convex.

Lemma 8.1.1. *A locally convex map $f : Y \to X$ between simply connected $C(6)$ complexes is injective.*

Proof. Let δ be a non closed path in Y which maps to a closed path γ in X. Since X is simply connected, γ bounds a disc diagram. Choose δ among all pathes between endpoints of δ such that its image γ has the minimal area disc diagram D in X. By Lemma 7.2.2, D has an i-shell called R where $1 \leqslant i \leqslant 3$. Let $\partial R = QS$ where S is the concatenation of i pieces $(1 \leqslant i \leqslant 3)$ and Q is the outer path of the shell. Since $Y \to X$ is locally convex, the map $R \to X$ induces a map $R \to Y$ otherwise, R will be a missing i-shell for Y . We show the image of R in Y by R. So Q is part of δ. Now push Q toward S in Y to get a new path $\acute{\delta}$ whose end points are the same as δ. The image of $\acute{\delta}$ in X bounds the disc diagram $\acute{D} = D - R$ where Area($\acute{D}$)= Area(D)-1 which is contradiction. $\square$

Lemma 8.1.2. *Let X be a $C(6)$ complex. Let Y_1 and Y_2 be subcomplexes of X such that each inclusion map $Y_i \hookrightarrow X$ is locally convex. Then $Y_1 \cap Y_2 \hookrightarrow X$ is also locally convex.*

Proof. This follows immediately from the definition. $\square$

Definition 29 (Strongly locally convex subcomplex)**.** Let X and Y be C(6) complexes. We call $\widetilde{Y} \hookrightarrow \widetilde{X}$ *strongly locally convex* if for any 2-cell R with $\bar{R} \cap \widetilde{Y} \neq \varnothing$, either $R \subseteq \widetilde{Y}$ or each component of $\bar{R} \cap \widetilde{Y}$ is the concatenation of at most 2 pieces. For example it is immediate that $\bar{R}$ is strongly locally convex whenever R is a single 2-cell. Honeycombs are also strongly locally convex.

Observe that strongly locally convex implies locally convex. Consequently, if $\widetilde{Y} \subseteq \widetilde{X}$ is strongly locally convex then $\bar{R} \cap \widetilde{Y}$ is actually connected by Lemma 8.1.2.

We emphasize that the definition requires that components of $\bar{R} \cap \widetilde{Y}$ be expressible as the concatenation of at most 2 pieces. It is possible that they are also expressible as the concatenation of more than two pieces.

Lemma 8.1.3. *If $\widetilde{Y} \subseteq \widetilde{X}$ is strongly locally convex and R_1, R_2 are 2-cells in $\widetilde{X}$ with $\bar{R}_1 \cap \bar{R}_2 \neq \varnothing$, $\bar{R}_1 \cap \widetilde{Y} \neq \varnothing$ and $\bar{R}_2 \cap \widetilde{Y} \neq \varnothing$ then $\bar{R}_1 \cap \bar{R}_2 \cap \widetilde{Y} \neq \varnothing$.*

Proof. We show that if $\bar{R}_1 \cap \bar{R}_2 \neq \varnothing$, $\bar{R}_1 \cap \widetilde{Y} \neq \varnothing$ and $\bar{R}_2 \cap \widetilde{Y} \neq \varnothing$ then $\bar{R}_1 \cap \bar{R}_2$ is a singleton or a piece that intersects $\widetilde{Y}$. Observe that $\bar{R}_1 \cap \bar{R}_2$ has one component. Assume $\bar{R}_1 \cap \bar{R}_2$ does not intersect $\widetilde{Y}$. Let D be the minimal area disc diagram whose boundary path consists of the pathes α, β and γ where $\alpha \subseteq \partial \bar{R}_1$, $\beta \subseteq \partial \bar{R}_2$ and $\gamma \subseteq \widetilde{Y}$. The disc diagram D is illustrated as the dark complex in Figure 8–2-A.

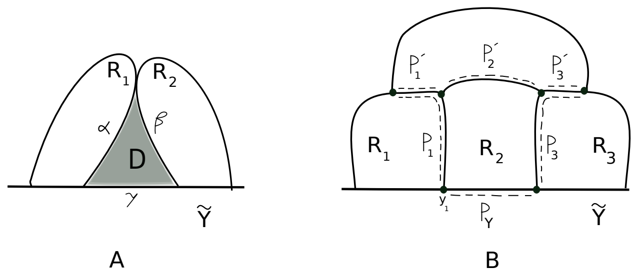

Figure 8–2: In Figure A, the complex $\widetilde{Y}$ is strongly locally convex and $\bar{R}_1, \bar{R}_2$ and $\widetilde{Y}$ should triply intersect. In Figure B, the 2-cells R_1, R_2, R_3 are subsets of $\mathsf{N}(\widetilde{Y})$ and $\bar{R} \cap \mathsf{N}(\widetilde{Y})$ is the concatenation of $P_1'P_2'P_3'$.

Observe that since the boundary path of D has at most 4 pieces, D is not a single 2-cell. By Lemma 7.2.2, since D does not have spurs, it must contain at least three i-shells where $i \leq 3$. Also since $\widetilde{Y}$ is strongly locally convex, if D has i-shells, they must lie in the corners. D can not have an i-shell in the corner corresponding to $\bar{R}_1$ and $\bar{R}_2$, therefore D has at most two i-shells which is contradiction. $\square$

We now show that a "nice extension" of a strongly locally convex subcomplex of a C(6) complex, is again strongly locally convex.

8.1.2 Thickening of C(6) Complexes

Lemma 8.1.4. *Let X be a $C(6)$ complex and $\widetilde{Y} \subseteq \widetilde{X}$ be a connected subcomplex. If $\widetilde{Y} \hookrightarrow \widetilde{X}$ is strongly locally convex then $\mathsf{N}(\widetilde{Y}) \hookrightarrow \widetilde{X}$ is also strongly locally convex.*

Proof. Let R be a 2-cell in $\widetilde{X}$ such that $\bar{R} \cap \mathsf{N}(\widetilde{Y}) \neq \varnothing$. Suppose a subpath P of $\bar{R} \cap \mathsf{N}(\widetilde{Y})$ is the concatenation $P_1'P_2'P_3'$ where each P_i' is a path in $\bar{R} \cap \bar{R}_i$ and each

$R_i \subseteq \mathsf{N}(\widetilde{Y}) - \widetilde{Y}$. The lemma follows easily from the following claim: The subpath P can be expressed as the concatenation of at most two pieces.

Proof of the claim: Without loss of generality assume that $\bar{R}_1 \cap \bar{R}_2$ and $\bar{R}_1 \cap \bar{R}_3$ are both nonempty. By Lemma 8.1.3 $\bar{R}_1$, $\bar{R}_2$ and $\widetilde{Y}$ triply intersect, also $\bar{R}_3$, $\bar{R}_2$ and $\widetilde{Y}$ triply intersect. Let P_1 be the shortest path containing $\bar{R}_1 \cap \bar{R}_2$ from a point y_1 in $\bar{R}_1 \cap \bar{R}_2 \cap \widetilde{Y}$ to the initial point of $P'_1P'_2P'_3$. Similarly let P_3 be the shortest path containing $\bar{R}_3 \cap \bar{R}_2$ from the terminal point of $P'_1P'_2P'_3$to a point y_3 in $\bar{R}_2 \cap \bar{R}_3 \cap \widetilde{Y}$. Let P_Y be a path in $\bar{R}_2 \cap \widetilde{Y}$ between y_3 and y_1. Consider the path $P_1(P'_1P'_2P'_3)P_3P_Y = (P_1P'_1)(P'_2)(P'_3P_3)P_Y$. By hypothesis P_Y is at most two pieces and thus (after removing the backtracks in $P_1P'_1$ and P'_3P_3) the path in $\partial\bar{R}_2$ is the concatenation of less than 6 pieces. Therefore the path can not travel around R_2 and thus travels through an arc A in $\partial\bar{R}_2$. We claim that $R \subset \mathsf{N}(\widetilde{Y})$ otherwise $P'_1P'_2P'_3 \cap P_Y = \varnothing$ and therefore P_1 and P_3 must intersect in A. But then $\bar{R}_1 \cap \bar{R}_3 \neq \varnothing$ and by Lemma 8.1.3, $\bar{R}_1 \cap \bar{R}_3 \neq \varnothing$ which implies that $P'_1P'_2P'_3$ is replaceable by $P''_1P''_3$. We refer the reader to Figure 8–2-B. □

8.2 Locally Convex Maps in C(3)-T(6)

8.2.1 Definition of Locally Convex Maps in C(3)-T(6)

Definition 30 (locally convex map)**.** A combinatorial map between C(3)-T(6) complexes is *immersion* if it is locally injective. An immersion $\phi : Y \to X$ between C(3)-T(6) complexes is *locally convex* if it does not have a missing 1-shell and a missing fan with two 2-shells. A subcomplex $Y \subseteq X$ is *locally convex subcomplex* if the the inclusion map from Y to X is locally convex map.

Note that for any 2-cell R, the $R \subseteq X$ is locally convex. Also, any triangular flat plane is locally convex. Figure 8–1 illustrates an example in which the inclusion of the shaded subcomplex Y is not a locally convex map, indeed it has a missing 1-shell R_1 and a missing fan with two 2-shells R_2.

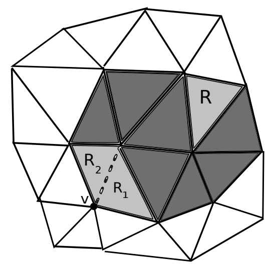

Figure 8–3: In the complement of the dark shaded complex, the 2-cell R is a 1-shell and is missing along its outer path in the shaded complex. Also, $F = R_1 \cup^e R_2$ is a missing fan with two 2-shells in the complement of the dark shaded complex. So the inclusion map from the dark complex to the whole complex is not locally convex. The dot-line is the common piece of the missing fan and v is the tip.

Lemma 8.2.1. *Let X be a $C(3)$-$T(6)$ complex. Let Y_1 and Y_2 be subcomplexes of X such that each inclusion map $Y_i \hookrightarrow X$ is locally convex. Then $Y_1 \cap Y_2 \hookrightarrow X$ is also locally convex.*

Proof. If $Y_1 \cap Y_2 \hookrightarrow X$ has a missing 1-shell or a missing fan with two 2-shells in X, then clearly Y_1 and Y_2 both will have a missing 1-shell or a missing 2-shell. □

Lemma 8.2.2. *Let X and Y be $C(3)$-$T(6)$ complexes and and let $f : \widetilde{Y} \to \widetilde{X}$ be locally convex. Then f is injective.*

Proof. Let δ be a non closed path in $\widetilde{Y}$ which maps to a closed path γ in $\widetilde{X}$. γ bounds a disc diagram. Choose δ among all paths between endpoints of δ such that its image γ has the minimal area disc diagram D in $\widetilde{X}$. By Theorem 7.2.3, since D is not a single 0-cell or a 2-cell, it contains 1-shell and/or fans with consecutive 2-shells. Let R be a 1-shell and let $\partial R = QS$ where S is one piece and Q is the outer path of the shell. Since $\widetilde{Y} \to \widetilde{X}$ is locally convex, the map $R \to \widetilde{X}$ induces a map $R \to \widetilde{Y}$ otherwise, R will be a missing 1-shell for $\widetilde{Y}$. We show the image of R in $\widetilde{Y}$ by R. So Q is part of δ. Now push Q toward S in $\widetilde{Y}$ to get a new path $\acute{\delta}$ whose end points are the same as δ. The image of $\acute{\delta}$ in $\widetilde{X}$ bounds the disc diagram $\acute{D} = D - R$ where Area($\acute{D}$)= Area(D)-1 which is contradiction. Now if D contains fans with consecutive 2-shells, similar argument will give contradiction.

$\square$

8.2.2 Thickening of C(3)-T(6) Complexes

The following Lemma implies that the thickening of a locally convex subcomplex is also locally convex.

Lemma 8.2.3. *Let X be a $C(3)$-$T(6)$ complex and $\widetilde{Y} \subseteq \widetilde{X}$ be a connected subcomplex. If $\widetilde{Y} \hookrightarrow \widetilde{X}$ is locally convex then $\mathsf{N}(\widetilde{Y}) \hookrightarrow \widetilde{X}$ is also locally convex.*

Proof. We show that $\mathsf{N}(\widetilde{Y})$ does not have any missing 1-shell and any missing fan with two 2-shells in $\widetilde{X}$. By contradiction, assume R is a missing 1-shell of $\mathsf{N}(\widetilde{Y})$. Let $\partial R = PS$ such that S is one piece and $P = \bar{R} \cap \mathsf{N}(\widetilde{Y})$. We show that P can be expressed as one piece, which is contradiction because the 2-cell ∂R will then have only two pieces. If P contains more than one piece $P_1 P_2$ where P_1 and P_2 intersect in the 0-cell v then there are 2-cells $R_i \subseteq \mathsf{N}(\widetilde{Y}) - \widetilde{Y}$ for $i = 1, 2$, such that P_i is a path in $\bar{R} \cap \bar{R}_i$. The complex X is T(6), therefore the valence of v is more than 6 and hence there are two 2-cells R_1' and R_2' with the following properties:

$$\bar{R}_1' \cap \bar{R}_2' = e \quad , \quad R_i' \subseteq \mathsf{N}(\widetilde{Y}) - \widetilde{Y}$$

where e is an edge joining to v, we refer the reader to Figure 8–4-A. Now let $S = R_1' \cup^e R_2'$. S is a missing fan with two 2-shells for $\widetilde{Y}$ which is a contradiction. Note that $\bar{R}_i' \cap \widetilde{Y}$ is a nontrivial piece.

We now show that $\mathsf{N}(\widetilde{Y})$ does not have any missing fan with two 2-shells. By contradiction, assume R is a missing fan with two 2-shells of $\mathsf{N}(\widetilde{Y})$ with common piece e and tip vertex v. Let $R = R_1 \cup^e R_2$. The piece e has two end vertices v and w where w is a vertex in closure of $\widetilde{Y}$, see Figure 8–4-B. Valence of w is at least 6, therefore there are 2-cells $R_i' \subseteq \mathsf{N}(\widetilde{Y}) - \widetilde{Y}$ for $i = 1, 2$, such that $\bar{R}_1' \cap \bar{R}_2' = e'$ is a piece containing w and also $\bar{R} \cap \bar{R}_i'$ contains w. Now letting $S = R_1' \cup^{e'} R_2'$, S is a missing fan with two 2-shells for $\widetilde{Y}$ which is a contradiction.

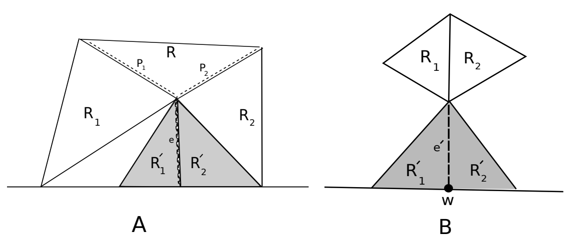

Figure 8–4: In Figure A, if R is a missing 1-shell of $\mathsf{N}(\widetilde{Y})$ then there is a fan with two 2-shells $R_1' \cup^e R_2'$ that is missing in $\widetilde{Y}$. In figure B, if $R_1 \cup R_2$ is a missing fan with two 2-shells in $\mathsf{N}(\widetilde{Y})$ then $\widetilde{Y}$ will have a missing fan with two 2-shells $R_1' \cup^{e'} R_2'$.

□

CHAPTER 9
Small-Cancellation Groups and $F_2 \times F_2$

In this chapter, we define 2-complexes called "bitori" which play an important role in this chapter. Also, we study locally convex maps $Y \to X$ that are associated with conjugacy classes of $F_2 \times \mathbb{Z}$ in $\pi_1 X$ where X is a C(6) complex or C(3)-T(6) complex. We then prove that C(3)-T(6) and C(6) groups do not contain a subgroup isomorphic to $F_2 \times F_2$.

9.1 Bitorus

In this section, we define bitorus for C(6) and C(3)-T(6) complexes and then we prove a flat annulus theorem for C(6) complexes.

Definition 31 (bitorus)**.** A *bitorus* is a compact and connected 2-complex homeomorphic with $B \times S^1$ where B is a finite connected leafless graph and $\chi(B) = -1$.

9.1.1 An Hexagonal Bitorus in C(6) Complexes

Definition 32 (Band and Slope)**.** Let X be a honeycomb in which all pieces have length 1. Two edges are *equivalent* if they are antipodal edges of a 2-cell in X. This generates an equivalence relation for 1-cells of X. A *band* is a minimal subcomplex of X containing an equivalence class. Note that a band corresponds to a sequence of hexagons inside a honeycomb where attaching 1-cells are antipodal. In a honeycomb we have three families of bands. Each band has two boundaries which we call *slopes*. So we have three different families of slopes. Let X be a complex whose universal cover $\widetilde{X}$ is a flat plane. An *immersed band* in X is the image of a band by the covering map. Note that interior of a band in X embeds but it is possible for slopes to get identified. Also note that bands do not cross themselves. Two distinct slopes are parallel if they do not cross. A bitorus that has C(6) structure is called an *hexagonal bitorus*.

A *flat annulus* is a concentric union of $n \geq 0$ bands. Equivalently, it is the complex obtained from a hexagonal torus by removing a single band.

Lemma 9.1.1. *Let A be a compact nonsingular annular $C(6)$ 2-complex with no spurs or i-shells with $i \leq 3$ along either of its boundary paths. Then A is a flat annulus.*

Proof. We assign a $\frac{2\pi}{3}$ angle to each internal corner of valence ≥ 3, a $\frac{\pi}{2}$ angle to each corner with a single boundary edge, and a π angle to all other corners. All internal 0-cells and all internal 2-cells have curvature ≤ 0. No closed 2-cell R intersects the same boundary path of A in two or more disjoint subpaths, since by Theorem 7.2.2 there would then be an i-shell with $i \leq 3$ in A at a subdiagram of A subtended by R, as indicated in Figure 9–1-(i). If some 2-cell R intersects both boundary paths of A, then by cutting along R, we obtain a disc diagram L with at most two i-shells (with $i \leq 3$) and hence L is a nonsingular "ladder" by Theorem 7.2.2, and consequently A was a "one band annulus" to begin with. See Figure 9–1-(ii).

Now we show that all 0-cells and 2-cells have curvature exactly 0. By Theorem 7.2.1, we have:

$$\sum_{f \in \text{2-cells(A)}} \text{curvature}(f) + \sum_{v \in \text{0-cells(A)}} \text{curvature}(v) = 2\pi \cdot \chi(A) \tag{9.1}$$

If there is an i-shell with $i \geq 5$ in one of the boundary paths, or there is an interior 2-cell with more than 6 pieces or an internal or external 0-cell of valence ≥ 4 then the left side of Equation (9.1) would be negative, but this would contradict that the right side is 0. Consider the 2-cells whose boundaries contain an edge in the outside boundary path of A. Since each of these 2-cells forms a 4-shell and since there are no valence 4 vertices on ∂A, we see that consecutive such 2-cells meet in a nontrivial piece, and this sequence of 2-cells forms a width 1 annular band. The subdiagram obtained by removing this band is again a nonsingular annular diagram with no i-shell with $i \leq 3$, for otherwise A would have had an internal 2-cell with ≤ 5 sides. The result now follows by induction, and A is a union of bands as in Figure 9–1-(iii). $\square$

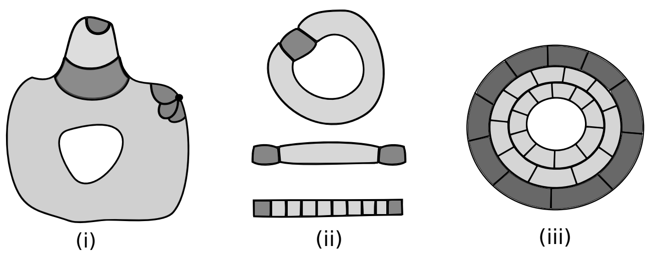

Figure 9–1: Figure (i) illustrates an i-shell which arises if a 2-cell is multiply external, and a (negatively curved) valence 4 vertex on ∂A. Figure (ii) illustrates the conclusion that can be drawn if a 2-cell contains an edge in both boundary paths. Figure (iii) illustrates the outer band of 4-shells that is sliced off to obtain a smaller annular diagram.

Remark 9.1.2. There are three families of hexagonal bitori:

The first family which is homeomorphic to a complex constructed by attaching a flat annulus to two tori along some slope, is the union of bands attached along parallel slopes. (As mentioned there are three families of slopes in a torus). Figure 9–2-(i) illustrates an example of this family where the attaching slope does not wrap around the torus. The second family is homeomorphic to a complex constructed by attaching two tori along some slope. Figure 9–2-(ii) illustrates this family but in general, a slope can wrap around a torus several times. The third is homeomorphic to the 2-complex obtained by attaching a flat annulus to a torus along two parallel and separate slopes.

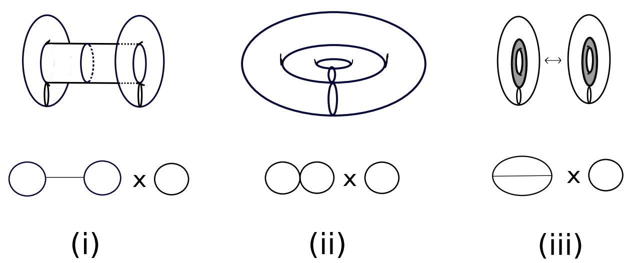

Figure 9–2: We illustrate the three types of hexagonal and triangular bitori. Figure (iii) is a bitorus obtained by attaching two hexagonal tori along the shaded regions which is the union of bands.

9.1.2 A Triangular Bitorus in C(3)-T(6) Complexes

Definition 33 (Zigzag-Band and Slope)**.** Let X be a triangular flat plane in which all pieces have length 1. A *zigzag-band* is a connected subcomplex of X homeomorphic to $\mathbb{R} \times [0, 1]$ which does not have any interior 0-cell and the valence of boundary 0-cells are 4. There are three families of zigzag-bands in a triangular flat plane. Each band has two boundaries which is called *slopes*. So we have three different families of slopes. Let X be a complex whose universal cover $\widetilde{X}$ is a flat plane. An *immersed zigzag-band* in X is the image of a zigzag-band by the covering map. Note that interior of a zigzag-band in X embeds but it is possible for slopes to get identified. Also note that zigzag-bands do not cross themselves. Two distinct slopes are parallel if they do not cross.

A *flat annulus* is a concentric union of $n \geq 0$ zigzag-bands, on the other hand it is the complex obtained from a triangular torus by removing a single zigzag-band.

Now we prove a lemma about flat annulus:

Lemma 9.1.3. *Let A be a compact nonsingular annular $C(3)$-$T(6)$ 2-complex with no zero cell of valence ≤ 3, along either of its boundary paths. Then A is a flat annulus.*

Proof. We make A an angled 2-complex by assigning a $\frac{\pi}{3}$ angle to each corner of valence ≥ 3. All 0-cells and all 2-cells have curvature ≤ 0. Since the right side of Equation (7.1) is zero, all 0-cells and 2-cells have curvature exactly 0, otherwise the left side of Equation (7.1) will be nonzero. Since each of the boundary 2-cells is triangle and since the boundary vertices have valence 4, we see that consecutive such 2-cells meet in a nontrivial piece, and this sequence of 2-cells forms a width 1 annular zigzag-band. The subdiagram obtained by removing this zigzag-band is again a nonsingular annular diagram with no valence ≥ 3 0-cell in the boundary, for otherwise A would have had an internal 2-cell with 2 sides. The result now follows by induction, and A is a union of zigzag-bands as in Figure 9–1-(iii). $\square$

Definition 34. A *triangular bitorus* is a bitorus that is a C(3)-T(6) complex.

Similar to C(6) case, there are three families of these complexes: The first family which is homeomorphic to a complex constructed by attaching a flat annulus to two tori along some slope, is the union of zigzag-bands attached along parallel slopes. (As mentioned there are three families of slopes in a torus). Figure 9–2-(i) illustrates an example of this family where the attaching slope does not wrap around the torus. The second family is homeomorphic to a complex constructed by attaching two tori along some slope. Figure 9–2-(ii) illustrates this family but in general, a slope can wrap around a torus several times. The third is homeomorphic to the 2-complex obtained by attaching a flat annulus to a torus along two parallel and separate slopes.

9.1.3 Locally Convex Maps to A Hexagonal Bitorus in C(6) Complexes

We now prove a lemma that plays an important role in Theorem 9.3.1.

Lemma 9.1.4. *Let X be a hexagonal bitorus. Let Y be a compact and connected $C(6)$ complex and $f: Y \to X$ a combinatorial map which is π_1-injective and locally convex. Then either $\pi_1 Y \cong 1$ or $\pi_1 Y \cong \mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z} \subseteq \pi_1 Y$.*

Example 9.1.5. The statement of Lemma 9.1.4 does not hold if we replace the "hexagonal bitorus" by an analogous complex Z that is constructed from three tori instead of two such that $\pi_1 Z \cong \langle a_1, a_2, a_3, a_4 \mid [a_i, a_{i+1}] = 1 : 1 \leq i \leq 4 \rangle$. Indeed let Z be the 2-complex obtained by attaching the 2-cells A, B and C to Z^1, as indicated in Figure 9–3. Let Y be the graph indicated in Figure 9–3 and observe that the inclusion map $i : Y \hookrightarrow Z$ is locally convex.

Example 9.1.6. For $C(4)$-$T(4)$ complexes, an immersion is *locally convex* if it has no missing i-shell for $i = 0, 1, 2$. But Lemma 9.1.4 fails in this case. Indeed $F_2 \times F_2 \cong \pi_1 X$ where X is the $C(4)$-$T(4)$ complex that is the product of two graphs.

Proof of Lemma 9.1.4. Consideration of all circles that are in the same slope as attaching circles yields a graph of spaces Γ_X whose vertex spaces are circles and whose edge spaces are bands. Figure 9–4-B illustrates Γ_X. Let X_v and X_e be

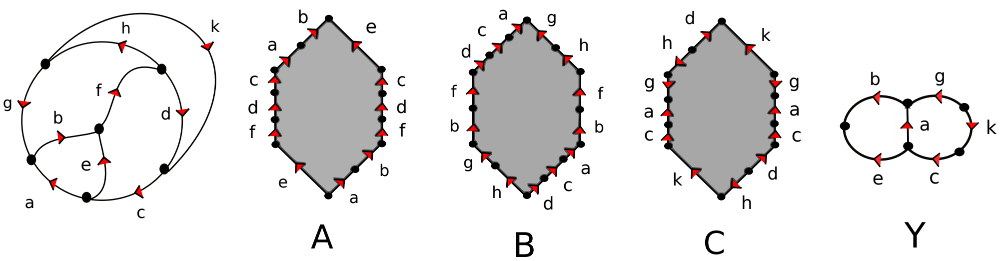

Figure 9–3: After attaching the 2-cells A, B and C to Z^1 we get a C(6) complex Z and the inclusion map $i : Y \hookrightarrow Z$ is locally convex.

respectively a vertex space and an edge space where $v \in \Gamma_X^0$ and $e \in \Gamma_X^1$. The graph of spaces for X will induce a graph of spaces Γ_Y for Y where:

$$Y_v = f^{-1}(X_v) \qquad\qquad Y_e = f^{-1}(X_e)$$

First, assume there is a vertex space in Γ_Y which is a circle called C. If there is no 2-cell attaching to C, since f is π_1-injective and locally convex, $Y = C$ and $\pi_1 Y = \mathbb{Z}$. Otherwise since there is no missing 3-shell, the edge space attached to C is a band and therefore all edge spaces in Y are bands. Figure 9–4-A illustrates an example of cylindrical edge space. In this case, if Γ_Y contains a circle then $\mathbb{Z} \times \mathbb{Z} \subseteq \pi_1 Y$ and if Γ_Y does not contain a circle then Y is homotopy equivalent to a circle and $\pi_1 Y \cong \mathbb{Z}$.

We have a map $g : Y \to \Gamma_Y$.

Case 1: Y does not contain a 2-cell and no vertex space of Γ_Y is a circle. Specifically each vertex space is each vertex space is a point or a subcomplex of a circle which is not closed. We show that Γ_Y has no valence 3 vertex. If there exists

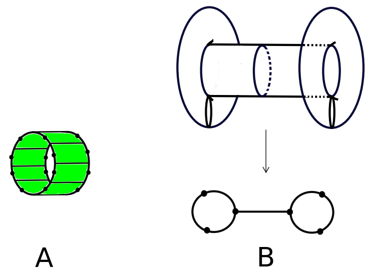

Figure 9–4: Figure A illustrates an edge space which is a band and in Figure B, the 2-complex is obtained by attaching a flat annulus to two tori and each torus is a union of three bands.

a valence 3 vertex in Γ_Y, then the image of Y by f locally looks like the dark path in 1-skeleton of X in Figure 9–5. In this case we will have a missing 3-shell which contains 0-cell with valence 3 in X and this is contradiction. So in this case, the valence of each vertex in Γ_Y is ≤ 2. Either Γ_Y is a circle and $\pi_1 Y \cong \mathbb{Z}$ or Γ_Y is not a circle in which Y is contractible and $\pi_1 Y \cong 1$.

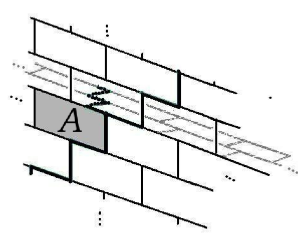

Figure 9–5: A is a missing 3-shell for the dark complex inside X.

Case 2: Y contains some 2-cells and no vertex space of Γ_Y is a circle. We show that $\pi_1 Y \cong \mathbb{Z}$ or 1. First note that since there is no missing 3-shell, the difference between the number of 2-cells on two adjacent edge spaces of Γ_Y is at most one. Assume a length 3 path in the graph Γ_Y then the corresponding 2-cells of edge spaces can not retreat and then extend. Therefore in a length 3 path in

Γ_Y the number of corresponding 2-cells of edge spaces can not decrease and then increase. Figure 9–11-B illustrates an example in which the number of the edge spaces corresponding to a path in Γ_Y retreat and then extend. If there is no valence 3 vertex in Γ_Y then either Γ_Y is a circle and $\pi_1 Y \cong \mathbb{Z}$ or Γ_Y does not contain a circle and $\pi_1 Y \cong \mathbb{Z}$.

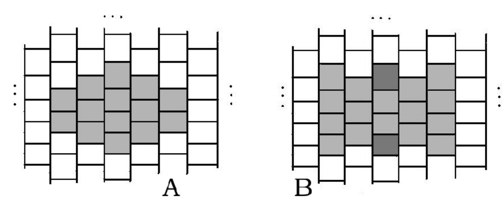

Figure 9–6: In Figure A, the number of 2-cells in edge spaces decreases after increasing (from right to left), so there is no missing 3-shell. But, in Figure B, this number first decreases and then increases and we have two very dark missing 3-shells.

Now, we will discuss the case in which we have a valence 3 vertex in Γ_Y. Consider 3 edge spaces ε_1, ε_2 and ε_3 meeting at a vertex space ν. Assume that they have the same number of 2-cells n. We know that the vertex space ν is a segment of a circle. Therefore it has two vertices of valence 1 called ω_1 and ω_2. One of the edge spaces ε_1, ε_2 and ε_3 contains none of ω_1 and ω_2. Assume ε_1 does not contain ω_1 and ω_2. In Figure 9–7-A the dark edge in the back is ε_1.

Considering ε_1 and ε_2, the edge space ε_1 is retreating in one side, also considering ε_1 and ε_3, the edge space ε_1 is retreating. Therefore we should not have missing 3-shell, the number of 2-cells in the edge space attached to ε which does not have intersection with ν is $n-1$. Also, since the number of 2-cells of edge spaces in this branch decreased from n to $n-1$, it should decrease in the next edge spaces. So we showed that if we have a valence 3 vertex in Γ_Y called υ and all edge spaces attached

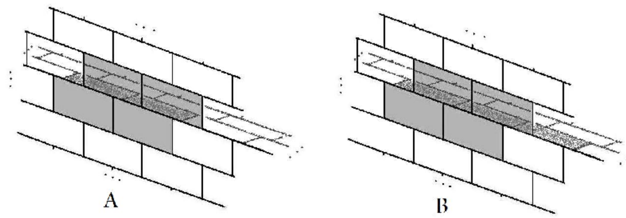

Figure 9–7: Three edge spaces $\varepsilon_1, \varepsilon_2, \varepsilon_3$ meet along a vertex space ν. The rear edge space is ε_1 Figure A, and ε_3 in Figure B.

to υ have the same number of 2-cells in Y, then in one of the paths (branches) called τ when we travel far from υ, the number of 2-cells of edge spaces will decrease. Therefore in this case, the image of τ in Γ_Y will not result a loop in Γ_Y and we can collapse τ without changing $\pi_1 Y$.

Now, assume the 3 edge spaces ε_1, ε_2 and ε_3 having intersection in the vertex space ν, do not have the same number of 2-cells. So two of them should have the same number of 2-cells m and the third one $m-1$ or $m+1$. Let ε_1 and ε_2 have m 2-cells. If ε_3 has $m-1$ 2-cells, then by the same argument in the previous case, we can collapse this branch without changing the fundamental group. Now assume that ε_3 has $m+1$ 2-cells. Since the number of 2-cells from the ε_3 to ε_1 and ε_2 decreases, the image of both of them in Γ_Y are not part of a loop. Figure 9–7-B illustrates this case.

Therefore if case 1 or case 2 occurs then either Γ_Y is homotopy equivalent to a circle in which case $\pi_1 Y \cong \mathbb{Z}$, or Y is contractible so $\pi_1 Y \cong 1$. $\square$

We will employ Lemma 9.1.4 in the following contrapositive form:

Corollary 9.1.7. *There is no locally convex, π_1-injective map $Y \to X$ where X is a hexagonal bitorus and Y is a compact connected 2-complex with $\pi_1 Y \cong F_2$.*

9.1.4 Locally Convex Maps to A Triangular Bitorus in C(3)-T(6) Complexes

The following Lemma is similar to Lemma 9.1.4, but for C(3)-T(6) complexes. This Lemma is important in the proof of Theorem 9.3.1.

Lemma 9.1.8. *Let X be a triangular bitorus. Let Y be a compact and connected $C(3)$-$T(6)$ complex and $f : Y \to X$ a combinatorial map which is π_1-injective and locally convex. Then either $\pi_1 Y \cong 1$ or $\pi_1 Y \cong \mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z} \subseteq \pi_1 Y$.*

Example 9.1.9. In Lemma 9.1.8, if we let X to be an analogous complex that is constructed from three tori instead of two such that $\pi_1 X \cong \langle a_1, a_2, a_3, a_4 \mid [a_i, a_{i+1}] = 1 : 1 \leq i \leq 4\rangle$, then the statement does not hold. Indeed let X be the 2-complex obtained by attaching the 2-cells A, B, C, D, E and F to X^1, as indicated in Figure 9–8. Let Y be the graph indicated in Figure 9–8 and observe that the inclusion map $i : Y \hookrightarrow Z$ is locally convex.

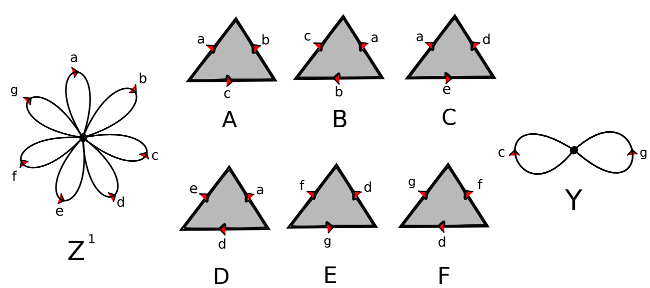

Figure 9–8: After attaching the 2-cells A, B, D, E and F to Z^1 we get a C(3)-T(6) complex Z and the inclusion map $i : Y \hookrightarrow Z$ is locally convex.

As we mentioned before, the statement of the Lemma fails for $C(4)$-$T(4)$ complexes.

Proof of Lemma 9.1.8. First consider the angle $\frac{\pi}{3}$ for each corner of each 2-cell in X. Let Γ_X be a graph of spaces whose vertex spaces are circles which are in the same slope as attaching circles of X and whose edge spaces are zigzag-bands. Figure 9–9-B illustrates Γ_X. Denote the vertex space corresponding to $v \in \Gamma_X^0$ by X_v and the edge space corresponding to $e \in \Gamma_X^1$ by X_e. The graph of spaces for X will induce a graph of spaces Γ_Y for Y where $Y_v = f^{-1}(X_v)$ and $Y_e = f^{-1}(X_e)$.

First, assume there is a vertex space in Γ_Y which is a circle called C. If there is no 2-cell attaching to C, since f is π_1-injective and locally convex, $Y = C$ and $\pi_1 Y = \mathbb{Z}$. Otherwise since Y has no missing 1-shell or missing doubled-2-shell, the edge space attached to C is a zigzag-band and therefore all edge spaces in Y are zigzag-bands. (Figure 9–9-A illustrates a complex containing two zigzag-bands which is a graph of spaces with tree vertices and two edges.) In this case, if Γ_Y contains a circle then $\mathbb{Z} \times \mathbb{Z} \subseteq \pi_1 Y$ and if Γ_Y does not contain a circle then Y is homotopy equivalent to a circle and $\pi_1 Y \cong \mathbb{Z}$.

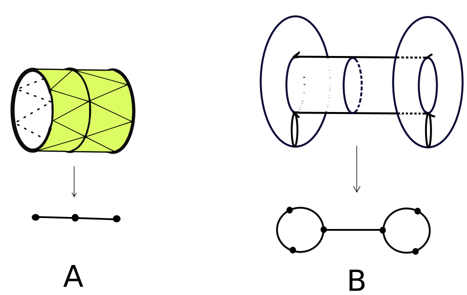

Figure 9–9: Figure A illustrates an edge space which is a band and in Figure B, the 2-complex is obtained by attaching a flat annulus to two tori and each torus is a union of three bands.

Now assume no vertex space of Γ_Y is a circle. We have two cases:

Case 1: Y does not contain a 2-cell. In this case, each vertex space of Γ_Y is a non-closed subcomplex of a circle. We show that Γ_Y has no valence 3 vertex. If there exists a valence 3 vertex in Γ_Y, then the image of Y by f locally looks like the dark path in 1-skeleton of X in Figure 9–10. Therefore, in this case we will have a missing fan with two 2-shells R_1 and R_2 containing the 0-cell with valence 3 which is a contradiction. So in this case, the valence of each vertex in Γ_Y is ≤ 2. Either Γ_Y is a circle and $\pi_1 Y \cong \mathbb{Z}$ or Γ_Y is not a circle in which Y is contractible and $\pi_1 Y \cong 1$.

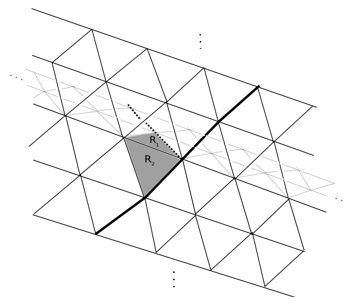

Figure 9–10: $F = R_1 \cup R_2$ is a missing fan with two 2-shells R_1 and R_2 for the dark complex inside X.

Case 2: Y contains some 2-cells. We show that in this case $\pi_1 Y \cong \mathbb{Z}$ or 1. First note that since there is no missing 1-shell or missing fan with two 2-shells, the difference between the number of 2-cells on two adjacent edge spaces of Γ_Y is at most two. Consider a length 3 path in Γ_Y and the corresponding 2-cells of edge spaces in Y. The number of these edge spaces cannot decrease and then increase from one edge space to another, otherwise there will be a missing 1-shell or missing fan with two 2-shells. This situation is illustrated in Figure 9–11-B, where the number of the edge spaces corresponding to a path in Γ_Y decrease and then increase. Note that indeed increasing and then decreasing the number of 2-cells yields a change in the slope of boundary which results an angle $\frac{2\pi}{3}$ in the boundary and therefore results a missing fan with two 2-shells. If there is no valence 3 vertex in Γ_Y then either Γ_Y is a circle and $\pi_1 Y \cong \mathbb{Z}$ or Γ_Y does not contain a circle and $\pi_1 Y \cong \mathbb{Z}$.

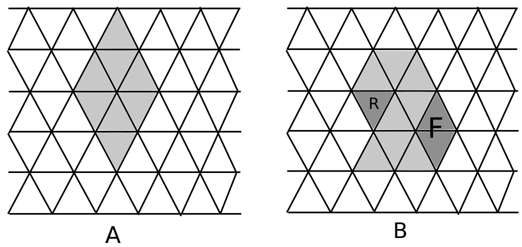

Figure 9–11: The number of 2-cells in Figure A decreases after increasing, so there is no missing 1-shell or missing fan. On the other hand, in Figure B, this number first decreases and then increases and we have a dark missing 1-shell R and a dark missing fan F.

We now consider the case in which there is a valence 3 vertex in Γ_Y. Let V be the vertex space which is adjacent to three edge spaces E_1, E_2 and E_3. Note that the vertex space V is a non-closed subcomplex of a circle.

Assume E_1, E_2 and E_3 have the same number of 2-cells n. For this case we have two subcases:

-n is even. In each E_i consider the first and the nth 2-cells. Each of these 2-cells have a side which intersects V and does not intersect any other 2-cell of E_i, we call these two sides, the boundaries of E_i. Since there is no missing fan with two 2-shells each boundary of E_1 intersects the corresponding boundary of E_2 with the angle π, by the same reason each boundary of E_1 intersects the corresponding boundary of E_2 with the angle π but then one of the boundaries of E_2 intersects the corresponding boundary of E_3 with the angle $\frac{2\pi}{3}$ which yields a missing fan with two 2-shells, Figure 9–12-A illustrative this situation.

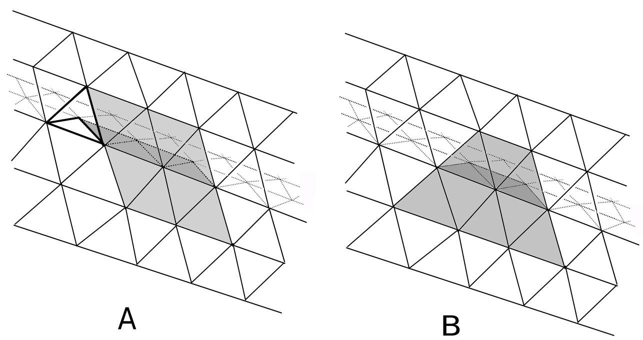

Figure 9–12: Three edge spaces E_1, E_2, E_3 meet along a vertex space V. In Figure A the fan whose boundary is heavy dark is missing in the dark subcomplex containing E_1, E_2 and E_3.

-n is odd. Since there is no missing fan with two 2-shells, for each i the edge space attached to E_i which does not have intersection with V, has $n-2$ 2-cells. So we showed that if we have a valence 3 vertex in Γ_Y whose corresponding vertex space in Y is V and all edge spaces attached to V have the same number of 2-cells $n = 2k+1$ in Y, then in all three branches, when we travel far from V, the number of 2-cells decrease. Therefore in this case we can collapse all these three branches without changing $\pi_1 Y$ and $\pi_1 Y \cong 1$.

Now, we assume the 3 edge spaces E_1, E_2 and E_3 which have intersection in the vertex space V, do not have the same number of 2-cells. Therefore without loss of generality we can assume the number of 2-cells in E_1 is m and the number of 2-cells in E_2 is $m-1$ or $m-2$. Now by the same argument in the previous case, we can collapse the branch containing E_2 without changing the fundamental group of Y, since its image in Γ_Y can not be part of a cycle. Figure 9–12-B illustrates this case.

Therefore in these two cases, case 1 or case 2 either Γ_Y is homotopy equivalent to a circle in which case $\pi_1 Y \cong \mathbb{Z}$, or Y is contractible so $\pi_1 Y \cong 1$. □

Lemma 9.1.8 shows the following:

Corollary 9.1.10. *There is no locally convex, π_1-injective map $Y \to X$ where X is a triangular bitorus and Y is a compact connected 2-complex with $\pi_1 Y \cong F_2$.*

9.2 The existence of a Bitorus, and C(6) and C(3)-T(6) Complexes

In this section, we prove that if X is a C(6) and C(3)-T(6) complex where $F_2 \times \mathbb{Z} \subseteq \pi_1 X$, then there is a locally convex map from a bitorus to X.

Lemma 9.2.1. *Let $f : \widetilde{Y} \to \widetilde{X}$ be a map where Y is a hexagonal bitorus and X is a C(6) complex. Then f is strongly locally convex. In particular f is locally convex.*

Proof. We first show that f is locally convex. Suppose that $Q \to \widetilde{Y}$ is the outer path of a missing i-shell R with $i \leq 3$. Since the inner path of R is the concatenation of $i \leq 3$ pieces, the C(6) condition applied to R shows that Q cannot be the concatenation of ≤ 2 pieces in $\widetilde{Y}$. It follows that Q must fully contain two consecutive maximal pieces in the boundary of a single 2-cell R' of $\widetilde{Y}$. But this violates the C(6) condition for R'.

Having proven local convexity, we turn to strong local convexity. Suppose R is a 2-cell that is not in $\widetilde{Y}$ such that $\bar{R} \cap \widetilde{Y} \neq \varnothing$. Assume that $P = \bar{R} \cap \widetilde{Y}$ cannot be expressed as the concatenation of at most two pieces. As before, consideration of paths in the honeycomb $\widetilde{Y}$, shows that P contains two consecutive maximal pieces in the boundary of a single 2-cell R'. This violates the C(6) condition for R'. □

Lemma 9.2.2. *Let $f : \widetilde{Y} \to \widetilde{X}$ be a map where Y is a triangular bitorus and X is a $C(3)$-$T(6)$ complex. Then f is locally convex.*

Proof. let R be a missing 1-shell for Y. Since the inner path of R is just one piece, C(3) condition shows that the outer path of R cannot be just one piece as well. Therefore since the pieces have length one in X, the outer path fully contain two consecutive pieces in the boundary of a single 2-cell R' of $\widetilde{Y}$. But this violates the T(6) condition.

Now assume that Y have a missing fan with two 2-shells $R_1 \cup^e R_2$, where e is the intersection of $\bar{R}_1 \cap \bar{R}_2 = e$ i.e. the common edge which has two vertices v and w. Let $\partial R = PS$ where S is the concatenation of 2 pieces joining along the vertex v such that one of the pieces is in the boundary of R_1 and the other in the boundary of R_2. Note that for each i, $\bar{R}_i \cap Y$ cannot be more than a piece (edge), otherwise the T(6) condition violates. Now consider w, it has a corner in R_1 and a corner in R_2, therefore it should have 4 corners in 2-cells of Y, but then there will be a disc diagram $D \to Y$ where w has valence 4, which violates T(6) condition. Therefore there is no missing fan with two 2-shells. $\square$

Lemma 9.2.3. *Let X be a $C(6)$ complex (or a $C(3)$-$T(6)$ complex) such that $F_2 \times \mathbb{Z} \subseteq \pi_1 X$. There exists a 2-complex Y' equal to $Y \vee Q$ where Y is a hexagonal bitorus (triangular bitorus), $Q = [0, n]$ and 0 is identified with a 0-cell in Y and n is the basepoint. And there exists a basepoint preserving map $f : Y' \to X$ such that $f\mid_Y$ is* *locally convex and the following diagram commutes:*

$$\begin{array}{ccc} & & \pi_1 Y' \\ & \overset{\cong}{\swarrow} & \downarrow \\ F_2 \times \mathbb{Z} & \hookrightarrow & \pi_1 X \end{array}$$

Proof. We will construct $Y' = Y \vee Q$ and an immersion $f : Y' \to X$, by Lemma 9.2.1 (Lemma 9.2.2), $f\mid_Y$ is locally convex.

Let $\nu \in X^0$ be the basepoint. Let $F_2 \times \mathbb{Z} \cong \langle a_1, a_2, c \mid [a_1, c], [a_2, c] \rangle$. For $i = 1, 2$, let A_i and C_i be closed, based paths in X, such that A_i represents a_i and C_1 and C_2 both represent c. For $i = 1, 2$, let D_i be a minimal area disc diagram with boundary path $A_i C_i A_i^{-1} C_i^{-1}$. Moreover we shall make the above choices such that D_i has minimal area among all such choices of A_i and C_i. By identifying the top and bottom C_i paths and identifying the left and right A_i paths we obtain a quotient T_i of D_i. Observe that $T_i = T_i' \vee [0, n_i]$ is the wedge of a torus with $[0, n_i]$. Moreover there exists an induced combinatorial map $T_i \to X$. The minimality of D_i ensures that $T_i' \to X$ is an immersion. Let $V_i = [0, n_i]$. By possibly folding, we can shorten V_i to assume that $T_i \to X$ is also an immersion. Note that our original paths $C_i \to X$ corresponds to two paths $C_i \to D_i$ which are then identified to a single path $C_i \to T_i$ which we shall now examen. For $i = 1, 2$, let U_i be an embedded closed path in T_i' such that $C_i = V_i U_i V_i^{-1}$. The complex T_i is illustrated in Figure 9–13-A.

Let A be an annular diagram whose boundary paths P_1, P_2 are respectively homotopic to the image of U_1 in $U_1 \to T_1' \to X$ and the image of U_2 in $U_2 \to T_2' \to X$ and whose conjugator is path-homotopic to the image of $V_1^{-1} V_2$ in $V_1^{-1} V_2 \to X$. Moreover choose A such that it has minimal area with these properties. Note that

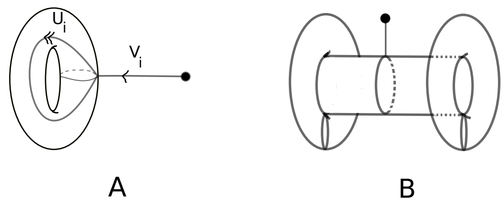

Figure 9–13: On the left is T_i which is an hexagonal torus with an arc attached to it. On the right is N.

A is non-singular since U_1 is path-homotopic to U_2. Consider the base lifts of $\widetilde{T}_1$ and $\widetilde{T}_2$ to $\widetilde{X}$. Note that these determine lifts of $\widetilde{T_1}'$ and $\widetilde{T_2}'$. Either $\widetilde{T_1}'$ and $\widetilde{T_2}'$ intersect or do not intersect.

We first consider the case where $\widetilde{T_1}'$ and $\widetilde{T_2}'$ do not intersect. Observe that in C(6) case, for $0 \leq i \leq 3$, A has no missing i-shell along either boundary path. Indeed ${T_1}'$ and ${T_2}'$ do not have missing i-shell and so if A had an i-shell, we could reduce its area. Moreover, in C(3)-T(6) case, A does not have a boundary 0-cell whose degree in A is ≤ 2. Since if there is a boundary 0-cell in A with valence ≤ 2, then we could reduce the area of A. Therefore by Lemma 9.1.1 (9.1.3 in C(3)-T(6) case), A is a flat annulus. Let Y be the 2-complex obtained by attaching A to ${T_1}' \sqcup {T_2}'$ along P_1, P_2. Note that the annulus is nonsingular and it is a strip whose one side is identified in ${T_1}'$ and the other is identified in ${T_2}'$. Since A has minimal area, by the above argument, there is no folding and S is a bitorus of the first type. Moreover since the conjugator of A is path-homotopic to ${V_1}^{-1}V_2$, there is a path $Q = [0, n]$ where 0 is identified in A and n is identified with the basepoint $n_1 = n_2$ in V_1 and V_2. Figure 9–13-B shows the 2-complex $Y \vee Q$. In conclusion, in this case Y' equals

$Y \vee Q$ where Y is a bitorus of the first type and Q corresponds to a basepath $[0, n]$ where 0 is identified in A and n is the basepoint.

We now consider the case where $\widetilde{T_1'}$ and $\widetilde{T_2'}$ intersect. By Lemma 8.1.2 (8.2.1 in C(3)-T(6) case), $\widetilde{T_1'} \cap \widetilde{T_2'}$ is a locally convex subcomplex of $\widetilde{X}$ and it is also infinite since the element c stabilizes both $\widetilde{T_1'}$ and $\widetilde{T_2'}$. Observe that since $\widetilde{T_1'} \cap \widetilde{T_2'}$ is locally convex and infinite, it is a slope or union of consecutive bands (zigzag-bands). Therefore $\widetilde{T_1'} \cap \widetilde{T_2'}$ contains a periodic line. Let $A_1 = \widetilde{A}/c$ and let $S = T_1' \sqcup T_2'/A_1$.

If there is no folding between T_1' and T_2', then A_1 is a slope and Y' equals $Y \vee Q$ where $Y = S$ is a bitorus of the second type and Q corresponds to a basepath $[0, n]$ where 0 is identified in the common slope and n is the basepoint. Otherwise, we start to fold T_1' with T_2'. Note that the folding process cannot identify T_1' with T_2', since $\pi_1 T_1'$ and $\pi_1 T_2'$ are not commensurable in $\pi_1 X$. By considering all slopes parallel to a given slope, it is natural to regard T_i' as a graph of spaces whose vertex spaces are circles and whose edge spaces are bands (zigzag-bands). If a 2-cell R_1 in T_1' is folded with a 2-cell R_2 in T_2' then the entire band containing R_1 will be folded with the band (zigzag-band) containing R_2. Moreover, note that the C(6) and C(3)-T(6) structure of S ensures that these bands consist of the same number of 2-cells. As a result T_1' and T_2' will be identified along a union of consecutive bands (zigzag-band) and we call the obtained complex Y. In conclusion the result of folding process in this case is a complex Y' equal to $Y \vee Q$ where Y is a bitorus of the third type and Q corresponds to a basepath $[0, n]$ where 0 is identified in one of T_1' or T_2' and n is the basepoint. Moreover in all cases, there exists an induced combinatorial map $f : Y' \to X$ such that $f\mid_Y$ is an immersion and therefore locally convex by Lemma 9.2.1 (9.2.2 in C(3)-T(6) case). $\square$

9.3 No $F_2 \times F_2$ in C(6) and C(3)-T(6) Groups

In this section, we prove the main theorem of this chapter:

Theorem 9.3.1. *A $C(6)$ group cannot contain $F_2 \times F_2$. The same statement holds for a $C(3)$-$T(6)$ group.*

Proof. Let X be a based C(6) complex (a based C(3)-T(6) complex) whose fundamental group is G. Suppose that G contains $F_2 \times F_2 \cong \langle a, b\rangle \times \langle c, d\rangle$, we will reach a contradiction. Without loss of generality, we can assume that each 1-cell of X lies on a 2-cell. Indeed, since we are arguing by contradiction, we can replace X by a smallest π_1-injective subcomplex X_o whose fundamental group contains $\langle a, b\rangle \times \langle c, d\rangle$. Since $F_2 \times F_2$ does not split as a free product, if X_o contained a 1-cell not on the boundary of a 2-cell, then we could pass to a smaller π_1-injective subcomplex whose fundamental group contains $F_2 \times F_2$.

Consider the subgroups $G_1 = \langle a, b\rangle \times \langle c\rangle \cong F_2 \times \mathbb{Z}$ and $G_2 = \langle a, b\rangle \times \langle d\rangle \cong F_2 \times \mathbb{Z}$. Since $G_i \subset G$, by Lemma 9.2.3, there exists a 2-complex Y_i' that equals $Y_i \vee Q_i$ where Y_i is a bitorus, $Q_i = [0, n_i]$ and n_i is identified with a point in Y_i and 0 is the basepoint. Moreover, there exists a basepoint preserving map $f : Y_i' \to X$ such that $f\mid_{Y_i}$ is locally convex and the following diagram commutes:

$$\begin{tikzcd} & \pi_1 Y_i' \arrow[dl, "\cong"'] \arrow[d] \\ G_i \arrow[r, hook] & \pi_1 X \end{tikzcd}$$

By Remark 8.0.5, there exists k such that $\widetilde{Y}_1 \cap \mathsf{N}^k(\widetilde{Y}_2) \neq \varnothing$.

Since by Lemma 8.1.4 (Lemma 8.2.3), $\widetilde{Y}_1 \to \widetilde{X}$ and $\mathsf{N}^k(\widetilde{Y}_2) \to \widetilde{X}$ are both locally convex, by Lemma 8.1.2 (Lemma 8.2.1), $\widetilde{Y}_1 \cap \mathsf{N}^k(\widetilde{Y}_2) \to \widetilde{X}$ is also locally convex. Now observe that $\langle a, b\rangle = stab(\widetilde{Y}_1) \cap stab(\mathsf{N}^k(\widetilde{Y}_2)) \subseteq stab(\widetilde{Y}_1 \cap \mathsf{N}^k(\widetilde{Y}_2))$. Moreover the quotient space $Z = (stab(\widetilde{Y}_1 \cap \mathsf{N}^k(\widetilde{Y}_2)) \setminus (\widetilde{Y}_1 \cap \mathsf{N}^k(\widetilde{Y}_2)))$ is compact, since Z is a component of the fibre product of the maps $Y_1 \to X$ and $\mathsf{N}^k(Y_2) \to X$. Since $F_2 \cong \langle a, b\rangle \subseteq \pi_1 Z$ and $Z \to Y_i'$ is locally convex with Y_i' a bitorus, this contradicts Corollary 9.1.7 (Corollary 9.1.10). $\square$

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