A MARKOV MODEL FOR FATIGUE CRACK GROWTH AND THE PREDICTION OF COMPONENT RELIABILITY

by

STEVEN R. BOHN

Department of Mechanical Engineering

McGill University

Montreal, Canada.

A thesis submitted to the Faculty of Graduate Studies and Research

in partial fulfillment of the requirements for the degree of

Master of Engineering

May, 1990.

Abstract

This thesis presents a probabilistic model, specifically a nonhomogeneous Markov model, for the description of fatigue crack growth. This model is capable of predicting the statistics of crack growth, the mean crack size and the variance in the crack size, at future times given two empirically determined system parameters, λ and κ, and an initial crack size distribution. Prior to the mathematical development of this model a brief review of existing crack growth models is presented. After the mathematical development of the model has been presented the results from a previous application in the study of pitting corrosion are presented. Following this, the results from the modelling of several existing fatigue data sets are examined in order to determine the necessary empirical material parameters and also to verify the model's applicability. Once the model has been verified and the empirical parameters determined one data set is chosen in order to illustrate the uses of this model for component reliability predictions. These uses include reliability at a future time, inspection optimization and the effect of changes in repair policy on reliability.

Résumé

Cette thèse présente un modèle probabiliste, en particulier un modèle Markov non-homogène, qui décrit la propagation d'une fissure en fatigue. Ce modèle est capable de prédire les statistiques de propagation d'une fissure, la grandeur moyenne et la variance de la fissure, étant donné deux paramètres du système, λ et κ,qui ont été déterminés empiriquement, et la distribution initiale de la grandeur des fissures. Avant de procéder au développement mathématique de ce modèle, une brève révision des modèles actuels en propagation de fissure est presentée. Après la présentation du développement, les résultats provenant d'une application précédente d'une étude de piqûres de corrosion sont présentés. Ensuite, les résultats qui ont été produits par le modelage des séries actuelles de données sont évalués dans le but de déterminer les paramètres empiriques nécessaires du système et aussi afin de vérifier l'application du modèle. Une fois que le modèle est verifié et que les paramètres empiriques sont déterminés, une série de données est choisie afin d'étudier les usages de ce modèle pour prédire la fiabilité des composants. Ces usages comprennent la fiabilité dans un temps futur, l'optimization des inspections et l'effet que des changements de la politique de réparation produisent sur la fiabilité.

Acknowledgements

The author would like to thank the following people for their assistance:

My wife Tina, whose constant support and encouragement have made the completion of this thesis possible.

Prof. J.W. Provan, my thesis supervisor, whose support and enthusiasm have never faltered and whose confidence in me gave me confidence in myself.

Dr. S.D. Manning, of General Dynamics, for providing the data used in this report.

Nadia Kosseim, a fellow student, for her help in editing the French version of the abstract.

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Chapter 1

INTRODUCTION AND MOTIVATION

1.1 The Importance of Fatigue and Reliability

The phenomenon known as fatigue is the primary cause of failure of components and structures. Fatigue is the process whereby a component will fail when subjected to alternating stresses at a level below the limit stress of the material. These failures can have a wide variety of effects, ranging from inconvenience to loss of life. Since almost all engineering structures are subjected, to some degree, to fluctuating loads, it is vital that the design engineer have the ability to account for them.

It is accepted engineering practice to treat a metal as a homogeneous continuum. In the case of static stress analysis this is often a valid assumption which does not usually lead to any serious errors. When performing a fatigue analysis, however, this is no longer true. It is precisely because the material is no homogeneous that scatter is observed in fatigue test data even in strictly controlled laboratory experiments. The scatter is due to microscopic defects and necessitates the use of probability theory to accurately describe both fatigue crack initiation and propagation.

This use of probabilistic methods leads directly to the concept of reliability, which has been defined by Bompas-Smith [1] as *the probability that a component will perform*

satisfactorily for a specified period of time under specified operating conditions. By making use of this concept engineers are able to compare their designs on a quantitative basis and can make necessary modifications in order to reach a desired reliability level.

Petroski [2] has reviewed a recent study by the National Bureau of Standards and Battelle Columbus Laboratories [3] which came to the conclusion that the total costs of fracture, which includes many diverse phenomenon, amount to well over *$100 billion annually* in the United States alone. These costs arise not only from the replacement of broken components but are also due to overdesign, inventory costs, inspection costs, insurance against failure and many other sources. This report goes on to conclude that these costs could be reduced by one-half by making better use of available technology and also by the use of the improved techniques of fracture control expected to come from future research. Petroski also states that it is estimated that 50 to 90 % of all structural failures are a result of crack growth. These statements illustrate the importance of the study of fatigue crack growth and are most definitely a source of motivation for this thesis.

It is the task of the reliability engineer to assist the design engineer with design review procedures and statistical analyses. However, the designer still remains the key person to ensure component and system reliability[4]. With the advent of today's increasingly complex structures it is no longer sufficient to depend on good design practices to ensure structural integrity. Reference [5] lists some of the motivating factors for the study of the reliability of mechanical components :

1. **Lack of Design Experience.** Changes in technology occur quite rapidly and consequently mechanical designers no longer have the time to master the design, especially when complex equipment is designed for use in aerospace or military applications.

2. **Cost and Time Constraints.** Due to the costs and time involved, the designer cannot learn from past mistakes. In other words, the trial and error approach cannot be used.

3. **Optimization of resources.** The workable design is no longer considered sufficient. The design must be optimized subject to the constraints on reliability, cost, weight, performance, size, etc.

4. **Stringent Requirements and Severe Environments.** Because of the large-scale investments in developing systems to be used under severe environments, such as the military or space, the reliability problem becomes more important.

5. **Influence from Electronic Reliability.** The vastly improved techniques for predicting electronic reliability, and their success, stimulated similar developments in mechanical engineering.

1.2 Thesis Objectives

It was proposed by Forsyth [6] that there are two stages in the fatigue process, this is illustrated in Figure 1.1. *Stage I* is called crack initiation where the crack, which often forms at a local discontinuity, is extremely small. When the crack has been initiated it propagates in a direction perpendicular to the applied stress, this is *Stage II*. It is this stage that is of great interest in practical applications as the crack is large enough to be detected and its growth monitored. It is, therefore, with this stage that this report is concerned.

Not only do engineering materials contain discontinuities but each one of a group of supposedly identical components will have a different number and distribution of them. This is largely responsible for the uncertainty involved with fatigue crack size predictions. Because of this, simply predicting the mean crack size by deterministic methods is not sufficient, probabilistic methods should be employed. This thesis will detail the development of a non-homogeneous Markov model that will predict not only the mean crack size but its variance as well. This mathematical model will assist the engineer who is trying to determine: the reliability of a component at some future time; the optimum time for an inspection procedure; and the inspection schedule necessary

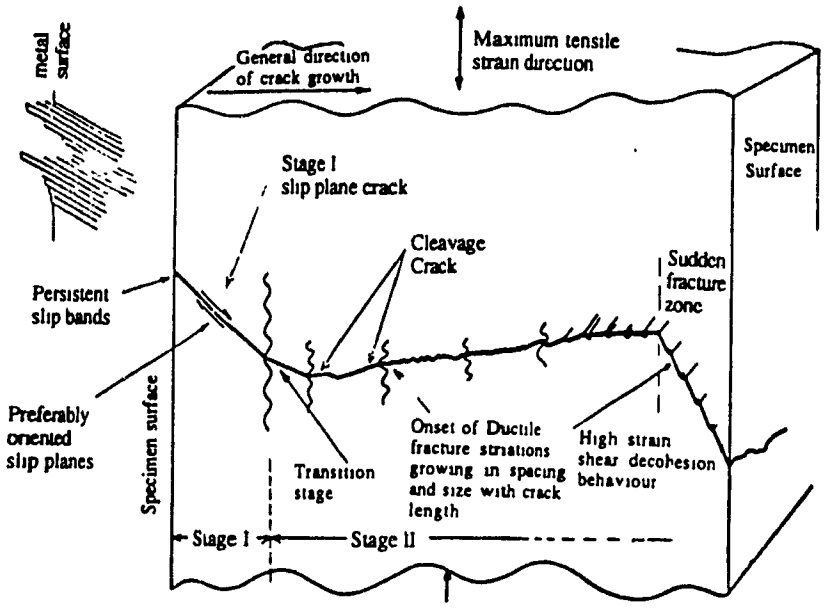


Figure 1.1: The Two–Stage Fatigue Process [6]

to maintain a certain level of reliability. By allowing the engineer to determine how component reliability would be affected by various changes in inspection schedule, allowable flaw size, quality of inspection procedure, etc. this model can become a valuable tool for reliable designs.

1.3 Thesis Organization

Chapter 2 This chapter will present an introduction to fatigue and reliability, including some of the methods that have been used for reliability calculations.

Chapter 3 This chapter will detail the mathematical development of the Markov model to be used in this thesis.

Chapter 4 The specific research objectives of the current work will be briefly presented here.

Chapter 5 In this chapter the Markov model will be applied to several data sets (OFHC Cu, WPF, XWPF) and the results for the modelling of crack growth will be given.

Chapter 6 The reliability methodology used in this thesis and the results of the reliability calculations will be presented and discussed in this chapter.

Chapter 7 This chapter will contain the conclusions and the recommendations for future work.

Chapter 2

INTRODUCTION TO FATIGUE AND RELIABILITY

2.1 Fracture Mechanics and Fatigue

2.1.1 Introductory Fracture Mechanics

Fracture Mechanics concepts are used today by design engineers in order to anticipate and control both brittle fracture and fatigue failure. The first attempts at solving problems involving fracture were made by Griffith [7,8] and for this he is known today as the father of fracture mechanics theory. The Griffith energy criterion, as it is known, considers an infinite plate of unit thickness with a central transverse crack of length $2a$. With the ends fixed, a stress, σ, is induced over the plate and the crack extends by a distance, da. Figure 2.1 illustrates the changes in the load-displacement diagram, as well as a schematic diagram of the plate.

Originally the energy contained in the plate is given by the area OAB. As the crack grows by da the elastic energy stored in the plate decreases to OCB due to a decrease in plate stiffness. Therefore, there is a release of energy equal to the area OAC. What Griffith stated was that the crack will propagate if the energy released by crack growth

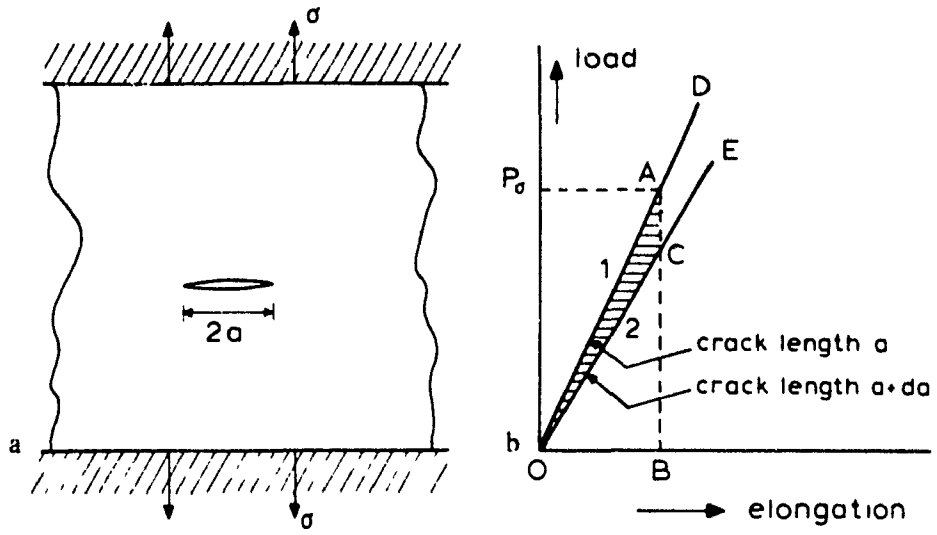


Figure 2.1: The Griffith plate and load-displacement diagram [9]

provides the necessary energy for the crack to grow. This can be written as:

$$\frac{dU}{da} = \frac{dW}{da}\,, \tag{2.1}$$

where:

U = elastic energy of the plate, and

W = energy required for crack growth or fracture energy.

Making use of the work of Inglis [10], Griffith calculated dU/da per unit thickness and per half crack length, a, as :

$$\frac{dU}{da} = \frac{\pi\sigma^2 a}{E} = G\,, \tag{2.2}$$

where:

E = Young's Modulus

G = Energy release rate per unit crack extension, whose units are energy per unit thickness, per unit of crack extension

The energy spent in propagating a crack (dW/da) can also be denoted by R, the crack resistance of the material. Using this, Equation 2.1 states that G must be at least equal to R for crack propagation to occur, or, the condition for propagation can be written as:

$$G \geq R\,. \tag{2.3}$$

Griffith derived his equation for glass, which is extremely brittle. He assumed, therefore, that R was related to surface energies only. In ductile materials, however, plastic deformation occurs at the crack tip. The energy associated with the production of this plastic zone may be considered as the energy required for crack growth.

As a result, Linear Elastic Fracture Mechanics (LEFM) is invalidated by the elastic-plastic behavior and formation of large plastic zones in engineering materials. Because of this it is necessary to determine what is meant by a *large* plastic zone. Provan[11] has discussed the mathematical development of LEFM stress fields where he shows the existence of a stress singularity term that is inversely proportional to the square root of the radius of the plastic zone ($r^{-1/2}$). Irwin[12] made his first plastic zone size estimation as:

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r_y}} = \sigma_{ys}\,, \tag{2.4}$$

where:

$K_I =$ $Y\sigma\sqrt{\pi a}$ (and Y is a constant of proportionality)

rearranging, we have:

$$r_y = \frac{1}{2\pi}\left(\frac{K}{\sigma_{ys}}\right)^2 = \frac{a}{2}\left(\frac{\sigma}{\sigma_{ys}}\right)^2\,. \tag{2.5}$$

The actual size of the plastic zone must be larger than r_y because the load represented by the shaded area in Figure 2.2 must be sustained. Irwin therefore modified the plastic zone size and arrived at a value of plastic zone size, r_p, of

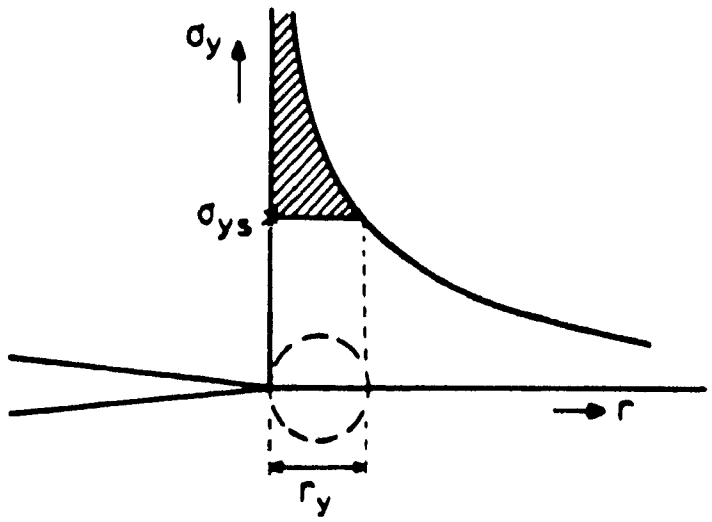


Figure 2.2: Irwin's first estimate of plastic zone size [12]

$$r_p = 2r_y = \frac{1}{\pi}\left(\frac{K}{\sigma_{ys}}\right)^2 , \tag{2.6}$$

where $\left(\frac{K}{\sigma_{ys}}\right)^2$ is called the plastic constraint factor.

The plastic region for Plane Stress is larger than that for Plane Strain(for the mathematical development see [11]), this means that, in general, plane stress failure will be ductile while plane strain fracture will be brittle, even for a material that is generally ductile. Considering the plastic region in front of a three–dimensional crack front as in Figure 2.3, it is easily observed that the plastic zone is larger at the free surfaces. This is expected since no matter how thin the specimen, plane stress must exist at a free surface.

This explains the observed results of laboratory tests where thin samples show higher values of fracture toughness, K_{Ic}. Since plane stress fracture toughness is influenced by specimen geometry it is important that in testing for a materials K_{Ic} value plane strain conditions are maintained. This can be accomplished by using specimens that have a thickness larger than a limiting thickness where the critical value of stress intensity factor reaches its minimum plane strain value.

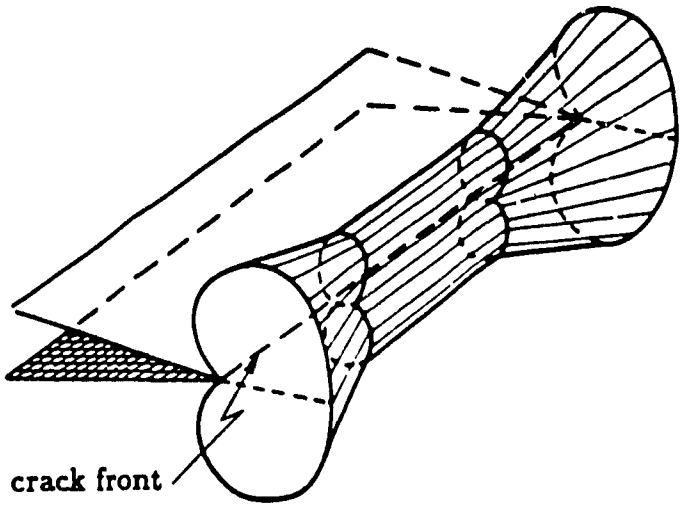


Figure 2.3: The Three Dimensional Plastic Zone [11]

2.1.2 Constant Amplitude Fatigue Crack Growth

In order to be able to predict how a fatigue crack will propagate under constant amplitude cyclic loading, experiments are performed. These experiments monitor the size of the crack, a, as a function of the number of load cycles, N. The results of a typical experiment of this type can be shown schematically as in Figure 2.4.

From the experimental data obtained, the rate of change of crack size with respect to cycles, da/dN, can be determined. When da/dN is plotted against the stress intensity factor ΔK_I on a log-log scale there are three distinct regions on the graph as shown in Figure 2.5. Region I contains the "*threshold*" value, K_{th}, below which the crack will not propagate. Region III is where the crack approaches its critical size and where the stress intensity factor approaches its critical value, K_{Ic}. The region of interest in this section is Region II, where there is a linear relationship between the log of da/dN and the log of ΔK_I.

As discussed by Provan [13], when the plastic zone at the crack tip is small compared to the crack size, the crack growth rate is governed by the stress intensity factor, or:

$$\frac{da}{dN} = f(\Delta K)\,. \tag{2.7}$$

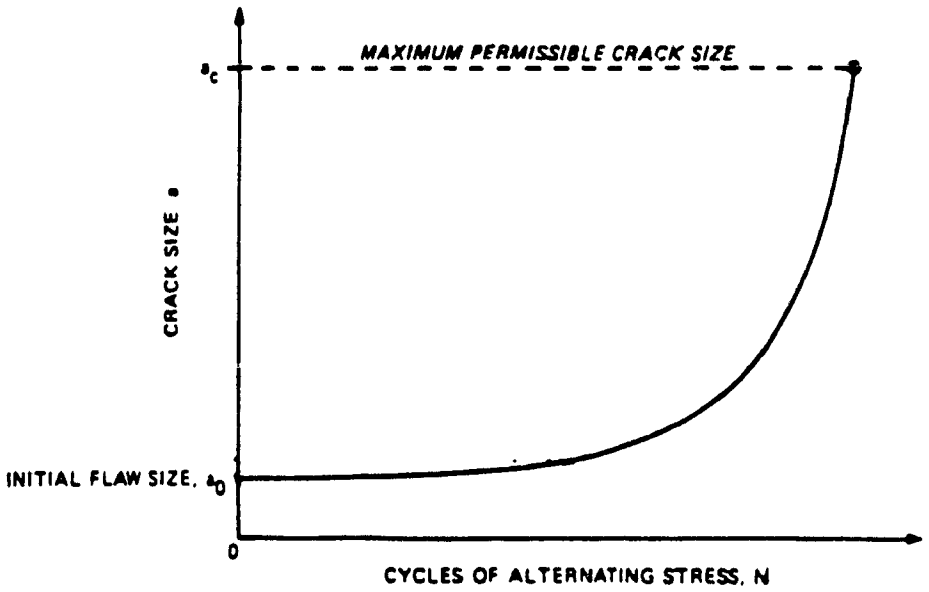


Figure 2.4: Crack size as a function of load cycles

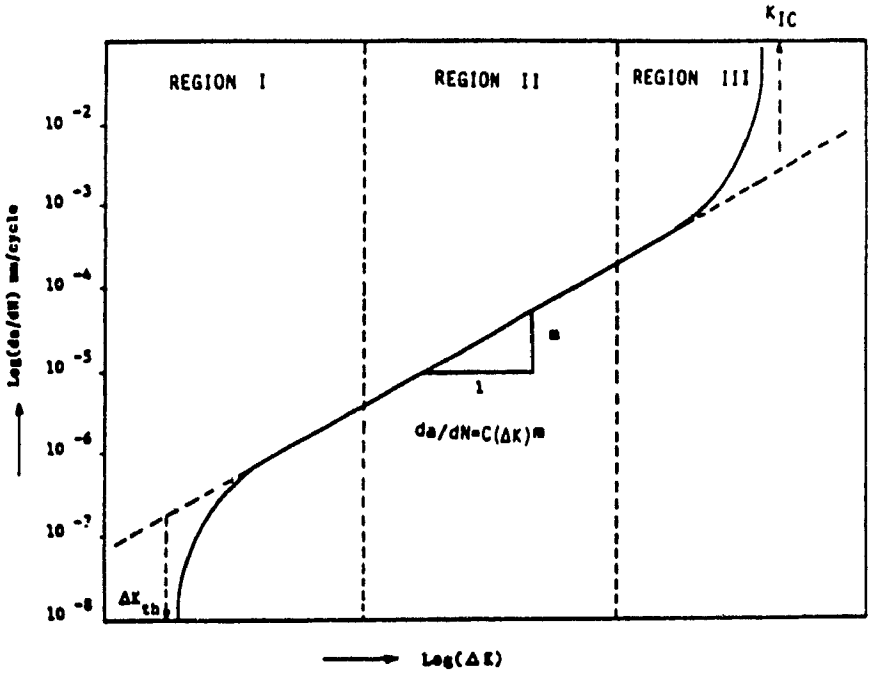


Figure 2.5: A Typical Crack Growth Rate Curve

Recalling from Section 2.1.1 that :

$$K = Y\sigma\sqrt{\pi a}\,, \tag{2.8}$$

we have that:

$$\Delta K = Y\Delta\sigma\sqrt{a}\,, \tag{2.9}$$

or,

$$\frac{da}{dN} = f\left(Y\Delta\sigma\sqrt{a}\right)\,. \tag{2.10}$$

The crack growth rate can be influenced by many factors including material microstructure, mean stress, loading frequency, and environment among others. This being the case there have been many attempts to describe the da/dN vs. ΔK curve by empirically determined "crack growth laws". The two that are most familiar are those of Paris-Erdogan [14]:

$$\frac{da}{dN} = C\,(\Delta K)^m\,, \tag{2.11}$$

and of Forman [15]:

$$\frac{da}{dN} = \frac{C(\Delta K)^m}{(1-R)K_{Ic} - \Delta K}\,, \tag{2.12}$$

where C and m are material constants that are determined from experimental data, and R is called the stress ratio, given by :

$$R = \frac{\sigma_{min}}{\sigma_{max}} = \frac{K_{min}}{K_{max}}\,. \tag{2.13}$$

The Paris-Erdogan law can only be used in Region II of the crack growth rate curve while the Forman law can be used in Regions II and III.

2.2 Reliability

2.2.1 Introduction

Although the idea of reliability, especially when human life is at risk, is not new, the study of reliability has a relatively recent origin, its significance began to be recognized after World War II. As discussed by Shooman [16] several studies at the end of the war, between 1945 and 1950, revealed some startling results:

- A Navy study made during maneuvers showed that the electronic equipment was operative only 30% of the time.
- An Army study revealed that between two-thirds and three-quarters of their equipment was out of commission or under repairs.
- An Air Force study conducted over a 5 year period disclosed that repair and maintenance costs were about 10 times the original cost.

These findings motivated much research into the causes and prevention of failure.

In recent times there have been much publicized accounts of some catastrophic failures, notably in the commercial aircraft industry. These failures are by no means the first of their kind and, unfortunately, they will not be the last. They have, however, increased public awareness of the fact that all designs are not perfect and that there is a degree of uncertainty involved. This has led, in turn, to a public demand for greater safety precautions. Just what steps should be taken are a matter of considerable difficulty since the same public that wants lower risks associated with flying do not wish to have airline fares raised. In order to decide which measures provide the greatest increase in reliability for the least cost we must be able to quantify reliability just as we can quantify costs.

2.2.2 Empirical Reliability Distributions

One method for predicting component reliability is through the use of exhaustive laboratory testing. A component or a standardized test specimen is placed in a laboratory environment which simulates the one in which it is to be used. The behavior of the component or its operating characteristic, which in the case of fatigue is crack length, is monitored as a function of time. The experiment can then be repeated several times and the data obtained can be used to develop an empirical reliability distribution.

The functions used for these empirical distributions are chosen solely for their applicability to the data obtained and their ease of application, *they are not based on theoretical concepts*. This, in fact, is the definition of an empirical model. Typically, the number of cycles a component can withstand before failure is recorded. This data set is then plotted as a Cumulative Distribution Function (CDF), where the probability of failure at a given time is equal to that percentage of components that failed prior to that time during the experiment. This results in a figure such as Figure 2.6. The next step is to find a function that will produce a curve that fits the obtained experimental data well. The resulting distribution is known as time – to – failure distribution.

The rest of this section will present several of the most common empirical distributions that can be used for reliability predictions. The following information serves only as an introduction to these distributions and it is left to texts such as Bompas - Smith [1], Mann et.al. [17], O'Connor [18] and others to provide further information regarding them.

THE EXPONENTIAL DISTRIBUTION

Perhaps the most commonly used time – to – failure distribution is the exponential distribution. It is often chosen not for its applicability to the problem at hand but rather because it is easy to use. This distribution has been used for life studies in the past by Davis [20] and by Epstein [21]. The CDF for the exponential distribution is given as :

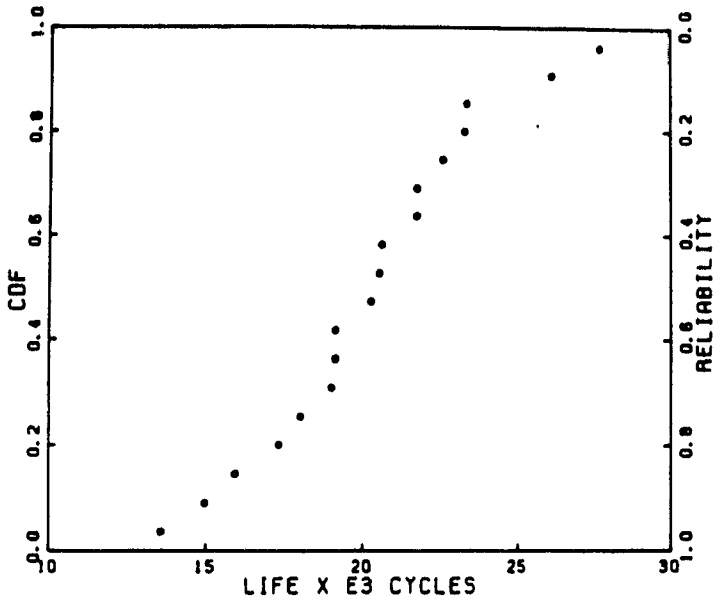


Figure 2.6: Typical Experimental Life–Data [19]

$$F(x) = \int_{-\infty}^{x} \frac{1}{\delta} e^{-x/\delta} \, dx \,, \tag{2.14}$$

which is simply:

$$F(x) = 1 - e^{-x/\delta} \,, \tag{2.15}$$

and the reliability is:

$$R(x) = e^{-x/\delta} \,. \tag{2.16}$$

The quantity $1 \; / \; \delta$ is known as the hazard rate, which is a constant for this distribution. This quantity, the hazard rate, is the conditional probability that a component will fail in a given interval $(x, x + \Delta x)$, as $\Delta x \rightarrow 0$ given that it has not failed prior to x [18]. This can be written, with h(x) being the hazard rate, as:

$$h(x) = \frac{f(x)}{1 - F(x)}, \tag{2.17}$$

where the denominator is the probability density function and the numerator is the reliability at x. In order to choose a form of hazard function for a particular process it is first necessary to decide whether or not the failure rate is time homogeneous. In the case of the exponential distribution the failure rate is time homogeneous but for many physical phenomenon the hazard rate is not. Failure rates that are time dependent usually fall into three categories, initial failure, chance failure and wear – out. These can be shown using the well known bath - tub curve shown in Figure 2.7. The first region of the curve, the initial failures, are those that appear quickly when a component is put into service but are of decreasing frequency. This type of failure is often due to fabrication defects and quality control limitations. The second region, the chance failures, is usually a result of the unpredictable nature of the operating environment. This type of failure is found in almost every area of engineering. The third region, the wear–out failures, are due to normal material and structural degradation. This type of failure has a small initial failure rate which increases due to some time - dependent degradation process such as is caused by fatigue, corrosion, wear, etc.

THE GAMMA DISTRIBUTION

The gamma distribution is an extension of the exponential distribution. In fact, the exponential distribution can be thought of as a limiting case of the gamma distribution. This distribution is used to predict the time to failure where this quantity is the time it takes for K subfailures to occur, and the occurrence of this K^{th} subfailure leads to a system failure. It has been used by Gupta and Groll [22] as a model in lifetest problems. The gamma distributions CDF is :

$$F(x) = \frac{\alpha^k}{\Gamma(k)} \int_0^x t^{k-1} e^{-\alpha t} \, dt, \tag{2.18}$$

where $\Gamma(k)$ is the well known gamma function given by:

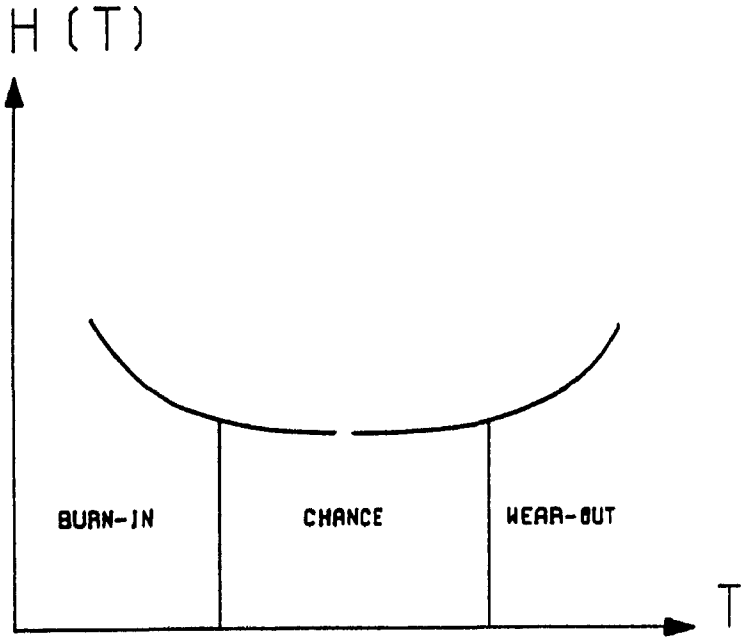


Figure 2.7: The Classical Bath-Tub Curve

$$\Gamma(k) = \int_0^k x^{k-1} e^{-x} dx , \tag{2.19}$$

which can be evaluated using the tables found in Pearson[23]. From Equation 2.18 it can be seen that the exponential distribution is the special case where k = 1, the CDF then becomes :

$$\begin{aligned} F(x) &= \alpha \int_0^x e^{-\alpha t} dt, \\ &= 1 - e^{-\alpha x} , \end{aligned} \tag{2.20}$$

where α is the hazard rate, 1 / δ, from Equation 2.15.

As the value of the parameter k changes the shape of the gamma distribution varies dramatically (for example see Reference [17]), this wide variety of shapes allows this distribution to be used effectively as an empirical model.

THE NORMAL DISTRIBUTION

The normal distribution is the most frequently used statistical model. An important reason for this is that the distributions of many observed phenomena approach it as the sample size increases. A well - documented example of this is the distribution of the strength of materials (see for example Reference [24]). The CDF of the normal distribution is :

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{[-\frac{1}{2}(\frac{\xi-\mu}{\sigma})^2]}\, d\xi \,, \tag{2.21}$$

where (μ, σ) are the mean and standard deviation of the data.

This function can be solved more readily by making the substitution:

$$s = \frac{\xi - \mu}{\sigma} \,,$$

resulting in the equation:

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-s^2/2}\, ds \,. \tag{2.22}$$

This function is tabulated in almost all elementary books on statistics and so the reliability can be found as :

$$R(x) = 1 - F(x) \,. \tag{2.23}$$

This distribution is of questionable validity as a time - to - failure model since the variate, x, includes negative values. This is not true, however, for distributions where $\mu \geq 3\sigma$ since:

$$\frac{1}{\sigma\sqrt{2\pi}} \int_{0}^{\infty} e^{[-\frac{1}{2}(\frac{x-\mu}{\sigma})^2]}\, dx \approx 1 \,, \tag{2.24}$$

to within about 0.14 %. For cases where $\mu < 3\sigma$ the distribution can be written as

$$\frac{1}{K\,\sigma\sqrt{2\pi}} \int_{0}^{x} e^{[-\frac{1}{2}(\frac{\xi-\mu}{\sigma})^2]}\, dx \,, \tag{2.25}$$

where K is a normalizing constant such that when $x \rightarrow \infty$ the integral approaches 1.

This distribution is not commonly used, however, since it is not very flexible and cannot be used for data that exhibit a skew distribution.

THE LOG – NORMAL DISTRIBUTION

The log – normal distribution is more commonly used than the normal distribution since it is more versatile. It has been used previously to describe fracture problems by Howard and Dobson [25] and by Peck [26]. It gives a better fit to reliability data than the normal distribution for populations with wear - out characteristics and does not have the normal distributions disadvantage of including negative numbers. This distribution is used for data where the logarithms of the lifetimes are normally distributed, its CDF is given by :

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_0^x \frac{1}{y} e^{[-\frac{1}{2}(\frac{\ln(y)-\mu}{\sigma})^2]} dy \ , \tag{2.26}$$

where (μ, σ) are the mean and standard deviation of the log data.

This distribution has been derived more fundamentally by Kao [27] by considering a physical process where failure is due to fatigue cracks. This derivation seems to justify the use of the lognormal distribution for failure problems.

THE WEIBULL DISTRIBUTION

Swedish research engineer W. Weibull proposed his statistical distribution function in 1951 [28]. It has found wide acceptance in the engineering community due to its broad range of applicability. The distribution can have many varied shapes and can be used to model data with a variable hazard rate, *which is the case for fatigue.* Another reason why it can be used for the fatigue phenomenon is that it can account for failures that take time to develop, i.e. the crack initiation stage. It has been successfully applied to data from a wide range of fields such as ball - bearings [29], electron tubes [30] and transistors [31]. The Weibull distributions CDF is :

$$F(x) = 1 - e^{\left[-\left(\frac{x-\gamma}{\eta}\right)^{\beta}\right]}, \tag{2.27}$$

with $\beta, \eta > 0$; $\gamma, x \geq 0$. These three Weibull parameters, β , η , γ, are each determined depending on the type of data used.

The parameter γ , known as the datum parameter, is the cause of most of the complication associated with the Weibull distribution. If the failure mode takes some time to develop (creep, fatigue) the distribution of failures take place at some finite time after this latent period. The length of this period is the value of γ and this is then used to adjust the data such that the distribution starts at x = 0. If the datum parameter is 0 then the distribution is often called the 2 parameter Weibull distribution.

The parameter η , known as the characteristic life, is the time when 63.2 % of the population will have failed. This parameter affects the scale of distribution in the x direction without affecting the shape of the distribution.

The parameter β , which is the shape parameter, is the one that allows the Weibull distribution to take on so many different forms, i.e. :

- $\beta = 1$, the distribution has a constant failure rate and becomes the exponential distribution with a mean life of η.
- $\beta < 1$, the distribution has an *decreasing* failure rate.
- $\beta > 1$, the distribution has an *increasing* failure rate.
- $\beta \approx 3.2$, the distribution approximates the normal distribution.

The combination of these factors make this distribution quite flexible and applicable to a wide range of engineering problems. Weibull probability paper is available to facilitate the plotting of the failure curves for the evaluation of the model parameters. However, when the Weibull distribution approximates another distribution, such as the exponential, the latter may be " accurate enough " and will most likely be easier to apply.

THE GUMBEL DISTRIBUTION

When the failure of a component or system can be related to causes that depend on the smallest or largest value (extreme value) of a variable, the Gumbel distribution can be used. This is the case for fatigue where failure depends on the weakest element, or the element with the "smallest" strength. It has been used by Lloyd and Lipow[32] for the study of corrosion where failure depends on the depth of the largest pit.

This distribution has been used extensively by Gumbel [33] for the study of many diverse phenomena. The Gumbel distribution can have two forms depending on the form of the initial distribution. If the initial distribution tends to 0 exponentially as its random variate, x, tends to $-\infty$ then this is called *The Gumbel distribution of the smallest extreme* , and the CDF is given by :

$$F(x) = 1 - exp[-e^{\frac{x-\alpha}{\delta}}], \tag{2.28}$$

where $-\infty < x < \infty$; $\delta > 0$; $-\infty < \alpha < \infty$.

If, however, the initial distribution tends exponentially to 0 as the random variable tends to $+\infty$ then we have *The Gumbel distribution of the largest extreme* , whose associated CDF is :

$$F(x) = exp[-e^{\frac{x-\alpha}{\delta}}], \tag{2.29}$$

where $-\infty < x < \infty$; $\delta > 0$; $-\infty < \alpha < \infty$.

These results are asymptotic, meaning they are derived for a sample size n where $n \rightarrow \infty$. The applicability of these functions for smaller sample sizes depend on the initial distributions. If the initial distribution approaches the exponential one less observations are needed than if the distribution approaches the normal one.

2.2.3 Probabilistic Reliability

The previous section introduced some of the empirical models that have been used for reliability predictions. A second method that can be used for component reliability

predictions is probabilistic mathematical modelling using a model that is based on both probability theory and the basic principles of micromechanics. This method, which will be utilized in this report, involves three important steps : the first step is to describe the physical phenomenon in appropriate mathematical terms; second, the model is analyzed using the laws of probability theory; and the third step is to determine how the model results can be used in the context of engineering analysis.

Siddall [34] said, "in order to deal with uncertainty we must first be able to measure it." In order to do this, the method of probabilistic mathematical modelling will be used here. The complete mathematical development of reliability including the necessary background in probability theory cannot be fully explained here and must be left to existing literature such as References [17,18,35]. What follows is merely a brief explanation of what is meant by mechanical reliability and some of the reasons why its study is important.

Reliability, in fatigue situations, is a function of time, even in the laboratory where the loading cycle does not change. The continuous application of the alternating load causes cracks in a component to grow, which then reduces its strength. This degradation of strength increases with time and is shown schematically in Figure 2.8. So, in terms of fatigue, reliability can be described as the probability that a component will still have sufficient strength to perform its function. The probability of failure, $P_f(t)$, which is simply expressed mathematically as (1 - Reliability), is equal to that proportion of the components whose strength is no longer sufficient at the given time to carry the applied load. In other words, $P_f(t)$ is equal to the area between the two curves in Figure 2.8. From this the three basic properties of the function $P_f(t)$ can be given as[36]:

- $P_f(t)$ is an increasing function of time,
- $0 < P_f(t) < 1$, and
- $P_f(0) = 0 \qquad and \qquad P_f(\infty) = 1$.

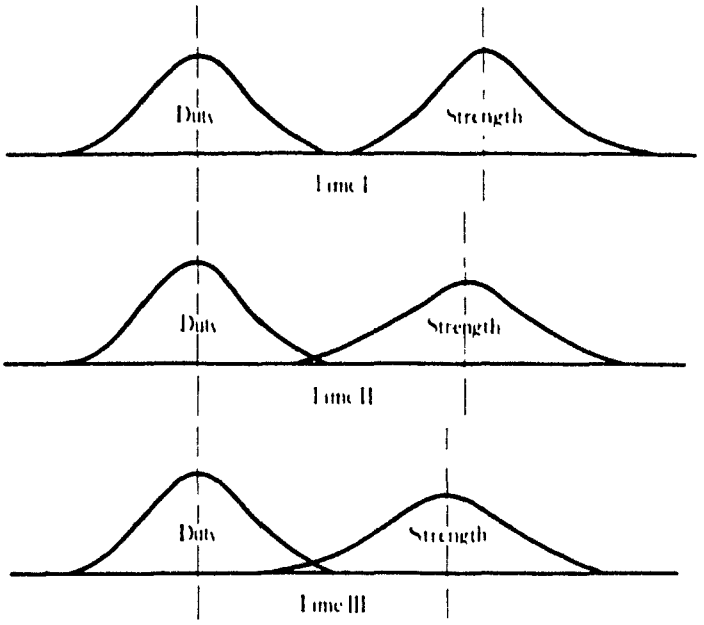


Figure 2.8: The Degradation in Strength as a Function of Time [1]

Alternatively, the probability of failure of a component in the case of fatigue crack growth. can also be expressed as *the likelihood that a crack will exceed the critical length.* This is shown graphically in Figure 2.9 where the area of the region to the right of the line is the probability of failure and the area to the left is the reliability. This representation can be quite useful since, as long as it is possible to predict the distribution of crack size at a future time and the critical crack size, the reliability can be determined. This can be an advantage over the strength – duty interference model since it is not necessary to predict the distribution of two quantities but only of one. *It is this method of predicting reliability which will be utilized later in this report.*

A DURABILITY METHODOLOGY

Another method of assessing the reliability of components is the one that has been developed by Manning, Yang and Rudd et.al[38,39,40,41]. They have performed both analytical and experimental analyses in developing a methodology for predicting exces-

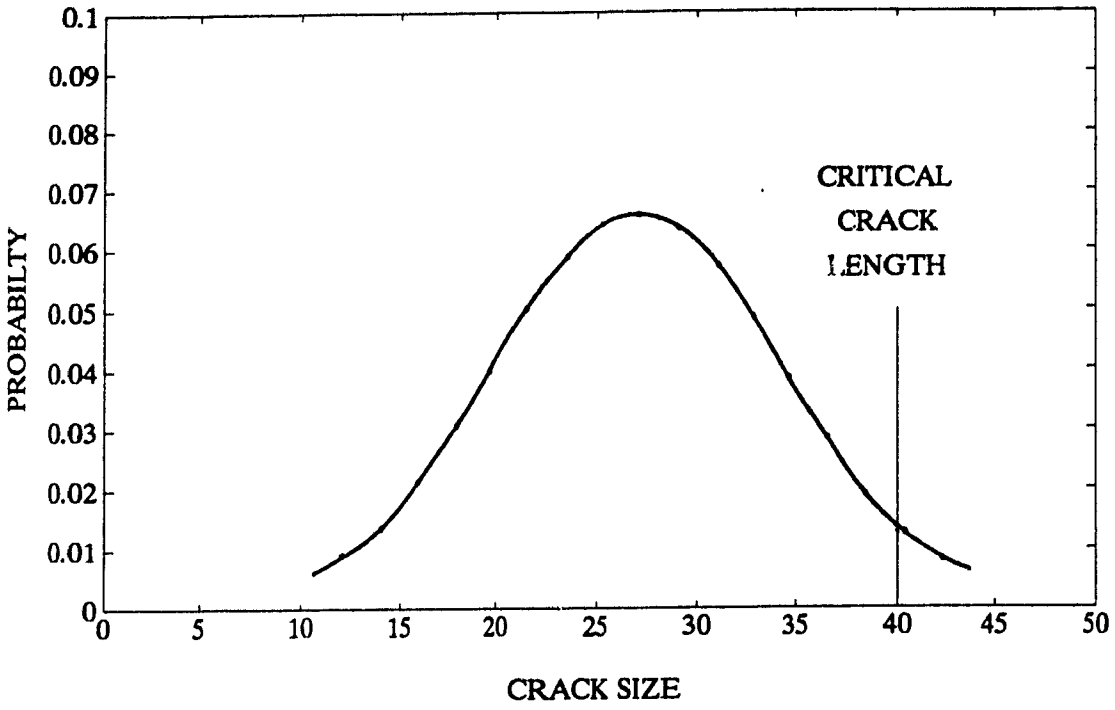


Figure 2.9: Fatigue Reliability – the Probability of Crack Exceedance

sive crack growth in metallic airframes. Their methodology is based on a probabilistic fracture mechanics approach and has been developed for fatigue cracks around fastener holes, but, the theoretical approach should apply to details like fillets, lugs, cutouts, etc.

The objective of this methodology is to describe, quantitatively, the extent of damage as a function of service time. This extent of damage is the probability of a crack exceeding a certain size. There are two essential steps in this analysis: one, quantifying the Initial Fatigue Quality(IFQ) of the structural details considered; two, predicting the probability of crack exceedance using this IFQ and the design conditions (loading, stress level, % load transfer, etc.) There are also several necessary assumptions and limitations.

1. Crack length, measured in the direction of propagation, is the fundamental measure of durability damage.

2. Each detail in an aircraft structure has a single dominant fatigue crack which governs the durability of the structure. The size of this crack is considered to be a random variable.

3. The largest fatigue crack in each detail is relatively small (e.g. ≤ 1.27 *mm* corner crack in a fastener hole) and they are statistically independent, hence, the growth of a crack in one detail does not affect other details. Therefore the binomial distribution can be used to quantify the extent of damage for different details, parts, components, or the entire airframe.

4. An Equivalent Initial Flaw Size (EIFS) distribution can be found by back extrapolating fractographic data using a deterministic crack growth curve. *This EIFS is a mathematical quantity describing the IFQ for a given detail* not *necessarily an actual initial crack size.*

5. This EIFS is determined for a given crack size range and can be grown from time zero using a single deterministic crack growth curve.

6. A suitable Service Crack Growth Master Curve (SCGMC) can be determined, either analytically or experimentally, for specific analysis conditions.

The IFQ is defined as the initially manufactured state of a structural detail or details. The IFQ for a group of components can be represented as an Equivalent Initial Flaw Size (EIFS) distribution. The EIFS is a mathematical quantity, it is an artificial initial crack which results in an actual crack size at an actual time, when grown forward. The EIFS is a hypothetical flaw that is used as a convenient tool for analysis purposes.

Manning, Yang, Rudd et.al.[39] have used two different methods for 'growing' flaws backward in time to determine the EIFS, these are; the deterministic crack growth approach and the stochastic crack growth approach. The deterministic approach uses a single deterministic (average) crack growth rate equation to back extrapolate the fractographic data. This equation is known as the EIFS master curve and is given by:

$$\frac{da(t)}{dt} = Q[a(t)]^b , \tag{2.30}$$

where da(t)/dt is the crack growth rate; a(t) is the crack size at time t; and Q,b are empirical constants which depend on loading and design parameters and are different for each specimen. The stochastic crack growth approach uses a stochastic crack growth rate equation to back extrapolate the fractographic data, this equation is:

$$\frac{da(t)}{dt} = X\ Q[a(t)]^b , \tag{2.31}$$

where X is a lognormal random variable with a median of 1.0. Therefore, Equation 2.30 is the average crack growth behavior while Equation 2.31 considers the variability by using the lognormal random variable X. These authors have found that both methods give reasonable results.

The EIFS is found, for the deterministic model, using Equation 2.30 and the Time To Crack Initiation (TTCI) distribution for a group of specimens. For a given reference crack size, a_u, the TTCI reference distribution is:

$$F_T(t) = P[T < t] = 1 - exp[-(\frac{t-\varepsilon}{\beta})^{\alpha}] , \tag{2.32}$$

where t > ε , T = TTCI and $\alpha, \beta, \varepsilon$ are the three Weibull parameters determined for the data set. The EIFS distribution, $F_{a(0)}(x)$, is found from the expression:

$$a(0) = \text{EIFS} = a_0\ exp(-Q\ T) , \tag{2.33}$$

where T = the TTCI and a(T) = a_0 . The upper bound on a(0) and the lower bound of T are given respectively as:

$$x_u = a_0\ exp(-Q\ \varepsilon) , \tag{2.34}$$

$$\varepsilon = (1/Q)\ ln(a_0/x_u) . \tag{2.35}$$

Therefore, the expression for the EIFS distribution is:

$$F_{a(0)}(x) = \begin{cases} exp[-(\frac{ln(x_u/x}{Q\beta})^{\alpha}] & 0 < x, x_u \ , \\ 1.0 & x > x_u \ . \end{cases} \tag{2.36}$$

For each component the structural details are grouped into m stress regions, where the maximum stress in each region is assumed to be equal for every location or detail. For the i^{th} stress region the corresponding EIFS value, $y_{1i}(\tau)$, is that value which grows to a crack size x_1 at time τ. The crack growth rate expression, Equation 2.31, can then be integrated from $a(0) = y_{1i}(\tau)$ to $a(\tau) = x_1$ to obtain the value for $y_{1i}(\tau)$ as:

$$y_{1i}(\tau) \ = x_1 \ exp(-Q_i \ \tau) \ , \tag{2.37}$$

where, if suitable fractographic data is available, Q_i may be expressed by the following power function:

$$Q_i \ = \ \xi \ \sigma^{\gamma} \ . \tag{2.38}$$

In this equation, σ is the maximum stress applied in the loading spectrum and ξ, γ are constants that are determined from the data. If this suitable data is not available, the parameter Q_i can be found by fitting Equation 2.31 to predict the crack size a(t) at service time t.

This leads to the following expression for the probability of crack exceedance at a time τ $p(i,\tau) = P[a(\tau) > x_1] = 1 - F_{a(\tau)}(x_1)$:

$$p(i,\tau) \ = \ 1 - exp[-(\frac{ln(x_u/y_{1i}(\tau))}{Q\beta})^{\alpha}] \ , \tag{2.39}$$

for all $0 < y_{1i}(\tau) \leq x_u$, else the probability of exceedance is 0.

Extensive investigation of this model, both from an analytical as well as an experimental viewpoint, has been performed. A comparison of actual data and predictions indicates a very good correlation. This type of model holds much promise but is, at present, limited to the specific application to aircraft durability analysis. Time will tell whether or not this procedure can be used for other fatigue situations or possibly extended to other degradation phenomena.

STOCHASTIC MODELS

Over the past decade or so other probabilistic methods have been developed for dealing with reliability. These methods are firmly based on the fundamentals of probability theory, specifically, stochastic processes. One of these approaches, which is based on a nonhomogeneous Markov process, has been developed by Provan et.al. [42,43,44,45] and is the basis for the model developed later in this report. Another method has been developed by Bogdanoff and Kozin [46], this method has been the subject of extensive work and will be briefly dealt with here.

The model of Bogdanoff and Kozin is a discrete time - discrete state Markov process called a Markov chain. A brief introduction to Markov processes is presented in Chapter 3 and a more detailed mathematical explanation of the model of Bogdanoff and Kozin is left to their book [46]. The aim of this model is to evaluate life data and to predict the growth of fatigue cracks. The mean and variance of the number of cycles required for a crack to reach a certain size are used to determine two parameters, b_j and r_j , for each interval j. These parameters are: b_j , the number of states in the interval j (which must be an integer); r_j , a parameter used by the model to predict mean and variance of crack size as a function of cycles. Note that these parameters are different for each interval j.

Bogdanoff and Kozin have shown that this model can be used to produce an empirical distribution such as the one in Figure 2.10 which shows the actual data as well as the model prediction. The information from this model can be applied to reliability and maintainability calculations. The model of Bogdanoff and Kozin has been shown to be a potentially valuable method of examining the variability that exists in fatigue data.

The major difference between this model and the one presented in this thesis are: *i*), the Bogdanoff and Kozin model considers the variation in the number of cycles to reach a given crack size as opposed to the distribution of crack size at a given time; and *ii*), that their model uses different parameters for each data interval whereas the model of this thesis uses two parameters for the whole data range.

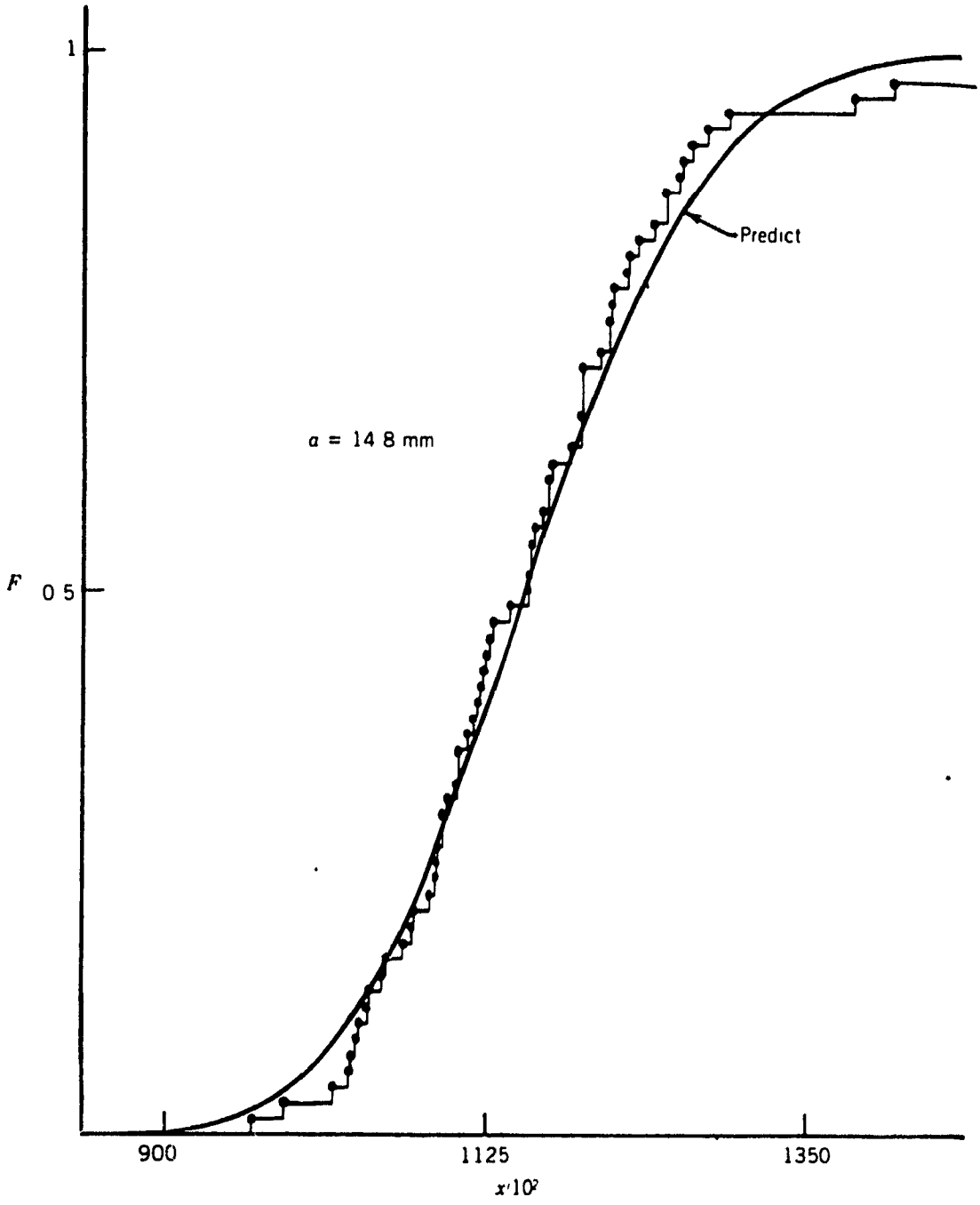


Figure 2.10: Comparison of Data and Model of Bogdanoff & Kozin [46]

One of the aims of this thesis is to apply a probabilistic model, based on the concepts of micromechanics, to describe the growth of fatigue cracks. One of the advantages of this model is that it can conceivably be applied to the modelling of degradation processes in general with only the determination of two empirical parameters. Once developed, this model can then be used to determine how component reliability will change with time. The use of this model will facilitate the prediction of reliability at future times. In addition, other predictions that may be useful to engineers, such as the optimization of inspection schedule, can also be made. The probabilistic model to be used is a Markov model and it will be developed in the following chapter.

Chapter 3

MATHEMATICAL DEVELOPMENT

3.1 Stochastic Processes

3.1.1 Introduction

In fields such as engineering, the physical sciences, economics, and others, there exist random phenomena which vary with time. It is of interest to investigators in these fields to be able to predict the future behavior of such phenomena. It is this desire that has led to the development of the stochastic process, which Doob [47] has defined as the *mathematical abstraction of an empirical process whose development is governed by probabilistic laws.* Bharucha-Reid [48] points out that one should be well aware that this term, stochastic process, refers to the mathematical model and not to the empirical process itself. This emphasizes to the engineer that sound engineering judgement must always accompany the use of these processes or models.

3.1.2 Basic Probability Background

The basic background in probability theory is not presented (as it can be found in many introductory level texts [49,50,51]), with the exception of the next section which

introduces those concepts which are central to stochastic processes.

Due to the random nature of the phenomena which stochastic processes are used to model, the processes are based on probability theory. This thesis will assume a certain familiarity with probability and set theory (see, for example, references [49,50, 51,52]). However, a few concepts which are central to the mathematical development of stochastic processes will be briefly introduced in this section. These concepts are conditional probability and distribution and density functions.

Conditional Probability Often two or more events are connected, meaning that their occurrences are related in some way. This means that the occurrence of one of these events will affect the likelihood of occurrence of the other(s). If we have two events of this type, say A and B, the probability that A occurs, given that event B has already occurred, is written as:

$$P\{A|B\} . \tag{3.1}$$

This can be read, the probability of A given B. This is called the conditional probability since only cases favourable to event B are considered, as opposed to all cases. This probability is equal to the probability of both A and B occurring divided by the probability of B occurring (since it has already happened) or:

$$P\{A|B\} = \frac{P\{AB\}}{P\{B\}} . \tag{3.2}$$

This probability will be undefined if $P\{B\} = 0$ and will be zero if events A and B are mutually exclusive ($P\{AB\} = 0$).

An important extension of this is for the case where there exists a set of mutually exclusive events, $B_1, B_2, \ldots, B_n$, where one of them necessarily occurs (i.e., the union of all the events B_i is the whole sample space). The probability of event A occurring can then be written as [50]:

$$P\{A\} = \sum_{i=1}^{n} P\{A|B_i\}P\{B_i\} . \tag{3.3}$$

This is called the *theorem of total probability* and is very important for the development of stochastic processes, as shall be shown later in this chapter.

Distribution and density functions The distribution function characterizes the probability distribution of a random variable, X. Its domain is the set of real numbers, for a continuous process, or a set of integers, for a discrete one, and its range is from 0 to 1. If x is a number and the event is defined as [X $\leq$ x], the probability of [X $\leq$ x] is the distribution function and is represented as $F_x(x)$ where [49]:

$$F_x(x) = Prob.[X \leq x] \qquad -\infty < x < \infty \,. \tag{3.4}$$

This function has four properties [52]:

1. $F_x(-\infty) = 0$.
2. $F_x(\infty) = 0$.
3. F_x is a nondecreasing function, i.e. for $x_1 \leq x_2$ $F_x(x_1) \leq F_x(x_2)$.
4. F_x is continuous from the right in the sense that
   $lim_{\alpha \to \alpha_0} F_x(\alpha) = F_x(\alpha_0^+) = F_x(\alpha_0)$.

The distribution function is related to the density function and is written as f_x where:

$$f_x(x) = Prob[X = x] \,. \tag{3.5}$$

where X and x are as defined for distribution functions and $f_x(x)$ has the following properties for discrete variables [49] :

1. $f_x(x) = 0$ if x is not in the set of points $x_0, x_1, x_2, \ldots$.
2. $0 \leq f_x(x_i) \leq 1$ for all x_i in the range.
3. $\sum_i f_x(x_i) = 1$.

The distribution and density functions are related as follows:

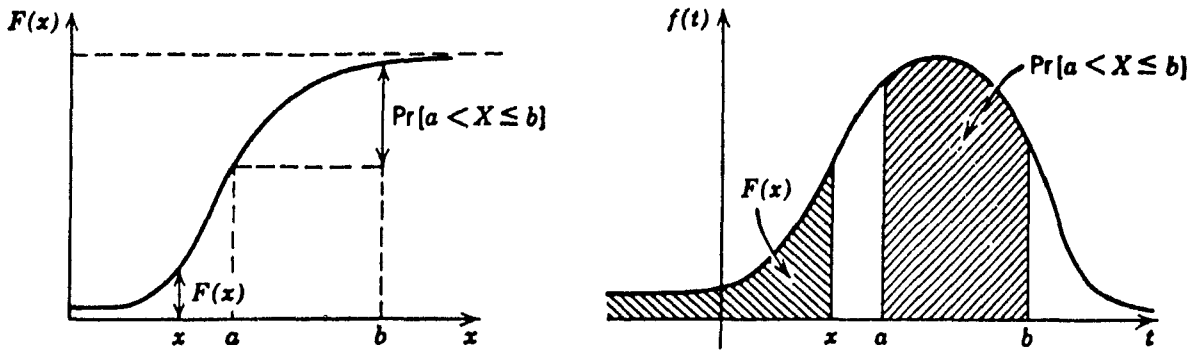


Figure 3.1: The Relationship Between Distribution and Density Functions [49]

$$F_x(\alpha) = \int_{-\infty}^{\alpha} f_x(\beta) d\beta \,, \tag{3.6}$$

for continuous processes, or:

$$F_x(\alpha) = \sum_{i=0}^{\alpha} f_x(i) \,, \tag{3.7}$$

for discrete processes. Figure 3.1 illustrates the relationship between the distribution and density functions. These probabilistic concepts will be used in the next section to help define a stochastic process.

3.1.3 Mathematical Definition

The following mathematical description of a stochastic process was developed with the help of many references [47,48,49,53,54], with many of the ideas being common to more than one source.

In order to define a stochastic process a set of times, T, must be specified. This set can be defined as:

$$\mathrm{T} = \{t \mid \mathrm{t} \geq 0\}$$

or

$$T = \{t \mid -\infty < t < \infty\}$$

which is called a continuous parameter process. It can also be defined as:

$$T = \{0, 1, 2, \ldots\}$$

or

$$T = \{\ldots, -1, 0, 1, \ldots\}$$

which is called a discrete parameter process.

For any time, t, in the set T the random variable, X, can be observed. If this is done over the range of t that comprise one experiment, a function, X_t, is obtained. If a sample point is denoted by x then the function can be written as:

$$\{X_t(x), t \in T\}\,, \tag{3.8}$$

this is, in mathematical terms, a random or stochastic process. The range of possible values of X_t is called the state space of the stochastic process. The values that x can have are known as the *states*.

The values of $X_{t_1}, X_{t_2}, \ldots$ can be observed over a range of times, $t_1 < t_2 < \ldots < t_n$ where all $t \in T$. Using this information the joint distribution function or density function can be defined. It is written as, $f_{X_{t_n},\ldots,X_{t_1}}(x_n, \ldots, x_1)$, and this distribution function can then be used to predict future behavior if past behavior is known. This is represented mathematically by:

$$f_{X_{t_n}|X_{t_{n-1}},\ldots,X_{t_1}}(x_n|x_{n-1}, \ldots, x_1) = \frac{f_{X_{t_n},\ldots,X_{t_1}}(x_n, \ldots, x_1)}{f_{X_{t_{n-1}},\ldots,X_{t_1}}(x_{n-1}, \ldots, x_1)}\,, \tag{3.9}$$

where the left hand side of Equation 3.9 is the conditional density of X_{t_n}, given the past behavior of the process.

As mentioned earlier, the process can have either discrete or continuous parameters, in addition, the process itself can be either discrete or continuous. An example of a discrete process would be a numerical count of objects; for example, the number of

items produced by a certain machine. An example of a continuous process might be a record of temperature in a certain location. This means that there are four types of stochastic processes:

- discrete processes with discrete parameters,
- discrete processes with continuous parameters,
- continuous processes with discrete parameters, and
- continuous processes with continuous parameters.

In the next section an example of the second type, the discrete process with a continuous parameter, also known as a discrete random process, will be developed.

3.2 Markov Processes

In the development of this section many of the references from the previous section were used, additional references are also cited where appropriate.

The Markov process is a special case of stochastic process. Its distinguishing feature is that it is a stochastic process whose future value depends only on its current state, it is independent of all previous values. This statement explains why Markov models have the potential to be extremely useful in engineering applications. If a thorough inspection of a structure is carried out the current state of damage is then completely known. Using a Markov process or model the future damage state can be predicted without any knowledge of how the structure arrived at the current damage state. In other words, it is not necessary to know through what damage states the structure passed before arriving at its current one.

In order to apply this idea in the form of a stochastic model we write the mathematical definition of a Markov process as:

$$f_{X_{t_{n+1}}|X_{t_n},\ldots,X_{t_1}}(x_{n+1}|x_n,\ldots,x_1) = f_{X_{t_{n+1}}|X_{t_n}}(x_{n+1},x_n)\,, \tag{3.10}$$

where the left-hand side of Equation 3.10 is known as the conditional density and also as the transition density for the Markov process. This equation says that knowledge of the present state (X_{t_n}) means the future state ($X_{t_{n+1}}$) is independent of the past ($X_{t_{n-1}}, \ldots, X_1$).

Now, suppose that for any time, t_n , the initial distribution, $f_{X_n}(x_n)$, is known. Given this distribution in addition to a knowledge of the transition density given by Equation 3.10 the following distribution can be obtained from:

$$f_{X_{t_{n+1}}}(x_{n+1}) = f_{X_{t_{n+1}}|X_{t_n}}(x_{n+1}|x_n) f_{X_n}(x_n) \,. \tag{3.11}$$

Future distributions can be obtained in this manner, for example:

$$f_{X_{t_{n+2}}}(x_{n+2}) = f_{X_{t_{n+2}}|X_{t_{n+1}}}(x_{n+2}|x_{n+1}) f_{X_{n+1}}(x_{n+1}) \,. \tag{3.12}$$

Where this procedure can be carried out for any future time of interest by simply repeating the process as many times as necessary.

3.2.1 Transition Probability

The distribution of interest in this report is the probability mass distribution. Using a Markov process this may be found from the initial distribution and the transition probabilities. The transition probabilities are given by [44] :

$$P(X_{t_{n+1}} = x_{n+1} | X_{t_n} = x_n) \,. \tag{3.13}$$

This can be read *the probability of* $X_{t_{n+1}}$ *being at state* x_{n+1} *given that* X_{t_n} *is at state* x_n. For the discrete-state, continuous parameter Markov process we write this transition probability as :

$$P(X_t = j | X_\tau = i) = p_{ij}(\tau, t) \,, \tag{3.14}$$

where i,j are integer states and $0 \leq \tau <$ t. This is the probability of going from state i to state j between time τ and time t.

There are several conditions that the transition probability, $p_{ij}(\tau, t)$ must satisfy, namely:

$$0 \leq p_{ij}(\tau, t) \leq 1 \,, \tag{3.15}$$

for all i,j and $\tau \geq 0$, $t \geq 0$.

$$\sum_j p_{ij}(\tau, t) = 1 \,, \tag{3.16}$$

for all i and τ , t both ≥ 0 ,

and

$$p_{ij}(0,0) = \begin{cases} 1 & \text{for i = j} \,, \\ 0 & \text{for i} \neq \text{j} \,, \end{cases} \tag{3.17}$$

where Equation 3.17 means the process cannot change states in an interval of 0 time units. The transition probability must also satisfy the following equality :

$$p_{ij}(\tau, t) = \sum_k p_{ik}(\tau, s) p_{kj}(s, t) \,, \tag{3.18}$$

where $\tau < s < t$, and Equation 3.18 is the time-continuous Chapman - Kolmogrov equation.

Finally, the condition that the probability of two or more transitions between states in a small increment of time Δ t is $0(\Delta\ t)$ where $0(\Delta\ t)$ is defined if $f(\Delta\ t) = 0(\Delta\ t)$ and:

$$\lim_{\Delta t \to 0} \frac{f(\Delta t)}{\Delta t} = 0 \tag{3.19}$$

3.2.2 The Kolmogrov Differential Equations

In order to solve for the transition probability, $p_{ij}(\tau, t)$, which satisfies Equations 3.15 - 3.19, two functions are introduced. These functions are called the *intensity functions* and they describe the infinitesimal transition scheme.

For an infinitely small increment of time, Δt , the probability of transition from state i at time t to state j at time t + Δt is given by :

$$p_{ij}(t, t+\Delta t) \;=\; q_{ij}(t)\Delta t \;+\; 0(\Delta t)\,, \tag{3.20}$$

where $0(\Delta\ t)$ is as described in Equation 3.19.

Using the result from Equation 3.16 and summing over j for all $j \neq i$ we arrive at the result :

$$p_{ii}(t, t+\Delta t) \;=\; 1 \;-\; \Delta t \sum_{j \neq i} q_{ij}(t) \;+\; 0(\Delta t)\,, \tag{3.21}$$

or:

$$p_{ii}(t, t+\Delta t) \;=\; 1 \;-\; q_i(t)\Delta t \;+\; 0(\Delta t)\,, \tag{3.22}$$

where $q_i(t) \;=\; \sum_{j \neq i} q_{ij}(t)$.

Now, following the development in the references,(Feller [50] for example), these functions $q_i(t)$ and $q_{ij}(t)$ which are known as the *intensity functions* can be defined more precisely. For every state i in the sample space assume there exists a time continuous function $q_i(t)$ such that :

$$q_i(t) \;=\; \lim_{\Delta t \to 0} \frac{1 \;-\; p_{ii}(t, t+\Delta t)}{\Delta t}\,. \tag{3.23}$$

In addition, assume that for every pair of states i and j (for $i \neq j$) there is a time continuous function $q_{ij}(t)$ such that :

$$q_{ij}(t) \;=\; \lim_{\Delta t \to 0} \frac{p_{ij}(t, t+\Delta t)}{\Delta t}\,. \tag{3.24}$$

As mentioned earlier, these intensity functions govern the infinitesimal transition scheme. In modelling a physical phenomenon using a Markov process it is often this infinitesimal transition scheme that is specified rather than the transition probability itself.

Recalling Equation 3.18 it then follows :

$$p_{ij}(\tau, t+\Delta t) \;=\; \sum_{k} \; p_{ik}(\tau, t) p_{kj}(t, t+\Delta t)\,, \tag{3.25}$$

using the definition of the intensity functions we obtain the two Kolmogrov equations. By holding j and t constant where i and τ are variables we obtain the **backward differential equation:**

$$\frac{\partial p_{ij}(\tau,t)}{\partial \tau} = q_i(\tau)p_{ij}(\tau,t) - \sum_{k\neq i} q_{ik}(\tau)p_{kj}(\tau,t)\,. \tag{3.26}$$

If i and τ are now kept constant with j and t as the variables the resulting equation is the **forward differential equation:**

$$\frac{\partial p_{ij}(\tau,t)}{\partial t} = -\,p_{ij}(\tau,t)q_j(t) + \sum_{k\neq i} p_{ik}(\tau,t)q_{kj}(t)\,. \tag{3.27}$$

Both of these equations must satisfy the initial conditions :

$$p_{ij}(t,t) = \begin{cases} 1 & \text{for i} = \text{j}\,, \\ 0 & \text{for i} \neq \text{j}\,. \end{cases} \tag{3.28}$$

These Kolmogrov Differential Equations are the equations that need to be solved in order to determine the transition probabilities. This can be done, either analytically or numerically, only after the intensity functions presented in this section have been specified.

3.3 The Nonhomogeneous Markov Law, A Recent Application

Recently, the type of model presented in the previous section was used by Provan and Rodriguez [44,54] for the study the phenomenon of pitting corrosion. Their model was based on the model developed by Provan[43,55]. In order to apply this model the intensity functions governing the infinitesimal transition scheme needed to be specified. Several sources, Bharucha - Reid [48] and Parzen [51] for example, have presented various forms of intensity functions as well as the solutions of the resulting Kolmogrov Differential Equations. The pitting corrosion study began with an examination of four

of these forms and ended with the postulation of a new form. The four forms examined were the Poisson process, the Furry - Yule or pure birth process, the Nonhomogeneous Linear Birth process and the Polya process. The following sections briefly outline these processes.

3.3.1 The Poisson Process

The Poisson process is the simplest of the nonhomogeneous, time continuous Markov processes. There are three assumptions which determine the development of this process, they are :

- The probability of a change from state i to state i + 1 in the interval of time Δt is given by $\lambda\ \Delta t + 0(\Delta t)$ where λ is a positive constant.
- The probability of a change of two or more states in the interval Δ t is $0(\Delta t)$.
- The probability of staying in the same state is $1\ -\ \lambda\ \Delta t\ +\ 0(\Delta t)$.

where all these probabilities are independent of the state, i, of the process.

With these three assumptions the intensity functions can be specified as follows:

$$q_i(t) = \lambda \quad , \qquad \text{for all i} > 0\ , \tag{3.29}$$

$$q_{ij}(t) = \begin{cases} \lambda & \text{for j} = \text{i} + 1\ , \\ 0 & \text{for j} \neq i\ ,\ i+1\ . \end{cases} \tag{3.30}$$

With this transition scheme the Kolmogrov Differential Equations become :

$$\frac{dp_{ij}(t)}{dt} = \lambda p_{ij}(t) + \lambda p_{i+1,j}(t)\ , \tag{3.31}$$

$$\frac{dp_{ij}(t)}{dt} = -\lambda p_{ij}(t) + \lambda p_{i,j-1}(t)\ , \tag{3.32}$$

whose solution is:

$$p_{ij}(t) = \begin{cases} \frac{(\lambda t)^{j-1}}{(j-1)!} e^{-\lambda t} & \text{for j} \geq \text{i}\,, \\ 0 & \text{for j} < \text{i}\,. \end{cases} \tag{3.33}$$

3.3.2 The Furry - Yule or Pure Birth Process

This process can be considered to be a generalized Poisson process where the intensity functions *are* dependent on the state of the process.

The assumptions for this process are :

- The probability of transition from state i to state i + 1 in the interval of time Δt is given by $\lambda_i\ \Delta t + 0(\Delta t)$.
- The probability of a transition from state i to a state other than i + 1 in the interval Δ t is $0(\Delta t)$.
- The probability of no change in state is $1 - \lambda_i\ \Delta t + 0(\Delta t)$.

This results in the following intensity functions:

$$q_i(t) = \lambda\, i \quad ; \quad \text{for i} = 1,2,\ldots, \tag{3.34}$$

$$q_{ij}(t) = \begin{cases} \lambda\, i & \text{for j} = \text{i} + 1\,, \\ 0 & \text{for j} \neq i\,,\ i+1\,, \end{cases} \tag{3.35}$$

and the Kolmogrov Differential Equations become :

$$\frac{dp_{ij}(t)}{dt} = -\lambda\, i\, p_{ij}(t) + \lambda\, i\, p_{i+1,j}(t)\,, \tag{3.36}$$

$$\frac{dp_{ij}(t)}{dt} = -\lambda\, j\, p_{ij}(t) + \lambda\, (j-1)\, p_{i,j-1}(t)\,, \tag{3.37}$$

which has a solution:

$$p_{ij}(t) = \begin{pmatrix} j-1 \\ j-i \end{pmatrix} e^{-i\lambda t} (1 - e^{-\lambda t})^{j-i}\,. \tag{3.38}$$

3.3.3 The Nonhomogeneous Linear Birth Process

The nonhomogeneous linear birth process differs from the two previously presented in that is *does* depend on time. In this case the probability of transition depends not only of the state of the process but on time as well. This results in a form similar to the pure birth process with the important difference that the intensity functions are now functions of time, they are given as :

$$q_i(t) = i\,\lambda(t) \quad ; \quad \text{for i} = 1,2,\ldots, \tag{3.39}$$

$$q_{ij}(t) = \begin{cases} i\,\lambda(t) & \text{for j} = \text{i} + 1\,, \\ 0 & \text{for j} \neq i\,,\ i+1\,, \end{cases} \tag{3.40}$$

which make the Kolmogrov Differential Equations:

$$\frac{dp_{ij}(t)}{dt} = -\lambda(t)\,i\,p_{ij}(t) + \lambda(t)\,(i\ +\ 1)\,p_{i+1,j}(t)\,, \tag{3.41}$$

$$\frac{dp_{ij}(t)}{dt} = -\lambda(t)\,j\,p_{ij}(t) + \lambda(t)\,(j\ -\ 1)\,p_{i,j-1}(t)\,, \tag{3.42}$$

for which the solution is:

$$p_{ij}(\tau,t) = \begin{pmatrix} j\ -\ 1 \\ j\ -\ i \end{pmatrix} q^i\,(1\ -\ q^i)^{j\ -\ i}\,, \tag{3.43}$$

where $q = e^{(-\,(h(t)\,-h(\tau)))}$ and $h(k) = \int_0^k \lambda(k)\,dk$.

3.3.4 The Polya Process

A second example of a state and time dependent process is the Polya process. The intensity functions for this process are:

$$q_i(t) = \frac{1\ +\ \kappa i}{1\ +\ \kappa\lambda t} \quad ; \quad \text{for i} = 1,2,\ldots, \tag{3.44}$$

$$q_{ij}(t) = \begin{cases} \frac{1+\kappa i}{1+\kappa\lambda t} & \text{for j} = \text{i} + 1\ , \\ 0 & \text{for j} \neq i\ ,\ i+1\ , \end{cases} \tag{3.45}$$

resulting in Kolmogrov Differential Equations of the following form :

$$\frac{dp_{ij}(t)}{dt} = -\lambda\ \frac{1\ +\ \kappa j}{1\ +\ \kappa\lambda t}\ p_{ij}(t) + \lambda\ \frac{1\ +\ \kappa(j\ -1)}{1\ +\ \kappa\lambda t}\ p_{i,j-1}(t)\ , \tag{3.46}$$

$$\frac{dp_{ij}(t)}{dt} = -\lambda\ \frac{1\ +\ \kappa i}{1\ +\ \kappa\lambda t}\ p_{ij}(t) + \lambda\ \frac{1\ +\ \kappa(i\ +\ 1)}{1\ +\ \kappa\lambda t}\ p_{i+1,j}(t)\ , \tag{3.47}$$

which can be solved numerically using a Runge - Kutta technique.

3.3.5 A New Intensity Function Form and the Results of its Application to Pitting Corrosion

In order to determine which form, if any, was appropriate for the modelling of pitting corrosion each was applied, in turn, to the same data set. After this analysis was carried out it was decided to use a new form of intensity functions in order to achieve better results. Intensity functions were chosen that could incorporate the best features of those previously examined into a more useful form. Upon examination, a set of intensity functions that were both state and time dependent were deemed most suitable. The form that gave the best results was found to be :

$$q_j(t) = \lambda_j \quad ; \quad \text{for j} = 1,2,\ldots\ , \tag{3.48}$$

$$q_{ij}(t) = \begin{cases} \lambda_{j-1} & \text{for i} = \text{j - 1}\ , \\ 0 & \text{otherwise}\ , \end{cases} \tag{3.49}$$

with:

$$\lambda_j = \lambda j \frac{1+\lambda t}{1+\lambda t^{\kappa}}\ , \tag{3.50}$$

where λ and κ, which are positive constants known as the empirical system parameters, are found by determining the best fit to experimental data. These two parameters include all the various effects that influence the process, such as, temperature, material, geometry, etc. In this way all the variables are conveniently described. Therefore, if these parameters can be found the system can be modeled, of course if the system is changed then new system parameters must be found.

These intensity functions are then used in the Kolmogrov forward differential equation, which becomes:

$$\frac{dp_{ij}(\tau,t)}{dt} = -\lambda_j p_{ij}(\tau,t) + \lambda_{j-1} p_{i,j-1}(\tau,t) \; . \tag{3.51}$$

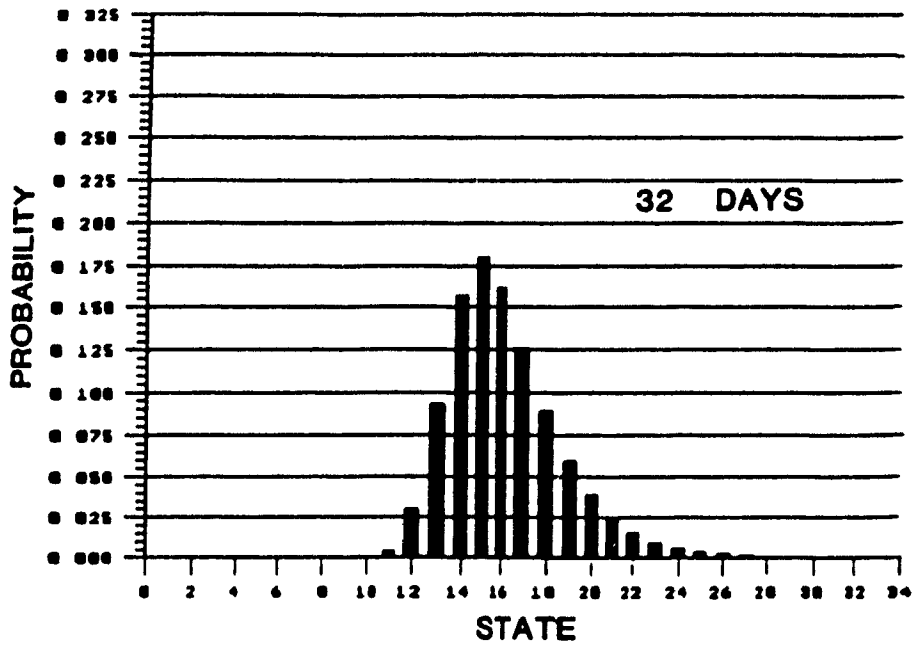
In order to solve this differential equation a numerical technique, the Runge-Kutta method [56,57], was used.

The model in this form was found to give a very good description of the actual behavior of a pitting corrosion system. This model was then applied to the results of an experimental program involving the pitting corrosion of CA-15 stainless steel in a simulated white-water environment (see Rodriguez [54] for details). By an iterative procedure it was found that the values of the system parameters, λ and κ, for this experiment which gave the best fit for the experimental results were:

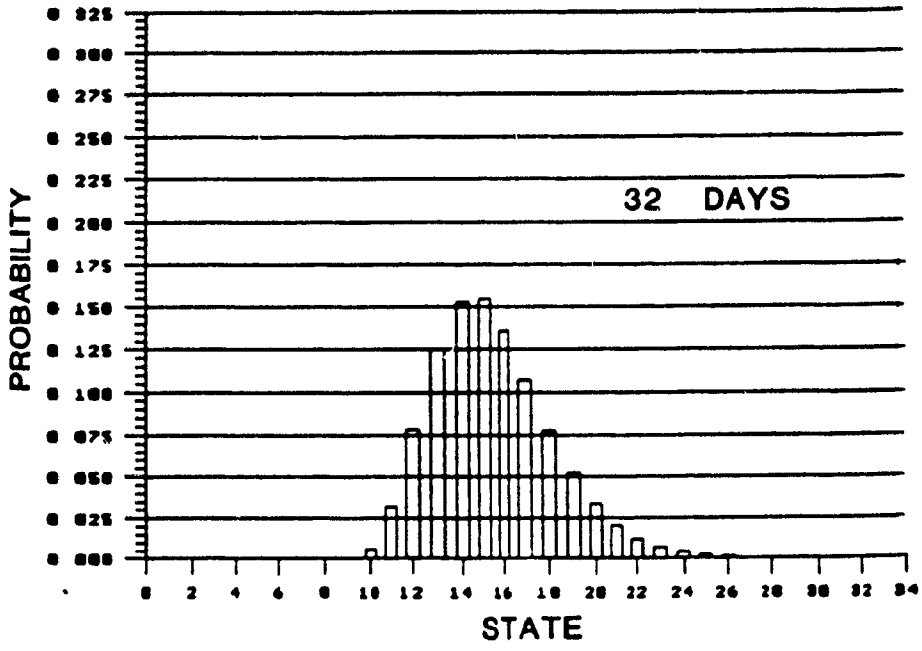
$$\lambda = 0.015 \qquad ; \qquad \kappa = 1.7$$

Using these values the transition probabilities were determined and then used to find the probability of a corrosion pit being a certain size. This is represented in the histograms in Figure 3.2 showing the probability of a pit being in a certain state (depth) for both the experimental data and for the model prediction. From these histograms the mean and variance of pit depth are found and are shown, along with the actual data, in Figure 3.3. The model was also used for reliability calculations in Reference [58].

These results have shown that a non-homogeneous Markov model can be used effectively to model the growth of corrosion pits. This success has raised the question of

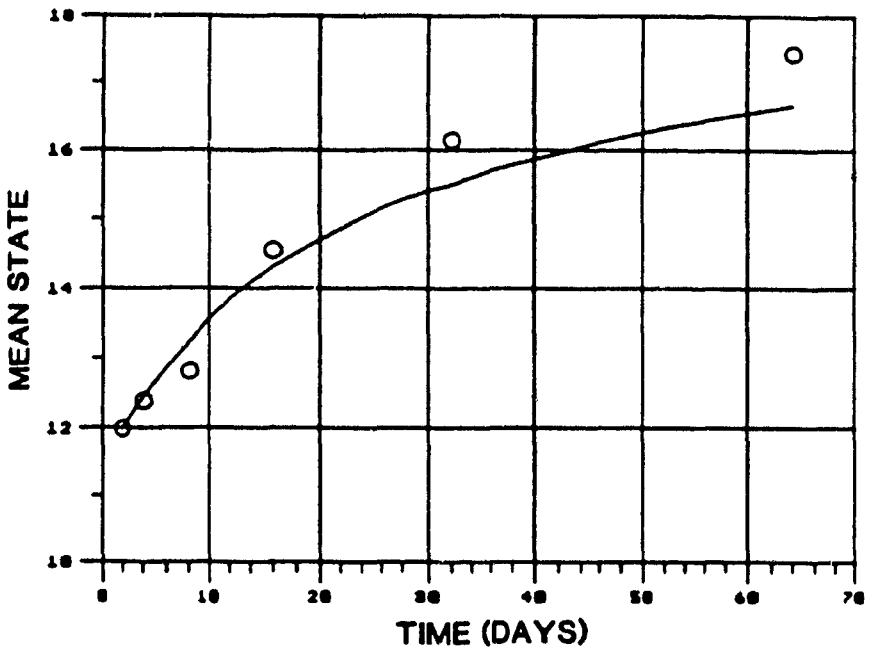


(e) Actual

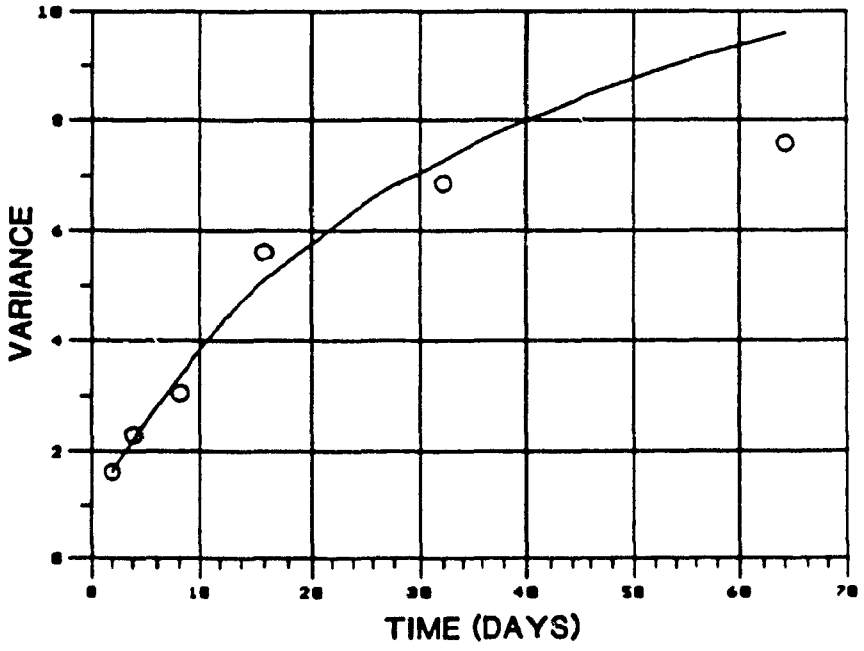


(e)' Predicted

Figure 3.2: Actual Data Histogram and Model Histogram [44]



(a) Mean VS. Exposure Time



(b) Variance VS. Exposure Time

Figure 3.3: Mean and Variance of Pit Depth, from Data and Model [44]

whether or not this method may be applied to other degradation processes, specifically to the fatigue phenomenon. If this model can indeed be used for the modelling of fatigue crack growth what information can the engineer obtain, in the form of reliability predictions ? These are the questions that will be addressed in the remaining chapters of this thesis.

Chapter 4

SPECIFIC RESEARCH OBJECTIVES

The previous chapters have presented a brief review of some of the information available in the literature concerning the fatigue phenomenon, reliability, and stochastic processes. The aim of the chapters that follow is to examine the use of the specific stochastic process mentioned earlier, namely the Markov process. This process can be used for the modelling of the mean and variance of fatigue crack growth as well as for reliability predictions.

The first objective of the current investigation is the use of the Markov model outlined in Chapter 3 to predict the statistics of fatigue crack growth. In order to do this the empirical system parameters, λ and κ, must be established for a set of experimental data. The following chapter will detail the attempt to determine these system parameters for several specific data sets. Once these system parameters have been found they will be used with the Markov model to predict the statistics of fatigue crack growth.

The second objective of this research is the examination of the potential applications of this model. One of the uses of this model is for the solution of a practical engineering problem, namely reliability calculations. There are many different applications that fall under this general heading, some of those which will be examined (in Chapter 6 of

this thesis) include the following:

- the prediction of component reliability at a future time,
- the scheduling of the necessary maintenance procedures to maintain a certain minimum level of reliability, and
- the optimization of inspection and repair scheduling for the maximization of reliability (or the minimization of the probability of failure) and,
- the effect of a change in repair policy on reliability.

One of the long term objectives of this research is the development of computer programs that will not only be of use in the current investigation but that will facilitate further work with this type of model. To this end, the program developed by Rodriguez [54] has been extensively modified for use in this investigation and several new programs have been written for the reliability calculations found in chapter 6. These programs can be found, along with brief explanations of how they work, in Appendix A. It is hoped that they will be of use for future investigations.

Chapter 5

APPLICATION TO FATIGUE

5.1 Introduction

In Chapter 2 the deterministic laws for fatigue crack growth were presented. Since these laws cannot account for the scatter found in laboratory results we turned to the probabilistic model of Chapter 3, the Markov process. It has been shown that this type of model can be used to describe pitting corrosion and it is the aim of this chapter to examine its applicability to fatigue crack growth. Several specific aspects of the Markov process which will be used here will be explained briefly in the following section before moving on to the application of the model to a data set.

5.2 Fatigue Modelling

For this investigation a Markov process with discrete states is used. Although it may appear that crack growth should be thought of as a continuous process it is reasonable to treat crack size as a discrete quantity due to the built-in limitations of crack detection and measuring systems. As Bogdanoff and Kozin [46] point out, this restriction (treating crack size as discrete) is modest in terms of loss of physical reality when compared to computational advantages gained.

An example of how crack length can be discretized is illustrated in Figure 5.1.

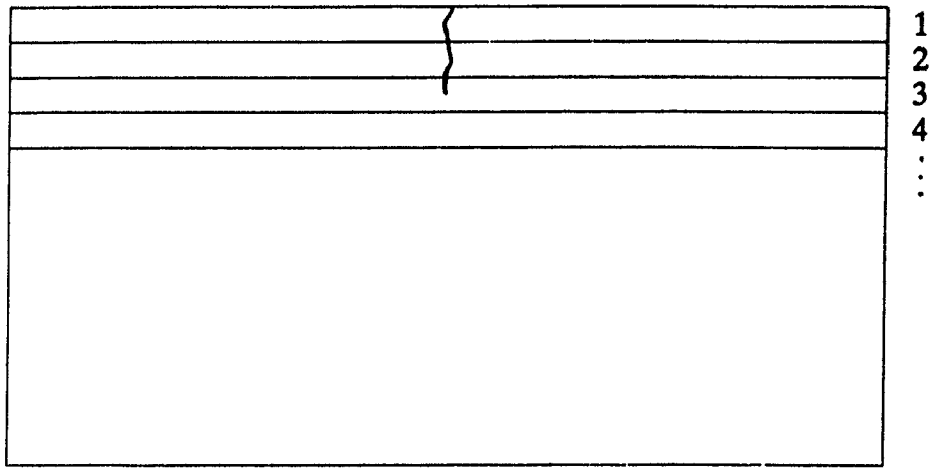


Figure 5.1: A Schematic Representation for Discrete Crack Size

The crack size variable, a_t , can only be measured to within equipment and operator limitations, in other words, the crack size can be given as:

$$a_m - \Delta a \leq a_t \leq a_m + \Delta a , \tag{5.1}$$

where a_m is the measured crack size and Δ a is the range of error in the measurement.

By considering the observable zones, i, this can be written as:

$$x_i < a_t \leq x_i + \Delta x_i , \tag{5.2}$$

where x_i is the state number and Δx_i is the width of a state or state size. This state size, as well as the number of states that need to be considered can be determined by the engineer.

In order to apply this model the infinitesimal transition scheme must be specified. This transition scheme is governed by the intensity functions, $q_j(t)$ and $q_{ij}(t)$. In

Section 3.3 the intensity functions used for pitting corrosion were presented. In the current investigation these same functions will be used to describe the infinitesimal transition scheme, these were given as:

$$q_j(t) = \lambda_j \quad ; \quad \text{for j = 1,2,...} , \tag{5.3}$$

$$q_{i,j}(t) = \begin{cases} \lambda_{j-1} & \text{for i = j - 1 ,} \\ 0 & \text{otherwise ,} \end{cases} \tag{5.4}$$

with:

$$\lambda_j = \lambda\, j\, \frac{1+\lambda t}{1+\lambda t^{\kappa}} , \tag{5.5}$$

where the parameters, λ and κ, can be determined for any data set.

Using these intensity functions the Kolmogrov forward differential equation was:

$$\frac{dp_{i,j}(\tau,t)}{dt} = -\lambda_j p_{i,j}(\tau,t) + \lambda_{j-1} p_{i,j-1}(t) . \tag{5.6}$$

In order to apply the Markov model this equation must be solved to obtain the transition probabilities, $p_{i,j}\ (\tau,t)$. This was accomplished in the present investigation using the program **SOLUTION** found in Appendix A. This Appendix also presents a brief description of what this program does and how it works.

5.3 Specific Applications

5.3.1 OFHC Copper

In order to fully explore the capabilities of this process it was decided to try and use it to predict the statistics of the growth of a single crack front. By examining fatigue fractographs the penetration of a crack front at a given time, or cycle, can be measured at many points along its length (the x_3 direction as shown in Figure 5.2). From these observations the mean and variance of crack length can be determined.

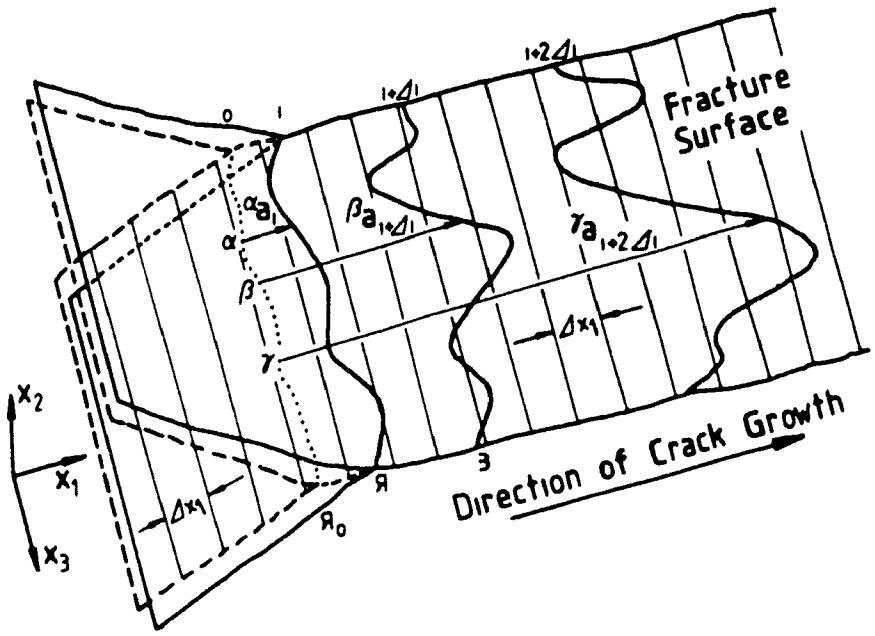


Figure 5.2: A Schematic of Fatigue Cracks at a Given Cycle [55]

Work done by Provan and Mbanugo [59] included an experimental program designed to measure the mean and variance of crack front penetration into OFHC Copper. These results were then reported in terms of the mean and variance at several cycle numbers. This data is given in Table 5.1 [59] after converting to discrete states. In addition, the initial distribution needed for the Markov model was reported in Reference [60] and is shown in Figure 5.3.

Applying the model the empirical material parameters, λ and κ , that best describe this system were found to be:

$$\lambda = 0.032 \quad ; \quad \kappa = 1.1 \tag{5.7}$$

These parameters were found to give an adequate description of the mean crack penetration but were quite inaccurate with regards to the variance. The results for the mean crack size are shown in Figure 5.4. As for the variance the Markov process greatly overestimates the amount of scatter. As pointed out by Provan and Mbanugo [59] this

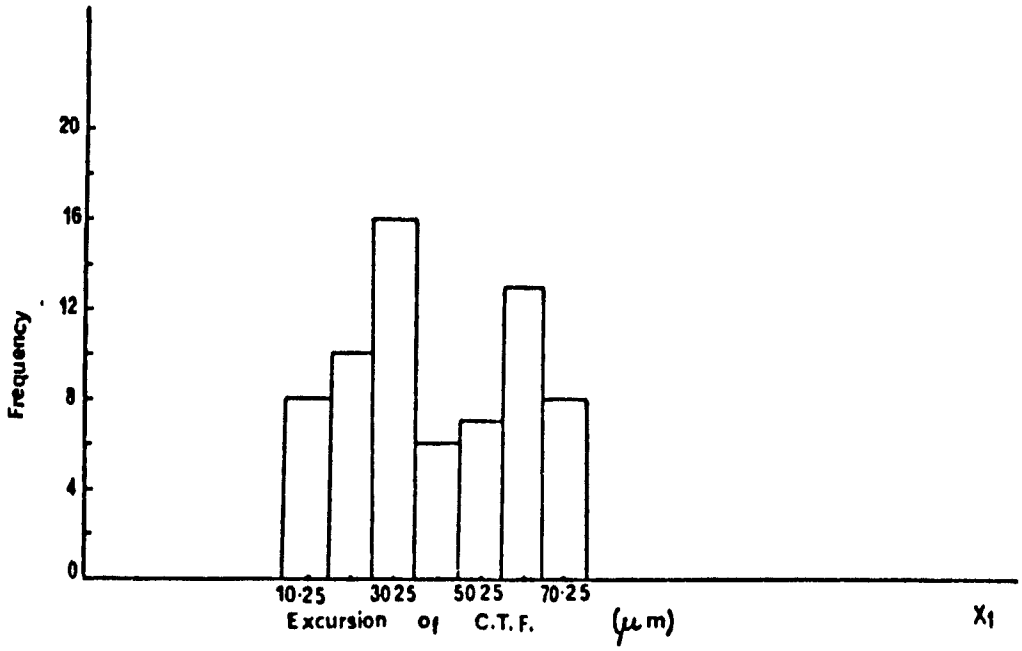


Figure 5.3: The Initial Histogram of Crack Size [60]

Cycle Number	Mean State	Variance
$i_s + 5$	2.69	1.07
$i_s + 36$	22.54	2.77
$i_s + 49$	32.47	3.66
$i_s + 69$	49.88	2.05

Table 5.1: OFHC Copper Data [59], where i_s is the reference cycle number

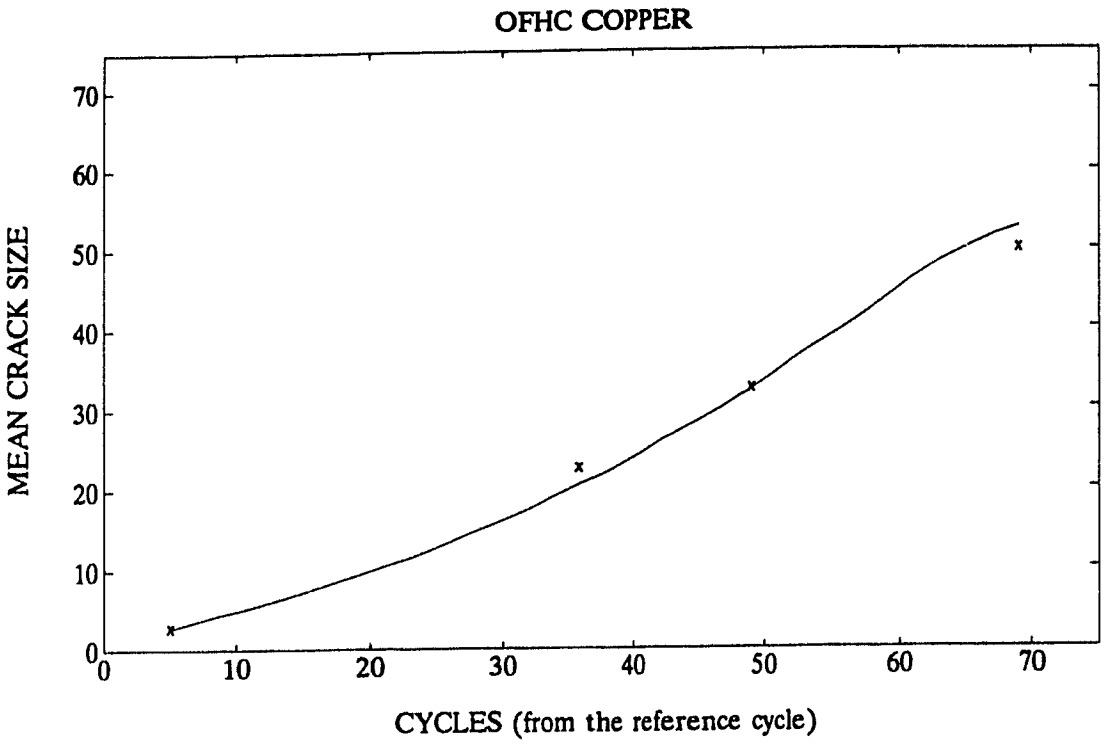


Figure 5.4: Mean Crack Penetration, From Data and Markov Model

is not unexpected since, by definition, it does not take account of either boundary effects or the effect of spatial interaction between neighboring points along the crack front.

These findings indicate that the Markov model, *in the current form,* is not appropriate for the modelling of the mean and variance of crack penetration for a single crack front. It was decided that instead of modifying the model for this application it would be used for the description of the statistics of crack growth for a group of components.

The findings for this work are presented in the following sections.

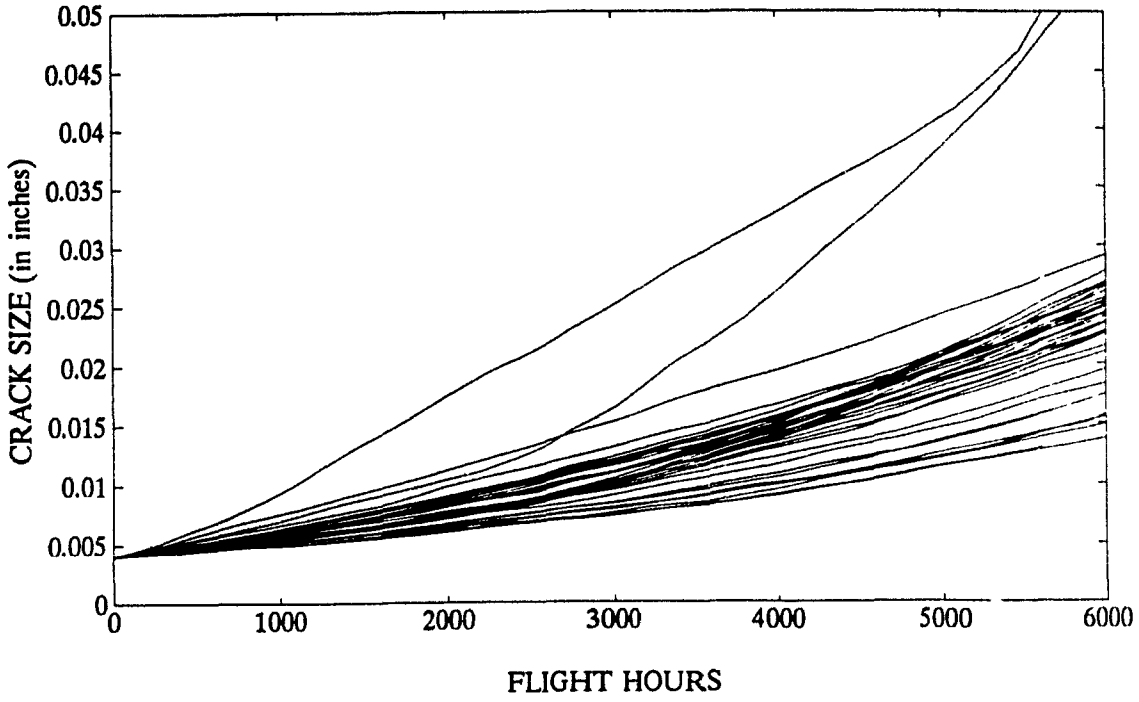


Figure 5.5: Raw data WPF, Normalized to 0.004" (0.102 *mm*) [61]

5.3.2 Data Set WPF

In order to use the non-homogeneous Markov model to predict the statistics of crack growth for a group of components a fairly large number of specimens are needed. There are several such fractographic data sets available in the literature. In this investigation the data set **WPF** [61], see Figure 5.5, which describes the growth of cracks originating around aircraft fastener holes, has been used.

WPF contains the results for 33 specimens whose geometry is given in Figure 5.6 and whose material is 7475 - T7351 aluminum. The specimens were tested in a laboratory air environment under fighter spectrum loading, with a maximum gross stress of 234.4 MPa, for 16,000 flight hours or until failure. There was no pre - cracking and the largest crack in each specimen was evaluated fractographically after testing [39].

The intensity functions used here are the same ones presented in Section 5.2. The

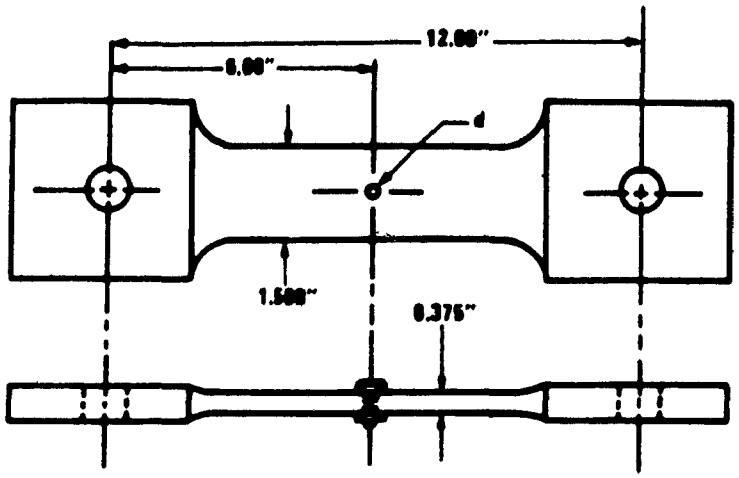


Figure 5.6: WPF Test Specimen Geometry [39]

use of these functions necessitates the determination of the empirical material parameters, λ and κ . These are found by an iterative process of fitting the model prediction to the experimental results.

Several steps had to be taken before this iterative process was performed. The first step is the normalization of the data to an initial crack length of 0.004" (0.102 *mm*) at time t = 0. This is done to eliminate the crack initiation stage. Secondly, the data need to be discretized into states of width Δx. This Δx should be chosen small enough so that the discrete data still closely resembles the continuous data yet large enough so that the number of states needed is kept to a reasonable amount.

The reason for limiting the number of states is that the number of non–zero transition probabilities grows as the sum of the number of states, i.e. :

$$\text{Number of non–zero } p_{ij} \text{ 's } = \sum_{j=1}^{N} j \,, \tag{5.8}$$

which can be shown to be :

$$\sum_{j=1}^{N} j \;=\; \frac{(N\,+\,1)\,N}{2} \,, \tag{5.9}$$

where N = the number of states.

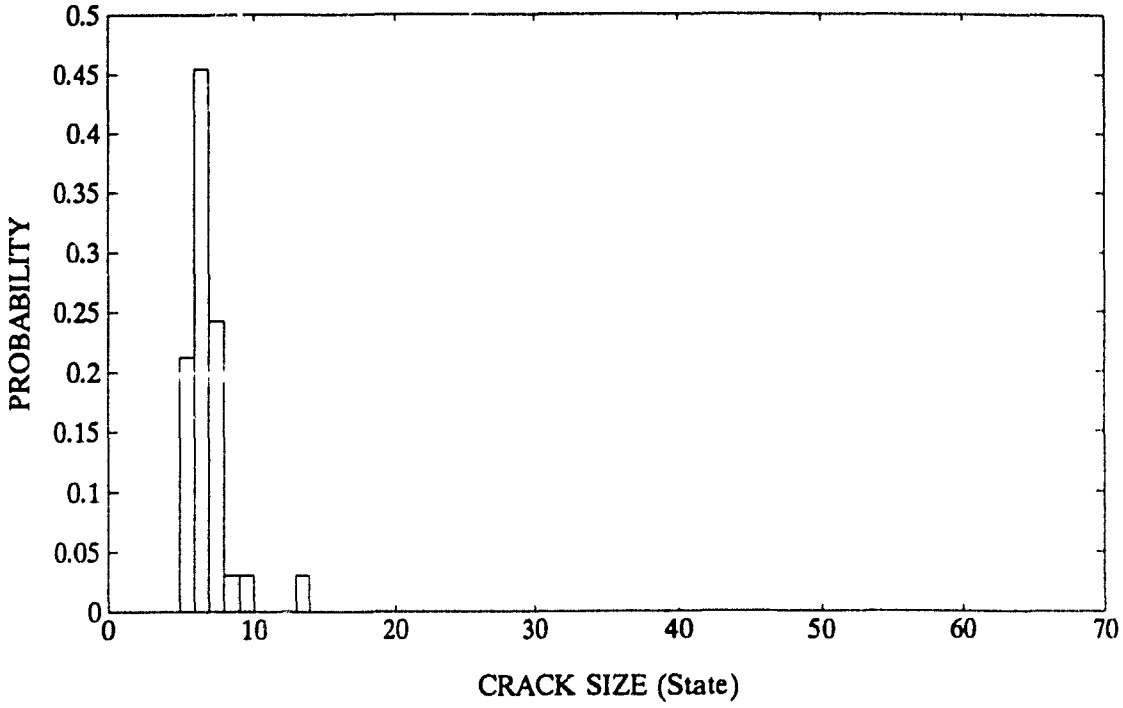


Figure 5.7: Initial Flaw Size Distribution, at 1500 Flight Hours

From Equation 5.9 it can be seen that the size of the matrix of transition probabilities increases as the square of the number of states. This is important since the size of the transition probability matrix has a significant effect on both computer storage space used and computing time. For this data set a state size of 0.001" (0.0254 mm) was chosen. A total of 70 states were used since this was judged as a sufficient number for the interval of time used, t = 0 flight hours through t = 5500 flight hours.

With this information the material parameters can be determined. Using the program **SOLUTION** from Appendix (A) with the initial flaw size distribution at 1500 flight hours (see Figure 5.7), Number of states = 70, T initial = 1500 hours, and T final = 1500 to 5500 hours, λ and κ are found by iteration. The values of λ and κ which give a good fit to experimental data were determined to be :

$$\lambda = 0.26 \quad ; \quad \kappa = 0.96 \tag{5.10}$$

Using these parameters with the program **SOLUTION** the probability histograms

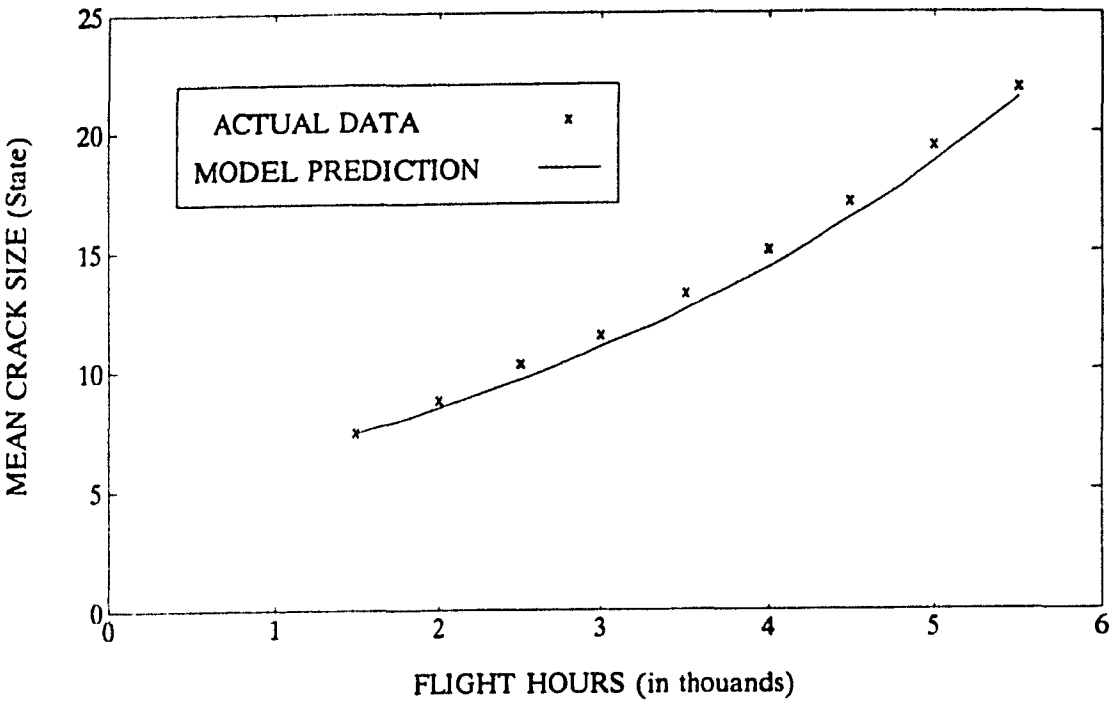


Figure 5.8: WPF – Mean Crack Size, From Data and Markov Model

at future times were generated and the mean (Equation 5.11) and variance (Equation 5.12) calculated from :

$$\mu_j(t) = \sum_j j\, p_j(t)\,, \tag{5.11}$$

$$\sigma_j^2(t) = \sum_j (j - \mu_j(t))^2\, p_j(t)\,. \tag{5.12}$$

Figures 5.8 and 5.9 show the results for the mean and variance from the model as well as the experiment and these results are also presented in Table 5.2. Figure 5.10 presents a probability histogram generated by the model as well as one from the experimental results. From a comparison between the actual and predicted histograms it would appear that given a larger data set the model would be able to give quite good predictions for these distributions.

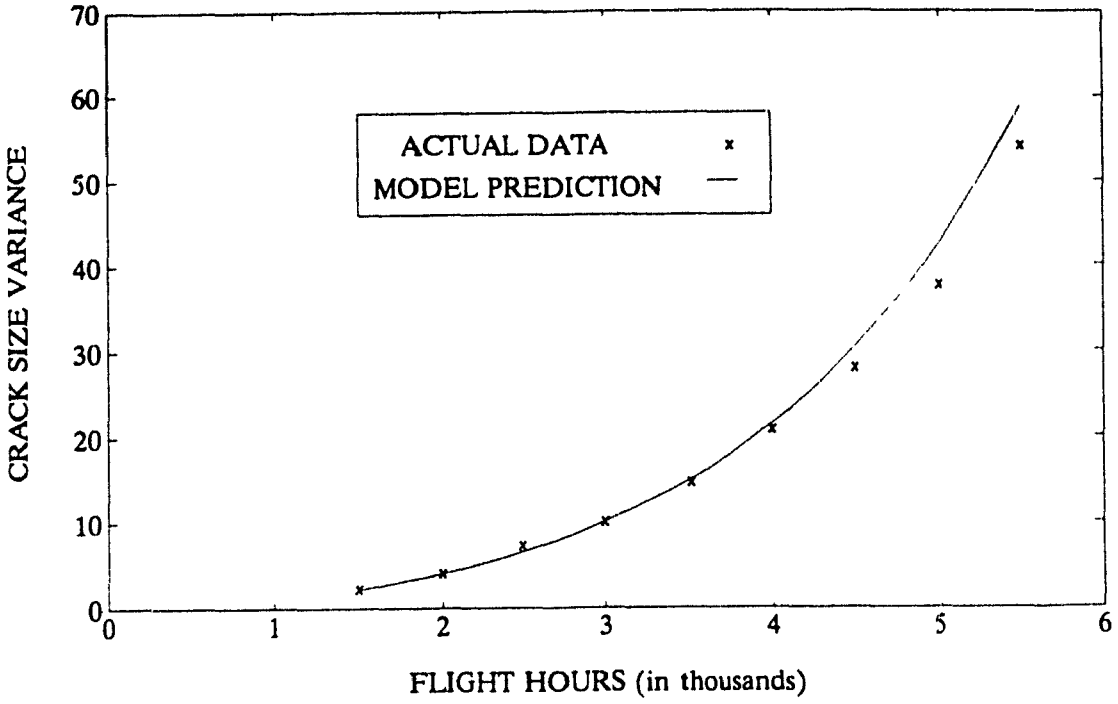
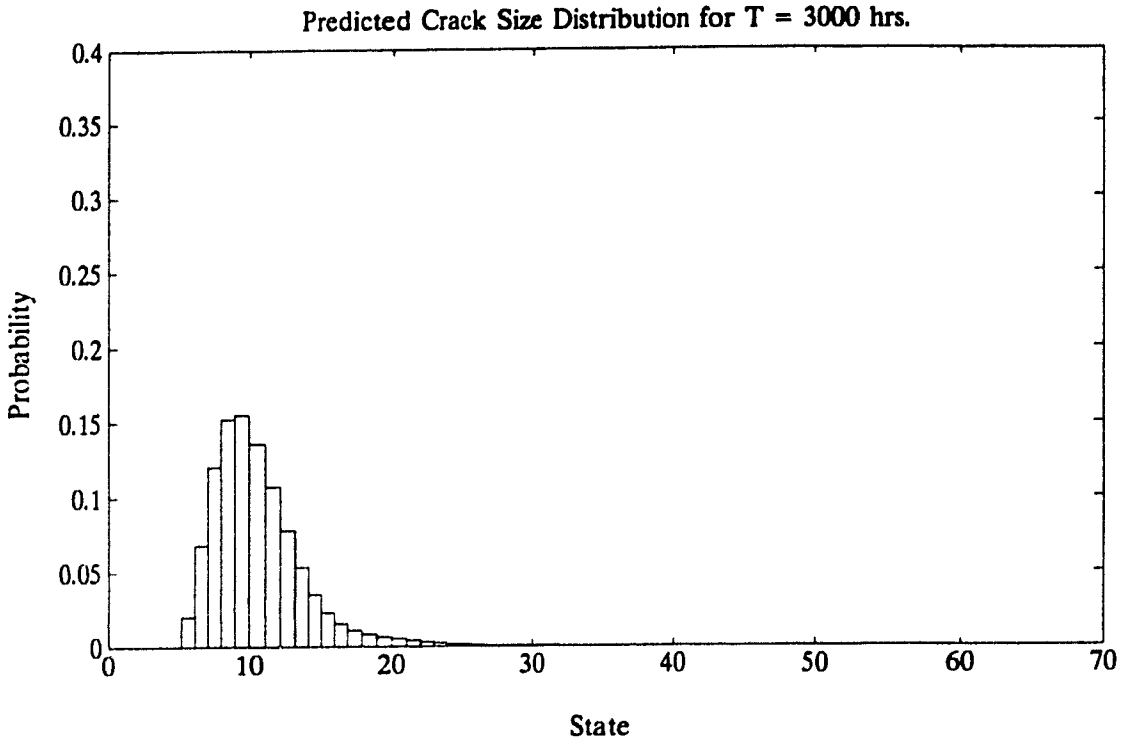


Figure 5.9: WPF – Crack Size Variance, From Data and Markov Model

FLIGHT	MEAN		VARIANCE	
HOURS	ACTUAL	MODEL	ACTUAL	MODEL
1500	7.39	7.39	2.18	2.18
2000	8.70	8.43	3.97	4.01
2500	10.24	9.61	7.17	6.57
3000	11.42	10.97	9.94	10.10
3500	13.18	12.53	14.51	14.96
4000	15.03	14.32	20.76	21.57
4500	17.03	16.37	27.91	30.55
5000	19.36	18.73	37.57	42.66
5500	21.82	21.43	53.97	58.77

Table 5.2: Actual Data and Model Prediction For Data Set WPF



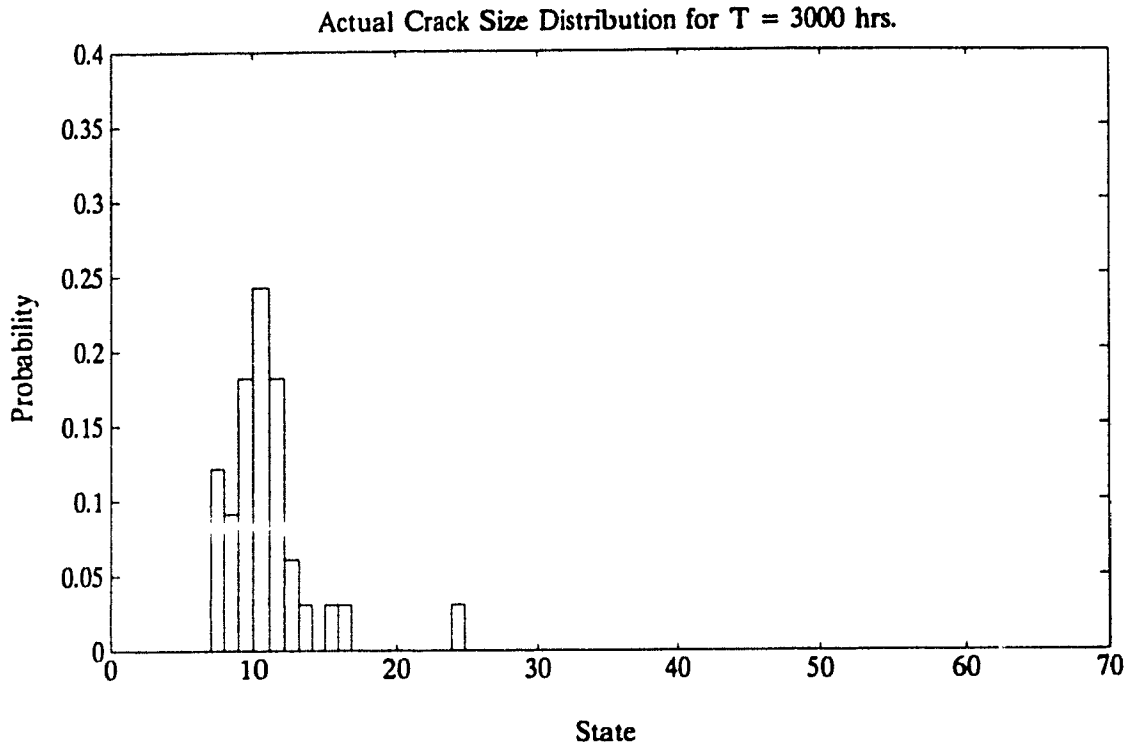


Figure 5.10: Probability histograms from Markov model and Actual data

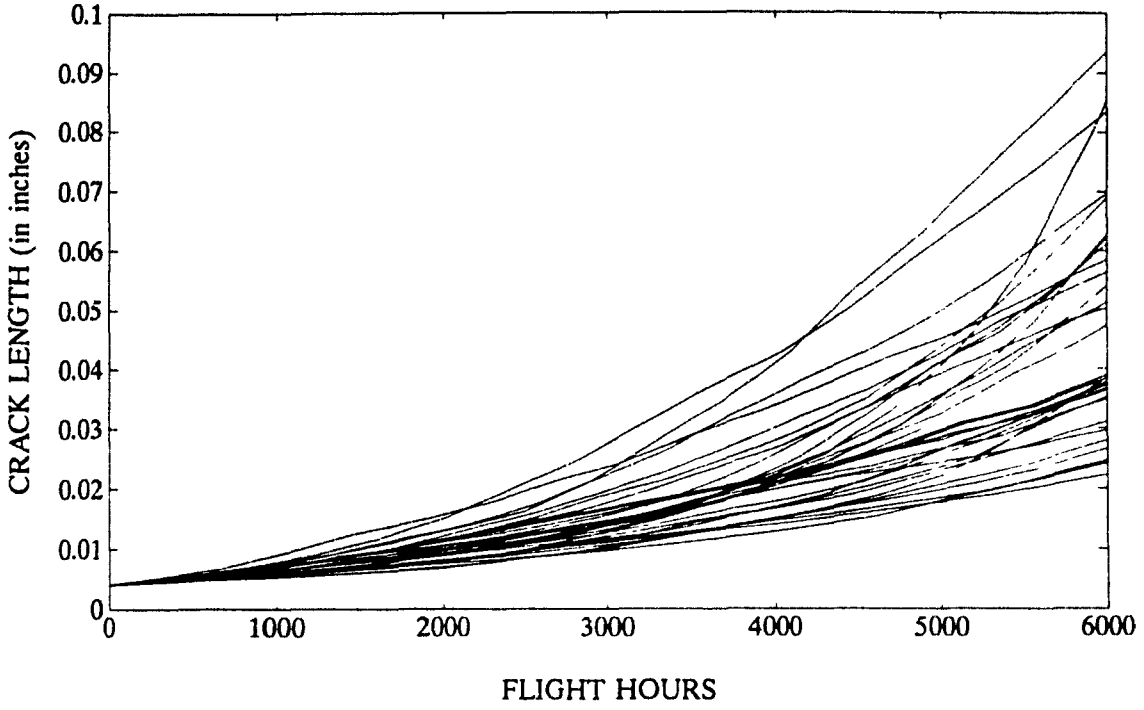


Figure 5.11: Raw Data Set XWPF, Normalized to 0.004" (0.102 *mm*) [61]

5.3.3 Data Set XWPF

One further and final demonstration of the ability of the Markov model to predict fatigue crack growth a second data set has been used. The data set **XWPF** [61] is shown in Figure 5.11. This data was obtained for tests performed on the specimen shown in Figure 5.12. This specimen was made of the same material as the one from the data set **WPF** and was tested under the same conditions (see previous section). The data set was again normalized to an initial flaw size of 0.004"(0.102 *mm*) and the same state size of 0.001" (0.0254 *mm*) was used.

Since there has been a change in the system, in this case a change in the specimen geometry, new empirical system parameters, λ and κ, need to be found. Values of λ and κ were found, using the program **SOLUTION,** to be:

$$\lambda = 0.35 \quad ; \quad \kappa = 0.79 \,. \tag{5.13}$$

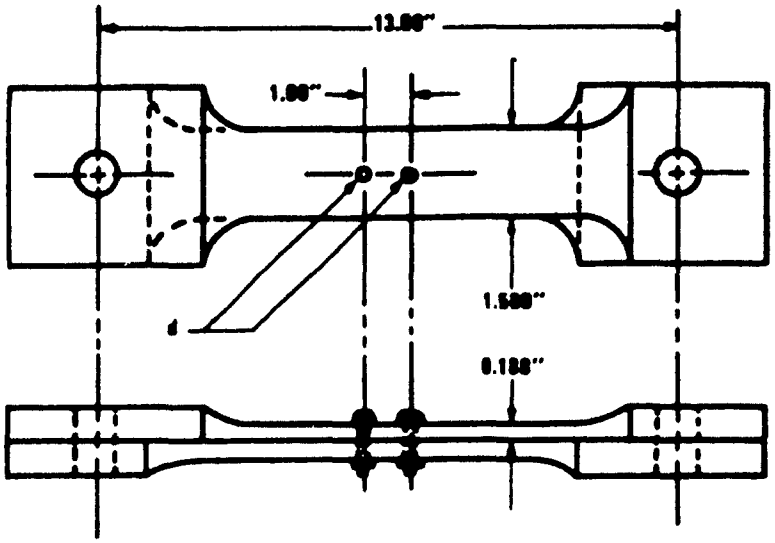


Figure 5.12: XWPF Test Specimen Geometry [39]

As expected, the system parameters are indeed different due to the change in specimen geometry. These values yielded the results for the mean and variance as given in Table 5.3. These results are also shown graphically in Figures 5.13 and 5.14 and again illustrate the fact that the Markov model is indeed capable of making excellent predictions for the mean and variance of fatigue crack size as a function of time.

These figures and the ones from the previous section are an indication of the ability of this model to predict the statistics of fatigue crack growth. Thus, one of the primary objectives of this thesis, the development of a stochastic process to model the growth of fatigue cracks *including the inherent scatter*, has been achieved. The second major objective of this report, the application of this model to component reliability predictions, will be examined in the following chapter. The reliability calculations in the remainder of this thesis make use of the data set **WPF**, since only one data set was necessary to illustrate the capabilities of this model, the procedures used, however, could have been applied equally to the second data set.

FLIGHT	MEAN		VARIANCE	
HOURS	ACTUAL	MODEL	ACTUAL	MODEL
1500	8.50	8.50	2.31	2.31
2000	10.38	10.20	4.67	5.38
2500	12.69	12.31	9.09	10.38
3000	15.50	14.93	17.38	18.45
3500	18.00	18.21	30.19	31.41
4000	22.84	22.31	54.63	52.14
4500	27.53	27.42	89.25	84.48

Table 5.3: Actual Data and Model Prediction For Data Set XWPF

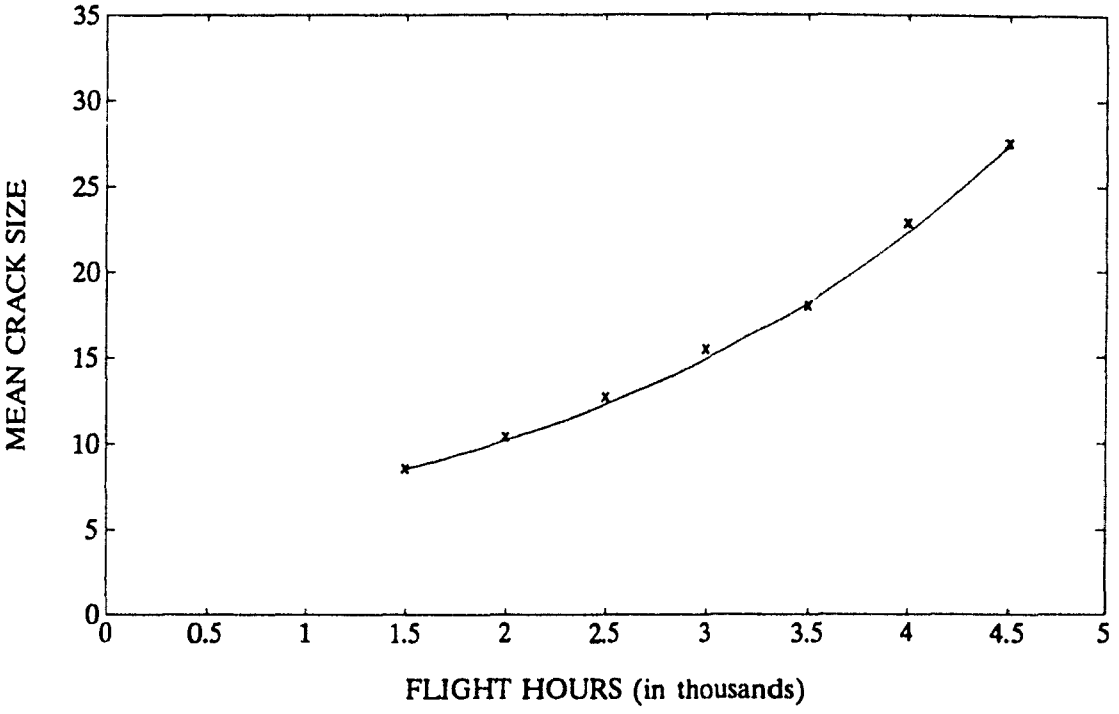


Figure 5.13: XWPF - Mean Crack Size, From Data and Markov Model

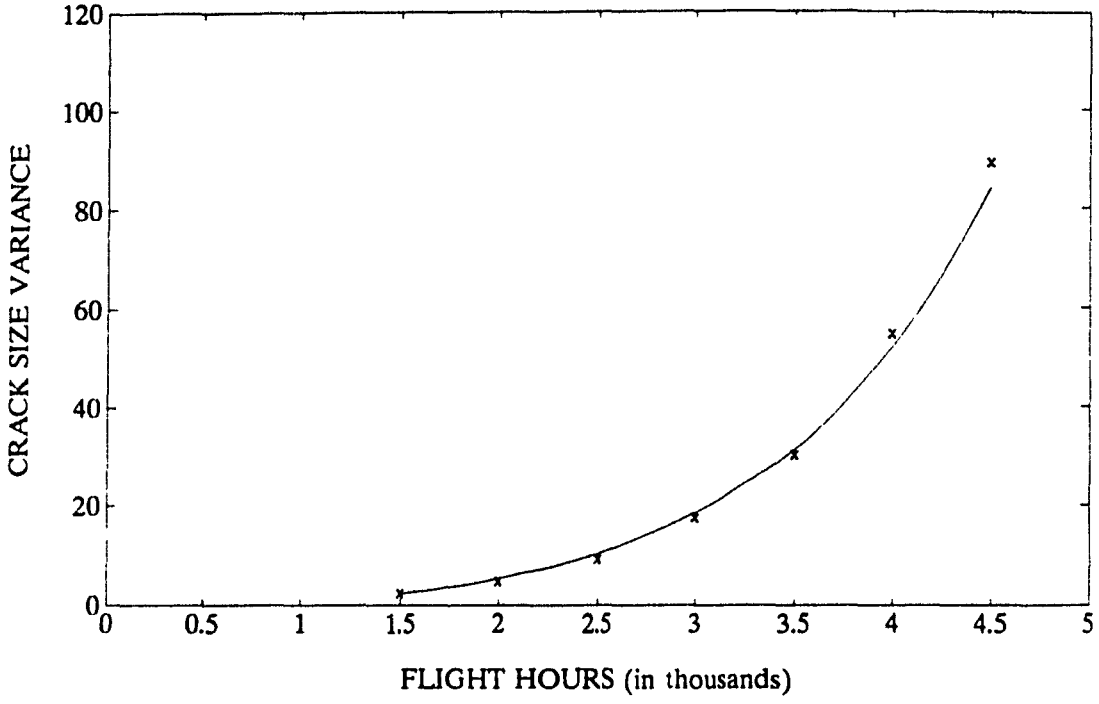


Figure 5.14: XWPF - Crack Size Variance, From Data and Markov model

Chapter 6

RELIABILITY ANALYSIS

6.1 A Failure Control System

Modern engineering designs require the use of high reliability components. This is especially true in the case of hazardous material containers, petroleum pipelines, nuclear power plants, aircraft structures and other applications where failure can be life-threatening. One of the major causes of the failures of these components is the unavoidable weakening associated with aging. This weakening of a component or structure can be caused by fatigue, wear, corrosion and other phenomena. The result of one or more of these processes is the degradation in the strength of the component. In order to combat this degradation the operator of the structure needs to implement a **Failure Control System** such as the one introduced by Hay et all. [62] and further developed by Rodriguez et all. [63].

There are two essential parts of the Failure Control System, namely, the *degradation module,* for which the model discussed in this report can be used, and what Hay et all. have termed the *"upgradation" module.* This upgradation module can also be called the inspection-correction module as it allows for the location and repair or replacement of components with "significant" flaws, where the operator must decide what is meant by significant. Schematically this failure control system can be shown as in Figure 6.1.

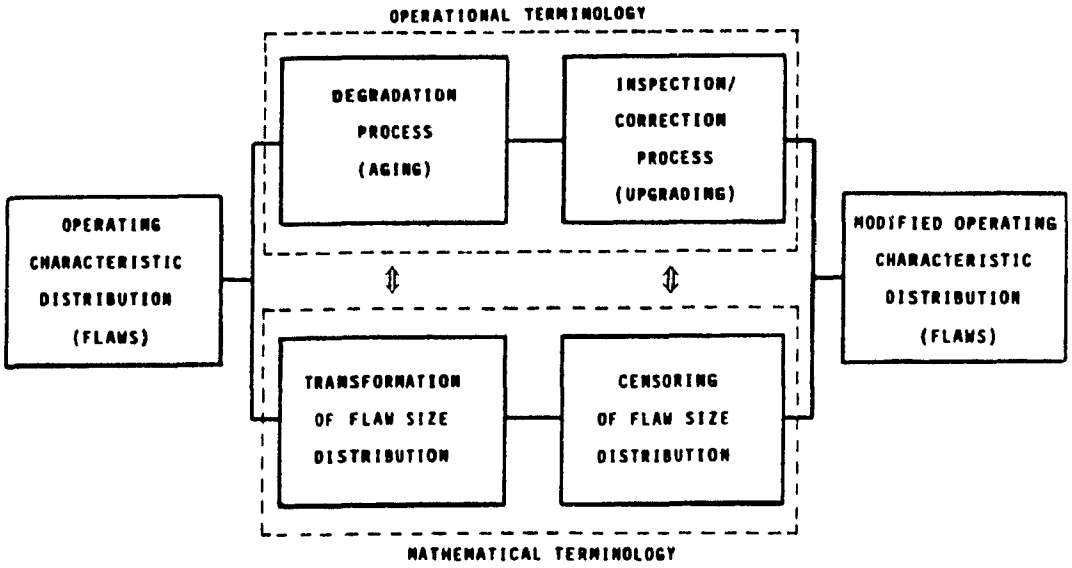


Figure 6.1: Schematic of a Failure Control System [63]

6.1.1 The Inspection/Correction Process

This section of the failure control system involves stopping the degradation process, locating components that pose a risk to structural integrity, and carrying out the necessary maintenance procedure. In mathematical terms it is the censoring of the flaw size distribution. This corrective procedure increases confidence that the structure will continue to perform satisfactorily. The increase in confidence depends on such things as the quality of the inspection apparatus and personnel, the definition of what is meant by a significant flaw and the quality of the repair procedure. All of these are variables which can be controlled, to some degree, by the operator of the structure.

INSPECTION

There are many non-destructive inspection procedures available such as, X-ray, flux-leakage, ultrasonic, acoustic emission, eddy current, dye penetrant and others. All of these techniques have different accuracy, time needed for testing, complexity, limitations and costs. They also introduce a degree of uncertainty which is in itself a complex

problem and is beyond the scope of the present work.

There are four possible outcomes of an inspection procedure and their respective probabilities depend on factors such as quality of inspection apparatus, flaw orientation and, perhaps most importantly, the quality of inspection personnel. The four possibilities are :

- correct identification of an existing flaw,
- failure to locate an existing flaw,
- correct identification of a component as defect-free,
- incorrectly labeling a defect-free component as flawed.

What is of interest to the engineer is the probability that the operator will be able to make a correct identification of an existing flaw in an in-service component. Many authors, references [64] - [68] for example, have proposed functions to describe the probability of detecting an existing flaw in a structure. For example, that of Davidson [66] is given by :

$$P_D(d) = \begin{cases} C\{1 - exp[-b(d - d_{th})]\} & ;\mathrm{d} \geq d_{th} \\ 0 & ;d < d_{th} \end{cases} \tag{6.1}$$

where:

C = a constant, less than 1, which indicates that even large flaws have a probability of detection that is less than 100%,

b = a parameter which depends on the critical crack criterion, i.e. the probability of detecting a crack of a certain critical size, d_c, should exceed a certain value, and

d_{th} = the detection threshold, which is the minimum size defect that can be found.

With a knowledge of the particular inspection process in question the constants C,b and d_{th} are determined and the probability of detection expression can then be used for reliability calculations. In the present investigation this function has been used, with the following values for the constants: the constant C = 0.99, which shows that even large flaws can be missed; b was obtained using the boundary condition that $P_d(0.030") = 0.98$; and the detection threshold, $d_{th} = 0.010"$ $(0.254mm)$.

CORRECTION

This step concerns the actual removal of the flawed components that were located by the inspection and their repair and/or replacement. This stage depends heavily on the judgement of the engineer, who must decide what is meant by a significant flaw and how to correct it. One option is to remove all flawed components and replace them with new ones. This would result in the structure being "as good as new" **if the inspection procedure was capable of detecting all flaws.** Certainly, in practice, these procedures are not perfect and often it is not economically feasible to replace all flawed components, no matter how small the flaw. The result is that the engineer decides when a flaw is large enough so that it may grow to critical size before the next scheduled inspection-correction procedure. The size is usually chosen to conform to either company policy or a design code, but it may be considered as a variable within certain limitations.

MODIFIED FLAW SIZE DISTRIBUTION

The modified flaw size distribution is the combination of the components that remain from the initial population and the components that have been repaired or their replacements. This can be seen from Figures 6.2[a]-[d] which are a graphical representation of this failure control system as presented by Rodriguez and Provan[58]. Figure 6.2[a] shows the initial flaw distribution where d_r is the flaw size at which components require correction and d_f is the size at which failure occurs. After some time, t, has elapsed the components have undergone degradation, which can be predicted us-

ing the stochastic law already developed. The result of this degradation is the flaw size distribution of Figure 6.2[b], note that a small number of components have exceeded the failure size, d_f. The hatched area of this figure represents the probability of failure. If a non-destructive inspection is carried out at this point some of the flaws present will be found while others will be missed, depending on the inspection procedure. This is shown in Figure 6.2[c] where the shaded region represents the flaws that are found. Knowing the probability of detection the number of flaws missed can be inferred. It can be seen that the smaller the flaw size the smaller the proportion of flaws actually located is. The correction procedure is then carried out where all components that have been found to have flaws greater than d_r are removed and replaced (in this case with new components). Figure 6.2[d] shows the modified distribution where the new components are considered as a separate group from the original population. The new components could be considered as part of the same group for the case of a time-homogeneous process. It is important to observe that there still exists a small number of components with flaws greater than d_r but which have not been found and replaced due to inspection shortcomings. This modified flaw size distribution can now be considered as the initial distribution for another degradation and inspection-correction cycle.

In these figures the replacement components have been considered with the smallest possible flaw size, this, however, does not have to be the case. The flawed components could have been repaired so that there were still some flaws existing but of varying sizes. A flaw size distribution could then be specified that allows for some of these components to have larger flaws present.

6.2 Reliability Analysis

The combination of the failure control system outlined in the previous section with the Markov model presented in Chapters 3 & 5 can be a very powerful tool for practical engineering reliability calculations. In order to illustrate this two specific uses will be

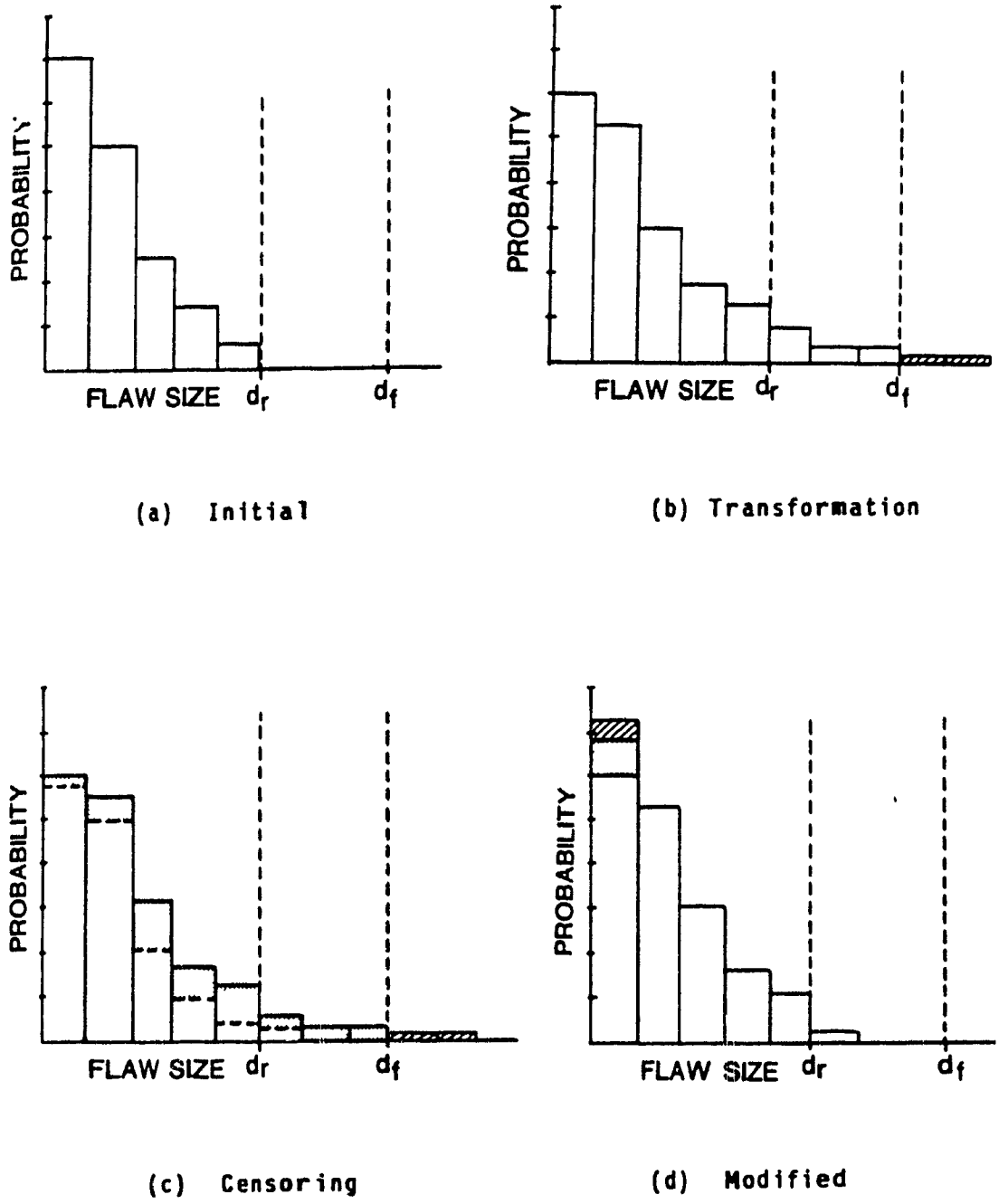


Figure 6.2: A Graphical Representation of the Failure Control System [58]

explained in the following sections. The first use is for the prediction of repair times in order to maintain a certain level of reliability (section 6.2.2), and the second is the determination of the optimum time for an inspection-correction procedure (section 6.2.3). Before these are examined a method that is central to both, the method of predicting reliability at a future time, is presented.

6.2.1 Reliability as a Function of Time

Reliability, as was discussed earlier, has been defined as the probability a component will perform satisfactorily for a specified period in a specified environment. A method for determining this probability is through the use of the mathematical model developed in this report. The first step is to use the model to generate probability histograms for crack size at given future times. The reliability can then be found if the critical crack length is known, it is the probability that the crack does not exceed this critical length. This quantity can be obtained by summing up the probability of a crack being any sub-critical size. This is illustrated in Figure 6.3 where the reliability is equal to the sum of the areas to the left of the critical size. This method of assessing reliability is used in the analysis contained in the remainder of this chapter.

RELIABILITY vs. TIME FOR WPF

For the data set WPF, as described in Chapter 5, predictions of reliability as a function of time can be made following the method of Figure 6.3. To facilitate this the computer program **SOLUTION** from Appendix A was used to generate crack size histograms for times from 1500 to 5500 flight hours. This data was then used to generate reliability as a function of time curves such as Figure 6.4. A family of curves of this nature can be obtained by varying the critical crack size or failure state, NF. This variation allows the user to examine the effect that changing the failure state has on reliability. For the figure shown, as well as for the analysis in the following sections, the failure state, NF, was taken to be a crack size of 0.040" (1.02 *mm*).

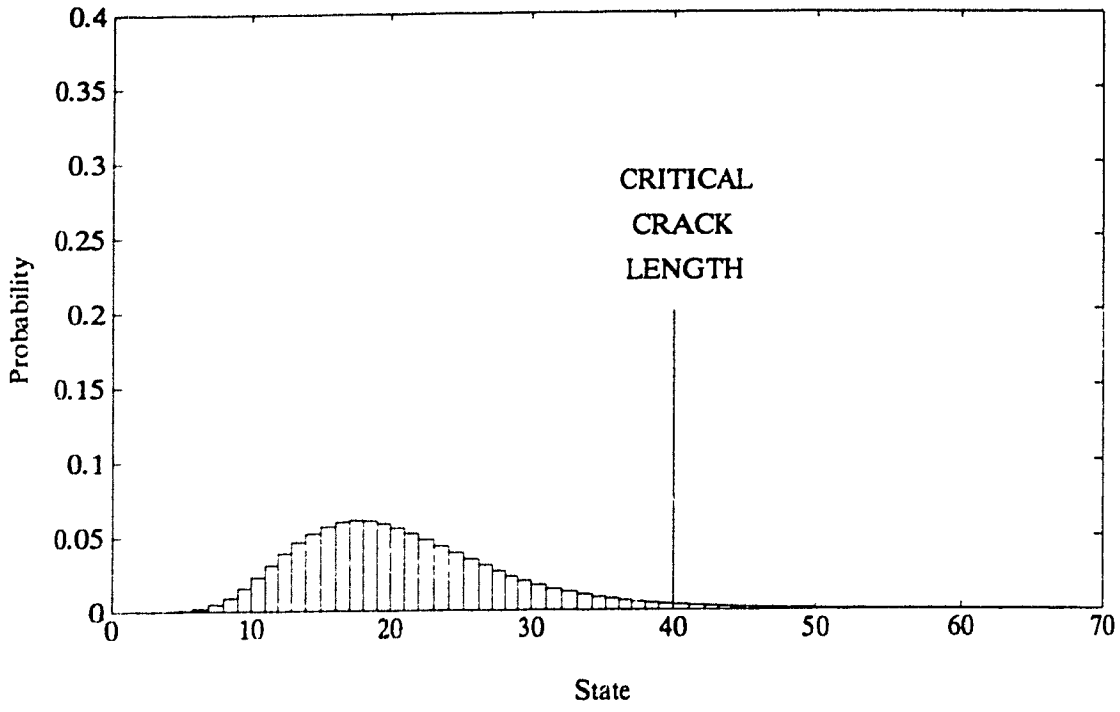


Figure 6.3: Determination of Reliability at a Given Time

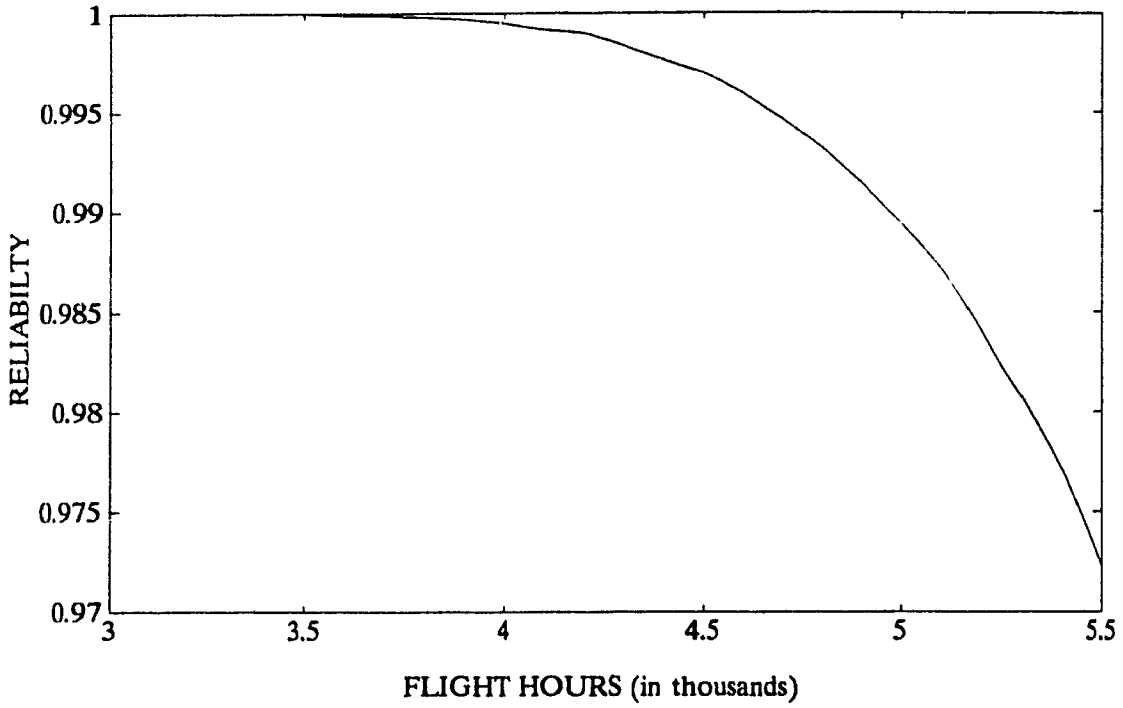


Figure 6.4: Reliability vs. Time for WPF, where $a_{cr} = 0.040$"

6.2.2 Reliability Maintenance

The operator of a structure will often decide upon a desired level of reliability. Once this level has been determined, perhaps by company policy, standard industry practices or other means, it becomes necessary to determine when to schedule the maintenance procedures that will ensure it. This can be done by using the Markov model to predict when the probability of failure will reach the desired limit and then calling for an inspection-correction procedure.

Before the analytical analysis is carried out the repair or maintenance policy must be determined. This policy consists of the answers to two questions: first, 'which components will be replaced?' and second, 'what will the quality of the replacement components be?'. In the present investigation it was decided that the size at which components would be replaced would be smaller than the critical size. The use of a replacement size smaller than the critical one allows for the removal of some components that are in greater danger of failure before the next maintenance procedure. The actual replacement size used was allowed to vary in order to illustrate the usefulness of the model for examining the effect of changes in repair policy. For all examples in this report the replacement components were assumed to have small initial flaws of 0.004"(0.102 mm) in order to eliminate the scatter due to initiation.

Analytically, as a result of the removal and replacement of some components, there are two distinct populations in the structure. These populations will be referred to as Population I, which consists of the remaining components from the initial group, and Population II, which is the group of replacement components. The reason this distinction is necessary is that the Markov process developed here is time-dependent or non-homogeneous. This means that for an inspection-correction at time $T_{inspect}$ the fatigue process continues for Population I while for Population II it starts at time T = 0 and ends at time T = T_{final} - $T_{inspect}$. As long as the fatigue loading situation remains the same the system parameters, λ and κ, can be used for both populations.

After the inspection-correction procedure has been carried out the probability of failure for the two populations is monitored in order to determine the total probability

of failure. Following the development of Rodriguez and Provan[58], this total probability of failure is a weighted combination of the probabilities of failure of Populations I and II. The weighting constant, K_t, is simply equal to the fraction of the initial population remaining in service and the total probability of failure is given by:

$$P_{fTOTAL}(t) = (1 - R_t^I)\, K_t + (1 - R_t^{II})(1 - K_t)\,, \tag{6.2}$$

where R_t^I and R_t^{II} are the reliabilities of Populations I and II respectively at time t. R_t^I is calculated by first normalizing the histogram of the remaining initial components by dividing by K_t. This new population is allowed to "age" using the Markov model until time t when its reliability is calculated. R_t^{II} is obtained by using the Markov model with the initial distribution being all components with an initial flaw of 0.004" (0.102 *mm*) and starting at time t and finishing at time t - T_{inspec}. This process can easily be extended to include as many inspection- correction procedures as desired. For example, the extension to include a second procedure would result in the following expression for the total probability of failure at a final time t, with inspections at times t_1 and t_2:

$$\begin{aligned} P_{fTOTAL}(t) = {} & (1 - R_t^I)K_{t_1}K_{t_2}^I + (1 - R_t^{II})(1 - K_{t_1})K_{t_2}^{II} + \\ & (1 - R_t^{III})[K_{t_1}(1 - K_{t_2}^I) + (1 - K_{t_1})(1 - K_{t_2}^{II})]\,; \end{aligned} \tag{6.3}$$

where $K_{t_2}^I$ and $K_{t_2}^{II}$ are the predicted reliabilities of Populations I and II at time t_2 and R_t^I , R_t^{II} , and R_t^{III} are the predicted reliabilities of Populations I, II and III, respectively, at time t.

For the case where it is desired to replace components that have not yet failed but which have flaws greater than the replacement size, the inspection procedure used must be considered. The components that will be removed will only be those whose flaws are detected. In other words, the normalizing constant will be the sum of the components with flaws less than the replacement size *plus* the components that have larger flaws but are missed due to the probability of non-detection associated with the

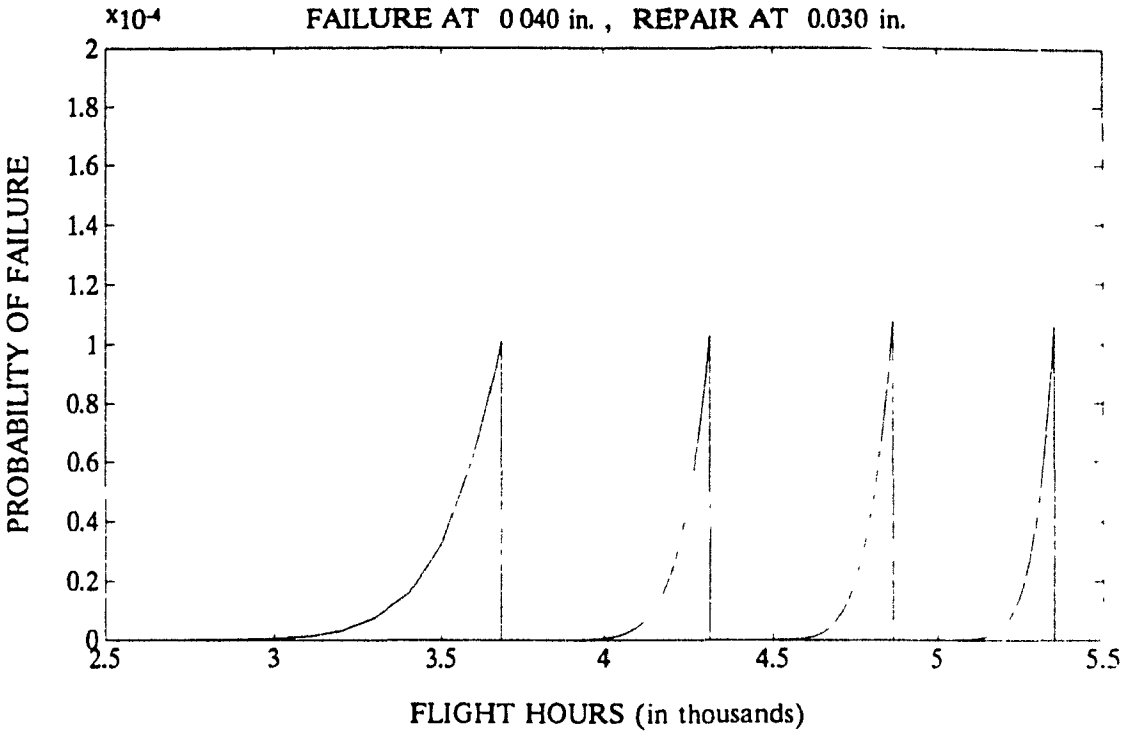


Figure 6.5: Determining the Inspection Schedule to Maintain Reliability at .9999

inspection process.

RELIABILITY MAINTENANCE FOR WPF

Applying this method of analysis, which is contained in the program **MAINTAIN** in Appendix A, is perhaps the best way to illustrate its capabilities. Defining the replacement size, NR, to be .030"(0.762 *mm*) and the desired reliability level to be .9999 or $P_{fTOTAL}(t) = 1.0 \times 10^{-4}$ the total probability of failure for times 2500 to 5500 Flight Hours were obtained as in Figure 6.5. From this figure the times for inspection-correction procedures can be determined.

One of the interesting uses of this methodology is the examination of the effect on repair schedule if, for all other variables constant, the desired level of reliability is changed. In Figure 6.5 the probability of failure was 1.0×10^{-4}, or 1 failure in 10,000, what if the operator decides that a probability of failure of, say, 1 in 2,000 would be acceptable? Obviously, they would like to know how the inspection schedule would

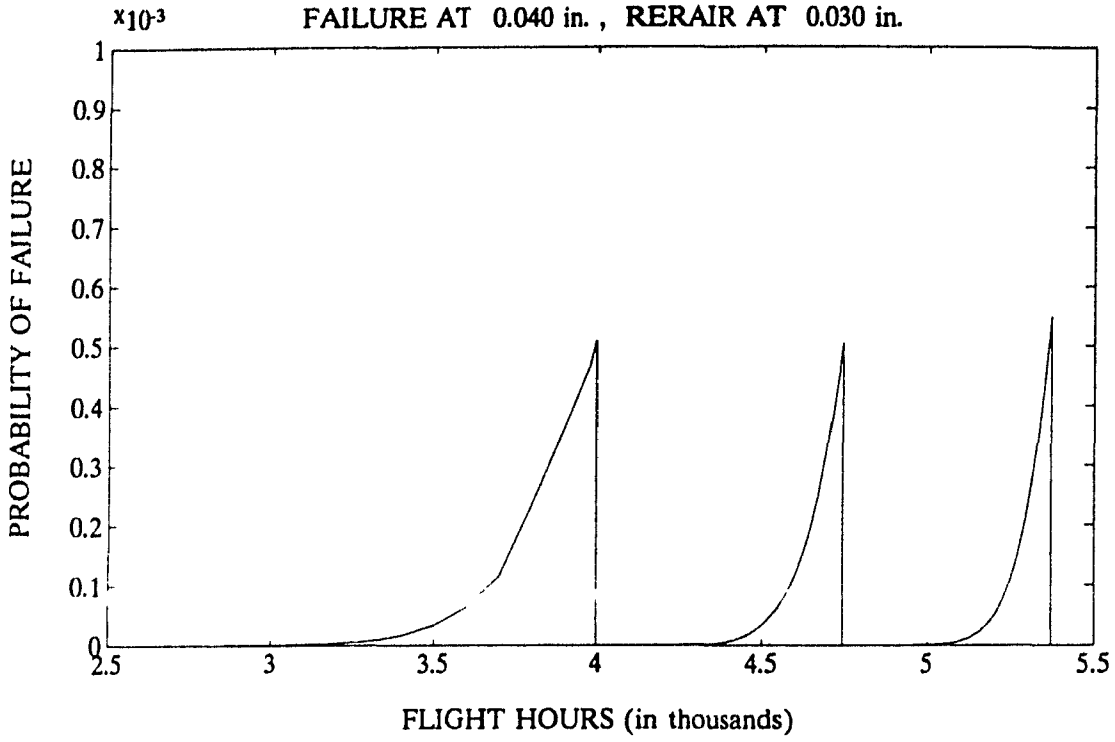


Figure 6.6: Determining the Inspection Schedule to Maintain Reliability at .9995

vary, i.e. *will there be a substantial enough savings in maintenance costs to justify the increased risks.* The results for this change in acceptable reliability level, to .9995, are presented in Figure 6.6. From a comparison with the previous figure it is immediately apparent that not only will the first maintenance procedure be carried out at a later time but that one fewer procedure will be necessary.

The effect of repair size on inspection interval is another useful application of this process. Figure 6.7 shows how inspection interval is affected by a change in repair size from 0.030"(0.762 *mm*) to 0.025"(0.635 *mm*). The first inspection will occur at the same time for both since this does not depend on repair size. After this first inspection, however, there is a noticeable difference between the two curves. There are a total of four inspection-corrections for NR = 0.030" before 5500 flight hours while for NR = 0.025" the third one would be scheduled *after* this time! The ability to generate information of this nature is a powerful tool for the system operator. These results help answer the question "what is the savings in maintenance time and costs

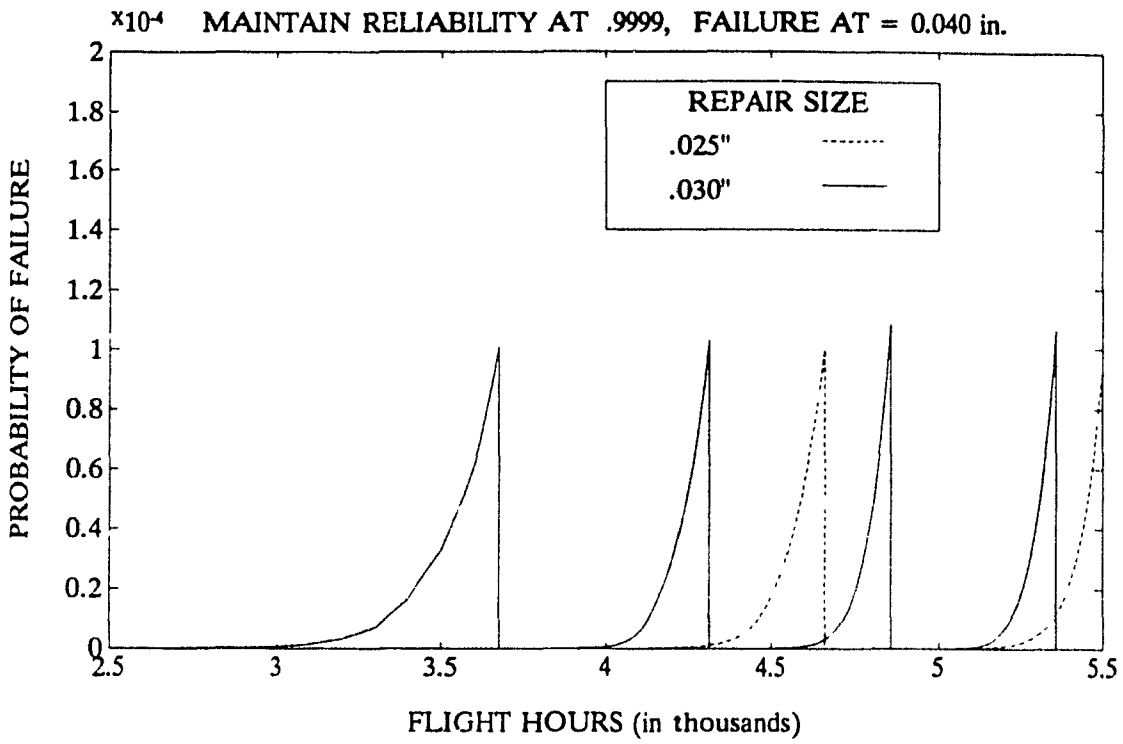


Figure 6.7: Influence of Repair Policy on Inspection Scheduling

corresponding to an increase in expenditure for the replacement of more components?"

These figures (6.5,6.6,6.7) illustrate the type of information that can be obtained from this model concerning inspection intervals. The analysis has shown that the engineer can make hypothetical changes in the repair and/or reliability policy and examine, quantitatively, the effects of these changes and the potential benefits. All of these analyses can be performed, with minimal necessary input from the operator, using the program **MAINTAIN** found in Appendix A.

6.2.3 Inspection Optimization

Another useful form of reliability analysis which can be carried out with this model is the optimization of the inspection time. As an example, suppose that it is desired to minimize the total probability of failure at a future time, and further that there will be only one inspection-correction process in a given interval of time. The question that the operator must ask and answer is "when is the optimum time for this procedure?"

If there is no repair of the structure the probability of failure can be obtained from a curve such as the one presented in Figure 6.4 (section 6.2.1). In order to decrease this probability of failure an inspection-correction procedure will be carried out at some time. An inspection too early in the service life will, on the one hand, remove few components that may subsequently fail while a later inspection may be too late to remove components that will have failed. The optimum time for inspection will depend on several variables such as: critical crack size, repair size, inspection process, and the quality of replacement components [45]. The quantity to be minimized, the total probability of failure at some time t_{final} , is given by [58] :

$$P_{fTOTAL}(t_{final}) = (1 - R_t) + (1 - R^I_{t_{final}})K_t + (1 - R^{II}_{t_{final}})(1 - K_t)\,, \qquad (6.4)$$

where t is the time of the inspection, R_t is the reliability of the initial population at the time of inspection, K_t is the normalizing constant as explained previously, and $R^I_{t_{final}}$ and $R^{II}_{t_{final}}$ are the reliabilities of Populations I and II, respectively, at the final time.

In order to carry out this analysis the computer program **OPINSPEC** was developed (Appendix A). This program allows the user to control such variables as NF, NR and the Probability of Detection. By using this program with the same probability of detection function and with NF = 0.040"(1.016 *mm*) and with NR = 0.030"(0.762 *mm*) the results presented in Figure 6.8 are obtained. From this figure it is apparent that the optimum time for inspection-correction is at 4700 hours and that the total probability of failure is decrease by 60 % over the no inspection case.

By changing repair policy, such that NR is varied, different curves are obtained. By examining several of these, as in Figure 6.9, the change in optimum inspection time as well as total probability of failure at 5500 hours can be observed. By changing from a repair size of 0.030"(0.762 *mm*) to one of 0.025"(0.635 *mm*) the optimum inspection time becomes about 300 flight hours earlier, which is to be expected since more cracks will reach this smaller size at an earlier time. In addition, the minimum achievable probability of failure is decreased by a further 50% to about 1/5 the value for no

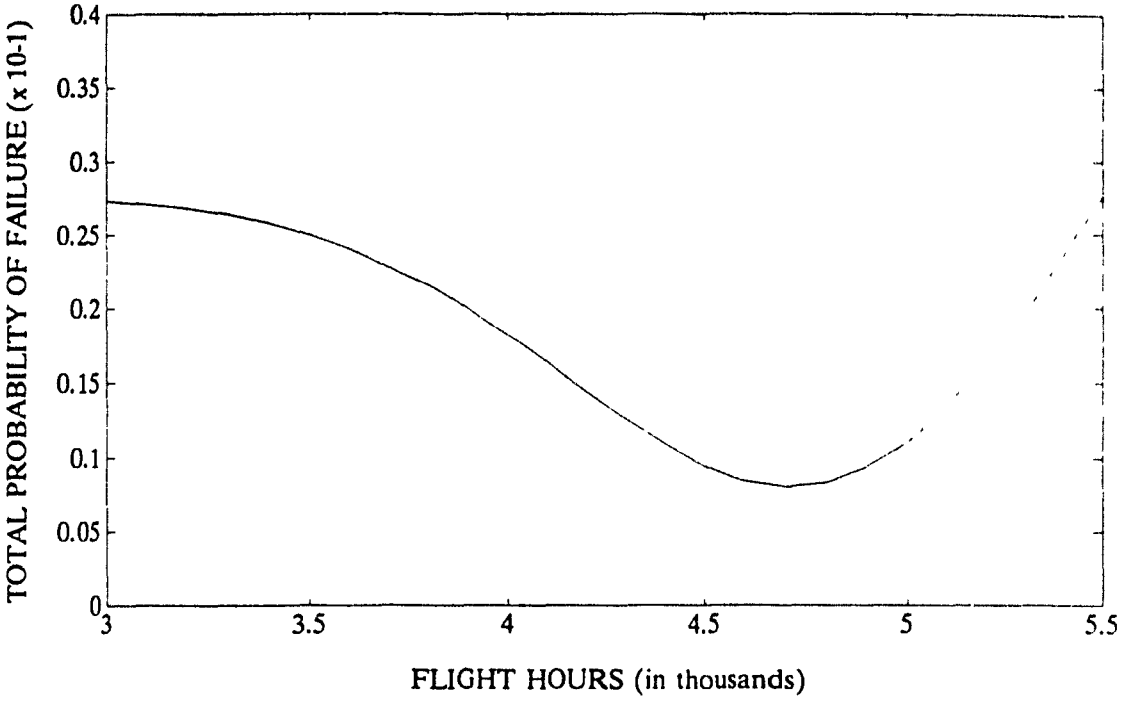


Figure 6.8: Total Probability of Failure as a Function of Inspection Time

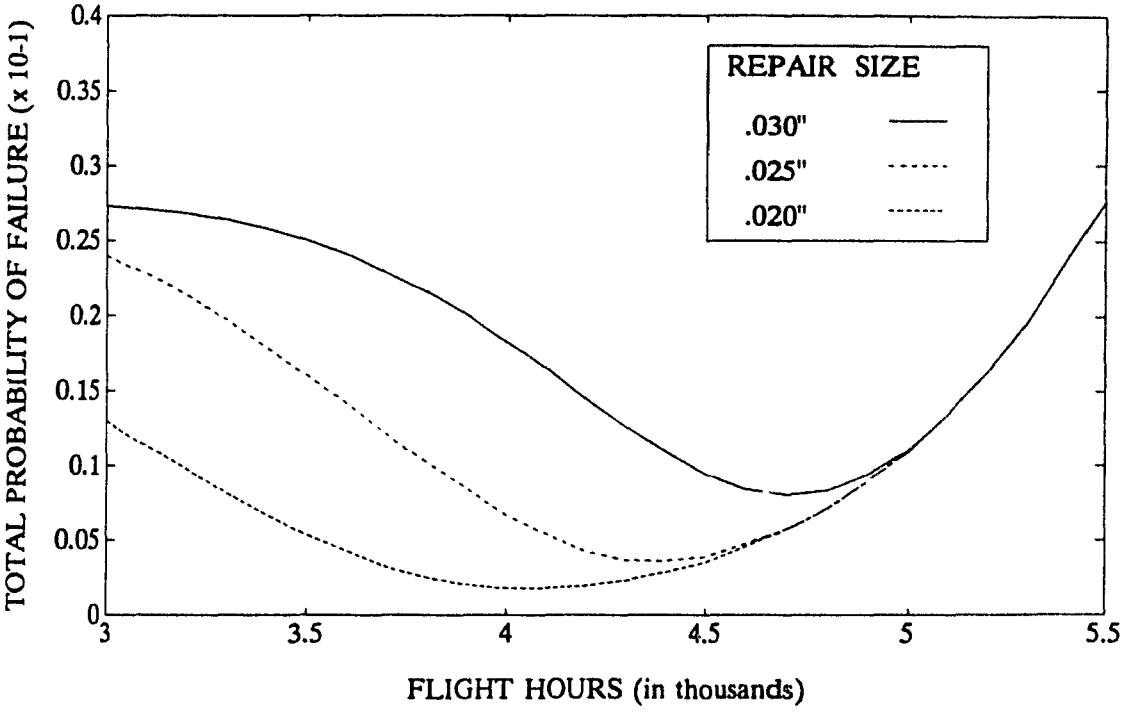


Figure 6.9: Influence of Repair Policy on Optimum Inspection Time

inspection-correction. These trends are also observable for a further decrease in the repair size to 0.020"(0.508 *mm*).

This analysis can of course be extended to allow for two or more inspection-corrections during the given interval. This is conceptually just a simple extension of Equation 6.4 to include as many as desired. Each new inspection-correction adds considerably to the amount of computer time needed so, for the present investigation, this analysis was not performed. It could, however, be easily done through some straight-forward modifications of the program **OPINSPEC.**

Chapter 7

CONCLUSIONS

It has been the objective of this thesis to develop a probabilistic model for fatigue crack growth and apply this model to reliability analyses. The nonhomogeneous Markov model developed has been applied to several fatigue data sets that are available in the literature. The following can be concluded based upon the results of the current research:

- The nonhomogeneous Markov model can be used, with the choice of appropriate intensity functions, to provide a valid prediction of fatigue crack growth since it can predict not only the mean crack size as a function of time but also crack size variance as a function of time.

- A change in the fatigue system, consisting of the material, loading, temperature, loading etc., means a change in the parameters λ and κ. This has been shown by the different values for the two data sets **WPF** and **XWPF** where the only difference in the system was a different specimen geometry.

- From the results it can be seen that the Markov model, used in combination with a failure control methodology based upon the methodology presented in Chapter 6, can be a useful tool for obtaining valuable reliability information. The types of information obtainable can include:

    i. estimates of reliability as a function of time,

ii. the determination of the maintenance schedule which will ensure a desired level of reliability,

iii. the optimal time for a maintenance procedure, and

iv. the examination of the effect of varying quantities such as the minimum reliability level and the repair size.

In light of these findings this method of modelling fatigue crack growth certainly merits further investigation. It is hoped that the computer programs developed for this research can be of use in compiling a table of system parameters, λ and κ, to be used for future fatigue and reliability analysis.

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Appendix A

COMPUTER PROGRAMS

The computer programs presented here are written for a PC - based FORTRAN compiler. They require user interaction as well as the use of input and output data files. Some of the syntax would need to be changed for different computer systems, compilers or for mainframe use, but the overall structure and the algorithms would remain the same.

A.1 Program - SOLUTION

This program is instrumental in the numerical solution of the Kolmogrov Differential Equation 5.6 for the transition probabilities, $p_{i,j}$ (τ,t). The algorithm for the solving of this equation is the one developed by Rodriguez[54] These transition probabilities, which are determined in subroutine TRANSP, are then used in subroutine EVOLVE to calculate the probability histograms for crack size distribution. The program also calculates the mean and variance (subroutine MOMENT) and the reliability (subroutine EVOLVE) for each of these histograms.

The necessary inputs are :

TOL	the tolerance for error control.
NSTATE	the number of states in the model.
PI(NSTATE)	a vector of the initial probability distribution

λ , κ	the empirical material parameters
T1,T2,DT	these are τ , t_{final} , and the step size
NF	the failure state, to use for reliability estimates

The listing of the program is as follows:

```
      PROGRAM SOLUTION
C
C       COMPUTES THE MEAN AND VARIANCE OF THE DISTRIBUTION
C       AT A LATER TIME, GIVEN THE INITIAL PROBABILITY
C       DISTRIBUTION ON FILE INITIAL.DAT (the file contains the
C       probability of being in each state from 1 to NSTATE).
C       IT ALSO OUTPUTS THE FINAL DISTRIBUTION ON THE FILE
C       PROB.OUT AND THE RELIABILITY TO THE FILE RELI.OUT
C
        DIMENSION PI(70),PJ(70)
        DOUBLE PRECISION PI,PJ,TINIT,TFINAL,DT,T1,T2
        COMMON/VAR/JJJ,III,ALAMBD,AKAPP,TOL
        DOUBLE PRECISION TOL,AVER,VAR,RELI
        OPEN(1,"INITIAL.DAT")
        OPEN(2,"PROB.OUT")
        OPEN(3,"RELI.OUT")
        PRINT*," ENTER TOLERANCE FOR SOLUTION :  "
        READ*, TOL
        PRINT*,"ENTER NUMBER OF STATES :  "
        READ*, NSTATE
        DO 10 I = 1,NSTATE
   10   READ(1,200) PI(I)
        CALL MOMENT(PI,NSTATE,AVER,VAR)
        PRINT*," MEAN STATE = ",AVER
        PRINT*," VARIANCE = ",VAR
    4   PRINT*,"   ENTE.. LAMBDA (if you want to quit type 0)  "
        READ*, ALAMBD
        IF (ALAMBD.LE.0.) GOTO 999
        PRINT*,"   ENTER KAPPA  "
        READ*, AKAPP
        PRINT*,"   ENTER INTIAL AND FINAL TIMES AND INCREMENT  "
        READ*,T1,T2,DT
        PRINT*," "
        PRINT*,"   ENTER FAILURE STATE, N =  "
        READ*,NF
        IF(NF.LE.0) GOTO 999
        TINIT = T1
        TFINAL = T1
```

```
      WRITE(2,99)ALAMBD,AKAPP
  3   TFINAL = TFINAL + DT
      IF(TFINAL.GT.T2) GOTO 4
      PRINT*,"  AT TIME = ",TFINAL
      CALL EVOLVE(TINIT,TFINAL,PI,PJ,NF,NSTATE,RELI)
      DO 20 I = 1,NSTATE
 20   WRITE(2,200)PJ(I)
      CALL MOMENT(PJ,NSTATE,AVER,VAR)
      PRINT*,"   MEAN STATE =  ",AVER
      PRINT*,"   VARIANCE   =  ",VAR
      PRINT*,"   RELIABILITY IS  ",RELI
      WRITE(2,100)AVER
      WRITE(2,101)VAR
      WRITE(2,102)TFINAL
      WRITE(3,150)TFINAL,RELI,NF
      GOTO 3
999   STOP
 99   FORMAT(' LAMBDA  ',F6.4,'    KAPPA  ',F6.4)
100   FORMAT('  THE MEAN      ',F8.5)
101   FORMAT('  THE VARIANCE  ',F10.7)
102   FORMAT('  AT TIME       ',F6.3)
150   FORMAT("T = ",F6.4,"  RELIABILITY ",F10.8,"   NF = ",I5)
200   FORMAT(E16.9)
      END

      SUBROUTINE MOMENT(PX,NSTATE,AVER,VAR)
      DIMENSION PX(1)
      DOUBLE PRECISION PX,SUM,VAR,AVER
      SUM = 0.D0
      DO 10 I = 1,NSTATE
 10   SUM = SUM + FLOAT(I) * PX(I)
      AVER = SUM
      SUM = 0.D0
      DO 20 I = 1,NSTATE
 20   SUM = SUM + PX(I)*(FLOAT(I) - AVER)**2
      VAR = SUM
      RETURN
      END

      SUBROUTINE EVOLVE(TINIT,TFINAL,PI,PJ,NF,NSTATE,RELI)
      DIMENSION TP(2485),PI(1),PJ(1)
      DOUBLE PRECISION TP,PI,PJ,SUMJ,TINIT,TFINAL,SUM,RELI
      CALL TRANSP(TP,TINIT,TFINAL,NSTATE)
      DO 40 J = 1,NSTATE
      SUMJ = 0.D0
      DO 30 I = 1,J
```

```
      IPOS = J + NSTATE * (I-1) - I * (I-1)/2
      SUMJ = SUMJ + PI(I) * TP(IPOS)
30    CONTINUE
      PJ(J) = SUMJ
40    CONTINUE
      SUM = 0.DO
      NFM1 = NF -1
      DO 50 I = 1,NFM1
50    SUM = SUM + PJ(I)
      RELI = SUM
      RETURN
      END

      SUBROUTINE TRANSP(TP,TINIT,TFINAL,NSTATE)
      DIMENSION TP(1),Y(70)
      DOUBLE PRECISION TP,Y,TINIT,TFINAL
      DO 10 I = 1,NSTATE
      CALL PIJ(I,NSTATE,TINIT,TFINAL,Y)
      NN = NSTATE - I + 2
      DO 5 J = I,NSTATE
      NN = NN - 1
      IPOS = J + NSTATE * (I-1) - I * (I-1)/2
      TP(IPOS) = Y(NN)
5     CONTINUE
10    CONTINUE
      RETURN
      END


      SUBROUTINE PIJ(II,JJ,TINIT,TFINAL,Y)
      INTEGER N,IND,NW,KK
      DIMENSION W(70,10),Y(1),C(24)
      COMMON/VAR/JJJ,III,ALAMBD,AKAPP,TOL
      DOUBLE PRECISION W,X,XEND,TOL,Y,C,TINIT,TFINAL
      EXTERNAL FCN
      III = II
      JJJ = JJ
      NW = JJ - II + 1
      N = NW
      X = TINIT
      NM1 = N - 1
111   DO 10 KK = 1,NM1
10    Y(KK) = 0.DO
      Y(N) = 1.DO
      XEND = TFINAL
      CALL SOLVEDE(N,FCN,X,Y,XEND,TOL,C,NW,W)
```

```
      RETURN
      END

      SUBROUTINE FCN(N,X,Y,YPRIME)
C
C     THE KOLMOGROV DIFFERENTIAL EQUATION
C
      COMMON/VAR/JJJ,III,ALAMBD,AKAPP,TOL
      DOUBLE PRECISION TOL,X,Y(N),YPRIME(N),A,B,C
      NM1 = N - 1
      DO 10 I = 1,NM1
      IP1 = I + 1
      A= (1.D0+ALAMBD*X)*FLOAT(JJJ-I+1)/(1.D0+ALAMBD*X**AKAPP)
      B = (1.D0+ALAMBD*X)*FLOAT(JJJ-I)/(1.D0+ALAMBD*X**AKAPP)
      YPRIME(I) = -A*ALAMBD*Y(I) + B*ALAMBD*Y(IP1)
10    CONTINUE
      C = (1.D0+ALAMBD*X)*FLOAT(III)/(1.D0+ALAMBD*X**AKAPP)
      YPRIME(N) = -C*ALAMBD*Y(N)
      RETURN
      END
```

A.2 Program - MAINTAIN

This is the program that is used for the reliability maintenance calculations following the method outlined in section 6.2.2. The version presented here neglects the probability of failure of the replacement population(s). This has been done since i was found that the probability of failure of these populations is several orders of magnitude smaller than that for the initial population *for the time interval studied in this report.* This being the case, and considering the fact that each new population significantly increases computing time, these populations have not been considered

This program uses the subroutines presented in the previous program for the solving of the Kolmogrov Differential Equation. It also uses the subroutine REPAIR which can be modified by the user in order to set the probability of detection function.

The necessary inputs are the same as for **SOLUTION** with the addition of the following :

INC1	an initial increment of time which allows the user to skip over some increment during which previous results have shown that reliability is high enough. If there are no previous results of this nature set INC1 = DT
NR	the crack size at which parts will be replaced
RLIMIT	the level of reliability to be maintained.

The program listing is ...

```
      PROGRAM MAINTAIN
C
C       COMPUTES THE RELIABILITY OF THE DISTRIBUTION
C       AT A LATER TIME, GIVEN THE INITIAL PROBABILITY
C       DISTRIBUTION ON FILE INITIAL.DAT (the file contains the
C       probability of being in each state from 1 to NSTATE).
C       THE PROGRAM ASKS FOR THE LEVEL OF RELIABILITY THAT
C       IS TO BE MAINTAINED AND WHEN THE RELIABILITY OF THE
C       POPULATION DROPS BELOW THIS LEVEL, IT CARRIES OUT
C       AN INSPECTION/REPAIR OPERATION.  THE INITIAL INCREMENT
C       SKIPS THE INITIAL INSPECTIONS WHEN WE KNOW, FROM
C       PREVIOUS RESULTS, THAT RELIABILITY IS HIGH ENOUGH.
C
C       IT OUTPUTS THE RELIABILITY TO THE FILE RELI.OUT
C
      DIMENSION PI(70),PJ(70)
      DOUBLE PRECISION PI,PJ,TINIT,TFINAL,DT,T1,T2
      COMMON/VAR/JJJ,III,ALAMBD,AKAPP,TOL
      DOUBLE PRECISION TOL,AVER,VAR,RELI,INC1,RLIMIT
      OPEN(1,"INITIAL.DAT")
```

```
      OPEN(2,"MAINTAIN.OUT")
      PRINT*," ENTER TOLERANCE FOR SOLUTION :  "
      READ*, TOL
      PRINT*,"ENTER NUMBER OF STATES :  '
      READ*, NSTATE
      REWIND 1
      DO 10 I = 1,NSTATE
10    READ(1,200) PI(I)
      CALL MOMENT(PI,NSTATE,AVER,VAR)
      PRINT*," MEAN STATE = ",AVER
      PRINT*," VARIANCE = ",VAR
4     PRINT*,"   ENTER LAMBDA (if you want to quit type 0)   "
      READ*, ALAMBD
      IF (ALAMBD.LE.0.) GOTO 999
      PRINT*,"   ENTER KAPPA   "
      READ*, AKAPP
      PRINT*,"   ENTER INITIAL AND FINAL TIMES AND INCREMENT
"
      READ*,T1,T2,DT
C
C          THIS ALLOWS A JUMP IN TIME INITIALLY TO SKIP OVER
C          SOME INITIAL TIMES WHERE RELIABILITY IS HIGH.
C
      PRINT*," "
      PRINT*,"      ENTER THE INITIAL INCREMENT "
      READ*,INC1
      PRINT*,"  "
      PRINT*,"   ENTER FAILURE STATE, N =  "
      READ*,NF
      IF(NF.LE.0) GOTO 999
      PRINT*,"   ENTER THE REPLACEMENT STATE, NR "
      READ*,NR
      PRINT*," "
      PRINT*,"  MAINTAIN RELIABILITY AT  ?  "
      READ*,RLIMIT
      KOUNT = 0
      TINIT = T1
      TFINAL = T1 + INC1 - DT
      REWIND 2
      WRITE(2,99)ALAMBD,AKAPP
      WRITE(2,100)NF,NR
3     TFINAL = TFINAL + DT
      IF(TFINAL.GT.T2) GOTO 4
      PRINT*,"  AT TIME = ",TFINAL
      CALL EVOLVE(TINIT,TFINAL,PI,PJ,NF,NSTATE,RELI)
      CALL MOMENT(PJ,NSTATE,AVER,VAR)
```

```
      PRINT*,"   MEAN STATE =  ",AVER
      PRINT*,"   VARIANCE   =  ",VAR
      PRINT*,"   RELIABILITY IS  ",RELI
      WRITE(2,150)TFINAL,RELI
C
C          CHECK RELIABILITY LEVEL, IF O.K. THEN INCREMENT
C          TIME AND CONTINUE, IF NOT THEN INSPECT AND
C          REPLACE COMPONENTS USING SUBROUTINE "REPAIR"
C
      IF(RELI.GE.RLIMIT) GOTO 3
      KOUNT = KOUNT + 1
      PRINT*,"INSPECTION NUMBER ",KOUNT
      CALL REPAIR(NR,PJ)
      DO 25 I = 1,NSTATE
 25   PI(I) = PJ(I)
      RELI = 1.D0
      DO 30 K = NF,NSTATE
 30   RELI = RELI - PJ(K)
      TINIT = TFINAL
      WRITE(2,150)TFINAL,RELI
      GOTO 3
999   STOP
 99   FORMAT(' LAMBDA  ',F6.4,'    KAPPA  ',F6.4)
100   FORMAT('FAILURE STATE  ',I5,'     REPLACEMENT STATE',I5)
150   FORMAT(F6.4,'   ',F15.12)
200   FORMAT(E16.9)
      END

      SUBROUTINE REPAIR(NR,PJ)
      DIMENSION PJ(70)
      DOUBLE PRECISION PJ,REMOVED,PDETECT
      REMOVED = 0.D0
      DO 10 I = NR,70
      PDETECT = .99D0*(1.D0 - DEXP(-.2298D0*FLOAT(I-10)))
      REMOVED = REMOVED + PJ(I)
 10   PJ(I) = PJ(I) * (1.D0 - PDETECT)
      RETURN
      END
```

A.3 Program - OPINSPEC

This program finds the total probability of failure at some time, t_{final} , given that there is one inspection-correction between $t_{initial}$ and t_{final}. The program is based on the algorithm presented in section 6.2.3. At a time $t_{inspect}$, between $t_{initial}$ and t_{final} , it calls for an inspection-correction procedure (subroutine DETECT) and then assesses the total reliability at t_{final} using the subroutine TOTFAIL which is merely Equation 6.4. This time, $t_{inspect}$, is allowed to vary from $t_{initial}$ to t_{final} by the increment DT. The program also makes use of the Kolmogrov Differential Equation solving subroutines of the program **SOLUTION.**

The necessary inputs are TOL,NSTATE,λ , κ , T1, T2, DT, NF and NR, which have been explained for the previous programs, as well as the following :

PI(NSTATE)	a vector of the probability distribution for the initial components at time $t_{inspect}$. These are read from a file in order to save computer time.
PI(NSTATE)	a vector of the probability distribution for the replacement components at time $t_{inspect}$. The are also read from a file in order to save computer time.

The listing of the program is :

```
      PROGRAM OPINSPEC
C
C     COMPUTES THE RELIABILITY AT A GIVEN FUTURE TIME GIVEN
C     THAT THERE IS ONE INSPECTION / CORRECTION DURING THE
C     INTERVAL TINITIAL TO TFINAL.  THE INITIAL DISTRIBUTION
C     OF PROBABILITIES(at several times) IS READ FROM THE
C     FILE INSPEC.IN ( to save computer time it has been saved
C     in a file) AND THE REPLACEMENT SIZE DISTRIBUTION IS
C     READ FROM THE FILE REPLACE.IN.  THE FINAL DISTRIBUTIONS
C     ARE WRITTEN TO THE FILE FINPROB.OUT AND THE RELIABILITY
C     IS WRITTEN TO THE FILE OPINSPEC.REL
C
      DIMENSION PI(70),PJ(70),PI2(70)
      DOUBLE PRECISION PI,PJ,PI2,R1,R2,R3,FAIL
      DOUBLE PRECISION TINIT,DT,T1,T2,REMOVED
      DOUBLE PRECISION ALAMBD,AKAPP,TOL
      COMMON/VAR/JJJ,III,ALAMBD,AKAPP,TOL
      OPEN(1,"INSPEC.IN")
      OPEN(2,"REPLACE.IN")
      OPEN(3,"OPINSP2.REL")
      OPEN(4,"FINPROB.OUT")
      PRINT*," ENTER TOLERANCE FOR SOLUTION :  "
```

```
      READ*, TOL
      PRINT*,"ENTER NUMBER OF STATES :  "
      READ*, NSTATE
C
   1  PRINT*,"   ENTER LAMBDA (if you want to quit type 0)  "
      READ*, ALAMBD
      IF (ALAMBD.LE.0.) GOTO 999
      PRINT*,"   ENTER KAPPA  "
      READ*, AKAPP
      PRINT*,"   ENTER INITIAL AND FINAL TIMES AND INCREMENT
"
      READ*,T1,T2,DT
      PRINT*," "
      PRINT*,"   ENTER FAILURE STATE, NF =  "
      READ*,NF
      IF(NF.LE.0) GOTO 999
      PRINT*,"   ENTER THE REPLACEMENT STATE, NR = "
      READ*,NR
      IF(NR.LE.0) GOTO 999
      TINIT = T1
      WRITE(3,99)NR,NF
   3  TINIT = TINIT + DT
      IF(TINIT.GE.T2) GOTO 1
      DO 4 I = 1,NSTATE
   4  READ(2,200)PI2(I)
      DO 10 I = 1,NSTATE
  10  READ(1,200)PI(I)
      R1 = 0.D0
      DO 15 I = NF,NSTATE
  15  R1 = R1 + PI(I)
C
C      CALL THE DETECTION AND REMOVAL SUBROUTINE
C
      CALL DETECT(NSTATE,NR,PI,REMOVED)
C
C      'AGE' THE INITIAL POPULATION, AFTER REMOVAL OF 'FAILED'
C      COMPONENTS.
C
      PRINT*,"INSPEC. TIME =",TINIT,"FINAL TIME =",T2
      PRINT*,"  PROCESS THE REMAINING INITIAL COMPONENTS"
      CALL EVOLVE(TINIT,T2,PI,PJ,NSTATE)
      R2 = 0.D0
      DO 25 I = NF,NSTATE
  25  R2 = R2 + PJ(I)
C
C      USE REPLACEMENT POPULATION
```

```
C         - BUT ONLY IF THERE HAVE BEEN COMPONENTS REPLACED
C         IF REMOVED = 0 THEN THERE HAS BEEN NO REPLACEMENT SO
C         LET R3 = 0.0 AND GO TO TOTAL RELIABILITY CALCULATION
C         ALLOWING ALL PI2 = 0.0
C
          IF(REMOVED.GT.0.0) GOTO 30
          R3 = 0.D0
          DO 28 I = 1,NSTATE
   28     PI2(I) = 0.D0
          GOTO 36
C
    30    R3 = 0.D0
          DO 35 I = NF,NSTATE
    35    R3 = R3 + PI2(I)
C
C         WRITE THE FINAL PROBABILITIES TO THE OUTPUT FILE.
C
    36    WRITE(4,240)TINIT
          DO 40 I = 1,NSTATE
    40    WRITE(4,250)I,PJ(I),PI2(I)
C
C         CALL THE SUBROUTINE 'TOTFAIL' WHICH WILL COMPUTE THE
C         OVERALL FAILURE PROBABILITY AT THE FINAL TIME
C
          CALL TOTFAIL(R1,R2,R3,REMOVED,FAIL)
C
          WRITE(3,150)TINIT,FAIL
          GOTO 3
  999     STOP
   99     FORMAT("REPAIR STATE = ",I4,"  FAILURE STATE = ",I4)
  110     FORMAT(A40)
  150     FORMAT("INSPEC. TIME  ",F6.4," PROB(fail) = ",E16.9)
  200     FORMAT(E16.9)
  240     FORMAT(" FOR INSPEC. AT T = ",F6.4,"  FINAL "
     &"PROBABILITIES ARE: ")
  250     FORMAT("STATE ",I3," INIT.PARTS ",E16.9," REPL. ",E16.9)
          END

          SUBROUTINE DETECT(NSTATE,NR,PI,REMOVED)
          DIMENSION PI(70)
          DOUBLE PRECISION REMOVED,PDETECT,PI
          REMOVED = 0.D0
          DO 20 I = NR,NSTATE
          PDETECT = .99D0*(1.0D0-DEXP(-.2298D0*FLOAT(I-10)))
          REMOVED = REMOVED + PDETECT * PI(I)
          PI(I) = PI(I) * (1.0D0 - PDETECT)
```

```
20      CONTINUE
        DO 30 I = 1,NSTATE
30      PI(I) = PI(I)/(1.0D0 - REMOVED)
        RETURN
        END

        SUBROUTINE TOTFAIL(R1,R2,R3,REMOVED,FAIL)
        DOUBLE PRECISION R1,R2,R3,REMOVED,FAIL
        FAIL =  R1 + R2 * (1.D0-REMOVED) + R3 * REMOVED
        RETURN
        END
```