THE APPLICATION OF MELLIN TRANSFORMS TO STATISTICS

by

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In the present essay the theory of this application is developed and many examples are discussed. The theory, originally developed for positive random variables only, is made to include any real variable. This is done by replacing the product of Mellin transforms by the product of certain 2×2 matrices whose elements are Mellin transforms.

Some functional equations between frequency functions and between Mellin transforms are solved, sometimes using arguments drawn from Statistics.

The problem of interpolating between the moments to find the Mellin transform is solved, in the case where all the moments exist.

CONTENTS

Introduction

The use of Mellin transforms in the field of Statistics as given in this essay, parallels somewhat the use of Fourier transforms in this field. In the latter case the addition of random variables is studied in terms of the corresponding characteristic functions which are the Fourier transforms of the frequency functions. The characteristic function of a sum of independent random variables is the product of the characteristic functions of the random variables making up the sum.

The product of independent random variables is studied in a similar way through products of Mellin transforms. Although the frequency function for the product had often been calculated (e.g., in Student's "t" distribution) the use of Mellin transforms was first introduced by Epstein[1].

In the present essay the theory of this application is developed and many examples are discussed. The theory, originally developed for positive random variables only, is made to include any real variable. This is done by replacing the product of Mellin transforms by the product of certain 2×2 matrices whose elements are Mellin transforms.

Some functional equations between frequency functions and between Mellin transforms are solved, sometimes using

(1) Epstein, 1948. See Bibliography on p.35

arguments drawn from Statistics. The random variable ξ^{μ} is discussed, and the closure of a certain set of random variables is shown to hold under the operations ξ_1^{μ}, $\xi_1\xi_2$ where ξ_1 and ξ_2 are in the set. The examples given in § 22 show the results of multiplying random variables from well known distributions.

The Mellin transform of a frequency function is a moment of arbitrary complex order. The problem of interpolating between the moments to find the Mellin transform is solved, in the case where all the moments exist, in example vi of § 22.

Chapter 1

Positive Random Variables

In order to discuss the relation between the theory of Mellin transforms and Mathematical Statistics, it will be necessary to give a brief discussion of these subjects. The proofs of the properties of the Mellin transform are to be found in Titchmarsh's work (1). In that text the properties are derived from those of the Fourier transform, for the Mellin and Fourier integrals differ essentially by a change of variable and a different line of integration. The discussion of Probability and Statistics is more self-contained since proofs may be indicated without difficulty.

1. The Mellin transform and its properties

All the integrals that will be used are Lebesgue integrals, and a knowledge of the classes L(L^1), L^2, etc., is assumed. Consider a function of the real variable x, f(x) which is such that $f(x)x^{k-1} \in L(0,\infty)$, for some real k. Then the Mellin transform $\mathcal{F}(s)$ of f(x) is defined as follows

$$\mathcal{F}(s) = \int_0^\infty f(x)\, x^{s-1}\, dx \qquad R(s), \text{ the real part of } s, = k$$

It is shown by Titchmarsh § 1.29 that the following inversion formula holds

$$f(x) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \mathcal{F}(s)\, x^{-s}\, ds$$

An example is provided by $f(x)=e^{-x}$, $\mathcal{F}(s)=\Gamma(s)$. In this case $k>0$. For purposes of calculation parameters may

(1) Titchmarsh, 1948

be introduced into known pairs of Mellin transforms in order to extend their range. The only three formulae which seem to be needed in practice are, $\mathcal{F}(s)$ being the Mellin transform of f(x),

i) $f(x^\alpha)$ $\qquad \frac{1}{|\alpha|}\mathcal{F}(\frac{s}{\alpha})$

ii) $f(\frac{x}{a})$ $\qquad a^s\mathcal{F}(s)$

iii) $x^\lambda f(x)$ $\qquad \mathcal{F}(s+\lambda)$

The operations of inserting these parameters are not commutative and must be done in the order given. The result of the three is $\qquad x^\lambda f[(\frac{x}{a})^\alpha] \qquad \frac{a^{\lambda+s}}{|\alpha|}\mathcal{F}(\frac{\lambda+s}{\alpha})$.

The numbers α, a, λ are not specified and the only requirement is that $x^\lambda f[(\frac{x}{a})^\alpha]x^{k-1}\in L(0,\infty)$ for some real k.

The integral of the product of two functions may be expressed as an integral of the product of their Mellin transforms. The result may be deduced from the corresponding theorem for Fourier transforms and is as follows:

2. Convolution Theorem If $x^k g(x)$ and $x^k f(x)\in L(0,\infty)$ and if $h(x)=\int_0^\infty f(y)g(\frac{x}{y})\frac{dy}{y}$ then $x^k h(x)\in L(0,\infty)$ and the Mellin transform of h(x), $\mathcal{H}(s)$ is equal to $\mathcal{G}(s)\mathcal{F}(s)$.

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\mathcal{F}(s)\mathcal{G}(s)x^{-s}ds=\int_0^\infty f(y)g(\frac{x}{y})\frac{dy}{y}$$

Using formula i) above this may be written

$$\frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\mathcal{F}(s)\mathcal{G}(2-s)x^{-s}ds=\int_0^\infty y f(y)\, g(xy)dy$$

The proof is given by Titchmarsh in § 2.7.

3. Random Variables and Distribution Functions

The notion of probability must be introduced in terms of a finite set. If ξ is a variable which may equal any member of the set, and x is a given element, then the probability that ξ takes the value x is defined to be the number of times x appears in the set, divided by the total number of elements in the set. The variable ξ is called a random variable. The sets we shall deal with are sets of real numbers. If the set is countable, but infinite, or uncountable, another measure of the number of elements in the set must be introduced and in the case of the real numbers this is always taken to be the Lebesgue measure of the set.

The distribution function F(x) is the probability that $-\infty < \xi < x$ where ξ is a random variable defined over the set. Since the measure of a point is zero some other means must be used to indicate the incidence of a single element. This is similar to the problem of a continuous distribution of mass in a solid, and in this case also the notion of density is introduced. The probability that ξ lies between x, $x+\delta x$, viz., $F(x+\delta x)-F(x)$, is divided by δx. The limit as $\delta x \to 0$ is called the frequency function of the set and is denoted by f(x). Thus $f(x)=F'(x)$. The probability that ξ lies between a and b is $\int_a^b f(x)\,dx$.

4. <u>The Fundamental Laws</u>

The following are the fundamental laws of probability:

i) If P(A), P(B) are the probabilities of elements A, B respectively, then P(A or B) = P(A) + P(B), if A and B are mutually exclusive.

ii) P(A and B) = P(A)P(B) if P(A) does not depend on B, and P(B) does not depend on A.

Clearly, the probability of an unspecified element is 1. In the case of the real numbers

$$P(\text{ any element }) = \int_{-\infty}^{\infty} f(x)\,dx = 1$$

Since $f(x) \geqslant 0$, $F(x)$ is monotone increasing.

5. Bivariate Distributions

It is possible to consider morethan one random variable, e.g., the pair ξ, η. The distribution function $F(x,y)$ is defined as the probability that $-\infty < \xi < x$ and $-\infty < \eta < y$. The frequency function is defined as $f(x,y)$ $f(x,y) = \frac{\partial^2 F(x,y)}{\partial x \partial y}$. Extending ii) above to this joint frequency function, we find that $f(x,y) = f_1(x) f_2(y)$, where $f_1(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$ $f_2(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$, in the case where ξ and η are independent. The probability that ξ and $\eta \in E$, a set in the space of all pairs (x,y), is $\int_E f(x,y)\,dx\,dy$.

6. Frequency functions of $\xi\eta$ and ξ/η.

We will now find the frequency function of the random variable $\xi\eta$. It is necessary to find the distribution function. Let E be the set of (x,y) such that $xy \leqslant z$. Then

$$P(\xi\eta < z) = \int_E f(x,y)\,dx\,dy = \int_0^{\infty} dx \int_0^{z/y} f(x,y)\,dy + \int_{-\infty}^{0} dy \int_{z/y}^{\infty} f(x,y)\,dx$$

The frequency function of $\zeta = \xi\eta$ is the derivative of this with respect to z.

$$\int_0^{\infty} dy\, \frac{1}{y} f\left(\frac{z}{y}, y\right) + \int_{-\infty}^{0} dy \left(-\frac{1}{y}\right) f\left(\frac{z}{y}, y\right)$$

$$= \int_{-\infty}^{\infty} \frac{1}{|y|} f\left(\frac{z}{y}, y\right) dy$$

In a similar way the frequency function of ξ/η may be found,

$$\int_{-\infty}^{\infty} |y|\, f(zy, y)\,dy$$

It is the similarity of these formulae to the ones given above on page 2 that leads to the applications with which this essay deals.

7. Distribution of ξ^n

When the frequency function ~~of~~ f(x) of ξ is known, it may be desirable to have the frequency function of ξ^n, where n is an integer. There are four cases, according to whether n is an even or odd, positive or negative integer. Let $f_n(x)$ be the frequency function of ξ^n. Recall that

$$P(\xi < x) = F(x) = \int_{-\infty}^{x} f(t)\,dt$$

i) n positive, even $\quad P(\xi^n < 0) = 0$

$$x \geqslant 0 \quad P(\xi^n < x) = P(\xi < x^{\frac{1}{n}}) = F(x^{\frac{1}{n}})$$

$$f_n(x) = 0 \qquad x < 0$$

$$f_n(x) = \frac{1}{n} x^{\frac{1}{n}-1} f(x^{\frac{1}{n}}) \qquad x \geqslant 0$$

ii) n positive, odd $\quad P(\xi^n < x) = P(\xi < x^{\frac{1}{n}}), \; -\infty < x < \infty$

$$f_n(x) = \frac{1}{n} x^{\frac{1}{n}-1} f(x^{\frac{1}{n}}) \qquad -\infty < x < \infty$$

iii) n negative, even $\quad P(\xi^n < 0) = 0$

$$x \geqslant 0 \quad P(\xi^n < x) = P(\xi > x^{\frac{1}{n}}) = 1 - P(\xi < x^{\frac{1}{n}}) = 1 - F(x^{\frac{1}{n}})$$

$$f_n(x) = 0 \qquad x < 0$$

$$f_n(x) = -\frac{1}{n} x^{\frac{1}{n}-1} f(x^{\frac{1}{n}}) \qquad x \geqslant 0$$

n positive or negative, even $\quad \begin{cases} f_n(x) = 0 & x < 0 \\ f_n(x) = \frac{1}{|n|} x^{\frac{1}{n}-1} f(x^{\frac{1}{n}}) & x \geqslant 0 \end{cases}$

iv) n negative, odd $\quad P(\xi^n < x) = P(\xi > x^{\frac{1}{n}}) \quad -\infty < x < \infty$

$$= 1 - P(\xi < x^{\frac{1}{n}}) = 1 - F(x^{\frac{1}{n}})$$

$$f_n(x) = -\frac{1}{n} x^{\frac{1}{n}-1} f(x^{\frac{1}{n}})$$

n positive or negative, odd $\quad f_n(x) = \frac{1}{|n|} x^{\frac{1}{n}-1} f(x^{\frac{1}{n}}) \quad -\infty < x < \infty$

8. Mean Values and Moments

The mean value of a function g(x), defined over the set in consideration, is $\sum_{x \in E} P(x)\, g(x)$ for countable sets

or $\int_{-\infty}^{\infty} g(x) f(x)\, dx$ for the set of all real numbers.

These are sometimes called the expedtation of g(x) and are written E(g(x)). The n^{th} moment of a set is defined as $E(x^n)$.

We will calculate the moments of the product and ratio of two independant random variables ξ and η. Let these have frequency functions $f_1(x)$ and $f_2(x)$, and moments μ_n and μ'_n, respectively. Let π_n and ρ_n be the n^{th} moments of $\xi\eta$ and ξ/η, respectively. In order to calculate π_n and ρ_n we may use the frequency functions found on page 4, or we may calculate the n^{th} moment of xy, x/y using the joint distribution $f_1(x)f_2(y)$.

$$\pi_n = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (xy)^n f_1(x)\, f_2(y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x^n f_1(x)\, dx \cdot \int_{-\infty}^{\infty} y^n f_2(y)\, dy = \mu_n \mu'_n$$

Thus $\pi_n = \mu_n \mu'_n$. In a similar it may be shown that $\rho_n = \mu_n \mu'_{-n}$.
In the next section we shall require that these last equations be true for a complex variable s instead of the integer n. It is only necessary to restrict the real part of s so that the integrals converge.

9. The Use of Mellin Transforms

For the rest of chapter 1 we shall consider positive random variables only. The Mellin transform $\mathcal{F}(s)$ of the frequency function f(x) of the positive random variable ξ is

$$\mathcal{F}(s) = \int_0^\infty f(x)\, x^{s-1}\, dx = \mu_{s-1}$$

The results of the last section may be written in terms of Mellin transforms. Let $f_1(x)$, $f_2(x)$ be the frequency functions of ξ, η ; let $h_1(x)$, $h_2(x)$ be the frequency functions of $\xi\eta$, ξ/η . Since ξ, η are positive, so are $\xi\eta$, ξ/η . Thus $h_1(x)=0$, $h_2(x)=0$ for $x<0$. The Mellin transforms of these functions are $\mathcal{F}_1(s)$, $\mathcal{F}_2(s)$, $\mathcal{H}_1(s)$, $\mathcal{H}_2(s)$.

$$\mathcal{H}_1(s) = \pi_{s-1} = \mu_{s-1}\, \mu'_{s-1} = \mathcal{F}_1(s)\, \mathcal{F}_2(s)$$

$$\mathcal{H}_2(s) = \rho_{s-1} = \mu_{s-1}\, \mu'_{1-s} = \mathcal{F}_1(s)\, \mathcal{F}_2(2-s)$$

$$h_1(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \mathcal{F}_1(s)\, \mathcal{F}_2(s)\, x^{-s}\, ds = \int_0^\infty f_1(t)\, f_2\left(\frac{x}{t}\right)\frac{dt}{t}$$

$$h_2(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \mathcal{F}_1(s)\, \mathcal{F}_2(2-s)\, x^{-s}\, ds = \int_0^\infty t\, f_1(t)\, f_2(xt)\, dt$$

Here we have used the convolution theorem given on page 2. These are special cases of the formulae found using elementary considerations on page 4. The advantage of using Mellin transforms is that the products and ratios of any number of random variables may be handled with equal ease. The elementary method requires the use of multiple integrals. If the positive independent random variables $\xi_1, \xi_2 \dots, \eta_1, \eta_2 \dots$ lead to the Mellin transforms $\mathcal{F}_1(s), \mathcal{F}_2(s) \dots \mathcal{G}_1(s), \mathcal{G}_2(s) \dots$, then the Mellin transforms for the variables $\xi_1\xi_2\cdots\eta_1\eta_2\cdots$, $\dfrac{\xi_1\xi_2\cdots}{\eta_1\eta_2\cdots}$ are $\mathcal{F}_1(s)\,\mathcal{F}_2(s)\cdots\mathcal{G}_1(s)\,\mathcal{G}_2(s)\cdots$ and $\mathcal{F}_1(s)\,\mathcal{F}_2(s)\cdots\mathcal{G}_1(2-s)\,\mathcal{G}_2(2-s)\cdots$.

10. An example in the use of Mellin transforms will now be given, using the χ^2 distribution. Let x_i have the frequency function $\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$; then the positive random variable $\chi^2 = \Sigma_1^{\nu} x_i^2$ has the frequency function[1] $f(x) = \dfrac{x^{\frac{\nu}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2})}$
The integer ν is called the number of degrees of freedom. We will now find the frequency functions of the product and ratio of the independent positive variables χ_1^2, χ_2^2 ; each one has a χ^2 distribution where the number of degrees of freedom is ν and λ , respectively. Thus $f_1(x)$, $f_2(x)$ are given by

$$f_1(x) = \frac{x^{\frac{\nu}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2})} \qquad f_2(x) = \frac{x^{\frac{\lambda}{2}-1} e^{-\frac{x}{2}}}{2^{\frac{\lambda}{2}} \Gamma(\frac{\lambda}{2})} \qquad \begin{matrix} f_1(x)=0 \\ f_2(x)=0 \end{matrix} \ x<0$$

We may calculate $\mathfrak{F}_1(s)$ directly:

$$\mathfrak{F}_1(s) = \int_0^{\infty} f_1(x) x^{s-1} dx = \frac{2^{\frac{\nu}{2}+s-1}}{2^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2})} \int_0^{\infty} \left(\frac{x}{2}\right)^{\frac{\nu}{2}+s-2} e^{-\frac{x}{2}} d\left(\frac{x}{2}\right) = \frac{2^{s-1} \Gamma(\frac{\nu}{2}+s-1)}{\Gamma(\frac{\nu}{2})}$$

Similarly, $\mathfrak{F}_2(s) = \dfrac{2^{s-1} \Gamma(\frac{\lambda}{2}+s-1)}{\Gamma(\frac{\lambda}{2})}$

According to our theory the Mellin transform of the frequency function of $\chi_1^2 \chi_2^2$ is

$$\mathfrak{F}_1(s) \mathfrak{F}_2(s) = \frac{2^{2s-2} \Gamma(\frac{\nu}{2}+s-1) \Gamma(\frac{\lambda}{2}+s-1)}{\Gamma(\frac{\nu}{2}) \Gamma(\frac{\lambda}{2})}$$

In order to invert this, i.e., to find the frequency function itself, we will make use of a formula given by Titchmarsh in eq. 7.9.12. Throughout this essay we shall have occasion to use the large number of pairs of Mellin transforms calculated by Titchmarsh. Eq. 7.9.12 is

$$x^{\nu} K_{\nu}(x) \ , \quad 2^{s+\nu-2} \Gamma(\tfrac{1}{2}s) \Gamma(\tfrac{1}{2}s+\nu), \ R(s) > \max(0, -2\nu)$$

where $K_{\nu}(x) = \frac{\pi i}{2} e^{\frac{1}{2}\nu\pi i} H_{\nu}^{(1)}(ix) = \frac{\pi i}{2} e^{\frac{1}{2}\nu\pi i} [J_{\nu}(ix) + iY_{\nu}(ix)]$

$J_{\nu}(x)$, $Y_{\nu}(x)$ are the Bessel functions of the first and second kind, respectively, of order ν.

(1) Weatherburn §74

In order to obtain the desired pair of functions, we make use of the formulae given on page 2.

Change s to 2s: $x^{\frac{\nu}{2}} K_\nu(x^{\frac{1}{2}})$, $2^{2s+\nu-1}\Gamma(s)\,\Gamma(s+\nu)$

Change s to $s+\frac{\lambda}{2}-1$: $x^{\frac{\nu+\lambda}{2}-1} K_\nu(x^{\frac{1}{2}})$, $2^{2s+\lambda+\nu-3}\,\Gamma(s+\frac{\lambda}{2}-1)\,\Gamma(s+\nu+\frac{\lambda}{2}-1)$

Change ν to $\frac{\nu-\lambda}{2}$: $x^{\frac{\nu+\lambda}{4}-1} K_{\frac{\nu-\lambda}{2}}(x^{\frac{1}{2}})$, $2^{2s+\frac{\lambda+\nu}{2}-3}\,\Gamma(s+\frac{\lambda}{2}-1)\,\Gamma(s+\frac{\nu}{2}-1)$

Multiply by a constant: $\frac{2^{1-\frac{\lambda+\nu}{2}}\,x^{\frac{\nu+\lambda}{4}-1}}{\Gamma(\frac{\nu}{2})\,\Gamma(\frac{\lambda}{2})} K_{\frac{\nu-\lambda}{2}}(x^{\frac{1}{2}})$, $\frac{2^{2s-2}\,\Gamma(s+\frac{\lambda}{2}-1)\,\Gamma(s+\frac{\nu}{2}-1)}{\Gamma(\frac{\nu}{2})\,\Gamma(\frac{\lambda}{2})}$

Thus the frequency function of $\chi_1^2\chi_2^2$ is

$$\frac{2^{1-\frac{\lambda+\nu}{2}}\,x^{\frac{\nu+\lambda}{4}-1}}{\Gamma(\frac{\nu}{2})\,\Gamma(\frac{\lambda}{2})} K_{\frac{\nu-\lambda}{2}}(x^{\frac{1}{2}})$$

This takes a simple form when $\nu\sim\lambda$ is an odd integer, for then $K_{\frac{\nu-\lambda}{2}}$ is a finite sum. In particular, when $\nu\sim\lambda=1$ we have

$$\frac{x^{\frac{\lambda}{2}-1}\,e^{-x^{\frac{1}{2}}}}{2\,\Gamma(\lambda)}$$

The frequency function of the ratio χ_1^2/χ_2^2 is the inverse of the Mellin transform

$$\mathfrak{F}_1(s)\,\mathfrak{F}_2(2-s)=\frac{\Gamma(\frac{\nu}{2}+s-1)\,\Gamma(\frac{\lambda}{2}+1-s)}{\Gamma(\frac{\nu}{2})\,\Gamma(\frac{\lambda}{2})}$$

We start with eq.7.7.9 in Titchmarsh: $\frac{1}{(1+x)^a}$, $\frac{\Gamma(s)\,\Gamma(a-s)}{\Gamma(a)}$

We find as in the above case the frequency function

$$\frac{\Gamma(\frac{\lambda+\nu}{2})\;x^{\frac{\nu}{2}-1}}{\Gamma(\frac{\nu}{2})\,\Gamma(\frac{\lambda}{2})\,(1+x)^{\frac{\lambda+\nu}{2}}}$$

This is occasionally used in practice and is known as the F distribution[1].

1) See Bennet and Franklin, 1954

Weatherburn §92

11. The Distribution of ξ^n

If f(x) is the frequency function of ξ and is such that f(x) = 0, $x < 0$ (ξ is a positive random variable), then the frequency function of ξ^n, as found on page 5, is

$f_n(x) = \frac{1}{|n|} x^{\frac{1}{n}-1} f(x^{\frac{1}{n}})$ for all integral values of n.

If $\mathcal{F}(s)$ is the Mellin transform of f(x), then the Mellin transform of $f_n(x)$ is $\mathcal{F}(ns-n+1)$. In particular the frequency function of $1/\xi$ is $\frac{1}{x^2} f(\frac{1}{x})$, whose Mellin transform is $\mathcal{F}(2-s)$.

We will now consider the class of positive random variables each of which has the same frequency function as its reciprocal. If $1/\xi$ has the same frequency function f(x) as ξ , then f(x) satisfies the functional equation

$$f(x) = \frac{1}{x^2} f\left(\frac{1}{x}\right) \tag{1}$$

This functional equation is equivalent to the one involving the Mellin transforms of these functions, viz.,

$$\mathcal{F}(s) = \mathcal{F}(2-s) \tag{2}$$

for if $\mathcal{F}(s)$ converges on the line $s = 1+i\tau$, then so does $\mathcal{F}(2-s)$. Every frequency function has a Mellin transform convergent on this line since

$$\int_0^\infty |f(x)x^{s-1}|dx = \int_0^\infty |f(x)||x^{s-1}|dx = \int_0^\infty f(x)|x^{i\tau}|dx = \int_0^\infty f(x)dx = 1$$

Therefore $f(x)x^{s-1} \in L(0,\infty)$. If f(x) is of bounded variation in the neighbourhood of x the inversion formula holds[1]

(1) Titchmarsh § 1.29

$$f(x) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \mathfrak{F}(s)\, x^{-s}\, ds$$

We may solve eqn.(1) in two ways, both of which give all the solutions.

i) Let $g(x)$ be any function such that

$$\begin{aligned} &g(x) \geqslant 0 \qquad 0 \leq x \leq 1 \\ &g(x) = 0 \qquad x < 0,\ x > 1 \\ &\int_0^1 g(x)\,dx = \tfrac{1}{2} \end{aligned} \tag{3}$$

Then a solution of (1) is

$$\begin{aligned} f(x) &= g(x) && -\infty \leq x \leq 1 \\ &= \frac{1}{x^2}\, g\left(\frac{1}{x}\right) && x > 1 \end{aligned} \tag{4}$$

for,

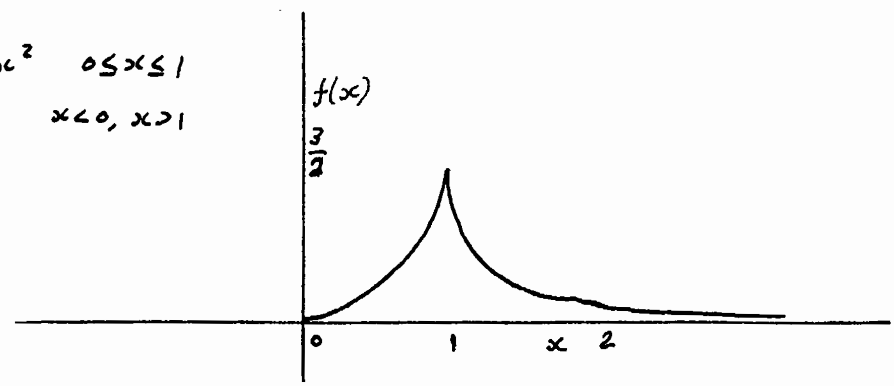
$$x < 1 \qquad \frac{1}{x^2} f\left(\frac{1}{x}\right) = \frac{1}{x^2}\, x^2 g(x) = g(x) = f(x)$$

$$x > 1 \qquad \frac{1}{x^2} f\left(\frac{1}{x}\right) = \frac{1}{x^2}\, g\left(\frac{1}{x}\right) = f(x)$$

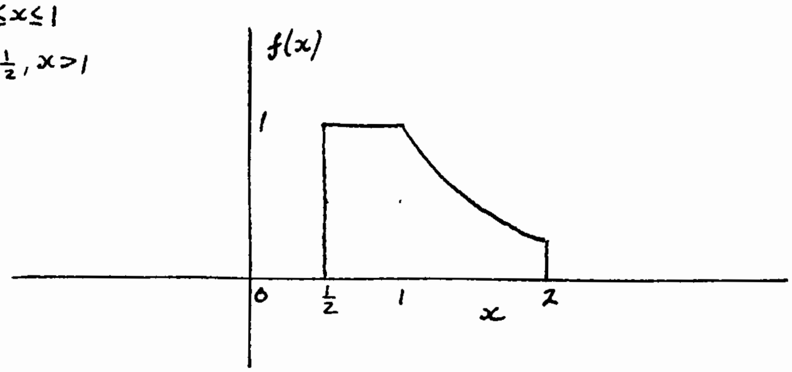
Clearly $f(x) \geqslant 0$, $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 g(x)\,dx + \int_1^{\infty} \frac{1}{x^2} g\left(\frac{1}{x}\right) dx = \frac{1}{2} + \int_0^1 g(x)\,dx = 1$

Some examples will now be given.

$$\begin{aligned} g(x) &= \tfrac{3}{2} x^2 && 0 \leq x \leq 1 \\ &= 0 && x < 0,\ x > 1 \end{aligned}$$



$$\begin{aligned} g(x) &= 1 && \tfrac{1}{2} \leq x \leq 1 \\ &= 0 && x < \tfrac{1}{2},\ x > 1 \end{aligned}$$



Any solution of eqn.(1) which is the frequency function of a positive random variable may be given in the form (4), where the function g(x) has the properties (3). Suppose f(x) is such a solution. Then define

$$h(x) = f(x) \qquad -\infty < x \leq 1$$
$$= 0 \qquad x > 1$$

Then $\quad x > 1 \qquad f(x) = \frac{1}{x^2} f(\frac{1}{x}) = \frac{1}{x^2} h(\frac{1}{x})$

We see that $h(x)$ has the properties (3):

$$h(x) \geqslant 0 \qquad 0 \leq x \leq 1$$
$$h(x) = 0 \qquad x < 0,\ x > 1$$

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^1 h(x)dx + \int_1^{\infty} \frac{1}{x^2} h(\tfrac{1}{x})dx = 2\int_0^1 h(x)dx$$

Thus we have the following result:

Theorem 1 In order that the equation (1) have a solution which is a frequency function of a positive random variable it is necessary and sufficient that there be a function g(x) such that

$$g(x) \geqslant 0 \qquad 0 \leq x \leq 1$$
$$g(x) = 0 \qquad x < 0,\ x > 1$$
$$\int_0^1 g(x)dx = \frac{1}{2}$$

where the solution is given by $\quad f(x) = g(x) \qquad -\infty \leq x \leq 1$

$$= \frac{1}{x^2} g(\tfrac{1}{x}) \qquad x > 1$$

The Mellin transform of f(x) may be expressed in terms of that of g(x). Write $\quad \mathcal{G}(s) = \int_0^{\infty} g(x)x^{s-1}dx = \int_0^1 g(x)x^{s-1}dx$

Then $\quad \mathcal{F}(s) = \int_0^{\infty} f(x)x^{s-1}dx = \int_0^1 f(x)x^{s-1}dx + \int_1^{\infty} f(x)x^{s-1}dx$

$$= \int_0^1 g(x)x^{s-1}dx + \int_1^{\infty} \frac{1}{x^2} g(\tfrac{1}{x})x^{s-1}dx$$

$$= \mathcal{G}(s) + \int_0^1 g(x)x^{2-s-1}dx = \mathcal{G}(s) + \mathcal{G}(2-s) \quad (5)$$

We have shewn that the existence of $\mathcal{G}(s)$ implies that of $\mathcal{G}(2-s)$. Clearly $\mathcal{F}(s)$ satisfies eqn.(2)

$$\mathcal{F}(2-s) = \mathcal{G}(2-s) + \mathcal{G}(s) = \mathcal{F}(s)$$

The second solution of eqn.(1) will now be given.

ii) Let $h(x)$ be any function such that

$$\left.\begin{array}{r} h(x) \geqslant 0 \\ h(x) = h(-x) \end{array}\right\} -\infty < x < \infty \qquad (6)$$
$$\int_{-\infty}^{\infty} h(x)\, dx = 1$$

A solution to (1) is given by
$$\left.\begin{array}{ll} f(x) = \frac{1}{x} h(\ln x) & x > 0 \\ f(0) = f(0+0) \qquad f(x) = 0 & x < 0 \end{array}\right\} \quad (7)$$

for, $\frac{1}{x^2} f\left(\frac{1}{x}\right) = \frac{1}{x^2}\, x\, h\left(\ln \frac{1}{x}\right) = \frac{1}{x} h(-\ln x) = \frac{1}{x} h(\ln x) = f(x)$

Clearly, $f(x) \geqslant 0$

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{0}^{\infty} f(x)\, dx = \int_{0}^{\infty} \frac{1}{x} h(\ln x)\, dx = \int_{-\infty}^{\infty} h(y)\, dy = 1$$

Any solution may be put in the form (7), where $h(x)$ satisfies (6). Suppose f(x) is a solution. Then define

$$k(x) = e^{x} f(e^{x})$$

We see that $k(x)$ has the properties (6):

$$k(x) \geq 0 \qquad -\infty < x < \infty$$

$$k(-x) = e^{-x} f(e^{-x}) = \frac{1}{e^{x}} f\left(\frac{1}{e^{x}}\right) = e^{x} f(e^{x}) = k(x)$$

$$\int_{-\infty}^{\infty} k(x)\, dx = \int_{-\infty}^{\infty} e^{x} f(e^{x})\, dx = \int_{0}^{\infty} f(y)\, dy = 1$$

<u>Theorem 2</u> In order that the equation (1) have a solution which is the frequency function of a positive random variable it is necessary and sufficient that there be a function h(x) such that

$$\left.\begin{array}{r} h(x) \geqslant 0 \\ h(x) = h(-x) \end{array}\right\} -\infty < x < \infty$$
$$\int_{-\infty}^{\infty} h(x)\, dx = 1$$

where the solution is given by (7).

The Mellin transform $\mathcal{F}(s)$ of f(x) may be written in terms of h(x).

$$\mathcal{F}(s) = \int_0^\infty f(x) x^{s-1} dx = \int_0^\infty h(\ln x) x^{s-2} dx = \int_{-\infty}^\infty h(y) e^{y(s-1)} dy$$
$$= \int_0^\infty h(y) e^{y(s-1)} dy + \int_0^\infty h(y) e^{y(1-s)} dy$$

The convergence of these integrals is required on the line $s = 1 + i\tau$. On account of (6) $h(x) \in L(0,\infty)$, so that convergence is assured. Therefore

$$\mathcal{F}(s) = H(1-s) + H(s-1) \tag{8}$$

where $H(s)$ is the Laplace transform of h(y).

Clearly $\mathcal{F}(s)$ satisfies (2),

$$\mathcal{F}(s) = H(1-s) + H(s-1) = \mathcal{F}(2-s)$$

We may obtain all the solutions to the equation (2). If $g(s)$ is an even function of s, then $g(1-s)$ is a solution. However, we cannot infer from this the solution of (1). The difficulty is that we do not know what properties of $\mathcal{F}(s)$ are important if f(x) is to be non-negative. The problem is nowhere broached in the literature and the present author cannot make any progress. To determine the class of Mellin transforms whose inverses are frequency functions is the most important problem in the theory of the application of Mellin transforms to Statistics.

12. The Multiplication of Random Variables

We have so far considered two types of multiplication between random variables. These have bean denoted by $\xi_1 \xi_2 \cdots \xi_n$ and ξ^n. In one case the multiplication is between independant variables, and in the other the variables are dependent, in fact, identical. The difference may be illustrated in the case of the Gaussian variable. We shall anticipate the method of Chapter 2 and give the results. Let ξ_1 and ξ_2 have the frequency function

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \qquad -\infty < x < \infty$$

Then $\xi_1 \xi_2$, the product of independent variables, has frequency function

$$g(x) = \frac{1}{\pi} K_0(x) \qquad -\infty < x < \infty$$

where[1] $K_0(x) = \frac{\pi i}{2} H_0^{(1)}(ix) = \frac{\pi i}{2} \lim_{\nu \to 0} \frac{J_{-\nu}(ix) - e^{-\nu\pi i} J_\nu(ix)}{i \sin \nu\pi}$

ξ_1^2, the product of dependent variables, has the frequency function

$$h(x) = \frac{1}{(2\pi x)^{\frac{1}{2}}} e^{-\frac{x}{2}} \qquad x > 0$$
$$= 0 \qquad x < 0$$

Let us consider the set of positive random variables each of which has the same frequency function as its reciprocal. Denote it by R. We shall show that R is closed under both types of multiplication.

<u>Theorem 3</u> If $\xi_1 \in R$, $\xi_2 \in R$ then $\xi_1 \xi_2 \in R$ and $\xi_1^n, \xi_2^n \in R$, where n is a positive or negative integer.

Proof: Let ξ_1, ξ_2 lead to the Mellin transforms $\mathfrak{F}_1(s)$, $\mathfrak{F}_2(s)$. Then

$$\begin{aligned} \mathfrak{F}_1(s) &= \mathfrak{F}_1(2-s) \\ \mathfrak{F}_2(s) &= \mathfrak{F}_2(2-s) \end{aligned} \qquad (9)$$

$\xi_1 \xi_2$ leads to the Mellin transform $\mathfrak{G}(s) = \mathfrak{F}_1(s)\mathfrak{F}_2(s)$ and from (9) we see that

$$\mathfrak{G}(s) = \mathfrak{F}_1(s)\mathfrak{F}_2(s) = \mathfrak{F}_1(2-s)\mathfrak{F}_2(2-s) = \mathfrak{G}(2-s)$$

Therefore $\xi_1 \xi_2 \in R$.

We have in § 11. found that ξ_1^n leads to the Mellin transform $\mathfrak{G}(s) = \mathfrak{F}_1(ns-n+1)$. Now, from (9),

(1) Watson 1922, § 3.61, § 3.7

$$g(s) = \mathcal{F}_1(ms-m+1) = \mathcal{F}_1(2-(ms-m+1)) = \mathcal{F}_1(m-ms+1)$$
$$= \mathcal{F}_1(m(2-s)-m+1) = g(2-s)$$

Therefore $\xi_1^m \in R$.

13. <u>The Delta Function as a Frequency Function</u>

Consider the problem of inverting $\mathcal{F}(s)=1$. The integral

$\int_{c-i\infty}^{c+i\infty} x^{-s} ds$ does not converge.

However $\int_0^x x^{-s} ds = \frac{x^{1-s}}{1-s} = \frac{x^{1-c-i\tau}}{1-c-i\tau}$ and this belongs to

$L^2(-\infty,\infty)$, assuming that $c<1$. Let $G(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{1-s}}{1-s} ds$, $x>0$

$= 0$, $x \leq 0$

It is clear that $G(x)$ converges conditionally when $|x| \neq 1$. When $0<x<1$, we may close the contour of integration on the left and exclude the pole at $s=1$; then $G(x)=0$. When $1<x$, we may close the contour on the right, including the pole; then $G(x)=1$. If we could differentiate under the integral sign $G'(x)$ would be the desired inverse. However, we shall interpret the inverse of $\mathcal{F}(s)=1$ as the formal derivative of $G(x)$. Since $G(x)$ is a step function the derivative at $x=1$ does not exist.

$$f(x) = \frac{d}{dx}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{1-s}}{1-s} ds = \delta(x-1) \quad x>0$$
$$= 0 \quad x \leq 0$$

The derivative of a step function is known as a delta function. It has the following properties: $\delta(x-1) = 0 \quad x \neq 1$

$= \infty \quad x = 1$

$\int_{-\infty}^{\infty} \delta(x-1) = 1$

In terms of Statistics $\delta(x-1)$ is the frequency function of a set all of whose members equal one; that is, a unit point mass distribution at $x = 1$.

In general the inverse of $\mathcal{F}(s) = a^{s-1}$ is taken to be

$$f(x) = \frac{d}{dx}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\left(\frac{x}{a}\right)^{1-s}}{1-s}\,ds = \delta(x-a). \tag{11}$$

$\delta(x-a)$ is a unit point mass distribution at $x = a$. The derivative of the integral exists at all points except $x = a$, where $f(x)$ is considered to be a formal derivative.

14. <u>Some Functional Equations Leading to Frequency Functions</u>

The following functional equations are between the Mellin transforms of frequency functions.

i) $$\mathcal{F}(s) = [\mathcal{F}(s)]^2 \tag{12}$$

There are two obvious solutions: $\mathcal{F}(s) = 0 \qquad f(x) = 0$

$\mathcal{F}(s) = 1 \qquad f(x) = \delta(x-1)$

Only the latter is a frequency function.

ii) $$\mathcal{F}(s)\,\mathcal{G}(s) = 1 \tag{13}$$

This states that the product of dispersed distributions is a point distribution at $x = 1$. This is impossible. Thus the only solution is that $f(x)$, $g(x)$ are point distributions.

$$\mathcal{F}(s) = a^{s-1} \qquad f(x) = \delta(x-a)$$
$$\mathcal{G}(s) = a^{1-s} \qquad g(x) = \delta\left(x-\tfrac{1}{a}\right),\ a>0$$

iii) If ξ_i all have the same frequency function $f(x)$, can $\xi_1\xi_2\cdots\xi_m$ ever have the same frequency function as ξ_1^m? This leads to the equation

$$[\mathcal{F}(s)]^m = \mathcal{F}(ms-m+1) \tag{14}$$

We shall find the coefficients in the Taylor series for $\mathcal{F}(s)$

at s = 1. Let $\mathcal{F}(s) = \sum_{n=0}^{\infty} \frac{a_n}{n!}(s-1)^n \quad (a_n = \mathcal{F}^{(n)}(1))$.
Differentiate (14) indefinitely and in each case put s = 1, using the fact that $\mathcal{F}(1) = \int_0^{\infty} f(x)\,dx = 1$

$$n[\mathcal{F}(s)]^{n-1}\,\mathcal{F}'(s) = n\mathcal{F}'(ns-n+1)$$

$$\therefore \quad \mathcal{F}'(1) = \mathcal{F}'(1) = a_1 \quad , \text{ arbitrary}$$

$$(n-1)[\mathcal{F}(s)]^{n-2}[\mathcal{F}'(s)]^2 + [\mathcal{F}(s)]^{n-1}\mathcal{F}''(s) = n\mathcal{F}''(ns-n+1)$$

$$\therefore \quad (n-1)\,a_1^2 + a_2 = n a_2$$

$$\therefore \quad a_2 = a_1^2$$

$$\text{in general} \quad a_i = a_1^i$$

$$\mathcal{F}(s) = \sum_0^{\infty} \frac{a_1^n (s-1)^n}{n!} = e^{a_1(s-1)} = a^{s-1}, \quad a = e^{a_1}$$

Thus f(x) = $\delta(x-a)$. Thus the answer is that the two products have different frequency functions unless all have point distributions.

Chapter 2

Extension of the Theory and Examples

15. The Use of Mellin Transforms With Frequency Functions Defined Over the Entire Real Axis

Let the random variables ξ_1, ξ_2 have the frequency functions $f_1(x)$, $f_2(x)$ respectively. Let $h(x)$ be the frequency function of $\xi_1 \xi_2$. Then, by §6

$$h(x) = \int_{-\infty}^{\infty} \frac{1}{|u|} f_1(u) f_2\left(\frac{x}{u}\right) du$$

$$= \int_0^{\infty} \frac{1}{u} f_1(u) f_2\left(\frac{x}{u}\right) du + \int_0^{\infty} \frac{1}{u} f_1(-u) f_2\left(-\frac{x}{u}\right) du \qquad (15)$$

Let

$$\begin{aligned} f_1(x) &= f_{11}(x) + f_{12}(x) & \text{where} \quad & f_{11}(x) = 0 \quad x < 0 \\ f_2(x) &= f_{21}(x) + f_{22}(x) & & f_{12}(x) = 0 \quad x > 0 \\ h(x) &= h_1(x) + h_2(x) & & f_{21}(x) = 0 \quad x < 0 \\ & & & f_{22}(x) = 0 \quad x > 0 \\ & & & h_1(x) = 0 \quad x < 0 \\ & & & h_2(x) = 0 \quad x > 0 \end{aligned} \qquad (16)$$

Let $\mathfrak{F}_{11}(s)$ be the Mellin transform of $f_{11}(x)$

$$\begin{array}{lcl} \mathfrak{F}_{12}(s) & " & f_{12}(-x) \\ \mathfrak{F}_{21}(s) & " & f_{21}(x) \\ \mathfrak{F}_{22}(s) & " & f_{22}(-x) \\ \mathfrak{H}_1(s) & " & h_1(x) \\ \mathfrak{H}_2(s) & " & h_2(-x) \end{array} \qquad (17)$$

Then it follows from (15) and §2 that

$$\mathcal{H}_1(s) = \mathcal{F}_{11}(s)\,\mathcal{F}_{21}(s) + \mathcal{F}_{12}(s)\,\mathcal{F}_{22}(s)$$
$$\mathcal{H}_2(s) = \mathcal{F}_{11}(s)\,\mathcal{F}_{22}(s) + \mathcal{F}_{12}(s)\,\mathcal{F}_{21}(s) \qquad (18)$$

$$h(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{H}_1(s)\, x^{-s}\, ds \qquad x > 0$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathcal{H}_2(s)\, (-x)^{-s}\, ds \qquad x < 0 \qquad (19)$$

The frequency function of ξ_1/ξ_2 is given by (19) if we change (18) to

$$\mathcal{H}_1(s) = \mathcal{F}_{11}(s)\,\mathcal{F}_{21}(2-s) + \mathcal{F}_{12}(s)\,\mathcal{F}_{22}(2-s)$$
$$\mathcal{H}_2(s) = \mathcal{F}_{11}(s)\,\mathcal{F}_{22}(2-s) + \mathcal{F}_{12}(s)\,\mathcal{F}_{21}(2-s) \qquad (20)$$

16. <u>The Use of Matrices</u>

In order to obtain the results for the product of three random variables $\xi_1\,\xi_2\,\xi_3$ the previous results will be written in a more tractable form. Let the following matrices be defined:

$$\bar{\mathcal{H}}(s) = \begin{pmatrix} \mathcal{H}_1(s) & \mathcal{H}_2(s) \\ \mathcal{H}_2(s) & \mathcal{H}_1(s) \end{pmatrix}, \quad \bar{\mathcal{F}}_1(s) = \begin{pmatrix} \mathcal{F}_{11}(s) & \mathcal{F}_{12}(s) \\ \mathcal{F}_{12}(s) & \mathcal{F}_{11}(s) \end{pmatrix}, \quad \bar{\mathcal{F}}_2(s) = \begin{pmatrix} \mathcal{F}_{21}(s) & \mathcal{F}_{22}(s) \\ \mathcal{F}_{22}(s) & \mathcal{F}_{21}(s) \end{pmatrix} \qquad (21)$$

Then it is clear from (18) that

$$\bar{\mathcal{H}}(s) = \bar{\mathcal{F}}_1(s)\,\bar{\mathcal{F}}_2(s)$$

where the multiplication is ordinary matrix multiplication. For the random variable ξ_1/ξ_2 we make the calculation

$$\bar{\mathcal{H}}(s) = \bar{\mathcal{F}}_1(s)\,\bar{\mathcal{F}}_2(2-s)$$

The general rule is now clear. In order to find the frequency function of the random variable $\eta = (\xi_1\,\xi_2\cdots)/(\zeta_1\,\zeta_2\cdots)$ where ξ_i has the frequency function $f_i(x)$ and ζ_i has the frequency function $g_i(x)$, we calculate the matrix product

$$\bar{\mathcal{H}}(s) = \bar{\mathcal{F}}_1(s)\,\bar{\mathcal{F}}_2(s)\cdots\,\bar{\mathcal{G}}_1(2-s)\,\bar{\mathcal{G}}_2(2-s)\cdots \qquad (22)$$

and the frequency function of η is $h(x)$ given by (19).

These 2×2 matrices are commutative and may be multiplied in any order. The proof of (22) is carried out by the repeated use of (18) and (20).

17. <u>The Matrix for the Delta Function Distrbution</u>

Let ξ have the frequency function $f(x) = \delta(x-a)$. In order to calculate $\bar{\mathfrak{F}}(s)$ we must consider two cases, $a > 0$ and $a < 0$. Let $f(x) = f_1(x) + f_2(x)$ as in (16).

i) $a \geq 0$
$$f(x) = \delta(x-a)$$
$$f_1(x) = \delta(x-a)$$
$$f_2(x) = 0$$

According to §13 $\quad \mathfrak{F}_1(s) = a^{s-1}, \quad \mathfrak{F}_2(s) = 0$

Therefore
$$\bar{\mathfrak{F}}(s) = \begin{pmatrix} a^{s-1} & 0 \\ 0 & a^{s-1} \end{pmatrix} \tag{23}$$

In particular, if $a = 1$, $\bar{\mathfrak{F}}(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the identity matrix.

ii) $a < 0$
$$f(x) = \delta(x-a)$$
$$f_1(x) = 0$$
$$f_2(x) = \delta(x-a)$$

According to §13 $\quad \mathfrak{F}_1(s) = 0, \quad \mathfrak{F}_2(s) = (-a)^{s-1}$

Therefore
$$\bar{\mathfrak{F}}(s) = \begin{pmatrix} 0 & (-a)^{s-1} \\ (-a)^{s-1} & 0 \end{pmatrix} \tag{24}$$

Consider a variable η with frequency function $g(x)$. The matrix for $\xi\eta$ is $(a > 0)$
$$\begin{pmatrix} a^{s-1} & 0 \\ 0 & a^{s-1} \end{pmatrix} \begin{pmatrix} g_1(s) & g_2(s) \\ g_2(s) & g_1(s) \end{pmatrix} = \begin{pmatrix} a^{s-1}g_1(s) & a^{s-1}g_2(s) \\ a^{s-1}g_2(s) & a^{s-1}g_1(s) \end{pmatrix}.$$

By the rule in §1 concerning parameters in Mellin transforms we see that the frequency function of $\xi\eta$ is $\frac{1}{a} g(\frac{x}{a})$. If $a < 0$ we may show that the frequency function is $-\frac{1}{a} g(\frac{x}{a})$.

18. The Matrix for the Distribution of ξ^n

If the frequency function f(x), of ξ, is not zero for $x<0$, as we are assuming in this chapter, the consideration of ξ^n leads to two cases, depending on whether n be odd or even. Let $f(x) = f_1(x) + f_2(x)$

where $f_1(x) = 0 \quad x<0 \quad , \quad f_2(x) = 0 \quad x>0$

For ξ^n, let $f_n(x) = f_{n1}(x) + f_{n2}(x)$

where $f_{n1}(x) = 0 \quad x<0 \quad , \quad f_{n2}(x) = 0 \quad x>0$

Let the matrix of ξ be $\bar{\mathfrak{F}}(s) = \begin{pmatrix} \mathfrak{F}_1(s) & \mathfrak{F}_2(s) \\ \mathfrak{F}_2(s) & \mathfrak{F}_1(s) \end{pmatrix}$

Let the matrix of ξ^n be $\bar{\mathfrak{F}}_n(s) = \begin{pmatrix} \mathfrak{F}_{n1}(s) & \mathfrak{F}_{n2}(s) \\ \mathfrak{F}_{n2}(s) & \mathfrak{F}_{n1}(s) \end{pmatrix}$

i) n even
$$f_{n1}(x) = \frac{1}{|n|} x^{\frac{1}{n}-1} f(x^{\frac{1}{n}}) \qquad (\S 7)$$
$$f_{n2}(x) = 0$$
$$\mathfrak{F}_{n1}(s) = \mathfrak{F}_1(ns-n+1) \quad , \quad \mathfrak{F}_{n2}(s) = 0$$
$$\bar{\mathfrak{F}}_n(s) = \begin{pmatrix} \mathfrak{F}_1(ns-n+1) & 0 \\ 0 & \mathfrak{F}_1(ns-n+1) \end{pmatrix} \tag{25}$$

ii) n odd
$$f_{n1}(x) = \frac{1}{|n|} x^{\frac{1}{n}-1} f(x^{\frac{1}{n}}) \qquad (\S 7)$$
$$f_{n2}(x) = \frac{1}{|n|} x^{\frac{1}{n}-1} f(x^{\frac{1}{n}})$$
$$\mathfrak{F}_{n1}(s) = \mathfrak{F}_1(ns-n+1) \quad , \quad \mathfrak{F}_{n2}(s) = \mathfrak{F}_2(ns-n+1)$$
$$\bar{\mathfrak{F}}_n(s) = \begin{pmatrix} \mathfrak{F}_1(ns-n+1) & \mathfrak{F}_2(ns-n+1) \\ \mathfrak{F}_2(ns-n+1) & \mathfrak{F}_1(ns-n+1) \end{pmatrix} \tag{26}$$

In particular the matrix for $1/\xi$ is
$$\bar{\mathfrak{F}}_{-1}(s) = \begin{pmatrix} \mathfrak{F}_1(2-s) & \mathfrak{F}_2(2-s) \\ \mathfrak{F}_2(2-s) & \mathfrak{F}_1(2-s) \end{pmatrix}$$

The problem we considered earlier, on page 10, may be generalized now to include random variables in general. Thus ξ has the same frequency function as $1/\xi$ if and only if $\bar{\mathfrak{F}}(s) = \bar{\mathfrak{F}}_{-1}(s)$. This leads to the two equations

$$\mathfrak{F}_1(s) = \mathfrak{F}_1(2-s)$$
$$\mathfrak{F}_2(s) = \mathfrak{F}_2(2-s)$$

These may be solved by the methods given in § 11.

Let the set of random variables each of which has the same frequency function as its reciprocal be denoted by R_1. Then the closure of R_1 may be shewn to hold for both types of multiplication (§ 12) in the case of positive and negative random variables.

<u>Theorem 4</u> Let $\xi_1 \in R_1$, $\xi_2 \in R_1$ then $\xi_1\xi_2 \in R$ and ξ_1^n, $\xi_2^n \in R_1$, if n is a positive or negative integer.

Proof: Let $\bar{\mathfrak{F}}_1(s)$, $\bar{\mathfrak{F}}_2(s)$ be the matrices for ξ_1, ξ_2.

Then $\bar{\mathfrak{G}}(s) = \bar{\mathfrak{F}}_1(s)\bar{\mathfrak{F}}_2(s)$ is the matrix for $\xi_1\xi_2$.

$$\bar{\mathfrak{G}}(s) = \bar{\mathfrak{F}}_1(s)\,\bar{\mathfrak{F}}_2(s) = \bar{\mathfrak{F}}_1(2-s)\,\bar{\mathfrak{F}}_2(2-s) = \bar{\mathfrak{G}}(2-s)$$

Therefore $\xi_1\xi_2 \in R_1$.

Let $\bar{\mathfrak{H}}(s) = \begin{pmatrix} \mathfrak{F}_{11}(ns-n+1) & \mathfrak{F}_{12}(ns-n+1) \\ \mathfrak{F}_{12}(ns-n+1) & \mathfrak{F}_{11}(ns-n+1) \end{pmatrix}$ be the matrix for ξ_1^n

n being odd.

$$\bar{\mathfrak{H}}(s) = \begin{pmatrix} \mathfrak{F}_{11}(ns-n+1) & \mathfrak{F}_{12}(ns-n+1) \\ \mathfrak{F}_{12}(ns-n+1) & \mathfrak{F}_{11}(ns-n+1) \end{pmatrix}$$
$$= \begin{pmatrix} \mathfrak{F}_{11}[2-(ns-n+1)] & \mathfrak{F}_{12}[2-(ns-n+1)] \\ \mathfrak{F}_{12}[2-(ns-n+1)] & \mathfrak{F}_{11}[2-(ns-n+1)] \end{pmatrix}$$
$$= \begin{pmatrix} \mathfrak{F}_{11}(n-ns+1) & \mathfrak{F}_{12}(n-ns+1) \\ \mathfrak{F}_{12}(n-ns+1) & \mathfrak{F}_{11}(n-ns+1) \end{pmatrix} = \bar{\mathfrak{H}}(2-s)$$

Therefore $\xi_1^n \in R_1$, when n is odd. If n is even then the $\mathfrak{F}_{12}$ elements are put equal to zero and the proof follows in the same way.

19. The Moments of the Distribution Over the Whole Axis

The moments of the frequency function f(x), $-\infty < x < \infty$ may be written in terms of the Mellin transforms. Let

$f(x) = f_1(x) + f_2(x)$ as usual;

$\mathfrak{F}_1(s)$ is the Mellin transform of $f_1(x)$

$\mathfrak{F}_2(s)$ " " $f_2(-x)$

Then μ_n, the n^{th} moment of f(x) is

$$\mu_n = \int_{-\infty}^{\infty} x^n f(x)\,dx = \int_0^{\infty} x^n f(x)\,dx + \int_{-\infty}^{0} x^n f(x)\,dx$$

$$= \mathfrak{F}_1(n+1) + (-1)^n \int_0^{\infty} x^n f(-x)\,dx$$

$$\mu_n = \mathfrak{F}_1(n+1) + (-1)^n \mathfrak{F}_2(n+1) \qquad (27)$$

20. Diagonalization of the Matrices

The matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ satisfies the equation $A^2 = 1$. Thus the normal form is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. It is found that the matrix giving the implicit change of basis vectors is $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ which has the inverse $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$. Thus

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = a\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$+ b\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix}$$

Thus if we multiply the following equation on the left by $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$ and on the right by $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, two scalar equations may be obtained.

$$\bar{\mathfrak{H}}(s) = \bar{\mathfrak{F}}_1(s)\,\bar{\mathfrak{F}}_2(s) \cdots \quad \mathfrak{G}_1(2-s)\,\mathfrak{G}_2(2-s) \cdots$$

The form after the diagonalization is

$$\begin{pmatrix} \mathcal{H}_1(s)+\mathcal{H}_2(s) & 0 \\ 0 & \mathcal{H}_1(s)-\mathcal{H}_2(s) \end{pmatrix} = \begin{pmatrix} \mathcal{F}_{11}(s)+\mathcal{F}_{12}(s) & 0 \\ 0 & \mathcal{F}_{11}(s)-\mathcal{F}_{12}(s) \end{pmatrix} \times \cdots \begin{pmatrix} \mathcal{G}_{11}(2-s)+\mathcal{G}_{12}(2-s) & 0 \\ 0 & \mathcal{G}_{11}(2-s)-\mathcal{G}_{12}(2-s) \end{pmatrix}$$

which gives the two scalar equations

$$\begin{aligned} \mathcal{H}_1(s)+\mathcal{H}_2(s) &= \{\mathcal{F}_{11}(s)+\mathcal{F}_{12}(s)\} \times \cdots \{\mathcal{G}_{11}(2-s)+\mathcal{G}_{12}(2-s)\} \times \cdots \\ \mathcal{H}_1(s)-\mathcal{H}_2(s) &= \{\mathcal{F}_{11}(s)-\mathcal{F}_{12}(s)\} \times \cdots \{\mathcal{G}_{11}(2-s)-\mathcal{G}_{12}(2-s)\} \times \cdots \end{aligned} \qquad (28)$$

These may be solved for $\mathcal{H}_1(s)$ and $\mathcal{H}_2(s)$. At present no meaning is known for $\mathcal{H}_1(s) \pm \mathcal{H}_2(s)$.

Since the Mellin transforms considered are always non-negative, no interpretation can be given the matrix $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Since $\mu_n = \mathcal{F}_1(n+1) + (-1)^n \mathcal{F}_2(n+1)$ we may regard (28) as the general form of the result that the moment of the product of independent random variables is the product of the moments (§ 8)

21. <u>Symmetrical Distributions</u>

A symmetrical distribution is one for which f(x) is even, $f(x) = f(-x)$. In the usual manner we have $f(x) = f_1(x) + f_2(x)$, where , since f(x) is even, $f_1(x) = f_2(-x)$. Thus $\mathcal{F}_1(s) = \mathcal{F}_2(s)$. The matrix of the distribution is

$\bar{\mathcal{F}}(s) = \begin{pmatrix} \mathcal{F}_1(s) & \mathcal{F}_2(s) \\ \mathcal{F}_2(s) & \mathcal{F}_1(s) \end{pmatrix}$ which has a determinant equal to zero.

$$|\bar{\mathcal{F}}(s)| = (\mathcal{F}_1(s))^2 - (\mathcal{F}_2(s))^2 = 0$$

This is clearly a sufficient condition that $f(x) = f(-x)$.

<u>Theorem 5</u> The necessary and sufficient condition that a distribution be symmetric is that the matrix have zero determinant.

22. Examples

Some examples of the use of the various formulae will now be given.

i) The gaussian distribution with mean zero.

$$f_1(x) = \frac{1}{\sigma_1\sqrt{2\pi}}\, e^{-\frac{x^2}{2\sigma_1^2}} \qquad f_2(x) = \frac{1}{\sigma_2\sqrt{2\pi}}\, e^{-\frac{x^2}{2\sigma_2^2}}$$

$$\mathfrak{F}_{11}(s) = \frac{1}{\sigma_1\sqrt{2\pi}}\int_0^\infty e^{-\frac{x^2}{2\sigma_1^2}} x^{s-1}\,dx = \pi^{-\frac{1}{2}} 2^{\frac{s-3}{2}} \sigma_1^{s-1}\Gamma\left(\frac{s}{2}\right) \qquad (29)$$

$$\mathfrak{F}_{12}(s) = \mathfrak{F}_{11}(s)$$

$$\mathfrak{F}_{21}(s) = \mathfrak{F}_{22}(s) = \pi^{-\frac{1}{2}} 2^{\frac{s-3}{2}} \sigma_2^{s-1}\Gamma\left(\frac{s}{2}\right)$$

In order to find the frequency function of $\xi_1\xi_2$ we will calculate $\mathfrak{H}_1(s), \mathfrak{H}_2(s)$ using (18)

$$\mathfrak{H}_1(s) = \mathfrak{H}_2(s) = \pi^{-1} 2^{s-2} (\sigma_1\sigma_2)^{s-1}\left[\Gamma\left(\frac{s}{2}\right)\right]^2$$

Using 7.9.11. in Titchmarsh: $K_0(x)$, $2^{s-2}\left[\Gamma\left(\frac{s}{2}\right)\right]^2$

we have $h(y) = \dfrac{1}{\pi\sigma_1\sigma_2} K_0\left(\dfrac{y}{\sigma_1\sigma_2}\right), \; -\infty < y < \infty$ (30)

In order to find the frequency function of $\eta = \xi_1/\xi_2$ we use (20)

$$\mathfrak{H}_1(s) = \mathfrak{H}_2(s) = (2\pi)^{-1}\left(\frac{\sigma_1}{\sigma_2}\right)^{s-1}\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)$$

$$= \left(\frac{\sigma_1}{\sigma_2}\right)^{s-1}\frac{1}{2\sin\frac{\pi}{2}s}$$

Using 7.7.8. in Titchmarsh: $\dfrac{1}{1+x}$, $\dfrac{\pi}{\sin\pi s}$

we have

$$h(u) = \frac{1}{\pi}\frac{\sigma_2}{\sigma_1}\frac{1}{1+\left(\frac{\sigma_2 u}{\sigma_1}\right)^2} \qquad (31)$$

which is the Cauchy distrbution. If $\sigma_1 = \sigma_2$, $h(u)$ satisfies the equations $h_1(u) = \frac{1}{u^2}h_1\left(\frac{1}{u}\right)$ and $h_2(u) = \frac{1}{u^2}h_2\left(\frac{1}{u}\right)$ so that the distribution of η is the same as that of $1/\eta$ when $\sigma_1 = \sigma_2$

It is important to note that none of the moments of the Cauchy distribution exist. Consider the ratio of two Gaussian variables having any mean, and standard deviation. We shall prove the following theorem.

Theorem 6 None of the moments of the ratio of two Gaussian variables exist.

Proof: Let ξ_1 have the frequency function $f_1(x) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{(x-\mu_1')^2}{2\sigma_1^2}}$

Let ξ_2 " $f_2(x) = \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{(x-\mu_1'')^2}{2\sigma_2^2}}$

Let $f_1(x) = f_{11}(x) + f_{12}(x)$, $f_2(x) = f_{21}(x) + f_{22}(x)$ in the usual way. Let h(x) be the frequency function of ξ_1/ξ_2 The n^{th} moment ρ_n of h(x) is from (27), (20)

$$\begin{aligned} \rho_n &= \mathfrak{H}_1(n+1) + (-1)^n \mathfrak{H}_2(n+1) \\ &= \mathfrak{F}_{11}(n+1)\,\mathfrak{F}_{21}(1-n) + \mathfrak{F}_{12}(n+1)\,\mathfrak{F}_{22}(1-n) \\ &\quad + (-1)^n \mathfrak{F}_{11}(n+1)\,\mathfrak{F}_{22}(1-n) + (-1)^n \mathfrak{F}_{12}(n+1)\,\mathfrak{F}_{21}(1-n) \end{aligned}$$

Consider $\mathfrak{F}_{21}(1-n) = \int_0^\infty \frac{1}{\sigma_2\sqrt{2\pi}}\, x^{-n} e^{-\frac{(x-\mu_1'')^2}{2\sigma_2^2}}\, dx$

This does not converge at $x = 0$ for any integral $n > 0$. In fact none of the Mellin transforms of argument 1-n converge. Thus ρ_n does not exist for any $n > 0$.

When there is a measure of dependence between the Gaussian variables the ratio still has no moments, as was shewn by E. C. Fieller[1], who actually found the frequency function of the ratio- a very complicated expression. The difficulty lies in the assumption of a non-zero probability of a Gaussian variable no matter how large the mean. This is physically untrue, and Fieller considers the so-called

(1) Fieller, Biometrika, 24

curtailed distribution which has a frequency function equal to zero beyond a certain interval containing the mean. As this interval is increased the moments of the ratio tend to practical values, but if the interval is infinite so are the moments.

Example ii) The Resolution of the Gaussian variable into the product of two independent variables may be carried out using the duplication theorem for the Gamma function. The distributions are assumed to be symmetric. Let h(x) be resolved into $f_1(x)$ and $f_2(x)$, with the same notation as already used for the Mellin transforms.

$$\mathcal{H}_1(s) = 2\,\mathcal{F}_{11}(s)\,\mathcal{F}_{21}(s) \qquad \mathcal{F}_{11}(s) = \mathcal{F}_{12}(s)$$

$$\mathcal{H}_2(s) = 2\,\mathcal{F}_{11}(s)\,\mathcal{F}_{22}(s) \qquad \mathcal{F}_{21}(s) = \mathcal{F}_{22}(s)$$

The product distribution is Gaussian so that

$$\mathcal{H}_1(s) = \pi^{-\frac{1}{2}}\, 2^{\frac{s-3}{2}}\, \sigma^{s-1}\, \Gamma\left(\tfrac{s}{2}\right)$$

$$= A\, \sigma^{\frac{s-1}{2}}\, 2^{\frac{s}{2}}\, \Gamma\left(\tfrac{s}{4}\right) \times B\, \sigma^{\frac{s-1}{2}}\, 2^{\frac{s}{2}}\, \Gamma\left(\tfrac{s+2}{4}\right).$$

$\mathcal{H}(1) = \mathcal{H}_1(1) + \mathcal{H}_2(1) = 1$ Thus for a symmetric distribution

$$\mathcal{H}_1(1) = \mathcal{H}_2(1) = \tfrac{1}{2}, \quad AB = \pi^{-1}\, 2^{-s/2}$$

$$\mathcal{F}_{11}(s) = A\, (2\sigma)^{\frac{s-1}{2}}\, \Gamma\left(\tfrac{s}{4}\right)$$

$$\mathcal{F}_{21}(s) = B\, (2\sigma)^{\frac{s-1}{2}}\, \Gamma\left(\tfrac{s+2}{4}\right)$$

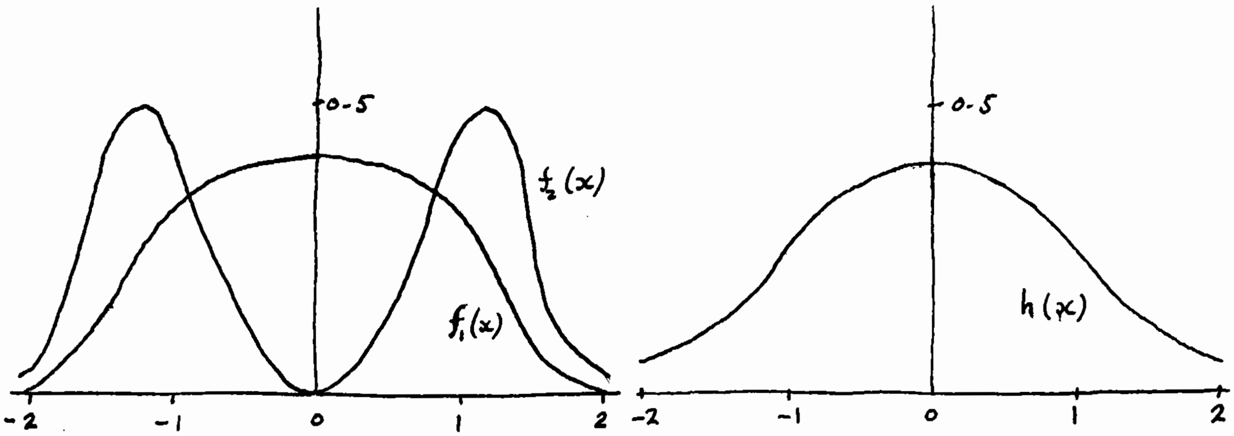
But $\mathcal{F}_{11}(1) = \mathcal{F}_{21}(1) = \frac{1}{2}$ thus $A = \dfrac{1}{2\,\Gamma(\frac{1}{4})}$, $B = \dfrac{1}{2\,\Gamma(\frac{3}{4})}$

$$f_1(x) = \frac{2}{\Gamma(\frac{1}{4})\,(2\sigma)^{1/2}}\, e^{-\frac{x^4}{4\sigma^2}}$$

$$f_2(x) = \frac{2x^2}{\Gamma(\frac{3}{4})\,(2\sigma)^{3/2}}\, e^{-\frac{x^4}{4\sigma^2}}$$

It is seen that a distribution with two peaks may give rise to a distribution with only one peak.

The functions h(x), $f_1(x)$, $f_2(x)$ are sketched below for the case $\sigma = 1$.



Example iii) The Cauchy distribution is

$$f(x) = \frac{1}{\pi}\,\frac{1}{1+x^2} \qquad -\infty < x < \infty$$

A variable ξ having this distribution has the same distribution as $1/\xi$. Let g(x) be frequency function of the product $\xi_1\xi_2$ of independent Cauchy variables. g(x) is also the frequency function of ξ_1/ξ_2. Using the result of § 6 we have

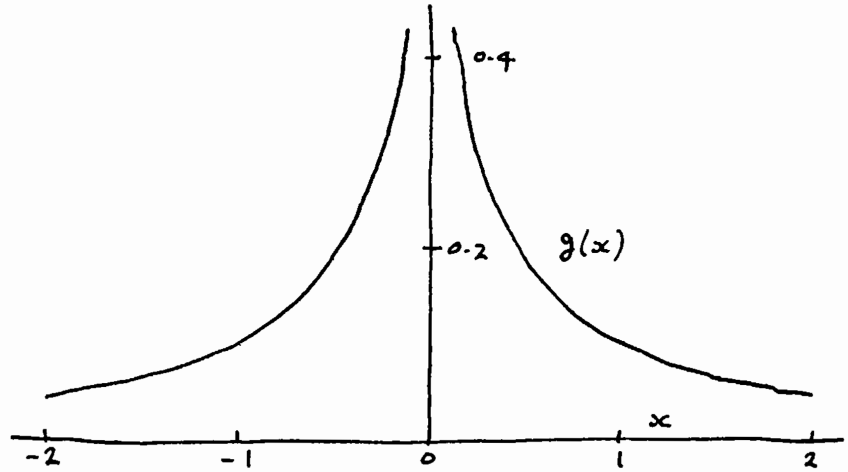
$$g(y) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{1}{|x|}\,\frac{1}{1+\left(\frac{y}{x}\right)^2}\,\frac{1}{1+x^2}\,dx$$

$$= \frac{1}{\pi^2}\,\frac{\ln y^2}{y^2-1} \qquad -\infty < y < \infty$$

None of the moments of this distribution exist. The Mellin transform matrix is

$$\tilde{g}(s) = \begin{pmatrix} \dfrac{1}{2\sin\frac{\pi}{2}s} & \dfrac{1}{2\sin\frac{\pi}{2}s} \\ \dfrac{1}{2\sin\frac{\pi}{2}s} & \dfrac{1}{2\sin\frac{\pi}{2}s} \end{pmatrix}^2 = \begin{pmatrix} \dfrac{1}{2\sin^2\frac{\pi}{2}s} & \dfrac{1}{2\sin^2\frac{\pi}{2}s} \\ \dfrac{1}{2\sin^2\frac{\pi}{2}s} & \dfrac{1}{2\sin^2\frac{\pi}{2}s} \end{pmatrix}.$$

The function g(x) is drawn below.



It is easier to evaluate the integral for g(x) directly than to use the Mellin inversion formula. We have proved the inversion of $\mathcal{G}_1(s)$ to be

$$\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\frac{y^{-s}}{2\sin^2\frac{\pi}{2}s}\,ds = \frac{1}{\pi^2}\quad\frac{\ln y^2}{y^2-1}$$

<u>Example</u> iv) <u>Student's Distribution</u>

Consider the variable $t = \frac{u\sqrt{\nu}}{v}$, where u is distributed normally with mean zero and standard deviation one, and v^2 has a χ^2 distribution with ν degrees of freedom. The random variable t has what is known as a Student's distribution. We shall find the frequency function of t using the method of Mellin transforms. The frequency function of $\sqrt{\nu}u$ is

$$f_1(x) = \frac{1}{\sqrt{2\pi\nu}}\, e^{-\frac{x^2}{2\nu}} \quad -\infty < x < \infty$$

The corresponding matrix elements are (see (29))

$$\mathcal{F}_{11}(s) = \mathcal{F}_{12}(s) = \pi^{-\frac{1}{2}}\,\nu^{\frac{s-1}{2}}\,2^{\frac{s-3}{2}}\,\Gamma(\tfrac{s}{2})$$

The frequency function of v^2 is $f(x) = \dfrac{x^{\frac{\nu}{2}-1}\,e^{-\frac{x}{2}}}{2^{\nu/2}\,\Gamma(\frac{\nu}{2})} \quad x > 0$

$$= 0 \qquad x \leq 0$$

The frequency function of v is, according to § 7, $f_2(x) = 2x\,f(x^2)$

The Mellin transform of f(x), x ⩾ 0, is $\dfrac{2^{s-1}\,\Gamma(\frac{\nu}{2}+s-1)}{\Gamma(\frac{\nu}{2})}$

Therefore the matrix elements of v are

$$\mathfrak{F}_{21}(s) = \frac{2^{\frac{s-1}{2}}\,\Gamma(\frac{\nu}{2}+\frac{s-1}{2})}{\Gamma(\frac{\nu}{2})} \qquad \mathfrak{F}_{22}(s) = 0$$

The matrix elements of t are, according to (20)

$$\mathfrak{H}_1(s) = \mathfrak{H}_2(s) = \frac{\pi^{-\frac{1}{2}}\,\nu^{\frac{s-1}{2}}\,2^{-1}\,\Gamma(\frac{s}{2})\,\Gamma(\frac{\nu}{2}+\frac{1-s}{2})}{\Gamma(\frac{\nu}{2})}$$

The inverse Mellin transform may be found using the pair 7.7.9 in Titchmarsh: $\dfrac{1}{(1+x)^a}$, $\dfrac{\Gamma(s)\,\Gamma(a-s)}{\Gamma(a)}$, whence the frequency function of t is

$$h(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})}\sqrt{\frac{\nu}{\pi}}\,\frac{1}{(1+\frac{x^2}{\nu})^{\frac{1+\nu}{2}}} \qquad -\infty < x < \infty \tag{32}$$

This exemplifies the fact that the ratio or product of a symmetrical and a positive variable is symmetrically distributed.

<u>Example v</u>) There are two distributions which are not so well known, but which are related in an interesting way with the χ^2 distribution.

<u>Beta Distribution</u>[1]

$$f(x) = \frac{\Gamma(\frac{n+m}{2})}{\Gamma(\frac{n}{2})\,\Gamma(\frac{m}{2})}\,x^{\frac{n}{2}-1}(1-x)^{\frac{m}{2}-1} \qquad 0<x<1$$
$$= 0 \qquad x<0,\ x>1 \tag{33}$$

$$\mathfrak{F}(s) = \frac{\Gamma(\frac{n+m}{2})}{\Gamma(\frac{n+m}{2}+s-1)}\;\frac{\Gamma(\frac{n}{2}+s-1)}{\Gamma(\frac{n}{2})} \qquad n,m>0 \tag{34}$$

This is given by the equations following 7.8.6 in Titchmarsh. The Beta distribution is frequently used in

(1) See Weatherburn Chapter VIII

Biological Statistics.

"Bessel" function distribution

Let $f(x) = \left(\frac{2}{b}\right)^{1+\mu} \frac{\Gamma(\nu-\mu)}{\pi^{1/2}\,\Gamma(1+\nu+\mu)\,\Gamma(-\frac{1}{2}-\mu)} e^{-\frac{x}{b}} x^{\mu} I_\nu\left(\frac{x}{b}\right)$ (35)

$b > 0,\ x > 0,\ \mu < -\frac{1}{2},\ \mu+\nu > -1$

where $I_\nu(x) = e^{-\frac{1}{2}\nu\pi i} J_\nu(x\, e^{\frac{1}{2}\pi i}),\ -\pi < \arg z \leq \frac{1}{2}\pi$

The Mellin transform of f(x) may be obtained from the pair 7.10.7 in Titchmarsh $e^{-x} I_\nu(x)$ $\quad \frac{\Gamma(s+\nu)\,\Gamma(\frac{1}{2}-s)}{2^s \pi^{1/2}\,\Gamma(1+\nu-s)}$

$$\mathfrak{F}(s) = \left(\frac{b}{2}\right)^{s-1} \frac{\Gamma(\nu-\mu)}{\Gamma(1+\nu-\mu-s)} \frac{\Gamma(s+\mu+\nu)}{\Gamma(1+\nu+\mu)} \frac{\Gamma(\frac{1}{2}-\mu-s)}{\Gamma(-\frac{1}{2}-\mu)} \qquad (36)$$

A distribution very similar to f(x), viz., the case $\mu = \nu$ has been considered by A. T. McKay[1]. In that case the factor $e^{-x/b}$ is replaced by $e^{-cx/b}$, $c > 1$. The Mellin transform is much more complcatedq and the case will not be considered here.

Now consider the ratio of a χ^2 variable to a Beta variable:

$$\chi^2 \ :\ f_1(x) = \frac{x^{\frac{\lambda}{2}-1} e^{-\frac{x}{2}}}{2^{\lambda/2}\,\Gamma(\frac{\lambda}{2})}, \quad \lambda \text{ degrees of freedom}$$

$$\mathfrak{F}_1(s) = \frac{2^{s-1}\,\Gamma(\frac{\lambda}{2}+s-1)}{\Gamma(\frac{\lambda}{2})}$$

$$\beta \ :\ \mathfrak{F}_2(s) = \frac{\Gamma(\frac{m+n}{2})\,\Gamma(\frac{m}{2}+s-1)}{\Gamma(\frac{m+n}{2}+s-1)\,\Gamma(\frac{m}{2})}$$

$$\mathfrak{F}_1(s)\,\mathfrak{F}_2(2-s) = \frac{2^{s-1}\,\Gamma(\frac{\lambda}{2}+s-1)\,\Gamma(\frac{m}{2}+1-s)\,\Gamma(\frac{m+n}{2})}{\Gamma(\frac{\lambda}{2})\,\Gamma(\frac{m}{2})\,\Gamma(\frac{m+n}{2}+1-s)}$$

In the special case where $n = \lambda + m$ this is the Mellin transform of the "Bessel" distribution with parameters

$b = 4,\ \mu = -\frac{1}{2} - \frac{m}{2},\ \nu = \frac{\lambda+m-1}{2}$

(1) A. T. McKay 1932, p. 39

Example vi) Orthonormal Series[1]

In certain cases it is possible to expand in a convergent or asymptotic series the given frequency function. It is usual to use the orthonormal Hermite or Laguerre polynomials which have the weighting factors $e^{-\frac{x^2}{2}}$ and e^{-x} respectively. The Gram-Charlier series uses the Hermite polynomials and this is the series which is used when the random variable takes both positive and negative values. However, in our present work where the frequency function is decomposed into the functions $f_1(x)$ and $f_2(x)$, it is permissible to use the Laguerre polynomials, which lead to somewhat simpler results.

The Laguerre polynomial[2] is defined as follows:

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!} \left(\frac{d}{dx}\right)^n \left(e^{-x} x^{n+\alpha}\right) \quad \alpha > -1$$

The orthogonality relation is

$$\int_0^\infty e^{-x} x^\alpha L_n^{(\alpha)}(x)\, L_m^{(\alpha)}(x)\, dx = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\, \delta_{nm}$$

where $\delta_{nm} = 1 \quad n = m$

$\qquad\quad = 0 \quad n \neq m$

A function f(x) may be expanded in a series of the following form

$$f(x) = \sum_0^\infty a_n L_n^{(\alpha)}(x)\, e^{-x} x^\alpha \qquad (37)$$

where $a_n = \dfrac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} \displaystyle\int_0^\infty f(x)\, L_n^{(\alpha)}(x)\, dx$

If the series (37) is uniformly convergent for $x \geqslant 0$ the Mellin transform $\mathfrak{F}(s)$ may be found by term-by-term integration.

(1) See M. G. Kendall p. 147

(2) See G. Szego §5.1

Let $g(x) = e^{-x} x^{m+\alpha}$, $g^{(m)}(x)$ be the n^{th} derivative of g(x) and $\mathcal{G}(s)$ the Mellin transform of g(x). We require the Mellin transform of $\frac{1}{m!} g^{(m)}(x) = \frac{1}{m!}\left(\frac{d}{dx}\right)^m \left(e^{-x} x^{m+\alpha}\right)$ (38)

Consider $\int_0^\infty g^{(m)}(x) x^{s-1} dx$. We may integrate by parts until g(x) appears under the integral sign. The behaviour of (38) as $x \to 0$ and $x \to \infty$ insures that the integrated parts vanish. The Mellin transform of (38) is

$$\frac{(-1)^m (s-1)(s-2)\cdots(s-m)\,\Gamma(s+\alpha)}{\Gamma(m+1)}$$

Thus

$$\int_0^\infty f(x) x^{s-1} dx = \mathfrak{F}(s) = \Gamma(s+\alpha)\left\{a_0 + \sum_{m=1}^{\infty} (-1)^m \frac{a_m (s-1)\cdots(s-m)}{m!}\right\} \quad (39)$$

The N^{th} moment is given by

$$\mu_N = \mathfrak{F}(N+1) = \Gamma(N+\alpha+1)\left\{a_0 + \sum_{m=1}^{N} (-1)^m \frac{a_m N(N-1)\cdots(N-m+1)}{m!}\right\} \quad (40)$$

With these formulae, we are in a position to find the Mellin transform $\mathfrak{F}(s)$ when we know only $\mathfrak{F}(N+1)$; i.e., to interpolate between the moments. Even if not all the moments exist we may still approximate to the Mellin transform. Eq.(40) may be solved for a_n:

$$a_m = \sum_{r=0}^{m} (-1)^r \frac{\mu_r\, m(m-1)\cdots(m-r+1)}{\Gamma(\alpha+r+1)\; r!}$$

We may then find $\mathfrak{F}(s)$ in terms of $\mathfrak{F}(N+1)$

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