Lattices and Theta Functions

Terry Gannon

Department of Mathematics and Statistics

McGill University, Montreal

June 27, 1991

A Thesis submitted to the Faculty of

Graduate Studies and Research

in partial fulfillment of the requirements

for the degree of Doctor of Philosophy

Abstract

The gluing and tensor product constructions of lattices are reviewed. A third construction, called *shifting*, is defined and its relationship to gluing and rational equivalence is investigated. There follows a discussion of a method, based on gluing, for systematically generating theta function identities. Gluing is used also as a device for constructing phenomenologically realistic superstring theories. Finally, these lattice methods are applied to recent work concerning vanishing cosmological constants in string theories. In addition, scattered throughout the thesis are a number of smaller results on lattices.

Les constructions collage et produit tensoriel des réseaux sont revues. Une troisième construction, appelé *déplacement*, est définie et sa connexité au collage et la équivalence rationnelle est examinée. Suit une discussion sur une methode, basée sur le collage, pour engendrer systématiquement des identités fonction thêta. Aussi, le collage est employé comme un instrument pour construire des théories de supercordes qui sont réalistes phénoménologiquement. Finalement, ces methodes de réseaux sont appliquées à des travaux récents regardant la constante cosmologique dans les théories de supercordes. Nombre de plus petits résultats sur les réseaux sont éparpillés dans toute la thèse.

PREFACE

This thesis is concerned with lattices, their construction, and their applications to theta function identities and superstrings. It is based primarily on five papers [GL1-5].

Four different ways of constructing lattices are addressed: direct sums, direct products, gluing, and shifting. Direct sums are elementary and hence pervasive enough to recur throughout the entire subject. On the other hand, not much work has been done on direct products; I review some of what has been done and rederive some of this from more elementary arguments. Central to most of the work contained herein are the gluing and shifting constructions, and their inter-relationship. Much of this material has been taken from [GL2]. Gluing is well known to the mathematical community, and played a pivotal role for example in Niemeier's classification of the 24-dimensional even self-dual lattices. Shifting is less well known, though mathematicians have confronted a special case of it (namely, neighbours) in Borcherds' classification of the 24-dimensional odd self-dual lattices, and physicists working in string theory have for the last few years used what is called here the self-dual shift to construct new string theories from old ones. One of the major accomplishments of this thesis (and of [GL2]) is the generalization of this self-dual shift and the derivation of a number of interesting results concerning shifting (with consequences both for the theory of neighbours, and the self-dual shift). It turns out for example that a lattice Λ can be constructed by shifting another iff the two lattices are rationally equivalent. Moreover, the neighbourhood graph for self-dual lattices of any given dimension is connected.

We apply these constructions to two different areas. The gluing construction is useful in finding systematic geometrical derivations of theta function identities. This I cover in Chapters 4 and 5 — the results are taken from [GL3] and [GL4]. The gluing construction can also be used in the construction of strings — see Sec.3

of Chapter 6, which is condensed from [**GL1**]. Another application to string theory concerns the existence of string theories with zero cosmological constant. Dienes recently found in [**DIEN**] a class of partition functions which correspond to vanishing cosmological constant; lattice techniques turn out to be ideally suited to investigate whether a string theory can be found with such a partition function. I discuss this in Sec.4 of Chapter 6 — it covers material in [**GL5**].

Chapter 1 reviews basic properties of lattices. Section 1 sets the stage for subsequent developments by including several important definitions and by proving results such as the finiteness of the automorphism groups of Euclidean lattices. Section 2 is concerned with the direct sum operation, and Section 3 discusses self-duality. Section 4 establishes a number of results, some of which do not seem to be generally known. Sections 5 and 6 discuss root lattices and introduce the gluing construction. Chapter 1 is intended to be primarily a survey of fundamental aspects of the theory of lattices — the remainder of the thesis is built upon it. Although most of the theorems may not be new, their proofs (with few exceptions, and those are clearly identified in the text) are all my own.

Standard references for Chapter 1 include [**CS1**], [**CAS**], [**SER**], [**MH**], and [**GO**].

Chapter 2 discusses two other ways of constructing lattices: the tensor product and shifting. Sections 1 and 2 cover tensor products: Sec.1 describes its basic properties; and Sec.2 investigates the question of the minimal norm of tensor products. Tensor products do not arise again in this thesis. Included in these sections is some recent work by Kitaoka (see [**KIT1-4**]) — his tools are fairly sophisticated, and so I have included there some of my own material, derived from first principles and without knowledge of his work, even though much of it is a special case of his. Shifting, on the other hand, recurs throughout this thesis. It is stated in its most most general form in Sec.3, and some basic results of the self-dual shift and of neighbours are given in Sec.4.

Relevant references for Secs.1 and 2 are the papers by Kitaoka quoted in the bibliography. Most of Secs.3 and 4 is new; see Chapter 17 of [**CS1**] for the theory of neighbouring lattices, and [**GL1**] for references on self-dual shifting used in string theory construction. Secs.3 and 4 were based on parts of [**GL2**].

A self-dualizable lattice is one which can be glued to a self-dual one. Analysis of the properties of self-dualizable lattices has led to the notion of *similarity* defined in Sec.1 of Chapter 3. Its close connections with rational equivalence are discussed there. Sec.2 provides a recipe for determining whether two lattices are similar or not; it can also be thought of as a geometrical (as opposed to algebraic) derivation of the analogous question for rational equivalence. Sec.3 reviews and extends some work (on embedding a lattice in some orthonormal lattice I_m) I included in my M.Sc. Thesis, and did independently of more far reaching research by Conway and Sloane. This material is interesting in its own right, but is also useful in Sec.4 where I use many of the results obtained in previous sections to find theorems of relevance to similarity, neighbours, *etc.*

The bulk of this chapter (except for Sec.3) came from [**GL2**]). References on rational equivalence (especially its p-adic analysis) include Chapter 3 of [**CAS**] and Chapter 15 of [**CS1**]. The relevant paper for Sec.3 is [**CS4**]. Although a little of Sec.4 also overlaps [**CS4**], most of it is new, obtained first in [**GL2**].

Theta functions are useful in the theory of elliptic functions, the theory of modular and Jacobi forms, analytic number theory, the study of Riemann surfaces, and the representation of affine Lie algebras. They arise in physics in the partition functions of strings and two-dimensional conformal field theories (see [**GL3**] for references). The theta functions considered here are exclusively of genus $g = 1$ (see [**MUM**] for definitions). However, to some extent the results and techniques obtained here should generalize quite naturally to higher g.

In Chapter 4 we discuss theta constants — *i.e.* functions only of $\tau \in \mathcal{H}$. Sec.1 reviews the basic properties of the Jacobi θ-functions $\theta_2, \theta_3, \theta_4$ and ψ_k. Sec.2 covers the theta constants of lattices and their glue classes and introduces the main strategy of Chapters 4 and 5: to use glue decompositions of lattices to obtain theta functions identities. Sec.3 explicitly does this, after discussing some general material on theta function identities. In there for example we find all linear identities in the Jacobi functions as well as all quadratic identities in θ_3 derivable from this lattice method. We also discover that the famous (quartic) Jacobi identity $\theta_2(\tau)^4+\theta_4(\tau)^4 = \theta_3(\tau)^4$ is derivable from one of these degree two identities. In Sec.4 we investigate polynomials that the ψ_k's and the theta constants of glue classes are roots of and we include several interesting consequences of the existence of these polynomials.

Chapter 5 discusses theta series, where the extra complex variable z or variables $\vec{z}$ involved both complicate the analysis and strengthen the conclusions. The structure of this chapter is analogous to that of the previous one (*e.g.* the famous quartic Riemann identity is derived from a quadratic theta series identity), except that it goes further. For example, it is shown there that any theta series identity can be derived from this lattice method. Also, whereas it is possible for two lattices (*e.g.* $E_8 \oplus E_8$ and D_{16}^+) to have the same theta *constants*, this is not so for their theta series.

The material for Chapters 4 and 5 came mostly from [**GL3**] and [**GL4**]. Some of the basic results in Secs 1 and 2 of these chapters can be found in Chapter 4 of [**CS1**] and in [**MUM**]. The idea of using gluing decompositions for theta function identities is not new (see both those references), but no source we have found goes into nearly the depth we have in Secs.3 and 4. In fact, we have found only three independent theta *constant*, and only one full rank theta *series*, quadratic identities in literature searches (all these can be found in [**TM**] and [**KT**]); included in Tables 8 and 11, respectively, are at least 33 and 24 independent quadratic theta constant and theta series identities.

Chapter 6 discusses the application of the previous work on lattices to string theory. Sec.1 provides a brief description of those aspects of string theory needed in the remaining sections. Sec.2 treats the bosonic lattice string. At the end of it the shifting method used by string theorists is reviewed and its mathematical limitations are discussed. Sec.3 introduces the 'bottom-up' construction of strings, a method giving physicists greater control over constructing string theories with phenomenologically correct zero mass particle spectra and gauge groups. Sec.4 discusses a recent proposal by Dienes for finding strings with a zero cosmological constant. The results obtained in earlier chapters help to show that his partition functions cannot be realized by a lattice string with certain desired properties.

The standard reference on strings is [**GSW**]. The lattice formalism of strings can be found in [**KLT**] and [**LAM**1,3]. Sec.3 is condensed from [**GL**1]. Our analysis [**GL**5] of Dienes' partition function [**DIEN**] has not yet been completed in complete generality, but we have shown that the most natural class of physically acceptable candidate strings (which may or may not exhaust all possibilities) cannot possess his partition functions.

In summary, scattered throughout the thesis are many results which seem to be new. Also, even when the theorems are familiar, the proofs are my own (unless clearly stated to the contrary in the text) — they may differ from the standard ones by involving for example more elementary or more geometric (as opposed to algebraic) arguments. The greatest concentrations of original results are in Sec.6 of Chapter 1, Sec.4 of Chapter 3, Secs.3 and 4 of Chapters 2, 4, 5, and 6, and the end of Sec.2 of Chapter 4.

I would like to thank Professor C. S. Lam — not only has he been my advisor (with all that entails), but he has co-authored with me the five papers this thesis is based on. Also, I greatly appreciate the assistance of Rudelle Hall in proof-reading, typing, *etc.* Financial support for the research contained herein was provided in part by the Natural Sciences and Engineering Research Council.

Tables and Figures

Table of Contents

GLOSSARY

$\mathrm{Aut}(\Lambda)$	the automorphism group of Λ	p.5
average theta series	$\hat{\vartheta}(z\|\tau) = \frac{1}{2}\{\vartheta(z\|\tau) + \vartheta(-z\|\tau)\}$	p.163
background space	$V(\Lambda)$	p.1
$\mathbf{C}$	the complex numbers	
C_n	the cyclic group of order n	
Dedekind eta function	$\eta(\tau)$	p.154
degree n identity		p.119
determinant	$\|\Lambda\|$	p.8
direct sum (external, internal)	$\oplus$	p.12
discriminant	(see *determinant*)	
dual basis	β^*	p.9
dual lattice	Λ^*	p.9
even lattice	integral and all norms are even	p.3
Euclidean lattice		p.5
$\mathcal{F}_3^{(n)}, \mathcal{F}_{1,3}^{(n)}$, *etc.*		p.157
$\mathcal{F}^{(n)}, \mathcal{F}$		p.162
full-rank identities		p.156
generator matrix	M	p.3
genus		p.89
glue class	$[g]\Lambda = g + \Lambda$	p.44
glue gauge		p.73
gluing	$\Lambda_0[G]$	p.44
Gram matrix	A	p.4
$\mathcal{H}$	$\{z \in \mathbf{C} \mid Im\, z > 0\}$	
I_n	the n-dimensional cubic lattice	p.5
$I_{n,m}$	the indefinite Type I lattices	p.5
$II_{n,m}$	the indefinite Type II lattices	p.18
indecomposable lattice		p.13
indefinite lattice		p.5
independent with respect to Λ *with orders n*		p.30
integral lattice	all dot products are integers	p.3
integrally equivalent	$\approx$	p.7

integral shift	p.73
Jacobi θ-functions	$\theta_3, \vartheta_3, \psi_k, \Psi_k$ *etc.* pp.117,153
lattice	Λ p.1
Leech lattice	Λ_{24} p.22
LR-decomposition	$\{\Lambda_L; \Lambda_R\}$ p.30
$\mu(\Lambda)$	the minimal nonzero norm of Λ p.6
neighbours	p.80
Niemeier lattice	p.23
non-singular lattice	p.5
odd lattice	integral but not even p.3
orthogonal complement	$\Lambda^\perp$ p.29
orthogonal decomposition	$\{(k_1), \ldots, (k_m); (\ell_1), \ldots, (\ell_n)\}$ p.29
orthogonal lattice	has a basis of orthogonal vectors p.28
$\phi(n)$	the Euler phi function p.147
primitive vector	p.3
$\mathbf{Q}$	the rational numbers
$\mathbf{Q}$-*matrix*	matrix over $\mathbf{Q}$
quadratic form	p.8
quadratic residue	p.92
$\mathbf{R}$	the real numbers
rational equivalence	$\overset{Q}{\approx}$ p.33
rationally 3-solvable	p.121
root lattices	A_n, D_n, E_6, E_7, E_8 p.35
saturated sublattice	p.25
self-dual	$\Lambda^* = \Lambda$ p.16
self-dualizable	glues to a self-dual lattice p.46
self-dual shift	integral shift with Λ_1 self-dual p.75
shift	$\Lambda_1(\{\omega_1, \ldots, \omega_m\}, \zeta_{ij}) = \Lambda_1(\Omega, \zeta)$ p.67
signature	(n_+, n_-) p.5
similarity	$\sim$ p.84
singular lattice	p.4
sublattice	p.3
$\mathcal{T}_3^{(n)}, \mathcal{T}_3^*$	p.121
$\mathcal{T}^{(n)}, \mathcal{T}^*$	p.124
$\mathcal{T}_L^{(n)}, \mathcal{T}_R^{(n)}$, *etc.*	p.146
$\mathcal{T}_L(k)^{(n)}, \mathcal{T}_R(k)^{(n)}$, *etc.*	p.185

tensor product	$\otimes$ p.57
$\Theta(\Lambda), \Theta([g]\Lambda)$	lattice/glue theta constants p.122
theta constants	$\theta(\tau)$ p.118
theta series	$\vartheta(z\vert\tau)$ p.153
3-solvable	p.121
3-Solvable, (1,3)-Solvable, etc.	p.157
Type I lattice	self-dual and odd p.17
Type II lattice	self-dual and even p.17
unimodular	(see *self-dual*)
$V_0(\Lambda)$	p.5
$\mathbf{Z}$	the integers
$\mathbf{Z}$-*matrix*	matrix over $\mathbf{Z}$
zero lattice	Λ_Z p.1

$\cong$	group isomorphism
$\langle S\rangle$	the lattice generated by the set S p.2
$\Lambda^{(\ell)}$	the scaled-up lattice $\sqrt{\ell}\Lambda$ p.9
Λ^{ℓ}	$\Lambda \oplus \cdots \oplus \Lambda$
$\Vert G\Vert$	the order of group G
$(\frac{p}{q})$	the Legendre symbol p.92

Chapter 1 **LATTICES**

1.1 Introduction

'Lattice' is a mathematical homonym, a single term representing fundamentally different mathematical structures. A lattice, to many mathematicians, involves two binary operations on a partially ordered set, obeying certain properties. This algebraic structure has absolutely nothing to do with the type of lattice considered in this work.

Definition 1.1.1 A *lattice* Λ is a non-empty nowhere dense set of points in some finite dimensional real inner product space $V = V(\Lambda)$ (called the *background space*) such that $a, b \in \Lambda$ and $k, \ell \in \mathbf{Z}$, implies $ka + \ell b \in \Lambda$ (we say Λ is closed under $\mathbf{Z}$-linear combinations).

In other words, a lattice is a discrete additive subgroup Λ of some real inner product space V. Equivalently, a lattice is a free finitely generated abelian $\mathbf{Z}$-module on which is defined a symmetric real-valued bilinear form. Indeed, that is how it is usually defined. Although unfortunate, it will be necessary at times for us to explicitly include the background space V in our considerations.

The trivial example of a lattice is the *zero lattice*, $\Lambda_Z \stackrel{\text{def}}{=} \{0\} = V(\Lambda_Z)$ consisting of exactly one point. Unless explicitly stated to the contrary, by 'lattice' we exclude the zero lattice. Also, by 'lattice' we mean, in this work, 'non-singular lattice' — see later in this section for the definition.

The familiar $a \cdot b$ will be used to denote the inner product=bilinear form=dot product, and $a^2 \stackrel{\text{def}}{=} a \cdot a$ will be called the *norm*. It is well known that up to isomorphism, $\mathbf{R}^{\ell,m}$ is the unique indefinite real space with dimension $\ell + m$ and signature (ℓ, m). $\mathbf{R}^{\ell} \stackrel{\text{def}}{=} \mathbf{R}^{\ell,0}$ is the unique positive definite, or *Euclidean*, inner product space of dimension ℓ.

Note that Def.1 requires that V be considered as a topological space. This can be done in a number of (topologically) equivalent ways (*e.g.* by choosing a basis $\{v_1, \dots, v_n\}$ of V and defining $\| \sum a_i v_i \| = \sum a_i^2$).

Theorem 1.1.1: There exists a set $\beta = \{b_1, \dots, b_n\} \subset \Lambda$, linearly independent (over $\mathbf{R}$), whose $\mathbf{Z}$-span $\langle b_1, \dots, b_n \rangle \stackrel{\text{def}}{=} \mathbf{Z} \otimes \beta \stackrel{\text{def}}{=} \{\sum_{i=1}^{n} \ell_i b_i \mid \ell_i \in \mathbf{Z}\}$ equals Λ Moreover, any such set β also has cardinality n.

Of course, β is called a *basis*, and n the *dimension*, of the lattice. The dimension of the lattice may be less than, but never more than, the dimension of the background space. The zero lattice is said to have dimension 0.

Thm. 1 can be proved in the following way. From the structure theorem of finitely generated modules over a PID (see pp.218-226 of [**HUN**]), we know Λ is isomorphic (as a $\mathbf{Z}$-module) to $\mathbf{Z}^n = \prod \mathbf{Z}$ for some n. So, there exists a finite subset $\beta = \{b_1, \dots, b_n\}$ of Λ whose $\mathbf{Z}$-span equals Λ and whose elements b_i are linearly independent over $\mathbf{Z}$. Suppose $\sum \alpha_i b_i = 0$ for $\alpha_i \in \mathbf{R}$. Then the following lemma tells us that because Λ is nowhere dense, all α_i must be 0, thus concluding the proof of Thm.1.

Lemma 1.1.2: For any $x \in \mathbf{R}$ let $n(x)$ denote the integer satisfying $x - n(x) \in (-\frac{1}{2}, \frac{1}{2}]$. Let $\alpha_1, \dots, \alpha_n$ be real numbers. Then for any $\epsilon > 0$, there exists an integer $N_\epsilon \neq 0$ such that $|N_\epsilon \alpha_i - n(N_\epsilon \alpha_i)| < \epsilon$, for each $i = 1, \dots, n$.

Proof If all α_i are rational the lemma clearly holds — just choose N_ϵ to be the greatest common denominator of all α_i.

In the remaining case, where at least one α_i is irrational, define the map $\alpha : \mathbf{Z} \to \mathbf{R}^n$ taking $k \in \mathbf{Z}$ to the vector $\alpha(k) \stackrel{\text{def}}{=} (k\alpha_1 - n(k\alpha_1), \dots, k\alpha_n - n(k\alpha_n))$. Since $|\alpha(k) - \alpha(\ell)| \geq |\alpha(k - \ell)|$, the map is one-to-one. Because each $\alpha(k)$ lies in the unit cube $[-\frac{1}{2}, \frac{1}{2}]^n$, the image of $\mathbf{Z}$ contains accumulation points. In fact, any point $\alpha(k)$ is an accumulation point of $\alpha(\mathbf{Z})$ (again using the above inequality). Hence 0 is an accumulation point. QED

It is easy to show (see *e.g.* p.21 of [**LEK**]) that if x is any *primitive vector* of Λ (*i.e.* if $x/k \in \Lambda$ for some $k \in \mathbf{Z}$ holds only when $k = \pm 1$), then there can be found a basis $\beta = \{b_1, \ldots, b_n\}$ for Λ such that $b_1 = x$.

Λ_0 is said to be a *sublattice* of a lattice Λ if Λ_0 is a lattice in its own right, and $\Lambda_0 \subseteq \Lambda$; the bilinear form on Λ_0 is induced by that on Λ. Note that any lattice has ***proper*** sublattices (in fact infinitely many) of equal dimension to it. If S is any subset of Λ, then $\langle S \rangle$ is a sublattice of Λ.

For the most part we will be concerned here only with *rational lattices*, *i.e.* lattices whose dot products $a \cdot b$ are all rational. A lattice is called *integral* (sometimes called *classically integral* — see [**CAS**]) if all of its dot products are integers. An integral lattice is called *even* if all norms a^2 are even integers; otherwise it is called *odd*. Hence an odd lattice always has both even and odd norms.

Because the background space V is isomorphic to some $\mathbf{R}^{\ell,m}$, there exists a basis e_i of V (called here an *orthonormal basis*) such that

$$e_i \cdot e_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \leq \ell \\ -1 & i = j > \ell \end{cases}.$$

Let $\beta = \{b_1, \ldots, b_n\}$ be any basis of Λ, and suppose we have

$$b_i = \sum_{j=1}^{\ell+m} M_{ij} e_j \quad \text{for } i = 1, \ldots, n.$$

Then M is an $n \times (\ell + m)$ matrix with real entries and is called a *generator matrix* of Λ corresponding to β. Thm.1 tells us M is always of rank n.

Write $A = M\, G^{\ell,m}\, M^t$, where

$$G^{\ell,m} = diag\{\underbrace{+1, \ldots, +1}_{\ell}, \underbrace{-1, \ldots, -1}_{m}\}. \tag{1.1.1}$$

Note that $A_{ij} = b_i \cdot b_j$. In fact, let $x = \sum_{i=1}^n x_i b_i$ and $y = \sum_{j=1}^n y_j b_j$ be any two vectors in Λ. Then $x \cdot y = \vec{x}\, A\, \vec{y}^t$, where $\vec{x}$ and $\vec{y}$ denote the row vectors $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ respectively. Hence A, unlike M, is independent of the choice of basis for the background space V (though it is dependent on the choice of basis for the lattice).

Thus the dot product of Λ is encapsulated in the symmetric $n \times n$ matrix A, which is called the *Gram matrix* of Λ corresponding to β. Λ is rational iff all of the entries of A are rational, *i.e.* iff A is a **Q**-matrix; Λ is integral iff all of the entries of A are integers, *i.e.* iff A is a **Z**-matrix. Λ is even iff A is a **Z**-matrix whose diagonal entries are all even; it is odd iff its Gram matrix is a **Z**-matrix with at least one odd diagonal entry.

There is no unique Gram matrix corresponding to a given lattice. In particular, let $\beta = \{b_1, \ldots, b_n\}$ and $\beta' = \{b'_1, \ldots, b'_m\}$ be any two bases of the lattice Λ, and let A_{ij} and A'_{ij} be their corresponding Gram matrices. Thm.1 tells us $n = m$ equals the dimension of Λ. Let $b_i = \sum_{j=1}^{n} U_{ij} b'_j$ and $b'_i = \sum_{j=1}^{n} V_{ij} b_j$. Then U and V are two $n \times n$ **Z**-matrices. In addition, they are inverses of each other. This implies that the determinants $|U| = |V| = \pm 1$. Moreover, we get

$$A' = V\, A\, V^t, \quad \text{and} \quad A = U\, A'\, U^t. \tag{1.1.2}$$

A is symmetric, which means it can be expressed as

$$A = B^t\, G\, B, \tag{1.1.3}$$

where B is an invertible (real) $n \times n$ matrix, and where G is an $n \times n$ diagonal matrix whose entries are either 0, +1 or −1 (see [**MH**], p.6). When G (or A) fails to be invertible, Λ is called *singular* (or *null*). For example, take $V = \mathbf{R}^{1,1}$ and Λ to be the 1-dimensional lattice generated by $b_1 = e_1 + e_2$. Then

$$G^{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad M = (\,1 \quad 1\,), \quad \text{and} \quad A = G = (\,0\,).$$

Hence in this case Λ is singular.

Singular lattices are briefly discussed on pp.27-28 and pp.108-109 of [**CAS**]. There it is shown that a singular lattice is essentially a non-singular lattice of smaller dimension, in disguise.

From this point on, all lattices will be assumed to be *non-singular* (or *regular*), so G will be of the form eq.(1). Let n_+ be the number of $+1$ entries in G, and n_- the number of -1 entries. Then $n_+ + n_- = n$. The Sylvestor law of inertia (see [**MH**], p.61) says that n_+ and n_- are well-defined, *i.e.* independent of the particular decomposition chosen in eq.(3). This proves, using eq.(2), the independence of n_+ and n_- on the specific choice of β (and hence A). If n_+ and n_- are both nonzero, Λ is said to be an *indefinite* lattice; if $n_- = 0$, Λ is said to be *Euclidean* (or *positive definite*). Most of the lattices considered here will be Euclidean. The *signature* of Λ is defined to be (n_+, n_-) (this convenient definition is from [**SER**] — most writers define the signature to be $n_+ - n_-$).

The simplest examples of lattices are the cubic, or orthonormal, lattices $I_{m,n}$ consisting of those points in $\mathbf{R}^{m,n}$ with integral coordinates relative to some orthonormal basis $e_1, \ldots, e_{m+n}$ of $\mathbf{R}^{m,n}$. These lattices are of dimension $m+n$ and signature (m, n). The vectors e_i represent one possible choice of basis of $I_{m,n}$, and corresponding to this basis the Gram matrix $A = G^{m,n} = G$. $I_{m,n}$ is an odd lattice. $I_m \stackrel{\text{def}}{=} I_{m,0}$ is Euclidean.

Define $V_0 = V_0(\Lambda)$ to be the subspace $\mathbf{R} \otimes \Lambda$ of V; its dimension and signature will equal that of Λ. By an *automorphism* of Λ we mean a linear map $T : \Lambda \longrightarrow \Lambda$ preserving dot products: $(Tu) \cdot (Tv) = u \cdot v$, $\forall u, v \in \Lambda$. Then each automorphism induces an orthogonal map on V_0. Define $\mathrm{Aut}(\Lambda)$ to be the set of all automorphisms of Λ; it is easy to verify that it is a group.

Theorem 1.1.3: Suppose Λ is a Euclidean lattice. Then:

(i) for any $\ell \in \mathbf{R}$, the number of vectors v in Λ with norm $v^2 \leq \ell$ is finite (in fact *odd* for $\ell \geq 0$); and

(ii) $\mathrm{Aut}(\Lambda)$ is finite.

Proof V_0 will also be Euclidean, so the closed ball in V_0 of radius ℓ will be compact. Hence Thm.3(i) follows because Λ must be nowhere dense (obviously the number of norm $k \neq 0$ vectors is even since $(-v)^2 = v^2$).

To show Thm.3(ii), first choose some basis β of Λ and let N be the largest norm of the basis vectors. An automorphism of Λ is uniquely specified by its behaviour on β, and preserves norms; if we let $M < \infty$ be the number of vectors in Λ of norm $\leq N$, then there clearly cannot be more than M^2 automorphisms of Λ. QED

Thm.3(i) tells us that the set $\{v^2 \mid v \in \Lambda\}$ of all norms of Λ is countable and has an enumeration n_k, $k = 0, 1, 2, \ldots$, satisfying

$$0 = n_0 < n_1 < n_2 < \cdots \to \infty. \tag{1.1.4}$$

Moreover, if we let N_k denote the number of vectors in Λ with norm n_k, then each N_k is finite, $N_0 = 1$, and N_k is even for all $k > 0$.

We call n_1 =min$\{v^2 \mid v \in \Lambda, v \neq 0\}$ the *minimal norm* of Λ and denote it by $\mu = \mu(\Lambda)$. Note that if Λ is an n-dimensional Euclidean lattice with minimal norm μ, then the number of vectors v in Λ with norm $v^2 \leq r$ is at most

$$[(1 + 2r/\mu)^n], \tag{1.1.5}$$

where $[x]$ here denotes the greatest integer not more than x.

The assumption in Thm.3 that Λ be Euclidean is necessary — in fact most indefinite lattices serve as counterexamples to it. However, if Λ is indefinite and rational, it is easy to see that the set $\{v^2 \mid v \in \Lambda\}$ of all norms of Λ is countable and has an enumeration n_k, $k = 0, \pm 1, \pm 2, \ldots$, satisfying

$$-\infty \leftarrow \cdots n_{-2} < n_{-1} < 0 = n_0 < n_1 < n_2 < \cdots \to \infty. \tag{1.1.6}$$

(The assumption here that Λ be rational is necessary, as the choice $\Lambda = \langle e_+, \pi e_- \rangle$ shows, where $\{e_+, e_-\}$ is an orthonormal basis of $\mathbf{R}^{1,1}$.) Indeed, let A be any Gram matrix of Λ, and choose a nonzero $m \in \mathbf{Z}$ so that mA is a $\mathbf{Z}$-matrix. Then $v^2 \in \frac{1}{m}\mathbf{Z}$, $\forall v \in \Lambda$.

Definition 1.1.2: Two lattices Λ and Λ' are said to be *integrally equivalent*, written $\Lambda \approx \Lambda'$, if there exists an orthogonal (*i.e.* dot product-preserving) transformation $T : V_0(\Lambda) \to V_0(\Lambda')$ such that, for any basis $\beta = \{b_1, \ldots, b_n\}$ of Λ, $T(\beta) \stackrel{\text{def}}{=} \{T\, b_1, \ldots, T\, b_n\}$ is a basis for Λ'.

Note that if $T\,\beta$ is a basis of Λ' for *one* choice of basis β of Λ, $T\,\beta'$ will be a basis of Λ' for *any other* basis β' of Λ. It is straightforward to verify that 'integral equivalence' is an equivalence relation.

Theorem 1.1.4: Let Λ and Λ' be any two lattices. Then:

(i) if they have Gram matrices A and A' satisfying $A = A'$, we have $\Lambda \approx \Lambda'$;

(ii) if $\Lambda \approx \Lambda'$ and A is any Gram matrix of Λ, then A is also a Gram matrix of Λ'.

For (i), let $\{b_1, \ldots, b_n\}$ and $\{b'_1, \ldots, b'_n\}$ denote the bases which produce the Gram matrices A and A' for Λ and Λ', respectively. The equivalence in (i) is induced by mapping b_j to b'_j for each j. The proof of (ii) is similar.

Two integrally equivalent lattices can thus be thought of as being essentially identical: they have the same dimension and signature; they have the same set of possible Gram matrices (see Thm.4); one is rational/integral/even iff the other is; *etc.* Of course, $\Lambda \approx \Lambda'$ does not imply the background spaces V and V' are isomorphic. In fact:

Theorem 1.1.5: Given any lattice Λ of signature (n_+, n_-), there exists an integrally equivalent lattice Λ' in a background space $V' = V(\Lambda')$ of signature (n_+, n_-).

Indeed, the simplest such construction is to take $\Lambda' = \Lambda$ and $V' = V_0$. Nevertheless, we will find that it is most convenient in some cases to choose a background space of larger dimension. For example, the root lattices A_n, E_6 and E_7 defined in Sec.5 are of dimensions n, 6 and 7 respectively, but are usually defined using background spaces of dimensions $n+1$, 8 and 8 respectively.

Historically, lattices were often expressed in the language of quadratic forms. Lattices have been studied in number theory mostly in that disguise. Given any $n \times n$ Gram matrix A, we may construct the quadratic form of n variables $x_1, \ldots, x_n$ by computing the product $x A x^t$, where $x = (x_1, \ldots, x_n)$. For example, $I_{m,n}$ corresponds to the form $x_1^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_{m+n}^2$, among others. There is a one-to-one correspondence between quadratic forms and Gram matrices, so any lattice generates several quadratic forms and any quadratic form corresponds to several lattices. However, there is a one-to-one, onto correspondence between integral equivalence classes of lattices and integral equivalence classes of quadratic forms (see Thm.4). Thus the two languages can be treated as isomorphic; that of lattices is preferred here because it is more geometric.

Indeed, often in the literature 'lattice' means an 'integral equivalence class of lattices'. For reasons that will become clearer in the subsequent chapters, it is more convenient, for our purposes, to use the definition given in Def.1.

The *fundamental region* for a lattice, given a basis $\{b_1, \ldots, b_n\}$, is the subset of the background space V consisting of all points of the form $t_1 b_1 + \cdots + t_n b_n$, for $t_i \in [0, 1)$. It is a building block for the lattice (or, more precisely, for V_0) and when stacked *ad infinitum* fills V_0 with precisely 1 lattice point per block (namely, at one of the corners).

Define the ***determinant*** (also called the ***discriminant***) of Λ to be $|\Lambda| \stackrel{\text{def}}{=} ||A||$, the absolute value of the determinant of a Gram matrix A of Λ. We see from eq.(2) and the observation given there (namely, that $|U| = \pm 1$) that $|\Lambda|$ is independent of the choice of A (and hence the basis). $|\Lambda|$ is simply the volume-squared of any fundamental region (shape, but not volume, of the fundamental region is affected by basis transformations). Note that the determinant of a rational lattice is in **Q**, and that of an integral lattice is in **Z** (however, we shall see in Section Three that the determinant of an even lattice may be odd).

Define Λ^*, the *dual* (or *polar*) of Λ, to be:

$$\Lambda^* \stackrel{\text{def}}{=} \{y \in V_0 \mid x \cdot y \in \mathbf{Z} \;\; \forall x \in \Lambda\}.$$

Then Λ and Λ^* are of equal dimension and signature, and $(V_0 =)\, \mathbf{R} \otimes \Lambda = \mathbf{R} \otimes \Lambda^*$. In fact, let $\{e_i\}$ be an orthonormal basis of V_0 (*i.e.* $e_i \cdot e_j = \pm\delta_{ij}$), let G be as in eq.(1), and let M be the generator matrix corresponding to some basis $\{b_1, \ldots, b_n\}$ of Λ (expressed with respect to $\{e_i\}$). Then $M^* \stackrel{\text{def}}{=} (M^{-1})^t G$ is a generator matrix of Λ^* corresponding to the *dual basis* $\{b_1^*, \ldots, b_n^*\}$ of Λ^* which satisfies $b_i^* \cdot b_j = \delta_{ij}$. In this dual basis, the Gram matrix for Λ^* becomes $A^* \stackrel{\text{def}}{=} A^{-1}$. Hence $|\Lambda^*| = |\Lambda|^{-1}$ and $(\Lambda^*)^* = \Lambda$, and the dual lattice of any rational lattice is rational. Clearly, $\Lambda \subseteq \Lambda^*$ iff Λ is integral.

Finally, for any lattice Λ and positive number ℓ, we will be occasionally interested in the *scaled-up* lattice $\Lambda^{(\ell)} \stackrel{\text{def}}{=} \{\sqrt{\ell}x \mid \forall x \in \Lambda\}$. We may extend this definition to negative ℓ by defining $\Lambda^{(-1)}$ to be Λ with the signature flipped (*i.e.* with the dot product multiplied by -1).

Note that for $\ell > 0$ the minimal norms obey $\mu(\Lambda^{(\ell)}) = \ell\mu(\Lambda)$. Both lattices Λ and $\Lambda^{(\ell)}$ have equal dimension n, and for $\ell \in \mathbf{Z}$, $\Lambda^{(\ell)}$ is integral if Λ is, and $\Lambda^{(\ell^2)}$ is a sublattice of Λ. The Gram matrices of $\Lambda^{(\ell)}$ are ℓ times those of Λ, so $|\Lambda^{(\ell)}| = |\ell|^n|\Lambda|$. Also, for any rational lattice Λ, there exists a positive integer N (*e.g.* choose the least common denominator of the entries in a Gram matrix) such that $\Lambda^{(N)}$ is integral — hence many of the results for integral lattices trivially extend to rational lattices.

Because $M^* = (M^{-1})^t G$ is a generator matrix of Λ^*, we immediately get that $\Lambda^* \subseteq \Lambda^{(\ell)}$ for integral Λ, where $\ell = 1/|\Lambda|^2$.

Let Λ and Λ' be two lattices with Gram matrices A and A'. Then if $A = \lambda A'$, we know from Thm.4(i) that $\Lambda \approx \Lambda'^{(\lambda)}$.

Using this, we can classify all 1-dimensional lattices as such: Λ is 1-dimensional iff there exists a nonzero real number r, such that $\Lambda \approx I_1^{(r)}$.

The classification of all 2-dimensional lattices was first accomplished by Gauss, using *reduced forms* (see *e.g.* [**CS1**], pp 356-366; the classification can also be easily accomplished by gluing orthogonal lattices). Some important 2-dimensional lattices are shown in Fig.2. The classification of higher dimensional lattices is unfortunately more complicated and incomplete. See [**CS1**] and [**CS2**] for up-to-date tables of the lattices of small determinant and dimension. For example, there are precisely 1, 2, 4, 7, 9, 13, and 18 integral Euclidean lattices of determinant 25 and dimension 1, 2, 3, 4, 5, 6 and 7, respectively. However, we do know that there is a finite number of integral equivalence classes of integral lattices with a given determinant and dimension. This is what we will now proceed to prove

Theorem 1.1.6 *(Minkowski's Theorem)*: Let Λ be any n-dimensional rational Euclidean lattice. Then

$$\mu(\Lambda) \leq 4\omega_n^{-2/n}|\Lambda|^{1/n}, \text{ where } \omega_n \stackrel{\text{def}}{=} \pi^{n/2}/\Gamma(1 + \frac{1}{2}n).$$

Γ is just the gamma function and ω_n equals the volume of the unit sphere in n dimensions. Thm.6 is actually a special case of Minkowski's Convex Body Theorem (discussed in detail in Ch.2 of [**LEK**]). Incidently, by Stirling's formula, for large n Minkowski's bound behaves like $|\Lambda|^{1/n}2n/2\pi e$. Rogers in [**ROG**] has a much sharper upper bound: it behaves like $|\Lambda|^{1/n}n/\pi e$ for large n. Minkowski also showed that there exist rational lattices Λ with $\mu(\Lambda) \geq \omega_n^{-2/n}|\Lambda|^{1/n}$.

A . raightforward inductive calculation (given on pp.36-7 of [**MH**]), and using the above upper bound for μ, gives us:

Corollary 1.1.7: Inductively define the positive constants c_n, $n = 1, 2, \ldots$ by the formula $c_1 = 1$, $c_n = (4/3)^{n-2}c_{n-1} + 4^n\omega_n^{-2}$. Then given any n-dimensional Euclidean lattice Λ, a basis $b_1, \ldots, b_n$ can be found which satisfies

$$|b_i \cdot b_j| \leq c_n|\Lambda|/\mu(\Lambda)^{n-1}.$$

From Cor.7 (it is also possible to reason more directly from Thm.6) we immediately get an important result:

Theorem 1.1.8 For each $n = 1, 2, \ldots$ and $d = 1, 2, \ldots$, there are only a finite number of (integral) equivalence classes of Euclidean *integral* lattices of dimension n and determinant d.

(This result apparently is originally due to Eisenstein and Hermite — see [**MH**].) In fact, we can use Cor.7 to find a (crude) upper bound for the number of such classes:

$$4(d/\omega_n^2)^{1/n}(2c_n d + 1)^{n^2}, \tag{1.1.7}$$

using the notation of Cor.7 and Thm.8. We shall see in Section Three that the assumption that the lattices be integral is necessary (*e.g.* there are infinitely many classes of *rational* lattices of determinant 1 and dimension 2). However, the assumption that the lattices be Euclidean may be dropped: a similar argument shows that the class number for indefinite lattices of fixed dimension and determinant is also finite. In particular, Gauss (using cycles of reduced forms) and Eichler (using the spinor genus — see *e.g.* Ch.11 of [**CAS**]) have completely classified all indefinite integral forms of given dimension (see for example the discussion in [**CS1**], pp.352-405).

1.2 Direct Sums

There are three basic ways of building higher dimensional lattices up from smaller ones:

(i) *direct sums* (discussed in this section);

(ii) *direct products* (discussed in Secs.2.1 and 2.2); and

(iii) *lamination* (not dealt with in this work).

Lamination is discussed for example in Sec.1.8 of [GAN] and in Chapter 6 of [CS1]. It provides an interesting construction of the 24-dimensional Leech lattice Λ_{24}, which we discuss in the following section.

Two ways of constructing lattices from other ones of *equal* signature are *gluing* and *shifting*. This will be introduced and investigated in some detail later in this work.

Let $\Lambda_1, \dots, \Lambda_k$ be lattices of signature $(m_1, n_1), \dots, (m_k, n_k)$. Consider the set $\Lambda = \{(x_1, \dots, x_k) \mid x_i \in \Lambda_i\}$. For any two points $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$ in Λ, define $x + y \stackrel{\text{def}}{=} (x_1 + y_1, \dots, x_k + y_k)$, $x \cdot y \stackrel{\text{def}}{=} x_1 \cdot y_1 + \cdots + x_k \cdot y_k$, and for $\lambda \in \mathbf{R}$, $\lambda x \stackrel{\text{def}}{=} (\lambda x_1, \dots, \lambda x_k)$.

Obviously this makes Λ a lattice of signature $(m_1 + \cdots + m_k, n_1 + \cdots + n_k)$. It is called the *direct sum* of the *components* Λ_i and is denoted by $\Lambda_1 \oplus \cdots \oplus \Lambda_k$. Note that it is an orthogonal sum — *i.e.* loosely speaking $\Lambda_i \perp \Lambda_j$ for $i \neq j$. If V_i is the background space for the component Λ_i, $V = V_1 \oplus \cdots \oplus V_k$ is the background space for Λ. Let $\iota_i : V_i \to V$ and $\pi_i : V \to V_i$ be the obvious embeddings and projections. Then any $x \in \Lambda$ can be uniquely written as $x = \sum_{i=1}^{k} \iota_i(x_i)$, where $x_i = \pi_i(x) \in \Lambda_i$

Let β_i be a basis for Λ_i and let $\tilde{\beta}_i = \iota_i(\beta_i)$ denote the embedding of β_i into V. A basis β for Λ consists of the union of these $\tilde{\beta}_i$. For this basis,

$$M = \begin{pmatrix} M_1 & & & 0 \\ & M_2 & & \\ & & \ddots & \\ 0 & & & M_k \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_k \end{pmatrix}$$

using obvious notation. Thus Λ is Euclidean, integral, even or self-dual iff each component Λ_i is. Also:

$$\mu(\Lambda_1 \oplus \cdots \oplus \Lambda_k) = \min\{\mu(\Lambda_1), \dots, \mu(\Lambda_k)\} \tag{1.2.1a}$$

$$|\Lambda_1 \oplus \cdots \oplus \Lambda_k| = \prod_{i=1}^{k} |\Lambda_i|. \tag{1.2.1b}$$

We will often write Λ^ℓ (not to be confused with the scaled-up lattice $\Lambda^{(\ell)}$) for

$\underbrace{\Lambda \oplus \cdots \oplus \Lambda}_{\ell}$. Note also that the order of the summands Λ_i do not matter: *e.g.* $\Lambda_1 \oplus \Lambda_2 \approx \Lambda_2 \oplus \Lambda_1$.

Such a direct sum may be termed *external.* We will also be interested in the *internal* direct sum: let Λ be any lattice and let $\Lambda_1, \ldots, \Lambda_k$ be sublattices of Λ. If every $x \in \Lambda$ can be uniquely written as $x = \sum_{i=1}^{k} x_i$ for $x_i \in \Lambda_i$, and if $\Lambda_i \cdot \Lambda_j = \{0\}$ for $i \neq j$, then we call Λ the *internal* direct sum of the sublattices Λ_i, and write (as before) $\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_k$. It should be obvious from the context whether we are referring to internal or external direct sum.

Definition 1.2.1: Call Λ *indecomposable* if $\Lambda = \Lambda_1 \oplus \Lambda_2$ implies either Λ_1 or Λ_2 is the zero lattice $\{0\}$, *i.e.* if Λ cannot be expressed as the (internal) direct sum of proper sublattices.

Hence, every one-dimensional lattice is indecomposable. Indecomposable lattices are the basic building blocks of lattices. Direct sums can be defined for vector spaces; the only indecomposable (real) vector space is (up to isomorphism) $\mathbf{R}^1$. Lattices are much less trivial in this respect. For example, all root lattices (see Sec.5) are indecomposable (Thm.5.2). Clearly, I_k is indecomposable only for $k = 1$; in fact, the only indecomposable integral lattice containing unit vectors (*i.e.* vectors of norm 1) is I_1:

Theorem 1.2.1: Any integral Euclidean lattice Λ can be uniquely expressed as the (external) direct sum $\Lambda \approx I_k \oplus \Lambda'$, where the integral lattice Λ' contains no unit vectors and where Λ has exactly $2k$ unit vectors.

Proof Let $\{b_1, \ldots, b_n\}$ be a basis for Λ. Let $2k$ be the number of unit vectors in Λ (this number is even because $u \cdot u = (-u) \cdot (-u)$) and let $u_1, \ldots, u_k$ be k linearly independent unit vectors in Λ (*i.e.* $u_i \neq \pm u_j$ for all $i \neq j$). Then for $i \neq j$, $u_i \cdot u_j$ must be an integer (as Λ is integral) and also must satisfy $-1 = -u_i^2 u_j^2 < (u_i \cdot u_j)^2 < u_i^2 u_j^2 = 1$. Therefore $u_i \cdot u_j = \delta_{ij}$, so $\langle u_1, \ldots, u_k \rangle \approx I_k$.

Let $b'_i = b_i - \sum_{j=1}^{k}(b_i \cdot u_j)u_j$. Then $b_i \cdot u_j = 0$ for all i, j. Note that if $\sum \alpha_i u_i + x = \sum \beta_j u_j + y$, where $\alpha_i, \beta_j \in \mathbf{Z}$ and $x, y \in \langle b'_1, \ldots, b'_n \rangle$, then dotting this with u_ℓ gives $\alpha_\ell = \beta_\ell$ for each ℓ, and hence also $x = y$. Finally, the $\mathbf{Z}$-span $\langle u_1, \ldots, u_k, b'_1, \ldots, b'_n \rangle$ must equal Λ. Thus, Λ equals the (internal) direct sum $\langle u_1, \ldots, u_k \rangle \oplus \langle b'_1, \ldots, b'_n \rangle$, and so is integrally equivalent to the (external) direct sum $I_k \oplus \Lambda'$, for $\Lambda' \stackrel{\text{def}}{=} \langle b'_1, \ldots, b'_n \rangle$.

Of course, Λ' can contain no unit vectors, for such a vector (or its negative) would have to be contained in the list $u_1, \ldots, u_k$, *and* be orthogonal to all such vectors, which is absurd. QED

A similar proof (apart from the first paragraph) establishes the indefinite case:

Theorem 1.2.2: Any integral lattice Λ can be expressed as the (external) direct sum $\Lambda \approx I_{k,\ell} \oplus \Lambda'$, where the integral lattice Λ' contains no vectors of norm ± 1.

Of course, whereas Λ nowhere dense implies Euclidean lattices can only have finite numbers of unit vectors, this is not so for indefinite lattices. For example, $I_{2,1}$ has infinitely many unit vectors. Moreover, k and ℓ (and hence Λ') are not uniquely determined, given an indefinite Λ (*e.g.* Thm.3.2 tells us that $E_8 \oplus I_{0,1} \approx I_{8,1}$).

Of course, the hypothesis that Λ be integral is crucial. These results are exploited in the various enumerations of lattices. A slightly weaker theorem (Witt's Theorem in Sec.5) applies to vectors of norm 2 (with the role of direct sums being taken by gluings and with I_k being replaced by the root lattices).

We will conclude this section with a discussion of the uniqueness of the direct sum decompositions.

Theorem 1.2.3: Let Λ be any Euclidean lattice. Then there exist indecomposable sublattices $\Lambda_1, \ldots, \Lambda_k$ of Λ such that Λ equals the *internal* direct sum $\Lambda_1 \oplus \cdots \oplus \Lambda_k$. Moreover, if $\Lambda = \Lambda'_1 \oplus \cdots \oplus \Lambda'_\ell$ is any other internal direct sum and each Λ'_i is indecomposable, then $k = \ell$ and there exists a permutation σ such that $\Lambda_i = \Lambda'_{\sigma i}$.

Proof First note that such indecomposable decompositions are certainly possible. Let $\Lambda_1 \oplus \cdots \oplus \Lambda_k$ and $\Lambda'_1 \oplus \cdots \oplus \Lambda'_\ell$ be as in the statement of the theorem. We wish to find the permutation σ with the desired properties.

Fix i and let $\Lambda_{ij} \stackrel{\text{def}}{=} \Lambda_i \cap \Lambda'_j \ \forall j$. Then $\Lambda_{i0} \stackrel{\text{def}}{=} \Lambda_{i1} \oplus \cdots \oplus \Lambda_{i\ell} \subseteq \Lambda_i$. Enumerate all vectors $v \in \Lambda_i$ in such a way that

$$0 = v_0^2 \leq v_1^2 \leq v_2^2 \leq \cdots$$

(this is possible by Thm.1.3(i) and eq.(1.4)). We will prove by induction that $\Lambda_{i0} = \Lambda_i$.

Clearly $v_0 = 0$ is in Λ_{i0}. Suppose for induction that $v_m \in \Lambda_{i0} \ \forall m < N$. We wish to show that $v_N \in \Lambda_{i0}$. If $v_N \in \Lambda_{ij}$ for some j, then we are done. Thus (since $\Lambda = \Lambda'_1 \oplus \cdots \oplus \Lambda'_\ell$) we may assume $v_N = u_1 + \cdots + u_\ell$ where $u_j \in \Lambda'_j$ and where $u_j^2 < v_N^2$.

Because $\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_k$, we may write each u_j as $\sum_{h=1}^k u_{ij}$, where $u_{hj} \in \Lambda_h$. Then $v_N = (\sum_{j=1}^\ell u_{1j}) + \cdots + (\sum_{j=1}^\ell u_{kj})$, where each $\sum_{j=1}^\ell u_{hj} \in \Lambda_h$ and where $u_{hj}^2 \leq u_j^2 < v_N^2$. But $v_N \in \Lambda_i$, so for $h \neq i$, $0 = v_N \cdot (\sum_{j=1}^\ell u_{hj}) = (\sum_{j=1}^\ell u_{hj})^2$; *i.e.* $v_N = \sum_{j=1}^\ell u_{ij}$. Since $u_{ij}^2 < v_N^2$, by the induction hypothesis $u_{ij} \in \Lambda_i$ implies $u_{ij} \in \Lambda_{i0}$. Therefore, $v_N \in \Lambda_{i0}$.

Thus, $\Lambda_i = \Lambda_{i0}$. But Λ_i is indecomposable by hypothesis, so all but one Λ_{ij}, say Λ_{ij_i}, must be the zero lattice. Therefore, $\Lambda_i \subseteq \Lambda'_{j_i}$. A similar argument to the above will then give $\Lambda'_{j_i} \subseteq \Lambda_i$, so we have $\Lambda_i = \Lambda'_{j_i}$. Define $\sigma(i) = j_i$. QED

For example, the vector space $\mathbf{R}^n$ does not decompose uniquely in this strong sense: $(\mathbf{R} \otimes e_1) \oplus (\mathbf{R} \otimes e_2) = (\mathbf{R} \otimes \{e_1 + e_2\}) \oplus (\mathbf{R} \otimes \{e_1 - e_2\})$. However it *does* satisfy the following easy consequence of Thm.3:

Corollary 1.2.4: If Λ_1, Λ_2, Λ'_1 and Λ'_2 are all Euclidean, and both $\Lambda_1 \approx \Lambda'_1$ and $\Lambda_1 \oplus \Lambda_2 \approx \Lambda'_1 \oplus \Lambda'_2$ hold, then $\Lambda_2 \approx \Lambda'_2$ also holds.

Both Thm.3 and Cor.4 fail for indefinite Λ. An example can be constructed from the equivalence $I_{0,1} \oplus I_{1,1} \approx I_{1,0} \oplus II_{1,1}$ we get from Thm.3.2 ($I_{1,1}$ is odd and $II_{1,1}$ is even, so they cannot be equivalent). There are many more counterexamples.

Another important consequence of Thm.3 concerns automorphisms:

Corollary 1.2.5: Let $\Lambda_1^{n_1} \oplus \Lambda_2^{n_2} \oplus \cdots \oplus \Lambda_k^{n_k}$ be a decomposition of a Euclidean lattice into indecomposable lattices Λ_i, where Λ_i and Λ_j are (integrally) inequivalent for $i \neq j$. Then the automorphism group $\mathrm{Aut}(\Lambda) = (\mathrm{Aut}(\Lambda_1))^{n_1} \times S_{n_1} \times \cdots \times (\mathrm{Aut}(\Lambda_k))^{n_k} \times S_{n_k}$.

Here S_n is the group of permutations on n elements and $(\mathrm{Aut}(\Lambda_1))^{n_1} = \mathrm{Aut}(\Lambda_1) \times \cdots \times \mathrm{Aut}(\Lambda_1)$ (n_1 times). This result tells us that it suffices to know only the automorphisms of indecomposable lattices. For example, $\mathrm{Aut}(\Lambda)$ has $\|\mathrm{Aut}(\Lambda')\|\, 2^k k!$ elements, where $I_k \oplus \Lambda'$ is the decomposition of Thm.1.

Cor.5 also fails in general for indefinite lattices. For example, the automorphism groups of $II_{25,1}$ (see the next section for its definition) and most other indefinite lattices are infinite, unlike those of $II_{1,1}$ and all Euclidean lattices.

1.3 Self-duality

Recall the definition of dual lattice Λ^* at the end of Sec.1, as well as some of the elementary results established there concerning duals.

Definition 1.3.1: Λ is called *self-dual* (or *unimodular*) if $\Lambda^* = \Lambda$.

Sometimes (*e.g.* **[MH]**) the term *unimodular* is used to denote lattices with determinant 1 (which is a weaker condition than self-duality — see Thm.1). Throughout this work only the term *self-dual* will be used.

Since $\Lambda \subseteq \Lambda^*$ iff Λ is integral, Λ is self-dual iff both it and its dual Λ^* are integral; *i.e.* iff both A and A^{-1} are **Z**-matrices for any Gram matrix A of Λ. Hence (by p.353 of **[HUN]**):

Theorem 1.3.1: Λ is self-dual iff Λ is integral and $|\Lambda| = 1$.

In other words, self-dual lattices have one lattice point per unit volume.

If an integral lattice Λ satisfies the relation $\Lambda \approx \Lambda^*$, Thm.1 tells us it is self-dual. On the other hand, there are several examples of *non-integral* (hence non-self-dual) lattices Λ satisfying $\Lambda \approx \Lambda^*$. For example, for any two-dimensional Euclidean lattice Λ_2 with determinant $d = |\Lambda_2|$ and basis $\{b_1, b_2\}$, it is easy to verify that $\Lambda_2 \approx (\Lambda_2^*)^{(d)}$ (see Thm.6.10(iii) — the equivalence takes $b_1 \to \sqrt{d}b_2^*$ and $b_2 \to -\sqrt{d}b_1^*$). Therefore $\Lambda \stackrel{\text{def}}{=} \Lambda_2^{(\sqrt{d})}$ satisfies the desired relation. $\Lambda = D_4^{(1/\sqrt{2})}$ (see Sec.5) is another example.

Less trivial examples involve the 12-dimensional Coxeter-Todd lattice K_{12} (the densest sphere packing known in 12 dimensions; see pp.127-9 of [**CS1**]) and the 16-dimensional Barnes-Wall lattice Λ_{16} (the densest packing known in 16 dimensions; see pp.129-131 of [**CS1**]). Let $k = (|K_{12}|)^{-1/12} = 729^{-1/12}$ and $\ell = (|\Lambda_{16}|)^{-1/16} = 256^{-1/16}$. It can be shown that $\Lambda = K_{12}^{(k)}$ and $\Lambda = \Lambda_{16}^{(\ell)}$ also satisfy the relation.

Needless to say, it is not a property shared by most lattices of determinant 1.

Odd self-dual lattices will usually be called Type I; even self-dual lattices will usually be called Type II. The cubic lattices $I_{m,n}$ are all Type I. There are no trivial examples of Type II lattices (but see eqs.(1),(2) below). There is no 'Type III' lattice, for example, corresponding to those self-dual lattices whose norms are all multiples of 3. In fact, it can be shown that if the norms of the vectors in a self-dual lattice are all multiples of some positive $k \in \mathbf{Z}$, then $k = 1$ or 2 (this was stated without proof on p.48 of [**CS1**]). More generally, suppose a positive $k \in \mathbf{Z}$ divides the norms of all vectors in an n-dimensional integral lattice Λ. Let $\ell = k$ if k is odd; otherwise let $\ell = k/2$. Then a straightforward argument shows ℓ must divide every element of any Gram matrix A of Λ. Thus $|\Lambda|$ must be an integral multiple of ℓ^n.

The root lattice E_8 (defined in Sec.5) is a Euclidean Type II lattice in 8 dimensions. The lattice $II_{1,1}$ is defined to have generator matrix and Gram matrix (in terms of an orthonormal basis of $\mathbf{R}^{1,1}$)

$$M = \begin{pmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.3.1}$$

$II_{1,1}$ is a Type II lattice of signature (1,1). Define

$$II_{n+8k,n} \stackrel{\text{def}}{=} \underbrace{II_{1,1} \oplus \cdots \oplus II_{1,1}}_{n} \oplus \underbrace{E_8 \oplus \cdots \oplus E_8}_{k}, \tag{1.3.2}$$

and $II_{n,n+8k} \stackrel{\text{def}}{=} (II_{n+8k,n})^{(-1)}$, for $n > 0$, $k \geq 0$. Then these $II_{m,n}$ are all Type II of signature (m, n).

It was mentioned at the end of Sec.1 that all indefinite integral lattices have been classified. The classification of all the indefinite *self-dual* lattices is as follows.

Theorem 1.3.2: Let m and n be any positive integers. Then Λ is an indefinite Type I lattice of signature (m, n) iff $\Lambda \approx I_{m,n}$, and Λ is an indefinite Type II lattice of signature (m, n) iff $m - n \equiv 0 \pmod 8$ and $\Lambda \approx II_{m,n}$.

Proof (due to Serre) We will prove only the Type I case (the Type II case is handled on p.57-8 of [**SER**]), and we will assume another result established on p.55-6 of [**SER**]: namely, that if Λ is indefinite and self-dual, then it *represents* zero (*i.e.* there exists a nonzero $x \in \Lambda$ such that $x^2 = 0$). To prove this theorem, we will explicitly find vectors in Λ with norm ± 1.

Let $x \in \Lambda$ satisfy $x \neq 0$ and $x^2 = 0$. We may suppose x is *primitive* (*i.e.* $x/k \in \Lambda$ for $k \in \mathbf{Z}$ implies $k = \pm 1$). Consider the set $N = \{x \cdot y \mid y \in \Lambda\}$. Then N is clearly an additive subgroup of $\mathbf{Z}$, so equals $\ell\mathbf{Z}$ for some $\ell \in \mathbf{Z}$. Since x is primitive and Λ is self-dual, $\ell = \pm 1$. Therefore there exists a $y \in \Lambda$ such that $x \cdot y = 1$.

Now if y^2 is *even*, take any $v \in \Lambda$ with v^2 odd (possible since Λ is Type I) and define $y' = v + (1 - x \cdot v)y$. Then $y' \in \Lambda$, $x \cdot y' = 1$, and $y' \cdot y'$ is odd. Thus we may assume y^2 is odd.

Let $y^2 = 2k + 1$, and put $e_1 = y - kx$ and $e_2 = y - (k+1)x$. It is easy to verify that $e_1^2 = -e_2^2 = +1$ and $e_1 \cdot e_2 = 0$, so we have shown Λ has a sublattice integrally equivalent to $I_{1,1}$.

Now use induction on the dimension $m+n$ of Λ. For $m+n=2$, the above argument shows $\Lambda \approx I_{1,1}$; suppose the theorem holds for all indefinite lattices with dimension $\leq M$, for $M \geq 2$, and consider the case $m+n=M+1$. Then by the above argument, $\Lambda \approx I_{1,1} \oplus \Lambda'$ where Λ' is self-dual and of dimension $M-1$. Now Λ' may be neither odd nor indefinite, but either $I_1 \oplus \Lambda'$ or $I_{0,1} \oplus \Lambda'$ must be indefinite (and both are Type I) — without loss of generality suppose $I_1 \oplus \Lambda'$ is indefinite. Since its dimension is M, the induction hypothesis tells us $I_1 \oplus \Lambda' \approx I_{a,b}$ for some $a, b > 0$. Thus $\Lambda \approx I_{0,1} \oplus I_{a,b} \approx I_{a,b+1}$. QED

(This proof was adapted from [**SER**], pp.53-8. Serre and many others use Γ_8 for E_8 and Γ_n for what we will later call D_n^+.)

Hence by Thm.2, $II_{1,1}$ is the only indecomposable indefinite self-dual lattice.

The Euclidean case is more complicated (though vaguely resembling Thm.2) and hence a little richer. All Euclidean self-dual lattices have been enumerated only for dimensions $n \leq 25$ and, for reasons to be given shortly, it is doubtful much more progress will be made along these lines.

There is at least one Type I Euclidean lattice in each dimension (namely I_n), and Type II Euclidean lattices exist only in dimensions which are multiples of 8 (two proofs of this are given in Sec.3.4).

In 1938, L.J. Mordell proved $E_8 = D_8^+$ was the unique Type II Euclidean lattice in 8 dimensions. In 1941 E. Witt showed $E_8 \oplus E_8$ and D_{16}^+ were the only such lattices in 16 dimensions and in 1968 H.-V. Niemeier found all 24 such lattices in 24 dimensions (although the most important of these, the Leech lattice Λ_{24}, was found in 1965 by J. Leech).

In 1957 M. Kneser enumerated all Type I lattices in dimensions $n \leq 16$. J.H. Conway and N. Sloane extended this to $n \leq 23$ in 1982, and in his Ph.D. dissertation in 1984, Borcherds handled $n = 24$ and $n = 25$ (see [**CS1**] for a complete list of references). See Table 1 (which is based on Table 2.2 in [**CS1**]) for a summary of the known results (Thm.2.1 implies the recursion $a_{n+1} = a_n + b_{n+1} + c_n$; the

Table 1: The n-dimensional Self-dual Euclidean Lattices

Dim. $= n$	Total Number Type I$= a_n$	Number With No Unit Vectors$=b_n$	Total Number Type II$= c_n$	Indecomposable$= d_n$
1	1	0	0	1 (I_1)
2	1	0	0	0
3	1	0	0	0
4	1	0	0	0
5	1	0	0	0
6	1	0	0	0
7	1	0	0	0
8	1	0	1	0+1 (E_8)
9	2	0	0	0
10	2	0	0	0
11	2	0	0	0
12	3	1	0	1 (D_{12}^+)
13	3	0	0	0
14	4	1	0	1
15	5	1	0	1
16	6	1	2	1+1
17	9	1	0	1
18	13	4.	0	4
19	16	3	0	3
20	28	12	0	11
21	40	12	0	12
22	68	28	0	27
23	117	49	0	48
24	273	156	24	154+22
25	665	368	0	367

values of d_n can also be derived from the other columns). This table shows that for smaller dimensions (less than 20 or so), well over half of all Type I Euclidean lattices contain unit vectors and hence fail to be indecomposable.

The Minkowski-Siegel 'mass' formulae can be used to show these enumerations are complete (apparently the original German is 'massformel', which actually means 'measure formula', but this mistranslation is now in standard usage). For example:

Theorem 1.3.3· Let Ω be the set of all integral equivalence classes of Type II Euclidean lattices of dimension n. Then

$$\sum_{\Lambda\in\Omega} \frac{1}{\|\mathrm{Aut}(\Lambda)\|} = \frac{|B_k|}{2k} \prod_{j=1}^{k-1} \frac{|B_{2j}|}{4j}$$

where $n = 2k$ is a multiple of 8.

Here, $\|\mathrm{Aut}(\Lambda)\|$ is the order of the automorphism group of Λ, and B_k is the kth Bernoulli number. A similar, but more complicated, result holds for Type I Euclidean lattices. There are several standard ways of finding these automorphism groups (see *e.g.* Cor.2.5), so Thm.3 provides a straightforward, if somewhat messy, way of verifying the completeness of the enumerations summarized in Table 1.

The mass formulae are discussed in much more detail in [**CS3**]. Incidently, there are also mass formulae for determinants other than 1 — the sum then is over all Λ in a given *genus* (see Sec.3.1 for the definition).

Note that $\|\mathrm{Aut}\| \geq 2$ since $x \to -x$ is always a symmetry. Thus, doubling the right-hand side of the formula in Thm.3 gives a (crude) lower bound for the number of Type II Euclidean lattices of dimension n. For example, this gives us $\approx 10^{-9}$ for $n = 8$, $\approx 5 \times 10^{-18}$ for $n = 16$, and $\approx 10^{-14}$ for $n = 24$ (instead of 1, 2 and 24 respectively). But for $n = 32$ it gives a (presumably crude) lower bound of 80 million. It seems rather doubtful Niemeier's work will ever be extended.

Using another mass formula, lower bounds can be similarly found for Type I Euclidean lattices. For $n = 20$ we get a bound of about 10^{-12} (instead of the actual number of 28). But for $n = 28$ we get about 200, for $n = 29$ we get about 40 000, for $n = 30$ about a billion, for $n = 31$ about a trillion, and for $n = 32$ about 10^{17} In general, we get from this argument that the number of Type I Euclidean lattices of dimension n grows more quickly than n^{n^2} — compare this with eq.(1.7).

Mathematically, the enumerations of self-dual lattices can be used in a fairly simple manner to find all lattices with other determinants (particularly the smaller determinants). See [**CS1**] and [**CS2**] for details.

In the remainder of this section we will briefly discuss some of the properties of the Euclidean self-dual lattices of smaller dimensions.

In Sec.1 we discussed bounds on the minimal norms μ of rational Euclidean lattices. Let $\mu_I(n)$ denote the minimal norm of the Type I Euclidean lattice of dimension n with the largest minimal norm; define $\mu_{II}(8k)$ similarly for the Type II lattices. Let $k(n)$ denote the closest integer, and $K(n)$ the closest even integer, to $(\frac{5}{3}\omega_n^{-1})^{2/n}$. Then for each n,

$$k(n) \leq \mu_I(n) \leq 1 + [n/8], \tag{1.3.3a}$$

$$K(8n) \leq \mu_{II}(8n) \leq 1 + [n/8] \tag{1.3.3b}$$

(see p.46 of [**MH**] and p.189 of [**CS1**]). In fact, the second '$\leq$' in both of these can be replaced with '$<$' when $n > 24$ (see Chapter 19 of [**CS1**]).

For example, $\mu_I(n) = 1$ for $1 \leq n \leq 11$ and $n = 13$; it equals 2 for $n = 12$, $14 \leq n \leq 22$, and $n = 25$; it equals 3 for $n = 23$, $n = 24$, $26 \leq n \leq 31$, $n = 33$ and perhaps a few more; and $\mu_I(32) = 4$. On the other hand, $\mu_{II}(n)$ equals 2 for $n = 8$ and $n = 16$; it equals 4 for $n = 24$, $n = 32$, and $n = 40$; and $\mu_{II}(48) = 6$ (these values were communicated by John Conway).

There is only one 24-dimensional Euclidean Type II lattice with minimal norm $\mu_{II}(24) = 4$ (compare with the over 10^{20} 33-dimensional Euclidean Type I lattices with minimal norm $\mu_I(33) = 3$). This lattice is called the Leech lattice Λ_{24}, and has a number of very interesting properties. Unfortunately we cannot discuss many of them here. A generator matrix for it is given in Figure 1 — for increased readability only the nonzero entries are displayed there. Its theta constant is given in eq.(4.2.10a). See [**BOR**] and throughout [**CS1**] (especially pp.131-5) for a more detailed treatment. For a simple, self-contained construction of Λ_{24}, see pp.135-138 of [**MH**]. The 23 other 24-dimensional Euclidean Type II lattices all have minimal norm 2, and are called the *Niemeier lattices* (we will discuss them in more detail in Sec.5). Their theta constants are given in eq.(4.2.10b).

Conway's analysis of the automorphism group of Λ_{24} in the late 1960's produced three previously unknown sporadic finite simple groups: $\cdot 1$, $\cdot 2$ and $\cdot 3$ (see for example [**GOR**], or Ch.10 in [**CS1**]).

We will conclude this section with a remarkable result concerning self-dual lattices which will be useful in later sections (most notably Sec.3.4).

Theorem 1.3.4: Let Λ be a self-dual lattice of signature (n_+, n_-). Then there exists a vector $u \in \Lambda$ such that $u \cdot x \equiv x^2 \pmod 2$ for any $x \in \Lambda$. Moreover, any such u has norm $u^2 \equiv n_+ - n_- \pmod 8$.

Proof First we prove the existence of such vectors u. Choose any basis $\beta = \{b_1, \ldots, b_{n_+ + n_-}\}$ of Λ and let $\beta^* = \{b_1^*, \ldots, b_{n_+ + n_-}^*\}$ be the dual basis. Then Λ self-dual implies

$$u \stackrel{\text{def}}{=} \sum_{i=1}^{n_+ + n_-} b_i^2 b_i^* \in \Lambda.$$

Also, for any $x = \sum \ell_j b_j \in \Lambda$, $u \cdot x = \sum \ell_i b_i^2 \equiv \sum \ell_i^2 b_i^2 \equiv x^2 \pmod 2$, because Λ is integral. Therefore such u exist.

Let u' be any other such vector. Then $(u - u') \cdot x \equiv 0 \pmod 2$ for all $x \in \Lambda$, *i.e.* $v = (u - u')/2 \in \Lambda^* = \Lambda$. Thus $u'^2 = (u + 2v)^2 = u^2 + 4u \cdot v + 4v^2 \equiv u^2$ (mod 8). Define $U(\Lambda)$ to be u^2 (mod 8) — we have just seen that it is well-defined.

Now consider $\Lambda' = \Lambda \oplus I_{1,1}$. By Thm.2, $\Lambda' \approx I_{n_+ + 1, n_- + 1}$. Let $e_1, \ldots, e_{n_+ + 1 + n_- + 1}$ be orthonormal basis vectors for Λ'. Then $u'' = \sum e_i$ works as a u for Λ', so $U(\Lambda') \equiv u''^2 \equiv n_+ + 1 - n_- - 1 \equiv n_+ - n_- \pmod 8$.

Figure 1: The Leech Lattice Generator Matrix

$$
\frac{1}{\sqrt{8}}\left(\begin{array}{cccccccccccccccccccccccc}
8 &&&&&&&&&&&&&&&&&&&&&&& \\
4 & 4 &&&&&&&&&&&&&&&&&&&&&& \\
4 && 4 &&&&&&&&&&&&&&&&&&&&& \\
4 &&& 4 &&&&&&&&&&&&&&&&&&&& \\
4 &&&& 4 &&&&&&&&&&&&&&&&&&& \\
4 &&&&& 4 &&&&&&&&&&&&&&&&&& \\
4 &&&&&& 4 &&&&&&&&&&&&&&&&& \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 &&&&&&&&&&&&&&&& \\
4 &&&&&&&& 4 &&&&&&&&&&&&&&& \\
4 &&&&&&&&& 4 &&&&&&&&&&&&&& \\
4 &&&&&&&&&& 4 &&&&&&&&&&&&& \\
2 & 2 & 2 & 2 &&&&& 2 & 2 & 2 & 2 &&&&&&&&&&&& \\
4 &&&&&&&&&&&& 4 &&&&&&&&&&& \\
2 & 2 &&& 2 & 2 &&& 2 & 2 &&& 2 & 2 &&&&&&&&&& \\
2 && 2 && 2 && 2 && 2 && 2 && 2 && 2 &&&&&&&&& \\
2 &&& 2 & 2 &&& 2 & 2 &&& 2 & 2 &&& 2 &&&&&&&& \\
4 &&&&&&&&&&&&&&&& 4 &&&&&&& \\
2 && 2 && 2 &&& 2 & 2 & 2 &&&&&&& 2 & 2 &&&&&& \\
2 &&& 2 & 2 & 2 &&& 2 && 2 &&&&&& 2 && 2 &&&&& \\
2 & 2 &&& 2 && 2 && 2 &&& 2 &&&&& 2 &&& 2 &&&& \\
& 2 & 2 & 2 & 2 &&&& 2 &&&& 2 &&&& 2 &&&& 2 &&& \\
&&&&&&&& 2 & 2 &&& 2 & 2 &&& 2 & 2 &&& 2 & 2 && \\
&&&&&&&& 2 && 2 && 2 && 2 && 2 && 2 && 2 && 2 & \\
-3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right)
$$

We may write $u'' = u_0'' + u_1''$ where $u_0'' \in \Lambda$ and $u_1'' \in I_{1,1}$. It is easy to see u_0'' and u_1'' are valid choices for u's for Λ and $I_{1,1}$. Hence $u_1''^2 \equiv U(I_{1,1}) \equiv 0 \pmod 8$, so $U(\Lambda) \equiv u_0''^2 = u''^2 - u_1''^2 \equiv n_+ - n_- \pmod 8$. QED

Apparently, van der Blij (see p.24 of [**MH**]) was the first to find this result. Incidently, given *any* even lattice (not necessarily self-dual) of signature (n_+, n_-), it can also be shown (see App.4 in [**MH**]) that

$$\sqrt{|\Lambda|}\exp[2\pi i(n_+ - n_-)/8] = \sum_{[g]\in\Lambda^*/\Lambda} \exp(\pi i g^2).$$

1.4 Useful Lemmas

In this section we will prove a number of results which will be used repeatedly in the following sections and chapters. The main results of this section are Lemma 1, Cors.2(i), 4, 5 and 6, and Thms.9 and 10. The terminology introduced in this section (see especially Defs.1, 2 and 3) also will be used throughout this work.

Definition 1.4.1: Λ_0 is said to be a *saturated sublattice* of Λ if it is a sublattice of Λ whose dimension (hence signature) equals that of Λ.

Equivalently, a sublattice Λ_0 of Λ is saturated iff the vector spaces $V_0(\Lambda_0) = \mathbf{R} \otimes \Lambda_0$ and $V_0(\Lambda) = \mathbf{R} \otimes \Lambda$ are identical, or iff Λ_0 has finite index in Λ (see Lemma 1). $\Lambda^{(\ell^2)}$ is a saturated sublattice of Λ, for any nonzero $\ell \in \mathbf{Z}$. Any integral lattice is saturated in its dual Λ^*. Any rational lattice contains a saturated integral sublattice. A lattice is rational iff *any* of its saturated sublattices is rational, iff *all* of its saturated sublattices are rational.

This terminology is not standard. For example, a saturated lattice sometimes in the mathematical literature refers to an integral lattice Λ whose norm 1 and 2 vectors span a saturated (in our sense) sublattice of Λ. In [**GL2**] we used the term 'dense' in place of 'saturated'.

Lemma 1.4.1: Suppose Λ' is a saturated sublattice of some lattice Λ. Then Λ/Λ' is an abelian group of order

$$\|\Lambda/\Lambda'\| = \sqrt{|\Lambda'|/|\Lambda|}. \tag{1.4.1}$$

Proof Let M, A and M', A' be the generator/Gram matrices of Λ and Λ', respectively, relative to the same basis of $V(\Lambda) = V(\Lambda')$. Then because Λ' is saturated in Λ, there exists a **Z**-matrix U with nonzero determinant satisfying $M' = U\,M$. Hence $A' = U\,A\,U^t$, so $|\Lambda'| = |U|^2|\Lambda|$ and it suffices to show that $\|\Lambda/\Lambda'\| = \big||U|\big|$.

Express U as the product $U = U_k \cdots U_1$ of elementary row matrices (see pp.335-347 of [**HUN**]) and define Λ_i for $i = 1, \ldots, k$ recursively as being the lattice whose generator matrix is $M_i = U_i\,M_{i-1}$, where $M_0 = M$. But $|\Lambda_{i-1}|/|\Lambda_i|$ trivially equals $\big||U_i|\big|$. Therefore

$$\|\Lambda/\Lambda'\| = \|\Lambda/\Lambda_1\| \times \cdots \times \|\Lambda_{k-1}/\Lambda_k\| = \big||U_1|\big| \times \cdots \times \big||U_k|\big| = \big||U|\big|.$$

QED

This lemma will be especially useful in the following section when we calculate the determinant of a glued lattice. One immediate consequence is that the *dual group* Λ^*/Λ of any integral lattice Λ has order $|\Lambda|$. Also, we see that the determinants of a lattice and any saturated sublattice must differ by a factor that is a perfect integer square. (Of course it was not assumed in Lemma 1 that Λ be rational.)

Corollary 1.4.2: Suppose Λ' is a saturated sublattice of some lattice Λ. Then:

(i) $\Lambda' = \Lambda$ iff $|\Lambda'| = |\Lambda|$; and

(ii) $\Lambda^{(\ell)} \subseteq \Lambda'$ where $\ell = |\Lambda'|/|\Lambda|$.

Cor.2(i) follows immediately from Lemma 1 (or from Cor.2(ii)), and Cor.2(ii) follows from Lemma 1 and Lagrange's Theorem (see p.39 of [**HUN**]).

Lemma 1.4.3: Let Λ be an n-dimensional rational lattice and suppose $v \in \Lambda$ is nonzero. Then $v^\perp \stackrel{\text{def}}{=} \{u \in \Lambda \mid u \cdot v = 0\}$ is an $(n-1)$-dimensional sublattice of Λ.

Proof Because $v \neq 0$, there exist $n-1$ vectors $b_1, \ldots, b_{n-1} \in \Lambda$ such that $\beta = \{b_1, \ldots, b_{n-1}, b_n = v\}$ is a linearly independent set. Define $\Lambda' = \langle \beta \rangle$. Then Λ' is a saturated sublattice of Λ.

Now consider the basis $\beta^* = \{b_1^*, \ldots, b_n^*\}$ of Λ'^* dual to β. Define $\Lambda'' = \langle b_1^*, \ldots, b_{n-1}^* \rangle$; it is an $(n-1)$-dimensional sublattice of Λ'^* orthogonal to $b_n = v$.

But Λ is rational, so so is Λ' and hence Λ'^*. Thus there exists an ℓ such that $(\Lambda'^*)^{(\ell)}$ is integral. This implies

$$(\Lambda'')^{(\ell^2)} \subset (\Lambda'^*)^{(\ell^2)} \subseteq \Lambda' \subseteq \Lambda.$$

Therefore $(\Lambda'')^{(\ell^2)} \subseteq v^\perp$, so $v^\perp$ is at least $(n-1)$-dimensional.

Because $(\Lambda'^*)^{(\ell^2)} \subseteq \Lambda$, we know $\ell b_n^* \in \Lambda$; since $\ell b_n^* \cdot v = \ell \neq 0$, $\Lambda / v^\perp$ is infinite and $v^\perp$ cannot be saturated in Λ. Thus its dimension must be $n-1$. QED

Of course if Λ has signature (n_+, n_-) and $v^2 > 0$, then $v^\perp$ has signature $(n_+ - 1, n_-)$; if $v^2 < 0$, then $v^\perp$ would have signature $(n_+, n_- - 1)$. In either of those cases, $\langle v \rangle \oplus v^\perp$ is a saturated sublattice of Λ. When $v^2 = 0$, v obviously lies in $v^\perp$, and so $v^\perp$ is a singular lattice (*i.e.* its determinant is 0).

The assumption that Λ be rational is necessary here. For example, because the subset $\mathbf{Q} \otimes \{1, \pi\}$ of real numbers is only countable, there exist real numbers α not in it. Then the indefinite (non-singular) nonrational lattice Λ given by the Gram matrix:

$$A = \begin{pmatrix} 1 & \alpha \\ \alpha & -\pi \end{pmatrix}, \tag{1.4.2}$$

has the property that for any nonzero $v \in \Lambda$, $v \cdot u \neq 0$ for all nonzero $u \in \Lambda$. Hence for any nonzero $v \in \Lambda$, $v^\perp = \{0\} \approx \Lambda_Z$ is 0-dimensional, not 1-dimensional.

The assumption that Λ be rational is necessary even in the Euclidean case: take

$$A = \begin{pmatrix} 1 & \alpha \\ \alpha & \pi \end{pmatrix}, \tag{1.4.3}$$

where the notation is as in eq.(2) (except to guarantee the positive definite-ness of A we must ensure $\alpha^2 < \pi$).

An *orthogonal lattice* is a lattice with a basis $\{b_1, \ldots, b_n\}$ satisfying $b_i \cdot b_j = 0$ for $i \neq j$. The Gram matrix corresponding to this basis is then diagonal. Every orthogonal lattice is clearly integrally equivalent to a lattice of the form $I_1^{(k_1)} \oplus \cdots \oplus I_1^{(k_n)}$, where the $k_i = b_i^2$ are nonzero real numbers. The following much-used theorem, called the Orthogonal Decomposition Theorem (first given in [**GL**1]), tells us that every rational lattice has a saturated orthogonal sublattice. For notational convenience, we will henceforth abbreviate the one-dimensional lattice $I_1^{(k)}$ to $\{(k)\}$, and the n-dimensional lattice $I_1^{(k_1)} \oplus \cdots \oplus I_1^{(k_n)}$ to $\{(k_1), \ldots, (k_n)\}$. Finally, $\{(k_1), \ldots, (k_m); (\ell_1), \ldots, (\ell_n)\}$ will denote the $(m+n)$-dimensional lattice $I_1^{(k_1)} \oplus \cdots \oplus I_1^{(k_m)} \oplus I_1^{(-\ell_1)} \oplus \cdots \oplus I_1^{(-\ell_n)}$ (so if each k_i and ℓ_j is positive, $(k_1, \ldots, k_m; \ell_1, \ldots, \ell_n)$ will be an indefinite lattice of signature (m, n)).

Corollary 1.4.4 (*Orthogonal decomposition*): Let Λ be any rational lattice of signature (m, n). Then there exist positive integers $k_1, \ldots, k_m, \ell_1, \ldots, \ell_n$ such that Λ contains a sublattice integrally equivalent to $\{(k_1), \ldots, (k_m); (\ell_1), \ldots, (\ell_n)\}$.

Cor.4 follows inductively from Lemma 3 once one realizes that any non-singular lattice (of dimension > 0) contains vectors of nonzero norm. This also can be shown using the well-known fact (see *e.g.* p.6 of [**MH**]) that symmetric **Q**-matrices can always be diagonalized over **Q**; however, the previous proof allows us to squeeze additional details out of it. For example, k_1 can be any (positive) norm in Λ, and if Λ_0 is any orthogonal sublattice of Λ equivalent to $\{(k'_1), \ldots, (k'_{m_0}); (\ell'_1), \ldots, (\ell'_{n_0})\}$, then we can choose k_i, ℓ_j in Cor.4 so that $k_i = k'_i$ for $i = 1, \ldots, m_0$, and $\ell_j = \ell'_j$ for $j = 1, \ldots, n_0$. Another proof of Cor.4 is given in [**GL**1]; it is constructive and lattice-theoretic, but has the disadvantage that complications arise in the indefinite case.

Although simple, this theorem is quite useful and will be exploited often in what follows. Eqs.(2) and (3) again show that it is necessary for Λ to be rational.

The sublattice equivalent to $\{(k_1),\ldots,(k_m);(\ell_1),\ldots,(\ell_n)\}$ is called an *orthogonal decomposition* of Λ; it is a saturated sublattice. Clearly, if Λ is even, so are all k_i, ℓ_j. Using the notation of eq.(1.1.5), each $k_i \geq n_1$ and each $\ell_i \geq |n_{-1}|$. If $\Lambda^{(1/k)}$ is integral for some integer k, then k must divide each k_i, ℓ_j.

There is no unique orthogonal decomposition; if k_i, ℓ_j defines one, so does $m_i^2 k_i, n_j^2 \ell_j$ for any nonzero integers m_i, n_j. In later sections we will address the possible existence of orthogonal decompositions that are in a sense particularly economical. For example, we eventually will prove that the k_i, ℓ_j for any self-dual lattice can be chosen to be powers of 4 (this is trivial for indefinite lattices).

As an example, orthogonal decompositions for the root lattices (see the next section) are computed in Table 4, given later in the next section.

The following generalization of Lemma 3 follows quickly from Lemma 3 and Cor.4.

Corollary 1.4.5: Let Λ_0 be a k-dimensional non-singular sublattice of an n-dimensional lattice Λ. Then $\Lambda_0^\perp \stackrel{\text{def}}{=} \{v \in \Lambda \mid v \cdot v_0 = 0 \ \forall v_0 \in \Lambda_0\}$ is a sublattice of Λ of dimension $n-k$. Moreover, if Λ has signature (m,n) and Λ_0 has signature (m_0, n_0), then $\Lambda_0^\perp$ has signature $(m - m_0, n - n_0)$.

$\Lambda_0^\perp$ is called the *orthogonal complement* of Λ_0 in Λ.

Of course, $\Lambda_0 \cap \Lambda_0^\perp = \{0\}$ since Λ_0 is non-singular, so $\Lambda_0 \oplus \Lambda_0^\perp$ is saturated in Λ. If Λ_0 *were* singular, the dimension of $\Lambda_0^\perp$ would still be $n-k$, but $\Lambda_0 \cap \Lambda_0^\perp$ would not be $\{0\}$ and $\Lambda_0^\perp$ would also be singular. We will be interested only in the non-singular case. We will address the sublattices $\Lambda_0^\perp$ in more detail in Sec.6. Note that Λ_0 is a saturated sublattice of $(\Lambda_0^\perp)^\perp$, but in general they may not be equal (*e.g.* for any nonzero $v \in \Lambda$, $(\langle 2v\rangle^\perp)^\perp = \langle v \rangle$). However, $((\Lambda_0^\perp)^\perp)^\perp = \Lambda_0^\perp$ always holds. Also, if $\Lambda = \Lambda_1 \oplus \Lambda_2$, then $\Lambda_1^\perp = \Lambda_2$ and $\Lambda_2^\perp = \Lambda_1$.

Eqs.(2) and (3) show that the assumption that Λ be rational is necessary.

The following result often will enable us to reduce a proof involving rational indefinite lattices to the Euclidean case. It follows immediately from Cor.4.

Corollary 1.4.6: Suppose Λ is a rational lattice of signature (m,n). Then there exist Euclidean integral lattices Λ_L and Λ_R of dimensions m and n respectively, such that $\{\Lambda_L;\Lambda_R\} \stackrel{\text{def}}{=} \Lambda_L \oplus \Lambda_R^{(-1)}$ is a saturated sublattice of Λ. Consider the projections $\pi_L : V(\Lambda) \to V_0(\Lambda_L)$ and $\pi_R : V(\Lambda) \to V_0(\Lambda_R)$, and for any $v \in \Lambda$ define $v_L \stackrel{\text{def}}{=} \pi_L(v)$ and $v_R \stackrel{\text{def}}{=} \pi_R(v)$. Then $u \cdot v = u_L \cdot v_L - u_R \cdot v_R$, where the dot products $u_L \cdot v_L$ and $u_R \cdot v_R$ are induced by those of Λ_L and Λ_R, respectively.

$\{\Lambda_L;\Lambda_R\}$ will be called an *LR-decomposition* of Λ. Of course $\Lambda_L \subseteq \Lambda_R^{\perp}$ and $\Lambda_R \subseteq \Lambda_L^{\perp}$. Additional properties possessed by these Λ_L and Λ_R will be discussed in Sec.6.

By using Lemma 3 we obtain some additional information. For example, for any Euclidean sublattice Λ' of Λ there can be found an LR-decomposition $\{\Lambda_L;\Lambda_R\}$ of Λ such that Λ' is a sublattice of Λ_L (similarly for any negative definite sublattice Λ'' of Λ). There is no unique LR-decomposition: *e.g.* there is a different LR-decomposition of $I_{8,1}$ for each of the infinitely many different choices of vectors $v \in I_{8,1}$ satisfying $v^2 = -1$ (just choose $\Lambda_R = \langle v\rangle^{(-1)} \approx I_1$), and both $\Lambda_L \approx I_8$ and $\Lambda_L \approx E_8$ (see the following section for the definition of E_8) are possible.

As before, eq.(2) shows it is necessary to assume Λ is rational.

Definition 1.4.2: Vectors $y_1, \ldots, y_m$ are said to be *independent with respect to* some lattice Λ *with orders* n_i, if the n_i are nonzero integers satisfying

$$\sum_{i=1}^{m} k_i y_i \in \Lambda \text{ for } k_i \in \mathbf{Z} \text{ iff } n_i \text{ divides } k_i \ \forall i.$$

The purpose of the following definitions is the statement of Lemma 7, which is used only in the proof of Lemma 8.

Let B be any $n \times m$ **Q**-matrix. Define the 'column space' B_c of B to be the set of n-tuples

$$B_c \stackrel{\text{def}}{=} \{\sum_{i=1}^{m} k_i B_c^{(i)} \mid k_i \in \mathbf{Z}\},$$

where $B_c^{(i)}$ is the ith column of B. Define the 'row space' B_r similarly. B_r and B_c are **Z**-modules.

Finally, let $\widetilde{B_c}$, which we will call the 'column group', be the additive group obtained by modding each component of each vector in B_c by 1; define the 'row group' $\widetilde{B_r}$ similarly. Specifically, we may write

$$\widetilde{B_c} \stackrel{\text{def}}{=} B_c \,/\, (\mathbf{Z}^n \cap B_c) \ , \ \widetilde{B_r} \stackrel{\text{def}}{=} B_r \,/\, (\mathbf{Z}^m \cap B_r).$$

It is important not to confuse the finite groups $\widetilde{B_c}$ and $\widetilde{B_r}$, the infinite **Z**-modules B_c and B_r, and the columns $B_c^{(i)}$ of B, with each other.

Lemma 1.4.7: The row group $\widetilde{B_r}$ and column group $\widetilde{B_c}$ of any **Q**-matrix B are isomorphic.

Proof Let $\mathbf{Z}(\ell)$ be the set of all $\ell \times \ell$ **Z**-matrices whose inverse is also a **Z**-matrix. Firstly, we clearly have that the 'column space' B_c of B equals the 'column space' of BU for any matrix $U \in \mathbf{Z}(m)$, so the groups $\widetilde{B_c}$ and $(\widetilde{BU})_c$ are also equal. Secondly, for any $V \in \mathbf{Z}(n)$, the correspondence $B_c^{(i)} \leftrightarrow V B_c^{(i)}$ induces an isomorphism between the groups $\widetilde{B_c}$ and $(\widetilde{VB})_c$. Similar comments apply to the row groups.

Let ℓ be any integer such that ℓB is a **Z**-matrix. We know then that there exist matrices $U \in \mathbf{Z}(m)$, $V \in \mathbf{Z}(n)$, such that

$$D = V(\ell B)U$$

is a $n \times m$ diagonal matrix. The above argument now shows that $\widetilde{B_c} \cong (\widetilde{\frac{1}{m} D})_c$ and $\widetilde{B_r} \cong (\widetilde{\frac{1}{m} D})_r$. But D diagonal trivially implies $(\widetilde{\frac{1}{m} D})_c \cong (\widetilde{\frac{1}{m} D})_r$. QED

Lemma 1.4.8: Let Λ be any n-dimensional lattice, and let $\{x_1, \ldots, x_m\}$ be a set of vectors such that the group of cosets

$$G = \{\sum_{i=1}^{m} k_i x_i + \Lambda^* \mid k_i \in \mathbf{Z}\}$$

is finite. Define the lattice

$$\Lambda_0 = \{q \in \Lambda \mid q \cdot x_i \in \mathbf{Z}, \ \ \forall i\}.$$

Then G and $G' \stackrel{\text{def}}{=} \Lambda \,/\, \Lambda_0$ are isomorphic as groups.

Proof Because G is finite, without loss of generality we may assume the x_i are independent generators of G, with orders n_i. Let $\{q_1, \ldots, q_n\}$ be a basis for Λ. Define the $n \times m$ matrix B by

$$B_{ij} = q_i \cdot x_j.$$

B is a $\mathbf{Q}$-matrix.

Note that $\sum_j k_j B_{ij} \in \mathbf{Z}\ \forall i$ iff $\sum_j q_i \cdot (k_j x_j) \in \mathbf{Z}\ \forall i$ iff $\sum k_j x_j \in \Lambda^*$ iff $n_j | k_j\ \forall j$ by definition of the x_j being independent generators. Thus G is isomorphic to the column group $\widetilde{B_c}$ defined before Lemma 7; similarly, $G' \cong \widetilde{B_r}$. The result follows immediately from Lemma 7. QED

A special case of Lemma 8 is that if Λ_1 is a saturated sublattice of some Λ, then the groups Λ/Λ_1 and Λ_1^*/Λ^* are isomorphic. Lemma 8's main import, however, lies in its role in the following proof, which is one of the principle results of this section.

Theorem 1.4.9: Let Λ be any lattice. Suppose there exists a set of vectors $\{y_1, \ldots, y_m\}$ independent with respect to Λ^*, with orders n_i. Then there exist vectors $\{r_1, \ldots, r_m\}$ in Λ such that

$$r_i \cdot y_j \equiv \frac{1}{n_i} \delta_{ij} \pmod 1.$$

Proof Let $\Lambda_0 = \{q \in \Lambda \mid q \cdot y_i \in \mathbf{Z}\ \forall i\}$. Define $G' = \Lambda / \Lambda_0$. Then for any $q, q' \in \Lambda$, $q \equiv q' \pmod{\Lambda_0}$ iff $q \cdot y_i = q' \cdot y_i \pmod 1$, for each i. Therefore there is a well-defined one-to-one mapping from each $[q] \in G'$ to the m-tuple $(q \cdot y_1 \pmod 1, \ldots, q \cdot y_m \pmod 1)$.

Since $q \cdot (n_i y_i) \in \mathbf{Z}$, we see that $q \cdot y_i \pmod 1$ can only take the values in $\{\frac{0}{n_i}, \frac{1}{n_i}, \ldots, \frac{n_i - 1}{n_i}\}$. Therefore, there are at most $\prod_{j=1}^m n_j$ possible m-tuples.

But $\prod_{j=1}^m n_j$ is precisely the order of G' (by Lemma 8 and the independence of the y_i). Since our mapping was one-to-one, we get that it is also *onto* the set of m-tuples

$$\left(\frac{k_1}{n_1}, \ldots, \frac{k_m}{n_m}\right),$$

where $0 \le k_i < n_i$.

Define $r_1 \in \Lambda$ to be a representative of that class of G' corresponding to the m-tuple $(\frac{1}{n_1}, 0, \ldots, 0)$; define $r_2 \in \Lambda$ to be a representative of that class of G' corresponding to $(0, \frac{1}{n_2}, 0, \ldots, 0)$; and similarly for $r_3, \ldots, r_m$. QED

Note that Λ may or may not be rational, and that m may or may not equal the dimension of Λ. Also, note that the orders n_i here are with respect to Λ^*, not Λ.

Definition 1.4.3: Let A and A' be Gram matrices for two lattices Λ and Λ', respectively. Λ_1 and Λ_2 are said to be ***rationally equivalent***, written $\Lambda_1 \overset{Q}{\approx} \Lambda_2$, if there exists an invertible $\mathbf{Q}$-matrix V such that $V^t A V = A'$.

Equivalently, $\Lambda \overset{Q}{\approx} \Lambda'$ iff the bilinear forms corresponding to Λ and Λ' are isomorphic over $\mathbf{Q}$ when tensored with $\mathbf{Q}$.

Because two Gram matrices A_1 and A_2 of a lattice Λ are related by $A_1 = U^t A_2 U$ for some $\mathbf{Z}$-matrix U with determinant ± 1, we see that this definition is well-defined (*i.e.* independent of the particular choice of Gram matrix). This also shows that $\Lambda \approx \Lambda'$ implies $\Lambda \overset{Q}{\approx} \Lambda'$.

For example, we have $\Lambda \overset{Q}{\approx} \Lambda^*$ for any rational lattice Λ (choose $V = A^{-1}$) in fact this is also a sufficient condition for Λ to be rational.

Def.3 defines rational equivalence in the usual way for quadratic forms. The following theorem interprets rational equivalence geometrically, *i.e.* in a manner more conducive for lattices. This simple theorem underlies all subsequent results concerning rational equivalence given in this work.

Theorem 1.4.10: $\Lambda_1 \overset{Q}{\approx} \Lambda_2$ iff there exists a lattice Λ_2' (integrally) equivalent to Λ_2 such that $\Lambda_1 \cap \Lambda_2'$ is saturated in both Λ_1 and Λ_2'.

Proof Let A_1 and A_2 be Gram matrices for Λ_1 and Λ_2, corresponding to bases $\{\beta_i\}$ and $\{\beta_j'\}$, respectively. Then $\Lambda_1 \overset{Q}{\approx} \Lambda_2$ iff $\exists$ an invertible **Q**-matrix U such that $A_2 = U^t A_1 U$.

"$\Rightarrow$ " Let Λ_2' be the lattice with basis vectors $\beta_i'' = U\beta_i$. Then $\Lambda_2' \approx \Lambda_2$ (the equivalence is given by $\beta_i' \leftrightarrow \beta_i''$). Let $\ell \in \mathbf{Z}$ be such that ℓU is a **Z**-matrix. Then $\Lambda_2'^{(\ell^2)}$, and hence $\Lambda_1 \cap \Lambda_2'$, is a saturated sublattice of both Λ_1 and Λ_2'.

"$\Leftarrow$ " It suffices to show that any saturated sublattice Λ_0 of Λ_1 is rationally equivalent to Λ_1. This follows because the matrix U expressing a basis for Λ_0 in terms of one for Λ_1 is a **Z**-matrix with nonzero determinant (since Λ_0 is saturated in Λ_1), and so is invertible as a **Q**-matrix. QED

Corollary 1.4.11: $\Lambda_1 \overset{Q}{\approx} \Lambda_2$ iff $\exists \ell \in \mathbf{Z}$ and a lattice Λ_2' (integrally) equivalent to Λ_2 such that $\Lambda_1^{(\ell^2)} \subset \Lambda_2'$ and $\Lambda_2'^{(\ell^2)} \subset \Lambda_1$.

In fact, it is possible to prove that $\ell = |\Lambda_1 \cap \Lambda_2'| / \sqrt{|\Lambda_1|\,|\Lambda_2|}$ works in Cor.11. Cor.11 follows immediately from the above proof of Thm.10.

1.5 The Root Lattices and Gluings

The theory of Lie groups and algebras is surely among the most elegant and useful of all mathematical theories. Its influence is felt in areas such as high energy physics and the classification of the finite simple groups (16 of the 18 infinite families of finite simple groups are of Lie type — see [**GOR**]). In this section its significant applications to lattice theory will be presented.

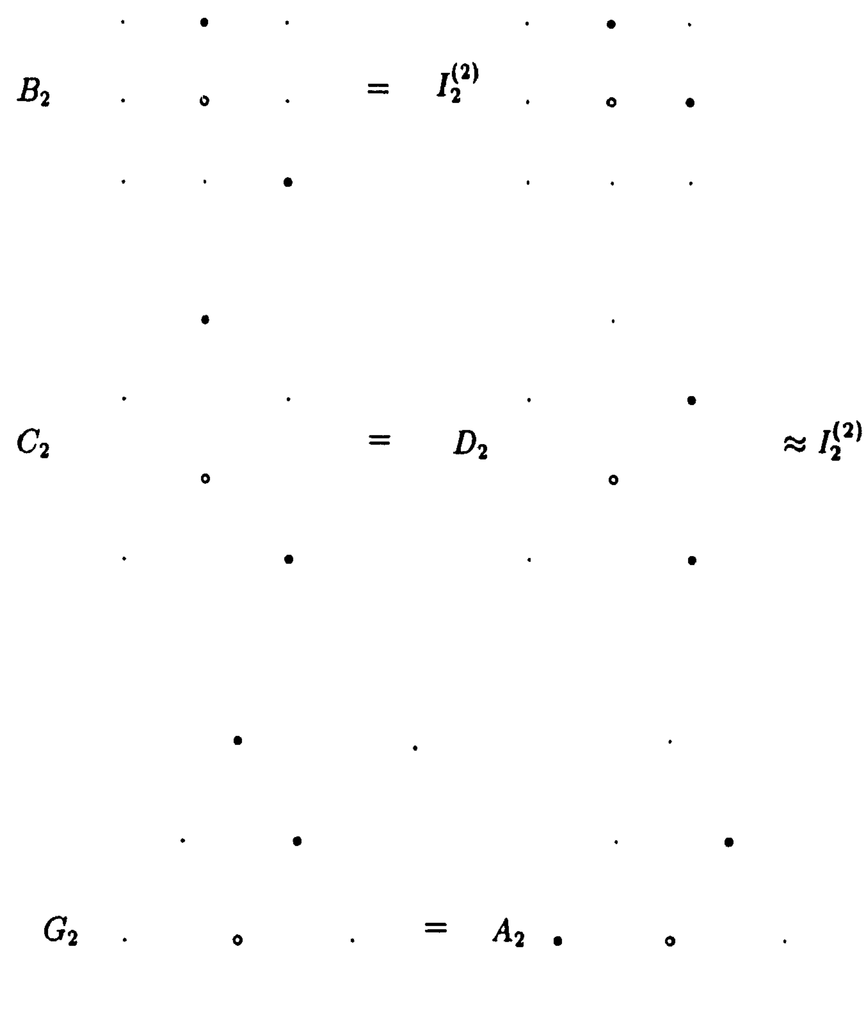
To every complex semi-simple Lie algebra there is associated a root system (see p.42 of [**HUM**]), *i.e.* a set of vectors $\{\alpha_i\}$ (called *root vectors*) satisfying various properties (*e.g.* $2\alpha_i \cdot \alpha_j/\alpha_i^2 \in \mathbf{Z}$). A basis for it can be found — these basis vectors are called *simple root vectors*. A convenient way of graphically representing a set of simple roots is with a *Dynkin diagram*: to each simple root there is associated a node in the diagram, and two nodes are connected by 0, 1, 2 or 3 (sometimes directed) segments depending on the dot product of the corresponding simple roots. All possible connected Dynkin diagrams are known (see *e.g.* [**BOU**] p.197) — they correspond to the complex simple Lie algebras A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, and G_2 (all other Dynkin diagrams are simply unions of these).

By a root lattice of some Lie algebra we simply mean the lattice generated by the simple root vectors of the Lie algebra. The simple roots are determined only up to a global rotation and global scale factor, but we shall fix them by adopting the conventions of Bourbaki (see [**BOU**], pp.250-262) and Conway and Sloane (see [**CS1**], pp.116-129). The dimension of the root lattice, *i.e.* the number of simple roots, is the rank of the Lie algebra and the value of the subscript (*e.g.* the root lattice A_n has dimension n). We will use the same symbol to denote the root lattice and the Lie algebra (no confusion should result).

See Figure 2 for the 2-dimensional root lattices (the origin is labelled with an "o" and the simple roots with a bullet). As can be seen there, many of the root lattices are integrally equivalent (perhaps using scale factors).

Theorem 1.5.1: $B_n = I_n$, $G_2 = A_2$, $C_n = D_n$, and $F_4 \approx D_4^{(2)}$, for all n. In addition, $A_1 \approx I_1^{(2)}$, $D_1 = I_1^{(4)}$, $D_2 \approx A_1 \oplus A_1 \approx I_2^{(2)}$, and $D_3 \approx A_3$. These completely exhaust the (possibly scaled) integral equivalences between root lattices.

Figure 2: The Two-Dimensional Root Lattices



The explicit proof of these equivalences was done in Thm.2.6.1 of [**GAN**]. That these exhaust all possible equivalences follows easily by computing determinants (see Table 2). Of course, if we had not adopted Bourbaki's conventions, Thm.1 would still hold, though perhaps with different scalings and provided we replaced all equalities, '=', with equivalences, '≈'.

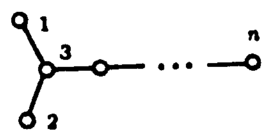
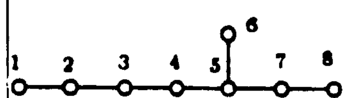
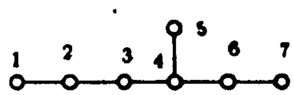
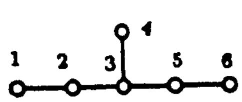
Thm.1 permits us to consider in what follows only the *simply-laced* root lattices, *i.e.* the ones whose simple roots all have equal length. Henceforth by root lattices we will mean A_n for $n = 1, 2, \ldots$, D_k for $k = 4, 5, \ldots$, E_6, E_7, and E_8 (usually I_n is not considered a root lattice). They all are Euclidean, have minimal norm $\mu = 2$, all are even, and in fact are spanned by a basis consisting of norm 2 vectors. They have determinants $|A_n| = n + 1$, $|D_n| = 4$, $|E_6| = 3$, $|E_7| = 2$, and $|E_8| = 1$. Hence only E_8 is self-dual. Many features of the root lattices are listed in Table 2 (note there that the vectors e_j, in terms of which the roots α_i are expressed, are orthonormal, and R^{n+1} *etc.* denote the Euclidean background spaces). Except for A_1, Bourbaki's choice of norm 2 for the simple roots of these simply-laced algebras is the smallest possible choice for which the root lattices are integral (*e.g.* $D_n^{(1/2)}$ is not integral).

Theorem 1.5.2: The root lattices A_n for $n \geq 1$, D_k for $k \geq 4$, E_6, E_7, and E_8 are all indecomposable.

Indeed, because these lattices Λ are even, if we had $\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_k$, then no Λ_i could contain unit vectors, so each root vector of Λ would have to lie completely in one component Λ_i; because the Dynkin diagram is connected for all these (unlike, for example, that of D_2), these root lattices must all be indecomposable.

Note that A_n is the n-dimensional sublattice of I_{n+1} consisting of all vectors whose coordinates (relative to the standard orthonormal basis of I_{n+1}) add up to zero. D_n is the (saturated) sublattice of I_n consisting of all vectors whose coordinates add up to an even integer.

Table 2: The Root Lattices

Symbol	Equiv.	Dynkin Diagrams	Basis Vectors	Det.
$I_n,\ n \geq 1$	B_n		$\alpha_i = e_i$	1
$A_n, n \geq 1$	$G_2 = A_2$		$\alpha_1 = -e_1 + e_2$ $\alpha_2 = -e_2 + e_3$ $\vdots$ $\alpha_n = -e_n + e_{n+1}$ $(\alpha_i \in R^{n+1})$	$n+1$
$D_n, n \geq 4$	C_n $F_4 \approx D_4^{(2)}$		$\alpha_1 = -e_1 - e_2$ $\alpha_2 = e_1 - e_2$ $\vdots$ $\alpha_n = e_{n-1} - e_n$ $(\alpha_i \in R^n)$	4
E_8	—		$\alpha_1 = e_2 - e_3$ $\alpha_2 = e_3 - e_4$ $\vdots$ $\alpha_6 = e_7 - e_8$ $\alpha_7 = e_7 + e_8$ $\alpha_8 = -\frac{1}{2}\sum_{i=1}^{8} e_i$ $(\alpha_i \in R^8)$	1
E_7	—		$\alpha_1 = e_3 - e_4$ $\vdots$ $\alpha_5 = e_7 - e_8$ $\alpha_6 = e_7 + e_8$ $\alpha_7 = -\frac{1}{2}\sum_{i=1}^{8} e_i$ $(\alpha_i \in R^8)$	2
E_6	—		$\alpha_1 = e_4 - e_5$ $\vdots$ $\alpha_4 = e_7 - e_8$ $\alpha_5 = e_7 + e_8$ $\alpha_6 = -\frac{1}{2}\sum_{i=1}^{8} e_i$ $(\alpha_i \in R^8)$	3

From the Dynkin diagram (also given in Table 2) we can read off the Gram matrix. Thus the Gram matrix has 2's down the diagonal, and -1's and 0's scattered elsewhere. In the cases considered here, the Gram matrix equals the Cartan matrix of the corresponding root system. The Dynkin diagram can also be used to recursively compute all determinants of the root lattices (see Sec.3.4 of [**GAN**]), and verify the values given in Table 2.

Similarly, the *weight lattice* corresponding to a Lie algebra consists of the lattice generated by the root vectors and the weight vectors. It turns out that the weight lattice is simply the dual of the corresponding root lattice.

We see from Lemma 4.1 that the dual group A_n^*/A_n has precisely $|A_n| = n+1$ elements. It can be shown to equal the cyclic group C_{n+1}. We let $[i]$ (or less ambiguously $[i]A_n$) for $i = 0, \ldots, n$ denote its elements (which are called *glue classes* for reasons we will later discuss). Similarly, D_n^*/D_n has order 4; its four glue classes are labelled [0], [1], [2] and [3]. The orders for E_6, E_7 and E_8 are 3, 2 and 1, with classes [0], [1] and [2], [0] and [1], and [0], respectively. In all cases the class [0] has been chosen to be the zero element in the dual group — *i.e.* the root lattice itself. The nonzero glue classes are listed in Table 3; they are of course all of the form $[i] = g_i + [0]$ where g_i is some vector in the dual of the root lattice. In that table, we have written for convenience, *e.g.*,

$$(\{\frac{i}{n+1}\}^j, \{\frac{-j}{n+1}\}^i) \text{ for } (\underbrace{\frac{i}{n+1}, \ldots, \frac{i}{n+1}}_{j}, \underbrace{\frac{-j}{n+1}, \ldots, \frac{-j}{n+1}}_{i}).$$

Also, in the third column the dot products of the representatives (chosen in the second column) of the glue classes are displayed in matrix form; these numbers also give of course the correct dot products (mod 2) for *any* choice of representatives.

The orthogonal decompositions of the root lattices are given in Table 4. In the second column the orthogonal lattice vectors β_i are expressed in terms of the root vectors α_j. In the third column their norms $m_i = \beta_i^2$ are given. In the fourth and fifth columns the root vectors α_j and weight vectors λ_j are expressed in terms of

Table 3: The Nonzero Glue Vectors of the Root Lattices

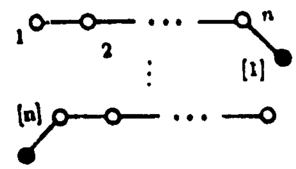
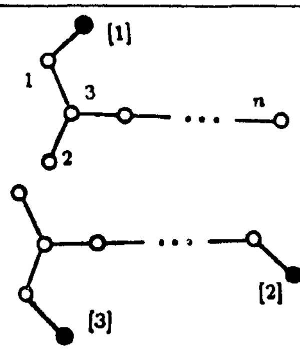
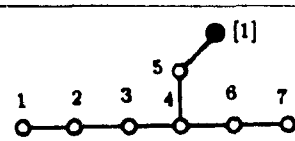
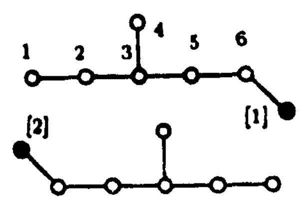
	Glue Vectors	$[i]\cdot[k]$	Dynkin Diagrams	Glue Gp
A_n	$[i] = A_n+$ $(\{\frac{i}{n+1}\}^j, \{\frac{-j}{n+1}\}^i)$ for $i = 1, \ldots, n$ $(j = n+1-i)$	For $i \geq k$, $[i]\cdot[k] = \frac{jk}{n+1}$		C_{n+1} $([i]+[k]$ $=[i+k])$
D_n	$[1]=(\{\frac{1}{2}\}^n) + D_n$ $[2]=e_n + D_n$ $[3]=(-\frac{1}{2}, \{\frac{1}{2}\}^{n-1}) + D_n$	$\begin{matrix} \frac{n}{4} & \frac{1}{2} & \frac{n-2}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{n-2}{4} & \frac{1}{2} & \frac{n}{4} \end{matrix}$		$C_2 \times C_2$ when n even C_4 $([1]+[3]=[0])$ for n odd
E_7	$[1]=e_1 + e_2 + \frac{1}{2}e_8$ $-\frac{1}{2}\sum_{k=3}^{7} e_k + E_7$	$\frac{7}{2}$		C_2
E_6	$[1]=E_6+$ $\frac{2}{3}(e_1+e_2+e_3)$ $[2]=E_6+$ $\frac{1}{3}(e_1+e_2+e_3) - e_4$	$\begin{matrix} \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{3} \end{matrix}$		C_3

Table 4: Orthogonal Decompositions of the Root Lattices

Λ	**Orthogonal Basis** β_i	m_i	**Roots** α_i	**Glues** g_i
A_n	$\beta_{2k+1} = \alpha_{2k+1}$	$m_{2k+1} = 2$	$\alpha_{2k+1} = \beta_{2k+1}$	$\frac{j-1}{2}\beta^*_{j+1} + \beta^*_j + j\sum_{\ell=1}^{\frac{j-1}{2}} \beta^*_{j-2\ell+1}$ ($j = n+1-i$ odd)
	$\beta_{2k} = \sum_{\ell=1}^{2k} \ell\alpha_\ell + k\alpha_{2k+1}$ if $2k < n$	$m_{2k} = 2k(k+1)$ if $2k < n$	$\alpha_{2k} = (1-k)\beta^*_{2k-2} - \beta^*_{2k-1} + (k+1)\beta^*_{2k} - \beta^*_{2k+1}$ if $2k < n$	$j\sum_{\ell=1}^{\frac{j}{2}} \beta^*_{2\ell}$ j even
	$\beta_{2k} = \sum_{\ell=1}^{2k} \ell\alpha_\ell$ if $n = 2k$	$m_{2k} = n(n+1)$ if $2k = n$	$\alpha_{2k} = (1-k)\beta^*_{2k-2} - \beta^*_{2k-1} + (2k+1)\beta^*_{2k}$ if $2k = n$	
D_n	$\beta_1 = \alpha_1 - \alpha_2$	$m_i = 4$	$\alpha_1 = 2\beta^*_1 + 2\beta^*_2$	$[1] = \sum_{i=1}^n \beta^*_i$
	$\beta_2 = \alpha_1 + \alpha_2$		$\alpha_2 = -2\beta^*_1 + 2\beta^*_2$	$[2] = 2\beta^*_n$
	$\beta_i = \alpha_1 + \alpha_2 + 2\sum_{k=3}^{i} \alpha_k$ if $i > 2$		$\alpha_i = -2\beta^*_{i-1} + 2\beta^*_i$ $i > 2$	$[3] = -\beta^*_1 + \sum_{\ell=2}^n \beta^*_\ell$
E_8	$\beta_{1,2,3,4} = \alpha_{2,4,6,7}$	$m_i = 2$	$\alpha_1 = -\beta^*_4 + \beta^*_5 - \beta^*_7 - \beta^*_8$	
	$\beta_5 = -\alpha_2 - 2\alpha_3 - 3\alpha_4 - 4\alpha_5 - 2\alpha_6 - 3\alpha_7 - 2\alpha_8$		$\alpha_{2,4,6,7} = \beta_{1,2,3,4}$	
	$\beta_6 = \alpha_4 + 2\alpha_3 + \alpha_6 + \alpha_7$		$\alpha_3 = -\beta^*_3 - \beta^*_4 - \beta^*_6 + \beta^*_7$	
	$\beta_7 = \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7$		$\alpha_5 = -\beta^*_1 - \beta^*_2 - \beta^*_3 + \beta^*_6$	
	$\beta_8 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 3\alpha_6 + 4\alpha_7 + 2\alpha_8$		$\alpha_8 = -\beta^*_2 - \beta^*_5 - \beta^*_6 - \beta^*_7$	

Table 4: Orthogonal Decompositions of the Root Lattices *(cont.)*

Λ	Orthogonal Basis β_i	m_i	Roots α_i	Glues g_i
E_7	$\beta_{1,2,3,4} = \alpha_{1,3,5,6}$	$m_i = 2$	$\alpha_{1,3,5,6} = \beta_{1,2,3,4}$	$[1] = \beta_1^* - 2\beta_5^* - \beta_6^* - \beta_7^*$
	$\beta_5 = -\alpha_1 - 2\alpha_2 - 3\alpha_3 - 4\alpha_4 - 2\alpha_5 - 3\alpha_6 - 2\alpha_7$		$\alpha_2 = -\beta_3^* - \beta_4^* - \beta_6^* + \beta_7^*$	
	$\beta_6 = \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$		$\alpha_4 = -\beta_1^* - \beta_2^* - \beta_3^* + \beta_6^*$	
	$\beta_7 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$		$\alpha_7 = -\beta_2^* - \beta_5^* - \beta_6^* - \beta_7^*$	
E_6	$\beta_{1,2,3} = \alpha_{2,4,6}$	$m_i = 2,\ i < 5$	$\alpha_{2,4,6} = \beta_{1,2,3}$	$[1] = \beta_1^* + \beta_4^* - \beta_5^* + \beta_6^*$
	$\beta_4 = \alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6$		$\alpha_1 = -\beta_3^* - \beta_4^* - \beta_5^* + 3\beta_6^*$	$[2] = -2\beta_5^* + 2\beta_6^*$
	$\beta_5 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$	$m_5 = 4$	$\alpha_3 = -\beta_2^* - \beta_4^* - \beta_5^* - 3\beta_6^*$	
	$\beta_6 = 2\alpha_1 + \alpha_2 + 2\alpha_5 + \alpha_6$	$m_6 = 12$	$\alpha_5 = -\beta_1^* + \beta_4^* + \beta_5^* + 3\beta_6^*$	

the β_i. See Tables 2 and 3 for our choices of root and glue vectors. Incidently, these m_i are not quite the best that can be done (see Sec.3.1 for the best).

The root lattices are extremely important for two reasons. For one thing, they provide us with a rich supply of lattices to use for examples and counterexamples. For example, the root lattices and their duals solve many packing-type problems (see Chapter 1 in [**CS1**]). The other reason is Witt's Theorem, and the lattice construction method called *gluing*:

Theorem 1.5.3 *(Witt's Theorem)*: Suppose some Euclidean integral lattice Λ is generated by vectors of norm 1 and 2. Then Λ is equivalent to the direct sum of I_n and various root lattices.

This can be proven as follows. By Thms.2.1 and 5.2 it suffices to show that an indecomposable Euclidean integral lattice Λ generated by norm 2 vectors alone must be one of the root lattices. It is trivial to verify that the norm 2 vectors in Λ constitute a root system (see p.42 of [**HUM**]). In a number of books on Lie algebras — *e.g.* see pp.42-76 of [**HUM**] for a clear geometric presentation — these root systems are classified, directly showing Λ is one of the root lattices. In fact the proof in our case is significantly simplified because all (root) vectors here have equal norm (namely 2).

Thm.3 fails for indefinite lattices, even if we include as possible summands $I_{0,1}$ and the root lattices likewise scaled by -1.

Throughout this work we will be primarily interested in two ways of constructing new lattices from old ones: *gluing* and *shifting*. The interplay between these two related methods produces some useful results, as we shall see. Shifting will be introduced in the following chapter. The remainder of this section will be devoted to gluing theory.

Let Λ_0 be a saturated sublattice of Λ. This implies by Lemma 4.1 that $\sqrt{|\Lambda_0|/|\Lambda|} \in \mathbf{Z}$. For any $g \in \Lambda$, let $[g] \stackrel{\text{def}}{=} g + \Lambda_0 = \{g + x \mid \forall x \in \Lambda_0\}$ be a conjugacy class of vectors. These conjugacy classes form an additive group $G = \{[g]\}$.

Conversely, given a lattice Λ_0 and a set of vectors $g_1, \ldots, g_k$ in $\mathbf{Q} \otimes \Lambda_0$, we can form the conjugacy classes $[g] \in (\mathbf{Q} \otimes \Lambda_0)/\Lambda_0$ as before, for each $g \in \langle g_1, \ldots, g_k \rangle$. Then the union of the vectors in all these classes $[g]$ will form a new lattice Λ. The (finite) group $G = \{[g] \mid g \in \langle g_1, \ldots, g_k \rangle\}$ is called the *glue group*, the vectors g_i are called the *generators* of G, the vectors g are called the *glue vectors*, the classes $[g]$ are called *glue classes*, and the lattice Λ_0 is called the *base lattice*. The lattice Λ so formed will be denoted by $\Lambda_0[G]$ or $\Lambda_0[\{g_1, \cdots, g_k\}]$.

To emphasize its dependence on the base lattice Λ_0, the glue classes $[g]$ will also be denoted as $[g]\Lambda_0$. The direct sum of the classes $[g_1]\Lambda_1$, $[g_2]\Lambda_2$, *etc.*, will be denoted by the following notation: $[g_1, \ldots, g_k]\{\Lambda_1, \ldots, \Lambda_k\}$.

Compare the discussion given earlier for the 'glue classes' of the root lattices.

The least positive integer n such that $n[g] = [0] \stackrel{\text{def}}{=} \Lambda_0$ is called the *order* of $[g]$. By the fundamental theorem of abelian groups (see p.76 of [**HUN**]) the glue group $G = \Lambda/\Lambda_0$, being abelian and finite, is isomorphic to $C_{n_1} \times \cdots \times C_{n_k}$. These n_j can be taken to be the orders of the *independent generators* of G (with respect to Λ_0 see Def.4.2). Hence we can always assume the glue generators g_i are independent generators in the sense of Def.4.2; in general of course those generators (nor even their glue classes) are not unique.

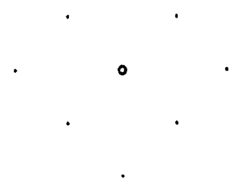
It is important not to confuse the symbol $\Lambda_0[g] = \Lambda_0[\{g\}]$ with $[g]\Lambda_0$. If the order of $[g]$ is n, then $\Lambda_0[g] = \bigcup_{i=1}^{n}[ig]\Lambda_0$.

Note that Λ_0 is a saturated sublattice of $\Lambda_0[G]$, for any glue group G.

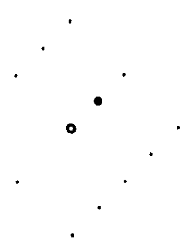
For example, $A_2 \approx (A_1, I_1^{(6)})[1,3]$, and for any n, m, $D_{m+n} = (D_m \oplus D_n)[2,2] = \{D_m, D_n\}[2,2]$ and $A_n[1] = A_n^*$. In glue class notations, these identities can be written as: $[0]A_2 \approx [0,0]\{A_1,(6)\} \cup [1,3]\{A_1,(6)\}$; $[0]D_{m+n} = [0,0]\{D_m, D_n\} \cup [2,2]\{D_m, D_n\}$; and $[0]A_n^* = \cup_{i=0}^{n}[i]A_n$. Note that the glue classes of any glue decomposition are pairwise disjoint. Further examples, the four gluings $D_2[0] \approx I_2^{(2)}$, $D_2[1] \approx D_2[3] \approx I_1^{(1/2)} \oplus I_1^{(2)}$ and $D_2[2] = I_2$ of D_2, are illustrated in Figure 3

Figure 3: The Gluings of D_2

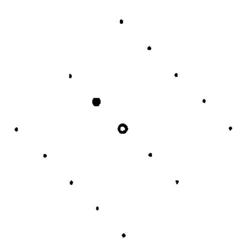
$D_2 = D_2[0] \approx I_2^{(1)}$



$D_2[1]$

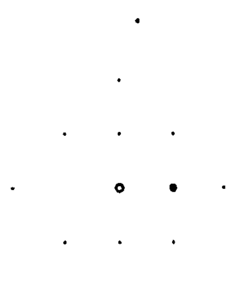


$\approx D_2[3]$



$\approx I_1^{(2)} \oplus I_1^{(\frac{1}{2})}$

$D_2[2] = I_2$



(where the origin is labelled "o" and the glue vectors are denoted by bullets).

Witt's Theorem tells us that whenever the norm 1 and 2 vectors in a Euclidean integral lattice Λ span a saturated sublattice of Λ, th n we can obtain Λ by gluing together root lattices and I_m. This is very important. It turns out that for smaller determinants and dimensions, this condition is often satisfied. For example, the first self-dual lattice not of this form is 19-dimensional. 26 of the 27 Type II lattices of dimension $n \leq 24$ can be obtained by gluing root lattices (the Leech lattice Λ_{24} is the sole exception).

Let g_i, for $i = 1, \ldots, k$, be independent glue generators of G with orders n_i. Then $\Lambda_0[G]$ is integral iff Λ_0 is integral, all g_i are in Λ_0^*, and each dot product $g_i \cdot g_j$ is an integer. Self-duality of the glued lattice $\Lambda_0[G]$ depends in addition on the size of G. In particular, eq.(4.1) immediately gives us

$$|\Lambda_0[G]| = |\Lambda_0|/\|G\|^2 = |\Lambda_0|/(\prod_{i=1}^{k} n_i)^2. \qquad (1.5.1)$$

Corollary 1.5.4: Let g_i be independent glue generators of G with orders n_i. Then $\Lambda_0[G]$ is self-dual iff Λ_0 is integral, the glue vectors g_i lie in Λ_0^* and have integral dot products with each other, and

$$|\Lambda_0| = (\prod_i n_i)^2.$$

A lattice Λ_0 is called ***self-dualizable*** if it has a self-dual gluing. Thus to be self-dualizable, for any prime p dividing $|\Lambda_0|$ there must exist a vector in Λ_0^* of order p and with integral norm. A lattice is self-dualizable iff it is a saturated sublattice of some self-dual lattice.

A self-dualizable lattice obviously must be integral. Moreover, Cor.4 tells us that its determinant must be a perfect (integral) square. Unfortunately, the converse fails: even if $|\Lambda_0|$ is a perfect square, no gluing of it may be self-dual. $\Lambda_0 = E_6 \oplus E_6$

is an example, as we shall see. Also, for any prime p, $\Lambda_0 = A_{p-1} \oplus A_{p-1}$ is self-dualizable iff either $p = 2$ or $p \equiv 1 \pmod 4$.

We can easily apply Cor.4 to the task of constructing new self-dual lattices. For example, consider the root lattices. The corollary tells us that for glues we must use their weight vectors. $D_n[2]$ is always self-dual, and in fact equals I_n. $D_n[1]$ and $D_n[3]$ are self-dual iff 4 divides n, in which case they are equivalent. We write D_{4k}^+ for $D_{4k}[1] \approx D_{4k}[3]$ — it is even/odd when k is even/odd. We also have $D_4^+ \approx I_4$ and $D_8^+ = E_8$.

The only other self-dualizable root lattice is A_{k^2-1}. Cor.4 again immediately implies that the lattice $A_{k^2-1}^+ \stackrel{\text{def}}{=} A_{k^2-1}[k]$ is self-dual. It is also even/odd when k is even/odd. Hence $A_3^+ \approx I_3$ and $A_8^+ \approx E_8$.

The following lattices are also all self-dual: $A_{3k^2-1}E_6[k1] \stackrel{\text{def}}{=} \{A_{3k^2-1}, E_6\}[k,1]$, $A_{2k^2-1}E_7[k1]$, and $A_{4k^2-1}D_{4\ell+1}[k1]$. Moreover, $A_2E_6[11] \approx A_1E_7[11] \approx A_3D_5[11] \approx E_8$ (these equivalences follow immediately because E_8 is the unique Type II lattice in 8 dimensions; Witt's Theorem also works, and a slightly more complicated argument from first principles based on Cor.4.2(i) can also establish them).

Finally, for any integral Euclidean lattice Λ, the indefinite lattice $\{\Lambda;\Lambda\} \stackrel{\text{def}}{=} \Lambda \oplus \Lambda^{(-1)}$ is self-dualizable (take as glues the vectors $(\alpha;\alpha)$, $\forall\alpha \in \Lambda^*$).

Gluing theory was used in Niemeier's classification of all 24 Type II Euclidean 24-dimensional lattices; the 23 Niemeier lattices (see Table 5; it was based on Table 16.1 of [CS1]) can be obtained by gluing various root lattices. The table includes those root lattices (given in the first column), as well as the glues (given in the second). There we use the short-hand [(122)], *e.g.* , to denote the three glues [122], [212], [221] obtained by cyclically permuting [122]. The total number of glue vectors, which equals the square root of the determinant of the root lattice, is given in the third column. The final column gives the so-called 'Coxeter number' h of the lattice: $24h$ is the number of root vectors (*i.e.* norm 2 vectors) in the lattice; and 196 560$-576h$ is the number of norm 4 vectors. In fact, two Niemeier lattices

Table 5: The 24-dimensional Type II (Niemeier and Leech) Lattices

Root Lattice	Glue Vector Generators g_i	Order of Glue Group	h
D_{24}	[1]	2	46
$D_{16}E_8$	[10]	2	30
E_8^3	[000]	1	30
A_{24}	[5]	5	25
D_{12}^2	[(12)]	4	22
$A_{17}E_7$	[31]	6	18
$D_{10}E_7^2$	[[110], [301]]	4	18
$A_{15}D_9$	[21]	8	16
D_8^3	[(122)]	8	14
A_{12}^2	[15]	13	13
$A_{11}D_7E_6$	[111]	12	12
E_6^4	[1(012)]	9	12
$A_9^2D_6$	[240], [501], [053]	20	10
D_6^4	[even perms of {0123}]	16	10
A_8^3	[(114)]	27	9
$A_7^2D_5^2$	[1112], [1721]	32	8
A_6^4	[1(216)]	49	7
$A_5^4D_4$	[2(024)0], [33001], [30302], [30033]	72	6
D_4^6	[111111], [0(02332)]	64	6
A_4^6	[1(01441)]	125	5
A_3^8	[3(2001011)]	256	4
A_2^{12}	[2(11211122212)]	729	3
A_1^{24}	[1(00000101001100110101111)]	4096	2
Λ_{24}	—	—	0

with equal Coxeter numbers have the same numbers of norm k vectors for each k, *i.e.* they have equal theta constants (see eq.(4.2.10b)). The proof that all these lattices are self-dual is straightforward from Cor.4. The proof that these are all distinct follows by applying Thms.2.3, 5.1, 5.2 and 5.3 to the sublattice in each generated by the norm 2 vectors (it is called the *root lattice* of the given Niemeier lattice). The proof that the Niemeier lattices and the Leech lattice exhaust all Type II Euclidean lattices of dimension 24 now follows from the Minkowski-Siegal mass formula.

The reverse process to gluing, called 'ungluing', will be considered in Sec.6.3.

1.6 Gluing Theory Continued

This section contains a number of basic results meant first of all to illustrate some of the results already presented, and, secondly, to prepare for some further developments in the following chapters.

The following simple but useful result is true. Though it probably can be proved directly, it is most naturally and eloquently proved using the shifting construction, and as such is an example of the consequences resulting from the inter-relationship between gluing and shifting to be discussed in the next chapter (it is proved after the statement of Thm.2.4.5 in the next chapter).

Theorem 1.6.1: Let Λ be self-dualizable. Let $G' \subset \Lambda^*$ be any set of vectors with pairwise integral dot products. Then there exists a self-dual gluing of Λ containing G'.

In Sec.4 we defined and discussed some properties of the orthogonal complement $\Lambda_0^\perp$ of a sublattice Λ_0 in a lattice Λ. We can now go a little further.

Theorem 1.6.2: Let Λ_0 be a k-dimensional non-singular sublattice of an n-dimensional integral lattice Λ. Then $\Lambda_0^\perp$ is a sublattice of Λ with a determinant

$|\Lambda_0^\perp|$ which divides $|\Lambda_0|\,|\Lambda|$ (in fact, $|\Lambda_0|\,|\Lambda|/|\Lambda_0^\perp|$ must be a perfect integer square). If instead Λ is a rational lattice, the quotient $|\Lambda_0|\,|\Lambda|/|\Lambda_0^\perp|$ must be a perfect *rational* square. Moreover, if Λ is self-dualizable, then Λ_0 is self-dualizable iff $\Lambda_0^\perp$ is.

Proof It suffices (by using a scaling argument) to consider the case when Λ is integral.

Note that $\Lambda_0 \oplus \Lambda_0^\perp$ glues to Λ. Let $g_i = g_i' + g_i''$ be the independent glues, with orders n_i, where $g_i' \in (\Lambda_0)^*, g_i'' \in (\Lambda_0^\perp)^*$. Then $\prod n_i = \sqrt{|\Lambda_0|\,|\Lambda_0^\perp|/|\Lambda|}$.

Because $\Lambda_0^\perp$ is the largest sublattice orthogonal to Λ_0, $\sum k_i g_i' \in \Lambda_0$ iff $\sum k_i g_i \in \Lambda_0 \oplus \Lambda_0^\perp$ iff $n_i | k_i$. Since the g_i''s span a subgroup of Λ_0^*/Λ_0, $\prod n_i$ must divide $|\Lambda_0|$, and hence $|\Lambda_0^\perp|$ divides $|\Lambda_0|\,|\Lambda|$.

The fact that $|\Lambda_0|\,|\Lambda|/|\Lambda_0^\perp|$ must be a perfect square now follows from the observation that $|\Lambda_0|\,|\Lambda_0^\perp|/|\Lambda| = \prod n_i^2$ is a perfect square.

Finally, suppose Λ is self-dualizable. Then so is $\Lambda_0 \oplus \Lambda_0^\perp$. Thm.1 now gives us the final statement of this theorem. QED

From the previous section, we know Λ can be obtained from gluing any of its orthogonal decompositions, or from gluing any LR-decomposition. This result allows us to investigate in a little more detail some properties of the orthogonal and LR-decompositions. In particular:

Theorem 1.6.3: Let $\{(k_1), \ldots, (k_m); (\ell_1), \ldots, (\ell_n)\}$ be any orthogonal decomposition of some lattice Λ. Then $(\prod k_i)(\prod \ell_j)/|\Lambda|$ must be the square of an integer. Moreover, if Λ is integral and some prime p divides k_1, say, but does divides no other k_i, ℓ_j, nor $|\Lambda|$, then p^2 divides k_1 and we may take $k_1' = k_1/p^2$ instead of m_1.

Proof The first statement follows immediately from Lemma 4.1.

Let $x_1, \ldots, x_{n+m}$ be the orthogonal vectors in Λ defining the given orthogonal decomposition. Now consider the sublattice Λ_0 of Λ defined to be $\langle x_2, \ldots, x_{n+m}\rangle^\perp$ (where the $^\perp$ is taken relative to Λ). It is 1-dimensional, say spanned by $x_1' \in \Lambda$, and contains $\langle x_1 \rangle$ (so $x_1 = Nx'_1$ for $N \in \mathbf{Z}$). Its determinant $|\Lambda_0|$ (by Thm.2 above) divides $|\Lambda| \cdot (\prod_{i=2}^n k_i) \cdot (\prod_{j=1}^m |\ell_j|)$. Hence p does not divide $|\Lambda_0| = x_1'^2 = k_1/N^2$, so it must divide N. $\langle (N/p)\, x'_1, x_2, \ldots, x_{n+m} \rangle$ is an orthogonal sublattice of Λ with the desired properties. QED

Call a LR-decomposition $\{\Lambda_L; \Lambda_R\}$ of Λ *maximal* if any other LR-decomposition $\{\Lambda'_L; \Lambda'_R\}$ of Λ satisfying $\Lambda_L \subseteq \Lambda'_L$ and $\Lambda_R \subseteq \Lambda'_R$ satisfies $\Lambda_L = \Lambda'_L$ and $\Lambda_R = \Lambda'_R$. Any LR-decomposition is clearly contained within a maximal one.

Theorem 1.6.4: Let $\{\Lambda_L; \Lambda_R\}$ be any maximal LR-decomposition of Λ. Then $\{\Lambda_L; 0\}^\perp = \{0; \Lambda_R\}$ and $\{0; \Lambda_R\}^\perp = \{\Lambda_L; 0\}$, using obvious notation. Moreover, if Λ is self-dual,

$$\Lambda_L^* = \pi_L(\Lambda) \stackrel{\text{def}}{=} \{x \in V(\Lambda_L) \mid \exists y \in V(\Lambda_R) \text{ such that } (x; y) \in \Lambda\},$$

$$\Lambda_R^* = \pi_R(\Lambda),$$

$|\Lambda_L| = |\Lambda_R|$ and in fact the dual groups Λ_L^*/Λ_L and Λ_R^*/Λ_R are isomorphic.

Proof The first statement is clear from the definition of maximality. Now assume Λ is self-dual.

Because $\{\Lambda_L; \Lambda_R\}$ is saturated in Λ, the glue group G defined by $\{\Lambda_L; \Lambda_R\}[G] = \Lambda$ is finite. Let $[g_k]\{\Lambda_L; \Lambda_R\} = (g_{kL}; g_{kR}) + \{\Lambda_L; \Lambda_R\}$, $k = 1, \ldots, N$, be the elements of G, using obvious notation. Then eq.(5.1) implies $\prod_{k=1}^N n_k^2 = |\Lambda_L|\,|\Lambda_R|$. Also, Λ integral implies each $g_{kL} \in \Lambda_L^*$. Hence $\Lambda_L[\{[g_{1L}], \ldots, [g_{NL}]\}] = \{\pi_L(v) \mid v \in \Lambda\}$ is a saturated sublattice of Λ_L^*.

Suppose $\ell_k \in \mathbf{Z}$ are found satisfying $\sum \ell_k g_{kL} \in \Lambda_L$. Then $\sum \ell_k (0; g_{kR}) = \sum \ell_k g_k - \sum \ell_k (g_{kL}; 0) \in \Lambda_L^\perp$, so the maximality of $\{\Lambda_L; \Lambda_R\}$ means $\sum \ell_k g_{kR} \in \Lambda_R$, *i.e.* $\sum \ell_k g_k \in \{\Lambda_L; \Lambda_R\}$, which implies n_k divides ℓ_k for each k.

Thus the maximality of $\{\Lambda_L; \Lambda_R\}$ has implied that the glues g_{kL} are independent of order n_k in Λ_L. Now $\pi_L(\Lambda) = \Lambda_L[\{g_{1L}, \ldots, g_{NL}\}] \subseteq \Lambda_L^*$, so

$$\frac{1}{|\Lambda_L|} \leq |\Lambda_L[\{g_{kL}\}]| = \frac{|\Lambda_L|}{\prod n_k^2} = \frac{1}{\Lambda_R}. \qquad (*)$$

Therefore $|\Lambda_R| \leq |\Lambda_L|$, so reversing the roles of L and R in the above argument gives us $|\Lambda_R| = |\Lambda_L|$. Eq.(*) now implies $1/|\Lambda_L| = |\Lambda_L[\{g_{kL}\}]|$, so $\Lambda_L[\{g_{kL}\}] = \Lambda_L^*$ by Cor.4.2.

Hence for any $x \in \Lambda_R^*$ there exists a vector $q \in \Lambda$ with $x = q_R \stackrel{\text{def}}{=} \pi_R(q)$. Let $q' \in \Lambda$ be any other vector with $q'_R \in [x]\Lambda_R$. Then by the maximality of $\{\Lambda_L; \Lambda_R\}$ we know that $q_L - q'_L \in \Lambda_L$. Thus the mapping from Λ_R^*/Λ_R to Λ_L^*/Λ_L defined by $[q_R]\Lambda_R \rightarrow [q_L]\Lambda_L$ is well-defined; it is straightforward to verify that it is in fact a group isomorphism. QED

The following theorem tells of one way to relate two lattices, provided their intersection is saturated.

Theorem 1.6.5: If $\Lambda_{12} \stackrel{\text{def}}{=} \Lambda_1 \cap \Lambda_2$ is saturated in both Λ_1 and Λ_2, then:

(i) $\Lambda_1 = \Lambda_{12}[G_1]$ and $\Lambda_2 = \Lambda_{12}[G_2]$, where the glue groups are defined by $G_i = \Lambda_i/\Lambda_{12}$;

(ii) if in addition Λ_1 and Λ_2 are self-dual, the groups G_1 and G_2 will be isomorphic.

Of course, Thm.5(i) follows easily from the discussion in the previous section. Thm.5(ii) follows from Thm.4.9 by letting y_i be the independent generators of G_2 (this is always possible) and letting Λ be Λ_1. Then the r_i will be independent generators of G_1, and the mapping $y_i \leftrightarrow r_i$ will induce an isomorphism between G_1 and G_2.

Theorem 1.6.6: Suppose Λ is integral and let n_1, n_2 be the orders of the glues $[g_1], [g_2] \in \Lambda^*/\Lambda$. Then $D\, g_1 \cdot g_2 \in \mathbf{Z}$, where $D = (n_1, n_2)$ is the greatest common divisor of n_1 and n_2. In particular, if n_1 and n_2 are relatively prime, $g_1 \cdot g_2 \in \mathbf{Z}$.

This follows because the vectors $n_1\, g_1$ and $n_2\, g_2$ are in Λ, so $n_1\, g_1 \cdot g_2 \in \mathbf{Z}$ and $n_2\, g_1 \cdot g_2 \in \mathbf{Z}$.

We see from Cor.5.4 that the determinant of a self-dualizable lattice must be a perfect square, and yet we gave examples in the last section of integral lattices with determinants that were perfect squares and which were not self-dualizable. We will address the question of self-dualizability much more completely later, but for now consider the following special cases.

Theorem 1.6.7: Suppose Λ is integral. Then Λ is self-dualizable if *either* of these conditions hold:

(i) Λ^*/Λ is isomorphic to $C_{n_1^2} \times \cdots C_{n_k^2}$, for certain integers n_i;

(ii) $|\Lambda|$ is a power of 4.

As usual C_m denotes the cyclic group of order m. (i) follows by letting $g_1, \ldots, g_k$ be independent generators of Λ^*/Λ of order $n_1^2, \ldots, n_k^2$, respectively. Then for each i and j, $n_i^2 g_i \cdot g_j \in \mathbf{Z}$. Without loss of generality we may suppose each n_i is a power of a prime p_i: $n_i = p_i^{\ell_i}$ (by using the fundamental theorem of abelian groups). Consider the gluing $\Lambda[\{n_1 g_1, \ldots, n_k g_k\}]$. Then each dot product $(n_i g_i) \cdot (n_j g_j)$ is integral, by Thm.6. Cor.5.4 now tells us this glued lattice is self-dual, which gives us Thm.7.

(ii) is proven similarly: let Λ^*/Λ be isomorphic to $C_{n_1} \times \cdots \times C_{n_k}$, where $n_i = 2^{\ell_i}$. By choosing glues as in the proof for (i) given above, it suffices to consider the case when all ℓ_i's are 1, *i.e.* when all $n_i = 2$. Then k must be even. Let $g_1, \ldots, g_k$ be independent generators of Λ^*/Λ, each of order 2. We may also assume their norms are all non-integral, and hence are $\frac{1}{2}$ times odd integers (otherwise use the g_i with integral norms g_i^2 as glues). By Thm.6, then, $g_1 + g_2$ has integral norm and so can be used as a glue. This amounts to reducing k by 2. Proceeding inductively (*i.e.* applying the same argument to $\Lambda[g_1 + g_2]^*/\Lambda[g_1 + g_2]$) establishes (ii).

Theorem 1.6.8: $\forall n \in \mathbf{Z}$, $n \neq 0$, $\{(n)^4\} \stackrel{\text{def}}{=} \{(n), (n), (n), (n)\}$ is self-dualizable.

Proof Without loss of generality assume n is positive (for $n < 0$, the following proof will show $\{(-n)^4\}$ is self-dualizable, and hence so is $\{(-n)^4\}^{(-1)} = \{(n)^4\}$).

From Lagrange's Theorem (p.47 of [**SER**]) we see that there exist integers i, j, k, ℓ such that $i^2 + j^2 + k^2 + \ell^2 = n$. Define the following glue vectors of $\{(n)^4\}$:

$$g_1 = [i, j, k, \ell],\ \ g_2 = [-j, i, -\ell, k],\ \ g_3 = [k, -\ell, -i, j] \text{ and } g_4 = [\ell, k, -j, -i]$$

Then it is trivial to verify that $g_i \cdot g_j = \delta_{ij}$ — in other words, we have that the gluing $\{(n)^4\}[\{g_1, g_2, g_3, g_4\}] \approx I_4$. QED

Theorem 1.6.9: Let Λ be any integral lattice. Then $\Lambda^4 \stackrel{\text{def}}{=} \{\Lambda, \Lambda, \Lambda, \Lambda\} \stackrel{\text{def}}{=} \Lambda \oplus \Lambda \oplus \Lambda \oplus \Lambda$ and $\{\Lambda; \Lambda\} \stackrel{\text{def}}{=} \Lambda \oplus \Lambda^{(-1)}$ are self-dualizable.

Proof Let Λ_0 be any orthogonal decomposition of Λ; Thm.8 tells us that Λ_0^4 is self-dualizable. Thm.1 now finishes the proof. QED

Theorem 1.6.10: Let Λ be a two-dimensional non-singular lattice. Then:

(i) if Λ is rational there exist nonzero integers m, n, ℓ, j such that $|\Lambda| = mn$, the greatest common divisor $(j, \ell) = 1$, and

$$\Lambda \approx \{(\ell m), (\ell n)\}[m, jn];$$

(ii) if Λ has a glue decomposition as given in (i), then $\Lambda^* \approx \{(\ell/m), (\ell/n)\}[-j/m, 1/n]$ (here m, n can be irrational and this result would still hold);

(iii) $(\Lambda^*)^{(\pm|\Lambda|)} \approx \Lambda$, where the '+' sign is chosen when Λ is definite and the '−' sign when Λ is indefinite;

(iv) $\mu(\Lambda^*) = \mu(\Lambda)/|\Lambda|$; and

(v) Suppose Λ is rational, and let Λ' be any other 2-dimensional lattice with a determinant $|\Lambda'|$ such that $|\Lambda|/|\Lambda'|$ is a perfect rational square. Then $\Lambda \stackrel{Q}{\approx} \Lambda'$ iff they have equal signature, Λ' is also rational, and there exists some nonzero vectors $x \in \Lambda$, $y \in \Lambda'$, with $x^2 = y^2$.

Proof (i) follows quickly from orthogonal decomposition.

Let x and y be the orthogonal vectors in Λ which define the given orthogonal decomposition (so $x^2 = \ell m$, $x \cdot y = 0$, and $y^2 = \ell n$). Then a glue vector g in the glue class $[m, \jmath n]\{(m),(n)\}$ is $g = x/\ell + \jmath\, y/\ell$. It is trivial to verify that x/m, y/n, and $g' = -\jmath\, x/(\ell m) + y/(\ell n)$ all have integral dot products with x, y, and g (*e.g.* $g' \cdot g = -\jmath/\ell + \jmath/\ell = 0$), so $\langle x, y\rangle[g'] \subseteq \Lambda^*$. But g' has order ℓ in $\langle x/m, y/n\rangle$, so $|\langle x/m, y/n\rangle[g']| = (\ell/m)(\ell/n)/(\ell^2) = 1/(mn) = \Lambda^*$. Cor.4.2 gives us (ii).

The most general way to see (iii) is to look at Gram matrices. Let A be the Gram matrix of Λ corresponding to a basis $\{b_1, b_2\}$. Without loss of generality suppose Λ is definite (so $|A| > 0$). Then we know A^{-1} is the Gram matrix for Λ^* corresponding to the dual basis $\{b_1^*, b_2^*\}$. Let $b_1' = \sqrt{|\Lambda|}b_2^*$ and $b_2' = -\sqrt{|\Lambda|}b_1^*$ — they form a basis for $(\Lambda^*)^{(|\Lambda|)}$ which, it is easy to verify, corresponds to a Gram matrix equal to A. Thus by Thm.1.4 we get (iii).

(iv) follows immediately from (iii).

Now for (v). It is clear that, since Λ is rational, $\Lambda \overset{Q}{\approx} \Lambda'$ can only occur when Λ' is also rational and has signature equal to that of Λ.

Suppose first that there exist vectors $x \in \Lambda$, $y \in \Lambda'$, with norms $x^2 = y^2 \neq 0$. Then $\{x, x^\perp\}$ and $\{y, y^\perp\}$ define orthogonal decompositions of Λ and Λ' respectively. By Thm.2 it is trivial to verify that these orthogonal sublattices are rationally equivalent, and so then are Λ and Λ'.

Next, if there exist nonzero vectors $x \in \Lambda$, $y \in \Lambda'$, with norms $x^2 = y^2 = 0$, then it can be shown in a number of ways that both Λ and Λ' are self-dualizable, and hence must glue to either $I_{1,1}$ or $II_{1,1}$. Since these two self-dual lattices are easily seen to be rationally equivalent, so must Λ and Λ' in this case. This concludes the proof of the '$\Leftarrow$' half of (v).

The other direction of (v) follows immediately from Cor.4.11. QED

Chapter 2 TENSOR PRODUCTS AND SHIFTING

2.1 The Tensor Product: Basic Properties

This chapter will address two means of constructing lattices from other ones. In this and the following section, we will discuss tensor products. The final two sections introduce and develop the *shifting construction*.

Let V and V' be two real inner product spaces. By the *tensor product* $V \otimes V'$ we mean the real vector space consisting of all points of the form $\sum_i (x_i, x'_i)$, for $x_i \in V$ and $x'_i \in V'$. We write $x_i \otimes x'_i$ for the ordered pair (x_i, x'_i). It is also required that $\otimes$ be a bilinear product; *i.e.* $(x + ry) \otimes x' = x \otimes x' + ry \otimes x'$ and $x \otimes (x' + sy') = x \otimes x' + sx \otimes y'$, for all $x, y \in V$, $x', y' \in V'$, and $r, s \in \mathbf{R}$. $V \otimes V'$ is made into an inner product space by defining $(x \otimes x') \cdot (y \otimes y') = (x \cdot y)(x' \cdot y')$, for all $x, y \in V$, and $x', y' \in V'$.

More formally, the tensor product can be defined category theoretically as a certain universal object $\otimes_R$ (see *e.g.* p.209 of [**HUN**] for details), but this level of abstraction is unnecessary here.

It is easy to show that $V \otimes V'$ has dimension nn', where n and n' are the dimensions of V and V' respectively. Also, $0 \otimes x' = x \otimes 0 = 0$ is the zero vector in the space.

For reasons soon to be made clear, define $A \otimes A'$, for any $m \times n$ and $m' \times n'$ $\mathbf{R}$-matrices A and A', to be the $mm' \times nn'$ $\mathbf{R}$-matrix whose $(i' + m'i)(j' + n'j)$th entry, for $i = 1, \ldots, m$, $j = 1, \ldots, n$, $i' = 1, \ldots, m'$ and $j' = 1, \ldots, n'$, is $A_{ij} A'_{i'j'}$. For example,

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad A' = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \text{ and } A \otimes A' = \begin{pmatrix} 5 & 6 & 10 & 12 \\ 7 & 8 & 14 & 16 \\ 15 & 18 & 20 & 24 \\ 21 & 24 & 28 & 32 \end{pmatrix}.$$

Definition 2.1.1: Let Λ and Λ' be two lattices with background spaces $V =$

$V(\Lambda)$ and $V' = V(\Lambda')$. The *tensor product* $\Lambda \otimes \Lambda'$ of Λ and Λ' is defined to be $\langle\{x \otimes x' \mid x \in \Lambda,\ x' \in \Lambda'\}\rangle \subseteq V \otimes V'$.

The following results are immediate.

Theorem 2.1.1: Let Λ and Λ' be any lattices. Then:

(i) if $\beta = \{b_1, \ldots, b_n\}$ and $\beta' = \{b'_1, \ldots, b'_{n'}\}$ are any bases for Λ and Λ' respectively, then $\beta \otimes \beta' \overset{\text{def}}{=} \{b_1 \otimes b'_1, b_1 \otimes b'_2, \ldots, b_n \otimes b'_{n'-1}, b_n \otimes b'_{n'}\}$ is a basis for $\Lambda \otimes \Lambda'$;

(ii) if A and A' are the Gram matrices corresponding to β and β', then $A \otimes A'$ is the Gram matrix for $\Lambda \otimes \Lambda'$ corresponding to $\beta \otimes \beta'$;

(iii) $V(\Lambda \otimes \Lambda') = V(\Lambda) \otimes V(\Lambda')$ and $V_0(\Lambda \otimes \Lambda') = V_0(\Lambda) \otimes V_0(\Lambda')$; and

(iv) $\Lambda \otimes \Lambda'$ is non-singular iff both Λ and Λ' are non-singular.

For example, Thm.1(iv) follows from Thm.1(ii) which follows from Thm.1(i). As before, we will assume for the remainder of this chapter that all lattices considered are non-singular.

The following theorem follows quickly from Thm.1 and the above definitions.

Theorem 2.1.2: Let $\Lambda_1, \ldots, \Lambda_4$ be any lattices of dimensions $n_1, \ldots, n_4$ respectively. Then:

(i) $\Lambda_1 \otimes \Lambda_2$ is of dimension $n_1 n_2$ and signature $(n_{1+}n_{2+} + n_{1-}n_{2-}, n_{1+}n_{2-} + n_{1+}n_{2-})$, where Λ_1 and Λ_2 are of signature (n_{1+}, n_{1-}) and (n_{2+}, n_{2-}) respectively;

(ii) $|\Lambda_1 \otimes \Lambda_2| = |\Lambda_1|^{n_2} |\Lambda_2|^{n_1}$;

(iii) $\Lambda_1 \otimes I_n \approx \Lambda_1^n \overset{\text{def}}{=} \Lambda_1 \oplus \cdots \oplus \Lambda_1$;

(iv) $\Lambda_1 \otimes (\Lambda_2 \oplus \Lambda_3) \approx (\Lambda_1 \otimes \Lambda_2) \oplus (\Lambda_1 \otimes \Lambda_3)$;

(v) $\Lambda_1 \otimes I_1^{(\ell)} \approx \Lambda_1^{(\ell)}$, for any $\ell \in \mathbf{R}$;

(vi) $(\Lambda_1 \otimes \Lambda_2)^{(\ell)} \approx \Lambda_1^{(\ell)} \otimes \Lambda_2 \approx \Lambda_1 \otimes \Lambda_2^{(\ell)}$, for any $\ell \in \mathbf{R}$;

(vii) $\Lambda_1 \approx \Lambda_3$ and $\Lambda_2 \approx \Lambda_4$ implies $\Lambda_1 \otimes \Lambda_2 \approx \Lambda_3 \otimes \Lambda_4$ (similarly for $\Lambda_1 \overset{Q}{\approx} \Lambda_3$, *etc.*);

(viii) $(\Lambda_1 \otimes \Lambda_2) \otimes \Lambda_3 \approx \Lambda_1 \otimes (\Lambda_2 \otimes \Lambda_3)$ and $\Lambda_1 \otimes \Lambda_2 \approx \Lambda_2 \otimes \Lambda_1$;

(ix) $\Lambda_1 \otimes \Lambda_Z \approx \Lambda_Z$, for the zero lattice Λ_Z; and

(x) $(\Lambda_1 \otimes \Lambda_2)^* = \Lambda_1^* \otimes \Lambda_2^*$.

Proof The dimension of $\Lambda_1 \otimes \Lambda_2$ follows from Thm.1(i) Its signature can be seen by diagonalizing $A_1 \otimes A_2$: if $U_1^t A_1 U_1$ =diag$\{+1,\ldots,+1,-1,\ldots,-1\}$ and $U_1^t A_1 U_1$ =diag$\{+1,\ldots,+1,-1,\ldots,-1\}$, then $U_1 \otimes U_2$ works for $A_1 \otimes A_2$. To obtain the formula in (ii), it suffices by Thm.1(ii) to evaluate the determinant $|A \otimes A'|$ This is a straightforward exercise.

The obvious choices of integral equivalences work in (iv), (v), (vi), (vii), (viii) and (ix); (iii) follows immediately from (iv) and (v) (with $\ell = 1$)

To verify (x), first note that for any $x \in \Lambda_1$, $y \in \Lambda_2$, $x' \in \Lambda_1^*$ and $y' \in \Lambda_2^*$, $(x \otimes y) \cdot (x' \cdot y') = (x \cdot x')(y \cdot y') \in \mathbf{Z}$, so $x' \otimes y' \in (\Lambda_1 \otimes \Lambda_2)^*$ and hence, $\Lambda_1^* \otimes \Lambda_2^* \subseteq (\Lambda_1 \otimes \Lambda_2)^*$. The conclusion now follows from Cor.1.4.2(i) by computing determinants. QED

Corollary 2.1.3: Let Λ_1 and Λ_2 be any lattices. Then:

(i) if Λ_1 and Λ_2 are Euclidean (resp. indefinite), so is $\Lambda_1 \otimes \Lambda_2$;

(ii) if Λ_1 and Λ_2 are integral (resp. rational), so is $\Lambda_1 \otimes \Lambda_2$;

(iii) if Λ_1 and Λ_2 are integral, $\Lambda_1 \otimes \Lambda_2$ is even iff Λ_1 and/or Λ_2 is even; and

(iv) if Λ_1 and Λ_2 are self-dual (resp. self-dualizable), so is $\Lambda_1 \otimes \Lambda_2$.

Cor.3(i) follows immediately from Thm.2(i); Cor.3(ii) and 3(iii) follow from Thm.1(ii); and Cor.3(iv) follows from Thm.1.3.1, Cor.3(ii), and Thm.2(ii)

Note that by Thm.2(vi), $\Lambda_1 \otimes \Lambda_2 \approx \Lambda_1^{(\ell)} \otimes \Lambda_2^{(1/\ell)}$ for any nonzero $\ell \in \mathbf{R}$, so the converses of Cor.3 cannot be expected to hold. However, we can say the following:

Theorem 2.1.4: Let Λ_1 and Λ_2 be any lattices. Then:

(i) if $\Lambda_1 \otimes \Lambda_2$ is integral (resp. rational), there exists a positive $\ell \in \mathbf{R}$ such that $\Lambda_1^{(\ell)}$ and $\Lambda_2^{(1/\ell)}$ are both integral (resp. rational); and

(ii) if $\Lambda_1 \otimes \Lambda_2$ is self-dual, there exists a positive $\ell \in \mathbf{R}$ such that $\Lambda_1^{(\ell)}$ and $\Lambda_2^{(1/\ell)}$ are both self-dual

Proof (i) Choose some $y \in \Lambda_2$ with nonzero norm L. Then for any $x, x' \in \Lambda_1$, $Lx \cdot x' = (x \otimes y) \cdot (x' \otimes y) \in \mathbf{Z}$. Therefore $\Lambda_1^{(L)}$ is an integral lattice.

Let ℓ be the smallest positive real number such that $\Lambda_1^{(\ell)}$ is integral. Then $\langle\{\ell x \cdot x' \mid x, x' \in \Lambda_1\}\rangle = \mathbf{Z}$ (because $\mathbf{Z}$ is a principle ideal domain). Choose $m_i \in \mathbf{Z}$ and $x_i, x_i' \in \Lambda_1$ so that $\sum \ell m_i x_i \cdot x_i' = 1$.

Choose any $y', y'' \in \Lambda_2$. Then $y' \cdot y''/\ell = (\sum m_i x_i \cdot x_i') y' \cdot y'' = \sum m_i (x_i \otimes y') \cdot (x_i' \otimes y'') \in \mathbf{Z}$. Thus $\Lambda_2^{(1/\ell)}$ is an integral lattice.

To prove (ii), let ℓ be as in (i). Then $\Lambda_1^{(\ell)}$ and $\Lambda_2^{(1/\ell)}$ are integral and therefore have integral determinants. Hence $1 = |\Lambda_1 \otimes \Lambda_2| = |\Lambda_1^{(\ell)} \otimes \Lambda_2^{(1/\ell)}| = |\Lambda_1^{(\ell)}|^{n_2} |\Lambda_2^{(1/\ell)}|^{n_1}$ implies that $|\Lambda_1^{(\ell)}|$ and $|\Lambda_2^{(1/\ell)}|$ both equal 1. Thm.1.3.1 tells us they are self-dual. QED

However, $\Lambda_1 \otimes \Lambda_2$ self-dualizable does not necessarily imply that there exists an $\ell \in \mathbf{R}$ such that $\Lambda_1^{(\ell)}$ and $\Lambda_2^{(1/\ell)}$ are both self-dualizable. Indeed, by Thm.2(iii) and Thm.1.6.9, the lattice $\Lambda \otimes I_4$ is self-dualizable for *any* integral lattice Λ.

Thm.2(iv) tells us that Λ_1 decomposable (with respect to direct sum) implies $\Lambda_1 \otimes \Lambda_2$ is decomposable. Does the converse hold? Also, Thms.2(v) and (vi) tells us that decompositions with respect to $\otimes$ are never unique.

Not much work has been done on tensor products. There are at least two reasons for this. First of all, it is not a very practical means of constructing new lattices. For example, we see from Thm.3(iv) that the tensor products of self-dual lattices are always self-dual. However, the first nontrivial self-dual lattice constructed in this manner is $E_8 \otimes E_8$ which has dimension 64. The second reason for the relative lack of results, is the complexity of the theory.

For the last couple decades Yoshiyuki Kitaoka has published a number of papers in which he addresses questions concerning tensor products. We will investigate one of them ([**KIT1**]) in some detail in the following section.

Three questions he has analysed in considerable detail are:

(i) if Λ_1 and Λ_2 are indecomposable (with respect to $\oplus$), then when is $\Lambda_1 \otimes \Lambda_2$ also indecomposable?;

(ii) when does $\Lambda_1 \otimes \Lambda_2 \approx \Lambda_1 \otimes \Lambda_3$ imply $\Lambda_2 \approx \Lambda_3$?; and

(iii) when is a decomposition with respect to $\otimes$ unique? *i.e.* when does

$$\Lambda_1 \otimes \Lambda_2 \cdots \otimes \Lambda_n \approx \Lambda'_1 \otimes \Lambda'_2 \cdots \otimes \Lambda'_m$$

imply $m = n$ and $\Lambda_i \approx \Lambda'_{\sigma i}$, for each i and some permutation σ.

(See for example [**KIT2**], [**KIT3**] and [**KIT4**] for details.) One interesting conclusion he reached in [**KIT4**] was that when the Λ_i in (iii) are all root lattices other than A_1 and when the Λ'_j there are all of dimension ≥ 2 and are indecomposable with respect to $\otimes$, we get that (iii) holds (up to the trivial scaling factors of Thm.2(vi) of course). (Clearly, 1-dimensional lattices such as A_1 can and must be excluded, by Thm.2(v).) In other words, for lattices Λ which can be obtained by tensoring together root lattices, their decomposition with respect to $\otimes$ is as unique as is conceivably possible.

2.2 The Minimum Norm of the Tensor Product

In this section we will restrict our attention to Euclidean lattices.

Let Λ_1 and Λ_2 be Euclidean. Then $\Lambda_1 \otimes \Lambda_2$ is also Euclidean, so the minimal norms $\mu(\Lambda_1)$, $\mu(\Lambda_2)$ and $\mu(\Lambda_1 \otimes \Lambda_2)$ are all positive and are actually attained. Choose $x \in \Lambda_1$ and $y \in \Lambda_2$ satisfying $x^2 = \mu(\Lambda_1)$ and $y^2 = \mu(\Lambda_2)$. Then $x \otimes y \in \Lambda_1 \otimes \Lambda_2$ and $(x \otimes y)^2 = \mu(\Lambda_1)\mu(\Lambda_2)$. Therefore we may write:

$$\mu(\Lambda_1 \otimes \Lambda_2) \leq \mu(\Lambda_1)\mu(\Lambda_2). \qquad (2.2.1)$$

Of course it is not *a priori* necessary that equality holds in eq.(1); the minimal vector in $\Lambda_1 \otimes \Lambda_2$ may look like $\sum x_i \otimes y_i$ for certain $x_i \in \Lambda_1$ and $y_i \in \Lambda_2$. In fact, R. Steinberg has shown (see pp.47-8 of [**MH**]) that in every sufficiently large dimension there exist self-dual lattices Λ_1 and Λ_2 such that $\mu(\Lambda_1 \otimes \Lambda_2) < \mu(\Lambda_1)\mu(\Lambda_2)$. His argument is as follows.

Consider any self-dual lattice Λ of dimension n. Let $\beta = \{b_1, \ldots, b_n\}$ be any basis for Λ and let $\beta^* = \{b_1^*, \ldots, b_n^*\}$ be the dual basis. Then $b \overset{\text{def}}{=} b_1 \otimes b_1^* + \cdots + b_n \otimes b_n^*$ lies in $\Lambda \otimes \Lambda$. Note that

$$b^2 = \sum_{i,j \leq n} (b_i \cdot b_j)(b_i^* \cdot b_j^*) = \sum_{i,j \leq n} A_{ij} A_{ij}^* = \mathrm{Tr}(AA^{-1}) = n,$$

where A and $A^* = A^{-1}$ are the Gram matrices corresponding to β and β^*. Thus $\mu(\Lambda \otimes \Lambda) \leq n$.

From eq (1.3 3a) we know that for any n there exist self-dual (Euclidean) lattices Λ of dimension n with minimal norm $\mu(\Lambda)$ at least as large as $k(n)$, the closest integer to $(\frac{5}{3}\omega_n^{-1})^{2/n}$. It is possible to show that $k(n) > \sqrt{n}$ for $n \geq 292$. Hence for such n, there exist n-dimensional self-dual lattices Λ satisfying $\mu(\Lambda \otimes \Lambda) < \mu(\Lambda)\mu(\Lambda)$. For these Λ, equality does not hold in eq.(1).

Nevertheless it seems to be very difficult to explicitly find lattices Λ_1 and Λ_2 which produce an inequality in eq.(1).

We shall let $\mathcal{TE}$ denote the set of all Euclidean lattices Λ_1 with the following property: if Λ_2 is any other Euclidean lattice, then $\mu(\Lambda_1 \otimes \Lambda_2) = \mu(\Lambda_1)\mu(\Lambda_2)$.

Obviously, if $\Lambda \in \mathcal{TE}$, so is any lattice integrally equivalent to Λ. The following results concerning lattices in $\mathcal{TE}$ are less trivial.

Theorem 2.2.1: Let Λ and Λ' be Euclidean and suppose $\Lambda \in \mathcal{TE}$. Then:

(i) $\Lambda \otimes \Lambda' \in \mathcal{TE}$; $\Lambda^{(\ell)} \in \mathcal{TE}$ for any positive $\ell \in \mathbf{R}$; any sublattice of Λ with minimal norm equal to that of Λ is in $\mathcal{TE}$; if in addition $\Lambda' \in \mathcal{TE}$, then so is $\Lambda \oplus \Lambda'$;

(ii) any orthogonal Euclidean lattice is in $\mathcal{TE}$;

(iii) if $\Lambda' \notin \mathcal{TE}$, then there exists a Euclidean Λ'' with dimension equal to that of Λ' such that $\mu(\Lambda' \otimes \Lambda'') < \mu(\Lambda')\mu(\Lambda'')$;

(iv) the root lattices A_n, D_n, E_6, E_7 and E_8 are all in $\mathcal{TE}$; and

(v) any lattice of dimension ≤ 3 is in $\mathcal{TE}$.

Proof (i) follows from the work in the previous section and eq.(1.2.1a), which says that $\mu(\Lambda_1 \oplus \Lambda_2) = \min\{\mu(\Lambda_1), \mu(\Lambda_2)\}$. If a sublattice Λ_0 of Λ has minimal norm $\mu(\Lambda_0) = \mu(\Lambda)$, then for any Λ', $\mu(\Lambda)\mu(\Lambda') = \mu(\Lambda \otimes \Lambda') \leq \mu(\Lambda_0 \otimes \Lambda') \leq \mu(\Lambda_0)\mu(\Lambda') = \mu(\Lambda)\mu(\Lambda')$, *i.e.* $\mu(\Lambda_0 \otimes \Lambda') = \mu(\Lambda_0)\mu(\Lambda')$. (ii) follows from (i) and some results of the previous section.

To obtain (iii), let $n = \dim(\Lambda')$ and $\mu(\Lambda' \otimes \Lambda_1) < \mu(\Lambda')\mu(\Lambda_1)$ for some Euclidean lattice Λ_1. Let $\sum_{i=1}^{N} x_i \otimes y_i$ be a minimal vector of $\Lambda' \otimes \Lambda_1$, where $x_i \in \Lambda'$ and $y_i \in \Lambda_1$. By the bilinear property of $\otimes$ discussed in the previous section, we may assume that the x_i are linearly independent, and the y_i are linearly independent. Thus $N \leq n$. Take $\Lambda'' = \langle y_i \rangle \oplus I_{n-N}^{(\ell)}$ for any $\ell \geq \mu(\Lambda_1)$ — it is trivial to verify that Λ'' has the desired properties.

(iv) is more difficult. First we will prove $E_8 \in \mathcal{TE}$. The argument we will use was first found by Steinberg (see pp.47-8 of [**MH**]).

Let $\{e_1, \ldots, e_8\}$ be an orthonormal basis of $\mathbf{R}^8$. Then E_8 can be thought of as the set of all vectors $\sum k_i e_i$ in $\mathbf{R}^8$ with $\sum k_i \in 2\mathbf{Z}$, and either all $k_i \in \mathbf{Z}$ or all $k_i \in \mathbf{Z} + \frac{1}{2}$. Thus, every element $u \in E_8 \otimes \Lambda'$ can be written as $\sum e_i \otimes u_i$, where $\sum u_i \in \Lambda'^{(4)}$, and either all $u_i \in \Lambda'$ or all $u_i \notin \Lambda'$ and $2u_i \in \Lambda'$

Let u be a minimal element of $E_8 \otimes \Lambda'$. Note that $u^2 = \sum u_i^2$. We must show that $\sum u_i^2 \geq 2\mu(\Lambda')$.

Obviously not all u_i can be in $\Lambda'^{(4)}$ (for then $\sum u_i^2 \geq 4\mu(\Lambda')$). Then if all u_i are in Λ', at least two — say u_1 and u_2 — must not be in $\Lambda'^{(4)}$ (because $\sum u_i$ must be). But then $\sum u_i^2 \geq u_1^2 + u_2^2 \geq 2\mu(\Lambda')$.

Now consider the case when no u_i is in Λ'. Then all u_i are nonzero and in $\Lambda'^{(1/4)}$. Thus $\sum u_i^2 \geq 8\mu(\Lambda'^{(1/4)}) = 2\mu(\Lambda')$.

Therefore $E_8 \in \mathcal{TE}$. Now, we know from Sec.1.5 that $E_7 \oplus A_1$ and $E_6 \oplus A_2$ are (integrally equivalent to) saturated sublattices of E_8. So E_7 and E_6 are (integrally equivalent to) sublattices of E_8 and their minimal norms equal that of E_8. Hence, by (i) we see that E_7 and E_6 must also be in $\mathcal{TE}$.

Steinberg's idea of embedding the root lattice in $\mathbf{Q} \odot I_n$ does not seem to work for A_n. Instead we do the following.

Let r_i, $i = 1, \ldots, n$, be the basis of 'simple roots' of A_n in Table 2. Let $u = \sum r_i \otimes x_i$, for $x_i \in \Lambda'$, be any minimal vector of $A_n \otimes \Lambda'$. Then, $u^2 = 2x_1^2 - 2x_1 \cdot x_2 + 2x_2^2 - \cdots - 2x_{n-1} \cdot x_n + 2x_n^2 = x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + \cdots + (x_{n-1} - x_n)^2 + x_n^2 \geq 2\mu(\Lambda')$, since $u \neq 0$.

The remaining root lattices D_n can be treated by the method used for A_n, or for E_8.

(v) Consider first any 2-dimensional lattice Λ. Let b_1 and b_2 generate it — without loss of generality we may suppose $b_1^2 = \mu(\Lambda)$ and $|b_1 \cdot b_2| \leq \mu/2$. Let $u = b_1 \otimes x_1 + b_2 \otimes x_2$, for $x_1, x_2 \in \Lambda'$, be a minimal vector in $\Lambda \otimes \Lambda'$. Then $u^2 = b_1^2 x_1^2 + 2(b_1 \cdot b_2)(x_1 \cdot x_2) + b_2^2 x_2^2 = (\sqrt{b_1^2 x_1^2} - \sqrt{b_2^2 x_2^2})^2 + 2(\sqrt{b_1^2 x_1^2 b_2^2 x_2^2} + (b_1 \cdot b_2)(x_1 \cdot x_2)) \geq 0 + 2\mu(\Lambda)\sqrt{x_1^2 x_2^2} - \mu(\Lambda)\sqrt{x_1^2 x_2^2} \geq \mu(\Lambda)\mu(\Lambda')$.

The proof for Λ being 3-dimensional is similar (though messier) and will not be given here. QED

It is possible to use the methods used in the proofs of Thm.1(iv,v) to extend those results. For example, $D_n^+ \in \mathcal{TE}$. However a recent paper makes all that unnecessary.

The proofs thus far in this section have all been from first principles. Y. Kitaoka also has investigated these questions and in [**KIT1**] he brought some heavier machinery to bear on them. (Actually, he restricts his attention to rational lattices Λ and Λ', but this is unnecessary in the present context.)

For each $n = 1, 2, 3, \ldots$ let

$$\mu_n = \sup\{\mu(\Lambda)/|\Lambda|^{1/n}\}, \tag{2.2.2a}$$

where the 'sup' runs over all Euclidean lattices of dimension n. It is known that for each n this sup is attained (in fact by an integral lattice -- see the lemma on p 29 of [**MH**]). Several upper bounds for μ_n are known: we gave one in Thm.1 1.6; the sharpest one is due to C.A. Rogers (see *e.g.* [**ROG**]); and Kitaoka used

$$\mu_n < \frac{2}{\pi}\Gamma(2+\frac{n}{2})^{2/n}. \qquad (2.2.2b)$$

(It is possible that if he used Rogers' bound, Thms.2 and 3 given below could be slightly strengthened.)

Theorem 2.2.2 (*Kitaoka*): Any Euclidean lattice Λ of dimension ≤ 42 is necessarily in $\mathcal{TE}$.

sketch of Proof Let Λ' be any Euclidean lattice and let $v = \sum_{i=1}^{N} x_i \otimes y_i$, for $x_i \in \Lambda$ and $y_i \in \Lambda'$ be any minimal vector in $\Lambda \otimes \Lambda'$. We may assume, by the bilinearity of $\otimes$, that the x_i vectors are linearly independent, as are the y_i. Define the N-dimensional Euclidean lattices $\Lambda_0 \stackrel{\text{def}}{=} \langle\{x_1,\ldots,x_N\}\rangle$ and $\Lambda_0' \stackrel{\text{def}}{=} \langle\{y_1,\ldots,y_N\}\rangle$, and let A_0 and A_0' be their Gram matrices relative to the bases $\{x_i\}$ and $\{y_i\}$. Then $\mu(\Lambda_0)\mu(\Lambda_0') \geq \mu(\Lambda)\mu(\Lambda') \geq \mu(\Lambda\otimes\Lambda') = v^2 = \mathrm{Tr}(A_0A_0')$.

Now it is a straightforward argument in matrix algebra to show that for any positive definite symmetric $n\times n$ $\mathbf{R}$-matrices B and C, we have $\mathrm{Tr}(BC) \geq n(|B|\,|C|)^{1/n}$ (just diagonalize, *etc.*).

Hence,
$v^2 \geq N(|A_0|\,|A_0'|)^{1/N} = N(|\Lambda_0|\,|\Lambda_0'|)^{1/N}$, so $\mu(\Lambda_0)/|\Lambda_0|^{1/N}\,\mu(\Lambda_0')/|\Lambda_0'|^{1/N} \geq N$. Thus $\mu_N^2 \geq N$.

Now it is possible to show (using eq.(2*b*)) that for $n \leq 42$, $\mu_n \geq \sqrt{n}$ implies $n = 1$.

But $N \leq$ the dimension of Λ, which is ≤ 42 by hypothesis. Therefore $N = 1$, $v = x_1 \otimes y_1$, and so $\mu(\Lambda\otimes\Lambda') = \mu(\Lambda)\mu(\Lambda')$. QED

Theorem 2.2.3 (*Kitaoka*): Let Λ be any Euclidean lattice with $\mu(\Lambda) \leq 6$ and such that any nonzero sublattice Λ_0 of it has determinant $|\Lambda_0| \geq 1$. Then Λ is in $\mathcal{TE}$.

Proof Let Λ' be any Euclidean lattice, and let $v = \sum_{i=1}^{N} x_i \otimes y_i$, Λ_0 and Λ_0' be as in the proof of Thm.2. As was done there, $6\mu(\Lambda_0') \geq \mu(\Lambda)\mu(\Lambda') \geq \mu(\Lambda \otimes \Lambda') \geq N(|\Lambda_0|\,|\Lambda_0'|)^{1/N} \geq N|\Lambda_0'|^{1/N}$. Hence $\mu_N \geq \mu(\Lambda_0')/|\Lambda_0'|^{1/N} \geq N/6$.

It is possible to show from eq.(2*b*) that $\mu_n > n/6$ for $n \geq 40$.

Thus, $N < 40$. So by the proof of Thm.2, $N = 1$. QED

For example, this implies that any integral Euclidean lattice Λ with $\mu(\Lambda) \leq 6$ — *e.g.* Λ_{24} (in fact all self-dual lattices of dimension < 56) and all the root lattices — must necessarily lie in $\mathcal{TE}$.

Actually, we have proven something slightly stronger in all three theorems (namely, that $N = 1$; *i.e.* that any minimal vector in $\Lambda \otimes \Lambda'$ is of the form $x \otimes y$), but this is not really relevant here.

There are many consequences that can be extracted from the above remarks. Some of these are collected below.

Corollary 2.2.4: (i) For any $\alpha > 1$, there exists a self-dual Euclidean lattice Λ satisfying $\mu(\Lambda \otimes \Lambda) < \mu(\Lambda)^{\alpha}$.

(ii) For any integral Euclidean lattice Λ and any Euclidean lattice Λ', $\mu(\Lambda \otimes \Lambda') \geq \mu(\Lambda')$, with equality only when $\mu(\Lambda) = 1$.

(iii) Suppose Λ is integral and Euclidean, and for some Euclidean lattice Λ' we have $\mu(\Lambda \otimes \Lambda') \leq 6\mu(\Lambda')$. Then $\mu(\Lambda) \leq 6$.

(iv) Let Λ and Λ' be any Euclidean lattices satisfying $\mu(\Lambda \otimes \Lambda') < \mu(\Lambda)\mu(\Lambda')$, and let $\sum_{i=1}^{N} x_i \otimes y_i$ be any minimal vector of $\Lambda \otimes \Lambda'$, with $x_i \in \Lambda$, $y_i \in \Lambda'$. Then $N > 42$.

Proof (i) follows from the argument given at the beginning of this section and the observation that $k(n)^{\alpha/2}$ asymptotically grows like $(n/2\pi e)^{\alpha/2}$, and so will eventually surpass $\sqrt{n}$.

(ii) follows from (iii) and Thm.3. The proof of Thm.3 can be used to prove (iii).

Finally, the proof of Thm.2 can be used to get (iv). QED

2.3 The Shifting Method: General

A second method for generating new lattices from a given one is called *shifting*. It will be proved that two lattices can be connected by a (rational) shift iff they are rationally equivalent. Hence in this way shifting generalizes gluing.

In its most general form shifting connects rational lattices. However, a special case of it (the *integral shift*, discussed later in this section) takes integral lattices only to other (rationally equivalent) integral lattices, and never increases the determinant. This integral shift specializes to the usual (self-dual) shift when one considers shifting only self-dual lattices. It takes self-dual lattices only to other self-dual lattices, and connects any two such lattices of equal dimension and signature. When shifting occurs in the mathematical and physical literature, it is usually the self-dual variety.

A simple example of shifting is provided by the theory of neighbouring lattices, discussed on pp.421-3 of [**CS1**] and in Secs.2.4 and 3.4. (Self-dual) shifting is also used in string theory to generate new theories from a given one (see *e.g.* Sec 6.2).

Although shifting has appeared before in the literature, it seems that it has never been systematically studied. and the connection between it and rational equivalence has never been explicitly made.

When shifting *has* appeared before, it has not always been without errors. For example, physicists have come up with a number of arguments (see *e.g.* [**LAM3**] and [**LL**]) for the self-duality of the (integral) shift of a self-dual lattice, but none of these arguments seem to be complete. Also, Borcherds on p.422 of [**CS1**] defined *neighbours* (an order 2 shift) in two different ways; the second way is not correct — $[\langle I_8, (\frac{1}{2})^8\rangle]$ (his notation) equals D_8, not E_8 as he claims it does.

Suppose we are given a (rational) lattice Λ_1, vectors $\omega_1, \ldots, \omega_m \in \mathbf{Q} \otimes \Lambda_1$, and an $m \times m$ $\mathbf{Q}$-matrix $\zeta = (\zeta_{ij})$.

Definition 2.3.1: By the *(rationally) shifted lattice* we mean the set

$$\Lambda_1(\{\omega_1, \ldots, \omega_m\}, \zeta) \stackrel{\text{def}}{=}$$

$$\{q + \sum_{i=1}^{m} \ell_i \omega_i \mid \ell_i \in \mathbf{Z}, q \in \Lambda_1, \text{ and } \forall j, \; q \cdot \omega_j \equiv \sum_{i=1}^{m} \zeta_{ji} \ell_i \pmod 1\}. \tag{2.3.1}$$

ω_i are called the *shift vectors*, and ζ_{ij} are called the *vacuum parameters*.

(This terminology is taken from string theory — see Sec.6.2.) We may write this more compactly in matrix notation in the following way. Choose some basis $\beta = \{b_1, \ldots, b_n\}$ of Λ_1 and let A be the Gram matrix of Λ_1 relative to β. Let W denote the $n \times m$ $\mathbf{Q}$-matrix whose ijth entry W_{ij} is the ith component $(\omega_j)_i$ of ω_j relative to β. Let $q \in I_n$ and $\ell \in I_m$ be column vectors with integer components (q represents a vector in Λ_1, relative to β, in the natural way). Then the shifted lattice can be written as

$$\Lambda_1(W, \zeta) \stackrel{\text{def}}{=} \{q + W\ell \mid \ell \in I_m, \; q \in I_n, \; W^t A q \equiv \zeta \ell \pmod 1\}. \tag{2.3.2}$$

It is trivial to verify from either eqs.(1) or (2) that the shifted 'lattice' is indeed always a lattice.

The following definitions are made with Thm.3 in mind. Consider the sets

$$S \stackrel{\text{def}}{=} \{\sum_{i=1}^{m} \ell_i \omega_i \in \Lambda_1 \mid \ell_i \in \mathbf{Z}\},$$

$$S^* \stackrel{\text{def}}{=} \{\sum_{i=1}^{m} \ell_i \omega_i \in \Lambda_1^* \mid \ell_i \in \mathbf{Z}\}, \text{ and}$$

$$\tilde{S} \stackrel{\text{def}}{=} \{\sum_{i=1}^{m} \ell_i \omega_i \in \Lambda_1 \mid \ell_i \in \mathbf{Z}, \text{ and } \forall j, \ \sum \ell_i(\zeta_{ji} + \omega_i \cdot \omega_j) \in \mathbf{Z}\}.$$

Define the *shift groups* to be

$$\Omega = \Omega(W) \stackrel{\text{def}}{=} \{\sum_{i=1}^{m} \ell_i \omega_i \mid \ell_i \in \mathbf{Z}\} \,/\, S,$$

$$\Omega^* = \Omega^*(W) \stackrel{\text{def}}{=} \{\sum_{i=1}^{m} \ell_i \omega_i \mid \ell_i \in \mathbf{Z}\} \,/\, S^*, \text{ and}$$

$$\tilde{\Omega} = \tilde{\Omega}(W, \zeta) \stackrel{\text{def}}{=} \{\sum_{i=1}^{m} \ell_i \omega_i \mid \ell_i \in \mathbf{Z}\} \,/\, \tilde{S}.$$

The following result is straightforward to verify.

Theorem 2.3.1: The shift groups Ω, Ω^* and $\tilde{\Omega}$ are all finite.

If the shift groups $\Omega(W)$ and $\Omega(W')$ of two different shifts are equal, then $S = S'$, and there exist an $m \times m'$ $\mathbf{Z}$-matrix K, and an $m' \times m$ $\mathbf{Z}$-matrix K', such that $\omega_i = \sum \omega_j' K'_{ji}$ and $\omega_i' = \sum \omega_j K_{ji}$. Also, $S^* = S^{*\prime}$, so $\Omega^*(W) = \Omega^*(W')$. However, as we shall see below, we do not have in general that $\Lambda_1(W, \zeta) = \Lambda_1(W', \zeta')$, even if in addition $\tilde{\Omega}(W, \zeta) = \tilde{\Omega}(W', \zeta')$. Conversely, suppose $\omega_i = \sum \omega_j' K'_{ji}$ and $\omega_i' = \sum \omega_j K_{ji}$, for $\mathbf{Z}$-matrices K, K'. If in addition $\zeta' = K^t \zeta K$ (mod 1) and $\zeta = K'^t \zeta' K'$ (mod 1), then $\Omega(W) = \Omega(W')$, $\Omega^*(W) = \Omega^*(W')$, $\tilde{\Omega}(W, \zeta) = \tilde{\Omega}(W', \zeta')$ and $\Lambda_1(W, \zeta) = \Lambda_1(W', \zeta')$.

Define the ℓth sector $\Lambda_{1,\ell}$ of the shifting to be

$$\Lambda_{1,\ell} \stackrel{\text{def}}{=} \{q + W\ell \mid q \in I_n, \ W^t A q \equiv \zeta \ell \ (\text{mod } 1)\}.$$

Then $\Lambda_1(W, \zeta) = \cup \Lambda_{1,\ell}$. $\Lambda_{1,0}$ is a nonempty sublattice of both Λ_1 and $\Lambda_1(W, \zeta)$; moreover, the shift vectors ω_i are all in $\Lambda_{1,0}^*$.

There is a relationship between shifting and gluing that will be developed in the remainder of this section and the next. The essence of it lies in the trivial observation that any nonempty sector $\Lambda_{1,\ell}$ can be written as $q + W\ell + \Lambda_{1,0}$ for any vector $q + W\ell$ in $\Lambda_{1,\ell}$.

For convenience write $\omega(\ell) \stackrel{\text{def}}{=} \sum \ell_i \omega_i$. Also, let $q(\ell)$ denote any vector satisfying $q(\ell) + \omega(\ell) \in \Lambda_{1,\ell}$, provided $\Lambda_{1,\ell}$ is nonempty.

An example will now be given to illustrate the above definitions.

Let $\Lambda_1 = I_8$, $\omega = \omega_1 = (\frac{1}{2}, \ldots, \frac{1}{2})$, and $\zeta = 0$. Then $S = \{2\ell\omega \mid \ell \in \mathbf{Z}\}$, $\Omega = \Omega^* = \tilde{\Omega} = \{S, S + \omega\} \cong C_2$, $\Lambda_{1,0} = D_8$, and $q(1)$ can be chosen to be 0. The shifted lattice $\Lambda_1(\omega, \zeta)$ is then $D_8[\omega] = D_8[1] = E_8$. (This is an example of the glue gauge, as we will see later in this section.)

Now consider $\omega' = \omega$ but $\zeta' = \frac{1}{2}$. Then again we have $S' = S$, $\Omega' = \Omega'^* = \tilde{\Omega}' = \Omega$, and $\Lambda'_{1,0} = \Lambda_{1,0}$, but $q'(1) \neq 0$ ($q'(1) = -e_1$ is a possible choice) and $\Lambda_1(\omega', \zeta') = D_8[-e_1 + \omega'] = D_8[3]$. Although $\Lambda_1(\omega', \zeta') \approx \Lambda_1(\omega, \zeta)$, these two lattices are not pointwise identical.

In general, $\Omega' = \Omega$ will not force $\Lambda_1(W, \zeta)$ and $\Lambda_1(W', \zeta')$ to be (integrally) equivalent to each other. In spite of this, the notation $\Lambda_1(\Omega)$ is sometimes used in the literature.

Given a shift matrix W, an $m \times m$ $\mathbf{Z}$-matrix N can be found with the property that $WM \equiv 0 \pmod 1$, for any other $\mathbf{Z}$-matrix M, iff $M = NM'$ for some $\mathbf{Z}$-matrix M' (in fact, we may choose generators ω_i' for $\Omega(W)$ so that the corresponding N is diagonal — these ω_i' are then independent with respect to Λ_1, as defined in Def.1.4.2). Clearly, the determinant $|N|$ equals the order $\|\Omega\|$ of Ω. Also, $S = \{\omega(N\ell) \mid \ell \in I_m\}$. Incidentally, changing the shift vector generators by $\omega_i' = \sum \omega_j K_{ji}$, $\omega_i = \sum \omega_j' K'_{ji}$ induces the changes $N' = K'NK'^t$ and $N = KN'K^t$.

Theorem 2.3.2: Let $\Lambda_2 = \Lambda_1(W, \zeta)$. Then:

$$\text{(i)} \qquad \Lambda_1 \cap \Lambda_2 = \bigcup_{\ell \in I_m} \Lambda_{1,N\ell} \quad \text{and}$$

$$\text{(ii)} \qquad \Lambda_{1,0} \stackrel{Q}{\approx} \Lambda_1 \stackrel{Q}{\approx} \Lambda_2.$$

Proof (i) Clearly the containment "$\subseteq$" holds

Suppose $q' \in \Lambda_1 \cap \Lambda_2$. Then $q' \in \Lambda_{1,\ell'}$ for some ℓ', so $q' = q(\ell') + \omega(\ell')$, where $q(\ell') \in \Lambda_1$. We also have $q' \in \Lambda_1$, which implies $\omega(\ell') \in \Lambda_1$. Therefore, by definition of N, $\ell' = N\ell$ for some ℓ, so $q' \in \Lambda_{1,N\ell}$.

(ii) It suffices to prove $\Lambda_{1,0} \overset{Q}{\approx} \Lambda_1$.

Λ_1 is rational so there exists a $L \in \mathbf{Z}$ such that $\Lambda_1^{(L^2)}$ is integral Let $\ell = L^2 \prod n_i$. Choose any $q \in \Lambda_1$. Then $\ell^2 q \cdot \omega_i \in \mathbf{Z}\ \forall i$. The desired result follows from Cor.1.4.11. QED

Theorem 2.3.3: Let $\Lambda_2 = \Lambda_1(W, \zeta)$. Then we have the following isomorphisms between glue groups and shift groups:

(i) $\Lambda_1 \,/\, \Lambda_{1,0} \cong \Omega^*$

(ii) $\Lambda_2 \,/\, \Lambda_{1,0} = \{\text{nonempty sectors } \Lambda_{1,\ell}\} \cong \{\omega(\ell) + \tilde{S} \in \tilde{\Omega} \mid \Lambda_{1,\ell} \text{ is nonempty }\}$

(iii) $\Lambda_2 \,/\, (\Lambda_1 \cap \Lambda_2) \cong \{\omega(\ell) + S \in \Omega \mid \Lambda_{1,\ell} \text{ is nonempty }\}$.

Thm.3(i) follows immediately from Lemma 1.4.8. Thm.3(ii) is proven by noting that for nonempty $\Lambda_{1,\ell}$, $q(\ell) + \omega(\ell) \in \Lambda_{1,0}$ iff $\omega(\ell) \in \tilde{S}$ Similarly, Thm 3(iii) results from the observation that for nonempty $\Lambda_{1,\ell}$, $q(\ell) + \omega(\ell) \in \Lambda_1 \cap \Lambda_2$ iff $\omega(\ell) \in S$

Theorem 2.3.4: $\Lambda_1 \overset{Q}{\approx} \Lambda_2$ iff $\exists W, \zeta$ such that $\Lambda_2 \approx \Lambda_1(W, \zeta)$.

Proof "$\Rightarrow$" By Thm.1.4.10 there exists a saturated integral sublattice Λ_0 of both Λ_1 and some lattice Λ_2' integrally equivalent to Λ_2. In fact we may choose it so that $\Lambda_1 \subset \Lambda_0^*$ and $\Lambda_2' \subset \Lambda_0^*$.

Let $\omega_i + \Lambda_0$, $i = 1, \ldots, k$, be generators of Λ_2'/Λ_0 and choose $\zeta = 0$. Then the shift $\Lambda_0(W, \zeta')$ has zero sector Λ_0 and in fact equals $\Lambda_0[\{\omega_i\}] = \Lambda_2'$.

Let $\omega_j' + \Lambda_1^*$, $j = 1, \ldots, \ell$, be generators of Λ_0^*/Λ_1^* and choose $\zeta_{ij}' = -\omega_i' \cdot \omega_j'$. Then the zero sector of the shift $\Lambda_1(W', \zeta')$ again is Λ_0 and in this case we have $\Lambda_1(W', \zeta') = \Lambda_0$, as a simple calculation reveals.

Now augment the shifts together:

$$\tilde{\omega}_i \stackrel{\text{def}}{=} \begin{cases} \omega_i & \text{for } i \leq k \\ \omega'_i & \text{for } i > k \end{cases}$$

$$\tilde{\zeta}_{ij} \stackrel{\text{def}}{=} \begin{cases} \zeta'_{ij} & \text{for } i, j > k \\ 0 & \text{otherwise} \end{cases},$$

and consider the shift $\Lambda_1(\tilde{W}, \tilde{\zeta})$. Then $\Lambda_{1,0}$ again turns out to be Λ_0, and the shift $\Lambda_1(\tilde{W}, \tilde{\zeta}) = \Lambda'_2$.

"$\Leftarrow$" follows immediately from Thm.2(ii). QED

Thm.4 is the reason the rational shift is interesting in its own right, and not interesting merely because it is the natural generalization of the self-dual shift. Of course, by no means are these W and ζ unique. For one thing, any other generators W' of $\Omega(W)$ will work as well (provided ζ is adjusted appropriately). A systematic source of redundancy that will become more important in the following section, bears striking resemblance to the gauge problems in modern physics:

Theorem 2.3.5: $\Lambda_1(W,\zeta) = \Lambda_1(W',\zeta')$, where

$$\omega_i{}' = \omega_i + x_i, \quad x_i \in \Lambda_1^*$$

$$\omega'(\ell) = \omega(\ell) + \sum \ell_i x_i$$

$$q'(\ell) = q(\ell) - \sum \ell_i x_i$$

$$\zeta'_{ij} = \zeta_{ij} - x_j \cdot (x_i + \omega_i).$$

Moreover, the shift groups $\Omega^*(W)$ and $\Omega^*(W')$ are isomorphic, and $\Lambda_{1,0} = \Lambda'_{1,0}$.

The obvious calculations establish Thm.5. See also Thm.12.

So far we have considered shifting in its full generality. In the remainder of this section we will investigate the consequences of imposing additional constraints on W and ζ.

Consider first the constraint that $(W^tAW + \zeta)N$ be a **Z**-matrix. Of course this constraint is independent of the choice of shift vector generators. When $N = \mathrm{diag}(n_1, \ldots, n_m)$, so that the shift vectors ω_i are independent with respect to Λ_1 with orders n_i (see Def.1.4 2), this condition becomes $n_j(\omega_i \cdot \omega_j + \zeta_{ij}) \in \mathbf{Z}$, $\forall i, j$.

Theorem 2.3.6: For a shift $\Lambda_2 = \Lambda_1(W, \zeta)$ such that $(W^tAW + \zeta)N$ is a **Z**-matrix, $\Lambda_1 \cap \Lambda_2 = \Lambda_{1,0}$. Also, $\Omega = \tilde{\Omega}$.

In proving this, it suffices of course to show that $\Lambda_{1,N\ell} \subseteq \Lambda_{1,0}$ $\forall \ell$ (in fact equality holds).

Theorem 2.3.7: Any two rationally equivalent integral lattices are (up to integral equivalence) connected by a (rational) shift satisfying the constraint that $(W^tAW + \zeta)N$ be a **Z**-matrix.

Proof Let $\Lambda_2' \approx \Lambda_2$ be as in Thm.1.4 10. Define $\Lambda_0 = \Lambda_1 \cap \Lambda_2'$. Then Λ_0 is a saturated sublattice of both Λ_2' and $\Lambda_2'^*$.

Choose a representative ω_i from each class in $\Lambda_2'^*/\Lambda_0$, and define $\zeta_{ij} = -\omega_i \cdot \omega_j$. Then the shift $\Lambda_1(W, \zeta)$ has zero sector $\Lambda_{1,0} = \Lambda_0$.

For $q \in \Lambda_1$, $q + \omega(\ell) \in \Lambda_1(W, \zeta)$ implies $q \cdot \omega_j \equiv \sum \ell_i \zeta_{ji}$ (mod 1) $\forall j$, which implies $(q + \omega(\ell)) \cdot \omega_j \in \mathbf{Z}$ $\forall j$. Therefore ($\Lambda_0 \subset$) $\Lambda_1(W, \zeta) \subseteq \Lambda_2'$.

Note that Λ_2' is the lattice obtained by gluing to Λ_0 all shift vectors $\omega(\ell) \in \Lambda_2'$. For these ℓ, choosing $q(\ell) = 0$ shows that $\Lambda_2' \subseteq \Lambda_1(W, \zeta)$. QED

In other words, when we are dealing with integral lattices (which is our main interest here), we can without loss of generality impose the simplifying constraint that $(W^tAW + \zeta)N$ be a **Z**-matrix.

Theorem 2.3.8: When $(W^tAW + \zeta)N$ is a **Z**-matrix, a gauge (see Thm.5) can always be chosen so that $\zeta'_{ij} = -\omega_i' \cdot \omega_j'$ (mod 1). Conversely, if there is such a choice of gauge for a given W and ζ, then $(W^tAW + \zeta)N$ is a **Z**-matrix.

This choice of shift vectors is called the *glue gauge*, for reasons that will become more apparent shortly. The proof of Thm.8 is similar to that of Thm.9 below.

As before, given a shift matrix W, we can find an $m \times m$ **Z**-matrix N^* with the following property: $M = (M_{ij})$ is a **Z**-matrix such that $\sum M_{ij}\omega_i \in \Lambda_1^*$ for all i, iff $M = N^*M'$ for some **Z**-matrix M'. When Λ_1 is integral, $M = N$ is such a matrix (since $\Lambda_1 \subseteq \Lambda_1^*$), so there would be a **Z**-matrix N' such that $N = N^*N'$. Hence the determinant $|N^*|$ divides the determinant $|N|$. When Λ_1 is self-dual, N^* can be taken to be N Of course, for any Λ_1 (integral or not), $|N^*|$ is the order of Ω^*.

Theorem 2.3.9: When $\zeta^t N^*$ is a **Z**-matrix, all sectors $\Lambda_{1,\ell}$ are nonempty.

Proof Assume ω_j are independent (relative to Λ_1^*) and let n_j^* be their orders. From Thm.1.4.9 there exist vectors $r_i \in \Lambda_1$, $i = 1, \ldots, \ell$, such that

$$r_i \cdot \omega_j \equiv \frac{1}{n_i^*}\delta_{ij} \pmod 1.$$

Define $q(\ell) \stackrel{\text{def}}{=} \sum\sum \ell_i n_j^* \zeta_{ji} r_j$. Then $q(\ell) \in \Lambda_1$ and $q(\ell) \cdot \omega_j \equiv \sum \ell_i \zeta_{ji} \pmod 1$, $\forall j$. Therefore, $q(\ell) + \omega(\ell) \in \Lambda_{1,\ell}$. QED

Thm.8 follows from Thm.1.4.9 in a similar manner: let $x_i = -\sum n_j(\omega_i \cdot \omega_j + \zeta_{ij})r_j$.

Hence, when $\zeta^t N^*$ and $(W^tAW + \zeta)N$ are both **Z**-matrices, we have $\Omega = \tilde{\Omega} \cong \Lambda_2/(\Lambda_1 \cap \Lambda_2)$ and $\Omega^* \cong \Lambda_1/(\Lambda_1 \cap \Lambda_2)$.

Thm.9 tells us that for each i, there exists a $q_i \in \Lambda_1$ such that $q_i + \omega_i \in \Lambda_2$. We may now set $q(\ell) = \sum \ell_i q_i$.

Now suppose Λ_1 is integral and $\zeta^t N^*$ is a **Z**-matrix. Then because of the important Thm.9, a simple calculation shows that the shifted lattice Λ_2 will be integral iff $\omega_i \cdot \omega_j + \zeta_{ij} + \zeta_{ji} \in \mathbf{Z}$ $\forall i, j$ — *i.e.* iff $W^tAW + \zeta + \zeta^t$ is a **Z**-matrix.

Definition 2.3.2: A shift is called an *integral shift* if both $\zeta^t N^*$ and $W^tAW + \zeta + \zeta^t$ are **Z**-matrices.

Note that for an integral shift, $(W^tAW+\zeta)N$ is always a $\mathbf{Z}$-matrix. Hence there is a glue gauge; in fact, in it $\omega_i\cdot\omega_j \in \mathbf{Z}$, $\zeta_{ij} \in \mathbf{Z}$ and $\Lambda_1(W,\zeta) = \Lambda_{1,0}[\{\omega_i\}] = \Lambda_{1,0}[\Omega]$. This is the reason for the name 'glue gauge'

Theorem 2.3.10: Let Λ_1 be integral, and suppose $\Lambda_2 = \Lambda_1(W,\zeta)$ is an integral shift. Then Λ_2 is an integral lattice with determinant

$$|\Lambda_2| = |\Lambda_1|\frac{|N^*|^2}{|N|^2}.$$

Proof Thm.3 becomes $\Omega \cong \Lambda_2/\Lambda_{1,0}$ and $\Omega^* \cong \Lambda_1/\Lambda_{1,0}$. Therefore

$$|N|^2 = \|\Omega\|^2 = |\Lambda_{1,0}|/|\Lambda_2| \text{ and}$$

$$|N^*|^2 = \|\Omega^*\|^2 = |\Lambda_{1,0}|/|\Lambda_1|.$$

The desired identity follows immediately. QED

Hence $|\Lambda_1| \geq |\Lambda_2| \geq |\Lambda_1|/\|\Omega\|^2$. Thus the determinant never increases when using the integral shift. If one starts with a non-self-dual but self-dualizable lattice Λ_1, a self-dual lattice of equal dimension and signature will eventually be obtained, by repeatedly shifting in this manner.

Incidently, the determinant formula in Thm.10 holds even if Λ_1 is not integral.

Thm.10 also tells us that it may not be possible to connect two rationally equivalent integral lattices with an integral shift (*e.g.* $\{4\} \overset{Q}{\approx} \{9\}$ but neither integral shift $\{4\} \to \{9\}$ nor $\{9\} \to \{4\}$ can exist). However, it is easy to show from the proof of Thm.4 that given integral lattices $\Lambda_1 \overset{Q}{\approx} \Lambda_2$, there exists a third integral lattice Λ_3 such that Λ_3 can be integrally shifted to both Λ_1 and Λ_2. Moreover, the proof of Thm.7 implies:

Theorem 2.3.11: Let Λ_1 be self-dualizable and Λ_2 be self-dual. Suppose they are of equal dimension and signature. Then there exists a $\Lambda_2' \approx \Lambda_2$ such that there is an integral shift of Λ_1 to Λ_2'. Moreover, if in addition $\Lambda_1 \cap \Lambda_2$ is saturated in Λ_1, then there is an integral shift of Λ_1 to Λ_2.

Using the existence of a glue gauge, which is unique up to vectors in $\Lambda_1 \cap \Lambda_2$, we quickly get this interesting result:

Theorem 2.3.12: Suppose Λ_1 is integral. Then the integrally shifted lattices $\Lambda_1(W,\zeta)$ and $\Lambda_1(W',\zeta')$ are equal iff the shifts are connected by a gauge transformation of the type defined in Thm.5.

Note that this theorem refers to lattice equality '=' and not the more general lattice equivalence '≈'.

2.4 The Shifting of Self-dual Lattices

By 'shift' we will mean throughout this section the 'self-dual shift', *i.e.* the integral shift (see Def.3.2) restricted to self-dual Λ_1.

As we shall shortly see, one can get self-dual lattices from others by 'shifting'. This procedure has been widely used in constructing new strings (see *e.g.* Sec.6.2). In fact, the results of this section can be used in an attempt to systematically produce, by shifting and gluing, Grand Unified Theories with a given gauge group, with supersymmetry, and with a superconformal current (see Sec.6.3). Shifting's general properties and its connection with the gluing method will be discussed in this section (see also Sec.3.4). The terminology given here is that used by string theorists (see Sec.6.2).

For convenience, we shall explicitly include here the definition of the self-dual shift. The shifting operation begins with a self-dual lattice Λ_1, a set of shift vectors ω_i, $1 \le i \le m$, satisfying $n_i \omega_i \in \Lambda_1 \;\; \forall i$, and an $m \times m$ **Q**-matrix $\zeta = (\zeta_{ij})$ of vacuum parameters. Without loss of generality we may assume that the shift vectors are independent with respect to Λ_1, with orders n_i (see Def.1.4.2).

The conditions the shift vectors and vacuum parameters must satisfy for (self-dual) shifting to take place become here:

$$\zeta_{ij} + \zeta_{ji} + \omega_i \cdot \omega_j \in \mathbf{Z} \quad \forall i,j \tag{2.4.1a}$$

$$n_j \zeta_{ji} \in \mathbf{Z} \quad \forall i, j. \tag{2.4.1b}$$

In the case considered here, the shift group Ω is isomorphic to the direct product $C_{n_1} \times C_{n_2} \times \cdots \times C_{n_m}$ of cyclic groups.

Let $\Lambda_2 = \Lambda_1(W, \zeta)$. Then we know from Sec.3 the following results:

Theorem 2.4.1: (i) Λ_2 is self-dual;

(ii) $\Lambda_1 \cap \Lambda_2 = \Lambda_{1,0}$;

(iii) $\omega_i \in \Lambda^*_{1,0}$, $\forall i$;

(iv) All sectors $\Lambda_{1,\ell}$ are nonempty; and

(v) $\Omega = \tilde{\Omega} = \Omega^* \cong \Lambda_1/(\Lambda_1 \cap \Lambda_2) \cong \Lambda_2/(\Lambda_1 \cap \Lambda_2)$.

We learned in the proof of Thm.3.10 that eq.(1a) is required for Λ_2 to be integral, while eq.(1b) is needed to show $|\Lambda_2| = 1$. Though it is widely quoted, rigorous proofs of Thm.1(i) are difficult to find. Physicists usually exploit modular invariance of a *partition function* (see Chapter 6). The proof given in the previous section is more self-contained and general, and produces other interesting results.

Thms.3.11 and 3.12, respectively, reduce to:

Theorem 2.4.2: $\Lambda_2 \approx \Lambda_1(W, \zeta)$ for some W, ζ, iff Λ_2 is self-dual and has the same dimension and signature as Λ_1.

Theorem 2.4.3: $\Lambda_1(W, \zeta) = \Lambda_1(W', \zeta')$ iff the shifts are connected by a gauge transformation of the type defined in Thm.3.5.

Note that eqs.(1a, b) imply the following constraints on ω_i: there must exist integers $F_i \in \{0, 1\}$ (called the *fermionic parameters*) such that

$$n_i({\omega_i}^2 - F_i) \in 2\mathbf{Z} \quad \forall i \tag{2.4.2a}$$

$$n_i n_j \omega_i \cdot \omega_j \in D_{ij}\mathbf{Z} \quad \forall i, j. \tag{2.4.2b}$$

The quantity $D_{ij} = (n_i, n_j)$ in eq.(2b) is the greatest common divisor of n_i and n_j. For n_i odd, eq.(2a) reduces to $n_i\omega_i{}^2 \in \mathbf{Z}$ with $F_i \equiv n_i\omega_i{}^2 \pmod 2$; for n_i even it becomes $n_i\omega_i{}^2 \in 2\mathbf{Z}$, with F_i arbitrary.

The constraints in eqs.(1) for ζ_{ij} can be solved in terms of a number of free parameters (the *discrete torsions*) $Q_{ij} \equiv -Q_{ji} \pmod{D_{ij}}$, $1 \leq i < j \leq m$, each taking arbitrary integral values between 0 and $D_{ij} - 1$. The solution (see [**LAM3**]) is:

$$\zeta_{ji} \equiv (-n_i Y_{ji} \omega_i \cdot \omega_j + Q_{ij})/D_{ij} \pmod 1 \quad i \neq j \tag{2.4.3a}$$

$$\zeta_{ii} \equiv (F_i - \omega_i{}^2)/2 \pmod 1, \tag{2.4.3b}$$

where $Y_{ij} \in \mathbf{Z}$ is defined so that $Y_{ij}n_j + Y_{ji}n_i = D_{ij}$ (it is an elementary theorem in number theory that such Y_{ij} exist — they are not unique, but any non-uniqueness can be absorbed in the parameters Q_{ij}).

A different set of discrete torsion parameters Q_{ij} and (when n_i is even) fermionic parameters F_i may correspond to a different shifted lattice Λ_2, so altogether we may have as many as $2^m \prod_{1 \leq i < j \leq m} D_{ij}$ possible self-dual lattices Λ_2 obtained from a given Λ_1 and a given shift group Ω, by varying ζ.

As we know from the work of the last section, there is a close connection between the gluing and the shifting operations. The gluing operation may actually be thought of as the shifting operation in a particularly simple gauge, the glue gauge. The glue gauge is unique only up to vectors in $\Lambda_{1,0}$. This is discussed in the next two theorems, which are just restatements of facts learned in the previous section.

Theorem 2.4.4: By choosing the gauge (*i.e.* the $x_i \in \Lambda_1$ in Thm.3.5) appropriately, we may assume without loss of generality that all $Q_{ij} = \zeta_{ij} = 0$, and that the shift vectors have integral dot products $\omega_i \cdot \omega_j$ (this choice of gauge is called the glue gauge). In the glue gauge, $\Lambda_2 = \Lambda_{1,0}[\{\omega_1, \ldots, \omega_m\}] = \Lambda_{1,0}[\Omega]$.

Theorem 2.4.5: Suppose you are given two self-dual lattices Λ_1 and Λ_2 with intersection $\Lambda_0 = \Lambda_1 \cap \Lambda_2$ saturated in Λ_1 and Λ_2. Let $\omega_i + \Lambda_0$ be the classes in Λ_2 / Λ_0. Then $\Lambda_1(W, 0) = \Lambda_2$.

Shifting gives us an elegant proof of Thm.1.6 1. Let Λ_2 be any self-dual lattice Λ_0 glues to. Shift Λ_2 by the vectors in G', choosing $\zeta = 0$ (so we are in the glue gauge). Then the resulting shifted lattice has the desired properties.

Let us turn to another property of the shift operations: namely, that they are generally not commutative. Let ω_1 and ω_2 be independent (with respect to Λ_1), and let ζ be any 2×2 **Q**-matrix. Then the following three self-dual lattices obtained from different orderings and combinations of the shifts are generally different: $\Lambda_{(1,2)} \stackrel{\text{def}}{=} (\Lambda(\omega_1, \zeta_{11}))(\omega_2, \zeta_{22})$, $\Lambda_{(2,1)} \stackrel{\text{def}}{=} (\Lambda(\omega_2, \zeta_{22}))(\omega_1, \zeta_{11})$, and $\Lambda_{(12)} = \Lambda_{(21)} \stackrel{\text{def}}{=} \Lambda(\{\omega_1, \omega_2\}, \zeta)$. In fact, the following theorem can be proved.

Theorem 2.4.6: (i) $\Lambda_{(1,2)} = \Lambda_{(2,1)}$ iff $\omega_1 \cdot \omega_2 \in \mathbf{Z}$;

(ii) $\Lambda_{(1,2)} = \Lambda_{(12)}$ iff $\zeta_{12} \in \mathbf{Z}$.

Proof Note that we have

$$\Lambda_{1,2} = \{q + \ell_1\omega_1 + \ell_2\omega_2 \mid (q + \ell_1\omega_1) \cdot \omega_2 \equiv \ell_2\zeta_{22} \pmod 1,$$

$$q \cdot \omega_1 \equiv \ell_1\zeta_{11}, \text{ where } q \in \Lambda\},$$

$$\Lambda_{2,1} = \{q + \ell_1\omega_1 + \ell_2\omega_2 \mid (q + \ell_2\omega_2) \cdot \omega_1 \equiv \ell_1\zeta_{12},$$

$$q \cdot \omega_2 \equiv \ell_2\zeta_{22}, \text{ where } q \in \Lambda\}.$$

Thm.1(iv) implies that there exists a $q \in \Lambda$, and an $\ell_1 \in \mathbf{Z}$ such that $q + \ell_1\omega_1 + \omega_2 \in \Lambda_{1,2}$. Then $\exists q' \in \Lambda$, and integers ℓ_1', ℓ_2' such that $0 \leq \ell_1' < n_1'$ and $0 \leq \ell_2' < n_2$, and $q' + \ell_1'\omega_1 + \ell_2'\omega_2 = q + \ell_1\omega_1 + \omega_2$. The independence of ω_1 and ω_2 implies $\ell_1' \equiv \ell_1 \pmod{n_1}$ and $\ell_2' \equiv 1 \pmod{n_2}$. Hence $\ell_2' = 1$. Using eq.(1*b*) and the above equations, a single subtraction yields that $\Lambda_{1,2} = \Lambda_{2,1}$ implies $\omega_1 \cdot \omega_2 \in \mathbf{Z}$. The converse follows immediately from the above equations.

This proves (i). A similar argument establishes (ii). QED

Thus, in the glue gauge of $\Lambda(\{\omega_1,\omega_2\},\zeta)$, $(\Lambda(\omega_1,\zeta_{11}))(\omega_2,\zeta_{22}) = (\Lambda(\omega_2,\zeta_{22}))(\omega_1,\zeta_{11}) = \Lambda(\{\omega_1,\omega_2\},\zeta)$.

Theorem 2.4.7: Let ω_i, $1 \leq i \leq m$, be independent with respect to Λ_1. If $\Lambda_2 = \Lambda_1(W,\zeta)$, then $\Lambda_1 = \Lambda_2(W,\zeta)$ iff both conditions below are satisfied:

(i) $\zeta_{ij} \equiv \zeta_{ji} \pmod 1$, $\forall i,j$;

(ii) $\sum_i k_i\zeta_{ij} \in \mathbf{Z}\ \forall j = 1,\cdots,m$, implies $n_i | k_i\ \forall i$.

Proof By Thm.1(iv), for each i there exists a $q_i \in \Lambda_1$ such that $q_i + \omega_i \in \Lambda_2$. Note that $\Lambda_1 = \Lambda_2(W,\zeta)$ iff both $q_i \in \Lambda_2(W,\zeta)$, $\forall i$, and the zero sectors $\Lambda_{1,0}$ and $\Lambda_{2,0}$ of the shifts $\Lambda_1(W,\zeta)$ and $\Lambda_2(W,\zeta)$, respectively, are equal. Note that $q' \in \Lambda_2$ implies that there exist integers k_i such that $q' \cdot \omega_j \equiv -\sum k_i\zeta_{ij} \pmod 1$, $\forall j$. It is trivial to verify that $\Lambda_{1,0} \subset \Lambda_{2,0}$. Thus $\Lambda_{1,0} = \Lambda_{2,0}$ iff each $q' \in \Lambda_2$ satisfying $q' \cdot \omega_j \in \mathbf{Z}$ $\forall j$ is necessarily in Λ_1, which is equivalent to condition (ii).

Also, $q_i \in \Lambda_2(W,\zeta)$ iff $(q_i + \omega_i)\cdot\omega_j \equiv -\zeta_{ji} \pmod 1$ $\forall j$ iff $q_i \cdot \omega_j \equiv \zeta_{ij}, \forall j$. But $q_i + \omega_i \in \Lambda_2$, so $q_i \cdot \omega_j \equiv \zeta_{ji}, \forall j$.

Therefore $q_i \in \Lambda_2(W,\zeta)$ iff $\zeta_{ij} \equiv \zeta_{ji} \pmod 1$, $\forall j$. QED

In the special case where $m = 1$, *i.e.* when the shift group has only one generator, this theorem simplifies to the following.

Corollary 2.4.8: Let ω have order n. If $\Lambda_2 = \Lambda_1(\omega,\zeta)$, then $\Lambda_1 = \Lambda_2(\omega,\zeta)$ iff $(n,k) = 1$, where $\omega^2 = 2k/n + F$. For n odd or $F = 0$, this 'iff' condition becomes $(2n, n\omega^2) = 1$ or 2.

(Here, as before, (n,k) denotes the greatest common divisor of n and k.) Related to this is the following result.

Theorem 2.4.9: Suppose $\Lambda_2 = \Lambda_1(W, \zeta)$. Then there exists a shift given by W', ζ' obtainable from W, ζ by means of a gauge transformation such that $\Lambda_2 = \Lambda_1(W', \zeta')$ and $\Lambda_1 = \Lambda_2(W', \zeta')$.

Proof Let $\tilde{W}, \tilde{\zeta}$ be the glue gauge satisfying $\Lambda_1(\tilde{W}, \tilde{\zeta}) = \Lambda_1(W, \zeta) \stackrel{\text{def}}{=} \Lambda_2$, and let $\tilde{\omega}_i$ be its independent generators. Define a new shift W', ζ' obtained from $\tilde{W}, \tilde{\zeta}$ by the gauge transformation given by $x_i = r_i$ (using the notation of Thm.1.4.9, where we take $y_i \stackrel{\text{def}}{=} \tilde{\omega}_i$). Then W', ζ' is connected to W, ζ by some gauge transformation. Let ω_i' be the shift generators. Using Thm.3.5, note that

$$\zeta'_{ij} = \tilde{\zeta}_{ij} - r_j \cdot \tilde{\omega}_i \equiv -\frac{1}{n_i}\delta_{ij} \pmod 1.$$

Thus, $\sum k_i \zeta'_{ij} \equiv -\frac{k_j}{n_j} \pmod 1$, so

(i) $\zeta'_{ji} \equiv \zeta'_{ij} \pmod 1$

(ii) $\sum k_i \zeta'_{ij} \in \mathbf{Z}\ \forall j$ iff $n_i | k_i,\ \forall i$.

Therefore by Thm.7, $\Lambda_2 = \Lambda_1(W', \zeta')$ and $\Lambda_1 = \Lambda_2(W', \zeta')$. QED

In the remainder of this section we will introduce a term which we will study in much more detail in the following chapter, where we exploit the interconnections between shifting and gluing touched on in this section.

Definition 2.4.1: Two self-dual lattices Λ, Λ' are called *neighbours* if their intersection $\Lambda \cap \Lambda'$ has index 2 in both of them.

Thus, Λ and Λ' are neighbours iff an order 2 shift connects them. This term is taken from [**CS1**]. A number of special properties are satisfied by neighbours, some of which we will touch on, but it is beyond the scope of this paper to list them all. They have proven to be useful in the enumeration of self-dual lattices (*e.g.* see pp.421-425 of [**CS1**]).

Up to integral equivalence, there are exactly two self-dual indefinite lattices of signature $(n_+, n_-) = (8k + \ell, \ell)$, namely $I_{8k+\ell,\ell}$ and $\mathrm{II}_{8k+\ell,\ell}$. These are neighbours: the shift operation $I_{8k+\ell,\ell} \rightarrow II_{8k+\ell,\ell}$ is given by the shift vector $\omega = (\frac{1}{2}, \ldots, \frac{1}{2}; \frac{1}{2}, \ldots, \frac{1}{2})$ expressed in the standard basis of $I_{8k+\ell,\ell}$. Of course, any other (self-dual) indefinite lattices connected by a shift must be (integrally) equivalent.

The following theorem shows how to find neighbouring lattices to a given one, and also gives some nontrivial examples of shifting.

Theorem 2.4.10: The neighbours of I_n are equivalent to $D_{4k}^{+} \oplus I_{n-4k}$ for $k = 1, \ldots, [n/4]$, while the Leech lattice Λ_{24} only has the neighbours $O_{23} \oplus I_1$ (where O_{23} is the so-called *shorter Leech lattice* — see p.179 of [**CS1**]) and the Niemeier lattice with root lattice A_1^{24} (see Table 5).

Proof Without loss of generality we may work in the glue gauge, and write *e.g.* $I_n(x)$ for the shifted lattice $I_n(\{x\}, 0)$. We are interested in shift vectors $x \notin I_n$ such that $2x \in I_n$. Note that $I_n(x) = I_n(y)$ iff $x - y \in I_n$, so we are interested in the conjugacy classes of $I_n^{(1/4)}/I_n$.

Every such class contains a vector whose coordinates (relative to the orthonormal basis of I_n) consist of k $\frac{1}{2}$'s and $n - k$ 0's for some k. It suffices to consider the results of shifting by such vectors. Moreover, up to integral equivalence (in fact a trivial rearranging of the basis vectors of I_n) it clearly suffices to consider the results of shifting I_n by the vectors $x_k \stackrel{\text{def}}{=} (\frac{1}{2}, \ldots, \frac{1}{2}, 0, \ldots, 0)$ (*i.e.* k k $\frac{1}{2}$'s and $n - k$ 0's) for $k = 1, 2, \ldots, n$.

The shift vector must satisfy the conditions eqs.($1a, b$) — they imply $x_k^2 \in \mathbf{Z}$, *i.e.* 4 divides k. Directly from the definition we can see $I_n(x_k) = D_k^{+} \oplus I_{n-k}$ (the zero sector is $D_k \oplus I_{n-k}$, *etc.*). This gives us the desired list of neighbours of I_n.

The Leech lattice is dealt with similarly: we need to know about the group $\Lambda_{24}^{(1/4)}/\Lambda_{24}$, or equivalently the group $\Lambda_{24}/\Lambda_{24}^{(4)}$. This is investigated in Thm.28 on p.289 of [**CS1**]. There we find that every vector of Λ_{24} is congruent modulo $\Lambda_{24}^{(4)}$ to either the zero vector (which corresponds to an order 1 shift, and so can be discarded), or a norm 4, 6 or norm 8 vector. In other words it suffices to consider shifting by a norm 1=4/4 vector or a norm 2=8/4 vector (a vector of norm 3/2=6/4 would fail to satisfy eq.(1*b*)). Thm.27 on p.288 of [**CS1**] implies that shifting by any norm 1 vector will produce the same lattice (up to integral equivalence), and shifting by any norm 2 vector will also produce the same lattice (up to integral equivalence).

Thus there are two cases: $\Lambda_{24}(x_1)$ for $2x_4 \in \Lambda_{24}$, $x_1^2 = 1$; and $\Lambda_{24}(x_2)$ for $2x_2 \in \Lambda_{24}$, $x_2^2 = 2$.

Consider first $\Lambda_{24}(x_1)$. By Thm.1.2.1 we can write this as $I_1 \oplus \Lambda'_{23}$, where Λ'_{23} is a self-dual 23-dimensional lattice. Any nonzero vector $u \in \Lambda'_{23}$ is either in Λ_{24} (*i.e.* in the zero sector), in which case its norm is $u^2 \geq 4$, or equal to $x_1 + v$ for some nonzero $v \in \Lambda_{24}$. Since $x_1 \cdot (x_1 + v) = 0$, $u^2 = 1 + 2x_1 \cdot v + v^2 = v^2 - 1 \geq 3$ Thus Λ'_{23} has minimal norm $\mu \geq 3$, so it must be integrally equivalent to O_{23}, the shorter Leech lattice.

Now consider $\Lambda_{24}(x_2)$. It must be Type II (since x_2^2 and the zero sector are both even), and it has a root vector (namely x_2), so it must be one of the 23 Niemeier lattices (see Table 5). Let r be a root vector (*i.e.* a norm 2 vector) in $\Lambda_{24}(x_2)$. Then $2r$ is a norm 8 vector in Λ_{24}, congruent to $2x_2$ mod $\Lambda_{24}^{(4)}$. But p.289 of [**CS1**] tells us there are only 48 such vectors. Thus $\Lambda_{24}(x_2)$ can have no more than 48 root vectors. However, Table 5 tells us that all Niemeier lattices have over 48 roots except the one with root lattice A_1^{24}, which has exactly 48.

Therefore $\Lambda_{24}(x_2)$ must be integrally equivalent to that lattice. QED

Let Λ_1 and Λ_2 be neighbours, and let $\Lambda_0 = \Lambda_1 \cap \Lambda_2$ be their intersection. Then $|\Lambda_0| = 4$, so Λ_0^*/Λ_0 has 4 elements. Let $\Lambda_1 = \Lambda_0[g_1]$ and $\Lambda_2 = \Lambda_0[g_2]$. Then $\Lambda_0^*/\Lambda_0 = \{[0]\Lambda_0, [g_1]\Lambda_0, [g_2]\Lambda_0, [g_1 + g_2]\Lambda_0\} \cong C_2 \times C_2$.

Note that $g_1^2 \in \mathbf{Z}$, $g_2^2 \in \mathbf{Z}$ and $2g_1 \cdot g_2 = (2g_1) \cdot g_2 \in \mathbf{Z}$. Therefore $(g_1 + g_2)^2 \in \mathbf{Z}$, so $\Lambda_3 \stackrel{\text{def}}{=} \Lambda_0[g_1 + g_2]$ also is self-dual. Λ_3 is a neighbour of both Λ_1 and Λ_2.

What this shows is that neighbours come in triples (in this case Λ_1, Λ_2 and

Λ_3).

It is easy to show, using the above notation, that $g_1^2 + g_2^2 + (g_1 + g_2)^2 \equiv 1 \pmod 2$. Hence either all three self-dual neighbours in the triple are odd, or exactly 2 of the 3 are even. This has proved useful in the enumeration of all Type I lattices of 24 dimensions (see Ch.17 of [**CS**1]).

Neighbouring lattices will also be discussed in Sec.3.4.

Chapter 3 RATIONAL EQUIVALENCE AND SIMILARITY

3.1 Similarity

Two lattices can be rationally equivalent only if their dimension and signature are equal. There are some circumstances (most notably when studying self-dualizability) where this restriction is inconvenient.

Definition 3.1.1: Two lattices Λ_1 and Λ_2 are said to be *similar*, written $\Lambda_1 \sim \Lambda_2$, if there exist integers $k, \ell, m, n \geq 0$ such that $\Lambda_1 \oplus I_{k,\ell} \overset{Q}{\approx} \Lambda_2 \oplus I_{m,n}$.

Thus two integrally or rationally equivalent lattices will always be similar.

We introduced the concept of similarity in [**GL2**] to study self-dualizability and shifting. In fact, it was only while writing up that paper that we realized its intimate connection with rational equivalence. In the previous chapter we found that shifting is closely related to rational equivalence; however, we will see in this chapter that similarity is more natural and useful for questions concerning self-dualizability.

Suppose some lattice Λ is similar to a rational lattice Λ_1. Then the direct sum of Λ with some orthonormal lattice, is rationally equivalent to some rational lattice. It is easy to show from this that Λ must be rational. Throughout this chapter we will only consider rational lattices.

The basic result for studying similarity is the following theorem.

Theorem 3.1.1: Let Λ_1, Λ_1', Λ_2 and Λ_2' be rational lattices. Suppose $\Lambda_1 \sim \Lambda_1'$. Then $\Lambda_1 \oplus \Lambda_2 \sim \Lambda_1' \oplus \Lambda_2'$ iff $\Lambda_2 \sim \Lambda_2'$.

Proof "$\Rightarrow$": Consider first $\Lambda_1 = \Lambda_1' = \{(n_0)\}$, $\Lambda_2 = \{(m_1), \ldots, (m_k)\}$, and $\Lambda_2' = \{(n_1), \ldots, (n_k)\}$, and suppose $\{(n_0), (m_1), \ldots, (m_k)\} \overset{Q}{\approx} \{(n_0), \quad (n_1), \ldots, (n_k)\}$. Let x_i and y_i, for $i = 0, \ldots, k$, be orthonormal bases for these lattices (so *e.g.* $y_i \cdot y_j = n_i \delta_{ij}$). Then by Cor.1.4.11 there exists a nonzero integer ℓ such that $(\Lambda_1' \oplus \Lambda_2')^{(\ell^2)} =$

$\langle \ell y_0, \ell y_1, \ldots, \ell y_k \rangle \subseteq \langle x_0, x_1, \ldots, x_k \rangle = \Lambda_1 \oplus \Lambda_2$. So, there exist integers a_{ij} for $0 \le i, j \le k$ such that, for these i, j,

$$(a_{i0} x_0 + \tilde{a}_i) \cdot (a_{j0} y_0 + \tilde{a}_j) = \ell^2 n_i \delta_{ij},$$

$$\text{where } \tilde{a}_i = \sum_{j=1}^{k} a_{ij} x_j,$$

for $0 \le i \le k$. Choose the sign of ℓ so that $\ell \ne a_{00}$. Define

$$\tilde{c}_i = (\ell - a_{00})\tilde{a}_i + a_{i0}\tilde{a}_0 \in \Lambda_1 \quad , \quad 1 \le i \le k.$$

Then simple arithmetic gives $\tilde{c}_i \cdot \tilde{c}_j = (\ell - a_{00})^2 \ell^2 n_i \delta_{ij}$ for $1 \le i, j \le k$. This is just the statement that $\{(m_1), \ldots, (m_k)\} \overset{Q}{\approx} \{(n_1), \ldots, (n_k)\}$.

By induction (on p=dim(Λ_1)) we quickly get that for any orthogonal lattices Λ_1, Λ_2 and Λ_2', $\Lambda_1 \oplus \Lambda_2 \overset{Q}{\approx} \Lambda_1 \oplus \Lambda_2'$ implies $\Lambda_2 \overset{Q}{\approx} \Lambda_2'$.

Now suppose $\Lambda_1 \overset{Q}{\approx} \Lambda_1'$, Λ_2 and Λ_2' are all rational lattices, and $\Lambda_1 \oplus \Lambda_2 \overset{Q}{\approx} \Lambda_1' \oplus \Lambda_2'$. Let $\Lambda_1'' \approx \Lambda_1$ have intersection $\Lambda_1'' \cap \Lambda_1'$ with Λ_1' which is saturated in Λ_1' (possible by Thm.1.4.10). Then $\Lambda_1'' \oplus \Lambda_2 \overset{Q}{\approx} \Lambda_1'' \oplus \Lambda_2'$, as can be seen *e.g.* by Cor.1.4.11. Now let Λ_{01}'', Λ_{02} and Λ_{02}' be orthogonal decompositions of Λ_1'', Λ_2 and Λ_2' respectively. Then, by the same reasoning we have $\Lambda_{01}'' \oplus \Lambda_{02} \overset{Q}{\approx} \Lambda_{01}'' \oplus \Lambda_{02}'$. But, by the above argument this implies $\Lambda_{02} \overset{Q}{\approx} \Lambda_{02}'$ and hence $\Lambda_2 \overset{Q}{\approx} \Lambda_2'$.

What we have shown so far is that for any rational lattices $\Lambda_1 \overset{Q}{\approx} \Lambda_1'$, Λ_2 and Λ_2', $\Lambda_1 \oplus \Lambda_2 \overset{Q}{\approx} \Lambda_1' \oplus \Lambda_2'$ implies $\Lambda_2 \overset{Q}{\approx} \Lambda_2'$. It is easy to see (directly from Def.1) that this remains true after replacing each '$\overset{Q}{\approx}$' with '$\sim$'.

"$\Leftarrow$": This direction is either trivial or reduces to the above direction. QED

Hence similarity is an equivalence relation (transitivity was not obvious before Thm.1). Thm.1 is essentially the Witt Cancellation Theorem, stated and proved in [**CS1**] (the above proof has the advantage in this context of being lattice-theoretic).

Let the signature of Λ_1 and Λ_2 be (n_{1+}, n_{1-}) and (n_{2+}, n_{2-}). Then Thm.1 also tells us that in Def.1 we may always suppose one of k or m is zero and the other is $|n_{1+} - n_{2+}|$, and one of ℓ or n is zero and the other $|n_{1-} - n_{2-}|$. Hence:

Theorem 3.1.2: For rational lattices Λ_1 and Λ_2, $\Lambda_1 \overset{Q}{\approx} \Lambda_2$ iff $\Lambda_1 \sim \Lambda_2$, and Λ_1 and Λ_2 have the same dimension and signature.

A revealing characterization of similarity is given in Cor.9. Thm.1.6.1 becomes:

Theorem 3.1.3: If $\Lambda_1 \sim \Lambda_2$ are two integral lattices, then Λ_1 is self-dualizable iff Λ_2 is.

In fact, Thm.3 hints at a strong relationship between similarity and self-dualizability that will be more fully developed in what follows. We will show ultimately (Cor.7) that two integral lattices Λ_1 and Λ_2 are similar iff they fall short of being self-dualizable to exactly equal extents.

One more elementary result is:

Theorem 3.1.4: For rational lattices Λ_1 and Λ_2, $\Lambda_1 \sim \Lambda_2$ implies $\sqrt{|\Lambda_1|\,|\Lambda_2|} \in \mathbf{Q}$. In fact, if in addition $\Lambda_1 \overset{Q}{\approx} \Lambda_2$, then $\ell = |\Lambda_1 \cap \Lambda_2'|/\sqrt{|\Lambda_1|\,|\Lambda_2|}$ works in Cor.1.4.11.

Proof Suppose that $\Lambda_1 \overset{Q}{\approx} \Lambda_0 = \Lambda_2 \oplus I_{k,\ell}$, for some non-negative integers k and ℓ (the other cases can all be treated similarly). We have then that $|\Lambda_0| = |\Lambda_2|$, and if $\Lambda_1 \overset{Q}{\approx} \Lambda_2$, then $k = 0$ and $\Lambda_0 = \Lambda_2$. For some $\Lambda_0' \approx \Lambda_0$, $\ell_1 = \sqrt{|(\Lambda_1 \cap \Lambda_0')|\,/\,|\Lambda_1|}$ is an integer and is in fact the order of the group $\Lambda_1/(\Lambda_1 \cap \Lambda_0')$ (Lemma 1.4.1 and Thm.1.4.10). Hence the order of any glue class $[g] \in \Lambda_1/(\Lambda_1 \cap \Lambda_0')$ must divide ℓ_1, so $\ell_1[g] = [0]$, and thus $\ell_1 g \in \Lambda_0'$.

Define ℓ_0 similarly. Then because $\ell \overset{\text{def}}{=} \ell_1\ell_0$ is an integer, $\sqrt{|\Lambda_1|\,|\Lambda_2|}$ must be rational. If $\Lambda_1 \overset{Q}{\approx} \Lambda_2$, ℓ has the desired properties. QED

Thus I_2 and the root lattice A_2 cannot be similar. The converse is not true; for example, $I_1 \oplus I_1^{(9)}$ and $I_2^{(3)} \overset{\text{def}}{=} I_1^{(3)} \oplus I_1^{(3)}$ are not similar.

A useful theoretical (as we saw in the proof of Thm.1) and practical (as we shall see in Sec.2) tool for analyzing similarity is ***orthogonal decomposition***, which we presented in Cor.1.4.4. A lattice is always similar to its orthogonal decomposition. The orthogonal decompositions of the root lattices are given in Table 4. Of the root lattices, only E_8 is self-dual. However, note that $D_n \sim \{(4), \ldots, (4)\}$, so by Thm.3 D_n is always self-dualizable. Indeed, we have already seen that $D_n[2] = I_n$ for all n. The converse is almost true; it turns out (see Thm.12) that the m_i's for any self-dual (but not self-dualizable) lattice can also be chosen to be powers of 4.

It will prove convenient to write $\{(k)^\ell\}$ and Λ^ℓ for $\{(k), \ldots, (k)\}$ (ℓ times) and $\Lambda \oplus \cdots \oplus \Lambda$ (ℓ times), respectively (not to be confused with the scaled lattice $\Lambda^{(\ell)}$). Thus, $D_n \sim \{(4)^n\}$ and $(I_1)^n = I_n$. It turns out that D_{2k} (and hence D_{2k}^+) has an orthogonal decomposition $\{(2)^{2k}\}$, while D_{2k+1} has $\{(2)^{2k}, (4)\}$. E_6, E_7 and E_8 have orthogonal decompositions $\{(2)^5, (12)\}$, $\{(2)^7\}$, and $\{(2)^8\}$, respectively (these are the 'smallest' decompositions of D_n, E_6, E_7 and E_8).

In Sec.2 we will be concerned with reducing a set of m_i's by replacing them essentially with primes in a way which preserves similarity at the expense of (rational or integral) equivalence. To determine whether two given lattices are similar (or rationally equivalent), we will reduce their orthogonal decompositions to 'primary' ones and compare the results. An example of this reduction process is Thm.1.6.8. Thm.1 allows us to simplify each m_i separately, thus making the task of finding a systematic reduction scheme that much easier.

One immediate consequence of Thm.1 (and the uniqueness of Type I indefinite lattices — see Thm.1.3.2) is:

Theorem 3.1.5: An integral lattice Λ is self-dualizable iff $\Lambda \sim \{(1)\} \stackrel{\text{def}}{=} I_1$.

Proof Let Λ be of signature (m, n) and first suppose Λ is self-dualizable. Then Λ glues to some self-dual lattice Λ' and hence, $\Lambda \sim \Lambda'$ (in fact $\Lambda \stackrel{Q}{\approx} \Lambda'$). But by Thm.1.3.2 $\Lambda' \oplus I_{1,1} \approx I_{m+1,n+1}$, so we get $\Lambda \sim \Lambda' \oplus I_{1,1} \sim I_{m+1,n+1} \sim I_1$.

Now suppose $\Lambda \sim I_1$. Then Thm.1 implies $\Lambda \overset{Q}{\approx} I_{m,n}$, so by Thm.1.4.10 there exists a lattice $\Lambda' \approx I_{m,n}$ such that $\Lambda \cap \Lambda'$ is saturated in both Λ and Λ'. Because $\Lambda \cap \Lambda'$ is saturated in the orthonormal (hence self-dual) lattice Λ', it is self-dualizable. Thus by Thm.1.6.1 (or Thm.3) Λ is self-dualizable (though of course it itself need not glue to an orthonormal lattice). QED

If we replace the condition that Λ be integral with the weaker condition that it merely be rational, then Λ self-dualizable will still imply $\Lambda \sim \{(1)\}$ (since Λ would have to be integral), but $\Lambda \sim \{(1)\}$ could now imply only that Λ contains a saturated lattice which is self-dualizable (this can be shown using the above proof). Thus Thm.1.6.9 becomes

Theorem 3.1.6: Let Λ be any rational lattice. Then $\Lambda^4 \overset{\text{def}}{=} \Lambda \oplus \Lambda \oplus \Lambda \oplus \Lambda \sim \{(1)\}$.

Thm.5 tells us that any two self-dualizable lattices are similar. In fact:

Corollary 3.1.7: For integral lattices Λ_1 and Λ_2, $\Lambda_1 \sim \Lambda_2$ iff there exists a lattice Λ_3 (which we may take to be Euclidean) such that $\Lambda_1 \oplus \Lambda_3$ and $\Lambda_2 \oplus \Lambda_3$ are both self-dualizable.

For example, $\Lambda_3 = \Lambda_1^3$ works.

Given the Chapter 1 theorems on LR-decomposition (*e.g.* Cor.1.4.5), we can reduce the indefinite case to the Euclidean one in the following way:

Theorem 3.1.8: Suppose Λ is indefinite and rational with an LR-decomposition $(\Lambda_L; \Lambda_R)$. Then $\Lambda \sim \Lambda_L \oplus (\Lambda_R)^3$. Λ is self-dualizable iff $\Lambda_L \sim \Lambda_R$. Moreover, if a second lattice Λ' has an LR-decompostion $(\Lambda'_L; \Lambda'_R)$, then $\Lambda \sim \Lambda'$ iff $\Lambda_L \oplus \Lambda'_R \sim \Lambda'_L \oplus \Lambda_R$.

The first statement in Thm.8 follows from Thm.5, Thm.6, and the fact that $(\Lambda_2; \Lambda_2)$ is self-dualizable. The other two statements follow from this one.

A number of applications of these results can be obtained from Thm.10, given below, which in turn arises from the following lemma. Before stating the lemma, we will define for completeness the term *genus*.

Two Euclidean integral lattices Λ_1 and Λ_2 are said to lie in the same genus if they are of equal dimension, if they are both odd or both even, and if $\Lambda_1^*/\Lambda_1 \cong \Lambda_2^*/\Lambda_2$, where the isomorphism preserves norms (mod 1). Hence all Type I lattices of equal dimension lie in the same genus; similarly for all Type II lattices of equal dimension.

This definition came from [**CS3**]. The more familiar definition (see p.128 of [**CAS**]) concerns the equivalence of the corresponding quadratic forms over the p-adic integers, but is less geometric.

Lemma 3.1.9: If Λ is odd, self-dual, Euclidean, and of dimension $n \geq 4$, then for any sufficiently large $k \in \mathbf{Z}$, $\exists x \in \Lambda$ such that $x^2 = k$. If instead Λ is even, then so must be k.

Proof From Thm.1.6 on p.204 of [**CAS**], it suffices to prove the lemma for I_n and E_8^ℓ, $\forall n \geq 4$, and $\forall \ell \geq 1$. But this follows immediately from the facts that I_4 represents all positive integers and E_8 represents all positive even integers. QED

This lemma is actually far stronger than we need in the proof of Thm.10. Unfortunately it was necessary to quote a nontrivial result from [**CAS**], so our proof of the following theorem is not completely self-contained.

Theorem 3.1.10: Suppose a lattice Λ is self-dualizable and of signature (k, ℓ). Then an orthogonal decomposition $\{(m_1), \ldots, (m_k); (n_1), \ldots, (n_\ell)\}$ for Λ can be found such that any prime p dividing $(\prod m_i)(\prod n_j)$ must also divide $2|\Lambda|$.

Proof First we shall prove this for Euclidean lattices Λ of dimension n.

First suppose that this theorem is known to be true for each self-dual lattice of a given dimension n and let Λ be a self-dualizable lattice of that dimension which glues to the self-dual lattice Λ_1. Then $\Lambda \overset{Q}{\approx} \Lambda_1$, so by Thm.4 $\Lambda_1^{(\ell)} \subset \Lambda$ for $\ell = |\Lambda|$. By assumption Λ_1 has an orthogonal decomposition given by m_i's which are powers of 2. Then $m_i' \overset{\text{def}}{=} \ell m_i$ defines an orthogonal decomposition of Λ with the desired property. Thus, if Thm.10 is satisfied by every *self-dual* lattice of a given dimension n, then it is satisfied by every *self-dualizable* lattice of that dimension.

The proof of Thm 10 for self-dual lattices Λ of dimension n will be by induction on n. Suppose that Λ is self-dual. If Λ is of dimension $n \leq 7$, then $\Lambda = I_n$ and the theorem is trivially satisfied; take each $m_i = 1$. Thus, any self-dualizable lattice of dimension ≤ 7 satisfies the theorem.

Now suppose the theorem has been confirmed for all self-dualizable lattices of dimension $< N$ and let Λ be any self-dual lattice of dimension N. Then Lemma 9 implies that for some positive $k \in \mathbf{Z}$ there exists a vector x in Λ of norm 4^k. Thm.1.6.2 tells us that the $(N-1)$-dimensional lattice $x^\perp$ is self-dualizable and hence, satisfies the theorem by assumption. Since $|x^\perp|$ divides 4^k (again by Thm.1.6.2), we have that $x^\perp$ has an orthogonal decomposition given by m_i's, $i = 1, \ldots, N-1$, which are powers of 2. Then $\{(m_1), \ldots, (m_{N-1}), (4^k)\}$ is a decomposition of Λ. Thus any self-dual, and hence self-dualizable, Euclidean lattice of dimension N must satisfy the theorem.

The proof for the indefinite case is similar; it suffices to verify the theorem for the self-dual lattices $I_{m,n}$ and $II_{m,n}$. But it holds trivially for $I_{m,n}$ (choose all $m_i = n_j = 1$) and E_8 (see Table 4 or the proof given above). $II_{1,1}$ has orthogonal decomposition $\{(2);(2)\}$ (if we let $\{b_1, b_2\}$ be its basis satisfying $b_1^2 = 0 = b_2^2$ and $b_1 \cdot b_2 = 1$, then $b_1 + b_2$ and $b_1 - b_2$ are the vectors defining the desired orthogonal decomposition). Thus, $II_{m,n}$ has a decomposition with $m_i = n_j = 2$ and the theorem is proved in the indefinite case. QED

$2|\Lambda|$, rather than merely $|\Lambda|$, is necessary in the statement of Thm.10, as otherwise E_8 for example would have an orthogonal decomposition $\{(1)^8\} = I_8$, which is absurd.

Sec.4 contains some of the interesting consequences of Thm.10.

3.2 The Primary Decomposition Procedure

In this section we discuss a method for finding which 'similarity class' a given rational lattice belongs to. It suffices, by orthogonal decomposition, to analyze all n-tuples m_i of positive integers. We will arrive at a method for simplifying a given choice of m_i's which preserves similarity at the expense of integral and rational equivalence.

The main points of this section are Thm.9 (the Primary Decomposition Theorem) and Thm.10; the earlier theorems provide an algorithm (summarized in Thm.10) for obtaining the primary decomposition of a given lattice. The section closes with a more standard characterization of similarity, which can also be shown to follow from the analysis of this section.

Note that because of Thms.1.1 and 1.6, similarity and direct sums define an abelian group; the identity is $\{(1)\}$ and the inverse of any group element Λ is $\Lambda^3 \sim \Lambda^{(-1)}$. Thm.1.1 allows us to reduce each $\{(m_i)\}$ individually; *i.e.* to consider without loss of generality only 1-dimensional lattices. By 'reducing' m_i we mean finding integers $n_{i1}, \ldots, n_{ik}$ such that $\{(m_i)\} \sim \{(n_{i1}), \ldots, (n_{ik})\}$, where each n_{ij} has as few prime divisors as possible. After reducing each $\{(m_i)\}$ as much as possible, Thm.1.6 (or, more precisely, Thm.2) will then enable us to treat the m_i's collectively again, ultimately yielding one of the 'canonical' sets $\{(m'_1), (m'_2), \ldots\}$ to be defined later.

Theorem 3.2.1: $\{(m_1 k_1^2), \ldots, (m_n k_n^2)\} \sim \{(m_1), \ldots, (m_n)\}$, for any $k_i \in \mathbf{Z}$.

The proof is simple: gluing the n glue vectors $[0, \ldots, 0, m_i k_i, 0, \ldots, 0]$ to the lattice $\{(m_1 k_1^2), \ldots, (m_n k_n^2)\}$ results in $\{(m_1), \ldots, (m_n)\}$.

Because we will be dealing so often with orthogonal decompositions, in this section we will let $\{m_1, \ldots, m_n\}$ denote the n-dimensional lattice $\{(m_1), \ldots, (m_n)\}$.

Moreover, Thm.1 tells us we may assume each m_i is *square-free*; *i.e.* p divides m_i implies p^2 does not. With this in mind we will let, for example, $\{m^4\}$ denote $\{(m)^4\} \stackrel{\text{def}}{=} \{m, m, m, m\}$.

For $k = 1, 3$ let P_k denote the set of all primes p congruent to k (mod 4), and define $P_{12} \stackrel{\text{def}}{=} P_1 \cup \{2\}$. We know these three sets are infinite (*e.g.* by Dirichlet's Theorem on primes in arithmetic progressions — see Chapter VI of [**SER**]).

Theorem 3.2.2: Suppose $m \neq 1$ is square-free. Then:

$$\{m^k\} \sim \{m^\ell\} \text{ iff } k \equiv \ell \begin{cases} \pmod 2 & \text{if no } p \in P_3 \text{ divides } m \\ \pmod 4 & \text{otherwise} \end{cases} .$$

To see this, note first that (Thm.1.6) $\{m^4\} \sim \{1\}$. Now, for m square-free, $\{m^2\} \stackrel{\text{def}}{=} \{m, m\}$ is self-dualizable iff it has an integer-normed glue vector of order m, *i.e.* iff $\exists k, \ell \in \mathbf{Z}$ relatively prime to m satisfying $k^2 + \ell^2 \equiv 0 \pmod m$, which in turn is equivalent to the statement that -1 be a quadratic residue of m. The square-free m's with quadratic residue -1 are precisely of the form given in Thm.2.

(Recall the definition of quadratic residues: m is a quadratic residue of n iff $\exists x \in \mathbf{Z}$ such that $x^2 \equiv m \pmod n$.)

This type of argument will recur throughout the remainder of this paper. In particular, quadratic residues will play a large role in what follows; for the basic theory see any number theory book (*e.g.* Ch.1 of [**SER**]). As usual, let the Legendre symbol be

$$\left(\frac{p}{q}\right) = \begin{cases} +1 & \text{if p is a quadratic residue of q} \\ -1 & \text{otherwise} \end{cases} ,$$

for distinct primes p and q. For example, $(\frac{p}{2}) = +1$ for any odd prime p.

Dirichlet's Theorem on primes in an arithmetic progression gives us the following useful result:

Lemma 3.2.3: For any collection of distinct primes $p_1, \ldots, p_k$ and any choice for each $\epsilon_i \in \{1, -1\}$, $i = 1, \ldots, k$, there exists a $p \in P_1$, $q \in P_3$ such that $(\frac{p_i}{p}) = (\frac{p_i}{q}) = \epsilon_i$ for each i.

The following theorem tells us how to reduce products $p \cdot q$ of primes.

Theorem 3.2.4: Let p and q be distinct primes. Then:

(i) if $p \in P_{12}$, $q \in P_1$, $\{pq\} \sim \{p,q\}$ iff $(\frac{p}{q}) = +1$;

(ii) if $p \in P_{12}$, $q \in P_3$, $\{pq\} \sim \begin{cases} \{p,q\} & \text{iff } (\frac{p}{q}) = +1 \\ \{p,q^3\} & \text{iff } p = 2 \text{ and } (\frac{2}{q}) = -1 \end{cases}$;

(iii) if $p, q \in P_3$, $\{pq\} \sim \begin{cases} \{p^3,q\} & \text{if } (\frac{p}{q}) = +1 \\ \{p,q^3\} & \text{otherwise} \end{cases}$.

This important theorem allows us to reduce any product $p \cdot q$ of two primes, except for when $q \in P_1$ and $(\frac{p}{q}) = -1$ — *e.g.* both $\{10\}$ and $\{15\}$ cannot be reduced. It turns out, though, that $\{15\} \sim \{2, 3^3, (2 \cdot 5)\}$ — in fact it is possible to 'reduce' $p \cdot q$ whenever $p \in P_3$ (see Thm.5 below).

To prove Thm.4 first note that $\{p, pq, q\} \sim \{1\}$ iff $\Lambda \stackrel{\text{def}}{=} \{p, pq, q\}$ is self-dualizable, iff Λ has two glues, one of order p and the other of order q, and both with integral norm. This is the case iff $(\frac{-q}{p}) = (\frac{-p}{q}) = +1$, using the kind of argument above to prove Thm.2. This immediately gives us Thm.4(i) and the bottom half of Thm.4(ii).

Next, $\{pq, p\} \sim \{q\}$ iff $qx^2 + y^2 = pz^2$ has integral solutions for x, y and z. From Legendre's Theorem (see p.80 of [**CAS**]) this happens iff $(\frac{-q}{p}) = (\frac{p}{q}) = +1$. This gives us Thm.4(iii) and the top half of Thm.4(ii).

Theorem 3.2.5: Suppose $p \in P_3$, $q \in P_1$, and $(\frac{p}{q}) = -1$. Then if $(\frac{2}{q}) = -1$,

$$\{pq\} \sim \{2, p^3, (2 \cdot q)\}; \tag{3.2.1a}$$

otherwise for any $q' \in P_1$ satisfying $(\frac{2}{q'}) = (\frac{q'}{q}) = -1$,

$$\{pq\} \sim \{2, p^3, (2 \cdot q'), (q' \cdot q)\}. \tag{3.2.1b}$$

Proof Suppose $p \in P_3$, $q \in P_1$ and $(\frac{p}{q}) = -1$. If $(\frac{2}{q}) = -1$, then $\{p, pq, 2q, 2\}$ has integral normed glues of order p (since $(\frac{-p}{q}) = +1$), of order q (since $(\frac{2p}{q}) = +1$) and of order 2 (since $(\frac{q}{2}) = +1$) and hence is similar to $\{1\}$. The case when $(\frac{2}{q}) = +1$ is dealt with similarly. QED

The existence of q' in Thm.5 is guaranteed by Lemma 3. Note that eq.(1b) is also satisfied when $(\frac{2}{q}) = -1$. The point of Thm.5 is that (together with the following theorem) it tells us how to remove any prime in P_3 from any product $p_1 \cdot p_2 \cdots p_k$.

Theorem 3.2.6: $\{m_1, \ldots, m_n\} \sim \{m'_1, \ldots, m'_n\}$ implies $\{\lambda m_1, \ldots, \lambda m_n\} \sim \{\lambda m'_1, \ldots, \lambda m'_n\}$ for any positive $\lambda \in \mathbf{Z}$.

The proof is obvious. Note that there must be the same number of m_i's as m'_i's: $\{6\} \sim \{2, 3^3\}$ but $\{12\} \sim \{3\}$ isn't similar to $\{4, 6^3\} \sim \{2, 3\}$.

Thms.4-6 allow us to reduce a large number of $\{m_i\}$'s. For example, using $\{2 \cdot 3\} \sim \{2, 3^3\}$ and $\{3 \cdot 5\} \sim \{2, 3^3, (2 \cdot 5)\}$, we get $\{2 \cdot 3 \cdot 5\} \sim \{5 \cdot 2 \cdot 3, 5^3, 5\} \sim \{5 \cdot 2, (5 \cdot 3)^3, 5\} \sim \{(2 \cdot 5), 2^3, 3^9, (2 \cdot 5), 5\} \sim \{2, 3, 5\}$ by applying Thm.4(ii), eq.(1a), Thm.2, Thm.6, and Thm.2 in that order.

In particular, the previous theorems allow us to reduce any single $\{m\}$ to a direct sum $\{n_1, \ldots, n_k\}$, where each n_i is square-free, has fewer prime divisors than m (except perhaps when Thm.5 is used) and is either:

(a) itself prime; or

(b) can be decomposed into distinct primes $n_i = p_1 \cdots p_\ell$, where $p_1 \in P_{12}$, $p_j \in P_1$ for $j > 1$, and $(\frac{p_j}{p_{j'}}) = -1$ for any $1 \leq j < j' \leq \ell$.

For example, $\{170\} = \{2 \cdot 5 \cdot 17\}$ is of type (b).

The following theorem will allow us to further reduce those n_i of type (b).

Theorem 3.2.7: Suppose $p \in P_{12}$ and $q, r \in P_1$. If $n = pqr$ is of type (b), then $\{pqr\} \sim \{p, q, r\}$.

(This can be proved in the manner of Thm.4.) Thms.6 and 7 allow us to reduce any type (b) n to a direct sum of a number of n_i, each of type (a), or of type (b) with only $\ell = 2$ prime divisors.

Let $\mathcal{P}' \stackrel{\text{def}}{=} \{pq \mid p \in P_{12}, q \in P_1, \ (\frac{p}{q}) = -1\}$. We immediately get:

Corollary 3.2.8: Let Λ be any rational lattice. Then $\Lambda \sim \{n_1, \ldots, n_k\}$, where either n_i is prime or $n_i \in \mathcal{P}'$.

It turns out that the decomposition given in Cor.8 is not quite unique: *e.g.* $\{2 \cdot 5, 5 \cdot 13, 13 \cdot 2\} \sim \{1\}$. Call a subset $\{s_1, s_2, \ldots, s_n\}$ of $\mathcal{P}'$ a *loop* if each $p \in P_{12}$ divides either 0 or 2 of the s_i. Loops look like:

$$p_1p_2, \ p_2p_3, \ \cdots p_{n-1}p_n, \ p_np_1.$$

It will be shown below that every loop is similar to $\{1\}$. In fact, they generate all subsets in $\mathcal{P}'$ similar to $\{1\}$.

Let $\mathcal{P}''$ be any subset of $\mathcal{P}'$ not containing any loops, and such that any larger subset does contain loops. The following two paragraphs give one way of doing this.

Write any element of $\mathcal{P}'$ as $p \cdot q$, where $p < q$, and define a lexicographic ordering $\prec$ on $\mathcal{P}'$ by

$$p \cdot q \prec p' \cdot q' \text{ iff either } q < q', \text{ or } q = q' \text{ and } p < p'.$$

Let $r_1 \prec r_2 \prec r_3 \prec \cdots$ be an enumeration of $\mathcal{P}'$. For example, $r_1 = 2 \cdot 5$ and $r_2 = 2 \cdot 13$.

Define subsets $\mathcal{P}'_k$ of $\mathcal{P}'$ recursively as follows: $\mathcal{P}'_0 = \emptyset$; and, for $k = 1, 2, \ldots$,

$$\mathcal{P}'_k = \begin{cases} \mathcal{P}'_{k-1} & \text{if } r_k \text{ belongs to a loop in } \mathcal{P}'_{k-1} \cup \{r_k\} \\ \mathcal{P}'_{k-1} \cup \{r_k\} & \text{otherwise} \end{cases}.$$

Define $\mathcal{P}'' = \cup \mathcal{P}'_k$. The elements in the relative complement $\mathcal{P}'/\mathcal{P}''$ can be expressed in terms of those in $\mathcal{P}''$. For example, $r_3 = 5 \cdot 13$ is the 'smallest' element of $\mathcal{P}'$ not in $\mathcal{P}''$. It can be written as $\{r_3\} \sim \{r_1, r_2\}$. Hence:

Theorem 3.2.9 (*The Primary Decomposition Theorem*): Let Λ be a rational lattice.

(i) Then $\Lambda \sim \{m_1, \ldots, m_n\}$, where either $m_i \in P_{12} \cup \mathcal{P}''$ and occurs only once, or $m_i \in P_3$ and occurs at most 3 times.

(ii) Suppose $m_i \in P_{12} \cup P_3 \cup \mathcal{P}''$, for $i \leq n$. Suppose further that no m_i's are the same. Then for any integers $k_i, \ell_j \geq 0$, $\{(m_1)^{k_1}, \ldots, (m_n)^{k_n}\} \sim \{(m_1)^{\ell_1}, \ldots, (m_n)^{\ell_n}\}$ iff, for each $i \leq n$,

$$k_i \equiv \ell_i \begin{cases} \pmod 4 & \text{if } m_i \in P_3 \\ \pmod 2 & \text{otherwise} \end{cases}.$$

Proof First note that by the usual reasoning any loop is self-dualizable. Thus Thm.9(i) follows immediately from Cor.8.

To prove Thm.9(ii) it suffices to show it for $\ell_j = 0$, $k_i \neq 0$, $\forall i, j$. The direction "$\Rightarrow$" is obvious from Thm.2. Assume $\Lambda = \{(m_1)^{k_1}, \ldots, (m_n)^{k_n}\}$ is self-dualizable. Suppose $m_i \in P_3$ and assume without loss of generality that $0 < k_i \leq 4$. Then there must be at least one glue of Λ of order m_i. This requires $k_i \geq 4$ (since m_i can divide no other m_j) and hence $k_i = 4$. Therefore we may assume each m_i is in $P_{12} \cup \mathcal{P}''$ and that each $k_i = 1$.

Since there are no loops in $\mathcal{P}''$ and $|\Lambda|$ must be a perfect square, we may assume for contradiction that $m_1 \stackrel{\text{def}}{=} q \in P_{12}$. Suppose $m_2, \ldots, m_{2\ell}$ are the remaining m_i's which q divides. Then for $i = 2, \ldots, 2\ell$, $m_i \in \mathcal{P}''$. Write $m_i = qp_i$, for $p_i \in P_{12}$, and let $q' \in P_1$ satisfy $(\frac{q}{q'}) = (\frac{p_i}{q'}) = -1$. Adding 2ℓ copies of $\{qq'\}$ to Λ and noting that $\{qq', qp_i\} \sim \{q'p_i\}$ allows us to assume without loss of generality that $\ell = 1$. It is now easy to verify that there can be no glue of of order q, so such a lattice Λ cannot be self-dualizable. This completes the proof of the Decomposition Theorem. QED

Part (i) of the Primary Decomposition Theorem characterizes the 'primary decompositions'. This theorem says that any similarity class contains exactly one of these. For convenience, $\{1\}$, rather than $\emptyset$, will be called the primary decomposition for the class self-dualizable lattices.

Thus the similarity classes of lattices are in a natural one-to-one correspondence with the finite subsets of

$$P_{12} \cup \mathcal{P}'' \cup (P_3 \times \{1,2,3\}).$$

In Table 6 can be found the primary decompositions of all (square-free) products of the primes ≤ 17. Use is made there of the enumeration r_i of $\mathcal{P}'$ given above.

Recall that there are two possibilities for an m_i in the primary decomposition of some lattice Λ: it can either be a prime, or the product of two primes. If a prime p equals one of the m_i in the primary decomposition of Λ, we say p is a *singlet* in that decomposition. Then $p \in P_{12}$ can be a singlet 0 or 1 times in the decomposition of Λ and $p \in P_3$ can be a singlet 0,1,2 or 3 times.

For example, $\{6\} \sim \{2, 3^3\}$, so 2 and 3 are the only singlets, occurring 1 and 3 times respectively. On the other hand, no prime is a singlet for $\{10\}$.

Inductively collecting the previous theorems, we get the following powerful result. Thm.10(a) tells us that to verify that two lattices are similar, it suffices to compare the number of times each prime is a singlet in their primary decompositions. Until now the test was that the two primary decompositions themselves be equal — this apparently is more work than is necessary. Thms.10(b-e) tell us how to quickly compute the number of times each prime will be a singlet, given an orthogonal decomposition for the lattice.

Theorem 3.2.10: Let Λ and Λ' be any rational lattices. Then:

(a) $\Lambda \sim \Lambda'$ iff $|\Lambda|\,|\Lambda'|$ is a perfect (rational) square and, for each $p \in P_1 \cup P_3$, p is a singlet in the primary decomposition of Λ the same number of times as it is a singlet in the primary decomposition of Λ';

(b) If $p \in P_1 \cup P_3$ and p does not divide N, then p cannot be a singlet in the prime decomposition of N;

Table 6: Primary Decompositions Involving Primes p Less Than 19

$p \le 3$: $2*3 \sim \{2\ \ 3^3\}$.

$p \le 5$: $r_1 \stackrel{\text{def}}{=} 2*5$ *irred.;* $3*5 \sim \{2\ \ 3^3\ \ r_1\}$; $2*3*5 \sim \{2\ \ 3\ \ 5\}$.

$p \le 7$: $2*7 \sim \{2\ \ 7\}$; $3*7 \sim \{3\ \ 7^3\}$; $5*7 \sim \{2\ \ r_1\ \ 7^3\}$;
$2*3*7 \sim \{2\ \ 3^3\ \ 7^3\}$; $2*5*7 \sim \{2\ \ 5\ \ 7^3\}$; $3*5*7 \sim \{3^3\ \ 5\ \ 7\}$;
$2*3*5*7 \sim \{3\ \ 7\ \ r_1\}$.

$p \le 11$: $2*11 \sim \{2\ \ 11^3\}$; $3*11 \sim \{3^3\ \ 11\}$; $5*11 \sim \{5\ \ 11\}$;
$7*11 \sim \{7\ \ 11^3\}$; $2*3*11 \sim \{2\ \ 3\ \ 11^3\}$; $2*5*11 \sim \{11^3\ \ r_1\}$;
$2*7*11 \sim \{2\ \ 7\ \ 11\}$; $3*5*11 \sim \{2\ \ 3\ \ 11\ \ r_1\}$; $3*7*11 \sim \{3^3\ \ 7^3\ \ 11^3\}$;
$5*7*11 \sim \{2\ \ r_1\ \ 7^3\ \ 11^3\}$; $2*3*5*11 \sim \{2\ \ 3^3\ \ 5\ \ 11^3\}$;
$2*3*7*11 \sim \{2\ \ 3\ \ 7^3\ \ 11\}$; $2*5*7*11 \sim \{2\ \ 5\ \ 7^3\ \ 11\}$;
$3*5*7*11 \sim \{3\ \ 5\ \ 7\ \ 11^3\}$; $2*3*5*7*11 \sim \{3^3\ \ 7\ \ 11\ \ r_1\}$.

$p \le 13$: $r_2 \stackrel{\text{def}}{=} 2*13$ *irred.;* $3*13 \sim \{3\ \ 13\}$; $r_3 \stackrel{\text{def}}{=} 5*13 \sim \{r_1\ \ r_2\}$;
$7*13 \sim \{2\ \ 7^3\ \ r_2\}$; $11*13 \sim \{2\ \ 11^3\ \ r_2\}$; $2*3*13 \sim \{3^3\ \ r_2\}$;
$2*5*13 \sim \{2\ \ 5\ \ 13\}$; $2*7*13 \sim \{2\ \ 7^3\ \ 13\}$; $2*11*13 \sim \{2\ \ 11\ \ 13\}$;
$3*5*13 \sim \{2\ \ 3^3\ \ 5\ \ r_2\}$ $3*7*13 \sim \{2\ \ 3\ \ 7\ \ r_2\}$ $3*11*13 \sim \{2\ \ 3^3\ \ 11^3\ \ r_2\}$
$5*7*13 \sim \{5\ \ 7\ \ 13\}$; $5*11*13 \sim \{2\ \ 11^3\ \ 13\ \ r_1\}$; $7*11*13 \sim \{7^3\ \ 11\ \ 13\}$
$2*3*5*13 \sim \{3\ \ 13\ \ r_1\}$; $2*3*7*13 \sim \{2\ \ 3^3\ \ 7\ \ 13\}$;
$2*3*11*13 \sim \{2\ \ 3\ \ 11\ \ 13\}$; $2*5*7*13 \sim \{2\ \ 7\ \ r_1\ \ r_2\}$;
$2*5*11*13 \sim \{5\ \ 11\ \ r_2\}$; $2*7*11*13 \sim \{7^3\ 11^3\ r_2\}$;
$3*5*7*13 \sim \{2\ \ 3^3\ \ 7^3\ \ 13\ \ r_1\}$; $3*5*11*13 \sim \{3\ \ 5\ \ 11^3\ \ 13\}$;
$3*7*11*13 \sim \{3^3\ \ 7\ \ 11\ \ 13\}$; $5*7*11*13 \sim \{2\ \ 5\ \ 7\ \ 11\ \ r_2\}$;
$2*3*5*7*13 \sim \{3\ \ 5\ \ 7^3\ \ r_2\}$; $2*3*5*11*13 \sim \{2\ \ 3^3\ \ 11\ \ r_1\ \ r_2\}$;
$2*3*7*11*13 \sim \{3\ \ 7\ \ 11^3\ \ r_2\}$; $2*5*7*11*13 \sim \{7\ \ 11^3\ \ 13\ \ r_1\}$;
$3*5*7*11*13 \sim \{3\ \ 7^3\ \ 11\ \ r_1\ \ r_2\}$ $2*3*5*7*11*13 \sim \{2\ \ 3^3\ \ 5\ \ 7^3\ \ 11^3\ \ 13\}$

$p \le 17$: $2*17 \sim \{2\ \ 17\}$; $3*17 \sim \{2\ \ 3^3\ \ r_1\ \ r_4\}$; $r_4 \stackrel{\text{def}}{=} 5*17$ *irred.;*
$7*17 \sim \{2\ \ 7^3\ \ r_1\ \ r_4\}$; $11*17 \sim \{2\ \ 11^3\ \ r_1\ \ r_4\}$; $13*17 \sim \{13\ \ 17\}$;
$2*3*17 \sim \{3\ \ r_1\ \ r_4\}$; $2*5*17 \sim \{5\ \ r_1\ \ r_4\}$; $2*7*17 \sim \{7^3\ \ r_1\ \ r_4\}$;
$2*11*17 \sim \{11\ \ r_1\ \ r_4\}$; $2*13*17 \sim \{17\ \ r_2\}$; $3*5*17 \sim \{3\ \ 5\ \ 17\}$;
$3*7*17 \sim \{3^3\ \ 7\ \ 17\}$; $3*11*17 \sim \{3\ \ 11^3\ \ 17\}$;
$3*13*17 \sim \{2\ \ 3^3\ \ 13\ \ r_1\ \ r_4\}$;

$5*7*17 \sim \{5\ \ 7\ \ 17\}$; $5*11*17 \sim \{2\ \ 11^3\ \ 17\ \ r_1\}$; $5*13*17 \sim \{5\ \ r_1\ \ r_2\ \ r_4\}$;
$7*11*17 \sim \{7^3\ \ 11\ \ 17\}$; $7*13*17 \sim \{7\ \ r_1\ \ r_2\ \ r_4\}$;
$11*13*17 \sim \{11\ \ r_1\ \ r_2\ \ r_4\}$; $2*3*5*17 \sim \{3^3\ \ 17\ \ r_1\}$;
$2*3*7*17 \sim \{2\ \ 3\ \ 7\ \ 17\}$; $2*3*11*17 \sim \{2\ \ 3^3\ \ 11\ \ 17\}$;
$2*3*13*17 \sim \{2\ \ 3\ \ r_1\ \ r_2\ \ r_4\}$; $2*5*7*17 \sim \{7\ \ 17\ \ r_1\}$;
$2*5*11*17 \sim \{2\ \ 5\ \ 11\ \ 17\}$; $2*5*13*17 \sim \{2\ \ 13\ \ r_4\}$;
$2*7*11*17 \sim \{2\ \ 7^3\ \ 11^3\ \ 17\}$; $2*7*13*17 \sim \{7\ \ 13\ \ r_1\ \ r_4\}$;
$2*11*13*17 \sim \{11^3\ \ 13\ \ r_1\ \ r_4\}$; $3*5*7*17 \sim \{3\ \ 7^3\ \ r_4\}$;
$3*5*11*17 \sim \{2\ \ 3^3\ \ 5\ \ 11^3\ \ r_1\ \ r_4\}$; $3*5*13*17 \sim \{3\ \ 17\ \ r_1\ \ r_2\}$;
$3*7*11*17 \sim \{2\ \ 3\ \ 7\ \ 11\ \ r_1\ \ r_4\}$; $3*7*13*17 \sim \{2\ \ 3^3\ \ 7^3\ \ 17\ \ r_2\}$;
$3*11*13*17 \sim \{2\ \ 3\ \ 11\ \ 17\ \ r_2\}$; $5*7*11*17 \sim \{2\ \ 5\ \ 7\ \ 11\ \ r_1\ \ r_4\}$;
$5*7*13*17 \sim \{2\ \ 7^3\ \ 13\ \ 17\ \ r_1\}$; $5*11*13*17 \sim \{5\ \ 11\ \ 13\ \ 17\}$;
$7*11*13*17 \sim \{7\ \ 11^3\ \ 13\ \ 17\}$; $2*3*5*7*17 \sim \{3^3\ \ 5\ \ 7^3\ \ r_1\ \ r_4\}$;
$2*3*5*11*17 \sim \{2\ \ 3\ \ 11\ \ r_4\}$; $2*3*5*13*17 \sim \{2\ \ 3^3\ \ 5\ \ 13\ \ 17\}$;
$2*3*7*11*17 \sim \{3^3\ \ 7\ \ 11^3\ \ r_1\ \ r_4\}$; $2*3*7*13*17 \sim \{2\ \ 3\ \ 7^3\ \ 13\ \ 17\}$;
$2*3*11*13*17 \sim \{2\ \ 3^3\ \ 11^3\ \ 13\ \ 17\}$; $2*5*7*11*17 \sim \{2\ \ 7\ \ 11^3\ \ r_4\}$;
$2*5*7*13*17 \sim \{5\ \ 7^3\ \ 17\ \ r_2\}$; $2*5*11*13*17 \sim \{2\ \ 11^3\ \ 17\ \ r_1\ \ r_2\}$;
$2*7*11*13*17 \sim \{7\ \ 11\ \ 17\ \ r_2\}$; $3*5*7*11*17 \sim \{2\ \ 3^3\ \ 7^3\ \ 11\ \ 17\ \ r_1\}$;
$3*5*7*13*17 \sim \{2\ \ 3\ \ 5\ \ 7\ \ 13\ \ r_1\ \ r_4\}$ $3*5*11*13*17 \sim \{3^3\ \ 11\ \ 13\ \ r_4\}$
$3*7*11*13*17 \sim \{2\ \ 3\ \ 7^3\ \ 11^3\ \ 13\ \ r_1\ \ r_4\}$ $5*7*11*13*17 \sim \{2\ \ 7^3\ \ 11^3\ \ r_2\ \ r_4\}$
$2*3*5*7*11*17 \sim \{2\ \ 3\ \ 5\ \ 7^3\ \ 11^3\ \ 17\}$ $2*3*5*7*13*17 \sim \{3^3\ \ 7\ \ r_2\ \ r_4\}$
$2*3*5*11*13*17 \sim \{2\ \ 3\ \ 5\ \ 11^3\ \ r_1\ \ r_2\ \ r_4\}$;
$2*3*7*11*13*17 \sim \{2\ \ 3^3\ \ 7^3\ \ 11\ \ r_1\ \ r_2\ \ r_4\}$;
$2*5*7*11*13*17 \sim \{5\ \ 7^3\ \ 11\ \ 13\ \ r_1\ \ r_4\}$;
$3*5*7*11*13*17 \sim \{2\ \ 3^3\ \ 5\ \ 7\ \ 11^3\ \ 17\ \ r_2\}$;
$2*3*5*7*11*13*17 \sim \{3\ \ 7\ \ 11\ \ 13\ \ 17\ \ r_1\}$.

(c) If $p \in P_3$ and p does not divide N, then in the primary decomposition of Np, p is a singlet

$$(\frac{N}{p}) \pmod 4 \text{ times;}$$

(d) If $p \in P_1$ and p does not divide N, then in the primary decomposition of Np, p is a singlet

$$\frac{1+(\frac{N}{p})}{2} \quad \text{times;}$$

(e) Assume N is square-free. Let n_2 be the number of times 2 divides N. Let p_i be the primes in P_1 dividing N and define $N_i = N/p_i$. Then in the primary decomposition of N, 2 is a singlet

$$n_2 + \sum_i \frac{1-(\frac{N_i}{p_i})}{2} \pmod 2 \text{ times.}$$

Proof Thm.10(a) follows from Thm.1.4 and the following observation:

Let $m_i = p_{2i-1}p_{2i} \in \mathcal{P}''$, $i = 1, \ldots, \ell$. Then for any $p \in P_1$ satisfying $(\frac{p}{p_i}) = -1 \; \forall i$,

$$\{m_1, \ldots, m_\ell\} \sim \{p_1 p, \ldots, p_{2\ell} p\} \sim \{p'_1 p, \ldots, p'_{2k} p\}$$

where p'_i are the distinct primes occurring an odd number of times in $\{p_1, \ldots, p_{2\ell}\}$.

Thm.10(b) follows immediately from the theorems leading up to Thm.9. Inductively using Thm.4(iii) and eq.(1*b*) gives us Thm.10(c). Thm.10(d) follows from Thm.4(i) and by noting that when $p_i \in P_{12}$ satisfy $(\frac{p_i}{p_j}) = -1 \; \forall i < j$, then

$$\{p_1 \cdots p_{2k}\} \sim \{p_1 p_2, \ldots, p_{2k-1} p_{2k}\}$$

$$\{p_1 \cdots p_{2k+1}\} \sim \{p_1, p_2, \ldots, p_{2k}\}.$$

Thm.10(e) is just combinatorics. QED

Examples of the power of Thm.10 are given in the proofs of Thms.4.1-4.4. Thm.10 can be used to construct the primary decomposition of any orthogonal decomposition, and hence of any lattice. Together with Thm.1.2 this gives a complete classification of rational equivalence. A more standard classification is given by the weak Hasse principle (namely, that two lattices are rationally equivalent iff they are equivalent over the p-adic rationals for all primes $p \leq \infty$ — *e.g.* see pp.76-78 of [**CAS**]), or equivalently by the following theorem (see *e.g.* p.372 of [**CS1**]):

Theorem 3.2.11: $\Lambda \overset{Q}{\approx} \Lambda'$ iff

(i) Λ and Λ' have the same dimension and signature,

(ii) $|\Lambda|\,|\Lambda'|$ is a perfect (rational) square, and

(iii) for each prime $p \geq 3$, Λ and Λ' have the same p-excess (mod 8).

This theorem is proved, and p-excess is defined, in pp.370-372 of [CS1]. It implies the following:

Theorem 3.2.12: $\Lambda \sim \Lambda'$ iff

(i) $|\Lambda|\,|\Lambda'|$ is a perfect (rational) square, and

(ii) for each prime $p \geq 3$, Λ and Λ' have the same p-excess (mod 8).

The 2-excesses of Λ and Λ' may differ here. What is interesting here is that, using the fact that $\Lambda \sim \Lambda'$ iff their primary decompositions are equal, Thm.10 immediately gives us Thm.12, and Thm.11 then follows from Thm.1.2. Thus we have derived a central result of the theory of quadratic forms from a more geometric lattice perspective. (Of course, conversely, Thm.11 can be used to derive the theorems in this section).

3.3 Integral Coordinates

In this section we consider a question addressed in [**GAN**] and (independently!) a couple years later in [CS4]. Some of the history of this problem is given in [CS4]. The title for this section comes from the Conway-Sloane paper. Although this section is interesting in its own right, some of the results obtained here will be useful in Sec.4.

Definition 3.3.1: An n-dimensional Euclidean lattice Λ is said to be (ℓ,k)-*integrable* if $\Lambda^{(\ell)}$ is integrally equivalent to some sublattice of I_{n+k}. A lattice Λ with signature (m,n) is called $(\ell;k_+,k_-)$-*integrable* if $\Lambda^{(\ell)}$ is integrally equivalent to some sublattice of $I_{m+k_+,n+k_-}$.

In other words, Λ is (ℓ,k)-integrable iff the vectors in Λ can be given coordinates $\ell^{-1/2}(x_1,\ldots,x_{k+n})$, where all x_i are integers. We will be most interested in the cases where $k,k_+,k_- \geq 0$ and $\ell > 0$ are integers, and Λ is integral.

As an example, consider $\Lambda = A_1 \approx \{(2)\}$. Clearly in this case Λ is not $(1,0)$-integrable. However it is $(1,1)$-integrable: *e.g.* map the norm 2 vectors $\pm r_1$ to $(\pm1,\pm1)$. In fact, Lagrange's Theorem (see *e.g.* p.47 of [**SER**]) is the statement that for any positive integer N, the one-dimensional lattice $\{(N)\}$ is $(1,3)$-integrable.

Note that A_1 is $(2,0)$-integrable; however A_n for $n \geq 2$ is not $(\ell,0)$-integrable for any ℓ. For example, let $r_1 = -e_1+e_2$ and $r_2 = -e_2+e_3$ be the usual basis of A_2 and suppose we can give them coordinates $\ell^{-1/2}(a,b)$ and $\ell^{-1/2}(c,d)$, respectively, for integers a,b,c,d (which we may take to be relatively prime). That means $a^2+b^2 = c^2+d^2 = 2\ell$ and $ac+bd = -\ell$. The second of these tells us that $\ell \in \mathbf{Z}$, so the first implies $a \equiv b$ and $c \equiv d \pmod 2$. But $\text{odd}^2+\text{odd}^2 \equiv 2 \pmod 4$ and $\text{even}^2+\text{even}^2 \equiv 2 \pmod 4$, so we must have $a \equiv b \equiv c \equiv d \equiv 1 \pmod 2$. The first equation then implies ℓ is odd and the second implies ℓ is even. This contradiction means that no such coordinates can be found for r_1 and r_2 — *i.e.* that A_2 is not $(\ell,0)$-integrable. The proof for $n > 2$ follows from this. However it is easy to see that each A_n is $(1,1)$-integrable (the definition of A_n given in Sec.1.5 expresses it as a sublattice of I_{n+1}; this can also be seen from the gluing $\{A_n,(n+1)\}[1,1] \approx I_{n+1}$).

Finally, consider the lattice E_6. By a determinant check it is easy to show that it also cannot be $(\ell,0)$-integral for any ℓ (see *e.g.* Thm.2(iv)). Suppose for contradiction that it is $(1,k)$-integrable for some $k \geq 0$. Let r_i, $i = 1,\ldots,6$, be the root vectors of E_6, as in Table 2, and let e_j, for $j = 1,\ldots,6+k$, be an orthonormal basis for I_{6+k}. Each of the basis vectors r_i has norm 2, so must have the form

$\pm e_i \pm e_j$ for $i \neq j$.

Without loss of generality let $r_2 \to e_1 - e_2$. Then $r_2 \cdot r_3 = -1$ tells us that we may take $r_3 \to e_2 - e_3$. Similarly, $r_4 \cdot r_3 = -1$ and $r_4 \cdot r_2 = 0$ implies either $r_4 \to -e_1 - e_2$ or $r_4 \to e_3 - e_4$. However, in the first case coordinates for r_5 cannot be found, because of both $r_5 \cdot r_4 = -1$ and $r_5 \cdot r_2 = 0$. Thus we must have $r_4 \to e_3 - e_4$.

Now look at r_6. From $r_6 \cdot r_3 = -1$ and $r_6 \cdot r_2 = r_6 \cdot r_4 = 0$ we get either $r_6 \to -e_1 - e_2$ (in which case coordinates for r_1 cannot be found) or $r_6 \to e_3 + e_4$ (in which case coordinates for r_5 cannot be found).

Therefore E_6 cannot be $(1, k)$-integrable. A similar argument establishes this for E_7 and E_8. However, see Thm.1 and Thm.3.

Theorem 3.3.1: Let Λ be any n-dimensional Euclidean rational lattice. Then Λ is (ℓ, k)-integral for some integers ℓ, k. Moreover, ℓ may be taken to be a perfect square, and k may be chosen to be $3n$.

Proof Let $\{b_i\}$ be any basis for the lattice and let $\{v_j\}$, $j = 1, \ldots, n$, be an orthogonal basis for any orthogonal decomposition of Λ (we may choose the norms $v_j^2 \in \mathbf{Z}$). Then there is an invertible $n \times n$ $\mathbf{Q}$-matrix $B = (B_{ij})$ satisfying

$$b_i = \sum_{j=1}^{n} B_{ij} v_j, \qquad \text{for } i = 1, \ldots, n$$

(in fact B^{-1} is the $\mathbf{Z}$-matrix expressing the v_j's as linear combinations of the b_i's).

Now any orthogonal lattice (*e.g.* $\langle v_1, \ldots, v_n \rangle$) is obviously $(1, k)$-solvable — in fact $k = 3n$ works by Lagrange's Theorem. Choose a nonzero $\ell' \in \mathbf{Z}$ so that $\ell' B$ is a $\mathbf{Z}$-matrix. Then providing the vectors v_j with integral coordinates induces coordinates $1/\ell'(x_{i1}, \ldots, x_{in})$ for the basis vectors b_i, where $x_{ij} \in \mathbf{Z}$. In other words, we have shown that Λ is (ℓ'^2, k)-integrable. QED

Of course, in this Euclidean case, the E_6 example considered above shows that in general we must have both $\ell > 1$ and $k > 0$. Contrast this with the indefinite case, considered in Thm.3.

Thms.1.10 and 4.3 allow us a quicker (but less elementary) proof of Thm.1 (in fact they allow us to strengthen Thm.1 — see Thm.2(vii)). See also the proof for Thm.3 given below.

Some easy consequences of the definitions and previous results are collected in Thm.2.

Theorem 3.3.2: Let Λ be any n-dimensional Euclidean rational lattice. Then:

(i) if Λ is (ℓ, k)-integrable, then it is (ℓ, k')-integrable for any $k' \geq k$; if Λ_0 is any sublattice of Λ, then it too is (ℓ, k)-integral;

(ii) if Λ is both (ℓ, k)-integral and (ℓ', k')-integral, then it is also $(\ell+\ell', k+k')$-integral;

(iii) Λ is self-dualizable iff Λ is $(4^m, 0)$-integrable for some m;

(iv) for integral Λ, Λ is $(\ell, 0)$-integral for some integer ℓ, iff $\Lambda^{(|\Lambda|)}$ is self-dualizable; if n is even, this condition reduces to Λ being self-dualizable;

(v) Λ self-dual and $(1, k)$-integral for some k, implies $\Lambda \approx I_n$;

(vi) A_n is $(1,1)$-integrable; D_n is $(1,0)$-integrable; E_6, E_7 and E_8 are (2,2)-, (2,1)- and (2,0)-integrable, respectively; D_{4n} is (2,0)-integrable;

(vii) Λ is $(\ell, 3)$-integrable (if Λ is integral we may also take ℓ to be a power of 4); and

(viii) Λ is $(1,0)$-integrable if Λ is integral and has dimension $n \leq 5$.

Proof (i) is obvious. To see (ii), let an arbitrary vector $v \in \Lambda$ have integral coordinates $1/\sqrt{\ell}(x_1, \ldots, x_k)$ and $1/\sqrt{\ell'}(x'_1, \ldots, x'_{k'})$. Then it is trivial to verify that it can be given integral coordinates $1/\sqrt{\ell+\ell'}(x_1, \ldots, x_k, x'_1, \ldots, x'_{k'})$ (*i.e.* that all dot products will be preserved).

Both (iii) and (iv) are immediate consequences of Thm.1.10.

To see (v), let $\Lambda' \approx \Lambda$ be such that $\Lambda' \subseteq I_{n+k}$. Define Λ'' to be the sublattice $\Lambda'^{\perp}$ of I_{n+k}. Then by Thm.1.6.2, $|\Lambda''| = 1$, so by Thm.1.4.2 $\Lambda' \oplus \Lambda'' = I_{n+k}$. The desired result now follows from the uniqueness of direct sum decomposition

(Thm.1.2.3).

To see that D_{4n}^{+}, and hence E_8, is $(2,0)$-integrable, see Table 4. (i) now tells us that E_6 and E_7 are $(2,2)$- and $(2,1)$-integrable, respectively.

(vii) follows from Thm.4.3 and Thm.1.10. (viii) now follows from (vii). QED

So far we have constructed only Euclidean lattices. The indefinite case is much simpler, because the indefinite self-dual lattices are basically unique.

Theorem 3.3.3: Let Λ be any integral lattice. Then Λ is $(1;1,2)$-integrable.

Proof By Thm.4.3, Λ is similar to a 3-dimensional indefinite lattice $\{(m_1),(m_2);(m_3)\}$, where the m_i are positive integers, not all even. Then $\Lambda \oplus \{(m_3);(m_1),(m_2)\}$ is indefinite, odd and self-dualizable; hence it glues to $I_{m+1,n+2}$. QED

Of course, Λ is also $(1;2,1)$-integrable. Λ in Thm.3 may be either Euclidean or indefinite.

Thms.1 and 3 can be expressed in the following equivalent way:

Corollary 3.3.4: Let A be any symmetric $n \times n$ **Z**-matrix and A' any symmetric positive definite **Z**-matrix. Then there exist **Z**-matrices M and M' (not necessarily square), a positive integer ℓ, and a matrix G of the form $\mathrm{diag}\{+1,\ldots,+1,-1,\ldots,-1\}$ such that

$$A = M\,G\,M^t \quad \text{and} \quad A' = \ell\, M'\, M'^t.$$

These results were also obtained in [**CS4**], although in most cases their proofs were different. Their main interest was in establishing the following theorem.

Theorem 3.3.5: Let $\varphi(\ell)$ be the smallest dimension n for which there exists an n-dimensional integral lattice which is not (ℓ,k)-integrable for any k. Then:

(i) $\varphi(1) = 6$;

(ii) $\varphi(2) = 12$;

(iii) $\varphi(3) = 14$;

(iv) $21 \le \varphi(4) \le 25$;

(v) $16 \le \varphi(5) \le 22$;

(vi) $\varphi(\ell) \le 4\ell + 2$, ($\ell$ odd);

(vii) $\varphi(\ell) \le 2\pi e\ell\ (1 + o(1))$, ($\ell$ even);

(viii) $\varphi(\ell) \ge 2\log\ \log\ell/\log\ \log\ \log\ell\ (1 + o(1))$; and

(ix) any 24-dimensional Type II lattice, and *any* self-dual lattice in dimensions < 24, is 4-integrable.

Thm.5(i)-(viii) is Theorem 1, and Thm.5(ix) is Theorems 17 and 18, in [**CS4**]. We will use Thm.5(ix) in the following section to obtain bounds for the m_i's in orthogonal decompositions of lattices.

3.4 Some Consequences and Examples

To help illustrate the usefulness of the preceding analysis, we have included here a number of its consequences. This section will thus be a little more disjointed than the previous ones.

We will generally adopt the convention of Sec.3.2 and write $\{m_1, \ldots, m_n\}$ for the n-dimensional lattice $\{(m_1), \ldots, (m_n)\}$.

Because root lattices are so effective at constructing other lattices, convenient expressions for their similarity classes should come in handy. Hence:

Example 3.4.1: *The root lattices* From Table 4 we can read off $D_n \sim E_8 \sim \{1\}$, $E_7 \sim \{2\}$, and $E_6 \sim \{3\}$.

It is trivial to prove $I_1^{(n+1)} \oplus A_n$ is self-dualizable (in fact it glues to I_{n+1}); hence $A_n \sim \{(n+1)^3\} \stackrel{\text{def}}{=} \{n+1, n+1, n+1\}$. (End of Ex.1)

Obviously, Thm.1.5 allows us to quickly test whether a given base lattice is self-dualizable. This is useful for example in the gluing construction of strings described in Sec.6.3. One simple but effective constraint was that the lattice must be integral and have a perfect (integral) square as its determinant (*e.g.* A_2 is not self-dualizable). But by no means is this a sufficient condition. For example, $A_n \oplus A_n$ has determinant $(n+1)^2$, but we see from Ex.1 that it is similar to $\{n+1, n+1\}$, which is not always similar to $\{1\}$ (see Thm.2.2). In fact, for any prime p, $A_{p-1} \oplus A_{p-1}$ is self-dualizable iff $p = 2$ or $p \equiv 1 \pmod 4$, in spite of the fact that its determinant is always a perfect square.

The Niemeier lattices (see Table 5) provide several other examples. For example, by Ex.1 $A_{17} \oplus E_7 \sim \{(18)^3, 2\} \sim \{(2)^3, 2\} \sim \{1\}$, so $A_{17}E_7$ is self-dualizable and indeed glues to one of the Niemeier lattices. This test quickly confirms that all Niemeier root lattices are self-dualizable, but it alone cannot distinguish between self-dual and self-dualizable.

Similarity also allows us a quick test for $\Lambda_1 \subseteq \Lambda_2$: Λ_1 is saturated in Λ_2 only if both $\Lambda_1 \sim \Lambda_2$ and $|\Lambda_1|/|\Lambda_2|$ is a perfect square. Unfortunately it is difficult to find sufficient conditions for Λ_1 being saturated in Λ_2 (of course, rational shifting provides one).

Unrelated to these considerations, the Hilbert symbol (see Chapter III of [**SER**]) $(a,b)_Z$ of a and b relative to $\mathbf{Z}$ is defined to be $+1$ if $z^2 - ax^2 - by^2 = 0$ has a solution $(z,x,y) \neq (0,0,0)$ in Z_3, and otherwise equals -1. We obviously have

$$(a,b)_Z = +1 \text{ iff } \{a,b\} \sim \{ab\}.$$

Thm.1.10 has many applications to the theory of self-dualizable lattices, as we shall soon see. It is natural to ask if it can be generalized to *all* lattices. The following example (and also Ex.3) shows that it cannot. The technique developed here will also be useful elsewhere in this section.

Example 3.4.2: Consider the two-dimensional lattice Λ' given by the Gram matrix

$$\begin{pmatrix} 3 & 1 \\ 1 & 5 \end{pmatrix}.$$

It has determinant $14 = 2 \cdot 7$. Note that $\Lambda' \sim \{3, 3 \cdot 14\} \sim \{2, (7)^3\}$.

Suppose it contained a vector x of norm $2k^2$ for some $k \in \mathbf{Z}$. Then there would be another lattice vector, orthogonal to x, of norm $2 \cdot 14\ell^2$ for some $\ell \in \mathbf{Z}$, by Thm.1.6.2. In this case $\Lambda' \sim \{2, 7\}$, contradicting the previous calculation.

Similar conclusions apply if we assume $\exists x \in \Lambda_2$ with norm k^2, and hence $7k^2$ or $14k^2$. Therefore, in particular there cannot lie in Λ' a vector of norm $2^k 7^\ell$, so the desired m_i's cannot be found. (End of Ex.2)

It is now natural to ask if we at least can find m_i's, $i = 1, \dots, k$, such that $\Lambda \sim \{m_1, \dots, m_k\}$ (which is much weaker than saying that the m_i's constitute an orthogonal decomposition for Λ) and such that any prime p dividing at least one of the m_i's must also divide $2|\Lambda|$. For example the lattice Λ' in Ex.2 satisfies this, as do all the root lattices (see Ex.1).

Example 3.4.3: Consider the two-dimensional lattice Λ'' given by the Gram matrix

$$\begin{pmatrix} 3 & 1 \\ 1 & 6 \end{pmatrix}.$$

It has determinant 17. Note that $\Lambda'' \sim \{3, 3 \cdot 17\} \sim \{3, 2, (3)^3, 2 \cdot 5, 5 \cdot 17\} \sim \{2, 2 \cdot 5, 5 \cdot 17\}$, using Thm.2.5(ii). Since $(\frac{2}{17}) = +1$, Sec.2 tells us this is not similar to any of $\{1\}$, $\{2\}$, $\{17\}$ or $\{2, 17\}$. Hence, this answers in the negative the question posed in the previous paragraph. (End of Ex.3)

However, the following result holds.

Theorem 3.4.1: Given a lattice Λ, there exists a $q \in P_1$ and integers $m_1, \dots, m_k$ such that $\Lambda \sim \{m_1, \dots, m_k\}$ and such that any prime p dividing at least one of the m_i's must also divide $2q|\Lambda|$.

Proof Let $\{m_1, \dots, m_\ell\}$ be the primary similarity class of Λ, as defined in Thm.2.9. The singlets p must, by Thm.2.10, divide $2|\Lambda|$.

Choose $q \in P_1$ to satisfy $(\frac{p}{q}) = -1$ $\forall p$ dividing $\prod m_i$. Since for any $m_i = p'p''$, $\{m_i\} \sim \{p'q, p''q\}$, we get the theorem. QED

For example in Ex.3 $q = 5$ may be chosen. We can also demand that all m_i's be in $P_{12} \cup P_3 \cup \mathcal{P}'$, and that q divides only those m_i's in $\mathcal{P}'$.

The technique used in Ex.2 can now be applied to two-dimensional (Euclidean) lattices of any determinant (or, equivalently, positive definite binary quadratic forms of any discriminant). For example, let Λ be any two-dimensional lattice of determinant 3. Then automatically we know Λ cannot contain any vectors of norm $7k^2$, $11k^2$, $14k^2$, ..., for any integer k, for otherwise Thm.1 would be violated. These results can be explicitly verified by checking the two lattices of dimension 2 and determinant 3: A_2 and $I_1 \oplus I_1^{(3)}$.

Similarly, let Λ be any two-dimensional lattice of determinant m^2 for some $m \in \mathbb{Z}$. When m is large we can expect there to be a large number of candidates for Λ. Nevertheless, none of them can contain vectors whose norm is $3k^2$ or $6k^2$ (unless 3 divides m), or $7k^2$ (unless 7 divides m), *etc.*

In other words, from inspecting the discriminant alone one can immediately determine infinite families of numbers that a given positive definite binary quadratic form cannot represent.

We already know that $\Lambda^4 \sim \{1\}$. We can do better than that. Thm.2.9 implies the following surprising generalization of Thm.2.2:

Theorem 3.4.2: Suppose $\Lambda_1 \sim \Lambda_2$ are rational. Then $\Lambda_1 \oplus \Lambda_2 \sim \{1\}$ iff $\{|\Lambda_1|, |\Lambda_2|\} \sim \{1\}$, *i.e.* iff for each $p \in P_3$, the exact power α_p that p occurs in the expansion of $|\Lambda_1|$ into a product of primes is even.

Of course, a special case of this is $\Lambda_1 = \Lambda_2$, which can be used to immediately prove Thm.1.5.

Theorem 3.4.3: Any rational lattice Λ is similar to an odd 3-dimensional lattice

Λ' with any desired signature $\{(3,0),(2,1),(1,2),(0,3)\}$ (*e.g.* you may choose Λ' to be Euclidean).

Proof Let $\{m_1,\ldots,m_n\}$ be the primary decomposition of Λ. Let p_i be the distinct primes in P_{12} dividing an odd number of m_i's, and let p_i' be the primes in P_{12} dividing an even, nonzero number of m_i's. Let q_i be the primes in P_3 that are singlets an odd number of times, and let q_i' be the primes in P_3 that are singlets an even, nonzero number of times (hence twice).

We are interested in choosing primes $p,q \in P_1 \cup P_3$ so that

$$\Lambda \sim \Lambda_3 \stackrel{\text{def}}{=} \{p(\prod p_i)(\prod p_j'),\ q(\prod q_i)(\prod q_j'),\ pq(\prod p_i')(\prod q_j')\}.$$

It suffices, by Thm.2.10(a), to verify that p,q,p_i,p_i',q_i and q_i' all are singlets for Λ_3 the same number of times they are singlets for Λ.

That Λ_3 has the correct number of singlets p_i,p_i',q_i and q_i' fixes, respectively,

$$(\frac{p}{p_i}),\ (\frac{q}{p_i'}),\ (\frac{q}{q_i}),\ \text{and}\ (\frac{p}{q_i'})\ \text{for each}\ i.$$

That p and q cannot be singlets for Λ_3 gives us values for $(\frac{p}{q})$ and $(\frac{q}{p})$, depending on whether $p \in P_1$ or P_3 and whether $q \in P_1$ or P_3. It is possible, using quadratic reciprocity and Lemma 2.3, to choose $p,q \in P_1 \cup P_3$ in such a way that they possess all the desired quadratic residues.

The 3-dimensional lattice Λ_3 constructed above was Euclidean. An analogous argument shows that we could have imposed on it the signature $(+-+)$, for example — this would affect of course the various quadratic residues calculated there, but as before primes p and q could be found to satisfy all the necessary constraints. Finally, applying the above constructions to $\Lambda^{(-1)}$ shows that 3-dimensional lattices similar to Λ can also be found with the signatures $(1,2)$ and $(0,3)$. QED

Hence, any rational quadratic form is rationally equivalent to a diagonal form, all but three of whose entries are ± 1.

In general, the full three dimensions are required, as we see in this next theorem.

Theorem 3.4.4: Suppose $p, q \in P_1$ satisfy $(\frac{p}{q}) = -1$. Then any lattice similar to

$$\{p, q, pq\}$$

is at least 3-dimensional.

Proof Suppose for contradiction that $\{p, q, pq\} \sim \{a, b\}$, for some $a, b \in \mathbf{Z}$. We may without loss of generality assume a and b are square-free. Then Thm.1.4 implies $a = \pm b$. Therefore $\{p, q, pq, a, \pm a\} \sim \{1\}$. Thm.2.10(d) tells us that p is a singlet in the primary decomposition of the LHS, while it is not a singlet on the RHS; Thm.2.10(a) tells us that this is a contradiction. QED

A similar sort of proof shows that any lattice similar to $\{q, q, q\}$ is at least 3-dimensional, for any $q \in P_3$ satisfying $(\frac{2}{q}) = +1$.

Theorem 3.4.5: Let Λ_1 and Λ_2 be any two rational lattices of equal dimension n and signature. Then there exists a $\Lambda_2' \approx \Lambda_2$ such that $\Lambda_1 \cap \Lambda_2'$ is of dimension $\geq n - 2$.

In general, $n-2$ is the best that can be done; for example, consider $\Lambda_1 \sim \{1\}$ and Λ_2 to be equal to certain direct sums of the lattices given in Thm.4.

Using techniques such as those applied in the above proofs, Thm.2.10 can also be used to prove Thm.5, but a far simpler proof is Meyer's Theorem (see *e g.* pp.20-22 of [**MH**]): any indefinite lattice of dimension ≥ 5 represents 0 nontrivially.

Let Λ_1 and Λ_2 be two n-dimensional Euclidean self-dual lattices. From Thm.2.4.2 and Thm.2.4.1 we know it is possible to shift between them, the shift group Ω being isomorphic to $\Lambda_1/(\Lambda_1 \cap \Lambda_2')$, for some $\Lambda_2' \approx \Lambda_2$. Now by Thm.1.10, Λ_1 and Λ_2 contain saturated orthogonal sublattices equivalent to $\{4^{N_1}, \ldots, 4^{N_1}\}$ and $\{4^{N_2}, \ldots, 4^{N_1}\}$, respectively, for certain positive integers N_1 and N_2. Let $N = \max\{N_1, N_2\}$. Choosing $\Lambda_2' \approx \Lambda_2$ to contain the sublattice of Λ_1 equivalent to $\{4^N, \ldots, 4^N\}$ (we now know such a Λ_2' can be found), the order of Ω, namely $\sqrt{|\Lambda_1 \cap \Lambda_2'|}$, is seen from Lemma 1.4.1 to be a power of 2. Hence:

Theorem 3.4.6: Any two self-dual lattices of equal dimension and signature can be connected by a shift whose shift group has order a power of 2.

This is an important result. Now consider shifting I_n by a single shift vector of order 2. The result is another self-dual lattice Λ_1 of dimension n. Shift Λ_1 by another order 2 (with respect to Λ_1) shift vector; the result is another self-dual n-dimensional lattice Λ_2. Continuing in this way, a large number of lattices $\Lambda_3, \ldots, \Lambda_k, \ldots$, can be constructed. By choosing these order 2 shifts sufficiently carefully, how many of the n-dimensional self-dual Euclidean lattices are left out? Thm.6 implies that all of them can be generated in this way:

Corollary 3.4.7: Any n-dimensional Euclidean self-dual lattice Λ can be obtained from I_n by successively applying order 2 shifts.

Proof From Thm.6 we have $\Omega \cong C_{2^{k_1}} \times \cdots \times C_{2^{k_\ell}}$. Let $\{w_i\}$ be the corresponding generators of Ω — choose these in the glue gauge. Define $\Lambda_1 = I_n(\{2^{k_1-1}\omega_1\},0)$, $\ldots$, $\Lambda_{k_1} = \Lambda_{k_1-1}(\{\omega_1\},0), \Lambda_{k_1+1} = \Lambda_{k_1}(\{2^{k_2-1}\omega_2\},0)$, $\ldots$, $\Lambda_s = \Lambda_{s-1}(\{\omega_\ell\},0)$, where we have written s for $\sum_{i=1}^{\ell} k_i$.

Note that the shift group taking Λ_i to Λ_{i+1} has order 2. Thm.2.4.6 shows that $\Lambda = \Lambda_s$. QED

Recall the definition of neighbouring lattices given in Sec.2.4. Consider the neighbourhood graph $\mathcal{G}_n$ of all self-dual lattices of a given dimension n. It consists of a node for each such lattice, with two nodes being connected iff the corresponding two lattices are neighbours. Then Cor.7 is precisely the statement that $\mathcal{G}_n$ is connected for each n. Similar reasoning shows that $\mathcal{E}_n$, the neighbourhood graph of all Type II lattices in dimension n, is connected. $\mathcal{E}_8$, $\mathcal{E}_{16}$ and $\mathcal{E}_{24}$ are given on p.423 of [**CS1**]. (More precisely, the graph of $\mathcal{E}_{24}$ there was made with the enumeration of the 24-dimensional Type I lattices in mind. It can be shown however that two different 24-dimensional Type II lattices are neighbours iff their corresponding nodes in the graph in [**CS1**] are connected; moreover, every Niemeier lattice is a neighbour of itself, and Λ_{24} is not.)

Moreover, it is not difficult to show that in $8k$ dimensions, every Type I lattice is the neighbour of a Type II lattice (this in fact is established in the proof of Thm.10 given later in this section), and vice versa. For example, a neighbour of I_{24} is D_{24}^{+}, while the Leech lattice Λ_{24} has $O_{23} \oplus I_1$ (where O_{23} is the so-called *shorter Leech lattice*) as a neighbour (see Thm.2.4.10).

A particularly crude upper bound on the m_i's chosen in the orthogonal decomposition of Λ can be obtained effortlessly from Thm.1.4 and Cor.7:

Theorem 3.4.8: If Λ self-dualizable, then an orthogonal decomposition $\{m_1, \ldots, m_n\}$ can be found for it so that all m_i's have absolute values which are less than or equal to $|\Lambda| 4^{N-1}$, where N is the number of self-dual lattices with the same dimension and signature as Λ.

Proof Let Λ glue to the self-dual lattice Λ_1. First note that Thm.1.4 shows that $\Lambda_1^{(\ell)} \subset \Lambda$ for $\ell = |\Lambda|$, so ℓ times the orthogonal decomposition of Λ_1 is an orthogonal decomposition for Λ. Thus it suffices to prove the bound for the self-dual lattice Λ_1. Similarly, if $\Lambda_2 = \Lambda_1(\Omega)$, $\Lambda_1^{(\lambda)} \subset \Lambda_2$ for $\lambda = \|\Omega\|^2$. In particular, an order 2 shift will quadruple the m_i's. Cor.7 then gives us the bound. QED

Thm.3.5(ix) together with Thm.1.4 immediately imply that an upper bound for the m_i's of any self-dualizable lattice Λ of dimension $n \leq 24$ is $16|\Lambda|$ and is $4|\Lambda|$ for $n \leq 23$ — considerably better than Thm.8.

The following famous and important result has several proofs (see *e.g.* pp.127-130 of [**MH**]), but what is remarkable is the ease with which it follows from Thm.1.10. This is given below as the first proof. The second proof (which can be found on p.51 of [**SER**]) follows quickly from Thm.1.3.4 (although Serre's proof of this result, on pp.49-50 of [**SER**], is quite different from the one given in Sec.1.3 of this work).

Theorem 3.4.9: When Λ is Type II and Euclidean, 8 must divide its dimension.

First Proof Let Λ be even, self-dual and of dimension n. Then by Thm.6 we may write

$$\Omega \cong C_{2^{k_1}} \times \cdots \times C_{2^{k_\ell}}$$

for $k_1 \geq \cdots k_\ell > 0$, where Ω is the glue gauge shift group taking I_n to Λ. Let $\omega_i = \frac{1}{2^{k_i}} \sum \omega_{ij} e_j$ be the independent generators of Ω (e_j is the orthonormal basis for I_n), so $\omega_{ij} \in \mathbf{Z}$.

By Thm.1.4.9, there exist vectors $r_i \in I_n$ such that

$$r_i \cdot \omega_j \equiv \frac{1}{n_i} \delta_{ij} \pmod 1.$$

Consider $x = \sum 2^{k_i - k_\ell} r_i^2 \omega_i = \frac{1}{2^{k_\ell}} \sum x_i e_i$, so $x_i \in \mathbf{Z}$. Being a linear combination of the shift vectors, $x \in \Lambda$. We shall now show that all x_i are odd.

Let $y = \sum \omega_{im} r_i - e_m$ for some $1 \leq m \leq n$. Then $y \in I_n$. But $\omega_i \cdot y \in \mathbf{Z}$ for each i, so $y \in \Lambda$. Thus y^2 must be even, so $(\sum \omega_{im} r_i)^2$ is odd, which means $\sum \omega_{im}^2 r_i^2$ and hence $\sum \omega_{im} r_i^2$ are odd. Therefore x_m is odd for each m.

Now x, being a linear combination of the shift vectors, is in Λ. Thus x^2 is even, so $\frac{1}{2^{2k_\ell+1}} \sum x_i^2 \in \mathbf{Z}$ and hence $\sum x_i^2 \equiv 0 \pmod 8$. But x_i odd implies $x_i^2 \equiv 1 \pmod 8$, so we get the desired conclusion: $n \equiv 0 \pmod 8$. QED

Second Proof By Theorem 1.3.4, any $u \in \Lambda$ satisfying $u \cdot x \equiv x^2 \pmod 2$ for all $x \in \Lambda$, has norm $u^2 \equiv n \pmod 8$. But Λ is even, so we may take $u = 0$. Hence $n \equiv 0 \pmod 8$. (The indefinite case can be treated similarly.) QED

Two other proofs of Thm.9 involve theta functions (see p.109 of [**SER**]) and Gauss sums (see Appendix 4 in [**MH**]).

Theorem 3.4.10: If Λ is even and self-dualizable, then it glues to some Type II lattice if 8 divides $n_+ - n_-$, where (n_+, n_-) is the signature of Λ.

Proof Suppose that Λ glues to a Type I lattice Λ_1. Let $u \in \Lambda$ be as in Thm.1.3.4. Then since $8|(n_+ - n_-)$ by hypothesis, $\Lambda_2 = \Lambda_1(\{u/2\}, 0)$ is a Type II lattice.

$\Lambda \subset \Lambda_1$ and is even, so

$$\Lambda \subset \{x \in \Lambda_1 \mid x^2 \in 2\mathbf{Z}\} = \{x \in \Lambda_1 \mid x \cdot u \in 2\mathbf{Z}\} \subset \Lambda_2.$$

Since Λ and Λ_2 are both of dimension n, Λ is saturated in Λ_2 and hence must glue to it. QED

For example, D_n glues to I_n for all n; when (and only when) 8 divides n it also glues to the Type II lattice D_n^+.

Now consider even lattices Λ with odd determinant. A_{2k}, E_6 and E_8 are examples. It is trivial to show that such lattices are self-dualizable only in dimensions which are multiples of 8, since they cannot glue to odd lattices. By Thm.1.9 we get that $(A_2)^k$ and $(E_6)^k$ are self-dualizable iff 4 divides k. Similar reasoning gives us this surprising result:

Theorem 3.4.11: (a) Only in even dimensions can a lattice be both even and of odd determinant.

(b) Suppose in addition the lattice Λ has the property that for each $p \in P_3$, the exact power that p occurs in the expansion of $|\Lambda|$ into a product of primes, is even. Then the dimension must be a multiple of 4.

Note that the additional condition in Thm.11(b) implies that $|\Lambda| \equiv 1 \pmod 4$. We can illustrate this theorem with the lattices A_{n-1}: they are always even and have an odd determinant precisely when they are of even dimension. When the determinant condition in Thm.11(b) is satisfied, $n \equiv 1 \pmod 4$, so 4 divides the dimension $n - 1$.

An alternate proof of Thm.11(a) involves reducing the lattice mod 2. The result is a nondegenerate alternating bilinear form, and so is of even dimension (Prof. Jacques Hurtubise brought this point to my attention).

Chapter 4 THETA CONSTANTS

4.1 Jacobi θ-functions

The Jacobi θ-functions which we need are defined by:

$$\vartheta_3(z \mid \tau) \stackrel{\text{def}}{=} \sum_{m=-\infty}^{\infty} \exp(2m\pi i z + \pi i \tau m^2) \tag{4.1.1a}$$

$$\theta_2(\tau) \stackrel{\text{def}}{=} \sum_{m=-\infty}^{\infty} q^{(m+1/2)^2} = \exp(\pi i \tau/4)\,\vartheta_3(\frac{\tau}{2} \mid \tau) \tag{4.1.1b}$$

$$\theta_3(\tau) \stackrel{\text{def}}{=} \sum_{m=-\infty}^{\infty} q^{m^2} = \vartheta_3(0 \mid \tau) \tag{4.1.1c}$$

$$\theta_4(\tau) \stackrel{\text{def}}{=} \sum_{m=-\infty}^{\infty} (-q)^{m^2} = \vartheta_3(\frac{1}{2} \mid \tau) \tag{4.1.1d}$$

$$\psi_k(\tau) \stackrel{\text{def}}{=} \sum_{m=-\infty}^{\infty} q^{(m+1/k)^2} = \exp(\pi i \tau/k^2)\,\vartheta_3(\frac{\tau}{k} \mid \tau) \tag{4.1.1e}$$

$$q \stackrel{\text{def}}{=} \exp(\pi i \tau). \tag{4.1.1f}$$

An important and closely related function, the *Dedekind eta function* $\eta(\tau)$, is discussed in Sec.5.1 (see eqs.(5.1.2d,e)).

The theta functions considered here (and in Chapter 5) are of genus $g = 1$. In algebraic geometry and elsewhere it is convenient to generalize these functions to sums over $m \in I_g$, $g \geq 1$, where the complex parameter τ becomes a $g \times g$ complex matrix whose imaginary part is positive definite (see Chapter II of [**MUM**], or App.C of [**LSW**]). The results obtained here and in Ch.5 for $g = 1$ should generalize in a natural way to higher g — in particular, see pp.211-226 of [**MUM**].

This notation is taken from [**CS1**], with the exception that their $\theta_3(\xi \mid z)$ is our $\vartheta_3(\xi/\pi \mid z)$. The relationship between our notation and that in [**MUM**] is:

$$\vartheta_3(z \mid \tau) = \vartheta(z, \tau) \tag{4.1.2a}$$

$$\theta_2(\tau) = \vartheta_{10}(0, \tau) \stackrel{\text{def}}{=} \vartheta_{\frac{1}{2},0}(0, \tau) \tag{4.1.2b}$$

$$\theta_3(\tau) = \vartheta_{00}(0,\tau) \stackrel{\text{def}}{=} \vartheta_{0,0}(0,\tau) = \vartheta(0,\tau) \tag{4.1.2c}$$

$$\theta_4(\tau) = \vartheta_{01}(0,\tau) \stackrel{\text{def}}{=} \vartheta_{0,\frac{1}{2}}(0,\tau) \tag{4.1.2d}$$

$$\psi_k(\tau) = \vartheta_{\frac{1}{k},0}(0,\tau). \tag{4.1.2e}$$

Note that all these functions converge for $\tau \in \mathcal{H} \stackrel{\text{def}}{=} \{w \in \mathbf{C} \mid \text{Im } w > 0\}$: in fact, for Im $\tau > \epsilon > 0$,

$$|q^{n^2}| < \left(\exp[-\pi\epsilon]\right)^{n^2},$$

so by the Weierstrass M test (see p.343 of [**LEV**]) the series for $\theta_3(\tau)$ converges uniformly in each half plane Im $\tau > \epsilon$. Thus θ_3 is analytic in $\mathcal{H}$ (see p.336 of [**LEV**]). Similar arguments apply to the other functions in eq.(1). Therefore, throughout the rest of this paper τ will be taken to lie in $\mathcal{H}$.

By *theta constants* (the term *Thetanullwert* is sometimes used in the literature) is meant the restriction to $z = 0$. This chapter will be concerned primarily with the above theta constants θ_2, θ_3, θ_4, and ψ_k. In the following chapter we will extend these techniques and results to $z \neq 0$.

For convenience, define $\psi_\infty \stackrel{\text{def}}{=} \theta_3$. Note that

$$\psi_k = \theta_3 \text{ iff } 1/k \in \mathbf{Z} \tag{4.1.3a}$$

$$\psi_k = \psi_\ell \text{ iff either } 1/k \pm 1/\ell \in \mathbf{Z}. \tag{4.1.3b}$$

(See eq.(3f) below.) In general, we will thus be interested in ψ_k where k is rational and ≥ 2.

From the definitions the following basic identities can be readily verified:

$$\theta_4(\tau) = 2\theta_3(4\tau) - \theta_3(\tau) \tag{4.1.3c}$$

$$\psi_2(\tau) = \theta_2(\tau) \tag{4.1.3d}$$

$$\sum_{\ell=1}^{k} \psi_{k/\ell}(\tau) = \theta_3(\tau/k^2) \tag{4.1.3e}$$

$$\psi_{k/\ell}(\tau) = \psi_{k/(k-\ell)}(\tau). \tag{4.1.3f}$$

In fact, in the next section we will find that eqs.(3e, f) are special cases of much more general relations which reflect basic facts about lattices and their glues.

These first identities allow us to establish the following:

$$\psi_1(\tau) = \theta_3(\tau) \tag{4.1.4a}$$

$$\psi_2(\tau) = \theta_2(\tau) = \theta_3(\tau/4) - \theta_3(\tau) \tag{4.1.4b}$$

$$\psi_3(\tau) = \frac{1}{2}\{\theta_3(\tau/9) - \theta_3(\tau)\} \tag{4.1.4c}$$

$$\psi_4(\tau) = \frac{1}{2}\theta_2(\tau/4) \tag{4.1.4d}$$

$$\psi_6(\tau) = \frac{1}{2}\{\theta_2(\tau/9) - \theta_2(\tau)\}. \tag{4.1.4e}$$

For example, eq.(4b) is a consequence of eq.(3d, e) with $k = 2$.

Using eqs.(3c) and (4b), identities involving θ_2, θ_3 and θ_4 can always be reduced to identities involving only θ_3. For that reason we will discuss from now on only those identities involving θ_3 alone. We will also be interested in identities involving ψ_k, *provided* they lead to algebraic equations that can be used to *solve* for ψ_k explicitly in terms of θ_3. Thus (until Sec.4) we will be talking primarily about identities of the type given in eq.(4), but not those of the type eq.(3e, f).

The equations considered so far will be called *linear relations*. More generally, we refer to an identity involving the θ_i and the ψ_k functions as *of degree n*, if each term of the identity is composed of the product of n such functions. For example, the celebrated Jacobi identity $\theta_3(\tau)^4 = \theta_2(\tau)^4 + \theta_4(\tau)^4$, which is derived in Sec.3 in two different ways, is of degree 4. Thm.3.1 tells us that any *fundamental* identity is homogeneous in degree.

Linear relations for the remaining ψ_k, *e.g.* ψ_5, which can be used to solve ψ_k explicitly in terms of θ_3 as was done in eq.(4) for $k = 1, 2, 3, 4, 6$, cannot be obtained from eq.(3) alone. This is because eq.(3e) generally contains more than one 'unknown' function $\psi_{k/\ell}$ for the remaining values of k. For example, for $k = 5$ both $\psi_5 = \psi_{5/4}$ and $\psi_{5/2} = \psi_{5/3}$ are unknowns. Another way of saying this is that the Euler ϕ-function $\phi(k)$ is ≤ 2 iff $k = 1, 2, 3, 4, 6$. See also Thm.1.

Identities of higher degrees are discussed in Sec.3.

Let Λ be any lattice, and f be any 'rapidly decreasing smooth funtion' on $V_0(\Lambda)$. The Poisson summation formula directly implies

$$\sum_{x\in\Lambda} f(x) = \frac{1}{\sqrt{|\Lambda|}} \sum_{y\in\Lambda^*} \hat{f}(y), \tag{4.1.5a}$$

where $\hat{f}(x)$ is the Fourier transform of f. Choosing $\Lambda = I_1$ and using the fact that the Fourier transform of $g(x) = \exp[-\pi x^2]$ is $\hat{g}(y) = \exp[-\pi y^2]$, we get (see p.109 of [**SER**] for details)

$$\vartheta_3(z \mid -\frac{1}{\tau}) = (\tau/i)^{1/2} \exp(\pi i z^2 \tau)\, \vartheta_3(z\tau \mid \tau). \tag{4.1.5b}$$

This immediately implies

$$\theta_2(-\frac{1}{\tau}) = (\tau/i)^{1/2}\, \theta_4(\tau) \tag{4.1.6a}$$

$$\theta_3(-\frac{1}{\tau}) = (\tau/i)^{1/2}\, \theta_3(\tau) \tag{4.1.6b}$$

$$\theta_4(-\frac{1}{\tau}) = (\tau/i)^{1/2}\, \theta_2(\tau) \tag{4.1.6c}$$

$$\begin{aligned}\psi_{n/k}(-\frac{1}{\tau}) &= (\tau/i)^{1/2}\, \vartheta_3(\frac{k}{n} \mid \tau) \\ &= (\tau/i)^{1/2} \sum_{\ell=0}^{n-1} \zeta^{\ell k}\, \psi_{n/(\ell m)}(n^2\tau),\end{aligned} \tag{4.1.6d}$$

where in eq.(6d) $\zeta = e^{2\pi i/n}$ and $m \in \mathbf{Z}$ satisfies $mk \equiv 1 \pmod{n}$. For example, these allow eq.(4b) to imply eq.(3c).

Equations (1b-f) immediately give us

$$\begin{gathered}\theta_3(\tau+1) = \theta_4(\tau), \quad \theta_4(\tau+1) = \theta_3(\tau), \\ \theta_2(\tau+1) = \sqrt{i}\theta_2(\tau), \quad \psi_{n/k}(\tau+2n) = \psi_{n/k}(\tau).\end{gathered} \tag{4.1.7}$$

The question about which functions (*e.g.* which ψ_k) can be expressed in terms of θ_3 is a recurring one in this chapter. With this in mind, make the following definition. Let $\mathcal{T}_3^{(n)}$ denote the **C**-module of functions

$$\sum_{j=1}^{N} \alpha_j \theta_3(k_{1j}\tau) \cdots \theta_3(k_{nj}\tau), \tag{4.1.8}$$

where each $\alpha_j \in \mathbf{C}$, $k_{ij} \in \mathbf{R}$, and $k_{ij} > 0$ (otherwise convergence would fail for $\tau \in \mathcal{H}$). Define $\mathcal{T}_3$ to be the sum of all $\mathcal{T}_3^{(n)}$. For example, $\mathcal{T}_3^{(1)}$ and hence $\mathcal{T}_3$ contains $\theta_2, \theta_3, \theta_4$, and $\psi_2, \psi_3, \psi_4, \psi_6$. Define $\mathcal{T}_3^*$ to be the field of fractions of $\mathcal{T}_3$. We will be interested in using lattices to find functions in $\mathcal{T}_3$ which are identically zero. If the function lies in $\mathcal{T}_3^{(n)}$, it is said to be a *degree n identity*.

Definition 4.1.1: Call $F(\tau)$ *3-solvable* if it lies in $\mathcal{T}_3$ — *i.e.* if it can be expressed polynomially in terms of θ_3. Call $F(\tau)$ *rationally 3-solvable* if it lies in $\mathcal{T}_3^*$ — *i.e.* if it can be expressed as a fraction involving only θ_3.

We will also be interested in determining the 3-solvability of various theta functions/constants.

Theorem 4.1.1: For $k \geq 2$, ψ_k is 3-solvable iff $k = 2, 3, 4, 6$ and ∞.

One direction of this theorem is already known; the other follows as a special case of Thm.3.3.

Lemma 4.1.2: Let $n_1, n_2, \ldots$ and $n'_1, n'_2, \ldots$ be two increasing unbounded sequences of real numbers. Let $p_1(\tau), p_2(\tau), \ldots$ and $p'_1(\tau), p'_2(\tau), \ldots$ be two sequences of nonzero polynomials with complex coefficients. Then provided the series converge for $\tau \in \mathcal{H}$,

$$\sum_{k=1}^{\infty} p_k(\sqrt{\tau}) \exp(n_k \pi i \tau) = \sum_{k=1}^{\infty} p'_k(\sqrt{\tau}) \exp(n'_k \pi i \tau)$$

iff, for each k, $n_k = n'_k$ and $p_k = p'_k$.

This trivial result has some important consequences. Of course to prove it, suppose without loss of generality that $0 = n_1 \leq n'_1$ and consider the limit $z \to +\infty i$, *etc.*

As a final remark, eqs.(1b-f) allow us to directly confirm the validity of the following checks. For example, as $\tau \to +\infty i$ in the limit, $q \to 0^+$ so $\theta_2(k\tau) \to 0$,

$\theta_3(k\tau) \to 1$, $\theta_4(k\tau) \to 1$, and $\psi_\ell(k\tau) \to 0$ for any $k > 0$ and any ℓ for which $1/\ell \notin \mathbf{Z}$ (if $1/\ell \in \mathbf{Z}$, then $\psi_\ell = \theta_3$). Thus any identity in θ_2, θ_3, θ_4 and ψ_ℓ will remain a (numerical) identity after the above substitutions. For example the Jacobi identity gives $1^4 = 0^4 + 1^4$.

Now consider the limit $\tau \to i0^+$. To do this make the substitution $\tau \to -1/\tau$. Then from eqs.(6) and the previous test we get (provided our identity is homogeneous in degree) that in the original identity we can replace $\theta_2(k\tau) \to 1/\sqrt{k}$, $\theta_3(k\tau) \to 1/\sqrt{k}$, $\theta_4(k\tau) \to 0$, and $\psi_\ell(k\tau) \to 1/\sqrt{k}$ for any $k > 0$, and any $\ell \in \mathbf{Q}$.

As trivial as these tests seem, they have some practical value (*e.g.* in double-checking a conjectured identity), and some theoretical value. For example, we know that $\theta_3(\tau) = 2\theta_2(4\tau) + \theta_4(\tau)$, *i.e.* that θ_3 can be linearly solved for using *both* θ_2 *and* θ_4, but the above two tests show that this cannot be done using *only* θ_2 *or* θ_4.

It is possible of course to derive further tests. In Sec.3 we obtain some additional conditions identities in the Jacobi functions must obey.

4.2 Theta Constants of Lattices and their Glue Classes

In this section and chapter we will be most concerned with *integral Euclidean* lattices.

Given a class $[g]\Lambda_0$, its *theta constant* is defined to be

$$\Theta([g]\Lambda_0) \stackrel{\text{def}}{=} \sum_{x\in\Lambda_0} \exp[\pi i\tau(g+x)^2]$$
$$\Theta(\Lambda_0) \stackrel{\text{def}}{=} \Theta([0]\Lambda_0). \tag{4.2.1}$$

(We reserve the term *theta series* for the $z \neq 0$ case — see the following chapter.) As before $\exp(\pi i\tau)$ is usually written simply as q. The argument of the function $\Theta([g]\Lambda_0)$ is understood to be τ. If this is not the case, the argument will be explicitly included, as in $\Theta([g]\Lambda_0)(2\tau)$.

A similar argument to that given at the beginning of Sec.1 (using as well eqs.(1.1.4)) shows that these lattice theta constants converge and in fact are analytic in $\mathcal{H}$.

Note that Λ is even iff $\Theta(\Lambda)(\tau+1) = \Theta(\Lambda)(\tau)$ (see eq.(11)).

In this notation, the Jacobi θ-functions in eqs.(1.1) can be written as

$$\theta_2(2\tau) = \Theta([1]\{(2)\}) \stackrel{\text{def}}{=} \Theta([1]I_1^{(2)}) = \Theta([1]A_1) \tag{4.2.2a}$$

$$\theta_3(k\tau) = \Theta(\{(k)\}) \tag{4.2.2b}$$

$$\psi_{k/\ell}(k\tau) = \Theta([\ell]\{(k)\}). \tag{4.2.2c}$$

Theorem 4.2.1: (i) The theta constant of a direct sum of glue classes is the product of the theta constants of the individual classes:

$$\Theta([g_1,\ldots,g_k]\{\Lambda_1,\ldots,\Lambda_k\}) = \prod_{i=1}^{k} \Theta([g_i]\Lambda_i);$$

(ii) the theta constant for the disjoint union of glue classes is the sum of the theta constants of the individual classes:

$$\Theta\{\cup_{i=1}^{k}[g_i]\Lambda_i\} = \sum_{i=1}^{k} \Theta([g_i]\Lambda_i) \text{ provided } ([g_i]\Lambda_i)\cap([g_j]\Lambda_j) \neq \emptyset \text{ when } i \neq j;$$

(iii) $\Theta([\sqrt{\ell}g]\Lambda^{(\ell)})(\tau) = \Theta([g]\Lambda)(\ell\tau);\ \ \Theta(\Lambda^{(\ell)})(\tau) = \Theta(\Lambda)(\ell\tau).$

This theorem encapsulates the technique for generating identities investigated in this and the following chapter. Note that the glue classes of any glue decomposition are pairwise disjoint — hence the value of Thm.1(ii).

By Thm.1(i) we get that the theta constant of I_n is $\theta_3(\tau)^n$. In fact,

$$\Theta(\{(m_1),\ldots,(m_k)\})(\tau) = \theta_3(m_1\tau)\cdots\theta_3(m_k\tau) \tag{4.2.3a}$$

$$\Theta([\ell_1,\ldots,\ell_k]\{(m_1),\ldots,(m_k)\})(\tau) = \psi_{m_1/\ell_1}(m_1\tau)\cdots\psi_{m_k/\ell_k}(m_k\tau). \tag{4.2.3b}$$

Orthogonal decomposition (*i.e.* Cor.1.4.4) has the following immediate consequences.

Corollary 4.2.2: The theta constant of any glue class of any rational Euclidean lattice can be expressed polynomially in terms of θ_3 and ψ_k (with arguments τ scaled appropriately).

Corollary 4.2.3: The theta constants of the glue classes of all rational Euclidean lattices are all rationally solvable iff all ψ_k are rationally solvable.

Define $\mathcal{T}^{(n)}$ to be the **C**-module consisting of functions such as

$$\sum_{j=1}^{N} \alpha_j \psi_{k_{1j}}(m_{1j}\tau) \cdots \psi_{k_{nj}}(m_{nj}\tau).$$

Define $\mathcal{T}$ to be the sum of all $\mathcal{T}^{(n)}$. Then $\mathcal{T}^{(n)}$ contains $\mathcal{T}_3^{(n)}$ (defined in Sec.1), and Thm.1.1 tells us that $\mathcal{T}^{(1)}$ *properly* contains $\mathcal{T}_3^{(1)}$. Cor.2 tells us that the theta constant of any glue class of any n-dimensional rational Euclidean lattice lies in $\mathcal{T}^{(n)}$.

In this way, given a glue decomposition of a class or lattice, one obtains relations between the theta constants. This is how the identities between the Jacobi θ-functions are obtained in Sec.3. For example, some simple examples of gluing decompositions that we have already seen are as follows: $E_8 = D_8[1]$; $D_{m+n} = \{D_m, D_n\}[2,2]$; $A_2 \approx \{A_1, I_1^{(6)}\}[1,3]$. In glue class notations, these identities can be written as: $[0]E_8 = [0]D_8 \cup [1]D_8$; $[0]D_{m+n} = [0,0]\{D_m, D_n\} \cup [2,2]\{D_m, D_n\}$; $[0]A_2 \approx [0,0]\{A_1,(6)\} \cup [1,3]\{A_1,(6)\}$. These translate to the identities

$$\Theta(E_8) = \Theta(D_8) + \Theta([1]D_8) \tag{4.2.4a}$$

$$\Theta(D_{m+n}) = \Theta(D_m)\cdot\Theta(D_n) + \Theta([2]D_m)\cdot\Theta([2]D_n) \tag{4.2.4b}$$

$$\Theta(A_2) = \Theta(A_1)\cdot\Theta(\{(6)\}) + \Theta([1]A_1)\cdot\Theta([3]\{(6)\}). \tag{4.2.4c}$$

Finally, we note that an outer automorphism of the Lie algebra (a symmetry operation of the Dynkin diagram) preserves the theta constants. Hence,

$$\Theta([m]A_r) = \Theta([r+1-m]A_r) \tag{4.2.5a}$$

$$\Theta([1]D_r) = \Theta([3]D_r) \tag{4.2.5b}$$

$$\Theta([1]E_6) = \Theta([2]E_6) \tag{4.2.5c}$$

$$\Theta([1]D_4) = \Theta([2]D_4) = \Theta([3]D_4). \tag{4.2.5d}$$

Indeed, for any Λ and any glue $[g]$ in a glue group G, $[-g]$ is also in G, and:

$$\Theta([-g]\Lambda) = \Theta([g]\Lambda). \tag{4.2.5e}$$

Related to this is the useful fact that two (integrally) equivalent lattices have the same theta constants. Incidently, eqs.(5a-e) apply only to theta constants; their failure to hold for $z \neq 0$ is the reason the analysis and identities in the following chapter are more complicated than here. For example, the specific transformation connecting the two integrally equivalent lattices must be built into the identities derived from their equivalence: $\vec{z}$ is a vector while τ is a scalar.

We will use gluing decompositions to construct various identities; others (see *e.g.* App.C of [**LSW**]) have used automorphisms of lattices to obtain identities, but they have derived only a fraction of those we have.

An analogous calculation to that given in Sec.1 (see also p.109 of [**SER**]) shows that, given an n-dimensional lattice Λ,

$$\Theta(\Lambda)(-1/\tau) = \frac{(\tau/i)^{n/2}}{\sqrt{|\Lambda|}}\Theta(\Lambda^*)(\tau). \tag{4.2.6a}$$

From this it is possible to prove

$$\Theta([g]\Lambda)(-1/\tau) = \frac{(\tau/i)^{n/2}}{\sqrt{|\Lambda|}}\sum_{k=0}^{n-1}\zeta^k\Theta([kr]\Lambda_0)(\tau). \tag{4.2.6b}$$

Here, $[g]\Lambda$ is a glue of Λ of order n, $\zeta = e^{2\pi i/n}$, $r \in \Lambda^*$ satisfies $r \cdot g \equiv \frac{1}{n} \pmod 1$ (such a vector r always exists by Thm.1.4.9), and Λ_0 is the largest sublattice of Λ^* satisfying $g \cdot \Lambda_0 \subseteq \mathbf{Z}$. A special case of eq.(6b) is eq.(1.6d). Eq.(6b) can be used to derive the related formula given in [**OS**] (the converse is not obviously true).

Because $\Lambda^* = \Lambda[G]$ where $G = \Lambda^*/\Lambda$,

$$\Theta(\Lambda^*) = \sum_{g\in G} \Theta([g]\Lambda). \tag{4.2.7}$$

The usefulness of eqs.(6) and (7) follows if $\Theta(\Lambda)$ can be calculated independently of eq.(6) — *e.g.* if it can be expressed in terms of θ_3's (*i.e.* '3-solved'). Note that eq.(1.3e) is a special case of eq.(7), taking $\Lambda = I_1^{(k)}$ and using eqs.(2b, c). Also, eq.(1.3f) is a special case of eq.(5e).

The theta constants of the root lattices are all known (see for example pp.108-127 of [**CS1**]):

$$\Theta(D_n)(\tau) =\frac{1}{2}\{\theta_3(\tau)^n + \theta_4(\tau)^n\}, \tag{4.2.8a}$$

$$\Theta([1]D_n)(\tau) =\Theta([3]D_n)(\tau) = \frac{1}{2}\theta_2(\tau)^n, \tag{4.2.8b}$$

$$\Theta([2]D_n)(\tau) =\frac{1}{2}\{\theta_3(\tau)^n - \theta_4(\tau)^n\}, \tag{4.2.8c}$$

$$\Theta(D_n^*)(\tau) =\theta_2(\tau)^n + \theta_3(\tau)^n, \tag{4.2.8d}$$

$$\Theta(E_6)(\tau) =\phi_0(\tau)^3 + \frac{1}{4}\{\phi_0(\tau/3) - \phi_0(\tau)\}^3, \tag{4.2.8e}$$

$$\Theta(E_6^*)(\tau) =\frac{1}{3}[\phi_0(\tau/3)^3 + \frac{1}{4}\{3\phi_0(\tau) - \phi_0(\tau/3)\}^3], \tag{4.2.8f}$$

$$\Theta([1]E_6)(\tau) =\Theta([2]E_6)(\tau) = \frac{1}{2}\{\Theta(E_6^*)(\tau) - \Theta(E_6)(\tau)\}, \tag{4.2.8g}$$

$$\Theta(E_7)(\tau) =\theta_3(2\tau)^7 + 7\theta_3(2\tau)^3\theta_2(2\tau)^4, \tag{4.2.8h}$$

$$\Theta([1]E_7)(\tau) =\theta_2(2\tau)^7 + 7\theta_2(2\tau)^3\theta_3(2\tau)^4, \tag{4.2.8i}$$

$$\Theta(E_7^*)(\tau) =\theta_3(2\tau)^7 + \theta_2(2\tau)^7 + 7\theta_2(2\tau)^3\theta_3(2\tau)^3\theta_3(\tau/2), \tag{4.2.8j}$$

$$\Theta(E_8)(\tau) =\frac{1}{2}\{\theta_2(\tau)^8 + \theta_3(\tau)^8 + \theta_4(\tau)^8\}. \tag{4.2.8k}$$

Here $\phi_0(\tau) \stackrel{\text{def}}{=} \Theta(A_2)(\tau) = \theta_2(2\tau)\theta_2(6\tau) + \theta_3(2\tau)\theta_3(6\tau)$, as in [**CS1**].

The situation for A_n and its glue classes is by far the most complicated:

$$\Theta(A_{n-1})(\tau) = \frac{\sum_{k=0}^{n-1} \vartheta_3(\frac{k}{n}|\tau)^n}{n\theta_3(n\tau)}. \tag{4.2.9a}$$

Let $\zeta \stackrel{\text{def}}{=} e^{2\pi i/n}$. Then similarly we have for any $\ell = 1, \ldots, n-1$,

$$\Theta([\ell]A_{n-1})(\tau) = \frac{\sum_{k=0}^{n-1} \zeta^{-k\ell} \vartheta_3(\frac{k}{n}|\tau)^n}{n\psi_{n/\ell}(n\tau)}. \tag{4.2.9b}$$

Note that eq.(9a) is a special case of eq.(9b). Also, the $\vartheta_3(\frac{k}{n}|\tau)$'s occurring in eqs.(9) can be expressed if desired as linear combinations of ψ_ℓ's (see eq.(1.6d)). All of the expressions in eqs.(8) and (9) will be proved in the following section.

Call a lattice or glue class *3-solvable* if its theta constant is. Eqs.(8) show that each glue class of D_n, E_6, E_7, and E_8 is 3-solvable. However, it is not clear from eq.(9) that this is true of A_n. Indeed, we have been unable to prove the (rational) 3-solvability of its theta constant. However, for smaller n explicit '3-solutions' can be found, and are listed in Table 7.

More generally, it would be interesting to know if all lattices are 3-solvable. Certainly all 1-dimensional lattices (see eq.(2b)), and hence all orthogonal lattices (see eq.(3a)), are. Moreover, by Hecke's Theorem (see pp.187 and 192 of [**CS1**]) the same holds for all self-dual lattices:

Theorem 4.2.4 (*Hecke's Theorem*): Let Λ be any self-dual Euclidean lattice. Then:

(i) the theta constant $\Theta(\Lambda)(\tau)$ is a polynomial in $\Theta(I_1)(\tau) = \theta_3(\tau)$ and $\Theta(E_8)(\tau)$;

(ii) if Λ is Type II, $\Theta(\Lambda)(\tau)$ is also a polynomial in $\Theta(E_8)(\tau)$ and $\Theta(\Lambda_{24})(\tau)$.

These are usually proved using the theory of modular forms. (A discussion of general results concerning lattice theta constants as modular forms can be found on pp.382-8 of [**CAS**].) Note that in Thm.4(ii) any Niemeier lattice other than E_8^3 and $E_8 \oplus D_{16}^+$ will do instead of Λ_{24}.

To illustrate Thm.4, note that

$$\begin{aligned}\Theta(\Lambda_{24})(\tau) =& \Theta(E_8)(\tau)^3 \\ &- \frac{45}{16}(\Theta(E_8)(\tau)^2\theta_3(\tau)^8 - 2\Theta(E_8)(\tau)\theta_3(\tau)^{16} + \theta_3(\tau)^{24}) \end{aligned}\tag{4.2.10a}$$

$$\Theta(\Lambda_h)(\tau) = \frac{h}{30}\Theta(E_8)(\tau)^3 + (1 - \frac{h}{30})\Theta(\Lambda_{24})(\tau), \tag{4.2.10b}$$

Table 7: Known Theta Constants of A_n

Lattice	Weights	Name	Theta series
A_1	[0]	$\Theta_{(1,0)}$	$\theta_3(2)$
	[1]	$\Theta_{(1,1)}$	$\theta_3(\frac{1}{2})$
A_2	[0]	$\Theta_{(2,0)}$	$\theta_3(2)\theta_3(6)+\theta_2(2)\theta_2(6)$
	[1], [2]	$\Theta_{(2,1)}$	$\theta_2(2)\psi_6(6)+\theta_3(2)\psi_3(6)$
A_3	[0]	$\Theta_{(3,0)}$	$\frac{1}{2}\{\theta_3(1)^3+\theta_4(1)^3\}$
	[1], [3]	$\Theta_{(3,1)}$	$\frac{1}{2}\theta_2(1)^3$
	[2]	$\Theta_{(3,2)}$	$\frac{1}{2}\{\theta_3(1)^3-\theta_4(1)^3\}$
A_4	[0]	$\Theta_{(4,0)}$	$\sum_{k=0}^{3}\psi_{4/k}(20)\,\Theta_{(3,k)}(1)$
	[1], [4]	$\Theta_{(4,1)}$	??
	[2], [3]	$\Theta_{(4,2)}$	??
A_5	[0]	$\Theta_{(5,0)}$	$\Theta_{(2,0)}(1)^2\,\theta_3(6)+2\Theta_{(2,1)}(1)^2\,\psi_3(6)$
	[1], [5]	$\Theta_{(5,1)}$	$2\Theta_{(2,0)}(1)\,\Theta_{(2,1)}(1)\,\psi_6(6)+\Theta_{(2,1)}(1)^2\,\psi_2(6)$
	[2], [4]	$\Theta_{(5,2)}$	$2\Theta_{(2,0)}(1)\,\Theta_{(2,1)}(1)\,\psi_3(6)+\Theta_{(2,1)}(1)^2\,\theta_3(6)$
	[3]	$\Theta_{(5,3)}$	$\Theta_{(2,0)}(1)^2\,\psi_2(6)+2\Theta_{(2,1)}(1)^2\,\psi_6(6)$
A_6	[0]	$\Theta_{(6,0)}$	$\sum_{k=0}^{5}\psi_{6/k}(42)\,\Theta_{(5,k)}(1)$
	[1], [6]	$\Theta_{(6,1)}$	??
	[2], [5]	$\Theta_{(6,2)}$	??
	[3], [4]	$\Theta_{(6,3)}$	??
A_7	[0]	$\Theta_{(7,0)}$	$\sum_{k=0}^{3}\psi_{4/k}(8)\,\Theta_{(3,k)}(1)^2$
	[1], [7]	$\Theta_{(7,1)}$	??
	[2], [6]	$\Theta_{(7,2)}$	$\frac{1}{2}\{\Theta_{E_7^*}(1)-\Theta_{E_7}(1)\}$
	[3], [5]	$\Theta_{(7,3)}$	??
	[4]	$\Theta_{(7,4)}$	$\Theta_{E_7}(1)-\Theta_{(7,0)}(1)$
A_8	[0]	$\Theta_{(8,0)}$	$\sum_{k=0}^{5}\psi_{6/k}(18)\,\Theta_{(2,k)}(1)\,\Theta_{(5,k)}(1))$
	[1], [8]	$\Theta_{(8,1)}$	??
	[2], [7]	$\Theta_{(8,2)}$	??
	[3], [6]	$\Theta_{(8,3)}$	$\frac{1}{2}\{\Theta_E(1)-\Theta_{(8,0)}(1)\}$
	[4], [5]	$\Theta_{(8,4)}$	??
A_9, A_{10}			??
A_{11}	[0]	$\Theta_{(11,0)}$	$\sum_{k=0}^{5}\psi_{6/k}(12)\,\Theta_{(5,k)}(1)^2$
	[1], [11]	$\Theta_{(11,1)}$	??
	[2], [10]	$\Theta_{(11,2)}$	$\sum_{k=0}^{5}\Theta_{(5,k+2)}(1)\,\Theta_{(5,k)}(1)\,\psi_{6/(k+1)}(12)$
	[3], [9]	$\Theta_{(11,3)}$	??
	[4], [8]	$\Theta_{(11,4)}$	$\sum_{k=0}^{5}\Theta_{(5,k-2)}(1)\,\Theta_{(5,k)}(1)\,\psi_{6/(k+2)}(12)$
	[5], [7]	$\Theta_{(11,5)}$	??
	[6]	$\Theta_{(11,6)}$	$\sum_{k=0}^{5}\Theta_{(5,k)}(1)^2\,\psi_{6/(k-3)}(12)$

where in eq.(10b) Λ_h is a Niemeier lattice (*i.e.* even, 24-dimensional) with Coxeter number h.

An immediate consequence of Thm.4(i), using eqs.(2b) and (8k), is:

Corollary 4.2.5: Any Euclidean self-dual lattice is 3-solvable.

Let Λ be any integral Euclidean lattice; define Λ_e to be the set of all even-normed vectors in Λ. Then Λ_e is a saturated sublattice of Λ, of index 1 or 2. Note that

$$\Theta(\Lambda_e)(\tau) = \frac{1}{2}\{\Theta(\Lambda)(\tau) + \Theta(\Lambda)(\tau+1)\}. \qquad (4.2.11)$$

Hence if Λ is integral and 3-solvable, then (usually) so will Λ_e.

In addition, let Λ be integral and of even dimension, and suppose all of its glue classes $[g]\Lambda \in \Lambda^*/\Lambda$ are of order 2 (this last condition would be satisfied for example when $|\Lambda| = 2$). Then similar modular form arguments, based on results found for example around Cor.10.2 in [**MUM**], can be used to show that $\Theta(\Lambda)(\tau)$, and in fact each $\Theta([g]\Lambda)(\tau)$, can be written as a polynomial in $\theta_2(\tau)^2$, $\theta_3(\tau)^2$ and $\theta_4(\tau)^2$ Hence in this case each glue class $[g]\Lambda$ is 3-solvable.

However, it is unlikely that all 2-dimensional lattices are, as we shall see in the following two sections.

[**KT**] investigated the sublattices Λ_{24}^m of Λ_{24} invariant under an element $m \in M_{24} \subset \mathrm{Aut}(\Lambda_{24})$. There are 21 such sublattices, varying in dimension from 2 to 24. They also computed the theta constants of these. All of these turn out to be 3-solvable.

Generalizations of Hecke's Theorem for integral lattices of determinant higher than one are possible in some cases (*e.g.* $|\Lambda| = 2, 4$). For example, consider odd Λ with determinant $|\Lambda| = 4$, whose glues $g \in \Lambda^*$ all have integral norms g^2. $I_k \oplus D_{4\ell}$ are examples of such lattices (but there are many more). It can be shown from the above results that, for an n-dimensional lattice Λ with those properties, $\Theta(\Lambda)(\tau)$

lies in an $[\frac{n-1}{4}]+1$-dimensional vector space, generated by the theta constants of $I_{n-4}\oplus D_4,\ I_{n-8}\oplus D_8,\ldots$.

For another example, let Λ be any even Euclidean lattice of determinant 4, and suppose that Λ^* contains an odd normed vector. Then $\Theta(\Lambda)(\tau)$ is an element of an $[\frac{n-1}{8}]+1$-dimensional space, and is generated by $D_n,\ E_8\oplus D_{n-8},\ E_8^2\oplus D_{n-16},\ldots$.

Note that the condition that Λ^* contain an odd normed vector follows from the other conditions whenever 8 does not divide n. As an example, let Λ be any 22-dimensional even Euclidean lattice with $|\Lambda|=4$. Then the group Λ^*/Λ consists of four glue classes: one, $[0]\Lambda$, containing even normed vectors only; one, which we will call $[g_1]\Lambda$, containing odd norms only; and two, $[g_2]\Lambda$ and $[g_3]\Lambda$, which only contain vectors of norm $\equiv \frac{3}{2}$ (mod 2). It is not difficult to verify the following formulae:

$$\begin{aligned}\Theta([0]\Lambda) &= \frac{1-a-b}{2}(\theta_3^{22}+\theta_4^{22})+\frac{a}{2}(\theta_3^{14}+\theta_4^{14})(\theta_3^8-\theta_3^4\theta_4^4+\theta_4^4)\\ &+\frac{b}{2}(\theta_3^6+\theta_4^6)(\theta_3^8-\theta_3^4\theta_4^4+\theta_4^4)^2\\ &=\frac{1}{2}(\theta_3^{22}+\theta_4^{22})+\frac{-a-2b}{2}(\theta_3^{18}\theta_4^4+\theta_3^4\theta_4^{18}) \qquad (4.2.12a)\\ &+\frac{a+3b}{2}(\theta_3^{14}\theta_4^8+\theta_3^8\theta_4^{14})-b(\theta_3^{10}\theta_4^{12}+\theta_3^{12}+\frac{b}{2}(\theta_3^6\theta_4^{16}+\theta_3^{16}\theta_4^6),\end{aligned}$$

$$\begin{aligned}\Theta([g_1]\Lambda) &= \frac{1}{2}(\theta_3^{22}-\theta_4^{22})+\frac{-a-2b}{2}(\theta_3^{18}\theta_4^4-\theta_3^4\theta_4^{18}) \qquad (4.2.12b)\\ &+\frac{a+3b}{2}(\theta_3^{14}\theta_4^8-\theta_3^8\theta_4^{14})-b(\theta_3^{10}\theta_4^{12}-\theta_3^{12}\theta_4^{10})+\frac{b}{2}(\theta_3^6\theta_4^{16}-\theta_3^{16}\theta_4^6),\end{aligned}$$

$$\begin{aligned}\Theta([g_2]\Lambda) &= \Theta([g_3]\Lambda)=\theta_2^2[\frac{1}{2}(\theta_3^{22}-\theta_4^{22})+\frac{a+2b-5}{2}(\theta_3^{16}\theta_4^4-\theta_3^4\theta_4^{16})\\ &+\frac{10-3a-5b}{2}(\theta_3^{12}\theta_4^8-\theta_3^8\theta_3^{12})], \qquad (4.2.12c)\end{aligned}$$

where a,b are real parameters. This is explicitly given because it will be used in Sec.6.4. Of course Λ being 22-dimensional is not significant — analogous formulae exist for other dimensions. The difficult part of the derivation of these lies in showing that $\Theta([g_2]\Lambda)=\Theta([g_3]\Lambda)$ — this can be done using eq.(6b).

It is possible for two integrally inequivalent lattices to have the same theta constant. We shall call such lattices *theta equivalent*. An example is $E_8 \oplus E_8$ and D_{16}^+, as can be seen either by using the Jacobi identity and eqs.(8a, b, k), or by using Thm.4(ii). Two theta equivalent integral lattices are necessarily of the same type (*i.e.* even or odd). Putting $-1/\tau$ in their theta constants tell us that they are of the same dimension (see Thm.3.1) and determinant and that the theta constants of their duals also are equal.

This suggests that two such lattices lie in the same genus, or at least are rationally equivalent. We have been unable to prove this, however.

It is possible to show the following results using Hecke's Theorem.

Corollary 4.2.6: Let $N_k(\Lambda)$ be the number of norm k vectors in Λ. Let Λ and Λ' be two different Euclidean self-dual lattices of equal dimension n, and let N_k, N_k' be the numbers of norm k vectors in them. Define $\ell = [n/8]$. Then:

(i) for each integer $k \geq 0$ there exists a function $F_{n,k}$ of ℓ arguments such that $F_{n,k}(N_1, \ldots, N_\ell) = N_k$; and

(ii) Λ and Λ' are theta equivalent iff $N_1 = N_1', \ldots, N_\ell = N_\ell'$.

For example, for $n = 22$ $\ell = 2$ and it is possible to compute

$$F_{22,3}(N_1, N_2) = 248N_1 - 4N_2 + 5104, \tag{4.2.13a}$$

$$F_{22,4}(N_1, N_2) = 960N_1 - 12N_2 + 85932. \tag{4.2.13b}$$

Also, it is possible to show using Cor.6(ii) that it is not at all rare for *integrally* inequivalent lattices to be *theta* equivalent. In particular, from the explicit tables of self-dual lattices of dimensions < 24 (see *e.g.* Table 16.7 of [**CS1**]) it can be easily seen now that there are precisely 40 pairs of integrally inequivalent yet theta equivalent self-dual Euclidean lattices of dimension $n < 24$ (12 of these are without unit vectors); 10 triples (6 without unit vectors); 3 quadruples (2 without unit vectors); 1 quintuple (none have unit vectors); and 3 sextuples (none have unit vectors). The often quoted E_8^2, D_{16}^+ pair is the one with minimal dimension, but it is by no means unique (other pairs are the trivial $E_8^2 \oplus I_k, D_{16}^+ \oplus I_k$ for $k = 1, 2, \ldots, 8$, and the less trivial $A_{17}A_1[31]$ and $D_{10}E_7A_1[110, 301]$).

On the other hand, in the following chapter we show (in Thm.5.2.5) that the theta *series* of two lattices are equal only when the lattices are integrally equivalent.

4.3 Identities of the Jacobi Functions

If an integral lattice admits several glue decompositions, then the theta series computed from those decompositions (using Thm.2.1) must be equal. This is how identities between the Jacobi θ-functions can be obtained. If the lattice in question is of dimension n, then the identities will be of degree n. We are interested in *algebraically independent* identities — *i.e.* identities that cannot be obtained from each other and ones of lower degree *arithmetically* (*i.e.* through multiplication and addition) and/or by transforming τ (*e.g.* $\tau \to \tau + 1$, $\tau \to k\tau$, $\tau \to -1/\tau$). For example, eqs.(1.3*c*) and (1.4*b*) are not algebraically independent in this sense. Restricting attention to algebraically independent identities cleans up most of the maze of identities that can be found scattered throughout [**TM**], for example. We shall find shortly that the (quartic) Jacobi identity also is not algebraically independent, as it can be derived from first and second degree ones.

4.3.1 A general discussion of identities:

Recall the definition of the **C**-modules $\mathcal{T}$ (given in Sec.2 — it is generated by products of arbitrarily scaled ψ_k's) and $\mathcal{T}_3$ (given in Sec.1 — it is generated by products of arbitrarily scaled θ_3's). This subsection will concern functions in $\mathcal{T}$ which are identically zero for all $\tau \in \mathcal{H}$; in the remainder of the section we will focus on identities in $\mathcal{T}_3$.

We can write any identity in $\mathcal{T}$ in the form

$$\sum_{\ell=0}^{N} \alpha_\ell \Theta([g_\ell]\Lambda_\ell)(\tau) = 0, \tag{4.3.1}$$

where each $\alpha_\ell \in \mathbf{C}$, where the Λ_ℓ are orthogonal lattices, and where the glues g_ℓ are of finite order: $g_\ell \in \mathbf{Q} \otimes \Lambda_\ell$. For example, eqs.(2.6) and (1.6-7) are not identities in this sense.

Can Thm.2.1 (*i.e.* the gluing method) algebraically generate all identities in $\mathcal{T}$? This is an unsolved problem (see p.117 of [**MUM**]), and we will not be able to answer it in this chapter. However, in the following chapter we show that the analogous question for *theta series of full rank* is true (see Thm.5.3.6 there), but the analysis here is more difficult and the proofs do not carry over.

A few simple results are possible which allow us to simplify the form of the identity eq.(1).

Theorem 4.3.1: Any given identity in $\mathcal{T}$ can be expressed as the (finite) sum of identities in $\mathcal{T}$, each homogeneous in degree.

Theorem 4.3.2: Any given identity in $\mathcal{T}$ is a linear combination (over $\mathbf{C}$) of identities in $\mathcal{T}$ whose coefficients α_ℓ are all rational.

These theorems would have to hold if Thm.2.1 were to generate all identities in $\mathcal{T}$. Both these results follow from Lemma 1.2. In particular:

Consider a theta function identity, and look what happens when $-1/\tau$ is placed in it. Each dimension (*i.e.* each θ_3 or ψ_k) will contribute a factor $(\tau/i)^{1/2}$, by eqs.(1.6). Applying Lemma 1.2 gives us Thm.1.

Now for Thm.2. Let $\alpha_1, \ldots, \alpha_N$ be the non-zero coefficients. Say that α_i and α_j are equivalent if $\alpha_i/\alpha_j \in \mathbf{Q}$. Let $A_1, \ldots, A_k$ be the corresponding equivalence classes of coefficients, and let a_i be any element in A_i. Without loss of generality (*e.g.* by multiplying the identity by $1/a_i$) we may assume that $a_1 \in \mathbf{Q}$.

Suppose first that the a_i are linearly independent over $\mathbf{Q}$: *i.e.* $\sum r_i a_i = 0$ for $r_i \in \mathbf{Q}$ can only happen for $r_i = 0$. Then Lemma 1.2 implies that for each i, the sum of the terms whose coefficient is in A_i must be identically equal to zero. This is because, when we expand any $\theta_3(m\tau)$ or $\psi_k(m\tau)$ in terms of powers of q, the coefficients are rational (in fact they are 0,1 or 2). Thus, the original identity is a linear combination (with coefficients $a_i \in \mathbf{R}$) of identities with purely rational coefficients.

The case where the a_i are not linearly independent can easily be reduced to the case where they are by rewriting one of the a_i, say a_k, in terms of the others, and thus rewriting the terms with coefficient $\alpha \in A_k$ as a linear combination (over $\mathbf{Q}$) of terms with coefficients in the other A_i. The net result is that A_k is absorbed into the other A_i. If $a_1, \ldots, a_{k-1}$ are linearly independent, the argument in the above paragraph will apply. Otherwise, proceed again as in this paragraph.

An immediate consequence of Thm.2 (obtained by replacing τ with $-1/\tau$ in an identity) is that we may assume that the pairwise products of the determinants $|\Lambda_i|\,|\Lambda_j|$ are always a perfect (rational) square. This observation, together with Thm.1, suggests the possibility that an arbitrary identity in $\mathcal{T}$ can be written as a sum of identities eq.(1) whose lattices Λ_ℓ are all rationally equivalent (compare Thm.5.3.4) Indeed, we can show that this is so for all linear (see Thm.3) and quadratic (this follows from Thm.1.6.10(v)) identities in $\mathcal{T}$.

4.3.2 Linear identities:

Earlier in this section we dismissed as unsolved the question as to whether the gluing method algebraically generates all identities in $\mathcal{T}$. In $\mathcal{T}^{(1)}$ the situation is simple enough to tackle directly:

Theorem 4.3.3: Any linear identity in $\mathcal{T}$ is generated by eqs.$(1.3a, b, e)$.

Proof Consider the linear identity

$$\nu_1(\tau) = \sum_{i=1}^{N} \nu_{n_i}(\tau), \qquad \text{where} \tag{4.3.2a}$$

$$\nu_1(\tau) = \sum_{j=1}^{N_0} \alpha_{0j}\theta_3(m_j\tau), \quad \nu_{n_i}(\tau) = \sum_{j=1}^{N_i} \alpha_{ij}\psi_{\frac{n_i}{k_{ij}}}(m'_{ij}\tau), \tag{4.3.2b}$$

where all $\alpha_{ij} \neq 0$, $m'_{ij} > 0$, $0 < m_1 < \cdots < m_{N_0}$, and where $n_i, k_{ij} \in \mathbf{Z}$. We can assume the greatest common divisor $(n_i, k_{ij}) = 1$, and $1 < k_{ij} < \frac{n_i}{2}$. It clearly suffices to show that no such identity eq.(2) can exist.

It is not difficult to show that without loss of generality all $m_j, m'_{ij} \in \mathbf{Q}$, and hence that we may take all m'_{ij} to be integer multiples of n_i^2 and all m_j to be integers.

Suppose for now that $N_0 \geq 0$ — the case where there are no θ_3 can be handled similarly.

Define $\mathcal{P}(n_i)$ to be the set of all primes $p \equiv \pm 1 \pmod{n_i}$. We know that these sets are all infinite. Note that for each i, and any $M \in \mathbf{Z}$, $\nu_{n_i}(\tau)$ cannot represent Mp^2 (*i.e.* when expanded out, it cannot contain a term βq^{Mp^2}) for any sufficiently large $p \in \mathcal{P}(n_i)$.

Let $\mathcal{P} \stackrel{\text{def}}{=} \cap_{i=1}^{N} \mathcal{P}(n_i)$; by Dirichlet's Theorem on primes in arithmetic progressions (see Chapter VI of [**SER**]), $\mathcal{P}$ is infinite. Note that for all sufficiently large $p \in \mathcal{P}$, ν_1 represents $m_1 p^2$. However, by the preceding paragraph, $\sum \nu_{n_i}$ *cannot* represent $m_1 p^2$ for such p. This contradicts eq.(2a). QED

In proving Thm.3 we have also established Thm.1.1.

4.3.3 Quadratic identities:

It is possible to generate an infinite number of identities in $\mathcal{T}$, independent of each other and of the linear ones discussed earlier. For example, consider the gluing $\{(k), (k^2 - k)\}[1, k-1] \approx I_1 \oplus I_1^{(k-1)}$. It gives us the quadratic identity

$$\sum_{\ell=1}^{k} \psi_{k/\ell}(k\tau)\psi_{k/\ell}(\{k^2 - k\}\tau) = \theta_3(\tau)\theta_3(\{k-1\}\tau). \qquad (4.3.5)$$

This will be true for $k = 2, 3, 4, \ldots$. A multitude of other identities can be found.

For this reason we will consider in the remainder of this section those identities in $\mathcal{T}_3$, *i.e.* those that can be expressed as polynomials of θ_3 with scaled arguments (and, of course, θ_2, θ_4, and ψ_k for $k = 3, 4, 6$). The following section will address more general identities, when θ_3 for example enters non-polynomially.

Consider the gluings $\{(2),(2)\}[1,1] \approx I_2$ and $\{(8),(8)\}[2,2] \approx I_1 \oplus I_1^{(4)}$. They give us, respectively, the quadratic identities

$$\theta_2(\tau)^2 + \theta_3(\tau)^2 = \theta_3(\frac{1}{2}\tau)^2, \tag{4.3.6a}$$

$$\theta_3(8\tau)^2 + \theta_2(8\tau)^2 + \frac{1}{2}\theta_2(2\tau)^2 = \theta_3(\tau)\theta_3(4\tau), \tag{4.3.6b}$$

where in eq.(6b) we have used eq.(1.4d). Using eqs.(6a) and (1.4b), we can rewrite eq.(4.3.6b) as

$$\frac{1}{2}\theta_2(\frac{1}{2}\tau)^2 = \theta_2(\tau) \cdot \theta_3(\tau). \tag{4.3.6c}$$

But note that eq.(6c) is also obtainable from eqs.(6a) and (1.4b). This calculation shows that two different *lattice* equalities may result in algebraically equivalent *theta* identities.

Eq.(6a) is interesting for another reason. Replacing τ in it by $2 - 1/\tau$ and $-1/\tau$, respectively, gives us

$$-\theta_4(\tau)^2 + \theta_3(\tau)^2 = 2\theta_2(2\tau)^2, \tag{4.3.7a}$$

$$\theta_4(\tau)^2 + \theta_3(\tau)^2 = 2\theta_3(2\tau)^2. \tag{4.3.7b}$$

Then using eqs.(7a, b) and (6c) we can write

$$\theta_3^4(\tau) - \theta_4^4(\tau) = \{2\theta_2(2\tau)^2\}\{2\theta_3(2\tau)^2\} = \theta_2(\tau)^4. \tag{4.3.7c}$$

Hence *the Jacobi identity is not fundamental* in our sense, and can be derived from the *two*-dimensional lattice gluing $\{(2),(2)\}[1,1] \approx I_2$ (to some extent, this was recently realized as well by [**KT**]).

Note that any two-dimensional (integral) gluing

$$\{(m_1),(m_2)\}[G] \approx \{(n_1),(n_2)\}[G'] \tag{4.3.8a}$$

gives us an identity in $\mathcal{T}^{(2)}$ (using Thm.2.1 and eqs.(2.3)), and if the orders of all glues in G and G' are in $\{1,2,3,4,6\}$ this identity also lies in $\mathcal{T}_3^{(2)}$. Unfortunately, there are infinitely many gluing equivalences eq.(8a), even when we restrict the orders of the glues to $\{1,2,3,4,6\}$. However, we are interested in gluings eq.(8a) which lead to algebraically independent identities. The one-dimensional identities eqs.(1.3b,e) allow us to restrict our attention to gluings of the form

$$\{(ka),(kb)\}[a,b] \approx \{(\ell c),(\ell d)\}[c,d], \tag{4.3.8b}$$

where the glue orders $k,\ell \in \{1,2,3,4,6\}$ (so the identity lies in $\mathcal{T}_3^{(2)}$), where $a \leq b$ and $c \leq d$, and where $\{ka,kb\} \neq \{\ell c,\ell d\}$ (so the corresponding identity is not trivial). Moreover, Thm.2.2(iii) tells us it suffices to consider only the case where a, b and $[a,b]^2 = (a+b)/k$ have greatest common divisor 1.

Theorem 4.3.4: There are precisely 51 gluings of form eq.(8b); all these are listed in Table 8.

A sketch of the proof of this is given below. It turns out that all but one of these gluings (namely $\{(16),(32)\}[4,8] \approx \{(12),(96)\}[2,16]$) are of the following form:

Consider any order k gluing $\Lambda \stackrel{\text{def}}{=} \{(ka),(kb)\}[a,b]$, where $k \in \{2,3,4,6\}$. The above gluing defines an orthogonal decomposition $\{(ka),(kb)\}$ of Λ. Λ has a second orthogonal decomposition $\{(\ell c),(\ell d)\}$, obtained by choosing the vector there of norm ℓc to be the glue vector $(a/\sqrt{ka}, b/\sqrt{kb}) \in [a,b]\{(ka),(kb)\}$. This defines a second gluing $\{(\ell c),(\ell d)\}[c,d]$ of Λ. We are interested in the situations where this second glue also has its order ℓ in $\{1,2,3,4,6\}$.

We will now sketch a proof of Thm.4. It is a straightforward exercise to find all the gluings of the form described above (we must have $c=1$ there, as otherwise $\Lambda^{(1/c)}$ is still integral). The remainder of the proof consists in showing that apart from the one exception $\{(16),(32)\}[4,8] \approx \{(12),(96)\}[2,16]$, any gluing eq.(8$b$) must be of that form.

The key observation is that the minimal norm μ of the lattice on the left-hand side of eq.(8b) must be either ka or $(a+b)/k$; a similar comment holds for the minimal norm μ' of the lattice on the right-hand side of eq.(8b). Since μ must equal μ' (the two lattices are integrally equivalent), we get four (non-mutually exclusive) cases. If $\mu = ka$ and $\mu' = (c+d)/\ell$, or $\mu = (a+b)/k$ and $\mu' = \ell c$, then the gluing is of the form already considered (or else we are also in one of the other two cases). If $\mu = ka$ and $\mu' = \ell c$, a simple argument shows that $kb = \ell d$, so by Thm 1.4.1 $k = \ell$, $a = c$, $b = d$, and the corresponding identity trivially holds.

Therefore we may consider $\mu = (a+b)/k$ and $\mu' = (c+d)/\ell$. From Thm.1.4.1 we get $ab = cd$. If $k = \ell$ we get $a = c$ and $b = d$, which also yields a trivial identity. It is also easy to eliminate the case $k = 1$. Thus we may assume $1 < k < \ell$.

Let $x = (ma/\sqrt{ka}, nb/\sqrt{nb})$ be the vector in the lattice on the left-hand side of eq.(8b) which corresponds (under the integral equivalence) to the vector $(\sqrt{\ell c}, 0)$ on the right-hand side. Then $m \equiv n \pmod{k}$ and $x^2 = \ell c$. It is not difficult to eliminate the possibility that $x \in \{(ka)\} \oplus \{(kb)\}$ (*i.e.* $m \equiv n \equiv 0 \pmod{k}$). The equations $x^2 = \ell c$, $(a+b)/k = (c+d)/\ell$, and $ab = cd$, and the fact that $(k,\ell) \in \{(2,3),(2,4),(2,6),(3,4),(3,6),(4,6)\}$ can also be used to force either $k = 2$ and $m, n = \pm 1$ (which turns out to be equivalent to the gluing $\{(4),(8)\}[2,4] \approx \{(3),(24)\}[1,8]$, which is of the already considered form), or $k = 4$, $\ell = 6$, and $m, n = \pm 2$. This final case is realized only by the gluing $\{(16),(32)\}[4,8] \approx \{(12),(96)\}[2,16]$. This concludes the proof of Thm.4.

In Table 8 we list all of the identities obtainable from the gluings eq.(8b). In the appendix we discuss their algebraic independence from one another. The identities in the table have been divided into 33 groups. Identities are in the same group (*e.g.* the second and third identities in the table) if we have been unable to determine their algebraic dependence/independence from each other. Any identities in separate groups are known to be algebraically independent. Thus, Table 8 lists for us at least 33 algebraically independent quadratic identities. Included in the table are

Table 8: The Quadratic Theta Constant Identities

Lattices	Identity
$\{2,2\}[1,1] \approx I_2$ or $\{8,8\}[2,2] \approx \{1,4\}$	$\theta_2(1)^2 + \theta_3(1)^2 = \theta_3(\frac{1}{2})^2$
$\{3,6\}[1,2] \approx \{1,2\}$, $\{4,8\}[2,4] \approx \{3,24\}[1,8]$, or $\{12,24\}[2,4] \approx \{1,8\}$	$\theta_3(3)\,\theta_3(6) + 2\psi_3(3)\,\psi_3(6) = \theta_3\,\theta_3(2)$
$\{16,32\}[4,8] \approx \{3,96\}[1,32]$, $\{12,96\}[2,16] \approx \{3,96\}[1,32]$, $\{16,32\}[4,8] \approx \{12,96\}[2,16]$	$\frac{1}{2}\,\theta_2(1)\,\theta_2(2) =$ $\theta_2(3)\,\theta_3(24) + 2\psi_6(3)\,\psi_3(24)$
$\{4,12\}[1,3] \approx \{1,3\}$	$\theta_2(1)\,\theta_2(3) + \theta_4(1)\,\theta_4(3) = \theta_3(1)\,\theta_3(3)$
$\{9,9\}[3,3] \approx \{2,18\}[1,9]$ or $\{18,18\}[3,3] \approx \{1,9\}$	$\theta_3(1)^2 + 2\psi_3(1)^2 =$ $\theta_3(\frac{2}{9})\,\theta_3(2) + \theta_2(\frac{2}{9})\,\theta_2(2)$
$\{3,15\}[1,5] \approx \{2,10\}[1,5]$, $\{8,40\}[2,10] \approx \{3,60\}[1,20]$, or $\{6,30\}[1,5] \approx \{1,5\}$	$\theta_3(3)\,\theta_3(15) + 2\psi_3(3)\,\psi_3(15) =$ $\theta_3(2)\,\theta_3(10) + \theta_2(2)\,\theta_2(10)$
$\{12,15\}[4,5] \approx \{3,60\}[1,20]$, $\{48,60\}[8,10] \approx \{3,240\}[1,80]$	$\theta_3(4)\,\theta_3(5) + 2\psi_3(4)\,\psi_3(5) =$ $\theta_3(1)\,\theta_3(20) + 2\psi_3(1)\,\psi_3(20)$
$\{6,21\}[2,7] \approx \{3,42\}[1,14]$, $\{24,84\}[4,14] \approx \{3,168\}[1,56]$	$\theta_3(2)\,\theta_3(7) + 2\psi_3(2)\,\psi_3(7) =$ $\theta_3(1)\,\theta_3(14) + 2\psi_3(1)\,\psi_3(14)$
$\{4,44\}[1,11] \approx \{3,33\}[1,11]$, $\{6,66\}[1,11] \approx \{2,22\}[1,11]$	$\frac{1}{2}\,\theta_3(1)\,\theta_3(11) + \frac{1}{2}\,\theta_2(1)\,\theta_2(11) + \frac{1}{2}\,\theta_4(1)\,\theta_4(11) =$ $\theta_3(3)\,\theta_3(33) + 2\psi_3(3)\,\psi_3(33)$

Table 8: The quadratic identities (cont.)

Lattices	Identity
$\{4,28\}[1,7] \approx$ $\{2,14\}[1,7]$	$\frac{1}{2}\,\theta_3(1)\,\theta_3(7) + \frac{1}{2}\,\theta_2(1)\,\theta_2(7) + \frac{1}{2}\,\theta_4(1)\,\theta_4(7) =$ $\theta_3(2)\theta_3(14) + \theta_2(2)\theta_2(14)$
$\{12,20\}[3,5] \approx$ $\{2,30\}[1,15]$	$\frac{1}{2}\,\theta_3(3)\theta_3(5) + \frac{1}{2}\,\theta_2(3)\theta_2(5) + \frac{1}{2}\,\theta_4(3)\theta_4(5) =$ $\theta_3(2)\theta_3(30) + \theta_2(2)\theta_2(30)$
$\{6,10\}[3,5] \approx$ $\{4,60\}[1,15]$	$2\theta_3(6)\theta_3(10) + 2\theta_2(6)\theta_2(10) =$ $\theta_3(1)\,\theta_3(15) + \theta_4(1)\,\theta_4(15) + \theta_2(1)\,\theta_2(15)$
$\{20,28\}[5,7] \approx$ $\{3,105\}[1,35]$, $\{10,14\}[5,7] \approx$ $\{6,210\}[1,35]$	$\frac{1}{2}\,\theta_3(5)\theta_3(7) + \frac{1}{2}\,\theta_2(5)\theta_2(7) + \frac{1}{2}\,\theta_4(5)\theta_4(7)$ $= \theta_3(3)\theta_3(105) + 2\psi_3(3)\psi_3(105)$
$\{30,42\}[5,7] \approx$ $\{2,70\}[1,35]$, $\{15,21\}[5,7] \approx$ $\{4,140\}[1,35]$	$\theta_3(15)\theta_3(21) + \theta_2(15)\theta_2(21) + 2\psi_3(15)\psi_3(21) +$ $2\psi_6(15)\psi_6(21) = \theta_3\,\theta_3(35) + \theta_2\,\theta_2(35)$
$\{20,44\}[5,11] \approx$ $\{4,220\}[1,55]$	$\theta_2(5)\theta_2(11) + \theta_3(5)\theta_3(11) + \theta_4(5)\theta_4(11) =$ $\theta_2(1)\,\theta_2(55) + \theta_3(1)\,\theta_3(55) + \theta_4(1)\,\theta_4(55)$
$\{28,36\}[7,9] \approx$ $\{4,252\}[1,63]$	$\theta_2(7)\theta_2(9) + \theta_3(7)\theta_3(9) + \theta_4(7)\theta_4(9) =$ $\theta_2(1)\,\theta_2(63) + \theta_3(1)\,\theta_3(63) + \theta_4(1)\,\theta_4(63)$
$\{12,52\}[3,13] \approx$ $\{4,156\}[1,39]$	$\theta_2(3)\theta_2(13) + \theta_3(3)\theta_3(13) + \theta_4(3)\theta_4(13) =$ $\theta_2(1)\,\theta_2(39) + \theta_3(1)\,\theta_3(39) + \theta_4(1)\,\theta_4(39)$

Table 8: The quadratic identities (cont.)

Lattices	Identity
{30,186}[5,31] ≈ {6,210}[1,155]	$\theta_3(5)\,\theta_3(31) + \theta_2(5)\,\theta_2(31) + 2\psi_3(5)\,\psi_3(31) + 2\psi_6(5)\,\psi_6(31) =$ $\theta_3(1)\,\theta_3(155) + \theta_2(1)\,\theta_2(155) + 2\psi_3(1)\,\psi_3(155) + 2\psi_6(1)\,\psi_6(155)$
{42,174}[7,29] ≈ {6,1218}[1,203]	$\theta_3(7)\,\theta_3(29) + \theta_2(7)\,\theta_2(29) + 2\psi_3(7)\,\psi_3(29) + 2\psi_6(7)\,\psi_6(29) =$ $\theta_3(1)\,\theta_3(203) + \theta_2(1)\,\theta_2(203) + 2\psi_3(1)\,\psi_3(203) + 2\psi_6(1)\,\psi_6(203)$
{66,150}[11,25] ≈ {6,1650}[1,275]	$\theta_3(11)\,\theta_3(25) + \theta_2(11)\,\theta_2(25) + 2\psi_3(11)\,\psi_3(25) + 2\psi_6(11)\,\psi_6(25)$ $= \theta_3(1)\,\theta_3(275) + \theta_2(1)\,\theta_2(275) + 2\psi_3(1)\,\psi_3(275) + 2\psi_6(1)\,\psi_6(275)$
{78,138}[13,23] ≈ {6,1794}[1,299]	$\theta_3(13)\,\theta_3(23) + \theta_2(13)\,\theta_2(23) + 2\psi_3(13)\,\psi_3(23) + 2\psi_6(13)\,\psi_6(23)$ $= \theta_3(1)\,\theta_3(299) + \theta_2(1)\,\theta_2(299) + 2\psi_3(1)\,\psi_3(299) + 2\psi_6(1)\,\psi_6(299)$
{102,114}[17,19] ≈ {6,1938}[1,323]	$\theta_3(17)\,\theta_3(19) + \theta_2(17)\,\theta_2(19) + 2\psi_3(17)\,\psi_3(19) + 2\psi_6(17)\,\psi_6(19)$ $= \theta_3(1)\,\theta_3(323) + \theta_2(1)\,\theta_2(323) + 2\psi_3(1)\,\psi_3(323) + 2\psi_6(1)\,\psi_6(323)$
{18,54}[3,9] ≈ {2,54}[1,27], {9,27}[3,9] ≈ {4,108}[1,27]	$\psi_3(1)\,\psi_3(3) + \psi_6(1)\,\psi_6(3) = \psi_3(1)\,\theta_3(3) + \psi_6(1)\,\theta_2(3)$
{6,102}[1,17] ≈ {3,51}[1,17]	$\theta_3(2)\,\theta_3(34) + \theta_2(2)\,\theta_2(34) + 2\psi_3(2)\,\psi_3(34) +$ $2\psi_6(2)\,\psi_6(34) = \theta_3(1)\,\theta_3(17) + 2\psi_3(1)\,\psi_3(17)$
{30,78}[5,13] ≈ {3,195}[1,65], {15,39}[5,13] ≈ {6,390}[1,65]	$\theta_3(10)\,\theta_3(26) + \theta_2(10)\,\theta_2(26) + 2\psi_3(10)\,\psi_3(26) +$ $2\psi_6(10)\,\psi_6(26) = \theta_3(1)\,\theta_3(65) + 2\psi_3(1)\,\psi_3(65)$

Table 8: The quadratic identities (cont.)

Lattices	Identity
{42,66}[7,11]≈ {3,231}[1,77], {21,33}[7,11]≈ {6,462}[1,77]	$\theta_3(14)\theta_3(22)+\theta_2(14)\theta_2(22)+2\psi_3(14)\psi_3(22)+$ $2\psi_6(14)\psi_6(22)=\theta_3(1)\,\theta_3(77)+2\psi_3(1)\,\psi_3(77)$
{6,138}[1,23]≈ {4,92}[1,23]	$\theta_3(6)\theta_3(138)+\theta_2(6)\theta_2(138)+2\psi_3(6)\psi_3(138)+$ $2\psi_6(6)\psi_6(138)=\frac{1}{2}\,\theta_3(1)\,\theta_3(23)+\frac{1}{2}\,\theta_2(1)\,\theta_2(23)+\frac{1}{2}\,\theta_4(1)\,\theta_4(23)$
{18,126}[3,21]≈ {4,252}[1,63]	$\theta_3(18)\theta_3(126)+\theta_2(18)\theta_2(126)+2\psi_3(18)\psi_3(126)+$ $2\psi_6(18)\psi_6(126)=\frac{1}{2}\,\theta_3(1)\,\theta_3(63)+\frac{1}{2}\,\theta_2(1)\,\theta_2(63)+\frac{1}{2}\,\theta_4(1)\,\theta_4(63)$
{30,114}[5,19]≈ {4,380}[1,95]	$\theta_3(30)\theta_3(114)+\theta_2(30)\theta_2(114)+2\psi_3(30)\psi_3(114)+$ $2\psi_6(30)\psi_6(114)=\frac{1}{2}\,\theta_3(1)\,\theta_3(95)+\frac{1}{2}\,\theta_2(1)\,\theta_2(95)+\frac{1}{2}\,\theta_4(1)\,\theta_4(95)$
{20,76}[5,19]≈ {6,570}[1,95]	$\frac{1}{2}\theta_3(5)\theta_3(19)+\frac{1}{2}\,\theta_2(5)\theta_2(19)+\frac{1}{2}\,\theta_4(5)\theta_4(19)=$ $\theta_3(6)\,\theta_3(570)+\theta_2(6)\theta_2(570)+2\psi_3(6)\psi_3(570)+2\psi_6(6)\psi_6(570)$
{42,102}[7,17]≈ {4,476}[1,119]	$\theta_3(42)\theta_3(102)+\theta_2(42)\theta_2(102)+2\psi_3(42)\psi_3(102)+$ $2\psi_6(42)\psi_6(102)=\frac{1}{2}\,\theta_3(1)\,\theta_3(119)+\frac{1}{2}\,\theta_2(1)\,\theta_2(119)+\frac{1}{2}\,\theta_4(1)\,\theta_4(119)$
{28,68}[7,17]≈ {6,714}[1,119]	$\frac{1}{2}\theta_3(7)\theta_3(17)+\frac{1}{2}\,\theta_2(7)\theta_2(17)+\frac{1}{2}\,\theta_4(7)\theta_4(17)=$ $\theta_3(6)\theta_3(714)+\theta_2(6)\theta_2(714)+2\psi_3(6)\psi_3(714)+2\psi_6(6)\psi_6(714)$
{54,90}[9,15]≈ {4,540}[1,135]	$\theta_3(54)\theta_3(90)+\theta_2(54)\theta_2(90)+2\psi_3(54)\psi_3(90)+$ $2\psi_6(54)\psi_6(90)=\frac{1}{2}\,\theta_3(1)\,\theta_3(135)+\frac{1}{2}\,\theta_2(1)\,\theta_2(135)+\frac{1}{2}\,\theta_4(1)\,\theta_4(135)$
{66,78}[11,13]≈ {4,572}[1,143]	$\theta_3(66)\theta_3(78)+\theta_2(66)\theta_2(78)+2\psi_3(66)\psi_3(78)+$ $2\psi_6(66)\psi_6(78)=\frac{1}{2}\,\theta_3(1)\,\theta_3(143)+\frac{1}{2}\,\theta_2(1)\,\theta_2(143)+\frac{1}{2}\,\theta_4(1)\,\theta_4(143)$
{44,52}[11,13]≈ {6,858}[1,143]	$\frac{1}{2}\theta_3(11)\theta_3(13)+\frac{1}{2}\,\theta_2(11)\theta_2(13)+\frac{1}{2}\,\theta_4(11)\theta_4(13)$ $=\theta_3(6)\theta_3(858)+\theta_2(6)\theta_2(858)+2\psi_3(6)\psi_3(858)+2\psi_6(6)\psi_6(858)$

the lattice gluings that produced these identities. Each gluing produces an identity; when two gluings (such as $\{(2),(2)\}[1,1] \approx I_2$ and $\{(8),(8)\}[2,2] \approx \{(1),(4)\}$) are known to correspond to algebraically equivalent identities, both gluings are written adjacent to that identity.

We have found several theta constant identities in the mathematical literature ([**TM**] is the richest source of these, and [**KT**] also contains many – both used predominantly "Schroter's formula"), but all of those turn out to be algebraically equivalent to either the first, fourth or tenth identities in Table 8, or to be derivable solely from the linear identities.

In Table 8 we write for convenience $\theta_3(k)$ for $\theta_3(k\tau)$, *etc.*.

By the algebraic equivalence of θ_3 identities we mean in this paper that one can be obtained from the other and from lower degree identities through any combination of the following algebraic manipulations: (1) arithmetic (*i.e.* through multiplication and linear combinations of the other identities; and (2) transforming τ by $\tau \to -1/\tau$ or $\tau \to k\tau + \ell$, for $k \in \mathbf{Q}$, $\ell \in \mathbf{Z}$.

It is not difficult to show that, for example, the quadratic identities

$$\theta_2(2\tau)^2 + \theta_3(2\tau)^2 = \theta_3(\tau)^2, \qquad (4.3.9a)$$

$$\theta_2(\tau)\theta_2(3\tau) + \theta_4(\tau)\theta_4(3\tau) = \theta_3(\tau)\theta_3(3\tau), \qquad (4.3.9b)$$

are algebraically independent, by writing both identities entirely in terms of θ_3, and considering the ratios of the scalings of each of the terms. The ratios for eq.(9a) are 1 and 4; those for eq.(9b) are 4/3, 3, and 12. The transformations in (2) will at most affect these ratios by powers of 4. On the other hand, the identity eq.(6b) also has ratios 1 and 4; as we saw earlier it is algebraically equivalent to eq.(9a). This type of argument provides us with the 33 groupings in Table 8 discussed above.

Moreover, an identity in T_3 is independent of the linear relations iff, when it is expressed entirely in terms of θ_3 (using for example eqs.(1.4)), it does not reduce to the triviality '0=0'. This is because of Thm.3.

4.3.4 Higher-dimensional identities:

Lattice considerations can easily be used to derive the theta constant expressions for the root lattices. One way to derive these for D_n, E_6, E_7 and E_8 is to use their orthogonal decompositions (these are calculated in Table 4). Also, eq.(2.11) yields the constant for D_n, and those for its glue classes follow from $I_n = D_n[2]$ and eqs (2.5-7) Since $D_8^+ = E_8$, we immediately get that for E_8. The theta constant for E_6 follows from the equivalence $E_6 \approx \{A_2, A_2, A_2\}[1,1,1]$; the constants for its glue classes now follow from eqs.(2.5-7). Alternate expressions, and hence theta function identities, can be obtained using for example $E_8 \approx \{E_7,(2)\}[1,1] \approx \{E_6, A_2\}[1,1]$, $E_6 \approx \{D_5,(12)\}[1,3]$ and $E_7 \approx \{E_6,(6)\}[1,2]$.

Of course, A_n is trickier. Since their orthogonal decompositions are known (see Table 4), their theta constants (like that of any rational lattice) are in $\mathcal{T}$ and can be written down explicitly (but messily). Eq.(2.9) itself can be derived (by putting $-1/\tau$ in it) from $A^{*}_{n-1}{}^{(n)} \oplus I_1 \approx I_n^{(n)}[1,\ldots,1]$. But the easiest proof comes from looking at the standard embedding of A_{n-1} in $\mathbf{R}^n$ and using the projection operator

$$\frac{1}{n}\sum_{\ell=1}^{n} \zeta^{k\ell} = \begin{cases} 1 & \text{if } k \equiv 0 \pmod{n} \\ 0 & \text{otherwise} \end{cases}, \qquad (4.3.10)$$

where $\zeta = e^{2\pi i/n}$.

The expressions given in Table 7 were found using lattice equivalences such as $E_8 \approx A_8[3]$, $E_7 = A_7[4]$, $E_7^* = A_7[2]$, and

$$A_{n+m} \approx \{A_{n-1},(L),A_m\}[n-1,\ell,1], \qquad (4.3.11)$$

where $\ell = \frac{n+m+1}{(n+m+1,n)}$, and $L = \ell^2 n(m+1)/(n+m+1)$.

Of course each of the expressions in Table 7 imply identities when compared with eq.(2.9). Also, by Hecke's Theorem (Thm.2.4) self-dual gluings such as $E_7^2[1,1]$ produce other identities. Further sources of identities are:

$$\{A_n,(n+1)\}[1,1] \approx I_{n+1}, \qquad (4.3.12a)$$

$$\text{and } D_4^+ \approx I_4. \qquad (4.3.12b)$$

For example, the simplest derivation of Jacobi's identity is to read it off from eq.(12*b*).

Of course, root lattices are not the only source of lattice identities. The gluings of orthogonal lattices can yield great numbers of them in each dimension, using the methods used earlier for generating quadratic identities. These are too numerous and messy to explicitly write down. In addition, the results of the following section allow in some cases glues of order other than 2,3,4, or 6 to be used. An example will be given in the next section.

4.4 Theta Constants of Glue Classes

In this section we investigate the existence of polynomials that ψ_k satisfies. We will be particularly interested in polynomials with coefficients in $\mathcal{T}_3$. We will find that the results for ψ_k generalize very naturally to the theta constant of any glue class.

Let us look at the lowest 'forbidden' k, *i.e.* $k = 5$. Recall from Thm.1.1 that ψ_5 cannot be 3-solved — *i.e.* it cannot be expressed as a polynomial of θ_3. However, as we shall presently show, ψ_5 can be expressed as an irrational function of θ_3.

To see that, start with the following equation obtained from eqs.(1.3*e*,*f*):

$$\psi_{5/2}(\tau) = \frac{1}{2}\theta_3(\tau/25) - \frac{1}{2}\theta_3(\tau) - \psi_5(\tau). \tag{4.4.1a}$$

Using $\{(5),(5)\}[1,2] \approx I_2$, we find a quadratic identity ψ_5 must satisfy, namely

$$0 = \psi_5(\tau)^2 + \frac{1}{2}\{\theta_3(\tau) - \theta_3(\tau/25)\}\psi_5(\tau) + \frac{1}{4}\{\theta_3(\tau/5)^2 - \theta_3(\tau)^2\}. \tag{4.4.1b}$$

This enables us to solve for ψ_5 in terms of θ_3 using the quadratic formula:

$$\psi_5(\tau) = \frac{\{\theta_3(\tau) - \theta_3(\tau/25)\} + \sqrt{\{\theta_3(\tau) - \theta_3(\tau/25)\}^2 - 4\theta_3(\tau/5)^2 + 4\theta_3(\tau)^2}}{4}. \tag{4.4.1c}$$

Of course, the quadratic equation (**3.3***b*) has a second root — it is easy to see that $\psi_{5/2}$ is that second root:

$$\psi_{5/2}(\tau) = \frac{\{\theta_3(\tau) - \theta_3(\tau/25)\} - \sqrt{\{\theta_3(\tau) - \theta_3(\tau/25)\}^2 - 4\theta_3(\tau/5)^2 + 4\theta_3(\tau)^2}}{4}. \tag{4.4.1d}$$

Thus if we allow these non-polynomial dependences, we may derive more θ_3 identities by allowing for order 5 glues (*e.g.* when extending Table 7).

However, in this work we are interested in identities in $\mathcal{T}_3$. Hence eqs.(1c, d) do not mean we may in higher dimensions consider ψ_5 and $\psi_{5/2}$ to be 'derived' quantities, in the sense that θ_2 and ψ_6, for example, are. In fact, in two dimensions we shall see ψ_5 involved in a number of independent identities. We will make this point more graphic later in this section when we discuss the 3-solvability of the lattices Λ_k.

We shall investigate how eq.(1b) can be generalized to other ψ_k's. First (*i.e.* in Thm.1) we will discuss the polynomial equations the ψ_k's satisfy, and later (*i.e.* in Thm.2 and beyond) we shall see when the coefficients of those polynomials are functions of θ_3. Higher dimensional generalizations will be discussed in Thm.4, and various illustrations and remarks will be presented in the remainder of the section.

Let $\mathcal{T}_L^{(n)}$ be the **C**-module spanned by the theta constants of n-dimensional Euclidean lattices, and let $\mathcal{T}_L$ be their sum. Define $\mathcal{T}_R^{(n)}$ and $\mathcal{T}_R$ similarly, except use only *rational* lattices. Then both $\mathcal{T}_L$ and $\mathcal{T}$ contain $\mathcal{T}_R$, which contains $\mathcal{T}_3$. We have no examples of lattices whose theta constants are not in $\mathcal{T}_3$, but for reasons to be discussed below we suspect there are 2-dimensional counterexamples.

Theorem 4.4.1: For each $n = 1, 2, 3, \ldots$, there exists a monic polynomial $f_n(\psi) = \psi^k + s_{n,1}\psi^{k-1} + \cdots + s_{n,k}$ of degree $k = \lceil \frac{1}{2}\phi(n) \rceil$, with coefficients $s_{n,\ell} \in \mathcal{T}_R^{(\ell)}$, whose k roots are precisely $\psi_{n/j} = \psi_{n/(n-j)}$, for all j, $1 \le j \le n$, relatively prime to n.

Proof Fix n and let $1 = K_1 < \ldots < K_k \le \frac{n}{2}$ be the k numbers relatively prime to n. Define $\psi_{(n,i)}(\tau) \stackrel{\text{def}}{=} \psi_{n/K_i}(\tau)$. To prove Thm.1 it suffices to show that

$$(-1)^\ell s_{n,\ell} \stackrel{\text{def}}{=} \sum_{1\leq i_1<\cdots<i_\ell\leq k} \psi_{(n,i_1)}(\tau)\cdots\psi_{(n,i_\ell)}(\tau), \qquad (4.4.2a)$$

for $\ell = 1,\ldots,k$ can be expressed as a linear combination (over $\mathbf{Q}$) of the theta constants of various ℓ-th dimensional lattices. This follows by induction on ℓ and n and by considering the sum of the theta constants corresponding to the lattice gluings

$$\{(n),\ldots,(n)\}[K_{i_1},\ldots,K_{i_\ell}] \quad \text{for every choice of } 1\leq i_1<\cdots<i_\ell\leq k. \quad (4.4.2b)$$

QED

Here $\phi(n)$ is the Euler ϕ-function, *i.e.* the number of numbers less than n and relatively prime to n, and $\lceil x\rceil$ is the least integer $\geq x$.

For example, for $n = 2,5,7$ we have $k = 1,2,3$:

$$f_2(\psi)(\tau) = \psi(\tau) - \theta_3(\tau/4) + \theta_3(\tau), \qquad (4.4.3a)$$

$$f_5(\psi)(\tau) = \psi^2 - \frac{1}{2}\{\theta_3(\tau/25) - \theta_3(\tau)\}\psi + \frac{1}{4}\{\theta_3(\tau/5)^2 - \theta_3(\tau)^2\}, \qquad (4.4.3b)$$

$$\begin{aligned} f_7(\psi)(\tau) =& \psi^3 - \frac{1}{2}\{\theta_3(\frac{1}{49}) - \theta_3\}\psi^2 + \\ & \frac{1}{4}[\frac{1}{2}\{\theta_3(\frac{1}{49}) - \theta_3\}^2 - \theta_3(\frac{2}{7})\theta_3(14) + \{\theta_3(\frac{1}{4}) - \theta_3\}\{\theta_3(\frac{1}{196}) - \theta_3\}]\psi \\ & - \frac{1}{4}\theta_3^2\psi - \frac{1}{6}[\Theta_{(2,0)}\theta_3(21) + \Theta_{(2,1)}\{\theta_3(\frac{7}{3}) - \theta_3(21)\} - \theta_3^3], \end{aligned} \qquad (4.4.3c)$$

where for convenience we dropped τ from some arguments, and for eq.(3c) we used the notation of Table 7.

Recall that we have called a lattice (rationally) 3-solvable if its theta constant is. Call a polynomial *3-solvable/rationally 3-solvable* if all of its coefficients are, respectively, in $\mathcal{T}_3$ or $\mathcal{T}_3^*$. For example, eqs.(3) show that f_2, f_5 and f_7 are all 3-solvable.

The proof of Thm.1 is constructive, explicitly showing us how to find f_n. From a related construction we get the following results:

Theorem 4.4.2: Fix n, and suppose f_d is (rationally) 3-solvable for all $d < n$ dividing n. Then f_n is (rationally) 3-solvable iff $\forall \ell$ satisfying $1 \leq \ell \leq k \stackrel{\rm def}{=} \lceil \frac{1}{2}\phi(n) \rceil$, the lattice $\Lambda_{n,\ell} \stackrel{\rm def}{=} \{(\ell n^2), A_{\ell-1}\}[n^2, n]$ is (rationally) 3-solvable.

Corollary 4.4.3: f_n is (rationally) 3-solvable for all n, if every lattice is (rationally) 3-solvable. If every f_n is rationally 3-solvable, then every A_n is rationally 3-solvable. If every f_n is 3-solvable, then for each k, either A_k is 3-solvable or there are an infinite number of independent identities in $\mathcal{T}_3^{(k+2)}$.

Thm.2 and hence Cor.3 follow from the observation that $\{(n^2)^\ell\}[n, \ldots, n] \approx \{(\ell), A_{\ell-1}^{(n^2)}\}[n, n[1]] = \{\Lambda_{n,\ell}\}^{*(n^2)}$, and from the facts that the symmetric polynomials

$$p_{\ell,k} \stackrel{\rm def}{=} \sum_{i=1}^{\ell} x_i^\ell \text{ and } p'_{\ell,k} \stackrel{\rm def}{=} \sum_{1 \leq i_1 < \cdots < i_\ell \leq k} x_{i_1} \cdots x_{i_k},$$

for $\ell = 1, \ldots, k$, in the indeterminants $x_1, \ldots, x_k$, both generate any symmetric polynomial in the x_i.

Using Table 7 and eqs.(1.4), we immediately get from Thm.2 that f_n is 3-solvable (and can be explicitly — if somewhat messily — written down) if $\phi(n) \leq 8$, *i.e.* $\forall n \leq 10$, or n=12, 14, 15, 16, 18, 20, 24, or 30.

Now, A_{n-1} is rationally 3-solvable iff $\sum \psi^n_{n/k}$ is rationally 3-solvable, by eqs.(1.6d) and (2.9a). Therefore A_9, A_{13}, A_{14}, A_{15}, A_{19}, A_{23} and A_{29} are all rationally 3-solvable and their theta constants can be explicitly given.

A significant generalization of Thm.1 is possible:

Theorem 4.4.4: Consider any N-dimensional glue class $[g]\Lambda$ of order n. Then there exists a monic polynomial f of degree $k = \lceil \frac{1}{2}\phi(n) \rceil$ whose coefficients s_ℓ, $0 \leq \ell \leq k$ are in $\mathcal{T}_L^{(\ell N)}$ and whose k roots are the theta constants of the glue classes $[jg]\Lambda = [(n-j)g]\Lambda$ for all j, $1 \leq j < n$, relatively prime to n. Moreover, if Λ is rational then the coefficients s_ℓ will also lie in $\mathcal{T}_R^{(\ell N)}$.

Thm.4 follows from arguments analogous to those used in proving Thm.1: in particular, in place of eq.(2b) use

$$\{\Lambda,\ldots,\Lambda\}\left[\{[K_{i_1}g],\ldots,[K_{i_\ell}g]\}\right] \quad \text{for every choice of } 1\leq i_1<\cdots<i_\ell\leq k.$$

Obviously, Thm.1 is a special case of Thm.4.

Any order 2 glue $[g]\Lambda$ has theta constant $\Theta([g]\Lambda)$ which satisfies:

$$f(\psi)=\psi+\{\Theta(\Lambda)-\Theta(\Lambda[g])\}, \tag{4.4.4a}$$

while the theta constants of the non-trivial glue classes $[1]A_4$, $[2]A_4$, $[3]A_4$, and $[4]A_4$ of A_4 satisfy:

$$f(\psi)=\psi^2-\frac{1}{2}\{\Theta(A_4^*)-\Theta(A_4)\}\psi+\frac{1}{4}\{\Theta(E_8)-\Theta(A_4)^2\}. \tag{4.4.4b}$$

Note that the polynomial in eq.(4b) is 3-solvable.

At the beginning of this section we 'solved' for ψ_5 and $\psi_{5/2}$ in terms of θ_3. Unfortunately, the existence of this 'solution' does not simplify our analysis of expressions with ψ_5 in them, because of our decision to stay in $\mathcal{T}$ or $\mathcal{T}_3$.

In particular, in Table 9 we investigate the 3-solvability of

$$\psi_5(\tau)\psi_5(k\tau)+\psi_{5/2}(\tau)\psi_{5/2}(k\tau). \tag{4.4.5a}$$

In the following discussions we will refer repeatedly to such functions. It is easy to verify (using eq.(1a)) that eq.(5a) is 3-solvable/rationally 3-solvable iff $\psi_5(\tau)\psi_{5/2}(k\tau)+\psi_{5/2}(\tau)\psi_5(k\tau)$ is 3-solvable/rationally 3-solvable, iff the lattice $\Lambda_k\stackrel{\text{def}}{=}\{(25),(25k)\}[5,5k]$ is 3-solvable/rationally 3-solvable, iff the lattice $\{(25),(25k)\}[10,5k]$ is 3-solvable/rationally 3-solvable. Table 9 was derived by rewriting Λ_k as the gluing of an orthogonal lattice with a single glue of order 1, 2, 3, 4 or 6. This is possible for example whenever the minimum norm in Λ_k is 1, 2, 3, 4 or 6. Λ_7 is the first such lattice whose theta constant cannot be '3-solved' in this way (its minimum norm is 8).

Table 9: $\psi_5(\tau)\,\psi_5(k\tau) + \psi_{5/2}(\tau)\,\psi_{5/2}(k\tau)$

k	**Expression**
$k=1$	$\frac{1}{4}\theta_3(\tau/5)^2 + \frac{3}{4}\theta_3(5\tau)^2 - \frac{1}{2}\theta_3(\tau/5)\,\theta_3(5\tau) - \frac{1}{2}\theta_3(\tau)^2$
$k=2$	$\frac{1}{2}\theta_3(3\tau/5)\,\theta_3(30\tau) - \frac{1}{2}\theta_3(5\tau)\,\theta_3(10\tau) + \psi_3(\tau/5)\,\psi_3(10\tau)$
$k=3$	$\frac{1}{2}\theta_3(4\tau/5)\,\theta_3(60\tau) - \frac{1}{2}\theta_3(5\tau)\,\theta_3(15\tau) + \frac{1}{4}\theta_2(\tau/20)\,\theta_2(15\tau/4) + \frac{1}{2}\theta_2(2\tau/5)\,\theta_2(30\tau)$
$k=4$	$\frac{1}{2}\theta_3(\tau)\,\theta_3(4\tau) - \frac{1}{2}\theta_3(5\tau)\,\theta_3(20\tau)$
$k=5$	$\frac{1}{2}\theta_3(6\tau/5)\,\theta_3(150\tau) - \frac{1}{2}\theta_3(5\tau)\,\theta_3(25\tau) + \psi_6(\tau/5)\,\psi_6(25\tau) + \psi_3(2\tau/5)\,\psi_3(50\tau) + \theta_2(3\tau/5)\,\theta_2(75\tau)$
$k=6$	$\frac{1}{4}\{\theta_3(\tau/5) - \theta_3(5\tau)\}\{\theta_3(6\tau/5) - \theta_3(30\tau)\} - \frac{1}{2}\theta_3(2\tau)\,\theta_3(3\tau)$
$k=7$	??
$k=8$	??
$k=9$	$\frac{1}{2}\theta_3(2\tau)\,\theta_3(18\tau) - \frac{1}{2}\theta_3(5\tau)\,\theta_3(45\tau) + \theta_2(\tau)\,\theta_2(9\tau)$

It would be interesting to see if its theta constant is 3-solvable — it is one of the simplest lattices which is not obviously 3-solvable.

Eqs.(1) tell us that Λ_k is (rationally) 3-solvable iff $S_k(\tau) \stackrel{\text{def}}{=} \sqrt{\Delta(\tau)\Delta(k\tau)}$ is, where $\Delta(\tau) = \{\theta_3(\tau) - \theta_3(\tau/25)\}^2 - 4\theta_3(\tau/5)^2 + 4\theta_3(\tau)^2$ is the discriminant of the quadratic polynomial eq.(1b). Note that

$$S_{k\ell}(\tau) = \Delta(\tau)S_\ell(k\tau)/S_k(\tau), \qquad (4.4.5b)$$

so if Λ_k and Λ_ℓ are rationally 3-solvable, so is $\Lambda_{k\ell}$. It is also easy to verify

$$S_{1/k}(\tau) = S_k(\tau/k). \qquad (4.4.5c)$$

These two equations tell us that S_k is rationally 3-solvable for any *rational* k iff S_k is rationally 3-solvable for any *integral* k, iff S_p is rationally 3-solvable for any *prime* p.

From Table 9 we learn that Λ_k is 3-solvable for $k \leq 6$ and $k = 9$. Then eq.(5b) tells us Λ_8, Λ_{10}, Λ_{12}, and Λ_{15}, to name a few, are all rationally 3-solvable. Their 3-solvability, however, is an open question.

Table 9 implies some degree four identities: one for each k that Λ_k is 3-solvable. These take the form $S_k(\tau)^2 = \Delta(\tau)\Delta(k\tau)$. This idea tells us that if Λ_k is 3-solvable for all k, then we will get an infinite number of algebraically independent degree 4 identities for θ_3. The unlikelihood of this suggests that not all Λ_k, and hence not all 2-dimensional lattices, are 3-solvable. In fact, it hints that only finitely many Λ_k may be 3-solvable.

On the other hand, if all Λ_k were *rationally* 3-solvable, we again would have an infinite number of algebraically independent θ_3 identities but they would be of arbitrarily high degrees.

Note that although we learn from eq.(5b) that, for example, Λ_{25} is rationally 3-solvable and hence get an identity reflecting this, this identity (at least if it is derived solely from eq.(5b)) would not be algebraically independent of the one for Λ_5. Something cannot be gained from nothing.

It is obvious that $\mathcal{T}_3$ is an integral domain. It would be extremely useful if $\mathcal{T}_3$ was known to be in addition a unique factorization domain (see p.137 of **[HUN]** for definitions). However, consider the identity for $k = 2$, for example — it has the form $S_2(\tau)^2 = \Delta(\tau)\Delta(2\tau)$ — and assume for contradiction that $\mathcal{T}_3$ *is* a unique factorization domain. Now if $S_2(\tau)$ were irreducible in $\mathcal{T}_3$, then because $\deg(\Delta(\tau)) = 2 = \deg(\Delta(2\tau))$, we must have $S_2(\tau) = \Delta(\tau) = \Delta(2\tau)$ (at least up to real number proportionality constants — the units in $\mathcal{T}_3$). This is of course false. Thus $S_2(\tau)$ is reducible, say $S_2(\tau) = u_1(\tau)u_2(2)$ where $\deg(u_1(\tau)) = \deg(u_2(\tau)) = 1$ and $u_1(\tau)$, $u_2(\tau)$ are irreducible. Then $\Delta(\tau) = u_1(\tau)^2$ and $\Delta(2\tau) = u_2(\tau)^2$ (interchanging u_1 and u_2 if necessary). But $\psi_5(\tau)$ is not 3-solvable by Thm.1.1, so neither is $\sqrt{\Delta(\tau)}$ by eq.(1c). This is another contradiction. Hence:

Theorem 4.4.5: The integral domain $\mathcal{T}_3$ is not a unique factorization domain.

In Thm.2 we are interested in the theta constants of gluings such as $\{(5k), A_4\}[k, 1]$ and $\{(5k), A_4\}[k, 2]$ for various k. Using the lattice equivalence $\{(5), A_4\}[1, 2] \approx I_5$, as well as eqs.(1) and (4b), we get that $\{(5k), A_4\}[k, 1]$ is rationally 3-solvable iff $\{(5k), A_4\}[k, 2]$ is, iff Λ_k is. In this way the rational 3-solvability of a 5-dimensional lattice is reduced to the rational 3-solvability of a 2-dimensional one.

As a final note, these results are of interest even if the polynomials f are not (rationally) 3-solvable, for they seem to represent the minimal polynomials that the lattice gluing method can be expected to generate. This is discussed in more detail in the following chapter.

Problems in the full-rank case seem to be more tractible than here in the theta constant (0-rank) case For example, in Chapter 5 we prove that the full-rank theta series of a glue class is 3-solvable only when the order of the glue class is 1, 2, 3, 4 or 6 (compare Thm.1.1). The proof does not carry over to this case, however.

Chapter 5 THETA SERIES OF FULL RANK

5.1 Jacobi θ-functions

The Jacobi θ-functions which we need are defined by:

$$\vartheta_3(z \mid \tau) \stackrel{\text{def}}{=} \sum_{m=-\infty}^{\infty} \exp[2m\pi iz + \pi i m^2 \tau] \tag{5.1.1a}$$

$$\vartheta_1(z \mid \tau) \stackrel{\text{def}}{=} \sum_{m=-\infty}^{\infty} \exp[2\pi i(m+\frac{1}{2})(z+\frac{1}{2}) + \pi i(m+\frac{1}{2})^2\tau]$$

$$= \exp[\pi i\tau/4 + \pi iz + \frac{1}{2}\pi i]\ \vartheta_3(z+\frac{1}{2}+\frac{1}{2}\tau \mid \tau) \tag{5.1.1b}$$

$$\vartheta_2(z \mid \tau) \stackrel{\text{def}}{=} \sum_{m=-\infty}^{\infty} \exp[2\pi i(m+\frac{1}{2})z + \pi i(m+\frac{1}{2})^2\tau]$$

$$= \exp[\pi i\tau/4 + \pi iz]\ \vartheta_3(z+\frac{1}{2}\tau \mid \tau) \tag{5.1.1c}$$

$$\vartheta_4(z \mid \tau) \stackrel{\text{def}}{=} \sum_{m=-\infty}^{\infty} \exp[2m\pi i(z+\frac{1}{2}) + \pi i m^2\tau]$$

$$= \vartheta_3(z+\frac{1}{2} \mid \tau) \tag{5.1.1d}$$

$$\Psi_k(z \mid \tau) \stackrel{\text{def}}{=} \sum_{m=-\infty}^{\infty} \exp[2\pi i(m+\frac{1}{k})z + \pi i(m+\frac{1}{k})^2\tau]$$

$$= \exp[\pi i\tau/k^2 + \frac{2}{k}\pi iz]\ \vartheta_3(z+\frac{1}{k}\tau \mid \tau). \tag{5.1.1e}$$

Note that all these series converge for $\tau \in \mathcal{H} \stackrel{\text{def}}{=} \{w \in \mathbf{C} \mid \text{Im } w > 0\}$, and any $z \in \mathbf{C}$. In fact it is possible to prove using arguments similar to that given at the beginning of Sec.4.1 (see p.1 of [**MUM**] for details) that for each fixed $z \in \mathbf{C}$, they are analytic functions (of τ) in $\mathcal{H}$, and for each fixed $\tau \in \mathcal{H}$ they are entire functions (of z). Moreover, for fixed $\tau \in \mathcal{H}$, ϑ_2, ϑ_3 and ϑ_4 are *even* functions of z, while ϑ_1 is *odd*. As in the previous chapter, τ will be always taken to lie in $\mathcal{H}$.

The following functions (first given by Hermite) can be defined for each $a, b \in \mathbf{Q}$:

$$\vartheta_{a,b}(z \mid \tau) = \sum_{m=-\infty}^{\infty} \exp[\pi i(m+a)^2\tau + 2\pi i(m+a)\cdot(z+b)]. \tag{5.1.2a}$$

Let n be the smallest positive integer such that $nb \in \mathbf{Z}$. Then there is a k such that $kb \equiv \frac{1}{n} \pmod 1$. Let $\zeta = \exp[2\pi i/n]$. A simple calculation (see eq.(2.3b)) now shows

$$\vartheta_{a,b}(z \mid \tau) = \exp[2\pi \imath ab] \sum_{\ell=0}^{n-1} \zeta^{\ell} \Psi_{n/(\ell k+a)}(nz \mid n^2\tau). \tag{5.1.2b}$$

For this reason it will suffice in what follows to consider only the Ψ functions. rather than all $\vartheta_{a,b}$. A special case of eq.(2b) is

$$\vartheta_1(z \mid \tau) = i\Psi_4(2z \mid 4\tau) - \imath\Psi_{4/3}(2z \mid 4\tau), \tag{5.1.2c}$$

which we will use later in deriving eq.(5e), and in proving Thm.4.4.

Another important function is the Dedekind eta function $\eta(\tau)$:

$$\eta(\tau) \stackrel{\text{def}}{=} \exp[\pi\imath\tau/12] \prod_{m=1}^{\infty} (1 - \exp[2\pi\imath m\tau]). \tag{5.1.2d}$$

$\eta(\tau)$ is interesting in the theory of modular forms; we will see it also in Ch.6 in the partition functions of strings (*e.g.* see eq (6.2.6)). It is possible to show using modular form arguments (*e.g.* p.72 of [**MUM**]) that $\eta^{24}(\tau)$ and $\vartheta_{1/6,1/2}(0|3\tau)^{24}$ both equal the cusp form $\Delta_{24}(\tau)$. From this and eq.(2b) it is trivial to derive

$$\begin{aligned} \eta(\tau) &= \exp[-\pi\imath/6]\vartheta_{1/6,1/2}(0 \mid 3\tau) \\ &= \psi_{12}(12\tau) - \psi_{12/5}(12\tau). \end{aligned} \tag{5.1.2e}$$

Therefore, Thm.4.3.3 immediately implies that η is not 3-solvable, and that eq.(2e) is the unique expression for η in terms of the theta functions in eqs.(4.1.1). Moreover, the identity $\eta^{24} = \theta_2^8\theta_3^8\theta_4^8/2^8$ shows that $\eta(\tau)^3$ *is* 3-solvable, and hence that η satisfies a degree 3 polynomial in the sense of Sec.4.4.

By *theta constants* is meant the restriction to $z = 0$. The previous chapter considered exclusively the theta constants $\theta_2(\tau) \stackrel{\text{def}}{=} \vartheta_2(0|\tau)$, $\theta_3(\tau) \stackrel{\text{def}}{=} \vartheta_3(0|\tau)$, $\theta_4(\tau) \stackrel{\text{def}}{=} \vartheta_4(0|\tau)$, and $\psi_k(\tau) \stackrel{\text{def}}{=} \Psi_k(0|\tau)$ ($\vartheta_1(0|\tau)$ is identically zero). In this chapter we will extend the techniques and results developed there to $z \neq 0$.

For convenience, define $\Psi_\infty \stackrel{\text{def}}{=} \vartheta_3$. Note that

$$\Psi_k = \vartheta_3 \text{ iff } 1/k \in \mathbf{Z} \tag{5.1.3a}$$

$$\Psi_k = \Psi_\ell \text{ iff } 1/k - 1/\ell \in \mathbf{Z}. \tag{5.1.3b}$$

(See eq.(3f) below.) In general, we will thus be interested in Ψ_k where k is rational and ≥ 1. Ψ_k is never an *odd* function of z for fixed τ; it is *even* iff $1/k \equiv 0 \pmod{\frac{1}{2}}$.

From the definitions the following basic identities can be readily verified:

$$\vartheta_4(z \mid \tau) = 2\vartheta_3(2z \mid 4\tau) - \vartheta_3(z \mid \tau) \tag{5.1.3c}$$

$$\Psi_2(z \mid \tau) = \vartheta_2(z \mid \tau) \tag{5.1.3d}$$

$$\sum_{\ell=1}^{k} \Psi_{k/\ell}(z \mid \tau) = \vartheta_3(z/k \mid \tau/k^2) \tag{5.1.3e}$$

$$\Psi_{k/\ell}(-z \mid \tau) = \Psi_{k/(k-\ell)}(z \mid \tau). \tag{5.1.3f}$$

In fact, in the next section we will find that eqs.(3e, f) are special cases of much more general relations (namely, eqs.(2.11) and (2.7d) respectively) which reflect basic facts about lattices and their glues.

Eq.(3f) above (compare with eq.(4.1.3f) in the previous chapter) is our first hint of the difficulties facing us in our attempt to generalize the results of the previous chapter. Indeed, in what follows it will be common to use, instead of Ψ_k, its *symmetrization*:

$$\hat{\Psi}_k(z \mid \tau) \stackrel{\text{def}}{=} \frac{1}{2}\{\, \Psi_k(z \mid \tau) + \Psi_k(-z \mid \tau)\,\}. \tag{5.1.4a}$$

Hence

$$\hat{\Psi}_{k/\ell} = \hat{\Psi}_{k/(k-\ell)} \tag{5.1.4b}$$

$$\hat{\Psi}_k(0 \mid \tau) = \Psi_k(0 \mid \tau) = \psi_k(\tau). \tag{5.1.4c}$$

These first identities allow us to establish the following:

$$\Psi_1(z \mid \tau) = \hat{\Psi}_1(z \mid \tau) = \vartheta_3(z \mid \tau) \tag{5.1.5a}$$

$$\Psi_2(z \mid \tau) = \hat{\Psi}_2(z \mid \tau) = \vartheta_2(z \mid \tau) = \vartheta_3(z/2 \mid \tau/4) - \vartheta_3(z \mid \tau) \tag{5.1.5b}$$

$$\hat{\Psi}_3(z \mid \tau) = \frac{1}{2}\{\, \vartheta_3(z/3 \mid \tau/9) - \vartheta_3(z \mid \tau) \,\} \tag{5.1.5c}$$

$$\hat{\Psi}_4(z \mid \tau) = \frac{1}{2}\vartheta_2(z/2 \mid \tau/4) \tag{5.1.5d}$$

$$\Psi_4(z \mid \tau) = \frac{1}{2}\vartheta_2(z/2 \mid \tau/4) - \frac{i}{2}\vartheta_1(z/2 \mid \tau/4) \tag{5.1.5e}$$

$$\hat{\Psi}_6(z \mid \tau) = \frac{1}{2}\{\, \vartheta_2(z/3 \mid \tau/9) - \vartheta_2(z \mid \tau) \,\}. \tag{5.1.5f}$$

For example, eq.($5b$) is a consequence of eqs.($3d, e$) with $k = 2$. In fact, the derivations of eqs.($5a$-d) and ($5f$) are identical to the corresponding ones in the previous chapter.

Using the linear identities eqs.($3c$) and ($5b$), identities involving ϑ_2, ϑ_3 and ϑ_4 can always be reduced to identities involving only ϑ_3. We will also be interested in identities of higher degree: *e.g.* the (degree 4) Riemann identity $\vartheta_1^4(\vec{z}|\tau) + \vartheta_2^4(\vec{z}|\tau) + \vartheta_3^4(\vec{z}|\tau) + \vartheta_4^4(\vec{z}|\tau) = 2\vartheta_3^4(\vec{z}'|\tau)$ (see eq.(3.8e)). Thm.3.1 tells us that any *fundamental* identity is homogeneous in degree. We will be interested in identities which are of *full rank*. The precise meaning of this will become clearer later (see for example the definition of $\mathcal{F}_3$ later in this section, and especially the discussions after Thm.3.2 and before Thm.4.1), but suffice it to say here that theta constant identities are of rank 0 (and hence not of full rank), while the identities considered in this section are of rank and degree 1 (and so *are* of full rank).

Identities of higher degrees are discussed in Sec.3.

Using eq.(4.1.5b), we immediately get

$$\vartheta_1(z \mid -\frac{1}{\tau}) = -(\tau/i)^{1/2} \exp[\pi i z^2 \tau]\ \vartheta_1(z\tau \mid \tau) \tag{5.1.6a}$$

$$\vartheta_2(z \mid -\frac{1}{\tau}) = (\tau/i)^{1/2} \exp[\pi i z^2 \tau]\ \vartheta_4(z\tau \mid \tau) \tag{5.1.6b}$$

$$\vartheta_4(z \mid -\frac{1}{\tau}) = (\tau/i)^{1/2} \exp[\pi i z^2 \tau]\ \vartheta_2(z\tau \mid \tau) \tag{5.1.6c}$$

$$\Psi_{n/k}(z \mid -\frac{1}{\tau}) = (\tau/i)^{1/2} \exp[\pi i z^2 \tau] \; \vartheta_3(\frac{k}{n} + z\tau \mid \tau)$$

$$= (\tau/i)^{1/2} \exp[\pi i z^2 \tau] \; \sum_{\ell=0}^{n-1} \zeta^{\ell k} \Psi_{n/\ell m}(nz \mid n^2 \tau), \qquad (5.1.6d)$$

where in eq.(6d) $\zeta = e^{2\pi i/n}$ and $m \in \mathbf{Z}$ satisfies $mk \equiv 1 \pmod{n}$. For example, these allow eq.(5b) to imply eq.(3c).

Eqs.(1a-e) immediately give us

$$\vartheta_1(z \mid \tau + 1) = \sqrt{i}\vartheta_1(z \mid \tau), \quad \vartheta_2(z \mid \tau + 1) = \sqrt{i}\vartheta_2(z \mid \tau) \qquad (5.1.7a)$$

$$\vartheta_3(z \mid \tau + 1) = \vartheta_4(z \mid \tau), \qquad \vartheta_4(z \mid \tau + 1) = \vartheta_3(z \mid \tau) \qquad (5.1.7b)$$

$$\vartheta_1(z + \frac{1}{2} \mid \tau) = -\vartheta_2(z \mid \tau), \quad \vartheta_2(z + \frac{1}{2} \mid \tau) = \vartheta_1(z \mid \tau) \qquad (5.1.7c)$$

$$\vartheta_3(z + \frac{1}{2} \mid \tau) = \vartheta_4(z \mid \tau), \qquad \vartheta_4(z + \frac{1}{2} \mid \tau) = \vartheta_3(z \mid \tau) \qquad (5.1.7d)$$

$$\Psi_{n/k}(z + 1 \mid \tau) = e^{2\pi i k/n} \Psi_{n/k}(z \mid \tau) \quad \Psi_{n/k}(z \mid \tau + 2n) = \Psi_{n/k}(z \mid \tau) (5.1.7e)$$

Let $\mathcal{F}_3^{(n)}$ denote the *full rank homogeneous polynomials of degree n in* ϑ_3, *i.e.* the $\mathbf{C}$-module of functions generated by the monomials $\vartheta_3(\vec{z} \cdot \vec{a}_1 \mid b_1\tau) \cdots \vartheta_3(\vec{z} \cdot \vec{a}_n \mid b_n\tau)$, where the variable $\vec{z} = (z_1, \ldots, z_n) \in \mathbf{C}^n$, where the vectors $\vec{a}_i \in \mathbf{R}^n$ are *linearly independent*, and where the real numbers b_i are positive. Define $\mathcal{F}_3$ to be the sum of all $\mathcal{F}_3^{(n)}$. Then $\mathcal{F}_3^{(1)}$ and hence $\mathcal{F}_3$ contain $\vartheta_2, \vartheta_3, \vartheta_4$, and $\Psi_2 = \hat{\Psi}_2, \hat{\Psi}_3, \hat{\Psi}_4, \hat{\Psi}_6$. Define $\mathcal{F}_{1,3}^{(n)}$ and $\mathcal{F}_{1,3}$ similarly, except the monomials consist of products of ϑ_1 as well as ϑ_3. Then $\mathcal{F}_{1,3}$ contains $\mathcal{F}_3$, as well as ϑ_1 and Ψ_4. Define $\mathcal{F}_3^*$ and $\mathcal{F}_{1,3}^*$ to be the field of fractions of $\mathcal{F}_3$ and $\mathcal{F}_{1,3}$, respectively. We will be interested in using lattices to find functions in $\mathcal{F}_3$ and $\mathcal{F}_{1,3}$ which are identically zero.

Definition 5.1.1: Call $F(\vec{z}|\tau)$ *3-Solvable* if it lies in $\mathcal{F}_3$ — *i.e.* if it can be expressed polynomially in terms of ϑ_3. Call $F(\vec{z}|\tau)$ *rationally 3-Solvable* if it lies in $\mathcal{F}_3^*$ — *i.e.* if it can be expressed as a fraction involving only ϑ_3. Similarly, call $F(\vec{z}|\tau)$ *(1,3)-Solvable/rationally (1,3)-Solvable* if it lies in $\mathcal{F}_{1,3}$ or $\mathcal{F}_{1,3}^*$, respectively.

We will also be interested in determining the Solvability of various theta functions/series.

Theorem 5.1.1: For $k \geq 1$, Ψ_k is *3-Solvable* iff $k = 1, 2$ and ∞, and *(1,3)-Solvable* iff $k = 1, 2, 4, \infty$. $\hat{\Psi}_k$ is *(1,3)-Solvable* iff *3-Solvable* iff $k = 1, 2, 3, 4, 6$ and ∞.

One direction of this theorem is already known; the other is an immediate consequence of Thm.3.6. Compare with Thm.4.4.

5.2 Theta Series of Integral Lattices and their Glue Classes

Given a glue class $[g]\Lambda \stackrel{\text{def}}{=} g + \Lambda$, where $g \in \mathbf{Q} \otimes \Lambda$, its *theta series* is defined to be

$$\vartheta([g]\Lambda)(\vec{z} \mid \tau) = \sum_{x \in \Lambda} \exp[\pi i \tau (g + x)^2 + 2\pi i (x + g) \cdot \vec{z}] \qquad (5.2.1a)$$

$$\vartheta(\Lambda) = \vartheta([0]\Lambda). \qquad (5.2.1b)$$

Here, the variable $\vec{z}$ is in the n-dimensional *complex vector space* $\mathbf{C} \otimes \Lambda$, where n is the dimension of the lattice Λ (the arrow will be used to emphasize its vectorial nature). In the one-dimensional case, we will sometimes write z for $\vec{z}$. Of course, the theta constants are obtained by setting $\vec{z} = \vec{0}$: $\Theta([g]\Lambda)(\tau) = \vartheta([g]\Lambda)(\vec{0} \mid \tau)$. The argument of the function $\vartheta([g]\Lambda)$ is understood to be $(\vec{z} \mid \tau)$. If this is not the case, the argument will be explicitly included, as in $\vartheta([g]\Lambda)(\sqrt{2}\vec{z} \mid 2\tau)$.

By the usual analyticity arguments, these are analytic in $\mathcal{H}$ for fixed $\vec{z}$, and for fixed $\tau \in \mathcal{H}$ and fixed $z_i \in \mathbf{C}$, $i \neq j$, they are entire in z_j.

It will occasionally be convenient to rewrite eq.(1b), for example, as

$$\vartheta(\Lambda)(\vec{z} \mid \tau) = \vartheta(I_n)(\vec{z} M^t \mid A\tau)$$
$$\stackrel{\text{def}}{=} \sum_{\vec{m} \in I_n} \exp[\pi i \tau \vec{m} A \vec{m}^t + 2\pi i \vec{z} \vec{m}^t], \qquad (5.2.1c)$$

where M is a generator matrix, and $A = MM^t$ is the corresponding Gram matrix, for Λ.

In the notation of eqs.(1a, b), the Jacobi θ-functions in eqs.(1.1) can be written as

$$\vartheta_2(\sqrt{2}z \mid 2\tau) = \vartheta([1]\{(2)\})(z|\tau) \stackrel{\text{def}}{=} \vartheta([1]I_1^{(2)})(z|\tau)$$
$$= \vartheta([1]A_1)((z/\sqrt{2}, -z/\sqrt{2}) \mid \tau) \qquad (5.2.2a)$$

$$\vartheta_3(\sqrt{k}z \mid k\tau) = \vartheta(I_1^{(k)}) = \vartheta(\{(k)\}) \qquad (5.2.2b)$$

$$\Psi_{k/\ell}(\sqrt{k}z \mid k\tau) = \vartheta([\ell]I_1^{(k)}) = \vartheta([\ell]\{(k)\}). \qquad (5.2.2c)$$

Suppose $y \in \mathbf{Q} \otimes \Lambda^*$. Then there exists some integer $n > 0$, called the *order* of y with respect to Λ^*, such that $ky \in \Lambda^*$ iff n divides k. By Thm.1.4.9 we can find an $r \in \Lambda$ such that $r \cdot y \equiv \frac{1}{n}$ (mod 1). Moreover, if we let Λ_0 denote the lattice of all vectors $x \in \Lambda$ with integral dot products with y, then $\Lambda = \Lambda_0[r]$. A simple calculation from eq.(1) then enables us to write

$$\vartheta(\Lambda)(\vec{z} + y \mid \tau) = \sum_{k=0}^{n-1} \zeta^k \vartheta([kr]\Lambda_0)(\vec{z} \mid \tau), \qquad (5.2.3a)$$

$$\vartheta([g]\Lambda)(\vec{z} + y \mid \tau) = \exp[2\pi i g \cdot y] \sum_{k=0}^{n-1} \zeta^k \vartheta([kr + g]\Lambda_0)(\vec{z} \mid \tau), \qquad (5.2.3b)$$

where $[g]$ is any glue of Λ, and where $\zeta = e^{2\pi i/n}$.

From eq.(1) we also immediately get information about the $\vec{z}$-quasi-period of these theta series:

$$\vartheta(\Lambda)(\vec{z} + \tau\lambda \mid \tau) = \exp[-\pi i \lambda^2 \tau - 2\pi i \lambda \cdot \vec{z}]\, \vartheta(\Lambda)(\vec{z} \mid \tau) \qquad (5.2.4a)$$

$$\vartheta([g]\Lambda)(\vec{z} + \tau\lambda \mid \tau) = \exp[-\pi i \lambda^2 \tau - 2\pi i \lambda \cdot \vec{z}]\, \vartheta([g]\Lambda)(\vec{z} \mid \tau) \qquad (5.2.4b)$$

for any $\lambda \in \Lambda$. Similar reasoning gives us the following interesting relation between the theta series of the glue classes and the base lattice:

$$\vartheta([g]\Lambda)(\vec{z} \mid \tau) = \exp[\pi i g^2 \tau + 2\pi i g \cdot \vec{z}]\, \vartheta(\Lambda)(\vec{z} + \tau g \mid \tau). \qquad (5.2.4c)$$

These straightforward calculations lead to this useful result:

Theorem 5.2.1: Suppose $0 = \sum_{j=1}^{N} \alpha_i \, \vartheta([g_i]\Lambda)(\vec{z} \mid \tau)$ is satisfied for all $\vec{z}, \tau$, and assume the glue classes $[g_i]\Lambda$ are all distinct. Then $\alpha_1 = \cdots = \alpha_N = 0$.

Proof Suppose $g_0 \in \Lambda$. Choose $N > 0$ so that $Ng_k \in \Lambda$ for all k. By Thm.1.4.9 we see that there exists an $r \in \Lambda^*$ such that $r \cdot g_k \in \mathbf{Z}$ iff $g_k \in \Lambda$, *i.e.* iff $k = 0$. From the expression (see eq.(3*b*))

$$\vartheta([g_k]\Lambda)(\vec{z} + r \mid \tau) = \exp[2\pi i g_k \cdot r] \, \vartheta([g_k]\Lambda)(\vec{z} \mid \tau) \qquad \text{we obtain}$$

$$0 = \sum_{\ell=1}^{N} \sum_{k=0}^{M} \alpha_k \, \vartheta([g_k]\Lambda)(\vec{z} + \ell r \mid \tau) = N\alpha_0 \, \vartheta(\Lambda)(\vec{z} \mid \tau), \qquad (*)$$

and hence get $\alpha_0 = 0$.

Now note that, for any $g \in \mathbf{Q} \otimes \Lambda$, eq.(4*c*) gives us

$$\vartheta([g_k]\Lambda)(\vec{z} + g\tau \mid \tau) = \exp[-\pi i g^2 \tau - 2\pi i g \cdot \vec{z}] \, \vartheta([g_k + g]\Lambda)(\vec{z} \mid \tau).$$

Therefore $\sum_{k=1}^{M} \alpha_\ell \, \vartheta([g_k + g]\Lambda)(\vec{z} \mid \tau) = 0$ for each such g. Choosing $g = -g_k$ implies by $(*)$ that $\alpha_k = 0$. QED

Eqs.(1.7) tell us about the periodicity of ϑ_3 etc. in the z variable. This can now be generalized, using eqs.(3): for any $y \in \Lambda^*$,

$$\vartheta(\Lambda)(\vec{z} + y \mid \tau) = \vartheta(\Lambda)(\vec{z} \mid \tau), \qquad (5.2.5a)$$

and for any $y \in (\Lambda[g])^*$,

$$\vartheta([g]\Lambda)(\vec{z} + y \mid \tau) = \vartheta([g]\Lambda)(\vec{z} \mid \tau). \qquad (5.2.5b)$$

In fact, by Thm.1 and eqs.(3), these can be shown to exhaust all periods of $\vartheta([g]\Lambda)$ (and hence $\vartheta(\Lambda)$):

Theorem 5.2.2: For $y \in \mathbf{C} \otimes \Lambda$, y is a $\vec{z}$-period of $\vartheta([g]\Lambda)$ — *i.e.* $\vartheta([g]\Lambda)(\vec{z} + y \mid \tau) = \vartheta([g]\Lambda)(\vec{z} \mid \tau)$ — iff $y \in (\Lambda[g])^*$.

Let $\vec{z}$ be a vector with n complex components z_i, $i = 1, \ldots, n$. By $\vec{z}^{(k_1, \ldots, k_\ell)}$ we mean the vector whose ℓ components are $z_{k_1}, \ldots, z_{k_\ell}$. We use the short-hand $\vec{z}^{(k-\ell)}$ for $\vec{z}^{(k,k+1,\ldots,\ell)} = (z_k, \ldots, z_\ell)$. The following easily verified theorem is the basis of the lattice derivation of theta function identities developed in this chapter.

Theorem 5.2.3: (i) The theta series of a direct sum of glue classes is the product of the theta series of the individual classes:

$$\vartheta([g_1, \ldots, g_k]\{\Lambda_1, \ldots, \Lambda_k\})(\vec{z} \mid \tau) = \prod_{i=1}^{k} \vartheta([g_i]\Lambda_i)(\vec{z}^{(M_i - N_i)} \mid \tau),$$

where $M_i = 1 + \sum_{\ell=1}^{i-1} \dim \Lambda_\ell$, and $N_i = \sum_{\ell=1}^{i} \dim \Lambda_\ell$;

(ii) the theta series of the disjoint union of glue classes is the sum of the theta series of the individual classes:

$$\vartheta\{\cup_{i=1}^{k}[g_i]\Lambda_i\} = \sum_{i=1}^{k} \vartheta([g_i]\Lambda_i) \text{ provided } [g_i]\Lambda_i \cap [g_j]\Lambda_j \neq \emptyset \text{ when } i \neq j;$$

(iii) $\vartheta([\sqrt{\ell}g]\Lambda^{(\ell)})(\vec{z} \mid \tau) = \vartheta([g]\Lambda)(\sqrt{\ell}\vec{z} \mid \ell\tau)$; $\quad \vartheta(\Lambda^{(\ell)})(\vec{z} \mid \tau) = \vartheta(\Lambda)(\sqrt{\ell}\vec{z} \mid \ell\tau)$.

Note that the glue classes of any glue decomposition are pairwise disjoint - hence the value of Thm.3(ii).

By Thm.3(i) we get that the theta series of I_n is

$$\vartheta_3^n(\vec{z} \mid \tau) \stackrel{\text{def}}{=} \vartheta_3(z_1 \mid \tau) \cdots \vartheta_3(z_n \mid \tau), \tag{5.2.6a}$$

where $(z_1, \ldots, z_n) \in \mathbf{C}^n$ are the coordinates of $\vec{z} \in \mathbf{C} \otimes I_n$ relative to the orthonormal basis of I_n. In fact,

$$\begin{aligned} &\vartheta(\{(m_1), \ldots, (m_k)\})(\vec{z} \mid \tau) \\ &\qquad = \vartheta_3(\sqrt{m_1}z_1 \mid m_1\tau) \cdots \vartheta_3(\sqrt{m_k}z_k \mid m_k\tau) \end{aligned} \tag{5.2.6b}$$

$$\begin{aligned} &\vartheta([\ell_1, \ldots, \ell_k]\{(m_1), \ldots, (m_k)\})(\vec{z} \mid \tau) \\ &\qquad = \psi_{m_1/\ell_1}(\sqrt{m_1}z_1 \mid m_1\tau) \cdots \psi_{m_k/\ell_k}(\sqrt{m_k}z_k \mid m_k\tau), \end{aligned} \tag{5.2.6c}$$

using the obvious orthogonal basis of the lattice $\{(m_1), \ldots, (m_k)\}$.

As in Chapter 4, the *Orthogonal Decomposition Theorem* (Cor.1.4.4) has the following immediate consequence.

Theorem 5.2.4: The theta series of any glue class of any rational Euclidean lattice can be expressed polynomially in terms of ϑ_3 and Ψ_k (with arguments τ and z_i scaled appropriately).

More explicitly, recall the definition of $\mathcal{F}_3^{(n)}$ given in the preceding section; define $\mathcal{F}^{(n)}$ and $\mathcal{F}$ similarly, except that the monomials are products of $\Psi_{k_i}(\vec{z} \cdot \vec{a}_i \mid b_i)$'s, for $k_i \in \mathbf{Q}$, $i = 1, \ldots, n$. Then $\mathcal{F}^{(n)}$ contains $\mathcal{F}_3^{(n)}$ and $\mathcal{F}_{1,3}^{(n)}$. We learn in Thm.4 that $\mathcal{F}^{(n)}$ also contains the theta series of the glue classes of any *rational* lattice.

The vectorial nature of $\vec{z}$ introduces a non-trivial complication into the analysis (in this chapter) of theta *series*, which is not present in the corresponding analysis (in Chapter 4) of theta *constants*. In particular, if two lattices Λ and Λ' are integrally equivalent, their corresponding theta constants $\Theta(\Lambda)(\tau)$ and $\Theta(\Lambda')(\tau)$ are numerically equal. This was used repeatedly in the previous chapter. For theta series we must be more careful: the vector $\vec{z}$ will in general have to be rotated to preserve the dot product $\vec{z} \cdot (x + g)$ in eq.(1). An equivalence $\Lambda \leftrightarrow \Lambda'$ induces an equivalence $\mathbf{C} \otimes \Lambda \leftrightarrow \mathbf{C} \otimes \Lambda'$; write this symbolically as $\vec{z} \leftrightarrow \vec{z}'$. Explicitly, let $\beta_\ell \leftrightarrow \beta'_\ell$, $\ell = 1, \ldots, n$ be corresponding bases of Λ, Λ' respectively. Then for $\vec{z} = \sum_{\ell=1}^n z_\ell \beta_\ell$, we have $\vec{z} \leftrightarrow \vec{z}' = \sum_{\ell=1}^n z_\ell \beta'_\ell$ and

$$\vartheta(\Lambda)(\vec{z} \mid \tau) = \vartheta(\Lambda')(\vec{z}' \mid \tau). \tag{5.2.7a}$$

Moreover, if the equivalence takes glue class $[g]$ of Λ to glue class $[g']$ of Λ', we similarly have

$$\vartheta([g]\Lambda)(\vec{z} \mid \tau) = \vartheta([g']\Lambda')(\vec{z}' \mid \tau). \tag{5.2.7b}$$

A special case of these gives:

$$\vartheta(\Lambda)(-\vec{z} \mid \tau) = \vartheta(\Lambda)(\vec{z} \mid \tau) \tag{5.2.7c}$$

$$\vartheta([-g]\Lambda)(-\vec{z} \mid \tau) = \vartheta([g]\Lambda)(\vec{z} \mid \tau), \tag{5.2.7d}$$

for any glue g of any lattice Λ. These generalize eq.(1.3f) and the even-ness of ϑ_2 and ϑ_3, discussed in the previous section.

It is natural (and frequently done) to identify all integrally equivalent lattices. This chapter is one of the reasons we chose not to do so in this work. For each n, fix an orthonormal basis $\beta_n = \{e_i\}$ of $\mathbf{R}^n$. The coordinatization of any lattice Λ considered below will be defined relative to the β_n of some $\mathbf{R}^n$. Moreover, we will take β_n to be the basis of $\mathbf{C} \otimes \Lambda$, when we speak of coordinates of $\vec{z}$. By I_n we mean precisely the subset of $\mathbf{R}^n$ with integer coordinates relative to β_n, so the basis of the lattice is also β_n. Any other n-dimensional (Euclidean) orthogonal lattice of form $\{(m_1), \ldots, (m_n)\}$ is similarly defined to have basis $\sqrt{m_i}e_i$ in $\mathbf{R}^n$ Hence the eqs.(6) are automatically satisfied; it is unnecessary to specify further what the coordinatization of $\vec{z}$ there is taken to be, for this is implicit in the above discussion.

We defined in Sec.1.5 the root lattices. All that need be added here is that these lattices are to be defined as in Sec.1.5, but relative to one of the specific basis choices β_n. For example, D_n is the set of vectors in I_n whose components (relative to the basis β_n) sum up to an even integer.

Note that from eq.(7d) we know $\vartheta([g]\Lambda)(-\vec{z} \mid \tau) = \vartheta([g]\Lambda)(\vec{z} \mid \tau)$ iff $[g]$ is an order 2 glue of Λ (see Thm.4.4). A consequence of this is that only order 2 glues have 3-Solvable theta series. To facilitate calculations, it is often convenient to introduce the *average* theta series:

$$\hat{\vartheta}([g]\Lambda)(\vec{z} \mid \tau) = \frac{1}{2}\{\vartheta([g]\Lambda)(-\vec{z} \mid \tau) + \vartheta([g]\Lambda)(\vec{z} \mid \tau)\},$$

as was done in the previous section with Ψ_k in eq.(1.4a).

As in the previous chapter, identities between the theta series of the glue classes of root lattices arise by considering the outer automorphisms of the Lie algebras corresponding to those root lattices:

$$\vartheta([i]A_n)(\vec{z} \mid \tau) = \vartheta([n+1-i]A_n)(-\vec{z} \mid \tau) \qquad (5.2.8a)$$

$$\vartheta([1]D_n)(\vec{z} \mid \tau) = \vartheta([3]D_n)(\vec{z}' \mid \tau) \qquad (5.2.8b)$$

$$\vartheta([1]E_6)(\vec{z} \mid \tau) = \vartheta([2]E_6)(-\vec{z} \mid \tau) \qquad (5.2.8c)$$

$$\vartheta([1]D_4)(\vec{z} \mid \tau) = \vartheta([2]D_4)(\vec{z}' \mid \tau) = \vartheta([3]D_4)(\vec{z}'' \mid \tau), \qquad (5.2.8d)$$

where $\vec{z}'$ in eq.(8b) has coordinates $(z_1, \ldots, z_{n-1}, -z_n)$, and where in eq.(8$d$), $\vec{z}' = \frac{1}{2}(z_1 + z_2 + z_3 + z_4, z_1 + z_2 - z_3 - z_4, -z_1 + z_2 - z_3 + z_4, z_1 - z_2 - z_3 + z_4)$ and $\vec{z}'' = (z_1, z_2, z_3, -z_4)$.

The principle way we generated identities for the theta constants in Chapter 4 was by finding one or more gluings a given lattice is (integrally) equivalent to, and equating the theta constants corresponding to each of these. For example, we considered there the equivalences $E_8 = D_8[1]$; $D_{m+n} = \{D_m, D_n\}[2,2]$; and $A_2 \approx \{A_1, (6)\}[1,3] = \{(2), (6)\}[1,3]$. Using Thm.3, here we get (compare eqs.(4.2.4) in the previous chapter):

$$\vartheta(E_8) = \vartheta(D_8) + \vartheta([1]D_8) \qquad (5.2.9a)$$

$$\vartheta(D_{m+n}) = \vartheta(D_m) \cdot \vartheta(D_n) + \vartheta([2]D_m) \cdot \vartheta([2]D_n) \qquad (5.2.9b)$$

$$\begin{aligned} \vartheta(A_2)(\vec{z} \mid \tau) &= \vartheta(A_1)(\vec{z}' \mid \tau) \cdot \vartheta(\{(6)\})(z'' \mid \tau) \\ &\quad + \vartheta([1]A_1)((z', -z') \mid \tau) \cdot \vartheta([3]\{(6)\})(z'' \mid \tau) \\ &= \vartheta(\{(2)\})(\sqrt{2}z' \mid \tau) \cdot \vartheta(\{(6)\})(z'' \mid \tau) \\ &\quad + \vartheta([1]\{2\})(\sqrt{2}z' \mid \tau) \cdot \vartheta([3]\{(6)\})(z'' \mid \tau), \end{aligned} \qquad (5.2.9c)$$

where in eq.(9c) $z' = \frac{1}{2}\vec{z} \cdot (1, -1, 0)$ and $z'' = \frac{1}{\sqrt{6}}\vec{z} \cdot (1, 1, -2)$.

As before, the usual calculation implies that for any lattice Λ,

$$\vartheta(\Lambda)(\vec{z} \mid -1/\tau) = \frac{(\tau/i)^{n/2}}{\sqrt{|\Lambda|}} \exp[\pi i \vec{z}^2 \tau] \vartheta(\Lambda^*)(\tau \vec{z} \mid \tau). \qquad (5.2.10a)$$

Thus we can also write:

$$\vartheta([g]\Lambda)(\vec{z} \mid -1/\tau) = \frac{(\tau/i)^{n/2}}{\sqrt{|\Lambda|}} \exp[\pi i \vec{z}^2] \sum_{k=0}^{n-1} \zeta^k \vartheta([kr]\Lambda_0)(\tau\vec{z} \mid \tau), \quad (5.2.10b)$$

$$\hat{\vartheta}([g]\Lambda)(\vec{z} \mid -1/\tau) = \frac{(\tau/i)^{n/2}}{\sqrt{|\Lambda|}} \exp[\pi i \vec{z}^2] \sum_{k=0}^{n-1} \zeta^k \hat{\vartheta}([kr]\Lambda_0)(\tau\vec{z} \mid \tau), \quad (5.2.10c)$$

Here, $[g]$ is a glue of Λ of order n (*i.e.* $kg \in \Lambda$ iff n divides k), $\zeta = e^{2\pi i/n}$, $r \in \Lambda^*$ satisfies $r \cdot g \equiv \frac{1}{n} \pmod 1$, and Λ_0 is the largest sublattice of Λ^* satisfying $g \cdot \Lambda_0 \subseteq \mathbf{Z}$. A special case of eq.(10b) is eq.(1.6e).

Because $\Lambda^* = \Lambda[G]$ where $G = \Lambda^*/\Lambda$,

$$\vartheta(\Lambda^*) = \sum_{g\in G} \vartheta([g]\Lambda) = \sum_{g \in G} \hat{\vartheta}([g]\Lambda). \quad (5.2.11)$$

The usefulness of eqs.(10) and (11) follows if $\vartheta(\Lambda)$ can be calculated independently of eq.(10) — *e.g.* if it can be expressed in terms of ϑ_3's (*i.e.* '3-Solved'). Note that eq.(1.3e) is a special case of eq.(11), taking $\Lambda = I_1^{(k)}$ and using Thm.3.

The theta series of the root lattices can be computed from the methods of the previous chapter (compare eqs.(4.2.8) there):

$$\vartheta(D_n)(\vec{z} \mid \tau) = \frac{1}{2}\{ \vartheta_3^n(\vec{z} \mid \tau) + \vartheta_4^n(\vec{z} \mid \tau) \}, \quad (5.2.12a)$$

$$\vartheta([1]D_n)(\vec{z} \mid \tau) = \frac{1}{2} \vartheta_2^n(\vec{z} \mid \tau) + \frac{1}{2}(-i)^n \vartheta_1^n(\vec{z} \mid \tau), \quad (5.2.12b)$$

$$\vartheta([2]D_n)(\vec{z} \mid \tau) = \frac{1}{2}\{ \vartheta_3^n(\vec{z} \mid \tau) - \vartheta_4^n(\vec{z} \mid \tau) \}, \quad (5.2.12c)$$

$$\vartheta([3]D_n)(\vec{z} \mid \tau) = \frac{1}{2} \vartheta_2^n(\vec{z} \mid \tau) - \frac{1}{2}(-i)^n \vartheta_1^n(\vec{z} \mid \tau), \quad (5.2.12d)$$

$$\vartheta(D_n^*)(\vec{z} \mid \tau) = \vartheta_2^n(\vec{z} \mid \tau) + \vartheta_3^n(\vec{z} \mid \tau), \quad (5.2.12e)$$

$$\begin{aligned}\vartheta(E_6)(\vec{z} \mid \tau) = &\frac{1}{2}\{ \vartheta_3^5(\vec{z}'|\tau) + \vartheta_4^5(\vec{z}'|\tau) \}\, \vartheta_3(\sqrt{12}z''|12\tau) \\ &+ \frac{1}{4}\{ \vartheta_2^5(\vec{z}'|\tau) - i\vartheta_1^5(\vec{z}'|\tau) \} \cdot \\ &\cdot \{ \vartheta_2(\sqrt{3}z''|3\tau) - i\vartheta_1(\sqrt{3}z''|3\tau) \}, \quad (5.2.12f)\end{aligned}$$

$$\vartheta(E_6^*)(\vec{z} \mid \tau) = \frac{1}{4\sqrt{3}}\{ \vartheta_3^5(\vec{z}'|\tau) + \vartheta_2^5(\vec{z}'|\tau) \}\, \vartheta_3(z''/\sqrt{12}|\tau/12)$$

$$+ \frac{1}{4\sqrt{3}} \{ \vartheta_4^5(\vec{z}'|\tau) - i\vartheta_1^5(\vec{z}'|\tau) \} \cdot$$

$$\cdot \{ \vartheta_4(z''/\sqrt{3}|\tau/3) - i\vartheta_1(z''/\sqrt{3}|\tau/3) \}, \qquad (5.2.12g)$$

$$\hat{\vartheta}([1]E_6)(\vec{z} \mid \tau) = \hat{\vartheta}([2]E_6)(\vec{z}|\tau) = \frac{1}{2}\{ \vartheta(E_6^*)(\vec{z}|\tau) - \vartheta(E_6)(\vec{z}|\tau) \}, \qquad (5.2.12h)$$

$$\begin{aligned} \vartheta(E_7)(\vec{z} \mid \tau) =& \vartheta_3^7(\sqrt{2}\vec{z}'|2\tau) + \vartheta_2^4(1-4)\vartheta_3^3(5-7) + \vartheta_2^4(3-6)\vartheta_3^3(1,2,7) \\ &+ \vartheta_2^4(2,3,6,7)\vartheta_3^3(1,4,5) + \vartheta_2^4(1,2,5,6)\vartheta_3^3(3,4,7) \\ &+ \vartheta_2^4(1,4,6,7)\vartheta_3^3(2,3,5) + \vartheta_2^4(2,4,5,7)\vartheta_3^3(1,3,6) \\ &+ \vartheta_2^4(1,3,5,7)\vartheta_3^3(2,4,6), \end{aligned} \qquad (5.2.12i)$$

$$\vartheta([1]E_7)(\vec{z} \mid \tau) = \{ \text{ interchange each } \vartheta_2 \text{ and } \vartheta_3 \text{ in eq.}(12i)\}, \qquad (5.2.12j)$$

$$\vartheta(E_7^*)(\vec{z} \mid \tau) = \{ \text{ sum eqs } (12i, j)\}, \qquad (5.2.12k)$$

$$\vartheta(E_8)(\vec{z} \mid \tau) = \frac{1}{2}\{ \vartheta_1^8(\vec{z}|\tau) + \vartheta_2^8(\vec{z}|\tau) + \vartheta_3^8(\vec{z}|\tau) + \vartheta_4^8(\vec{z}|\tau) \}. \qquad (5.2.12\ell)$$

In eqs.$(12f, g, h)$ the (row) vector $(\vec{z}', z'')$ can be obtained from $\vec{z}$ by the formula $(\vec{z}', z'') = \vec{z}T$, where

$$T = \frac{1}{\sqrt{8}} \begin{pmatrix} -1 & 1 & 1 & 1 & 1 & -\sqrt{3} \\ -1 & 1 & 1 & 1 & -1 & \sqrt{3} \\ -2 & 0 & 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & 0 & -2 & 0 \\ 0 & 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In eqs.$(12i, j, k)$, the (row) vector $\vec{z}'$ can be obtained from $\vec{z}$ by the formula $\vec{z}' = \vec{z}T$, where

$$T = \frac{1}{\sqrt{8}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}.$$

In eq.(12i) we also used the short-hand $\vartheta_2^4(1-4)\vartheta_3^3(5-7)$ for $\vartheta_2^4(\sqrt{2}\vec{z}'^{(1-4)} \mid 2\tau)\,\vartheta_3^3(\sqrt{2}\vec{z}'^{(5-7)} \mid 2\tau)$, *etc.*.

The situation for A_n and its glue classes is by far the most complicated:

$$\vartheta(A_{n-1})(\vec{z} \mid \tau) = \frac{\sum_{k=0}^{n-1} \vartheta_3^n(\{\frac{k}{n} - \frac{z}{\sqrt{n}}\}\vec{e} + \vec{z} \mid \tau)}{n\vartheta_3(\sqrt{n}z \mid n\tau)}. \tag{5.2.13a}$$

Here, $\vec{e}$ is the norm n vector with components $(1, \ldots, 1)$. Let $\zeta = e^{2\pi i/n}$ Then we similarly have for any $\ell = 1, \ldots, n-1$,

$$\vartheta([\ell]A_{n-1})(\vec{z} \mid \tau) = \frac{\sum_{k=0}^{n-1} \zeta^{-k\ell}\vartheta_3^n(\{\frac{k}{n} - \frac{z}{\sqrt{n}}\}\vec{e} + \vec{z} \mid \tau)}{n\Psi_{n/\ell}(\sqrt{n}z \mid n\tau)}. \tag{5.2.13b}$$

Note that eq.(13b) can be obtained from eqs.(4a, c) by substituting $\vec{z} \rightarrow \vec{z} + [\ell]\tau$ and $z \rightarrow \ell\tau/\sqrt{n}$ into eq.(13a). The expressions in eqs.(12) and eqs.(13) will all be proved in the following section. Of course in eqs.(13) it may be most useful to consider the case when $z = 0$.

Equations (12) show that the theta series of each glue class of D_n, E_6, E_7, and E_8 is (1,3)-Solvable (except for $[1]E_6$ and $[2]E_6$, whose *average* $\hat{\vartheta}$ is (1,3)-Solvable). However, it is not clear from eqs.(13) that this is true of A_n. Indeed, we have been unable to prove the (rational) (1,3)-Solvability of its theta series. However, for smaller n explicit '(1,3)-Solutions' can be found, and are listed in Table 10. Note that by Thm.4.4(d) those classes denoted there by a '??' are not (1,3)-Solvable, even in the average.

Thm.4.4 discusses in more generality the 3- and (1,3)-Solvability of glue classes.

There are integrally inequivalent lattices whose theta constants are equal: for example $E_8 \oplus E_8$ and D_{16}^+, as can be seen using the Jacobi identity. However, the theta series stores more information about the lattice (or glue class):

Theorem 5.2.5: The equality $\vartheta([g]\Lambda)(\vec{z}|\tau) = \vartheta([g']\Lambda')(\vec{z}|\tau)$ holds for all $\vec{z}$ and τ iff the glue classes $[g]\Lambda = [g']\Lambda'$, *i.e.* iff they are pointwise identical.

Of course a special case of this shows that the theta series of two lattices are equal iff the two lattices are equal.

Table 10: Known Theta Series of A_n

Lattice	Weights	Name	Theta series
A_1	[0]	$\Theta_{(1,0)}$	$\vartheta_3(2)$
	[1]	$\Theta_{(1,1)}$	$\vartheta_3(\frac{1}{2})$
A_2	[0]	$\Theta_{(2,0)}$	$\vartheta_3(2)\ \vartheta_3(6) + \vartheta_2(2)\ \vartheta_2(6)$
	[1], [2]	$\Theta_{(2,1)}$	$\vartheta_2(2)\ \Psi_6(6) + \vartheta_3(2)\ \hat{\Psi}_3(6)$
A_3	[0]	$\Theta_{(3,0)}$	$\frac{1}{2}\ \{\vartheta_3^3(1) + \vartheta_4^3(1)\}$
	[1]	$\Theta_{(3,1)}$	$\frac{1}{2}\ \{\vartheta_2^3(1) + i\vartheta_1^3(1)\}$
	[2]	$\Theta_{(3,2)}$	$\frac{1}{2}\ \{\vartheta_3^3(1) - \vartheta_4^3(1)\}$
	[3]	$\Theta_{(3,3)}$	$\frac{1}{2}\ \{\vartheta_2^3(1) - i\vartheta_1^3(1)\}$
A_4	[0]	$\Theta_{(4,0)}$	$\sum_{k=0}^{3} \Psi_{4/k}(20)\, \Theta_{(3,k)}(1)$
	[1], [4]	$\Theta_{(4,1)}$	??
	[2], [3]	$\Theta_{(4,2)}$	??
A_5, A_6			??
A_7	[0]	$\Theta_{(7,0)}$	$\sum_{k=0}^{3} \Theta_{(3,k)}(1)\ \Psi_{4/k}(8)\, \Theta_{(3,4-k)}(1)$
	[1], [7]	$\hat{\Theta}_{(7,1)}$	??
	[2]	$\Theta_{(7,2)}$	$\sum_{k=0}^{3} \Theta_{(3,k+2)}(1)\ \Psi_{4/(k-2)}(8)\, \Theta_{(3,4-k)}(1)$
	[2], [6]	$\hat{\Theta}_{(7,2)}$	$\frac{1}{2}\ \{\Theta_{E_7^*}(1) - \Theta_{E_7}(1)\}$
	[3], [5]	$\hat{\Theta}_{(7,3)}$	??
	[6]	$\Theta_{(7,6)}$	$\sum_{k=0}^{3} \Theta_{(3,k)}(1)\ \Psi_{4/(k-2)}(8)\, \Theta_{(3,2-k)}(1)$
	[4]	$\Theta_{(7,4)}$	$\Theta_{E_7}(1) - \Theta_{(7,0)}(1)$

Thm.5 can be shown in the following way. We get from Thm.2 that $\Lambda[g] = \Lambda'[g']$ — this immediately does it when $g \in \Lambda$ and $g' \in \Lambda'$. Otherwise, we may apply Thm.1 after expressing $\vartheta([g]\Lambda)$ and $\vartheta([g']\Lambda')$ as sums of glue classes of $\Lambda \cap \Lambda'$.

5.3 Identities of the Jacobi Functions

Theta series identities are obtained here in a completely analogous manner to that described in the previous chapter — *i.e.* by comparing different glue decompositions of lattices. If the lattice in question is of dimension n, then the identities will be of degree n. We are interested in *algebraically independent* identities *i.e.* identities that cannot be obtained from each other and ones of lower degree *arithmetically* (*i.e.* through multiplication and addition) and/or by transforming τ (*e.g.* $\tau \to \tau + 1$, $\tau \to k\tau$, $\tau \to -1/\tau$) or $\vec{z}$ (*e.g.* through a rotation). For example, eqs.(1.3c) and (1.5b) are not algebraically independent in this sense. Nor is the celebrated Riemann identity, as we shall soon see.

5.3.1 The general theory of full rank identities:

Recall the definition of the **C**-module $\mathcal{F}$ given in Sec.2 (it is generated by products of Ψ_k's). We will consider in this section only identities in $\mathcal{F}$, although the results can be extended to more general identities. Although eqs.(2.10), (1.6) and (1.7a,b) are not identities in this sense, almost all other equations considered in the first three sections of this chapter are, including everything in Table 11.

We can write any identity in $\mathcal{F}$ in the form

$$\sum_{\ell=0}^{N} \alpha_\ell \exp[\pi i \vec{z}_{\ell 2}^{\,2}\tau + 2\pi i(\vec{z}M_\ell') \cdot \vec{z}_{\ell 2}]\, \vartheta(\Lambda_\ell)(\vec{z}M_\ell' + \vec{z}_{\ell 1} + \vec{z}_{\ell 2}\tau \mid \tau) = 0. \quad (5.3.1a)$$

Here, each $\alpha_\ell \in \mathbf{C}$, $\vec{z}_{\ell 1} \in \mathbf{Q} \otimes \Lambda_\ell^*$, $\vec{z}_{\ell 2} \in \mathbf{Q} \otimes \Lambda_\ell$, and where M_ℓ' is any real (constant) matrix whose rank equals the dimension of Λ_ℓ. Note that by eq.(2.1c)

we may rewrite eq.(1a) as

$$\sum_{\ell=0}^{N} \alpha_\ell \exp[\pi i \vec{z}_{\ell 2} A_\ell^{-1} \vec{z}_{\ell 2}^t \tau + 2\pi i \vec{z} M_\ell A_\ell^{-1} \vec{z}_{\ell 2}^t] \vartheta(I_{n_\ell})(\vec{z} M_\ell + \vec{z}_{\ell 1} + \vec{z}_{\ell 2}\tau \mid A_\ell \tau) = 0, \tag{5.3.1b}$$

where $\vec{z}_{\ell 1} \in \mathbf{Q}^n$, $\vec{z}_{\ell 2} A_\ell^{-1} \in \mathbf{Q}^n$, and M_ℓ is a real matrix of rank n_ℓ (A_ℓ is a Gram matrix for Λ_ℓ, hence must be positive definite and symmetric). Note that because we are in $\mathcal{F}$ we can choose in eq.(1a) all Λ_ℓ to be orthogonal lattices, and in eq.(1b) all A_ℓ to be diagonal.

The analogs of Thms.4.3.1-2 in Chapter 4 continue to apply here:

Theorem 5.3.1: Any given identity in $\mathcal{F}$ can be expressed as the (finite) sum of identities in $\mathcal{F}$, each homogeneous in degree.

Theorem 5.3.2: Any given identity in $\mathcal{F}$ is a linear combination (over $\mathbf{R}$) of identities whose terms all have translates $\vec{z}_{\ell 1} = 0$. An identity in $\mathcal{F}$ whose translates $\vec{z}_{\ell 1}$ all vanish is a linear combination (over $\mathbf{C}$) of identities whose terms all have rational coefficients α_ℓ and translates $\vec{z}_{\ell 1} = 0$.

Thm.1 tells us that in eq.(1a) we may assume all the lattices Λ_ℓ are of equal dimension n; this dimension is the degree of the identity. We will be interested in this section only in identities of *full rank*, *i.e.* identities whose matrices A_ℓ and M_ℓ all have rank equal to the degree n.

It is eqs.(2.3a) and (2.4c) that allow us to disregard $\vec{z}_{\ell 1}$. Note that in Thm.2 the translates $\vec{z}_{\ell 2}$ will in general not vanish. The proofs of Thms.1.2 are as in the previous chapter.

The following simple result will turn out to be useful for the next two proofs.

Lemma 5.3.3: For any $x \in \mathbf{R}^n$ and any n-dimensional lattices $\Lambda_1, \ldots, \Lambda_N$ in $\mathbf{R}^n$, we have for $T \in \mathbf{R}$

$$\text{liminf}_{T\to\infty} \max_{\ell=1,\ldots,N} \text{dist}(Tx, \Lambda_\ell) = 0,$$

where $\text{dist}(Tx, \Lambda_\ell) \stackrel{\text{def}}{=} \min_{q\in\Lambda_\ell}(Tx-q)^2$.

Proof By considering $\Lambda_1 \oplus \cdots \oplus \Lambda_N$, it suffices to prove the lemma for $N = 1$.

Let $\Lambda \subset \mathbf{R}^n$ be an n-dimensional lattice and take any $y \in \mathbf{R}^n$. Suppose the set $\Lambda + \langle y \rangle$ is not dense at $\vec{0}$. Then it is nowhere dense, and so is itself an n-dimensional lattice. But this implies $y \in \mathbf{Q} \otimes \Lambda$, in which case the above liminf is attained and equals zero. QED

Note that, for example, in eqs.(1.5), (2.6) and Thm.2.3(iii) we consistently see that if τ is scaled by k, then z is scaled by $\sqrt{k}$. This is true in general:

Theorem 5.3.4: We may assume in eq.($1b$) that $M_\ell^t M_\ell = A_\ell$ for each ℓ.

Prrof Consider the degree n identity

$$0 = \sum_{\ell=0}^{N} \alpha_\ell t_\ell(\vec{z} \mid \tau), \qquad \text{where} \qquad (*a)$$

$$t_\ell(\vec{z} \mid \tau) = \exp[\pi i \vec{z}_\ell A_\ell \vec{z}_\ell^{\,t} \tau + 2\pi i \vec{z} M_\ell \vec{z}_\ell^{\,t}]\, \vartheta(I_n)(\vec{z} M_\ell + \vec{z}_\ell A_\ell \tau \mid A_\ell \tau), \qquad (*b)$$

for $\vec{z}_\ell \in \mathbf{Q}^n$ and where M_ℓ and A_ℓ are $n \times n$ real matrices of rank n (A_ℓ must in addition be positive definite). We may and will assume each A_ℓ is diagonal. Choose M_0 so that $M_0 = \sqrt{A_0}$.

Clearly, to prove Thm.4 it suffices to prove that the sum of all $t_\ell(\vec{z} \mid \tau)$ in eq.($*$) satisfying $M_\ell^t M_\ell = A_\ell$ is also identically 0 for all $\vec{z}, \tau$.

First note that each t_ℓ has 'quasi-period' $\vec{m}_\ell \stackrel{\text{def}}{=} \vec{m} A_\ell M_\ell^{-1} \tau$, for all $\vec{m}_\ell \in I_n$, *i.e.*

$$t_\ell(\vec{z} + \vec{m}_\ell \tau \mid \tau) = \exp[-\pi i \vec{m} A_\ell \vec{m}^t \tau - 2\pi i \vec{z} M_\ell \vec{m}^t]\, t_\ell(\vec{z} \mid \tau). \qquad (**a)$$

Because A_ℓ is diagonal, we know explicitly all the zeros of $\vartheta(I_n)(\vec{z} \mid A_\ell \tau)$. In particular, consider the following set:

$$S_\ell = \{x \in \mathbf{R}^n \mid \exists y \in \mathbf{R}^n \text{ such that } t_\ell(x + y\tau \mid \tau) = 0\}.$$

For each ℓ, it can be shown that the intersection of S_ℓ with any compact set in $\mathbf{R}^n$ has Lebesgue measure 0 Choose any $\vec{x} \notin \cup_{\ell=0}^{N} S_\ell$ — the set of such $\vec{x}$ is dense in $\mathbf{R}^n$.

Write $\tau = \tau_1 + \imath\tau_2$, and choose any $\vec{y} \in \mathbf{R}^n$. Then we have by eq.$(**a)$ that there exist positive constants m, M such that

$$0 < m \exp[\pi \tau_2 T^2 \vec{y} B_\ell \vec{y}^{\,t}] < | \, t_\ell(\vec{x} + T\vec{y}\tau \mid \tau) \, | < M \exp[\pi\tau_2 T^2 \vec{y} B_\ell \vec{y}^{\,t}] < \infty, \quad (**b)$$

for all $T \in \mathbf{R}$, where $B_\ell \stackrel{\text{def}}{=} M_\ell A_\ell^{-1} M_\ell^t$.

Suppose $B_\ell = B_0'$ for $\ell = 0, \ldots, N_0$, $B_\ell = B_1'$ for $\ell = N_0 + 1, \ldots, N_1$, ..., and $B_\ell = B_k'$ for $\ell = N_{k-1} + 1, \ldots, N_k = N$, where the B_i' are all distinct. Then there exist $\vec{y}$ dense in $\mathbf{R}^n$ such that the 'norms' $\vec{y} B_0' \vec{y}^{\,t}, \ldots, \vec{y} B_k' \vec{y}^{\,t}$ are all distinct. Now using Lemma 3 and continuity of, for example, $\sum_{\ell=0}^{N_0} \alpha_\ell t_\ell$, we get from eq.$(**a, b)$ (by sending $T \to \infty$) that for such $\vec{y}$

$$\sum_{\ell=0}^{N_0} \alpha_\ell t_\ell(\vec{x} \mid \tau) = \cdots = \sum_{\ell=N_{k-1}+1}^{N_k} \alpha_\ell t_\ell(\vec{x} \mid \tau) = 0$$

for all τ. Since these $\vec{x}$ are dense in $\mathbf{R}^n$, we can write

$$\sum_{\ell=0}^{N_0} \alpha_\ell t_\ell(\vec{z} \mid \tau) = \cdots = \sum_{\ell=N_{k-1}+1}^{N_k} \alpha_\ell t_\ell(\vec{z} \mid \tau) = 0$$

for all $\vec{z}, \tau$. QED

This proof worked because we knew something about the zeros of $\vartheta(I_n)(\vec{z}|A\tau)$ for A diagonal. Thm.4 and hence Thm.6 should hold for more general identities than merely those in $\mathcal{F}$, but to increase their generality we must know something about the zeros for non-diagonal A. At this time we have not completed this generalization.

Note that any identity derived using these lattice techniques will automatically satisfy the conclusion of this theorem. In this case, M_ℓ^t will be the generator matrix for the corresponding lattice.

Consider again an identity of form eq.($1b$), and assume it satisfies $M_\ell^t M_\ell = A_\ell$. Define Λ_ℓ to be the lattice with generator matrix M_ℓ^t. We may assume by Thm.1 that these lattices are all of equal dimension n.

Theorem 5.3.5: We may assume in addition that $\Lambda \stackrel{\text{def}}{=} \cap_{\ell=0}^{N} \Lambda_\ell$ also has dimension n.

Proof Thanks to Thm.4 we may assume in eq.($*$) that $M_\ell^t M_\ell = A_\ell$ for all $\ell = 1, \ldots, N$. Define Λ_ℓ to be the lattice with generator matrix M_ℓ^t. To prove Thm.5 it suffices to show that the sum over all ℓ in eq.($*$) for which $\mathbf{Q} \otimes \Lambda_0^* = \mathbf{Q} \otimes \Lambda_\ell^*$, also is identically 0 for all $\vec{z}, \tau$.

The periods of t_ℓ comprise the lattice $\Lambda_\ell' \stackrel{\text{def}}{=} (\Lambda_\ell[\vec{z}_\ell A_\ell])^*$, which is an n-dimensional sublattice of Λ_ℓ^*. Suppose $\mathbf{Q} \otimes \Lambda_\ell^* = \bar{\Lambda}_0$ for $\ell = 0, \ldots, N_0, \ldots$, and $\mathbf{Q} \otimes \Lambda_\ell^* = \bar{\Lambda}_k$ for $\ell = N_{k-1} + 1, \ldots, N_k = N$, where all the $\bar{\Lambda}_i$ are distinct. Then $\Lambda_k'' \stackrel{\text{def}}{=} \cap_{\ell=N_{k-1}+1}^{N_k} \Lambda_\ell'$ is n-dimensional, for $k = 0, \ldots, m$.

Define $s_k(\vec{z} \mid \tau) = \sum_{\ell=N_{k-1}+1}^{N_k} \alpha_\ell t_\ell(\vec{z} \mid \tau)$, $k = 0, \ldots, m$. Then $s_k(\vec{z} + \vec{z}' \mid \tau) = s_k(\vec{z} \mid \tau)\ \forall \vec{z}' \in \Lambda_k''$. Eq.($*$) becomes

$$s_0(\vec{z} \mid \tau) = -\sum_{k=1}^{m} s_k(\vec{z} \mid \tau). \qquad (***)$$

We will show $s_0 = 0$ by inducting on m.

For $m = 1$, s_1 has period in both Λ_0'' and Λ_1'', and so has periods in $\Lambda_0 + \Lambda_1$. By Lemma 3, such a set is dense. Therefore s_1 must be constant for all $\vec{z}$, so sending $\vec{z} \to \tau\infty$ implies s_1, and hence s_0, must both be identically 0.

For $m > 1$ note that eq.($***$) implies $s_0(\vec{z}|\tau) - s_0(\vec{z} + \vec{z}'|\tau) = -\sum_{k=1}^{m-1}\{s_k(\vec{z}|\tau) - s_k(\vec{z} + \vec{z}'|\tau)\}$ for any $\vec{z}' \in \Lambda_m''$, so induction immediately gives us $s_0(\vec{z} \mid \tau) = s_0(\vec{z} + \vec{z}' \mid \tau)$ for any such $\vec{z}'$. This is precisely the statement that s_0 has periods in Λ_m'', which by the preceding argument implies $s_0 = 0$. QED

One consequence of Thm.5 is that all Λ_ℓ are rationally equivalent, but it says much more. In particular, it allows us to replace the identity eq.($1a$) with several identities, each of the form

$$\sum_{\ell=0}^{N} \alpha_\ell \vartheta([g_\ell]\Lambda)(\vec{z} \mid \tau) = 0. \tag{5.3.2}$$

But now we can apply Thm.2.1. The conclusion of our analysis is the main result of this subsection:

Theorem 5.3.6: Eqs.(2.3), (2.4), (2.7) and Thm.2.3 generate all identities in $\mathcal{F}$.

Results corresponding to Thms.2.1, 3.4 and 3.5 for theta constants are not yet known to be true, at least in their full generality. These theorems are proven here using information about $\vec{z}$-periods and $\vec{z}$-quasi-periods of the theta series of glue classes of lattices; these tools are not immediately available in the analysis of theta constant identities.

5.3.2 Linear identities:

Thm.6 tells us that the only linear identities in $\mathcal{F}$ are the obvious ones: eqs.($1.3b, c, f$) generate all of them (the translates $z_1 + z_2\tau$ for $z_1, z_2 \in \mathbf{Q}$ can of course be handled by the one-dimensional analogs of eqs.($2.3a$) and ($2.4a, c$)).

Thm.1.1 is obviously a special case of this remark.

5.3.3 Quadratic identities:

As we have seen, the vectorial nature of $\vec{z}$ is responsible for most of the complications introduced when we generalize the results of Chapter 4 to this chapter. With this in mind, consider the gluing $\Lambda = \{(m_1),(m_2)\}[k_1,k_2]$, $k_1, k_2 > 0$. Then if the glue vector $(k_1/\sqrt{m_1}, k_2/\sqrt{m_2}) \in [k_1,k_2]$ is primitive, we can also write $\Lambda \approx \{(n_1),(n_2)\}[\ell_1,\ell_2]$, where n_i, ℓ_i are chosen so that the lattice equivalence be a rotation T taking $(k_1/\sqrt{m_1}, k_2/\sqrt{m_2})$ to $(\sqrt{n_1}, 0)$ (so we must have $n_1 = k_1^2/m_1 + k_2^2/m_2$).

Let the orders of the glues be M and N, respectively. Then the lattice expression $\{(m_1),(m_2)\}[k_1,k_2] \approx \{(n_1),(n_2)\}[\ell_1,\ell_2]$ implies the identity:

$$\sum_{i=1}^{M} \Psi_{m_1/ik_1}(\sqrt{m_1}z_1 \mid m_1\tau)\Psi_{m_2/ik_2}(\sqrt{m_2}z_2 \mid m_2\tau) =$$

$$\sum_{j=1}^{N} \Psi_{n_1/j\ell_1}(\sqrt{n_1}z_1' \mid n_1\tau)\Psi_{n_2/j\ell_2}(\sqrt{n_2}z_2' \mid n_2\tau), \quad (5.3.3a)$$

$$\text{for } (z_1',z_2') = (z_1,z_2)T, \text{ where } T = \begin{pmatrix} \frac{k_1}{\sqrt{n_1 m_1}} & \frac{-k_2}{\sqrt{n_1 m_2}} \\ \frac{k_2}{\sqrt{n_1 m_2}} & \frac{k_1}{\sqrt{n_1 m_1}} \end{pmatrix} = (T^t)^{-1} \quad (5.3.3b)$$

We will write eq.(3a) in the more compact form

$$\sum_{i=1}^{M} \Psi_{m_1/ik_1}(m_1)\Psi_{m_2/ik_2}(m_2) = \sum_{j=1}^{N} \Psi_{n_1/j\ell_1}(n_1')\Psi_{n_2/j\ell_2}(n_2'). \quad (5.3.3c)$$

Note that the *order* of the Ψ's in each term in eq.(3c) is important: the left-most Ψ's in each product take z_1 (or z_1') as an argument, while the right one takes z_2 (or z_2').

Of course *any* lattice equivalence $\{(m_1),(m_2)\}[k_1,k_2] \approx \{(n_1),(n_2)\}[\ell_1,\ell_2]$ will induce an identity of the form eq.(3a), but in general $\vec{z}'$ will not be related to $\vec{z}$ as in eq.(3b). As in Chapter 4 it turns out (see Thm.7 below) that all but one of the lattice equivalences we need to consider for Table 11 belong to the subclass of gluing equivalences defined above.

As in the previous chapter, it is possible in this way to generate an infinite number of identities in $\mathcal{F}^{(2)}$, independent of each other and of the linear ones discussed earlier. For this reason we will consider in the remainder of this section only those identities in $\mathcal{F}^{(2)}_{1,3}$, *i.e.* those involving only ϑ_1 and ϑ_3 (and, of course, ϑ_2, ϑ_4, Ψ_2, $\hat{\Psi}_3$, Ψ_4 and $\hat{\Psi}_6$). The following section will address some more general identities.

In particular, we may already anticipate one problem with eqs.(3). There we find Ψ's and not $\hat{\Psi}$'s. This can be accommodated for in the following way.

Consider the following transformations:

$$\vec{z}^{\perp} = -\vec{z} + 2\{\vec{z}\cdot(\frac{k_1}{\sqrt{n_1 m_1}}, \frac{k_2}{\sqrt{n_1 m_2}})\}(\frac{k_1}{\sqrt{n_1 m_1}}, \frac{k_2}{\sqrt{n_1 m_2}})$$

$$= (z_1', -z_2')T^t \tag{5.3.5a}$$

$$(z_1'', z_2'') = ((\frac{-k_1}{\sqrt{n_1 m_1}}, \frac{k_2}{\sqrt{n_1 m_2}}) \cdot \vec{z}, \ (\frac{k_2}{\sqrt{n_1 m_2}}, \frac{k_1}{\sqrt{n_1 m_1}}) \cdot \vec{z} \ )$$
$$= (z_1, -z_2)T \tag{5.3.5b}$$

Both these represent reflections (*e.g.* eq.(5a) represents reflecting $-\vec{z}$ through the glue vector). Then in the spirit of eq.(3c) we get the following identities:

$$\frac{1}{4}\{\sum_{i=1}^{M} \Psi_{\frac{m_1}{ik_1}}(m_1)\Psi_{\frac{m_2}{ik_2}}(m_2) + \sum_{i=1}^{M} \Psi_{\frac{m_1}{ik_1}}(-m_1)\Psi_{\frac{m_2}{ik_2}}(-m_2) + \perp\}$$
$$= \sum_{j=1}^{N} \hat{\Psi}_{\frac{n_1}{j\ell_1}}(n_1')\hat{\Psi}_{\frac{n_2}{j\ell_2}}(n_2'), \tag{5.3.6a}$$

$$\sum_{i=1}^{M} \hat{\Psi}_{\frac{m_1}{ik_1}}(m_1)\hat{\Psi}_{\frac{m_2}{ik_2}}(m_2)$$
$$= \frac{1}{4}\{\sum_{j=1}^{N} \Psi_{\frac{n_1}{j\ell_1}}(n_1')\Psi_{\frac{n_2}{j\ell_2}}(n_2') + \sum_{j=1}^{N} \Psi_{\frac{n_1}{j\ell_1}}(-n_1')\Psi_{\frac{n_2}{j\ell_2}}(-n_2') + {}''\}, \tag{5.3.6b}$$

where by '$+\perp$' we mean to add all the previous terms within the brackets $\{\,\}$ with z_1 and z_2 replaced with $z_1^\perp$ and $z_2^\perp$, respectively; similarly by '$+{}''$' we mean to add the previous terms within $\{\,\}$ with z_1' and z_2' replaced with z_1'' and z_2''.

The situation is unfortunately more complicated when it is necessary to get $\hat{\Psi}$'s on both sides of the identity. In particular, when T is not idempotent (*i.e.* when the rotation it represents is not through a rational angle) the above maneuver cannot be extended so as to get $\hat{\Psi}$'s on both sides.

As in Table 8, we are interested for Table 11 here in (rank 2) quadratic identities in ϑ_1 and ϑ_3 which are algebraically independent of each other and of the one-dimensional identities. Because of this, it suffices to consider a subset of the infinite set of all two-dimensional gluing equivalences $\{(m_1),(m_2)\}[G] \approx \{(m_1'),(m_2')\}[G']$. This subset is identical to that defined in Subsection 4.3.3, except that $m_1 = m_1'$, $m_2 = m_2'$ and $G = G'$ can be allowed here. In particular, it suffices (for reasons given in Subsec.4.3.3) to consider equivalences of the form

$$\{(ka),(kb)\}[a,b] \approx \{(\ell c),(\ell d)\}[\pm c, d],$$

Table 11: The Quadratic Theta Series Identities

Lattices	Identity
$\{2,2\}[1,1] \approx I_2$, or $\{8,8\}[2,2] \approx \{1,4\}$	$\vartheta_2^2(1) + \vartheta_3^2(1) = \vartheta_3^2(\frac{1}{2}')$
$\{3,6\}[1,2] \approx \{1,2\}$, $\{4,8\}[2,4] \approx \{3,24\}[1,8]$, $\{16,32\}[4,8] \approx \{12,96\}[2,16]$, $\{12,24\}[2,4] \approx \{1,8\}$, or $\{16,32\}[4,8] \approx \{3,96\}[2,32]$	$\vartheta_3(3)\,\vartheta_3(6) + 2\Psi_3(3)\,\Psi_3(6) = \frac{1}{2}\,\{\vartheta_3(1')\,\vartheta_3(2') + ''\}$
$\{2,6\}[1,3] \approx \{2,6\}[1,3]$, or $\{4,12\}[1,3] \approx \{1,3\}$	$\vartheta_2(1)\,\vartheta_2(3) + \vartheta_3(1)\,\vartheta_3(3) = \vartheta_2(1')\,\vartheta_2(3') + \vartheta_3(1')\,\vartheta_3(3')$
$\{9,9\}[3,3] \approx \{2,18\}[1,9]$, or $\{18,18\}[3,3] \approx \{1,9\}$	$\vartheta_3^2(9) + 2\Psi_3^2(9) = \frac{1}{2}\,\{\vartheta_3(2')\,\vartheta_3(18') + \vartheta_2(2')\,\vartheta_2(18') + ''\}$
$\{3,15\}[1,5] \approx \{2,10\}[1,5]$, $\{6,30\}[1,5] \approx \{1,5\}$, or $\{8,40\}[2,10] \approx \{3,60\}[1,20]$	$\vartheta_3(3)\,\vartheta_3(15) + 2\Psi_3(3)\,\Psi_3(15) = \frac{1}{2}\,\{\vartheta_3(2')\,\vartheta_3(10') + \vartheta_2(2')\,\vartheta_2(10') + ''\}$
$\{3,33\}[1,11] \approx \{4,44\}[3,11]$, $\{6,66\}[1,11] \approx \{2,22\}[1,11]$	$\vartheta_3(3)\,\vartheta_3(33) + 2\Psi_3(3)\,\Psi_3(33) = \frac{1}{4}\{\vartheta_3(1')\,\vartheta_3(11') + \vartheta_2(1')\,\vartheta_2(11') + \vartheta_1(1')\,\vartheta_1(11') + \vartheta_4(1')\,\vartheta_4(11') + ''\}$
$\{4,28\}[1,7] \approx \{2,14\}[1,7]$	$\vartheta_3(1)\,\vartheta_3(7) + \vartheta_2(1)\,\vartheta_2(7) - \vartheta_1(1)\,\vartheta_1(7) + \vartheta_4(1)\,\vartheta_4(7) = 2\vartheta_3(2')\,\vartheta_3(14') + 2\vartheta_2(2')\,\vartheta_2(11')$
$\{12,20\}[3,5] \approx \{2,30\}[1,15]$	$\vartheta_3(3)\,\vartheta_3(5) + \vartheta_2(3)\,\vartheta_2(5) - \vartheta_1(3)\,\vartheta_1(5) + \vartheta_4(3)\,\vartheta_4(5) = 2\vartheta_3(2')\,\vartheta_3(30') + 2\vartheta_2(2')\,\vartheta_2(30')$
$\{6,10\}[3,5] \approx \{4,60\}[1,15]$	$2\vartheta_3(6)\,\vartheta_3(10) + 2\vartheta_2(6)\,\vartheta_2(10) = \vartheta_3(1')\,\vartheta_3(15') + \vartheta_4(1')\,\vartheta_4(15') + \vartheta_2(1')\,\vartheta_2(15') - \vartheta_1(1')\,\vartheta_1(15')$

Table 11: The quadratic identities (cont.)

Lattices	Identity
$\{20,28\}[5,7] \approx$ $\{3,105\}[1,35]$, $\{10,14\}[5,7] \approx$ $\{6,210\}[1,35]$	$\frac{1}{4}\{\vartheta_3(5)\,\vartheta_3(7) + \vartheta_2(5)\,\vartheta_2(7) - \vartheta_1(5)\,\vartheta_1(7) + \vartheta_4(5)\,\vartheta_4(7)$ $+ \perp\} = \vartheta_3(3')\,\vartheta_3(105') + 2\dot{\Psi}_3(3')\,\dot{\Psi}_3(105')$
$\{30,42\}[5,7] \approx$ $\{2,70\}[1,35]$, $\{15,21\}[5,7] \approx$ $\{4,140\}[3,35]$	$\vartheta_3(15)\,\vartheta_3(21) + \vartheta_2(15)\,\vartheta_2(21) + 2\dot{\Psi}_3(15)\,\dot{\Psi}_3(21) +$ $2\Psi_6(15)\,\Psi_6(21) = \frac{1}{2}\,\{\vartheta_3(1')\,\vartheta_3(35') + \vartheta_2(1')\,\vartheta_2(35') + ''\}$
$\{4,60\}[1,15] \approx$ $\{4,60\}[3,15]$	$-\vartheta_1(1)\,\vartheta_1(15) + \vartheta_2(1)\,\vartheta_2(15) + \vartheta_3(1)\,\vartheta_3(15) + \vartheta_4(1)\,\vartheta_4(15) =$ $\vartheta_1(1')\,\vartheta_1(15') + \vartheta_2(1')\,\vartheta_2(15') + \vartheta_3(1')\,\vartheta_3(15') + \vartheta_4(1')\,\vartheta_4(15')$
$\{20,44\}[5,11] \approx$ $\{4,220\}[3,55]$	$-\vartheta_1(5)\,\vartheta_1(11) + \vartheta_2(5)\,\vartheta_2(11) + \vartheta_3(5)\,\vartheta_3(11) + \vartheta_4(5)\,\vartheta_4(11) =$ $\vartheta_1(1')\,\vartheta_1(55') + \vartheta_2(1')\,\vartheta_2(55') + \vartheta_3(1')\,\vartheta_3(55') + \vartheta_4(1')\,\vartheta_4(55')$
$\{28,36\}[7,9] \approx$ $\{4,252\}[1,63]$	$-\vartheta_1(7)\,\vartheta_1(9) + \vartheta_2(7)\,\vartheta_2(9) + \vartheta_3(7)\,\vartheta_3(9) + \vartheta_4(7)\,\vartheta_4(9) =$ $-\vartheta_1(1')\,\vartheta_1(63') + \vartheta_2(1')\,\vartheta_2(63') + \vartheta_3(1')\,\vartheta_3(63') + \vartheta_4(1')\,\vartheta_4(63')$
$\{12,52\}[3,13] \approx$ $\{4,156\}[1,39]$	$-\vartheta_1(3)\,\vartheta_1(13) + \vartheta_2(3)\,\vartheta_2(13) + \vartheta_3(3)\,\vartheta_3(13) + \vartheta_4(3)\,\vartheta_4(13) =$ $-\vartheta_1(1')\,\vartheta_1(39') + \vartheta_2(1')\,\vartheta_2(39') + \vartheta_3(1')\,\vartheta_3(39') + \vartheta_4(1')\,\vartheta_4(39')$
$\{18,54\}[3,9] \approx$ $\{2,54\}[1,27]$, $\{9,27\}[3,9] \approx$ $\{4,108\}[1,27]$	$\vartheta_3(9)\,\vartheta_3(27) + 2\dot{\Psi}_3(9)\,\dot{\Psi}_3(27) + 2\dot{\Psi}_6(9)\,\dot{\Psi}_6(27) +$ $\vartheta_2(9)\,\vartheta_2(27) = \frac{1}{2}\,\{\vartheta_3(1')\,\vartheta_3(27') + \vartheta_2(1')\,\vartheta_2(27') + ''\}$

Table 11: The quadratic identities (cont.)

Lattices	Identity
$\{6,138\}[1,23] \approx$ $\{4,92\}[3,23]$	$\vartheta_3(6)\vartheta_3(138) + \vartheta_2(6)\vartheta_2(138) + 2\dot{\Psi}_3(6)\dot{\Psi}_3(138)$ $+2\hat{\Psi}_6(6)\hat{\Psi}_6(138) = \frac{1}{4}\{\vartheta_3(1')\ \vartheta_3(23')+$ $\vartheta_2(1')\ \vartheta_2(23') + \vartheta_1(1')\ \vartheta_1(23') + \vartheta_4(1')\ \vartheta_4(23') + ''\}$
$\{18,126\}[3,21] \approx$ $\{4,252\}[1,63]$	$\vartheta_3(18)\vartheta_3(126) + \vartheta_2(18)\vartheta_2(126) + 2\dot{\Psi}_3(18)\Psi_3(126)$ $+2\hat{\Psi}_6(18)\Psi_6(126) = \frac{1}{4}\{\vartheta_3(1')\ \vartheta_3(63')+$ $\vartheta_2(1')\ \vartheta_2(63') - \vartheta_1(1')\ \vartheta_1(63') + \vartheta_4(1')\ \vartheta_4(63') + ''\}$
$\{30,114\}[5,19] \approx$ $\{4,380\}[3,95]$	$\vartheta_3(30)\vartheta_3(114) + \vartheta_2(30)\vartheta_2(114) + 2\dot{\Psi}_3(30)\Psi_3(114)$ $+2\hat{\Psi}_6(30)\hat{\Psi}_6(114) = \frac{1}{4}\{\vartheta_3(1')\ \vartheta_3(95')+$ $\vartheta_2(1')\ \vartheta_2(95') + \vartheta_1(1')\ \vartheta_1(95') + \vartheta_4(1')\ \vartheta_4(95') + ''\}$
$\{20,76\}[5,19] \approx$ $\{6,570\}[1,95]$	$\frac{1}{4}\{\vartheta_3(5)\vartheta_3(19) + \vartheta_2(5)\vartheta_2(19) - \vartheta_1(5)\vartheta_1(19)$ $+\vartheta_4(5)\vartheta_4(19) + \perp\} = \vartheta_3(6')\ \vartheta_3(570')+$ $\vartheta_2(6')\vartheta_2(570') + 2\Psi_3(6')\Psi_3(570') + 2\Psi_6(6')\Psi_6(570')$
$\{42,102\}[7,17] \approx$ $\{4,476\}[1,119]$	$\vartheta_3(42)\vartheta_3(102) + \vartheta_2(42)\vartheta_2(102) + 2\dot{\Psi}_3(42)\dot{\Psi}_3(102)$ $+2\hat{\Psi}_6(42)\hat{\Psi}_6(102) = \frac{1}{4}\{\vartheta_3(1')\ \vartheta_3(119')+$ $\vartheta_2(1')\ \vartheta_2(119') - \vartheta_1(1')\ \vartheta_1(119') + \vartheta_4(1')\ \vartheta_4(119') + ''\}$
$\{28,68\}[7,17] \approx$ $\{6,714\}[5,119]$	$\frac{1}{4}\{\vartheta_3(7)\vartheta_3(17) + \vartheta_2(7)\vartheta_2(17) - \vartheta_1(7)\vartheta_1(17)$ $+\vartheta_4(7)\vartheta_4(17) + \perp\} = \vartheta_3(6')\vartheta_3(714')+$ $\vartheta_2(6')\vartheta_2(714') + 2\Psi_3(6')\dot{\Psi}_3(714') + 2\dot{\Psi}_6(6')\Psi_6(711')$
$\{54,90\}[9,15] \approx$ $\{4,540\}[3,135]$	$\vartheta_3(54)\vartheta_3(90) + \vartheta_2(54)\vartheta_2(90) + 2\dot{\Psi}_3(54)\dot{\Psi}_3(90)$ $+2\hat{\Psi}_6(54)\dot{\Psi}_6(90) = \frac{1}{4}\{\vartheta_3(1')\ \vartheta_3(135')+$ $\vartheta_2(1')\ \vartheta_2(135') + \vartheta_1(1')\ \vartheta_1(135') + \vartheta_4(1')\ \vartheta_4(135') + ''\}$
$\{66,78\}[11,13] \approx$ $\{4,572\}[1,143]$	$\vartheta_3(66)\vartheta_3(78) + \vartheta_2(66)\vartheta_2(78) + 2\dot{\Psi}_3(66)\dot{\Psi}_3(78)$ $+2\hat{\Psi}_6(66)\dot{\Psi}_6(78) = \frac{1}{4}\{\vartheta_3(1')\ \vartheta_3(143')+$ $\vartheta_2(1')\ \vartheta_2(143') - \vartheta_1(1')\ \vartheta_1(143') + \vartheta_4(1')\ \vartheta_4(143') + ''\}$
$\{44,52\}[11,13] \approx$ $\{6,858\}[1,143]$	$\frac{1}{4}\{\vartheta_3(11)\vartheta_3(13\{) + \vartheta_2(11)\vartheta_2(13) - \vartheta_1(11)\vartheta_1(13)+$ $\vartheta_4(11)\vartheta_4(13) + \perp\} = \vartheta_3(6')\vartheta_3(858')+$ $\vartheta_2(6')\vartheta_2(858') + 2\hat{\Psi}_3(6')\hat{\Psi}_3(858') + 2\hat{\Psi}_6(6')\Psi_6(858')$

where the gluing orders $k, \ell \in \{1,2,3,4,6\}$, where $0 < a \leq b$ and $0 < c \leq d$ are all integers, where the sign before c in the right-hand glue is chosen so that the transformation T involved in the equivalence is a rotation, and lastly, where whenever $ka = \ell c$ and $kb = \ell d$, the rotation T involved is nontrivial.

Theorem 5.3.7: There are exactly 55 such lattice equivalences; 38 of these are given in Table 11, and the remainder are:

$$\{(3),(24)\}[1,8] \approx \{(3),(24)\}[-1,8], \quad \{(12),(96)\}[2,16] \approx \{(3),(96)\}[1,32],$$

$$\{(6),(21)\}[2,7] \approx \{(3),(42)\}[1,14], \quad \{(24),(84)\}[4,14] \approx \{(3),(168)\}[-1,56],$$

$$\{(12),(15)\}[4,5] \approx \{(3),(60)\}[-1,20], \quad \{(48),(60)\}[8,10] \approx \{(3),(240)\}[1,80],$$

$$\{(6),(210)\}[1,35] \approx \{(6),(210)\}[-1,35], \quad \{(30),(186)\}[5,31] \approx \{(6),(210)\}[1,155],$$

$$\{(42),(174)\}[7,29] \approx \{(6),(1218)\}[-1,203], \quad \{(21),(33)\}[7,11] \approx \{(6),(462)\}[-1,77],$$

$$\{(78),(138)\}[13,23] \approx \{(6),(1794)\}[-1,299], \quad \{(6),(102)\}[1,17] \approx \{(3),(51)\}[-1,17],$$

$$\{(66),(150)\}[11,25] \approx \{(6),(1650)\}[1,275], \quad \{(30),(78)\}[5,13] \approx \{(3),(195)\}[1,65],$$

$$\{(15),(39)\}[5,13] \approx \{(6),(390)\}[1,65], \quad \{(42),(66)\}[7,11] \approx \{(3),(231)\}[-1,77],$$

$$\text{and } \{(102),(114)\}[17,19] \approx \{(6),(1938)\}[1,323].$$

The proof of Thm.7 is similar to that of Thm.4.3.4 in Chapter 4. The reason the 17 equivalences explicitly listed in Thm.6 were not included in Table 11 was because those equivalences had $k, \ell \in \{3,6\}$ and the rotations T involved were *not* through a rational angle (see the discussion after eqs.(6)). All but one of these 55 equivalences are of the type considered in eqs.(3); $\{(16),(32)\}[4,8] \approx \{(12),(96)\}[2,16]$ is the only exception, and its T is defined by

$$T = \begin{pmatrix} \frac{-1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \end{pmatrix}.$$

The notation in eqs.(6) is used throughout Table 11. To illustrate all the preceding notation, the second identity in Table 11, written out in full, becomes

$$\begin{aligned}\vartheta_3(\sqrt{3}z_1|3\tau)\,\vartheta_3(\sqrt{6}z_2|6\tau) + 2\hat{\Psi}_3(\sqrt{3}z_1|3\tau)\,\hat{\Psi}_3(\sqrt{6}z_2|6\tau) \\ = \frac{1}{2}\{\vartheta_3(z_1'|\tau)\,\vartheta_3(\sqrt{2}z_2'|2\tau) + \vartheta_3(z_1''|\tau)\,\vartheta_3(\sqrt{2}z_2''|2\tau)\}.\end{aligned} \tag{5.3.6c}$$

Note that eq.(3c) tells us that, corresponding to the gluings $\{(2),(2)\}[1,1] \approx I_2$ and $\{(8),(8)\}[2,2] \approx \{(1),(4)\}$, are the identities

$$\vartheta_2^2(1) + \vartheta_3^2(1) = \vartheta_3^2(\frac{1}{2}'), \tag{5.3.7a}$$

$$\vartheta_3^2(8) + \vartheta_2^2(8) + \frac{1}{2}\vartheta_2^2(2) - \frac{1}{2}\vartheta_1^2(2) = \vartheta_3(1')\vartheta_3(4'), \tag{5.3.7b}$$

where as can be seen by eq.(3b) the $\vec{z}'$ in eq.(7a) equals the $\vec{z}'$ in eq.(7b). Using eq.(7a), we may rewrite eq.(7b) as

$$\vartheta_2^2(1) - \vartheta_1^2(1) = 2\vartheta_2(2')\,\vartheta_3(2'). \tag{5.3.7c}$$

This is interesting for three reasons. Firstly, it shows that, although ϑ_1 *cannot* be expressed as a *linear* expression in ϑ_3's (scaled in various ways), ϑ_1^2 *can* be expressed as a *quadratic* expression in ϑ_3's — we say that ϑ_1 ***is not*** 3-Solvable but ϑ_1^2 ***is*** (see Sec.1 and eq.(4.1a)).

Also, we may derive eq.(7c) (and hence eq.(7b)) by substituting $\vec{z} + (\frac{1}{2}, \frac{1}{2})$ for $\vec{z}$ into eq.(7a) ($\frac{1}{\sqrt{2}}\vec{z}'$ becomes $\frac{1}{\sqrt{2}}\vec{z}' + (\frac{1}{2}, 0)$). Hence two different gluings may yield the same identity (we also saw this in the previous chapter).

Finally, replacing τ with $-1/\tau$, and z_1 with $z_1 + 1$, in eq.(7a) gives us, respectively,

$$\vartheta_3^2(1) + \vartheta_4^2(1) = 2\vartheta_3^2(2') \tag{5.3.8a}$$

$$\vartheta_3^2(1) - \vartheta_2^2(1) = \vartheta_4^2(\frac{1}{2}'). \tag{5.3.8b}$$

Now, multiply eq.(7a) (with variables z_1 and z_2) by the same equation (this time with variables z_3 and z_4), do the same to eq.(8b), and add the two products together.

The result is

$$2\vartheta_3^4(1) + 2\vartheta_2^4(1) = \vartheta_3^4(\frac{1}{\sqrt{2}}\vec{z}' \mid \frac{1}{2}\tau) + \vartheta_4^4(\frac{1}{\sqrt{2}}\vec{z}' \mid \frac{1}{2}\tau), \qquad (5.3.8c)$$

where $\vec{z}'$ denotes here $\frac{1}{\sqrt{2}}(z_1 + z_2, z_1 - z_2, z_3 + z_4, z_3 - z_4)$. Now, into eq.(8c) replace each z_i with $z_i + \frac{1}{2}$, and add the result to eq.(8c). We get

$$\begin{gathered} 2\vartheta_1^4(1) + 2\vartheta_2^4(1) + 2\vartheta_3^4(1) + 2\vartheta_4^4(1) = \\ \vartheta_3^4(\frac{1}{2}') + \vartheta_4^4(\frac{1}{2}') + (\vartheta_3\vartheta_4\vartheta_3\vartheta_4)(\frac{1}{2}') + (\vartheta_4\vartheta_3\vartheta_4\vartheta_3)(\frac{1}{2}') \end{gathered} \qquad (5.3.8d)$$

using obvious notation. We may write the RHS of eq.(8d) as

$$\{\vartheta_3^2(\frac{1}{\sqrt{2}}\vec{z}_1|\frac{1}{2}\tau) + \vartheta_4^2(\frac{1}{\sqrt{2}}\vec{z}_1|\frac{1}{2}\tau)\}\,\{\vartheta_3^2(\frac{1}{\sqrt{2}}\vec{z}_2|\frac{1}{2}\tau) + \vartheta_4^2(\frac{1}{\sqrt{2}}\vec{z}_2|\frac{1}{2}\tau)\},$$

where $\vec{z}_1 = (z_1 + z_2, z_3 + z_4)$ and $\vec{z}_2 = (z_1 - z_2, z_3 - z_4)$. Now, applying eq.(8$a$) twice gives us precisely Riemann's formula (see p.17 of [**MUM**]):

$$\vartheta_1^4(1) + \vartheta_2^4(1) + \vartheta_3^4(1) + \vartheta_4^4(1) = 2\vartheta_3^4(1''), \qquad (5.3.8e)$$

where here $\vec{z}'' = \frac{1}{2}(z_1+z_2+z_3+z_4, z_1+z_2-z_3-z_4, z_1-z_2+z_3-z_4, -z_1+z_2-z_3+z_4)$.

What we have shown is that a *second degree* identity (namely eq.(7a)) can be used to derive the *fourth degree* Riemann identity eq.(8e). Hence the Riemann identity is not an algebraically independent identity in our terms. All of the very useful information stored in the Riemann identity is present, though perhaps in not so accessible a form, in the much simpler eq.(7a) (A simpler derivation of the Riemann identity will be given later in this section.)

In Table 11 we list all the ϑ_1 and ϑ_3 quadratic identities derivable using the above techniques. The discussion of their algebraic independence from one another and from the linear identities is similar to the theta constant case (*i.e.* by looking at the ratios of the scalings in each term). Included in the table are the gluings that produced these identities. We have been able to find only one of these (namely the first one, which can be found for example in [**TM**]), in its fullest generality — *i.e.* with rank 2 — in the literature.

5.3.4 Higher-dimensional identities:

Lattice considerations can easily be used to derive the theta series expressions for the root lattices. One way to derive these for D_n, E_7 and E_8 is to use their orthogonal decompositions (these are calculated in Table 4). Also, the series for D_n follows immediately from the observation that D_n consists of all the even normed vectors in I_n, and those for $[2]D_n$ and $[1]D_n \cup [3]D_n$ follow from $I_n = D_n[2]$ (no rotation necessary) and eqs.(2.8*b*) and (2.10*a*). Finally, the series for $[1]D_n$ and $[3]D_n$ can be obtained inductively from $D_{n+1}^+ = D_n D_1[11]$ for odd n, and $D_{n+2}^+ = D_n D_2\left[\{[11],[22]\}\right]$ for even $n > 0$. Since $D_8^+ = E_8$, we immediately get it for E_8. The theta series for E_6 can be found from $E_6 \approx \{D_5,(12)\}[1,3]$ (the average series for its glue classes now follow from eq.(2.10*a*)). Alternate expressions, and hence theta function identities, can be obtained by using other gluings, as in Ch.4.

Of course, A_n is trickier. Since their orthogonal decompositions are known (see Table 4), their theta series (like that of any rational lattice) are in $\mathcal{F}^{(n)}$ and can be written down explicitly (but messily). Eq.(2.13*a*) and hence eq.(2.13*b*) can be derived from ${A_{n-1}^*}^{(n)} \oplus I_1 \approx I_n^{(n)}[1,\ldots,1]$, but the simplest derivation is probably the projection operator argument given in eq.(4.3.10) of the previous chapter.

The expressions given in Table 10 were found using the lattice equivalences given at the end of Sec.4.3. $E_7 = A_7[4]$ and $E_7^* = A_7[2]$ give us the $\Theta_{(7,0)}$, $\Theta_{(7,2)}$ and $\Theta_{(7,6)}$ entries in the table (no rotations are involved). Eq.(4.3.11) implies the theta function:

$$\vartheta(A_{n+m})(\vec{z} \mid \tau) = \sum_{k=0}^{N-1} \vartheta([-k]A_{n-1})(\vec{z}\,' \mid \tau)\, \Psi_{L/k\ell}(\sqrt{L}z'' \mid L\tau)\, \vartheta([k]A_m)(\vec{z}\,''' \mid \tau), \tag{5.3.9a}$$

where $N = n(m+1)/(n+m+1,n)$ is the order of the glue. Define the following unit vectors: $\vec{e}_{n+m+1} = \frac{1}{\sqrt{n+m+1}}(\{1\}^{n+m+1})$ and $\vec{e}_{n+m+1}{}' = \frac{1}{\sqrt{n}}(\{1\}^n,\{0\}^{m+1})$.

Then in eq.(9a),

$$\vec{z}' = \vec{z}^{(1-n)} - (\vec{z} \cdot \vec{e}_{n+m+1}{}')\vec{e}_n, \quad z'' = \vec{z} \cdot \vec{e}_{n+m+1}{}',$$
$$\vec{z}''' = \vec{z}^{(n+1-n+m+1)} + \vec{z} \cdot \vec{e}_{n+m+1}{}' \, \vec{e}_{m+1}. \tag{5.3.9c}$$

This gives us the $\Theta_{(4,0)}$ entry in Table 10. There clearly is no rotation involved in the A_1 entries in Table 10. For the rotation involved in the A_2 entries, see the discussion after eq.(2.9c). And for A_3, T is

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.3.9d}$$

Of course each of the expressions in Table 10 imply identities when compared with eq.(2.13). Further sources of identities are:

$$\{A_n, (n+1)\}[1,1] \approx I_{n+1}, \tag{5.3.10a}$$

$$\text{and} \quad D_4^+ \approx I_4. \tag{5.3.10b}$$

For example, the simplest proof of the Riemann identity (eq.(8e)) is to read it off from eq.(10b).

Of course, root lattices are not the only source of lattice identities. The gluings of orthogonal lattices can yield great numbers of them in each dimension, using the methods used earlier for generating quadratic identities. These are too numerous and complicated to explicitly write down.

5.4 Theta Series of Glue Classes

In this section we investigate the existence of polynomials that Ψ_k satisfies. We will find that the results for Ψ_k generalize very naturally to the theta series of general glue classes. We will find that the theta series of the glue classes are roots of polynomials whose coefficients are linear combinations of the theta series of lattices, and we will investigate their minimal polynomials.

$\vartheta_1(z \mid \tau)$ is an odd function of z while $\vartheta_3(z \mid \tau)$ is even. Therefore ϑ_1 cannot be '3-Solved'. However, from eq.(3.7c) we see that we can write

$$\vartheta_1(z \mid \tau) = -\sqrt{\vartheta_2(z \mid \tau)^2 - 2\vartheta_2(2z \mid 2\tau)\,\theta_3(2\tau)}. \tag{5.4.1a}$$

Similarly, by Thm.1.1 we know that $\hat{\Psi}_5$ cannot be 'Solved'. However, from $\{(5),(5)\}[1,2] \approx I_2$ and eq.(1.3f) we see that it also satisfies a quadratic equation:

$$\begin{aligned}\hat{\Psi}_5(z|\tau)^2 &+ \frac{1}{2}\{\vartheta_3(z|\tau) - \vartheta_3(z/5|\tau/25)\}\hat{\Psi}_5(z|\tau)\\ &+ \frac{1}{2}\{\vartheta_3(z/5|\tau/5)\vartheta_3(3z/5|\tau/5) - \vartheta_3(z|\tau)^2\} = 0,\end{aligned} \tag{5.4.1b}$$

so we may write

$$\hat{\Psi}_5(z \mid \tau) = \frac{\vartheta_3(z|\tau) - \vartheta_3(z/5|\tau/25)}{4} + \tag{5.4.1c}$$
$$\frac{\sqrt{\{\vartheta_3(z|\tau) - \vartheta_3(z/5|\tau/25)\}^2 + 4\vartheta_3(z|\tau)^2 - 4\vartheta_3(z/5|\tau/5)\vartheta_3(3z/5|\tau/5)}}{4}.$$

Of course, the quadratic equation eq.(1b) has a second root — it is easy to see that $\hat{\Psi}_{5/2}(z \mid \tau)$ is that second root.

Throughout this section, we will continue to write, for example, $\vartheta_3(k)$ for $\vartheta_3(\sqrt{k}z \mid k\tau)$. Until now, we have been interested in this paper only in *full rank* expressions. However, eq.(1b), for example, is rank 1 but degree 2. In this section we will be interested in polynomials that the (average) theta series of glue classes satisfy; these polynomial identities will have the same rank as the dimension of the glue class in question, but their degrees will be some multiple (depending on the order of the glue) of that rank. With this in mind, consider the following definitions.

Define $\mathcal{T}_L(k)^{(n)}$, for $0 \leq k \leq n$, to be the **R**-module of rank k spanned by the theta series of n-dimensional lattices. Explicitly, it will be the (finite) linear combination of terms $\vartheta(\Lambda)(\vec{z}M|\tau)$, where Λ is an n-dimensional lattice and where the $n \times n$ real matrices M are of rank k. Define $\mathcal{T}_R(k)^{(n)}$ similarly, except that the lattices Λ must be rational. Let $\mathcal{T}_L(k)^{(n)*}$ and $\mathcal{T}_R(k)^{(n)*}$ be their field of fractions. Define $\mathcal{T}_L(k)$ and $\mathcal{T}_R(k)$ as the sums over n of $\mathcal{T}_L(k)^{(n)}$ and $\mathcal{T}_R(k)^{(n)}$, respectively. Then we know that $\mathcal{T}_L(k)^{(n)}$ always contains $\mathcal{T}_R(k)^{(n)}$ and (Thms.1 1 and 2.4) $\mathcal{T}^{(n)}$ properly contains $\mathcal{T}_R(n)^{(n)}$.

Theorem 5.4.1: For each $n = 1, 2, 3, \ldots$, there exist monic polynomials $f_n(\psi) = \psi^k + s_{n,1}\psi^{k-1} + \cdots + s_{n,k}$ and $\hat{f}_n(\psi) = \psi^{k'} + \hat{s}_{n,1}\psi^{k'-1} + \cdots + \hat{s}_{n,k'}$ of degrees $k = \phi(n)$ and $k' = \lceil \frac{1}{2}\phi(n) \rceil$, respectively, with coefficients $s_{n,\ell}, \hat{s}_{n,\ell} \in \mathcal{T}_R(1)^{(\ell)}$, whose k and k' roots are precisely $\Psi_{n/m}$ and $\hat{\Psi}_{n/m} = \hat{\Psi}_{n/(n-m)}$, respectively, for all m relatively prime to n.

Here $\phi(n)$ is the Euler ϕ-function, *i.e.* the number of numbers relatively prime to n, and $\lceil x \rceil$ is the least integer $\geq x$.

For example, the polynomial given in eq.(1b) is $\hat{f}_5$. For $n = 2, 3$ we have $k = k' = 1$ and $k = 2$, respectively:

$$f_2(\psi)(z \mid \tau) = \hat{f}_2(\psi)(z \mid \tau) = \psi - \vartheta_3(1/4) + \vartheta_3(1), \tag{5.4.2a}$$

$$f_3(\psi)(z \mid \tau) = \psi^2 - \frac{1}{2}\{\vartheta_3(1/9) - \vartheta_3(1)\}\psi + \frac{1}{2}\{\Theta(1) - \vartheta_3(1)^2\}, \tag{5.4.2b}$$

where by $\Theta(1)$ in eq.(2b) we mean $\vartheta(\{(3),(3)\}[1,2])(z/\sqrt{3}, z/\sqrt{3} \mid \tau/3)$.

A significant generalization of Thm.1 is possible:

Theorem 5.4.2: Consider any N-dimensional glue class $[g]\Lambda$ of order n. Then there exist monic polynomials f and $\hat{f}$ of degrees $\phi(n)$ and $\lceil \phi(n) \rceil$, respectively, whose coefficients s_k and $\hat{s}_k$ are in $\mathcal{T}_L(N)^{(\ell N)}$ and whose $\phi(n)$ and $\lceil \phi(n) \rceil$ roots are $\vartheta([mg]\Lambda)$ and $\hat{\vartheta}([mg]\Lambda) = \hat{\vartheta}([-mg]\Lambda)$, respectively, for all m relatively prime to n. Moreover, if Λ is rational, the coefficients will in fact be in $\mathcal{T}_R(N)^{(\ell N)}$.

Of course here $\vec{z}$ will be in $\mathbf{C} \otimes \Lambda$, so will have N independent complex components. Thm.1 is a special case of Thm.2. Their proofs are similar to those of

Thms.4.4.1 and 4.4.4 in Chapter 4, and will not be given here. Later in this section we will discuss how minimal these polynomials f and $\hat{f}$ are.

For example, any order 2 glue $[g]\Lambda$ has theta series $\vartheta([g]\Lambda)(\vec{z} \mid \tau)$ which satisfies:

$$f(\psi)(\vec{z} \mid \tau) = \psi - \{\vartheta(\Lambda[g])(1) + \vartheta(\Lambda)(1)\}, \tag{5.4.3a}$$

while the average series of the non-trivial glue classes $[1]A_4$, $[2]A_4$, $[3]A_4$, and $[4]A_4$ of A_4 satisfy:

$$\hat{f}(\psi)(\vec{z} \mid \tau) = \psi^2 - \frac{1}{2}\{\vartheta(A_4^*)(1) - \vartheta(A_4)(1)\}\psi + \frac{1}{4}\{\vartheta(E_8)(\vec{z}' \mid \tau) - \vartheta(A_4)(1)^2\}, \tag{5.4.3b}$$

where $\vec{z} \in \mathbf{C} \otimes A_4 \subset \mathbf{C}^5$, (1) as usual stands for $(\vec{z}|\tau)$, and $\vec{z}' = (\vec{z}, \vec{z})T$ for

$$T = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thms.1 and 2 tell us the theta series of glue classes are algebraic in $\mathcal{T}_L(N)$. A natural question to ask at this point is whether the polynomials given in those theorems are the ones of smallest possible degree, *i.e.* whether they can be factored over $\mathcal{T}_L(N)$ or $\mathcal{T}_L(N)^*$. An easy argument (see below) following from the results of Sec.3 tells us they cannot be factored over $\mathcal{T}_L(N)$:

Theorem 5.4.3: Given any glue class $[g]\Lambda$, any monic polynomial over $\mathcal{T}_L(N)$ having its theta series or average theta series as a root is a multiple of the polynomial f or $\hat{f}$, respectively, given in Thm.2.

The analogous result over $\mathcal{T}_L(N)^*$ — *e.g.* the question of whether the (average) theta series lies in $\mathcal{T}_L(N)^*$ — is more complicated and has not yet been obtained.

However we do know that $\vartheta([g]\Lambda) \in \mathcal{T}_L(N)^*$ only if $[g]\Lambda$ is of order 1 or 2, in which case it is in $\mathcal{T}_L(N)$ as well.

Thm.3 follows from Thm.2.1 and the observation that the sum of the roots of any monic polynomial dividing f or $\hat{f}$ will be in $\mathcal{T}_L(N)^{(N)}$. Similar arguments, together with eq.(1.2c), give us:

Theorem 5.4.4: a) $\vartheta([g]\Lambda)$ is 3-Solvable only if the order of $[g]\Lambda$ is 1 or 2,

b) $\hat{\vartheta}([g]\Lambda)$ is 3-Solvable only if the order of $[g]\Lambda$ is 1, 2, 3, 4 or 6;

c) $\vartheta([g]\Lambda)$ is (1,3)-Solvable only if the order of $[g]\Lambda$ is 1, 2 or 4; and

d) $\hat{\vartheta}([g]\Lambda)$ is (1,3)-Solvable only if the order of $[g]\Lambda$ is 1, 2, 3, 4 or 6.

Chapter 6 APPLICATIONS TO STRING THEORY

6.1 Introduction to String Theory

Many of the results obtained in the previous chapters were motivated by string theory. In Secs.3 and 4 of this chapter we focus on two applications to strings of the foregoing analysis. However, we will first (in this section) give a brief sketch of the theory of superstrings, emphasizing those aspects of particular relevance to the remaining three sections. In the second section we will review the so-called lattice string

This is primarily intended to be a mathematics thesis, so some of the arguments in the first three sections of this chapter may be found to be too sketchy (the reader should find the fourth section sufficiently self-contained) This is unfortunate but unavoidable Wherever possible, references will be given

The standard reference on superstring theory is [**GSW**] — please refer to it for any missing details or for clarifications on the following material.

A field in physics is simply an object (*e.g.* a 4-vector $\hat{A}(x)$ of Hermitian operators) defined at each point x in space-time The quantum field obeys the action principle $\delta I = 0$, where the action I can be obtained by integrating the Lagrangian. This leads to the Euler-Lagrange equations. The important quantities that must be computed are the transition amplitudes, from which the probabilities are obtained. These amplitudes are given by a Feynman path-integral, a weighted sum over all possible (classical) paths going from the initial to the final state, the weight factor being $\exp(iI/h)$ These transition amplitudes are usually calculated perturbatively using the Feynman rules and Feynman diagrams.

There are some formal similarities between a quantum field theory of strings and the standard ones of point particles, but the finite extension of strings introduces some important new features, as we shall see.

The fundamental object in string theory is the string; in the heterotic theory we will be considering it is a closed loop (parametrized say by σ, with period 2π) about 10^{-33} cm long - something like 10^{20} times smaller than the 'diameter' of a proton. An electron or any other elementary particle is realized whenever a string vibrates, rotates, *etc.* in a certain particular way. The string has an infinite number of such modes, with arbitrarily high frequency and angular momentum To each of these corresponds a particle (more precisely, a state), whose masses and spins are arbitrarily large. The natural mass scale in string theory is the Planck mass, $\approx 10^{19}$ GeV, so the elementary particles familiar to us would presumably correspond to the massless modes. Of course we do not want all of these particles to remain truly massless; as in the Standard Model they could acquire relatively small masses using spontaneous symmetry breaking.

Incidently, [**GSW**] give on pp.59-60 a 'no-go theorem', which shows the difficulties of constructing a physics based on extended objects of dimension > 1.

As a particle moves it traces out in space-time a (one-dimensional) world line. A string similarly traces out a two-dimensional world sheet. The action I of a string is proportional to the surface area of its world sheet (at least in the simplest bosonic case). An action of this form is clearly invariant under the choice of parametrization of the world sheet; the fact that we are dealing with *strings* (rather than, say, *membranes*) permits I to be *conformally invariant* as well. These symmetries are central to string theory, as we shall see.

Thus, the transition amplitudes in quantum string theory require summing over all the possible surfaces joining the initial and final states of the string(s) — the topology of each surface specifies as we shall see the precise nature of the interaction represented in each 'summand'. Two of the appeals of string theory are the simplicity of its interactions — two strings may join into one, one string may split into two (and of course any combination of these may also occur) and the fact that there is no Lorentz-invariant interaction point (so that the form of the interactions are uniquely determined from the free theory). The world sheets resulting from, say, the scattering of two strings will, topologically speaking, include world sheets looking like an 'H', like a ladder with two rungs, *etc.*. Quantum mechanically, we must take the weighted sum of these possibilities. The result is a perturbation series.

In quantum mechanics it is possible to start with nothing and end with nothing, but to do so in a very complicated way. It does this through the so-called virtual processes. In string theory, for example, what could happen is that a virtual string would spontaneously appear, split, rejoin, and then disappear.

These processes contribute to what is known in quantum field theory as the 'vacuum-to-vacuum amplitude'. Its first order term corresponds to the physical process described in the previous paragraph, whose Feynman diagram looks like a torus. This term is usually called the *partition function* of the string. The corresponding torus can be characterized by a complex number τ called the modular parameter, and so the partition function will be a function $Z(\tau)$, usually written as $Z(\tau, \bar{\tau})$, where $\bar{\tau}$ is the complex conjugate of τ but is treated as an independent variable (more accurately, the first term of the vacuum-to-vacuum amplitude is actually the sum over all tori: $\int d\tau d\bar{\tau} Z(\tau, \bar{\tau})/Im\tau$, where the integrals are over the fundamental domain $\mathcal{H}$). From this arises as well the cosmological constant corresponding to the theory (see Sec.6.4).

Now, the partition function should be invariant under reparametrization of the torus. Some of these are global transformations not continuously connected to the identity. These transformations induce mappings on τ; these mappings form a discrete group isomorphic to the *modular group* SL(2,$\mathbf{Z}$)/$\{\pm 1\}$. Thus $Z(\tau)$ must be invariant under the modular group — this is how the powerful constraint of modular invariance enters into string theory. Modular invariance constrains the dimension of the theory Modular invariance guarantees the vacuum-to-vacuum amplitude (to first order) is finite. Some plausibility arguments exist which seem to show that modular invariance also guarantees that strings are free of *all* divergences, so that string theory is completely finite. The modular invariance of the partition function will play a recurring role throughout the remaining sections. For example, in Sec.4 we try to determine, given a certain class of (modularly invariant) functions which *look like* partition functions, whether an acceptable string could be found whose partition function belongs to that class.

One of the qualities of string theory that many physicists found appealing was the apparent near uniqueness of a string satisfying the desired properties (although the large number of different 'compactifications', as can be seen in Sec.3, shows that uniqueness to be illusory in some ways). An important example is the demand that the theory be chiral. It has long been known that the weak force can distinguish between a system and its mirror image. The Standard Model (the accepted quantum field theory, with gauge group SU(3)× SU(2)×U(1)) of course is chiral: the left handed leptons e_L and ν_{eL} form an SU(2) doublet, while the right-handed e_R is a singlet and the right-handed electron-neutrino ν_{eR} does not exist. We shall find chirality in Sec.3 to be a strong constraint on a string theory.

String theory is a gauge theory. In more usual quantum field theories what must eventually be done is to *fix the gauge*. The situation is similar in string theory. Throughout the remainder of this chapter we will use the so-called *light-cone gauge* (see *e.g.* pp.93-95 of **[GSW]**). In a D-dimensional string theory, there are D bosonic fields $X^0,\ldots,X^{D-1}$; the light-cone gauge non-covariantly singles out two of these: X^0 and X^{D-1}. The net effect is that the only independent oscillators α (see eq.(2.3a)) are the transverse ones (*i.e.* those associated with X^i for $i=1,\ldots,D-2$). The oscillators are creation and annihilation operators acting on the Fock space of states (see *e.g.* p.76 of **[GSW]**); this means that we should apply only the transverse oscillators to the ground states $|0\rangle$ to get physical states.

Ghosts are states $|ghost\rangle$ with negative norm $\langle ghost|ghost\rangle < 0$. This is a very undesirable situation because in quantum field theories (including string theories) these norms are interpreted as transition probabilities, and so should be ≥ 0. It turns out that in the light-cone gauge there are no ghosts. However, the gauge fixing obviously broke Lorentz invariance. We must make sure the resulting theory is Lorentz invariant. It turns out that Lorentz invariance holds only in, *e.g.* for the bosonic string, $D = 26$ dimensions, and for the so-called Type II string, $D = 10$ dimensions, consistent with the constraints derived from modular invariance.

We just referred to the dimension in string theories. Unfortunately, nature appears to be only 4- (rather than 26- or 10-) dimensional. The most obvious way to explain the discrepancy is to *compactify* the extra dimensions — that is, to make them so small (*e.g.* on the order of the Planck length of 10^{-33} cm) that we have no hope of observing them. This will be discussed in the following section.

In 1971 Ramond, Neveu and Schwarz found a fermionic string that was later discovered to have built into it a previously unknown symmetry called supersymmetry. Supersymmetry is the only symmetry that can mix bosons and fermions, and thus is the only hope to truly unify all the particles found in nature. It can do this because it has a fermionic generator Q which changes the spin of particles by $\frac{1}{2}$. (See [**FRE**] for a complete introduction to supersymmetry). Locally supersymmetric theories (called supergravity) automatically include general relativity in the appropriate limit, but have serious problems which seem to prevent them from accurately describing nature.

There is no experimental evidence yet that nature is supersymmetric (in fact if it is, the supersymmetry must be badly broken), but many physicists are nevertheless convinced that supersymmetry is just too promising not to somehow play an important role in reality. Supersymmetry comes in two versions in string theory: space-time supersymmetry and world sheet supersymmetry.

The 26-dimensional bosonic string is not supersymmetric (and has the more serious flaw of having tachyons). There are several classes of worldsheet supersymmetric strings — *i.e.* superstrings. *Type I* superstrings are both open and closed, while *Type II* strings are only closed. The former seems to hold some promise as a possible theory of physics. The latter has difficulty either with chirality or with supporting an adequate gauge group.

But the most promising superstring today is called the *heterotic string*. It is a closed string, and hence its right- and left-moving modes are independent (there are no endpoints to reflect its wave). It is a hybrid of the old bosonic string and the Type II string: its left-movers live in 26 dimensions while its right-movers are only in 10, but this is rectified by making 8 of these transverse and 16 of them internal (2 are eliminated by the light-cone gauge). Only right-movers are (world sheet) supersymmetric (the heterotic string may or may not be space-time supersymmetric— see Sec.3). The low-energy limit of the theory is $D = 10$, $N = 1$ supergravity (D is the dimension of space-time, N is the number of fermionic generators Q and the number of spin $\frac{3}{2}$ supersymmetric partners of the graviton, called gravitinos) coupled to the gauge group $\mathrm{Spin}(32)/Z_2$ or $E_8 \times E_8$. These gauge groups, and the way to rectify the difference of 16 dimensions are closely related to each other and to the 16-dimensional even self-dual lattices, and is one of the main ways lattices enter into string theory. The heterotic string is anomaly-free, free of ghosts and tachyons, and there is reason to believe it is entirely finite. Its lowest mass states (and there are many of them!) are all massless (prior, that is, to symmetry breaking). In the remaining three sections of this chapter we will consider only the heterotic string.

The most promising of the heterotic strings is the $E_8 \times E_8$ one. It seems to have the best hope of predicting the observed particles. It has been speculated that one of these E_8's might give rise to another type of matter (called shadow matter) which can interact with our matter only gravitationally. We will address some questions related to shadow matter in Sec.3.

6.2 The Lattice String Formalism

We review in this section how strings in the light-cone gauge (see the previous section) can be constructed from lattices. Our goal is to construct heterotic strings whose dynamical degrees of freedom are completely described by world-sheet free bosons quantized in a lattice, and to see how the various constraints (such as the modular invariance of the partition function — see Sec.1) restrict the physically allowable theories. This is discussed in much more detail in a number of papers by the "Cornell group" (see *e.g.* [**KLT**]) and, independently, by Lam (see [**LAM1-3**] — the equivalence of the two approaches was made explicit in [**LAM1**]). We will also discuss how shifting arises naturally in such theories, and apply some of the results obtained in Chapters 2 and 3 to the study of this shifting construction of strings.

An alternate lattice formalism is the so-called 'covariant lattice' approach, which uses Lorentzian lattices (*e.g.* $I_{25,1}$). It will not be discussed here (but see *e.g.* [**LL**]).

The lattice string is a tiny subclass of all string theories; they represent the simplest and most accessible strings, and yet most features of general strings are reflected in these lattice ones. Hence the real interest of the lattice string is not so much in finding among them a totally adequate physical theory, but more conservatively, to help us understand general properties of string theories. This theme runs throughout the lengthy survey article [**LSW**] on lattices and strings.

Let X and Ψ be boson and fermion fields, respectively. They are functions of (σ^0, σ^1) (in fact, of $\sigma^{\pm} \stackrel{\text{def}}{=} \sigma^0 \pm \sigma^1$), where $\sigma^0 = t$ is 'time', and where $\sigma^1 = \sigma$ is a parameter that runs along the string. Now, our strings are all closed, so σ is a periodic coordinate, say with period 2π. How X and Ψ behave as we wrap around the string — *i.e.* when we replace σ with $\sigma + 2\pi$ — constitute their boundary conditions. As they are not themselves directly observable, they do not have to be periodic.

Because our strings are all closed, the fields can be functions of σ^+ alone (and are called *left-movers*), or functions of σ^- alone (called *right-movers*) see *e.g.* eq.(1.1a). (If the string had endpoints, waves could reflect off them and reverse direction.)

The *conformal currents* (which generate the conformal transformations discussed in the previous section) look something like

$$T(\sigma,t) = -\frac{1}{2}\partial X^\mu \cdot \partial X_\mu - \frac{1}{2}\partial\Psi^\lambda \cdot \Psi_\lambda.$$

They are physical, so must be periodic (*i.e.* $T(\sigma+2\pi,t) = T(\sigma,t)$), which suggests the boundary conditions

$$X(\sigma+2\pi,t) = \exp(-2\pi i\hat{\omega})X(\sigma,t) \text{ or } X(\sigma+2\pi,t) = X(\sigma,t) + 2\pi u$$
$$\text{and } \Psi(\sigma+2\pi,t) = \exp(-2\pi i\omega)\Psi(\sigma,t), \tag{6.2.1}$$

for constants u, ω and $\hat{\omega}$. These are called the *twist* (by phases ω and $\hat{\omega}$) and *shift* (by u) boundary conditions.

A heterotic string must also possess a conserved *superconformal current* in order to eliminate all the remaining 'ghosts' (*i.e.* negative normed states). It looks like

$$T_F = \partial_+ X^\mu \Psi_\mu - f_{\alpha\beta\gamma}\Psi^\alpha\Psi^\beta\Psi^\gamma$$

(where $f_{\alpha\beta\gamma}$ is the structure constant of a Lie algebra), but takes a different form in the bosonized framework used in Sec.3 (see [**GL1**] for a more detailed treatment of the superconformal current, including references). Because only the right-hand side of such a string is supersymmetric, only it has a superconformal current (both sides are conformally invariant, however). We will discuss the superconformal current more in the following section. The superconformal current (being a fermionic quantity) must be periodic or antiperiodic as we travel around the string: *i.e.* when we replace $\sigma \to \sigma + 2\pi$. This is highly nontrivial (leading *e.g.* to the so-called triplet constraint) and, as we shall see, strongly restricts the right-hand side of the theory.

Ψ can be related to X through a process called *bosonization*, which is given by the formula

$$\Psi =: \exp(-iX) :, \tag{6.2.2}$$

where the colons denote the normal ordered product, and tells one how to interpret products of fields in the Taylor expansion of 'exp'. This clearly relates (complex) fermions with the twisted boundary conditions to (real) bosons satisfying the shifted boundary conditions The existence of this bosonization process is intimately connected with the fact that the world sheet is two-dimensional.

There are two possible strategies. Those bosons could be *fermionized*, or the fermions could be bosonized (in both cases using eq.(2)). In the lattice string considered here, the latter is the approach taken.

Whereas the shifted bosons (and hence the twisted fermions) live on a lattice, as we shall see, the twisted bosons on the other hand required something more complicated, called an *orbifold*. An orbifold is obtained from a lattice by identifying the lattice points in the orbits of automorphisms of the lattice. It is flat, except for isolated singularities, so its curvature looks like the sum of Dirac deltas. We will not discuss orbifolds further here, and will assume all bosons in the theory obey the shifted boundary conditions.

In the heterotic string, there are 24 left-moving degrees-of-freedom (*i.e.* real bosons), and 12 right-moving degrees-of-freedom. The dimension D of space-time in the theory is not simply the number $(24 + 12)$ of boson fields, but rather is determined from the number of bosons playing the role of space-time coordinates. In particular, for a D-dimensional string (*i.e.* a string representing a universe with a D-dimensional space-time) $D-2$ left-moving bosons X_L^β pair up with $D-2$ right-moving bosons X_R^β — they are uncompactified: the center-of-mass coordinates $X_0^\beta = X_{L0}^\beta + X_{R0}^\beta$ and momenta $p_0^\beta = p_{L0}^\beta + p_{R0}^\beta$ (see below) of the resulting fields $X^\beta(\sigma, t) = X_L^\beta(\sigma^+) + X_R^\beta(\sigma^-)$ are unconfined (the number of such fields is $D-2$ rather than D because we are in the light-cone gauge — see the previous section).

In the remainder of this discussion we will consider only the remaining compactified real bosons.

Let $X(\sigma,t) = (X_L(\sigma^+); X_R(\sigma^-))$ denote the compactified real bosonic coordinates. The m fields in X_L are left-moving, and the n fields in X_R are right-moving. The previous paragraphs tell us that, for a D-dimensional heterotic string, these numbers are $m = 26 - D$ and $n = 14 - D$. The two cases of greatest interest are $D = 10$ and $D = 4$; in the following two sections we will focus on 4-dimensional strings.

Consider the normal mode expansion (see *e.g.* eq.(1.1))

$$X_{L/R}(\sigma,t) = X_0 + P_{L/R}\sigma^{\pm} + i\sum_{k\neq 0}\frac{1}{k}\alpha_k^{\pm}\exp(-ik\sigma^{\pm}). \qquad (6.3.3a)$$

X_0 is (classically) the center-of-mass coordinates of the string and divided by 2π is assumed to lie in a torus defined by an $(m+n)$-dimensional lattice Λ — *i.e.* it is invariant under translations by vectors in Λ. The boundary conditions satisfied by X is then of the form

$$X(\sigma+2\pi,t) = X(\sigma,t) + 2\pi u \pmod{2\pi\Lambda} \qquad (6.2.3b)$$

for some constant vector $u \in \mathbf{R}\otimes\Lambda$.

Let $p = (p_L; p_R) \in \mathbf{R}\otimes\Lambda$ denote the momentum of a state (it is a vector of eigenvalues of $P = (P_L; P_R)$). Its dot products (it turns out) are defined by $p_1 \cdot p_2 = p_{1L}\cdot p_{2L} - p_{1R}\cdot p_{2R}$ — in other words, Λ has signature (m,n). The boundary condition eq.(3*b*) means $p - u \in \Lambda$.

A state with mass M has a momentum $p = (p_L; p_R)$ satisfying

$$M^2 = p_R^2/2 + M_R - \frac{1}{2} = p_L^2/2 + M_L - 1, \qquad (6.2.4)$$

where $M_{L,R}$ are non-negative integers describing the states of excitations of the bosonic oscillators. In particular, zero mass states — which are the experimentally relevant ones — require $p_R^2 = 1$ and $p_L^2 = 2$ or 0. The mass scale of these string theories is on the order of the Planck mass, $\approx 10^{19}$ Gev, so massive particles would have masses comparable to bacteria — far beyond the scope of modern accelerators.

Tachyons have imaginary 'mass', and hence travel faster than light. They occur when both $p_L^2 < 2$ and $p_R^2 < 1$. For a number of reasons their presence in a theory is not desired. We will confront them in the following section.

Space-time has a preferred role in quantum field theories. But here everything is a world sheet field of operators — including space-time. The Lorentz group is on an equal footing with all other symmetries of the theory. The spin-statistics theorem, giving the correct relationship between spin and statistics, is automatic in (space-time) quantum field theories, but in string theory its validity is not guaranteed: it is imposed, and not derived. We shall now discuss the consequences in this formalism of insisting upon this spin-statistics connection.

Even though all fermions Ψ have been bosonized *via* eq.(2), fermionic states must still exist in this purely bosonic formalism. The fermionic number $\mathcal{F}$ is conserved and additive, and in this framework the fermionic parity $(-1)^{\mathcal{F}}$ is given by $(-1)^{2p\cdot v}$, where v is called the *fermionic vector*.

We are interested in maintaining the correct connection between spin and statistics. Spin is governed by the representation of a state in the transverse space-time rotational group SO($D-2$) (the Lorentz group is SO($D-1$,1), but in the light-cone gauge this collapses to SO($D-2$) — this is discussed on p.41 of [**GL**1]). For heterotic strings, this group operates only on the right-moving coordinates, which means the root lattice $D_{(D-2)/2}$ for this group must be contained in the right-hand side of Λ (explicitly, $(D_{(D-2)/2})^{(-1)} \subset \Lambda$). The usual connection between spin and statistics is obtained therefore if the fermionic vector v is allowed to have nonzero components only in the coordinates $\mathbf{R} \otimes (D_{(D-2)/2})^{(-1)}$, and it should transform like a vector with respect to the transverse space-time group SO($D-2$) — in other words, its component in $(D_{(D-2)/2})^{(-1)}$ should lie in the glue class $[2](D_{(D-2)/2})^{(-1)}$. We can write $v = (0; v_R)$, and we can choose

$$-v^2 = v_R^2 = 1. \qquad (6.2.5)$$

The root lattice for SO(2) is one-dimensional (in fact equal to $D_1 = I_1^{(4)} = (4)$), so in the case of a four-dimensional string the fermionic vector v lies in the direction defined by this one-dimensional lattice.

So far there have been no restrictions on the choice of lattice Λ (other than that for a four-dimensional string it has signature (22, 10)). We shall see that the modular invariance of the partition function restricts it quite significantly.

The partition function (see Sec.1 for its definition) of this lattice string is given by

$$\begin{aligned} Z_\Lambda(vu|\tau,\bar\tau) &= \eta(\tau)^{-m}\eta(\bar\tau)^{-n}\sum_{r\in\Lambda}\exp[\pi i\tau(r_L+u_L)^2 - \pi i\bar\tau(r_R+u_R)^2](-1)^{\mathcal{F}} \\ &= \eta(\tau)^{-m}\eta(\bar\tau)^{-n}\sum_{r\in\Lambda}\exp[\pi i\tau(r_L+u_L)^2 - \pi i\bar\tau(r_R+u_R)^2 + 2\pi i(r+u)\cdot v] \\ &\stackrel{\text{def}}{=} \eta(\tau)^{-m}\eta(\bar\tau)^{-n}\Theta_\Lambda(vu|\tau,\bar\tau) \end{aligned} \qquad (6.2.6)$$

where $\eta(\tau)$ is the Dedekind eta function of the modular parameter τ (see eq.(5.1.2d)). The η factors arise from the contributions of the bosonic oscillators. Θ_Λ contains the contributions of the momentum states $p = r + u$, together with the statistical factor $(-1)^{\mathcal{F}}$. In these expressions, r_L and r_R are the left- and right-hand parts of r, so $r = (r_L; r_R)$.

We saw in the previous section that this partition function $Z_\Lambda(vu|\tau,\tau)$ has to be invariant under the modular transformations. This is so iff it is invariant under both $\tau \to \tau + 1$ (hence $\bar\tau \to \bar\tau + 1$) and $\tau \to -1/\tau$ (hence $\bar\tau \to -1/\bar\tau$).

Invariance under $\tau \to \tau + 1$ implies both

$$r^2 + 2r\cdot v \equiv 0 \pmod 2 \quad \forall r \in \Lambda \qquad (6.2.7a)$$

$$v^2 \equiv (m-n)/12 \pmod 2. \qquad (6.2.7b)$$

For the heterotic string, $m-n=12$, so eq.(7b) is fortunately consistent with eq.(5). Note also that because eq.(7a) must be valid for both $\pm r$, we see that we must have both $2r \cdot v \in \mathbf{Z}$ and $r^2 \in \mathbf{Z}$ — *i.e.* that $2v \in \Lambda^*$ and Λ is integral.

Invariance under $\tau \to -1/\tau$ (using the Poisson summation formula) implies

$$\Lambda = \Lambda^* \tag{6.2.8a}$$

$$u \equiv \pm v \pmod{\Lambda}. \tag{6.2.8b}$$

Thus we may take $v = u$, and we have $2v \in \Lambda$. Note that eqs.(7) imply p^2 must always be odd, and r^2 is even for bosons and odd for fermions. Note also that the component in $\mathbf{R} \otimes (D_{(D-2)/2})^{(-1)}$ of any momentum p must lie in $(D_{(D-2)/2})^{(-1)*}$, and will be in $[1](D_{(D-2)/2})^{(-1)}$ or $[3](D_{(D-2)/2})^{(-1)}$ if the corresponding particle is a fermion, or $[2]D^{(-1)}_{(D-2)/2}$ or $D^{(-1)}_{(D-2)/2}$ if the corresponding particle is a boson.

To summarize, the momenta p for such a D-dimensional lattice string lie in $\Lambda + v$, where Λ is a Type I lattice of signature $(26-D, 14-D)$, and where v satisfies eqs.(4), (6a), and $2v \in \Lambda$. We will be interested in the choice $D=4$.

A generalization of string theory is *conformal field theory*; lattices also can be used in their construction — for a brief review see [**CN**] or [**LSW**].

We will end this section with a brief discussion of the shifting method applied to the construction of lattice strings. This is also touched upon for example on pp.110-111 and pp.118-123 of [**LSW**].

The basic idea is that after choosing more complicated shift boundary conditions than the earlier ones, which were given by a single u satisfying, it turned out, eq.(8b), the theory can be rewritten in terms of those boundary conditions after throwing away the additional nonphysical states. But different boundary conditions mean a different string theory. In other words, this rewriting of the theory amounts to changing the lattice Λ which defined the theory. The new lattice is precisely the shift of Λ, in the sense of Sec.2.4.

This is usually expressed in the fermionic formulation, so we shall turn now to that. Consider first the following string.

The lattice $\Lambda = I_{m,n}$ corresponds to m left-moving and n right-moving world-sheet fermions satisfying periodic boundary conditions, as can be seen through bosonization. In this case the fermionic direction v is $v = ((1/2)^m; (1/2)^n)$, thanks to eq.(7a). This v however is not the vector representation of SO$(D-2)$ (in fact it is not even on the right-hand side of Λ), so spin and statistics are not connected. To restore the spin-statistics connection we must convert it into a string of the earlier type by incorporating antiperiodic boundary conditions, in the following way.

Next, consider a fermionic string given by $(m+n)$ complex fermion fields $(\Psi_1^+(\sigma^+), \ldots, \Psi_m^+(\sigma^+); \Psi_1^-(\sigma^-), \ldots, \Psi_n^-(\sigma^-))$ satisfying the aperiodic twisted boundary condition:

$$\Psi_a^{\pm}(\sigma^{\pm} + 2\pi) = \exp(-2\pi i \omega_{\pm}^{a}) \Psi_a^{\pm}(\sigma^{\pm}). \qquad (6.2.9)$$

(The RNS string briefly referred to in the previous section is a special case of such a string.) From these $\omega_{\pm}^{a}$, we can form a vector $\omega \in \mathbf{R} \otimes \Lambda$, where Λ is $I_{m,n}$.

Now consider A such string theories, each defined by a different choice of twist vectors $\omega_k \in \mathbf{R} \otimes \Lambda$, $k = 1, \ldots, A$. It is usually assumed that each of these twist vectors are of finite order in Λ, *i.e.* there exist positive $n_k \in \mathbf{Z}$ such that $n_k \omega_k \in \Lambda$ for each k. Then the twist group Ω generated by linear combinations of these vectors (modulo Λ) is finite, of order at most Πn_k.

Each $s \in \Omega$ can be written as $\sum_{k=1}^{A} s_k \omega_k$ for integers s_k satisfying $0 \leq s_k < n_k$. Each $s \in \Omega$ defines a Hilbert space (called a sector) of states. The string theory consisting of all these sectors is not physically acceptable. For one thing, it is not modularly invariant (its bosonic lattice is $\Lambda[\Omega]$, which is not self-dual). But there is another problem.

Physical states (which are obtained by applying the field operators Ψ to the vacuum $|0\rangle$, as in eq.(6.1.2)) must be periodic under $\sigma \to \sigma + 2\pi$. We thus must project out of the physical Hilbert space all aperiodic ones. This is achieved by the ***Gliozzi-Scherk-Olive (GSO) projection***, which is built into eq.(10) below.

Eq.(9) tells us the phase the fields Ψ gain as $\sigma \to \sigma + 2\pi$. To determine the phase gained by an arbitrary state, it is necessary (and sufficient) to know in addition the phase gained by the vacuums $|0\rangle_s$ of each sector $s \in \Omega$. Call this phase $\zeta(s) \in \mathbf{R} \otimes \Lambda$. It is essentially a linear function of s.

We can now write down the partition function for the system:

$$Z_{\Lambda,\Omega,\zeta}(v|\tau,\bar{\tau}) = (1/\Pi_{k=1}^{A} n_k) \sum_{s,t \in \Omega} Z_{\Lambda}(s+v, t+v|\tau,\bar{\tau}) \exp[2\pi i \phi(s,t)],$$
$$\phi(s,t) = t \cdot \zeta(s) - v \cdot (t+v), \tag{6.2 10}$$

and Z_Λ is as in eq.(6). The sum over s in eq.(10a) is the sum over sectors. The sum over t, together with the $(1/\Pi n_k)$ factor, gives the GSO condition $(q(s)+s+\zeta(s))\cdot t \in \mathbf{Z}$. (Specifically, the GSO projection is the normalized sum of twist operators — see *e.g.* [**LAM**1-3].)

Imposing the condition on $Z_{\Lambda,\Omega,\zeta}$ that it be modularly invariant turns out to entail a number of constraints on the twist vectors ω_k and vacuum phases $\zeta(s)$. In particular, defining ζ_{ij} to be the jth component of $\zeta(\omega_i)$, we get precisely the conditions eq.(2.4.1a,b) in Section 2.4.

Indeed, it is not difficult to see that the GSO projection is such that the bosonic lattice Λ' corresponding to the string theory with partition function $Z_{\Lambda,\Omega,\zeta}$ is simply the shifted lattice $\Lambda' = \Lambda(\Omega,(\zeta_{ij}))$ discussed in Sec.2.4. Modular invariance, not surprisingly, is the condition needed to ensure that that shifted lattice be self-dual.

What we have seen is that imposing arbitrary aperiodic twisted boundary conditions on complex fermion fields is equivalent (after bosonization and GSO projection) to shifting the original lattice Λ by those twist vectors. The treatment of shifting in Ch.2 was partially motivated by the widespread use of shifting in string physics. Most of the theorems in Sec.2.4 were designed for that purpose.

We know that any odd indefinite self-dual lattice is integrally equivalent to some $I_{m,n}$. We are interested here in odd self-dual lattices with signature $(m,n) = (26-D, 14-D)$. However, strings corresponding to integrally equivalent lattices are not necessarily physically equivalent. In particular, an arbitrary rotation in

SO(m,n) will in general mix up the left-movers and the right-movers, and so could for example change the mass spectrum.

In other words, for lattices used in constructing strings, the projections π_L and π_R are physically relevant. This is another reason why in this work we do not necessarily equate (as many do) integrally equivalent lattices

On the other hand, since rotations of SO$(m)\times$SO(n) do not mix up the left movers with the right-movers, they describe the same physics. Only rotations in SO$(m,n)/($SO$(m)\times$SO$(n))$ give rise to different models, so the number of continuous parameters specifying different physical models is mn.

Results in Chs.2 and 3 tell us the following important result:

Let Λ and Λ' be bosonic lattices for two different D-dimensional strings, and let π_L, π_R, π'_L, and π'_R be the projections satisfying $\pi_L(p_L;p_R) = p_L$, *etc.*

Theorem 6.2.1: Let $(\Lambda_L;\Lambda_R)$ and $(\Lambda'_L;\Lambda'_R)$ be the LR-decompositions of Λ and Λ' defined by π_L, π_R, π'_L, and π'_R. Then it is possible to shift in the above manner between the two strings theories (up to a physically irrelevant rotation in SO(22) $\times$ SO(10)), iff $\Lambda_L \sim \Lambda'_L$ (*i.e.* iff $\Lambda_L \overset{Q}{\approx} \Lambda'_L$).

For example, it is impossible, starting with the fermionic string $I_{22,10}$, to get at (through shifting in this string-theoretic manner) the strings constructed in the following section (*e.g.* the $\mathbf{Z}_3$ orbifold discussed there).

6.3 The Bottom-up Construction of Strings

String theory is the only known grand unified theory where quantum gravity can be meaningfully incorporated. And, because of that, the theory is most easily formulated at the Planck scale. From that scale down, as the universe cools, dynamics intervene to give rise to spontaneous symmetry breakings (these are what give many of the 'massless' states a small mass) and phase transitions. Our inability to deal with the dynamical issues makes it difficult to predict anything from first principles at the present (extremely low) energies — in fact this absence of tangible experimental predictions is one of the most serious flaws of string theories.

The usual way in the physics literature for constructing new strings is the shifting method described at the end of Sec.2. The physicist starts with some string, usually one with far too much symmetry, then imposes on it new aperiodic boundary conditions to break down that symmetry, and sees what phenomenological characteristics the resulting string possesses. It is very difficult to decide which boundary conditions to try, with the result that this becomes a hit-and-miss kind of activity. It has not produced any phenomenologically acceptable theories. This gives rise to two natural questions: (i) can another method for constructing strings be found which gives the physicist more control over the phenomenological properties of the resulting string; and (ii) is the lack of success of the shifting method an indication that there simply are not many (if any) physically acceptable (lattice) string theories. We shall see in this section that the answer to (i) is 'yes', and that (ii) there does seem to be an inherent scarcity of phenomenologically reasonable (lattice string) theories (in particular, 'shadow matter' does not seem to be a miracle cure of the 'rank 22' problem — see below).

This section is a summary of the results and techniques developed in [**GL1**]. That paper is far too long to include here without major abbreviations, but the reader who finds the treatment given here confusing or incomplete should consult [**GL1**]. In it we focus on a method (based on the gluing construction of lattices, and called the 'bottom-up construction') for constructing strings with desired low energy spectrum and gauge group. The strings considered will be 4-dimensional, and will have, as they should, a super(conformal) current. We wish them to be chiral and without tachyons. What tachyons are in the lattice formalism was discussed after eq.(2.4). The forms chirality and space-time supersymmetry take in this formalism will be addressed after Thm.1.

Four-dimensional heterotic lattice strings suffer a drawback in that the gauge group is of rank 22 (see below), which is definitely too large for phenomenological application. One possible way out of this difficulty is to try to arrange some of these 22 ranks to be in a *shadow group*, thereby effectively reducing the rank of the observed gauge group. By definition, a shadow group is one in which the observed particles have zero quantum numbers, and that any massless particles with nontrivial quantum numbers in the shadow group have no quantum numbers in the observed gauge group. Thus matter in the observed gauge group and matter in the shadow group can interact only through gravity, and the shadow group is effectively decoupled and unobservable at the present energies. In this section we will prove some general theorems concerning the shadow-group scenarios.

We would like to construct four-dimensional heterotic strings by constructing self-dual lattices Λ of signature (22,10) (see the previous section for the basic relationship between Λ and the string theory). Moreover, we would like to tailor it to possess a given gauge group $\bar{\mathcal{G}}$ and to contain a given massless spectrum $\mathcal{S}$. We will next discuss the constraint on the construction of the self-dual lattice put by these two requirements.

Let $\{\Lambda_L;\Lambda_R\}$ be the maximal LR-decomposition of Λ discussed at the end of Sec.2. The lattice Λ_L is even, by eq.(2.7a) and $v = (0; v_R)$. We will construct Λ (and hence the string theory) by gluing Λ from a base lattice $\Lambda_0 = \{\Lambda_{0L};\Lambda_{0R}\}$. There is no natural choice for this base lattice; we have found it most convenient to choose it to be an orthogonal decomposition of $\{\Lambda_L;\Lambda_R\}$. The strategy is that choosing a superconformal current will fix Λ_{0R} (see Thm.1); the gauge group will constrain Λ_{0L}; and the massless particle spectrum will provide us with a set of glues. The conditions of chirality and absence of tachyons further constrain Λ_{0L} and the remaining glue vectors.

Let us recall two facts. The momentum $p = (p_L;p_R)$ of a massless state must satisfy $p_L^2 = 0$ or 2, and $p_R^2 = 1$ (see eq.(2.4)). Moreover, the rank of the gauge group $\bar{\mathcal{G}}$ for the four-dimensional strings constructed this way is always 22 (the dimension of Λ_L). Consequently if $\mathcal{G}$ is the largest semisimple group in $\bar{\mathcal{G}}$ and it is of rank r, the gauge group must be of the form $\bar{\mathcal{G}} = \mathcal{G} \times \mathrm{U}(1)^{22-r}$, with the U(1) groups generated by the left-moving bosonic oscillators.

$\mathcal{G}$ is the experimentally accessible gauge group. Its gauge bosons have (ground state) momenta $(p_L; v)$, where $p_L^2 = 2$, and so are massless (and hence experimentally relevant). Let Λ_g be the root lattice corresponding to $\mathcal{G}$. Then p_L transforms like the adjoint representation of the gauge group, and so lies in Λ_g. Hence Λ_g is an r-dimensional sublattice of Λ_L. By Witt's Theorem (Thm.1.5.3), Λ_g is precisely the sublattice of Λ_L generated by its norm 2 vectors.

The Standard Model (the currently accepted quantum field theory) has the gauge group SU(3)×SU(2)×U(1). This means two things. First of all, we are interested preferably in theories with r as small as possible (because the rank of the Standard Model is so small). One possible way of getting around this is *shadow matter*, as we have seen. Secondly, $A_2 \oplus A_1 \subset \Lambda_L$.

The choice of the glue group G is partially determined by the massless spectrum $\mathcal{S}$. Each particle in $\mathcal{S}$ gives us a (possibly redundant) glue vector $p - v = g \in G$. Unfortunately we only know some of the components of p in Λ_0. For example, whether it is a fermion or a boson determines its glue class in $(D_1)^{(-1)}$: *e.g.* a 'vector boson' lies in [2]. Also, we know how it transforms under $\mathcal{G}$, so we know which weight classes it lies in (*e.g.* a gauge boson must lie in $[0]\Lambda_g$). The other components of p are determined only by the massless condition $p_L^2 = 0, 2$ and $p_R^2 = 1$, and the constraints that the glues must form an additive glue group with integral dot products. These constraints are surprisingly strong as evidenced by the arguments given in **[GL1]** for the proof of Theorem 5.11 there (reproduced as Thm.8 here). Usually the massless states in $\mathcal{S}$ will not yield a large enough glue group to make $\Lambda_0[G]$ self-dual. In that case, additional glues, hopefully massive, would have to be supplied to complete the job.

A heterotic string must possess a conformal and a superconformal current (=supercurrent) in order to eliminate all the negative-norm states. Conformal invariance has already been used in the light-cone gauge construction under discussion, but we must still make sure of the existence of a supercurrent constructed out of the right-moving variables. A detailed discussion of the supercurrent is beyond the scope of this work (see *e.g.* the discussion in [**GL1**] and the references therein).

Unfortunately, a complete list of supercurrents constructable from lattice bosons is not known, though special solutions are available. Among these known supercurrents, some of them can never lead to a four-dimensional chiral theory and can be rejected on phenomenological grounds. Of the remaining ones, we choose for the sake of concreteness to discuss in this section only the supercurrent used in the $\mathbf{Z}_3$ orbifold at a particularly symmetric moduli (see in particular Tables III and IV in [**GL1**]) and consider its consequences. The method employed below is general enough to be usable for other supercurrents as well. At this point in time, however, we have not yet applied our methods to alternate supercurrents.

This particular supercurrent solution may be used to constrain the right-hand part Λ_R of the lattice Λ.

Theorem 6.3.1: (i) Λ_R contains the sublattice

$$\Lambda_{0R} \stackrel{\text{def}}{=} \{I_1^{(12)}, I_1^{(12)}, I_1^{(12)}, I_1^{(12)}, I_1^{(12)}, I_1^{(12)}, I_1^{(12)}, I_1^{(12)}, I_1^{(12)}, I_1^{(4)}\}$$

$$\stackrel{\text{def}}{=} \{(12),(12),(12),(12),(12),(12),(12),(12),(12),(4)\}$$

and the glue classes $[(q_3)_i] = [0,\ldots,0,6,0,\ldots,0,2]$, where the '6' occurs at the ith place and $1 \le i \le 9$.

(ii) Let $q = (q_L; q_R) = p - v$ represent a massless fermion. Then in the basis of Λ_{0R}, each of the 10 components of p_R must be equal to $+1$ or -1.

(iii) Suppose a space-time supersymmetric theory contains a massless fermion $q = (q_L; q_R)$ not in the adjoint representation (*i.e.* glue class [0]) of the gauge group. If $p_R = q_R + v_0$ is chosen conventionally to be $p_R = ((-1)^{10}) \stackrel{\text{def}}{=} \eta$, then the gaugino glue vector $\delta = (1^6(-1)^4) \in \Lambda_R$ is uniquely determined up to permutations of its first nine coordinates. In particular, there can be no $N > 1$ space-time supersymmetric theory that is chiral.

(iv) We may take the left-hand base lattice to be the 22-dimensional orthogonal lattice

$$\Lambda_{0L} = \{(3k^2), (k^2), (k^2), \ldots, (k^2)\}, \tag{6.3.1}$$

for some (as yet unknown) nonzero integer k.

Note that the final component of Λ_{0R} is $\{(4)\} = D_1$, the helicity or space-time group. Thm.1(i) says that any string with the given supercurrent in it must necessarily contain Λ_{0R}.

By a *nonchiral* theory, we mean here one in which every left-handed massless fermion with fixed quantum numbers in $\bar{\mathcal{G}}$ has a right-handed partner. Let us define 'left-handed' helicity to be in the class $[3]D_1$ (for p) of the SO(2) (the tenth coordinate on the right-handed side), and 'right-handed' helicity to be the ones in class [1]. So a left-handed fermion $q = (q_L; q_R)$ is a chiral partner of a right-handed fermion $q' = (q'_L; q'_R)$ iff $q_L = q'_L$, the tenth coordinate of q_R belongs to class [3], and the tenth coordinate of q'_R belongs to class [1]. Note that there are no requirements for the other components of q_R and q'_R. Note also that if a theory is nonchiral in our sense, it is nonchiral in a more usual sense as well (which is concerned only with quantum numbers in $\mathcal{G}$).

It is also necessary to explain what is meant here by a space-time supersymmetric theory in this context. (Recall that the heterotic string necessarily has world sheet supersymmetry on the right-hand side, but may or may not have space-time supersymmetry.) We mean by this that a glue — a gaugino — of the form $(0; \delta)$ exists, with δ belonging to glue class [3] in the helicity group SO(2) (the tenth coordinate of Λ_{0R}), and with $\delta^2 = (\delta + v_0)^2 = 1$. We also mean by this that every massless vector boson must be in the adjoint representation (*i.e.* the glue class [0]) of the gauge group. Beyond this, no supersymmetric operator algebra is assumed.

With this definition, we note that neither $(q_L; 0)$ nor $(0; q_R)$ in (iii) of Thm.1 may be in the self-dual lattice Λ for the string.

The proof of Thm.1 is like the proof of most of the theorems in this section very straightforward once the physical terms are successfully translated into the lattice language.

The determinant of Λ_{0R} in Thm.1 is $|\Lambda_{0R}| = (12)^9 4$. Incorporating the right-hand glues $(q_3)_i$ and δ, the right-hand glued lattice $\Lambda_{1R} \stackrel{\text{def}}{=} \Lambda_{0R}[(q_3)_1, \ldots, (q_3)_9, \delta]$ has a determinant equal to $|\Lambda_{1R}| = (12)^9 4/(2^9 6)^2 = 3^7 \geq |\Lambda_R|$. We thus obtain immediately Thm.2 below. If the theory is not space-time supersymmetric, then the gaugino glue δ should not be included, and the bound for $|\Lambda_R|$ is increased to $3^9 2^2$.

Theorem 6.3.2: $|\Lambda_L| = |\Lambda_R| = 3^7$, 3^5, 3^3, and 3 if the theory is space-time supersymmetric. Otherwise, the allowed v7alues for $|\Lambda_L| = |\Lambda_R|$ are all of the above, plus each of the above multiplied by 2^2, plus 3^9 and $3^9 2^2$.

This theorem will be used later to find the allowed gauge groups.

Note that an immediate consequence of Thm.2.1 and Thm.3.1 is that any two strings with this supercurrent can be obtained from each other by shifting. In particular, all of them can be obtained by shifting the $\mathbf{Z}_3$ orbifold of Example 3.1 in [**GL1**].

In the remainder of this section we give a number of results concerning this class of string theories. The proofs of the theorems are given in [**GL1**], but again are straightforward once the physical terminology is expressed in the language of lattices and glue vectors.

It should be borne in mind that all we are starting from is a self-dual lattice containing the sublattice Λ_{0R}. Thus although some of the theorems stated below (*e.g.* Thm.3) may be familiar to workers in string theory, they are usually proven with the help of the supersymmetric algebra, which is absent in the present approach. In our framework these 'familiar' theorems are not immediately obvious, and it is gratifying that with much less input in the gluing string framework, they remain valid.

Theorem 6.3.3: (i) Tachyons are absent in space-time supersymmetric theories.

(ii) To each massless scalar boson in a supersymmetric theory there is one and only one massless fermion with the same gauge quantum numbers.

(iii) To each massless gauge vector boson in such a theory there are two massless fermions (gauginos), one left-handed and the other right-handed.

The following theorem shows the absence of low-mass and high-spin elementary particles — something apparent experimentally.

Theorem 6.3.4: Other than the gravitons and gravitinos (their supersymmetric partners), which have spins 2 and $\frac{3}{2}$ respectively, there can be no massless particles of spin greater than one.

Thms.3-4 hold for more general choices of supercurrents than that used in Thm.1.

The following theorem about the Higgs boson makes it impossible to construct conventional grand unified theories in this framework (the Higgs boson is involved in spontaneous symmetry breaking, necessary to give the electrons, *etc.* a small but nonzero mass and also, at another energy scale, to break the GUT gauge group down to the Standard Model). Supersymmetry is not required for the following proof.

Theorem 6.3.5: No massless scalar particles in a chiral theory can lie in the adjoint representation of the gauge group.

Recall from Thm.2 that $|\Lambda_L|$ may take on values 3^7, 3^5, 3^3, and 3 for a space-time supersymmetric theory. The following theorem deals with the relationship between these values and the chirality of the theory (see Sec.1 for a discussion of chirality).

Theorem 6.3.6: A $|\Lambda_L| = 3^7$ space-time supersymmetric theory is necessarily chiral. A $|\Lambda_L| = 3$ theory is necessarily nonchiral. When $|\Lambda_L| = 3^3$ or 3^5, the theory can be either chiral or nonchiral.

Note that $A_2 \oplus \Lambda_{0L}$ and $A_2 \oplus \Lambda_L$ can both be glued to one of the 23 Niemeier lattices. Moreover, all the glues for $A_2 \oplus \Lambda_L$ are of order 3. This can be used to get at the possible gauge groups, as we shall shortly see.

The next two theorems give a classification of the allowed gauge groups in the presence of shadow matter. Recall from earlier in this section that a shadow group is one in which no glue is allowed to have nonzero components both inside and outside of the confines of the corresponding root lattice. For that reason a shadow group lattice has to be self-dual in order for the whole string lattice to be so, and this confines the allowed shadow group lattices to E_8, $E_8 \oplus E_8$, and D_{16}^+. The possible Niemeier lattices that can be built out of $A_2 \oplus \Lambda_L$ are greatly reduced. Then it can be shown that:

Theorem 6.3.7: (i) When shadow matter is present, we may take k in eq.(2) to be products of 2's and 3's only.

(ii) If the shadow group lattice is E_8, then Λ_L is a sublattice of either $E_6 \oplus E_8$ or $\{D_{13}, (12)\}[13] \oplus E_8$, and hence the gauge group $\mathcal{G}$ is a subgroup of either $E_6 \times E_8$ or $D_{13} \times$ U(1).

(iii) If the shadow group is of rank 16, then Λ_L is a sublattice of either $E_6 \oplus$(the shadow group lattice), and hence the gauge group $\mathcal{G}$ is a subgroup of E_6.

Define a ***first-level order 3 ungluing*** of a Euclidean lattice Λ to be any lattice Λ' for which there exists a glue g of order 3 satisfying $\Lambda = \Lambda'[g]$. Define a ***second-level order 3 ungluing*** of Λ to be a first-level order 3 ungluing of any first-level order 3 ungluings of Λ, *etc.*. For example, the first-level order 3 ungluings of E_8 are A_8, $E_6 \oplus A_2$, $\{D_7, (36)\}[19]$, and $\{E_7, (18)\}[19]$. E_7 itself has five first-level order 3 ungluings

The earlier results imply Λ_L must be a first-, second-, or third-level order 3 ungluing of E_6, $E_6 \oplus E_8$, or $\{D_{13}, (12)\}[13]$, if the resulting string is to be chiral and have shadow matter. Conversely, any such ungluing determines Λ_L and hence the gauge group of a supersymmetric string with the supercurrent of this section.

Similarily, with or without shadow matter, it can be shown using [**CS2**] that Λ_R is a first-, second-, or third-level order 3 ungluing of $\{D_9, (12)\}[13]$, $E_6 \oplus I_4$, $I_9 \oplus I_1^{(3)}$, or $E_8 \oplus I_1 \oplus I_1^{(3)}$. In particular, because there is only a finite number of integral Euclidean lattices with a given determinant and dimension, there is a finite number of possible strings whose supercurrent is as in this section. Moreover, in theory at least, these could be systematically and completely enumerated.

The Standard Model group SU(3)×SU(2)×U(1) has rank 4. It is, therefore, conceivable to have a rank-16 shadow matter group in the theory. If this were possible, the standard model would effectively be contained in a rank $22 - 16 = 6$ grand-unified gauge group. Unfortunately, the following theorem shows that this is impossible, at least when the theory is *grand unifiable* and contains the supercurrent of this section. By a grand unifiable standard model, we mean one in which all the observed fermions (quarks and leptons) are constructed out of glues $q = (q_L; q_R)$ with all $q_R = \eta$. We call that grand unifiable because if this were not the case, it is hard to imagine how the fermions can be unified into a larger gauge group.

Theorem 6.3.8: A grand unifiable standard model, with an arbitrary number of generations, with or without the usual choice of Higgs boson, will be nonchiral if it contains 16 dimensions of shadow matter and the supercurrent of this section.

A corollary to this result is that no chiral GUT string, containing the Standard Model in a natural way, may have 16 dimensions of shadow matter.

To summarize, we have discussed the bottom-up construction of lattice strings, and have arrived at a number of conclusions concerning strings with low-energy behaviour similar to that of the Standard Model. We have failed to construct a phenomenologically correct string.

The conclusions of this section are reached for lattice strings with the supercurrent of the $\mathbf{Z}_3$ orbifold example. Some of these conclusions (*e.g.* Thms.3-4) hold in much greater generality, whereas others (*e.g.* Thm.5) do not. The theorems concerning shadow matter (Thms.7 and 8) are probably the most far-reaching physical results of this section. [**GL1**] does not solve the all-important question of whether a string containing the standard model can be constructed. It only gives some answers to the specific cases of lattice strings with the specified supercurrent. Nevertheless, it provides a systematic analysis at least for this limited class of strings. Much more work is necessary to systematically analyze the cases of other supercurrents, and to determine whether the present methods are helpful in constructing more general conformal field theories along the lines of [**CN**].

6.4 Zero Cosmological Constants in String Theories

One of the most serious problems of the current theory of particle physics is its inability to account in a natural way for the large size of the universe, or more technically, for the smallness of the cosmological constant Λ ($|\Lambda| < 10^{-122} M_P^2$, where $M_P \approx 10^{19} GeV$ is the Planck mass). One interesting mechanism for producing such a small cosmological constant to one-loop order is to arrange to have the contribution from the fermion loops to cancel that from the boson loops. This would be the case for a supersymmetric theory (where there is a symmetry between the fermions and bosons — see Sec.2), but unfortunately our world is not supersymmetric — not to the required accuracy anyway. Nevertheless, there are still infinitely many other ways to arrange such a cancellation.

In a superstring, the bosonic and fermionic mass spectra are highly constrained (as we saw for instance in the last section), so it becomes possible and interesting to ask whether a non-supersymmetric string theory can give rise to a zero cosmological constant.

There has been some work in this direction. It turns out that the cosmological constant corresponding to a string theory can be obtained by integrating (over τ in the fundamental region of the modular group) the partition function of the theory. Dienes in [**DIEN**] has found a class of partition functions which gives rise to a zero cosmological constant to one-loop order. Moreover his partition functions satisfy a number of additional constraints (*e.g.* they have no on-shell tachyons) which physically acceptable strings are expected to obey. The partition functions he found are the kind that one would obtain from a lattice string, but after looking over more than 120 000 such strings with the help of a computer, he reports in [**DIEN**] that he was still unable to find a consistent string with such a partition function.

This final section reviews research done in [**GL5**]; it applies the techniques included in the earlier chapters to investigate the question whether any (consistent) lattice string can be found with Dienes' partition function. We will be able to quickly show (in Cor.4) that no such string exists. However, it appears that Dienes' constraints are harsher than they have to be; we have generalized his class of partition functions, and most of this section concerns this larger class. The main conclusion of our work (Cor.6) is that no such string exists, even for the larger class of possible partition functions, provided that the string must in addition satisfy the ***half-norm property*** (given in eq.(11)). This property is extremely natural given the class of possible partition functions, and indeed is consistent with the type of string Dienes seems to be most interested in, but we have not attempted yet to generalize our solution to all conceivable strings. The approach developed in this section should be applicable to the general case.

Dienes' one-loop partition functions are $\eta(\tau)^{-22}\eta(\bar\tau)^{-12}T(\tau\bar\tau)$, for T given by

$$T(\tau\bar\tau) = Q(\tau\bar\tau) + I(\tau\bar\tau), \qquad (6.4.1a)$$

where Q is given by

$$\begin{aligned} Q(\tau\bar\tau) = {} & \bar\theta_2^2\theta_2^2\{\theta_2^4\theta_3^4\theta_4^4[2\theta_3^4\theta_4^4\bar\theta_3^4\bar\theta_4^4 - \theta_3^8\bar\theta_4^8 - \bar\theta_3^8\theta_4^8] + \theta_2^{12}[4\theta_2^8\bar\theta_3^4\bar\theta_4^4 + 13\theta_3^4\theta_4^4\bar\theta_3^4\bar\theta_4^4]\} \\ & + \bar\theta_3^2\theta_3^2\{\theta_2^4\theta_3^4\theta_4^4[2\theta_2^4\theta_4^4\bar\theta_2^4\bar\theta_4^4 - \theta_2^8\bar\theta_4^8 - \bar\theta_2^8\theta_4^8] + \theta_3^{12}[4\theta_3^8\bar\theta_2^4\bar\theta_4^4 - 13\theta_2^4\theta_4^4\bar\theta_2^4\bar\theta_4^4]\} \\ & + \bar\theta_4^2\theta_4^2\{\theta_2^4\theta_3^4\theta_4^4[2\theta_2^4\theta_3^4\bar\theta_2^4\bar\theta_3^4 - \theta_2^8\bar\theta_3^8 - \bar\theta_2^8\theta_3^8] + \theta_4^{12}[4\theta_4^8\bar\theta_2^4\bar\theta_3^4 + 13\theta_2^4\theta_3^4\bar\theta_2^4\bar\theta_3^4]\}, \end{aligned}$$

$$(6.4.1b)$$

and where I is an unknown function of τ and $\bar\tau$ with the property that the Taylor expansion $\sum_{m,n} a_{mn}\bar q^m q^n$ of $\eta(\tau)^{-22}\eta(\bar\tau)^{-12}I(\tau\bar\tau)$ satisfies $a_{mn} = -a_{nm}$. Here and throughout this section the theta functions $\bar\theta_2$, *etc.* are functions of τ, and $\bar q = \exp(-\pi i\bar\tau)$. Note that eq.(1$b$) implies the string has 22 left-moving bosonic degrees of freedom and 10 right-moving ones.

For reasons that will become clear shortly, we will generalize the partition functions in eq.(1a) to

$$T(\tau\bar\tau) = cQ(\tau\bar\tau) + I(\tau\bar\tau), \qquad (6.4.1c)$$

where q and I are given above and where c is any nonzero real number. It is clear that any such partition function will also have zero cosmological constant. Dienes' class corresponds of course to the choice $c = 1$. Only two of his constraints seem to restrict the possible values of c. First of all, his graviton/gravitino constraint on p.1980 of **[DIEN]** says that c must be a positive rational number with a denominator which divides 64. We will be able to derive this ourselves from more general considerations (see Thm.3 below, so we can for now ignore this constraint). More importantly, in the discussion after his eq.(1), he demands (without explanation) that a_s (his notation) must be an integer. This implies that $2c \in \mathbf{Z}$. Cor.4 will show that no string can be found for such values of c. However, we see no serious reason (other than simplicity) for maintaining this demand.

Therefore our main interest will be focused on the class of partition functions given in eq.(1c) for any $c \neq 0$. Dienes' class can be recovered by choosing $c = 1$.

A lattice Λ shall be called *v-even* for some vector v (not necessarily in Λ) if

$$r^2 + 2r \cdot v \equiv 0 \pmod{2} \; \forall r \in \Lambda. \tag{6.4.2}$$

Thm.1.3.4 tells us that any self-dual lattice Λ is v-even for some $v \in \frac{1}{2}\Lambda$. In fact: any lattice Λ is v-even for some $v \in \mathbf{R} \otimes \Lambda$; if Λ is rational then v can be chosen in $\mathbf{Q} \otimes \Lambda$; and if Λ is integral then $v \in \frac{1}{2}\Lambda^*$.

For any Euclidean lattice Λ^E define the 'shifted theta constant' $\Theta_{\Lambda^E}(vu|\tau)$ to be

$$\begin{aligned}\Theta_{\Lambda^E}(vu|\tau) &= \sum_{r \in \Lambda^E} \exp[\pi i \tau (r+u)^2 + 2\pi i (r+u) \cdot v] \\ &= \vartheta([u]\Lambda^E)(v \mid \tau).\end{aligned} \tag{6.4.3}$$

We will call $\Theta(\Lambda^E)(\tau)$ a 'pure theta constant' — it corresponds to $v = u = 0$. For indefinite Λ^I, define

$$\Theta_{\Lambda^I}(vu|\tau\bar{\tau}) = \sum_{r \in \Lambda^I} \exp[\pi i \tau (r_L + u_L)^2 - \pi i \bar{\tau} (r_R + u_R)^2 + 2\pi i (r+u) \cdot v], \tag{6.4.4}$$

where we write $r \in \Lambda^I$ in the usual way as $r = (r_L; r_R)$ (so dot products in Λ^I are given by $r \cdot r' = r_L \cdot r'_L - r_R \cdot r'_R$). We will also use the short-hand $\Theta(\Lambda^I)(\tau\bar{\tau})$ for $\Theta_{\Lambda^I}(00|\tau\bar{\tau})$

Then we know from Sec.2 that if Dienes' partition function corresponds to a (consistent) lattice string, then

$$T(\tau\bar{\tau}) = \Theta_\Lambda(vv|\tau\bar{\tau}), \tag{6.4.5}$$

where Λ is an odd indefinite v-even self-dual lattice of signature (22,10), where $v = (0; v_R)$ (*i.e.* v lies entirely on the RHS), and where $v^2 = -v_R^2 = -1$.

Given any indefinite lattice Λ^I of signature (m,n), define the n-dimensional dimensional Euclidean lattice $(\Lambda^I)_R$ by:

$$(\Lambda^I)_R \stackrel{\mathrm{def}}{=} \{r_R | (0;r_R) \in \Lambda^I\}, \tag{6.4.6}$$

where dot products in $(\Lambda^I)_R$ are (obviously) defined by $r_R \cdot r'_R = -(0;r_R)\cdot(0,r'_R)$ Λ_R is called the *RHS* of Λ^I.

We would like to find the lattice Λ responsible (in the sense of eq.(5)) for the partition function eq.(1), or show that no such lattice exists. One glaring difficulty is that the partition function is not precisely known. Thm.1, given below, overcomes that difficulty.

Theorem 6.4.1: Suppose Λ satisfies eq.(5). Then:

$$\begin{aligned}
\Theta_{\Lambda_R}(v_R v_R|\bar\tau) =&4c[\bar\theta_3^8\bar\theta_4^2 + \bar\theta_3^6\bar\theta_4^4 - \bar\theta_3^4\bar\theta_4^6 - \bar\theta_3^2\bar\theta_4^8] \\
=&16c\{8q - 896q^5 + 5184q^9 + \cdots\},
\end{aligned} \tag{6.4.7a}$$

$$\begin{aligned}
\widetilde{\Theta_\Lambda}(vv|\tau\bar\tau) =&c\theta_3^{22}[4\bar\theta_3^{14}\bar\theta_4^8 - 8\bar\theta_3^{10}\bar\theta_4^{12} + 4\bar\theta_3^6\bar\theta_4^{16}] + c\theta_2^2\theta_3^{20}[4\bar\theta_2^2\bar\theta_3^{12}\bar\theta_4^8 - 4\bar\theta_2^2\bar\theta_3^8\bar\theta_4^{12}] \\
&+ c\theta_3^{18}\theta_4^4[-13\bar\theta_3^{14}\bar\theta_4^8 + 24\bar\theta_3^{10}\bar\theta_4^{12} - 11\bar\theta_3^6\bar\theta_4^{16}] \\
&+ c\theta_2^2\theta_3^{16}\theta_4^4[-7\bar\theta_2^2\bar\theta_3^{12}\bar\theta_4^8 + 5\bar\theta_2^2\bar\theta_3^8\bar\theta_4^{12} + 2\bar\theta_2^2\bar\theta_3^4\bar\theta_4^{16}] \\
&+ c\theta_3^{16}\theta_4^6[11\bar\theta_3^8\bar\theta_4^{14} - 11\bar\theta_3^4\bar\theta_4^{18} + 4\bar\theta_4^{22}] \\
&+ c\theta_3^{14}\theta_4^8[4\bar\theta_3^{22} - 13\bar\theta_3^{18}\bar\theta_4^4 + 30\bar\theta_3^{14}\bar\theta_4^8 - 28\bar\theta_3^{10}\bar\theta_4^{12} + 11\bar\theta_3^6\bar\theta_4^{16}] \\
&+ c\theta_2^2\theta_3^{12}\theta_4^8[4\bar\theta_2^2\bar\theta_3^{20} - 7\bar\theta_2^2\bar\theta_3^{16}\bar\theta_4^4 + 6\bar\theta_2^2\bar\theta_3^{12}\bar\theta_4^8 - 4\bar\theta_2^2\bar\theta_3^8\bar\theta_4^{12} + 5\bar\theta_2^2\bar\theta_3^4\bar\theta_4^{16} - 4\bar\theta_2^2\bar\theta_3^{20}] \\
&+ c\theta_3^{12}\theta_4^{10}[4\bar\theta_3^{12}\bar\theta_4^{10} - 28\bar\theta_3^8\bar\theta_4^{14} + 24\bar\theta_3^4\bar\theta_4^{18} - 8\bar\theta_4^{22}] + sym.
\end{aligned} \tag{6.4.7b}$$

where '*sym.*' in eq.($7b$) denotes all of the previous terms with $\theta_2 \leftrightarrow \bar\theta_2$, $\theta_3 \leftrightarrow \bar\theta_4$ and $\theta_4 \leftrightarrow \bar\theta_3$, where Λ_R is the RHS of the lattice Λ, and where the notation $\widetilde{\Theta_\Lambda}(vv|\tau\tau)$ denotes the function

$$\bar\theta_2^4(\bar\tau)\bar\theta_3^4(\bar\tau)\bar\theta_4^4(\bar\tau)\Theta_\Lambda(vv|\tau\bar\tau) + \theta_2^4(\tau)\theta_3^4(\tau)\theta_4^4(\tau)\Theta_\Lambda(vv|\bar\tau\tau).$$

Eq.($7a$) follows by putting q equal to zero (*i.e.* considering the limit $\tau \rightarrow +\infty\, i$) in Dienes' partition function eq.($1c$), once we show that any acceptable I in eq.(1) vanishes in this limit (this easily follows from Dienes' 'pole strength a' constraint given in [**DIEN**]). Eq.($7b$) follows from the observation that by definition the I in eq.(1) satisfy $\tilde{I} = 0$.

Eq.($7b$) is very messy — we are being far more explicit here than is necessary, to help make as unambiguous as possible the following argument, and also to prepare for a future generalization of Cor 6. A third way to eliminate the ambiguity in eq.($1c$) due to the I is to 'Euclidean-ize' Λ (*i.e.* put $q = \bar{q}$). The resulting equation is much simpler than eq.($7b$), but much information is accordingly lost. We will come back to this later.

It will turn out that all we really need to consider is the third term in eq.($7b$). In particular, note that the coefficients in eq.($7b$) of $\theta_3^{18}\theta_4^4\bar{\theta}_3^{14}\bar{\theta}_4^8$ and $\theta_3^{18}\theta_4^4\bar{\theta}_3^6\bar{\theta}_4^{16}$ are not equal. We will show in Thm.6 that those coefficients must be equal for any lattice satisfying the half-norm property eq.(11). Cor.6 is based on a test (Thm.5) which can be used to show a given Λ_R is not the RHS of a Λ satisfying eq.($7b$). To help motivate the proof of Thm.5 we will shortly work out one example (eq.($10b$)) in complete detail. The proof of Thm.5 in the general case follows from that argument by simply generalizing.

Rather than directly trying to solve eq.(1) for Λ, we will try to solve eqs.($7a$) and ($7b$) for v_R, Λ_R and ultimately Λ (or show that no such solution exists).

One obvious difficulty with this strategy is that eqs.(7) involve 'shifted theta constants' $\Theta_{\Lambda_R}(v_R v_R|\tau)$ *etc.*, rather than *pure* theta constants. This makes it much harder to read off information about Λ_R and Λ_E. Thm.2 given below is designed to overcome this complication. With this in mind, make the following definitions.

Let Λ_B be the sublattice of Λ consisting only of the even norm vectors (*i.e.* the *bosons*). Then it can be shown that $|\Lambda_B| = 4$. Λ_B is also of signature (22,10). Let $\Lambda_{BR} \stackrel{\text{def}}{=} (\Lambda_B)_R$. Let $u = (0; u_R) \in \Lambda$ be any odd-normed vector living entirely in the right-hand side of Λ (such vectors will always exist, by eq.(7a)). Then $\Lambda = \Lambda_B[u]$ and $\Lambda_R = \Lambda_{BR}[u_R]$. Note also that $2v, 2u \in \Lambda_B$ (so $2v_R, 2u_R \in \Lambda_{BR}$).

Now define lattices $\Lambda'_R \stackrel{\text{def}}{=} \Lambda_{BR}[v_R + u_R]$ and $\Lambda''_R \stackrel{\text{def}}{=} \Lambda_{BR}[v_R]$; define Λ' and Λ'' similarly. Note that Λ_R, Λ'_R and Λ''_R are all integral (in fact, odd) and have equal determinant. Λ_R and Λ'_R are v_R-even, and Λ''_R is both u_R- and $(u_R + v_R)$-even.

Theorem 6.4.2:

$$\Theta_{\Lambda_R}(v_R v_R|\bar{\tau}) = \Theta(\Lambda''_R)(\bar{\tau}) - \Theta(\Lambda'_R)(\bar{\tau}), \qquad (6.4.8a)$$

$$\Theta_{\Lambda}(vv|\tau\bar{\tau}) = \Theta(\Lambda'')(\tau\bar{\tau}) - \Theta(\Lambda')(\tau\bar{\tau}). \qquad (6.4.8b)$$

Thm.2 can be verified by explicit calculation, using eqs.(3), (5.2.3) and (5.2.4).

First we will outline the approach taken to find all solutions to eqs.(7a) and (7b).

By using the known transformation property of lattice theta constants under $\tau \to -1/\tau$ (see eqs.(4.2.6)), we get:

$$\Theta_{\Lambda_R^*}(v_R v_R|\tau) = \Theta_{\Lambda''^*_R}(\tau) - \Theta_{\Lambda'^*_R}(\tau)$$
$$= \sqrt{|\Lambda_R|}(16c)\{q^{\frac{1}{2}} + 4q + 0q^{\frac{3}{2}} - 16q^2 - 14q^{\frac{5}{2}} + 0q^3 + 64q^4 + \cdots\}. \qquad (6.4.9)$$

The following are a sample of the kinds of information that can be squeezed out of eqs.(7a), (8a), and (9). They are not necessary for the proof of Cor.6, but could be useful in any generalization of it.

Theorem 6.4.3: (i) $c > 0$;

(ii) $c \in \{1/64, 1/32, 3/64, 1/16, 5/64, 3/32, 7/64, 1/8\}$;

(iii) $\sqrt{|\Lambda_R|} \cdot 8c \in \mathbf{Z}$;

(iv) Λ''_R contains $128c$ unit vectors (so can be written as $\Lambda''_R = I_\ell \oplus \Lambda_0$, for $\ell = 64c$). Λ_R and Λ'_R contain no unit vectors.

(v) $|\Lambda_R| = |\Lambda'_R| = |\Lambda''_R| = 4k^2$, for some integer $k \geq 2$.

We will not write down in detail the proof of Thm.3, because it is not important for what follows, but a small outline of one can easily be made. For example, $c < 0$ was ruled out by showing that $v_R^2 = 1$ implied Λ''_R must contain more unit vectors than Λ'_R (the result then follows from eq.(8a) and looking at the q^1 term in eq.(7a)). It suffices to show Λ''_R has at least as many unit vectors as Λ'_R (since $c \neq 0$); let $u_1, u_2 \in \Lambda'_R$ be unit vectors; then $u_i \cdot v_R = \pm\frac{1}{2}$, so for some choice of signs $(\pm u_1 \pm u_2) + v_R \in \Lambda''_R$ is a unit vector (the desired conclusion now follows from the obvious counting argument). Also, the coefficients of all the terms in eqs.(7a) and (9) must be even integers, which helps to give us (ii) and (iii). (v) follows from (ii), (iii), eq.(1.5.1), and the fact that v_R is an order 2 glue of Λ_R.

Many other simple results can be found.

Thm.3(ii) immediately tells us:

Corollary 6.4.4: No string exists having a partition function T in Dienes' class (*i.e.* either with $c = 1$ or even with $2c \in \mathbf{Z}$).

Basically, the reason is that Λ''_R and Λ'_R, being integral and 10-dimensional, can only have 20 unit vectors, but eqs.(7a) and (8a) imply, for $c = \frac{1}{2}$ say, that Λ''_R has at least 64 unit vectors. Because of this result we will consider for now on the more general class of partition functions given in eq.(1c).

As an example, all possible solutions Λ_R to eq.(7a) with determinant $|\Lambda_R| = 16$ (the smallest possible allowed determinant) can be found. By (iii), we see that $32c$ must be an integer, so by Thm.1.2.1 Λ''_R can be written $\Lambda''_R = I_2 \oplus \Lambda_8$ for some 8-dimensional integral lattice Λ_8 with determinant 16 (see Thm.3(iv)). Fortunately, all such lattices can be explicitly found using [CS2] (by way of comparison, their tables do not give all 9-dimensional integral lattices of determinant 16). It is then possible to verify that the only determinant 16 possibilities are (up to integral equivalence):

$$\Lambda_R = A_1A_1D_4D_4[1111], \ \ v_R = [0020], \ \ c = 1/16; \tag{6.4.10a}$$

$$\Lambda_R = A_1A_1A_1A_1D_6[01111], \ \ v_R = [11000], \ \ c = 1/32; \tag{6.4.10b}$$

$$\Lambda_R = A_1A_1A_1A_1D_6[01113], \ \ v_R = [00002], \ \ c = 3/32. \tag{6.4.10c}$$

However, a complete enumeration of all Λ_R satisfying eq.($7a$) could be beyond our power. Nevertheless a large class of solutions Λ_R to eq.($7a$), which includes the three in eqs.(10), satisfies the following property, which we shall call the *half-norm property*:

$$g \in \Lambda_R^* \Rightarrow g^2 \in \frac{1}{2}\mathbf{Z} \tag{6.4.11}$$

(indeed, it may turn out that *any* solution of eq.($7a$) must satisfy this additional property, since the glues violating eq.(11) would otherwise have to conveniently cancel out in eqs.($7b$) and (9)). It seems to be automatically satisfied by the strings/spin structures Dienes in [**DIEN**] is interested in. In any event, by Cor.10.2 in [**MUM**], this assumption means that the theta constants of all glue classes of Λ_R' and Λ_R'' can be expressed as polynomials in $\bar{\theta}_2^2, \bar{\theta}_3^2, \bar{\theta}_4^2$ (see the comments made in Sec.4.2 concerning modular forms and theta constants of lattices). The determinant of any such (integral) lattice must be a power of 2, since its glues (by Thm.1.6.9) must be of order 1,2 or 4, and if it is to also be a solution of eq.($7a$) the determinant (by Thm.3(v)) must be a power of 4. Also note that if one of Λ_R, Λ_R', Λ_R'' satisfies eq.(11), all do (see the correspondences discussed after eqs.(12)).

There are several simultaneous solutions Λ_R to eqs.($7a$) and (11). Their determinants range from $4^2 = 16$ (for which there are 3 solutions) to $4^8 = 16\ 384$ (with 4 solutions).

The strategy here is to investigate the LHS Λ_L of Λ — it will be 22-dimensional, even, and will have determinant $|\Lambda_L| = |\Lambda_R|$ (hence by Thm.1.6.7(ii) Λ_L must be self-dualizable and so must glue to one of the 68 self-dual lattices of dimension 22; these are all explicitly known — see pp.416-7 of [**CS1**]). Moreover, the glue groups Λ_L^*/Λ_L and Λ_R^*/Λ_R are isomorphic (by Thm.1.6.4).

By way of a concrete illustration of our findings, we will consider the choice of Λ_R in eq.(10b). What is important is the ideas behind the following equations, rather than the equations themselves. In particular, the group isomorphisms between Λ_L^*/Λ_L, Λ_R^*/Λ_R, *etc.* induced by the correspondences $h_i \leftrightarrow g_i \leftrightarrow g_i' \leftrightarrow g_i''$, which preserve dot products (mod 1), are important and hold in general, as are eqs.(13).

Consider the glue classes in $\Lambda_R'^*/\Lambda_{BR}$ and $\Lambda_R''^*/\Lambda_{BR}$ (because we are interested by eq.(8a) in the difference of their theta constants, it turns out that it does not matter whether we use glue classes of Λ_{BR} or ones of Λ_R', Λ_R'' — it is more convenient sometimes to use Λ_{BR}). There are $32 = 2 \cdot |\Lambda_R|$ of them. Only 16 of them have the property that $u_R = [10111]$ dotted with any of their vectors is not an integer (this is necessary if these classes are to lie in $\Lambda_R'^*$ resp. $\Lambda_R''^*$). These are listed below:

$$[11000]\Lambda_{BR} \leftrightarrow [01111]\Lambda_{BR} \qquad (6.4.12a)$$

$$[00100]\Lambda_{BR} \leftrightarrow [10011]\Lambda_{BR} \qquad (6.4.12b)$$

$$[00010]\Lambda_{BR} \leftrightarrow [10101]\Lambda_{BR} \qquad (6.4.12c)$$

$$[00002]\Lambda_{BR} \leftrightarrow [10113]\Lambda_{BR} \qquad (6.4.12d)$$

$$[00001]\Lambda_{BR} \leftrightarrow [10110]\Lambda_{BR} \qquad (6.4.12e)$$

$$[11003]\Lambda_{BR} \leftrightarrow [01112]\Lambda_{BR} \qquad (6.4.12f)$$

$$[11110]\Lambda_{BR} \leftrightarrow [01001]\Lambda_{BR} \qquad (6.4.12g)$$

$$[11113]\Lambda_{BR} \leftrightarrow [01002]\Lambda_{BR} \qquad (6.4.12h)$$

$$[00111]\Lambda_{BR} \leftrightarrow [10000]\Lambda_{BR} \qquad (6.4.12i)$$

$$[00112]\Lambda_{BR} \leftrightarrow [10003]\Lambda_{BR} \qquad (6.4.12j)$$

$$[00103]\Lambda_{BR} \leftrightarrow [10012]\Lambda_{BR} \qquad (6.4.12k)$$

$$[00013]\Lambda_{BR} \leftrightarrow [10102]\Lambda_{BR} \qquad (6.4.12l)$$

$$[11101]\Lambda_{BR} \leftrightarrow [01010]\Lambda_{BR} \qquad (6.4.12m)$$

$$[11102]\Lambda_{BR} \leftrightarrow [01013]\Lambda_{BR} \qquad (6.4.12n)$$

$$[11011]\Lambda_{BR} \leftrightarrow [01100]\Lambda_{BR} \qquad (6.4.12o)$$

$$[11012]\Lambda_{BR} \leftrightarrow [01103]\Lambda_{BR} \tag{6.4.12p}$$

(Fortunately it is not really necessary to be nearly this explicit!) Note that Λ_{BR} here is $A_1A_1A_1A_1D_6$. The left glues in eqs.(12) lie in $\Lambda_R''^*$, while the right ones are in $\Lambda_R'^*$. These are paired so that $u_R = [10111]$ added to one yields the corresponding one. Call these $[g_i'']\Lambda_{BR}$ and $[g_i']\Lambda_{BR}$, for $i = 1, \ldots, 16$ (so that $g_i'' \leftrightarrow g_i'$).

Define $g_i = g_i' + v_R \in \Lambda_R^*$. It is important to note that $g_i \cdot g_j \equiv g_i' \cdot g_j' \equiv g_i'' \cdot g_j''$ (mod 1), as well as $g_i^2 \equiv g_i'^2 \equiv g_i''^2$ (mod 2). Moreover, this correspondence defines a group isomorphism between Λ_R^*/Λ_R, $\Lambda_R'^*/\Lambda_R'$ and $\Lambda_R''^*/\Lambda_R''$ (*i.e.* $(g_i+g_j)' \equiv g_i'+g_j'$ (mod Λ_R'), *etc.* — note that $[g_i]\Lambda_R + [g_j]\Lambda_R = [g_k]\Lambda_R$, where $g_k \equiv g_i + g_j + u_R$ (mod Λ_{BR}). These relationships, central to what follows, will hold in general.

Suppose

$$\Lambda' = \bigcup_{i=1}^{16}([h_i]\Lambda_L; [g_i']\Lambda_R'), \tag{6.4.13a}$$

where $[h_i]\Lambda_L$ are the 16 glue classes in Λ_L^*/Λ_L (note that $h_1 = 0$). Then it is not difficult to see from the definitions that

$$\Lambda'' = \bigcup_{i=1}^{16}([h_i]\Lambda_L; [g_i'']\Lambda_R''). \tag{6.4.13b}$$

(This is the motivation for the pairings in eqs.(12).) Moreover,

$$\Theta_{\Lambda'}(\tau) = \sum_{i=1}^{16}\Theta([h_i]\Lambda_L)(\tau)\cdot\Theta([g_i']\Lambda_R')(\bar{\tau}), \tag{6.4.13c}$$

with a similar expression for $\Theta_{\Lambda''}$. Incidently, the analogues of all these equations of course will hold in general.

It is straightforward to verify the following expressions:

$$\Theta([g_i'']\Lambda_R'') - \Theta([g_i']\Lambda_R') \stackrel{\text{def}}{=} \Delta_i, \qquad \text{where} \tag{6.4.14a}$$

$$\Delta_1 = \frac{1}{3}\Delta_4 = -\Delta_{13} = -\Delta_{15} = \frac{1}{8}[\bar\theta_3^8\bar\theta_4^2 + \bar\theta_3^6\bar\theta_4^4 - \bar\theta_3^4\bar\theta_4^6 - \bar\theta_3^2\bar\theta_4^8], \tag{6.4.14b}$$

$$\Delta_2 = \Delta_3 = -\Delta_9 = \frac{1}{8}\bar\theta_2^2[\bar\theta_3^6\bar\theta_4^2 + 2\bar\theta_3^4\bar\theta_4^4 + \bar\theta_3^2\bar\theta_4^6], \tag{6.4.14c}$$

$$\Delta_5 = -\Delta_8 = \frac{3}{8}\bar\theta_2^2[\bar\theta_3^6\bar\theta_4^2 - \bar\theta_3^2\bar\theta_4^6], \qquad (6.4.14d)$$

$$\Delta_6 = -\Delta_{14} = -\Delta_{16} = \frac{1}{8}\bar\theta_2^2[\bar\theta_3^6\bar\theta_4^2 - 2\bar\theta_3^4\bar\theta_4^4 + \bar\theta_3^2\bar\theta_4^6], \qquad (6.4.14e)$$

$$\frac{1}{3}\Delta_7 = \Delta_{10} = -\Delta_{11} = -\Delta_{12} = \frac{1}{8}[-\bar\theta_3^8\bar\theta_4^2 + \bar\theta_3^6\bar\theta_4^4 + \bar\theta_3^4\bar\theta_4^6 - \bar\theta_3^2\bar\theta_4^8]. \; (6.4.14f)$$

Because Λ is v-even and self-dual, we know a lot about the glue classes $[h_i]\Lambda_L$. Important is the realization that the correspondence $h_i \leftrightarrow g_i$ induces a group isomorphism between Λ_L^*/Λ_L and Λ_R^*/Λ_R, however $h_i^2 \equiv 1 + g_i^2$ (mod 2). From this we get that $\Lambda_1 \stackrel{\text{def}}{=} \Lambda_L[h_{10}, h_{12}, h_{13}]$, $\Lambda_2 \stackrel{\text{def}}{=} \Lambda_L[h_{10}, h_{11}, h_{15}]$, and $\Lambda_3 \stackrel{\text{def}}{=} \Lambda_L[h_{10}, h_4, h_7]$ are all self-dual (since by direct inspection $\Lambda_R''[g_{10}'', g_{12}'', g_{13}'']$, *etc.* are); hence by Hecke's Theorem we may write

$$\Theta(\Lambda_i) = e_i\theta_3^{22} + f_i(\theta_3^8 - \theta_3^4\theta_4^4 + \theta_4^8)\theta_3^{14} + (1 - e_i - f_i)(\theta_3^8 - \theta_3^4\theta_4^4 + \theta_4^8)^2\theta_3^6$$

for $i = 1, 2, 3$, for (as yet unknown) real parameters e_i, f_i. Moreover, the analysis at the end of Sec.4.2 (apply eq.(4.2.11) to the first 'generalized Hecke's Thm.' example there) tells us that, for (as yet unknown) real parameters a, b, c, d, we may write

$$\Theta(\Lambda_L) = \frac{1-a-b-c-d}{4}(\theta_3^4 + \theta_4^4)(\theta_3^{18} + \theta_4^{18}) + \frac{a}{4}(\theta_3^8 + \theta_4^8)(\theta_3^{14} + \theta_4^{14})$$
$$+ \frac{b}{4}(\theta_3^{12} + \theta_4^{12})(\theta_3^{10} + \theta_4^{10}) + \frac{c}{4}(\theta_3^{16} + \theta_4^{16})(\theta_3^6 + \theta_4^6) + \frac{d}{4}(\theta_3^{20} + \theta_4^{20})(\theta_3^2 + \theta_4^2).$$

Incidently, the ten parameters we have just introduced are not independent: they satisfy the equations $e_1 + e_2 + e_3 = 3 - 2a - 3b - 2c$ and $f_1 + f_2 + f_3 = 2a + 3b - 5d$ (of course, the c used here has nothing directly to do with the c in eq.(1c)). It turns out, again using the analysis and results included at the end of Sec.4.2, that the theta constants of all the 16 glue classes of Λ_L can be expressed in terms of these parameters. The analysis is lengthy and messy, using equations such as eqs.(4.2.12), (4.2.6) and (4.2.7), and since eqs.(15) are not needed in the proof of Thm.5 we will not include here the intermediate calculations. In particular, we get:

$$\Theta([h_1]\Lambda_L) = \frac{1}{4}(\theta_3^{22} + \theta_4^{22}) + \frac{1-a-b-c-d}{4}(\theta_3^{18}\theta_4^4 + \theta_3^4\theta_4^{18}) + \frac{a}{4}(\theta_3^{14}\theta_4^8 + \theta_3^8\theta_4^{14})$$

$$+\frac{b}{4}(\theta_3^{10}\theta_4^{12}+\theta_3^{12}\theta_4^{10})+\frac{c}{4}(\theta_3^{6}\theta_4^{16}+\theta_3^{16}\theta_4^{6})+\frac{d}{4}(\theta_3^{2}\theta_4^{20}+\theta_3^{20}\theta_4^{2}); \quad (6.4.15a)$$

$$\Theta([h_2]\Lambda_L)=\Theta([h_{16}]\Lambda_L)=\frac{1}{4}\theta_2^2[(\theta_3^{20}-\theta_4^{20})$$
$$+(-9+2a+3b+4c+5d+4e_1+2f_1)(\theta_3^{16}\theta_4^4-\theta_3^4\theta_4^{16})$$
$$+(20-6a-9b-10c-10d-10e_1-4f_1)(\theta_3^{12}\theta_4^8-\theta_3^8\theta_4^{12}); \quad (6.4.15b)$$

$$\Theta([h_4]\Lambda_L)=\frac{1}{4}(\theta_3^{22}+\theta_4^{22})+\frac{-3+a+b+c+d+2e_3-2f_3}{4}(\theta_3^{18}\theta_4^4+\theta_3^4\theta_4^{18})$$
$$+\frac{6-a-6e_3-4f_3}{4}(\theta_3^{14}\theta_4^8+\theta_3^8\theta_4^{14})+\frac{-4-b+4e_3+4f_3}{4}(\theta_3^{10}\theta_4^{12}+\theta_3^{12}\theta_4^{10})$$
$$+\frac{2-c-2e_3-2f_3}{4}(\theta_3^6\theta_4^{16}+\theta_3^{16}\theta_4^6)+\frac{-d}{4}(\theta_3^2\theta_4^{20}+\theta_3^{20}\theta_4^2); \quad (6.4.15c)$$

$$\Theta([h_5]\Lambda_L)=\Theta([h_8]\Lambda_L)=\frac{1}{4}\theta_2^2[(\theta_3^{20}+\theta_4^{20}) \quad (6.4.15d)$$
$$+(-3+b+2c+3d)(\theta_3^{16}\theta_4^4+\theta_3^4\theta_4^{16})+(2-b-2c-2d)(\theta_3^{12}\theta_4^8+\theta_3^8\theta_4^{12})];$$

$$\Theta([h_7]\Lambda_L)=\frac{1}{4}(\theta_3^{22}-\theta_4^{22})+\frac{-3+a+b+c+d+2e_3-2f_3}{4}(\theta_3^{18}\theta_4^4-\theta_3^4\theta_4^{18})+$$
$$\frac{6-a-6e_3-4f_3}{4}(\theta_3^{14}\theta_4^8-\theta_3^8\theta_4^{14})+\frac{-4-b+4e_3+4f_3}{4}(\theta_3^{10}\theta_4^{12}-\theta_3^{12}\theta_4^{10})$$
$$+\frac{2-c-2e_3-2f_3}{4}(\theta_3^6\theta_4^{16}-\theta_3^{16}\theta_4^6)+\frac{-d}{4}(\theta_3^2\theta_4^{20}-\theta_3^{20}\theta_4^2); \quad (6.4.15e)$$

$$\Theta([h_{10}]\Lambda_L)=\frac{1}{4}(\theta_3^{22}-\theta_4^{22})+\frac{1-a-b-c-d}{4}(\theta_3^{18}\theta_4^4-\theta_3^4\theta_4^{18})$$
$$+\frac{a}{4}(\theta_3^{14}\theta_4^8-\theta_3^8\theta_4^{14})+\frac{b}{4}(\theta_3^{10}\theta_4^{12}-\theta_3^{12}\theta_4^{10})$$
$$+\frac{c}{4}(\theta_3^6\theta_4^{16}-\theta_3^{16}\theta_4^6)+\frac{d}{4}(\theta_3^2\theta_4^{20}-\theta_3^{20}\theta_4^2); \quad (6.4.15f)$$

$\Theta([h_3])=\Theta([h_{14}])$, resp. $\Theta([h_6])=\Theta([h_9])$, are the same as eq.(15$b$) with e_2 and f_2, resp. e_3 and f_3, instead of e_1 and f_1; $\Theta([h_{11}])$, resp. $\Theta([h_{12}])$, are the same as eq.(15e) with e_2 and f_2, resp. e_1 and f_1, instead of e_3 and f_3; and finally $\Theta([h_{13}])$, resp. $\Theta([h_{15}])$, are the same as eq.(15c) with e_1 and f_1, resp. e_2 and f_2, instead of e_3 and f_3.

Now, straightforward arithmetic gives us:

$$\widetilde{\Theta_\Lambda}(vv|\tau\bar\tau)=\frac{1}{8}\theta_3^{22}[\bar\theta_3^{14}\bar\theta_4^8-2\bar\theta_3^{10}\bar\theta_4^{12}+\bar\theta_3^6\bar\theta_4^{16}]+\frac{1}{8}\theta_2^2\theta_3^{20}[\bar\theta_2^2\bar\theta_3^{12}\bar\theta_4^8-\bar\theta_2^2\bar\theta_3^8\bar\theta_4^{12}]$$
$$+\frac{1}{8}\theta_3^{18}\theta_4^4[(-2+2e+f)\bar\theta_3^{14}\bar\theta_4^8+(4-4e-2f)\bar\theta_3^{10}\bar\theta_4^{12}+(-2+2e+f)\bar\theta_3^6\bar\theta_4^{16}]$$

$$+\frac{1}{8}\theta_2^2\theta_3^{16}\theta_4^4[\alpha\bar\theta_2^2\bar\theta_3^{12}\bar\theta_4^8-\alpha\bar\theta_2^2\bar\theta_3^8\bar\theta_4^{12}]$$
$$+\frac{1}{8}\theta_3^{16}\theta_4^6[(2-2e-2f)\bar\theta_3^{16}\bar\theta_4^6+(-4+4e+4f)\bar\theta_3^{12}\bar\theta_4^{10}$$
$$+(4-4e-3f)\bar\theta_3^8\bar\theta_4^{14}+(-2+2e+f)\bar\theta_3^4\bar\theta_4^{18}+\bar\theta_4^{22}]$$
$$+\frac{1}{8}\theta_3^{14}\theta_4^8[\bar\theta_3^{22}+(-2+2e+f)\bar\theta_3^{18}\bar\theta_4^4+(6-6e-4f)\bar\theta_3^{14}\bar\theta_4^8$$
$$+(-8+8e+6f)\bar\theta_3^{10}\bar\theta_4^{12}+(4-4e-3f)\bar\theta_3^6\bar\theta_4^{16}]$$
$$+\frac{1}{8}\theta_2^2\theta_3^{12}\theta_4^8[\bar\theta_2^2\bar\theta_3^{20}+\alpha\bar\theta_2^2\bar\theta_3^{16}\bar\theta_4^4+2\beta\bar\theta_2^2\bar\theta_3^{12}\bar\theta_4^8-2\beta\bar\theta_2^2\bar\theta_3^8\bar\theta_4^{12}-\alpha\bar\theta_2^2\bar\theta_3^4\bar\theta_4^{16}-\bar\theta_2^2\bar\theta_3^{20}]$$
$$+\frac{1}{8}\theta_3^{12}\theta_4^{10}[(-4+4e+4f)\bar\theta_3^{16}\bar\theta_4^6+(8-8e-8f)\bar\theta_3^{12}\bar\theta_4^{10}$$
$$+(-8+8e+6f)\bar\theta_3^8\bar\theta_4^{14}+(4-4e-2f)\bar\theta_3^4\bar\theta_4^{18}-2\bar\theta_4^{22}]+sym. \qquad (6.4.16)$$

where '*sym.*' and $\widetilde{\Theta_\Lambda}$ here is as in eq.(7b). Note that there are four parameters in eq.(16): $e\stackrel{\text{def}}{=}3e_3-e_1-e_2$, $f\stackrel{\text{def}}{=}3f_3-f_1-f_2$, $\alpha\stackrel{\text{def}}{=}-9+2a+3b+4c+5d+4e_1+4e_2-4e_3+2f_1+2f_2-2f_3$ and $\beta\stackrel{\text{def}}{=}20-6a-9b-10c-10d-10e_1-10e_2+10e_3-4f_1-4f_2+4f_3$.

Cor.10.2 in [**MUM**] tells us that a polynomial in θ_2^2, θ_3^2 and θ_4^2 vanishes for all τ iff it is a polynomial in $\theta_3^4-\theta_4^4-\theta_2^4$. Since eqs.(7$b$) (with c=1/32) and (16) must be equal, this implies, for example, that the coefficients of $\theta_3^{18}\theta_4^4$ must be equal, which in turn implies $-13/4=-2+2e+f=-11/4$, which is clearly impossible. Thus the choice of Λ_R given in eq.(10b), although it satisfies eq.(7a), cannot be the RHS of a permissible Λ which satisfies eq.(5) (and hence eq.(7b)).

Incidently, if we had instead 'Euclidean-ized' Λ (as was discussed after eqs.(7)) and carried out all of the analogous work that would have been entailed, we would have found no inconsistency, but rather the constraints: $e=-9/4$, $\alpha=-7/4$ and $\beta=1/4$. So it would seem that too much information is lost through Euclideanization.

Of course, all of the work used in deriving the inconsistency was not really necessary in hindsight. It would have sufficed to have shown that the coefficients of $\theta_3^{18}\theta_4^4\bar\theta_3^{14}\bar\theta_4^8$ and $\theta_3^{18}\theta_4^4\bar\theta_3^6\bar\theta_4^{16}$ — call them A and B — in eq.(16) are equal. Note that $A=A_1-A_2+A_3$, where A_1,A_2,A_3 are, resp., the coefficients of $\theta_3^{18}\theta_4^4\bar\theta_3^6\bar\theta_4^4$, $\theta_3^{18}\theta_4^4\bar\theta_3^{10}$ and $\theta_3^{14}\theta_4^8\bar\theta_3^{10}$ in $\Theta_\Lambda(vv|\tau\bar\tau)$; similarly, $B=-B_1+B_2$ where B_1,B_2 are, resp., the coefficients of $\theta_3^{18}\theta_4^4\bar{\theta}_3^2\bar{\theta}_4^8$ and $\theta_3^6\theta_4^{16}\bar{\theta}_3^{10}$ in $\Theta_\Lambda(vv|\tau\bar{\tau})$.

Now, consider any $\Delta_k = \Theta([g_k'']\Lambda_R'') - \Theta([g_k']\Lambda_R')$ for which $g_k^2 \in \mathbf{Z}$. Then each Δ_k can be expressed as a polynomial in $\bar{\theta}_3^2$ and $\bar{\theta}_4^2$. Let Δ_k' consist of those terms in Δ_k in which $\bar{\theta}_3^2$ occurs to odd power. For example, $\Delta_1' = (\Theta_{\Lambda_R})' = \frac{1}{8}\bar{\theta}_3^6\bar{\theta}_4^4 - \frac{1}{8}\bar{\theta}_3^2\bar{\theta}_4^8$. A priori one would expect these Δ_k' to look like $r\bar{\theta}_3^{10} + s\bar{\theta}_3^6\bar{\theta}_4^4 + t\bar{\theta}_3^2\bar{\theta}_4^8$, for arbitrary $r, s, t \in \mathbf{R}$. However, it is easy to verify from eqs.(14) that for all these k, Δ_k' is a real multiple of $\bar{\theta}_3^6\bar{\theta}_4^4 - \bar{\theta}_3^2\bar{\theta}_4^8$. From that, eqs.(13$c$) and (8$b$) immediately imply $A_1 = -B_1$ and $A_2 = A_3 = B_2 = 0$, *i.e.* that $A = B$.

These comments can be easily generalized. The result is:

Theorem 6.4.5: Using the notation described in the preceding paragraph, suppose that for each k for which $g_k^2 \in \mathbf{Z}$, both

(i) Δ_k is expressible as a polynomial in $\bar{\theta}_3^2$ and $\bar{\theta}_4^2$, and

(ii) there exists an $\ell_k \in \mathbf{R}$ such that $\Delta_k' = \ell_k(\bar{\theta}_3^6\bar{\theta}_4^4 - \bar{\theta}_3^2\bar{\theta}_4^8)$,

hold. Then eqs.(5) and (7b) will necessarily be inconsistent, and no acceptable string will exist with RHS Λ_R.

The choice of Λ_R considered earlier succumbs of course to Thm 5. The point of Thm.5 is the following corollary, which is the main result of this section.

Corollary 6.4.6: There is no string theory with partition function of the type given in eq.(1c), based on a lattice Λ whose RHS Λ_R satisfies the half-norm property, *i.e.* eq.(11).

Proof First note that by Cor.10.2 in [**MUM**], condition (i) of Thm.5 is always satisfied when the half-norm property is satisfied.

We will begin by making some general observations about the theta constants of lattices satisfying eq.(11). Only in the final paragraph of the proof will it be applied to Λ_R' and Λ_R''.

Let Λ_1 be any 10-dimensional (integral) lattice satisfying eq.(11). Let $D \stackrel{\text{def}}{=} |\Lambda_1|$. We can write

$$\Theta(\Lambda_1) = a\theta_3^{10} + b\theta_3^8\theta_4^2 + c\theta_3^6\theta_4^4 + d\theta_3^4\theta_4^6 + e\theta_3^2\theta_4^8 + f\theta_4^{10}, \tag{6.4.17a}$$

where a, b, c, d, e, f are real, and $f = 1 - a - b - c - d - e$. Then by eqs.(4.1.6) and (4.2.6a),

$$\begin{aligned}\Theta(\Lambda_1^*) =& Da\theta_3^{10} + Db\theta_3^8\theta_2^2 + Dc\theta_3^6(\theta_3^4 - \theta_4^4)\\ &+ Dd\theta_3^4\theta_2^6 + De\theta_3^2(\theta_3^4 - \theta_4^4)^2 + Df\theta_2^{10} \qquad (6.4.17b)\\ =&(Da + Dc + De)\theta_3^{10} + Db\theta_3^8\theta_2^2 + (-Dc - 2De)\theta_3^6\theta_4^4\\ &+ Dd\theta_3^4\theta_2^6 + De\theta_3^2\theta_4^8 + Df\theta_2^{10}. \qquad (6.4.17c)\end{aligned}$$

Because Λ_1^* only has one zero vector, eq.(17b) implies $a = 1/D$.

Now let $g \in \Lambda_1^*$, $g^2 \in \mathbf{Z}$. Then g will be order 1,2 or 4, and $\Theta([g]\Lambda_1)$ will be of the same form as eq.(17a): *i.e.* $\Theta([g]\Lambda_1) = a_g\theta_3^{10} + b_g\theta_3^8\theta_4^2 + \cdots$. Of course, $\Lambda_1[g]$ will also satisfy eq.(11). Consider first the case where g is of order 2. Then the previous paragraph applied to both Λ_1 and $\Lambda_1[g]$ immediately implies that $a_g = 1/D$. Hence the same conclusion must apply to g of order 4 (and trivially to order 1 glues) — *i.e.* the leading coefficient for any integral-normed glue g of Λ_1 is $a_g = 1/D$. Moreover, note that the coefficient of θ_3^{10} in the theta constant of a glue class of non-integral norm must be 0.

Now by eq.(4.2.7), we can rewrite eq.(17c) as the sum of $\Theta([g]\Lambda_1)$ for all glue classes $[g]\Lambda_1$. We then get N/D as the coefficient of θ_3^{10} there. Hence $c + e = (N - D)/D^2$. The same technique as in the previous paragraph allows us to find a similar formula for $c_g + e_g$, for glues g of Λ_1 with integral norm.

Finally, consider the two lattices Λ_R' and Λ_R'' corresponding to a Λ_R satisfying eq.(11). Their glues g_i', g_j'' can be paired as was done in eqs.(12). Consider integral-normed glues $g_k' \leftrightarrow g_k''$. From the previous two paragraphs (and the dot product-preserving group isomorphism mentioned after eqs.(12)), two things should be clear:

$a_{g'_k} = a_{g''_k}$ and $c_{g'_k} + e_{g'_k} = c_{g''_k} + e_{g''_k}$. Hence

$$\Delta'_k = (a_{g''_k} - a_{g'_k})\bar{\theta}_3^{10} + (c_{g''_k} - c_{g'_k})\bar{\theta}_3^6 \bar{\theta}_4^4 + (e_{g''_k} - e_{g'_k})\bar{\theta}_3^2 \theta_4^8$$

necessarily satisfies condition (ii) of Thm.5. QED

Obviously Cor.6 itself cannot be generalized to take care of Λ_R which do not satisfy eq.(11). However, in the more general case an analogue of eq.(16) can still be found (although it may be most convenient to consider the first few terms $q'''q''$ of its Taylor expansion and use identities such as eqs.(4.2.13)) and compared with eq.(7*b*). In other words, the general approach used here, rather than the specific details in the proof of Cor.6, should be very useful in generalizing Cor.6.

CONCLUSION

This thesis has been concerned with lattices, particularly ways of constructing them, and applying those lattice techniques and results to two areas: the theory of theta functions; and the theory of superstrings. Much of the material has been taken from five papers ([**GL**1-5]).

Chapter One presented a general overview of lattice theory Although most of the material included was not new, the proofs were my own (except for a couple, whose sources were clearly given there). Some new (or at least not generally known) findings can also be found in that chapter (see especially Sec.4 and 6).

Chapter Two developed two constructions (namely, tensor products and shifting) that were not considered in the first chapter. Most of the material on shifting was original. The sections on tensor products considered in some detail their minimal norms; most results obtained there are known.

Chapter Three treated rational equivalence/similarity from a geometric, rather than algebraic, perspective (*i.e.* it used the language of lattices rather than the more common one of quadratic forms). Among other things what resulted was a new derivation of the 'weak Hasse principle' (see Sec.2). In the final section this geometrical perspective is used to prove a variety of results (some old, some new) about lattices.

Chapter Four applies the gluing construction (discussed primarily in Chapter One) to the theory of theta functions in one (complex) variable τ. One of the highlights of this chapter was the generation of at least 33 independent quadratic identities in the Jacobi functions. Any identity of this kind we have been able to find in the mathematical literature (and there are lots of them) can be shown to be algebraically/modularly equivalent to one of three of ours, so Table 8 contains at least 30 new identities. These identities exhaust all that can be derived by the lattice method, so they may constitute a complete list. Many other findings lie in this chapter, including some generalizations of Hecke's Theorem in Sec.2.

Chapter Five generalizes the results of the previous chapter to theta *series*, *i.e.* functions of some complex vector $\vec{z}$ as well as τ. Although more complicated, the extra variables allow the analysis to be more thorough than that in Chapter Four, and the conclusions in general are stronger. For example, whereas different lattices can have different theta *constants*, their theta *series* must be different. Also, we proved that any identity of this type (*i.e.* of full rank) can be derived using lattices. The analogue of Table 8 is Table 11, which includes at least 24 independent quadratic identities. Our literature search has shown that all such previously known identities are modularly equivalent to exactly one of ours.

Chapter Six applies the previous material on lattices to two problems in string theory. One concerns an attempt to construct strings using the gluing method (see Sec.3), and the other concerns the possibility of finding a physically reasonable string theory having zero cosmological constant (see Sec.4). These two sections show the power of lattices in handling some questions in string theory, and are both original contributions.

Of course, there are many directions for future research. For example, it would be valuable to know if Tables 8 and 11 are complete (because of Thm.5.3.6 this seems to be particularly tractible in the latter case). Along the same lines, it would be interesting to know if similar tables could be constructed for identities of higher degree (degree 3 should be readily accessible). Further extensions of Hecke's Theorem would be quite useful. We could go on and on.

The analysis of Sec.6.4 culminated in Cor.6.4.4, which showed that Dienes' partition functions eq.(6.4.1a) can never be realized, and especially in Cor.6.4.6, which proves that any function in an even broader family of partition functions which also would lead to zero cosmological constant (*i.e.* those given in eq.(6.4.1c)), can be realized by a string satisfying the 'half-norm property': eq.(11). It should be emphasized that the class of strings ruled out by Cor.6.4.6 is both large and extremely natural — see Sec.6.4 for some discussion of this point. Nevertheless, my immediate project will be to try to generalize these corollaries to cover every possible lattice string (an intermediate step may be to consider all those with the supercurrent used in Sec.6 3, say). The work in Sec.6.4 suggests this should be possible. Completing this analysis would be valuable for mathematical reasons, too — indeed, much of the material at the end of Sec.4.2 arose in our attempts to make progress on Dienes' problem.

BIBLIOGRAPHY

[BOR] Borcherds, R.E. (1985), The Leech Lattice, *Proc. R. Soc Lond.* **398A**, 365-376.

[BOU] Bourbaki, N. (1968), *Groupes et Algebres de Lie*, Ch.4-9, (Hermann)

[CAS] Cassels, J.W.S. (1978) *Rational Quadratic Forms*, (Academic Press)

[CN] Caselle, M. and Narain, K.S. (1989), The New Approach to the Construction of Conformal Field Theories, *Nucl. Phys.* **B323**, 673-718.

[CS1] Conway, J.H. and Sloane, N.J.A. (1988), *Sphere Packings, Lattices and Groups*, (Springer-Verlag).

[CS2] Conway, J.H. and Sloane, N.J.A. (1988), Low-dimensional Lattices I: Quadratic Forms of Small Determinant, *Proc. R. Soc. Lond.* **A418**, 17-41.

[CS3] Conway, J.H. and Sloane, N.J.A. (1988), Low-dimensional Lattices IV: The Mass Formula, *Proc. R. Soc. Lond.* **A419**, 259-286.

[CS4] Conway, J.H. and Sloane, N.J.A. (1989), Low-dimensional Lattices V. Integral Coordinates for Integral Lattices, *Proc. R. Soc. Lond.* **A426**, 211-232.

[DIEN] Dienes, Keith R. (1990), New String Partition Functions with Vanishing Cosmological Constant, *Phys. Rev. Lett.* **65**, 1979-1982.

[GAN] Gannon, Terry (1989), *Lattices and Superstrings*, M.Sc. Thesis, McGill University.

[GL1] Gannon, Terry and Lam, C.S. (1990), Construction of Four-dimensional Strings, *Phys. Rev.* **D41**, 492-506.

[GL2] Gannon, Terry and Lam, C.S. (1991), Gluing and Shifting Lattice Constructions and Rational Equivalence, (to be publ. in Rev.Math.Phys.**3** No 3).

[GL3] Gannon, Terry and Lam, C.S. (1991), Lattices and Θ-function Identities I Theta Constants, (submitted to J.Math.Phys.).

[GL4] Gannon, Terry and Lam, C.S. (1991), Lattices and Θ-function Identities II: Theta Series, (submitted to J.Math.Phys.).

[GL5] Gannon, Terry and Lam, C.S., Can a Non-supersymmetric String Have a Zero Cosmmological Constant?, (in progress).

[GO] Goddard, P. and Olive, D. (1984), Algebras, Lattices and Strings, *Vertex Operators in Mathematics and Physics*, (Springer-Verlag), 51-96.

[GOR] Gorenstein, Daniel (1982), *Finite Simple Groups: An Introduction to Their Classification*, (Plenum Press).

[GSW] Green. M.B., Schwarz, J.H., and Witten, E. (1987), *Superstring Theory*, Vols. I and II, (Cambridge University Press).

[HUM] Humphreys, James E. (1972), *Introduction to Lie Algebras and Representation Theory*, (Springer-Verlag).

[HUN] Hungerford, Thomas W. (1974), *Algebra*, (Springer-Verlag).

[KIT1] Kitaoka, Yoshiyuki (1977), Scalar Extension of Quadratic Lattices II, *Nagoya Math. Jour.* **67**, 159-164.

[KIT2] Kitaoka, Yoshiyuk (1978), Tensor Products of Posiitive Definite Quadratic Forms III, *Nagoya Math. Jour.* **70**, 173-181.

[KIT3] Kitaoka, Yoshiyuk (1981), Tensor Products of Positive Definite Quadratic Forms V, *Nagoya Math. Jour.* **82**, 99-111.

[KIT4] Kitaoka, Yoshiyuk (1984), Tensor Products of Positive Definite Quadratic Forms VII, *Nagoya Math. Jour.* **96**, 133-137.

[KLT] Kawai, Hikaru, Lewellen, David C. and Tye, S.-H. Henry (1987), Construction of Fermionic String Models in Four Dimensions, *Nucl. Phys.* **288B**, 1-76.

[KT] Kondo, T. and Tasaka, T. (1986), The Theta Functions of Sublattices of the Leech Lattice, *Nagoya Math. Jour.* **101**, 151-179.

[LAM1] Lam, C.S. (1988), GSO Projections of Modular-Invariant Aperiodic Strings, *Int. J. Mod. Phys.* **3A**, 913-942.

[LAM2] Lam, C.S. (1986), Direct Compactification of Heterotic Strings From 26 to 4 Dimensions via T^{22}/G, *Commun. Theor. Phys.* **5**, 113-122.

[LAM3] Lam, C.S. (1987), Strings Constructed From Free Bose and Fermi Fields, *Pro-*

ceedings of Beijing String Workshop, (World Scientific) (unpublished).

[LEK] Lekkerkerker, C.G. (1969), *Geometry of Numbers*, (Wolters-Noordhoff Publishing).

[LEV] Levinson, Norman and Redheffer, Raymond M. (1970), *Complex Variables*, (Holden-Day).

[LL] Lerche, Wolfgang and Lust, Dieter (1987), Covariant Heterotic Strings and Odd Self-dual Lattices, *Phys. Lett.* **187B**, 45-50.

[LSW] Lerche, W., Schellekens, A.N., and Warner, N.P. (1989), Lattices and Strings, *Phys. Rep.* **177**, 1-140.

[MUM] Mumford, D. (1984), *Tata Lectures on Theta*, Vol. I, (Birkhauer).

[OS] Odlyzko, A.M. and Sloane, N.J.A. (1980), A Theta-Function Identity for Nonlattice Packings, *Studia Sci. Math. Hung.* **15**, 461-465.

[ROG] Rogers, C.A. (1958), The Packing of Equal Spheres, *Proc. Lond. Math. Soc* **8**, 609-620.

[SER] Serre, J.-P. (1973), *A Course in Arithmetic*, (Springer-Verlag).

[TM] Tannery, Jules and Molk, Jules (1893, 1972), *Eléments de la Théorie des Fonctions Elliptiques*, Vols.II and IV, (2nd ed.: Chelsea NY).