Hyperfiniteness of the actions of relatively hyperbolic groups on their Bowditch boundaries

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Abstract

We show that if G is a finitely generated group hyperbolic relative to a finite collection of subgroups $\mathcal{P}$, then the natural action of G on the geodesic boundary of the associated relative Cayley graph induces a hyperfinite equivalence relation. As a corollary of this, we obtain that the natural action of G on its Bowditch boundary $\partial(G, \mathcal{P})$ also induces a hyperfinite equivalence relation. This strengthens a result of Ozawa obtained for $\mathcal{P}$ consisting of amenable subgroups and uses a recent work of Marquis and Sabok.

Résumé

Nous montrons que si G est un groupe de type fini hyperbolique relatif à une collection finie de sous-groupes $\mathcal{P}$, alors l'action naturelle de G induit une rélation d'équivalence hyperfini sur le bord géodésique du graphe de Cayley relatif associé. Comme corollaire de cela, nous obtenons que l'action naturelle de G sur son bord de Bowditch $\partial(G, \mathcal{P})$ induit une rélation d'équivalence hyperfini. Ceci renforce un résultat d'Ozawa obtenu pour $\mathcal{P}$ constitué de sous-groupes moyennables et utilise un travail récent de Marquis et Sabok.

Claim of Originality

The following contributions are presented in this thesis:

- Establishment of the finite sections property of geodesic ray bundles in relative Cayley graphs, giving rise to local finiteness of geodesic ray bundles in relative Cayley graphs.
- Establishment of the finite symmetric difference property of modified geodesic ray bundles in relative Cayley graphs.
- Hyperfiniteness of the action of a relatively hyperbolic group on the geodesic boundary of some (equivalently, any) relative Cayley graph.
- Hyperfiniteness of the action of a relatively hyperbolic group on its Bowditch boundary.

Contribution of Authors

Christopher Karpinski has authored each section of this thesis. The topic of this thesis was selected by Dr. Marcin Sabok. Marcin suggested trying to generalize the finite sections property to relative Cayley graphs and using binary coding to generalize parts of Section 6 of his and Marquis' paper [26].

Acknowledgements

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Contents

1. Introduction

This thesis studies equivalence relations induced by boundary actions of relatively hyperbolic groups. The study of boundary actions began with the work of Connes, Feldman and Weiss in [12] and Vershik in [37] who studied the actions of free groups on their boundaries. They showed that for a free group, its action on the Gromov boundary is μ-hyperfinite for every Borel quasi-invariant probability measure μ on the boundary. Adams [1] later generalized this result to all hyperbolic groups.

Relatively hyperbolic groups were introduced by Gromov [20]; see also the monograph of Osin [29]. Given a relatively hyperbolic group G with a collection of parabolic subgroups $\mathcal{P}$ there is a natural boundary called the Bowditch boundary, denoted $\partial(G, \mathcal{P})$, which is a compact metrizable space on which G acts naturally by homeomorphisms.

In [31], Ozawa generalized the work of Adams [1] to the actions of relatively hyperbolic groups on their Bowditch boundary under the assumptions that the parabolic subgroups are exact. When the parabolic subgroups of G in $\mathcal{P}$ are amenable, Ozawa [31] proved that the action of G on $\partial(G, \mathcal{P})$ is topologically amenable, and, more generally, when the parabolic subgroups are exact, Ozawa [31] proved that the group G is exact. Alternative proofs of the exactness of the group were given by Osin [28] who worked with parabolic subgroups with finite asymptotic dimension and by Dadarlat and Guentner [13] who worked with parabolic subgroups that are uniformly embeddable into a Hilbert space.

In [38], Zimmer introduced the notion of amenability of equivalence relations; see also the work of Connes, Feldman and Weiss [12]. By [2, Theorem 5.1], a measurable action of a countable group G on a standard probability space (X, μ) is μ-amenable if and only if μ-almost all stabilizers are amenable and the orbit equivalence relation is μ-amenable.

In this thesis we generalize the result of Ozawa and work with relatively hyperbolic groups without any assumptions on the parabolic subgroups. In fact, we consider boundary actions from the Borel perspective. A countable Borel equivalence relation is called *hyperfinite* if it is a countable increasing union of finite Borel sub-equivalence relations. Dougherty, Jackson and Kechris showed in [17, Corollary 8.2] that the boundary action of any free group induces a hyperfinite orbit equivalence relation. The result of Dougherty, Jackson and Kechris was generalized to cubulated hyperbolic groups by Huang, Sabok and Shinko in [22], and later to all hyperbolic groups by Marquis and Sabok in [26]. In this thesis, we prove the following:

Theorem A. *Let G be a finitely generated group hyperbolic relative to a finite collection of subgroups $\mathcal{P}$ and let $\hat{\Gamma}$ be the associated relative Cayley graph. Then the natural action of G on the geodesic boundary $\partial\hat{\Gamma}$ induces a hyperfinite orbit equivalence relation.*

Corollary B. *Let G be a finitely generated group hyperbolic relative to a finite collection of subgroups $\mathcal{P}$. Then the natural action of G on the Bowditch boundary $\partial(G, \mathcal{P})$ induces a hyperfinite orbit equivalence relation.*

Corollary B in particular strengthens the result of Ozawa [31] in case the parabolic subgroups are amenable. Indeed, hyperfiniteness implies μ-amenability for every invariant Borel probability measure μ and by [3, Theorem 3.3.7], an action of a countable group on a locally compact space by homeomorphisms is topologically amenable if and only if it is μ-amenable for every invariant Borel probability measure μ.

We proceed by following a similar approach to [22] and [26], studying *geodesic ray bundles* $\text{Geo}(x, \eta)$ in relative Cayley graphs (Definition 2.23). For the case of a cubulated hyperbolic group G studied in [22], the crucial property from which the hyperfinitess of the boundary action of G follows is the finite symmetric difference of geodesic ray bundles: for any

$x, y \in G$ and any $\eta \in \partial G$, $\text{Geo}(x,\eta) \triangle \text{Geo}(y,\eta)$ is finite (see [22, Theorem 1.4]). In [36], Touikan showed that this symmetric difference need not be finite in Cayley graphs of general hyperbolic groups, although in [25], Marquis provides many examples of groups acting geometrically on locally finite hyperbolic graphs where this finite symmetric difference property does hold. In [26], Marquis and Sabok define a modified version of the geodesic ray bundle, denoted $\text{Geo}_1(x,\eta)$ for $x \in G$ and $\eta \in \partial G$ (see [26, Definition 5.5] and Definition 2.41 in our thesis) and show ([26, Theorem 5.9]) that these modified geodesic ray bundles satisfy a finite symmetric difference property: $|\text{Geo}_1(x,\eta) \triangle \text{Geo}_1(y,\eta)| < \infty$ for each $x, y \in G$ and for each $\eta \in \partial G$. Marquis and Sabok then deduce hyperfiniteness of the boundary action as a consequence of this finite symmetric difference property of the modified bundles (see [26, Section 6])

Local finiteness of the Cayley graph plays a crucial role in establishing the finite symmetric difference property of the Geo_1 bundles in [26]. However, relative Cayley graphs of relatively hyperbolic groups are not locally finite. To make up for this loss of local finiteness, we rely on finiteness results about relative Cayley graphs of relatively hyperbolic groups from [29] (namely, [29, Theorem 3.26]).

The main difference between this thesis and [26] is in Section 3. In Section 3, we prove the crucial "finite sections property" (Definition 3.1) of geodesic ray bundles in relative Cayley graphs (Theorem 3.2, whose proof is the main source of new content in this thesis) which yields the uniform local finiteness of these bundles (Corollary 3.6), and we establish the finite symmetric difference property of the modified (Geo_1) bundles (Theorem 3.9), which is the main goal of the section. Equipped with the results of Section 3, in Section 4 we show the hyperfiniteness of the action of G on $\partial\hat{\Gamma}$ as a consequence of the finite symmetric difference property of Geo_1 bundles, closely following the approach in [26, Section 6]. The main difference between Section 4 of our thesis and [26, Section 6] is our different way of coding labels of geodesic rays in the non locally finite relative Cayley graph $\hat{\Gamma}$. We finish

Section 4 by showing Corollary B, which follows immediately from Theorem A. We conclude by discussing further work and open problems in Section 5.

2. Preliminaries

2.1 Geodesics and Quasi-Isometry

Definition 2.1. *A **geodesic path** p between two points x, y in a metric space (X, d) is an isometric embedding of an interval $p : [0, \ell] \to X$ such that $p(0) = x$ and $p(\ell) = y$.*

We parameterize all paths to unit speed, so the length $\ell(p)$ of a path $p : [0, \ell] \to X$ equals ℓ. In this thesis, we frequently abuse notation and identify a path in a metric space with its image. For a path $p : [a, b] \to X$, we will denote its endpoints $p_- = p(a)$ and $p_+ = p(b)$.

We will sometimes denote a geodesic path between two points $x, y \in X$ by $[x, y]$. Note that every subpath of a geodesic path is a geodesic path.

Definition 2.2. *A **geodesic ray** γ based at a point x in a metric space (X, d) is an isometric embedding $\gamma : [0, \infty) \to X$ with $\gamma(0) = x$.*

Definition 2.3. *We say that two geodesic rays γ_1, γ_2 are **asymptotic** if $d_{Haus}(\gamma_1, \gamma_2) < \infty$ (where d_{Haus} denotes Hausdorff distance with respect to the metric d).*

Definition 2.4. *A **geodesic metric space** is a metric space (X, d) such that any two points $x, y \in X$ can be joined by a geodesic path.*

Definition 2.5. *A **geodesic triangle** in a geodesic metric space is a closed path that is the concatentation of three geodesic paths.*

Geometric group theory is often concerned with the coarse structure of spaces. The maps which preserve distances coarsely are known as quasi-isometric embeddings.

Definition 2.6. *For $\lambda \geq 1, \varepsilon \geq 0$, a map $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is a:*

- (λ, ε)***-quasi-isometric embedding*** *if $\forall x, y \in X$, we have:*

$$\frac{1}{\lambda} d_X(x, y) - \varepsilon \leq d_Y(f(x), f(y)) \leq \lambda d_X(x, y) + \varepsilon$$

- (λ, ε)***-quasi-isometry*** *if it is a (λ, ε)-quasi-isometric embedding and $\exists C \geq 0$ such that $Y = N_C(f(X))$.*

Supressing constants, a map $f : X \to Y$ as above is a ***quasi-isometric embedding*** *if it is a (λ, ε)-quasi-isometric embedding for some $\lambda \geq 1$ and $\varepsilon \geq 0$ and a* ***quasi-isometry*** *if it is a (λ, ε)-quasi-isometry for some $\lambda \geq 1$ and $\varepsilon \geq 0$.*

Note that if f is a (λ, ε)-quasi-isometric embedding (resp. quasi-isometry), then it is a (λ', ε')-quasi-isometric embedding (resp. quasi-isometry) for any $\lambda' \geq \lambda$ and $\varepsilon' \geq \varepsilon$. Note also that isometric embeddings are precisely $(1, 0)$-quasi-isometric embeddings.

It is not hard to show that the relation of being quasi-isometric is an equivalence relation on the class of all metric spaces. For symmetry, if $f : X \to Y$ is a quasi-isometry and $Y = N_C(f(X))$, then we can define $g : Y \to X$ by $g(y) = x$ where x is any element of X such that $d(f(x), y) \leq C$. Then g is a quasi-isometry.

Definition 2.7. *For $\lambda \geq 1$ and $\varepsilon \geq 0$, a path p in a metric space (X, d) is a (λ, ε)**-quasi-geodesic** if for each subpath q of p we have:*

$$\ell(q) \leq \lambda d(q_-, q_+) + \varepsilon$$

Note that if p is a (λ, ε)-quasi-geodesic, then it is a (λ', ε')-quasi-geodesic for any $\lambda' \geq \lambda$ and $\varepsilon' \geq \varepsilon$. Note also that geodesics are precisely $(1,0)$-quasi-geodesics.

2.2 Graphs

In this subsection we introduce some basic graph-theoretic terminology that we will use throughout this thesis.

Definition 2.8. *A* ***graph*** *consists of a vertex set X and a set of edges $E \subseteq X \times X$ connecting pairs of vertices in X (we allow loop edges about vertices but we do not allow multiple edges connecting two vertices). For a graph Γ, we say that $x, y \in X$ are* ***Γ-adjacent*** *if x, y are connected by an edge in Γ. If e is an edge connecting vertices x, y, we will denote the* ***vertex set*** *of e by $V(e) = \{x, y\}$. We can orient the edges by defining a* ***source*** *vertex e_- and* ***target*** *vertex e_+ for each edge e. A graph with oriented edges is called an* ***oriented graph****.*

Given a graph Γ, we denote the vertex set of Γ by $\Gamma^{(0)}$.

Definition 2.9. *A* ***combinatorial path*** *in a graph Γ from a vertex x to a vertex y is a sequence of edges $p = e_1, ..., e_n$ of Γ from x to y. The* ***length*** *of such a path p is the number of edges in the path. We say that a graph Γ is* ***connected*** *if for any two vertices of Γ there exists a path in Γ having endpoints those vertices. The* ***length*** *of a path p is the number of edges comprising it. We say that a graph Γ is* ***locally finite*** *if every vertex is adjacent to finitely many vertices and we say that Γ is* ***uniformly locally finite*** *if there exists a constant B such that for every vertex v of Γ the number of vertices adjacent to v is at most B.*

Given a connected graph Γ, we can turn Γ into a geodesic metric space as follows. For any two vertices x, y of Γ, we define $d(x, y)$ to be the infimal length of paths from x to y. We can extend this metric to be defined on edges themselves as follows. Isometrically identify each edge e, having vertices x, y, with $[0,1]$ by a map ϕ_e such that $x \mapsto 0$, $y \mapsto 1$. Then for points x, y on different edges e, f, respectively, put $d(x,y) = d(x,v_x) + d(v_x,v_y) + d(v_y,y)$, where v_x, v_y are the nearest vertices to x, y, respectively, that minimize the sum $d(x,v_x) + d(v_x,v_y) + d(v_y,y)$.

Definition 2.10. *The metric d on the graph Γ as above is called the **combinatorial metric** on Γ.*

Note that in the combinatorial metric, the inclusion $\Gamma^{(0)} \hookrightarrow \Gamma$ is a (1,1)-quasi-isometry. By default, we will always equip connected graphs with their combinatorial metric.

Definition 2.11. *Let Γ be a connected graph. A **combinatorial geodesic ray (CGR)** in Γ is a geodesic ray based at a vertex of Γ.*

CGRs are isometric embeddings of $\mathbb{N}$ into $\Gamma^{(0)}$. If λ is a CGR, we often write $\lambda = (x_n)_n$, where x_n are the vertices on λ.

In connected graphs, we are free to move the starting vertex of a combinatorial geodesic ray to any other vertex to obtain a geodesic ray with the same tail as the original ray. This is shown by the following lemma.

Lemma 2.12. *([26, Lemma 3.1]) Let Γ be a connected graph. If $\gamma = (x_n)_{n\geq 0}$ is a geodesic ray in Γ and if $y \in \Gamma^{(0)}$, then there exists N such that for any geodesic path $[y, x_N]$ connecting y and x_N, the concatenation $[y, x_N](x_n)_{n\geq N}$ of $[y, x_N]$ with the geodesic ray $(x_n)_{n\geq N}$ is a geodesic ray.*

Since we will use this lemma often and since its proof is straightforward, let us prove it.

Proof. Consider the map $g : \mathbb{N} \to \mathbb{Z}$ given by $g(n) = d(y, x_n) - d(x_n, x_0) = d(y, x_n) - n$. Our goal is to show that g is eventually constant, i.e. that $\exists N$ such that for all $n \geq N$, $g(n) = g(N)$, which will show that $d(y, x_n) = d(y, x_N) + d(x_N, x_n)$ for all $n \geq N$, which will show that the concatenation $[y, x_N](x_n)_{n\geq N}$ is a geodesic for any geodesic path $[y, x_N]$.

Note that g is non-increasing. Indeed, for any $n \in \mathbb{N}$,

$$\begin{aligned} g(n+1) - g(n) &= d(y, x_{n+1}) - (n+1) - d(y, x_n) + n \\ &= d(y, x_{n+1}) - d(y, x_n) - 1 \\ &\leq d(x_{n+1}, x_n) - 1, \text{ by the triangle inequality} \\ &= 0 \end{aligned}$$

Also, g is bounded. Indeed, for any $n \in \mathbb{N}$,

$$\begin{aligned} |g(n)| &= |d(y, x_n) - d(x_n, x_0)| \\ &\leq d(y, x_0), \text{ by the triangle inequality} \end{aligned}$$

Therefore, since g is $\mathbb{Z}$-valued, it is eventually constant.

□

Let us illustrate Lemma 2.12 with an example.

Example 2.13. *Consider the standard $\mathbb{Z}^2$ lattice (i.e. the graph with vertex set $\mathbb{Z}^2$ and vertical and horizontal edges joining the vertices). Consider the diagonal geodesic ray γ starting from the origin $(0,0)$ and the vertices $y = (4,0)$ and $z = (-3,-1)$ shown in Figure 2.1. Joining y to the vertex $(4,3)$ on γ by the dotted geodesic path results in a geodesic ray from y. Similarly, joining z to the vertex $(0,0)$ on γ by the dotted geodesic path results in a geodesic ray from z. Notice that joining y to γ via any geodesic path to $(0,0)$ (or any vertex on γ before $(4,3)$) does not result in a geodesic ray from y.*

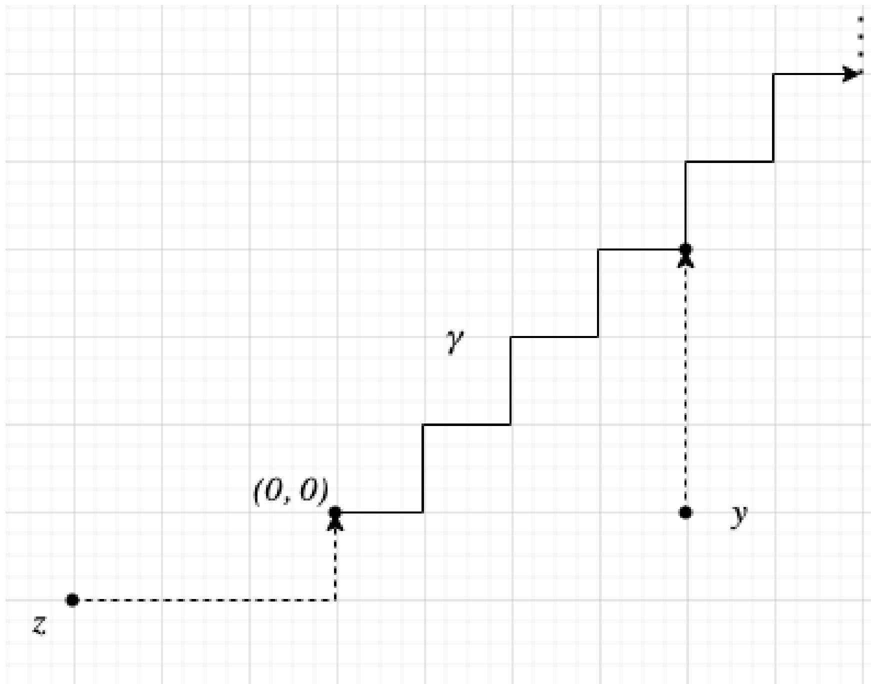

Figure 2.1: An illustration of Lemma 2.12.

2.3 Hyperbolic Metric Spaces

In this subsection, we introduce the notion of hyperbolicity of metric spaces and the geodesic boundary of a metric space.

Definition 2.14. *Given a geodesic triangle Δ with sides p, q, r, we say that Δ is δ-**slim** for a constant $\delta \geq 0$ if each of its sides is contained in the closed δ-neighbourhood of the union of the other two sides: $p \subseteq N_\delta(q \cup r)$, $q \subseteq N_\delta(p \cup r)$ and $r \subseteq N_\delta(p \cup q)$ (here, for $R \geq 0$ and a subset A of a metric space X, the closed R-neighbourhood of A is $N_R(A) := \{x \in X : \exists a \in A \text{ such that } d(x, a) \leq R\}$).*

Recall that a metric space is **proper** if all finite radius closed balls are compact. Note that a connected graph is a proper metric space if and only if it is locally finite.

Definition 2.15. *For $\delta \geq 0$, a geodesic metric space is called δ-**hyperbolic** if each of its* *geodesic triangles are δ-slim. We say that a geodesic metric space is **hyperbolic** if it is δ-hyperbolic for some $\delta \geq 0$.*

An important fact about hyperbolicity of metric spaces is that it is quasi-isometry invariant, which is a consequence of the following theorem.

Theorem 2.16. *([9, Theorem III.H.1.9]) Let X, Y be geodesic metric spaces with Y hyperbolic. If there exists a quasi-isometric embedding $f : X \to Y$, then X is hyperbolic.*

Let us look at some examples and non-examples of hyperbolic metric spaces.

Example 2.17. *(a) Any bounded geodesic metric space is hyperbolic (take δ to be the diameter of the metric space).*

(b) Trees (connected graphs with no closed loops) are 0-hyperbolic. Indeed, geodesic triangles in trees are "tripod" shaped so that each side is contained in the union of the other two sides.

(c) Euclidean space $\mathbb{R}^n$ is hyperbolic if and only if $n = 1$. Indeed, $\mathbb{R}$ is 0-hyperbolic as every side of a geodesic triangle is contained in the union of the other two, and if $n > 1$, then for any $\delta \geq 0$, any isosceles right angled triangle with legs of length $2(\delta + 1)$ is not δ-slim (see Figure 2.1).

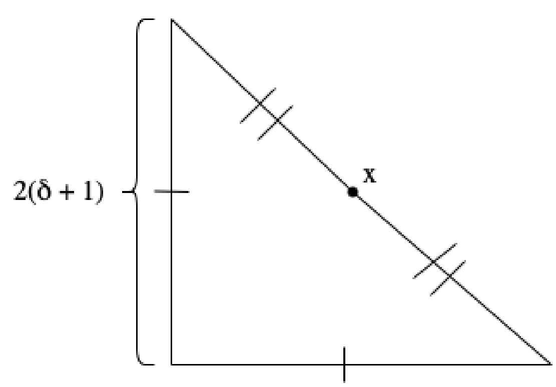

Figure 2.2: A geodesic triangle in $\mathbb{R}^n$, $n > 1$, that is not δ-slim. The midpoint x of the hypotenuse is not δ-close to the other two sides of the triangle.

In a hyperbolic metric space (X, d), we have the following theorem, which says that asymptotic geodesic rays are *uniformly close*.

Theorem 2.18. *([9, Lemma III.H.3.3(1)]) Let (X, d) be a δ-hyperbolic metric space and let γ_1, γ_2 be asymptotic geodesic rays based at a point $x \in X$. Then for each $t \in \mathbb{R}$, we have $d(\gamma_1(t), \gamma_2(t)) \leq 2\delta$.*

Let us give a proof of this theorem to illustrate the definition of δ-hyperbolicity and some techniques involved in geometry of hyperbolic metric spaces.

Proof. Let C be an upper bound on the Hausdorff distance between γ_1, γ_2 and let $t \in \mathbb{R}$. Choose $T > C + t + \delta$. Then there exists $s \in \mathbb{R}$ such that $d(\gamma_1(T), \gamma_2(s)) \leq C$. By the triangle inequality, we have that $s > t + \delta$.

Now connect $\gamma_1(T)$ and $\gamma_2(T)$ with a geodesic α. The sides γ_1 from x to $\gamma_1(T)$, γ_2 from x to $\gamma_2(s)$ and α form a geodesic triangle Δ. Consider the point $\gamma_1(t)$. By δ-hyperbolicity, Δ is δ-slim, so there exists some point p on γ_2 or α such that $d(p, \gamma_1(t)) \leq \delta$. Note that p cannot be on α because then we would have $d(\gamma_1(t), \gamma_1(T)) \leq C + \delta$, which implies $T - t \leq C + \delta$, yielding $T \leq C + t + \delta$, a contradiction. Therefore, p must be on γ_2. Write $p = \gamma_2(r)$ for some $r < s$.

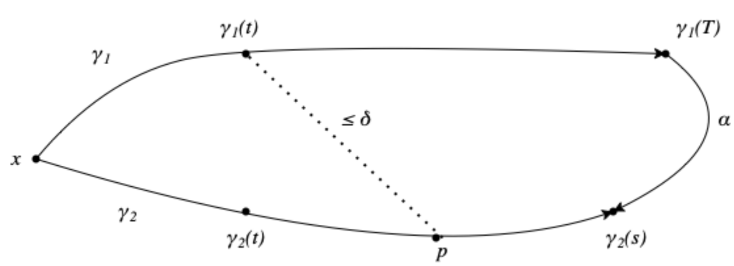

Figure 2.3: The geometric arrangement of the proof of Theorem 2.18.

We will show that $d(p, \gamma_2(t)) \leq \delta$, yielding $d(\gamma_1(t), \gamma_2(t)) \leq 2\delta$ by the triangle inequality. We consider two cases on where r can lie:

Case (i): $r \leq t$. Then we have

$$\begin{aligned} d(x, \gamma_2(t)) &= d(x, \gamma_1(t)) \\ &\leq d(x, \gamma_2(r)) + \delta \end{aligned}$$

Therefore, we obtain $d(p, \gamma_2(t)) = d(x, \gamma_2(t)) - d(x, \gamma_2(r)) \leq \delta$.

Case (ii): $r > t$: Then we have:

$$\begin{aligned} d(x, \gamma_1(r)) &= d(x, \gamma_2(r)) \\ &\leq \delta + d(x, \gamma_1(t)) \end{aligned}$$

So, we obtain $d(p, \gamma_2(t)) = d(x, \gamma_2(r)) - d(x, \gamma_2(t)) = d(x, \gamma_1(r)) - d(x, \gamma_1(t)) \leq \delta$.

We conclude that $d(p, \gamma_2(t)) \leq \delta$ in all cases and so $d(\gamma_1(t), \gamma_2(t)) \leq 2\delta$.

□

If (X, d) is any metric space, we can associate to it a natural object called its geodesic boundary, defined as follows. Note that the relation on geodesic rays defined by $\gamma \sim \lambda$ if γ, λ are asymptotic, is an equivalence relation.

Definition 2.19. *For a metric space X and $x \in X$, the* ***geodesic boundary based at*** **x***, $\partial_x X$, is the set of all asymptotic equivalence classes of geodesic rays in X starting at x. Removing reference to a basepoint, the* ***geodesic boundary*** *∂X is defined as the set of all geodesic rays in X modulo the equivalence relation of being asymptotic.*

If X is a proper δ-hyperbolic space or a hyperbolic graph (i.e. a graph that is a hyperbolic metric space equipped with its combinatorial metric), then $\partial X = \partial_x X$ for each $x \in X$.

For proper δ-hyperbolic metric spaces, this follows from [9, Lemma III.H.3.1] and if X is a hyperbolic graph, then this follows from Lemma 2.12.

If X is δ-hyperbolic, for each $x \in X$, we can equip $\partial_x X$ with a natural topology with neighbourhood base $V(p,r)^x = \{q \in \partial_x X : \exists \gamma \in q$ and $\exists \lambda \in p$ such that $d(\gamma(t), \lambda(t)) \leq 2\delta$ for each $t \leq r\}$ for each $p \in \partial_x X$ and each $r \in \mathbb{N}$ (see [11, Page 10] for this exact defiinition or [23, Definition 2.12] for a different but equivalent definition). If X is a proper hyperbolic metric space then the topology on ∂X is defined having neighbourhood base $V(p,r)^x$ for any choice of basepoint $x \in X$, i.e. $q \in V(p,r)$ if there exist geodesic rays $\gamma \in p$ and $\lambda \in q$ that start at the same point x and are 2δ close for a distance of at least r (the independence of the topology on the basepoint is shown in [9, Lemma III.H.3.7]).

It can be shown (see [16, Proposition 3.4.18]) that if X is a separable hyperbolic space, then the topology on $\partial_x X$ defined above is a Polish topology (i.e. second countable and completely metrizable), and that ∂X is a quasi-isometry invariant for proper hyperbolic spaces: if $f : X \to Y$ is a quasi-isometry, then there is an induced homeomorphism $f_\partial : \partial X \to \partial Y$ (see [9, Theorem III.H.3.9]).

Example 2.20. *(Examples of geodesic boundaries)*

(a) If X is a bounded geodesic metric space, then $\partial X = \emptyset$ because there are no geodesic rays in X.

(b) If $X = \mathbb{R}$, then $\partial X = \{\pm\infty\}$.

(c) If X is a d-regular tree with $d < \infty$ (i.e. each vertex has d edges coming out of it), then ∂X is homeomorphic to the Cantor space $\{0,1\}^{\mathbb{N}}$ (where $\{0,1\}$ is equipped with the discrete topology, and the product space has the product topology).

(d) If X is a regular tree with countably infinite valence (i.e. the set of edges from each vertex is countably infinite), then ∂X is homeomorphic to the Baire space $\mathbb{N}^{\mathbb{N}}$ (where $\mathbb{N}$ is equipped with the discrete topology, and the product space has the product topology).

(e) *If X is a wedge of countably infinitely many rays $[0, \infty)$ (which is a tree, hence hyperbolic), then $\partial X = \mathbb{N}$.*

(f) *In general, if X is a proper hyperbolic space, then ∂X is compact (see [9, Proposition III.H.3.7 (3), (4)]). If X is not proper, then $\partial_x X$ need not be compact as examples $(d), (e)$ above show.*

Remark 2.21. *If X is a hyperbolic space and if a group G acts on X by isometries, then G acts on ∂X by putting $g[\gamma] := [g\gamma]$ for every $g \in G$ and every geodesic ray γ in X. This action induces a homeomorphism $\partial_x X \to \partial_{gx} X$ for every $x \in X$. In particular, if X is a hyperbolic graph and G acts transitively on $X^{(0)}$, then $\partial_x X$ is homeomorphic to $\partial_y X$ for every $x, y \in X^{(0)}$. In this case, we can define a topology on ∂X by putting $\partial X = \partial_x X$ for each $x \in X^{(0)}$.*

2.4 Horoboundary

In this subsection, we introduce a refinement of the geodesic boundary called the **horoboundary** that will prove to be a crucial geometric tool. The following terminology is from [10, Section 3].

Let X be a connected graph equipped with its combinatorial metric d. Fix a basepoint $z \in X^{(0)}$ and define $\mathcal{F}(X, z) = \{f : X^{(0)} \to \mathbb{R} : |f(x) - f(y)| \le d(x, y)$ for all $x, y \in X^{(0)}$ and $f(z) = 0\}$. We equip $\mathcal{F}(X, z)$ with the topology of pointwise convergence.

For each $x \in X^{(0)}$, we associate the *Busemann function* $f_x \in \mathcal{F}(X, z)$ defined by $f_x(y) = d(x, y) - d(x, z)$. Define $\iota : X^{(0)} \to \mathcal{F}(X, z)$ by $\iota(x) = f_x$

Definition 2.22. *The **horofunction space** of X, denoted $C_{horo}(X)$ is the closure of $\iota(X^{(0)})$ in $\mathcal{F}(X, z)$. The **horoboundary** of X is $C_{hb}(X) := C_{horo}(X) \setminus \iota(X^{(0)})$.*

It can be shown that neither $C_{horo}(X)$ nor $C_{hb}(X)$ depend on the basepoint z, since $\mathcal{F}(X, z)$

can be identified with the space of all 1-Lipschitz functions $X^{(0)} \to \mathbb{R}$ modulo the constant functions.

2.5 Combinatorial Geodesic Ray Bundles

In this section, we discuss the important notions of combinatorial geodesic rays (CGRs) and collect lemmas and definitions from [26, Sections 3 and 4] that will come in handy later. We omit most proofs of the lemmas we present and we refer the reader to [26, Sections 3 and 4] for any omitted proofs.

Let X be a connected graph equipped with its natural combinatorial metric.

Definition 2.23. *For $x \in X^{(0)}$ and $\eta \in \partial X$, the* ***combinatorial geodesic ray bundle*** *$CGR(x, \eta)$ is the set of all combinatorial geodesic rays based at x and having asymptotic equivalence class η. Define $Geo(x, \eta)$ to be the set of all vertices on combinatorial geodesic rays based at x and in the equivalence class η.*

Combinatorial geodesic rays determine unique limit points in $C_{hb}(X)$ by the following lemma:

Lemma 2.24. *[26, Lemma 3.2] Let X be a connected graph and $\gamma = (x_n)_n$ a CGR. The sequence of functions $(f_{x_n})_n$ converges in $C_{horo}(X)$ to some $\xi \in C_{hb}(X)$.*

Proof. This follows from Lemma 2.12. Let $z \in X^{(0)}$ be a basepoint and let $y \in X^{(0)}$. By Lemma 2.12, there exists N such that $[y, x_N](x_n)_{n \geq N}$ and $[z, x_N](x_n)_{n \geq N}$ are geodesics. We then have for all $n \geq N$:

$$\begin{aligned} f_{x_n}(y) &= d(x_n, y) - d(x_n, z) \\ &= d(x_N, y) + d(x_N, x_n) - (d(x_N, z) + d(x_N, x_n)), \text{ by choice of } N \\ &= d(x_N, y) - d(x_N, z) \\ &= f_{x_N}(y) \end{aligned}$$

Therefore, $(f_{x_n}(y))$ converges to $f_N(y)$. We conclude that $(f_{x_n})_n$ converges pointwise in $C_{horo}(X)$. Since $(x_n)_n$ is a geodesic ray, the limit $\xi = \lim f_{x_n}$ is not of the form f_y for some $y \in X^{(0)}$, so $\xi \in C_{hb}(X)$.

□

In the above lemma, we denote such limit of a CGR γ by $\xi = \xi_\gamma$.

Definition 2.25. *([26, Definition 3.3]) Fixing a basepoint $z \in X^{(0)}$, for $\eta \in \partial X$, define the **limit set** $\Xi(\eta) = \{\xi_\gamma : \gamma \in CGR(z, \eta)\}$.*

By Lemma 2.12, the definition of $\Xi(\eta)$ is independent of the basepoint (i.e. for any $z_1, z_2 \in X^{(0)}$ and $\xi \in \Xi(\eta)$, we have $\xi = \xi_\gamma$ for some $\gamma \in \mathrm{CGR}(z_1, \eta)$ if and only if $\xi = \xi_{\gamma'}$ for some $\gamma' \in \mathrm{CGR}(z_2, \eta)$).

Definition 2.26. *([26, Definition 3.4]) For $x \in X^{(0)}, \eta \in \partial X$ and $\xi \in \Xi(\eta)$, define the **combinatorial sector** $Q(x, \xi) = \{y \in X^{(0)} : y \in \gamma$ for some $\gamma \in CGR(x, \eta)$ with $\xi_\gamma = \xi\}$.*

Definition 2.27. *([26, Definition 4.8]) For $\eta \in \partial X$, a vertex $x \in X^{(0)}$ is η-**special** if $\bigcap_{\xi \in \Xi(\eta)} Q(x, \xi)$ contains a CGR γ. The set of all η-**special** vertices is denoted $X_{s,\eta}$.*

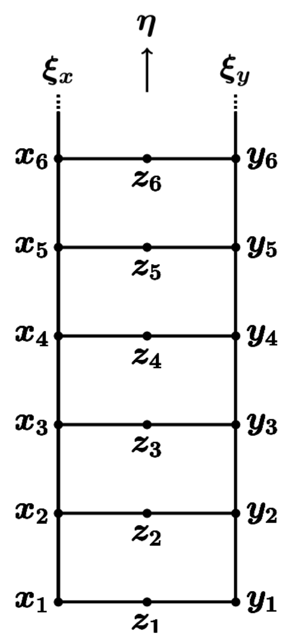

Figure 2.4: The "bad ladder" X. Taken from Figure 2 of [26].

Example 2.28. *Consider the graph X as in Figure 2.3. We have $\partial X = \{\eta\}$ and $\Xi(\eta) = \{\xi_x, \xi_y\}$. We see that for each $n \in \mathbb{N}$,*

- $Q(x_n, \xi_x) = \{x_m : m \geq n\}$,
- $Q(y_n, \xi_y) = \{y_m : m \geq n\}$,
- $Q(y_n, \xi_x) = Q(x_n, \xi_y) = \{x_m : m \geq n\} \cup \{y_m : m \geq n\} \cup \{z_m : m \geq n\}$,
- $Q(z_n, \xi_x) = \{z_n\} \cup \{x_m : m \geq n\}$, *and*
- $Q(z_n, \xi_y) = \{z_n\} \cup \{y_m : m \geq n\}$.

Therefore, $X_{s,\eta} = \{x_n : n \in \mathbb{N}\} \cup \{y_n : n \in \mathbb{N}\}$. Also, note that $Geo(x_1, \eta) = X^{(0)}$ while $Geo(z_1, \eta) = X^{(0)} \setminus \{z_n : n \geq 2\}$, so that $Geo(x_1, \eta) \triangle Geo(z_1, \eta) = \{z_n : n \geq 2\}$, which is infinite.

Lemma 2.29. *([26, Lemma 4.3]) Let $x \in X^{(0)}$, $\eta \in \partial X$, $\xi \in \Xi(\eta)$ and $y \in Q(x, \xi)$. Then $Q(y, \xi) \subseteq Q(x, \xi)$.*

Lemma 2.30. *([26, Lemma 4.5]) Let $x \in X^{(0)}$, $\eta \in \partial X$, $\xi \in \Xi(\eta)$ and $\gamma \in CGR(x,\eta)$. If $\gamma \subseteq Q(x,\xi)$ and if ξ_γ denotes the limit of ξ, then $Q(x,\xi_\gamma) \subseteq Q(x,\xi)$.*

Lemma 2.31. *([26, Lemma 4.6]) Let $x \in X^{(0)}$ and $\eta \in \partial X$. If $Q(x,\xi) = Q(x,\xi')$ for some $\xi, \xi' \in \Xi(\eta)$, then $\xi = \xi'$.*

Lemma 2.32. *([26, Lemma 4.7]) Let $x \in X^{(0)}$ and $\eta \in \partial X$. If $\bigcap_{\xi\in\Xi(\eta)} Q(x,\xi)$ contains a CGR from x to η, then there exists a unique limit point $\xi_{x,\eta} \in \Xi(\eta)$ such that $Q(x,\xi_{x,\eta}) = \bigcap_{\xi\in\Xi(\eta)}$.*

By Lemma 2.30 and Lemma 2.32, if $x \in X^{(0)}$ is η-special, then for any CGR γ contained in $\bigcap_{\xi\in\Xi(\eta)} Q(x,\xi)$ with limit ξ_γ, we have $\xi_\gamma = \xi_{x'\eta}$.

For any $x, y \in X^{(0)}$, denote $\gamma(x,y)$ to be the union of all geodesic paths from x to y in X.

Definition 2.33. *([26, Definition 4.9]) A geodesic ray $\gamma \in CGR(x,\eta)$ is **straight** if $\gamma \subseteq \bigcup_{n\in\mathbb{N}} \gamma(x,y_n)$ for any $(y_n)_n \in CGR(x,\eta)$.*

The following lemma gives a characterization of straight geodesics.

Lemma 2.34. *([26, Lemma 4.12]) Let $x \in X^{(0)}$, $\eta \in \partial X$ and $\gamma = (x_n)_{n\in\mathbb{N}} \in CGR(x,\eta)$. Then the following assertions are equivalent:*

(a) γ is straight

(b) $\gamma \subseteq \bigcap_{\xi\in\Xi(\eta)} Q(x,\xi)$

(c) For any $(y_n)_{n\in\mathbb{N}} \in CGR(y_0,\eta)$ and any $n \in \mathbb{N}$, there exists $M \in \mathbb{N}$ such that $d(x,x_n) + d(x_n,y_m) = d(x,y_m)$ for all $m \geq M$.

(d) For each m, the CGR $(x_n)_{n\geq m}$ is straight.

There is a relationship between special vertices and straight geodesics. It is expressed by the lemma below.

Lemma 2.35. *([26, Lemma 4.13]) Let $x \in X^{(0)}$ and $\eta \in \partial X$. The following are equivalent.*

(a) x is η-special.

(b) There exists a straight $\gamma \in CGR(x, \eta)$.

If any of the above assertions hold, then $\xi_{x,\eta} = \xi_\gamma$.

Lemma 2.36. *([26, Lemma 4.14]) Let $x \in X^{(0)}$, $\eta \in \partial X$ and let $y \in Geo(x, \eta)$ be η-special. Then $[x, y]\gamma \in CGR(x, \eta)$ for any geodesic $[x, y]$ between x, y and any CGR $\gamma \in CGR(y, \eta)$ converging to $\xi_{y,\eta}$.*

Lemma 2.37. *([26, Lemma 4.15]) Let $\eta \in \partial X$ and let $x \in X_{s,\eta}$. Then $Q(x, \xi_{x,\eta}) \subseteq X_{s,\eta}$ and $\xi_{y,\eta} = \xi_{x,\eta}$ for every $y \in Q(x, \xi_{x,\eta})$.*

The following functions will prove to be an important tool in the geometry that we do in Section 3.

Definition 2.38. *([26, Definition 4.16]) For $x \in X^{(0)}$ and $\xi \in \Xi(\eta)$ define the function $d_{x,\xi} : X^{(0)} \to \mathbb{Z}$ by $d_{x,\xi}(a) = d(x, a) + \xi(a) - \xi(x)$.*

Lemma 2.39. *([26, Lemma 4.17]) Let $x \in X^{(0)}$, $\eta \in \partial X$ and $\xi \in \Xi(\eta)$. Let $\gamma = (x_n)_{n\in\mathbb{N}} \in CGR(x, \eta)$. Then the following hold.*

(a) The sequence $(d_{x,\xi}(x_n))_{n\in\mathbb{N}}$ is non-decreasing.

(b) $d_{x,\xi}(x_n) \leq d_{Haus}(\gamma, \gamma')$ for any $n \in \mathbb{N}$ and any $\gamma' \in CGR(x, \eta)$ converging to ξ.

(c) There is some $N \in \mathbb{N}$ such that $d_{x,\xi}(x_n) = d_{x,\xi}(x_N)$ for all $n \geq N$.

(d) If N is as in (c), then for any $\gamma' = (y_n)_{n\in\mathbb{N}} \in CGR(y_0, \eta)$ converging to ξ, and for any $n \geq N$, there exists $M \in \mathbb{N}$ such that $d(x_N, x_n) + d(x_n, y_m) = d(x_N, y_m)$ for all $m \geq M$.

Lemma 2.40. *([26, Lemma 4.18]) Let $x \in X^{(0)}$, $\eta \in \partial X$ and $(x_n)_{n\in\mathbb{N}} \in CGR(x, \eta)$. If $\Xi(\eta)$ is finite, then there is some $N \in \mathbb{N}$ such that the CGR $(x_n)_{n\geq N}$ is straight.*

Our main objects of interest will be the following modified geodesic ray bundles, first defined in [26, Definition 5.5].

Definition 2.41. *Let $x \in X^{(0)}$ and $\eta \in \partial X$. For $\xi \in \Xi(\eta)$, let $Y(x,\xi)$ be the set of η-special vertices $y \in Geo(x,\eta)$ with $\xi_{y,\eta} = \xi$ at minimal distance to x. Put*

$$Geo_1(x,\eta) = \bigcup_{\xi \in \Xi(\eta)} \bigcup_{y \in Y(x,\xi)} Q(y,\xi)$$

Note that if $\Xi(\eta)$ is finite, then by Lemma 2.40, each CGR eventually consists of η-special vertices, so that $Y(x,\xi)$ is non-empty. If in addition $\mathrm{Geo}(x,\eta)$ is locally finite, then $Y(x,\xi)$ is a finite, non-empty set.

2.6 Cayley Graphs

In this subsection, we introduce the notion of a Cayley graph of a group and establish some terminology.

Definition 2.42. *Let G be a group generated by some set S. The* ***Cayley graph*** *of G with respect to S is the graph $\Gamma(G;S)$ whose vertices are elements of G and such that two vertices x, y are connected by an edge if there exists $s \in S$ such that $y = xs$.*

Definition 2.43. *If G is a group and S is a generating set for G, then for each path $p = (x_n)_{0 \le n \le k}$ (resp $p = (x_n)_{n \in \mathbb{N}}$), define the* ***label*** *of p to be $(x_n^{-1}x_{n+1})_{0 \le n \le k-1} \in (S^{\pm})^k$ for finite paths and $(x_n^{-1}x_{n+1})_{n \in \mathbb{N}} \in (S^{\pm})^{\mathbb{N}}$ for infinite paths, where $S^{\pm} := \{t : t = s \text{ or } t = s^{-1} \text{ for some } s \in S\}$ is the symmetrized generating set.*

Given an element $g \in G$, we denote $|g|_S$ the distance between the vertices 1 and g in the Cayley graph $\Gamma(G;S)$ (equivalently, $|g|_S$ is the minimal length of a word over $S^{\pm}$ representing g). Then $d(g,h) = |g^{-1}h|_S$ for any $g, h \in G$ defines a metric on G, which coincides with the combinatorial metric on $\Gamma(G;S)$.

Definition 2.44. *A metric on G induced from a generating set S as above is called the **word metric** on G with respect to S.*

Example 2.45. *(Examples of Cayley graphs)*

(a) $Cay(\mathbb{Z}/n\mathbb{Z}, \{1\})$ is a directed n-gon.

(b) For $n \geq 1$ (including $n = \omega$), if S is a free generating set for F_n, then $Cay(F_n; S)$ is a $2n$-regular tree.

(c) $Cay(\mathbb{Z}^n; \{e_1, ..., e_n\})$, where e_i are the standard basis for $\mathbb{R}^n$, is the lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$.

If X, Y are two finite generating sets for a group G, then the Cayley graphs $\Gamma_1 = \Gamma(G; X)$ and $\Gamma_2 = \Gamma(G; Y)$ are quasi-isometric. Indeed, it suffices to show that G with the word metrics from X, Y are quasi-isometric spaces, since the vertex set of a graph is quasi-isometric to the graph (see Section 2.2). Let $C = \max\{\max_{x \in X}\{|x|_Y\}, \max_{y \in Y}\{|y|_X\}\}$. Then for any $g \in G$, $\frac{1}{C}|g|_Y \leq |g|_X \leq C|g|_Y$. It follows that for any $g, h \in G$, $\frac{1}{C}d_Y(g, h) \leq d_X(g, h) \leq C d_Y(g, h)$. Thus, the identity map $G \to G$ is a $(C, 0)$-quasi-isometry from (G, d_X) to (G, d_Y).

In particular, by Theorem 2.16, hyperbolicity of Cayley graphs with respect to finite generating sets is an invariant of finite generating set (i.e. if the Cayley graph with respect to one finite generating set is hyperbolic, then the Cayley graph with respect to any other finite generating set is hyperbolic).

Remark 2.46. *If a Cayley graph Γ of a group is hyperbolic, then since the group acts isometrically on the Cayley graph and transitively on the vertex set, by Remark 2.21, up to homeomorphism, $\partial_x \Gamma$ is independent of x for every $x \in \Gamma^{(0)}$ and so we can define a topology on $\partial\Gamma$ to be the topology on $\partial_x \Gamma$ for any choice of $x \in \Gamma^{(0)}$.*

2.7 Hyperbolic and Relatively Hyperbolic Groups

In this subsection, we discuss the notions of hyperbolic group and relatively hyperbolic groups. Both of these objects were introduced by Gromov in his seminal paper [20], though for relatively hyperbolic groups, we will work with a different (but equivalent) definition than that given in [20].

Definition 2.47. *A finitely generated group G is called* ***hyperbolic*** *if its Cayley graph with respect to some (equivalently, any; see Section 2.6) finite generating set is hyperbolic.*

Definition 2.48. *The* ***boundary*** *of a hyperbolic group G, denoted ∂G, is the boundary of any of its Cayley graphs with respect to finite generating sets.*

Note that the boundary of a hyperbolic group does not depend, up to homeomorphism, on the choice of finite generating set because the Cayley graphs for different finite generating sets are all quasi-isometric, hence their boundaries are homeomorphic; see Section 2.3.

Let us now turn to defining relatively hyperbolic groups.

Definition 2.49. *Let G be a group, $\{H_\lambda\}_{\lambda\in\lambda}$ a collection of subgroups of G, with $\mathcal{H} := \bigcup_{\lambda\in\Lambda} H_\lambda$, and suppose X is a set such that G is generated by $X \cup \mathcal{H}$, (such a set X is called a relative generating set with respect to the collection $\{H_\lambda\}_{\lambda\in\Lambda}$). The* ***relative Cayley graph*** *associated with $\{H_\lambda\}_{\lambda\in\Lambda}$ and X is the Cayley graph $\hat{\Gamma} := \Gamma(G; X \cup \mathcal{H})$. The relative Cayley graph can be identified (quasi-isometrically) with the* ***coned-off Cayley graph*** *obtained by starting with the Cayley graph Γ of G with respect to X, adjoining to Γ a vertex v_{gH_i} for each left coset gH_i and connecting each vertex of gH_i in Γ to v_{gH_i} by an edge of length $\frac{1}{2}$.*

In $\hat{\Gamma}$, edges may be labeled by an element from X or an element from a subgroup H_λ.

Definition 2.50. *A subpath p of a path α in $\hat{\Gamma}$ whose label is a word in some H_λ is called an* ***$\mathbf{H}_\lambda$-subpath*** *of α. A maximal H_λ-subpath of α is called an* ***$\mathbf{H}_\lambda$-component*** *of α. By a* ***(parabolic) component*** *of α, we mean an H_λ-component of α for some $\lambda \in \Lambda$.*

Definition 2.51. *Two H_λ-components p, q of a path α in $\hat{\Gamma}$ are called **connected** if there exist $h_1, h_2 \in H_\lambda$ such that $p_- h_1 = q_-$ and $p_+ h_2 = q_+$. If an H_λ-component p of α is not connected to any other H_λ-component of α, then p is called an **isolated $\mathbf{H}_\lambda$-component** of α. A path α that contains two connected H_λ-components for some $\lambda \in \Lambda$ is said to **backtrack**, otherwise it is said to be a path **without backtracking**.*

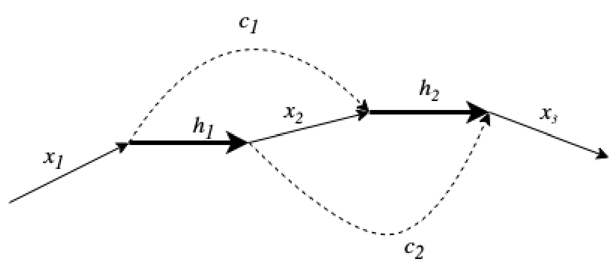

Figure 2.5: An example of a backtracking path in $\hat{\Gamma}$.

Definition 2.52. *We say that a group G is **weakly hyperbolic relative to** the collection $\{H_\lambda\}_{\lambda \in \Lambda}$ of subgroups if there exists a finite generating set X of G such that $\hat{\Gamma}$ is a hyperbolic metric space. In such a case, we call the subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$ the **parabolic subgroups**.*

We refer to the word metric on G from the generating set X as d_X (as opposed to the metric from the generating set $X \cup \mathcal{H}$ which we denote by d). Note that $d \leq d_X$ because $X \subseteq X \cup \mathcal{H}$.

Definition 2.53. *For a group G, a collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$ and a relative generating set X, the relative Cayley $\hat{\Gamma}$ satisfies the **bounded coset penetration property** (BCP) if for any $C \geq 1$ and $D \geq 0$ there exists a constant k depending only on C, D such that for any (C, D)-quasi-geodesics α, β without backtracking and with the same endpoints (i.e. $\alpha_- = \beta_-$ and $\alpha_+ = \beta_+$), if p is a component of α or β which is not connected to any component of α or β then $d_X(p_-, p_+) \leq k$.*

Definition 2.54. *We say that G is* ***hyperbolic relative to*** $\{H_\lambda\}_{\lambda\in\Lambda}$ *if G is weakly hyperbolic relative to $\{H_\lambda\}_{\lambda\in\Lambda}$ and some associated relative Cayley graph $\hat{\Gamma}$ satisfies the BCP. We say that a group is* ***relatively hyperbolic*** *if it is hyperbolic relative to a collection of proper subgroups.*

Remark 2.55. *Note that relative hyperbolicity is invariant under change of finite relative generating set, that is, if X, Y are finite relative generating sets of G with respect to $\{H_\lambda\}_{\lambda\in\Lambda}$, then G is hyperbolic relative to $\{H_\lambda\}_{\lambda\in\Lambda}$ with respect to X if and only if G is hyperbolic relative to $\{H_\lambda\}_{\lambda\in\Lambda}$ with respect to Y, i.e., $\Gamma(G; X\cup\mathcal{H})$ is hyperbolic and satisfies the BCP if and only if $\Gamma(G; Y\cup\mathcal{H})$ is hyperbolic and satisfies the BCP. Indeed, this follows because the metrics $d_{X\cup\mathcal{H}}$ and $d_{Y\cup\mathcal{H}}$ are Lipschitz equivalent: $\exists C\geq 1$ such that $\frac{1}{C}d_{Y\cup\mathcal{H}}\leq d_{X\cup\mathcal{H}}\leq Cd_{Y\cup\mathcal{H}}$, which is proved similarly to how the quasi-isometry type of a Cayley graph does not change when changing finite generating set as we saw below Example 2.45 (see [29, Proposition 2.8]).*

Throughout this thesis, we fix a group G generated by a finite set X hyperbolic relative to subgroups $\{H_1, ..., H_N\}$.

The notation $B_r(x)$ denotes the closed ball of radius r about the vertex $x\in G$ in $\hat{\Gamma}$ with its natural combinatorial metric and similarly $B_r^X(x)$ denotes the closed ball of radius r about the vertex $x\in G$ in $\Gamma(G;X)$ with its natural combinatorial metric.

Example 2.56. *(Examples of Hyperbolic and Relatively Hyperbolic Groups)*

(a) Finite groups are hyperbolic as their Cayley graphs with respect to the generating set that is the whole group is a bounded, geodesic metric space, hence hyperbolic.

(b) Finitely generated free groups are hyperbolic as their Cayley graphs with respect to a free generating set is a tree, which is hyperbolic.

(c) $\mathbb{Z}^n$ for $n>1$ is not hyperbolic as its Cayley graph with respect to the standard basis of $\mathbb{Z}^n$, is quasi-isometric to $\mathbb{R}^n$, which is not hyperbolic. In fact, any group containing $\mathbb{Z}^2$

as a subgroup is not hyperbolic (this follows from the fact that if g is an infinite order element in a hyperbolic group, then $\langle g\rangle$ has finite index in the centralizer $C(g)$; see [9, Corollary III.Γ.3.10 (2)]).

(d) *Consider a finitely generated group G which is hyperbolic relative the collection consisting of only the trivial subgroup $\{1\}$. Then G is a hyperbolic group. Indeed, the left cosets of the trivial subgroup are just singletons consisting of each group element. So, the relative Cayley graph is obtained from the non-relative Cayley graph by adding loops to each vertex, hence it is quasi-isometric to the non-relative Cayley graph. Conversely, this reasoning also shows that hyperbolic groups are hyperbolic relative to the trivial subgroup (or more generally, hyperbolic relative to finite subgroups).*

(e) *Consider the free abelian group $G = \langle a, b | [a, b]\rangle$ and subgroup $H = \langle a\rangle$. Then G is weakly hyperbolic relative to $\{H\}$. Indeed, the map $f : \mathbb{Z} \to \Gamma(G; \{a, b\} \cup \langle a\rangle)^{(0)}$ given by $f(n) = b^n$ is a quasi isometry since it is a (1,0)-quasi-isometric embedding as $d_{\{a,b\}\cup H}(f(n), f(m)) = d_{\{a,b\}\cup H}(b^n, b^m) = |n - m|$ for all $n, m \in \mathbb{Z}$. In addition, $G \subseteq N_1(f(\mathbb{Z}))$ since for any $n, m \in \mathbb{Z}$, $d_{\{a,b\}\cup H}(a^n b^m, b^m) = 1$. Therefore, f is a quasi-isometry. However, the pair (G, H) with the generating set $\{a, b\}$ does not satisfy the BCP because we may produce a geodesic bigon as in the picture below and the isolated H-component η in the lower geodesic portion of the bigon has $d_{\{a,b\}}(\eta_-, \eta_+) = n \to \infty$. This implies that G is not hyperbolic relative to $\{H\}$ (as relative hyperbolicity is an invariant of finite generating set by Remark 2.55).*

(f) *If H is a hyperbolic group, then $\mathbb{Z}^2 * H$ is hyperbolic relative to $\mathbb{Z}^2$ (the relative Cayley graph is quasi-isometric to the Cayley graph of H which is hyperbolic and the BCP holds because there cannot be isolated components in any bigon in the relative Cayley graph due to the free product structure), but is not hyperbolic since it contains $\mathbb{Z}^2$ as a subgroup. This, together with (d) above, shows that relatively hyperbolic groups form a strictly larger class than the class of hyperbolic groups.*

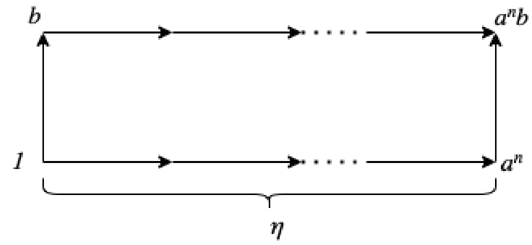

Figure 2.6: The bigon and parabolic component η from Example (e).

(g) Every finitely generated group G is hyperbolic relative to $\{G\}$. Indeed, let X be a finite generating set for G. We see that the relative Cayley graph $\Gamma(G; G \cup X)$ is a bounded metric space (it has diameter 1 as we may join any two vertices labeled by elements $g, h \in G$ by an edge labeled $g^{-1}h \in X \cup G$), so it is hyperbolic. To show the BCP, suppose we have a (λ, ε)-quasi-geodesic bigon $c = \alpha\beta$ in $\Gamma(G; X \cup G)$. Then a component η of c is isolated iff it is the only component in c (because if there were another component of c, then this would be connected to the first component because any two vertices in $\Gamma(G; X \cup G)$ are connected by an edge labeled by an element of G). However, note that we have $\ell(\alpha) \leq \lambda d_{X \cup G}(\alpha_-, \alpha_+) + \varepsilon \leq \lambda + \varepsilon$ because $d_{X \cup G}(\alpha_-, \alpha_+) \leq 1$ and similarly $\ell(\beta) \leq \lambda + \varepsilon$. Therefore, by the triangle inequality, for any component η of, say, β we have $d_X(\eta_-, \eta_+) \leq d_X(\alpha_-, \eta_-) + d_X(\alpha_-, \alpha_+) + d_X(\eta_+, \alpha_+) \leq \ell([\alpha_-, \eta_-]_\beta) + \ell([\alpha_-, \alpha_+]) + \ell([\eta_+, \alpha_+]_\beta) \leq 3(\lambda + \varepsilon)$. Therefore, the BCP property holds for $C(\lambda, \varepsilon) = 3(\lambda + \varepsilon)$. Thus, we conclude that any group G is hyperbolic relative to itself. This is the reason why we demand hyperbolicity relative to proper subgroups in the definition of a relatively hyperbolic group.

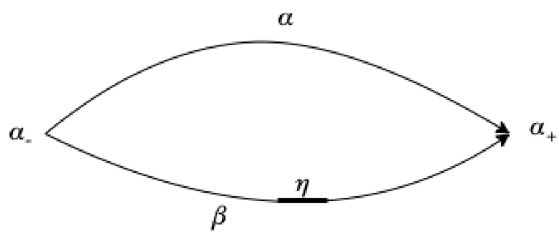

Figure 2.7: The (λ, ε)-quasi-geodesic bigon from (g) above.

In this thesis, we will use the following result from Osin's book [29]:

Theorem 2.57. *(Theorem 3.26 in [29]) Let G be a group hyperbolic relative to a collection of subgroups $\{H_1, ..., H_n\}$. There exists a finite generating set X of G such that the following holds. There exists a constant ν such that for any geodesic triangle pqr in the relative Cayley graph $\hat{\Gamma}$ and any vertex u on p, there exists a vertex v on $q \cup r$ such that $d_X(u, v) \leq \nu$.*

Theorem 2.57 implies that geodesic triangles in $\hat{\Gamma}$ are ν-slim with respect to d_X and hence also with respect to d. Therefore, we can take ν as our hyperbolicity constant for $\hat{\Gamma}$, so that vertex sets on geodesic triangles are ν-slim with respect to both d_X and d. We will do so throughout this thesis whenever we apply Theorem 2.57.

Let us also briefly discuss Bowditch's definition of relative hyperbolicity. Bowditch gives a more dynamical definition of relative hyperbolicity, in terms of actions on so-called *fine* hyperbolic graphs.

Definition 2.58. *Let G be a group together with a collection $\mathcal{P}$ of subgroups of G. We say that G is* ***hyperbolic relative*** *to $\mathcal{P}$ (in the sense of Bowditch, see [8]) if G admits an action on a connected graph K such that the following hold:*

(i) K is hyperbolic and each edge of K is contained in only finitely many circuits of a given length n (a ***circuit*** *in a graph is a closed path that does not have repeated edges) for every $n \in \mathbb{N}$. (The property of a graph having each edge contained in only finitely*

*many circuits of a given length is called **fineness**).*

(ii) There are only finitely many orbits of edges and each edge stabilizer is finite.

(iii) The conjugates of elements of $\mathcal{P}$ are precisely the infinite vertex stabilizers of K.

(iv) Every element of $\mathcal{P}$ is finitely generated.

From this fine hyperbolic graph K that G acts on, we obtain a proper hyperbolic metric space $X(K)$ on which G acts isometrically and such that ∂K embeds G-equivariantly as a topological subspace into $\partial X(K)$, whose complement can be identified with the countable set of all left cosets of subgroups in $\mathcal{P}$ (see [8, Page 26] for the isometric action of G on $X(K)$, [8, Page 17] for the construction of $X(K)$, [8, Proposition 8.5] for the embedding of ∂K into $\partial X(K)$ and see [8, Proposition 9.1] for the countable complement of ∂K in $\partial X(K)$).

Definition 2.59. *For G, $\mathcal{P}$, K, $X(K)$ as above, the **Bowditch boundary** $\partial(G, \mathcal{P})$ of the pair $(G, \mathcal{P})$ is defined to be $\partial X(K)$.*

We see therefore that $\partial(G, \mathcal{P})$ is a compact metrizable space on which G acts naturally by homeomorphisms. We can embed $\partial\hat{\Gamma}$ into $\partial(G, \mathcal{P})$, as the next theorem shows.

Theorem 2.60. *Let G be hyperbolic relative to a collection of subgroups $\mathcal{P}$, with relative Cayley graph $\hat{\Gamma}$. Then $\partial\hat{\Gamma}$ embeds G-equivariantly and homeomorphically into $\partial(G, \mathcal{P})$ with countable complement.*

Proof. In [15, Proposition 1, Section A.2], it is shown that the coned-off Cayley graph $\hat{\Gamma}$ is a fine hyperbolic graph satisfying Definition 2.58. Therefore, using the notation above, $\partial(G, \mathcal{P}) = \partial X(\hat{\Gamma})$, so $\partial\hat{\Gamma}$ embeds G-equivariantly and topologically into $\partial(G, \mathcal{P})$ with countable complement consisting of all left cosets of conjugates of elements of $\mathcal{P}$. $\square$

2.8 Descriptive Set Theory

In this subsection, we go over the necessary descriptive set theory background that we will need. A standard reference includes [24].

The main objects of study in descriptive set theory are *standard Borel spaces*.

Definition 2.61. *A **standard Borel space** is a Polish space (i.e. separable, completely metrizable topological space) equipped with its Borel σ-algebra.*

Examples of standard Borel spaces include countable discrete spaces, compact metrizable spaces and boundaries of separable hyperbolic spaces. The class of standard Borel spaces is closed under taking closed subsets and countable products ([24, Proposition 3.3 (ii) and (iii)]).

Definition 2.62. *An **analytic set** subset of a standard Borel space Z is an image of a Borel set under a Borel measurable function. A **co-analytic set** is the complement of an analytic set.*

Definition 2.63. *A **Borel equivalence relation** E on a standard Borel space Z is an equivalence relation on Z that is a Borel set as a subset of $Z \times Z$.*

Definition 2.64. *Given a Borel subset A of a standard Borel space X, if E is a Borel equivalence relation on X, then $E|_A$ denotes the **restriction** of E to A defined by $E|_A := E \cap (A \times A)$.*

Definition 2.65. *Given a Borel equivalence relation E on a standard Borel space X and a Borel subset $A \subseteq X$, the **E-saturation** of A is the set $[A]_E = \{x \in X : xEa \text{ for some } a \in A\}$.*

Definition 2.66. *A Borel equivalence relation is* ***finite*** *(resp.* ***countable****) if its equivalence classes are all finite (resp. countable).*

Definition 2.67. *A Borel equivalence relation is* ***hyperfinite*** *if it is a countable increasing union of finite Borel equivalence relations.*

Definition 2.68. *For two Borel equivalence relations E on X and F on Y (for X, Y standard Borel spaces), a map $f : X \to Y$ that satisfies $xEy \implies f(x)Ff(y)$ for all $x, y \in X$ is called a* ***Borel homomorphism****.*

Definition 2.69. *For two Borel equivalence relations E on X and F on Y (for X, Y standard Borel spaces), a map $f : X \to Y$ is called a* ***Borel reduction*** *from E to F if it satisfies $xEy \iff f(x)Ff(y)$ for all $x, y \in X$.*

Definition 2.70. *A Borel equivalence relation E is* ***Borel reducible*** *to a Borel equivalence relation F if there exists a Borel reduction from E to F.*

Definition 2.71. *A Borel equivalence relation is called* ***smooth*** *if it is Borel reducible to the equality relation (i.e. $x \sim y$ iff $x = y$) on some standard Borel space.*

Definition 2.72. *A Borel equivalence relation is called* ***hypersmooth*** *if it is a countable increasing union of smooth Borel equivalence relations.*

Definition 2.73. *An* ***analytic Borel equivalence relation*** *E on a standard Borel space X is a Borel equivalence relation that is an analytic subset of $X \times X$.*

In Section 4 of this thesis, we will make use of the following results.

Theorem 2.74. *(Lusin-Novikov Theorem) Let X, Y be standard Borel spaces and let $B \subseteq X \times Y$ be a Borel set whose vertical sections $B_x = \{y \in Y : (x, y) \in B\}$ have size at most r. Then there are Borel functions $f_i : X \to Y$, $i = 1, ..., r$ such that B is the union of the graphs of the f_i.*

Theorem 2.75. *[24, Theorem 18.18] Let X, Y be standard Borel spaces and let $B \subseteq X \times Y$ be a Borel set whose vertical sections $B_x = \{y \in Y : (x, y) \in B\}$ are σ-compact (i.e. a countable union of compact sets). Then the projection $\pi_X(B) \subseteq X$ is Borel.*

In particular, Theorem 2.75 can be applied when the vertical sections are countable.

Theorem 2.76. *[24, Theorem 14.12] Let $f : X \to Y$ be a map between standard Borel spaces X, Y. Then f is Borel if and only if its graph $\{(x, f(x)) : x \in X\} \subseteq X \times Y$ is Borel.*

The following lemma is an application of [24, Theorem 35.16].

Lemma 2.77. *([22, Lemma 4.1]) Let Z be a standard Borel space, $A \subseteq Z$ be analytic and let E be an analytic equivalence relation on Z such that there is some $K > 1$ such that every $E|_A$-class has size less than K. Then there is a Borel equivalence relation F on Z with $E|_A \subseteq F$ such that every F-class has size less than K.*

3. Geodesic Ray Bundles in Relative Cayley Graphs

In this section, we examine modified geodesic ray bundles and prove that these modified bundles have finite symmetric difference for a fixed boundary point. This section generalizes [26, Theorem 5.9].

We will be interested in the following property of a graph.

Definition 3.1. *A hyperbolic graph* Γ *has the* ***finite sections property*** *if there exists a constant* B *such that for any* $x \in \Gamma^{(0)}$*, for any* $\eta \in \partial\Gamma$ *and for any* $i \in \mathbb{N}$*, we have*

$$|\{\gamma(i) : \gamma \in CGR(x, \eta)\}| \leq B$$

Note that any uniformly locally finite hyperbolic graph has the finite sections property.

Here is the main result of this section, which is the main source of original content in this thesis.

Theorem 3.2. *Let* G *be a group hyperbolic relative to a collection of subgroups* $\{H_1, ..., H_n\}$*. There exists a finite generating set* X *of* G *such that the associated relative Cayley graph* $\hat{\Gamma}$ *has the finite sections property.*

Proof. Take the finite generating set X to be as in Theorem 2.57. Choose a hyperbolicity constant $\nu \in \mathbb{N}$ for $\hat{\Gamma}$ as in Theorem 2.57. Let $i \in \mathbb{N}$. Fix any $\gamma_0 \in CGR(x, \eta)$ and let

$k = i + 3\nu + 1$. We will show that for each $\gamma \in \mathrm{CGR}(x, \eta)$, there exists a vertex v on γ_0 with $d(v, \gamma_0(i)) \leq 3\nu$ and such that $d_X(\gamma(i), v) \leq \nu$.

Let $\gamma \in CGR(x, \eta)$ be arbitrary. Begin by joining $\gamma(k)$ and $\gamma_0(k)$ with a geodesic α (see Figure 3.1). By Theorem 2.18, we have that $d(\gamma(k), \gamma_0(k)) \leq 2\nu$, so α has length at most 2ν.

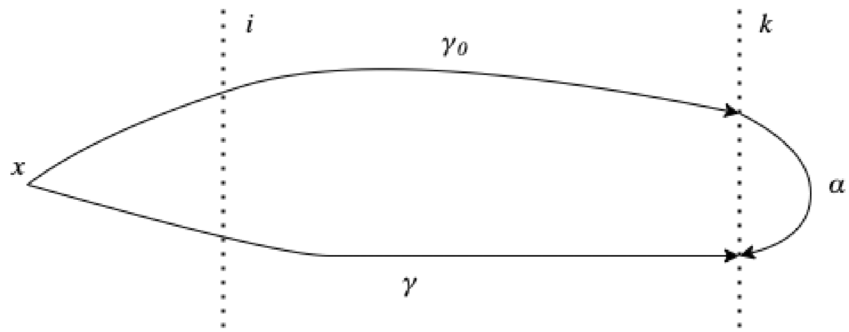

Figure 3.1: The arrangement of geodesics in the proof of Theorem 3.2.

Letting $|_k$ denote the restriction of a geodesic to $\{0, 1, ..., k\}$, we apply Theorem 2.57 to the geodesic triangle with sides $\gamma_0|_k, \alpha$ and $\gamma|_k$. By Theorem 2.57, there exists a vertex v on $\gamma_0|_k$ or on α such that $d_X(\gamma(i), v) \leq \nu$. We cannot have v on α because then we would have $d(\gamma(i), v) \leq \nu$ (since $d \leq d_X$), which would imply by the triangle inequality that

$$k - i = d(\gamma(i), \gamma(k)) \leq d(\gamma(i), v) + d(v, \gamma(k)) \leq d(\gamma(i), v) + \ell(\alpha) \leq \nu + 2\nu = 3\nu$$

contradicting our choice of k. Therefore, we must have that v is on $\gamma_0|_k$.

Lastly, let us show that $d(v, \gamma_0(i)) \leq 3\nu$. By Theorem 2.18, we have $d(\gamma(i), \gamma_0(i)) \leq 2\nu$, and note that $d_X(\gamma(i), v) \leq \nu$ implies $d(\gamma(i), v) \leq \nu$, so by the triangle inequality,

$$d(v, \gamma_0(i)) \leq d(v, \gamma(i)) + d(\gamma(i), \gamma_0(i)) \leq \nu + 2\nu = 3\nu$$

,

We conclude that for each $i \in \mathbb{N}$ and each $\gamma \in \mathrm{CGR}(x, \eta)$, $\gamma(i)$ must be ν-close in d_X to a vertex v on γ_0 with $d(v, \gamma_0(i)) \leq 3\nu$. There are at most $6\nu + 1$ such vertices on γ_0, so we obtain that $|\{\gamma(i) : \gamma \in \mathrm{CGR}(x, \eta)\}| \leq (6\nu + 1)|B^{\nu}_X(1)|$. Thus, we set $B = (6\nu + 1)|B^{\nu}_X(1)|$.

□

As a corollary of this theorem, we obtain the following results. We assume from now on that G is a relatively hyperbolic group generated a finite set X as in Theorem 3.2.

Corollary 3.3. *Let $(\gamma_n)_{n=0}^{\infty}$ be a sequence of elements of $CGR(x, \eta)$ for some $x \in G$ and some $\eta \in \partial\hat{\Gamma}$. Then $(\gamma_n)_{n=0}^{\infty}$ has a convergent subsequence which converges to a CGR $\gamma \in CGR(x, \eta)$ (where convergence of a sequence of CGRs is pointwise convergence).*

Proof. By Theorem 3.2, for each i, $\{\gamma_n(i) : n \in \mathbb{N}\}$ has cardinality bounded above by $B(\nu)$. We may view the sequence $(\gamma_n)_{n=0}^{\infty}$ as a sequence of elements of the product space $\prod_{i=0}^{\infty}\{\gamma_n(i) : n \in \mathbb{N}\}$, which is compact by Tychonoff's theorem since $\{\gamma_n(i) : n \in \mathbb{N}\}$ is a finite (hence, compact) metric space for each i (with the metric being the restriction of d to this set). Therefore, $(\gamma_n)_{n=0}^{\infty}$ has a subsequence which converges to some $\gamma \in \prod_{i=0}^{\infty}\{\gamma_n(i) : n \in \mathbb{N}\}$. Since $\gamma(i) \in \{\gamma_n(i) : n \in \mathbb{N}\}$ for each i, and $d(\gamma_n(i), \gamma_m(i)) \leq 2\nu$ for each m, n, it follows that γ is at bounded Hausdorff distance to each γ_n, so $\gamma \in CGR(x, \eta)$.

□

Corollary 3.4. *There exists a constant B such that for any $x \in G$ and $\eta \in \partial\hat{\Gamma}$, we have $|\Xi(\eta)| \leq B$.*

Proof. We follow the proof of [26, Proposition 5.2]. Let B be the constant from Theorem 3.2. Suppose for contradiction that there exists a subset $\Xi = \{\xi_0, \xi_1, ..., \xi_B\} \subseteq \Xi(\eta)$ of cardinality $B+1$. For each $\xi \in \Xi(\eta)$, choose a CGR $\gamma_\xi = (x^{\xi}_n)_{n \in \mathbb{N}} \in CGR(x, \eta)$ which converges to ξ. By Lemma 2.39(3), for each $\xi, \xi' \in \Xi(\eta)$, there exists $N_{\xi,\xi'} \in \mathbb{N}$ such that $d_{x,\xi}(x^{\xi'}_n) = d_{x,\xi}(x^{\xi}_{N_{\xi,\xi'}})$

for all $n \geq N_{\xi,\xi'}$. Taking the maximum of all these $N_{\xi,\xi'}$ as ξ, ξ' range over Ξ, we have that there exists $N \in \mathbb{N}$ such that $d_{x,\xi}(x_n^{\xi'}) = d_{x,\xi}(x_N^{\xi'})$ for all $n \geq N$ and for all $\xi, \xi' \in \Xi(\eta)$.

We claim that the vertices x_N^{ξ}, for $\xi \in \Xi(\eta)$, are pairwise distinct.

Suppose that $x_N^{\xi} = x_N^{\xi'} =: y$ for some $\xi, \xi' \in \Xi(\eta)$. We will show that $\xi = \xi'$. By Lemma 2.31, it is sufficient to show that $Q(y, \xi) = Q(y, \xi')$. It is sufficient to show $(x_n^{\xi})_{n \geq N} \subseteq Q(y, \xi')$, which will show $Q(y, \xi) \subseteq Q(y, \xi')$ by Lemma 2.30 (the opposite inclusion $Q(y, \xi') \subseteq Q(y, \xi)$ follows symmetrically). Let $n \geq N$. Since $d_{x,\xi}(x_n^{\xi'}) = d_{x,\xi}(x_N^{\xi'})$ for all $n \geq N$, by Lemma 2.39(4), there exists $M \in \mathbb{N}$ such that $d(y, x_n^{\xi}) + d(x_n^{\xi}, x_m^{\xi'}) = d(y, x_m^{\xi'})$ for all $m \geq M$. Therefore, $x_n^{\xi} \in Q(y, \xi')$. Hence $(x_n^{\xi})_{n \geq N} \subseteq Q(y, \xi')$, as desired, and we conclude that $Q(y, \xi) = Q(y, \xi')$ which yields $\xi = \xi'$ by Lemma 2.31.

This immediately yields a contradiction as by Theorem 3.2, $|\{x_N^{\xi} : \xi \in \Xi(\eta)\}| \leq B$. We conclude therefore that $|\Xi(\eta)| \leq B$.

□

Corollary 3.5. *Let $\gamma = (x_n)_{n \in \mathbb{N}} \in CGR(x_0, \eta)$. Then there is some $N \in \mathbb{N}$ such that $(x_n)_{n \geq N}$ is a straight CGR. In particular, $\gamma \setminus \hat{\Gamma}_{s,\eta}$ is finite.*

Proof. We have that $\Xi(\eta)$ is finite by Corollary 3.4, so by Lemma 2.40, there exists some $N \in \mathbb{N}$ such that $(x_n)_{n \geq N}$ is straight. By Lemma 2.34, we have $(x_n)_{n \geq m}$ is straight for each $m \geq N$ and so by Lemma 2.35, we have that x_m is special for each $m \geq N$, hence the second claim holds.

□

The next corollary does not appear in [26].

Corollary 3.6. *For any $g \in G$ and any $\eta \in \partial\hat{\Gamma}$, we have that $Geo(g, \eta)$ is uniformly locally finite (with respect to the metric d), where balls of radius r have cardinality at most $(2(r + 2\nu) + 1)B$, where B is the constant from Theorem 3.2.*

Proof. Let $h \in \text{Geo}(g, \eta)$ and choose $\gamma \in \text{CGR}(g, \eta)$ containing h. Write $h = \gamma(i)$, where $i = d(g, h)$. Let $x \in \text{Geo}(g, \eta)$ be such that $d(h, x) \leq r$. Write $x = \lambda(j)$ for some $\lambda \in \text{CGR}(g, \eta)$ and some $j \in \mathbb{N}$. By the triangle inequality and Theorem 2.18, we have:

$$|i - j| = d(\gamma(i), \gamma(j)) \leq d(\gamma(i), \lambda(j)) + d(\lambda(j), \gamma(j)) \leq r + 2\nu$$

Therefore, $x \in \bigcup_{j \geq 0 : |i-j| \leq r+2\nu} \{\lambda(j) : \lambda \in \text{CGR}(g, \eta)\}$. This shows that the ball of radius r about h in $\text{Geo}(g, \eta)$ is a subset of $\bigcup_{j \geq 0 : |i-j| \leq r+2\nu} \{\lambda(j) : \lambda \in \text{CGR}(g, \eta)\}$, which has cardinality at most $(2(r + 2\nu) + 1)B$ by Theorem 3.2. We conclude that balls of radius r in $\text{Geo}(g, \eta)$ have cardinality at most $(2(r + 2\nu) + 1)B$. Hence, $\text{Geo}(g, \eta)$ is uniformly locally finite.

□

Lemma 3.7. *Let $x \in \hat{\Gamma}_{s,\eta}$ and $y \in G$. Then $Q(x, \xi_{x,\eta}) \setminus Q(y, \xi_{x,\eta})$ is finite.*

Proof. We proceed as in the proof of [26, Lemma 5.4]. Suppose for contradiction that there exists an infinite sequence $(x_n)_{n \in \mathbb{N}} \subseteq Q(x, \xi_{x,\eta}) \setminus Q(y, \xi_{x,\eta})$. Then by Lemma 2.37, we have $x_n \in \hat{\Gamma}_{s,\eta}$ and $\xi_{x_n,\eta} = \xi_{x,\eta}$. For each $n \in \mathbb{N}$, choose a CGR $\gamma_n \in \text{CGR}(x, \eta)$ that converges to $\xi_{x,\eta}$ and passes through x_n. By Corollary 3.3, up to extracting a subsequence, we may assume that $(\gamma_n)_{n \in \mathbb{N}}$ converges to a CGR $\gamma = (y_n)_{n \in \mathbb{N}}$ in $\text{CGR}(x, \eta)$. Since $\gamma_n \to \gamma$, we have that $\gamma \subseteq Q(x, \xi_{x,\eta})$. By Lemma 2.30, this implies $Q(x, \xi_\gamma) \subseteq Q(x, \xi_{x,\eta})$. Therefore, by definition of $\xi_{x,\eta}$, we obtain that $Q(x, \xi_\gamma) = Q(x, \xi_{x,\eta})$, and so $\xi_\gamma = \xi_{x,\eta}$ by Lemma 2.31. By Lemma 2.12, we can find an $N \in \mathbb{N}$ such that $y_N \in Q(y, \xi_\gamma)$ and since $\gamma_n \to \gamma$ and since $x_n \in \gamma_n$ for all n, we can find $M \in \mathbb{N}$ large enough such that $y_N \in \gamma_M \cap \gamma(x, x_M)$ (i.e. y_N is before x_M on γ_M), which yields $x_M \in Q(y_N, \xi_{x,\eta})$. By Lemma 2.29, we have $Q(y_N, \xi_\gamma) \subseteq Q(y, \xi_\gamma)$. Therefore, we obtain $x_M \in Q(y, \xi_{x,\eta})$, contradicting that $(x_n)_{n \in \mathbb{N}} \subseteq Q(x, \xi_{x,\eta}) \setminus Q(y, \xi_{x,\eta})$.

□

Proposition 3.8. *Let $x \in G$ and $\eta \in \partial\hat{\Gamma}$. Then $Geo_1(x, \eta) \subseteq Geo(x, \eta) \cap \hat{\Gamma}_{s,\eta}$ and for any $\gamma \in CGR(x, \eta)$, $\gamma \setminus Geo_1(x, \eta)$ is finite.*

Proof. We proceed as in the proof of [26, Proposition 5.8].

$\mathrm{Geo}_1(x,\eta) \subseteq \mathrm{Geo}(x,\eta)$: Let $y \in \mathrm{Geo}_1(x,\eta)$. Then $y \in Q(h, \xi_{h,\eta})$ for some $h \in Y(x, \xi_{h,\eta})$, so there exists $\gamma \in CGR(h,\eta)$ such that $y \in \gamma$ and γ converges to $\xi_{h,\eta}$. Choosing any geodesic γ_{xy} joining x and h, by Lemma 2.36 we have that the concatenation $\gamma_{xy}\gamma \in CGR(x,\eta)$. Thus, $y \in \mathrm{Geo}(x,\eta)$.

$\mathrm{Geo}_1(x,\eta) \subseteq \hat{\Gamma}_{s,\eta}$: If $y \in \mathrm{Geo}_1(x,\eta)$, then $y \in Q(h,\xi)$ for some $h \in Y(x,\xi)$. But by definition of $Y(x,\xi)$, $h \in \hat{\Gamma}_{s,\eta}$ and $\xi = \xi_{h,\eta}$. By Lemma 2.37, $Q(h,\xi) \subseteq \hat{\Gamma}_{s,\eta}$, so $h \in \hat{\Gamma}_{s,\eta}$.

Now let $\gamma = (x_n)_n \in CGR(x,\eta)$. By Corollary 3.5, there exists $N \in \mathbb{N}$ such that $(x_n)_{n \geq N}$ is straight. By Lemma 2.35, we have $x_N \in \hat{\Gamma}_{s,\eta}$ and $\xi_\gamma = \xi_{x_N,\eta}$. Denote $\xi = \xi_\gamma = \xi_{x_N,\eta}$. By definition of $Q(x_N,\xi)$, we have $(x_n)_{n\geq N} \subseteq Q(x_N,\xi)$. For each $y \in Y(x,\xi)$, we have that $Q(x_N,\xi) \setminus Q(y,\xi)$ is finite by Lemma 3.7, and so $(x_n)_{n\geq N} \setminus \mathrm{Geo}_1(x,\eta)$ is finite, since $(x_n)_{n\geq N} \setminus \mathrm{Geo}_1(x,\eta) \subseteq Q(x_N,\xi) \setminus Q(y,\xi)$. Therefore, $\gamma \setminus \mathrm{Geo}_1(x,\eta)$ is finite.

□

We are now in a position to prove that the modified geodesic ray bundles Geo_1 have finite symmetric difference. Note that $Y(g,\xi)$ is finite in our case because $\mathrm{Geo}(x,\eta)$ is locally finite by Theorem 3.2. Also, by Corollary 3.5, $Y(g,\xi)$ is non-empty, since for any CGR γ in $CGR(x,\eta)$, we can find a special vertex on γ.

Theorem 3.9. *Let $x, y \in G$ and $\eta \in \partial\hat{\Gamma}$. Then $Geo_1(x,\eta) \setminus Geo_1(y,\eta)$ is finite.*

Proof. We follow the proof of [26, Proposition 5.9].

For contradiction, suppose that there exists an infinite sequence $(x_n)_n \subseteq \mathrm{Geo}_1(x,\eta) \setminus \mathrm{Geo}_1(y,\eta)$. Since $\Xi(\eta)$ is finite and $Y(x,\xi)$ is finite for each $\xi \in \Xi(\eta)$, up to taking a subsequence, we may assume that $(x_n)_n \subseteq Q(x', \xi_{x',\eta})$ for some $x' \in \hat{\Gamma}_{s,\eta}$. Then for any $y' \in Y(y, \xi_{x',\eta})$, we have $Q(y', \xi_{x',\eta}) \subseteq \mathrm{Geo}_1(y,\eta)$ by definition of $\mathrm{Geo}_1(y,\eta)$. By Lemma 3.7, we have that $Q(x',\xi_{x',\eta}) \setminus Q(y',\xi_{x',\eta})$ is finite. But $(x_n)_n = (x_n)_n \setminus \mathrm{Geo}_1(y,\eta) \subseteq$

$Q(x', \xi_{x',\eta}) \setminus Q(y', \xi_{x',\eta})$, giving that $(x_n)_n$ is finite, a contradiction. We conclude that $\mathrm{Geo}_1(x,\eta) \setminus \mathrm{Geo}_1(y,\eta)$ is finite.

Note that interchanging the roles of x, y, we also have that $\mathrm{Geo}_1(y,\eta) \setminus \mathrm{Geo}_1(x,\eta)$ is finite. Thus, $\mathrm{Geo}_1(x,\eta) \triangle \mathrm{Geo}_1(y,\eta)$ is finite.

□

Lemma 3.10. *For any $g, x \in G$ and $\eta \in \partial\hat{\Gamma}$ we have $gGeo_1(x,\eta) = Geo_1(gx, g\eta)$.*

Proof. This follows exactly as in the proof of [26, Lemma 5.10]:

Note that G acts by isometries on $\hat{\Gamma}$ (by left translation) and by homeomorphisms on $\mathbb{R}^G$ (via the shift $(gf)(h) = f(g^{-1}h)$ for $f \in \mathbb{R}^G$ and $g, h \in G$), where $\mathbb{R}^G$ has the topology of pointwise convergence.

Let us first verify that the G-action preserves the following sets: $\mathrm{CGR}(x,\eta)$, $\Xi(\eta)$, $Q(x,\xi)$, $\hat{\Gamma}_{s,\eta}$, $\xi_{x,\eta}$, $\mathrm{Geo}(x,\eta)$ and $Y(x,\xi)$.

$g\mathrm{CGR}(x,\eta) = \mathrm{CGR}(gx, g\eta)$: Let $\gamma \in \mathrm{CGR}(x,\eta)$. Then $g\gamma$ is a geodesic ray beginning at gx and pointing in the direction $g\eta$. Therefore, $g\gamma \in \mathrm{CGR}(gx, g\eta)$, showing that $g\mathrm{CGR}(x,\eta) \subseteq \mathrm{CGR}(gx, g\eta)$ for all $g \in G$ and for all $x \in G$, $\eta \in \partial\hat{\Gamma}$. The reverse inclusion follows from the forward inclusion. Indeed, for each $g, x \in G$ and each $\eta \in \partial\hat{\Gamma}$, by above we have $g^{-1}\mathrm{CGR}(gx, g\eta) \subseteq \mathrm{CGR}(g^{-1}gx, g^{-1}g\eta) = \mathrm{CGR}(x,\eta)$. Therefore, $\mathrm{CGR}(gx, g\eta) \subseteq g\mathrm{CGR}(x,\eta)$.

$g\Xi(\eta) = \Xi(g\eta)$: If $\gamma \in \mathrm{CGR}(x,\eta)$ converges to some $\xi \in \Xi(\eta)$, then since G acts by isometries on $\hat{\Gamma}$, for each $g \in G$, we have $g\gamma \in \mathrm{CGR}(gx, g\eta)$ and converges to $g\xi$. Therefore, $g\xi \in \Xi(g\eta)$. This shows that $g\Xi(\eta) \subseteq \Xi(g\eta)$ for each $g \in G$ and each $\eta \in \partial\hat{\Gamma}$. The reverse inclusion follows from the forward inclusion as above.

$gQ(x,\xi) = Q(gx, g\xi)$: If $y \in gQ(x,\xi)$, then $y \in g\gamma$ for some $\gamma \in \mathrm{CGR}(x,\eta)$ converging to ξ. But $g\gamma \in \mathrm{CGR}(gx, g\eta)$ and converges to $g\xi$, so $y \in Q(gx, g\xi)$. The reverse inclusion follows as above, acting by g^{-1}.

$g\hat{\Gamma}_{s,\eta} = \hat{\Gamma}_{s,g\eta}$: If $y \in \hat{\Gamma}_{s,\eta}$, then $\bigcap_{\xi\in\Xi(\eta)} Q(y,\xi)$ contains a CGR γ. Then $g\bigcap_{\xi\in\Xi(\eta)} Q(y,\xi) = \bigcap_{\xi\in\Xi(\eta)} Q(gy, g\xi) = \bigcap_{\xi\in\Xi(g\eta)} Q(gy,\xi)$ contains the CGR $g\gamma$, so $gy \in \hat{\Gamma}_{s,g\eta}$, showing that $g\hat{\Gamma}_{s,\eta} \subseteq \hat{\Gamma}_{s,g\eta}$. The reverse inclusion follows as above, acting by g^{-1}.

$g\xi_{x,\eta} = \xi_{gx,g\eta}$: If x is η-special and $\gamma \in \mathrm{CGR}(x,\eta)$ with $\gamma \subseteq \bigcap_{\xi\in\Xi(\eta)}$, hence converges to $\xi_{x,\eta}$, then $g\xi_{x,\eta}$ is the limit of $g\gamma \in \mathrm{CGR}(gx, g\eta)$. We have $gx \in \hat{\Gamma}_{s,g\eta}$ and by above $g\gamma \in \bigcap_{\xi\in\Xi(g\eta)} Q(gx,\xi)$, so $g\xi_{x,\eta}$, the limit of $g\gamma$, is equal to $\xi_{gx,g\eta}$. Thus, $g\xi_{x,\eta} \subseteq \xi_{gx,g\eta}$. The reverse inclusion follows as above, acting by g^{-1}.

$g\mathrm{Geo}(x,\eta) = \mathrm{Geo}(gx, g\eta)$: If $y \in \mathrm{Geo}(x,\eta)$, then $y \in \gamma$ for some $\gamma \in \mathrm{CGR}(x,\eta)$, so $gy \in g\gamma \in \mathrm{CGR}(gx, g\eta)$. Therefore, $gy \in \mathrm{Geo}(gx, g\eta)$. This shows that $g\mathrm{Geo}(x,\eta) \subseteq \mathrm{Geo}(gx, g\eta)$, which implies the reverse inclusion by acting by g^{-1} as above.

$gY(x,\xi) = Y(gx, g\xi)$: Let $y \in Y(x,\xi)$. Then $y \in \hat{\Gamma}_{s,\eta}$, $\xi_{y,\eta} = \xi$ and $d(x,y)$ is minimal. Then $gy \in \hat{\Gamma}_{s,g\eta}$, $\xi_{gy,g\eta} = g\xi_{y,\eta} = g\xi$, and $d(gx,gy)$ is minimal possible (because otherwise if there was a y' satisfying the same above properties as gy with $d(gx,y') < d(gx,gy)$, then $g^{-1}y$ would satisfy the same above properties as y but would have $d(x, g^{-1}y) < d(x,y)$, a contradiction). Therefore, $gY(x,\xi) \subseteq Y(gx, g\xi)$ and as above this also implies the reverse inclusion.

Finally, $g\mathrm{Geo}_1(x,\eta) = \mathrm{Geo}_1(gx, g\xi)$. Indeed, using our results above:

$$\begin{aligned} g\mathrm{Geo}_1(x,\eta) &= g \bigcup_{\xi\in\Xi(\eta)} \bigcup_{y\in Y(x,\xi)} Q(y,\xi) = \bigcup_{\xi\in\Xi(\eta)} \bigcup_{y\in Y(x,\xi)} gQ(y,\xi) = \bigcup_{\xi\in\Xi(\eta)} \bigcup_{y\in Y(x,\xi)} Q(gy, g\xi) \\ &= \bigcup_{\xi\in g\Xi(\eta)} \bigcup_{y\in gY(x,\xi)} Q(y,\xi) = \bigcup_{\xi\in\Xi(g\eta)} \bigcup_{y\in Y(gx,g\xi)} Q(y,\xi) = \mathrm{Geo}_1(gx, g\eta) \end{aligned}$$

□

4. Hyperfiniteness of the Boundary Action

In this section, we establish the hyperfiniteness of the boundary action of our relatively hyperbolic group G as a consequence of Theorem 3.9. Our arguments follow [26, Section 6]. The main difference here is in our coding of labels of geodesics. Recall that throughout we are working in the relative Cayley graph $\hat{\Gamma}$ with respect to a finite generating set X as in Theorem 3.2.

First, we give a binary coding to the symmetrized generating set $S := (X \cup \mathcal{H})^{\pm}$. Since S is countably infinite, we can fix a bijection $f : S \to 2^{<\mathbb{N}}$ from S to the set $2^{<\mathbb{N}}$ of all infinite binary sequences with only finitely many ones. The label of a geodesic ray is then coded as an element of $(2^{<\mathbb{N}})^{\mathbb{N}}$, which can be thought of as an "infinity by infinity" matrix with binary coefficients, with row i corresponding to the binary string representing s_i for each $i \in \mathbb{N}$ (see the figure below).

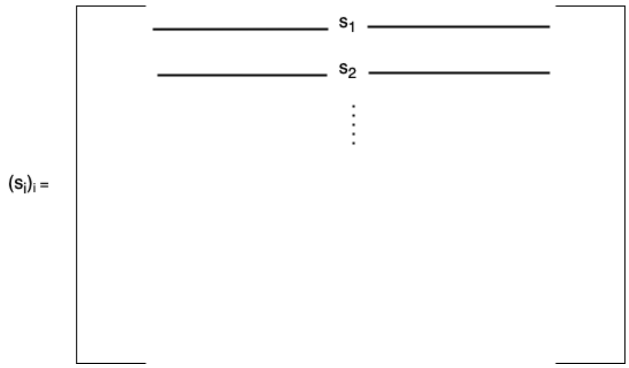

Figure 4.1: A matrix representation of an element of $(2^{<\mathbb{N}})^{\mathbb{N}}$

We can similarly also think of elements of $(2^n)^n$ (i.e. a length n sequence of length n binary sequences) as an $n \times n$ matrix by letting the ith row of the matrix be the ith-entry sequence. We will need to order elements of $(2^n)^n$ for each n.

We will need to order elements of $(2^n)^n$ (i.e. the set of length n sequences of length n binary strings) for each n. Following [17, Section 7], for each $m_1, m_2 \in \mathbb{N}$, each $w = (w_0, w_1, ..., w_{n-1}) \in (2^{m_1})^{m_2}$ and for each $n \in \mathbb{N}$ with $n \leq m_1, m_2$, we put $w|_n = ((w_0)|_n, (w_1)|_n, ..., (w_{n-1})|_n)$, where $(w_j)|_n$ is the restriction of the length m_1 binary sequence w_j to the first n entries. Similarly, if $w \in (2^{\mathbb{N}})^{\mathbb{N}}$, we put $w|_n = ((w_0)|_n, (w_1)|_n, ..., (w_{n-1})|_n)$. If we visualize $w \in (2^n)^n$ as an $n \times n$ matrix, then $w|_i$ is an $i \times i$ submatrix of the $n \times n$ matrix w, starting at the top left corner of w.

For each $n \in \mathbb{N}$, we fix a total order $<_n$ on $(2^n)^n$ as in [17, Section 7] such that for all $w, v \in (2^{n+1})^{n+1}$, $w|_n <_n v|_n \implies w <_{n+1} v$.

Given $\gamma \in CGR(g, \eta)$, we define $\text{lab}(\gamma) \in (2^{<\mathbb{N}})^{\mathbb{N}}$ its coded label. Therefore, according to above, $\text{lab}(\gamma)|_n \in (2^n)^n$ denotes the restricted label.

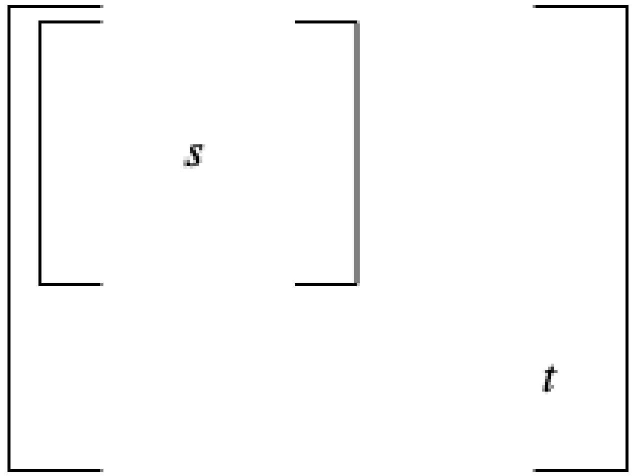

Figure 4.2: The element s, which is a restriction of t, represented as matrices.

Now, analogously to [26, Definition 6.1], we have:

Definition 4.1. *For $\eta \in \partial\hat{\Gamma}$, define:*

$$C^\eta = \{(g, s) \in G \times \bigcup_{m\in\mathbb{N}} (2^m)^m : g \in Geo_1(e, \eta), s = lab(\gamma)|_n \text{ for some } \gamma \in CGR(g, \eta), n \in \mathbb{N}\}$$

Definition 4.2. *An s in $(2^n)^n$ **occurs** in C^η if $(g, s) \in C^\eta$ for some $g \in Geo_1(e, \eta)$. An s in $(2^n)^n$ **occurs infinitely often** in C^η if $(g, s) \in C^\eta$ for infinitely many $g \in Geo_1(e, \eta)$.*

Note that for each $n \in \mathbb{N}$, there exists $s \in (2^n)^n$ which occurs infinitely often in C^η because taking any $\gamma \in CGR(e, \eta)$, by Proposition 3.8, $\gamma \setminus Geo_1(e, \eta)$ is finite, so there exists some N such that for all $k \geq N$, $\gamma(k) \in Geo_1(e, \eta)$. Then $(\gamma(k), \text{lab}((\gamma(i))_{i\geq k})|_n) \in C^\eta$ and $\text{lab}((\gamma(i))_{i\geq k})|_n \in (2^n)^n$ for each $k \geq N$. Since $(2^n)^n$ is finite, by the Pigeonhole Principle, some $s \in (2^n)^n$ must repeat infinitely often in C^η, that is, $(\gamma(k), s) \in C^\eta$ for infinitely many $k \geq N$. For each $n \in \mathbb{N}$, we can therefore choose the minimal (in the order $<_n$ defined above) such $s \in (2^n)^n$ occuring infinitely often in C^η. We shall denote this element by s_n^η.

For each $n \in \mathbb{N}$, note that $(s_{n+1}^\eta)|_n = s_n^\eta$. Indeed, s_{n+1}^η appears infinitely often in C^η, thus so does $(s_{n+1}^\eta)|_n$, so $s_n^\eta <_n (s_{n+1}^\eta)|_n$ or $s_n^\eta = (s_{n+1}^\eta)|_n$. If $s_n^\eta <_n (s_{n+1}^\eta)|_n$, then since there are only finitely many extensions of s_n^η to an element of $(2^{n+1})^{n+1}$ and since s_n^η appears infinitely often in C^η, there would exist $s \in (2^{n+1})^{n+1}$ such that $s|_n = s_n^\eta$ and s appears infinitely often in C^η. Since $s|_n <_n (s_{n+1}^\eta)|_n$, we obtain that $s <_{n+1} s_{n+1}^\eta$, contradicting the minimality of s_{n+1}^η. Therefore, $s_n^\eta = (s_{n+1}^\eta)|_n$.

We now fix a total order $\leq$ on the group G such that $g \leq h \implies d(e,g) \leq d(e,h)$ (for instance, fixing a total order on S, we can define $\leq$ to be lexicographic order on elements of G as words over S, where we choose for each element of G the lexicographically least word over S representing it).

Using the same notation as in [26, Section 6], we have:

Definition 4.3. *For each $n \in \mathbb{N}$ and $\eta \in \partial\hat{\Gamma}$, put $T_n^\eta = \{g \in Geo_1(e,\eta) : (g, s_n^\eta) \in C^\eta\}$ and put $g_n^\eta = \min T_n^\eta$ (where the minimum is with respect to the above total order on G). Put $k_n^\eta = d(e, g_n^\eta)$ for each $n \in \mathbb{N}$.*

Note that the minimum exists because $T_n^\eta \subseteq \mathrm{Geo}(e,\eta)$ and $\mathrm{Geo}(e,\eta)$ is locally finite by Corollary 3.6. By definition of $\leq$ and since $s_n^\eta = (s_{n+1}^\eta)|_n$ for each n, we have that $(T_n^\eta)_n$ is a non-increasing sequence of sets and therefore the sequence $(k_n^\eta)_{n\in\mathbb{N}}$ is a non-decreasing sequence of natural numbers.

We shall now generalize the remaining results of [26, Section 6]. We recall that the topology on G is the discrete topology induced by the relative metric d, the topology on $\partial\hat{\Gamma}$ is the topology having countable neighbourhood base $(V(\eta,m)^g)_{m\in\mathbb{N}}$ for each $\eta \in \partial\hat{\Gamma}$ and basepoint $g \in G$ (see Section 2.3; recall that $\partial\hat{\Gamma}$ is independent of the basepoint g by Remark 2.46 and Lemma 2.12), $G^\mathbb{N}$ has the product topology and $C_{hb}(\hat{\Gamma})$ has the topology of pointwise convergence.

Let us establish a link between the topology of $\partial\hat{\Gamma}$ and sequences of CGRs in $\hat{\Gamma}$. The condition in the following proposition is often used as the definition of the topology on ∂X when X is a proper hyperbolic space, but in general does not give the same topology on ∂X

that we defined in Section 2.3.

Proposition 4.4. *Suppose that $\eta_n \to \eta$ in $\partial\hat{\Gamma}$. Then for any $g \in G$, there exists a sequence of CGRs $(\gamma_n)_n$ such that $\gamma_n \in CGR(g, \eta_n)$ for each n and such that every subsequence of $(\gamma_n)_n$ has a subsequence converging to some CGR $\gamma \in CGR(g, \eta)$.*

Proof. Since $\eta_n \to \eta$, then by definition of the topology on $\partial\hat{\Gamma}$, we have that for each $m \in \mathbb{N}$, there exists a CGR $\gamma_m \in \mathrm{CGR}(g, \eta_m)$ and $\lambda_m \in \mathrm{CGR}(g, \eta)$ such that $d(\gamma_m(t), \lambda_m(t)) \leq 2\nu$ for every $t \leq m$. We can replace these λ_m with a single geodesic $\lambda \in \mathrm{CGR}(g, \eta)$ (such as λ_1) to obtain that $d(\gamma_m(t), \lambda(t)) \leq 4\nu$ for every $t \leq m$ and every m, since λ_m, λ are 2ν close for each m. We claim that every subsequence of $(\gamma_n)_n$ has a convergent subsequence.

First, let us argue as in the proof of Theorem 3.2 to show that for each i, $|\{\gamma_n(i) : n \in \mathbb{N}\}|$ is finite.

Given $i \in \mathbb{N}$, set $k = i + 5\nu + 1$. Then for $n \geq k$, we have $d(\gamma_n(k), \lambda(k)) \leq 4\nu$. Let u denote a geodesic between $\gamma_n(k)$ and $\lambda(k)$ (see Figure 4.3). Then arguing as in the proof of Theorem 3.2, there exists a vertex v on λ with $d_X(\gamma_n(i), v) \leq \nu$.Therefore, $|\{\gamma_n(i) : n \geq k\}|$ is finite, and so $|\{\gamma_n(i) : n \in \mathbb{N}\}|$ is finite. It follows that $\bigcup_n \gamma_n \cup \lambda$ is locally finite.

Since $\bigcup_n \gamma_n \cup \lambda$ is locally finite, arguing as in Corollary 3.3 it follows that every subsequence of $(\gamma_n)_n$ has a convergent subsequence.

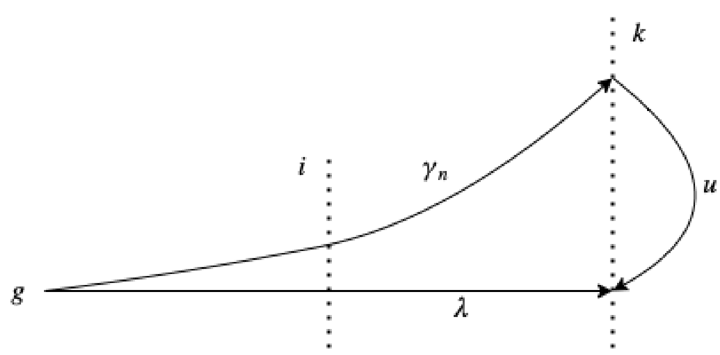

Figure 4.3: The geometry of the geodesics γ_n, λ.

□

We now generalize the claims of [26, Section 6] to relatively hyperbolic groups. We begin by generalizing Claim 1 of [26]. In Claim 1 in [26], the set C below is proved to be compact, while here it is only closed.

Claim 4.5. *The set $C = \{\gamma \in G^{\mathbb{N}} : \gamma$ is a CGR$\}$ is closed. Furthermore, for any $g \in G$ and any $\eta \in \partial\hat{\Gamma}$, the set $CGR(g, \eta) \subseteq G^{\mathbb{N}}$ is compact.*

Proof. Let $(\gamma_n)_n$ be a sequence of elements of C converging (pointwise) to some $\gamma \in G^{\mathbb{N}}$. We claim that γ is a geodesic. Indeed, since $\gamma_n \to \gamma$, for each $m \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\gamma_n|_m = \gamma|_m$. In particular, it follows that $\gamma|_m$ is a geodesic, since $\gamma_n|_m$ is a geodesic for each n. Thus, γ is a geodesic ray based at $\lim_n \gamma_n(0)$ and is hence a CGR, so $\gamma \in C$. Therefore, C is closed.

The "furthermore" statement follows immediately from Corollary 3.3.

□

The next claims are the exact relatively hyperbolic analogues of claims from [26] and their proofs are almost identical (most proofs are completely identical), however, we present all proofs for completeness.

Claim 4.6. *The set $R = \{(\eta, g, \gamma) \in \partial\hat{\Gamma} \times G^{\mathbb{N}} : \gamma \in CGR(g, \eta)\}$ is closed in $\partial\hat{\Gamma} \times G^{\mathbb{N}}$.*

Proof. Suppose that $(\eta_n, g_n, \gamma_n) \in R$ for all n and that $(\eta_n, g_n, \gamma_n) \to (\eta, g, \gamma)$. Then $\eta_n \to \eta \in \partial\hat{\Gamma}$, $g_n \to g$ in G (so that (g_n) is eventually equal to g, by discreteness of G) and $\gamma_n \to \gamma$ in $G^{\mathbb{N}}$, so that $\gamma \in CGR(g, \eta')$ for some $\eta' \in \partial\hat{\Gamma}$ (by Claim 4.5). We will show that $\eta = \eta'$. Since (g_n) is eventually g, we can assume that $g_n = g$ for all n.

As $\eta_n \to \eta$, by Proposition 4.4, there exists a sequence $(\gamma'_n)_n$ with $\gamma'_n \in CGR(g, \eta_n)$ which has a subsequence $(\gamma'_{n_k})_k$ that converges to some $\gamma' \in CGR(g, \eta)$.

We have $\gamma_{n_k}, \gamma'_{n_k} \in CGR(g, \eta_{n_k})$ for every k. so $d(\gamma_{n_k}(m), \gamma'_{n_k}(m)) \leq 2\nu$ for each m. Taking $k \to \infty$, we obtain that $d(\gamma(m), \gamma'(m)) \leq 2\nu$ for all m, and therefore that $\eta = \eta'$.

Thus, $(\eta_n, g_n, \gamma_n) \to (\eta, g, \gamma)$ with $\gamma \in CGR(g, \eta)$, so $(\eta, g, \gamma) \in R$ and so R is closed.

□

Claim 4.7. *The set $F = \{(\eta, g, (\gamma(0), \gamma(1)..., \gamma(n))) \in \partial\hat{\Gamma} \times G \times G^{<\mathbb{N}} : \gamma \in CGR(g, \eta)\}$ is Borel in $\partial\hat{\Gamma} \times G \times G^{<\mathbb{N}}$.*

Proof. Let $F' = \{(\eta, g, (\gamma(0), \gamma(1), ..., \gamma(n)), \gamma') \in \partial\hat{\Gamma} \times G \times G^{<\mathbb{N}} \times G^{\mathbb{N}} : (\eta, g, \gamma') \in R$ and $\gamma'(i) = \gamma(i)$ for each $0 \leq i \leq n\}$. By Claim 4.6, F' is closed in $\partial\hat{\Gamma} \times G \times G^{<\mathbb{N}} \times G^{\mathbb{N}}$. Note that F is the projection of F' to the first 3 coordinates. Note also that the section $F'_{(\eta,g,(\gamma(0),\gamma(1),...,\gamma(n)))}$ is compact for every $(\eta, g, (\gamma(0), \gamma(1), ..., \gamma(n))) \in \partial\hat{\Gamma} \times G^{<\mathbb{N}}$. Indeed, $F'_{(\eta,g,(\gamma(0),\gamma(1),...,\gamma(n)))} = \{\gamma' \in CGR(g, \eta) : \gamma'(i) = \gamma(i)$ for all $0 \leq i \leq n\}$. This is a closed subset of the compact set $CGR(g, \eta)$, hence it is compact. By Theorem 2.75, it follows that F is Borel in $\partial\hat{\Gamma} \times G \times G^{<\mathbb{N}}$. □

Claim 4.8. *The set $M = \{(\eta, \xi) \in \partial\hat{\Gamma} \times C_{hb}(\hat{\Gamma}) : \xi \in \Xi(\eta)\}$ is Borel in $\partial\hat{\Gamma} \times C_{hb}(\hat{\Gamma})$.*

Proof. We follow a similar proof to the proof of Claim 4 in [26].

We will show that M is both analytic and coanalytic, hence Borel by [24, Theorem 14.11].

By definition of $\Xi(\eta)$, we have that $(\eta, \xi) \in M$ if and only if

$$\exists \gamma \in G^{\mathbb{N}} : (\eta, \gamma(0), \gamma) \in R \text{ and } \xi_\gamma = \xi$$

We also have that

$$\xi_\gamma = \xi \iff \forall g \in G, \exists n \in \mathbb{N}\ \forall m \geq n\ f_{\gamma(m)}(g) = \xi(g)$$

which gives a Borel definition of the set $\{(\xi,\gamma) \in C_{hb}(\hat{\Gamma}) \times C : \xi_\gamma = \xi\}$. Thus, since R is closed, we have that M is analytic.

To show that M is coanalytic, we will show that:

$(\eta,\xi) \in M \iff \forall \lambda \in G^{\mathbb{N}}$ if $(\eta, e, \lambda) \in R,$ then $\forall k \in \mathbb{N}, \exists \gamma^k \in G^{k+1}$ a geodesic path with $\gamma^k(0) = e$ such that $\gamma^k \subseteq N_{2\nu}(\lambda)$ and such that $\forall g \in G, \exists n_g \in \mathbb{N}$ such that $\forall i, j > n_g$, $f_{\gamma^j(i)}(g) = \xi(g)$

This formula defines a coanalytic set since there is a single universal quantifier $\forall$ ranging over an uncountable standard Borel space $G^{\mathbb{N}}$.

For the forward direction, if $(\eta,\xi) \in M$, then there exists $\gamma \in CGR(e,\eta)$ converging to ξ. We simply take $\gamma^k = \gamma|_k$ (the restriction of Γ from 0 to k) for each $k \in \mathbb{N}$. Then for each $\lambda \in CGR(e,\eta)$, we have $d(\gamma(n),\lambda(n)) \leq 2\nu$ for each $n \in \mathbb{N}$, so $\gamma|_k \subseteq N_{2\nu}(\lambda)$ for each k. Furthermore, since γ converges to ξ, we have that for all $\forall g \in G$, there exists n_g such that for all $i, j > n_g$, $i \leq j$, we have $f_{\gamma|_j(i)}(g) = \xi(g)$.

For the reverse direction, let $\lambda \in CGR(e,\eta)$. Then by assumption, there exists a sequence $\gamma^k \in G^{k+1}$ of geodesic paths starting at e, each contained in $N_{2\nu}(\lambda)$ and such that $f_{\gamma^i(j)}(g) \to \xi(g)$. For each i, fix $k = i + 2\nu + 1$ and using Theorem 2.18, choose an N sufficiently large such that for all $n \geq N$, we have

$$d(\gamma_n(t),\lambda(t)) \leq 2\nu$$

for all $t \leq k$. Arguing as in the proof of Theorem 2.57, we have that$\{\gamma^j(i) : j \geq N\}$ is finite, so that $\{\gamma^j(i) : j \in \mathbb{N}\}$ is finite for each i. Therefore, arguing as in the proof of Corollary 3.3, $(\gamma^k)_k$ has a subsequence converging to some CGR γ based at e, and $\gamma \subseteq N_{2\nu}(\lambda)$, so $\gamma \in CGR(e,\eta)$. From $f_{\gamma^{i(j)}}(g) \to \xi(g)$ as $i,j \to \infty$, we have that $\xi_\gamma = \xi$. Since $\gamma \in CGR(e,\eta)$, we conclude that $(\eta,\xi) \in M$.

□

By Corollary 3.4, for each $\eta \in \partial\hat{\Gamma}$, we have that the section $M_\eta = \Xi(\eta)$ is finite, having cardinality bounded above by the constant $B = B(\nu)$ from Theorem 3.2. Since M is Borel and has finite sections of size at most B, by Theorem 2.74, we have Borel functions $\xi_1, ..., \xi_B : \partial\hat{\Gamma} \to C_{hb}(\hat{\Gamma})$ such that M is the union of the graphs $G_{\xi_i} = \{(\eta, \xi_i(\eta)) : \eta \in \partial\hat{\Gamma}\}$ of the ξ_i.

Claim 4.9. *For each $i = 1, ..., B$, $Q_i = \{(\eta, g, h) \in \partial\hat{\Gamma} \times G^2 : h \in Q(g, \xi_i(\eta))\}$ is Borel in $\partial\hat{\Gamma} \times G^2$.*

Proof. By [26, Lemma 4.2], we have $Q(g,\xi_i(\eta)) = \bigcup_{n\in\mathbb{N}} \gamma(g, x_n)$ for some, equivalently any, CGR $(x_n)_n \in \mathrm{CGR}(g,\eta)$ converging to $\xi_i(\eta)$.

From this, we obtain that $h \in Q(g, \xi_i(\eta)) \iff \exists\lambda \in C$ (resp. $\forall\lambda \in C$) with $\lambda(0) = g$ and $\xi_\lambda = \xi_i(\eta)$ and $\exists n \in \mathbb{N}$ such that $h \in \gamma(g, \lambda(n))$. This yields the analyticity (from the $\exists$ above) and coanalyticity (from the $\forall$ above) of Q_i, hence Borelness of Q_i.

□

Claim 4.10. *The set $P = \{(\eta, h) \in \partial\hat{\Gamma} \times G : h \in \hat{\Gamma}_{s,\eta}\}$ is Borel in $\partial\hat{\Gamma} \times G$.*

Proof. We have that $h \in \hat{\Gamma}_{s,\eta}$ if and only if:

$$\forall n \in \mathbb{N}, \exists \gamma^n \in G^{n+1} : (\eta, h, \gamma^n) \in F \text{ and } \forall i \leq B \; \forall k < n, (\eta, h, \gamma^n(k)) \in Q_i$$

Indeed, if $h \in \hat{\Gamma}_{s,\eta}$, then $\bigcap_{\xi\in\Xi(\eta)} Q(h,\xi)$ contains a CGR $\gamma \in \mathrm{CGR}(h,\eta)$, so we can take the restriction $\gamma^n = \gamma|_n$ for all $n \in \mathbb{N}$ and for all $k < n$, $\gamma^n(k) = \gamma(k)$ to satisfy the above condition.

Conversely, if the above condition holds, then by local finiteness of $\mathrm{Geo}(h,\eta)$ (Corollary 3.6), the sequence $(\gamma^n)_{n\in\mathbb{N}}$ with $(\eta, h, \gamma^n) \in F$ will have a subsequence converging to some

$\gamma \in \mathrm{CGR}(h,\eta)$ and the above condition yields that $\gamma \subseteq \bigcap_{\xi\in\Xi(\eta)} Q(h,\xi)$, so that $h \in \hat{\Gamma}_{s,\eta}$.

Then, because F and Q_i are Borel, we have that P is Borel.

□

Claim 4.11. *The set $P_1 = \{(\xi,\eta,h) \in C_{hb}(\hat{\Gamma}) \times \partial\hat{\Gamma} \times G : h \in \hat{\Gamma}_{s,\eta}$ and $\xi = \xi_{h,\eta}\}$ is Borel in $C_{hb}(\hat{\Gamma}) \times \partial\hat{\Gamma} \times G$.*

Proof. We have $(\xi,\eta,h) \in P_1$ if and only if $(\eta,h) \in P$ and $\exists i \leq B$ such that $(\eta,\xi) \in G_{\xi_i}$ and $\forall j \leq B$, $Q(h,\xi_i(\eta)) \subseteq Q(h,\xi_j(\eta))$. Since P is Borel (Claim 4.10), G_{ξ_i} is Borel (as ξ_i is Borel), and Q_i is Borel (Claim 4.9), the above yields that P_1 is Borel.

□

Claim 4.12. *The set $L = \{(h,\xi,\eta) \in G \times C_{hb}(\hat{\Gamma}) \times \partial\hat{\Gamma} : h \in Y(e,\xi), \xi \in \Xi(\eta)\}$ is Borel in $G \times C_{hb}(\hat{\Gamma}) \times \partial\hat{\Gamma}$.*

Proof. We have that $(h,\xi,\eta) \in L$ if and only if $(\eta,\xi) \in M$ and h is the closest element to e (in the metric d) such that $h \in \mathrm{Geo}(e,\eta)$ and $(\xi,\eta,h) \in P_1$. Thus, by Claims 4.8, 4.9, 4.11, L is Borel (note that $h \in \mathrm{Geo}(e,\eta) \iff (\eta,e,h) \in Q_i$ for some $i \leq B$, so $\{(h,\eta) \in G \times \partial\hat{\Gamma} : h \in \mathrm{Geo}(e,\eta)\}$ is Borel in $G \times \partial\hat{\Gamma}$ by Claim 4.9).

□

Claim 4.13. *The set $B = \{(g,h,\xi,\eta) \in G^2 \times C_{hb}(\hat{\Gamma}) \times \partial\hat{\Gamma} : g \in Q(h,\xi), h \in Y(e,\xi), \xi \in \Xi(\eta)\}$ is Borel in $G^2 \times C_{hb}(\hat{\Gamma}) \times \partial\hat{\Gamma}$.*

Proof. We have that $(g,h,\xi,\eta) \in B$ if and only if $\exists i \leq r$ such that $\xi = \xi_i(\eta)$ and $(\eta,h,g) \in Q_i$ and $(h,\xi,\eta) \in L$. Since L, Q_i and ξ_i are Borel, it follows that B is Borel.

□

Claim 4.14. *The set $A = \{(\eta,g) \in \partial\hat{\Gamma} \times G : g \in \mathrm{Geo}_1(e,\eta)\}$ is Borel in $\partial\hat{\Gamma} \times G$.*

Proof. Since $\mathrm{Geo}_1(e,\eta) = \bigcup_{\xi\in\Xi(\eta)} \bigcup_{h\in Y(e,\xi)} Q(h,\xi)$, we have $(\eta,g) \in A$ if and only if $\exists \xi \in \Xi(\eta), \exists h \in Y(e,\xi) : g \in Q(h,\xi)$ if and only if $\exists \xi \in \Xi(\eta), \exists h \in Y(e,\xi) : (g,h,\xi,\eta) \in B$. Thus,

A is the projection $(g, h, \xi, \eta) \mapsto (\eta, g)$ of B onto $\partial\hat{\Gamma} \times G$. By Claim 4.13, B is Borel. Also, the sections $\{(h, \xi) \in G \times C_{hb}(\hat{\Gamma}) : h \in Y(e, \xi), \xi \in \Xi(\eta)\}$ of B are finite by Theorem 3.2 and Corollary 3.4. Therefore, by Theorem 2.74, A is Borel.

□

Claim 4.15. *The set $D = \{(\eta, (\gamma(0), \gamma(1), ..., \gamma(n))) \in \partial\hat{\Gamma} \times G^{<\mathbb{N}} : \gamma(0) \in Geo_1(e, \eta)$ and $\gamma \in CGR(\gamma(0), \eta)\}$ is Borel in $\partial\hat{\Gamma} \times G^{<\mathbb{N}}$.*

Proof. We have that $(\eta, (\gamma(0), \gamma(1), ..., \gamma(n))) \in D$ if and only if $(\eta, \gamma(0), (\gamma(0), \gamma(1), ..., \gamma(n))) \in F$ and $(\eta, \gamma(0)) \in A$. By Claim 4.14, A is Borel in $\partial\hat{\Gamma} \times G$. Also, F is Borel by Claim 4.7. Therefore, D is Borel.

□

Claim 4.16. *For each n, the set $S_n := \{(\eta, s^n) \in \partial\hat{\Gamma} \times (2^n)^n : s^n = s^\eta_n\}$ is Borel in $\partial\hat{\Gamma} \times (2^n)^n$.*

Proof. We have that $(\eta, s^n) \in S_n$ if and only if s^n is the $<_n$-minimal element in $(2^n)^n$ for which the following holds:

$$\forall m \in \mathbb{N}, \exists(\gamma(0), \gamma(1), ..., \gamma(n)) \in G^{n+1} : d(\gamma(0), e) \geq m, (\eta, (\gamma(0), \gamma(1), ..., \gamma(n))) \in D \text{ and } \mathrm{lab}(\gamma)|_n = s^n$$

The "only if" holds by Corollary 3.6. Thus, S_n is Borel by Claim 4.15.

□

Now let E denote the orbit equivalence relation of the action of G on $\partial\hat{\Gamma}$.

Definition 4.17. *Let $Z = \{\eta \in \partial\hat{\Gamma} : k^\eta_n \nrightarrow \infty\}$.*

We can characterize elements $\eta \in Z$ in terms of geodesic rays having label s^η:

Proposition 4.18. *$Z = \{\eta \in \partial\hat{\Gamma} : \exists g \in Geo_1(e, \eta)$ and $\exists \gamma \in CGR(g, \eta) : lab(\gamma) = s^\eta\}$*

Proof. Suppose $\eta \in Z$. Then (k^η_n) converges, so $\exists N$ and $\exists r \in \mathbb{N}$ such that for all $n \geq N$, $k^\eta_n = r$. Therefore, $d(e, g^\eta_n) = r$ for all $n \geq N$. Since $g^\eta_n \in \mathrm{Geo}_1(e, \eta)$ for all n and since

$\mathrm{Geo}_1(e, \eta)$ is locally finite (by Corollary 3.6), we have that $\exists g_\eta \in \mathrm{Geo}_1(e, \eta)$ such that g_η equals g_n^η for infinitely many n. Therefore, $g_\eta \in T_n^\eta$ for infinitely many n, and so since $(T_n^\eta)_n$ is non-increasing, we have that $g \in T_n^\eta$ for all n. Therefore, for each n, there exists a $\gamma_n \in \mathrm{CGR}(g_\eta, \eta)$ with $\mathrm{lab}(\gamma_n)|_n = s_n^\eta$. By Corollary 3.3, the sequence $(\gamma_n)_n$ has a subsequence converging to some $\gamma^\eta \in \mathrm{CGR}(g_\eta, \eta)$. Since $(s_{n+1}^\eta)|_n = s_n^\eta$ for all n, we have that $\mathrm{lab}(\gamma) = s^\eta$.

Conversely, suppose $\exists g \in \mathrm{Geo}_1(e, \eta)$ and $\exists \gamma \in \mathrm{CGR}(g, \eta)$ such that $\mathrm{lab}(\gamma) = s^\eta$. Then $g \in T_n^\eta$ for all $n \in \mathbb{N}$, so $k_n^\eta \leq d(e, g)$ and therefore $\eta \in Z$.

□

Lemma 4.19. *The map $\alpha : (Z, E|_Z) \to (\partial\hat{\Gamma}, =)$ given by $\eta \mapsto g_\eta^{-1}\eta$ is a Borel reduction.*

Proof. We argue as in [26]. First, let us show that $s_n^\eta = s_n^{g\eta}$ for each $g \in G$, each $\eta \in \partial\hat{\Gamma}$ and each $n \in \mathbb{N}$.

If there are infinitely many couples $(h, s_n^\eta) \in C^\eta$, then since the left action of G on $\hat{\Gamma}$ preserves labels of geodesics, there are infinitely many couples (gh, s_n^η), where $s_n^\eta = \mathrm{lab}(\gamma)|_n$ for some $\gamma \in \mathrm{CGR}(\gamma(0), g\eta)$ and where $\gamma(0) \in g\mathrm{Geo}_1(e, \eta) = \mathrm{Geo}_1(g, g\eta)$ (using Lemma 3.10 in the last line).

By Theorem 3.9, the symmetric difference between $\mathrm{Geo}_1(g, g\eta)$ and $\mathrm{Geo}_1(e, g\eta)$ is finite and so there are infinitely many couples $(gh, s_n^\eta) \in \mathrm{Geo}_1(e, g\eta)$. Hence, there are infinitely many couples $(gh, s_n^\eta) \in C^{g\eta}$. Thus, as s_n^η is least in the order $<_n$ that appears infinitely often in C^η, we have that $s_n^\eta = s_n^{g\eta}$. As $s_n^\eta = s_n^{g\eta}$ for each n, we have $s^\eta = s^{g\eta}$.

This implies that α is constant on G-orbits. Indeed, suppose $\theta = g\eta$ for some $g \in G$, $\eta, \theta \in Z$. We have that α maps the boundary point $[\gamma^\theta]$ to the boundary point $[g_\theta^{-1}\gamma^\theta]$. Note that $g_\theta^{-1}\gamma^\theta \in \mathrm{CGR}(e, g_\theta^{-1}\theta)$ and $\mathrm{lab}(g_\theta^{-1}\gamma^\theta) = s^\theta$, because γ^θ has label s^θ and left multiplication preserves labels of geodesics.

On the other hand, α maps $\eta = [\gamma^\eta]$ to $g_\eta^{-1}\eta = [g_\eta^{-1}\gamma^\eta]$. We have that $g_\eta^{-1}\gamma^\eta \in \mathrm{CGR}(e, g_\eta^{-1}\eta)$ and $\mathrm{lab}(g_\eta^{-1}\gamma^\eta) = s^\eta$.

But by above, $s^\eta = s^{g\eta} = s^\theta$. Therefore, $g_\eta^{-1}\gamma^\eta$ and $g_\theta^{-1}\gamma^\theta$ both start at e and have the same label. Therefore, they are the same geodesic. Hence, $g_\theta^{-1}\theta = g_\eta^{-1}\eta$ i.e. $\alpha(\theta) = \alpha(\eta)$.

It follows that α is reduction to $=$ on $\partial\hat{\Gamma}$. Indeed, the above shows that $\theta E \eta \implies \alpha(\theta) = \alpha(\eta)$. Conversely, if $\alpha(\theta) = \alpha(\eta)$, then $g_\eta^{-1}\eta = g_\theta^{-1}\theta$, so $\theta = g_\theta g_\eta^{-1}\eta$, so $\theta E \eta$.

It remains to show that α is Borel. To show this, let us first show that the set $U := \{(\eta, s) \in Z \times (2^{\mathbb{N}})^{\mathbb{N}} : s = s^\eta\}$ is Borel. We have $s = s^\eta$ if and only if $(\eta, s|_n) \in S_n$ for each $n \in \mathbb{N}$, so $\{(\eta, s) \in Z \times (2^{\mathbb{N}})^{\mathbb{N}} : s = s^\eta\}$ is Borel by Claim 4.16 (note that the map $(\eta, s) \mapsto (\eta, s|_n)$ is continuous, hence Borel, for each $n \in \mathbb{N}$).

Now the Borelness of U implies the Borelness of the graph of α. Indeed, note that for $\eta \in Z$ and $\theta \in \partial\hat{\Gamma}$, we have $\theta = g_\eta^{-1}\eta \iff \exists \gamma \in C : \gamma \in CGR(e, \theta)$ and $\mathrm{lab}(\gamma) = s^\eta \iff \exists \gamma \in C : (\theta, e, \gamma) \in R$ and $(\eta, \gamma) \in \mathrm{Lab}^{-1}(U)$, where $\mathrm{Lab} : Z \times C \to Z \times (2^{\mathbb{N}})^{\mathbb{N}}$ is the continuous map $(\eta, \gamma) \mapsto (\eta, \mathrm{lab}(\gamma))$. Putting $T = \{(\eta, \theta, \gamma) \in Z \times \partial\hat{\Gamma} \times C : (\theta, e, \gamma) \in R$ and $(\eta, \gamma) \in \mathrm{lab}^{-1}(U)\}$, we have that T is Borel because R and U are Borel (see Claim 4.6 for the Borelness of R). By above, the graph of α is the projection $proj_{Z\times\partial\hat{\Gamma}}(T)$ of T onto the first two coordinates (η, θ). For each $(\eta, \theta) \in Z \times \partial\hat{\Gamma}$, the section $T_{(\eta,\theta)} = \{\gamma \in C : (\eta, \theta, \gamma) \in T\} = \{\gamma \in C : \gamma \in CGR(e, \theta)$ and $\mathrm{lab}(\gamma) = s^\eta\}$ is finite, being either a singleton or the empty set (because a geodesic ray is uniquely determined by its basepoint and label). Therefore, by Theorem 2.75, we have that $proj_{Z\times\partial\hat{\Gamma}}(T)$ is Borel. Thus, the graph of α is Borel, so α is Borel by Theorem 2.76.

□

Lemma 4.20. *E is smooth on the saturation $[Z]_E = \{\eta \in \partial\hat{\Gamma} : \exists \theta \in Z$ such that $\theta E \eta\}$.*

Proof. By Lemma 4.19, E is smooth on Z, hence it is smooth on its saturation. □

Definition 4.21. *The **shift action** of G on $2^G \cong \mathcal{P}(G)$ is the action $g \cdot A := gA = \{ga : a \in A\}$ for each $g \in G$ and $A \subseteq G$,*

Definition 4.22. *Let $Y = \partial\hat{\Gamma} \setminus [Z]_E$. For each $n \in \mathbb{N}$, define $H_n : \partial\hat{\Gamma} \to 2^G$ by $H_n(\eta) = (g_n^\eta)^{-1}T_n^\eta$. Let F_n be the restriction of the orbit equivalence relation of the shift action of G on 2^G to imH_n.*

The following lemma is a generalization of [26, Lemma 6.7].

Lemma 4.23. *There exists a constant K such that for each $n \in \mathbb{N}$, each equivalence class of F_n has size at most K.*

Proof. By Corollary 3.6, each closed ball of radius r has cardinality at most $(2(r+2\nu)+1)B$, where B is the constant from Theorem 3.2. We will show that we can take $K = (20\nu+1)B$.

Let $\eta, \theta \in Y$ and suppose that $H_n(\eta) = gH_n(\theta)$. By the proof of [26, Lemma 6.7] (which only relies on the hyperbolicity of the Cayley graph and local finiteness of geodesic ray bundles and so holds in our context when applied to $\hat{\Gamma}$), we have $d(e, g) \leq 8\nu$. For completeness, let us reproduce this proof.

By defiinition, T_n^η (resp. T_n^θ) is an infinite subset of $\mathrm{Geo}(e, \eta)$ (resp. $\mathrm{Geo}(e, \theta)$). Since $\mathrm{Geo}(e, \eta)$ is locally finite, this means that T_n^η (resp. T_n^θ) uniquely determines η (resp. θ). From $H_n(\eta) = gH_n(\theta)$, we have $(g_n^\eta)^{-1}T_n^\eta = g(g_n^\theta)^{-1}T_n^\theta$ and since T_n^η and T_n^θ determine their boundary points, this implies that $\sigma := (g_n^\eta)^{-1}\eta = g(g_n^\theta)^{-1}\theta$.

We have that $g, e \in (g_n^\eta)^{-1}T_n^\eta = g(g_n^\theta)^{-1}T_n^\theta \subseteq \mathrm{Geo}(g(g_n^\theta)^{-1}, \sigma)$, so there exists $\lambda \in \mathrm{CGR}(g(g_n^\theta)^{-1}, \sigma)$ passing through g and $\lambda' \in \mathrm{CGR}(g(g_n^\theta)^{-1}, \sigma)$ passing through e. Write $g = \lambda(m_1)$ and $e = \lambda'(m_2)$ for some $m_1, m_2 \in \mathbb{N}$. Note that by Theorem 2.18, we have $d(e, \lambda(m_2)) \leq 2\nu$. Also, we have $m_2 \geq m_1$. Indeed, since $g_n^\theta g^{-1} \in g_n^\theta g^{-1}(g_n^\eta)^{-1}T_n^\eta = T_n^\theta$, we have:

$$m_2 = d(e, g(g_n^\theta)^{-1}) = d(e, g_n^\theta g^{-1}) \geq d(e, g_n^\theta) = d(e, (g_n^\theta)^{-1}) = d(g, g(g_n^\theta)^{-1}) = m_1$$

Where $d(e, g_n^\theta g^{-1}) \geq d(e, g_n^\theta)$ by $\leq$ minimality of g_n^θ in T_n^θ.

Similarly, from $g, e \in (g_n^\eta)^{-1}T_n^\eta$, we have $g, e \in \text{Geo}((g_n^\eta)^{-1}, \sigma)$, and so there exists $\gamma \in \text{CGR}((g_n^\theta)^{-1}, \sigma)$ passing through g and $\gamma' \in \text{CGR}((g_n^\theta)^{-1}, \sigma)$ passing through e. Write $g = \gamma(m_3)$ and $e = \gamma'(m_4)$ for some $m_3, m_4 \in \mathbb{N}$. By Theorem 2.18, we have $d(e, \gamma(m_4)) \leq 2\nu$ and $m_4 \leq m_3$ because $g_n^\eta g \in g_n^\eta g(g_n^\theta)^{-1}T_n^\theta = T_n^\eta$ and so by the $\leq$-minimality of g_n^η in T_n^η, we have that:

$$m_3 = d((g_n^\eta)^{-1}, g) = d(e, g_n^\eta g) \geq d(e, g_n^\eta) = d(e, (g_n^\eta)^{-1}) = m_4$$

Let us now consider the sub-CGRs of λ and γ starting at g. Using Theorem 2.18, since $m_2 \geq m_1$, there exists $m_5 \geq m_3$ such that $d(\lambda(m_2), \gamma(m_5)) \leq 2\nu$. Then by the triangle inequality and our above estimates, we have:

$$d(\gamma(m_4), \gamma(m_5)) \leq d(\gamma(m_4), e) + d(e, \lambda(m_2)) + d(\lambda(m_2), \gamma(m_5)) \leq 6\nu$$

Therefore,

$$d(e, g) = d(e, \gamma(m_3)) \leq d(e, \gamma(m_4)) + d(\gamma(m_4), \gamma(m_3)) \leq 2\nu + d(\gamma(m_4), \gamma(m_5)) \leq 8\nu$$

where we have $d(\gamma(m_4), \gamma(m_3)) \leq d(\gamma(m_4), \gamma(m_5))$ because $m_5 \geq m_3 \geq m_4$. where we have $d(\gamma(m_4), \gamma(m_3)) \leq d(\gamma(m_4), \gamma(m_5))$ because $m_5 \geq m_3 \geq m_4$.

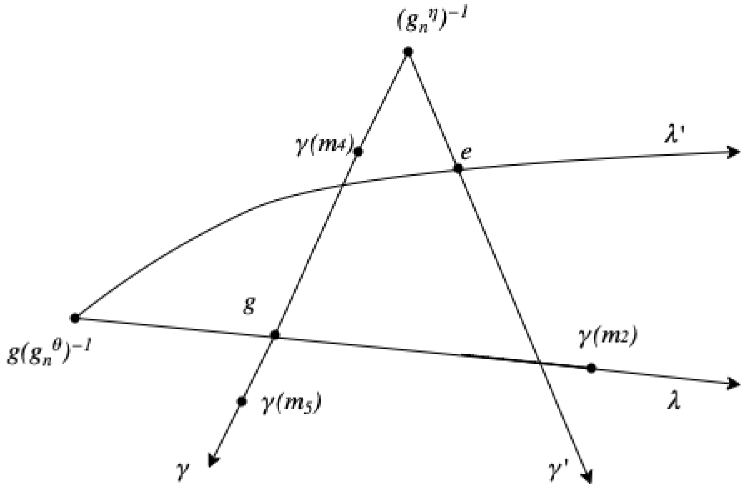

Figure 4.4: The geometry of the proof of Lemma 4.23

Thus, $H_n(\eta) = gH_n(\theta)$ implies that g is in the ball of radius 8ν about e in $\text{Geo}((g^n_\eta)^{-1}, \sigma)$, which has cardinality at most $(2(8\nu + 2\nu) + 1)B = (20\nu + 1)B = K$. Thus, F_n-classes have cardinality at most K. □

Claim 4.24. *For each $n \in \mathbb{N}$, the map $\phi : \partial\hat{\Gamma} \to G$ given by $\phi(\eta) = g^\eta_n$ is Borel.*

Proof. For each $g \in G$, we have $\phi^{-1}(g) = \{\eta \in \partial\hat{\Gamma} : g = g^\eta_n\}$. We have $g = g^\eta_n$ if and only if g is the least element in T^η_n. Now,

$$
\begin{aligned}
g \in T^\eta_n &\iff (g, s^\eta_n) \in C^\eta \\
&\iff \exists(\gamma(0), \gamma(1), ..., \gamma(n)) \in G^{n+1} : \gamma(0) = g \text{ and } (\eta, (\gamma(0), \gamma(1), ..., \gamma(n))) \in D \\
&\text{and } (\eta, \text{lab}((\gamma(0), \gamma(1), ..., \gamma(n))|_n) \in S_n
\end{aligned}
$$

Since G^{n+1} is countable, since D, S_n are Borel and since the label map $\partial\hat{\Gamma} \times G^{n+1} \to \partial\hat{\Gamma} \times (2^n)^n$ given by $(\eta, (\gamma(0), ..., \gamma(n))) \mapsto (\eta, \text{lab}(\gamma(0), ..., \gamma(n))|_n)$ is continuous (hence, Borel), we have that the set of all η which satisfy the latter condition is Borel. Therefore, $\phi^{-1}(g)$ is Borel for all $g \in G$. Therefore, since G is discrete, it follows that ϕ is Borel. □

Claim 4.25. *For each $n \in \mathbb{N}$, the map $\psi : \partial\hat{\Gamma} \to 2^G$ given by $\psi(\eta) = T_n^\eta$ is Borel.*

Proof. Note that 2^G has countable clopen basis for its topology consisting of the sets $U_g = \{A \subseteq G : g \in A\}$ for each $g \in G$. Thus, it suffices to show that $\psi^{-1}(U_g) = \{\eta \in \partial\hat{\Gamma} : g \in T_n^\eta\}$ is Borel for each $g \in G$.

From the proof of Claim 4.24, we have:

$$g \in T_n^\eta \iff \exists(\gamma(0), \gamma(1), ..., \gamma(n)) \in G^{n+1} : \gamma(0) = g \text{ and } (\eta, (\gamma(0), \gamma(1), ..., \gamma(n))) \in D$$
$$\text{and } (\eta, \mathrm{lab}((\gamma(0), \gamma(1), ..., \gamma(n))|_n) \in S_n$$

As noted in the proof of Claim 4.24, the set of all η which satisfy the latter condition is Borel, since G^{n+1} is countable and the sets S_n and D are Borel. Therefore, $\psi^{-1}(U_g)$ is Borel for every $g \in G$, and so ψ is Borel.

□

The following remaining results have the same proof as in [26].

Lemma 4.26. *Let $n \in \mathbb{N}$. Then the map H_n is Borel and so imH_n is analytic.*

Proof. By Claims 4.24 and 4.25, the maps $\phi : \eta \mapsto g_n^\eta$ and $\psi : \eta \mapsto T_n^\eta$ are Borel for each n.

Next, since multiplication $m : G \times 2^G \ni (g, \mathcal{S}) \mapsto g\mathcal{S}$ and inversion $\iota : g \mapsto g^{-1}$ are continuous (hence, Borel) and since H_n is the composition $H_n(\eta) = m(\phi(\eta)^{-1}, \psi(\eta))$, it follows that H_n is Borel.

Since H_n is Borel, we conclude that imH_n is analytic. □

By Lemma 4.23, the equivalence classes of F_n have size at most K, so applying Lemma 2.77 to F_n for each n, there exists a finite Borel equivalence relation F_n' on 2^G with $F_n \subseteq F_n'$. Since F_n' is finite Borel, there exists a Borel reduction $f_n : 2^G \to 2^{\mathbb{N}}$ from $F_n' \to E_0$ for each

$n \in \mathbb{N}$ (as finite Borel equivalence relations are smooth; see for instance [34, Proposition 1.4.4]), using which we define $f : \partial\hat{\Gamma} \to (2^{\mathbb{N}})^{\mathbb{N}}$ by $f(\eta) = (f_n(H_n(\eta)))_n$. Put $E' = f^{-1}(E_1)$ (so that $\theta E' \eta \iff f(\theta) E_1 f(\eta)$).

Lemma 4.27. *The equivalence relation E' is a hyperfinite countable Borel equivalence relation.*

Proof. Since H_n is Borel, we have that E' is Borel. We also have that E' is hypersmooth by definition, and so it is hyperfinite by [19, Theorem 8.1.5]. We follow the same proof as the proof of [26, Lemma 6.9], to show that E' is countable.

For each $n \in \mathbb{N}$, define the relation E'_n on $\partial\hat{\Gamma}$ by $\eta E'_n \theta$ if $f_m(H_m(\eta)) = f_m(H_m(\theta))$ for all $m \geq n$. Each E'_n is countable because if $\eta E'_n \theta$, then $f_n(H_n(\eta)) = f_n(H_n(\theta))$, but $f_n \circ H_n$ is countable-to-one (because H_n is countable-to-one because if $H_n(\eta) = H_n(\theta)$, then $\eta E \theta$ and f_n is finite-to-one since F'_n is finite), which implies that there are only countably many choices for η such that $\eta E'_n \theta$ once θ is fixed. Thus, E'_n is countable. Noting that $E' = \bigcup_{n\in\mathbb{N}} E'_n$, we obtain that E' is countable.

□

Lemma 4.28. *f is a homomorphism from $E|_Y$ to E_1.*

Proof. Suppose $\eta, \theta \in Y$ are E-related, as witnessed by $g \in G$ (so $g\eta = \theta$). By Theorem 3.9 and Lemma 3.10, we have that $g\mathrm{Geo}_1(e, \eta)$ and $\mathrm{Geo}_1(e, \theta)$ differ by a finite set. Since $\eta \in Y$, we have that $d(T^\eta_n, e) \to \infty$ as $n \to \infty$, so there exists $N_1 \in \mathbb{N}$ such that $gT^\eta_n \subseteq \mathrm{Geo}_1(e, \theta)$ for all $n \geq N_1$. By the proof of Lemma 4.19, we have $s^\eta_n = s^\theta_n$, so since $gT^\eta_n \subseteq \mathrm{Geo}_1(e, \theta)$ for all $n \geq N_1$ we have that $gT^\eta_n \subseteq T^\theta_n$ for all $n \geq N_1$. Repeating the above argument with the roles of η, θ reversed, we obtain $g^{-1}T^\theta_n \subseteq T^\eta_n$, i.e. $T^\theta_n \subseteq gT^\eta_n$ for all $n \geq N_2$, for some $N_2 \in \mathbb{N}$. Letting $N = \max\{N_1, N_2\}$, we then have $gT^\eta_n = T^\theta_n$ for all $n \geq N$. This yields $(g^\theta_n)^{-1} g g^\eta_n H^\eta_n = H^\theta_n$ for all $n \geq N$. Thus, we have $H_n(\eta) F_n H_n(\theta)$ and so $H_n(\eta) F'_n H_n(\theta)$ for all $n \geq N$. Thus, we have $f_n(H_n(\eta)) = f_n(H_n(\theta))$ for all $n \geq N$ and so $f(\eta) E_1 f(\theta)$.

□

Let us now establish Theorem A on the hyperfiniteness of E, following the proof of [26, Theorem A].

Proof of Theorem A:

Note that $E|_Y$ is a sub-relation of E'. Indeed, if $\theta, \eta \in Y$ and $\theta E \eta$, then by Lemma 4.28, $f(\theta) E_1 f(\eta)$, which implies $\theta E' \eta$. By Lemma 4.27, E' is hyperfinite, so $E|_Y$ is hyperfinite, since a sub-relation of a hyperfinite equivalence relation is hyperfinite. On $\partial\hat{\Gamma} \setminus Y = [Z]_E$, E is smooth by Lemma 4.20, and hence hyperfinite. Therefore, E is hyperfinite on $\partial\hat{\Gamma}$.

Recall that we fixed a finite generating set X as in Theorem 3.2. If we use any other finite generating set X' for G, then the boundary $\partial\hat{\Gamma}'$ of the relative Cayley graph $\hat{\Gamma}'$ corresponding to X' is G-equivariantly homeomorphic to $\partial\hat{\Gamma}$. One way to see this is to use Theorem 2.60. By Theorem 2.60, $\partial\hat{\Gamma}'$ and $\partial\hat{\Gamma}$ are both G-equivariantly homeomorphic to the same subspace of $\partial(G, \mathcal{P})$, hence are G-equivariantly homeomorphic. It follows that the orbit equivalence relation of G on $\partial\hat{\Gamma}'$ is also hyperfinite. □

As a corollary, we obtain Corollary B on the hyperfiniteness of the action of G on $\partial(G, \mathcal{P})$, where $\mathcal{P}$ is the collection of parabolic subgroups.

Proof of Corollary B:

By Theorem 2.60, the orbit equivalence relation of G on $\partial\hat{\Gamma}$ is a subrelation of the orbit equivalence relation of G on $\partial(G, \mathcal{P})$. Since the orbit equivalence relation of G on $\partial\hat{\Gamma}$ is hyperfinite (by Theorem A) and since $\partial(G, \mathcal{P}) \setminus \partial\hat{\Gamma}$ is countable, it follows that the orbit equivalence relation of G on $\partial(G, \mathcal{P})$ is also hyperfinite (because every equivalence relation on a countable standard Borel space is hyperfinite). □

5. Conclusion and Further work

In this thesis, we have used the machinary of relatively hyperbolic groups from [29] to generalize the hyperfiniteness of the boundary action of hyperbolic groups from [26] to relatively hyperbolic groups. We have shown that relatively hyperbolic groups admit a hyperfinite orbit equivalence relation through their action on the boundary of their relative Cayley graphs and as a quick corollary, we have obtained hyperfiniteness of the orbit equivalence relation of the action of relatively hyperbolic groups on their Bowditch boundary.

5.1 Boundary Actions of Acylindrically and Hierarchically Hyperbolic Groups

Beyond relative hyperbolicity, one can ask the question of hyperfiniteness of the action of the more general *acylindrically hyperbolic groups* on the boundaries of some or all *acylindrical Cayley graphs*.

Following [30], an isometric action of a group G on a metric space X is **acylindrical** if $\forall \varepsilon > 0, \exists R, N$ such that for all $x, y \in X$ with $d(x, y) \geq R$,

$$|\{g \in G : d(x, gx), d(y, gy) \leq \varepsilon\}| \leq N$$

Let G act isometrically on a hyperbolic metric space X. The **limit set** of G on the *sequential boundary* ∂X is $\Lambda(G) = \overline{Gx} \cap \partial X$ in the topology on $X \cup \partial X$ (defined using

Gromov product and sequences converging at infinity; see [16, Chapter 3.4] and [9, Pages 432-433] for the definition of sequential boundary and the topology) for any $x \in X$ (note that $\overline{Gx} \cap \partial X$ is the same set for any $x \in X$). An isometric action of a group G on a hyperbolic metric space X is called **non-elementary** if $|\Lambda(G)| > 2$.

A group G is called **acylindrically hyperbolic** if G admits a non-elementary acylindrical action on some hyperbolic space. A Cayley graph for a group G is called **acylindrical** (resp. **non-elementary**) if it is hyperbolic and the natural action of G on it is acylindrical (resp. non-elementary). The class of acylindrically hyperbolic groups contains many examples, in particular, all non-virtually cyclic relatively hyperbolic groups (hence all non-virtually cyclic hyperbolic groups).

Acylindrically hyperbolic groups can also be characterized in the language of relative Cayley graphs via *hyperbolically embedded subgroups.*

Let H be a subgroup of a group G. If X is a relative generating set of G with respect to H, then we can put a natural metric on H as follows. For any $x, y \in H$, a path p in the relative Cayley graph $\hat{\Gamma} = \Gamma(G; X \cup H)$ is **admissible** if p does not contain any edges from the complete subgraph associated to H in $\hat{\Gamma}$. We define a (potentially infinite valued) metric $\hat{d}$ on H by putting $\hat{d}(x, y)$ to be the least length of an admissible path between x, y in $\hat{\Gamma}$, if one exists, else we define $\hat{d}(x, y) = \infty$. Then H is hyperbolically embedded in G with respect to X (written $H \hookrightarrow_h (G, X)$) if $\hat{d}$ is a locally finite metric on H.

A subgroup H of G is called **non-degenerate** if it is proper and infinite. By [30, Theorem 1.2], G is acylindrically hyperbolic if and only if G has a non-degenerate hyperbolically embedded subgroup. It is shown in [14, Proposition 4.28] that G is hyperbolic relative to $\{H\}$ if and only if $H \hookrightarrow_h (G, X)$ for some finite relative generating set X.

With the characterization of acylindrically hyperbolic groups in terms of hyperbolically embedded subgroups, we can perform similar geometry as we did above for relatively hyperbolic groups in the relative Cayley graph $\hat{\Gamma}$. The main difficulty in the acylindrical case is that the relative generating set X need no longer be finite, so the proof of Theorem 3.2 (which was the main ingredient in showing hyperfiniteness of the boundary action) no longer works. So far, the author has not succeeded in further generalizing the results of this thesis to acylindrically hyperbolic groups and believes that new techniques might be required to establish hyperfiniteness of the action of acylindrically hyperbolic groups on the boundaries of their acylindrical or relative Cayley graphs (if the result is even true for acylindrically hyperbolic groups).

There are examples of acylindrically hyperbolic groups that are not relatively hyperbolic and induce hyperfinite orbit equivalence relations on boundaries of their Cayley graphs. Mapping class groups of surfaces of finite type are acylindrically hyperbolic ([32]) but not relatively hyperbolic in general ([4]) and are known to induce a hyperfinite equivalence relation on the boundary of the curve graph of the surface ([33]). Since the action on the curve graph is cobounded and by isometries, this translates to an isomorphic action of such mapping class groups on the boundary of some acylindrical Cayley graph quasi-isometric to the curve graph, hence establishing the hyperfiniteness of the action of the group on the boundary of this Cayley graph. However, it is not known if the action of the mapping class group on the boundary of every non-elementary acylindrical Cayley graph induces a hyperfinite equivalence relation. To the author's knowledge, mapping class groups of surfaces of finite type are the only known examples of non-relatively hyperbolic acylindrically hyperbolic groups inducing hyperfinite orbit equivalence relations on boundaries of their Cayley graphs.

A related problem to hyperfiniteness of the boundary action is *boundary amenability* of a group. A group is **boundary amenable** if it admits a *topologically amenable* action by homeomorphisms on a compact Hausdorff space (see [7] for the definition of topological amenability). Relatively hyperbolic groups with boundary amenable parabolic subgroups were shown to be boundary amenable by Ozawa in [31]. It is deduced in [33] from the hyperfiniteness of the action of a mapping class group on the space of complete geodesic laminations (which is compact Hausdorff) and the fact that point stabilizers are amenable that mapping class groups are boundary amenable. Also, it is shown in [7] that the outer automorphism group of a free group, $\mathrm{Out}(F_n)$, is boundary amenable for all n (note that for $n \geq 3$, $\mathrm{Out}(F_n)$ is acylindrically hyperbolic but not relatively hyperbolic; see [6] and [4]). However, in [27], Osajda has constructed a non-exact (hence not boundary amenable) acylindrically hyperbolic group. The author hopes to study Osajda's construction to see if it provides a counterexample to hyperfiniteness of the boundary action for acylindrically hyperbolic groups.

Aside from acylindrically hyperbolic groups, another generalization of hyperbolic groups are *hierarchically hyperbolic groups*. We shall not define these here but refer the reader to [5] for their definition and theory. Hierarchically hyperbolic groups are modeled off of the structure of mapping class groups and also have a notion of boundary, which is a compact metrizable space on which the group acts naturally (see [18]). Therefore, we may ask if hierarchically hyperbolic groups induce a hyperfinite orbit equivalence relation on their boundary. The author hopes to generalize the work of Przytycki and Sabok in [33] from mapping class groups to general hierarchically hyperbolic groups in the future.

5.2 Thin ends

If X is a graph, then we can define an equivalence relation on the set of all geodesic rays in X by putting, for geodesic rays γ_1, γ_2, $\gamma_1 \sim \gamma_2$ if for every finite subset $S \subseteq X^{(0)}$, γ_1, γ_2 have tails in the same connected component of $X^{(0)} \setminus S$. An **end** of X is an equivalence class with respect to $\sim$. The **degree** of an end is the maximum number of pairwise disjoint geodesic rays in the end. An end is **thin** if it has finite degree (see [21]).

Thin ends are used to show that accessibility of finitely generated groups is a quasi-isometry invariant (see [35]).

For the case of a relative Cayley graph $\hat{\Gamma}$ of a relatively hyperbolic group, Theorem 3.2 implies that if the set of ends of $\hat{\Gamma}$ equals $\partial\hat{\Gamma}$, then $\hat{\Gamma}$ has thin ends. This appears to not have been known for relatively hyperbolic groups.

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