INTERNAL WEAK OPENS, INTERNAL STABILITY AND MORSE THEORY FOR SYNTHETIC GERMS

by

Felipe Gago

Department of Mathematics
McGill University, Montreal

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— ¿Quérela, María Elena?
— Doucha.

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I have also profited from a series of lectures on Synthetic Differential Geometry given by A. Kock and G. E. Reyes at the Departamento de Alxebra of the Universidade de Santiago de Compostela; my thanks also to the members of this department, to whom I owe my Category Theory background.

Lastly, I cannot forget those people who taught me things other than mathematics: my family and my friends; I shall choose Mario as a representative to express my recognition to them all.

Abstract

In this work, we initiate the *classification of singularities* in the framework of Synthetic Differential Geometry; as in the classical setting, we restrict ourselves to the study of the stable mappings. To this end, we use an internalization of the classical Compact-Open topology of $C^\infty(M, \mathbb{R})$, particularly useful to show density results (*genericy aspect*). Using this topology we present and compare the internal versions of several notions of stability. Singularities, being a property of the "very near" to the point, are studied here using *infinitesimally represented synthetic germs*. We obtain a characterization of stable germs of functions in the synthetic context as being those with only non-degenerate singularities: *Morse germs*

The whole "building" has two cornerstones: Weierstrass' *Preparation Theorem*, and Sard's *Density of Regular Values Theorem*. On the other hand, the keystone is Thom's *Homotopy Method*. The three of them are shown to be valid in our test model: the topos of Dubuc.

Résumé

Ce travail est un pas en avant sur la route de la classification des singularités dans le contexte de la Géométrie Différentielle Synthétique. On utilise une version interne de la topologie des Compacts-Ouverts de $C^\infty(V, \mathbb{R})$ qui se révéle practique à l'heure de prouver des resultats de densité (l'aspect générique). Avec cette topologie on montre comment diverses notions de stabilité peuvent être internalissées et comparées.

Les singularités sont étudiées ici pour les germes synthétiques (représentés par des objects infinitésimals) car cette propiété appartienne au "tout près" du point. On obtient le resultat de characterisation suivant: les germes stables de fontions sur R sont exactement les germes de Morse.

Ce "bâtiment" a pour piliers le *Théorème de Preparation* de Malgrange et le*Théorème de Densité des valeurs regulières* de Sard. D'ailleurs, la pierre clef est la méthode homotopique de Thom. Tout les trois ont été validés dans le topos de Dubuc, le modèle de test.

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Jamais mon dessein ne s'est étendu plus avant que de tâcher à réformer mes propres pensées, et de bâtir dans un fonds qui est tout à moi...

...Mais, comme un homme qui marche seul et dans les ténèbres, je me résolus d'aller si lentement, et d'user de tant de circonspection en toutes choses, que, si je n'avançais que fort peu, je me garderais bien, au moins, de tomber.

(RENE DESCARTES: Discourse de la méthode)

Introduction

The mathematical setting to develop theories of geometry, analysis and continuum physics is usually considered to be the category of topological spaces or the category of Banach manifolds. In both cases, an increasing number of smoothness conditions have to be imposed to obtain some "technical theorems". Even so, there are some essential constructions whose physical or geometrical motivation is obscured by the deficiencies or inadequacy of this background. To mention just one example in continuum physics: the construction of the *function space*.

On the other side, one needs not go far reviewing the works of geometers to realize that the *synthetic reasoning* they use to discover or introduce concepts and axioms does not fit, without violence, into the *analytic* or set theoretical type of reasoning, on fashion since last century. We could go even further and say that the lack of an adequate language and formal setting has retarded, if not made imposible, the presentation of "known" solutions to some problems. In this sense, we quote from Kock's Synthetic Differential Geometry book the translation of a text of Lie: *The reason why I postponed for so long these investigations, which are basic to my other work in this field, is essentially the following. I found these theories originally by synthetic considerations. But I soon realized that, as expedient the synthetic method is for discovering, as difficult it is to give a clear exposition on synthetic investigations which deal with objects that till now have almost exclusively been considered analytically.*

With these or similar reasons in mind, in three lectures at the University of Chicago in 1967, Lawvere proposed a vast research program with three points:

1.- An axiomatic study of categories

2.- A direct axiomatization of the essence of differential topology using results and methods of the French work in algebraic geometry.

3.- An intrinsic axiomatization of continuum mechanics as developed by Walter Noll and others.

In order to get point number 3, the suggestion is to start off with the idea of smoothness as a property of how smooth spaces interact with each other, instead of basing everything on a definition of smooth object as a set of atoms with a given structure. This point of view leads us to taking the notion of map as the primitive concept and the way they compose; in other words, to considering a category as the basic datum, and therefore to point number 1. In Lawvere's words, *axiomatizing a category as a whole promises to be part of the simplest approach to certain calculations.*

The work presented here can be said to fit into point 2 of Lawvere's program. New axioms are added to the basic stock and used to develop the theory further, and then such axioms are tested in a specific well adapted model which already satisfies the other, previously introduced axioms and postulates. Before starting with its description, let us reverse the course of history and go backwards to the early days of calculus.We begin by the needs imposed by part 3, as explained by Lawvere.

Let E denote ordinary physical space, T a space which represents the notion of time, and B a space which represents a particular body. Then a particular motion of B may be represented as a map

$$B \times T \to E,$$

which is the right way if one wants to compute by composition how particles of the body, at various times, experience the values of some field defined on space. However, it is also necessary to construe the *same* motion as a map

$$T \to E^B,$$

where the space E^B of placements of the body is itself independent of T or a particular motion, if we want to compute by composition the temporal variation of quantities like the center of mass $E^B \to E$ of B. Still a third version

$$B \to E^T$$

of the *same* motion, where the space E^T of paths in space exists independently of B, is a necessary step if we want to compute by composition the velocity field on B induced by the motion.

The general possibility of such transformations within a given category is expressed by saying that the category in question is cartesian closed, and is much more fundamental for continuum physics than the precise determination of the concept of spaces-in-general, of which E, T and B should be examples

The basic framework to develop these ideas is the result of the efforts of many authors, and came to be called Synthetic Differential Geometry. Some of its features are described in Chapter 1 of this memoir, as well as the basic references to the promoters.

Lawvere's proposal (even some of his indications and basically the theory of models developed for this theory) of axiomatizing the original ideas of infinitesimal calculus finds its antecedents in the work of the Grothendieck school.

Prior to Lawvere's realization of the "curious" resemblance between the category of sets and the universe of discourse of algebraic geometers, there was no consistent language to accommodate and with which manipulate the infinitesimals. Even so, they managed to use these techniques by means of the *Théorie des Schémas*, and fruitfuly exploited the duality betweenthe category of affine algebraic varieties over an algebraically closed field K is the dual of the category of reduced K-algebras of finite type (cf. also [PENON: De l'infinitésimal au local]). Commutative algebra comes in to help to interpret geometrical objects. The reduced character of these geometrical objects (or rather of their duals) was an obstacle to the use of nilpotent elements which have an important role to play. With this state of things, the right decision seemed to be to "keep in mind" this duality and to consider (duals of) algebras as generalized algebraic manifolds. This way, a local algebra has a unique point (its maximal spectrum is just a point, or equivalently its dual has a unique global section,) yet it is different from K. So, local algebras are like *fat points*, i. e., points to which other "phantom points" infinitely close have stuck. In order to handle these "phantom" points of whose presence there was no doubt despite their incapacity to describe them, the algebraic geometers aimed to the *functorial machinery*: they preferred to study the structures by themselves rather that by their elements or points. So, infinitesimals were to be treated by "bunches" instead of isolated.

New difficulties confronted them when studying local properties with the brand new tools. For instance, a *morphism étale* between affine K-schemes of finite type is "infinitesimally invertible" though only in very few occasions it is "locally invertible". Then, after they had enlarged the scope of algebraic varieties by adding the infinitesimal machinery, Grothendieck decided to use the local machinery. The (much too big) Zariski neighborhoods were dropped in benefit of *étale* neighborhoods, which in turn are not in general subschemes. So, the local conceptions arising from general topology have to be substantially modified as the neighborhoods are "outside" rather than "inside" the space. This was the birth place of those huge categories where all imaginable constructions are possible and where the coverings are the sought: The U-toposes, their *universe of discourse*.

Algebraic geometers had learnt from Weil the advantages of using infinitesimals. In his *Théorie des point proches* he had proposed to go back to Fermat's methods of first-order infinitesimal calculus. Indeed, generalizing Ehresmann's theory of jets, he suggested that a point p of a smooth manifold M admits as *nearby* points certain $\mathbb{R}$-algebra morphisms from $C^\infty(M, \mathbb{R})$ into a given $\mathbb{R}$-algebra A which he called local: exactly those morphisms for which composition with the canonical morphism $A \to \mathbb{R}$ gives back the point p (in the sense that the composite morphism becomes evaluation at the point p.)

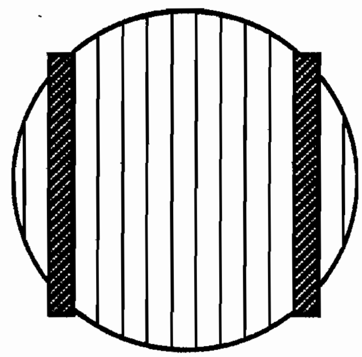
A typical example of nearby point to the point $p \in M$ is the morphism

$$\begin{array}{rcl} C^\infty(M, \mathbb{R}) & \to & \mathbb{R}[X]/_{(X^2)} \\ f & \mapsto & f(p) + \tau(f)\cdot\varepsilon, \end{array}$$

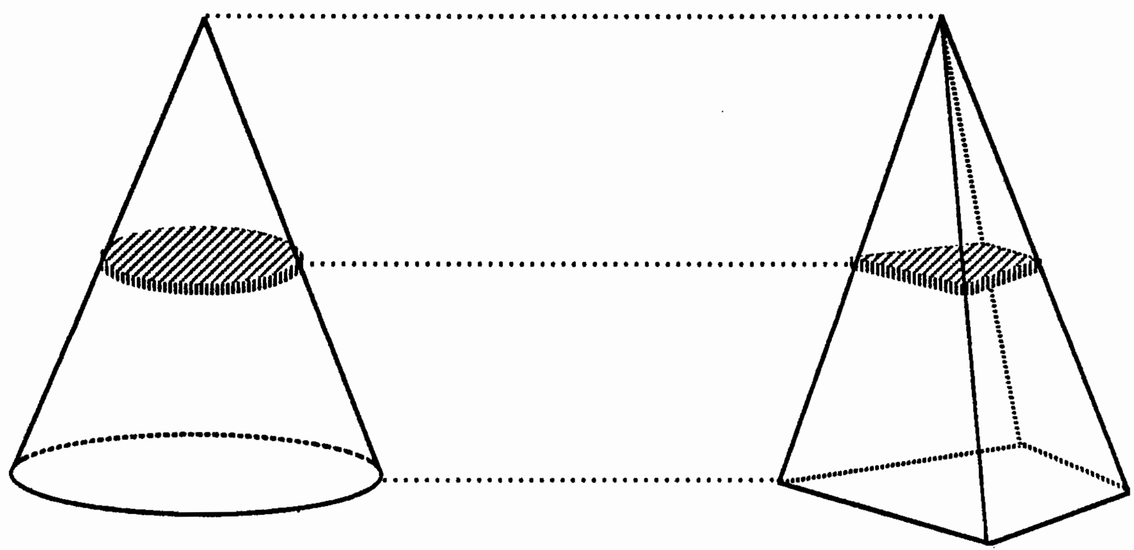
where τ is a tangent vector to the manifold at p, and ε (that is the generator of the $\mathbb{R}$-algebra $\mathbb{R}[\varepsilon] = \mathbb{R}[X]/_{(X^2)}$) is such that $\varepsilon^2 = 0$. Weil had already resorted on commutative algebra, in particular on local algebras which intended to generalize $\mathbb{R}[\varepsilon]$, to describe these entities which had been formally hidden almost since Fermat, one of the strongest advocates of the methods involving infinitesimally small numbers in the "umbral" of calculus.

The reason why the theory of infinitesimals had gradually fallen into disrepute must be sought in the fact that neither Fermat nor Leibnitz or any of their successors had been able to state with sufficient precision just what rules were supposed to govern the use of these infinitesimal quantities.

Probably the first explicit use made of infinitesimals in geometry is to be found in Keppler's *Nova stereometria doliorum vinariorum* (New solid geometry of wine barrels).



His approach consisted on the dissection of a given solid into an (apparently) infinite number of infinitesimal pieces, or solid "indivisibles" of a size and shape convenient to the solution of the particular problem. Cavalieri in his *Geometria indivisibilibus continuorum nova quadam ratione promota* devised a method of comparing two solids through their cross-sections, as well as another to calculate the volume of a single solid in terms of its cross-sections.



The latter led Cavalieri in his *Exercitationes geometricae sex* to a result equivalent to the basic integral

$$\int_0^a x^n dx = \frac{a^{n+1}}{n+1}.$$

However, Cavalieri was far from possessing the views which are expressed in terms "differential" and "integral". He himself appears to have regarded his method only as a pragmatic geometrical devise for avoiding Eudoxus' *method of exhaustion* (as described in Euclid XII,2;) the logical basis of this procedure did not interest him. He used to say that *rigor is the affair of philosophy rather than geometry.*

This lack of heed to demands of mathematical rigor made geometers chary of accepting the method of indivisibles as valid in demonstrations, although they employed it readily in preliminary investigations. As an example, let us mention Torricelli's twenty-one demonstrations of the quadrature of the parabola: ten of them are given following the method of the ancients, including the well-known proof by the method of exhaustion given by Archimedes in his *Quadrature of the parabola.* In the other eleven, he uses the new method of indivisibles; in one of these, he uses that it is possible to inscribe, within the parabolic segment, a figure, made up of parallelograms of equal height, which shall differ from the segment by less than any given magnitude.

Almost simultaneously, we have to consider the mathematical french triumvirate of Roberval, Fermat and Pascal. Whereas Cavalieri and Torricelli had proceeded on the basis of the purely geometrical considerations involved in the method of exhaustion and in the method of indivisibles, the french mathematicians combined their interest in the geometry of Archimedes with an enthusiasm for the theory of numbers, and this colors their work. For example, in order to determine how to subdivide a segment of length B into two segments of length A and $B - A$ whose product $A(B-A) = AB - A^2$ is maximal, Fermat proceeded as follows. First he substituted $A + E$ for the unknown A, and then he wrote down the following *pseudo-equality* (he used the Latin word *adequatio*) to compare the resulting expression with the original one

$$(A+E)B - (A+E)^2 = AB + EB - A^2 - 2AE - E^2 \sim AB - A^2.$$

After cancelling equal terms, one gets what he wrote "B *in* E *adaequabitur* A *in* E *bis* + Eq":

$$BE \sim 2AE + E^2 .$$

Then he divided through by E to obtain

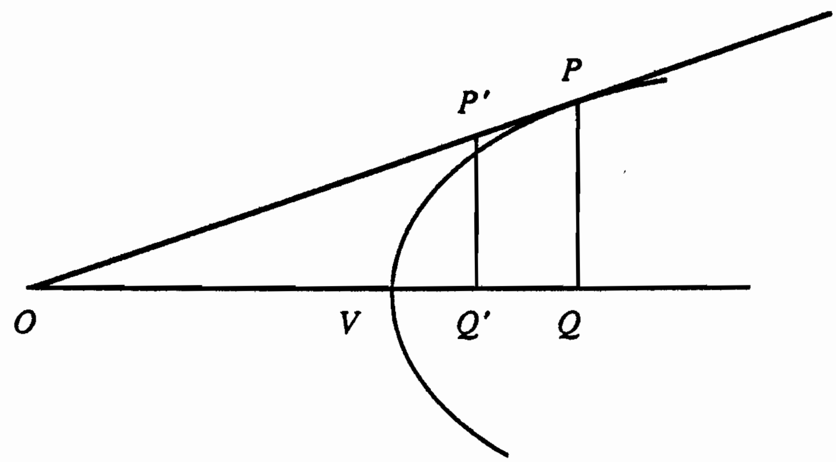
$$2A + E \sim B .$$

Finally he discarded the remaining term containing E, transforming the pseudo-equality into the equality

$$A = \frac{B}{2},$$

that gives the value of A which makes $AB - A^2$ maximal. The pseudo-equality for him conveyed the meaning that "near" of a maximum point the function takes different values though they should be equal; he then formed this pseudo-equality which would become an equality by letting E equal 0. It is up to us to admire the beauty of Fermat's method.

Fermat was led by the success of his method to apply it to the determination of tangent to curves. This he did as follows. He associates with each curve an equation in which all the properties of the curve are implied: the *specific property* of the curve. Let the curve be a parabola; then, from its "specific property" it is clear that if we set



$OQ = A$, $VQ = D$ and $QQ' = E$, we shall have

$$\frac{D}{D - E} > \frac{A^2}{(A - E)^2}.$$

For small values of E, the point P' may be regarded as practically on the curve as well as on the tangent line. This inequality becomes, as in the method for maximum values, a pseudo-equality, and by allowing E to vanish, this pseudo-equality becomes a true equality, and gives the desired result,

$$A = 2D.$$

From here on...Newton...D'Alembert...Cauchy...ε...δ.... Does it ring any bells?

Synthetic Differential Geometry, offering a subobject of R of the form

$$D = \{x \in R \mid x^2 = 0\},$$

comes in handy to the large number of mathematicians and physicists who like working with first-order approximations (neglecting higher-order terms.) The basic axiom of S.D.G. says precisely this for functions: a function (with values in R) defined on D is linear (i.e., determined by its value at 0 and the the value of its derivative). This axiom can be written as

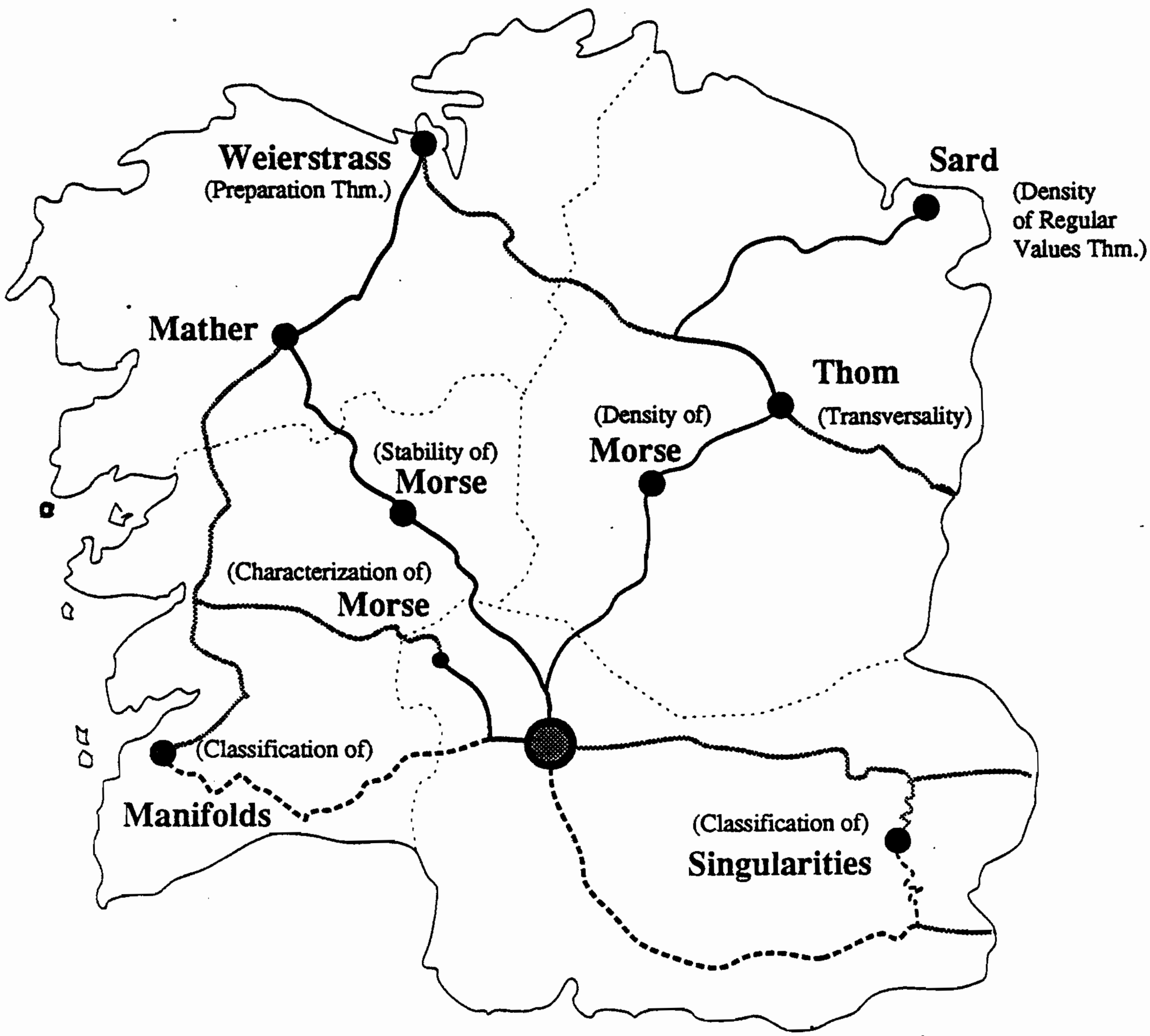
$$R^D \approx R \times R,$$

and is incompatible with classical logic (cf. [KOCK: Synthetic Differential Geometry].) For this reason we resort on intuitionistic logic, and of special manner on some of its models: the toposes.

In any topos, several notions of topology are available. Among them, the intrinsic topology defined by Penon (cf. [PENON: Intuitionism et topologie]) seems to be the most useful and widely used. The success of this topology resides on its logical nature; in particular in the use it makes of the double negation which is, in general, not equal to the identity. In particular, if U is a neighborhood of x , then $\neg\neg\{x\} \subset U$. This fact allows us "to consider" germs at x as maps defined on $\neg\neg\{x\}$ rather than equivalence classes of maps and neighborhoods of x. Synthetic germs, being infinitesimally represented, promise to be a powerful tool to attack those problems in which the "very close" to the point has a rôle to play. One such problem is the study of singularities, which we initiate in this work.

To arrive in Chapter 5 to the complete characterization of stable singularities of mappings into R from some part of a given R^n, we had to "travel" along several different roads and visit several "interesting places". To do this we have equipped our "vehicle" with the internal weak topology as well as the several achievements of S.D.G.

In the picture of next page we show a map of the territory (in our test model $\mathcal{G}$) which we have explored in this thesis



Weierstrass
(Preparation Thm.)
Sard
(Density
of Regular
Values Thm.)
Mather
Thom
(Transversality)
(Density of)
Morse
(Stability of)
Morse
(Characterization of)
Morse
(Classification of)
Manifolds
(Classification of)
Singularities

We distinguish two main roads to the Capital (*the classification of singularities of germs of maps into R*) coming from the northern territories. One of them starts at Sard (*the theorem of density of regular values*) follows by Thom (*the theorem of transversality*), and from there on to Morse (*density of Morse germs*). The other one begins at Weierstrass (*the theorem of preparation*) and passes by Mather (*criterium of stability*) to join the road coming from Morse (*characterization of Morse germs*) before entering the City. There are other secondary roads that we have not visited, as well as two roads leaving the City which we have left for a later exploration.

In Chapter 2, we investigate in this context a topological structure which turns out to be useful when proving density results. We call it the *Weak Topology*, and it has the property that its "observable opens" are the usual ones. In §1 we review the classical theory and give some technical results. Only the proofs of Proposition 1.3 and Corollary 1.4 seem to be new. In §3 we present some results about compactness in a topos which are needed to apply the definition to the space of germs. To the proof of the compactness of $\Delta(n)$ (Proposition 3.3, also in [BUNGE-GAGO: Synthetic aspects of C^∞-mappings, II: Mather's theorem for infinitesimally represented germs]) we have substantially contributed, and the boundness away from zero of functions defined on compact objects (Proposition 3.4) is new. In §4 we study the Weak topological structure and give a few properties; we prove that it is subintrinsic and analyze its separability properties in particular cases. In §5 we study the action of the global sections functor on weak opens, and obtain a bijection between the weak open parts of R^E and the weak C^∞- open subsets of $\Gamma(R^E)$. Propositions 3.4, 4.4, 4.5, 4.6 and 5.3 are entirely due to the author.

In Chapter 3, basically by adapting work done by Bunge, we "arrive" to the first important result to the Theory of Stability: *Thom's Transversality Theorem*. The machinery we use includes a proof of the axiom of *Density of Regular Values* of germs in the test model $\mathcal{G}$. This we did internally in §3, where we also do a complete study of immersions. In §2 we give a proof of *Sard's Theorem* in $\mathcal{G}$ which is used to internally derive the Theorem of Regular Values of §3. As the emphasis in this thesis is on the weak internal rather than on the Penon topology, there are some changes in the presentation of results also included in [BUNGE-GAGO: Synthetic aspects of C^∞-mappings, II: Mather's theorem for infinitesimally represented germs]. In addition, Propositions 3.4 and 3.5 are new.

Chapter 4 is dedicated to sightseeing several notions of *stability* in the framework of Synthetic Differential Geometry. Since everything in nature is subject to small disturbances, one expects that natural forms must be described by stable maps. In §1 we meet the basic definitions of equivalence for germs and stability. In §2 the concept of *infinitesimal stability* appears as an easy-to-check criterium for stability. In §3 we introduce in this context the original ideas of R. Thom to compare both notions, namely *homotopical stability*, and give a proof of the existence of solutions for time-dependent differential equations. §4 is consecrated to the second main result of the Theory of Stability: the *Malgrange-Weierstrass Preparation Theorem*. It is introduced as a postulate of the theory and shown to hold in $\mathcal{G}$. Finally, §5 is occupied by the celebrated *Mather's Theorem* which states the equivalence of all definitions of stability. We contributed substantially to the choice of definition of equivalence for germs and to discussions about the nature of unfoldings (in [BUNGE-GAGO: Synthetic aspects of C^∞-mappings, II: Mather's theorem for infinitesimally represented germs] not all included in this thesis.) The definition of stability (Definition 1.3) given here is, unlike that in [BUNGE-GAGO: op. cit.], a direct internalization of the classical one, made possible by my results in Chapter 2. Proposition 1.4 is also new, as well as all considerations of homotopical stability (§3) and its uses in Chapter 5. We also pointed out the need for a synthetic version of the theorem of existence and uniqueness of solutions to dynamical systems (Proposition 3.4 also in [BUNGE-GAGO: op. cit.]) and contributed to its proof.

In Chapter 5 we give a complete characterization of the stable singularities of germs of functions. The stable maps certainly form an open subobject of R^X but the questions were whether this subobject was dense and whether stable singularities could be classified. In §1 we introduce the notion of *non-degeneracy* for a singularity, and give some characterization results. In §2 we define a *Morse germ* as that one with only non-degenerate singularities, and prove that they form a dense subobject for the weak topology, and that their singularities are isolated. In §3 we find the *normal form* for a Morse germ and in §4 we prove that a singularity of a function is stable if and only if it is non degenerate. All the results in this chapter are due to the author and some of them are included in a paper [GAGO: Morse theory in S.D.G.] submitted to the proceedings of the Louvain-la-Neuve Category Theory Conference, July 1987.

Our intention was to make the text as readable and self-contained as possible. To the first end we have included a Chapter 0 with the basics of the formal language and the particular logical reasoning used in Synthetic Differential Geometry. In so doing we hope to gain the reader's mercy for our "loose talk" in later chapters. Unless explicitly stated to the contrary, all the work takes place at the internal level, though we very often make use of the naïve approach. For the sake of completeness, Chapter 1 contains a detailed introduction to the features of the background in which the theory develops. Nothing in Chapters 0 and 1, with the possible exception of the order of the presentation, is due to the author. Most of the results are stated without proofs, and references are scattered all along the text.

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The logic and the model

§1. Elementary topos

The present work makes use of intuitionistic logic (for details cf. [DUMMET: Elements of Intuitionism] or [TROELSTRA: Principles of Intuitionism]) to explain and/or understand situations which have no room within the scope of classical logic. What anyone who has ever heard of intuitionism knows is that the use of the law of "excluded middle" is forbidden, as well as any form of the axiom of choice. We do not hesitate to declare that our goal is not the substitution of a form of mathematic by another. Quite on the contrary, we take good profit of both of them to enrich our understanding of the realities we deal with. In this direction, by using a model for our theory, we intend not only to guarantee its non–contradiction but rather to clarify the interactions between the (at first sight) different situations.

Among the models for this kind of logic, we are particularly interested in toposes (cf. [LAWVERE: Quantifiers and sheaves], [KOCK-WRAITH: Elementary Toposes], [MIKKELSEN: Lattice Theoretic and Logical Aspects of Elementary Topoi], [JOHNSTONE: Topos Theory], [BARR-WELLS: Topos, Triples and Theories])

Definition 1.1 An (elementary) topos is a category E satisfying the following conditions:

— E has finite limits

— E is cartesian closed, i.e., the functor $A\times$— has a right adjoint $(\ )^A$.

— E has a subobject classifier; i.e., there exists an object, Ω, and an arrow $\mathbf{1}\to\Omega$, such that for each monomorphism $P\to A$ there exists a unique arrow $A\to\Omega$ making a pullback of the following square:[1]

$$\begin{array}{ccc} P & \to & A \\ \downarrow & & \downarrow \\ \mathbf{1} & \to & \Omega \end{array}$$

Apart from the category of sets (an obvious example, with $\Omega=\{0,1\}$), we will see later on, in certain detail, the case of Grothendieck toposes.

Every topos has its own "internal logic" which we will describe now. We begin with the first-order ingredients.

Given any object A in E, two monomorphisms, $P\to A$ and $Q\to A$, are called equivalent if there exists a (necessarily unique) isomorphism $P\to Q$ rendering commutative the right triangle. We call subobject of A to any of these equivalence classes, and denote $\wp(A)$ the set of subobjects of A..[2]

Clearly, $\wp(A)$ with the obvious order relation (denoted $\subset$) is a Heyting algebra:

1) It has a greatest element, denoted $TRUE_A$ (the one which corresponds to the identity monomorphism,) and a smallest element, $FALSE_A$ (corresponding to $\emptyset\to A$, where $\emptyset$ is the initial object of E, which always exists (cf. [MIKKELSEN: Lattice theoretic and logical aspects of elementary topoi], [PARE: Colimits in topoi])

2) Each pair P, Q of subobjects of A has an infimum, $P\cap Q$, (corresponding to the pullback) and a supremum, $P\cup Q$.

3) For each pair P, Q of subobjects of A there always exists a unique subobject $P\Rightarrow Q$ with the property

$$R\cap P\subset Q \qquad \text{iff} \qquad R\subset(P\Rightarrow Q) \qquad \text{for any} \in \wp(A).$$

[1] Intuitively, Ω represents the "set of truth values", and the unique arrow $A\to\Omega$ the "characteristic function" associated with the "part" P.

[2] Clearly $\wp(A)$ is in a one-to-one correspondence with the parts of A, which justifies the terminology.

Now, with each arrow $f:A \to B$ in E, we associate a monotone application

$$f^{-1}: \wp(B) \to \wp(A) \text{ (by pulling-back along } f).$$

This application preserves the structure of $\wp(B)$, i.e., $TRUE_B$, $FALSE_B$, intersection, union and implication. Moreover, there exists a monotone application,

$$\exists_f: \wp(A) \to \wp(B),$$

such that, for each subobject P of A and Q of B, we have $\exists_f(P) \subset Q$ iff $P \subset f^{-1}(Q)$.

Similarly, there exists a monotone application,

$$\forall_f: \wp(A) \to \wp(B),$$

such that, for each subobject P of A and Q of B, we have $f^{-1}(Q) \subset P$ iff $Q \subset \forall_f(P)$.

If we consider $\wp(A)$ as a category, then f^{-1}, $\exists_f$ and $\forall_f$ are functors, and we have the following adjunctions

$$\exists_f \dashv f^{-1} \dashv \forall_f .$$

In the case of sets, for a map of the form $\pi: X \times Y \to Y$, we have

$$\exists_\pi(P) = \{y \in Y \mid \exists x \in X\ (x,y) \in P\} \quad \text{and} \quad \forall_\pi(P) = \{y \in Y \mid \forall x \in X\ (x,y) \in P\}.$$

The intuitionistic character of the logic shows up immediately. Define $\neg P$ as $P \Rightarrow FALSE_A$ (this corresponds to the complement in sets): properties, such as the following

$$\neg\neg P = P \qquad \text{or} \qquad P \cup \neg P = TRUE_A,$$

have no grounds to hold on.

As for the higher-order properties, notice that for each object X of E, there exists an object PX and a subobject $\Sigma_X \to X \times PX$ (the membership relation of X) satisfying the following condition[1]:

For every pair of objects X and Y, and subobject $R \to X \times Y$, there exists a unique arrow $\uparrow_R : Y \to PX$, such that $R = (X \times \uparrow_R)^{-1}(\Sigma_X)$. Define $PX = \Omega^X$, and $\Sigma_X = ev^{-1}(T)$,

[1] Actually we could have defined a topos with this data, and define $\Omega = P1$.

where $ev : \Omega^X \times X \to \Omega$, and $T \to \Omega$ is the subobject corresponding to the map $1 \to \Omega$ in the definition of topos.

Before going into the main example for the purposes of this work, let us give a method for generating new examples. It is contained in the following theorem whose proof we omit (cf. [JOHNSTONE: Topos Theory] or [BARR-WELLS: Topos, Triples and Theories]).

Theorem 1.2 For any topos E, and any object X on it, $\mathsf{E}/_X$ (the slice category: its objects are arrows in E with target X, and its arrows are arrows in E making the right triangle to commute.) is itself a topos. Moreover, for any arrow $f : X \to Y$ in E, the functor "pullback along f", denoted f^*, is a logical functor (i.e., preserves the topos structure) and has both a left and a right adjoint. $\square$

§2. Grothendieck toposes

The concept of sheaf on a topological space is widely used in mathematics, and the category of these sheaves (on a fixed space) is an example of topos. We will use a generalization of this notion, due to Grothendieck (cf. [ARTIN-GROTHENDIECK-VERDIER: SGA 4], [JOHNSTONE: Topos Theory], [TIERNEY: Sheaf theory and the Continuum Hypothesis])

Definition 2.1 A pretopology Θ (of Grothendieck) on a left exact category, C, is given by a family Θ_A of families of arrows with codomain A, called coverings of A, for each A in C, satisfying the following:

a) Each singleton $(id_A : A \to A) \in \Theta_A$.

b) Coverings are stable under change of base, i.e., the pullback of a cover along any arrow is again a cover; i.e., if $(f_i : B_i \to B)_{i\in I} \in \Theta_B$, and $h : A \to B$ is any map in C, then $(A \times_B B_i \to A)_{i\in I} \in \Theta_A$.

c) Θ is closed under composition; i.e., if $(f_i : A_i \to A)_{i\in I} \in \Theta_A$ and, for each $i\in I$ $(g_{ik} : A_{ik} \to A_i)_{k\in I_i} \in \Theta_{A_i}$, then $(f_i \cdot g_{ik} : A_{ik} \to A)_{k\in I_i,\, i\in I} \in \Theta_A$.

The standard example is the *canonical topology* in *Set*, where $(f_i : A_i \to A)_{i\in I}$ is in Θ_A if and only if $\bigcup_{i\in I} Im(f_i) = A$. (i.e., it is a jointly epimorphic family.)

Definition 2.2 A sheaf on (C,Θ) is a functor $F{:}C^{op} \to \boldsymbol{Set}$, such that for every covering $(f_i : A_i \to A)_{i\in I}$ and a compatible family of elements $a_i \in F(A_i)$, there exists a unique element $a \in F(A)$ whose restriction (image via $F(f_i)$) to each $F(A_i)$ is a_i.

We denote by $\mathsf{Sh}(C)$ the category whose objects are sheaves for the pretopology, and whose arrows are the natural transformations between them. A topology for which every representable functor is a sheaf is called subcanonical.

Definition 2.3 A Grothendieck topos is a category which is equivalent to the category of sheaves on (C,Θ), for some small category C and pretopology Θ.

The result which makes Grothendieck toposes important for our work is the following:

Theorem 2.4 Every Grothendieck topos is an (elementary) topos.

Proof. After the theorem of Giraud which characterizes a Grothendieck topos as a category satisfying certain exactness conditions (cf. [JOHNSTONE: Topos Theory, page 16] or [BARR-WELLS: Topos, Triples and Theories; page 238]) we only have to show the definition of the adjoint to the product functor, and of Ω. As for the first of them, given F and G any two sheaves on C, define $G^F(A) = Nat(F \times A, G)$, where we identify A with the associated sheaf to the functor representable by A. For the second, define $\Omega(A) = \wp(A)$, where $\wp(A)$ denotes the set of subsheaves of A. □

Now that we know that any Grothendieck is a topos, let us take a look to the "logic" of Grothendieck toposes.For any fixed sheaf F, we have:

— $TRUE_F = (F \to F)$, $FALSE_F = (\emptyset \to F)$, where $\emptyset(A) = F(A) = 1$, if the empty family $(\to A)$ is a covering, otherwise $\emptyset(A) = \varnothing$.

— For any two subsheaves P and Q of F

- $(P\cap Q)(A) = P(A) \cap Q(A)$.
- $a \in (P\cup Q)(A)$ iff there exists a covering $(A_i \to A)_{i\in I}$, such that the restriction of a to $F(A_i)$ belongs either to $P(A_i)$ or $Q(A_i)$.
- $a \in (P \Rightarrow Q)(A)$ iff, for every arrow $B \to A$ in C, whenever the restriction of a to $F(B)$ is in $P(B)$ it happens to be also in $Q(B)$.

Given any $f : F \to G$ morphism of sheaves (natural transformation), if P is a subsheaf of F and Q is a subsheaf of G, then we have:

— $(f^{-1}Q)(A) = (f_A)^{-1}(QA)$

— $a \in \exists_f(P)(A)$ iff a admits locally an antecedent; by this we mean that there exists a covering $(A_i \to A)_{i \in I}$ and elements $a_i \in P(A_i)$ which go via f (rather, via f_{A_i}) to the restriction of a to a_i.

— $a \in \forall_f(P)(A)$ iff, for every arrow $B \to A$ in $\mathbf{C}$, $P(B)$ contains all the antecedents of a i.e., it contains all the elements of $F(B)$ which go via f to the restriction of a to B.

For the higher-order logic, $PF = \Omega^F$ is characterized by $PF(A) = \wp(F \times A)$, and the membership relation of F is characterized by $(a, R) \in \Sigma_F(A)$ iff $(a, id_A) \in R(A)$.

§3. A language for intuitionism

Trying to follow Lawvere's claim: "the notion of topos summarizes in objective categorical form the essence of higher-order logic" [LAWVERE, 1975], let us present a language suitable to deal with topos. Along with this language, we include a theory of types, also due to Joyal [BOILEAU-JOYAL: La logique des topos] which will do quite the same thing as what type hierarchies do in sets

Definition 3.1 A *similarity type* consists of the following data:

(1) A set S of *sorts*, from which we inductively form the set of all types T, as follows

i) any element of S is an element of T,

ii) if $S_1,\ldots, S_n$ are elements of T, then $\Omega\,(\underline{S})$ is in T, where Ω is not in S, and n may be 0.[1]

[1]Intuitively, the types represent sets, and $\Omega(T_1,\ldots,T_1)$ represents the set of parts of $T_1\times\cdots\times T_n$.

(2) A set of *function symbols*. To each function we associate a *source* (finite number of types), and a *target* (a single type). Functions with target $\Omega := \Omega(\,)$ are called *relational symbols*.

(3) The logical symbols, $\in$, $[\{\ |\ \}]$, $(\,)$, and an infinite set of variables of each type.

The terms and formulae are defined as follows:

a) Any variable of type S is a term of type S.

b) If f is a functional symbol with source $(\underline{S})$ and target R, and $(\underline{t})$ are terms of types $(\underline{S})$, then $f((\underline{t}))$ is a term of type R

c) If u and v are terms of the same type, then $u = v$ is a formula

d) If $(\underline{t})$ are terms of types $(\underline{S})$ and u is a term of type $\Omega((\underline{S}))$, then $(\underline{t}) \in u$ is a formula.

e) If φ is a formula and $(\underline{x})$ is a sequence of different variables with types $(\underline{S})$, then $[\{(\underline{x}) \in (\underline{S}) \mid \varphi\}]$ is a term of type $\Omega((\underline{S}))$.

f) If φ and γ are formulae, then $\varphi \wedge \gamma$ is a formula.

g) TRUE is a formula.

i) the procedure given in e) is the only binding-variable operator.

The interpretation of this language in any topos is given by associating to each sort S an object $[S]$, and to any functional symbol f a morphism in the topos; this correspondence is done according to the following rules:

1) $[\Omega(S_1, \ldots, S_n)] = P([S_1] \times \cdots \times [S_n])$

2) $[f] : [S_1] \times \cdots \times [S_n] \to [R]$, f being a function symbol with source $(\underline{S})$ and target R (in particular $[\Omega] = \Omega$).

To extend the interpretation to the whole language, the local character of the theory of topos must be taken into account. So, if $(\underline{x})$ is a finite number of variables of types $(\underline{S})$, and t is any term whose free variables are all among the $(\underline{x})$, one defines the interpretation of the term t relative to the variables $(\underline{x})$ as a morphism $[(\underline{x}) : t] : \prod([\underline{S}]) \to [R]$, where R is the type of t, according to the inductive rule:

a) If x is a variable of type S, then $[x] = [S]$

b) $[(\underline{x}) : x_i]$ = canonical projection $\prod([\underline{S}]) \to [S_i]$

c) $[(\underline{x}) : f(\underline{t})] = [f]\,([(\underline{x}) : t_1], \dots, [(\underline{x}) : t_m])$

d) $[\{(\underline{x}) : t_1 = t_2\}] = \Delta_{[U]}([(\underline{x}) : t_1], [(\underline{x}) : t_2])$, U being the type of the t's

e) $[\{(\underline{x}) : (\underline{t} \in t')\}] = ev_{\prod([\underline{s}])}([(\underline{x}) : t_1], \dots, [(\underline{x}) : t_n], [(\underline{x}) : t'])$

f) $[\{(\underline{x}) : [\{(\underline{y}) \mid \varphi\}]\}] = [\{(\underline{y}),(\underline{x}) \mid \varphi\}]$, provided that the y's do not occur in the x's

g) The interpretation of conjunction and TRUE is by composition with the corresponding maps.

Notice that all the data defining a topos are used in the interpretation. To present the theory, due to the lack of quantifiers (so far), the method will be *à la* Gentzen, i.e., by means of *sequents*.

Definition 3.2 A sequent (entailment) is any expression $\varphi \underset{U}{\Rightarrow} \gamma$, where φ and γ are formulae and U is any set of types containing those of all free variables in φ and γ.

We can think of a sequent as an implication, though the real representation is of the form

$$\bigwedge_{T\in V} \exists x \in T\,(x=x) \to (\varphi \to \gamma)$$

where $x \in T$ stands for x is a variable of type T (cf. [OUELLET: Axiomatisation de la logique interne du premier ordre des topos, version inclusive et multisorte].)

Definition 3.3 A sequent $\varphi \underset{U}{\Rightarrow} \psi$ is said to be valid, relative to the interpretation [ ] and we denote $[\ ] \models \varphi \underset{U}{\Rightarrow} \psi$, iff there exists a sequence $x_1, \ldots, x_n$ of distinct variables containing all free variables of φ and ψ, with types $S_1, \ldots, S_n$, and such that U is the set $\{S_1,\ldots, S_n\} \cap S$ (S is the set of all sorts in the similarity type), and $[\{(x_1,\ldots, x_n) \mid \varphi\}]$ equals $[\{(x_1,\ldots, x_n) \mid \varphi \wedge \gamma\}]$

Definition 3.4 By a *local higher-order theory* is meant any set Γ of sequents. A sequent is said to be valid in the theory Γ if and only if every interpretation in any topos which satisfies all sequents in Γ also satisfies it. We write $\Gamma \models \varphi \underset{U}{\Rightarrow} \gamma$

Let us now describe the deductive system complete with respect to this notion of validity, above defined. The system consists on six axioms and six rules of inference. The first two axioms and the the four first rules are propositional. Axioms III, IV and rule 5 deal with identity and substitution. Axioms V and VI, and rule 6 are topos theoretic versions of comprehension (abstraction) and extensionality.

Axioms

I) $\varphi \Rightarrow \varphi$

II) $\varphi \Rightarrow \mathrm{T}$

III) $\mathrm{TRUE} \Rightarrow x = x$

IV) $\varphi \wedge (x=t) \Rightarrow \varphi\,(t/x)$, provided t is free for x in φ

V) $\varphi \Rightarrow (x_1,\ldots, x_n) \in [\{(x_1,\ldots, x_n) \mid \varphi\}]$

VI) $(x_1,\ldots, x_n) \in [\{(x_1,\ldots, x_n) \mid \varphi\}] \Rightarrow \varphi$

In the above list of sequents, the set U is assumed to be exactly the set of types of free variables intervening on it.

Rules of inference

R.1 $$\frac{\varphi \underset{U}{\Rightarrow} \gamma,\ U \subset V}{\varphi \underset{V}{\Rightarrow} \gamma}$$

R.2 $$\frac{\varphi \underset{U}{\Rightarrow} \gamma \quad \gamma \underset{U}{\Rightarrow} \delta}{\varphi \underset{U}{\Rightarrow} \delta}$$

R.3 $$\frac{\varphi \underset{U}{\Rightarrow} \gamma \quad \varphi \underset{U}{\Rightarrow} \delta}{\varphi \underset{U}{\Rightarrow} \gamma \wedge \delta}$$

R.4 $$\frac{\varphi \underset{U}{\Rightarrow} \gamma \wedge \delta}{\varphi \underset{U}{\Rightarrow} \gamma} \qquad \frac{\varphi \underset{U}{\Rightarrow} \gamma \wedge \delta}{\varphi \underset{U}{\Rightarrow} \delta}$$

R.5 $$\frac{\varphi \underset{U}{\Rightarrow} \gamma}{(t/x) \underset{W}{\Rightarrow} \gamma\,(t/x)}$$ with t free for x, and W is the the set of sorts of the free variables of $\varphi(t/x)$ and $\gamma(t/x)$ plus those of U, except the corresponding to t.

R.6 $$\frac{\varphi \wedge ((\underline{x}) \in t_1) \underset{U}{\Rightarrow} \varphi \wedge ((\underline{x}) \in t_2), \quad \varphi \wedge ((\underline{x}) \in t_2) \underset{U}{\Rightarrow} \varphi \wedge ((\underline{x}) \in t_1)}{\varphi \underset{W}{\Rightarrow} (t_1 = t_2)}$$ the x's are different variables none of which appears free in φ, t_1 or t_2

One proves that these axioms are universally true and that the inference rules preserve the truth. This gives the soundness of the system (i.e., if a sequent is "derivable" from the axioms using the rules, then the sequent is valid), and since validity was defined in terms of the topos, the result becomes a theorem of adequacy.

An important fact which explains the interest of topos theory is that any topos E gives rise in a natural way to a language of the above form. The natural language of the topos E, denoted $L(\mathsf{E})$, comes equipped with a canonical interpretation. $L(\mathsf{E})$ has as sorts the objects of E, and the interpretation is the obvious one.

There are two remarks we want to make about these language and way of presenting theories. The first one is that the other logic connectives and quantifiers can be defined in terms of the ones presented. So, for instance, $\forall x{\in}T\ \varphi$ stands for $\{x \mid \varphi\} = \{x \mid \text{TRUE}\}$, and the canonical interpretation is $[(\underline{x}) : \forall y{\in}T\ \varphi] = \forall_\pi[(y,\underline{x}) : \varphi]$. The second one is that we could have presented the theory in an axiomatic way, instead of doing it *à la Gentzen*, and we would have found that the first-order axioms and rules of inference which are internally valid in any topos are those of the intuitionistic first-order predicate calculus with a unique restriction on the free variables in the rule of *Modus Ponens* (cf. [BOILEAU: Types vs. Topos], [COSTE: Logique d'ordre supérieur dans les topos élémentaires], [OSIUS: Logical and set theoretical methods in elementary topoi].)

Now we give some "tips" to determine the canonical interpretation of a formula in the natural language of a topos E.

Let $\varphi(x_1, \dots ,x_n)$ be a formula whose free variables are all among $x_1, \dots ,x_n$, of types $T_1, \dots ,T_n$, respectively, and let $a_i : \mathbf{1} \to |T_i|$ be global sections, for i=1, … ,n. We can consider these a_i's as being close terms in the language of the same type of the corresponding variables, and hence they can be substituted for the x_i's. In this case one proves that

$$\mathsf{E} \models \varphi(a_1, \dots ,a_n) \text{ iff } (a_1, \dots ,a_n) : \mathbf{1} \to |T_1| \times \cdots \times |T_n| \text{ factors through } [(x_1, \dots ,x_n) : \varphi].$$

We now generalize this result for the case when the a_i's are not necessary of domain $\mathbf{1}$ (we call them generalized elements, or elements defined at a given stage). First of all, recall that if X is any object in E, then $\mathsf{E}/_X$ is also a topos, and the functor $X^* : \mathsf{E} \to \mathsf{E}/_X$ is logical (Theorem 1.2) This functor, preserving the topos structure, preserves the internal logic; more specifically, it induces an application (we keep the same name) $X^* : L(\mathsf{E}) \to L(\mathsf{E}/_X)$.

If $a_i : X \to |T_i|$, then there are $\hat{a}_i : \mathbf{1} \to |X^*(T_i)|$ in the topos $\mathsf{E}/_X$, and we have

$$\mathsf{E}/_X \models X^*\varphi(\hat{a}_1, \dots ,\hat{a}_n) \text{ iff } (a_1,\dots,a_n) : X \to |T_1| \times \cdots \times |T_n| \text{ factors through } [(x_1,\dots,x_n) : \varphi].$$

This trick will be used over and over to characterize $[(x_1, \dots, x_n) : \varphi]$. Now, to finalize this section, we will give some rules which will enable us to determine the validity of a formula for the canonical interpretation in a Grothendieck topos of sheaves on a subcanonical pretopology. This set of rules are known as Joyal-Kripke (also functorial) semantics.

$\mathsf{E} \models \text{TRUE}$ is always true

$\mathsf{E} \models \text{FALSE}$ iff the empty family $(\to \mathbf{1})$ covers $\mathbf{1}$ (or $\mathbf{1} \approx \emptyset$)

$\mathsf{E} \models \varphi \wedge \psi$ iff $\mathsf{E} \models \varphi$ and $\mathsf{E} \models \varphi \wedge \psi$

$\mathsf{E} \models \varphi \vee \psi$ iff there exists a family of representable objects $(X_i)_{i \in I}$ such that $(X_i \to \mathbf{1})_{i \in I}$ is a covering, and $\mathsf{E}/_{X_i} \models X_i^* \varphi$ or $\mathsf{E}/_{X_i} \models X_i^* \psi$, for each $i \in I$.

$\mathsf{E} \models \exists x \in T\ \varphi$ iff there exists a family of representable objects $(X_i)_{i \in I}$ in the topos E such that $(X_i \to \mathbf{1})_{i \in I}$ is a covering, and there are global sections $a_i : \mathbf{1} \to |X_i^*(T)|$ in $\mathsf{E}/_{X_i}$, such that we have $\mathsf{E}/_{X_i} \models X_i^* \varphi(a_i)$.

$\mathsf{E} \models \forall x \in T\ \varphi$ iff for any representable object X, and any global section $a : \mathbf{1} \to |X^*(T)|$ then $\mathsf{E}/_X \models X^* \varphi(a)$.

In what follows we shall not make explicit use of these axioms and rules; we rather employ a sort of naïve intuitionistic logic, just in the same sense one uses a naïve classical logic in the practice of mathematics.

§4. The topos of Dubuc

We could say that Dubuc's topos is the result of applying the methods of algebraic geometry to differential geometry, and it will be the base model to test and develop our theory. In this chapter, we sketch the main results employed in its definition, as well as some of its properties. For more details on the subject we refer to [DUBUC: C^∞-schemes] and [DUBUC: Open covers and infinitary operations in C^∞-rings].

The first step will be the definition, in the context of differential geometry, of a notion equivalent to that of commutative $\mathbb{R}$-algebra in algebraic geometry. The original idea, as well as the means which made it possible, came from Lawvere [LAWVERE: Functorial semantics of algebraic theories]. He introduced a new way to define algebraic theories which enables the following definition

Definition 4.1 A C^∞-ring is a product preserving functor from the category C^∞, with objects all natural numbers, and arrows between n and m all smooth mappings between $\mathbb{R}^n$ and $\mathbb{R}^m$.

So, while in a commutative ring only polynomials can be interpreted, in a C^∞-ring every smooth mapping has an interpretation.

Examples 4.2 The main examples of C^∞-ring are the following

a) $C^\infty(\mathbb{R}^n)$, the set of smooth mappings $\mathbb{R}^n \to \mathbb{R}$. Actually this is the free C^∞-ring on n generators: the projections.

b) $C^\infty(M)$, for M a smooth manifold.

c) $C^\infty_x(M)$, the ring of germs at a point x of M of smooth maps.

d) Any Weil algebra [WEIL: Théorie des points proches sur les variétées différentiables] in particular the $\mathbb{R}$-algebra $\mathbb{R}[X]/_{(X^2)}$ of dual numbers.

Actually, most of the above are examples of the following important result

Theorem 4.3 If A is any C^∞-ring, and $I \subset A$ is any ideal (in the usual algebraic sense), then $A/_I$ can be endowed with a unique C^∞-structure making $A \to A/_I$ a morphism of C^∞-rings. $\square$

Definition 4.4 A C^∞-ring is said to be of finite type if there exist a natural number n and an ideal $I \subset C^\infty(\mathbb{R}^n)$, such that $A = C^\infty(\mathbb{R}^n)/_I$.

To give a morphism of C^∞-rings $C^\infty(\mathbb{R}^n)/_J \to C^\infty(\mathbb{R}^m)/_I$ is equivalent to giving a smooth mapping $\mathbb{R}^m \to \mathbb{R}^n$, modulo the ideal I, and such that

$$\forall \varphi \in C^\infty(\mathbb{R}^n)\ \varphi \in J \Rightarrow \varphi \circ f \in I.$$

What we mean by "modulo I" is an equivalence class with respect to the relation $f \sim g$ if and only if $\pi_i \circ f - \pi_i \circ g \in I$, $i = 1, \dots, n$.

Theorem 4.5 The category $A_{f.t.}$ of C^∞-rings of finite type has finite colimits, in particular we have the following descriptions:

a) initial objet $C^\infty(\mathbb{R}^0) \approx \mathbb{R} = \mathbf{0}$

b) Coproduct $C^\infty(\mathbb{R}^n)/_J \otimes C^\infty(\mathbb{R}^m)/_I = C^\infty(\mathbb{R}^{n+m})/_{p^*I + q^*J}$, where p^*I is the same ideal I, but now thought of as being in $n+m$ variables (similarly for p^*I) [1] □

Note that the set of arrows of C^∞-rings from $C^\infty(\mathbb{R}^n)/_I$ to $\mathbb{R}$ are in a one–to–one correspondence with the sets $Z(I)$ of zeros of the ideal I, i.e., the set of those $x \in \mathbb{R}^n$ on which vanishes every function of I. As in the case of algebraic geometry with the Galois connexion, it will be the dual category $A_{f.t.}^{op}$(or for that matter of a suitable full subcategory) which will be used to capture the geometric intuition. In such a category, we have that the set $Z(I)$ corresponds to the global elements (or points) of the object $\overline{C^\infty(\mathbb{R}^n)/_I}$, where by $\overline{(\ \ )}$ we denote the same object, this time in the dual category.

The idea is now to define a Grothendieck pretopology in $A_{f.t.}^{op}$ that retains the essence of classical open covers of $\mathbb{R}^n$. In a more precise manner, we define the so called open cover topology as the one generated by the empty family ($\mathbf{0} \rightarrow$) and the families in $A_{f.t.}$ of the form

$$(C^\infty(\mathbb{R}^n) \rightarrow C^\infty(U_\alpha))_\alpha$$

for all n and all open covers U_α of $\mathbb{R}^n$.

Once again, the similarity with algebraic geometry is present, as can be seen in the following proposition.

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[1] Actually, from this theorem it follows, just by "general reasons", that the category of all C^∞-rings has finite colimits.

Proposition 4.6 Let $U \subset \mathbb{R}^n$ be an open set, and let f be a characteristic function for U, i.e., $U = \{x \in \mathbb{R}^n \mid f(x) \neq 0\}$. Then

$$C^\infty(\mathbb{R}^n) \to C^\infty(U) \approx C^\infty(\mathbb{R}^n)\{f^{-1}\}$$

is the universal solution to the problem of making f invertible in the category $A_{f.t.}$.

Explicitly, any open cover $(A_\alpha \to A)_\alpha$ is obtained as a pullback (pushout in $A_{f.t.}$)

$$\begin{array}{ccc} C^\infty(\mathbb{R}^n) & \to & C^\infty(U_\alpha) \\ \downarrow & & \downarrow \\ A & \to & A\{a_\alpha^{-1}\} \end{array}$$

where a_α is the class, module the ideal of definition of A, of a characteristic function for U_α.

The basic open coverings, $\{C^\infty(\mathbb{R}^n) \to C^\infty(U_\alpha)\}$ are effective epimorphic, but they are not universal effective epimorphic. Indeed, if one looks at a representative example of the failure of this universallity (cf. [DUBUC: Open covers and infinitary operations in C^∞-rings]), namely the pushout along the arrow

$$C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)/_I$$

with $I = \{f \in C^\infty(\mathbb{R}^n) \mid f \text{ is of compact support}\}$, one is led to the following

Definition 4.7 An ideal $I \subset C^\infty(\mathbb{R}^n)$ is said of local character if whenever $f \in C^\infty(\mathbb{R}^n)$ it satisfies the following: if $f|_{U_\alpha} \in I|_{U_\alpha}$ for every α, where (U_α) is some open cover of $\mathbb{R}^n$, then $f \in I$. □

Dubuc has given several equivalent conditions for an ideal to be of local character. We collect them in the following proposition

Proposition 4.8 For an ideal $I \subset C^\infty(\mathbb{R}^n)$, the following conditions are all equivalent

i) I is of local character, in the sense of 4.7.

ii) $f|_p \in I|_p$, for all $p \in Z(I)$, implies $f \in I$. (Where $f|_p$ denotes the germ of f at p).[1]

iii) $\Phi_\alpha f \in I$, for some partition of unit $\{\Phi_\alpha\}_\alpha$, implies $f \in I$.

iv) $f_i \in I$ implies $\sum_i f_i \in I$, for every locally finite family f_i. (Locally finite means that each point of $\mathbb{R}^n$ has a n open neighborhood U such that $f_i|_U = 0$, except for a finite number of i's.) □

Corollary 4.9 (*Nullstellensatz*) For an ideal I of local character, $Z(I) = \varnothing$ iff $1 \in I$.

Examples 4.10 Among the many examples of ideals local character, we mention:

a) Any finitely generated ideal of $C^\infty(\mathbb{R}^n)$

b) Any closed ideal I of $C^\infty(\mathbb{R}^n)$ for the C^∞-strong topology (see Chapter 2, §1 and the references therein.)

c) The ideal of germs of functions $\mathfrak{m}_F^g = \{f \in C^\infty(\mathbb{R}^n) \mid \forall x \in F\ f|_x = 0\}$, for F a closed subset of $\mathbb{R}^n$.

d) An example of ideal which is *not* of local character is $\{f \mid f \text{ is of compact support}\}$. See comment above theorem 4.7.

To end this collection of results, let us mention that every ideal has a *closure* which is a local character ideal.[2] Moreover the category B^{op}, of duals of C^∞-rings of finite type presented by ideals of local character, has finite limits (he only difference appears in the product, where the ideal has to be substituted by its closure), and the open coverings form a universal effective epimorphic class; as a consequence, the pretopology is subcanonical, i.e., every representable functor is a sheaf.

Now we have fixed the problem pointed just before definition 4.7, in the sense of

[1]Due to this condition, sometimes these ideals are called *germ determined.*

[2]It is an actual closure operator for a well determined topology.

Proposition 4.11 The open covers of $\overline{A}$ are of the form $\{\ \overline{A_{U_i}} \to \overline{A}\ \}$ where the U_i form an open cover of $\Gamma(\overline{A})$, and $A_{U_i} = C^\infty(\mathbb{R}^n)/_{I|U_i}$. $\square$

Definition 4.12 The category of sheaves on B^{op}, $\widetilde{B^{op}} \hookrightarrow Set^B$, with the open cover topology is called the Dubuc topos, an will be denoted by $\mathcal{G}$.

As a consequence of Whitney's embedding theorem (cf. [GUILLEMIN-POLLACK: Differential Topology]) every manifold is a retract of some euclidean space. Therefore, the ideal of presentation of $C^\infty(M)$ is finitely generated, and by example 4.10 a), is of local character. The inclusion $C^\infty(\ ) : \mathcal{M} \to B^{op}$ is full, and composed with Yoneda embedding gives $\mathcal{M} \to \mathcal{G}$, full embedding ($\mathcal{M}$ denotes the category of smooth paracompact manifolds, and the functor lands in $\mathcal{G}$ by remark above definition 4.11). Moreover, this functor preserves open coverings (in the sense that they remain effective epimorphic families), transversal pullbacks, and the terminal object.

1

The framework of Synthetic Differential Geometry

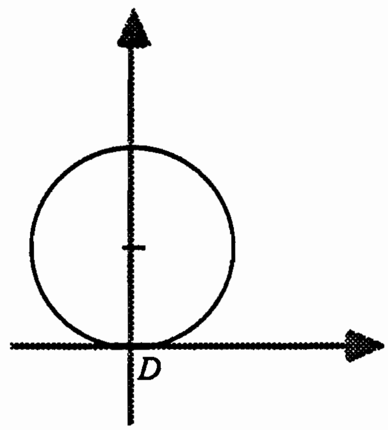
§1. First axioms

It has become a common practice among mathematicians and physicists to employ a sort of loose talking about first-order approximations. Very often they get to think that the quantities at hand are so small that their square are negligible (null for all practical purposes.) In those situations, it would be helpful to have at one's disposal a setting on which explicit considerations and rules for treating these *infinitesimal quantities* could be given. To that end, an object of the form

$$D = [\{d \in R \mid d^2 = 0\}]$$

is required, while on R we have a commutative ring structure and no need for every element to have an inverse.

This object, should Pythagoras' Theorem be true in R, is the intersection of the circle of radius 1 around $(0,1)$ with the x-axis, as in the picture below



where we identify R with the x-axis in R^2. If we extend this intuition to any function

$$f : D \to R,$$

by saying that f is a linear function, i.e., there is a unique $m \in R$ (the slope) so that, for any d

$$f(d) = f(0) + d \cdot m$$

we get an identification of tangent vectors with pairs of elements of R.

On the other hand, the concept of smooth manifold, as understood by Riemann's followers, is not suitable for many of the goals he proposed them for [RIEMANN: Uber diegen Hypothesen welche der Geometrie zu Grunde Leigen]. In particular, the set of all smooth mappings between two smooth manifold is not itself a manifold.

These considerations led Lawvere to propose, in a series of conferences given in 1967 at the University of Chicago [LAWVERE: Categorical dynamics], a new setting for the development of differential geometry. The proposal is to work in a category of "smooth spaces" (which at least would be a cartesian closed category) where there must be an object R, "the line", and $D \subset R$ an [infinitesimal neighborhood of $0 \in R$.]

A. Kock took up these ideas and stated the basic axiom in the following form: The map

$$R \times R \xrightarrow{\alpha} R^D,$$

defined by the rule $[(a,b) \mapsto [d \mapsto a + d{\cdot}m]]$, is invertible [KOCK: Synthetic Differential Geometry]: the object D is small enough to make the graph of any function a piece of a straight line, and big enough to make this line unique. Identifying m with $f'(0)$, the function is determined by its *1-jet* at 0, and the axiom can be seen as an axiom of *1-jet* representability.

Kock has also shown how to develop the entire basic calculus of derivatives and Taylor series expansions. For any function $f: R \to R$, and any given $p \in R$, it follows that, for all $d \in D$

$$f(p+d) = f(p) + d{\cdot}f'(p),$$

an we get a new function $f': R \to R$. [1]

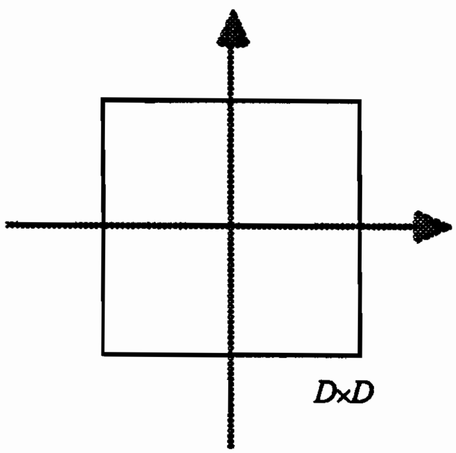
As a matter of fact, to determine $f'(p)$ is enough to have f defined on all elements of the form $p+d$, $\forall d \in D$. In a similar way, if $F \in R^{R^n}$ one can define the partial derivatives

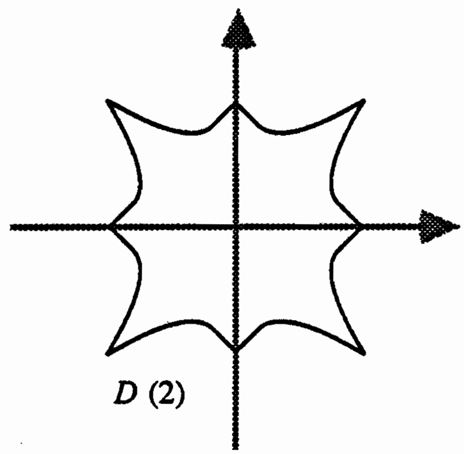
$$F(x_1, \dots ,x_i+d, \dots ,x_n) = F(x_1, \dots ,x_n) + d\,\frac{\partial F}{\partial x_i}(x_1, \dots ,x_n),$$

and the iteration of these processes allows the construction of the derivatives of higher order. So, to compute the, say, *2-jet* at 0, it is enough to have f defined at any element of the form d_1+d_2, with $d_1,d_2 \in D$. Notice that not always does one have that such an element belongs to D, yet its cube vanishes, and therefore somehow it is infinitesimal of second order. We could require that the information given by the *2-jet* were the same contained in the restriction of f to the object D_2 of elements x form R such that $x^3 = 0$. If F is defined on $D \times D$, we can determine $\frac{\partial F}{\partial x}(0,d)$, for all $d \in D$. Derive again, and get $\frac{\partial^2 F}{\partial y \partial x}(0,0)$.

Note that to define $\frac{\partial F}{\partial x}(0,0)$ one only needs F to be defined in a smaller object, $D(2) = \{[\,(x,y) \in D \times D \mid x^2 = y^2 = xy = 0]\}$.

[1] The same result holds for generalized (not just global) elements of R^R, for the existence, being unique, is "on the spot" [KOCK: Synthetic Differential Geometry, p. 140]





To determine $\dfrac{\partial^2 F}{\partial x^2}(0,0)$ and $\dfrac{\partial^2 F}{\partial y^2}(0,0)$ we need F defined on $D_2{\times}\{0\}$ and $\{0\}{\times}D_2$, respectively The information of the *2-jet* is contained in the restriction of F to $\{0\}{\times}D_2 \cup D{\times}D \cup D_2{\times}\{0\}$ and we impose, by axiom, that it extends to $D_2(2) = \{[(x,y)\in D \times D \mid$ *the product of any three of the coordinates vanishes*$]\}$. By doing so, we have that the *k-jet* at $\underline{0}$ of a function $F \in R^{R^n}$ is representable by $D_k(\mathrm{n})$. Following Bunge and Dubuc [BUNGE-DUBUC: Local concepts in S.D.G. and germ representability] we call these objects Ehresmann infinitesimal, to honor the introductor of the notion of jet in differential geometry [EHRESMANN: Les prolongements d'une variétée différentiable].

There is a problem with this class of objects, namely that although the product $D_r(\mathrm{n}){\times}D_s(\mathrm{m})$ is infinitesimal, it is not Ehresmann; in particular, the iterated jet bundle is not a jet bundle in the sense of Ehresmann. To treat this pathology, A. Weil [WEIL: Théorie des points proches sur les variétées diférentiables] introduced a class of algebras and developed a theory of jets that generalizes the work of Ehresmann for the algebras $R[D_r(\mathrm{n})]$. Essentially, a Weil algebra is a multiplication table on a finite dimensional module, and this information can be coded in a matrix with coefficients from the ground ring.

Definition 1.1 A Weil algebra W is an augmented commutative $\mathbb{R}$-algebra of finite dimension, whose augmentation ideal is nilpotent, i.e., W is equipped with a morphism $\pi : W \to \mathbb{R}$, such that:

a) W is local with maximal ideal $I = \pi^{-1}\{0\}$

b) W is a finite dimensional $\mathbb{R}$ vector space

c) I is nilpotent

Since there exists a unique integer l such that $W \approx \mathbb{R}^{l+1}$, if $\{e_0, \dots ,e_l\}$ is a linear base, then we have $e_i \cdot e_j = \sum_{k=0}^{l} \gamma_{ij}^{k}\, e_k$ in a unique way. The information contained in this matrix can be used to define an R algebra structure on any category (with finite limits); we denote it by $R[W]$ or, $R \otimes W$.[1] The presentation (h_i) of the $\mathbb{R}$ algebra can be used to carve out a subobject of R^n

$$Spec_R(W) = [[\, (x_1, \dots ,x_n) \in R^n \mid h_i(\underline{x}) = 0,\ \forall i\,]].$$

The restriction of a polynomial $\rho \in R[X_1, \dots ,X_n]$ to $R \otimes W$ (quotient of $R[X_1, \dots ,X_n]$ by the ideal generated by the h_i's.) This defines a morphism $R^{l+1} \approx R \otimes W \xrightarrow{\alpha} R^{Spec_R(W)}$, and the axiom takes now the form

AXIOM I (*Kock-Lawvere*) For any Weil algebra W, the morphism $R \otimes W \xrightarrow{\alpha} R^{Spec_R(W)}$, defined by $[\xi \mapsto [p \mapsto \xi(p)]]$, is an isomorphism.

Along with this axiom goes a companion axiom which states that those objects representing jets are tiny in the following sense

AXIOM II The functors $(-)^{D_r(n)}$ have right adjoints.

We state the following result [DUBUC: C^∞-schemes]

Proposition 1.2 AXIOM I and AXIOM II hold in the Dubuc topos $\mathcal{G}$, where R is the sheaf represented by $\overline{C^\infty(\mathbb{R})}$, and D is representable by $\overline{C^\infty(\mathbb{R})/(X^2)}$.[2] $\square$

An important "coincidence" is the following. Weil has shown how to see the tangent bundle to any manifold, in particular to $\mathbb{R}$, as representable by what he called the local algebra of dual numbers. He declared that his sources were *d'une part le retour aux méthodes de Fermat dans le calcul infinitésimal du premier order et d'autre la théorie des jets dévelopée dans ces dernières années par Charles Ehresmann*. In this aspect, Synthetic Differential Geometry is a natural continuation and completion of Ehresmann's foundational work of the 50's.

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[1]It is well stablished that $R[W]$ is independent of the particular presentation we chose.

[2]Note that (X^2) is the ideal of presentation of the Weil algebra of dual numbers.

To define the derivative of a function, another method that goes back, once more, to Fermat is available. We recall from [REYES (editor): Analyse C^∞] the following

Definition 1.3 The ring R is a Fermat ring if

$$\forall f \in R^R \; \exists! g \in R^{R^2} \; \forall x,y \in R \; [f(y)-f(x) = (y-x)\cdot g(x,y).$$

This unique g is denoted ∂f, and if R is also of line type, then $\partial f(x,x) = f'(x)$, and we adopt the following postulate (also discussed in [KOCK: Synthetic Differential Geometry], where it is related to an axiom of integration, and in [PENON: De l'infinitésimal au local]) [1]

AXIOM V (*Reyes-Fermat*) R is a Fermat ring.

Using this new approach, the two corollaries below easily follow

Corollary 1.4 Given $f: R^n \to R$, there exist n functions $g_1, \ldots, g_n : R^n \times R^n \to R$, such that

i) $\forall x \in R^n \forall y \in R^n \; [f(x) - f(y) = \sum_{i=1}^{n} g_i(x,y)\cdot(x_i - y_i)]$

ii) $\forall x \in R^n \; (g_i(x,x) = \frac{\partial f}{\partial x_i}(x))$. □

Corollary 1.5 Given $f: R^n \to R^p$, there exists $g: R^n \times R^n \to Mat(n \times p)$ such that

i) $\forall x \in R^n \forall y \in R^n \; [f(x) - f(y) = g(x,y)\cdot(x_i - y_i)]$

ii) $\forall x \in R^n \; (g(x,x) = D_x f)$. □

<u>NB</u> in both corollaries, everything is meant to be internal as the axiom itself. For instance, the last one reads $\forall f \in R^{pR^n} \; \exists g \in R^{p \cdot nR^{n+n}}$ …….

[1]Notice that, as Kock has shown, unique existence is decided on the spot, not locally, just the same result used to "describe" internal functions by means of rules, as in AXIOM I (cf. [KOCK: Synthetic Differential Geometry].)

As one would expect, AXIOM V holds in our test model G [PENON: De l'infinitésimal au local, p. 48]

§2. Linear algebra and order

We have indicated in §1 how to define, for any function, say $f: R^n \to R^p$, its Jacobian [1]. Now, if one wants to introduce the usual concepts which involve this matrix, at least the notions of field and linear independence are needed.

To introduce them, the first problem appears with the several (classically equivalent) notions of field, that turn out not to be equivalent at the intuitionistic level. We follow A. Kock [KOCK: Universal projective geometry via topos theory] to state

Definition 2.1 A commutative ring A in a topos $\mathcal{E}$ is a field (we will say Kock's field) if for each $n = 1,2,\ldots$

$$\neg(1 = 0) \text{ and}$$

$$\neg\Big(\bigwedge_{i=1}^{n}(x_i = 0)\Big) \Rightarrow \bigvee_{i=1}^{n}(x_i \# 0),$$

where, by $x\#0$ we mean that x is invertible.

Since we are definitely interested in notions of the type mentioned earlier, we impose in our setting the following

POSTULATE A The ring R is a field in the sense of Kock.

Fairly easy consequences of this postulate (cf.[KOCK: Synthetic Differential Geometry]) are collected without proof in the proposition below

Proposition 2.2 If R is a Kock field, then the following hold

a) R is a local ring, in the sense of that $\forall x,y \in A$ $(x+y \# 0 \Rightarrow x\#0 \vee y\#0$

[1] Of course, the same goes for internal functions.

b) $R^* = \neg\{0\}$, where R^* denotes the subobject of invertible elements in R.

c) For any Weil algebra W, $Spec_R(W) \subset \Delta(n) = \neg\neg\{0\} \subset R^n$, for appropriate n □

We give, now, the notion of linearly independent

Definition 2.3 A n-tuple of elements $\{v_1, \ldots, v_n\}$ (in an R module M) form a linearly independent set if the following holds

$$\forall \lambda_1, \ldots, \lambda_n \in R \left(\sum_{i=1}^{n} \lambda_i \cdot v_i = 0 \Rightarrow \lambda_1 = \cdots = \lambda_n = 0 \right).$$

Over a Kock field, this notion is equivalent to the following one [ROUSSEAU: Eigenvalues of symmetric matrices on topoi]

$$\forall \lambda_1, \ldots, \lambda_n \in R \left(\bigvee_{i=1}^{n} (\lambda_i \# 0) \Rightarrow \sum_{i=1}^{n} \lambda_i \cdot v_i \# 0 \right).$$

It is easy to see that POSTULATE A is exactly what is needed to have the "only if" part (the "if" part is straight forward) of the following

Proposition 2.4 For any matrix $A \in R^{n \cdot m}$ we have that *row-Rank*$(A) \geq r$ if and only if *determinant-Rank*$(A) \geq r$, where we say that the n-tuple of elements of R^m has *row-Rank* $\geq r$ if there exists a sub-r-tuple linearly independent. Similar result holds for *column-Rank*. In particular *row-Rank*$(A) \geq r$ if and only if *column-Rank*$(A) \geq r$.

In a different direction, if we want to introduce notions that utilize intervals in $\mathbb{R}$, an order relation has to be available. We require R to have defined an order relation, compatible with the ring structure, strict, local and separated, i.e. we adopt the following

POSTULATE WA1.2 (*Bunge-Dubuc*) On R there is defined an order relation "<", satisfying

R1. $\forall x, y \in R\ [(x>0) \wedge (y>0) \Rightarrow (x+y>0) \wedge (x \cdot y>0)$, and $1>0$

R2. $\forall x \in R\ \neg(x>x)$

R3. $\forall x,y \in R\ [(x>y) \Rightarrow \forall z \in R\ [(x>z) \vee (z>y)\ ]]$

R4. $\forall x_1, \dots ,x_n \in R\ \ [\neg(\bigwedge_{i=1}^{n}(x_i = 0)) \Rightarrow \bigvee_{i=1}^{n}((x_i>0) \vee (x_i<0))\ ].$

Compatibility gives transitivity, and from these conditions and the fact $R^* = \neg\{0\}$, we get an order relation on R, that is total on the units. A useful result can be derived [BUNGE-DUBUC: Local concepts in S.D.G. and germ representability]

Proposition 2.5 If R satisfies **R1 - R4** above, then the following holds

$$\forall x,y \in R\ [(x > 0) \wedge (y > 0) \Rightarrow \exists z \in R\ (z > 0) \wedge (z < x) \wedge (z < y)]. \quad \square$$

In algebraic geometry, the right topology has turned out to be the one determined by the ring structure of R. More precisely, the Zariski process consists on building up the open sets out of a basic one, namely $R^* \subset R$, by pulling back and forming unions. In our setting, since we have an (strict) order relation on R, another way is available: the basic opens in R are the open intervals $(a\text{-}\varepsilon, a+\varepsilon)$, for $a \in R$, $\varepsilon \in R$, $\varepsilon > 0$, and in R^n, just the products of these intervals; all other opens are formed with unions. This topological structure[1] is called Euclidean; given any part M of R^n, the Euclidean topological structure on M consists on those parts U of M for which the following holds:

$$\forall x \in M\ [x \in U \Leftrightarrow \exists \varepsilon>0\ (B(x,\varepsilon) \cap M \subset U)],$$

where $B(x,\varepsilon) = [\!|\ y \in R^n \mid \bigwedge_{i=1}^{n} (\ \text{-}\varepsilon < y_i\ \text{-}x_i < \varepsilon\ )\ |\!]$.

From POSTULATE WA1.1 we derive the following suggesting property for the ring R

$$\neg\neg\{x\} = \bigcap_{0<\varepsilon} (x\text{-}\varepsilon, x+\varepsilon)$$

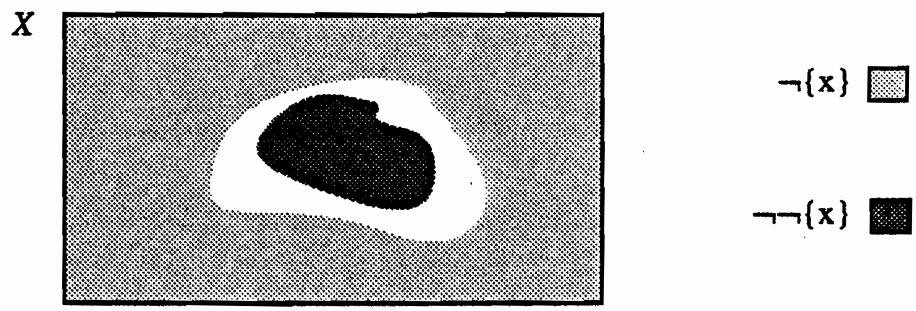
which talks about the infinitesimal nature of $\neg\neg\{x\}$, and that reminds us of the infinitesimal

---

[1]It is important not to confuse up the concepts of (pre) *topology* of Grothendieck and *topological structure* [PENON: De l'infinitésimal au local]. A topological structure on an object X of a topos $\mathcal{E}$ is a sublocale of $PX = \Omega^X$, i.e., a part closed under finite meets (including empty), and arbitrary unions.

monad of a point in Nonstandard analysis; however, notice that here the quantification is over all $\varepsilon > 0$, not only over the standard ones.

Once again, the richer meaning of the negation ($\neg$) in our setting can be exploited to analyze the meaning of the above equality. If one adds a metric content to the balls $B(x,\varepsilon)$, to assert the negation of $x{=}y$ amounts to saying that x and y are well separated, and $\neg\{x\}$ is the object of those elements which are well separated from x. This way, $\neg\neg\{x\}$ appears as the object of elements of X which are not well separated from x. The non validity of the statement $\neg\neg\{x\} \cup \neg\{x\} = X$ says that, in general, there is a part of X with no explicit description in the topos



This can be used to define a topological structure (see definition II.2.1) on any object X of a topos E [PENON: Topologie et intuitionisme]: a part U of X will be open if it contains $\neg\neg\{x\}$ and this "no man's land" for each of its elements.

Definition 2.6 Given X, an object in E, a part $U \subset X$ is Penon (or intrinsic) open if

$$\forall y \in U \ \forall x \in X \ (\neg(x=y) \vee x \in U).$$

Now we collect the most important properties of Penon opens, in the following proposition:

Proposition 2.7 Penon opens are stable under the following manipulations:

a) Change of base (i.e., if $f : X \to Y$ is an application, and $U \subset Y$ is any Penon open, then $f^{-1}(U)$ is Penon open in X.

b) composition (i.e., U open of X, V open of U, then V open of X)

c) arbitrary unions (even if indexed by objects of $\mathbb{G}$)

d) finite intersections. □

The requirements we have imposed on R confer to this topology some nice properties; among them, let us mention the fact that R is separated (T_1) [DUBUC-PENON: Objets compacts dans les topos]

Proposition 2.8 R is separated (T_1) for the Penon topological structure, i.e., it satisfies any of the following equivalent conditions:

i) $\forall x,y,z \in R\ (\neg(x=y) \Rightarrow \neg(z=x) \vee \neg(z=y))$

ii) $\forall x \in R\ \neg\{x\}$ *open of* R. □

Proposition 2.9 R is separated (T_2) for the Penon topological structure, i.e., $\neg\Delta_R$ is open in $R \times R$. □

Time has come for us to compare the two topologies we have so far introduced. In the presence of POSTULATE WA1.2, the euclidean topological structure is subintrinsic, i.e. $\mathbf{E}(X) \subset \mathbf{P}(X)$ (cf. [PENON: De l'infinitesimal au local].) The converse is not always true. It will be so if the object X satisfies the condition given in the following definition

Definition 2.10 The Euclidean topological structure on X satisfies the covering principle if the following condition holds:

$$\forall H,G \in \Omega^X (H \cup G = X \ \Rightarrow\ \iota(H) \cup \iota(G) = X),$$

where $\iota(\ )$ denotes the interior operator corresponding to $\mathbf{E}(X)$ as sublocale of Ω^X.

We impose this condition by adopting

POSTULATE **WA1.1** (*Bunge-Dubuc*) The euclidean topology satisfies the covering principle.

In our test model G, POSTULATE A, and POSTULATE WA1 (meaning 1 and 2) hold (cf. [KOCK: Synthetic Differential Geometry, p. 267] and []**)

In the Dubuc topos G, Penon (cf. [PENON: De l'infinitesimal au local]) has given interesting classifications for open parts in several cases. Among them we quote the following

Proposition 2.11 Let $\bar{A}$ be any representable object in G, say $\bar{A} = \overline{C^{\infty}(\mathbb{R}^n)/_I}$, then $X \subset \bar{A}$ is Penon open if and only if $\Gamma(X) \subset \Gamma(\bar{A}) = Z(I)$ is open in the usual sense with the induced topology. □

This result is a consequence of the existence of a right adjoint functor Λ to the global sections Γ considered as functors from $\wp(X)$ to $\wp(\Gamma(X))$. If $S \subset \Gamma X$, then $\Lambda(S) \subset X$ is characterized, in terms of generalized elements, as follows:

$$\bar{C} \cdots\cdots> \Lambda(S) \subset X \qquad \text{iff} \qquad \Gamma(\bar{C}) \cdots\cdots> S \subset \Gamma X.$$

Apart from the property $\Gamma\Lambda = id$, these functors have other nice properties in our setting, in particular in G. Among them we single out the following (cf. [DUBUC-PENON: Objects compacts dans les topos] or [PENON: De l'infinitésimal au local])

Proposition 2.12 Given any map $f : X \to Y$ in G, we have the commutative squares

$$\begin{tikzcd}
\wp(Y) \arrow[r, "f^{-1}"] \arrow[d, shift right, "\Gamma"'] & \wp(X) \arrow[r, "\neg"] \arrow[d, shift right, "\Gamma"'] & \wp(X) \arrow[d, shift right, "\Gamma"'] \\
\wp(\Gamma(Y)) \arrow[r, "f^{-1}"] \arrow[u, shift right, "\Lambda"'] & (\Gamma(X)) \arrow[r, "\neg"] \arrow[u, shift right, "\Lambda"'] & \wp(\Gamma(X)) \arrow[u, shift right, "\Lambda"']
\end{tikzcd}$$

In addition, as a consequence of the universal property of the functor Λ, we have $\Lambda(\Gamma X \times P) = X \times \Lambda P$ □

The result contained in 2.11 is that Γ and Λ establish a bijection between Penon open parts of $\bar{A}$ and usual open subsets of $\Gamma(\bar{A})$. $X \subset \bar{A}$ is Penon open if and only if $\Gamma(X) \subset \Gamma(\bar{A}) = Z(I)$ is open and in this case $X = \Lambda\Gamma(X)$.

For particular kinds of objects, this bijection admits a concrete interpretation. In this direction we have [DUBUC: Germ representability and local integration of vector fields in a well adapted model of S.D.G.]

> **Proposition 2.13** For any object of the form ιM (where $\iota : \mathsf{M} \to \mathsf{G}$ is the full embedding of comment after definition 0.4.12), Γ and ι establish a bijection between Penon open parts of ιM and (classical) open subsets of M. □

§3. Germs in S.D.G.

In §1 we saw the definition of the infinitesimal objects $D_r(n)$ ($r = 1, 2, \dots, n = 1, 2, \dots$), which played an important rôle in the synthetic theory of jets. For a fixed n, it is clear that one has a chain of inclusions

$$D_1(n) \subset D_2(n) \subset \cdots \subset D_\infty(n) \subset \Delta(n) = \neg\neg\{\underline{0}\},$$

where $D_\infty(n)$ denotes the inductive limit of the $D_r(n)$'s; The last inclusion follows from the identity $\Delta(n) = \Delta^n$, which in turn is a consequence of the two (intuitionistically) valid inference rules [DUMMET: Elements of intuitionism]

$$\frac{\neg(p \vee q)}{\neg p \wedge \neg q} \qquad \text{and} \qquad \frac{\neg p \vee \neg q}{\neg(p \wedge q)},$$

together with POSTULATE A (see §2.1). The objet $\Delta(n)$ is the largest infinitesimal object and has many interesting topological properties; for instance, not only it is true that $\Delta(n)$ is included in any Penon open part of R^n which contains $\underline{0}$, but also $\Delta(n)$ is the largest of such: $\Delta(n)$ is the intersection of all open parts of R^n containing $\underline{0}$. (the same result is true for any subintrinsic topological structure that satisfies the separation condition (T_1) of Proposition 2.8, [BUNGE-DUBUC: Local concepts in S.D.G. and germ representability].) Due to this fact, if we define the notion of germ at 0 as usual, namely as equivalence classes of pairs (f,U),

$f \in R^{R^n}$, $U \in \mathrm{P}(R^n)$ [1] the restriction to $\Delta(n)$ should be an invariant for each class. We would have a map

$$j : C^g_0(R^n, R) \to R^{\Delta(n)},$$

where $C^g_0(R^n, R)$ denoted the object of germs at 0 of functions R^{R^n}.

It has been emphasized by several authors (cf. [BUNGE: Synthetic aspects of C^∞-mapings], [BUNGE-DUBUC: Local concepts in S.D.G. and germ representability] and [PENON: De l'infinitésimal au local]) that the notion of germ in S.D.G. should be treated level with that of jet. In particular, claims have been made in the sense that the true setting to develop S.D.G. is a model on which germs are representable, just the same as jets are [BUNGE-DUBUC, op. cit.]. If that is to be so, a clear candidate to represent germs at 0 of functions in R^{R^n} is $\Delta(n)$, and this is the meaning of the following axiom

AXIOM III (*first version*). For each positive integer n, the restriction map j is invertible.

N.B. In presence of POSTULATE A, the same axiom applies to the euclidean topology.

We will need a stronger form of this axiom, namely a version identifying germs around some "closed" manifold rather than around an element of R^n. So, we require

AXIOM III (*Bunge-Dubuc*) For any pair of positive integers k, n, the restriction map

$$j : C^g_{R^k \times \{0\}}(R^k \times R^n, R) \to R^{R^k \times \Delta(n)},$$

is invertible.

Before discussing the validity of the axiom in our test model G, let us give some results concerning the behavior of Δ.

Proposition 3.1 For any $x \in R^n$ and $f \in R^{\Delta(n)}$ the following holds:

i) There exists an isomorphism $a_x : \Delta(n) \to x + \Delta(n)$, the addition by x.

---

[1] $(f,U) \sim (h,V)$ iff $\exists W \in \mathrm{P}(R^n)$ $[0 \in W \subset U \cap V \wedge f_{|W} = h_{|W}]$.

ii) $\forall x' \in \neg\neg\{x\}\ f(x') \in \neg\neg\{f(x)\}$

iii) $\forall x' \in \neg\neg\{x\}\ (\neg\neg\{x\} = \neg\neg\{x'\})$ □

Those (easily checked) assertions have a number of consequences in the form of simplified definitions.

Definition 3.2 A germ is an element of $\neg\neg\{y\}^{\neg\neg\{x\}}$, for some x and y.

We discuss now the validity of AXIOM III in the Dubuc topos, G. In this model, mappings $R^n \to R$ are essentially "smooth maps" in n variables, and the open neighborhoods in the axiom are euclidean opens (see N.B. in AXIOM III.) On the other hand, we have the following results [PENON: De l'infinitésimal au local]

Proposition 3.3 In G, $\Delta(n)$ is representable by the dual of the C^∞-ring $C_0^\infty(\mathbb{R}^n)$, of germs at 0 of functions $\mathbb{R}^n \to \mathbb{R}$. □

Note that this is a smooth C^∞-ring, as it has a presentation with an ideal of local character, namely the ideal $\mathfrak{m}^g_{\{0\}}$ of functions $\mathbb{R}^n \to \mathbb{R}$ whose germ at 0 vanishes.

Proposition 3.4 In G, the global elements are in bijective correspondence with germs at $0 \in \mathbb{R}^n$ of functions $\mathbb{R}^n \to \mathbb{R}$. □

Proposition 3.3 gives the epi part of the axiom; for the injective part see [DUBUC: Germ representability and local integration of vector fields in a well adapted model of S.D.G.]

AXIOM III comes with a companion axiom stating the tininess of Δ, namely

AXIOM IV The functor $(\ )^\Delta$ has a right adjoint $(\ )_\Delta$.

This axiom holds in G basically by the same reason AXIOM II does, due to the non existence of non-trivial covers for Δ.

§4. Vector fields in S.D.G.

The theory of vector fields makes clear the need of a cartesian structure on the category E of smooth sets, as claimed by Lawvere [LAWVERE: Categorical Dynamics]. In his own words, *the representability of tangent bundles by objects like* D, *leads to considerable simplifications of several concepts, constructions and calculations*. For instance, a first-order differential equation, or vector field, on R^n (we write E for R^n) is usually defined as a section $\hat{\xi}$ of the projection $\pi : E^D \to E$, i.e.,

$$\hat{\xi} : E \to E^D, \text{ satisfying } \pi \circ \hat{\xi} = id_E. \tag{4.1}$$

But, by the λ-conversion rule, $\hat{\xi}$ is equivalent to

$$\xi : E \times D \to E, \text{ satisfying } \xi(p, 0) = p, \forall p \in E, \tag{4.2}$$

which, in turn, is equivalent by a further λ-conversion to

$$\check{\xi} : D \to E^E, \text{ satisfying } \check{\xi}(0) = id_E. \tag{4.3}$$

that is, an infinitesimal path in the space E^E of all transformations of E, or an infinitesimal deformation of the identity map. This is a feature that the classical approach lacks though they do like talking about infinitesimal transformations as synonymous for vector fields.

In what follows we will use the following terminology

Definition 4.1 A vector field on E is any of the equivalent data (4.1), (4.3)

It is easy to check that the data of (4.2) and the properties of R force ξ to be an infinitesimal flow in the sense of the following definition

Definition 4.2 A flow in E is a family of curves, $f : U \subset E \times R \to E$, one for each $p \in E$, such that $f(p, -)$ is defined on some part of R that contains D and passes by p at time 0 and such that

$$f(p, t+s) = f(f(p, t), s).$$

Definition 4.3 A flow is an integral flow of a vector field $\xi : D \to E^E$, if the velocity vectors are the field vectors, i.e.,

$$f(p, t+d) = \xi(f(p, t), d), \forall (p, t) \in E \times R \mid (p, t+d) \in U \; \forall d \in D.$$

In presence of AXIOM I, $\xi(p, d) = p + d \cdot g(p)$, where $g : E \to R^n$, is called the principal part of the vector field. Thus, to have an integral flow for ξ means having a solution on U for the differential equation

$$\begin{cases} \dfrac{\partial f}{\partial t}(p, t) = g(f(p, t)), \; \forall (p, t) \in E \times R \mid (p, t+d) \in U \; \forall d \in D \\ f(p, 0) = p \end{cases}$$

Therefore, according to these definitions, any vector field $\xi : D \to E^E$, comes automatically integrated to an infinitesimal D-flow. It is easy to see [BUNGE-DUBUC: local concepts in SDG and germ representability] that this flow has a unique extension to a D_∞-flow $\zeta : E \times D_\infty \to E$,

i) $\zeta(p, d) = \xi(p, d), \; \forall d \in D$

ii) $\zeta(p, t+d) = \zeta(\zeta(p, t), d) \; \forall d, t \in D_\infty$

In [BUNGE-DUBUC, op. cit.] it was shown that AXIOM III is the key tool to pass from infinitesimal to local. In particular, to get a result of local integration of vector fields (or a local solution for a differential equation) it suffices to require

POSTULATE WA2 (*Δ integration of vector fields*) For any positive integer n, the restriction map, $Flow(R^n \times \Delta, R^n) \to Flow(R^n \times D, R^n)$, is invertible. (*Flow* denotes the object of flows.)

After what we have seen, an alternative formulation can be given in the following terms

POSTULATE WA2 (*Alternative formulation*) For any positive integer n, the map

$$Flow(R^n \times \Delta, R^n) \to R^{nR^n}, \; [f \mapsto \frac{\partial f}{\partial t}(x,t)|_{t=0}]$$

is invertible.

This means that, given a function $g : R^n \to R^n$, the differential equation $\frac{\partial f}{\partial t}(x, t) = g(f(x, t))$, with initial condition $f(x, 0) = x$, has a unique solution $f(x,t)$, defined for all $x \in R^n$ and $t \in \Delta$. So, the most suggestive formulation of the postulate is the following

POSTULATE WA2 (*solution of differential equations*) For all positive integers n,

$$\forall g \in R^{nR^n} \, \exists! f \in R^{nR^n \times \Delta} \, \forall x \in R^n \, \forall t \in \Delta \, [f(x, 0)=x \wedge \frac{\partial f}{\partial t}(x, t) = g(f(x, t))].$$

We will show, later on, that time dependent systems can also be integrated (see § 4.3.)

Definition 4.4 A local integral flow for a vector field $\xi : R^n \to R^{pD}$ is an integral flow $f : G \to R^n$, where G is a Penon open neighborhood of $R^n \times \{0\}$ in $R^n \times R$. □

In the presence of POSTULATE WA1, $f(x,t)$ is a solution to the differential equation associated to ξ, defined on some $U \times (-\varepsilon,\varepsilon)$ (for each $p \in R^n$). The result is now (cf. [BUNGE-DUBUC: Local concepts in S.D.G and germ representability])

Proposition 4.5 Given a Δ-flow, $\xi : R^n \times \Delta \to R^n$, there exists (uniquely) a local flow that extends ξ. □

Uniqueness means that two such extensions agree on a neighborhood of $R^n \times \{0\}$ in $R^n \times R$. Notice that the extension exists directly by AXIOM III and the only thing left to check is the flow equation, and for that, POSTULATE WA1 gives us a hand, giving that addition is open for the intrinsic topology (it is so for the euclidean.)

The last condition we require in our framework is an infinitesimal version of the Inverse Function Theorem, due to Penon [PENON: Le théoreme de inversion local en géométrie algébrique]

POSTULATE I.I. (*Infinitesimal inversion*) For every positive n , the following holds

$$\forall f \in \Delta(n)^{\Delta(n)} \, [f(0)=0 \wedge \frac{\partial f}{\partial x}(0) \neq 0 \Rightarrow f \textit{ iso}].$$

In our setting, an equivalent formulation is given by

POSTULATE I.I. (Alternative formulation)

$$\forall f \in R^{nR^n}\ [f(0)=0 \wedge \frac{\partial f}{\partial x}(0)\#0 \Rightarrow f \textit{ infinitesimal invertible at } 0],$$

where infinitesimal invertible at 0 means that the restriction to $\Delta(n)$ is an iso. The non trivial part of the equivalence of both formulations is a consequence of the fact that in $\mathcal{G}$, the following holds

$$\forall f \in R^{\Delta(n)}\ \exists g \in R^{nR^n}\ \forall x \in \Delta(n)\ (f(x) = g(x)\ ).$$

To end the section, let us say that POSTULATES WA2 and I.I. are valid in $\mathcal{G}$ (cf. [BUNGE-DUBUC: Local concepts in S.D.G. and germ representability] and [PENON: De l'infinitésimal au local].)

§5. Internal manifolds and tangent bundles

The idea of working in a topos is to consider every object as a generalized smooth space, and every map between two of such objects as a generalized smooth mapping. In particular, the notion of tangent vector to any object M at a point $p \in M$ is available: namely, a map $t : D \to M$ taking 0 to p. We can define the tangent vector bundle as the exponential object M^D, which comes equipped with a canonical projection $\pi : M^D \to M$, namely the evaluation at the point p; $\pi(t) = t(0)$. We have now that T_pM, the tangent space to M at the element p, is the fiber over p of $TM = M^D$.

$$T_pM = [\!|\ t \in M^D \mid t(0) = p\ |\!].$$

In the case of $M = R$, T_pM comes endowed with a natural structure of R-space; by AXIOM I, the map $[v \mapsto [d \mapsto p + d \cdot v\ ]]$ defines an isomorphism $R \to T_pR$. Therefore, for this object, the name of tangent *vector* bundle is fully justified. Are there any other objects for which this notion applies?. The objects of the form R^n, in presence of our axioms and postulates, behave very much as euclidean spaces, and so it seemed worthwhile introducing "euclidean-like" objects in the internal sense to play the rôle of smooth manifolds in classical differential geometry.

To begin with, in the well adapted models we already have that every smooth manifold can be faithfully embedded (see comment after definition 0.4.12). R itself is a particular example of these objects.

In our context, several notions of manifolds are available (cf. [KOCK: Formal manifolds and synthetic theory of jet bundles] and [PENON: Infinitésimaux et intuitionism].) The one that meets better our needs is the one of Penon, which is stronger.

Definition 5.1 An object M in a topos is said to be a (infinitesimal) manifold of dimension n if for every element $p \in M$, $\neg\neg\{p\}$ is isomorphic to Δ^n.

N.B. There is no difference if we require the isomorphism to carry p to 0 (proposition 3.1.)

Apart from the smooth manifolds (considered as included in $\mathcal{G}$) there are many other examples of manifolds; they appear as consequence of the following results.

Proposition 5.2 In any topos, if M and N are manifolds of dimensions m and n, respectively, then $M \times N$ is a manifold of dimension $m + n$. □

Corollary 5.3 R^n is a manifold of dimension n.

Proposition 5.4 Let $f : M \to N$ be infinitesimally invertible (see definition after POSTULATE I.I. (*alternative definition*).) The following are true in any topos:

a) If N is a manifold of dimension n, then so is M.

b) If f is surjective and M is a manifold of dimension n, then so is N. □

The interesting result is that for any manifold, the tangent bundle is also an R-space; indeed, (cf. [PENON: De l'infinitésimal au local, p. 58])

Proposition 5.5 Let M be a manifold and let $p \in M$. Then we have the chain of isomorphisms

$$T_pM \approx T_p(\neg\neg\{p\}) \approx T_0(\Delta^n) \approx T_0R^n \approx R^n. \qquad \square$$

The key fact is that Δ, hence any manifold, inherits from R the property which allows the introduction of an R-structure. This property, not exclusive of manifolds, is the following ([BERGERON: Objects infinitesimalement linéaires dans un modèle bien adapté de G.D.S.]

Definition 5.6 An object M is called infinitesimally linear if for each $n = 2,3,\ldots$ and each n-tuple of maps $t_i : D \to M$, such that $t_1(0) = \cdots = t_n(0)$, there exists a unique map $l : D(n) \to M$ with $l \circ inc_i = t_i$.

The following really makes a difference with the classical setting

> **Proposition 5.7** If M is infinitesimally linear, then so is any exponential M^X. Also, the inverse limit of a diagram of infinitesimally linear objects is infinitesimally linear. □

The result we have choosen to close the chapter is the following (see [REYES-WRAITH: A note on tangent bundles in a category with a ring object], [KOCK: Synthetic Differential Geometry, p. 35] or [LAVENDHOME: Leçons de Géométrie Différentielle Synthétique Naïve].)

> **Proposition 5.8** a) If M is infinitesimally linear, then T_pM is canonically endowed with an R-space structure.
> b) If $M \to N$ is any map between infinitesimally linear objects, then the induced map $T_pf : T_pM \to T_{f(p)}N$ is linear.

Proof. As for the addition in part a), given two tangent vectors ζ and ξ, we define $(\zeta+\xi)(d)$ as $l(d,d)$ for the unique l (see definition 5.6) such that $l(0,-) = \zeta$ and $l(-,d) = \xi$. □

2

The internal Weak Topology

§1. Preliminaries

In any object of a topos $\mathcal{E}$, in particular for the functional ones, several topologies are available. Among them, the intrinsic topology, defined by Penon in [PENON: Topologie et intuitionisme], seems to be the most useful and widely used.

In this chapter we present an internalization of the Weak C^∞-topology used by Wassermann [WASSERMANN: Stability of Unfoldings, page 17] for objects of the form R^E, with R an ordered ring of line type and suitable E which shows particularly helpful when proving certain density results.

We show that this topological structure is subintrinsic in the sense of Penon (cf. [BUNGE-DUBUC: Local concepts in SDG and germ representability, page 33]), and point out some of its properties.

Finally, we show that, in the Dubuc topos $\mathcal{G}$, the "global sections" functor establishes a bijective correspondence between internal weak open parts of R^X and usual weak C^∞-open subsets of $\Gamma(R^X)$.

Classically, the Weak topology on $C^\infty(\mathbb{R}^n)$ admits as a basis the collection of sets

$$V(K,r,g,U) = \{h \in C^\infty(\mathbb{R}^n) \mid J^r(g-h)K \subseteq U\},$$

where $K \subseteq \mathbb{R}^n$ is compact, $g \in C^\infty(\mathbb{R}^n)$, r is a positive integer, and U is an open neighborhood of 0 in $J^r(n) \approx \mathbb{R}^{\binom{n+r}{r}}$ [WASSERMANN: Stability of Unfoldings, page 15]. This topology is sometimes called Compact-Open topology, and can be characterized by sequences. A sequence of functions $\{f_n\} \subset C^\infty(\mathbb{R}^n)$ converges to a function $f \in C^\infty(\mathbb{R}^n)$, in this topology, if and only if in any compact set $K \subset \mathbb{R}^n$, $\{f_n\} \to f$ uniformly, and so do all the sequences of derivatives to the corresponding derivative of f (cf. [MICHOR: Manifolds of Differentiable Mappings, pp. 26-33] or [HIRSCH: Differential Topology, p. 34].)

This topology is different from the so called Strong topology (also Withney C^∞ topology) which is finer. A sequence $\{f_n\} \subset C^\infty(\mathbb{R}^n)$ converges to a function $f \in C^\infty(\mathbb{R}^n)$, in the Withney topology, if and only there exists a compact set $K_0 \subset \mathbb{R}^n$ on which $\{f_n\}$ and the sequences of derivatives converge uniformly to f and to the corresponding derivative, respectively (cf. [GOLUBITSKI-GUILLEMIN: Stable Mappings and their Singularities, page 43] or [MATHER: Stability of C^∞-mappings, III].)

Wherever needed, we will consider $C^\infty(U)$, for $U \subset \mathbb{R}^n$, endowed with the induced weak topology (actually the quotient topology $C^\infty(\mathbb{R}^n) \to C^\infty(U)$ induced by the restriction map.) Similarly we consider the sets $C^\infty(\mathbb{R}^n)/_I$, with $I \subset C^\infty(\mathbb{R}^n)$ any ideal, endowed with the quotient weak topology.

In the classical setting, among the advantages of working with the weak topology instead of the strong topology is the following fact [HIRSCH: Differential Topology, page 62]

> **Proposition 1.1** For every pair, (M, N), of smooth manifolds the set $C^\infty(M,N)$ with the weak topology has a complete metric. □

This result is not always valid for the strong topology; examples are known in which not even the first axiom of countability holds [GOLUBITSKI-GUILLEMIN: Stable Mappings and their Singularities, page 44].

As for countability, the result of the above proposition can be ameliorated for the case of $C^\infty(M)$, since we have [HIRSCH: Differential Topology, page 64]

> **Proposition 1.2** For every manifold M, $C^\infty(M)$ with the weak topology is separable.

A result of general topology gives that $C^\infty(M)$ is second countable, i.e., it has a countable basis. We prove now the following result:

> **Proposition 1.3** Let $I \subset C^\infty(\mathbb{R}^n)$ be any ideal. The map $\pi : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)/_I$ is open for the weak topologies.

Proof. Take any basic neighborhood of $f \in C^\infty(\mathbb{R}^n)$, say $V(K, r, f, U)$, where $K \subseteq \mathbb{R}^n$ is compact, r is a positive integer, and U is any open neighborhood of 0 in $J^r(n)$. We will show that $\pi(V(K, r, f, U))$ is a neighborhood of $\pi(f)$ in the quotient topology, in other words, that $\pi^{-1}\pi(V(K, r, f, U))$ is open in $C^\infty(\mathbb{R}^n)$. We claim that

$$\pi^{-1}\pi(V(K, r, f, U)) = \bigcup_{h \in f+I} V(K, r, h, U)$$

So, we have to show that if $g \in C^\infty(\mathbb{R}^n)$ is any function congruent with some function "close" to f, then g is "close" to some other function congruent with f.

To see this, if $g+I = f_1+I$, for some $f_1 \in V(K, r, f, U)$, then we define $h = g-(f-f_1)$. Clearly, g is as "close" to h as f is to f_1, and f and h are congruent module I. □

Corollary 1.4 For any ideal $I \subset C^\infty(\mathbb{R}^n)$, $C^\infty(\mathbb{R}^n)/_I$ with the induced weak topology satisfies the second axiom of countability.

Proof. Any open continuous image of a second countable space is also second countable [WILLARD: General Topology, page 108.] □

There is one more result that shows up as useful when using basic neighborhoods of elements in $\Gamma(R^{\bar{A}}) \approx C^\infty(\mathbb{R}^n)/_I \approx C^\infty(Z(I))$, where $\bar{A} = \overline{C^\infty(\mathbb{R}^n)/_I}$. It is the following: If $K \subset \mathbb{R}^n$ is compact, since $Z(I)$, the set of zeros of the ideal I, is a closed subspace, then in the induced topology $K \cap Z(I)$ is compact and $V(K, r, f, U)$ goes to $V(K \cap Z(I), r, f/_{Z(I)}, U)$.

§2. Topological structures in a topos

We begin this section by recalling that a locale is a partially ordered object L for which arbitrary suprema and finite infima exist and satisfy the following distributive law:

$$H \wedge (\bigvee_{i \in I} G_i) = \bigvee_{i \in I} (H \wedge G_i)$$

Examples of locales are Ω, the subobject classifier of any elementary topos, as well as the exponential object Ω^X, for X any object (the supremum and infimum being the internal union and intersection, respectively.)

We have now ([BUNGE-DUBUC: Local concepts in S.D.G. and germ representability] and [PENON: De l'infinitésimal au local]) the following definition:

Definition 2.1 A topological structure on an object X in a topos $\mathcal{E}$, is an object $S(X)$ in $\mathcal{E}$ which is a sublocale of Ω^X. By that, we mean a subobject $S(X) \subset \Omega^X$ closed under finite infima (including the empty ones) and arbitrary suprema.

If $U \subset X$ is so that $U \in S(X)$, then we call U an S-open part of X.

A base is an object $B(X)$ in $\mathcal{E}$ which is a sub inf-lattice of Ω^X; i.e., $B(X) \subset \Omega^X$ is closed under finite (including empty) infima.

Any base $B(X)$ generates a topological structure $S(X)$ as follows:

$$U \in S(X) \text{ if and only if } \forall x \in U\, \exists V \in B(X)\ (x \in V \subset U)$$

We say that $B(X)$ is a base for $S(X)$.

Using the internal logic, in any topos $\mathcal{E}$ a topological structure can be defined on any object X of $\mathcal{E}$. This is the intrinsic topology introduced by J. Penon [PENON: Topologie et intuitionisme].

Definition 2.2 A subobject $U \in \Omega^X$ is called intrinsic (or Penon) open if the following formula holds in $\mathcal{E}$:

$$\mathcal{E} \models \forall y \in U\, \forall x \in X\ \ (\neg(x = y) \vee x \in U)$$

N.B. We make the usual abuse of notation, and we shall very often omit the change of state.

When the topos under consideration is a Grothendieck topos, the functorial (Kripke-Joyal) semantics[1] says that

$$\mathcal{S}h(\mathcal{C}) \models \forall y \in U\, \forall x \in X\ (\neg(x = y) \vee x \in U) \qquad \text{iff}$$

for any representable functor C and any two maps $x : C^{\#} \to X$, $y : C^{\#} \to U$, $C^{\#}$ being the associated sheaf,

$$C^{\#} \Vdash \neg(x = y) \vee x \in U$$

[1] Several facts are involved, namely Yoneda Lemma, every presheaf is colimit of representable functors (effective epimorphic family), $(C_\alpha{}^{\#} \to C^{\#})_\alpha$ is an effective epimorphic family iff $(C_\alpha \to C)_\alpha$ is a covering, etc.

and this is true if and only if there is a covering in C of C, $(C_\alpha \to C)_\alpha$ so that, for each α

$$C_\alpha{}^{\#} \Vdash \neg(x = y) \quad \text{or} \quad C_\alpha{}^{\#} \Vdash x \in U$$

§3. Compactness in a topos

In their paper [DUBUC-PENON: Objects compacts dans les topos] Penon and Dubuc introduced a notion of compactness in an arbitrary topos. From the topological point of view, this notion recovers a well known property of compact spaces; on the other hand, from the logical side, it will yield the converse of a intuitionistically valid principle for certain objects of quantification.

Definition 3.1 An object K in a topos $\mathcal{E}$ is called compact iff

$$\mathcal{E} \models \forall A \in \Omega\ \forall \varphi \in \Omega^K\ [\forall k \in K\ (A \vee \varphi(k)) \to A \vee \forall k \in K\ \varphi(k)\ ]$$ [1]

The next proposition (cf. also [BUNGE-GAGO: Synthetic aspects of C^∞-mappings, II: Mather's theorem for infinitesimally represented germs]) gives the equivalent in this setting of a well-known classical result, and it will be used later

Proposition 3.2 The following holds for any object E of $\mathcal{E}$

$$\forall J, K \in \Omega^E\ [J \text{ compact} \wedge K \text{ compact} \Rightarrow J \cup K \text{ compact}].$$

Proof. We have the following chain of deductions, for $A \in \Omega$ and $B \in \Omega^{J \cup K}$

$$\frac{J \cup K = \pi^{-1}{}_{(J \cup K)} A \cup B}{J = \pi^{-1}{}_J A \cup (J \cap B)\ \wedge\ K = \pi^{-1}{}_K A \cup (K \cap B)}$$

which follows from the identities $\pi_{(J \cup K)} \circ i_J = \pi_J$ and $\pi_{(J \cup K)} \circ i_J = \pi_J$, where $i_J : J \to J \cup K$

---

[1] $A \in \Omega$ represents any truth value, and $B \in \Omega^K$ represents any formula with free values from K. Equivalent formulations can be given so to capture the intuition of compact objects in topology, via tubular neighborhoods. Indeed, it is the case that in the Dubuc topos, $C^\infty(M)$ is a compact object in this sense if and only if M is a compact manifold in the usual sense. For an arbitrary object $K = C^\infty(\mathbb{R}^n)/_I$, K is compact if and only if $\Gamma(K) = Z(I) \subset \mathbb{R}^n$ is compact in the usual sense.

$$\dfrac{\dfrac{1 = A \cup \forall_{\pi_J}(J \cap B) \quad \wedge \quad 1 = A \cup \forall_{\pi_K}(K \cap B)}{1 = A \cup (\forall_{\pi_J}(J \cap B) \cap \forall_{\pi_K}(K \cap B))}}{1 = A \cup \forall_{\pi_{(J \cup K)}}(B)}$$

The last derivation being a consequence of the above identities, which give

$$\forall_{\pi_J}(J \cap B) = \forall_{\pi_{(J \cup K)}}(\forall_{i_J}(J \cap B)) = \forall_{\pi_{(J \cup K)}}(B)$$

and similarly for K. □

We now prove a result (cf. also [BUNGE-GAGO: Synthetic aspects of C^∞-mappings, II: Mather's theorem for infinitesimally represented germs]) that will be of use later on, namely that the infinitesimal monad $\Delta = \neg\neg\{0\}$ is compact in the above sense.

Proposition 3.3 For any $n > 0$, $\Delta(n) \subset R^n$ is compact.

Proof. Let $A \in \Omega$, $B \in \Omega^{\Delta(n)}$, $\Delta(n) \xrightarrow{\pi} \mathbf{1}$ the unique morphism into the terminal object (epi, due to the existence of a global section $\ulcorner 0 \urcorner : \mathbf{1} \to \Delta(n)$.)

If we start with the assumption

$$\Delta(n) = \pi^{-1}A \cup B$$

POSTULATE WA1.1, the covering principle for the the intrinsic topological structure $P(\Delta(n))$ gives

$$\Delta(n) = \iota(\pi^{-1}A) \cup \iota(B), \text{ where } \iota \text{ denotes the interior operator.}$$

Now, the intrinsic topology in $\Delta(n)$ is trivial, if $0 \in \iota(\pi^{-1}A)$, then $\iota(\pi^{-1}A) = \Delta(n) = \pi^{-1}A$, and if $0 \in \iota(B)$, then $\iota(B) = \Delta(n) = B$. In the first case, from the epic character of the top arrow in the pull-back

$$\begin{tikzcd} \pi^{-1}A \arrow[r, "f"] \arrow[d] & \Delta(n) \arrow[d] \\ A \arrow[r, "f"] & \mathbf{1} \end{tikzcd}$$

it follows that $A \to \mathbf{1}$ is also epi, hence iso. In the second case, $\forall_\pi B = \mathbf{1}$. In either case, one concludes

$$\mathbf{1} = A \cup \forall_\pi B . \quad \square$$

We close this section with a handy result concerning functions defined on compact objects, in the sense of definition 3.1, with values in R. The result states that "every function defined on a compact object is bounded away from zero."

Proposition 3.4 For any compact object K in $\mathcal{G}$, the following holds

$$\mathcal{G} \models \forall f \in R^K \, [\forall x \in K \, f(x) > 0 \Rightarrow \exists \varepsilon \in R \; \varepsilon > 0 \;\; \forall x \in K \, (f(x) > \varepsilon)]$$

Proof. By definition, K is compact if and only if

$$\mathcal{G} \models \forall A \in \Omega \;\; \forall B \in \Omega^K \, [\forall k \in K \, (A \vee B(k) \Rightarrow A \vee \forall k \in K \, B(k)]$$

equivalently [DUBUC-PENON: Objects compacts dans les topos], for any object X

$$\mathcal{G} \models \forall x_o \in X \;\; \forall B \in \Omega^{K \times X} \, [\forall x \in X \; \forall k \in K \, (x \neq x_o \vee B(k,x)) \Rightarrow \forall x \in X \;\; (x \neq x_o) \vee \forall k \in K \, B(k,x)]$$

So, in particular, for $0 \in R$, we have

$$\frac{\neg \pi^{-1}\{0\} \cup B \; = R \times K}{\neg \{0\} \cup \forall_\pi B \; = R}, \text{ where } \pi : R \times K \to R.$$

Let, then, $f \in R^K$ be so that $\forall k \in K \, f(k) > 0$, and consider

$$B = [\mid (x,k) \in R \times K \;\mid f(k) > x \mid] \in \Omega^{R \times K}.$$

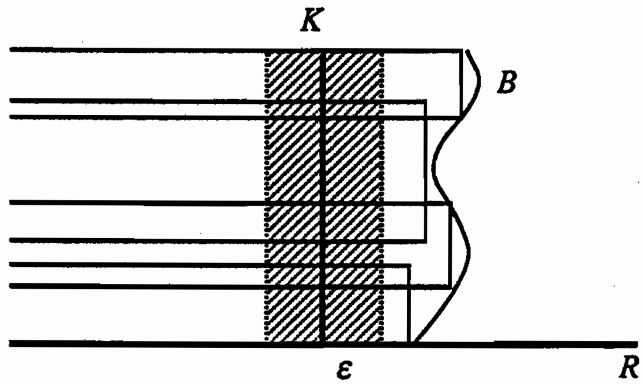
Clearly, we have $\neg \pi^{-1}\{0\} \cup B \; = R \times K$. Indeed, $\pi^{-1}\{0\} \subset \bigcup_{x \in R} (-\infty,x) \times f^{-1}[(x,\infty)] \subset B$, where $(-\infty,x) = [\mid y \in R \mid y < x \mid]$, and similarly (∞,x), which is euclidean open, hence Penon open, in the presence of POSTULATE WA1.1. The properties of Penon opens (Proposition 1.2.7) give us that $f^{1}[(x,\infty)]$ is open, as well as $\bigcup_{x \in R} (-\infty,x) \times f^{1}[(x,\infty)]$, and so the equation $\neg\pi^{-1}\{0\} \cup B = R \times K$ is the requirement for B to be a Penon neighborhood of $\pi^{-1}\{0\}$.

Compactness of K gives

$$\neg\{0\} \cup \forall_\pi B = R.$$

Since the euclidean topology has the covering property (definition 1.2.10), we must have

$$\neg\{0\} \cup \dot{\overline{\forall_\pi B}} = R.$$



which means that $0 \in \dot{\overline{\forall_\pi B}}$. Therefore, there exists $\varepsilon \in R$, $\varepsilon > 0$, such that $K \times (-\varepsilon,\varepsilon) \subset B$, as we wanted. □

§4. The Weak Topology

In this section we give (cf. also [BUNGE-GAGO: Synthetic aspects of C^∞-mappings, II: Mather's theorem for infinitesimally represented germs]) an internal version of the weak topology which will be used throughout the rest of this work.

Definition 4.1 Let $(\mathcal{E},R)$ be a ringed topos which satisfies the generalized Kock-Lawvere axiom, with R an ordered ring. Let $E \subset R^n$ be an object of $\mathcal{E}$ for which given any element $p \in E$, the objects $p+D_r(n)$ are contained in E. We define the "weak topological structure" as the one whose base is generated by the objects

$$V(K, r, g, (\varepsilon_\alpha)_{|\alpha|\leq r}) = [|\, f\in R^E \mid \forall x\in K \bigwedge_{|\alpha|\leq r} \frac{\partial^{|\alpha|}(f-g)}{\partial x^\alpha}(x) \in (-\varepsilon_\alpha,\varepsilon_\alpha)|]$$

where $K\subset E$ is a compact object, r is an (external) natural number, $g\in R^E$, and the ε_α's are in $R_{>0}$.

If E is as above, we denote by $W(R^E) \subset \Omega^{(R^E)}$ the subobject of weak opens of R^E. It can be characterized as follows:

$$\forall U\in \Omega^{(R^E)}\ [U\in W(R^E) \Leftrightarrow \forall g\in U\ \exists K\in \Omega^E\ \exists \varepsilon\in R\ [K \text{ comp.} \wedge \varepsilon>0 \wedge (\bigvee_{r=0}^{n} V(K,r,g,\varepsilon) \subset U)]$$

This characterization allows an easy way of seeing that W is indeed a topological structure.

Proposition 4.2 For any $n>0$ and $E\subset R^n$ closed under the addition by elements of the D_r(n), $W(R^E)$ is a topological structure, in the sense of definition 2.1.

Proof. We must show that $W(R^E) \subset \Omega^{(R^E)}$ is a sublocale, and for this it is enough to exhibit its closure under finite infima. Let $J,K\in\Omega^E$ be compact objects, $0 \leq r, s \leq n$, ε, $\delta > 0$ and $g\in R^E$; it is clear that

$$V(J\cup K, t, g, \gamma) \subset V(J, r, g, \varepsilon) \cap V(K, s, g, \delta)$$

where $t = \max(r, s)$ and $\gamma > 0$ is such that $\gamma < \varepsilon$, $\gamma < \delta$ (Proposition 1.2.5). The result now follows from Proposition 3.2 which asserts that $J\cup K$ is compact. □

We are now going to compare this easy-to-use topology we have just introduced with the one introduced by Penon for the objects upon which both are defined. The first result is the following (cf. also [BUNGE-GAGO: Synthetic aspects of C^∞-mappings, II: Mather's theorem for infinitesimally represented germs])

Proposition 4.3 For any $n>0$ and $E\subset R^n$ closed under the addition by elements of the $D_r(n)$ the weak topological structure on R^E is subintrinsic, i.e,

$$W(R^E) \subset P(R^E)$$

Proof. It is clearly enough to show that for any compact $K \in \Omega^E$, $0 \leq r \leq n$, and $\varepsilon \in R$, $\varepsilon > 0$, for $0 \in R^E$ (the proof would go just the same for any other $g \in R^E$)

$$V(K, r, 0, \varepsilon) \in P(R^E)$$

First of all, notice that

$$V(K, r, 0, \varepsilon) = \bigcap_{|\alpha| \leq r} \left(\frac{\partial^{|\alpha|}}{\partial x^{\alpha}}\right)^{-1} [[f \in R^E \mid \forall x \in K\, f(x) \in (-\varepsilon, \varepsilon)]],$$

and Penon opens are stable under finite infima and change of base; therefore, it would be enough to show that

$$W(K, \varepsilon) = [[f \in R^E \mid \forall x \in K\, (f(x) \in (-\varepsilon, \varepsilon)]] \in P(R^E)$$

in other words, we should show that

$$\mathcal{E} \models \forall h \in R^E\, \forall f \in W(K, \varepsilon)\, [\neg(f = h) \vee h \in W(K, \varepsilon)\,]$$

Now, $(-\varepsilon, \varepsilon) \subset R$ is an euclidean open, hence (see comment after proposition 1.2.9) Penon open and the following is true

$$\mathcal{E} \models \forall h \in R^E\, \forall f \in W(K, \varepsilon)\, \forall x \in K\, [\neg(f(x) = h(x)\,) \vee h(x) \in (-\varepsilon, \varepsilon)\,]$$

and, since

$$\frac{\forall x \in K\, [\, (f = g) \Rightarrow (f(x) = g(x))\,]}{\forall x \in K\, [\neg(f(x) = g(x)) \Rightarrow \neg(f = g)\,]}$$

is intuitionistically valid, from the above we get

$$\mathcal{E} \models \forall h \in R^E\, \forall f \in W(K, \varepsilon)\, \forall x \in K\, [\neg(f = h\,) \vee h(x) \in (-\varepsilon, \varepsilon)\,]$$

and compactness of K gives

$$\mathcal{E} \models \forall h \in R^E\, \forall f \in W(K, \varepsilon)\, [\neg(f = h\,) \vee \forall x \in K\, h(x) \in (-\varepsilon, \varepsilon)\,]$$

i.e.,

$$\mathcal{E} \models \forall h \in R^E\ \forall f \in W(K, \varepsilon)\ [\neg(f=h) \vee h \in W(K, \varepsilon)]$$

as required. □

As for the converse, it is not known to us whether or not it is true. We do know certain instances of Penon opens which can be shown to be weak opens, namely those which follow from the following two theorems:

Theorem 4.4 For any $n>0$ the object $R^{p^{\Delta(n)}}$ is separated (T_1) for the weak topological structure, i.e.:

$$\forall f \in R^{p^{\Delta(n)}}\ \neg\{f\} \subset R^{p^{\Delta(n)}} \textit{ is weak open.}$$

Proof. Let $f \in R^{p^{\Delta(n)}}$ be given at stage A, and let $h \in \neg\{f\}$ be an element at any later stage (though we do not make any distinction). We will show that there exist $\varepsilon \in R$, $\varepsilon > 0$, so that the basic neighborhood $V(\Delta(n), 1, h, \varepsilon)$ is contained in $\neg\{f\}$.

But, from $\neg(f=h)$, it follows that $\neg(f(0)=h(0))$; otherwise, we would have the following derivations (all of them known to be intuitionistically valid [DUMMET: Elements of Intuitionism],)

$$\frac{\dfrac{\neg\neg(f(0)=h(0))}{\forall x \in \Delta(n)\ \neg\neg(f(x)=h(x))}}{\neg\exists x \in \Delta(n)\ \neg(f(x)=h(x))}\ ,$$

and this would contradict our assumption $\neg(f=g)$, that reads $\neg\forall x \in \Delta(n)(f(x)=h(x))$ or equivalently $\neg\neg\exists x \in \Delta(n)\ \neg(f(x)=h(x))$.

Now, $\neg(f(0)=h(0)) \Leftrightarrow f(0)-h(0)>0 \vee f(0)-h(0)<0$ [by **R.4** in POSTULATE WA1.2, section 1.2]. Take $\varepsilon \in R$ to be the "positive" one of both possibilities in each member of the covering, and the result will follow[1]. □

---

[1]Notice that this is a local conclusion, as the existence of ε itself.

The same result can be shown to hold for any object E for which Kock's Principle is true [KOCK: Synthetic characterization of reduced algebras], that for representable objects in $\mathcal{G}$ correspond to point determined algebras.

Theorem 4.5 Let $E \subset R^n$ be any object closed under the addition by elements of the D_r(n), for which Kock's Principle holds, i.e.,

$$\forall f \in R^E \; \neg \forall x \in E \; \Phi(x)=0 \Rightarrow \exists x \in E \; \Phi(x)\#0.$$

Then R^E is weak- separated (T_1).

Proof. The proof follows a similar pattern to that of theorem 4.4. First of all, one shows that $\exists x \in E \; \neg(f(x)=h(x))$. Secondly, for this element, x_o, one considers the infinitesimal monad around it and follows the main line of the argument given above for the basic weak neighborhood $V(\neg\neg\{x_o\}, 1, h, \varepsilon)$. [2] □

If we were to have any kind of converse result, from the proof of Proposition 4.3 would follow the need for the basic neighborhoods, $V(K, r, f, \varepsilon)$ in definition 4.1, of being actually weak open. This is the content of our next proposition.

Proposition 4.6 Let $E \subset R^n$ be closed under addition by elements of the $D_r(n)$. For any compact $K \in \Omega^E$, any positive integer n, any map $g \in R^E$, $V(K, r, f, \varepsilon) \in W(R^E)$.

Proof. Let $h \in V(K, r, f, \varepsilon)$ at any given stage (which we do not consider since the result we are after is of local nature, the final condition of existence of a compact and a positive element of R are to hold in some cover of this stage at which we keep the same names for the objects). So, we have

$$\forall x \in K \quad \frac{\partial^{|\alpha|}(h-g)}{\partial x^\alpha}(x) \in (-\varepsilon, \varepsilon), \;\; |\alpha| \leq r.$$

We will show that there exists $\gamma \in R$, $\gamma > 0$, for which

$$V(K, r, h, \gamma) \subset V(K, r, g, \varepsilon).$$

---

[2] Notice that $\neg\neg\{x\} \subset E$, as E is closed under the addition by elements from $D_r(n)$.

Indeed, let α be so that $|\alpha| \leq r$; the function $[x \in R \mapsto \varepsilon - \frac{\partial^{|\alpha|}(h-g)}{\partial x^\alpha}(x)]$ is in the conditions of Proposition 3.4 and therefore there exists $\delta_1^\alpha \in R$, $\delta_1^\alpha > 0$, such that

$$\forall x \in K \quad \frac{\partial^{|\alpha|}(h-g)}{\partial x^\alpha}(x) < \varepsilon - \delta_1^\alpha .$$

Similarly, for the function $[x \in R \mapsto \varepsilon + \frac{\partial^{|\alpha|}(h-g)}{\partial x^\alpha}(x)]$, there exists $\delta_2^\alpha \in R$, $\delta_2^\alpha > 0$, such that

$$\forall x \in K \;\; 2\varepsilon > \varepsilon + \frac{\partial^{|\alpha|}(h-g)}{\partial x^\alpha}(x) > \delta_2^\alpha .$$

By proposition 1.2.5, there exists $\delta \in R$, $\delta > 0$, $\delta < \delta_i^\alpha$ ($|\alpha| \leq r$, $i = 1,2$), and for this δ

$$\bigwedge_{|\alpha| \leq r} \frac{\partial^{|\alpha|}(h-g)}{\partial x^\alpha}(x) \in (-\varepsilon + \delta, \varepsilon - \delta).$$

Now we can take $\gamma = \delta$, and linearity of derivative and compatibility of the order relation takes care of the rest, namely

$$\frac{\partial^{|\alpha|}(f-g)}{\partial x^\alpha}(x) = \frac{\partial^{|\alpha|}(f-h)}{\partial x^\alpha}(x) + \frac{\partial^{|\alpha|}(h-g)}{\partial x^\alpha}(x) \in (-\varepsilon, \varepsilon). \; \square$$

§5. Considerations of use

The reason why we introduced this topological structure, among others, is to use it in proving some density results about special classes of germs. With this goal in mind we should give the justification for such a decision. It is indeed the aim of this section to characterize this internal weak topology as corresponding to the classical weak C^∞-topology, via the global sections functor $\Gamma: \mathcal{G} \to$ **Set**. In the process, we will also show that in $\mathcal{G}$, for objects of interest, the internal weak topology agrees with the Penon topology, thus allowing us to rephrase important notions of Synthetic Stability Theory of [BUNGE-GAGO: Synthetic aspects of C^∞-mappings, II: Mather's theorem for infinitesimally represented germs] in the usual terms, while exploiting still their logical nature.

We begin by giving a result (also included in [BUNGE-GAGO: Synthetic aspects of C^∞-mappings, II: Mather's theorem for infinitesimally represented germs]) which characterizes Penon opens of $R^{\bar{X}}$, with $\bar{X}$ a representable object in $\mathcal{G}$, in terms of global sections:

Proposition 5.1 Let $\bar{X}$ be any representable object in $\mathcal{G}$. If $U \to R^{\bar{X}}$ is Penon open, then $\Gamma(U) \subset X$ is open in the (quotient) weak C^∞-topology.

Proof. Under these hypothesis we will show that, in the pullback diagram

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "h"'] & \Gamma(U) \arrow[d, "k"] \\ \Gamma(A) \arrow[r, "\Gamma(f)"] & \Gamma(R^{\bar{X}}) \end{tikzcd}$$

W is an open subset of $Z(I) = \Gamma(\bar{A})$, for any representable $\bar{A} = \overline{C^\infty(\mathbb{R}^n)/_I}$, I of local character and any $\bar{A} \xrightarrow{f} R^{\bar{X}}$.

Indeed, take any $a \in W$, i.e., a global section of $\bar{A}$, such that fa factors through U

$$\begin{tikzcd} 1 \arrow[r, dotted, "g"] \arrow[d, "a"'] & U \arrow[d, "i"] \\ \bar{A} \arrow[r] & R^{\bar{X}} \end{tikzcd}$$

and consider the $\bar{A}$-elements $f \in_{\bar{A}} R^{\bar{X}}$ and $g^* \in_A U$ (given by $\bar{A} \xrightarrow{f} R^{\bar{X}}$ and $\bar{A} \xrightarrow{!A} 1 \xrightarrow{g} U$, respectively.)

By definition of Penon open object U at stage $\bar{A}$ we have

$$\Vdash_{\bar{A}} \neg(f = g^*) \vee f \in U$$

Using functorial semantics, there exists an epimorphic family $(F_i \to \bar{A})_{i \in I}$ (a covering of $\bar{A}$ in the site, $(\bar{A_i} \to \bar{A})_{i \in I}$) such that, for each $i \in I$

$$\Vdash_{\bar{A_i}} \neg(f^{\#} = g^{*\#}) \qquad \text{or} \qquad \Vdash_{\bar{A_i}} f^{\#} \in U$$

where $^{\#}$ denotes the change of stage $\bar{A}_i \to \bar{A}$. Applying global sections we get a surjective family $(\Gamma(\bar{A}_i) \to \Gamma(\bar{A}))_{i\in I}$, and the V_i 's = $\Gamma(\bar{A}_i)$'s form an open cover of $Z(I)$; therefore, $a\in\Gamma(\bar{A})$ must be in $\Gamma(\bar{A}_{i_o})$, for some i_o . We claim that

$$\nVdash_{\bar{A}_{i_o}} \neg(f^{\#} = g^{*\#})$$

For we have

$$\Vdash_1 f = g \qquad \text{corresponding to the factorization } 1 \to \bar{A}_{i_o}$$

$$\begin{tikzcd} & 1 \arrow[r, "g"] \arrow[d, shift left, "a"] \arrow[dl, bend right] & U \arrow[d, "i"] \\ \bar{A}_{i_o} \arrow[r] & A \arrow[r, "f"] \arrow[u, shift left, "!_A"] & R^{\overline{X}} \end{tikzcd}$$

and this would imply $1 = 0$, contrary to the Nullstellensatz. Therefore, a belongs to $\Gamma(\bar{A}_{i_o})$ with $\Vdash_{\bar{A}_{i_o}} f^{\#}\in U$, which means $a \in \Gamma(\bar{A}_{i_o}) \subset W$, open, as claimed. [1]

To finish the proof we give a characterization of Weak C^∞-opens. A set $V\subset X = C^\infty(\mathbb{R}^p)/_J$ is weak open if for any given smooth path $[\hat{F}] : [0,1] \to X$ (i.e., induced by a smooth mapping $F : [0,1] \times \mathbb{R}^p \to \mathbb{R}$) $[\hat{F}]^{-1}(V)$ is open in $[0,1]$.[2] Indeed, if V is not weak open, then there exists a sequence $\{[g_n]\}$ in $X\backslash V$ such that $\{[g_n]\} \to [g]$ with the weak topology, and $[g] \in V$ By the definition of quotient topology, there exists a sequence $\{f_n\}$ weak converging to f in $C^\infty(\mathbb{R}^p)$, and such that $[g_n] = [f_n]$ and $[g] = [f]$. A result of Reyes-Van Quê [REYES & VAN QUE: Smooth functors and synthetic calculus] gives a subsequence $\{f_{n_k}\} \to f$, and a smooth map F: $[0,1] \times \mathbb{R}^p \to \mathbb{R}$ such that

---

[1]The argument given works just the same for any object E in the topos, not only for exponential ones.

[2]The converse result is also true, as smooth operators between spaces are continuous with respect to the Fréchet topology [FRÖLICHER: Applications lisses entre spaces et variétées de Fréchet]

$$F(0,\text{-}) \equiv f$$

$$F(\tfrac{1}{k},\text{-}) \equiv f_{n_k} \qquad \forall k$$

Thus $0 \in [\hat{F}]^{-1}(V)$ and $\frac{1}{k} \notin [\hat{F}]^{-1}(V)$, and $[\hat{F}]^{-1}(V)$ cannot be open.

Now we are done, for [0,1], being a closed set is the zero set of some smooth function φ [WITHNEY: Analytic extensions of differentiable functions defined in closed sets]; the ideal generated by φ has the same zero set and is of local character [DUBUC: C∞-schemes]; and we apply the result to the pullback

$$\begin{array}{ccc} W & \longrightarrow & \Gamma(U) \\ \downarrow & & \downarrow k \\ [0,1] & \xrightarrow{[\Gamma(\hat{F})]} & D \end{array}$$ [1] □

Corollary 5.2 Let $X \subset R^n$ be any representable object, closed under addition by elements of $D_r(n)$ in $\mathcal{G}$. If $U \to R^{\overline{X}}$ is Weak open, then $\Gamma(U) \subset X$ is open in the (quotient) weak C^∞-topology.

Proof. Immediate after Proposition 4.3 and the Lemma. □

So, for representable objects, closed under addition by elements of $D_r(n)$, in $\mathcal{G}$, Γ establishes an injective correspondence between internal weak open parts of $R^{\overline{X}}$ and weak C∞-subsets of $\Gamma(R^{\overline{X}})$. The question is now wether or not one has a similar result to that of Proposition 1.2.11. The answer is yes, and is contained in the following proposition

Proposition 5.3 Given any representable object, $X \subset R^n$, closed under addition by elements of the $D_r(n)$ in $\mathcal{G}$, if $V \subset \Gamma(R^{\overline{X}})$ is a weak C∞-open subset, then $\Lambda(V) \subset R^{\overline{X}}$ is internal weak open.

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[1]The reverse implication has been studied by Bruno [BRUNO: Logical opens of exponential objects]. He has proved that for any representable object $\overline{X}$, if $V \subset \Gamma(R^{\overline{X}})$ is weak C∞-open, then $\Lambda(V) \subset R^{\overline{X}}$ is Penon open (actually, this is just an instance of a more general result.)

Proof. Suppose that $\overline{X} = \overline{C^\infty(\mathbb{R}^n)/_J}$, and let $V \subset \Gamma(R^{\overline{X}})$ be a weak C^∞-open subset. If $\overline{A}$ is any representable object, say $\overline{A} = \overline{C^\infty(\mathbb{R}^r)/_K}$, and $g : \overline{A} \to R^{\overline{X}}$, factors through the subobject $\Lambda(V) \subset R^{\overline{X}}$, or equivalently (Proposition 1.2.11) $g : \Gamma(\overline{A}) \to \Gamma(R^{\overline{X}})$ factors through $V \subset \Gamma(R^{\overline{X}})$, we must show that, for some internal weak open W, $g \in W \subset \Lambda(V)$, i.e.,

$$\Vdash_{\overline{A}} \exists W \subset \Lambda(V)\ [W \textit{ weak open} \wedge g \in W\ ].$$

We keep the same name for $g : \Gamma(\overline{A}) \to \Gamma(R^{\overline{X}})$.

For each $x \in \Gamma(\overline{A})$, $g(x) \in V$, and therefore there exists a basic neighborhood around it; this means that $Im(g) \subset V$ is covered by a family $\{V(K_i, r_i, h_i, \varepsilon_i)\}_{i\in I}$, i.e., $K_i \subset Z(J)$ compact, $h_i \in \Gamma(R^{\overline{X}})$ (see remark after Corollary 1.4). Since $\Gamma(R^{\overline{X}}) = C^\infty(\mathbb{R}^n)/_J$ is second countable (see Corollary 1.4), the set of index I can be considered countable, and the family $\{A_i \to A\}_{i\in I}$, where $A_i = g^{-1}\,\Lambda(V(K_i, r_i, h_i, \varepsilon_i))$ is a cover of $\overline{A}$ (see Proposition 1.4.11); indeed $\Lambda(V(K_i, r_i, h_i, \varepsilon_i))$ is Penon open (footnote after proposition 5.1), and therefore each $A_i = g^{-1}\,\Lambda(V(K_i, r_i, h_i, \varepsilon_i))$ is Penon open (see Proposition 1.2.7), so $\Gamma(A_i) \subset Z(K)$ is open (Proposition 1.2.11).

The claim is that, for each $i \in I$ we have

$$\begin{tikzcd} A_i \arrow[r, dotted] \arrow[d] & V(\Lambda K_i, r_i, h_i, \varepsilon_i) \arrow[d] \\ \overline{A} \arrow[r, "g"] & R^{\overline{X}} \end{tikzcd}$$

Since $\Lambda K_i \subset \overline{X}$ is compact in the sense of definition 3.1 ($\Gamma\Lambda K_i = K_i \subset Z(I)$ compact), the claim will finish the proof.

We already know that

$$\begin{tikzcd} A_i \arrow[r, dotted] \arrow[d] & \Lambda(V(K_i, r_i, h_i, \varepsilon_i)) \arrow[d] \\ \overline{A} \arrow[r, "g"] & R^{\overline{X}} \end{tikzcd}$$

because we have $\Gamma(A_i) = g^{-1}(V(K_i, r_i, h_i, \varepsilon_i))$, and so

$$\begin{array}{ccc} \Gamma(A_i) & \cdots\cdots> & V(K_i, r_i, h_i, \varepsilon_i) \\ \downarrow & & \downarrow \\ \Gamma(\bar{A}) & \xrightarrow{\quad g \quad} & \Gamma(R^{\bar{X}}) \end{array}$$

So, it suffices to show that

$$\vdash_{\overline{A_i}} \Lambda(V(K_i, r_i, h_i, \varepsilon_i)) \subset V(\Lambda K_i, r_i, h_i, \varepsilon_i)$$

Actually we will show that this inclusion is true at any stage. This amounts to showing that, for all representable $\bar{B}$, any arrow $f : \bar{B} \to R^{\bar{X}}$ factors through $V(\Lambda K_i, r_i, h_i, \varepsilon_i)$, provided it does through $\Lambda(V(K_i, r_i, h_i, \varepsilon_i))$, or equivalently, provided that the corresponding global section $f : \Gamma(\bar{B}) \to \Gamma(R^{\bar{X}})$ factors through $V(K_i, r_i, h_i, \varepsilon_i)$ (see section 1.2). This last requirement gives the factorization

$$\frac{\partial^{|\alpha|}(f-h_i)}{\partial x^{\alpha}} : \Gamma(\bar{B}) \times K_i \ \cdots> \ (-\varepsilon_i\,, \varepsilon_i) \to \mathbb{R}, \text{ for every } \alpha \text{ with } |\alpha| \leq r$$

and from here, using the definition of Λ and its properties (Proposition 1.2.12) we get

$$\frac{\partial^{|\alpha|}(f-h_i)}{\partial x^{\alpha}} : \bar{B} \times \Lambda K_i \ \cdots> \ \Lambda((-\varepsilon_i\,, \varepsilon_i)) \to R, \text{ for every } \alpha \text{ with } |\alpha| \leq r,$$

which gives the required factorization

$$f : \bar{B} \ \cdots> V(\Lambda K_i, r_i, h_i, \varepsilon_i) \to R^{\bar{X}}. \qquad \square$$

3

Density of regular values

§1. Preliminaries

In [BUNGE: Synthetic aspects of C$^\infty$-mappings] some definitions concerning different aspects of C^∞-mappings were given for the synthetic context; among others, the notion of submersion, regular and critical values, transversality and submanifold cut out by independent functions.

Using the Dubuc topos, $\mathcal{G}$, as the test model, external versions of some theorems were given in order to achieve an (external) version of Thom's Transversality Theorem. One of the major difficulties in that study was the the lack of a positive (non negative) version of Sard's theorem in the internal sense as consequence of the intuitionistic character of the internal logic of the model.

In this chapter we present a way of overcoming these difficulties by postulating a version of the Theorem of Regular Values (for germs) (cf. also [BUNGE-GAGO: Synthetic aspects of C^∞-mappings,II] which can be internally derived from Sard's Theorem, which in turn we show to be internally valid in $\mathcal{G}$. We begin by recalling some definitions and notations from [BUNGE: Synthetic aspects of C$^\infty$-mappings].

Definition 1.1 An n-tuple of elements $v_1, \dots ,v_n$ in R^p forms a linearly independent set if the following holds

$$\forall \lambda_1, \ldots, \lambda_n \in R \; [\bigvee_{i=1}^{n} (\lambda_i \# 0) \Rightarrow \sum_{i=1}^{n} \lambda_i v_i \# 0 \;].$$

This notion (equivalent, over a field, to the one introduced by Kock in [KOCK: Universal projective geometry via topos theory]) allows the introduction of the basic notions of linear algebra in the usual manner, in particular the notion of *rank* $A \geq k$ (we write $=p$, if p is maximal) can be defined.

Definition 1.2 Let $f \in R^{pR^n}$ and $x \in R^n$. We say that f is a submersion at x if and only if $rank(D_x f) = p$, where $D_x f = \left(\frac{\partial f_i}{\partial x_j}(x) \right)_{ij}$.

f is called a submersion if, for every element $x \in R^n$, f is a submersion at x.

After Proposition 2.2.4, plain commutative algebra constructions give the existence of a right inverse for any matrix $A \in R^{p \cdot m}$ with $Rank(A) \geq p$.[1] With this result in mind it is easy to prove the following proposition

Proposition 1.3 Let $f \in R^{pR^n}$, $x \in R^n$. Then the following are equivalent:

i) f is a submersion at x.

ii) $\bigvee_{(i_1,\ldots,i_p) \in \binom{n}{p}} \left(\left\{ \frac{\partial f}{\partial x_{i_1}}(x), \; \ldots \;, \frac{\partial f}{\partial x_{i_p}}(x) \right\} \textit{linearly independent} \right)$

iii) df_x is locally surjective.

Part iii) just says that for the induced linear map between the tangent vector spaces (see Proposition 1.5.8) which corresponds to the Jacobian $D_x f$ (with respect to the canonical basis) the following holds

$$\forall v \in R^{pD} \; \pi(v) = f(x) \Rightarrow \exists u \in R^{nD} \; [\pi(u) = v \wedge f^D u = v]$$

Definition 1.4 Let $f \in R^{pR^n}$ and $y \in R^p$. We say that y is a critical value of f if

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[1] Of course, the existence is in the internal logic, as the notion of *Rank* itself is local.

$$\exists x \in R^n\ [f(x) = y \ \wedge \bigwedge_{H \in \binom{n}{p}} det(D_x f)_H = 0\ ],$$

where $\binom{n}{p}$ denotes the set of subsets of $\{1, \ldots ,n\}$ consisting of p elements.

As in the classical setting, we have the following

Definition 1.5 Let $f \in R^{pR^n}$ and $y \in R^p$. We say that y is a regular value of f if y is not a critical value.

In the presence of the field property (POSTULATE A in §1.2,) this condition is equivalent to the following one

$$\forall x \in R^n\ [f(x) \# y \ \vee \bigvee_{H \in \binom{n}{p}} det(D_x f)_H \# 0\ ],$$

i.e.,

$$\forall x \in R^n\ [f(x) \# y \vee f \text{ is a } \textit{submersion} \text{ at } x].$$

Corollary 1.6 Let $f \in R^{pR^n}$ be a submersion, then every element $y \in R^p$ is a regular value for f. $\square$

§2. Sard's Theorem

This section is dedicated to prove the negative version of the result we pursue in this chapter, namely a theorem of Sard's. This theorem is quoted in the classical context to derive several density results [GOLUBITSKI-GUILLEMIN: Stable Mappings and their Singularities, page 34] and establishes that the set of critical values of a smooth function has measure zero. However, what is actually used is the fact that in every non empty interval there are regular values, equivalent to the above within classical logic.

In our context, the internal logic of the topos follows the rules of intuitionistic logic and both results cannot be proven to be equivalent. We will show that when we restrict to functions defined in an infinitesimal domain, i.e., when we restrict to germs in the good models, the positive version follows from the negative one, which holds in our test model $\mathcal{G}$, as we are going to show.

Although we will not make use of the full strength of the result we give the proof in all generality, as there is no much difference. The result is given by the following theorem (cf. also [BUNGE-GAGO: Synthetic aspects of C^∞-mappings, II: Mather's theorem for infinitesimally represented germs])

Theorem 2.1 Sard's Theorem holds in $\mathcal{G}$, i.e.,

$$\forall f \in R^{pR^n}\ \forall U \in P(R^p)\ [\neg \forall y \in R^p\ [y \in U \Rightarrow y \textit{ critical value of } f\,]\,]$$

Proof. Let $f \in_{\bar{A}} R^{pR^n}$ be represented by $F : \mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}^p$, a smooth mapping defined modulo $I \bullet \pi_1$, provided $\bar{A} = \overline{C^\infty(\mathbb{R}^r)_{/I}}$. For our purposes it is certainly enough to suppose $U = (a,b)^p$ for $a,b \in R$, $\Vdash_{\bar{A}} a < b$. So, we have to show

$$\Vdash_{\bar{A}} \neg \forall y \in R^p\ [y \in (a,b)^p \Rightarrow y \textit{ critical value of } f\,].$$

If $a,b \in_{\bar{A}} R$ are represented by $\alpha, \beta : \mathbb{R}^r \to \mathbb{R}$, smooth mappings defined modulo J

$$\Vdash_{\bar{A}}\ a < b \quad \text{if and only if} \quad \forall t \in Z(J)\ (\ \alpha(t) < \beta(t)\ ),$$

Assume $\bar{B} = \overline{C^\infty(\mathbb{R}^s)_{/J}} \xrightarrow{\delta} \bar{A}$ in $\mathcal{G}$, such that

$$\Vdash_{\bar{B}} \forall y \in R^p\ [y \in (a,b)^p \Rightarrow y \textit{ critical value of } f\,]\,.$$

By functorial semantics, we have to show that $\bar{B} = 0$. If not, by the "nullstellensatz" (Corollary 0.4.9) $Z(J) \neq \emptyset$. Let $t_o \in Z(J)$; then

$$\alpha^{\#}(t_o) < \beta^{\#}(t_o),$$

where $\alpha^{\#}$ and $\beta^{\#}$ are the induced by α and β through the change of stage δ.

Take any $z \in (\alpha^{\#}(t_o)\,,\,\beta^{\#}(t_o)) \subset \mathbb{R}$. Then there exists $\lambda \in \mathbb{R}$ with

$$z = \lambda \cdot \alpha^{\#}(t_o) + (1-\lambda) \cdot \beta^{\#}(t_o).$$

Consider the equivalence class, module J of $\xi : \underline{x} \in \mathbb{R}^s \longmapsto \lambda \cdot \alpha^{\#}(\underline{x}) + (1-\lambda) \cdot \beta^{\#}(\underline{x}) \in \mathbb{R}$; it defines an element $c \in_{\overline{B}} (a,b)$, and therefore

$$\Vdash_{\overline{B}} c \textit{ critical value of } f\ .$$

This amounts to

$$\Vdash_{\overline{B}} \exists x \in R^n\, [f(x)=c \ \wedge \bigwedge_{H \in \binom{n}{p}} det\, (D_x f)_H = 0\,],$$

which means that there exists a covering $(\overline{B_\alpha} \to \overline{B})_\alpha$ and $\xi_\alpha : \mathbb{R}^{s\alpha} \to \mathbb{R}^n$ whose classes, module the ideals (of definition of the $\overline{B_\alpha}$'s) J_α, satisfy

$$\forall t \in Z(J_\alpha) \qquad F(t,0) = \xi(t), \text{ and}$$

$$\forall t \in Z(J_\alpha) \qquad \text{every subset of} \left\{ \frac{\partial F}{\partial x_i}(t,0),\ \ldots\ ,\ \frac{\partial F}{\partial x_n}(t,0) \right\} \text{ consisting of}$$

p vectors has zero determinant.

Now, since $Z(J) = \bigcup_\alpha Z(J_\alpha)$ (Proposition 0.4.11,) there must exist some α_o so that $t_o \in Z(J_{\alpha_o})$. Considering the mapping $F_o = F(t_o, -) : \mathbb{R}^n \to \mathbb{R}^p$, F_o is smooth and z is a critical value of F_o; but z is any point of the interval $(\alpha^{\#}(t_o)\,,\,\beta^{\#}(t_o))$, and that contradicts classical Sard's theorem. Therefore, we must have $\overline{B} = 0$. □

§3. Regular values of germs

As we mentioned in previous sections of this chapter, the key result to any study of transversality or stability seems to be the positive version of Sard's Theorem. We have also pointed out the difficulties in the synthetic context due to the incompatibility of classical logic with the basic axiom [KOCK: Synthetic Differential Geometry]. M. Bunge [BUNGE: Synthetic aspects of C^∞-mappings] has given a external version of this result which is valid in $\mathcal{G}$.

In this section we make use of very particular properties of the infinitesimal monad, $\Delta(n)$, to show that, when one reduces to spaces of the form $R^{\Delta(n)}$, the result is internally valid in $\mathcal{G}$, and can actually be derived from Sard's axiom. As a consequence, in the presence of the axiom of germ representability, this result will enable us to develop most of the theory of stability for germs.

We can now safely adopt POSTULATE D (cf. also [BUNGE-GAGO: Synthetic aspects of C^∞-mappings, II: Mather's theorem for infinitesimally represented germs]) and prove

Theorem 3.1 (*Density of regular values*) For $n,p > 0$ and $U \subset R^p$ open, the following holds in $\mathcal{G}$

$$\forall f \in R^{p^{\Delta(n)}}\ \exists y \in U\ [y \textit{ regular value of } f].$$

Proof. Spelling out the definition of regular value, and taking into account POSTULATE A (see §1.2) which $\mathcal{G}$ satisfies, what is to be proven takes the following form

$$\forall f \in R^{p^{\Delta(n)}}\ \exists y \in U\ \Big[\forall x \in \Delta(n)\ [\neg(f(x)=y) \vee \bigvee_{H \in \binom{n}{p}} \neg(det(D_x f)_H = 0)]\Big].$$

For a given $f \in_{\bar{A}} R^{p^{\Delta(n)}}$, we consider the map $\Phi \in_{\bar{A}} R^{p(\Delta(n)\times U)}$ defined as follows

$$\Phi(x,y) = (f(x)-y)^2 + \sum (det(D_x f)_H)^2.$$

Clearly, a sufficient condition for y to be a *regular value* of f is, for $y \in U$

$$\forall x \in \Delta(n)\ \neg(\Phi(x,y)=0), \tag{A}$$

and a necessary condition for y to be a *critical value* of f is, for $y \in U$

$$\exists x \in \Delta(n)\ (\Phi(x,y)=0). \tag{B}$$

Therefore, it suffices for our purposes to establish that (A) implies (B), as (A) is a consequence of Sard's which holds in $\mathcal{G}$ as we have shown in Theorem 3.6.

Using the rules of intuitionistic logic, valid in any topos (cf. CHAPTER 0 and the references therein) we first derive from (A), the following

$$\Vdash_{\bar{A}}\ \forall f \in R^{p^{\Delta(n)}}\ \neg \exists x \in \Delta(n)\ \forall y \in U\ [\Phi(x,y)=0],$$

and now we consider the equivalent statement [DUMMET: Elements of intuitionistic logic, p.29]

$$\Vdash_{\overline{A}} \forall f \in R^{p^{\Delta(n)}} \forall x \in \Delta(n)\ \neg \forall y \in U\ [\Phi(x,y)=0].$$

Since U is of the form $\iota(V)$ for some $V \subset \mathbb{R}^p$ open (Proposition 2.2.13), U is point determined in the sense of Kock in [KOCK: Synthetic characterization of reduced algebras] (see [KOCK: Synthetic Differential Geometry, p. 225]). For these objects, a sort of Marcov principle is available and allows us to derive from the above the following

$$\Vdash_{\overline{A}} \forall f \in R^{p^{\Delta(n)}} \forall x \in \Delta(n)\ \exists y \in U\ \neg[\Phi(x,y)=0].$$

Now we use the fact that in $\mathcal{G}$, the object representing the germs is in turn representable and the principle of local choice of Fourman (see [MOERDIJK-REYES: Smooth spaces vs. continuous spaces in the models for S.D.G.] for a proof for R, easily translated to any representable) can be applied to get

$$\Vdash_{\overline{A}} \forall f \in R^{p^{\Delta(n)}} \exists \mathcal{U} \in \Omega(\Omega^{\Delta(n)})[\mathcal{U} \text{ open cover of } \Delta(n)] \wedge \forall V \in \mathcal{U}\ \exists g \in U^V\ \forall x \in V \neg[\Phi(x,y)=0]$$

But, the intrinsic topology of $\Delta(n)$ is trivial (any Penon open object must contain the infinitesimal monad of each of its elements), and since $0 \in \Delta(n)$, and $\mathcal{U}$ is a covering, we must have $\Delta(n) = V$, for some $V \in \mathcal{U}$. In particular, we have

$$\Vdash_{\overline{A}} \forall f \in R^{p^{\Delta(n)}} \exists g \in U^{\Delta(n)}\ \forall x \in \Delta(n)\ \neg[\Phi(x,y)=0].$$

To finish the proof, we use the explicit description of $\Delta(n)$ in $\mathcal{G}$, i.e., $\Delta = \neg\neg\{0\}$, in the following sense. First of all, for a given $f \in_{\overline{A}} R^{p^{\Delta(n)}}$, the above gives the existence of a covering of $\overline{A}$ in the site, $(\overline{A_i} \to \overline{A})_{i \in I}$ such that, for each $i \in I$, there is a $g_i \in_{\overline{A_i}} R^{p^{\Delta(n)}}$ for which

$$\Vdash_{\overline{A_i}} \forall x \in \Delta(n)\ \neg[\Phi(x, g_i(x))=0].$$

Finally, since R is a local ring, from the definition of Φ we have the formulation

$$\Vdash_{\overline{A_i}} \forall x \in \Delta(n)\left[\neg(f(x)=g_i(x)) \vee \bigvee_{H \in \binom{n}{p}} \neg(det(D_x f)_H = 0)\right]$$

which gives, for any $x \in_{\overline{A}} \Delta(n)$,

$$\Vdash_{\overline{A_i}} \neg(f(x)=g_i(x)) \qquad \text{or} \qquad \Vdash_{\overline{A_i}} \bigvee_{H \in \binom{n}{p}} \neg(det(D_x f)_H = 0).$$

The second possibility does not depend on g_i. As for the first one, the assertion it makes is entirely equivalent to

$$\Vdash_{\overline{A_i}} \neg(f(x)=g_i(0)),$$

because of the monotonicity of $\neg\neg$, that guarantees $\neg\neg(f(x)=f(0))$ and $\neg\neg(g_i(x)=g_i(0))$, from $\neg\neg(x=0)$. Therefore for the $\overline{A_i}$ element $c_i = g_i(0)$ we have

$$\Vdash_{\overline{A_i}} \neg(f(x)=c_i),$$

and, the $\overline{A_i}$'s form a covering; thus, we have proved the wanted result

$$\Vdash_{\overline{A}} \exists y \in U \ \forall x \in \Delta(n) \ \neg(\Phi(x,y)=0). \quad \square$$

With such a handy tool, we can easily obtain the internal validity in $\mathcal{G}$ of the results of [BUNGE: Synthetic aspects of C^∞-mappings] for germs. We hope that this will contribute to reinforce the claim made by Bunge (cf. [BUNGE: Synthetic aspects of C^∞-mappings]), Bunge and Dubuc (cf. [BUNGE-DUBUC: Local concepts in S.D.G. and germ representability]) and Penon (cf. [PENON: De l'infinitésimal au local]) that the right place to do (interpret) Synthetic Differential Geometry is a model in which germs are infinitesimally represented. We subscribe the claim.

Definition 3.2 A germ $f \in R^{p^{\Delta(n)}}$ is said to be an immersion if $rank(D_0 f) = n$.

We now have the following result (cf. also [BUNGE-GAGO: Synthetic aspects of C^∞-mappings, II: Mather's theorem for infinitesimally represented germs]).

Corollary 3.3 If $p>2n$, then the class of immersions is dense in $R^{p^{\Delta(n)}}$ for the weak topology (see §2.4)

Proof. We will show that, for every basic neighborhood of the weak topology, there exists an immersion on it. In doing so, we use the fact that $\Delta(n)$ is compact (Proposition 2.3.3) to reduce ourselves to the consideration of objects of the form $V(\Delta(n), r, h, \varepsilon)$ with $h \in R^{p^{\Delta(n)}}$ $1 \leq r \leq n$, and $\varepsilon \in R$, $\varepsilon > 0$. For such an object, we will show that there exists a polynomial $\sigma \in R^{pR^n}$, of total degree 1 and coefficients $c_i \in (-\varepsilon, \varepsilon)^p$, such that $h + \sigma|_{\Delta(n)}$ is an immersion.

Let $s=rank(D_o h)$, and define $\Phi \in R^{p^{\Delta(s+n)}}$ as follows

$$\Phi(\lambda, x) = \sum_{i=1}^{s} \lambda_i \cdot \frac{\partial h}{\partial x_i}(x) - \frac{\partial h}{\partial x_{s+1}}(x),$$

by Theorem 3.1, we have

$$\exists c_{s+1} \in R^p\ [c_{s+1} \in (-\varepsilon,\varepsilon)^p \wedge c_{s+1} \textit{ regular value of } \Phi].$$

Define $g_1 \in R^{p^{\Delta(n)}}$ by $g_1(x) = h(x) + c_{s+1} \cdot x_{s+1}$. By ordinary differentiation [KOCK: Synthetic Differential Geometry, §1.2,] we get

$$\frac{\partial g_1}{\partial x_i}(x) = \frac{\partial h}{\partial x_i}(x), \qquad \text{for every } x \in \Delta(n), \text{ for } i \leq s,$$

$$\frac{\partial g_1}{\partial x_{s+1}}(x) = \frac{\partial h}{\partial x_{s+1}}(x) + c_{s+1} \qquad \text{for every } x \in \Delta(n).$$

Since c_{s+1} is a regular value of Φ, and $s \leq n$, $p \geq 2n$, Φ cannot be a submersion at (λ, x)

$$\forall (\lambda, x) \in \Delta(s+n)\ [\neg(\Phi \textit{ submersion at } (\lambda, x))],$$

and c_{s+1}, being a regular value of Φ, cannot be in the image of Φ. In particular $\neg(\Phi(0,0)=c_{s+1})$. Using this remark and that $s=rank(D_o h)$, it is easily seen (as in [BUNGE: Synthetic aspects of C^∞-mappings]) that

$$\forall \lambda_1 \ldots \lambda_s \in R\ [\neg(\sum_{i=1}^{s} \lambda_i \cdot \frac{\partial h}{\partial x_i}(0) - \frac{\partial h}{\partial x_{s+1}}(0) = c_{s+1})],$$

which means that

$$\forall \lambda_1 \ldots \lambda_s \in R\ [\neg(\sum_{i=1}^{s} \lambda_i \cdot \frac{\partial g_1}{\partial x_i}(0) = \frac{\partial g_1}{\partial x_{s+1}}(0))],$$

and this amounts to saying that the set

$$\left\{\frac{\partial g_1}{\partial x_1}(0), \ldots, \frac{\partial g_1}{\partial x_s}(0), \frac{\partial g_1}{\partial x_{s+1}}(0)\right\}$$

is linearly in dependent.

By repeating this procedure n-$(s+1)$ times, we get $c_{s+1}, c_{s+2}, \ldots, c_n \in (-\varepsilon,\varepsilon)^p$ the coefficients of $\sigma(x) = c_{s+1} \cdot x_{s+1} + \cdots + c_n \cdot x_n$, the wanted polynomial, as $h+\sigma$ is an immersion, and certainly $h+\sigma \in V(\Delta(n), r, h, \varepsilon)$. □

It is also possible to show that the immersions in $R^{p\Delta(n)}$ form a weak open object.

Proposition 3.4 If $p \geq n$, the object $[|f \in R^{p\Delta(n)} \mid f \textit{ is an immersion}|]$ is open for the weak topology in $R^{p\Delta(n)}$.

Proof. Since R is separated (T_1) for the Penon topology (Proposition 1.2.8) and $R^* = \neg\{0\}$ (Proposition 1.2.2), the object $[|A \in Mat(k \times k) \mid det(A) \# 0 |]$ is also Penon open (Proposition 1.2.7). By the standard equivalence between the definitions of $Rank(A)$ (Proposition 1.2.4) the object $[|A \in Mat(n \times p) \mid Rank(A) = n |] \subset R^{n \cdot p}$ is also Penon open (union of inverse images of opens), hence euclidean open in presence of POSTULATE WA1.1. Then, if a matrix A has rank n so does any other whose entries "differ" form those of A by less than ε, for some $\varepsilon \in R$, $\varepsilon > 0$

Now, $f \in R^{p^{\Delta(n)}}$ is an immersion if $Rank(D_0 f) = n$. By the above, there exists $\varepsilon \in R$, with $\varepsilon > 0$, for which $V(\Delta(n), 1, f, \varepsilon) \subset Imm(R^{p^{\Delta(n)}})$ as required. □

In our setting, germs of immersions behave particularly well. In this sense we have

Proposition 3.5 If $f \in R^{pR^n}$ is an immersion at $0 \in R^n$, then f is infinitesimally injective at 0, i.e.

$$\forall x,y \in \Delta(n)\ [(f(x) = f(y)) \Rightarrow (x = y)].$$

Proof. As a consequence of AXIOM V, for $f: R^n \to R^p$, there exists $g: R^n \times R^n \to Mat(n \times p)$ such that for every x and y in R^n, $f(x) - f(y) = g(x, y) \cdot (x_i - y_i)$, and for every given x of R^n, $g(x, x) = D_x f$ (see Corollary 1.1.5.) Now, f immersion at 0 means $Rank(D_0 f) = n$, and by proposition above $Rank(A) = n$, for every matrix A in $\neg\neg\{D_0 f\}$. But $D_0 f = g(0, 0)$, and if $x,y \in \Delta(n)$, then (x, y) is in $\Delta(n+n)$, and therefore $g(x, y) \in \neg\neg\{g(0, 0)\} = \neg\neg\{D_0 f\}$. This means that $g(x, y)$ is left invertible for all $x,y \in \Delta(n)$ which yields the wanted result. □

§4. Transversality

To end this chapter, we give one more consequence of the theorem of density of regular values, namely Thom's Transversality Theorem, the key result in the theory [GOLUBITSKI-GUILLEMIN: Stable mappings and their singularities, p. 54]

The notion of transversality is a generalization of that of regular value [GUILLEMIN-POLLACK: Differential Topology,] and can be defined in our context [BUNGE: Synthetic aspects of C^∞-mappings]. For this, recall that if X_1 and X_2 are R-submodules of a given R-module Y, by $X_1 + X_2$ we denote the following subobject of Y: $[[x_1 + x_2 \mid x_1 \in X_1 \wedge x_2 \in X_2]]$. It is an R–submodule of Y.

Recall also, that if $f \in R^{pX}$, with $x \in X \subset R^n$, there is induced $df_x \in (T_{f(x)} R^p)^{T_x X}$, an R-linear map whose image, $Im(df_x)$ is an R-submodule of $T_{f(x)} R^p$.

We have to choose a notion of manifold, among the several ones available in the synthetic context. For our general purposes, the right one seems to be the following (see Definition 1.5.1):

Definition 4.1 A subobject M of R^n is said to be a submanifold of R^n of dimension $r \le n$ (also, we could say of codimension n-r,) if for each $x \in M$ there is given an isomorphism $\alpha:\neg\neg\{x\} \to \neg\neg\{0\}^n$, such that the restriction of α to $\neg\neg\{x\}\cap M$ goes onto $\neg\neg\{0\}^r$, considered as subobject of $\neg\neg\{0\}^n$, the inclusion map being given by $(x_1, \ldots ,x_r) \mapsto (x_1,\ldots,x_r,0,\overset{n-r}{\cdots},0)$.

However, in stating and proving Thom's theorem, we seem to need a stronger notion

Definition 4.2 For $g_1, \ldots ,g_s \in R^{R^n}$, we say that they are independent functions if

$$\forall x \in \bigcap_{i=1}^{s} g_i^{-1}\{0\}\ \forall u \in T_x R^n\ [\{(dg_1)_x(u)\cdots(dg_s)_x(u)\}\ \textit{is linearly independent}].$$

We, now take from [BUNGE: Synthetic aspects of C^∞-mappings] the two following theorems

Theorem 4.3 (*Submersion theorem*) Let $f \in R^{pR^n}$, $x \in R^n$ with f a submersion at x. Then the germ of f at x is locally equivalent to the germ at 0 of the canonical projection $\pi^n_p:R^n \to R^p$.

Proof. Without loss of generality we can assume that x is the 0; indeed, the isomorphism $\neg\neg\{x\} \approx \neg\neg\{0\}$ does not affect the rank of f, nor the equivalence of germs. So, let $f\in R^{pR^n}$ be a submersion at $0 \in R^n$, i.e.,

$$\Vdash_{A} \bigvee_{(i_1,\ldots,i_p)\in\binom{n}{p}} \left(\left\{\frac{\partial f}{\partial x_{i_1}}(0),\ \ldots\ ,\frac{\partial f}{\partial x_{i_p}}(0)\right\}\ \textit{linearly independent}\right).$$

Kripke-Joyal semantics gives the existence of a jointly epimorphic family $\{\xi_i:A_i \to A\}_{i\in I}$ in $\mathcal{E}$, such that, for each $i\in I$, there is a p-tuple $(i_1, \ldots ,i_p)$ so that

$$\xi_i^*\Big(\frac{\partial f}{\partial x_{i_1}}(0)\Big), \ldots , \xi_i^*\Big(\frac{\partial f}{\partial x_{i_p}}(0)\Big) \text{ are linearly independent as vectors at stage } A_i,$$

or equivalently (by uniqueness in Kock-Lawvere axiom)

$$\Big(\frac{\partial(\xi_i^* f)}{\partial x_{i_1}}(0)\Big), \ldots , \Big(\frac{\partial(\xi_i^* f)}{\partial x_{i_p}}(0)\Big) \text{ are linearly independent}.$$

We intend to show that, for each $i \in I$, there is a jointly epimorphic family $\{B_{i_j} \to A_i\}_{j \in J_i}$ such that, for each $j \in J_i$, there are $\varphi \in_{B_{i_j}} R^{nR^n}$ infinitesimal invertible at 0, and $\psi \in_{B_{i_j}} R^{pR^p}$ infinitesimal invertible at $f(0)$ making commutative the following diagram

So, we could, and for simplicity we do now, assume that

$$\left\{ \frac{\partial f}{\partial x_{i_1}}(0) \ , \ \ldots \ , \ \frac{\partial f}{\partial x_{i_p}}(0) \right\} \textit{ linearly independent} \text{ at stage } A.,$$

Define $\varphi \in_A R^{nR^n}$ by $\varphi = \langle f, \pi^n_{n-p} \rangle$. Clearly, $\varphi(0) = 0$ and $rank(D_o\varphi) = n$. The inverse function theorem (POSTULATE I.I. [*alternative formulation*] §1.4) gives φ infinitesimally invertible at 0. Composing f with this inverse (at the corresponding stage) we get the required projection. □

Corollary 4.4 (*Preimage theorem*) The following holds in our setting

$$\forall f \in R^{pR^n} \forall y \in R^p \ [\textit{reg. val.} \text{ of } f \Rightarrow M = f^{-1}\{y\} \text{ submanifold of } R^n \text{ of codimension } p.$$

Proof. Assume f, y to be given both at stage A. If $x \in_A M$, then $f(x) = y$; therefore f is necessarily a submersion at x. By the theorem above, the germ of f at x is equivalent to that of π^n_p at 0. i.e., there is a jointly epimorphic family $\{B_i \to A\}_{i \in I}$, and for each $i \in I$ isomorphisms φ_i and ψ_i making commutative the square

$$\begin{tikzcd} \Delta(n) \arrow[r, "f"] \arrow[d, "\varphi_i"'] & \neg\neg f(0) \arrow[d, "\psi_i"] \\ \Delta(n) \arrow[r, "\pi"] & \Delta(p) \end{tikzcd} \qquad (*)$$

where we have omitted the notation indicating the change of stage and the arrows are to be interpreted at the corresponding level.[1] So, for instance, in the topos $\mathcal{E}/A$, we have the pullback diagram of next page

[1] The logical character of the functor involved in a change of stage enables us to make these abuses of notation. In particular, since these functors all preserve products, the canonical projections are also preserved.

$$\begin{tikzcd}
(\neg\neg\{x\} \cap M \arrow[r] \arrow[d] & \neg\neg\{x\} \arrow[d] \arrow[dd, bend left=60, "f|_x"] \\
f^{-1}\{y\} \arrow[r] \arrow[d] & R^n \arrow[d, "f"] \\
1 \arrow[r, "y"] & R^p
\end{tikzcd}$$

which says that $(f|_x)^{-1}\{y\} \approx \neg\neg\{x\} \cap M$.

Now, the result follows from the commutativity of (*), since we have the following chain of isomorphisms $(f|_x)^{-1}\{\psi_i(0)\} \approx \neg\neg\{x\} \cap M \approx (\pi^n_p)^{-1}\{0\} \approx \Delta(n\text{-}p)$.

The result we have just proved, establishes that the solutions of an equation $f(x) = y$ form a submanifold of R^n, provided $y \in R^p$ is a regular value of $f \in R^{pR^n}$. Very often, it is useful to conclude that the object of elements of R^n, whose functional values are constrained to be, not an element but, to satisfy a certain condition, form a submanifold. We give, in our context, an instance of condition to be imposed on $N \subset R^p$ such that the object of solutions for the equation $f(x) \in N$ forms a submanifold of R^n.

Definition 4.5 Let $f \in R^{pR^n}$, $x \in R^n$, $N \subset R^p$ be such that $f(x) \in N$. We say that is transversal to N at x (we write $f \pitchfork_x N$) if $T_{f(x)}R^p = Im(df_x) + T_{f(x)}N$. We say that is transversal to N (and we write $f \pitchfork N$) if $\forall x \in R^n\ (\neg(f(x) \in N) \vee f \pitchfork_x N$.

The following constitutes a generalization of the Preimage theorem (cf. also [BUNGE-GAGO Synthetic aspects of C^∞-mappings, II: Mather's theorem for infinitesimally represented germs]):

Theorem 4.6 Let $f \in R^{pR^n}$ and $N \subset R^p$ a submanifold cut out by independent functions, and of codimension $s \leq p$. Assume that $f \pitchfork N$. Then $M = f^{-1}(N) \subset R^n$ is submanifold of codimension s (also cut out by independent functions.)

Proof. Let $f \in R^{pR^n}$ and $N \subset R^p$ be both given at stage A, and assume that $f \pitchfork N$. By definition of submanifold cut out by independent functions, there is a jointly epimorphic family $\{A_i \to A\}_{i \in I}$ such that, for each $i \in I$, N is carved out of R^p by independent functions $g_1^i, \dots, g_s^i \in_{A_i} R^{R^p}$. Define a new function $g^i = (g_1^i, \dots, g_s^i) \in_{A_i} R^{sR^p}$. The claim is now, that $g^i \cdot f$ is a submersion at every $x \in_{A_i} R^n$, for which $g^i \cdot f(x) \in N$. To see this use the following diagram in $\mathcal{E}/A_i$,

$$\begin{tikzcd} T_x R^n \arrow[r, "d(g^i f)_x"] \arrow[d, "df_x"'] & T_{g^i f(x)} R^s \\ Im(df_x) \arrow[r] & T_{f(x)} R^p \arrow[u, "(dg^i)_{f(x)}"'] \end{tikzcd}$$

Now, since g^i is a submersion, $(dg^i)_{f(x)}$ is locally surjective, and the result will follow at once from the condition $T_{f(x)}R^p = Im(df_x) + Ker((dg^i)_{f(x)}) = Im(df_x) + T_{f(x)}N$, at stage A_i. But the second equality follows from definition of g^i, and $T_{f(x)}R^p = Im(df_x) + T_{f(x)}N$ is what transversality says at level A_i.[1]

Using Theorem 4.4, $(g^i \cdot f)^{-1}\{0\}$ is a submanifold of codimension s, and we have the equalities $(g^i \cdot f)^{-1}\{0\} = f^{-1}(g^{i\,-1}\{0\}) = f^{-1}(N)$, which end the proof. □

We are now in a position to state and prove the announced

Theorem 4.7 (*Thom's Transversality Theorem*) For $n, m > 0$, and $1 \le r \le n$, given any $N \subset R^{pD_r(n)} = R^s$ a submanifold cut out by independent functions, the class of germs $g \in R^{p\Delta(n)}$ with $J^r g \pitchfork N$ is dense for the weak topology.

Proof. With the same simplifications of Corollary 3.3, we will find a polynomial $\sigma \in R^{pR^n}$, of total degree l and coefficients $c_i \in (-\varepsilon, \varepsilon)^p$, such that $J^r(h + \sigma|_{\Delta(n)}) \pitchfork N$. Define the map γ_h at level A, given by the following law

$$[(x,f) \in \Delta(n) \times R^{pD_r(n)} \mapsto J^r(h+f)(x) \in R^{pD_r(n)}].$$

---

[1]Notice that the notion of transversality is stable in the sense of [KOCK: Synthetic Differential Geometry, p.141.]

Investigating the jacobian of γ_h we find that it is a submersion[1], and therefore $\gamma_h \pitchfork N$. Since N is cut out by independent functions, Theorem 4.6 gives that $M = \gamma_h^{-1}(N)$ is a submanifold of $\Delta(n) \times R^s$, and Theorem 3.1 gives

$$\exists (c_{i,\alpha})_{1\leq i\leq p,\ 1\leq\alpha\leq\binom{n+k}{k}} \in R^s\ [c_{i,\alpha} \in (-\varepsilon,\varepsilon) \wedge (c_{i,\alpha})\ \textit{regular value of}\ \pi^M|_o\,]$$

where $\pi^M|_o$ denotes the germ at 0 of the restriction to M of the projection $\pi\colon \Delta(n) \times R^s \to R^s$

Define $\sigma_i(x) = \sum_{|\alpha|\leq r} c_{i,\alpha} \cdot x^\alpha$, i=1, … ,$p$, and check that $\sigma = (\sigma_i)_{1\leq i\leq p}$ is the required polynomial. □

---

[1]It is useful the identification $R^{pD_r(n)} = R^s$.

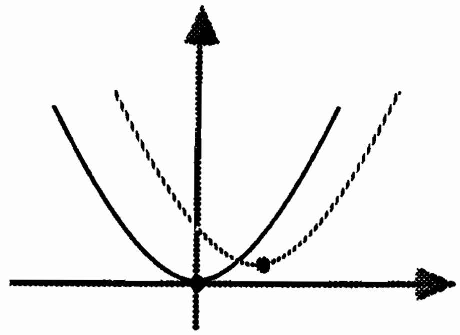
4

Stability

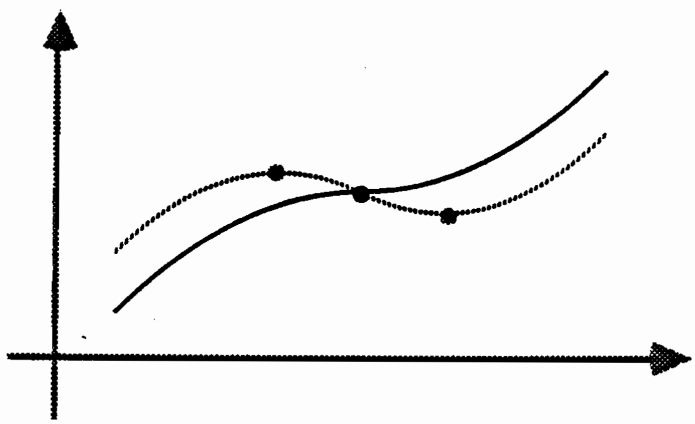
§1. Basic definitions

The aim of this chapter is the introduction of various notions of stability in the context of Synthetic Differential Geometry. To this end, we exhibit the second basic result (along with the Theorem of Density of Regular values) for the study of singularities, namely the Malgrange Theorem. We will see how these notions simplify and will present (internally) some useful theorems of characterization. In particular, we prove (cf. also [BUNGE-GAGO: Synthetic aspects of C^∞-mappings, II: Mather's theorem for infinitesimally represented germs]) that a version of Mather's theorem, which characterizes in algebraic terms the condition for a germ to be stable, holds in our test model $\mathcal{G}$. Finally, we point out (as we did more extensively in [BUNGE-GAGO: Synthetic aspects of C^∞-mappings, II: Mather's theorem for infinitesimally represented germs]) that, in this context, germs and unfoldings can receive the same treatment, contrary to what happens in the classical setting (cf. [WASSERMANN: Stability of Unfoldings]).

The basic notion in the theory of stable C^∞-mappings is that of equivalence and similarity for germs, unfoldings, vector fields, etc. (cf. [BRÖCKER: Differentiable Germs and Catastrophes], [GOLUBITSKI-GUILLEMIN: Stable Mappings and their Singularities], [POENARU: Analyse Différentielle], [WASSERMANN: Stability of Unfoldings]).



For mappings, equivalence means "looking alike" after some change of coordinates. So, for instance, in the picture above, the dotted function would be equivalent to the black one, while in the picture below the functions are not equivalent, yet they still are "near" to each other.



For germs, the definition is rather more complicated essentially for two reasons. First of all, representatives are to be taken; that implies the choice of some open neighborhood of the base point. Secondly, to avoid being too restrictive keeping fixed the base point, one has to allow variation; the equivalence is then established between germs at different points, within the open neighborhood we choose. So, there exist open neighborhoods $x \in U \subset \mathbb{R}^n$, $x' \in U' \subset \mathbb{R}^n$, $f(x) \in V \subset \mathbb{R}$ and $f'(x') \in V' \subset \mathbb{R}$, such that the following diagram of germs commutes:

$$\begin{tikzcd}
(U, x) \arrow[r, dashed, "f|_x"] \arrow[d, dashed, "h|_x"'] & (V, f(x)) \arrow[d, dashed, "h'|_{x'}"] \\
(U', x') \arrow[r, dashed, "f|_{x'}"] & (V', f'(x')).
\end{tikzcd}$$

In our setting these definitions simplify somehow in the following terms.

Definition 1.1 Given $f \in R^{nR^n}$ and $x \in R^n$, f is said to be infinitesimally invertible (respectively, surjective) at x if $f|_{\neg\neg\{x\}} : \neg\neg\{x\} \to \neg\neg\{f(x)\}$ is an isomorphism (respectively, a surjection).

In the presence of the axiom of germ representability (AXIOM III) it seems coherent to denote by $f|_x$ this restriction, and to call it the germ of f at x. If x is not a global section, this notation should not lead to confusion though there are no "external" grounds to interpret this $f|_x$ as a germ. However we will find useful to employ these intuitively conceived "phantom" germs to develop our theory.

Definition 1.2 Given $f, f' \in R^{\Delta(n)}$, we say that f is equivalent to f', and write $f \sim f'$ if the following holds:

$$\exists x, x' \in R^n \exists \varphi \in R^{nR^n} \exists \psi \in R^R\ [(\varphi\ \mathit{inf.inv.}\ \text{at}\ x) \wedge (\psi\ \mathit{inf.inv.}\ \text{at}\ f(0)) \wedge \varphi(x) = x' \wedge f' = \psi|_{f(0)} \circ f \circ (\varphi|_x)^{-1}$$

In a picture, we have the following commutative diagram

$$\begin{tikzcd}
\neg\neg\{x\} \arrow[r, "f|_x"] \arrow[d, "\varphi|_x"'] & \neg\neg\{y\} \arrow[d, "\psi|_y"] \\
\neg\neg\{x'\} \arrow[r, "f'|_{x'}"] & \neg\neg\{y'\}
\end{tikzcd}$$

where $f|_x$ denotes the composite $\neg\neg\{x\} \xrightarrow{a_x^{-1}} \Delta(n) \xrightarrow{f} \neg\neg\{y\}$, with $a_x : \Delta(n) \to \neg\neg\{x\}$ being the isomorphism adding by x of Proposition 1.3.1(i), and where $y = f(0)$.

With our terminology, the germ of f at x is equivalent to the germ of f' at x', were we to start with $f \in R^{\neg\neg\{x\}}$ and $f' \in R^{\neg\neg\{x'\}}$. If we interpret this fact internally, say in $\mathcal{G}$, we start with $x \in_{\overline{A}} R^n$ and $f \in_{\overline{A}} R^{\neg\neg\{x\}}$, where the arrow $\neg\neg\{x\} : \overline{A} \to \Omega^{R^n}$ corresponds to $\neg\neg\{x\} \subset \overline{A} \times R^n$, hence

$$\begin{tikzcd}
\neg\neg\{x\} \arrow[r, "f"] \arrow[d, tail] & R \\
\overline{A} \times R^n &
\end{tikzcd}$$

Now, if $f' \in_{\overline{A}} R^{\neg\neg\{x'\}}$, then to say that f is equivalent to f' as germs at x and x', respectively, amounts to saying that there exist [1] $\varphi \in_{\overline{A}} R^{nR^n}$ and $\psi \in_{\overline{A}} R^R$ infinitesimally invertible at x and $f(x)$, respectively, i.e.

$$\begin{tikzcd}
\neg\neg\{x\} \arrow[r, "\varphi|_x"] \arrow[d, tail] & \neg\neg\{\varphi(x)\} \arrow[d, tail] \\
\overline{A} \times R^n \arrow[r, "\langle id, \varphi \rangle"] & \overline{A} \times R^n
\end{tikzcd}$$

where $\varphi(x)$ is the composite $\overline{A} \xrightarrow{\langle id, x \rangle} \overline{A} \times R^n \xrightarrow{\varphi} R^n$ (and similarly for ψ) such that

$$\begin{tikzcd}
\neg\neg\{x\} \arrow[r, "f"] \arrow[d, "\varphi|_x"', "\approx"] & \neg\neg\{y\} \arrow[d, "\psi|_y", "\approx"'] \\
\neg\neg\{x'\} \arrow[r, "f'"] & \neg\neg\{y'\}
\end{tikzcd}$$

Let us now take a look to the picture for two particular germs defined at stage $\overline{A} = \Delta(r)$, one of which is constantly $\eta_0 \in {}_1R^{\Delta(n)}$, a germ at $0 : \Delta(r) \to R^n$, constantly $0 \in R^n$, and the other one is any $\eta : \Delta(r) \times \Delta(n) \to R^n$, a germ at $x : \Delta(r) \to R^n$, with $x(0) = 0$ and $\eta(0, -) = \eta_0$.

For u "near 0" we have each η_u a germ at $(u, x(u))$ equivalent to η_0 at $(u, 0)$, in fact the same germ as η_0 at $(0, 0)$, according to the picture below

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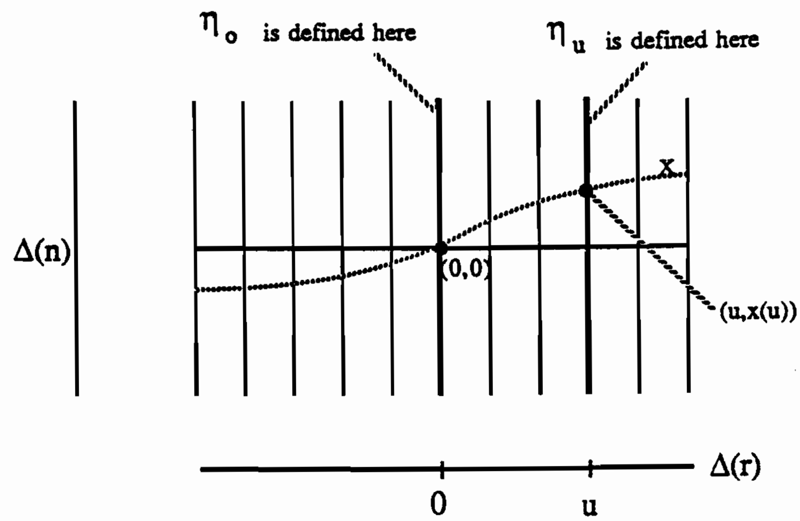
[1] Of course internal existence, hence on a cover, cover that we ignore for the moment.



This is exactly the situation for unfoldings (cf. [WASSERMANN: Stability of Unfoldings]) in our framework, which we obtain for free!.

Let us suppose now that $f \in R^{\Delta(n)}$ is so that $f(0) = 0$ [1]. We can (internally) define a map

$$G := Inf.inv._0(R^{n R^n}) \times Inf.inv._0(R^R) \xrightarrow{\gamma_f} R^{\Delta(n)}$$

by means of $\gamma_f(\varphi, \psi) = \psi|_0 \circ f \circ (\varphi|_0)^{-1} \circ a_{\varphi(0)}$.

Note that $\gamma_f(id_{R^n}, id_R) = f$, and that for any pair $(\varphi, \psi) \in G$, $\gamma_f(\varphi,\psi) = f' \in R^{\Delta(n)}$ is a germ at 0 which, when regarded as germ at $x = \varphi(0) \in R^n$, is equivalent to f as germ at 0.

The following definition (unlike the one given in [BUNGE-GAGO: Synthetic aspects of C^∞-mappings, II: Mather's theorem for infinitesimally represented germs]) reflects more faithfully the classical notion.

Definition 1.3 We say that $f \in R^{\Delta(n)}$ is stable if $Im(\gamma_f) \subset R^{\Delta(n)}$ is open for the weak topology and $\gamma_f(\varphi, \psi) = f$ implies $\varphi = id_{R^n}$ and $\psi = id_R$.

N.B. $\neg\neg\{(id_{R^n}, id_R)\} = \neg\neg\{id_{R^n}\} \times \neg\neg\{id_R\}$ is contained in G as a consequence of the following proposition

[1]This assumption is only to simplify matters and has no other significance

Proposition 1.4 For every n, the object $Inf.inv._0(R^nR^n) \subset R^nR^n$ is weak open, hence Penon open.

Proof. After Proposition 3.3.4, the object of immersions at 0 is weak open; by Proposition 3.3.5 every immersion at 0 is infinitesimally injective at 0, and by Proposition 1.1.5 it is infinitesimally invertible as we claimed.

Since $id_{R^n} \in R^nR^n$ is infinitesimally invertible at 0, the infinitesimal monad around it must be formed also of infinitesimal invertible maps; this completes the remark above, and gives sense to Definition 1.3.

It should also be pointed out that the second part of the condition in Definition 1.3 does not add any "observable" restriction; by this we mean that if one applies the global sections functor the condition is trivially satisfied, and f is stable in the usual sense (f has a neighborhood of equivalent mappings.)

§2. Infinitesimal stability

In the classical setting, the usual definition of stability proves difficult to apply in practice. However, using a criterion suggested by René Thom (cf. [THOM-LEVINE: Singularities of Differentiable Mappings, I, Bonn 1959],) John Mather (cf. [MATHER: Stability of C^∞-mappings, II: Infinitesimal stability implies stability]) has produced a theorem which provides a truly computable method for determining whether or not a mapping is stable. The intuition behind this useful result finds no room to accommodate within the classical theory of differentiable manifolds (see the introduction to §1.1) and has to be "disguised" with rather artificial formulae.

Definition 2.1 A map $f \in R^{\Delta(n)}$, with $f(0) = 0$, is infinitesimally stable if we have

$$\forall \omega \in Vect(f)\ \exists \sigma \in Vect(R^n)\ \exists \tau \in Vet(R)\ [\omega = \alpha_f(\sigma) \oplus \beta_f(\tau)].$$

To understand the notation, several comments (most of them from [BUNGE: Synthetical aspects of C^∞-mappings]) are in order; we proceed from left to right in the formula.

With $Vect(f) \subset R^{D^{\Delta(n)}}$ we denote the object of vector fields "along f", i.e., (internal) maps $\Delta(n) \rightarrow R^D$ making commutative the following diagram

$$\begin{tikzcd} & R^D \arrow[d, "\pi"] \\ \Delta(n) \arrow[ur] \arrow[r, "f"] & R \end{tikzcd}$$

where $\pi : D^D \to R$ is the canonical projection of the tangent bundle (see §1.4). The λ-rules valid in our models give the identifications $Vect(f) = T_f R^{\Delta(n)}$, as $R^{D^{\Delta(n)}} \approx R^{\Delta(n) \times D} \approx R^{\Delta(n)^D}$.

$\alpha_f : Vect(R^n) \to Vect(f)$ is defined by $\alpha_f(\sigma)\ (d) = f \circ (\sigma|_0(d))^{-1}$, for $\sigma \in R^{nD^{R^n}}$, and therefore $\sigma|_0 \in R^{nD^{\Delta(n)}}$. Once again, σ can be seen as an infinitesimal deformation of id_{R^n}, and for each $d \in D$, $\sigma(d) \in X^X$ has an inverse, namely, $\sigma(-d) \in X^X$ (see Definition 1.4.1).

β_f becomes the map between the tangent spaces $T_{id}(R^R) \to T_f(R^{\Delta(n)})$, induced by the mapping $[\psi \in R^R \longmapsto \psi \circ f \in R^{\Delta(n)}]$.

The key point is now that $T_{id}(R^R) \approx T_{id}(Inf.inv._0 R^R)$, as in Proposition 1.5.5, since it is a Penon open submanifold (Proposition 1.4) and α_f becomes the map induced by the following composition

$$\begin{array}{ccccc} Inf.inv._0\, R^{nR^n} & \to & Inf.inv._0\, R^{nR^n} & \to & R^{\Delta(n)} \\ \varphi & \mapsto & \varphi^{-1} \circ a_{\varphi(0)} & \mapsto & f \circ \varphi^{-1} \circ a_{\varphi(0)} \end{array}$$

Finally, $R^{\Delta(n)}$ is infinitesimally linear (Proposition 1.4.7,) and therefore $(R^{\Delta(n)})^D$ is fiberwise an R-module, which gives sense to $\alpha_f(\sigma) \oplus \beta_f(\tau)$ (see Definition 1.5.6 and Proposition 1.5.8.)

N.B. The observable part in $\mathcal{G}$ of this definition (i.e., when the global sections functor is applied) states that $\Gamma(f)$ is infinitesimally stable in the sense of [POENARU: Analyse Différentielle, pag. 168] (cf. [BUNGE: Synthetic aspects of C$^\infty$-mappings] and [BUNGE-GAGO: Synthetic aspects of C$^\infty$-mappings, II: Mather's theorem for infinitesimally represented germs]) .

As claimed before, we recover the lost intuition with the following proposition.

Proposition 2.2 Given $f \in R^{\Delta(n)}$, with $f(0) = 0$, f is infinitesimally stable if and only if $d\gamma_{f(id_{R^n}, id_R)}$ is surjective.

Proof. After what we have seen it is clear that $d\gamma_{f(id_{R^n}, id_R)}(d) = [\alpha_f(\sigma) \oplus \beta_f(\tau)]\,(d)$, for every $d \in D$.

We are now ready to state and prove the first part of the comparison theorem, which should come as no surprise.

Proposition 2.3 Let $f \in R^{\Delta(n)}$, with $f(0) = 0$, be stable. Then f is infinitesimally stable.

Proof. The call for "no surprise" is based on the following observation: f stable implies that γ_f is infinitesimally surjective, meaning that its restriction to $\neg\neg\{(id_{R^n}, id_R)\} \to \neg\neg\{f\}$ is surjective.[1] Indeed, by definition of stability, $f \in Im\gamma_f$ weak (hence Penon) open. This gives that $\neg\neg\{f\} \subset Im\gamma_f$, and by the second part of the condition, whenever $\gamma_f(\varphi, \psi) \in \neg\neg\{f\}$, we have $(\varphi, \psi) \in \neg\neg\{(id_{R^n}, id_R)\}$. Then we must show that $d\gamma_f$ at (id_{R^n}, id_R) is surjective, and we know that γ_f is infinitesimally surjective at (id_{R^n}, id_R).

But, from $\gamma_f : \neg\neg\{(id_{R^n}, id_R)\} \to \neg\neg\{f\}$ surjective and AXIOM II it follows that the map $(\gamma_f)^D : (\neg\neg\{(id_{R^n}, id_R)\})^D \to (\neg\neg\{f\})^D$ is also surjective.

Now, given $\xi \in (\neg\neg\{f\})^D$, there exists $\zeta \in (\neg\neg\{(id_{R^n}, id_R)\})^D$ such that, for any d in D, $\xi(d) = \gamma_f(\zeta(d))$; and if $\xi(0) = f$, i.e., $\xi \in T_f \neg\neg\{f\}$, then, part two on the condition for stability says that $\zeta(0) = (id_{R^n}, id_R)$, i.e., $\zeta \in T_{(id_{R^n}, id_R)} \neg\neg\{(id_{R^n}, id_R)\}$. In other words,

$$T_{(id_{R^n}, id_R)} \neg\neg\{(id_{R^n}, id_R)\} \to T_f \neg\neg\{f\}$$

is surjective, and this is what we wanted, for this is our map

$$d\gamma_{f(id_{R^n}, id_R)} : T_{(id_{R^n}, id_R)} G \to T_f R^{\Delta(n)}.$$

[1] Notice that (after proposition 1.4) G is weak (hence Penon) open, and $\neg\neg\{(id_{R^n}, id_R)\} \subset G$, as subobjects of $R^{nR^n} \times R^R$.

The reverse implication also holds and will be the subject of the last section of this chapter to prove it.

§3. Homotopical stability

This section is dedicated to extract the intuitive part of a technical procedure which will be employed in the proof of Mather's theorem in section 5. We also prove (cf. also [BUNGE-GAGO: Synthetic aspects of C^∞-mappings, II: Mather's theorem for infinitesimally represented germs]) a theorem of existence and uniqueness of solution for time dependent vector fields, provided we add to our stock of axioms a postulate of local integration of ordinary vector fields.

We begin with two definitions

Definition 3.1 Given $f: X \to Y$, and $\varepsilon \in R$, $\varepsilon > 0$, a map $F: X \times [0,\varepsilon] \to Y \times [0,\varepsilon]$ is called a deformation of f if the following two conditions are satisfied:

i) $F: X \times \{t\} \to Y \times \{t\}$, for each $t \in [0, \varepsilon]$

ii) $F_0 = f$.

It is clear that a deformation, in the above sense, gives rise (actually it is equivalent) to a unique family of maps $F_t: X \to Y$ indexed by $[0, \varepsilon]$, such that $F_0 = f$. We can even say more than that; in our context, this family (or the deformation) represents a curve on the functional space Y^X, $[0, \varepsilon] \to Y^X$, starting at f.

Definition 3.2 A deformation $F: X \times [0, \varepsilon] \to Y \times [0, \varepsilon]$, of a map $f: X \to Y$, is said to be trivial if there exist deformations of id_X, $G: X \times [0, \delta] \to X \times [0, \delta]$, and of id_Y, $H: Y \times [0, \delta] \to Y \times [0, \delta]$, for some $\delta \in R$, $0 < \delta \leq \varepsilon$, which are isomorphisms, and such that the diagram in next page

$$\begin{tikzcd} X \times [0, \delta] \arrow[r, "F"] \arrow[d, "G"'] & Y \times [0, \delta] \arrow[d, "H"] \\ X \times [0, \delta] \arrow[r, "f \times id_{[0,\delta]}"] & Y \times [0, \delta] \end{tikzcd}$$

commutes.

Viewing a deformation as a curve on Y^X, we have the condition on F of being trivial translated into the commutativity of

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ {\scriptstyle G_t}\downarrow & & \downarrow{\scriptstyle H_t} \\ X & \xrightarrow{\quad F_t \quad} & Y \end{array}$$

for all "small" t, which means that $f \sim F_t$.

We could now introduce the following definition

Definition 3.3 f is homotopically stable (or stable under deformations) if every deformation of f is trivial.

We are not interested in exploiting this definition (which can be proved to be equivalent to stable and infinitesimally stable) but in the homotopy method itself. For instance, to prove that $f \sim g$ we could join g to f by a deformation, and then prove that it is trivial. Also, to show that f is stable, we could define a weak neighborhood of f consisting of trivial deformed of f. All this will be used in section 5 and in chapter V.

A useful method to show that a deformation is trivial is to construct the G_t and the H_t as the integral flows of some time-dependent vector fields (or, in our context, a dynamical system.) For this reason we need to establish under what conditions these solutions exist.

By POSTULATE WA.2 (see §1.4) we have established that, given $g : R^n \to R^n$, there exists a unique $f : R^n \times \Delta \to R^n$, such that, for all $x \in R^n$ and $t \in \Delta$

$$\begin{cases} f(x, 0) = x \\ \dfrac{df}{dt}(x, t) = g(f(x, t)) \end{cases}$$

We now state the following

Proposition 3.4 (*Dynamical systems*) Let $n > 0$. Then the following holds

$$\forall g \in R^{n[0,1]\times R^n}\ \exists! f \in R^{nR^n \times \Delta}\ \forall x \in R^n\ \forall t \in \Delta \cap [0,1]\ \left(f(x,0)=x\ \wedge \frac{df}{dt}(x,t)=g(t,f(x,t))\right).$$

Proof. Given $g : [0, 1] \times R^n \to R^n$, define $\tilde{g} : [0, 1] \times R^n \to R \times R^n$ by $\tilde{g}(s, t) = (1, g(s, x))$. For this $\tilde{g}$, POSTULATE WA.2 gives the existence of a (unique) Δ-flow

$$\tilde{f} : [0, 1] \times R^n \times \Delta \to [0, 1] \times R^n$$

such that, for all $s \in [0, 1]$, $x \in R^n$ and $t \in \Delta$, we have

$$\begin{cases} \tilde{f}(s, x, t) = (s, x) \\ \dfrac{d\tilde{f}}{dt}(s, x, t) = \tilde{g}(\tilde{f}(s, x, t)) \end{cases}$$

Equivalently (by AXIOM I), for all $d \in D$, $s \in [0, 1]$, $x \in R^n$ and $t, r \in \Delta$,

$$\begin{cases} \tilde{f}(s, x, d) = (s, x) + d \cdot \tilde{g}(s, x) \\ \tilde{f}(s, x, t+r) = \tilde{f}(\tilde{f}(s, x, t), r) \end{cases}$$

For the first component of $\tilde{f}$,

$$\tilde{f}_1 : [0,1] \times R^n \times \Delta \to [0,1],$$

we get the following set of conditions:

$$\begin{cases} \tilde{f}_1(s, x, d) = s + d \\ \tilde{f}_1(s, x, t+r) = \tilde{f}_1(\tilde{f}_1(s, x, t), \tilde{f}_2(s, x, t), r) \end{cases}$$

where $\tilde{f}_2 : [0,1] \times R^n \times \Delta \to R^n$ denotes the second component.

Since $\tilde{f}_1(s, x, d)$ does not depend on x, the second part of the condition says precisely that, for each $s \in [0, 1]$, we have a D-flow, which extends (uniquely) to a Δ-flow. So, for all $t \in \Delta$, $s \in [0, 1]$ and $x \in R^n$, we must have $\tilde{f}_1(s, x, t) = s + t$.

With this result, the set of conditions for $\tilde{f}_2$ is now the following

$$\begin{cases} \tilde{f_2}(s, x, 0) = x \\ \tilde{f_2}(s, x, t+r) = \tilde{f_2}(s+t, \tilde{f_2}(s, x, t), r) \end{cases}$$

equivalently,

$$\begin{cases} \tilde{f_2}(s, x, 0) = x \\ \dfrac{d\tilde{f_2}}{dt}(s, x, t) = g(s+t, \tilde{f_2}(s, x, t)) \end{cases}$$

We can now define $f : R^n \times \Delta \to R^n$ by $f(x, t) = \tilde{f_2}(0, x, t)$. This f is unique with the conditions

$$\begin{cases} f(x, 0) = x & \text{for all } x \in R^n \\ \dfrac{df}{dt}(x, t) = g(t, f(x, t)) & \text{for all } x \in R^n \text{ and } t \in \Delta \cap [0, 1] \end{cases}$$

as we wanted.

Corollary 3.5 For every $n > 0$, the following holds

$$\forall g \in \left(R^{n[0,1]}\right)^{\Delta(n)} \exists ! f \in R^{n\Delta(n)\times\Delta}\ \forall x \in \Delta(n)\ \forall t \in \Delta \cap [0,1]\ (f(x,0)=x \ \wedge \frac{df}{dt}(x,t)=g(t,f(x,t)).$$

Proof. Given $g : \Delta(n) \to R^{n[0,1]}$, by AXIOM III (see section 1.3) there is an extension (locally unique) to a map $\hat{g} : R^n \to R^{n[0,1]}$. For this $\hat{g}$, the theorem above gives the existence of a unique $\hat{f} : R^n \times \Delta(n) \to R^n$, such that

$$\begin{cases} \hat{f}(x, 0) = x & \text{for all } x \in R^n \\ \dfrac{d\hat{f}}{dt}(x, t) = \hat{g}(t, \hat{f}(x, t)) & \text{for all } x \in R^n \text{ and } t \in \Delta \cap [0, 1] \end{cases}$$

Define $f : \Delta(n) \times \Delta \to R^n$ as the restriction (germ) of $\hat{f}$. This f is indeed the unique which satisfies the condition, and does not depend of the representative, for $\hat{f}(x, t) \in \neg\neg\{\hat{f}(x, 0)\}$ equal to $\neg\neg\{x\} = \Delta(n)$ (see Proposition 1.3.1. part iii),) $\forall t \in \Delta \cap [0,1]$, $\forall x \in \Delta(n)$; and all possible $\hat{g}$'s agree on $[0,1] \times \Delta(n)$.

There is a stronger result which will be used in the next sections, namely the <u>local</u> existence of solutions for differential equations. The result is a consequence of Proposition 1.4.5. In the case of the flows arising from a dynamical system, if we assume that the open part is of the form $\Delta^n \times (-\varepsilon,\varepsilon)$, the flow condition will be on $\Delta(n) \times ([0,1] \cap (-\varepsilon,\varepsilon))$, therefore on $\Delta(n) \times [0,\delta]$, for some $0 < \delta \leq 1$.

The last remark about flows, already used for D-flows (vector fields), is the following.

Proposition 3.6 If $\xi : M \times U \to M$ is a flow, with $0 \in U \subset R$, and $\xi(x,0) = x$, then the map $\overline{\xi} : U \to M^M$ factors through $Iso(M^M)$, provided that $-t \in U$, for each $t \in U$.

Proof. For each $t \in U$, define $(f(t))^{-1} = f(-t)$. □

§4. The Malgrange-Weierstrass preparation theorem

In this section we state the equivalent version of a technical theorem about smooth functions, used in the proof of Mather's Theorem and in establishing the existence of normal forms for singularities of certain stable mappings. We are talking about the Weierstrass Theorem (cf. [MALGRANGE: The preparation theorem for differentiable mappings] and [MATHER: Stability of C^∞-mappings, I: the division theorem].)

The statement of this theorem in the classical setting is in algebraic terms. The richness of our framework allows a geometrical formulation which encloses the essence of Mather's theorem, namely the passage from the infinitesimal to the local.

Recall from Proposition 2.2 and Proposition 3.1.3 that f is infinitesimal stable if γ_f is a submersion at $(id_{R^n}, id_R) \in G$. The objective of next postulate is to guarantee a similar condition for any germ in some neighborhood of f in the weak topological structure on $R^{\Delta(n)}$.

Just as in [BUNGE-GAGO: Synthetic aspects of C^∞-mappings, II: Mather's theorem for infinitesimally represented germs] we adopt the following postulate, except that we use weak opens here rather than intrinsic ones.

POSTULATE P (*Preparation postulate*) Let W be a weak neighborhood of f in $R^{\Delta(n)}$. Let $\Phi : W \to W^{[0,1]}$ be any map, such that $\Phi(f)(s) = f$, for all $s \in [0,1]$. Then if $d\gamma_{\Phi(f)}$ is surjective at $(\pi_{R^n} : [0,1] \times R^n \to R^n, \pi_R : [0,1] \times R \to R)$, it follows that $d\gamma_{\Phi|_{W'}}$ is surjective at $(\pi_{R^n} : W' \times [0,1] \times R^n \to R^n, \pi_R : W' \times [0,1] \times R \to R)$, for some weak neighborhood W' so that $f \in W' \subset W$.

We hope that now the rôle of the material displayed in section 3 becomes more clear. On the other hand, we should prove that our test model $\mathcal{G}$ is good enough for our purposes, in the sense that on it this last postulate also holds.

Proposition 4.1 (*Preparation theorem*) POSTULATE P holds in $\mathcal{G}$.

Proof. Let $f \in_{\overline{A}} R^{\Delta(n)}$ be infinitesimally stable, where A is represented by $C^\infty(\mathbb{R}^r)/_I$, and let $\Phi \in_{\overline{A}} (R^{\Delta(n)})^{R^{\Delta(n)} \times [0,1]}$ be so that $\Phi(f,s) = f$, for every $s \in [0,1]$.

Applying the global sections functor, we get a mapping $\Gamma(\Phi)$, which we call F,

$$F : Z(I) \times C_0^\infty(\mathbb{R}^n) \to C^\infty_{\{0\}\times[0,1]}(\mathbb{R}^n \times [0,1])$$

which is smooth in the first variable, seeing $Z(I) = \Gamma(\overline{A}) \subset \mathbb{R}^r$ (Proposition 0.4.5) as a submanifold, and continuous in the second variable, regarding $C_0^\infty(\mathbb{R}^n) = \Gamma(R^{\Delta(n)})$ and similarly $C^\infty_{\{0\}\times[0,1]}(\mathbb{R}^n \times [0,1]) = \Gamma(R^{\Delta(n)^{[0,1]}})$ (Proposition 1.3.4) endowed with the weak C^∞-topology.

The condition $\Phi(f, s) = f$ translates into $F(\lambda, f(\lambda))(s) = f(\lambda)$, for each $\lambda \in Z(I)$; moreover $f(\lambda)$ is infinitesimally stable (see remark before Proposition 2.2), and therefore $\alpha_{F(\lambda,f(\lambda))} \oplus \beta_{F(\lambda,f(\lambda))}$ is surjective. By [POENARU: Analyse Différentielle, Lemma 2.3], there exists some open W_λ in $Z(I) \times C_0^\infty(\mathbb{R}^n)$ such that $\alpha_{F|W_\lambda} \oplus \beta_{F|W_\lambda}$ is surjective, for each $\lambda \in Z(I)$.

We can consider that this open is of the form $W_\lambda = (U_\lambda \cap Z(I)) \times V_\lambda$, where $U_\lambda \subset \mathbb{R}^r$ is open in the usual sense, and $V_\lambda \subset C_0^\infty(\mathbb{R}^n)$ is open in the (quotient) weak C^∞-topology. We can also restrict ourselves to a countable family $\{U_\alpha\} \subset \{U_\lambda\}$ such that $\{U_\alpha \cap Z(I)\}$ covers $Z(I)$. Now, surjectivity of $\alpha_{F|W_\lambda} \oplus \beta_{F|W_\lambda}$ (at the corresponding projections), for the representable objects $A_\alpha = C^\infty(U_\alpha)/_{I|U_\alpha}$, gives that

$$\models_{\overline{A_\alpha}} \alpha_{\Phi|_{\Lambda(V_\lambda)}} \oplus \beta_{\Phi|_{\Lambda(V_\lambda)}} \; \textit{surjective} \, ,$$

But $\Lambda(V_\alpha) \subset R^{\Delta(n)}$ is weak open (see Proposition 2.5.3) and the $\{\overline{A_\alpha} \to \overline{A}\}$ form a cover (see Proposition 0.4.11). Therefore, we have proved that

$$\models_{\overline{A}} \exists W \in W(R^{\Delta(n)})\ [f \in W \wedge \alpha_{\Phi|_W} \oplus \beta_{\Phi|_W} \textit{ surjective }],$$

as we wanted.

§5. Mather's Theorem

In this section we will put together the material prepared in the previous sections to prove the following (cf. also [BUNGE-GAGO: Synthetic aspects of C^∞-mappings, II: Mather's theorem for infinitesimally represented germs])

> **Theorem 5.1** (*Mather's Theorem*) Let $f \in R^{\Delta(n)}$ with $f(0) = 0$. Then f is stable if and only if f is infinitesimally stable.

Proof. We have already seen that stability implies infinitesimal stability in an (almost) trivial way (see Proposition 2.3). For the "hard" part we will make use of POSTULATE P and the results about dynamical systems, given in section 3.

Consider the map $\Phi \in (R^{\Delta(n)})^{R^{\Delta(n)} \times [0,1]}$ given by the rule

$$\Phi(g, t) = t \cdot g + (1-t) \cdot f.$$

This map satisfies the following set of conditions

$$\begin{cases} \Phi(f,t) = f & \text{for any } t \in [0,1] \\ \Phi(g,0) = f & \text{for any } g \in R^{\Delta(n)} \end{cases}$$

If we consider a weak neighborhood of f in $R^{\Delta(n)}$ of the type $V = V(\Delta(n), 0, f, \varepsilon)$, for some $\varepsilon \in R$, $1 > \varepsilon > 0$, then the restriction of Φ to functions of V takes its values in V. Indeed, given $g \in V$, and $t \in [0,1]$, by the definition of V we have that

$$\forall x \in \Delta(n)\ [(g(x) - f(x)) \in (-\varepsilon, \varepsilon)\ ],$$

therefore, for all x in $\Delta(n)$ we have $\Phi(g,t)(x) - f(x) = t \cdot (g(x) - f(x)) \in (-t \cdot \varepsilon, t \cdot \varepsilon) \subset (-\varepsilon, \varepsilon)$. So we have the equality $\Phi(V \times [0,1]) = V$, as $\Phi_1 = id_V$. As a matterer of fact, for any other weak neighborhood, V', of f, and any $0 < \tau \leq 1$, $\Phi(V \cap V' \times [0,\tau])$ is also a weak neighborhood of f. Indeed, $V \cap V' \supset V(\Delta(n), 0, f, \delta)$, for some $\delta > 0$ (see the proof of Proposition 2.4.2), and we claim that

$$\Phi(V(\Delta(n), 0, f, \delta) \times [0, \tau]) = V(\Delta(n), 0, f, \tau\cdot\delta).$$

To see it, take any $h \in V(\Delta(n), 0, f, \tau\cdot\delta)$; then, for all x in $\Delta(n)$, $h(x)\text{-}f(x)) \in (-\tau\cdot\delta, \tau\cdot\delta)$, and Since $\tau > 0$, τ is invertible (see Proposition 1.2.2 and POSTULATE WA1.1) and we can define the map

$$g = \tau^{-1}\cdot(h\text{-}f) + f.$$

It is clear that $h = \tau\cdot g + (1\text{-}\tau)\cdot f$, and $g(x)\text{-}f(x) = \tau^{-1}\cdot(h\text{-}f)(x) \in (-\varepsilon,\varepsilon)$.

We will show that there exists a weak neighborhood V' of f, and some $\varepsilon' > 0$, such that if $f \in \Phi(V' \times [0,\varepsilon'])$, then h is equivalent to f and the result will follow.

For this Φ that we have defined, POSTULATE P gives some weak neighborhood V' of f in $R^{\Delta(n)}$ for which $\alpha_{\Phi|_{V'}} \oplus \beta_{\Phi|_{V'}}$ is surjective. This means that for any vector field along the map $\Phi|_{V'} : V' \times [0,1] \times \Delta(n) \to R$, in particular for the vector field given by $\frac{D\Phi|_{V'}}{dt}$, there exist $\sigma \in Vect(\pi_{R^n} : V' \times [0,1] \times \Delta(n) \to R^n)$ and $\tau \in Vect(\pi_R : V' \times [0,1] \times \Delta \to R)$, such that

$$\frac{D\Phi|_{V'}}{dt} = \alpha_{\Phi|_{V'}}(\sigma) \oplus \beta_{\Phi|_{V'}}(\tau).$$

The principal part of these vector fields are, respectively, $g_\sigma \in R^{n^{V' \times [0,1] \times \Delta(n)}}$ and $g_\tau \in R^{V' \times [0,1] \times \Delta}$, and, by the results on dynamical systems, we have unique $f_\sigma \in R^{n^{V' \times \Delta(n) \times [0,\varepsilon']}}$ and $f_\tau \in R^{V' \times \Delta \times [0,\varepsilon']}$ satisfying, respectively, the following sets of conditions, for each fixed $f' \in V'$,

$$\begin{cases} f_\sigma(f', x, 0) = x & \text{for all } x \in \Delta(n) \\ -\dfrac{df_\sigma}{dt}(f', x, t) = g_\sigma(f', t, f_\sigma(f', x, t)) & \text{for all } x \in \Delta(n) \text{ and } t \in [0,\varepsilon'] \end{cases}$$

and

$$\begin{cases} f_\tau(f', y, 0) = x & \text{for all } y \in \Delta \\ -\dfrac{df_\tau}{dt}(f', y, t) = g_\tau(f', t, f_\tau(f', y, t)) & \text{for all } y \in \Delta \text{ and } t \in [0,\varepsilon'] \end{cases}$$

Notice that, for $f = f'$, since $\Phi(f, t)(x) = f(x)$, for each $x \in \Delta(n)$ and $t \in [0, 1]$, we may take σ and τ satisfying

$$\sigma(f, t, x)(d) = x \qquad \forall d \in D, \forall x \in \Delta(n), \forall t \in [0, 1]$$

and

$$\tau(f, t, y)(d) = y \qquad \forall d \in D, \forall y \in \Delta, \forall t \in [0, 1].$$

Correspondingly, we will have

$$g_\sigma(f, t, x) = 0 \qquad \forall t \in [0, 1], \forall x \in \Delta(n)$$

and

$$g_\tau(f, t, y) = 0 \qquad \forall t \in [0, 1], \forall y \in \Delta.$$

Therefore, the unique solutions f_σ and f_τ satisfy

$$-\frac{df_\sigma}{dt}(f, x, t) = 0 \quad \text{and} \quad f_\sigma(f, x, 0) = x$$

and

$$\frac{df_\tau}{dt}(f, y, t) = 0 \quad \text{and} \quad f_\tau(f, y, 0) = y,$$

which means that f_σ and f_τ do not depend on t, and then we must have

$$f_\sigma(f, x, t) = x \qquad \forall x \in \Delta(n), \forall t \in [0, \varepsilon']$$

and

$$f_\tau(f, y, t) = y \qquad \forall y \in \Delta, \forall t \in [0, \varepsilon'].$$

In other words, the maps $f_\sigma(f,t) : \Delta(n) \to \Delta(n)$ and $f_\tau(f,t) : \Delta \to \Delta$ are both the identity. Also, for any $f' \in V'$, $f_\sigma(f', t) : \Delta(n) \to \neg\neg\{x\}$ and $f_\tau(f', t) : \Delta \to \neg\neg\{y\}$ are isomorphisms for every $t \in [0, \varepsilon']$, where $x = f_\sigma(f, t)(0)$ and $y = f_\tau(f, t)(0)$.

We now claim that if $\frac{D\Phi|_{V'}}{dt}(f',t) = \alpha_{\Phi|_{V'}}(\sigma) \oplus \beta_{\Phi|_{V'}}(\tau)$, then f is equivalent to $\Phi(f',t)$ as a germ at x, i.e., the diagram

$$
\begin{tikzcd}
\Delta(n) \arrow[r, "\Phi(f', 0)"] \arrow[d, "f_\sigma(f', t)"'] & \Delta \arrow[d, "f_\tau(f', t)"] \\
\neg\neg\{x\} \arrow[r, "\Phi(f', t)"] & \neg\neg\{y\}
\end{tikzcd}
$$

commutes: to put it with the words of section 3, the deformation is trivial. This commutativity can be reformulated as follows

$$\Phi_t = G_t \circ \Phi_0 \circ H_t^{-1}$$

where

$$G = \langle \pi_1, f_\tau, \pi_3 \rangle : V' \times \Delta \times [0,\varepsilon'] \to V' \times \Delta \times [0,\varepsilon'], \text{ and}$$

$$H = \langle \pi_1, f_\sigma, \pi_3 \rangle : V' \times \Delta(n) \times [0,\varepsilon'] \to V' \times \Delta(n) \times [0,\varepsilon'].$$

The condition reads now

$$\frac{D\Phi}{dt} = \alpha_\Phi\left(\frac{DG}{dt} \circ G^{-1}\right) \oplus \beta_\Phi\left(-\frac{DH}{dt} \circ H^{-1}\right)$$

because $\sigma = \frac{DG}{dt} \circ G^{-1}$ and $\tau = -\frac{DH}{dt} \circ H^{-1}$.

But, synthetically, we have

$$\frac{D}{dt}(G^{-1} \circ \Phi \circ H) = \frac{DG^{-1}}{dt} \circ \Phi \circ H \oplus (G^{-1})^D \circ \frac{D\Phi}{dt} \circ H \oplus (G^{-1})^D \circ \Phi^D \circ \frac{DH}{dt}$$

and, from $G^{-1} \circ G = id$, for any $t \in [0,\varepsilon']$, it follows that $\frac{D}{dt}(G^{-1} \circ G) = 0$, hence, synthetically,

$$\frac{D}{dt}(G^{-1} \circ G) \oplus (G^{-1})^D \circ \frac{DG}{dt} = 0, \quad \text{and so} \quad \frac{DG^{-1}}{dt} = -(G^{-1})^D \circ \frac{DG}{dt} \circ G^{-1}$$

which gives

$$\frac{D}{dt}(G^{-1}\circ\Phi\circ H) = -(G^{-1})^D\circ\frac{DG}{dt}\circ G^{-1}\circ\Phi\circ H\oplus(G^{-1})^D\circ\frac{D\Phi}{dt}\circ H\oplus(G^{-1})^D\circ\Phi^D\circ\frac{DH}{dt}=$$

$$=(G^{-1})^D\circ[-\frac{DG}{dt}\circ G^{-1}\circ\Phi\oplus\frac{D\Phi}{dt}\oplus\Phi^D\circ\frac{DH}{dt}\circ H^{-1}]\circ H=$$

$$=(G^{-1})^D\circ[-\beta_\Phi(\tau)\oplus\frac{D\Phi}{dt}\oplus-\alpha_\Phi(\sigma)]\circ H.$$

Now, $(G^{-1})^D$ and H are isomorphisms, hence the last member of the equality vanishes if and only if $\frac{D\Phi}{dt}=\alpha_\Phi(\sigma)\oplus\beta_\Phi(\tau)$, that is to say

$$\frac{d}{dt}(G_t^{-1}\circ\Phi_t\circ H_t)=0 \qquad \text{iff} \qquad \frac{D\Phi}{dt}=\alpha_\Phi(\sigma)\oplus\beta_\Phi(\tau);$$

in particular

$$G_t^{-1}\circ\Phi_t\circ H_t=G_0^{-1}\circ\Phi_0\circ H_0,\ \text{for all } t\in[0,\varepsilon] \qquad \text{iff} \qquad \frac{D\Phi}{dt}=\alpha_\Phi(\sigma)\oplus\beta_\Phi(\tau).$$

But H_0 and G_0 are identities, and therefore the result is that

$$G_t^{-1}\circ\Phi_t\circ H_t=\Phi_0,\ \text{for all } t\in[0,\varepsilon] \quad \text{iff} \quad \frac{D\Phi}{dt}=\alpha_\Phi(\sigma)\oplus\beta_\Phi(\tau),$$

as we wanted.

5

Morse theory

§1. Preliminaries

In this chapter we begin the study of Morse Theory (cf. [MILNOR: Morse Theory].) With our work, we believe that we lay down the first stone of two posible major avenues.

The first one will lead to the classification of singularities (those which are stable) in the synthetic framework. We completely characterize the singularities of functions into R, apart from other partial results about the stability of submersions and immersions which trivially follow from previous chapters.

The second one will eventually lead to a classification of manifolds via the Euler characteristic. In this direction, we give a proof of Morse's lemma characterizing Morse germs (or Morse functions in an infinitesimal neighborhood of a singularity.) A use of this result, among others, is to determine the behavior of a surface in R^3 with respect to its tangent plane at a given point (cf. [BERGER-GOSTIAUX: Géométrie Différentielle: variétées, courbes et surfaces, p. 136].)

In the same direction, it easily follows that the Poincaré Lemma (characterizing the De Rham cohomology groups) holds for $\Delta(n)$ (cf. [MOERDIJK-REYES: Cohomology theories in Synthetic Differential Geometry].) Our results yield a direct proof of Morse Inequalities (cf. [MILNOR: Morse Theory]) for the manifold $\Delta(n)$.

We begin our study by recalling some definitions.

Definition 1.1 Given $x \in R^n$, any map $x + D(n) \to R$ is called a 1-jet at x.[1]

Notice that in this context, as it happened with germs, there is no need to talk of equivalence classes of maps.

The legitimacy for this name comes from the basic axiom of Synthetic Differential Geometry (cf. section 1.1): For $x = 0 \in R^n$, in a map $D(n) \to R$ there is the same information than in its value at 0, and the value of its partial derivatives also at 0. In this line, $R^{D_r(n)} \to R$ is called the r-jet bundle, where the projection is the evaluation at 0.

N.B. We are only interested in maps landing on R, but all these notions can be extended in the same way to any power of R.

In section 1.3, we gave the synthetic basis for relating germs and jets, namely the inclusions, for each $n > 1$,

$$D(n) \subset D_2(n) \subset \cdots \subset D_\infty(n) \subset \Delta(n).$$

For instance, composition with $D(n) \to \Delta(n)$ induces a map, denoted

$$j_0^1 : R^{\Delta(n)} \to R^{D(n)},$$

which is said to assign, to a germ at 0, $\varphi \in R^{\Delta(n)}$, its 1-jet at 0, $j_0^1\varphi$.

Closely related to this map is the following. For each $f \in R^{\neg\neg\{x_o\}}$, we have the map

$$J^1 f : \neg\neg\{x_o\} \to R^{D(n)}$$

which is the one that to $x \in \neg\neg\{x_o\}$ associates the restriction of f to $x + D(n)$. This map can also be defined as follows: for $x \in \neg\neg\{x_o\}$, let $J^1 f(x) = j_0^1(f_x)$, where $f_x \in R^{\Delta(n)}$ is defined by $f_x(t) = f \cdot a_x(t)$, with a_x is the isomorphism of 1.3.1.

We can now give the definition

Definition 1.2 Given $f \in R^X$, where $X = \neg\neg\{x_o\}$, an element $x \in X$ is called a singularity of f if f is constant on $x + D(n)$, i.e., its 1-jet is constant.

---

[1] The definition is to be read internally, and it has similar extensions to k-jets through $D_k(n)$.

If one analyzes the definition of J^1f, it can easily be seen that, for $x \in X$, $d \in D(n)$, we have

$$J^1f(x)(d) = f(x+d) = f(x) + \frac{\partial f}{\partial x}(x)\cdot d,$$

and the isomorphism of AXIOM I, $R^{D(n)} \approx R \times R^n$, gives us

$$J^1f(x) \equiv (f(x), \frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x)).$$

Now, if we consider the subobject $S_1 \subset R^{D(n)}$, defined by

$$S_1 = [[g \in R^{D(n)} \mid \forall d \in D(n) \;\; g(d) = g(0)]],$$

which corresponds to $\pi^{-1}\{0\} \subset R \times R^n$, where $\pi : R \times R^n \to R^n$ is the canonical projection, we have that x is a singularity of f if and only if $J^1f(x) \in S_1$.

The first thing to be noticed is that S_1 is a submanifold of $R^{D(n)}$. Indeed, by Corollary 3.1.6, every value of a submersion is a regular value, and the preimage theorem (see Corollary 3.4.4) gives that S_1 is a submanifold of $R^{D(n)}$, of codimension n.

We are now ready for the key concept of this chapter, the notion of non-degeneracy:

Definition 1.3 Let $x \in X$ be a singularity of $f \in R^X$. We say that x is nondegenerate if $J^1f \pitchfork_x S_1$, i.e., $T_{J^1f(x)} R^{D(n)} = Im[d(J^1f)_x] + T_{J^1f(x)}S_1$ (see Definition 3.4.5.)

Our next goal is to give an internal characterization of non-degenerate singularities, and it is achieved at once in the following proposition:

Proposition 1.4 Let $f \in R^X$. An element $x \in X$ is a non-degenerate singularity of f if and only if $\pi \circ J^1f$ is a submersion at x, where $\pi : R \times R^n \to R^n$ is again the canonical projection.

Proof. Since the functor $(\cdot)^D$ preserves products[1], π^D is itself a projection, and we have that

$$\forall \tau \in T_{J^1f(x)}(R \times R^n) \; (\tau \in T_{J^1f(x)} S_1 \leftrightarrow \pi^D(\tau) = 0).$$

[1]General result, as it has a left adjoint cf. [MAC LANE: Categories for the Working Mathematician.]

Therefore, for any non-degenerate singularity x of f, we have that

$$T_{J^1f(x)}R^{D(n)} = Im[d(J^1f)_x] + Ker[d(\pi)_{J^1f(x)}].$$

Now, after Proposition 3.1.3, $\pi \cdot J^1f$ is a submersion at x if and only if $d(\pi \cdot J^1f)_x$ is locally surjective. But π^D, being a projection is a submersion and therefore (equivalently) $d\pi_{J^1f(x)}$ is locally surjective.Thus, the equality

$$T_{J^1f(x)}R^{D(n)} = Im[d(J^1f)_x] + Ker[d(\pi)_{J^1f(x)}]$$

holds if and only if $d(\pi \cdot J^1f)_x$ $(= d\pi_{J^1f(x)} \cdot d(J^1f)_x)$ is locally surjective, and the result follows. □

An immediate consequence of this proposition is the following property of the Hessian (the matrix of second-order partial derivatives) of a map at the non-degenerate singularities:

Corollary 1.5 Let $f \in R^X$. If $x \in X$ is a non-degenerate singularity of f, then the Hessian of f at $x = \left(\frac{\partial^2 f}{\partial x^2}(x)\right)$ is nonsingular.

Proof. By the theorem above, if x is a non-degenerate singularity of f, then the map $\pi \cdot J^1f$ is a submersion at x. The result now follows from 3.1.3 (ii), since the set of vectors is precisely the set of rows of the Hessian of f at x. □

§2. Morse germs

We begin this section with the following definition

Definition 2.1 A germ $f \in R^X$ is called a Morse germ if all its singularities are non-degenerated, i.e., the following is true

$$\forall x \in X\ [x \textit{ singularity of } f \Rightarrow x \textit{ non-degenerate singularity}].$$

As we mentioned in the introduction to this chapter, an important use of Morse functions (germs) is the analysis of the behavior of a manifold at a given point. In this sense, it would be useful to know whether there are any Morse germs at all, and "how much" a given germ is allow to differ from being Morse. The answer to the second question is that a germ $f \in R^X$, either it is Morse, or it is "close enough" to a Morse one. Indeed, we have

Proposition 2.2 The object of Morse germs is dense in R^X with respect to the weak topological structure (see Definition 2.4.1.)

The result follows at once from Thom Transversality Theorem (see Theorem 3.4.7) and the following two observations:

a) $S_1 \subset R^{D(n)}$, being $\pi^{-1}\{0\}$, is cut out by an independent function: every submersion is so (see Definition 3.4.2)

b) $f \in R^X$ is a Morse germ if and only if $J^1f \pitchfork S_1$. □

Before closing this section, we will prove one of the basic results of Morse theory in characterizing the behavior of manifolds at points. The result says that non-degenerate singularities are isolated.

Proposition 2.3 A Morse germ has at most one singularity.

Proof. Let $f \in R^X$ be a Morse germ. If $x \in X$ is a singularity of f, then it must be non-degenerate, and by Proposition 1.4, $\pi \cdot J^1 f$ is a submersion at x, and 0 is a regular value for this map. Now, the corresponding version of Preimage Theorem (see Corollary 3.4.4) for germs says precisely that

$$[[x \in X \mid x \textit{ singularity of } f]] = (\pi \cdot J^1 f)^{-1}\{0\} \subset X \text{ is a submanifold.}$$

Moreover, it has codimension n, hence dimension 0. □

It is worth pointing out that the object S_1 can be considered as the universal object of singularities of *co-rank 1*. In the case of considering the more general situation of germs ending in R^n, for some n, this definition can be extended to S_r, for $r \leq n$. The proof that this objects are submanifolds of the corresponding jet space is the fundamental result in the so called Thom-Boardman stratification theory (cf. [BOARDMAN: Singularities of differentiable maps]) and techniques such as the one of identifying Morse germs as those which are transversal to the corresponding universal object of singularities are "the" tools to employ towards a characterization of singularities (cf. also [GOLUBITSKI-GUILLEMIN: Stable mappings and their singularities].)

Our last result in this section will give an answer to the the first question posed after definition, concerning the actual existence of Morse germs.

Exercise 2.4 Let $f \in R^{\Delta(n)}$ be defined by the rule

$$[\underline{t} \longmapsto c + u_1 t_1^2 + \cdots + u_n t_n^2],$$

with the u_i's all invertible in R. Then f is a Morse germ with a non-degenerate singularity at 0.

Solution. The usual rules of derivation give us that the 1-jet of f at $\underline{0}$ is "codified" by the $(n+1)$-tuple $(c, 0, \ldots, 0)$. So $\underline{0} \in \Delta(n)$ is a singularity (see Definition 1.2 and the comments after it). As a matter of fact, $J^1 f : \Delta(n) \to R \times R^n$ has the following description

$$J^1 f(\underline{t}) = (c + \textstyle\sum u_i t_i^2 , 2u_1 t_1 + \cdots + 2u_n t_n),$$

which says that 0 is the only possible singularity, as the u_i's are invertible. So, we only have to check that 0 is non-degenerate, or equivalently, that $\pi \circ J^1 f$ is a submersion at 0.

Now, to prove this, we use Proposition 3.1.3 (ii), which says exactly what we want, just taking into account that the vectors

$$(u_1, 0, \ldots, 0), (0, u_2, \ldots, 0), \ldots , (0, \ldots, 0, u_n),$$

are linearly independent.

§3. Normal form of a Morse germ

One of the central results in classical Morse Theory (cf. [ARNOLD: Normal forms of functions in the neighbourhood of degenerate critical points] and [GOLUBITSKI-GUILLEMIN: Stable mappings and their singularities], [HIRSCH: Differential Topology], [MILNOR: Morse Theory], [MILNOR: Topology from the differentiable Viewpoint], [MORSE: The behavior of a function on its critical set] or [MUNKRES: Elementary Differential Topology]) is the construction of a local chart (for a manifold) or a change of coordinates (for a part of some

R^n) around a non-degenerate singularity making the function "look like" a non-degenerate quadratic form.

In our setting we count with the following result:

Theorem 3.1 Every Morse germ $g \in R^{\Delta(n)}$, with $g(0) = 0$, is right equivalent to a quadratic form.

Before we begin with the proof, let us make some remarks about the statement of the theorem and the proof itself.

(1) The condition $g(0) = 0$ is definitely unessential. We could modify the conclusion as require right-left equivalent, and apply the same method to the germ $g(x) - g(0)$.

(2) We may assume (see Lemma 3.2, below) that the 2-jet of g is of the form

$$a_1 x_1^2 + \cdots + a_n x_n^2 ,$$

with the a_i's invertible elements in R.

(3) With the above simplifications, we have $g = f + \phi$, where $f(x) = a_1 x_1^2 + \cdots + a_n x_n^2$ and ϕ is a germ in $\mathfrak{m}^3$ ($\mathfrak{m}$ is the object of germs at 0, ξ with $\xi(0) = 0$.) Notice that ϕ has a zero of order three at 0.

Proof of Theorem 3.1. We will prove that f is equivalent to $g = f + \phi$. For this we resort on the method introduced in section 4.3, the homotopy method.

First, we join f to g by the path $f + t \cdot \phi$, with $t \in [0,1]$. Then we show that it is possible to find a one-parameter family of "local diffeomorphisms"

$$x \in \Delta(n) \mapsto \Phi(x, t) \in \Delta(n)$$

such that

$$\begin{cases} (f + t\,\phi)\Phi(t, x) = f(x) & \forall x \in \Delta(n) \quad \forall t \in [0, 1] \\ \Phi(x, 0) = x & \forall x \in \Delta(n) \\ \Phi(0, t) = 0 & \forall t \in [0, 1] \end{cases}$$

Then, $\Phi(-,1)$ will do the job. In the words of section 4.3, the deformation is trivial.

We obtain these Φ_t's as the integral curves of suitable vector fields δ_t, or rather a time-dependent vector field δ (see Corollary 4.3.5.)

$$\frac{d\Phi}{dt}(x, t) = \delta(\Phi(x, t), t).$$

Now, to obtain the equations for δ, we can take derivatives, with respect to the parameter t, in the expression

$$(f + t\,\phi)\Phi(t, x) = f(x).$$

Since the right-hand term does not depend on t, we get, by the usual rules of derivation, that

$$\phi(\Phi(x, t)) + \frac{d(f + t\,\phi)}{dt}\Phi(x, t).\ \frac{d\Phi}{dt}(x, t). \equiv 0,$$

for each $t \in [0, 1]$.

So, if the principal part of δ_t is expressed by the functions $(\delta_{t_1}, \ldots, \delta_{t_n})$, we will have the following equation:

$$\phi_{|\Phi(x,t)} \equiv - \sum_{i=1}^{n} \delta_{t_i} y_{i\ |\Phi(x,t)}\,,$$

where $y_i = 2a_i x_i + t\,\phi_{x_i}$. Therefore,

$$-\phi \equiv \sum_{i=1}^{n} \delta_{t_i} y_i\,,$$

seeing both sides as functions on (x, t).

To end the proof we will show that $y = (y_1, \dots, y_n, t)$ defines an admissible change of coordinates. For this we use the Theorem of Infinitesimal Inversion (POSTULATE I.I, in §1.4) since

$$\left|\frac{\partial y}{\partial x}(0, t)\right| = \begin{vmatrix} 2a_1 & & 0 \\ & \ddots & \\ 0 & & 2a_n \end{vmatrix} \neq 0$$

because ϕ_{x_i} is in $\mathfrak{m}^2$, as ϕ was in $\mathfrak{m}^3$.

Then, for each t, y is an isomorphism, and so (y, t) defines a new system of coordinates. In this new system, ϕ takes the form

$$\phi(y, t) = \sum_{i=1}^{n} \psi_i(y, t)\, y_i$$

as ϕ has no component in t; moreover

$$\begin{cases} \psi_i(0, t) = 0 & \forall t \in [0, 1] \\ \phi(0) = 0 \end{cases}$$

Therefore $\delta_{t_i} = -\psi_i$ as functions on y, are the components of our time dependent vector field whose integral curves give the wanted solution. □

To close the section we will prove the claim we made in remark (2) after Theorem 3.1. This is aim of the following result

Lemma 3.2 Let $f \in R^{\Delta(n)}$ be a germ with a non-degenerate singularity at 0. Then we can find coordinates z_i such that the second-order Taylor polynomial at 0, of f, is

$$a_1 z_1^2 + \cdots + a_n z_n^2 .$$

Proof. The lemma states the existence of a linear isomorphism ϕ, represented by a non-singular matrix A, such that $f \circ A$ has the desired Taylor polynomial of degree two.

The usual rules of derivation give us the following equality

$$\frac{\partial(f\circ\phi)}{\partial x}(a) = \frac{\partial f}{\partial x}(\phi(a)) \cdot \phi'(a) = \frac{\partial f}{\partial x}(\phi(a)) \cdot A,$$

and therefore,

$$\frac{\partial^2(f\circ\phi)}{\partial x^2} = A^t \cdot \frac{\partial^2 f}{\partial x^2} \cdot A.$$

Therefore, the result will be proved if we show that there exists a matrix A which diagonalizes the non-degenerated (see Corollary 1.4) symmetric bilinear form associated to $\frac{\partial^2(f\circ\phi)}{\partial x^2}$, the Hessian of f.

One way of doing this in the classical setting (cf. [STOLL: Linear Algebra and Matrix Theory]) is by using elementary row operations (and the corresponding elementary column operations) until the matrix is in diagonal form. Multiplying together all the nonsingular matrices corresponding to the e.r.o.'s (and its transposed matrices, which correspond to the equivalent e.c.o.'s) we get the wanted nonsingular matrix.

Among the three basic elementary operations, namely,

$a \cdot R_i$	multiply row i by a
$R_i \leftrightarrow R_k$	interchange rows i and k
$a \cdot R_i + R_k$	substitute row k by a ·row i

only $a \cdot R_i$ might affect substantially the determinant of the original matrix, as far invertibility is concerned. So we have to make sure that the a 's we use are all invertible.

The proof goes in two steps:

step 1. Every non degenerate symmetric matrix S can be brought by e.r.o.'s (along with the corresponding e.c.o.'s) to a block diagonal matrix $\begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_n \end{pmatrix}$ where each A_i is either (a) or $\begin{pmatrix} t & a \\ a & s \end{pmatrix}$ with $a \# 0$ and t,s nilpotent, according to whether or not there are invertible entries in the diagonal.

step 2. Every matrix of the above form can be diagonalized along the following pattern

$$\begin{pmatrix} t & a \\ a & s \end{pmatrix} \longrightarrow\!\!\!\!\rightarrow \begin{pmatrix} t+a & a \\ a+s & s \end{pmatrix} \longrightarrow\!\!\!\!\rightarrow \begin{pmatrix} t+s+2a & a+s \\ a+s & s \end{pmatrix}$$

Now $t+s+2a$ is invertible, as a was, and $a+s$ can be cleared out. □

N.B. Since the elements in the diagonal are all invertible, they are positive or negative in our order relation. In order to obtain the validity of Sylvester's Law of Inertia, i.e., the entries of the diagonal are all $+1$ or -1, we need the existence of square roots for invertible elements in the sense of [JOYAL-REYES: Separably real closed local rings]. For our present purposes, it is enough to take into account the number of + (or —) signs.

§4. Stability and Morse germs

As mentioned in the introduction of this chapter, we open the road towards classification of singularities of stable germs: we completely characterize the singularity of maps into R.

We begin with the following proposition:

Proposition 4.1 Any germ $f \in R^X$ of the form

$$[(t_1,\ldots, t_n) \mapsto c + u_1 t_1^{\,2} + \cdots + u_n t_n^{\,2}\,],$$

with the u_i's invertible in R, is stable (see chapter 4.)

Proof. After Mather's Theorem (Theorem 4.5.1,) we have to check f for infinitesimal stability (Definition 5.2.1.) Without loss of generality, we can assume that $f \in R^{\Delta(n)}$. So, the requisite to meet is the following:

$$\forall \omega \in Vect(f)\ \exists \sigma \in Vect(R^n)\ \exists \tau \in Vect(R)\ [\omega = \alpha_f(\sigma) \oplus \beta_f(\tau)].$$

So, let $\omega : \Delta(n) \to R^D$ be any vector field along f, $\omega(x) = (f(x), \underline{\omega}(x))$, where $\underline{\omega}$ is the principal part of ω.

We may assume that the principal part of $\omega(0)$, $\underline{\omega}(0)$, equals 0. Otherwise we take any vector field τ on R such that $\underline{\tau}(f(0)) = \underline{\omega}(0)$, and then we consider $\omega - \tau \circ f$. So, the principal part is a map $\underline{\omega} : \Delta(n) \to R$, with $\underline{\omega}(0) = 0$.

After Corollary 1.1.4, there are unique functions

$$h_1, \ldots, h_n : \Delta(n) \to R,$$

such that $\underline{\omega}(\underline{t}) = \sum_{i=1}^{n} h_i(\underline{t}) \cdot t_i$.

We now claim that the required vector field σ on R^n is the one whose principal part is given by

$$\underline{\sigma}(\underline{t}) = \left(-\frac{h_1(\underline{t})}{2u_1}, \ldots, -\frac{h_n(\underline{t})}{2u_n}\right).$$

Indeed, by the remarks after Definition 4.2.1, $\alpha_f(\sigma)\,(d) = f \cdot \sigma(-d)$, and by the remarks after Definition 1.2, the principal part of $\alpha_f(\sigma)$ is given by

$$\frac{\partial f}{\partial x}(t) \cdot (-\underline{\sigma}(t))^t,$$

i.e.,

$$\begin{pmatrix} \frac{\partial f}{\partial x_1}(\underline{t}) & \cdots & \frac{\partial f}{\partial x_n}(\underline{t}) \end{pmatrix} \begin{pmatrix} \frac{h_1(\underline{t})}{2u_1} \\ \cdot \\ \cdot \\ \cdot \\ \frac{h_n(\underline{t})}{2u_n} \end{pmatrix}$$

as we wanted. □

At this point we have accumulated material enough to prove a part of the result we are after. We state the result in the following corollary:

Corollary 4.2 If $f \in R^X$ has a non degenerate singularity at x, then f is stable.

Proof. By Lemma 3.2, this f is equivalent to a Morse germ of the form $c + a_1 x_1^{\,2} + \cdots + a_n x_n^{\,2}$ On the other hand, by the theorem above, every germ of this form is stable. Now, from the very definition of stability (see Definition 4.1.3,) a germ equivalent to a stable one is itself stable, and the result follows. □

As for the other part, the result will be a consequence of the following easy lemma:

Lemma 4.3 Let f, $h \in R^{\Delta(n)}$ be such that $f \sim f'$ (see Definition 4.1.2.) If f is a Morse germ, then so is h.

Proof. If φ, ψ denote the changes of coordinates (infinitesimal invertible maps) of Definition 4.1.2, and x is a singularity of f, then, clearly, $\varphi(x)$ is a singularity of h. Just notice that, if Ψ denotes the Jacobian of ψ at $f(x)$, and Φ denotes the Jacobian of φ at x, then we have

$$\Psi\left(\frac{\partial f}{\partial x_1}(x) \;\cdots\; \frac{\partial f}{\partial x_n}(x)\right)=\left(\frac{\partial h}{\partial x_1}(x) \;\cdots\; \frac{\partial h}{\partial x_n}(\varphi(x))\right)\Phi,$$

and both Φ and Ψ are invertible. □

The same goes for nondegeneracy.

Corollary 4.4 Let $f \in R^X$ be a stable germ, and let $x \in X$ be a singularity of f. Then x is non degenerate, i.e., f is a Morse germ.

Proof. By definition of stability (see Definition 4.1.3) there is a weak open neighborhood of f in R^X such that every g in this neighborhood is equivalent to f. By density of Morse germs (Proposition 2.2) in this neighborhood there is a Morse germ. Therefore, f is equivalent to a Morse germ, hence Morse by the theorem. □

Bibliography

V.I. ARNOLD, Normal forms of functions in the neighborhood of degenerate critical points *Upekhi Matematicheskykh Nauk* **29** (1974), 11-49. *Russian Mathematical Surveys* **29**, 19-48.

M. ARTIN, A. GROTHENDIECK et J.L. VERDIER, *Théorie des Topos et Cohomologie Etale des Schémas*, Séminaire de Géométrie Algébrique du Bois-Marie SGA 4 (Tome 1). Second edition, Springer-Verlag, Lecture Notes in Mathematics **269**, 1972.

M. BARR and C WELLS, *Toposes, Triples and Theories*. Grundlehren der Mathematischen Wissenchaften **278**. Springer-Verlag, 1985.

M. BERGER et B. GOSTIAUX, *Géométrie Différentielle: Variétées, Courbes et Surfaces*. Presses Universitaires de France, 1987.

M. BERGER and B. GOSTIAUX, *Differential Geometry: Manifolds, Curves, and Surfaces*. Graduate Texts in Mathematics **115**, Springer-Verlag, 1988.

F. BERGERON, Objects infinitesimalement linéaires dans un modèle bien adapté de G.D.S. in G.E. REYES (edit.) Géométrie Différentielle Synthétique. Rapport de recherches, DMS 80-12, Univ. de Montréal (1980), 67-76.

A. BOILEAU, Types vs. Topos. Thése de Philosophiae Doctor, Univ. de Montréal, 1975.

A. BOILEAU et A. JOYAL, La logique des topos. *J. Symbolic Logic* **40** (1981), 6-16.

J.M. BOARDMAN, Singularities of differentiable maps. *Publ. Math. I.H.E.S.* **33** (1967), 21-57.

R. BOTT, *Morse Theory and its Application to Homotopy Theory*, Bonn, Sommersemester 1958. (Notes by A. VAN DER VEN.) Als Manuskript Vervielfältigt in Mathematischen Institut der Universität Bonn, 1960.

Th. BRÖCKER, *Differentiable Germs and Catastrophes*. (Transl. by L. Lander.) London Mathematical Society Lecture Notes Series **17**. Cambridge University Press, 1975.

O. BRUNO, Logical opens of exponential objects. *Cah. de Top. et Géo. Diff. Catégoriques* **XXVI-3** (1985), 311-323.

M. BUNGE, Synthetic aspects of C^∞-mappings. *J. Pure and Appl. Algebra* **28** (1983), 41-63

M. BUNGE and E.J. DUBUC, Local concepts in S.D.G. and germ representability, in E.G.K. López-Escobar & C. Smith (editors), *Mathematical Logic and Theoretical Computer Science*, pp. 39-158, M. Dekker, Inc., 1987.

M. BUNGE and F. GAGO, Synthetic aspects of C^∞-mappings, II: Mather's theorem for infinitesimally represented germs. Report from the Department of Mathematics and Statistics **87-6**. McGill university, 1987.(To appear in *J. Pure and Applied Algebra.*)

M. COSTE, Logique d'ordre supérieur dans les topos élémentaires. Séminaire Bénabou. Université Paris-Nord, 1974.

E.J. DUBUC, C^∞-schemes. *Amer. J. Math.* **103**-4 (1981), 683-690.

E.J. DUBUC, Germ representability and local integration of vector fields in WAM of SDG. Aarhus Univ. Math. Institute. Preprint Series 1985/86.

E.J. DUBUC, Open covers and infinitary operations in C^∞-rings. *Cahiers de Top. et Géo. Diff.* **22** (1981), 287-300.

E.J. DUBUC et J. PENON, Objets compacts dans les topos. *J. Austr. Math. Soc.* (Series A) **40** (1986), 203-217.

M. DUMMET, *Elements of Intuitionism*. Oxford: Clarendon Press, 1977.

C. EHRESMANN, Les prolongements d'une variétée différentiable, I, II, III. *C. R. Acad. Sci. Paris*, (1951).

A. FRÖLICHER, Applications lisses entre spaces et variétées de Fréchet. *C.R. Acad. Sci. Paris Sér I Math.* **293** (1981), 125-127.

F. GAGO, Morse germs in S.D.G. (submitted to the *Proceedings of the Louvain-la-Neuve Category Conference, July 1987*)

M. GOLUBITSKI and V. GUILLEMIN. *Stable Mappings and their Singularities*. Graduate Texts in Mathematics **14**, Springer-Verlag, 1973.

V. GUILLEMIN and A. POLLACK. *Differential Topology*. Prentice-Hall, 1974.

M.W. HIRSCH, *Differential Topology*. Graduate Texts in Mathematics **33**, Springer-Verlag, 1976.

P.T. JOHNSTONE,*Topos Theory*. London Mathematical Society Monographs **10** Academic Press, 1977.

A. JOYAL and G.E. REYES, Separably real closed local rings. *J. Pure Applied Algebra* **43** (1986), 274-279.

A. KOCK, Formal manifolds and synthetic theory of jet bundles. *Cahiers de Top. et Géo. Diff.* **21** (1980), 227-246.

A. KOCK, Synthetic characterization of reduced algebras. *J. Pure Appl. Algebra* **36** (1985), 273-279.

A. KOCK, *Synthetic Differential Geometry*. London Mathematical Society Lecture Notes Series **51**. Cambridge University Press, 1981.

A. KOCK, Universal projective geometry via topos theory. *J. Pure Appl. Algebra* **9** (1976), 1-24.

A. KOCK (edit.) Topos Theoretic Methods in Geometry. Århus Univ. Math. Institute. Var. Pub. Series **30**, 1979.

A. KOCK (edit.) *Category Theoretic Methods in Geometry*. Århus Univ. Mat. Institut Var. Pub. Series **35**, 1983.

A. KOCK and G.C. WRAITH *Elementary toposes*. Århus Univ. Math. Institute. Lecture Notes Series **30**, 1971.

R. LAVENDHOME, *Leçons de Géométrie Différentielle Synthétique Naïve*. Univ. Cat. Louvain-la-Neuve, Inst. de Math. Monographies de Mathématique **3**, 1987.

F.W. LAWVERE, Categorical dynamics, in A. KOCK (editor) Topos Theoretic Methods in Geometry. Århus Univ. Math. Institute. Var. Pub. Series **30** (1979), 1-28.

F.W. LAWVERE, Functorial semantics of algebraic theories. *Proc. Nat. Acad. Sci.* **63** (1963), 869-872.

F.W. LAWVERE, Introduction, *Model Theory and Topoi*. Springer Lecture Notes in Mathematics **445** (1975), 3-14.

F.W. LAWVERE, Quantifiers and sheaves. Actes Congrés International des Mathématiciens Nice 1970, 329-334.

B. MALGRANGE, The preparation theorem for differentiable mappings. Sém. H. Cartan, 15^{e} anné: 1962/63, Exposés 11, 12, 13. 22.

J. MARTINET, *Singularitées des Fonctions et Applications Différentiables*. Pontifícia Universidade Católica do Rio de Janeiro, Monografías de Matemática **1**, 1977.

J. MARTINET, *Singularities of Smooth Functions and Maps*. (Transl. by C. SIMON) Lond. Math. Soc. Lect. Not. Ser. **58**, Cambridge University Press, 1982.

J. MATHER, Stability of C^{∞}-mappings, I: The division theorem. *Ann. Math.* **87**-1 (1968), 89-104.

J. MATHER, Stability of C^∞-mappings, II: Infinitesimal stability implies stability. *Ann. Math.* **89-2** (1969), 254-291.

J. MATHER, Stability of C^∞-mappings, III: Finitely determined map germs. *Pub. Sci. I.H.E.S.* **35** (1969), 127-156.

P.W. MICHOR, *Manifolds of Differentiable Mappings*. Mathematics Series **3**, Shiva Publ. Lted., 1980.

C.J. MIKKELSEN, *Lattice Theoretic and Logical Aspects of Elementary Topoi*, Århus Univ. Mat. Institut Var. Pub. Series **25**, 1976.

J. MILNOR, *Morse Theory*. Annals of Mathematics Studies **51**, Princeton Univ. Press, 1963.

J. MILNOR, *Topology from the Differentiable Viewpoint*. University of Virginia Press, 1966.

I. MOERDIJK and G.E. REYES, Cohomology theories in Synthetic Differential Geometry, in A. KOCK (edit.) *Category Theoretic Methods in Geometry*. Århus Univ. Mat. Institut Var. Pub. Series **35** (1983), 1-29.

I. MOERDIJK and G.E. REYES, Smooth spaces vs. continuous spaces in the models for S.D.G., *J. Pure Appl. Algebra* **32** (1984), 143-176.

J.R. MUNKRES, *Elementary Differential Topology*. Annals of Mathematics Studies **54**. Princeton University Press, 1966.

G. OSIUS, Logical and set theoretical tools in elementary topoi, in *Model Theory and Topoi* Springer Lecture Notes in Mathematics **445** (1975), 297-346.

R. OUELLET, Axiomatisation de la logique interne du premier ordre des topos, version inclusive et multisorte, Thèse de Doctorat, Université de Montréal, 1974.

R. PARE, Colimits in topoi. *Bull. Amer. Math. Soc.* **80** (1974), 556-561.

J. PENON, De l'infinitésimal au local, Thèse de Doctorat d'Etat, Université Paris VII, 1985

J. PENON, Infinitésimaux et intuitionism. *Cahiers de Top. et Géo. Diff.* **22** (1980), 67-72.

J. PENON, Le théoreme de inversion local en géométrie algébrique. Journées Faisceaux et Logique, Mai 1982, Université Catholique Louvain-la-Neuve, pré-publications mathématiques. (1982)

J. PENON, Topologie et intuitionisme. Journées Faisceaux et Logique, Mai 1981, Univ. Paris-Nord, pré-publications mathématiques. (1982)

V. POENARU, *Analyse Différentielle*. Springer Lecture Notes in Mathematics **371**, 1970.

G.E. REYES, (editor).Géométrie Différentielle Synthétique, Rapport de Recherche, DMS 80-12, Univ. de Montréal, 1980.

G.E. REYES and G.C. WRAITH, A note on tangent bundles in a category with a ring object. *Math. Scand.* **42** (1978), 53-63.

G.E. REYES AND N. VAN QUE. Smooth functors and synthetic calculus, in *The L.E.J. Brower Centenary Symposium*, A.A. Troelstra & D. van Dalen (editors) North-Holland, 1982.

B. RIEMANN, Über die Hypothesen welche der Geometrie zu Grunde Leigen. *Gesammelte Mathematische Werke*, B.G. Teubner, Leipzig (1892), 272-287.

C. ROUSSEAU, Eigenvalues of symmetric matrices on topoi. Rapport de Recherche, DMS 80-5, Univ. de Montréal, 1980.

A. SARD, The measure of critical values of differentiable maps. *Bull. A.M.S.* **48** (1942), 883-890.

R.R. STOLL, *Linear Algebra and Matrix Theory*. Dover Publications, Inc., 1952.

R. THOM, Un lemme sur les applications différentiables. *Bol. Soc. Mat. Mex.* (2) **1** (1956), 59-71.

R. THOM and H.I. LEVINE, *Singularities of Differentiable Mappings, I, Bonn 1959*, reprinted in C.T.C.WALL (ed.) *Proceedings of Liverpool Singularities-Symposium* Springer Lecture Notes in Mathematics **192** (1972), 1-89.

M. TIERNEY, Sheaf theory and the Continuum Hypothesis, *Toposes, Algebraic Geometry and Logic* (ed. F.W. Lawvere) Springer Lecture Notes in Mathematics **274** (1972), 13-42.

A.S. TROELSTRA, *Principles of Intuitionism.* Springer Lecture Notes in Mathematics **95**, 1969.

C.T.C.WALL (ed.) *Proceedings of Liverpool Singularities-Symposium, I.* Springer Lecture Notes in Mathematics **192**, 1972.

G. WASSERMANN, *Stability of Unfoldings.* Springer Lecture Notes in Mathematics **393**, 1974.

A. WEIL, Théorie des points proches sur les variétées différentiables. Collq. Top. et Géom. Diff., Strasbourg (1953), 111-117.

S. WILLARD, *General Topology.* Addison-Wesley Publishing Company Inc., 1970

H. WHITNEY, Analytic extensions of differentiable functions defined in closed sets. *Trans. Amer. Math. Soc.* **36** (1934), 63-89.