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CONTRIBUTIONS TO THE KINEMATIC SYNTHESIS OF PARALLEL MANIPULATORS

Hamid Reza Mohammadi Daniali

B.Sc. (Mashhad University), 1986

M.Sc. (Tehran University), 1989

Department of Mechanical Engineering

McGill University

Montreal, Quebec, Canada

A thesis submitted to the Faculty of Graduate Studies and Research
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Doctor of Philosophy

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Abstract

This thesis is devoted to the kinematic synthesis of parallel manipulators at large, special attention being given to three versions of a novel class of manipulators, named *double-triangular*. These are conceived in planar, spherical and spatial double-triangular varieties.

The treatment of planar and spherical manipulators needs only planar and spherical trigonometry, a fact that inductively leads to the successful treatment of spatial varieties with methods of spatial trigonometry, wherein the relationships are cast in the form of *dual-number* algebraic expressions. Using the foregoing tools, the direct kinematics of the three types of double-triangular manipulators is formulated and resolved.

Moreover, a general three-group classification, to deal with *singularities* encountered in parallel manipulators, is proposed. The classification scheme relies on the properties of Jacobian matrices of parallel manipulators. It is shown that all singularities, within the workspaces of the manipulators of interest, are readily identified if their Jacobian matrices are formulated in an invariant form.

Finally, the optimal design of the manipulators is studied. These designs minimize the roundoff-error amplification effects due to the numerical inversion of the underlying Jacobian matrices. Such designs are called *isotropic*. Based on this concept the multi-dimensional isotropic design continua of several manipulators are derived.

Résumé

Cette thèse porte sur la synthèse cinématique des manipulateurs parallèles généraux, et plus particulièrement, sur une nouvelle classe de manipulateurs, dite à double-triangle. Ces manipulateurs se présentent en version planaire, sphérique et spatiale.

L'analyse de ces manipulateurs, en version planaire et sphérique, nécessite seulement des relations trigonométriques planaires et sphériques, induisant ainsi l'utilisation avec succès de relations trigonométriques spatiales pour la version spatiale de ces manipulateurs. Ces relations sont écrites sous forme d'expression algébrique à nombres duals. Le problème géométrique direct des trois versions de manipulateurs à double-triangle est formulé et résolu avec cet outil mathématique.

De plus, une classification générale des manipulateurs parallèles en trois groupes est proposée. Celle-ci repose sur les propriétés de la matrice Jacobienne des manipulateurs. Elle montre que toutes les singularités, situées à l'intérieur de l'espace de travail du manipulateur étudié sont facilement identifiées si la matrice Jacobienne est écrite sous forme invariante.

Finalement, la conception optimale des manipulateurs est étudiée, afin de minimiser les effets d'amplification des erreurs d'arrondissement lors de l'inversion de la matrice Jacobienne. Les manipulateurs ainsi conçus sont appelés isotropes. En se basant sur ce concept, l'auteur obtient le continuum multi-dimensionnel de plusieurs manipulateurs isotropes.

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The understanding, patience and support of my wife Masoumeh was unswerving. I am profoundly grateful for her great sacrifice on my behalf. The steadfast support and encouragement of my family gave me the confidence and determination to preserve.

Claim of Originality

The author claims the originality of ideas and results presented here, the main contributions being listed below:

- Introduction of three versions of a novel class of parallel manipulators, namely, planar, spherical and spatial double-triangular manipulators;
- solutions of the associated direct kinematic problems;
- derivation of the Jacobian matrices for these and other classes of parallel manipulators, based on an invariant representation;
- classification of singularities in parallel manipulators into three groups, and identification of all three groups within the workspaces of the manipulators;
- derivation of multi-dimensional continua of isotropic designs for some parallel manipulators;
- expression of the screw matrix and its invariant parameters in invariant form.

The material presented in this thesis has been partially reported in (Mohammadi Daniali, Zsombor-Murray and Angeles, 1993a, 1993b, 1994a, 1994b, 1994c, 1994d, 1995a, 1995b, 1995c, 1995d and Mohammadi Daniali and Zsombor-Murray, 1994).

Dedicated to:

the memory of my father;

my mother;

my wife and son.

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Chapter 1

Introduction

1.1 General Background

A *manipulator*, according to IFToMM¹'s Commission A Standards for Terminology (1991), is *a device for gripping and the controlled movement of objects*. Examples of manipulators appear in Figs. 1.1a–1.4a and 1.5. For the purpose of this thesis, we regard the manipulator as a kinematic chain of rigid links coupled by *kinematic pairs*. A kinematic pair is, in turn, the coupling of two links so as to constrain their relative motion. Kinematic chains are classified as simple or complex, open or closed. If the chain contains at least one link coupled to only one other link, the chain is called *open*, as depicted in Fig. 1.1b; otherwise it is *closed*, as depicted in Fig. 1.2b. Moreover, a *simple kinematic chain* is one with links coupled to at most two other links, while a *complex kinematic chain* is one with at least one link connected to three or more links. Both Figs. 1.1b and 1.2b show simple kinematic chains, while a complex kinematic chain is depicted in Fig. 1.3b.

Manipulators are classified here into four categories, namely, *serial*, *tree-type*, *parallel* and *hybrid*. The term serial manipulator denotes an open, simple kinematic chain structure, as shown in Fig. 1.1. A manipulator is said to have a tree-type

¹International Federation for the Theory of Machines and Mechanisms

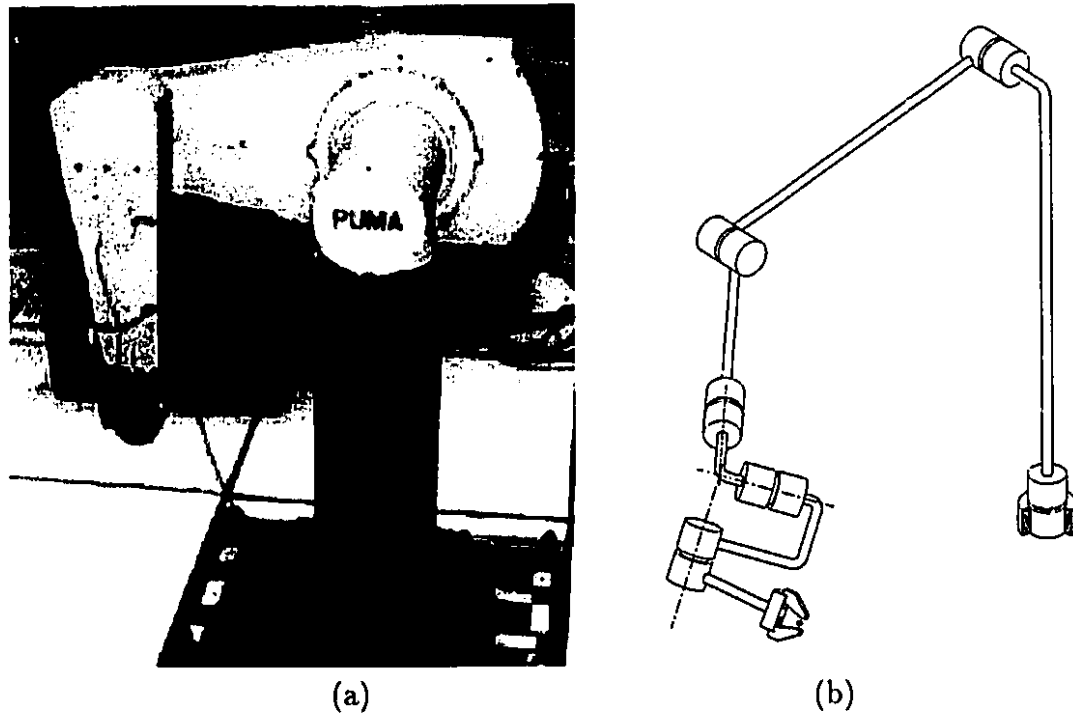


Figure 1.1: (a) A serial manipulator, and (b) its kinematic chain

structure if it has an open complex kinematic chain, while parallel manipulators have complex, closed-kinematic chains. The former is depicted in Fig. 1.4, while the manipulator of Fig. 1.3 has a parallel structure. Moreover, a hybrid manipulator contains both serial and parallel subchains, as shown in Fig. 1.5. The kinematic chain of this manipulator is a serial concatenation of that of the Stewart platform, like the one shown in Fig. 1.3b.

Consider now the large class of parallel manipulators wherein two bodies are connected to each other by several simple, open kinematic chains, called *legs*. It is proposed that these manipulators be classified into three subgroups based on the concept of *degree of parallelism* (dop), defined as:

$$\text{dop} = \frac{\text{number of legs}}{\text{degrees of freedom}} \quad (1.1)$$

A manipulator may have any number of legs from one to infinity, while the maximum

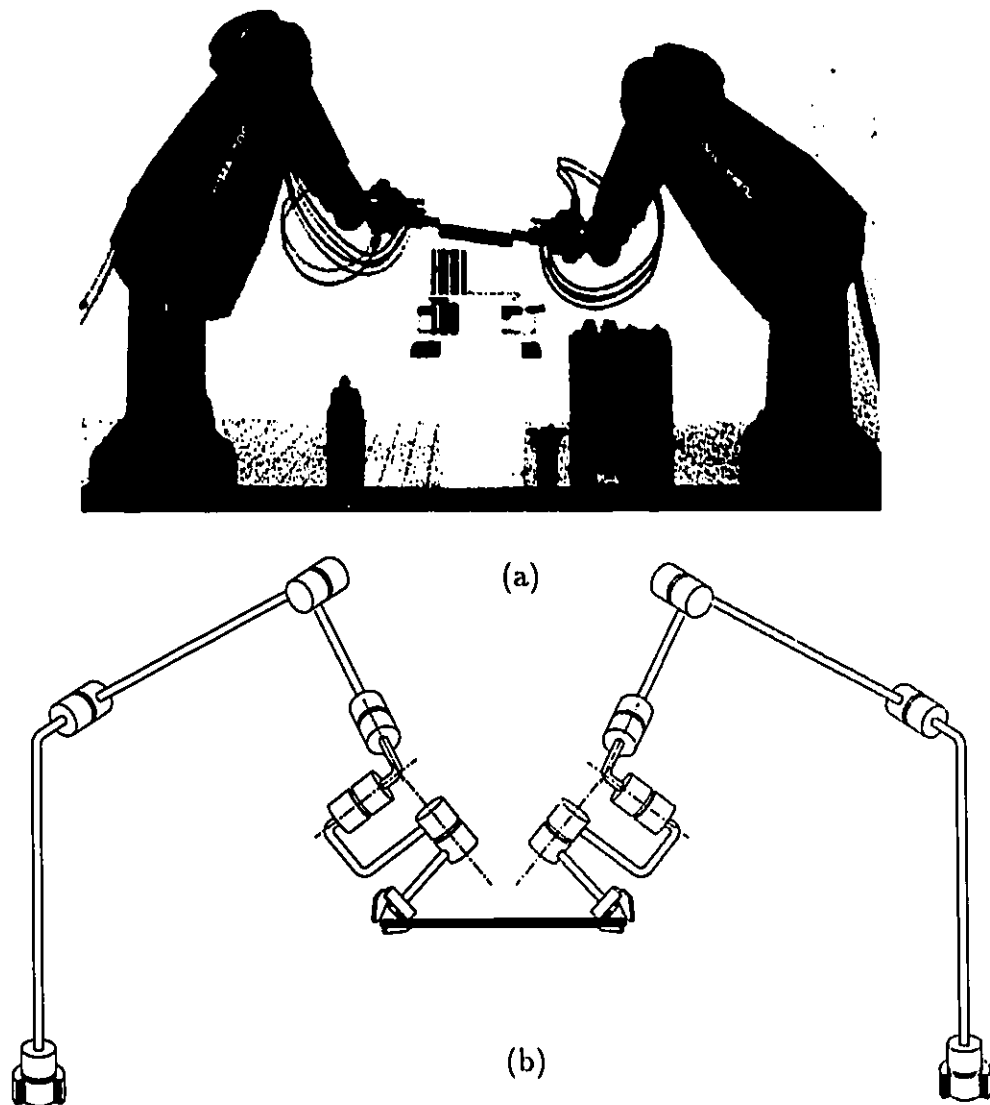


Figure 1.2: (a) Two cooperating manipulators, and (b) their kinematic chains

degree of freedom is six. Thus, for parallel manipulators, this number can be any integer fraction n/d , where $0 < d < 7$ and $n > 0$. If the fraction is less than unity, the manipulator is called *partially parallel*, while if it is greater than unity, we call the device *highly parallel*. Moreover, *fully parallel* manipulators are those with a d equal to unity. This thesis is mainly devoted to the kinematic synthesis of fully parallel manipulators, henceforth abbreviated parallel manipulators. However, the

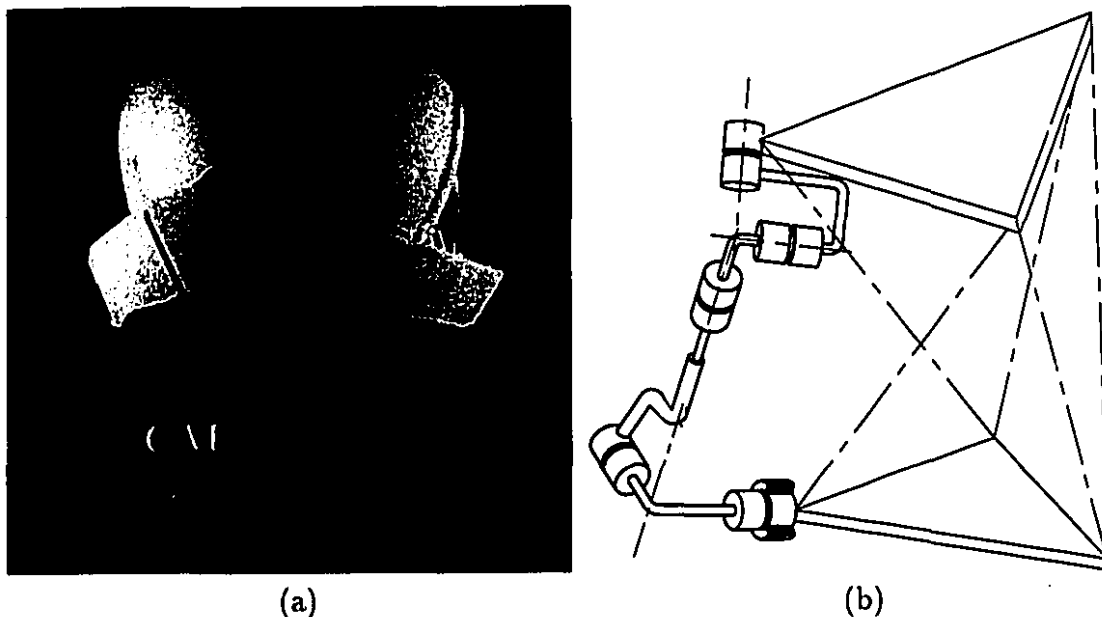
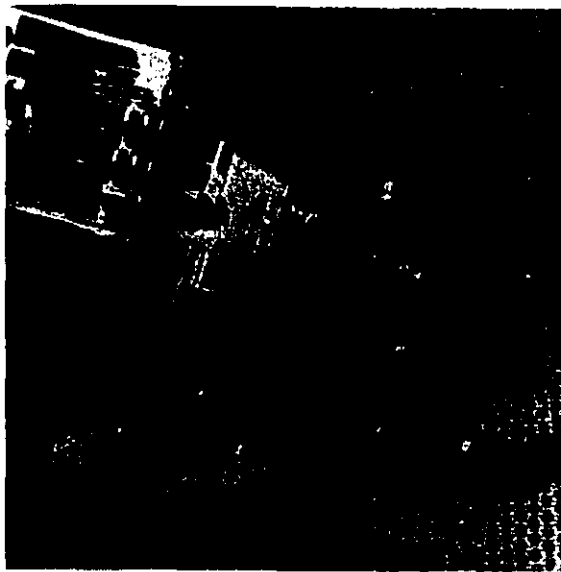


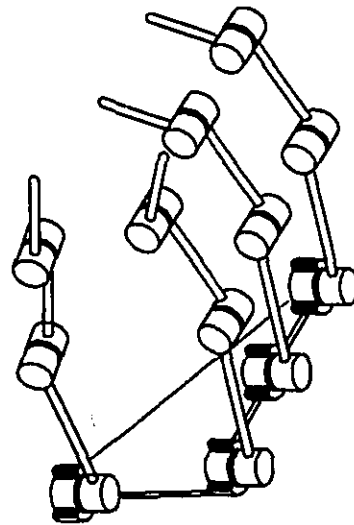
Figure 1.3: (a) Stewart platform, and (b) its kinematic chain

kinematics of some partially parallel manipulators is also included. The characteristics of the latter are slightly different from those of fully parallel manipulators, based on their dop.

Most industrial robots are serial manipulators. In general, these have the advantages of large workspace and design simplicity. However, they suffer from some drawbacks, such as lack of rigidity, operating inaccuracy, poor dynamic characteristics and small pay-load capacity. The source of the foregoing deficiencies of serial manipulators is their cantilever type of link loading. This indicates that providing the end-effector (EE) with multiple-point support could alleviate the aforementioned problems. Therefore, the obvious alternative is a parallel architecture. While load-to-weight ratios in serial manipulators are in the order of 5%, according to Merlet (1990), this ratio for some parallel manipulators like the flight simulator shown in Fig. 1.3a, is more than 500%. The simulator can shake its 10000 kg payload in a controlled manner at a frequency of 20 Hz and an amplitude of 50 mm, a performance



(a)



(b)

Figure 1.4: (a) A four-fingered hand (The Utah-MIT hand), and (b) its kinematic chain

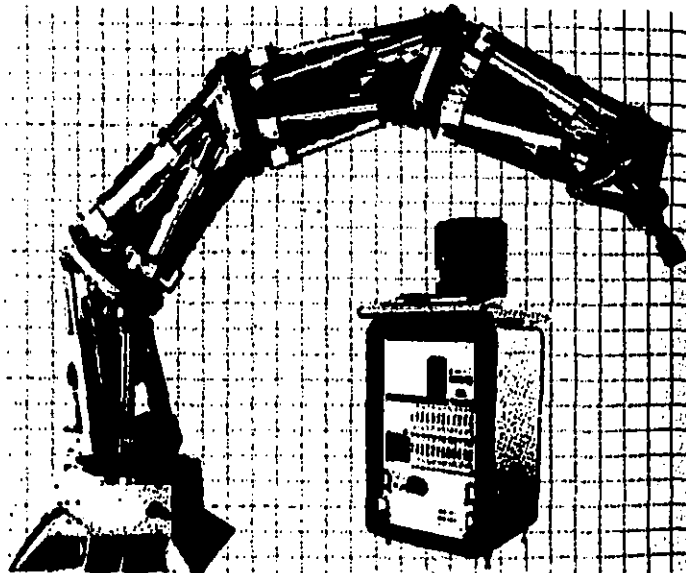


Figure 1.5: Hybrid manipulator; Logabex LX4 robot

that would be impossible with any known serial manipulator. A general comparison of some characteristics of serial and parallel manipulators is given in Table 1.1. Characteristics of a tree-type manipulator are similar to those of a serial one, while those of a hybrid manipulators constitute a compromise between serial and parallel manipulators.

Characteristics	Serial manipulators	Parallel manipulators
Accuracy	lower	higher
Workspace	larger	smaller
Stiffness	lower	higher
Load-to-weight ratio	lower	higher
Design complexity	simple	complex
Inertial load	higher	lower
Operating speed	lower	higher
Bandwidth	narrow	wide
Repeatability	lower	higher
Density of singularities	lower	higher

Table 1.1: General comparison of serial and parallel manipulators

1.2 Literature Survey and Motivation

1.2.1 Parallel Manipulators

For many applications, parallel manipulators are without rival. One application is micro-motion, where a very accurate robot in a limited workspace is required. Several micro-robots have been designed and built, e.g., a planar parallel three *degree-of-freedom* (dof) micro-robot was built by Behi et al. (1990). The length of each link of the manipulator is only $100\ \mu\text{m}$, while its workspace is $0.01\ \text{mm}^2$. Hara and Sugimoto (1989) built another parallel micro-manipulator whose range of motion is only in the order of $10\ \mu\text{m}$, but the micro-manipulator makes it possible to obtain

a resolution of $0.01\ \mu\text{m}$. The latter device is used in semiconductor manufacturing equipment and electron microscope. Moreover, NASA built a 6-dof parallel robot EE, for fine motion, to study tele robotic assembly of hardware in space (Nguyen and Pooran, 1989). Furthermore, a high-performance parallel drive micro-robot capable of nanometer-resolution movements has been developed and is being used at McGill University for micro-manipulation and mechanical testing (Hunter et al., 1989).

Another application of parallel manipulators is in manufacturing processes that require a rigid robot for holding and handling workpieces. In this case, a single rigid manipulator reduces set-up time if it can hold as well as manipulate the workpieces. Lee and Yien (1989) designed and built a 3-dof parallel robot for this purpose.

Many macro-motion parallel robots are available. The best known are probably those of the platform type (Stewart, 1965), like the one shown in Fig. 1.3, which is widely used in flight simulators to train pilots. Apart from this, several spherical 3-dof parallel manipulators have been designed and built (Hayward and Kurtz, 1991; Gosselin and Hamel, 1993). A novel class of parallel-manipulator architectures has been developed, namely, the 3-dof *DELTA* robot designed by R. Clavel (1988), of the *Ecole Polytechnique Fédérale de Lausanne*. ARIA, of Switzerland, designed, built and installed several versions of this robot, namely, the ARIA DELTA C300 and the ARIA DELTA C1000, for the handling of light objects at very high speeds, to be used in assembly lines. As an extension of the DELTA robot, researchers at *Laboratoire de Robotique, Informatique et Microélectronique de Montpellier* (Pierrot et al., 1991) designed and commissioned a 6-dof parallel manipulator, the HEXA robot, with an architecture resembling that of the DELTA robot. Furthermore, J. M. Hervé, of *Ecole Centrale de Paris*, designed and built a prototype of a 3-dof parallel manipulator, the Y-STAR robot, for 3-dimensional positioning tasks (Hervé and Sparacino, 1991).

Although all these parallel macro-robots are more rigid and move faster than their serial counterparts, they have long slender legs like a serial manipulator chain.

Long, slender legs produce undesirable flexibility and kinematic instabilities. Here, a novel class of architecture that certainly does not have this drawback is introduced. It is called *double-triangular* (DT), because it is based on a pair of triangles that move with respect to each other. The three obvious subclasses of this manipulator class are *planar*, *spherical* and *spatial* DT manipulators, based, respectively, on a pair of planar, spherical and spatial triangles. A common feature of DT manipulators is their short, possibly zero-length, legs, thereby avoiding the objectionable flexibility of long-legged robots like the Stewart platform, HEXA, Y-STAR and all versions of DELTA, while retaining desirable parallel manipulator features like high stiffness, load-carrying capacity and speed. Although they have a parallel architecture, they do not introduce the drawbacks of the conventional parallel manipulators, namely, extremely reduced workspace volume and high density of singularities within their workspaces. A very important issue here is the structural stiffness, which can be controlled at will, for the double-triangular architecture, similar to the *double tetrahedral mechanism* (Tarnai and Makai, 1988, 1989a, 1989b; Zsombor-Murray and Hyder, 1992), is free of long links and flexible joints that mar the performance of many parallel manipulators.

Double-triangular robotic devices do not exist; the concept is quite novel and offers many possibilities for innovation and can find many applications. In a flexible manufacturing system, the planar DT manipulator could be designed to manipulate workpieces or tools in a planar motion with one rotation about an axis perpendicular to the plane of motion. Moreover, augmented with an axis, to allow translation in a direction perpendicular to the plane of motion, this device can perform the motions of what are known as SCARA (Selective Compliance Assembly Robot Arm) robots. These are widely used, particularly to assemble printed circuit boards and other electronic hardware. The spherical device, in turn, may serve as a robotic wrist at the end of a positioning arm. A very large class of tasks involving spherical motion includes the orientation of antennas, radars and solar collectors, where very

heavy objects must be moved accurately. A spatial DT manipulator, by virtue of its 6-dof capabilities, can arbitrarily pose workpieces in 3D space. Alternatively, these manipulators can operate in a 3-dof mode, if orientation is either irrelevant or provided by other means, e.g., by a spherical wrist. The latter could be, in fact, the spherical DT manipulator.

1.2.2 Direct Kinematics

Manipulator kinematics is the study of the relationship between joint and EE motion, disregarding how the motion is caused. It provides a basis for the study and applications of robotics. There exist two basic problems in manipulator kinematics, namely, the *direct kinematic problem* and the *inverse kinematic problem*, as defined below:

Direct kinematics:

Given the actuator variables, find the Cartesian coordinates of the EE.

Inverse kinematics:

Given the Cartesian coordinates of the EE, find the actuator variables.

For most serial manipulators, the direct kinematics is straightforward, while the inverse kinematics is challenging. The literature on the latter is extensive (Pieper, 1968; Duffy and Derby, 1979; Duffy and Crane, 1980; Albala, 1982; Alizade et al., 1983; Primrose, 1986), but only recently has a systematic solution procedure, for general 6R architectures, been reported (Lee and Liang, 1988; Raghavan and Roth, 1990; Lee et al., 1991).

For parallel manipulators, as a rule, the inverse kinematics is straightforward, while the direct kinematic problem is quite challenging. A major issue in the control of manipulators with this architecture is their direct kinematics. The kinematics of several planar parallel manipulators was investigated by Gosselin and Angeles (1990a), Hunt (1983) and Gosselin and Sefrioui (1991). Moreover, the kinematics of a few spherical parallel manipulators was investigated by Gosselin and Angeles (1989,

1990a), Craver (1989) and Gosselin et al. (1994a, 1994b). The direct kinematics of the flight simulator admits up to sixteen different poses for a given set of leg extensions (Charentus and Renaud, 1989; Nanua et al. , 1990). Once this problem was solved, the next challenge to researchers became the direct kinematics of the most general platform. Numerical experiments conducted by Raghavan (1993) indicate that the direct kinematics of this device admits up to forty solutions. Recently, Husty (1994) proposed an algorithm for solving the problem and obtaining the characteristic 40th-degree polynomial.

Here, we solve the direct kinematics of all versions of DT manipulators. The kinematics of the planar and the spherical mechanisms require only the tools of planar and spherical trigonometries. This fact has inductively led us to expand the solution concept to three dimensions by invoking methods of *spatial trigonometry*. Although the latter is less known than its planar and spherical counterparts, its principles are well established and appear to be well suited to the direct kinematics of the spatial double-triangular mechanisms. Spatial trigonometric relationships are expressed in *dual-number algebra* (Clifford, 1873; Yang, 1963; Yang and Freudenstein, 1964). This tool is used to describe the geometric relations among lines in space by treating them as relations among points lying on the surface of a sphere centred at the origin of the *dual space*. While dual-number algebra was devised more than a century ago, owing its origins to Clifford (1873), it is not yet commonly used in the realm of kinematic design and analysis. However, it is the most suitable tool to handle the kinematics of rigid bodies in the context of screw theory, which owes its origins, in turn, to the work of Sir Robert Ball (1900). A milestone in the development of dual-number algebra, applied to mechanism analysis, is the work of Yang (1963) and of Yang and Freudenstein (1964). Yang extended the concept of dual number to that of dual vector and dual quaternion, thereby laying the foundations for the design and analysis of spatial kinematic chains. However, using this tool, he presented examples of application involving the kinematics of relatively simple problems. Here we derive

the dual screw matrix and its linear invariants in an invariant form. Moreover, building upon the work on dual-number algebra reported above, we analyze the spatial double-triangular mechanisms introduced here, thereby showing that more complicated direct kinematic problems can be solved conveniently with dual number algebra.

1.2.3 Singularities

A manipulator singularity occurs at the coincidence of different direct or inverse kinematic solutions. Algebraically, a singularity amounts to a rank deficiency of the associated Jacobian matrices while, geometrically, it is observed whenever the manipulator gains some additional, uncontrollable degrees of freedom, or loses some degrees of freedom.

The concept of singularity has been extensively studied in connection with serial manipulators (Sugimoto et al., 1982; Litvin and Parenti-Castelli, 1985; Litvin et al., 1986; Hunt, 1986, 1987; Lai and Yang, 1986; Angeles et al., 1988; Shamir, 1990). On the other hand, as regards manipulators with kinematic loops, the literature is more limited (Mohamed, 1983; Gosselin and Angeles, 1990b; Ma and Angeles, 1992; Sefrioui, 1992; Zlatanov et al., 1994a, 1994b; Notash and Podhorodeski, 1994; Husty and Zsombor-Murray, 1994). Mohamed (1983) classified singularities into three groups, based on the underlying Jacobian matrices, namely, *stationary configuration*, *uncertainty configuration* and *immovable structure*. Gosselin and Angeles (1990b) suggested a classification of singularities pertaining to parallel manipulators into three main groups. Later, Ma and Angeles (1992) introduced another classification for singularities, namely, *configuration singularities*, *architecture singularities* and *formulation singularities*. The latter is caused by the failure of a kinematic model at particular configurations of a manipulator and can be avoided by a proper formulation of the problem, while a configuration singularity is an inherent manipulator property and occurs at some configurations within the workspace of the manipulator.

An architecture singularity is caused by a particular architecture of a manipulator. Such a singularity prevails in all configurations inside the workspace. Moreover, Sefrioui (1992) considered architecture and configuration singularities, and classified the latter into two groups. Finally, Zlatanov et al. (1994a, 1994b) classified singularities of a non-redundant general mechanism into six groups. However, some of those groups always occur simultaneously. The above-mentioned singularity classifications fail in more general cases; the author has been unable to find reference to any other singularity classification methods for general kinematic chains with multiple kinematic loops. This motivated the study of singularities, which forms part of this thesis.

Here, an alternative classification of singularities encountered in parallel manipulators is proposed. Similar to the classification of singularities given in Gosselin and Angeles (1990b), the classification suggested here relies on the properties of the Jacobian matrices of the manipulator. These Jacobians, for the case of parallel manipulators, occur in kinematic relations of the form

$$\mathbf{J}\dot{\boldsymbol{\theta}} + \mathbf{K}\mathbf{t} = \mathbf{0} \quad (1.2)$$

where $\dot{\boldsymbol{\theta}}$ is the vector of joint rates, $\mathbf{t}$ is the *twist array* and $\mathbf{K}$ and $\mathbf{J}$ are the Jacobian matrices.

Deriving the Jacobian matrices for the manipulators of interest, in an invariant form, enables the detection of all singularities within the workspaces of the manipulators. Moreover, contrary to earlier claims, (Gosselin, 1988; Gosselin and Angeles, 1990b; Sefrioui, 1992), it is proven that the third type of singularity is not necessarily architecture-dependent.

1.2.4 Isotropy

An important property of robotic manipulators, which has attracted the attention of researchers for many years, is kinematic dexterity. However, dexterity bears different connotation in different contexts. One definition of dexterity is given as that fraction

of the workspace volume in which a manipulator can assume all orientations (Gupta and Roth, 1982). Dexterity has also been interpreted as a specification of the dynamic response of a manipulator (Yoshikawa, 1985), its joint range availability (Liegeois, 1977), and as global measures over a whole trajectory (Suh and Hollerbach, 1987).

With regard to dexterity in the context of local kinematic accuracy, a number of measures, based on the Jacobian matrices, have been proposed for quantification, namely, the *Jacobian determinant*, *manipulability*, *minimum singular value*, and *condition number*. For non-redundant manipulators, the determinant has been used to evaluate the accuracy of wrist configurations (Paul and Stevenson, 1983). Yoshikawa (1985) has extended the definition based on the Jacobian determinant to non-square matrices by using the determinant of the product of the Jacobian matrix by its transpose, thereby proposing the concept of manipulability. Klein and Blaho (1987) used the minimum singular value as a dexterity index.

If the determinant approaches zero, the value of the determinant cannot be used as a practical measure of ill-conditioning. This is true as well for the minimum singular value approaching zero. These two measures have dimensions of length to a certain power, their value thus depending on the choice of units. Nevertheless, to evaluate ill-conditioning, the matrix condition number has been recommended by numerical analysts (Issacson and Keller, 1966). This measure does not share the drawback of determinants and minimum singular value pointed out above.

The Jacobian matrices of parallel manipulators are configuration-dependent, and hence, a manipulator can be designed with an architecture that allows for postures entailing isotropic Jacobian matrices. An isotropic matrix, in turn, is a matrix with a condition number of unity. Such designs are called *isotropic*. The concept of isotropy was first introduced by Salisbury and Craig (1982), for the optimum design of multi-fingered hands. Later, isotropic Jacobian matrices were used as a design criterion to configure various manipulators (Gosselin, 1988; Gosselin and Angeles, 1988 and 1989; Klein and Miklos, 1991; Angeles and López-Cajún, 1992; Angeles

et al., 1992; Gosselin and Lavoie, 1993; Pittens and Podhorodeski, 1993; Husty and Angeles, 1994).

The foregoing concept is used here to find isotropic designs of several parallel manipulators. Parallel manipulators, contrary to their serial counterparts, have two Jacobian matrices, as expressed in eq.(1.2). Thus, the condition numbers of the two Jacobian matrices should be minimized. For DT and some other parallel manipulators, we find herein the complete set of isotropic designs. This is possible only because we could find the Jacobian matrices in an invariant form. Moreover, the continuum of design variables is at least one-dimensional, thereby allowing designers the freedom to investigate and incorporate optimality criteria other than isotropy, e. g., workspace volume and global dexterity.

1.3 Scope and Organization of the Thesis

The main theme of this thesis is the kinematic study of parallel manipulators. However, our focus will be on a novel class of parallel devices, namely, double triangular parallel manipulators. Other parallel robots are included in our study to show that some of the methods developed here are widely applicable and not confined to the DT class.

The kinematics of the planar and spherical mechanisms require only the tools of planar and spherical trigonometry. This fact has led us to expand the solution concept to three dimensions by invoking methods of spatial trigonometry. In Chapter 2, the analytic tools, including dual numbers, quaternions, dual quaternions and spatial trigonometry are studied.

Planar and Spherical DT manipulators, along with a generalization of the concept, are developed in Chapter 2, the spatial version of these manipulators being introduced in Chapter 3. Other parallel manipulators, namely, those to which our methods have been applied in the later chapters, are introduced in Chapter 3 as well. Among these, we have two general classes of planar parallel manipulators.

Chapter 4 is devoted to the direct kinematics of planar, spherical and spatial DT manipulators; for each, an example is included. The direct kinematics of all the manipulators mentioned in Chapter 3 is addressed here as well.

In Chapter 5, we introduce a classification of singularities in parallel manipulators, which is based on their Jacobian matrices. The Jacobian matrices of several parallel manipulators are derived in an invariant form, their singularities being identified according to the new scheme.

Chapter 6 is devoted to the study of the isotropic design of parallel manipulators. Using isotropy criteria, multi-dimensional parameters of isotropic designs are found, which we claim are exhaustive as regards the manipulators at hand.

Chapter 7 summarizes the work accomplished in this thesis and suggests further avenues of research.

Finally, five appendices provide additional theoretical and application depth. A brief account of Bezout's method to eliminate unknowns in a system of multivariate polynomials is included in Appendix A. The coefficients of long equations are tabulated in Appendices B, C and D. Appendix E contains mechanical designs to show how planar and spherical DT manipulators might be implemented.

Chapter 2

Analysis Tools

2.1 Introduction

The analysis tools on which this thesis relies are dual number algebra and spatial trigonometry. An extensive treatment of these can be found in Clifford (1873), Study (1903), Yang (1963), Yang and Freudenstein (1964), Ogino and Watanabe (1969), Pradeep et al. (1989), Funda and Paul (1990), Ge and McCarthy (1991), González-Palacios and Angeles (1993), Cheng (1993), Ge (1994) and Thompson and Cheng (1994). For quick reference, and also with the purpose of giving more insight into these concepts, while introducing new viewpoints, we give in this chapter an account of these valuable concepts.

2.2 Dual Numbers

A *dual number* $\hat{a}$, first introduced by Clifford (1873), is defined as an ordered pair, namely,

$$\hat{a} = (a, a_0) \tag{2.1}$$

with specific addition and multiplication rules. In the foregoing definition, a is the *primal* part and a_0 is the *dual* part, both being real numbers. Moreover, if $a_0 = 0$, $\hat{a}$

is called a *real* number; if $a = 0$, $\hat{a}$ is called a *pure dual number* and, if neither is zero, $\hat{a}$ is called a *proper dual number*.

Let ϵ denote the dual unit, which is a quasi-imaginary unit with two properties, namely,

$$\epsilon \neq 0, \quad \epsilon^2 = 0$$

Then, a dual number can be written as

$$\hat{a} = a + \epsilon a_0 \quad (2.2)$$

The dual number $\hat{a}$, whose Cartesian coordinates are a and a_0 as shown in Fig. 2.1, can be associated with a point in a plane called the *dual plane*. Each dual number corresponds to one point in that plane, and vice versa. Moreover, $\hat{a}$ can be represented as a vector from the origin of the dual plane to the point (a, a_0) .

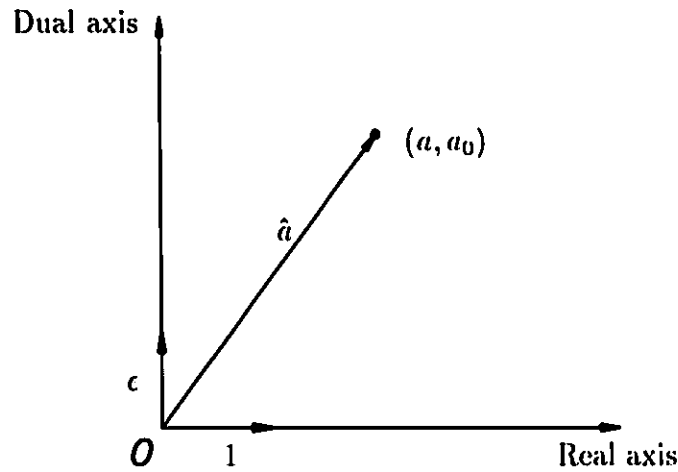


Figure 2.1: Dual plane

Let $\hat{b} = b + \epsilon b_0$ be another dual number. Equality, addition, multiplication and division are defined, respectively, as

$$\hat{a} = \hat{b} \iff a = b, \quad a_0 = b_0 \quad (2.3a)$$

$$\hat{a} + \hat{b} = (a + b) + \epsilon(a_0 + b_0) \quad (2.3b)$$

$$\hat{a}\hat{b} = ab + \epsilon(ab_0 + a_0b) \quad (2.3c)$$

$$\frac{\hat{a}}{\hat{b}} = \frac{a}{b} - \epsilon \left(\frac{ab_0 - a_0b}{b^2} \right), \quad b \neq 0 \quad (2.3d)$$

From eq.(2.3d) it is obvious that division by a pure dual is not defined. Hence, *dual numbers do not form a field*.

All formal operations involving dual numbers are identical to those of ordinary algebra while taking into account that $\epsilon^2 = \epsilon^3 = \dots = 0$. The series expansion of an analytic function of a dual number is of great importance, i.e.,

$$f(\hat{a}) = f(a + \epsilon a_0) = f(a) + \epsilon a_0 \frac{df(a)}{da} \quad (2.4)$$

where all higher terms vanish because of the foregoing property of ϵ .

2.2.1 Dual Quantities

The *dual angle* $\hat{\theta}$ between two skew lines $\mathcal{L}_1$ and $\mathcal{L}_2$, introduced by Study (1903), is defined as

$$\hat{\theta} = \theta + \epsilon s \quad (2.5)$$

where θ and s are, respectively, the twist angle and the distance between the two lines, as shown in Fig. 2.2.

Three trigonometric identities arise directly from eq.(2.4), namely,

$$\sin \hat{\theta} = \sin \theta + \epsilon s \cos \theta \quad (2.6a)$$

$$\cos \hat{\theta} = \cos \theta - \epsilon s \sin \theta \quad (2.6b)$$

$$\tan \hat{\theta} = \tan \theta + \epsilon s \sec^2 \theta \quad (2.6c)$$

Moreover, all ordinary trigonometric identities hold for dual angles.

A *dual vector* $\hat{\mathbf{a}}$ is defined as the sum of a primal part $\mathbf{a}$ and dual part $\mathbf{a}_0$, namely,

$$\hat{\mathbf{a}} = \mathbf{a} + \epsilon \mathbf{a}_0 \quad (2.7)$$

Moreover, a line $\mathcal{A}$ can be specified by a unit dual vector $\mathbf{a}^*$, whose 6 real coefficients in $\mathbf{a}$ and $\mathbf{a}_0$ are the Plücker coordinates of $\mathcal{A}$, namely,

$$\mathbf{a}^* = \mathbf{a} + \epsilon \mathbf{a}_0 \quad (2.8)$$

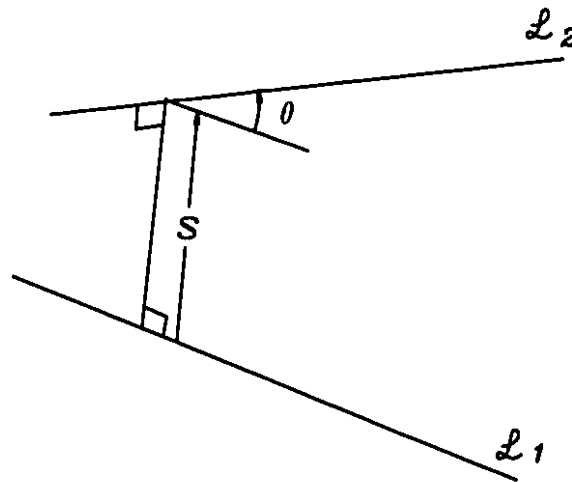


Figure 2.2: Dual angle between two skew lines

with $\mathbf{a} \cdot \mathbf{a} = 1$ and $\mathbf{a} \cdot \mathbf{a}_0 = 0$. Figure 2.3 shows $\mathbf{a}$ as defining the direction of $\mathcal{A}$, while $\mathbf{a}_0$ is the moment of $\mathbf{a}$ with respect to the origin O , namely,

$$\mathbf{a}_0 = \mathbf{p} \times \mathbf{a}$$

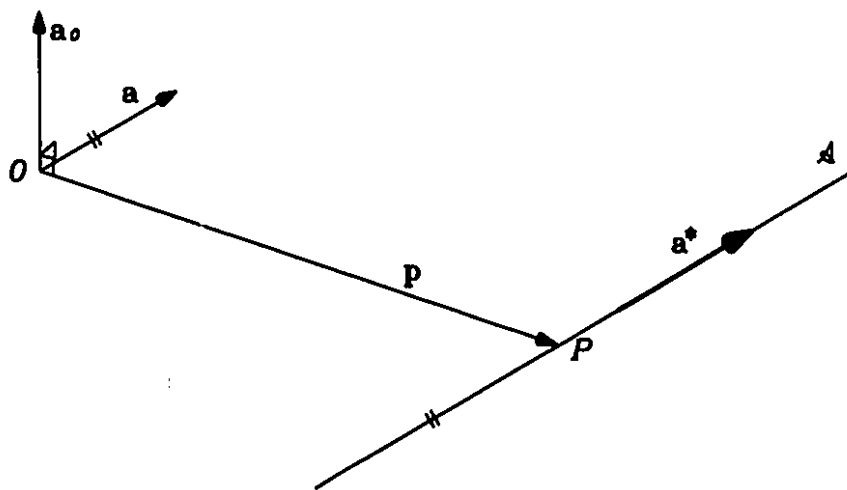


Figure 2.3: Plücker coordinates of a line

2.2.2 Quaternions

Quaternions were introduced by Hamilton (1844). They have recently played a significant role in several areas of science and engineering, namely, in differential geometry, in analysis and synthesis of mechanisms and machines, simulation of particle motion in molecular physics, and in the formulation of the relativistic equation of motion (Agrawal, 1987). Funda and Paul have shown that quaternions offer the most efficient alternative among point transformation formalisms (Funda and Paul, 1990). However, they have not received wide publicity in the area of kinematics and dynamics of mechanisms. This is mainly because quaternion algebra is complicated and leads to tedious operations.

The word *quaternion* is derived from the Latin word *quaterni* and means a set of four. It is a linear combination of four quaternion units, 1, *i*, *j* and *k*, namely,

$$q = d + ai + bj + ck \quad (2.9)$$

with the definitions

$$i^2 = j^2 = k^2 = -1$$

$$ij = k, \quad jk = i, \quad ki = j$$

Pre-multiplying both sides of $ij = k$, $jk = i$ and $ki = j$ by *i*, *j* and *k*, respectively, leads to

$$ik = -j, \quad ji = -k, \quad kj = -i$$

Moreover, *d*, *a*, *b* and *c* are all real numbers. The three quaternion units *i*, *j* and *k* can be considered as orthogonal unit vectors with respect to the scalar product. For this reason *i*, *j* and *k* are also identified as an orthogonal triad of unit vectors in a 3-dimensional Euclidean space.

A quaternion consists of two parts, the scalar part *s*, and the vector part *v*, namely,

$$q = s + v \quad (2.10a)$$

where

$$s \equiv d \quad (2.10b)$$

$$\mathbf{v} \equiv a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \quad (2.10c)$$

The *norm* of a quaternion is defined as

$$N \equiv \sqrt{qk(q)} \quad (2.11)$$

where $k(q)$ is called the conjugate of q , defined as

$$k(q) \equiv s - \mathbf{v} \quad (2.12)$$

Then, eq.(2.11) leads to

$$N = \sqrt{d^2 + a^2 + b^2 + c^2} \quad (2.13)$$

Furthermore, the reciprocal of a quaternion is defined as

$$q^{-1} \equiv \frac{k(q)}{N^2} \quad (2.14)$$

and, as a result, we have

$$q^{-1}q = qq^{-1} = 1 \quad (2.15)$$

A unit quaternion q^* is a quaternion whose norm is unity, and takes on the general form

$$q^* = \cos \theta + \mathbf{s} \sin \theta \quad (2.16a)$$

where

$$\begin{aligned} \cos \theta &= \frac{d}{N} \\ \sin \theta &= \frac{\sqrt{a^2 + b^2 + c^2}}{N} \end{aligned}$$

and $\mathbf{s}$ is the unit vector representing the axis of the unit quaternion q^* ; it is given as

$$\mathbf{s} = \frac{a\mathbf{i} + b\mathbf{j} + c\mathbf{k}}{\sqrt{a^2 + b^2 + c^2}} \quad (2.16b)$$

2.2.3 The Product of Two Vectors

Given two unit vectors $\mathbf{a}$ and $\mathbf{b}$, their *product* $\mathbf{ab}$, in this order, is defined as

$$\begin{aligned}\mathbf{ab} &\equiv -\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b} \\ &= -\cos \theta + \mathbf{s} \sin \theta\end{aligned}\tag{2.17}$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, and $\mathbf{s}$ is the unit vector perpendicular to them, as shown in Fig. 2.4.

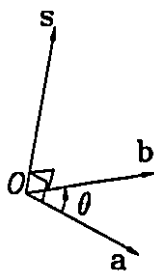


Figure 2.4: Vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{s}$

The conjugate of both sides of eq.(2.17) is readily calculated as

$$k(\mathbf{ab}) = -(\cos \theta + \mathbf{s} \sin \theta)\tag{2.18}$$

The left-hand-side, in the light of eq.(2.17), can be written as

$$\begin{aligned}k(\mathbf{ab}) &= k(-\mathbf{a} \cdot \mathbf{b} + \mathbf{a} \times \mathbf{b}) \\ &= -\mathbf{a} \cdot \mathbf{b} - \mathbf{a} \times \mathbf{b} \\ &= -\mathbf{b} \cdot \mathbf{a} + \mathbf{b} \times \mathbf{a} \\ &= (-\mathbf{b})(-\mathbf{a})\end{aligned}\tag{2.19}$$

But the negative of a vector is the same as its conjugate; hence, the foregoing equation leads to

$$k(\mathbf{ab}) = k(\mathbf{b})k(\mathbf{a})\tag{2.20}$$

Substitution of the value of $k(\mathbf{ab})$ from eq.(2.20) into eq.(2.18), and noting that the right-hand-side of the latter is equal to $-q^*$, leads to

$$k(\mathbf{b})k(\mathbf{a}) = -q^* \quad (2.21)$$

But, from eq.(2.14), we have

$$k(\mathbf{a}) = \mathbf{a}^{-1} \quad (2.22)$$

Substitution of the value of $k(\mathbf{a})$ from eq.(2.22) into eq.(2.21), with the fact that $k(\mathbf{b})$ is equal to $-\mathbf{b}$, yields

$$q^* = \mathbf{ba}^{-1} \quad (2.23)$$

Furthermore, post-multiplying both sides of the foregoing equation by $\mathbf{a}$ yields

$$q^*\mathbf{a} = \mathbf{b} \quad (2.24)$$

which implies that a unit quaternion q^* is a *rotation operator*. It rotates a vector $\mathbf{a}$ through an angle θ about an axis $\mathbf{s}$, called the quaternion axis that intersects the vector at right angles, as shown in Fig. 2.4. Moreover, this operation preserves the Euclidean norm of vectors.

The relation between a unit quaternion q^* and the corresponding rotation matrix $\mathbf{Q}$ follows directly from definitions (2.16a) and (2.16b), namely,

$$q^* = \frac{1}{2}[\text{tr}(\mathbf{Q}) - 1] + \text{vect}(\mathbf{Q}) \quad (2.25a)$$

where $\text{tr}(\mathbf{Q})$ and $\text{vect}(\mathbf{Q})$ are the linear invariants of $\mathbf{Q}$, as defined in (Angeles, 1988) as

$$\text{tr}(\mathbf{Q}) = 1 + 2 \cos \theta \quad (2.25b)$$

$$\text{vect}(\mathbf{Q}) = \sin \theta \mathbf{s} \quad (2.25c)$$

Moreover, $\mathbf{Q}$ is given in an invariant form in this reference as

$$\mathbf{Q} = \mathbf{ss}^T + \cos \theta (\mathbf{1} - \mathbf{ss}^T) + \sin \theta \mathbf{S} \quad (2.25d)$$

in which $\mathbf{s}$, θ and $\mathbf{S}$ are the axis of rotation, the rotation angle and the cross-product matrix of vector $\mathbf{s}$.

2.2.4 Dual Quaternions

The quaternion concept was combined with that of dual numbers by Yang (1963) to produce the dual quaternion. Moreover, he applied the latter to analyze four-bar linkages. Recently, Funda and Paul showed that dual quaternions provide the most compact and computationally efficient formalism for motion in parallel and serial screw computations (Funda and Paul, 1990).

A dual quaternion $\hat{q}$ is a quaternion with dual components, namely

$$\hat{q} = \hat{d} + \hat{a}\mathbf{i} + \hat{b}\mathbf{j} + \hat{c}\mathbf{k} \quad (2.26)$$

where $\hat{d}$, $\hat{a}$, $\hat{b}$ and $\hat{c}$ are dual numbers. Similar to an ordinary quaternion, a dual quaternion consists of two parts, the scalar part $\hat{s}$, and the vector part $\hat{\mathbf{v}}$, namely,

$$\hat{q} = \hat{s} + \hat{\mathbf{v}} \quad (2.27a)$$

where

$$\hat{s} \equiv \hat{d} \quad (2.27b)$$

$$\hat{\mathbf{v}} \equiv \hat{a}\mathbf{i} + \hat{b}\mathbf{j} + \hat{c}\mathbf{k} \quad (2.27c)$$

Moreover, the conjugate of a dual quaternion is defined as

$$k(\hat{q}) \equiv \hat{s} - \hat{\mathbf{v}} \quad (2.28)$$

while the norm of a dual quaternion is defined as

$$\hat{N} \equiv \sqrt{\hat{q}k(\hat{q})} = \sqrt{\hat{d}^2 + \hat{a}^2 + \hat{b}^2 + \hat{c}^2} \quad (2.29)$$

Furthermore, the reciprocal of a dual quaternion is defined as

$$\hat{q}^{-1} \equiv \frac{k(\hat{q})}{\hat{N}^2} \quad (2.30)$$

and, as a result, we have

$$\hat{q}^{-1}\hat{q} = \hat{q}\hat{q}^{-1} = 1 \quad (2.31)$$

A unit dual quaternion $\hat{q}^*$ is a dual quaternion with norm equal to unity, namely,

$$\hat{q}^* = \cos \hat{\theta} + \mathbf{s}^* \sin \hat{\theta} \quad (2.32a)$$

where

$$\begin{aligned} \cos \hat{\theta} &= \frac{\hat{d}}{\hat{N}} \\ \sin \hat{\theta} &= \frac{\sqrt{\hat{a}^2 + \hat{b}^2 + \hat{c}^2}}{\hat{N}} \end{aligned}$$

and hence, $\mathbf{s}^*$ represents the Plücker coordinates of the axis of the unit dual quaternion $\hat{q}^*$, which is given as

$$\mathbf{s}^* = \frac{\hat{a}\mathbf{i} + \hat{b}\mathbf{j} + \hat{c}\mathbf{k}}{\sqrt{\hat{a}^2 + \hat{b}^2 + \hat{c}^2}} \quad (2.32b)$$

2.2.5 The Product of Two Lines

Given the Plücker coordinates of two lines $\mathcal{A}$ and $\mathcal{B}$ in dual-vector form, $\mathbf{a}^*$ and $\mathbf{b}^*$, the product of $\mathcal{A}$ by $\mathcal{B}$, in this order, is defined as the product of $\mathbf{a}^*$ by $\mathbf{b}^*$, in the corresponding order, namely,

$$\mathbf{a}^* \mathbf{b}^* = -\mathbf{a}^* \cdot \mathbf{b}^* + \mathbf{a}^* \times \mathbf{b}^* \quad (2.33a)$$

where

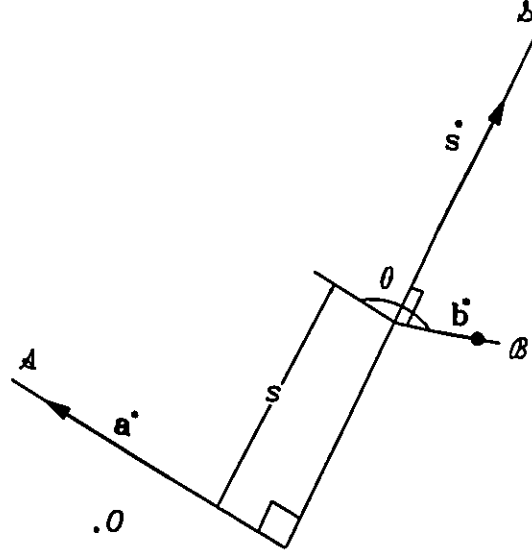
$$\mathbf{a}^* \cdot \mathbf{b}^* = \mathbf{a} \cdot \mathbf{b} + \epsilon(\mathbf{a} \cdot \mathbf{b}_0 + \mathbf{a}_0 \cdot \mathbf{b}) = \cos \hat{\theta} \quad (2.33b)$$

$$\mathbf{a}^* \times \mathbf{b}^* = \mathbf{a} \times \mathbf{b} + \epsilon(\mathbf{a} \times \mathbf{b}_0 + \mathbf{a}_0 \times \mathbf{b}) = \mathbf{s}^* \sin \hat{\theta} \quad (2.33c)$$

in which $\mathbf{s}^*$ is a unit dual vector representing the Plücker coordinates of the common perpendicular between $\mathbf{a}^*$ and $\mathbf{b}^*$. Moreover, $\hat{\theta}$ is given as

$$\hat{\theta} = \theta + \epsilon s$$

in which θ is the twist angle and s is the distance between the two lines, respectively, as shown in Fig. 2.5.

Figure 2.5: Layouts of lines $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{S}$

Equation (2.33a), in the light of eqs.(2.33b) and (2.33c), can be written as

$$\mathbf{a}^* \mathbf{b}^* = -\cos \hat{\theta} + \mathbf{s}^* \sin \hat{\theta} \quad (2.34)$$

Moreover, taking the conjugate of both sides of eq.(2.34) leads to

$$k(\mathbf{a}^* \mathbf{b}^*) = -(\cos \hat{\theta} + \mathbf{s}^* \sin \hat{\theta}) \quad (2.35)$$

The left-hand-side, in the light of eqs.(2.33b) and (2.33c), can be written as

$$\begin{aligned} k(\mathbf{a}^* \mathbf{b}^*) &= k(-\mathbf{a}^* \cdot \mathbf{b}^* + \mathbf{a}^* \times \mathbf{b}^*) \\ &= -\mathbf{a}^* \cdot \mathbf{b}^* - \mathbf{a}^* \times \mathbf{b}^* \\ &= -\mathbf{b}^* \cdot \mathbf{a}^* + \mathbf{b}^* \times \mathbf{a}^* \\ &= (-\mathbf{b}^*)(-\mathbf{a}^*) \end{aligned} \quad (2.36)$$

But the negative of a dual vector, based on eq.(2.28), is the same as its conjugate, the foregoing equation thus leading to

$$k(\mathbf{a}^* \mathbf{b}^*) = k(\mathbf{b}^*)k(\mathbf{a}^*) \quad (2.37)$$

Substitution of the value of $k(\mathbf{a}^*\mathbf{b}^*)$ from eq.(2.37) into eq.(2.35), with the fact that the right-hand-side of the latter is equal to $-\hat{q}^*$, leads to

$$k(\mathbf{b}^*)k(\mathbf{a}^*) = -\hat{q}^* \quad (2.38)$$

But, from eq.(2.30), we have

$$k(\mathbf{a}^*) = (\mathbf{a}^*)^{-1} \quad (2.39)$$

Substituting the value of $k(\mathbf{a}^*)$ from eq.(2.39) into eq.(2.38), and noting that $k(\mathbf{b}^*)$ is equal to $-\mathbf{b}^*$, we obtain

$$\hat{q}^* = \mathbf{b}\mathbf{a}^{-1} \quad (2.40)$$

Post-multiplying both sides of the above equation by $\mathbf{a}^*$ leads to

$$\hat{q}^*\mathbf{a}^* = \mathbf{b}^* \quad (2.41)$$

which implies that a unit dual quaternion $\hat{q}^*$ is a *screw operator*. It rotates a line $\mathcal{A}$ through an angle θ about an axis $\mathcal{S}$ that intersects the line at right angles, and slides it along that line through a distance s , as shown in Fig. 2.5.

As explained earlier, the main drawback of quaternions is that their algebra is quite involved, the complexity related to that of dual quaternions being even more so. To overcome this obstacle, the author implemented some user-defined functions in a MATHEMATICA environment to handle computations such as the product of two lines, the product of two dual quaternions, and the product of a line by a dual quaternion, to be used in Chapter 4.

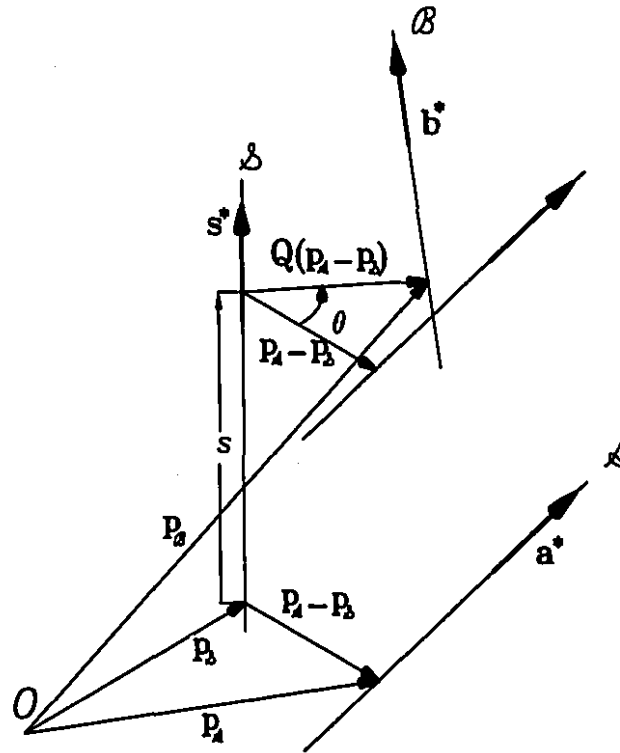
All the dual quantities and their properties are summarized in Table 2.1.

2.2.6 Dual Screw Matrices

Here, we combine the concept of rotation matrix with that of dual numbers, and will find a *dual screw matrix* in an invariant form. Let us define two lines $\mathcal{A}$ and $\mathcal{S}$ in dual-vector form as $\mathbf{a}^*$ and $\mathbf{s}^*$, respectively. Line $\mathcal{A}$ rotates through an angle θ

Quantity	Symbol	Primary part				Dual part				Constraints	No. of independent variables
		1	i	j	k	ϵ	ϵi	ϵj	ϵk		
Real number	a	a	0	0	0	0	0	0	0		1
Pure dual number	a_0	0	0	0	0	a_0	0	0	0		1
Proper dual number	$\hat{a}$	a	0	0	0	a_0	0	0	0		2
Distance	s	0	0	0	0	s	0	0	0		1
Angle	θ	0	0	0	0	0	0	0	0		1
Dual angle	$\hat{\theta}$	0	0	0	0	s	0	0	0		2
Vector	$\mathbf{a}$	0	a	b	c	0	0	0	0		3
Quaternion	q	d	a	b	c	0	0	0	0		4
Unit quaternion	q^*	d	a	b	c	0	0	0	0	$d^2 + a^2 + b^2 + c^2 = 1$	3
Dual vector	$\hat{\mathbf{a}}$	0	a	b	c	0	a_0	b_0	c_0		6
Line	$\mathbf{a}^*$	0	a	b	c	0	a_0	b_0	c_0	$a^2 + b^2 + c^2 = 1$ $aa_0 + bb_0 + cc_0 = 0$	4
Dual quaternion	$\hat{q}$	d	a	b	c	d_0	a_0	b_0	c_0		8
Unit dual quaternion	$\hat{q}^*$	d	a	b	c	d_0	a_0	b_0	c_0	$d^2 + a^2 + b^2 + c^2 = 1$ $dd_0 + aa_0 + bb_0 + cc_0 = 0$	6

Table 2.1: Dual quantities and their properties

Figure 2.6: Screw motion of line $\mathcal{A}$, about line $\mathcal{S}$

about $\mathcal{S}$ and slides along that line through a displacement s , as shown in Fig. 2.6, to coincide with line $\mathcal{B}$ whose direction $\mathbf{b}$ is given as

$$\mathbf{b} = \mathbf{Q}\mathbf{a} \quad (2.42)$$

$\mathbf{Q}$ being the rotation matrix given in eq.(2.25d). Moreover, the moment of line $\mathcal{B}$ with respect to the origin O , $\mathbf{b}_0$, is given as

$$\mathbf{b}_0 = \mathbf{p}_B \times \mathbf{b} \quad (2.43a)$$

where $\mathbf{p}_B$ is a vector directed from O to a point on line $\mathcal{B}$, and is given as

$$\mathbf{p}_B = \mathbf{p}_S + s\mathbf{s} + \mathbf{Q}(\mathbf{p}_A - \mathbf{p}_S) \quad (2.43b)$$

Substituting the values of $\mathbf{b}$ and $\mathbf{p}_S$ from eqs.(2.42) and (2.43b) into eq.(2.43a) leads to

$$\mathbf{b}_0 = [\mathbf{p}_S + \mathbf{s}\mathbf{s} + \mathbf{Q}(\mathbf{p}_A - \mathbf{p}_S)] \times (\mathbf{Q}\mathbf{a}) \quad (2.44)$$

Equation (2.44), upon expansion, yields

$$\mathbf{b}_0 = \mathbf{p}_S \times (\mathbf{Q}\mathbf{a}) + \mathbf{s}\mathbf{s} \times (\mathbf{Q}\mathbf{a}) + (\mathbf{Q}\mathbf{p}_A) \times (\mathbf{Q}\mathbf{a}) - (\mathbf{Q}\mathbf{p}_S) \times (\mathbf{Q}\mathbf{a}) \quad (2.45a)$$

Upon substitution of the value of $\mathbf{Q}$ from eq.(2.25d) into eq.(2.45a) and expansion, the first term of the foregoing equation leads to

$$\begin{aligned} \mathbf{p}_S \times (\mathbf{Q}\mathbf{a}) &= (\mathbf{p}_S \times \mathbf{s})\mathbf{s}^T \mathbf{a} + \cos \theta (\mathbf{p}_S \times \mathbf{a}) - \cos \theta (\mathbf{p}_S \times \mathbf{s})\mathbf{s}^T \mathbf{a} + \sin \theta [\mathbf{p}_S \times (\mathbf{s} \times \mathbf{a})] \\ &= \mathbf{s}_0 \mathbf{s}^T \mathbf{a} + \cos \theta (\mathbf{p}_S \times \mathbf{a}) - \cos \theta \mathbf{s}_0 \mathbf{s}^T \mathbf{a} + \sin \theta [(\mathbf{p}_S^T \mathbf{a})\mathbf{s} - (\mathbf{p}_S^T \mathbf{s})\mathbf{a}] \end{aligned} \quad (2.45b)$$

Similarly, for the second, third and fourth terms of eq.(2.45a), we have

$$\begin{aligned} \mathbf{s}\mathbf{s} \times (\mathbf{Q}\mathbf{a}) &= \mathbf{s}(\mathbf{s} \times \mathbf{s})\mathbf{s}^T \mathbf{a} + \mathbf{s} \cos \theta (\mathbf{s} \times \mathbf{a}) - \mathbf{s} \cos \theta (\mathbf{s} \times \mathbf{s})\mathbf{s}^T \mathbf{a} \\ &\quad + \mathbf{s} \sin \theta [\mathbf{s} \times (\mathbf{s} \times \mathbf{a})] \\ &= \mathbf{s} \cos \theta (\mathbf{s} \times \mathbf{a}) + \mathbf{s} \sin \theta [(\mathbf{s}^T \mathbf{a})\mathbf{s} - (\mathbf{s}^T \mathbf{s})\mathbf{a}] \end{aligned} \quad (2.45c)$$

$$(\mathbf{Q}\mathbf{p}_A) \times (\mathbf{Q}\mathbf{a}) = \mathbf{Q}(\mathbf{p}_A \times \mathbf{a}) = \mathbf{Q}\mathbf{a}_0 \quad (2.45d)$$

$$\begin{aligned} (\mathbf{Q}\mathbf{p}_S) \times (\mathbf{Q}\mathbf{a}) &= \mathbf{Q}(\mathbf{p}_S \times \mathbf{a}) \\ &= \mathbf{s}\mathbf{s}^T (\mathbf{p}_S \times \mathbf{a}) + \cos \theta (\mathbf{p}_S \times \mathbf{a}) - \cos \theta (\mathbf{s}\mathbf{s}^T) (\mathbf{p}_S \times \mathbf{a}) \\ &\quad + \sin \theta [\mathbf{s} \times (\mathbf{p}_S \times \mathbf{a})] \\ &= -\mathbf{s}\mathbf{s}_0^T \mathbf{a} + \cos \theta (\mathbf{p}_S \times \mathbf{a}) + \cos \theta \mathbf{s}\mathbf{s}_0^T \mathbf{a} \\ &\quad + \sin \theta [(\mathbf{s}^T \mathbf{a})\mathbf{p}_S - (\mathbf{s}^T \mathbf{p}_S)\mathbf{a}] \end{aligned} \quad (2.45e)$$

Substituting the values of $\mathbf{p}_S \times (\mathbf{Q}\mathbf{a})$, $\mathbf{s}\mathbf{s} \times (\mathbf{Q}\mathbf{a})$, $(\mathbf{Q}\mathbf{p}_A) \times (\mathbf{Q}\mathbf{a})$ and $(\mathbf{Q}\mathbf{p}_S) \times (\mathbf{Q}\mathbf{a})$ from eqs.(2.45b-e) into eq.(2.45a), upon simplification, yields

$$\begin{aligned}
\mathbf{b}_0 &= \mathbf{Q}\mathbf{a}_0 + (\mathbf{s}_0\mathbf{s}^T + \mathbf{s}\mathbf{s}_0^T)\mathbf{a} - s \sin \theta (\mathbf{1} - \mathbf{s}\mathbf{s}^T)\mathbf{a} - \cos \theta (\mathbf{s}_0\mathbf{s}^T + \mathbf{s}\mathbf{s}_0^T)\mathbf{a} + \\
&\quad \sin \theta [(\mathbf{p}_S^T\mathbf{a})\mathbf{s} - (\mathbf{s}^T\mathbf{a})\mathbf{p}_S] + s \cos \theta (\mathbf{s} \times \mathbf{a}) \\
&= \mathbf{Q}\mathbf{a}_0 + [(\mathbf{s}_0\mathbf{s}^T + \mathbf{s}\mathbf{s}_0^T) - s \sin \theta (\mathbf{1} - \mathbf{s}\mathbf{s}^T) - \cos \theta (\mathbf{s}_0\mathbf{s}^T + \mathbf{s}\mathbf{s}_0^T) + \\
&\quad \sin \theta \mathbf{S}_0 + s \cos \theta \mathbf{S}]\mathbf{a}
\end{aligned} \tag{2.46}$$

in which $\mathbf{S}_0$ is the cross-product matrix of vector $\mathbf{s}_0$. From eqs.(2.42) and (2.46), $\mathbf{b}^*$ can be written as

$$\begin{aligned}
\mathbf{b}^* \equiv \mathbf{b} + \epsilon \mathbf{b}_0 &= \mathbf{Q}\mathbf{a} + \epsilon \mathbf{Q}\mathbf{a}_0 + \epsilon [(\mathbf{s}_0\mathbf{s}^T + \mathbf{s}\mathbf{s}_0^T) - s \sin \theta (\mathbf{1} - \mathbf{s}\mathbf{s}^T) \\
&\quad - \cos \theta (\mathbf{s}_0\mathbf{s}^T + \mathbf{s}\mathbf{s}_0^T) + \sin \theta \mathbf{S}_0 + s \cos \theta \mathbf{S}]\mathbf{a}
\end{aligned} \tag{2.47}$$

Equation (2.47) leads to a simple form, namely,

$$\mathbf{b}^* = \hat{\mathbf{Q}}\mathbf{a}^* \tag{2.48a}$$

where $\hat{\mathbf{Q}}$ is the dual screw matrix in invariant form, i. e.,

$$\hat{\mathbf{Q}} = \mathbf{s}^*\mathbf{s}^{*T} + \cos \hat{\theta}(\mathbf{1} - \mathbf{s}^*\mathbf{s}^{*T}) + \sin \hat{\theta} \hat{\mathbf{S}} \tag{2.48b}$$

with $\mathbf{s}^{*T}$ and $\hat{\mathbf{S}}$ are defined as

$$\mathbf{s}^{*T} = \mathbf{s}^T + \epsilon \mathbf{s}_0^T \tag{2.48c}$$

$$\hat{\mathbf{S}} = \mathbf{S} + \epsilon \mathbf{S}_0 \tag{2.48d}$$

We have thus shown that a dual screw matrix can readily be derived by changing the real quantities of the rotation matrix of eq.(2.25d) into dual quantities. The same is true for its linear invariants, namely, $\text{tr}(\hat{\mathbf{Q}})$ and $\text{vect}(\hat{\mathbf{Q}})$, as we show below. From the invariant representations of the dual screw matrix, eq.(2.48b), it is clear that the first two terms of $\hat{\mathbf{Q}}$, namely, $\mathbf{s}^*\mathbf{s}^{*T}$ and $\cos \hat{\theta}(\mathbf{1} - \mathbf{s}^*\mathbf{s}^{*T})$ are symmetric, while the last term is skew-symmetric. Hence

$$\text{tr}(\hat{\mathbf{Q}}) = \text{tr}[\mathbf{s}^*\mathbf{s}^{*T} + \cos \hat{\theta}(\mathbf{1} - \mathbf{s}^*\mathbf{s}^{*T})] = 1 + 2 \cos \hat{\theta} \tag{2.49a}$$

$$\text{vect}(\hat{\mathbf{Q}}) = \text{vect}(\sin \hat{\theta} \hat{\mathbf{S}}) = \sin \hat{\theta} \mathbf{s}^* \tag{2.49b}$$

Therefore, the relation between a unit dual quaternion $\hat{q}^*$ and the corresponding dual screw matrix $\hat{\mathbf{Q}}$ follows directly from definitions (2.32a) and (2.32b), namely,

$$\hat{q}^* = \frac{1}{2}[\text{tr}(\hat{\mathbf{Q}}) - 1] + \text{vect}(\hat{\mathbf{Q}}) \quad (2.50)$$

Moreover, eqs.(2.49a) and (2.49b) should find extensive applications in the realm of motion determination whereby the displacement of some lines of a rigid body are given, and, from these, the screw parameters are to be determined.

2.3 Spatial Trigonometry

2.3.1 Spatial Triangle

A spatial triangle consists of three skew lines in space and their three common perpendiculars, as depicted in Fig. 2.7. In that figure, the three lines are labelled $\{\mathcal{L}_i\}_1^3$, their corresponding normals being $\{\mathcal{N}_i\}_1^3$, where $\mathcal{N}_1$ is the common normal between lines $\mathcal{L}_2$ and $\mathcal{L}_3$, $\mathcal{N}_2$ is that between $\mathcal{L}_1$ and $\mathcal{L}_3$, with a similar definition for $\mathcal{N}_3$. The lines are given by the three unit dual vectors $\{\lambda_i^*\}_1^3$, defined as

$$\lambda_i^* \equiv \lambda_i + \epsilon \lambda_{0i}, \quad i = 1, 2, 3 \quad (2.51)$$

where λ_i and λ_{0i} are, respectively, the direction and the moment vectors of $\mathcal{L}_i$ about origin.

Moreover, the three common perpendiculars of the foregoing lines, $\{\mathcal{N}_i\}_1^3$, are given by the three unit dual vectors $\{\nu_i^*\}_1^3$, defined as

$$\nu_i^* \equiv \nu_i + \epsilon \nu_{0i}, \quad i = 1, 2, 3 \quad (2.52)$$

with ν_i and ν_{0i} representing, respectively, the direction and the moment vectors of line $\mathcal{N}_i$ about the origin.

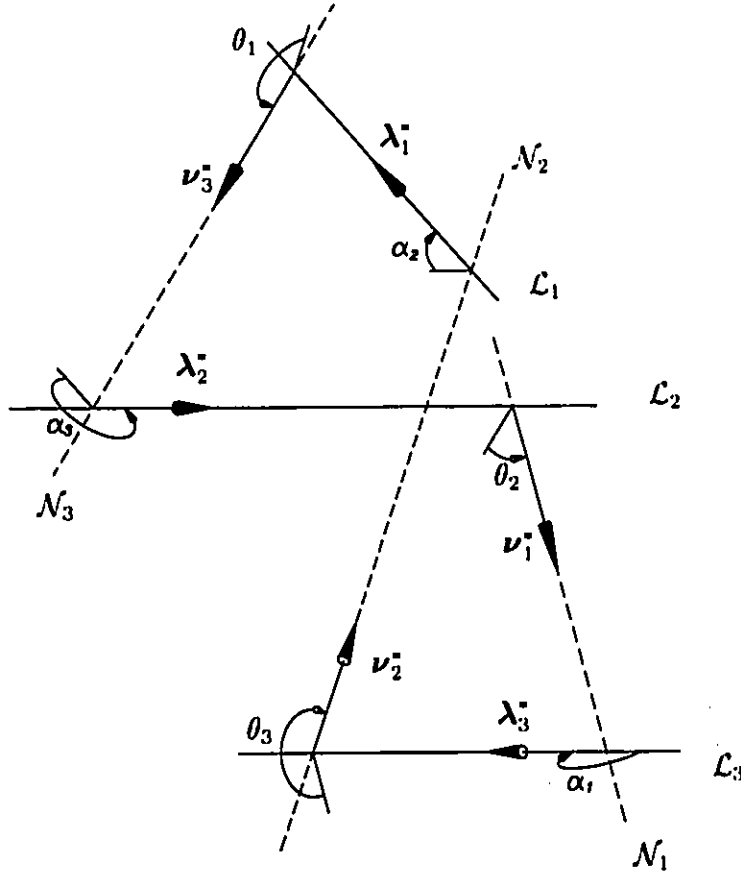


Figure 2.7: Spatial triangle

Similar to the planar and spherical trigonometries, one may define three *sides* of the spatial triangle as the associated dual angles, namely,

$$\hat{\alpha}_i = \alpha_i + \epsilon \nu_i, \quad i = 1, 2, 3 \quad (2.53)$$

where ν_i is the distance and α_i is the twist angle between $\mathcal{L}_{i+1}$ and $\mathcal{L}_{i-1}$, the sum and the difference in the subscripts throughout this thesis being understood as *modulo 3*.

The three *angles* of the triangle, similarly, are defined as

$$\hat{\theta}_i = \theta_i + \epsilon \lambda_i, \quad i = 1, 2, 3 \quad (2.54)$$

where λ_i is the distance and θ_i is the twist angle between $\mathcal{N}_{i+1}$ and $\mathcal{N}_{i-1}$, respectively.

2.3.2 Planar Triangle

If the three axes λ_1^* , λ_2^* and λ_3^* are parallel, the triangle reduces to a planar triangle, as shown in Fig. 2.8. Since the three lines are parallel, the twist angles between them, $\{\alpha_i\}_1^3$, of eq.(2.53), vanish, and the three sides of the triangle are represented by pure dual numbers, namely,

$$\hat{\alpha}_i = c\nu_i, \quad i = 1, 2, 3$$

With the three common perpendiculars represented by $\{\nu_i^*\}_1^3$ lying in the same

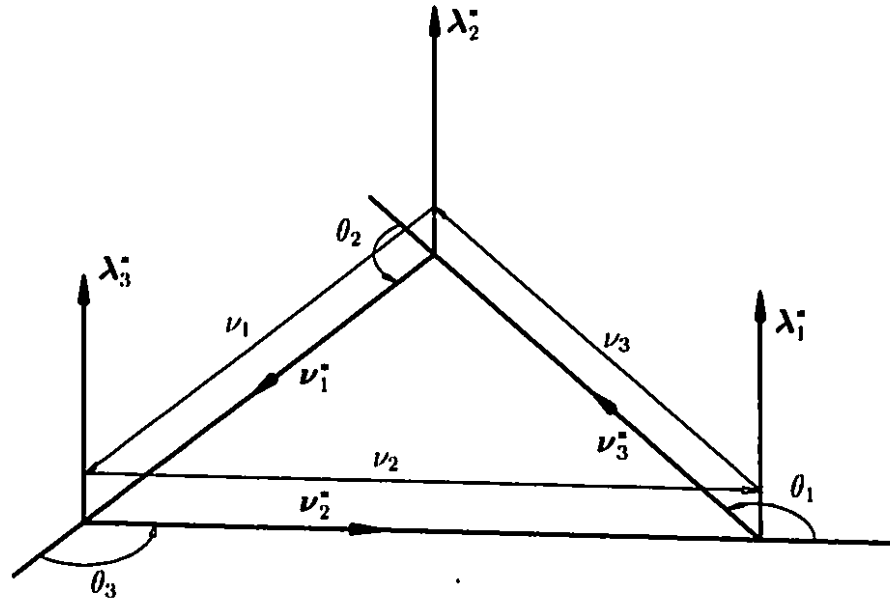


Figure 2.8: Planar triangle

plane, their common distances, $\{\lambda_i\}_1^3$, of eq.(2.54), vanish and the three angles of the triangle are given by real numbers, i.e.,

$$\hat{\theta}_i = \theta_i, \quad i = 1, 2, 3$$

2.3.3 Spherical Triangle

If the three axes λ_1^* , λ_2^* and λ_3^* intersect at a point, the triangle reduces to a spherical triangle, as shown in Fig. 2.9. Since the three lines intersect, the distances between

them, $\{\nu_i\}_1^3$, of eq.(2.53), vanish and the three sides of the triangle are represented by real numbers, namely,

$$\hat{\alpha}_i = \alpha_i, \quad i = 1, 2, 3$$

As the three lines intersect at a common point, the common perpendiculars intersect at the same point, as well. Then, the distances between them, $\{\lambda_i\}_1^3$ of eq.(2.54), vanish and the three angles of the triangle are given by

$$\hat{\theta}_i = \theta_i, \quad i = 1, 2, 3$$

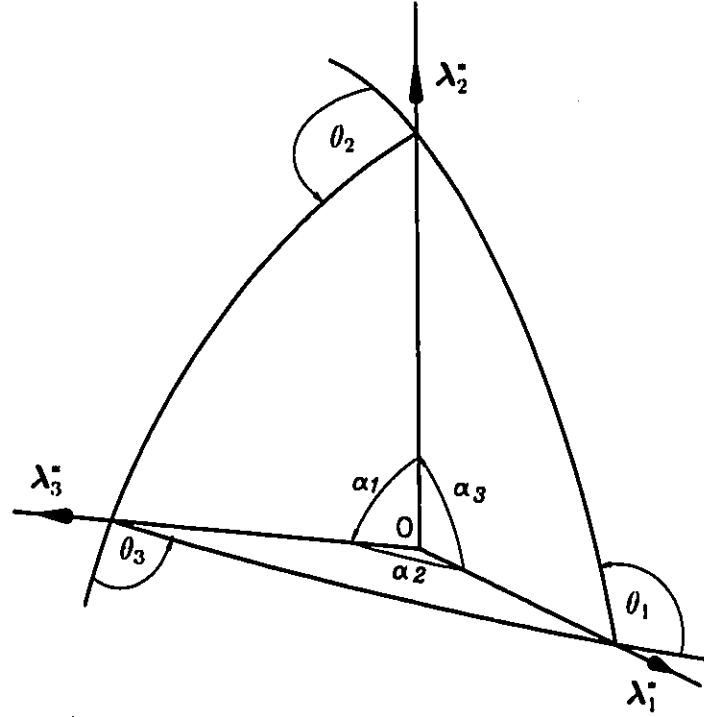


Figure 2.9: Spherical triangle

2.3.4 Trigonometric Identities

A unit dual quaternion is a screw operator that transforms a line into another line, as explained in Subsection 2.2.5. Then, the relationship between λ_1^* , λ_2^* and λ_3^* can

be expressed as

$$\lambda_1^* = \hat{\nu}_2^* \lambda_3^* \quad (2.55a)$$

$$\lambda_2^* = \hat{\nu}_3^* \lambda_1^* \quad (2.55b)$$

$$\lambda_3^* = \hat{\nu}_1^* \lambda_2^* \quad (2.55c)$$

where $\hat{\nu}_i^*$, for $i = 1, 2, 3$, is a unit dual quaternion, given as

$$\hat{\nu}_i^* = \cos \hat{\alpha}_i + \nu_i^* \sin \hat{\alpha}_i \quad (2.55d)$$

Substituting the value of λ_3^* from eq.(2.55c) into eq.(2.55a) yields

$$\lambda_1^* = \hat{\nu}_2^* \hat{\nu}_1^* \lambda_2^* \quad (2.56)$$

Moreover, substituting the value of λ_2^* from eq.(2.55b) into eq.(2.56), upon simplification, leads to

$$\hat{\nu}_2^* \hat{\nu}_1^* \hat{\nu}_3^* = 1 \quad (2.57)$$

The foregoing identity is called the *angular closure* equation for spatial triangles; it states that the three consecutive screw motions of λ_1^* , represented by $\hat{\nu}_2^*$, $\hat{\nu}_1^*$ and $\hat{\nu}_3^*$, transforms λ_1^* , via the intermediate poses λ_2^* and λ_3^* , back into itself.

Similarly, the *side closure* equation for spatial triangles transforms ν_1^* , via the intermediate poses ν_2^* and ν_3^* , back into itself, namely,

$$\hat{\lambda}_2^* \hat{\lambda}_1^* \hat{\lambda}_3^* = 1 \quad (2.58a)$$

where $\hat{\lambda}_i^*$, for $i = 1, 2, 3$, is a unit dual quaternion, given as

$$\hat{\lambda}_i^* = \cos \hat{\theta}_i + \lambda_i^* \sin \hat{\theta}_i \quad (2.58b)$$

One may conclude from eqs.(2.57) and (2.58a) the following spatial trigonometric identities (Yang, 1963):

Sine law:

$$\frac{\sin \hat{\theta}_1}{\sin \hat{\alpha}_1} = \frac{\sin \hat{\theta}_2}{\sin \hat{\alpha}_2} = \frac{\sin \hat{\theta}_3}{\sin \hat{\alpha}_3} \quad (2.59)$$

Cosine laws:

$$\cos \hat{\alpha}_1 = \cos \hat{\alpha}_2 \cos \hat{\alpha}_3 - \sin \hat{\alpha}_2 \sin \hat{\alpha}_3 \cos \hat{\theta}_1 \quad (2.60a)$$

$$\cos \hat{\theta}_3 = \cos \hat{\theta}_1 \cos \hat{\theta}_2 - \cos \hat{\alpha}_3 \sin \hat{\theta}_1 \sin \hat{\theta}_2 \quad (2.60b)$$

$$- \sin \hat{\alpha}_1 \cos \hat{\theta}_3 = \sin \hat{\alpha}_2 \cos \hat{\alpha}_3 + \cos \hat{\alpha}_2 \sin \hat{\alpha}_3 \cos \hat{\theta}_1 \quad (2.60c)$$

$$- \sin \hat{\alpha}_2 \sin \hat{\theta}_3 = \sin \hat{\theta}_1 \cos \hat{\theta}_2 + \cos \hat{\alpha}_3 \cos \hat{\theta}_1 \sin \hat{\theta}_2 \quad (2.60d)$$

The foregoing identities hold for spherical triangles. Indeed, if $(\hat{\cdot})$ is changed to $(\cdot)$, these identities become the sine and cosine laws of spherical triangles. Moreover, for planar triangles, the sine law and the cosine law reduce to the elementary sine and cosine laws of planar trigonometry.

Chapter 3

Parallel Manipulators

3.1 Introduction

Here, we introduce planar and spherical DT manipulators and, based on spatial trigonometry, which was introduced in Chapter 2, we generalize the concept of DT manipulators to that of spatial DT manipulators. Moreover, two general classes of planar manipulators are given, wherein the first class contains 20 manipulator types and the second contains four types. For the sake of completeness, the spherical 3-RRR manipulator is included as well.

3.2 Planar Manipulators

Planar tasks, whereby objects undergo two independent translations and one rotation about an axis perpendicular to the plane of the two translations, are common in manufacturing operations. These can be accomplished by planar parallel manipulators that consist of two rigid bodies connected to each other via several coplanar legs.

One of the general classes of planar parallel manipulators consists of two elements, namely the base ($\mathcal{P}$) and the moving ($\mathcal{Q}$) plates, connected by three legs, each with three degrees of freedom. These will be called three-legged (3L) manipulators. The

graph of such a manipulator is shown in Fig. 3.1 in which j_i , for $i = 1, 2, 3$, is a revolute (R) or a prismatic (P) pair.

The degree of freedom l of the manipulator is determined by means of the Chebyshev-Grübler-Kutzbach formula (Angeles, 1988), which for planar manipulators is given as

$$l = 3(n - 1) - 2p \quad (3.1)$$

where n and p are, respectively, the number of links and the number of R or P pairs.

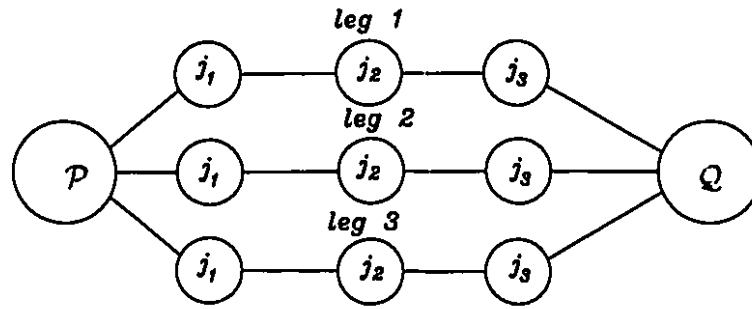


Figure 3.1: The graph of a general 3-dof, 3L parallel manipulator

For the manipulator of Fig. 3.1, we have $n = 8$ and $p = 9$, and hence, the dof of the manipulator is

$$l = 3 \times 7 - 2 \times 9 = 3$$

We can build several 3-dof manipulators with three legs, each leg containing three elementary pairs. These legs are PRR, PRP, PPR, RRR, RRP, RPR and RPP. Since we can choose to actuate any one of the three joints of the legs, we have $3 \times 7 = 21$ different legs and actuation modes. It is convenient, however, to actuate the joints attached to $\mathcal{P}$, in order to have stationary motors. This limits the choice to seven types of leg and actuation architectures. Moreover, we cannot have a 3-dof, 3L manipulator if more than one leg is of the RPP type. Therefore, this type of leg is left aside.

Let us classify the remaining legs into two categories, based on their third joints. Those legs attached to Q with a revolute joint form the first category, i.e., PRR, PPR, RRR and RPR. The 3L manipulators constructed with these legs are called manipulators of *class A*. Moreover, those legs attached to Q with a prismatic joint form the legs of another 3L parallel manipulator class that will be called *class B*. The joint sequences for the legs are PRP and RRP.

3.2.1 3-DOF Manipulators of Class A

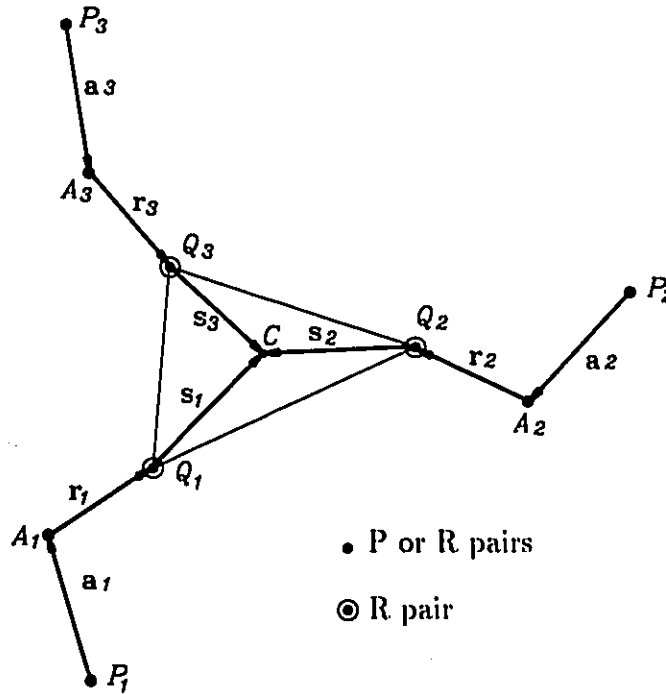
As explained above, the class-A manipulator has three legs and can be built with any three combinations of four types of legs, namely, PRR, PPR, RRR and RPR. Then, the number of manipulators in this class is given as

$$n = \sum_{i=1}^4 (4 - i + 1)i = 20$$

The geometric model of all of the foregoing manipulators is depicted as in Fig. 3.2, in which P_i represents the i th motor and C is the *operation point* of Q . Moreover, joint Q_i is a revolute, while joints P_i and A_i can be either revolute or prismatic. The axes of all revolute joints are perpendicular to the plane of motion, while the axes of prismatic joints lie in the plane. If P_i is a prismatic joint, its axis is given by a vector $\mathbf{a}_i$ directed from P_i to A_i . Similarly, if A_i is a prismatic joint, its axis is given by a vector $\mathbf{r}_i$ directed from A_i to Q_i .

Example 3.2.1.1: Planar 3-RRR Manipulator

An example of the manipulator of class A is the planar 3-RRR manipulator, which has been the subject of extensive research (Hunt, 1983; Mohamed, 1983; Gosselin, 1988; Gosselin and Angeles, 1989; Gosselin and Sefrioui, 1991). This manipulator is constructed with two bodies, P and Q , connected to each other via three RRR legs, as depicted in Fig. 3.3. Moreover, all three motors P_i are fixed to the ground.

Figure 3.2: The manipulator of class $\mathcal{A}$

3.2.2 3-DOF Manipulators of Class $\mathcal{B}$

The manipulators of this class have two bodies connected to each other via any three combinations of two types of legs, namely, PRP and RRP. Then, the number of manipulators in this class is given as

$$n = \sum_{i=1}^2 (2 - i + 1)i = 4$$

The geometric model of all of the foregoing manipulators is depicted as in Fig. 3.4, in which P_i represents the i th motor and C is the *operation point* of Q . Moreover, joints A_i and R_i are revolute and prismatic, respectively, while joint P_i can be either revolute or prismatic. The axes of all revolute joints are perpendicular to the plane of motion, while the axes of prismatic joints lie in the plane. If P_i is a prismatic joint, its axis is given by a unit vector $\mathbf{a}_i$ directed from P_i to A_i . Furthermore, the axis of joint R_i is given by the unit vector $\mathbf{b}_i$.

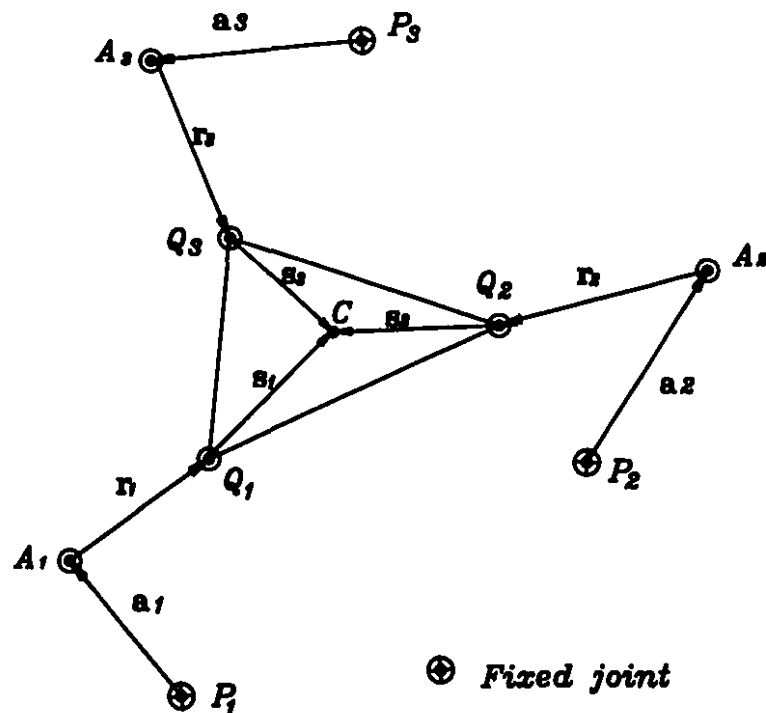


Figure 3.3: Planar 3-dof, 3-RRR parallel manipulator

Example 3.2.2.1: Planar DT Manipulator

A special type of class-B manipulator, which has three PRP legs, is the 3-dof planar DT manipulator. A sketch of the kinematic chain and a typical design of this device is depicted in Fig. 3.5. The manipulator consists of two rigid planar triangles connected to each other via three PRP legs. Moreover, the leg lengths are virtually zero, which enhances the structural stiffness of this manipulator. One of these triangles is fixed and is, thus, termed the *fixed triangle*; the other moves with respect to the fixed one and thus is called the *movable triangle*. Furthermore, the movable triangle is displaced by actuating three prismatic joints along the sides of the fixed triangle, denoted by three unit vectors, $\{\mathbf{a}_i\}_1^3$.

This manipulator is novel and offers some possibility for innovation. For example, augmented with a fourth axis, to allow translation in a direction perpendicular to the plane of the first 3-dof motion, the device can perform the motions of what are known as SCARA robots.

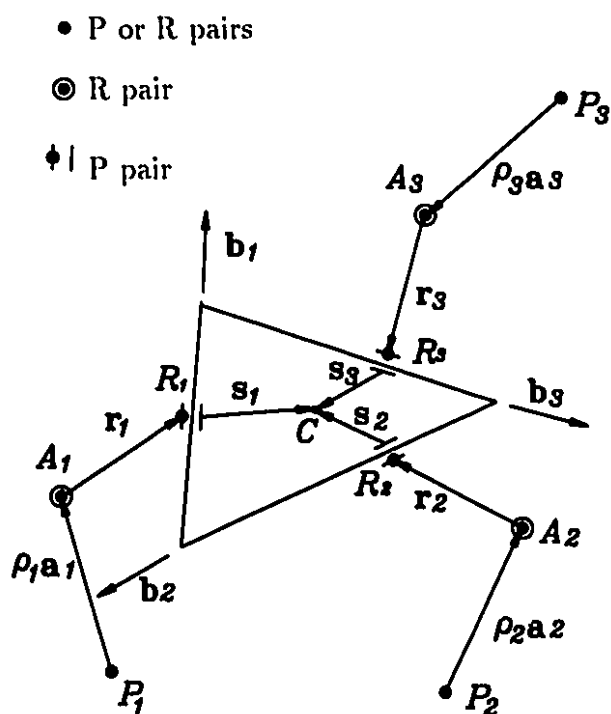
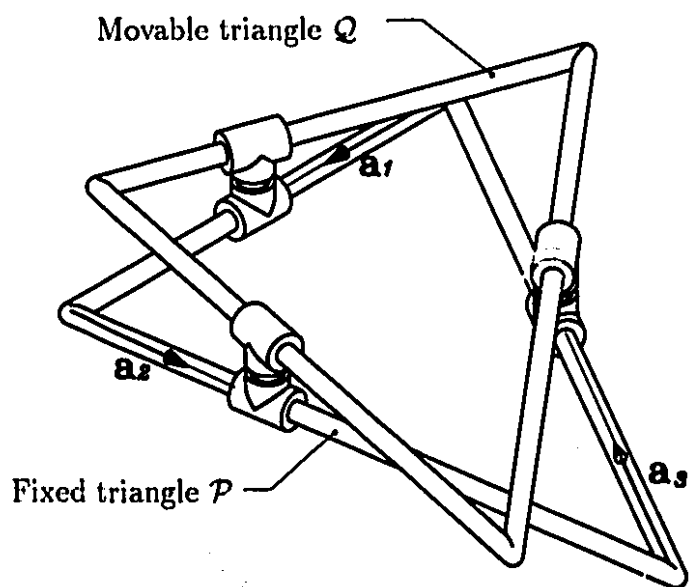
Figure 3.4: The manipulator of class $\mathcal{B}$ 

Figure 3.5: Planar 3-dof, DT manipulator

3.3 Spherical Manipulators

Spherical motion allows the arbitrary orientation of workpieces in 3D space. Here, two types of spherical parallel manipulators, namely, the spherical 3-RRR and the spherical DT manipulators, are included. Both of these devices consist of two bodies connected by three 3-dof legs, the graph of such a device being that of Fig. 3.1. Moreover, the dof of spherical manipulators is determined by means of the Chebyshev-Grübler-Kutzbach formula, as given in eq.(3.1). Here we have $n = 8$, $p = 9$ and, again, the dof, l , of the device is

$$l = 3(8 - 1) - 2 \times 9 = 3$$

3.3.1 Spherical 3-DOF, 3-RRR Manipulator

This manipulator consists of two bodies connected by three RRR legs, as shown in Fig. 3.6. Moreover, three actuators are attached to the base and rotate the links connected to the base about $\{u_i\}_1^3$. Similar to its planar counterpart, this manipulator is well documented in the research literature (Cox and Tesar, 1989; Craver, 1989; Gosselin and Angeles, 1989; Gosselin et al., 1994a, 1994b; Gosselin and Lavoie, 1993).

3.3.2 Spherical 3-DOF, DT Manipulator

The spherical 3-dof, DT manipulator consists of two spherical triangles connected by three legs. Similar to its planar counterpart, its leg lengths are virtually zero, which makes it particularly stiff. One of these triangles is fixed, and is thus termed the fixed triangle; the other moves with respect to the fixed one, and thus is called the movable triangle. Moreover, the movable triangle is driven by three actuators placed along the sides of the fixed triangle, with actuated-joint variables $\{\mu_i\}_1^3$.

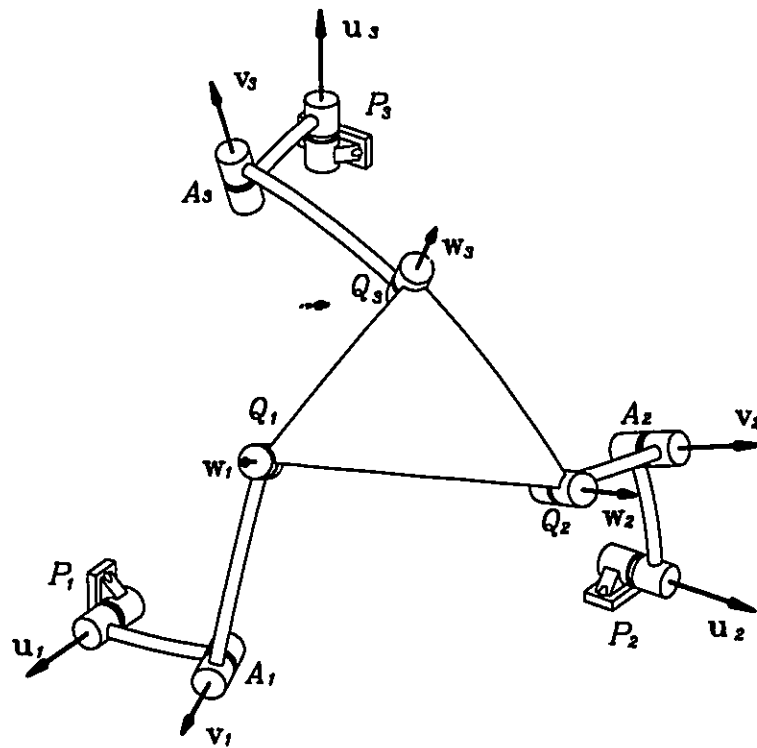


Figure 3.6: The spherical 3-dof, 3-RRR manipulator

3.4 Spatial DT Manipulators

Now, by way of generalization, the spatial DT manipulator can be introduced. A spatial DT manipulator consists of two spatial triangles connected by three multi-dof legs. Similar to its planar and spherical counterparts, one of these triangles is fixed, and thus termed the fixed triangle, the other moving with respect to the fixed one, and thus termed the movable triangle. Several versions of the spatial DT manipulator are possible, based on the topology of the connecting legs and the actuated joints, as discussed below.

3.4.1 Spatial 6-DOF, DT Manipulator

Consider two spatial triangles, $\mathcal{P}$ and $\mathcal{Q}$, with $\mathcal{P}$ connected to $\mathcal{Q}$ via three 6-dof $PRRPRP$ legs. The graph of such an array is shown in Fig. 3.8.

The degree of freedom l of the foregoing array is determined by means of the

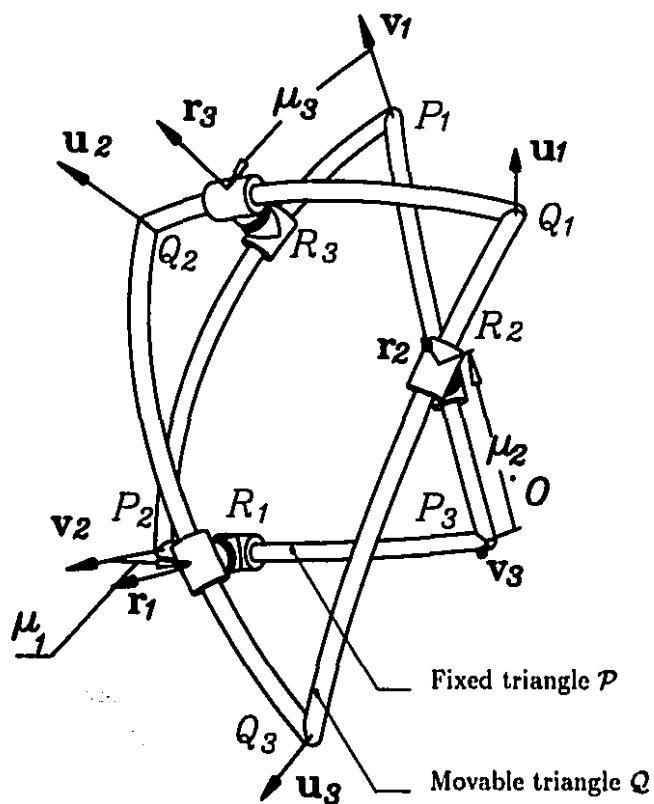


Figure 3.7: The spherical 3-dof, DT manipulator

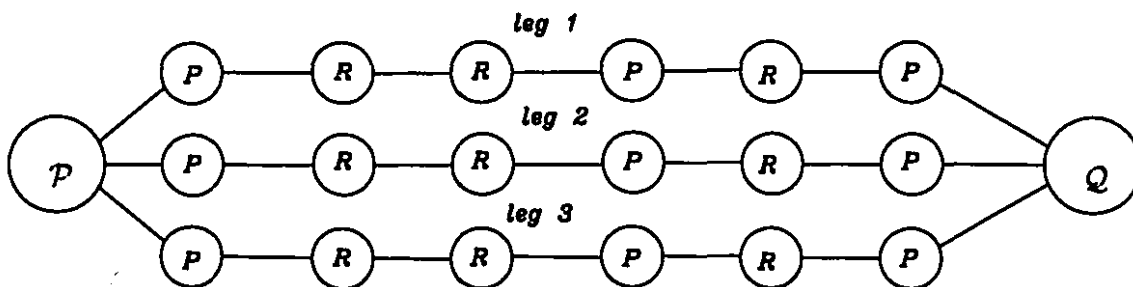


Figure 3.8: The graph of a general 6-dof, DT array

Chebyshev-Grübler-Kutzbach formula for spatial mechanisms (Angeles, 1988), namely,

$$l = 6(n - 1) - 5p \quad (3.2)$$

where n and p are, respectively, the numbers of links and of elementary R or P pairs. For this array, we have $n = 17$ and $p = 18$, and hence, the dof is

$$l = 6 \times 16 - 5 \times 18 = 6 \quad (3.3)$$

Triangle $\mathcal{P}$ is designated the fixed triangle, while $\mathcal{Q}$ is the movable triangle. Moreover, the previous array is, in fact, the graph of the manipulator depicted in Fig. 3.9. The manipulator has six degrees of freedom, but only three legs. Then, the degree of parallelism dop , based on eq.(1.1), is equal to 0.5, which means that we should actuate two joints per leg. If one chooses to actuate the first two joints in each leg, the manipulator is the general DT manipulator, of which the planar and spherical DT manipulators are special cases. Some alternative designs of this manipulator are given in the subsection below.

3.4.2 Other Versions of Spatial DT Manipulators

As explained in the previous subsection, one can choose alternative sets of actuating joints for the 6-dof, DT manipulator. One practical alternative would be to actuate the first two prismatic joints of each leg, instead of actuating the first two joints.

Moreover, we can also have a 3-dof spatial DT manipulator. The structure of this device is similar to that of its 6-dof counterpart, except that we omit the intermediate prismatic joint of each leg. The graph of such a manipulator is depicted in Fig. 3.10. The interesting feature of this manipulator is that we can make the distance between the two prismatic joints of each leg as small as possible. In this way, we can get rid of long legs, which are a major source of structural flexibility.

For this manipulator, we have a number of links $n = 14$ and the number of elementary joints $p = 15$. The dof of the manipulator, thus, is obtained by substituting

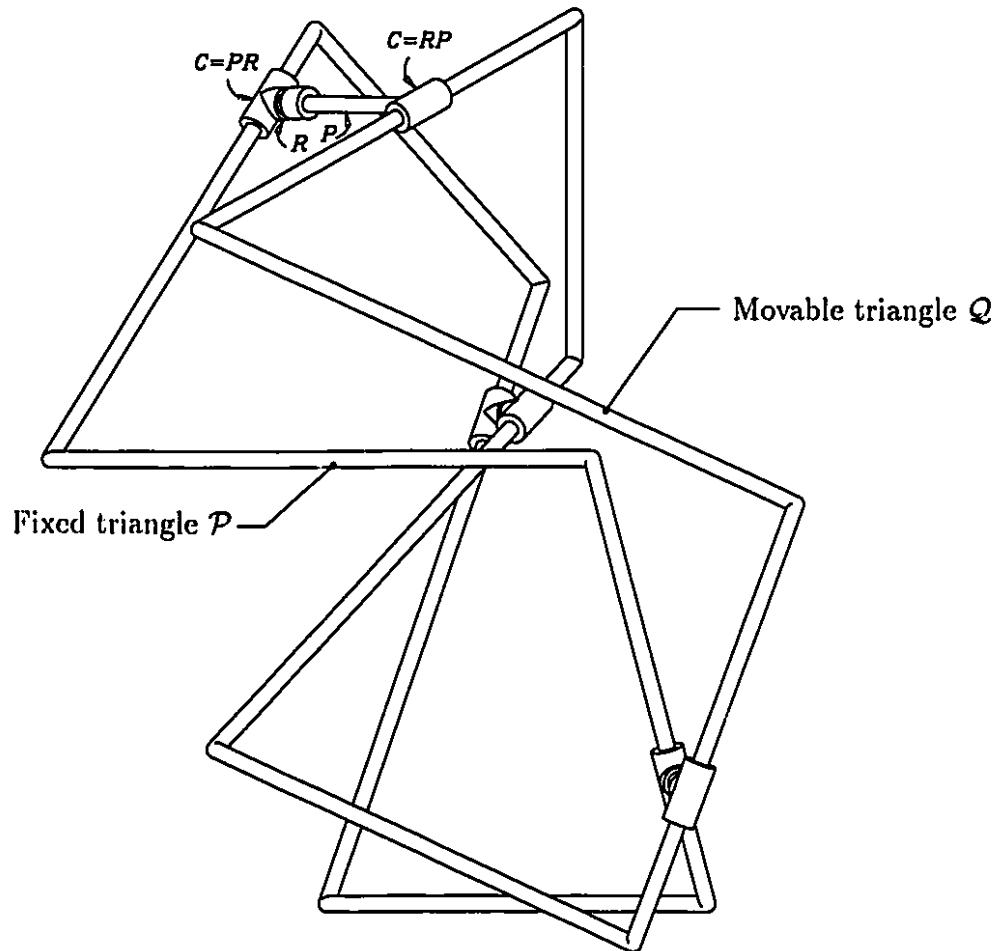


Figure 3.9: Spatial DT manipulator

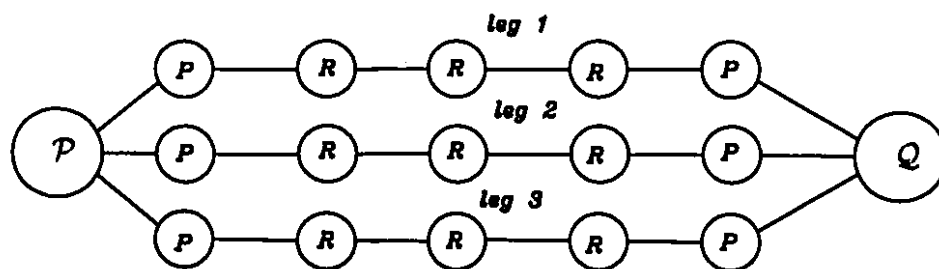


Figure 3.10: The graph of a general 3-dof, DT array

these values into eq.(3.2), namely,

$$l = 6 \times 13 - 5 \times 15 = 3 \quad (3.4)$$

Therefore, the degree of parallelism dop , based on eq.(1.1), is equal to one, which means that one can actuate one joint per leg to move triangle Q . This motion can be produced by actuating the first prismatic joints of the legs, so that all the motors remain conveniently fixed to the ground.

Chapter 4

Direct Kinematics

4.1 Introduction

The direct kinematics (DK) of the manipulators introduced in Chapter 3 is the subject of study of this chapter. The DK problem leads to a quadratic equation for the planar DT manipulator and to a polynomial of 16th degree for the spherical DT manipulator. Moreover, the DK of all versions of the spatial DT manipulators are formulated. For the sake of completeness, the solution of the DK problem for planar and spherical 3-RRR manipulators are included.

4.2 Planar Manipulators

4.2.1 Planar 3-DOF, 3-RRR Manipulator

The DK problem of the planar 3-RRR manipulator, introduced in Example 3.2.1.1, is the subject of this subsection. This problem is well represented in the literature. Hunt (1983) showed that the problem has at most six real solutions, but he failed to find the underlying polynomial. Merlet (1989) found a polynomial of degree 12 for the problem, which is not minimal. Recently, Gosselin and Sefrioui (1991) derived the minimal 6th-degree polynomial and gave an example having six real solutions.

4.2.2 Planar 3-DOF, DT Manipulator

The DK of the manipulator depicted in Fig. 3.5 is the subject of this subsection. The geometric model of the manipulator is shown in Fig. 4.1. It consists of two triangles, the fixed triangle $\mathcal{P}$ and the movable triangle $\mathcal{Q}$, with vertices $P_1P_2P_3$ and $Q_1Q_2Q_3$, respectively. Triangle $\mathcal{Q}$ can move on triangle $\mathcal{P}$ such that P_2P_3 intersects Q_2Q_3 at point R_1 , P_3P_1 intersects Q_3Q_1 at R_2 and P_1P_2 intersects Q_1Q_2 at R_3 . Moreover, R_i , for $i = 1, 2, 3$, cannot lie outside its corresponding vertices. Thus, feasible or admissible motions maintain R_i within edges $Q_{i+1}Q_{i-1}$ and $P_{i+1}P_{i-1}$, for $i = 1, 2, 3$.

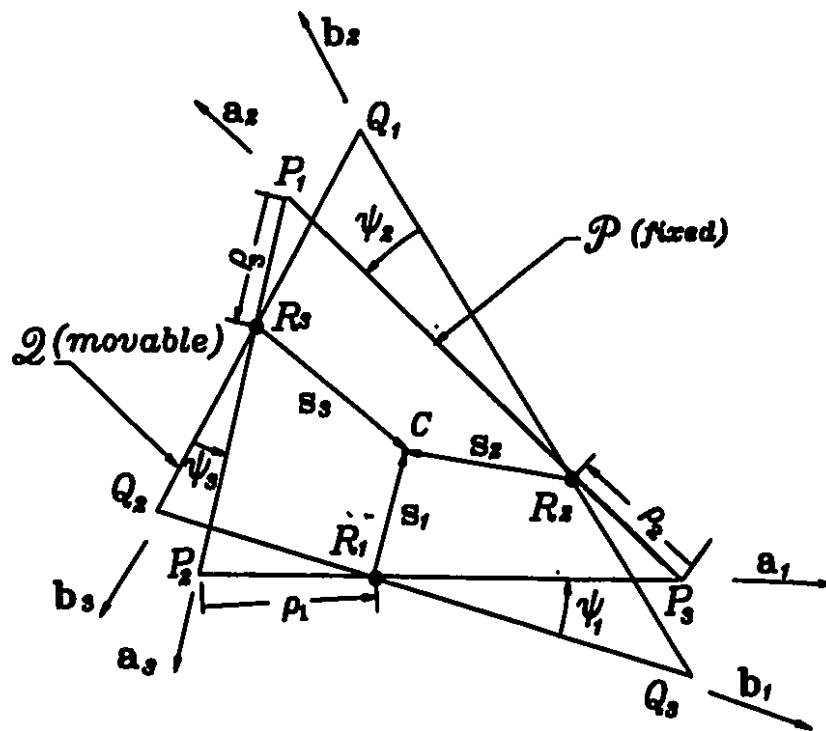


Figure 4.1: Geometric model of the planar 3-dof, DT manipulator

The motion of triangle $\mathcal{Q}$ can thus be described through changes in the edge length parameters, ρ_i , which locate R_i along a side of $\mathcal{P}$, measured from P_{i+1} , for $i = 1, 2, 3$. The non-negative displacements ρ_i are assumed to be produced by actuators, and hence, they are termed the *actuator coordinates*. The coordinates of the moving triangle $\mathcal{Q}$, in turn, are the set of variables used to define its pose. Note that the Cartesian coordinates of the three vertices of $\mathcal{Q}$ can be used to define this pose.

The problem may be formulated as: *Given the actuator coordinates ρ_i , for $i = 1, 2, 3$, find the Cartesian coordinates of the vertices of triangle Q .*

We solve this problem by *kinematic inversion*, i.e., by fixing the movable triangle Q and letting the fixed triangle P to accommodate itself to the constraints imposed. To this end, we define points R_i at given distances ρ_i , for $i = 1, 2, 3$, on the edges of P , thereby defining a triangle $R_1R_2R_3$, henceforth termed triangle $\mathcal{R}$, that is fixed to P . Next, we let d, e and f be the lengths of the sides of this triangle. The problem now consists of finding the set of all possible positions of triangle $\mathcal{R}$ for which vertex R_i lies within the side $Q_{i+1}Q_{i-1}$, for $i = 1, 2, 3$, as shown in Fig. 4.2. By carrying $\mathcal{R}$ back into its fixed configuration, while attaching Q rigidly to it, we determine the set of possible configurations of the movable triangle for the given values of actuator coordinates.

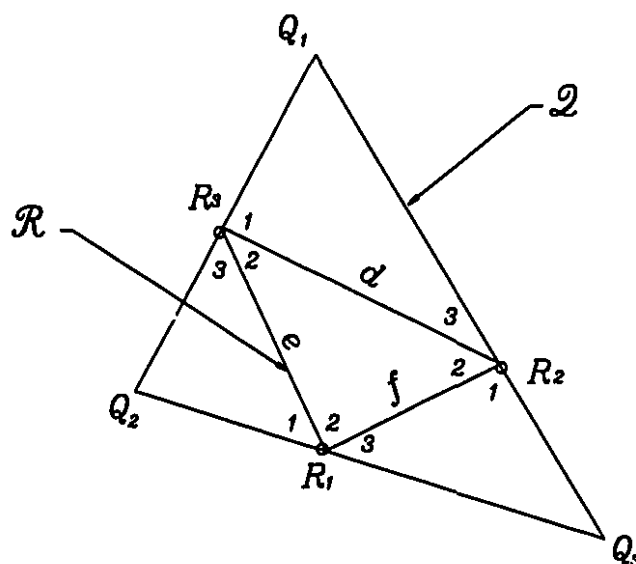


Figure 4.2: Triangles Q and $\mathcal{R}$

In Fig. 4.2 we note that each vertex R_i is common to three angles labeled 1, 2 and 3. We will denote these angles by a subscripted capital letter. The subscript indicates one of the three angles common to that vertex, while the capital letter corresponds to the lower-case label of the opposite side of the triangle $R_1R_2R_3$. We thus have at vertices R_1 , R_2 and R_3 the angles D_i , E_i and F_i , for $i = 1, 2, 3$.

Considering triangle $Q_1R_3R_2$, the law of sines for triangles yields

$$\overline{Q_1R_2} = a_1 \sin(F_1) \quad (4.1)$$

where

$$a_1 = \frac{d}{\sin(Q_1)}$$

Similarly, for triangle $Q_3R_2R_1$ we have

$$\overline{Q_3R_2} = a_2 \sin(D_3) \quad (4.2)$$

where

$$a_2 = \frac{f}{\sin(Q_3)}$$

Adding sidewise eq.(4.1) to eq.(4.2) gives

$$a_1 \sin(F_1) + a_2 \sin(D_3) = b \quad (4.3)$$

where

$$b = \overline{Q_1Q_3}$$

From triangle $Q_2R_1R_3$, we have

$$D_1 = \pi - F_3 - Q_2 \quad (4.4)$$

But

$$F_3 = \pi - F_1 - F_2 \quad (4.5)$$

Substitution of F_3 from eq.(4.5) into eq.(4.4) yields

$$D_1 = F_1 + F_2 - Q_2 \quad (4.6)$$

Again, we have

$$D_3 = \pi - D_1 - D_2 \quad (4.7)$$

Substitution of D_1 from eq.(4.6) into eq.(4.7) yields, in turn,

$$D_3 = G - F_1 \quad (4.8)$$

where

$$G = \pi - D_2 - F_2 + Q_2$$

Substituting the expression for $\sin(D_3)$ from eq.(4.8) into eq.(4.3), we obtain

$$b_1 \sin(F_1) + b_2 \cos(F_1) = b \quad (4.9)$$

with b_1 and b_2 defined as

$$b_1 = a_1 - a_2 \cos(G), \quad b_2 = a_2 \sin(G)$$

In eq.(4.9), we substitute now the equivalent expressions for cosines and sines given below:

$$\cos(F_1) = \frac{1 - x^2}{1 + x^2}, \quad \sin(F_1) = \frac{2x}{1 + x^2}$$

where x is the tangent of one half of the angle F_1 .

Upon simplification, eq.(4.9) leads to

$$c_1 x^2 + c_2 x + c_3 = 0 \quad (4.10)$$

with c_1 , c_2 and c_3 defined as

$$c_1 = -b_2 - b, \quad c_2 = 2b_1, \quad c_3 = b_2 - b$$

Solving eq.(4.10) for x gives

$$x = \frac{-b_1 \pm \sqrt{b_1^2 + b_2^2 - b^2}}{-(b_2 + b)} \quad (4.11)$$

The above expression thus leading to the result below:

Theorem 1: *Given two triangles $\mathcal{R}$ and $\mathcal{Q}$, we can inscribe $\mathcal{R}$ in $\mathcal{Q}$ in at most two poses such that vertex R_i is located on the edges $Q_{i+1}Q_{i-1}$ of triangle $\mathcal{Q}$, for $i = 1, 2, 3$.*

Example 4.2.2.1:

Consider the following sides assigned to the triangles $\mathcal{P}$ and $\mathcal{Q}$:

$$Q_1Q_2 = 0.4 \text{ m}, \quad Q_2Q_3 = 0.5 \text{ m}, \quad Q_3Q_1 = 0.6 \text{ m}$$

$$P_1P_2 = 0.29065 \text{ m}, \quad P_2P_3 = 0.5 \text{ m}, \quad P_3P_1 = 0.47875 \text{ m}$$

Choose three points, R_1 , R_2 and R_3 , located by three actuator coordinates specified as $\rho_1 = 0.2 \text{ m}$, $\rho_2 = 0.14161 \text{ m}$ and $\rho_3 = 0.03064 \text{ m}$. These values produce the lengths d , e and f given below:

$$d = 0.33166 \text{ m}, \quad e = 0.26458 \text{ m}, \quad f = 0.2 \text{ m}$$

The two roots of eq.(4.11) are:

$$x_1 = 1.0788, \quad x_2 = 0.4447$$

i.e., $(F_1)_1 = 94.34^\circ$, $(F_1)_2 = 48^\circ$. Equations (4.1–4.8) are used to compute the other parameters, which leads to two poses of the triangle, Fig. 4.3. The two triangles Q and Q' represent the two solutions that correspond to the assembly modes of the manipulator.

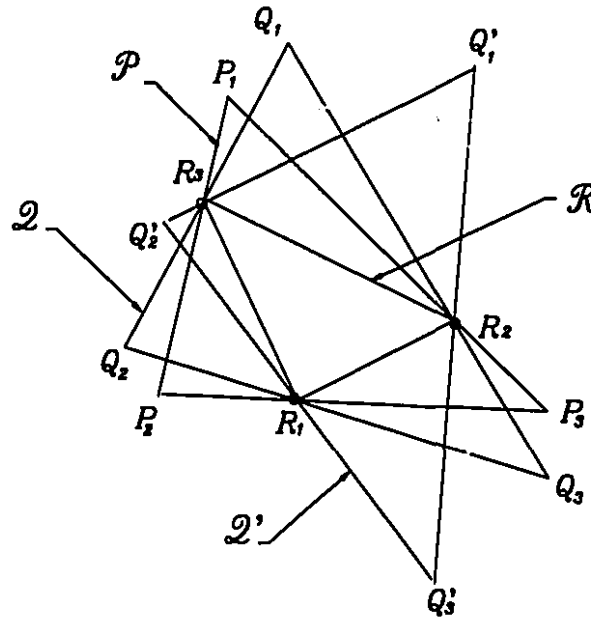


Figure 4.3: Triangles Q , Q' , P and R

4.3 Spherical Manipulators

4.3.1 Spherical 3-DOF, 3-RRR Manipulator

The solution of the DK problem of the manipulator of Fig. 3.6 can be found in the literature. Gosselin et al. (1994a, 1994b) derived a polynomial of eighth degree and gave an example having eight real solutions, the polynomial thus being minimal.

4.3.2 Spherical 3-DOF, DT Manipulator

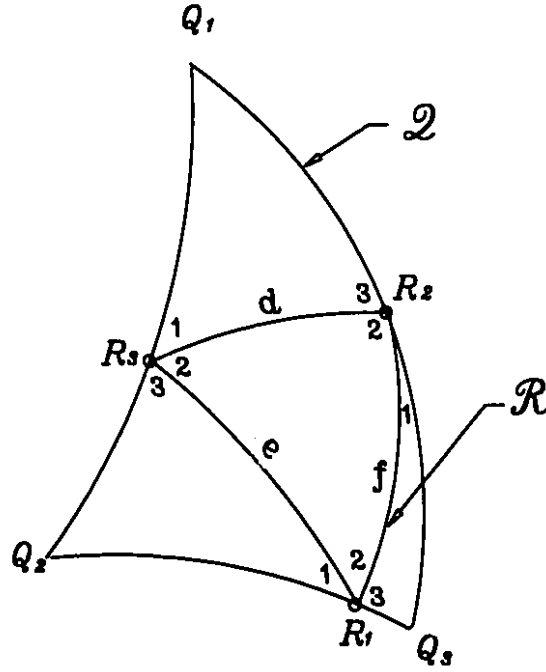
The DK of the manipulator depicted in Fig. 3.7 is the subject of this subsection. The manipulator consists of two triangles, the fixed triangle $\mathcal{P}$ and the movable triangle $\mathcal{Q}$, with vertices $P_1 P_2 P_3$ and $Q_1 Q_2 Q_3$, respectively. Moreover, the side $P_2 P_3$ of $\mathcal{P}$ intersects the arc $Q_2 Q_3$ of $\mathcal{Q}$ at point R_1 . We denote by R_2 and R_3 the other intersection points, that are defined correspondingly. Moreover R_i , for $i = 1, 2, 3$, cannot lie outside its corresponding vertices. Thus, feasible or admissible motions maintain R_i within edges $Q_{i+1} Q_{i-1}$ and $P_{i+1} P_{i-1}$, for $i = 1, 2, 3$.

Thus, the motion of triangle $\mathcal{Q}$ can be described through the arc lengths μ_i of Fig. 3.7, or *actuator coordinates*, for $i = 1, 2, 3$. Likewise, the Cartesian coordinates of the moving triangle $\mathcal{Q}$ are the set of variables defining its orientation. Note that the Cartesian coordinates of the three vertices of $\mathcal{Q}$ can be determined once its orientation is given.

Similar to the direct kinematics of the planar DT manipulator, the same problem, as pertains to the spherical manipulator, may be formulated as: *Given the actuator coordinates μ_i , for $i = 1, 2, 3$, find the Cartesian coordinates of the vertices of triangle $\mathcal{Q}$.*

Again, we solve this problem by *kinematic inversion*, i.e., by fixing the movable triangle $\mathcal{Q}$ and letting the fixed triangle $\mathcal{P}$ accommodate itself to the constraints imposed. To this end, we define points R_i at given arc lengths μ_i , for $i = 1, 2, 3$, on the edges of $\mathcal{P}$, thereby defining a triangle $R_1 R_2 R_3$, henceforth termed triangle $\mathcal{R}$, that is fixed to $\mathcal{P}$. Next, we let d, e and f be the sides of this triangle. The problem now consists of finding the set of all possible orientations of triangle $\mathcal{R}$ for which vertex R_i lies within the side $Q_{i+1} Q_{i-1}$, for $i = 1, 2, 3$, as shown in Fig. 4.4. By carrying $\mathcal{R}$ back into its fixed configuration, while attaching $\mathcal{Q}$ rigidly to it, we determine the set of possible configurations of the movable triangle for the given values of actuator coordinates.

In Fig. 4.4 we note that each vertex R_i is common to the three spherical angles

Figure 4.4: Spherical triangles $\mathcal{Q}$ and $\mathcal{R}$

labelled with numbers 1, 2 and 3. Similar to the planar mechanism, we label them D_i , E_i and F_i , for $i = 1, 2, 3$.

We introduce now the definitions below:

$$s \equiv \frac{d + e + f}{2} \quad (4.12a)$$

$$k \equiv \sqrt{\frac{\sin(s - d) \sin(s - e) \sin(s - f)}{\sin(s)}} \quad (4.12b)$$

From spherical trigonometry we have

$$D_2 = 2 \arctan\left(\frac{k}{\sin(s - d)}\right) \quad (4.13a)$$

$$E_2 = 2 \arctan\left(\frac{k}{\sin(s - e)}\right) \quad (4.13b)$$

$$F_2 = 2 \arctan\left(\frac{k}{\sin(s - f)}\right) \quad (4.13c)$$

Consider now the spherical triangle $Q_1 R_3 R_2$. Using the law of cosines for spherical

triangles, we have

$$\cos Q_1 = -\cos F_1 \cos E_3 + \sin F_1 \sin E_3 \cos d \quad (4.14a)$$

Similarly, for the spherical triangles $Q_2 R_1 R_3$ and $Q_3 R_2 R_1$ we have

$$\cos Q_2 = -\cos D_1 \cos F_3 + \sin D_1 \sin F_3 \cos e \quad (4.14b)$$

$$\cos Q_3 = -\cos E_1 \cos D_3 + \sin E_1 \sin D_3 \cos f \quad (4.14c)$$

However,

$$E_3 = \pi - E_1 - E_2 \quad (4.15a)$$

$$F_3 = \pi - F_1 - F_2 \quad (4.15b)$$

$$D_3 = \pi - D_1 - D_2 \quad (4.15c)$$

Substitution of the expressions for $\cos E_3$ and $\sin E_3$ from eq.(4.15a) into eq.(4.14a), we obtain

$$a_{11}c_1c_2 + a_{12}c_1s_2 + a_{13}s_1s_2 + a_{14}s_1c_2 + a_{15} = 0 \quad (4.16a)$$

where

$$\begin{aligned} a_{11} &= \cos E_2 & a_{12} &= -\sin E_2 \\ a_{13} &= \cos d \cos E_2 & a_{14} &= \cos d \sin E_2 \\ a_{15} &= -\cos Q_1 & c_1 &= \cos F_1 \\ s_1 &= \sin F_1 & c_2 &= \cos E_1 \\ s_2 &= \sin E_1 \end{aligned}$$

Similarly, substitution of eq.(4.15b) into eq.(4.14b) yields:

$$a_{21}c_3c_1 + a_{22}c_3s_1 + a_{23}s_3s_1 + a_{24}s_3c_1 + a_{25} = 0 \quad (4.16b)$$

where

$$\begin{aligned}
 a_{21} &= \cos F_2 & a_{22} &= -\sin F_2 \\
 a_{23} &= \cos e \cos F_2 & a_{24} &= \cos e \sin F_2 \\
 a_{25} &= -\cos Q_2 & c_3 &= \cos D_1 \\
 s_3 &= \sin D_1
 \end{aligned}$$

Likewise, substitution of eq.(4.15c) into eq.(4.14c) yields:

$$a_{31}c_2c_3 + a_{32}c_2s_3 + a_{33}s_2s_3 + a_{34}s_2c_3 + a_{35} = 0 \quad (4.16c)$$

where

$$\begin{aligned}
 a_{31} &= \cos D_2 & a_{32} &= -\sin D_2 \\
 a_{33} &= \cos f \cos D_2 & a_{34} &= \cos f \sin D_2 \\
 a_{35} &= -\cos Q_3
 \end{aligned}$$

Equations (4.16a-c) must be solved simultaneously to determine the values of angles D_1 , E_1 and F_1 . In the above equations, we substitute now the equivalent expressions for cosines and sines given below:

$$c_i = \frac{1 - x_i^2}{1 + x_i^2}, \quad s_i = \frac{2x_i}{1 + x_i^2}$$

where x_i , for $i = 1, 2, 3$, are the tangents of one half of the angles F_1 , E_1 and D_1 , respectively.

Upon simplification, eqs.(4.16a-c) lead to these trivariate polynomial equations in x_1 , x_2 and x_3 , namely,

$$d_1 x_2^2 + d_2 x_2 + d_3 = 0 \quad (4.17a)$$

$$d_4 x_2^2 + d_5 x_2 + d_6 = 0 \quad (4.17b)$$

$$d_7 x_3^2 + d_8 x_3 + d_9 = 0 \quad (4.17c)$$

where

$$d_1 = (a_{11} + a_{15})x_1^2 - 2a_{14}x_1 + (a_{15} - a_{11})$$

$$d_2 = -2a_{12}x_1^2 + 4a_{13}x_1 + 2a_{12}$$

$$d_3 = (a_{15} - a_{11})x_1^2 + 2a_{14}x_1 + (a_{15} - a_{11})$$

$$d_4 = (a_{31} + a_{35})x_3^2 - 2a_{34}x_3 + (a_{35} - a_{31})$$

$$d_5 = -2a_{32}x_3^2 + 4a_{33}x_3 + 2a_{32}$$

$$d_6 = (a_{35} - a_{31})x_3^2 + 2a_{34}x_3 + (a_{35} - a_{31})$$

$$d_7 = (a_{21} + a_{25})x_1^2 - 2a_{24}x_1 + (a_{25} - a_{21})$$

$$d_8 = -2a_{22}x_1^2 + 4a_{23}x_1 + 2a_{22}$$

$$d_9 = (a_{25} - a_{21})x_1^2 + 2a_{24}x_1 + (a_{25} - a_{21})$$

We now eliminate x_2 from eqs.(4.17a) and (4.17b), using Bezout's method (Salmon, 1964). A short account of this method is given in Appendix A. The resulting equation thus contains only x_1 and x_3 , namely,

$$\det \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{11} \end{bmatrix} = 0 \quad (4.18)$$

where quantities Δ_{11} , Δ_{12} and Δ_{21} are defined below:

$$\Delta_{11} \equiv \det \begin{bmatrix} d_1 & d_3 \\ d_4 & d_6 \end{bmatrix}, \quad \Delta_{12} \equiv \det \begin{bmatrix} d_5 & d_2 \\ d_4 & d_1 \end{bmatrix}, \quad \Delta_{21} \equiv \det \begin{bmatrix} d_2 & d_3 \\ d_5 & d_6 \end{bmatrix}$$

After expansion and simplification, eq.(4.18) reduces to

$$A_1 x_3^4 + A_2 x_3^3 + A_3 x_3^2 + A_4 x_3 + A_5 = 0 \quad (4.19a)$$

where

$$A_i = \sum_{p=0}^4 A_{ip} x_1^p, \quad i = 1, \dots, 5 \quad (4.19b)$$

and the coefficients A_{ip} , given in Appendix B, depend only on the data.

Now, x_2 is eliminated from eqs.(4.17a) and (4.17b), while x_3 is likewise eliminated from eqs.(4.17c) and (4.19a), thereby obtaining a single equation in x_1 , namely,

$$\det \begin{bmatrix} d_{11} & d_{12} & A_4 d_7 & A_5 d_7 \\ d_{21} & d_{22} & A_4 d_8 + A_5 d_7 & A_5 d_8 \\ d_7 & d_8 & d_9 & 0 \\ 0 & d_7 & d_8 & d_9 \end{bmatrix} = 0 \quad (4.20)$$

where

$$\begin{aligned} d_{11} &= A_2 d_7 - A_1 d_8, & d_{12} &= A_3 d_7 - A_1 d_9 \\ d_{21} &= A_3 d_7 - A_1 d_9, & d_{22} &= A_3 d_8 - A_2 d_9 + A_4 d_7 \end{aligned}$$

The foregoing determinant is now expanded and simplified, which then leads to

$$\sum_{i=0}^{16} k_i x_1^i = 0 \quad (4.21a)$$

where k_i depend only on kinematic parameters, and are related by

$$k_i = (-1)^i k_{16-i}, \quad i = 1, \dots, 7 \quad (4.21b)$$

The detailed expressions for k_i are not given here because these expansions would be too large (more than 100 pages in the most compact form) to serve any useful purpose. What is important to point out here is that the above equation admits 16 solutions, whether real or complex, among which we are interested only in the

real positive solutions. The real negative solutions lead to the same configurations as the positive ones, with the exception that the sides of the triangle $\mathcal{R}$, d , e and f , are replaced by another triangle with the same vertices $R_1 R_2 R_3$, but different sides, namely, $2\pi - d$, $2\pi - e$ and $2\pi - f$. So the negative solutions can be discarded. The upper bound for the number of real positive solutions of a polynomial is given by Descartes theorem (Householder, 1970), namely,

The number of real positive solutions of a polynomial is given by the number of change of signs of the coefficients $k_0, k_1, \dots, k_n$ minus $2m$, where $m \geq 0$.

The maximum of change of sign in the foregoing polynomial is eight. Therefore, the problem leads to a maximum of eight real positive solutions and, as a result, triangle $\mathcal{Q}$ of Fig. 3.7 admits up to eight different orientations, for the specified values of μ_1 , μ_2 and μ_3 .

Example 4.3.2.1:

Consider the spherical triangles $\mathcal{P}$ and $\mathcal{Q}$ given as:

$$\begin{aligned} Q_1 Q_2 &= 60^\circ, & Q_2 Q_3 &= 70^\circ, & Q_3 Q_1 &= 50^\circ \\ P_1 P_2 &= 70^\circ, & P_2 P_3 &= 58.6^\circ, & P_3 P_1 &= 81.5^\circ \end{aligned}$$

and three points, R_1 , R_2 and R_3 , located by the three values $\mu_1 = 10^\circ$, $\mu_2 = 49.5^\circ$ and $\mu_3 = 40^\circ$. These values correspond to the angles D_2 , E_2 and F_2 given below:

$$D_2 = 43.4745^\circ, \quad E_2 = 37.9120^\circ, \quad F_2 = 106.7287^\circ$$

Equation (4.21a) is now solved for x_1 , the solutions being shown in Table 4.1. For this particular problem, we were able to find two real positive solutions. These solutions, which are depicted in Fig. 4.5, correspond to the assembly modes of the manipulator.

No.	x_1	D_1 (deg.)	E_1 (deg.)	F_1 (deg.)
1	-3.52853659	$180^\circ + (D_1)_{13}$	$180^\circ + (E_1)_{13}$	$180^\circ + (F_1)_{13}$
2	-1.81493883	$180^\circ + (D_1)_{16}$	$180^\circ + (E_1)_{16}$	$180^\circ + (F_1)_{16}$
3	$-0.7122360 - j0.9461246$	-	-	-
4	$-0.7122360 + j0.9461246$	-	-	-
5	$-0.4987636 - j1.6448662$	-	-	-
6	$-0.4987636 + j1.6448662$	-	-	-
7	$-0.0110361 - j1.7618928$	-	-	-
8	$-0.0110361 + j1.7618928$	-	-	-
9	$0.00355500 - j0.5675491$	-	-	-
10	$0.00355500 + j0.5675491$	-	-	-
11	$0.1688234 - j0.5567607$	-	-	-
12	$0.1688234 + j0.5567607$	-	-	-
13	0.28340360	31.64584216	76.17273858	42.53021089
14	$0.5078577 - j0.6746313$	-	-	-
15	$0.5078577 + j0.6746313$	-	-	-
16	0.55098275	57.70801252	99.32576667	64.91849185

Table 4.1: The sixteen solutions of Example 4.3.2.1

4.4 Spatial DT Manipulators

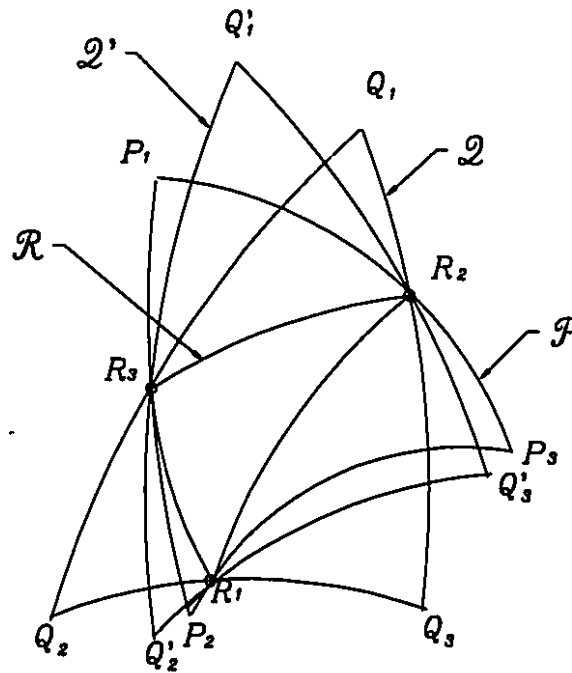
4.4.1 6-DOF Manipulator

The DK of the spatial manipulator discussed in Subsection 3.4.1 is the subject of this subsection. The manipulator consists of two spatial triangles, the fixed triangle $\mathcal{P}$ and the movable triangle $\mathcal{Q}$. Triangle $\mathcal{P}$ consists of three lines given by $\{\mathbf{v}_i^*\}_1^3$ and their three common perpendiculars given by $\{\mathbf{a}_i^*\}_1^3$, with $\mathbf{v}_i^*$ defined as

$$\mathbf{v}_i^* \equiv \mathbf{v}_i + \epsilon \mathbf{v}_{0i}, \quad i = 1, 2, 3 \quad (4.22)$$

where $\mathbf{v}_i$ and $\mathbf{v}_{0i}$ are the direction and the moment vectors of the i th line of $\mathcal{P}$ with respect to the origin, respectively. In the foregoing discussion, $\mathbf{a}_i^*$, the common perpendicular between $\mathbf{v}_{i+1}^*$ and $\mathbf{v}_{i-1}^*$, is defined as

$$\mathbf{a}_i^* \equiv \mathbf{a}_i + \epsilon \mathbf{a}_{0i}, \quad i = 1, 2, 3 \quad (4.23)$$

Figure 4.5: Spherical triangles Q , Q' and P

where $\mathbf{a}_i$ and $\mathbf{a}_{0i}$ are, respectively, the direction and the moment vectors of the line represented by $\mathbf{a}_i^*$ with respect to the origin.

Similarly, triangle Q consists of three lines given by $\{\mathbf{u}_i^*\}_1^3$ and their three common perpendiculars given by $\{\mathbf{b}_i^*\}_1^3$, with $\mathbf{u}_i^*$ defined as

$$\mathbf{u}_i^* \equiv \mathbf{u}_i + \epsilon \mathbf{u}_{0i}, \quad i = 1, 2, 3 \quad (4.24)$$

where $\mathbf{u}_i$ and $\mathbf{u}_{0i}$ are the direction and the moment vectors of the i th line of Q with respect to the origin, respectively. In the foregoing discussion, $\mathbf{b}_i^*$, the common perpendicular between $\mathbf{u}_{i+1}^*$ and $\mathbf{u}_{i-1}^*$, is defined as

$$\mathbf{b}_i^* \equiv \mathbf{b}_i + \epsilon \mathbf{b}_{0i}, \quad i = 1, 2, 3 \quad (4.25)$$

where $\mathbf{b}_i$ and $\mathbf{b}_{0i}$ are, respectively, the direction and the moment vectors of the line represented by $\mathbf{b}_i^*$ with respect to the origin.

Moreover, the movable triangle can move freely on the fixed triangle, so that $\mathbf{r}_i^*$, for $i = 1, 2, 3$, does not lie outside its corresponding line segments, Fig. 4.6. Thus,

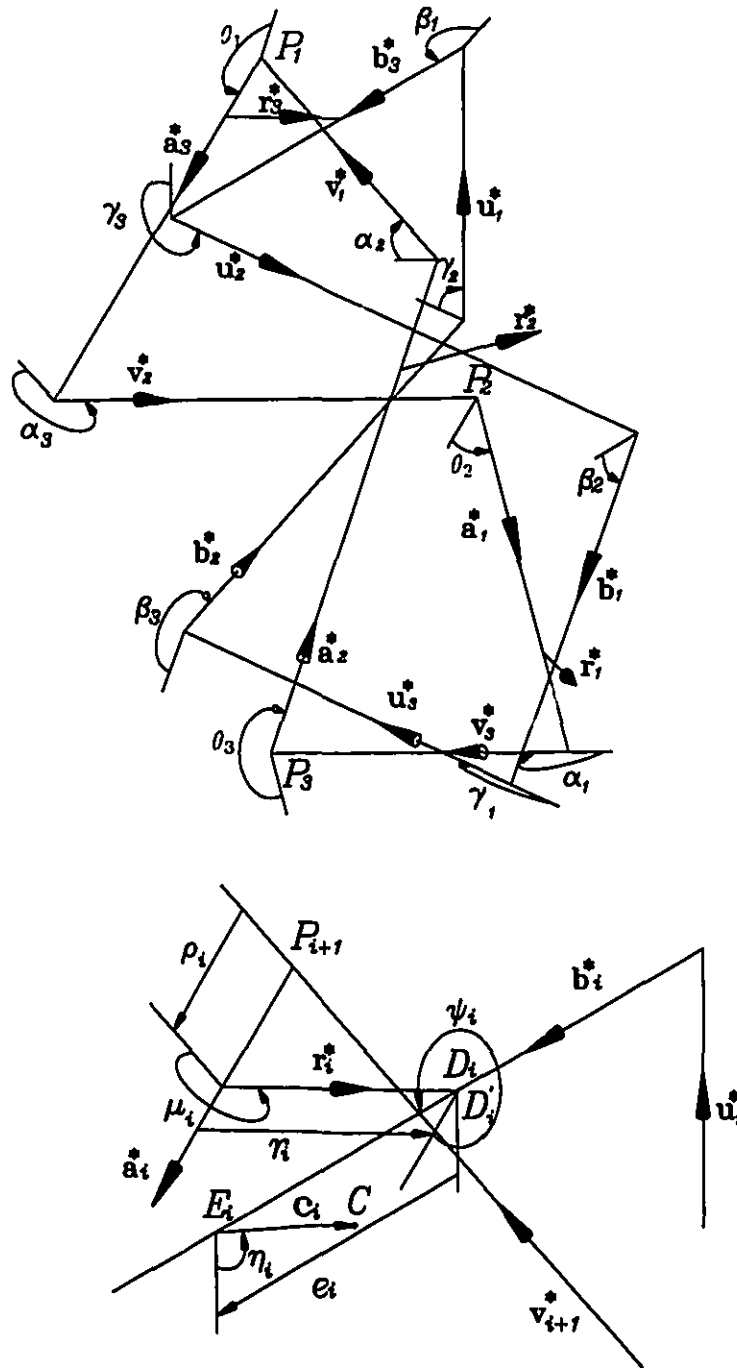


Figure 4.6: Geometric model of spatial DT manipulators

for feasible or admissible motions, $\mathbf{r}_i^*$ must intersect $\mathbf{a}_i^*$ and $\mathbf{b}_i^*$ within their line-segments. The motion of triangle $\mathcal{Q}$ can thus be described through changes in the edge-length parameters ρ_i , which locate $\mathbf{r}_i^*$ along a side of $\mathcal{P}$, measured from P_{i+1} , and changes in the twist angle between $\mathbf{v}_{i+1}^*$ and $\mathbf{r}_i^*$, μ_i , for $i = 1, 2, 3$. In other words, this motion can be described through changes in the dual angles $\hat{\mu}_i$, for $i = 1, 2, 3$. In this discussion, $\mathbf{r}_i^*$ is the dual representation of a line whose direction and moment vectors are specified by $\mathbf{r}_i$ and $\mathbf{r}_{0i}$, respectively, i.e.,

$$\mathbf{r}_i^* \equiv \mathbf{r}_i + \epsilon \mathbf{r}_{0i}, \quad i = 1, 2, 3 \quad (4.26a)$$

and $\hat{\mu}_i$ is the dual angle defined as

$$\hat{\mu}_i \equiv \mu_i + \epsilon \rho_i, \quad i = 1, 2, 3 \quad (4.26b)$$

The changes in $\hat{\mu}_i$, for $i = 1, 2, 3$, are assumed to be produced by actuators, and hence, they are termed the *actuator coordinates*. The three lines $\{\mathbf{b}_i^*\}_1^3$ of the moving triangle, in turn, are the set of variables used to define its pose. Note that three lines can be used to define a spatial triangle.

The DK problem may be formulated as: *Given the actuator coordinates $\hat{\mu}_i$, for $i = 1, 2, 3$, find the three lines of triangle $\mathcal{Q}$, namely, $\mathbf{b}_i^*$, for $i = 1, 2, 3$.* Thus, given $\{\hat{\mu}_i\}_1^3$, we define a spatial triangle whose three axes are $\{\mathbf{r}_i^*\}_1^3$. The DK problem thus consists of finding all triangles $\mathcal{Q}$ whose three common perpendiculars, $\{\mathbf{b}_i^*\}_1^3$, intersect these three axes at right angles.

The problem can be formulated in the same way that was formulated for the spherical DT manipulators, given in eqs.(4.14a–4.15c), by changing all the angles to the corresponding dual angles. Thus, we would have 12 equations in 18 unknowns, namely, $\hat{D}_i$, $\hat{E}_i$ and $\hat{F}_i$, for $i=1,2,3$. Therefore, we need, at least, six extra equations, which makes the problem more complicated. Here, an alternative formulation is given.

Note that $\mathbf{a}_i^*$ can be transformed into $\mathbf{b}_i^*$ via a screw motion represented by a

unit dual quaternion $\hat{r}_i^*$, namely,

$$\mathbf{b}_i^* = \hat{r}_i^* \mathbf{a}_i^*, \quad i = 1, 2, 3 \quad (4.27a)$$

where $\hat{r}_i^*$ is defined as

$$\hat{r}_i^* \equiv \cos \hat{\psi}_i + \mathbf{r}_i^* \sin \hat{\psi}_i, \quad i = 1, 2, 3 \quad (4.27b)$$

in which $\hat{\psi}_i$ is the dual angle defined as

$$\hat{\psi}_i \equiv \psi_i + \epsilon r_i, \quad i = 1, 2, 3 \quad (4.27c)$$

with ψ_i and r_i defined, in turn, as the twist angle and the distance between lines $\mathbf{a}_i^*$ and $\mathbf{b}_i^*$, respectively.

Substitution of the value of $\hat{r}_i^*$ from eq.(4.27b) into eq.(4.27a), upon simplification, leads to

$$\mathbf{b}_i^* = \cos \hat{\psi}_i \mathbf{a}_i^* + \mathbf{r}_i^* \sin \hat{\psi}_i \mathbf{a}_i^*, \quad i = 1, 2, 3 \quad (4.28)$$

Moreover, $\mathbf{r}_i^*$ is a transformation of $\mathbf{v}_{i+1}^*$ via a screw motion represented by a unit dual quaternion $\hat{a}_i^*$, as shown in Fig. 4.6, namely,

$$\mathbf{r}_i^* = \hat{a}_i^* \mathbf{v}_{i+1}^*, \quad i = 1, 2, 3 \quad (4.29a)$$

where

$$\hat{a}_i^* \equiv \cos \hat{\mu}_i + \mathbf{a}_i^* \sin \hat{\mu}_i, \quad i = 1, 2, 3 \quad (4.29b)$$

Substitution of the value of $\mathbf{r}_i^*$ from eq.(4.29a) into eq.(4.28), upon simplification, leads to

$$\mathbf{b}_i^* = \cos \hat{\psi}_i \mathbf{a}_i^* + \cos \hat{\mu}_i \sin \hat{\psi}_i \mathbf{v}_{i+1}^* \mathbf{a}_i^* + \sin \hat{\mu}_i \sin \hat{\psi}_i \mathbf{a}_i^* \mathbf{v}_{i+1}^* \mathbf{a}_i^*, \quad i = 1, 2, 3 \quad (4.30)$$

Equation (4.30) leads to 18 scalar equations in 24 unknowns, namely, the three lines represented by $\{\mathbf{b}_i^*\}_1^3$ and the three dual quantities $\{\hat{\psi}_i\}_1^3$.

Moreover, we recall the angular closure equation from eq.(2.57), which, for movable triangle, leads to

$$\hat{b}_2^* \hat{b}_1^* \hat{b}_3^* = 1 \quad (4.31a)$$

where $\hat{b}_i^*$, for $i = 1, 2, 3$, are unit dual quaternions, defined as

$$\hat{b}_i^* \equiv \cos \hat{\gamma}_i + \mathbf{b}_i^* \sin \hat{\gamma}_i, \quad i = 1, 2, 3 \quad (4.31b)$$

in which $\hat{\gamma}_i$ is the dual angle defined as

$$\hat{\gamma}_i \equiv \gamma_i + \epsilon b_i, \quad i = 1, 2, 3 \quad (4.31c)$$

where γ_i and b_i are the twist angle and the distance between lines $\mathbf{u}_{i+1}^*$ and $\mathbf{u}_{i-1}^*$, respectively. Moreover, pre-multiplying both sides of eq.(4.31a) by $k(\hat{b}_2^*)$, leads to

$$\hat{b}_1^* \hat{b}_3^* = k(\hat{b}_2^*) \quad (4.32)$$

Equation (4.32) thus leads to eight extra equations to give a total of 26 equations in 24 unknowns, whose roots are the solutions of the DK problem at hand.

Moreover, substituting the values of $\hat{b}_1^*$, $\hat{b}_2^*$ and $\hat{b}_3^*$ from eq.(4.31b) into eq.(4.32), upon simplification, leads to

$$\begin{aligned} \cos \hat{\gamma}_1 \cos \hat{\gamma}_3 + \mathbf{b}_3^* \cos \hat{\gamma}_1 \sin \hat{\gamma}_3 + \mathbf{b}_1^* \sin \hat{\gamma}_1 \cos \hat{\gamma}_3 + \\ \mathbf{b}_1^* \mathbf{b}_3^* \sin \hat{\gamma}_1 \sin \hat{\gamma}_3 - \cos \hat{\gamma}_2 + \mathbf{b}_2^* \sin \hat{\gamma}_2 = 0 \end{aligned} \quad (4.33)$$

Finally, substituting the values of $\mathbf{b}_i^*$, for $i = 1, 2, 3$, from eq.(4.30) into eq.(4.33), leads to eight equations in six unknowns, namely, six parameters in three dual quantities $\hat{\psi}_i$, for $i = 1, 2, 3$. Among the eight equations, only six are independent, and the problem should admit some solutions.

Example 4.4.1.1:

The fixed triangle is given by three dual vectors $\mathbf{v}_i^*$, for $i = 1, 2, 3$, via their direction and moment vectors, as explained in eq.(4.22), i.e.,

$$\begin{aligned} \mathbf{v}_1 &= [1, 0, 0]^T, & \mathbf{v}_{10} &= [0, 0, 0]^T \\ \mathbf{v}_2 &= [0, 0, 1]^T, & \mathbf{v}_{20} &= [1, 0, 0]^T \\ \mathbf{v}_3 &= [0, -1, 0]^T, & \mathbf{v}_{30} &= [1, 0, 1]^T \end{aligned} \quad (4.34)$$

The direction and moment vectors of the three common perpendiculars to the foregoing given lines, $\{\mathbf{a}_i^*\}_1^3$, are

$$\begin{aligned}\mathbf{a}_1 &= [-1, 0, 0]^T, & \mathbf{a}_{10} &= [0, -1, 1]^T \\ \mathbf{a}_2 &= [0, 0, -1]^T, & \mathbf{a}_{20} &= [0, -1, 0]^T \\ \mathbf{a}_3 &= [0, 1, 0]^T, & \mathbf{a}_{30} &= [0, 0, 0]^T\end{aligned}\quad (4.35)$$

Moreover, the moving triangle is given by its three sides, namely,

$$\begin{aligned}\hat{\gamma}_1 &= 1.99133 + \epsilon 0.37268 \\ \hat{\gamma}_2 &= 0.876816 + \epsilon 0.737494 \\ \hat{\gamma}_3 &= 1.74577 + \epsilon 0.123211\end{aligned}\quad (4.36)$$

Finally, six actuator coordinates are given in dual form as

$$\begin{aligned}\hat{\mu}_1 &= -\pi + \epsilon 0.5 \\ \hat{\mu}_2 &= -2.15873528 + \epsilon 0.75 \\ \hat{\mu}_3 &= -3\pi/4 + \epsilon 0.25\end{aligned}\quad (4.37)$$

Substitution of the foregoing data into eq.(4.33), upon simplification, leads to

$$\mathbf{q} = \mathbf{0} \quad (4.38)$$

where $\mathbf{q}$ is an 8-dimensional vector with only six independent components. The eight components of $\mathbf{q}$ are given in Appendix C.

Solving eq.(4.38) for r_i and ψ_i , for $i = 1, 2, 3$, leads to the six real solutions in Table 4.2.

Substitution of the data from eqs.(4.34 – 4.37) and the foregoing values for r_i and ψ_i , for $i = 1, 2, 3$, into eq.(4.30), gives $\mathbf{b}_i^*$, for $i = 1, 2, 3$. For example, for solution No. 4, one may obtain three lines of the moving triangle as

$$\begin{aligned}\mathbf{b}_1^* &= [-0.894427, -0.447214, 0]^T + \epsilon[0.223608, -0.447214, 1.11803]^T \\ \mathbf{b}_2^* &= [0.5547, -0.83205, 0]^T + \epsilon[0.208013, 0.138676, 0.416026]^T \\ \mathbf{b}_3^* &= [0.707107, 0, 0.707107]^T + \epsilon[0.176777, 0.353553, -0.176777]^T\end{aligned}$$

No.	r_1 m	r_2 m	r_3 m	ψ_1 (deg.)	ψ_2 (deg.)	ψ_3 (deg.)
1	0.0558852	1.0740471	0.4072191 6	-154.806	8.32983	21.746
2	0.34708068	1.17104435	-0.20553218	62.084	-7.16017	152.624
3	0.46406054	1.03938365	0.28097901	135.155	-39.8682	101.556
4	0.4999920	0.4160031	0.3535622	-26.5655	-89.999	-90.0004
5	0.74620074	-0.10399796	1.54681873	-46.5857	146.239	9.86011
6	1.34650493	1.13581347	0.01708774	-10.8435	-161.372	-46.3652

Table 4.2: The six solutions of Example 4.4.1.1

With the foregoing data, which are the three mutual perpendiculars to the three lines given by $\{\mathbf{u}_i^*\}_1^3$, we obtain

$$\mathbf{u}_1^* = [0.639602, 0.426401, -0.639602]^T + \epsilon[-0.193819, -0.218046, -0.339183]^T$$

$$\mathbf{u}_2^* = [-0.408249, 0.816496, 0.408249]^T + \epsilon[-0.0340218, 0.0680398, -0.170101]^T$$

$$\mathbf{u}_3^* = [0, 0, 1]^T + \epsilon[-0.75, -1, 0]^T$$

which correspond to the pose of the moving triangle.

4.4.2 Other Versions of Spatial DT Manipulators

The structure of the 3-dof spatial DT manipulator is similar to that of its 6-dof counterpart, except that the distances between the three common perpendiculars of the movable triangle, given by $\{\mathbf{a}_i^*\}_1^3$, and the three common perpendiculars of the fixed triangle, given by $\{\mathbf{b}_i^*\}_1^3$, namely, r_i , for $i = 1, 2, 3$, are fixed. In other words, we omit the prismatic joints along $\mathbf{r}_i^*$, for $i = 1, 2, 3$.

Contrary to the 6-dof device, we need only three actuators to move triangle $\mathcal{Q}$. This motion can be described through changes in the edge-length parameters, ρ_i , which locate $\mathbf{r}_i^*$ along a side of $\mathcal{P}$, measured from P_{i+1} , as shown in Fig. 4.6, for $i = 1, 2, 3$. The changes in ρ_i , for $i = 1, 2, 3$, are assumed to be produced by actuators, and hence, they are termed the *actuator coordinates*. The three lines of the moving triangle, $\{\mathbf{u}_i^*\}_1^3$, in turn, are the set of variables used to define its pose.

The direct kinematic problem of the manipulator described above is the subject of this subsection. This problem may be formulated, similarly, as: *Given the actuator coordinates ρ_i , for $i = 1, 2, 3$, find the three lines of triangle Q , namely, $\mathbf{b}_i^*$, for $i = 1, 2, 3$.* In order to solve this problem, we define three circles in planes of normals $\{\mathbf{a}_i^*\}_1^3$, their radii being given by r_i , for given value of ρ_i , for $i = 1, 2, 3$. The DK problem thus consists of finding all triangles Q whose three common perpendiculars, $\mathbf{b}_i^*$, for $i = 1, 2, 3$, intersect these three circles and are perpendicular to $\mathbf{r}_i^*$.

The governing equations are the same as described in eq.(4.33), in which we have eight equations in six unknowns. However, r_i , for $i = 1, 2, 3$, are given and our six unknowns are the six angles μ_i and ψ_i , for $i = 1, 2, 3$. Again, among these eight equations, only six are independent, and the problem should admit some solutions.

Example 4.4.2.1:

Given are the same fixed and movable triangles as in Example 4.4.1.1, and three lengths r_i , for $i = 1, 2, 3$, as

$$r_1 = 0.5, \quad r_2 = 0.416024, \quad r_3 = 0.353553 \quad (4.39)$$

Moreover, the three actuator coordinates are given as

$$\rho_1 = 0.5, \quad \rho_2 = 0.75, \quad \rho_3 = 0.25 \quad (4.40)$$

Substitution of the foregoing data and the data from eqs.(4.34–4.36) into eq.(4.33), upon simplification, leads to

$$\mathbf{q} = \mathbf{0} \quad (4.41)$$

where $\mathbf{q}$ is an 8-dimensional vector with only six independent components. The eight components of $\mathbf{q}$ are given in Appendix D.

Solving eq.(4.41) for μ_i and ψ_i , for $i = 1, 2, 3$, leads to 26 sets of solutions, as given in Table 4.3.

Substitution of the data from eqs.(4.34–4.36) and the foregoing values for μ_i and ψ_i , for $i = 1, 2, 3$, into eq.(4.30), gives $\{\mathbf{b}_i^*\}_1^3$. For example, for solution No. 1, we

obtain three lines of the moving triangle as

$$\mathbf{b}_1^* = [-0.894427, -0.447214, 0]^T + \epsilon[0.223608, -0.447214, 1.11803]^T$$

$$\mathbf{b}_2^* = [0.5547, -0.83205, 0]^T + \epsilon[0.208013, 0.138676, 0.416026]^T$$

$$\mathbf{b}_3^* = [0.707107, 0, 0.707107]^T + \epsilon[0.176777, 0.353553, -0.176777]^T$$

No.	μ_1 (deg.)	μ_2 (deg.)	μ_3 (deg.)	ψ_1 (deg.)	ψ_2 (deg.)	ψ_3 (deg.)
1	-179.999	-123.686	-134.999	-26.5618	-89.999	-90.0028
2	-176.718	1.31301	97.1191	-75.3075	18.6912	65.2472
3	-126.093	-107.015	118.478	-78.2416	-50.4778	81.853
4	-125.473	139.856	87.5191	-119.658	-75.6438	-16.4706
5	-85.2722	-123.952	-159.37	40.829	-69.3651	-130.402
6	-75.3172	-45.2051	22.5639	-91.6247	71.1107	-58.0193
7	-51.2519	-152.467	-6.54424	85.3781	-114.432	100.82
8	-45.2596	59.4541	91.8851	-128.594	-61.7801	-92.9408
9	-37.785	-18.5311	8.64666	105.239	102.775	83.1113
10	-28.1238	-35.9133	-110.721	78.6906	43.5577	-69.116
11	-26.3322	-132.262	-108.315	-100.215	28.9265	113.547
12	-8.45381	-15.3744	-72.1787	-76.5418	-44.0751	144.75
13	81.177	105.925	-76.7579	43.9773	56.8596	-125.883
14	91.8349	117.557	124.249	70.1942	74.6192	130.785
15	100.096	135.56	135.103	-83.4981	108.845	-67.097
16	101.174	143.636	-73.8456	-112.476	85.1327	62.6571
17	103.32	-90.5657	115.96	78.7748	-96.1684	115.42
18	117.739	-50.666	-84.423	-117.533	-101.001	73.2703
19	122.777	116.778	56.217	125.037	-38.9503	-68.0906
20	132.959	-116.331	173.181	-57.1777	-124.547	-80.606
21	136.783	175.627	-14.0032	85.7295	-81.1182	-86.3128
22	159.15	-100.046	63.5271	107.301	14.3701	-104.853
23	170.861	-135.938	-144.949	-39.9405	-98.5047	-79.2944
24	175.467	-93.879	-138.774	3.25529	-89.8733	-110.663
25	176.191	-151.89	89.169	128.299	25.691	-90.8829
26	177.005	-21.5867	17.7063	-104.144	84.8916	47.7

Table 4.3: The 26 solutions of Example 4.4.2.1

The foregoing data define the three mutual perpendiculars to the three lines, $\mathbf{u}_i^*$, for $i = 1, 2, 3$, namely,

$$\mathbf{u}_1^* = [0.639602, 0.426401, -0.639602]^T + \epsilon[-6.193819, -0.218046, -0.339183]^T$$

$$\mathbf{u}_2^* = [-0.408249, 0.816496, 0.408249]^T + \epsilon[-0.0340218, 0.0680398, -0.170101]^T$$

$$\mathbf{u}_3^* = [0, 0, 1]^T + \epsilon[-0.75, -1, 0]^T$$

thereby defining the pose of the moving triangle.

Similar to the DK problem of the 3-dof spatial DT manipulator, the DK problem of the 6-dof spatial DT manipulator with prismatic actuators can be formulated, the result being the same as that given in eq.(4.33), except that the unknown variables are ρ_i and r_i , for $i=1,2,3$, and the other parameters are known. Then, the equations can be solved similarly.

Chapter 5

Singularity Analysis

5.1 Introduction

The Jacobian matrices of several manipulators, introduced in Chapter 3, are derived here in an invariant form. A classification of singularities in parallel manipulators into three groups, which is based on the characteristics of their Jacobian matrices, is proposed and the singularities are identified for these manipulators. Deriving the Jacobian matrices in an invariant form allows us to detect all singularities within the manipulator workspaces.

5.2 Jacobian Matrices

The differential kinematic relations pertaining to parallel manipulators take on the form

$$\mathbf{J}\dot{\boldsymbol{\theta}} + \mathbf{K}\mathbf{t} = \mathbf{0} \quad (5.1a)$$

where $\mathbf{J}$ and $\mathbf{K}$ are the two Jacobian matrices of the manipulator at hand. Moreover, $\dot{\boldsymbol{\theta}}$ is the vector of joint rates and $\mathbf{t}$ is the *twist array*, which assumes different forms, depending on the nature of the task space, namely,

$$\mathbf{t} \equiv \begin{bmatrix} \omega \\ \dot{\mathbf{c}} \end{bmatrix}, \quad \mathbf{t} \equiv \omega, \quad \mathbf{t} \equiv \begin{bmatrix} \omega \\ \dot{\mathbf{c}} \end{bmatrix} \quad (5.1b)$$

where the first form corresponds to planar; the second to spherical; and the third to spatial tasks. Moreover, in the foregoing forms, ω is the scalar angular velocity of the moving platform and $\dot{\mathbf{c}}$ is the two-dimensional velocity vector of the *operation point* C of the moving platform for the planar case. Likewise, for spherical and spatial tasks, ω denotes the three-dimensional angular-velocity vector of the moving platform. Finally, for 6-dof positioning and orienting tasks, $\dot{\mathbf{c}}$ denotes the three-dimensional velocity of the operation point of the moving body. Therefore, $\mathbf{t}$ is a three-dimensional array for planar and spherical tasks, while it is six-dimensional for spatial tasks. So, the Jacobian matrices are of 3×3 for the planar and spherical devices. For six-dof spatial manipulators both $\mathbf{J}$ and $\mathbf{K}$ are 6×6 matrices.

Below we derive expressions for $\mathbf{J}$ and $\mathbf{K}$ for the manipulators introduced in Chapter 3. These are the two general classes of planar parallel manipulators, spherical 3-RRR and DT manipulators, and spatial 6-dof, DT manipulators.

5.2.1 Planar Manipulators of Class $\mathcal{A}$

Here, the Jacobian matrices of the 20 different class- $\mathcal{A}$ manipulators, discussed in Subsection 3.2.1, are derived. The manipulator, in general form, is depicted in Fig. 3.2.

The velocity $\dot{\mathbf{c}}$ can be written for the i th leg as

$$\dot{\mathbf{c}} = \dot{\mathbf{a}}_i + (\dot{\mathbf{q}}_i - \dot{\mathbf{a}}_i) + (\dot{\mathbf{c}} - \dot{\mathbf{q}}_i), \quad i = 1, 2, 3 \quad (5.2)$$

where $\mathbf{a}_i$ and $\mathbf{q}_i$ are the vector directed from P_i to A_i and from A_i to Q_i , respectively. Moreover, we have

$$\dot{\mathbf{a}}_i = \dot{\theta}_i \mathbf{A}_i \mathbf{a}_i, \quad i = 1, 2, 3 \quad (5.3a)$$

where $\dot{\theta}_i$ is the rate of the i th actuator and $\mathbf{A}_i$ is defined as

$$\mathbf{A}_i \equiv \begin{cases} \mathbf{E}, & \text{if the first joint is revolute} \\ (1/\|\mathbf{a}_i\|)\mathbf{1}, & \text{if the first joint is prismatic} \end{cases} \quad (5.3b)$$

in which $\mathbf{1}$ is the 2×2 identity matrix and $\mathbf{E}$ is the 2×2 orthogonal matrix rotating vectors in a plane through an angle of 90° counterclockwise, i. e.,

$$\mathbf{E} \equiv \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Moreover, we have

$$\dot{\mathbf{q}}_i - \dot{\mathbf{a}}_i = \dot{\gamma}_i \mathbf{B}_i \mathbf{r}_i + \dot{\theta}_i \mathbf{C}_i \mathbf{r}_i, \quad i = 1, 2, 3 \quad (5.4a)$$

where $\dot{\gamma}_i$ is the rate of the second joint, $\mathbf{r}_i$ is the vector directed from A_i to Q_i and $\mathbf{B}_i$ and $\mathbf{C}_i$ are defined as

$$\mathbf{B}_i \equiv \begin{cases} \mathbf{E}, & \text{if the second joint is revolute} \\ (1/\|\mathbf{r}_i\|)\mathbf{1}, & \text{if the second joint is prismatic} \end{cases} \quad (5.4b)$$

$$\mathbf{C}_i \equiv \begin{cases} \mathbf{E}, & \text{if the first joint is revolute} \\ \mathbf{O}, & \text{if the first joint is prismatic} \end{cases} \quad (5.4c)$$

with $\mathbf{O}$ denoting the 2×2 zero matrix. Furthermore, $\dot{\mathbf{c}} - \dot{\mathbf{q}}_i$ is given as

$$\dot{\mathbf{c}} - \dot{\mathbf{q}}_i = \omega \mathbf{E} \mathbf{s}_i, \quad i = 1, 2, 3 \quad (5.5)$$

with vector $\mathbf{s}_i$ directed from Q_i to C , as shown in Fig. 3.2.

Substitution of the values of $\dot{\mathbf{a}}_i$, $\dot{\mathbf{q}}_i - \dot{\mathbf{a}}_i$ and $\dot{\mathbf{c}} - \dot{\mathbf{q}}_i$ from eqs.(5.3a), (5.4a) and (5.5) into eq.(5.2), and simplification of the expression thus resulting leads to

$$\dot{\theta}_i \mathbf{A}_i \mathbf{a}_i + \dot{\gamma}_i \mathbf{B}_i \mathbf{r}_i + \dot{\theta}_i \mathbf{C}_i \mathbf{r}_i + \omega \mathbf{E} \mathbf{s}_i - \dot{\mathbf{c}} = \mathbf{0}, \quad i = 1, 2, 3 \quad (5.6)$$

where $\dot{\gamma}_i$, being associated with an unactuated joint, should be eliminated. To this end, we define $\mathbf{E}_i$ as

$$\mathbf{E}_i \equiv \begin{cases} \mathbf{1}, & \text{if the second joint is revolute} \\ \mathbf{E}, & \text{if the second joint is prismatic} \end{cases} \quad (5.7)$$

Upon multiplication of the two sides of eq.(5.6) by $\mathbf{r}_i^T \mathbf{E}_i$, we obtain

$$\dot{\theta}_i \mathbf{r}_i^T \mathbf{E}_i (\mathbf{A}_i \mathbf{a}_i + \mathbf{C}_i \mathbf{r}_i) + \omega \mathbf{r}_i^T \mathbf{E}_i \mathbf{E} \mathbf{s}_i - \mathbf{r}_i^T \mathbf{E}_i \dot{\mathbf{c}} = 0, \quad i = 1, 2, 3 \quad (5.8)$$

Parameters	Leg Type			
	PRR	PPR	RRR	RPR
$\mathbf{A}_i$	$(1/\ \mathbf{a}_i\)\mathbf{1}$	$(1/\ \mathbf{a}_i\)\mathbf{1}$	$\mathbf{E}$	$\mathbf{E}$
$\mathbf{B}_i$	$\mathbf{E}$	$(1/\ \mathbf{r}_i\)\mathbf{1}$	$\mathbf{E}$	$(1/\ \mathbf{r}_i\)\mathbf{1}$
$\mathbf{C}_i$	$\mathbf{O}$	$\mathbf{O}$	$\mathbf{E}$	$\mathbf{E}$
$\mathbf{E}_i$	$\mathbf{1}$	$\mathbf{E}$	$\mathbf{1}$	$\mathbf{E}$

Table 5.1: $\mathbf{A}_i$, $\mathbf{B}_i$, $\mathbf{C}_i$ and $\mathbf{E}_i$ for different legs of the manipulators of class $\mathcal{A}$

Moreover, eq.(5.8) written for $i = 1, 2, 3$, produces

$$\mathbf{J}\dot{\boldsymbol{\theta}} + \mathbf{K}\mathbf{t} = \mathbf{0} \quad (5.9a)$$

where $\mathbf{t}$, the *twist vector*, was defined above, and the 3×3 matrices $\mathbf{J}$ and $\mathbf{K}$ are given as

$$\mathbf{J} = \begin{bmatrix} \mathbf{r}_1^T \mathbf{E}_1 (\mathbf{A}_1 \mathbf{a}_1 + \mathbf{C}_1 \mathbf{r}_1) & 0 & 0 \\ 0 & \mathbf{r}_2^T \mathbf{E}_2 (\mathbf{A}_2 \mathbf{a}_2 + \mathbf{C}_2 \mathbf{r}_2) & 0 \\ 0 & 0 & \mathbf{r}_3^T \mathbf{E}_3 (\mathbf{A}_3 \mathbf{a}_3 + \mathbf{C}_3 \mathbf{r}_3) \end{bmatrix} \quad (5.9b)$$

and

$$\mathbf{K} = \begin{bmatrix} \mathbf{r}_1^T \mathbf{E}_1 \mathbf{E} \mathbf{s}_1 & -\mathbf{r}_1^T \mathbf{E}_1 \\ \mathbf{r}_2^T \mathbf{E}_2 \mathbf{E} \mathbf{s}_2 & -\mathbf{r}_2^T \mathbf{E}_2 \\ \mathbf{r}_3^T \mathbf{E}_3 \mathbf{E} \mathbf{s}_3 & -\mathbf{r}_3^T \mathbf{E}_3 \end{bmatrix} \quad (5.9c)$$

in which $\mathbf{A}_i$, $\mathbf{B}_i$, $\mathbf{C}_i$ and $\mathbf{E}_i$, for $i = 1, 2, 3$, are chosen for each row of the foregoing matrices based on the corresponding leg, as explained in eqs.(5.3b), (5.4b), (5.4c) and (5.7) and summarized in Table 5.1.

5.2.2 Planar Manipulators of Class $\mathcal{B}$

Here, the Jacobian matrices of the four different manipulators of class $\mathcal{B}$, discussed in Subsection 3.2.2, are derived. The manipulator, in general form, is depicted in Fig. 3.4.

The velocity $\dot{\mathbf{c}}$ can be now written for the i th leg as

$$\dot{\mathbf{c}} = \mathbf{v}_{Ai} + (\mathbf{v}_{Ri} - \mathbf{v}_{Ai}) + (\dot{\mathbf{c}} - \mathbf{v}_{Ri}), \quad i = 1, 2, 3 \quad (5.10)$$

Moreover, we have

$$\mathbf{v}_{Ai} = \dot{\theta}_i \mathbf{E}_i \mathbf{a}_i, \quad i = 1, 2, 3 \quad (5.11a)$$

where $\mathbf{a}_i$ is the unit vector directed from P_i to A_i and $\dot{\theta}_i$ is the rate of the i th actuator, while $\mathbf{E}_i$ is defined as

$$\mathbf{E}_i \equiv \begin{cases} \rho_i \mathbf{E}, & \text{if the first joint is revolute} \\ \mathbf{1}, & \text{if the first joint is prismatic} \end{cases} \quad (5.11b)$$

in which $\rho_i = \overline{P_i A_i}$.

Furthermore, we have

$$\mathbf{v}_{Ri} - \mathbf{v}_{Ai} = \omega \mathbf{E} \mathbf{r}_i, \quad i = 1, 2, 3 \quad (5.12)$$

where $\mathbf{r}_i$ is the vector directed from A_i to R_i . Finally, $\dot{\mathbf{c}} - \mathbf{v}_{Ri}$ is given as

$$\dot{\mathbf{c}} - \mathbf{v}_{Ri} = \dot{\lambda}_i \mathbf{b}_i + \omega \mathbf{E} \mathbf{s}_i, \quad i = 1, 2, 3 \quad (5.13)$$

with $\dot{\lambda}_i$ denoting the rate of the third joint. A unit vector $\mathbf{b}_i$ represents the direction of the third joint, which is prismatic, and $\mathbf{s}_i$ is vector directed from R_i to C , as shown in Fig. 3.4.

Substitution of the values of $\mathbf{v}_{Ai}$, $\mathbf{v}_{Ri} - \mathbf{v}_{Ai}$ and $\dot{\mathbf{c}} - \mathbf{v}_{Ri}$ from eqs.(5.11a), (5.12) and (5.13) into eq.(5.10), and simplification of the expression thus resulting leads to

$$\dot{\theta}_i \mathbf{E}_i \mathbf{a}_i + \dot{\lambda}_i \mathbf{b}_i + \omega \mathbf{E} (\mathbf{r}_i + \mathbf{s}_i) - \dot{\mathbf{c}} = \mathbf{0}, \quad i = 1, 2, 3 \quad (5.14)$$

where $\dot{\lambda}_i$ is associated with an unactuated joint and should be eliminated. Multiplication of the above equation by $\mathbf{b}_i^T \mathbf{E}$ accomplishes this, namely,

$$\dot{\theta}_i \mathbf{b}_i^T \mathbf{E} \mathbf{E}_i \mathbf{a}_i - \omega \mathbf{b}_i^T (\mathbf{r}_i + \mathbf{s}_i) - \mathbf{b}_i^T \mathbf{E} \dot{\mathbf{c}} = 0, \quad i = 1, 2, 3 \quad (5.15)$$

Upon writing eq.(5.15) for $i = 1, 2, 3$, we obtain

$$\mathbf{J} \dot{\boldsymbol{\theta}} + \mathbf{K} \mathbf{t} = \mathbf{0} \quad (5.16a)$$

where matrices $\mathbf{J}$ and $\mathbf{K}$ are given as

$$\mathbf{J} = \begin{bmatrix} \mathbf{b}_1^T \mathbf{E} \mathbf{E}_1 \mathbf{a}_1 & 0 & 0 \\ 0 & \mathbf{b}_2^T \mathbf{E} \mathbf{E}_2 \mathbf{a}_2 & 0 \\ 0 & 0 & \mathbf{b}_3^T \mathbf{E} \mathbf{E}_3 \mathbf{a}_3 \end{bmatrix} \quad (5.16b)$$

and

$$\mathbf{K} = \begin{bmatrix} -\mathbf{b}_1^T (\mathbf{r}_1 + \mathbf{s}_1) & -\mathbf{b}_1^T \mathbf{E} \\ -\mathbf{b}_2^T (\mathbf{r}_2 + \mathbf{s}_2) & -\mathbf{b}_2^T \mathbf{E} \\ -\mathbf{b}_3^T (\mathbf{r}_3 + \mathbf{s}_3) & -\mathbf{b}_3^T \mathbf{E} \end{bmatrix} \quad (5.16c)$$

5.2.3 Spherical 3-RRR Manipulator

A 3-RRR spherical parallel manipulator is depicted in Fig. 3.6. All the joints of this manipulator are revolute and the three motors P_1 , P_2 and P_3 are fixed to the base. The angular velocity $\boldsymbol{\omega}$ of the EE can be written as

$$\dot{\theta}_i \mathbf{u}_i + \dot{\alpha}_i \mathbf{v}_i + \dot{\gamma}_i \mathbf{w}_i = \boldsymbol{\omega}, \quad i = 1, 2, 3 \quad (5.17)$$

where $\mathbf{u}_i$, $\mathbf{v}_i$ and $\mathbf{w}_i$ are the unit vectors pointing from the center of the sphere to points P_i , A_i and Q_i , respectively. Moreover, $\dot{\theta}_i$, $\dot{\alpha}_i$ and $\dot{\gamma}_i$ are the rates of the joint attached to the base, the intermediate joint and the joint attached to EE, respectively. Below we eliminate the rates of the unactuated joints by dot-multiplying both sides of the foregoing equation by $\mathbf{v}_i \times \mathbf{w}_i$, thereby obtaining

$$\dot{\theta}_i \mathbf{u}_i \cdot (\mathbf{v}_i \times \mathbf{w}_i) = \boldsymbol{\omega} \cdot (\mathbf{v}_i \times \mathbf{w}_i), \quad i = 1, 2, 3 \quad (5.18)$$

which can be written in turn as

$$\dot{\theta}_i (\mathbf{v}_i \times \mathbf{u}_i) \cdot \mathbf{w}_i + (\mathbf{v}_i \times \mathbf{w}_i)^T \boldsymbol{\omega} = 0, \quad i = 1, 2, 3 \quad (5.19)$$

The above equations, for $i = 1, 2, 3$, are now assembled in the form

$$\mathbf{J} \dot{\boldsymbol{\theta}} + \mathbf{K} \boldsymbol{\omega} = \mathbf{0} \quad (5.20a)$$

where the 3×3 matrices $\mathbf{J}$ and $\mathbf{K}$ are defined as

$$\mathbf{J} \equiv \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix} \quad (5.20b)$$

and

$$\mathbf{K} \equiv \begin{bmatrix} (\mathbf{v}_1 \times \mathbf{w}_1)^T \\ (\mathbf{v}_2 \times \mathbf{w}_2)^T \\ (\mathbf{v}_3 \times \mathbf{w}_3)^T \end{bmatrix} \quad (5.20c)$$

in which

$$a_i \equiv (\mathbf{v}_i \times \mathbf{u}_i) \cdot \mathbf{w}_i, \quad i = 1, 2, 3$$

5.2.4 Spherical DT Manipulator

The Jacobian matrices of a spherical parallel manipulator, as depicted in Fig. 3.7, are derived here. Let us introduce the normalized vectors $\mathbf{a}_i$ and $\mathbf{b}_i$, for $i = 1, 2, 3$, which are perpendicular to the planes of arcs $P_{i+1}P_{i+2}$ and $Q_{i+1}Q_{i+2}$, respectively, as shown in Fig. 5.1.

Thus,

$$\mathbf{a}_i = \frac{\mathbf{v}_{i+1} \times \mathbf{v}_{i+2}}{\|\mathbf{v}_{i+1} \times \mathbf{v}_{i+2}\|}, \quad \mathbf{b}_i = \frac{\mathbf{u}_{i+1} \times \mathbf{u}_{i+2}}{\|\mathbf{u}_{i+1} \times \mathbf{u}_{i+2}\|} \quad (5.21)$$

where $\mathbf{u}_i$ and $\mathbf{v}_i$ are both unit vectors directed from O to Q_i and P_i , respectively.

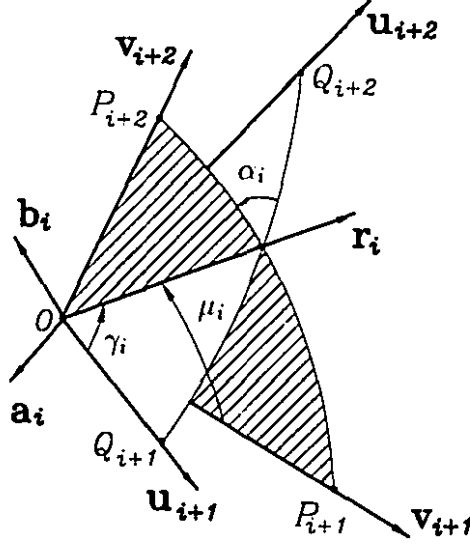
The angular velocity $\boldsymbol{\omega}$ of the EE can now be written as

$$\dot{\mu}_i \mathbf{a}_i + \dot{\alpha}_i \mathbf{r}_i - \dot{\gamma}_i \mathbf{b}_i = \boldsymbol{\omega}, \quad i = 1, 2, 3 \quad (5.22)$$

where $\mathbf{r}_i$ is the unit vector directed from the center of the sphere to R_i . Moreover, α_i is the angle between planes of P_{i+1} , P_{i+2} and Q_{i+1} , Q_{i+2} , while γ_i is the angle between $\mathbf{u}_{i+1}$ and $\mathbf{r}_i$.

The inner product of both sides of eq.(5.22) with $\mathbf{r}_i \times \mathbf{b}_i$, upon simplification, leads to an equation free of unactuated joint rates, namely,

$$\dot{\mu}_i (\mathbf{r}_i \times \mathbf{b}_i) \cdot \mathbf{a}_i - (\mathbf{r}_i \times \mathbf{b}_i) \cdot \boldsymbol{\omega} = 0, \quad i = 1, 2, 3 \quad (5.23)$$

Figure 5.1: The i th leg of spherical DT manipulator

The above equations, for $i = 1, 2, 3$, are now assembled and expressed in vector form as

$$\mathbf{J}\dot{\boldsymbol{\theta}} + \mathbf{K}\boldsymbol{\omega} = \mathbf{0} \quad (5.24a)$$

where $\mathbf{J}$ and $\mathbf{K}$ are as defined below:

$$\mathbf{J} \equiv \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix} \quad (5.24b)$$

and

$$\mathbf{K} \equiv \begin{bmatrix} -(\mathbf{r}_1 \times \mathbf{b}_1)^T \\ -(\mathbf{r}_2 \times \mathbf{b}_2)^T \\ -(\mathbf{r}_3 \times \mathbf{b}_3)^T \end{bmatrix} \quad (5.24c)$$

in which

$$c_i \equiv (\mathbf{r}_i \times \mathbf{b}_i) \cdot \mathbf{a}_i, \quad i = 1, 2, 3$$

5.2.5 Spatial 6-DOF, DT Manipulator

Here, the Jacobian matrices of the spatial 6-dof, DT manipulator, introduced in Subsection 3.4.1, are derived. The geometric model of the manipulator, in general,

is depicted in Fig. 4.6.

The angular velocity ω of the moving triangle can be written, for the i th leg, as

$$\dot{\mu}_i \mathbf{a}_i + \dot{\psi}_i \mathbf{r}_i + \dot{\eta}_i \mathbf{b}_i = \omega, \quad i = 1, 2, 3 \quad (5.25)$$

Below we eliminate the rates of the unactuated joints by dot-multiplying both sides of the foregoing equation by $\mathbf{b}_i \times \mathbf{r}_i$, thereby obtaining

$$-\dot{\mu}_i \mathbf{a}_i \cdot (\mathbf{b}_i \times \mathbf{r}_i) + \omega \cdot (\mathbf{b}_i \times \mathbf{r}_i) = 0, \quad i = 1, 2, 3 \quad (5.26)$$

Moreover, the velocity $\dot{\mathbf{c}}$ of the operation point of the EE can be written, for the i th leg, as shown in Fig. 4.6, namely,

$$\dot{\mathbf{c}} = \dot{\mathbf{d}}_i + (\dot{\mathbf{d}}'_i - \dot{\mathbf{d}}_i) + (\dot{\mathbf{c}} - \dot{\mathbf{d}}'_i), \quad i = 1, 2, 3 \quad (5.27a)$$

where $\mathbf{d}_i$ and $\mathbf{d}'_i$ are the position vectors of D_i and D'_i , in which D_i is fixed to the line $\mathbf{r}_i^*$ while D'_i is attached to the prismatic joint along that line; so, for $i = 1, 2, 3$ we have

$$\begin{aligned} \dot{\mathbf{d}}_i &= \dot{\rho}_i \mathbf{a}_i + \dot{\mu}_i r_i (\mathbf{a}_i \times \mathbf{r}_i) \\ \dot{\mathbf{d}}'_i - \dot{\mathbf{d}}_i &= \dot{r}_i \mathbf{r}_i \\ \dot{\mathbf{c}} - \dot{\mathbf{d}}'_i &= \dot{e}_i \mathbf{b}_i + \omega \times (e_i \mathbf{b}_i + \mathbf{c}_i) \end{aligned} \quad (5.27b)$$

where $\mathbf{c}_i$ is a vector whose end-point is the operation point and is normal to line $\mathbf{b}_i^*$, and $e_i = D_i E_i$.

Substituting $\dot{\mathbf{d}}_i$, $\dot{\mathbf{d}}'_i - \dot{\mathbf{d}}_i$ and $\dot{\mathbf{c}} - \dot{\mathbf{d}}'_i$ from eq.(5.27b) into eq.(5.27a), upon simplification, leads to

$$\dot{\mathbf{c}} = \dot{\rho}_i \mathbf{a}_i + \dot{\mu}_i r_i (\mathbf{a}_i \times \mathbf{r}_i) + \dot{r}_i \mathbf{r}_i + \dot{e}_i \mathbf{b}_i + e_i \omega \times \mathbf{b}_i + \omega \times \mathbf{c}_i, \quad i = 1, 2, 3 \quad (5.28)$$

where $\dot{r}_i$ and $\dot{e}_i$, the velocity of the unactuated joints, should be eliminated. This can be done by post-multiplying both sides of eq.(5.28) by $(\mathbf{b}_i \times \mathbf{r}_i)$, i.e.,

$$\begin{aligned} \dot{\mathbf{c}}^T (\mathbf{b}_i \times \mathbf{r}_i) &= \dot{\rho}_i \mathbf{a}_i^T (\mathbf{b}_i \times \mathbf{r}_i) + \dot{\mu}_i r_i (\mathbf{a}_i \times \mathbf{r}_i)^T (\mathbf{b}_i \times \mathbf{r}_i) + e_i (\omega \times \mathbf{b}_i)^T \\ &\quad (\mathbf{b}_i \times \mathbf{r}_i) + (\omega \times \mathbf{c}_i)^T (\mathbf{b}_i \times \mathbf{r}_i) = 0, \quad i = 1, 2, 3 \end{aligned} \quad (5.29)$$

Dividing the foregoing equation by r_i , upon simplification, leads to

$$-\dot{\mathbf{c}}^T(\mathbf{b}_i \times \mathbf{r}_i)/r_i + \dot{\rho}_i \mathbf{a}_i^T(\mathbf{b}_i \times \mathbf{r}_i)/r_i + \dot{\mu}_i(\mathbf{a}_i \times \mathbf{r}_i)^T(\mathbf{b}_i \times \mathbf{r}_i) + \omega^T(-c_i \mathbf{r}_i + \mathbf{c}_i \times (\mathbf{b}_i \times \mathbf{r}_i))/r_i = 0, \quad i = 1, 2, 3 \quad (5.30)$$

Writing eqs.(5.26) and (5.30) for $i = 1, 2, 3$, we obtain

$$\mathbf{J}\dot{\boldsymbol{\theta}} + \mathbf{K}\mathbf{t} = \mathbf{0} \quad (5.31a)$$

where $\mathbf{t}$ is the *twist* or Cartesian-velocity vector, and $\dot{\boldsymbol{\theta}}$ is the joint-velocity vector, defined below as

$$\mathbf{t} \equiv [\omega^T, \dot{\mathbf{c}}^T]^T$$

$$\dot{\boldsymbol{\theta}} \equiv [\dot{\rho}_1, \dot{\rho}_2, \dot{\rho}_3, \dot{\mu}_1, \dot{\mu}_2, \dot{\mu}_3]^T \quad (5.31b)$$

Moreover, the 6×6 Jacobian matrices $\mathbf{J}$ and $\mathbf{K}$ are given as

$$\mathbf{J} \equiv \begin{bmatrix} 0 & 0 & 0 & -\mathbf{a}_1^T \mathbf{m}_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathbf{a}_2^T \mathbf{m}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathbf{a}_3^T \mathbf{m}_3 \\ \mathbf{a}_1^T \mathbf{m}_1/r_1 & 0 & 0 & p_1 & 0 & 0 \\ 0 & \mathbf{a}_2^T \mathbf{m}_2/r_2 & 0 & 0 & p_2 & 0 \\ 0 & 0 & \mathbf{a}_3^T \mathbf{m}_3/r_3 & 0 & 0 & p_3 \end{bmatrix} \quad (5.31c)$$

$$\mathbf{K} \equiv \begin{bmatrix} \mathbf{m}_1^T & \mathbf{0}^T \\ \mathbf{m}_2^T & \mathbf{0}^T \\ \mathbf{m}_3^T & \mathbf{0}^T \\ \mathbf{q}_1^T & -\mathbf{m}_1^T/r_1 \\ \mathbf{q}_2^T & -\mathbf{m}_2^T/r_2 \\ \mathbf{q}_3^T & -\mathbf{m}_3^T/r_3 \end{bmatrix} \quad (5.31d)$$

in which $\mathbf{m}_i$, p_i and $\mathbf{q}_i$, for $i = 1, 2, 3$, are defined as

$$\mathbf{m}_i \equiv \mathbf{b}_i \times \mathbf{r}_i$$

$$p_i \equiv (\mathbf{a}_i \times \mathbf{r}_i)^T \mathbf{m}_i$$

$$\mathbf{q}_i \equiv (-c_i \mathbf{r}_i + \mathbf{c}_i \times \mathbf{m}_i)/r_i \quad (5.31e)$$

One may write eq.(5.31a) in dual form as

$$\hat{\mathbf{J}}\hat{\boldsymbol{\mu}} + \hat{\mathbf{K}}\hat{\omega} = \mathbf{0} \quad (5.32a)$$

where

$$\hat{\mathbf{J}} \equiv \begin{bmatrix} \mathbf{a}_1^T \mathbf{m}_1 + c p_1 & 0 & 0 \\ 0 & \mathbf{a}_2^T \mathbf{m}_2 + c p_2 & 0 \\ 0 & 0 & \mathbf{a}_3^T \mathbf{m}_3 + c p_3 \end{bmatrix} \quad (5.32b)$$

$$\hat{\mathbf{K}} \equiv \begin{bmatrix} -\mathbf{m}_1^T + c \mathbf{q}_1^T \\ -\mathbf{m}_2^T + c \mathbf{q}_1^T \\ -\mathbf{m}_3^T + c \mathbf{q}_1^T \end{bmatrix} \quad (5.32c)$$

$$\hat{\boldsymbol{\mu}} \equiv [\hat{\mu}_1', \hat{\mu}_2', \hat{\mu}_2']^T \quad (5.32d)$$

$$\hat{\omega} \equiv \omega + c\dot{c}/r_i \quad (5.32e)$$

in which

$$\dot{\mu}_i' \equiv \dot{\mu}_i + \dot{\rho}_i/r_i \quad (5.32f)$$

Equations (5.32a-c) reduce to velocity relationships of the spherical DT manipulator as expressed in eqs.(5.24a-c) by omitting the dual parts of the foregoing equations.

5.3 Classification of Singularities

In parallel manipulators, singularities occur whenever $\mathbf{J}$, $\mathbf{K}$, or both become singular. Thus, for these manipulators, a distinction can be made among three types of singularities, which have different kinematic interpretations, namely,

1) The first type of singularity occurs when $\mathbf{J}$ becomes singular but $\mathbf{K}$ is invertible, i. e., when

$$\det(\mathbf{J}) = 0 \quad \text{and} \quad \det(\mathbf{K}) \neq 0 \quad (5.33)$$

This type of singularity consists of the set of points where at least two branches of the inverse kinematic problem meet. Since the nullity of $\mathbf{J}$ is not zero, we can find a set of non-zero actuator velocity vectors $\dot{\boldsymbol{\theta}}$ for which the Cartesian velocity vector $\mathbf{t}$ is zero. Then, nonzero Cartesian velocity vectors $\mathbf{K}\mathbf{t}$, those lying in the nullspace of $\mathbf{J}^T$, cannot be produced, the manipulator thus losing one or more degrees of freedom.

2) The second type of singularity, occurring only in closed kinematic chains, arises when $\mathbf{K}$ becomes singular but $\mathbf{J}$ is invertible, i. e., when

$$\det(\mathbf{J}) \neq 0 \quad \text{and} \quad \det(\mathbf{K}) = 0 \quad (5.34)$$

This type of singularity consists of a point or a set of points whereby different branches of the direct kinematic problem meet. Since the nullity of $\mathbf{K}$ is not zero, we can find a set of nonzero Cartesian velocity vectors $\mathbf{t}$ for which the actuator velocity vector $\dot{\boldsymbol{\theta}}$ is zero. Then, the mechanism gains one or more uncontrollable degrees of freedom or, equivalently, cannot resist forces or moments in one or more directions, even if all the actuators are locked.

3) The third type of singularity occurs when both $\mathbf{J}$ and $\mathbf{K}$ are simultaneously singular, while none of the rows of $\mathbf{K}$ vanishes. Under a singularity of this type, configurations arise for which link $\mathcal{Q}$ of the manipulator can undergo finite motions even if the actuators are locked or, equivalently, it cannot resist forces or moments in one or more directions over a finite portion of the workspace, even if all the actuators are locked. As well, a finite motion of the actuators produces no motion of $\mathcal{Q}$ and some of the Cartesian velocity vectors cannot be produced. This type of singularity, as shown here, is not necessarily architecture-dependent, contrary to earlier claims (Gosselin, 1988; Gosselin and Angeles, 1990b; Sefrioui, 1992).

Furthermore, depending on the formulation, it can happen that one or more rows of $\mathbf{K}$ vanish. It turns out, then, that the corresponding rows of $\mathbf{J}$ vanish as well, $\mathbf{J}$ and $\mathbf{K}$ thus becoming singular simultaneously. In other words, the formulation leads to the third type of singularity. In this case, it is possible to reformulate the problem, and the new formulation may lead to any of the three types of singularities. If this

is not the case, we do not have a singular configuration at all. Therefore, this type of singularity, which arises merely from the way in which the kinematic relations are formulated, is, in fact, a formulation singularity.

5.3.1 Planar Manipulators of Class $\mathcal{A}$

In this subsection the three types of singularities discussed above are investigated for the case of manipulators of class $\mathcal{A}$.

1) It is recalled that the first type of singularity occurs when the determinant of $\mathbf{J}$ vanishes. From eq.(5.9b) this condition yields

$$\mathbf{r}_i^T \mathbf{E}_i (\mathbf{A}_i \mathbf{a}_i + \mathbf{C}_i \mathbf{r}_i) = 0, \quad i = 1 \text{ or } 2 \text{ or } 3 \quad (5.35)$$

This type of configuration is reached whenever either $\mathbf{E}_i^T \mathbf{r}_i$ is perpendicular to $(\mathbf{A}_i \mathbf{a}_i + \mathbf{C}_i \mathbf{r}_i)$ or $\mathbf{A}_i \mathbf{a}_i + \mathbf{C}_i \mathbf{r}_i = \mathbf{0}$, for $i = 1$ or 2 or 3 . Then, the motion of one actuator does not produce any motion of $\mathcal{Q}$ and the manipulator loses one dof.

2) The second type of singularity occurs when the determinant of $\mathbf{K}$ vanishes. This type of configuration can be inferred from eq.(5.9c) by imposing the linear dependence of the columns or the rows of $\mathbf{K}$.

Let us define

$$\mathbf{v}_i^T \equiv \mathbf{r}_i^T \mathbf{E}_i, \quad i = 1, 2, 3 \quad (5.36)$$

Then, $\mathbf{K}$ of eq.(5.9c) can be written as

$$\mathbf{K} = \begin{bmatrix} \mathbf{v}_1^T \mathbf{E} \mathbf{s}_1 & -\mathbf{v}_1^T \\ \mathbf{v}_2^T \mathbf{E} \mathbf{s}_2 & -\mathbf{v}_2^T \\ \mathbf{v}_3^T \mathbf{E} \mathbf{s}_3 & -\mathbf{v}_3^T \end{bmatrix} \quad (5.37)$$

Inspection of eq.(5.37) reveals two instances of this type of singularity. The first occurs when the three vectors $\mathbf{v}_i$ are parallel, the second and third columns of $\mathbf{K}$ thus becoming linearly dependent. Then, the nullspace of $\mathbf{K}$ represents the set of pure translations of $\mathcal{Q}$ along a direction normal to $\mathbf{v}_i$. Platform $\mathcal{Q}$ can move in

that direction even if the actuators are locked; likewise, a force applied to Q in that direction cannot be balanced by the actuators.

The second case in which $\mathbf{K}$ is singular occurs when each of the three vectors $\mathbf{v}_i$ passes through Q_i and all three intersect at a common point D . This is proven as follows:

Let us define the three vectors $\mathbf{t}_i \equiv \overrightarrow{Q_i D}$, for $i = 1, 2, 3$, as shown in Fig. 5.2. Since

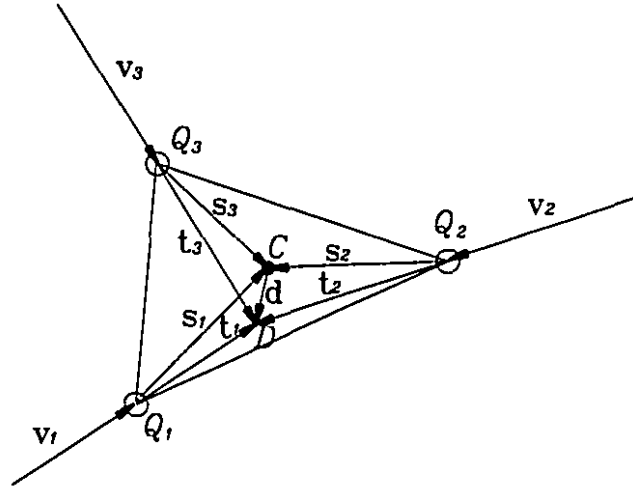


Figure 5.2: Example of the second type of singularity for the manipulators of class $\mathcal{A}$ in which the three vectors $\mathbf{v}_i$ intersect at a point

the three vectors $\mathbf{v}_i$, for $i = 1, 2, 3$, are coplanar, we can express $\mathbf{v}_3$ as a linear combination of the first two, namely,

$$\mathbf{v}_3 = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 \quad (5.38)$$

The inner product of eq.(5.38) by vector $\mathbf{E}d$ leads to

$$\mathbf{v}_3^T \mathbf{E}d = \alpha_1 \mathbf{v}_1^T \mathbf{E}d + \alpha_2 \mathbf{v}_2^T \mathbf{E}d \quad (5.39)$$

where $\mathbf{d} = \overrightarrow{CD}$. But we have

$$\mathbf{v}_i^T \mathbf{E}t_i = 0, \quad i = 1, 2, 3$$

So, eq.(5.39) can be written as

$$\mathbf{v}_3^T \mathbf{E}(t_3 - \mathbf{d}) = \alpha_1 \mathbf{v}_1^T \mathbf{E}(t_1 - \mathbf{d}) + \alpha_2 \mathbf{v}_2^T \mathbf{E}(t_2 - \mathbf{d}) \quad (5.40)$$

which, upon simplification, yields

$$\mathbf{v}_3^T \mathbf{E} \mathbf{s}_3 = \alpha_1 \mathbf{v}_1^T \mathbf{E} \mathbf{s}_1 + \alpha_2 \mathbf{v}_2^T \mathbf{E} \mathbf{s}_2 \quad (5.41)$$

From eqs.(5.38) and (5.41), it is obvious that we can write the third row of $\mathbf{K}$ as a linear combination of the first two rows, hence proof is demonstrated.

Then, the nullspace of $\mathbf{K}$ represents the set of pure rotations of $\mathcal{Q}$ about the common intersection point D . The platform $\mathcal{Q}$ can rotate about that point even if the actuators are locked; likewise, a moment applied to $\mathcal{Q}$ cannot be balanced by the actuators.

3) The third type of singularity occurs when the determinants of $\mathbf{J}$ and $\mathbf{K}$ both vanish. We have this type of singularity whenever the two previous types of singularities occur simultaneously.

By inspection of eq.(5.37) it is obvious that the i th row of $\mathbf{K}$ vanishes only if $\mathbf{v}_i = \mathbf{0}$. In this case we have a degenerate manipulator. Such a manipulator is irrelevant to our study and is thus left aside.

Example 5.3.1.1: Planar 3-RRR Manipulator

The three types of singularities discussed above are investigated here, for a particular case of class- $\mathcal{A}$ manipulator, with three RRR legs, as shown in Fig. 3.3.

It is recalled that the first type of singularity occurs when the determinant of $\mathbf{J}$ vanishes. Assigning $\mathbf{A}_i = \mathbf{E}$, $\mathbf{C}_i = \mathbf{E}$ and $\mathbf{E}_i = \mathbf{1}$, for $i = 1, 2, 3$ from Table 5.1, eq.(5.35) yields

$$\mathbf{r}_i^T \mathbf{E} \mathbf{a}_i = 0, \quad i = 1 \text{ or } 2 \text{ or } 3 \quad (5.42)$$

This type of configuration is reached whenever $\mathbf{r}_i$ and $\mathbf{a}_i$, for $i = 1$ or 2 or 3 , are parallel, which means that one or some of the legs are fully extended, Fig. 5.3a, or fully folded, Fig. 5.3b¹. At each of these configurations the motion of one actuator,

¹Whenever a pair of rigid-body lines are overlapping they will be depicted, as in Fig. 5.3b, merely close to each other.

that corresponding to the fully extended or fully folded leg, does not produce any motion of $\mathcal{Q}$ along the axis of the corresponding leg.

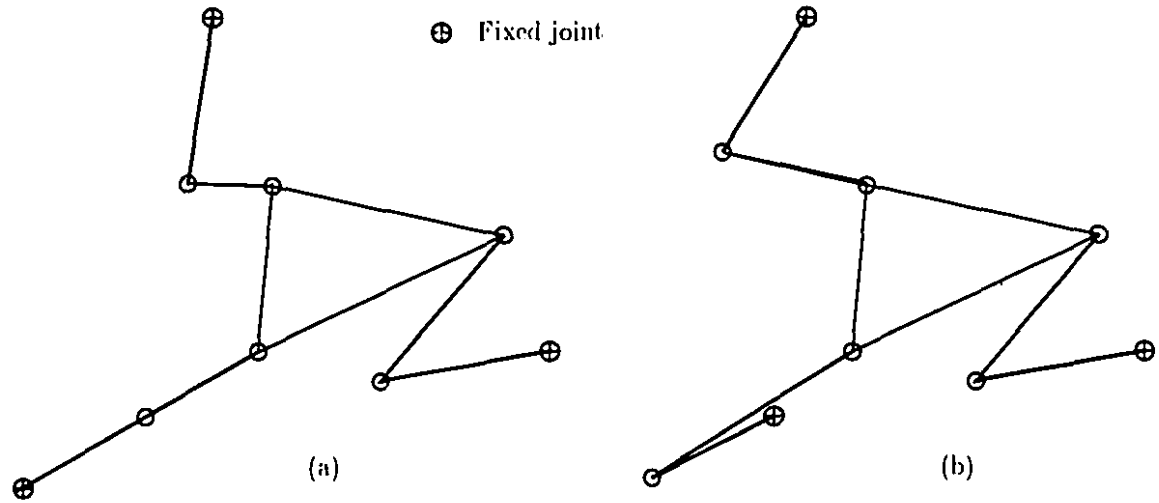


Figure 5.3: Examples of the first type of singularity for the planar 3-RRR manipulator with (a) one leg fully extended, and (b) one leg fully folded

The second type of singularity occurs when the determinant of $\mathbf{K}$ vanishes. Assigning $\mathbf{E}_i = \mathbf{1}$, for $i = 1, 2, 3$, eq.(5.36) yields

$$\mathbf{v}_i = \mathbf{r}_i, \quad i = 1, 2, 3 \quad (5.43)$$

Hence, all the reasoning set forth in the second part of Subsection 5.3.1 applies again if we exchange the roles of $\mathbf{v}_i$ and $\mathbf{r}_i$. Similarly, this type of singularity can arise in two ways. The first occurs when the three vectors $\mathbf{r}_i$ are parallel. Therefore, the second and third columns of $\mathbf{K}$ are linearly dependent, and the nullspace of $\mathbf{K}$ represents the set of pure translations of $\mathcal{Q}$ in a direction normal to $\mathbf{r}_i$, indicated by vector $\mathbf{u}$ of Fig. 5.4a. The platform $\mathcal{Q}$ can move along the direction of $\mathbf{u}$ even if the actuators are locked; likewise, a force applied to $\mathcal{Q}$ in that direction cannot be balanced by the actuators.

The second way in which $\mathbf{K}$ is singular occurs when the three vectors $\mathbf{r}_i$ intersect

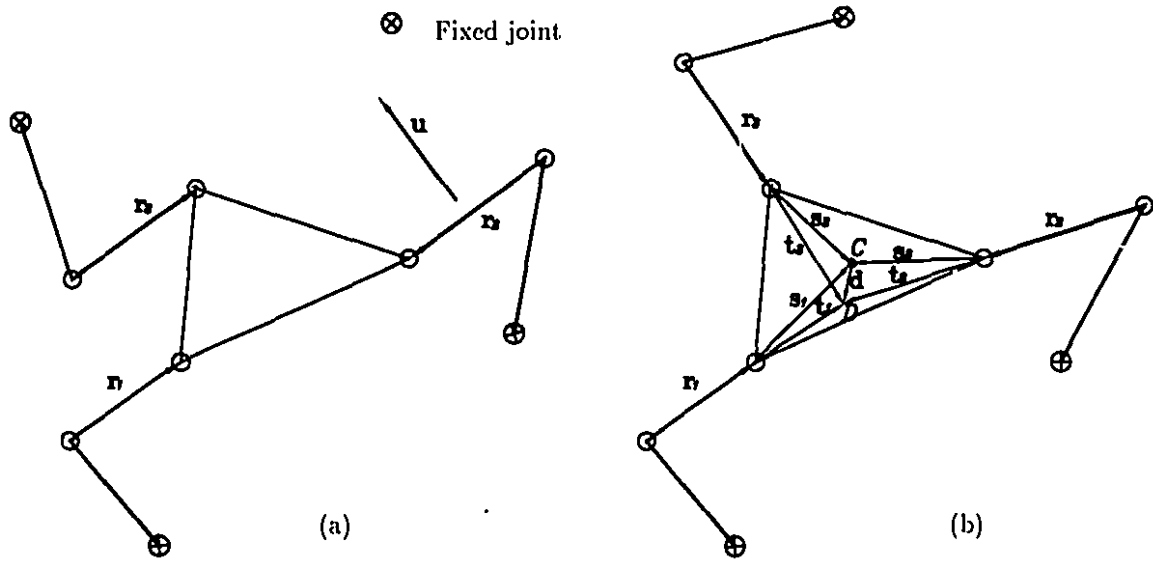


Figure 5.4: Examples of the second type of singularity for the planar 3-RRR manipulator in which (a) the three vectors $\mathbf{r}_i$ are parallel, and (b) the three vectors $\mathbf{r}_i$ intersect at a point

at a common point D , as shown in Fig. 5.4b. Then, the nullspace of $\mathbf{K}$ represents the set of pure rotations of $\mathcal{Q}$ about the common intersection point D . The platform $\mathcal{Q}$ can rotate about that point even if the actuators are locked; likewise, a moment applied to $\mathcal{Q}$ cannot be balanced by the actuators.

The third type of singularity occurs when the determinants of $\mathbf{J}$ and $\mathbf{K}$ both vanish, such that none of the rows of $\mathbf{K}$ vanishes. We have this type of singularity whenever the three vectors $\mathbf{r}_i$ are either parallel or concurrent at a common point and at least one leg is fully extended or fully folded. In the case in which one leg is fully extended, the manipulator might be configured as in Fig. 5.5a or, correspondingly, as in Fig. 5.5b. At these configurations the motion of at least one actuator does not produce any Cartesian velocity along the corresponding leg axis. As well, $\mathcal{Q}$ can move freely in one or more directions even if all actuators are locked and some forces or torque applied to $\mathcal{Q}$ cannot be balanced by the actuators.

By inspection of Figs. 5.5a and 5.5b it is obvious that this type of singularity is not architecture-dependent, because we can change the lengths attached to the base

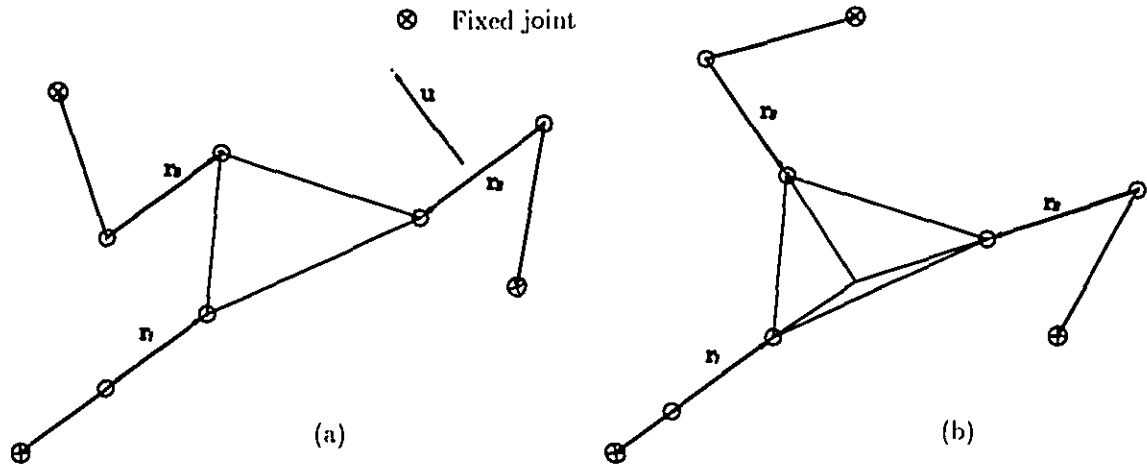


Figure 5.5: Examples of the third type of singularity for the planar 3-RRR manipulator in which (a) the three vectors r_i are parallel, and (b) the three vectors r_i intersect at a point

and intermediate links, while maintaining the third type of singular posture.

5.3.2 Planar Manipulators of Class $\mathcal{B}$

Here, the three types of singularities discussed above are investigated for manipulators of class $\mathcal{B}$.

1) It is recalled that the first type of singularity occurs when the determinant of $\mathbf{J}$ vanishes. From eq.(5.16b), this condition yields

$$\mathbf{b}_i^T \mathbf{E} \mathbf{E}_i \mathbf{a}_i = 0, \quad i = 1 \text{ or } 2 \text{ or } 3 \quad (5.44)$$

This type of configuration is reached whenever $\mathbf{b}_i$ is parallel to $\mathbf{E}_i \mathbf{a}_i$, for $i = 1$ or 2 or 3 . Then, the motion of one actuator does not produce any motion of $\mathcal{Q}$ and the manipulator loses one dof.

2) The second type of singularity occurs when the determinant of $\mathbf{K}$ vanishes. This type of configuration can be inferred from eq.(5.16c) by imposing the linear dependence of the columns or the rows of $\mathbf{K}$. By inspection of this equation, two different cases for which we have this type of singularity can be identified. The first

one occurs when the three vectors $\mathbf{b}_i$ are parallel. Therefore, the second and third columns of $\mathbf{K}$ are linearly dependent, the nullspace of $\mathbf{K}$ thus representing the set of pure translations of $\mathcal{Q}$ along a direction parallel to $\mathbf{b}_i$. Platform $\mathcal{Q}$ can move along that direction even if the actuators are locked; likewise, a force applied to $\mathcal{Q}$ in that direction cannot be balanced by the actuators.

We will show that the second case in which $\mathbf{K}$ is singular occurs when the three vectors $\mathbf{t}_i$ through point A_i and perpendicular to $\mathbf{b}_i$ intersect at a common point. Let us call the intersection point D , as shown in Fig. 5.6.

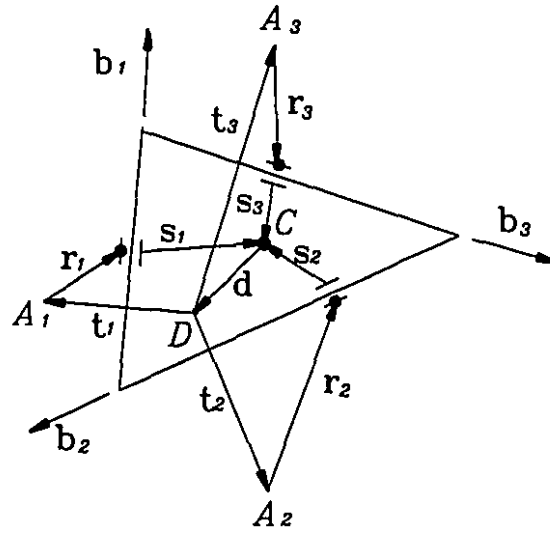


Figure 5.6: Example of the second type of singularity for the manipulators of class $\mathcal{B}$ in which the three vectors $\mathbf{t}_i$ intersect at a point

Since the three vectors $\mathbf{b}_i$, for $i = 1, 2, 3$, are coplanar, we can write $\mathbf{b}_3$ in terms of the first two, namely,

$$\mathbf{b}_3 = \alpha_1 \mathbf{b}_1 + \alpha_2 \mathbf{b}_2 \quad (5.45)$$

Moreover, the inner product of both sides of eq.(5.45) by vector $\mathbf{d}$, leads to

$$\mathbf{b}_3^T \mathbf{d} = \alpha_1 \mathbf{b}_1^T \mathbf{d} + \alpha_2 \mathbf{b}_2^T \mathbf{d} \quad (5.46)$$

where $\mathbf{d} = \overrightarrow{CD}$. But we have

$$\mathbf{b}_i^T \mathbf{t}_i = 0, \quad i = 1, 2, 3$$

Then, eq.(5.46) can be written as

$$\mathbf{b}_3^T(\mathbf{d} + \mathbf{t}_3) = \alpha_1 \mathbf{b}_1^T(\mathbf{d} + \mathbf{t}_1) + \alpha_2 \mathbf{b}_2^T(\mathbf{d} + \mathbf{t}_2) \quad (5.47)$$

Moreover, we have

$$\mathbf{d} + \mathbf{t}_i = -(\mathbf{r}_i + \mathbf{s}_i), \quad i = 1, 2, 3$$

Substituting the values of $\mathbf{d} + \mathbf{t}_i$, for $i = 1, 2, 3$, from the foregoing equation into eq.(5.47), yields

$$\mathbf{b}_3^T(\mathbf{r}_3 + \mathbf{s}_3) = \alpha_1 \mathbf{b}_1^T(\mathbf{r}_1 + \mathbf{s}_1) + \alpha_2 \mathbf{b}_2^T(\mathbf{r}_2 + \mathbf{s}_2) \quad (5.48)$$

Moreover, from eq.(5.45), it is apparent that

$$\mathbf{b}_3^T \mathbf{E} = \alpha_1 \mathbf{b}_1^T \mathbf{E} + \alpha_2 \mathbf{b}_2^T \mathbf{E} \quad (5.49)$$

From eqs.(5.48) and (5.49), it is obvious that one can write the third row of $\mathbf{K}$ as a linear combination of the first two rows, thereby completing the proof.

Then, the nullspace of $\mathbf{K}$ represents the set of pure rotations of $\mathcal{Q}$ about the common intersection point D . The platform $\mathcal{Q}$ can thus rotate about that point even if the actuators are locked; likewise, a moment applied to $\mathcal{Q}$ cannot be balanced by the actuators.

3) The third type of singularity occurs when the determinants of $\mathbf{J}$ and $\mathbf{K}$ both vanish. This type of singularity occurs whenever the two types of singularities arises simultaneously.

Inspection of eq.(5.16c) reveals that the rows of $\mathbf{K}$ cannot vanish, because $\|\mathbf{b}_i\| = 1$, for $i = 1, 2, 3$.

Example 5.3.2.1: Planar DT Manipulator

The three types of singularities discussed above are investigated here for a special type of class- $\mathcal{B}$ manipulator that has three PRP legs, namely the *double-triangular* (DT) manipulator shown in Fig. 3.5.

It is recalled that the first type of singularity occurs when the determinant of $\mathbf{J}$ vanishes. Assigning $\mathbf{E}_i = \mathbf{1}$ from eq.(5.11b), for $i = 1, 2, 3$, eq.(5.44) leads to

$$\mathbf{b}_i^T \mathbf{E} \mathbf{a}_i = 0, \quad i = 1 \text{ or } 2 \text{ or } 3 \quad (5.50)$$

This type of configuration is reached whenever $\mathbf{a}_i$ and $\mathbf{b}_i$, for $i = 1$ or 2 or 3 , coincide, which means that one or more edges of the triangles coincide, as shown in Fig. 5.7. In this configuration the motion of the i th actuator does not produce any motion of Q , the moving triangle, and the manipulator cannot move in a direction perpendicular to the coincident edges.

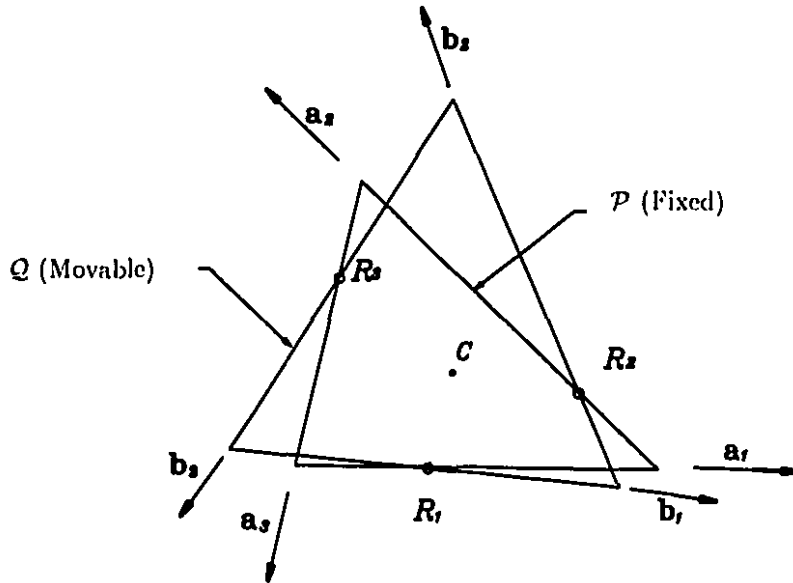


Figure 5.7: Example of the first type of singularity for the planar DT manipulator

The second type of singularity occurs when the determinant of $\mathbf{K}$ vanishes. As we explained in Subsection 5.3.2, this type of singularity arises in two cases. The first occurs when the three vectors $\mathbf{b}_i$ are parallel, but such a manipulator is not a DT manipulator and is thus left aside. The second case in which $\mathbf{K}$ is singular occurs when the three vectors $\mathbf{t}_i$, perpendicular to $\mathbf{b}_i$, intersect at a common point D , as shown in Fig. 5.8. In this configuration the moving triangle Q can undergo a finite

rotation about D , even if the actuators are locked; likewise, a torque applied to Q cannot be balanced by the actuators.

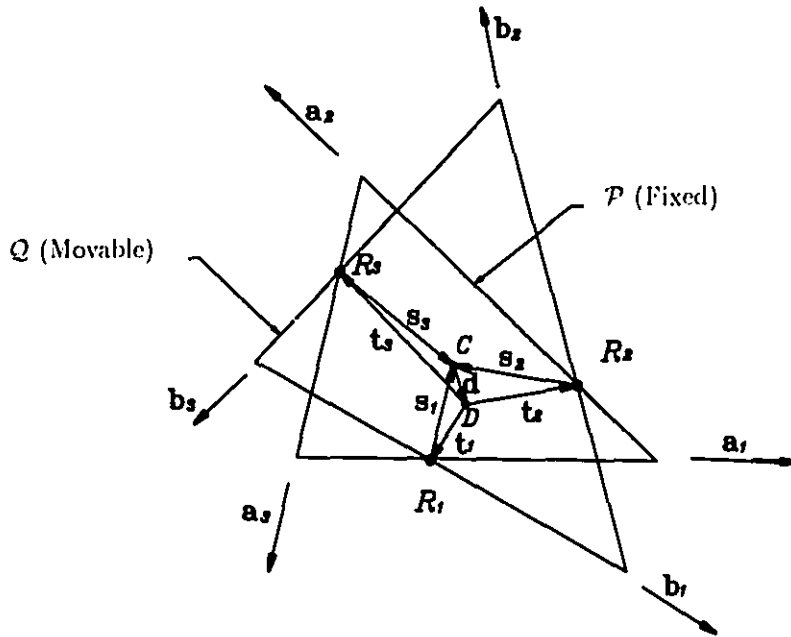


Figure 5.8: Example of the second type of singularity for the planar DT manipulator

The third type of singularity occurs when the determinants of $\mathbf{J}$ and $\mathbf{K}$ both vanish. We have this type of singularity whenever the three perpendiculars to the three edges of the moving triangle intersect at a common point and at least one pair of the edges of the two triangles coincide, as shown in Fig. 5.9. At this configuration the motion of one actuator does not produce any Cartesian velocity and the manipulator loses one dof. As well, the moving triangle Q can undergo a finite rotation about D , even if the actuators are locked; likewise, a torque applied to Q cannot be balanced by the actuators.

Again, for DT manipulators, this type of singularity is not architecture-dependent, since we can find one point in the plane of the moving triangle Q from which we can draw three perpendiculars to the three edges. Let us call the intersection points R_i ,

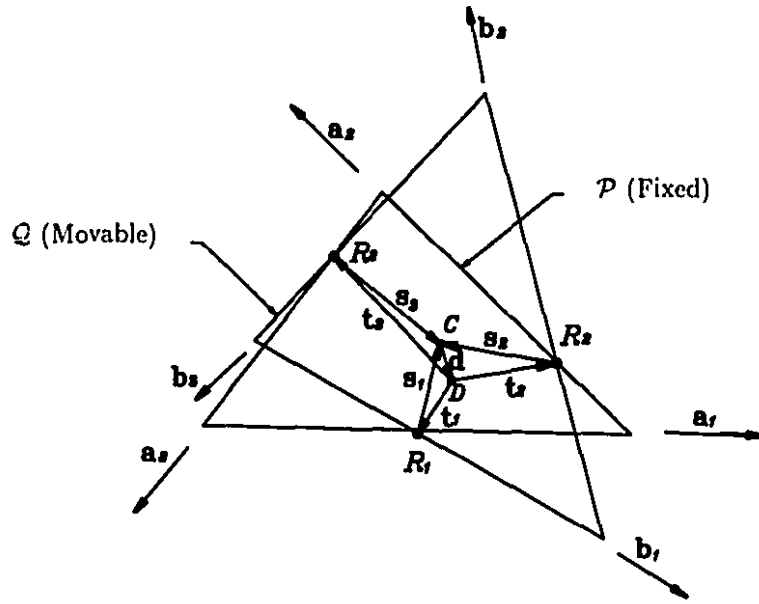


Figure 5.9: Example of the third type of singularity for the planar DT manipulator for $i = 1, 2, 3$, as shown in Fig. 5.9. It is obvious that any three lines passing through points R_i such that one of them coincides with one of the edges of the moving triangle can form the fixed triangle $\mathcal{P}$. Needless to say, such a triangle is not unique. In other words, we can choose the fixed and moving triangles arbitrarily.

5.3.3 Spherical 3-RRR Manipulator

In this subsection, the three types of singularities discussed above are investigated for the manipulator of Fig. 3.6. It is recalled that the first type of singularity occurs when the determinant of $\mathbf{J}$ vanishes. From eq.(5.20b), this condition yields

$$(\mathbf{v}_i \times \mathbf{u}_i) \cdot \mathbf{w}_i = 0, \quad i = 1 \text{ or } 2 \text{ or } 3 \quad (5.51)$$

This type of configuration is reached whenever $\mathbf{u}_i$, $\mathbf{v}_i$ and $\mathbf{w}_i$, for $i = 1$ or 2 or 3 , are coplanar, which means that one or some of the legs are fully extended, Fig. 5.10, or

fully folded, Fig. 5.11. At each of these configurations the motion of one actuator, that corresponding to the fully extended or folded leg, does not produce any motion of the EE.

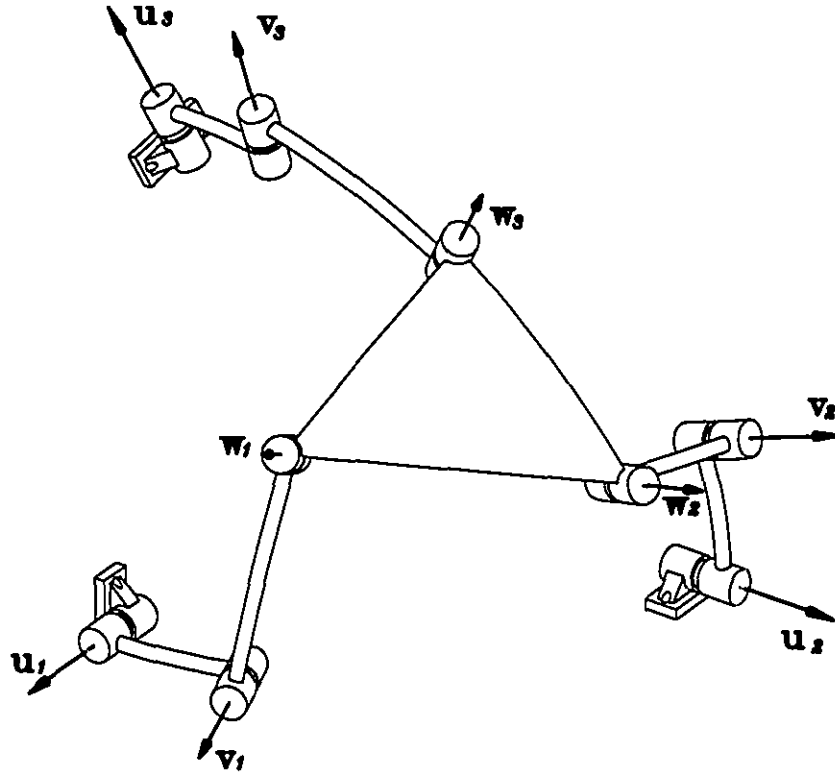


Figure 5.10: The first type of singularity of the spherical 3-RRR manipulator with one leg fully extended

The second type of singularity occurs when the determinant of $\mathbf{K}$ vanishes, which, in turn, occurs when the rows or columns of $\mathbf{K}$ are linearly dependent. By inspection of eq.(5.20c), we now show that this type of singularity occurs when the three planes defined by the axes of the revolute parallel to the unit vectors $\{v_i, w_i\}_1^3$ intersect at a common line. This can be readily seen by noting that the three vectors $v_i \times w_i$, for $i = 1, 2, 3$, which are perpendicular to the plane of v_i and w_i , are perpendicular to the intersection line. Then, these vectors are coplanar and each of them, which represents a row of $\mathbf{K}$, can be written as a linear combination of the other two. This is what we set out to show. This type of singularity is depicted in Fig. 5.12.

The third type of singularity occurs when the determinants of $\mathbf{J}$ and $\mathbf{K}$ both

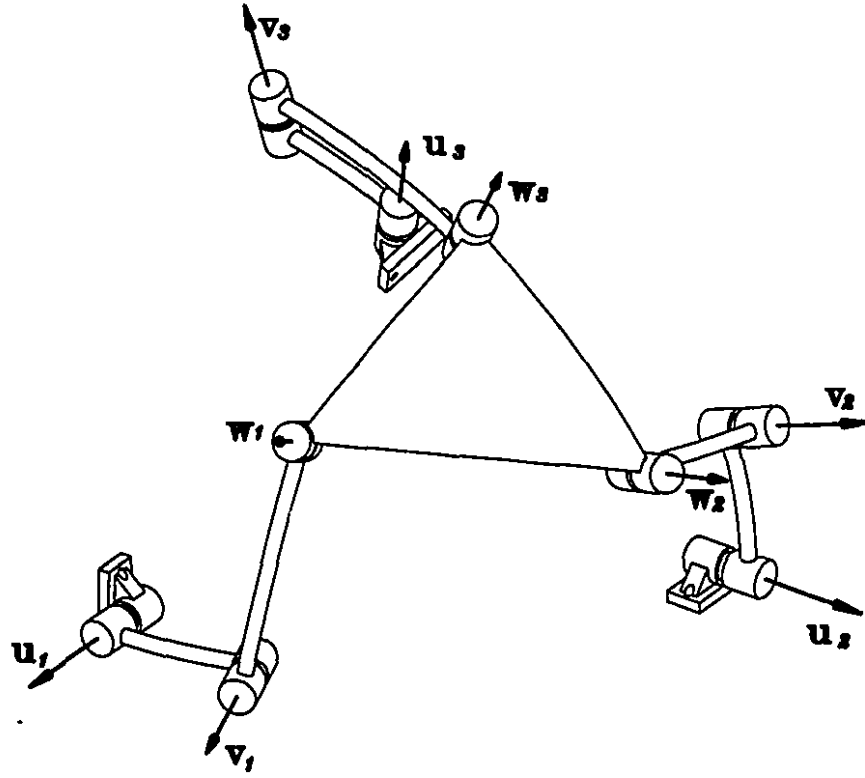


Figure 5.11: The first type of singularity of the spherical 3-RRR manipulator with one leg folded

vanish. We have this type of singularity whenever the two foregoing singularities occur simultaneously. In this case $\mathbf{k}_i \neq \mathbf{0}$, where $\mathbf{k}_i^T$, for $i = 1, 2, 3$, is the i th row of $\mathbf{K}$, the manipulator would then be configured as in Fig. 5.13. At this configuration, at least one actuator cannot produce any Cartesian velocity. As well, the gripper can rotate freely about the common intersection line of the planes defined by the axes of the revolute parallel to the unit vectors $\{\mathbf{v}_i, \mathbf{w}_i\}_1^3$, even if all of the actuators are locked and certain torques applied to the gripper cannot be balanced by the actuators.

Inspection of eq.(5.20c) reveals that the i th row of $\mathbf{K}$ vanishes only if $\mathbf{v}_i = \pm \mathbf{w}_i$. In this case we have a degenerate case of a 3-RRR manipulator with one leg of zero or π length. Such a manipulator is irrelevant to our study and is thus left aside.

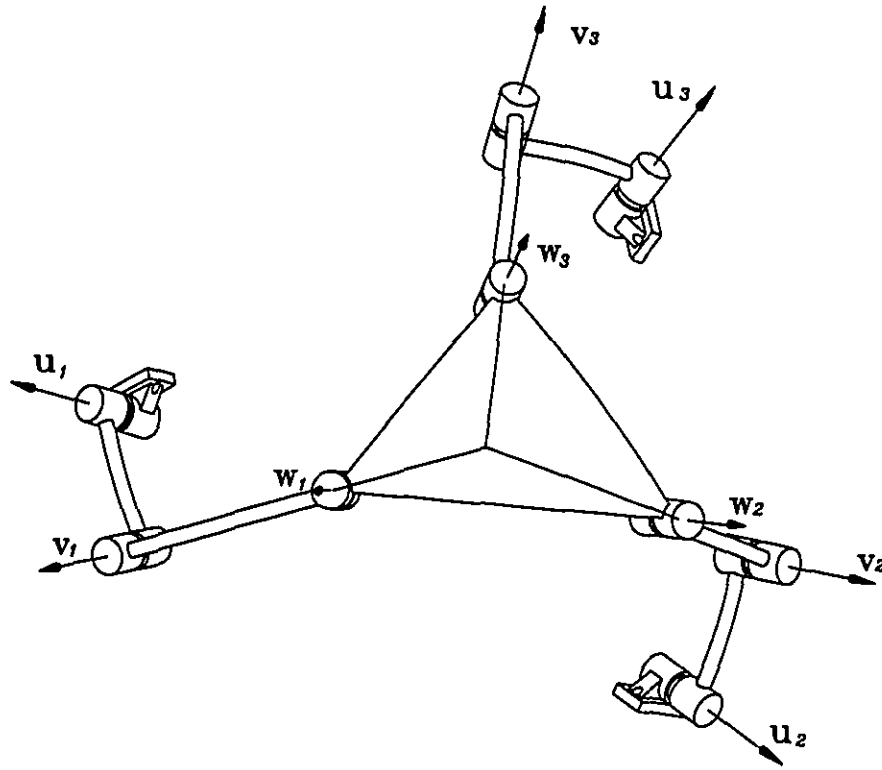


Figure 5.12: The second type of singularity of the spherical 3-RRR manipulator

5.3.4 Spherical DT Manipulator

In this subsection, the three types of singularities are investigated for the manipulator of Fig. 3.7. It is recalled that the first type of singularity occurs when the determinant of $\mathbf{J}$ vanishes. From eq.(5.24b), this condition yields

$$(\mathbf{r}_i \times \mathbf{b}_i) \cdot \mathbf{a}_i = 0, \quad i = 1 \text{ or } 2 \text{ or } 3 \quad (5.52)$$

This type of configuration is reached whenever $\mathbf{a}_i$ is perpendicular to $\mathbf{r}_i \times \mathbf{b}_i$, but $\mathbf{r}_i$ lies in the plane whose normal is $\mathbf{a}_i$, as shown in Fig. 5.1. Then, this type of singularity occurs whenever $\mathbf{b}_i$ and $\mathbf{a}_i$ coincide. In other words, each pair of two sides of two triangles lie in the same plane, as shown in Fig. 5.14. In this case the actuator along $\mathbf{a}_i$ does not produce any Cartesian velocity.

The second type of singularity occurs when the determinant of $\mathbf{K}$ vanishes, which occurs when the rows or columns of $\mathbf{K}$ are linearly dependent. By inspection of

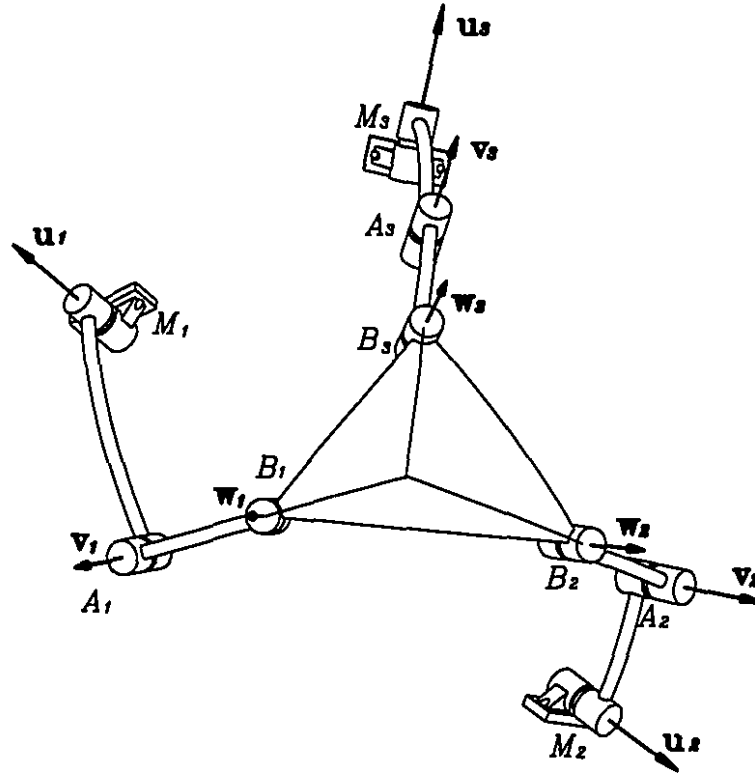


Figure 5.13: The third type of singularity of the spherical 3-RRR manipulator

eq.(5.24c), we will show that this type of singularity occurs when the three planes containing vectors $\mathbf{r}_i$ and $\mathbf{b}_i$ intersect at a common line. This can be readily seen by noting that the three vectors $\mathbf{r}_i \times \mathbf{b}_i$, for $i = 1, 2, 3$, which are perpendicular to the planes, are perpendicular to the intersection line as well. Then, these vectors are coplanar and each of them, which represents a row of $\mathbf{K}$, can be written as a linear combination of the other two, thereby completing the proof. This type of singularity is depicted in Fig. 5.15.

The third type of singularity occurs when the determinants of $\mathbf{J}$ and $\mathbf{K}$ both vanish. We have this type of singularity whenever the two foregoing singularities occur simultaneously. In this case, $\mathbf{k}_i \neq \mathbf{0}$, where $\mathbf{k}_i^T$, for $i = 1, 2, 3$, is the i th row of $\mathbf{K}$, the manipulator would then be configured as in Fig. 5.16. In this configuration the motion of at least one actuator does not produce any Cartesian velocity. As well, the gripper can rotate freely about the common intersection line of the planes

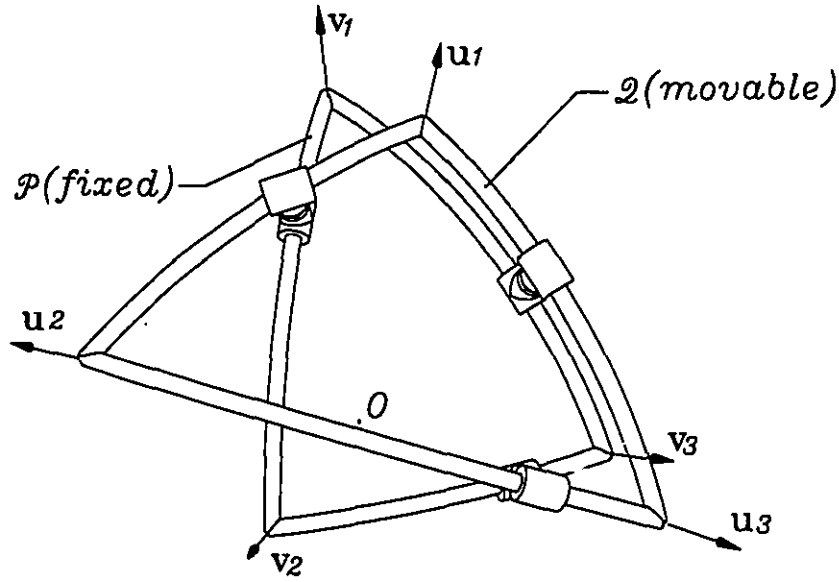


Figure 5.14: Spherical DT manipulator at the first type of singularity

defined by $\{\mathbf{r}_i \times \mathbf{b}_i\}_1^3$, even if all of the actuators are locked, and certain torques applied to the gripper cannot be balanced by the actuators.

Inspection of eq.(5.24c) reveals that the rows of $\mathbf{K}$ cannot vanish, because $\mathbf{b}_i$ is always perpendicular to $\mathbf{r}_i$, both being unit vectors.

Moreover, this type of singularity is not architecture-dependent, since we can find one point in the moving triangle Q from which we can draw three perpendiculars to the three edges. Let us call the intersection points R_i , for $i = 1, 2, 3$, as shown in Fig. 5.16. It is obvious that any three arcs passing through points R_i , for $i = 1, 2, 3$, such that one of them coincides with one of the edges of the moving triangles, can form an edge of the fixed triangle P . Needless to say, such a triangle is not unique. In other words, we can choose the fixed and moving triangles arbitrarily.

5.3.5 Spatial 6-DOF, DT Manipulator

In this subsection, the three types of singularities are investigated for the manipulator introduced in Subsection 3.4.1. It is recalled that the first type of singularity occurs

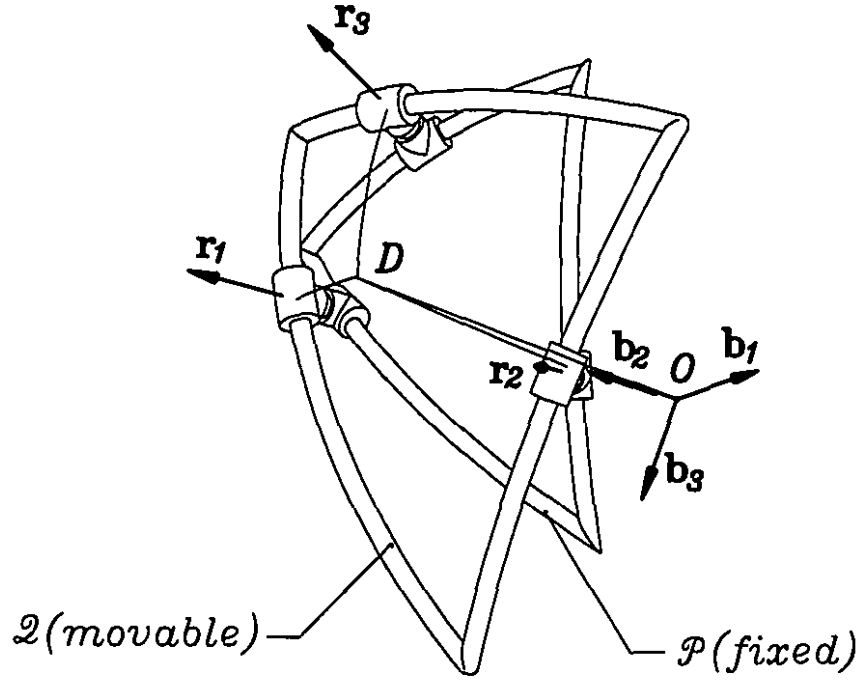


Figure 5.15: Spherical DT manipulator at the second type of singularity

when the determinant of $\mathbf{J}$ vanishes. From eq.(5.31c), this condition yields

$$\mathbf{a}_i^T \mathbf{m}_i = \mathbf{a}_i^T (\mathbf{b}_i \times \mathbf{r}_i) = 0, \quad i = 1 \text{ or } 2 \text{ or } 3 \quad (5.53)$$

This type of configuration is reached whenever $\mathbf{a}_i$, $\mathbf{b}_i$ and $\mathbf{r}_i$ lie in a plane. But, $\mathbf{r}_i$ is perpendicular to $\mathbf{a}_i$ and $\mathbf{b}_i$. Then, this type of singularity occurs whenever $\mathbf{a}_i$ and $\mathbf{b}_i$ are parallel, and the prismatic actuator along $\mathbf{a}_i$ does not produce any Cartesian velocity, as shown in Fig. 5.17.

The second type of singularity occurs when the determinant of $\mathbf{K}$ vanishes, which occurs when the rows or columns of $\mathbf{K}$ are linearly dependent. By inspection of eq.(5.31d), three different cases in which this type of singularity arises can be identified. The first occurs when the three vectors $\mathbf{b}_i$ are parallel. Since $\mathbf{b}_i$ is perpendicular to $\mathbf{r}_i$, $\{\mathbf{r}_i\}_1^3$ are coplanar. Therefore, $\{\mathbf{m}_i = \mathbf{b}_i \times \mathbf{r}_i\}_1^3$ lie in a plane, and we can write

$$\mathbf{m}_3/r_3 = \alpha_1 \mathbf{m}_1/r_1 + \alpha_2 \mathbf{m}_2/r_2 \quad (5.54)$$

As a result, the sixth column of $\mathbf{K}$ is a linear combination of the fourth and the fifth columns. This will render $\det(\mathbf{K}) = 0$. In this type of singularity, the movable

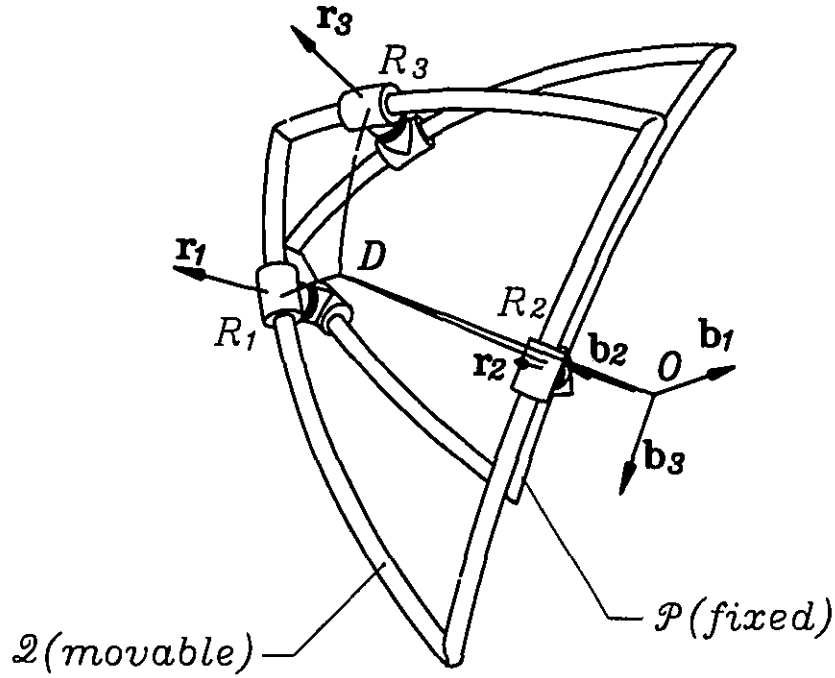


Figure 5.16: Spherical DT manipulator at the third type of singularity

triangle Q can move along $\mathbf{b}_i$, even if all the actuators are locked, and any force applied to Q along $\mathbf{b}_i$ cannot be balanced by the actuators, as shown in Fig. 5.18.

The second case, in which this type of singularity occurs, arises when $\mathbf{b}_i$ of two legs are parallel to $\mathbf{r}_i$ of the third leg. The reasoning set forth in the foregoing discussions for $\{\mathbf{m}_i = \mathbf{b}_i \times \mathbf{r}_i\}_1^3$ applies here if we exchange the roles of $\mathbf{r}_i$ and $\mathbf{b}_i$ of the third leg. Then, eq.(5.54) holds, $\{\mathbf{m}_i\}_1^3$ are coplanar, and, as a result, the sixth column of $\mathbf{K}$ is a linear combination of the fourth and the fifth columns and the movable triangle Q can move along the three parallel axes, even if all the actuators are locked.

The third case, in which we have this type of singularity, occurs when $\mathbf{r}_i$ of two legs are parallel to $\mathbf{b}_i$ of the third leg. The reasoning set forth in the foregoing discussions for $\{\mathbf{m}_i = \mathbf{b}_i \times \mathbf{r}_i\}_1^3$, again, applies if we exchange the roles of $\mathbf{r}_i$ and $\mathbf{b}_i$. Then, eq.(5.54) holds as well, $\{\mathbf{m}_i\}_1^3$ are coplanar and, similarly, $\mathbf{K}$ is singular.

The third type of singularity occurs when the determinants of $\mathbf{J}$ and $\mathbf{K}$ both vanish. We have this type of singularity whenever three of the six vectors $\{\mathbf{b}_i\}_1^3$ and

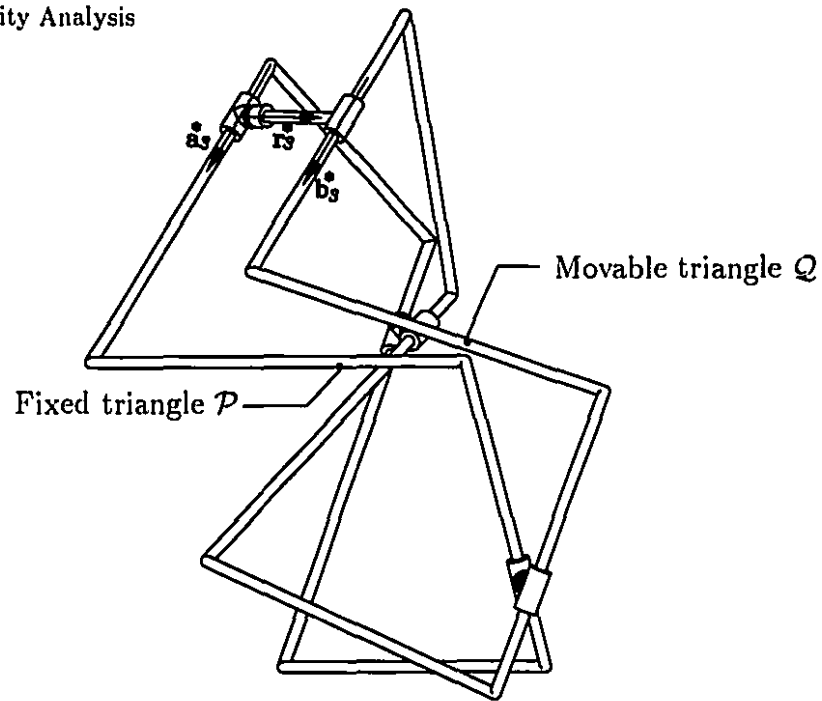


Figure 5.17: Spatial 6-dof, DT manipulator at the first type of singularity

$\{\mathbf{r}_i\}_1^3$ are parallel, and $\mathbf{a}_i$ and $\mathbf{b}_i$, for $i = 1$ or 2 or 3 , are parallel as well. In this case the movable triangle Q can move freely about an axis parallel to the three parallel axes, even if all actuators are locked and any force applied to Q in that direction cannot be balanced by the actuators. Moreover, at least one actuator cannot produce any Cartesian velocity along the corresponding leg axis, as shown in Fig. 5.19.

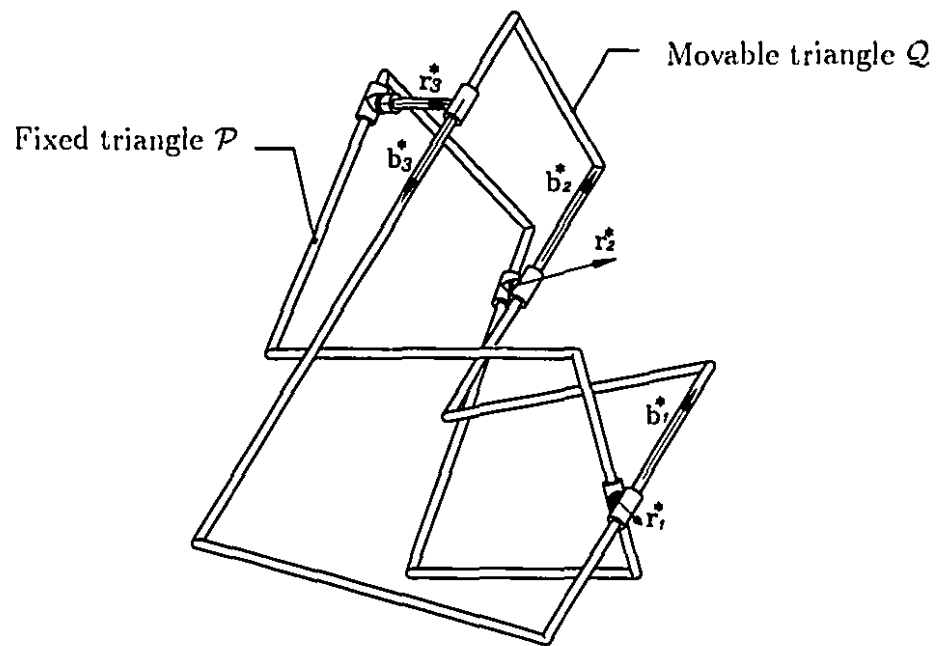


Figure 5.18: Spatial 6-dof, DT manipulator at the second type of singularity

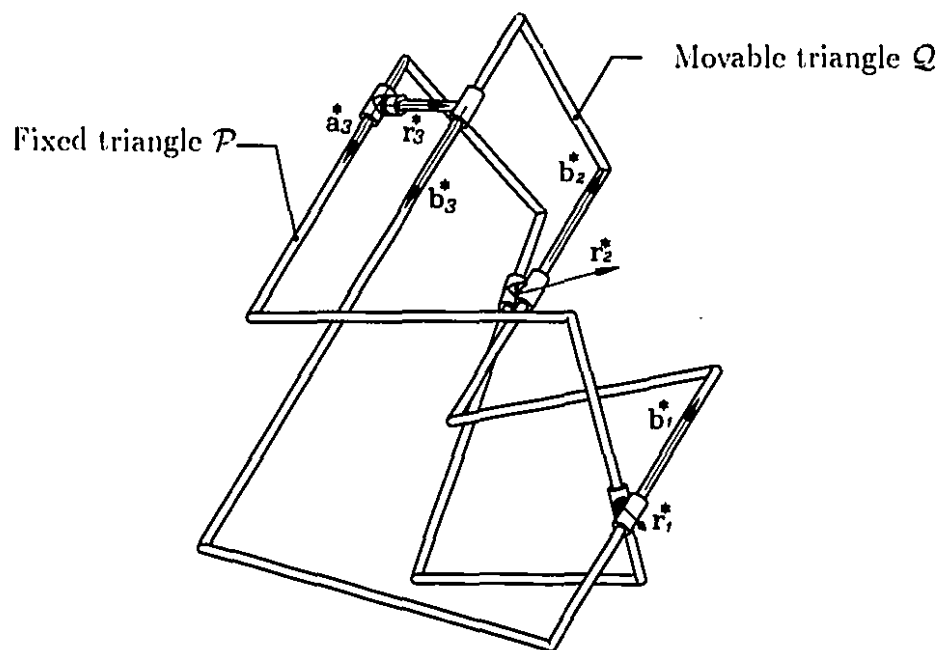


Figure 5.19: Spatial 6-dof, DT manipulator at the third type of singularity

Chapter 6

Isotropic Designs

6.1 Introduction

The concept of manipulator isotropy, based on the condition numbers of the Jacobian matrices, is now explained, as pertaining to parallel manipulators. Using this concept, the isotropic designs of two general classes of planar parallel manipulators, of spherical DT and 3-RRR parallel manipulators, and of spatial 6-dof, DT mechanism, introduced in Chapter 3, are found. Having derived the Jacobian matrices of the manipulators, in an invariant form in Chapter 5, allow us to find all isotropic designs.

6.2 Isotropic Designs

Mechanism control accuracy depends upon the condition number of the Jacobian matrices $\mathbf{J}$ and $\mathbf{K}$. The condition number is based on a concept common to all matrices, whether square or not, i.e., their *singular values*. For an $m \times n$ matrix $\mathbf{A}$, with $m < n$, we can define its m singular values as the non-negative square roots of the non-negative eigenvalues of the $m \times m$ matrix $\mathbf{A}\mathbf{A}^T$. Because $\mathbf{A}\mathbf{A}^T$ is square, symmetric and at least positive-semidefinite, its eigenvalues are all real and non-negative. Also, if the matrix under investigation is dimensionally homogeneous

which, in our case, happens for $\mathbf{J}$ and $\mathbf{K}$ only in the spherical case, then we can meaningfully order the singular values of these matrices from smallest to largest. If, on the other hand, these matrices are not dimensionally homogeneous, which is the case for planar and spatial tasks involving both positioning and orienting, or manipulators with both prismatic and revolute actuators; then we can redefine these matrices by recalling the concept of *characteristic length*, first introduced in (Tandirci et al., 1992), and dividing the elements that have units of length by this quantity. Therefore, we can always produce a dimensionally-homogeneous Jacobian matrix, which enables a meaningful ordering of its singular values from smallest to largest. Thus, if σ_m and σ_M denote the smallest and the largest singular values of a matrix, its condition number is then defined as

$$\kappa(\theta) \equiv \frac{\sigma_M}{\sigma_m} \quad (6.1)$$

and hence, the larger the variance of the singular values, the larger the condition number. The significance of the condition number of a matrix pertains to the numerical inversion of this matrix when solving a system of linear equations associated with the matrix. Clearly, in the case of non-square matrices, this inversion is understood as a *generalized inversion*. Indeed, when inverting a matrix with finite precision, a roundoff error is always present, and hence, a roundoff-error amplification affects the accuracy of the computed results. Furthermore, this amplification is bounded by the condition number of the matrix. It is apparent that a singular matrix has a minimum singular value of zero, and hence, its condition number becomes infinite. Conversely, if the singular values of a matrix are identical, then the condition number of the matrix attains a minimum value of unity, matrices with such a property being called *isotropic*. The reason why isotropic matrices are desirable is that they can be inverted at no cost because the inverse of an isotropic matrix, or the generalized inverse of a rectangular isotropic matrix for that matter, is proportional to its transpose, the proportionality factor being the reciprocal of its multiple singular value.

From the above discussion, and considering that the Jacobian matrices are configuration-dependent, it is apparent that the condition number of the Jacobian matrices of a manipulator is configuration-dependent as well, and hence, a manipulator can be designed with an architecture that allows for postures entailing isotropic Jacobian matrices, such a design being called isotropic. However, this property disappears in all other postures. This is a fact of life and nothing can be done about it, but one can design for postures that are isotropic, and then plan tasks that lie well within a region where the condition number is acceptable. For manipulators with isotropic designs, such regions cover a substantial percentage of the overall workspace, the condition number degenerating only for postures very close to singularities, which should be avoided in trajectory planning, in any event.

Below we will find the isotropic designs of several manipulators introduced in Chapter 3.

6.2.1 Planar Manipulators of Class $\mathcal{A}$

In this subsection we find isotropic designs for planar manipulators of class $\mathcal{A}$. It is recalled that a design is isotropic if both $\mathbf{J}$ and $\mathbf{K}$ are isotropic, i.e., if positive scalars σ and τ exist such that

$$\mathbf{J}\mathbf{J}^T = \sigma^2 \mathbf{1} \quad (6.2a)$$

$$\mathbf{K}\mathbf{K}^T = \tau^2 \mathbf{1} \quad (6.2b)$$

where $\mathbf{J}$ and $\mathbf{K}$ are given in eqs.(5.9b) and (5.9c), respectively. But $\mathbf{J}$ is not dimensionally-homogeneous if we have different types of actuators, i.e., if some actuators are revolute and the others are prismatic. If this is the case, in order to render $\mathbf{J}$ dimensionally-homogeneous we divide the i th column of $\mathbf{J}$ by a length l_i , the characteristic length of the i th leg of the manipulator, understood here as defined

in (Tandirci et al., 1992) for serial manipulators, and redefine $\mathbf{J}$ as

$$\mathbf{J} \leftarrow \begin{bmatrix} \mathbf{r}_1^T \mathbf{E}_1 (\mathbf{A}_1 \mathbf{a}_1 + \mathbf{C}_1 \mathbf{r}_1) / l_1 & 0 & 0 \\ 0 & \mathbf{r}_2^T \mathbf{E}_2 (\mathbf{A}_2 \mathbf{a}_2 + \mathbf{C}_2 \mathbf{r}_2) / l_2 & 0 \\ 0 & 0 & \mathbf{r}_3^T \mathbf{E}_3 (\mathbf{A}_3 \mathbf{a}_3 + \mathbf{C}_3 \mathbf{r}_3) / l_3 \end{bmatrix} \quad (6.3)$$

where $l_i = 1$, for $i = 1, 2, 3$, if we have the same types of actuators in all legs or the actuator of the i th leg is prismatic.

Matrix $\mathbf{K}$ is not dimensionally-homogeneous either. To render $\mathbf{K}$ dimensionally-homogeneous we divide the first column of $\mathbf{K}$ by a length L , the characteristic length of the manipulators, and redefine the Jacobian $\mathbf{K}$ as

$$\mathbf{K} \leftarrow \begin{bmatrix} \mathbf{r}_1^T \mathbf{E}_1 \mathbf{E} \mathbf{s}_1 / L & -\mathbf{r}_1^T \mathbf{E}_1 \\ \mathbf{r}_2^T \mathbf{E}_2 \mathbf{E} \mathbf{s}_2 / L & -\mathbf{r}_2^T \mathbf{E}_2 \\ \mathbf{r}_3^T \mathbf{E}_3 \mathbf{E} \mathbf{s}_3 / L & -\mathbf{r}_3^T \mathbf{E}_3 \end{bmatrix} \quad (6.4)$$

Substitution of the values of $\mathbf{J}$ and $\mathbf{K}$ of eqs.(6.3) and (6.4) into eqs.(6.2a) and (6.2b), respectively, upon simplification yields

$$\begin{bmatrix} (\mathbf{b}_1^T (\mathbf{A}_1 \mathbf{a}_1 + \mathbf{C}_1 \mathbf{r}_1) / l_1)^2 & 0 & 0 \\ 0 & (\mathbf{b}_2^T (\mathbf{A}_2 \mathbf{a}_2 + \mathbf{C}_2 \mathbf{r}_2) / l_2)^2 & 0 \\ 0 & 0 & (\mathbf{b}_3^T (\mathbf{A}_3 \mathbf{a}_3 + \mathbf{C}_3 \mathbf{r}_3) / l_3)^2 \end{bmatrix} = \sigma^2 \mathbf{1} \quad (6.5a)$$

and

$$\begin{bmatrix} a_1^2 / L^2 + \mathbf{b}_1^T \mathbf{b}_1 & a_1 a_2 / L^2 + \mathbf{b}_1^T \mathbf{b}_2 & a_1 a_3 / L^2 + \mathbf{b}_1^T \mathbf{b}_3 \\ a_1 a_2 / L^2 + \mathbf{b}_2^T \mathbf{b}_1 & a_2^2 / L^2 + \mathbf{b}_2^T \mathbf{b}_2 & a_2 a_3 / L^2 + \mathbf{b}_2^T \mathbf{b}_3 \\ a_1 a_3 / L^2 + \mathbf{b}_3^T \mathbf{b}_1 & a_2 a_3 / L^2 + \mathbf{b}_3^T \mathbf{b}_2 & a_3^2 / L^2 + \mathbf{b}_3^T \mathbf{b}_3 \end{bmatrix} = \tau^2 \mathbf{1} \quad (6.5b)$$

where a_i and $\mathbf{b}_i$, for $i = 1, 2, 3$, are defined as

$$a_i \equiv \mathbf{r}_i^T \mathbf{E}_i \mathbf{E} \mathbf{s}_i \quad (6.5c)$$

$$\mathbf{b}_i \equiv \mathbf{E}_i^T \mathbf{r}_i \quad (6.5d)$$

Equations (6.5a) and (6.5b) lead to the conditions for isotropic design of this class of manipulators, namely,

$$(\mathbf{b}_1^T(\mathbf{A}_1\mathbf{a}_1 + \mathbf{C}_1\mathbf{r}_1)/l_1)^2 = \sigma^2 \quad (6.6a)$$

$$(\mathbf{b}_2^T(\mathbf{A}_2\mathbf{a}_2 + \mathbf{C}_2\mathbf{r}_2)/l_2)^2 = \sigma^2 \quad (6.6b)$$

$$(\mathbf{b}_3^T(\mathbf{A}_3\mathbf{a}_3 + \mathbf{C}_3\mathbf{r}_3)/l_3)^2 = \sigma^2 \quad (6.6c)$$

$$\mathbf{b}_1^T\mathbf{b}_1 = \mathbf{b}_2^T\mathbf{b}_2 = \mathbf{b}_3^T\mathbf{b}_3 \quad (6.6d)$$

$$q + \frac{p^2}{L^2} = 0 \quad (6.6e)$$

where

$$a_1 = a_2 = a_3 = p \quad (6.6f)$$

$$\mathbf{b}_1^T\mathbf{b}_2 = \mathbf{b}_1^T\mathbf{b}_3 = \mathbf{b}_2^T\mathbf{b}_3 = q \quad (6.6g)$$

and hence

$$L^2 = \frac{p^2}{-q} \quad (6.7a)$$

$$l_i = \left\| \frac{\mathbf{b}_i^T(\mathbf{A}_i\mathbf{a}_i + \mathbf{C}_i\mathbf{r}_i)}{\sigma} \right\|, \quad i = 1, 2, 3 \quad (6.7b)$$

Example 6.2.1.1: Planar 3-RRR Manipulator

Here, we find isotropic designs for a particular case of class- $\mathcal{A}$ manipulator, with three RRR legs, as shown in Fig. 3.3. The isotropic design of this manipulator has been addressed in the literature, namely, by Gosselin (1988) and Gosselin and Angeles (1988). By resorting to numerical methods, they found a number of discrete isotropic designs for the manipulator.

Assigning $\mathbf{A}_i = \mathbf{E}$, $\mathbf{C}_i = \mathbf{E}$, $\mathbf{E}_i = \mathbf{1}$, from Table 5.1, and $l_i = 1$, for $i = 1, 2, 3$, the conditions for isotropic design, namely, eqs.(6.6a-e) yield

$$(\mathbf{r}_1^T\mathbf{E}\mathbf{a}_1)^2 = (\mathbf{r}_2^T\mathbf{E}\mathbf{a}_2)^2 = (\mathbf{r}_3^T\mathbf{E}\mathbf{a}_3)^2 = \sigma^2 \quad (6.8a)$$

$$\mathbf{r}_1^T\mathbf{r}_1 = \mathbf{r}_2^T\mathbf{r}_2 = \mathbf{r}_3^T\mathbf{r}_3 \quad (6.8b)$$

$$q + \frac{p^2}{L^2} = 0 \quad (6.8c)$$

where

$$a_i = \mathbf{r}_i^T \mathbf{E} \mathbf{s}_i = p, \quad i = 1, 2, 3 \quad (6.8d)$$

$$\mathbf{r}_1^T \mathbf{r}_2 = \mathbf{r}_1^T \mathbf{r}_3 = \mathbf{r}_2^T \mathbf{r}_3 = q \quad (6.8e)$$

Equations (6.8a– 6.8c), the conditions for isotropic design, produce manipulators with the following characteristics:

- 1) The base and the EE triangles are equilateral and share a common centroid at the isotropic configuration;
- 2) corresponding leg links have the same length;
- 3) the angles between the leg links are equal.

The foregoing characteristics lead to a three-parameter continuum for isotropic designs of the manipulator. The three-dimensional design parameters ρ , α and β are defined as follows:

$$\begin{aligned} a &= \|\mathbf{a}_1\| = \|\mathbf{a}_2\| = \|\mathbf{a}_3\| \\ \rho &= \frac{\|\mathbf{r}_1\|}{a} = \frac{\|\mathbf{r}_2\|}{a} = \frac{\|\mathbf{r}_3\|}{a} \\ \alpha &= \frac{l_E}{a} \\ \beta &= \frac{l_F}{a} \end{aligned} \quad (6.9)$$

where l_E and l_F are the side lengths of the EE and the fixed triangles, respectively. The continuum of isotropic designs is given as

$$\begin{aligned} 0 &\leq \alpha < \infty \\ 0 &\leq \beta < \infty \\ 0 &\leq \rho < \infty \end{aligned} \quad (6.10)$$

But the condition for feasibility of the design leads to the only constraint, i.e.,

$$\frac{\sqrt{3}(\alpha - \beta)}{3(\rho + 1)} \leq 1 \quad (6.11)$$

A typical isotropic design of the manipulator is depicted in Fig. 6.1. The manipulator remains in an isotropic configuration while the centroids of the two triangles coincide. The orientation of the EE triangle is not important, unless the three lines along the second links of the legs intersect at a common point, and hence, the manipulator assumes a singular configuration, as explained in Subsection 5.3.1.

It is now apparent that the set of eight specific isotropic designs of the manipulator reported by Gosselin (1988) and Gosselin and Angeles (1988) is a subset of the three-dimensional continuum derived above.

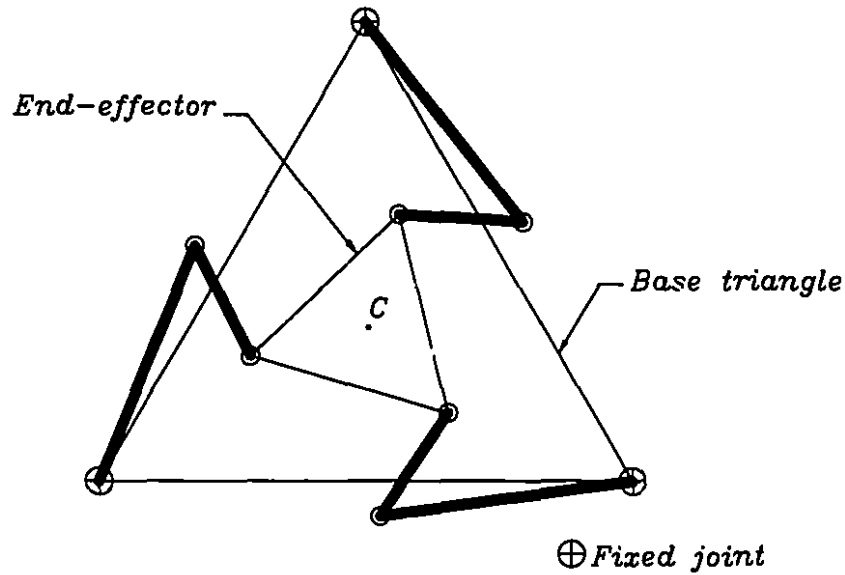


Figure 6.1: An isotropic design of a planar 3-RRR manipulator

6.2.2 Planar Manipulators of Class $\mathcal{B}$

In this subsection we find isotropic designs for planar manipulators of class $\mathcal{B}$. It is recalled that a design is isotropic if the manipulator Jacobians satisfy eqs.(6.2a) and (6.2b), where $\mathbf{J}$ and $\mathbf{K}$ are given in eqs.(5.16b) and (5.16c), respectively. But $\mathbf{J}$ is not dimensionally-homogeneous if we have different types of actuators, i.e., if some actuators are revolute and the others are prismatic. If this is the case, in order to render $\mathbf{J}$ dimensionally-homogeneous, we divide the i th column of $\mathbf{J}$ by a length l_i ,

the characteristic length of the i th leg of the manipulator, understood here, again, as defined in (Tandirci et al., 1992) for serial manipulators, and redefine $\mathbf{J}$ as

$$\mathbf{J} \leftarrow \begin{bmatrix} \mathbf{b}_1^T \mathbf{E} \mathbf{E}_1 \mathbf{a}_1 / l_1 & 0 & 0 \\ 0 & \mathbf{b}_2^T \mathbf{E} \mathbf{E}_2 \mathbf{a}_2 / l_2 & 0 \\ 0 & 0 & \mathbf{b}_3^T \mathbf{E} \mathbf{E}_3 \mathbf{a}_3 / l_3 \end{bmatrix} \quad (6.12)$$

where $l_i = 1$, for $i = 1, 2, 3$, if we have the same types of actuators in all legs or the actuator of the i th leg is prismatic.

Matrix $\mathbf{K}$ is not dimensionally-homogeneous either. To render $\mathbf{K}$ dimensionally-homogeneous we divide the first column of $\mathbf{K}$ by a length L , the characteristic length of the manipulator, and redefine the Jacobian $\mathbf{K}$ as

$$\mathbf{K} \leftarrow \begin{bmatrix} -\mathbf{b}_1^T (\mathbf{r}_1 + \mathbf{s}_1) / L & -\mathbf{b}_1^T \mathbf{E} \\ -\mathbf{b}_2^T (\mathbf{r}_2 + \mathbf{s}_2) / L & -\mathbf{b}_2^T \mathbf{E} \\ -\mathbf{b}_3^T (\mathbf{r}_3 + \mathbf{s}_3) / L & -\mathbf{b}_3^T \mathbf{E} \end{bmatrix} \quad (6.13)$$

Substitution of the values of $\mathbf{J}$ and $\mathbf{K}$ from eqs.(6.12) and (6.13) into eqs.(6.2a) and (6.2b), respectively, upon simplification yields

$$\begin{bmatrix} (\mathbf{b}_1^T \mathbf{E} \mathbf{E}_1 \mathbf{a}_1 / l_1)^2 & 0 & 0 \\ 0 & (\mathbf{b}_2^T \mathbf{E} \mathbf{E}_2 \mathbf{a}_2 / l_2)^2 & 0 \\ 0 & 0 & (\mathbf{b}_3^T \mathbf{E} \mathbf{E}_3 \mathbf{a}_3 / l_3)^2 \end{bmatrix} = \sigma^2 \mathbf{1} \quad (6.14a)$$

and

$$\begin{bmatrix} a_1^2 / L^2 + 1 & a_1 a_2 / L^2 + \mathbf{b}_1^T \mathbf{b}_2 & a_1 a_3 / L^2 + \mathbf{b}_1^T \mathbf{b}_3 \\ a_1 a_2 / L^2 + \mathbf{b}_1^T \mathbf{b}_2 & a_2^2 / L^2 + 1 & a_2 a_3 / L^2 + \mathbf{b}_2^T \mathbf{b}_3 \\ a_1 a_3 / L^2 + \mathbf{b}_1^T \mathbf{b}_3 & a_2 a_3 / L^2 + \mathbf{b}_2^T \mathbf{b}_3 & a_3^2 / L^2 + 1 \end{bmatrix} = \tau^2 \mathbf{1} \quad (6.14b)$$

where

$$a_i = \mathbf{b}_i^T (\mathbf{r}_i + \mathbf{s}_i), \quad i = 1, 2, 3 \quad (6.14c)$$

Equations (6.14a) and (6.14b) lead to the conditions for isotropic design of this class of manipulators, namely,

$$(\mathbf{b}_1^T \mathbf{E} \mathbf{E}_1 \mathbf{a}_1 / l_1)^2 = (\mathbf{b}_2^T \mathbf{E} \mathbf{E}_2 \mathbf{a}_2 / l_2)^2 = (\mathbf{b}_3^T \mathbf{E} \mathbf{E}_3 \mathbf{a}_3 / l_3)^2 = \sigma^2 \quad (6.15a)$$

$$q + \frac{p^2}{L^2} = 0 \quad (6.15b)$$

where

$$a_1 = a_2 = a_3 = p \quad (6.15c)$$

$$\mathbf{b}_1^T \mathbf{b}_2 = \mathbf{b}_1^T \mathbf{b}_3 = \mathbf{b}_2^T \mathbf{b}_3 = q \quad (6.15d)$$

and hence

$$L^2 = \frac{p^2}{-q} \quad (6.16a)$$

$$l_i = \left\| \frac{\mathbf{b}_i^T \mathbf{E} \mathbf{E}_i \mathbf{a}_i}{\sigma} \right\|, \quad i = 1, 2, 3 \quad (6.16b)$$

Example 6.2.2.1: Planar DT Manipulator

Here, we find isotropic designs, for a particular case of class- $\mathcal{B}$ manipulator, i.e., the planar DT manipulator shown in Fig. 3.5.

Assigning $\mathbf{E}_i = \mathbf{I}$ of eq.(5.11b), $l_i = 1$ and $\mathbf{r}_i = \mathbf{0}$, for $i = 1, 2, 3$, the conditions for isotropic designs, namely, eqs.(6.15a) and (6.15b) yield

$$(\mathbf{b}_1^T \mathbf{E} \mathbf{a}_1)^2 = (\mathbf{b}_2^T \mathbf{E} \mathbf{a}_2)^2 = (\mathbf{b}_3^T \mathbf{E} \mathbf{a}_3)^2 = \sigma^2 \quad (6.17a)$$

$$q + \frac{p^2}{L^2} = 0 \quad (6.17b)$$

where, again

$$a_1 = a_2 = a_3 = p \quad (6.17c)$$

$$\mathbf{b}_1^T \mathbf{b}_2 = \mathbf{b}_1^T \mathbf{b}_3 = \mathbf{b}_2^T \mathbf{b}_3 = q \quad (6.17d)$$

in which

$$\mathbf{a}_i = \mathbf{b}_i^T \mathbf{s}_i, \quad i = 1, 2, 3 \quad (6.17e)$$

Considering the foregoing conditions and the geometry of the problem, we find that isotropic designs are only possible for an equilateral DT manipulator. This type of manipulator has a one-parameter continuum of isotropic designs. This parameter is

$$\alpha = \frac{l_E}{l_F} \quad (6.18)$$

where l_E and l_F are the side lengths of the EE and the fixed triangle, respectively. The continuum of isotropic designs is given as

$$0.5 < \alpha < 2 \quad (6.19)$$

A typical isotropic design of the manipulator is depicted in Fig. 6.2. The manipulator remains in an isotropic configuration while the centroids of the two triangles coincide. The orientation of the movable triangle is not important, unless the two triangles coincide, where the manipulator assumes a singular configuration, as explained in Subsection 5.3.2.

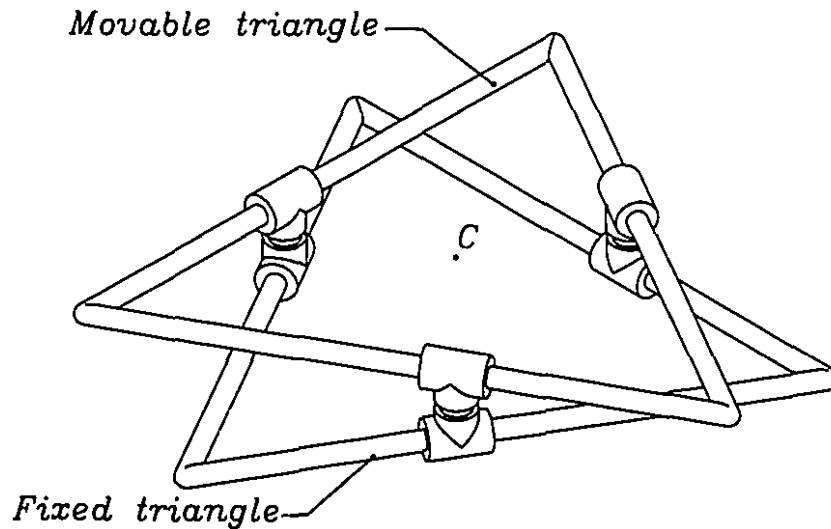


Figure 6.2: An isotropic design of a planar DT manipulator

6.2.3 Spherical 3-RRR Manipulator

In this subsection we find isotropic designs for the spherical 3-RRR manipulator, as shown in Fig. 3.6. The isotropic design of this manipulator has been addressed in the literature. Gosselin (1988) claimed that an isotropic design is impossible for this type of manipulators. Later, Gosselin and Lavoie (1993) found some isotropic designs for this class.

It is recalled that a design is isotropic if both $\mathbf{J}$ and $\mathbf{K}$ are isotropic, i.e., if eqs.(6.2a) and (6.2b) hold. Equation (6.2a), upon substituting the value of $\mathbf{J}$ of eq.(5.20b) and simplification, yields a diagonal matrix, namely,

$$\begin{bmatrix} a_1^2 & 0 & 0 \\ 0 & a_2^2 & 0 \\ 0 & 0 & a_3^2 \end{bmatrix} = \sigma^2 \mathbf{1} \quad (6.20)$$

Similarly, eq.(6.2b), upon substituting the value of $\mathbf{K}$ from eq.(5.20c) and simplification, yields

$$\begin{bmatrix} \mathbf{b}_1^T \mathbf{b}_1 & \mathbf{b}_1^T \mathbf{b}_2 & \mathbf{b}_1^T \mathbf{b}_3 \\ \mathbf{b}_1^T \mathbf{b}_2 & \mathbf{b}_2^T \mathbf{b}_2 & \mathbf{b}_2^T \mathbf{b}_3 \\ \mathbf{b}_1^T \mathbf{b}_3 & \mathbf{b}_2^T \mathbf{b}_3 & \mathbf{b}_3^T \mathbf{b}_3 \end{bmatrix} = \tau^2 \mathbf{1} \quad (6.21)$$

where

$$\mathbf{b}_i \equiv \mathbf{v}_i \times \mathbf{w}_i, \quad i = 1, 2, 3$$

Equations (6.20) and (6.21) lead to the conditions for isotropy, i.e.,

$$a_1^2 = a_2^2 = a_3^2 = \sigma^2 \quad (6.22a)$$

$$\mathbf{b}_i^T \mathbf{b}_j = \begin{cases} \tau^2, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \quad (6.22b)$$

The foregoing equations are the necessary and sufficient conditions for an isotropic design, which lead to manipulators with the following characteristics:

- 1) The middle links, A_1Q_1 , A_2Q_2 and A_3Q_3 lie on the arcs of an equilateral spherical

triangle $\mathcal{W}$ whose sides are all 90° , as shown in Fig. 6.3;

2) corresponding leg links have the same length;

3) the angles between the leg-link planes are equal.

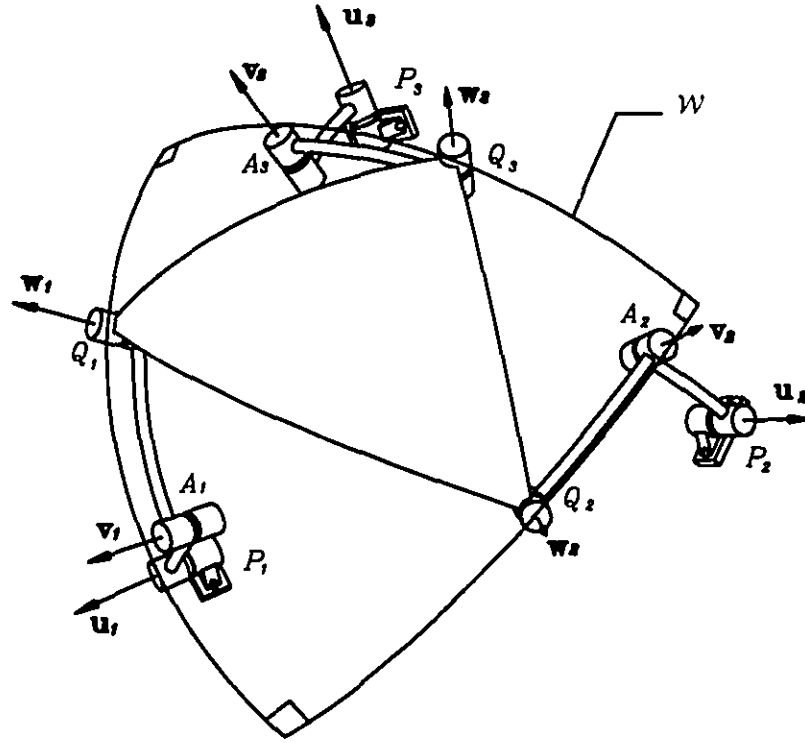


Figure 6.3: An isotropic design of a spherical 3-RRR manipulator

The foregoing characteristics lead to a six-parameter continuum for isotropic designs of the manipulator. First, one may choose the movable triangle $\mathcal{Q}$ such that its vertices $\{Q_i\}_1^3$ lie anywhere on the sides of triangle $\mathcal{W}$. In other words, one is free to choose the three independent vertices $\{Q_i\}_1^3$ to define triangle $\mathcal{Q}$.

Second, one may choose two arc lengths and the angle between the planes of the first and the second links α in the ranges specified as

$$0 < \widehat{A_i Q_i} < 2\pi, \quad \widehat{A_i Q_i} \neq \pi \quad (6.23a)$$

$$0 < \widehat{A_i P_i} < 2\pi, \quad \widehat{A_i P_i} \neq \pi \quad (6.23b)$$

$$0 < \alpha < 2\pi, \quad \alpha \neq 0, \alpha \neq \pi \quad (6.23c)$$

In the case in which we choose three vertices of triangle $\mathcal{Q}$ such that it cannot be inscribed in triangle $\mathcal{W}$, an isotropic design is impossible. That is why Gosselin (1988) could not find any. A typical isotropic design of the manipulator is depicted in Fig. 6.3. Clearly, the set of isotropic designs of the manipulator reported by Gosselin and Lavoie (1993) is a subset of the six-dimensional continuum, described above, which constitutes the complete set of isotropic designs for the manipulator at hand.

6.2.4 Spherical DT Manipulator

In this subsection, we find isotropic designs for the spherical DT manipulator, as shown in Fig. 3.7. It is recalled that a design is isotropic if both $\mathbf{J}$ and $\mathbf{K}$ are isotropic, i.e., if eqs.(6.2a) and (6.2b) hold. Equation (6.2a), upon substituting the value of $\mathbf{J}$ of eq.(5.24b) and simplification, yields a diagonal matrix, namely,

$$\begin{bmatrix} c_1^2 & 0 & 0 \\ 0 & c_2^2 & 0 \\ 0 & 0 & c_3^2 \end{bmatrix} = \sigma^2 \mathbf{1} \quad (6.24)$$

Similarly, eq.(6.2b), upon substituting the value of $\mathbf{K}$ of eq.(5.24c) and simplification, yields

$$\begin{bmatrix} \mathbf{d}_1^T \mathbf{d}_1 & \mathbf{d}_1^T \mathbf{d}_2 & \mathbf{d}_1^T \mathbf{d}_3 \\ \mathbf{d}_1^T \mathbf{d}_2 & \mathbf{d}_2^T \mathbf{d}_2 & \mathbf{d}_2^T \mathbf{d}_3 \\ \mathbf{d}_1^T \mathbf{d}_3 & \mathbf{d}_2^T \mathbf{d}_3 & \mathbf{d}_3^T \mathbf{d}_3 \end{bmatrix} = \tau^2 \mathbf{1} \quad (6.25)$$

where

$$\mathbf{d}_i \equiv \mathbf{r}_i \times \mathbf{t}_i, \quad i = 1, 2, 3$$

Equations (6.24) and (6.25) lead to the conditions for isotropy, namely,

$$c_1^2 = c_2^2 = c_3^2 = \sigma^2 \quad (6.26a)$$

$$\mathbf{d}_i^T \mathbf{d}_j = \begin{cases} \tau^2 & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \quad (6.26b)$$

Considering the foregoing conditions and the geometry of the problem, we find that isotropic designs are only possible for an equilateral DT manipulator in which the sides of the movable triangle $\mathcal{Q}$ are all equal to 90° . The one-dimensional continuum of isotropic designs comprises a single variable whose range is given as

$$60^\circ < l_P < 109.4^\circ \quad (6.27)$$

in which l_P is the side of the fixed triangle $\mathcal{P}$. A typical isotropic design of the manipulator is depicted in Fig. 6.4.

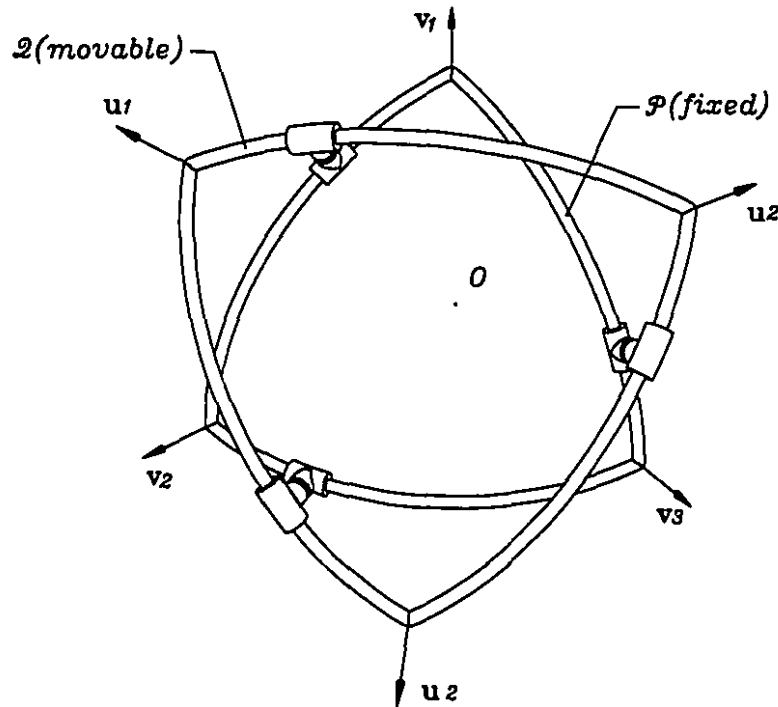


Figure 6.4: An isotropic design of a spherical DT manipulator

6.2.5 Spatial 6-DOF, DT Manipulator

Here, we derive the isotropic designs of the spatial 6-dof, DT manipulator, as shown in Fig. 3.9. It is recalled that a design is isotropic if the manipulator Jacobians satisfy eqs.(6.2a) and (6.2b), where $\mathbf{J}$ and $\mathbf{K}$ are given in eqs.(5.31c) and (5.31d), respectively. But $\mathbf{J}$ is not dimensionally homogeneous. To render $\mathbf{J}$ dimensionally-homogeneous we divide the $(i + 3)$ th column of $\mathbf{J}$ by a length l_i , for $i = 1, 2, 3$, and

redefine the Jacobian $\mathbf{J}$ as

$$\mathbf{J} \leftarrow \begin{bmatrix} 0 & 0 & 0 & -\mathbf{a}_1^T \mathbf{m}_1/l_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\mathbf{a}_2^T \mathbf{m}_2/l_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathbf{a}_3^T \mathbf{m}_3/l_3 \\ \mathbf{a}_1^T \mathbf{m}_1/r_1 & 0 & 0 & p_1/l_1 & 0 & 0 \\ 0 & \mathbf{a}_2^T \mathbf{m}_2/r_2 & 0 & 0 & p_2/l_2 & 0 \\ 0 & 0 & \mathbf{a}_3^T \mathbf{m}_3/r_3 & 0 & 0 & p_3/l_3 \end{bmatrix} \quad (6.28)$$

where l_i is the characteristic length of the i th leg of the manipulator.

Matrix $\mathbf{K}$ is not dimensionally-homogeneous either. To render $\mathbf{K}$ dimensionally-homogeneous we divide the first three columns of $\mathbf{K}$ by a length L , the characteristic length of the manipulator, and redefine the Jacobian $\mathbf{K}$ as

$$\mathbf{K} \leftarrow \begin{bmatrix} \mathbf{m}_1^T/L & \mathbf{0}^T \\ \mathbf{m}_2^T/L & \mathbf{0}^T \\ \mathbf{m}_3^T/L & \mathbf{0}^T \\ \mathbf{q}_1^T/L & -\mathbf{m}_1^T/r_1 \\ \mathbf{q}_2^T/L & -\mathbf{m}_2^T/r_2 \\ \mathbf{q}_3^T/L & -\mathbf{m}_3^T/r_3 \end{bmatrix} \quad (6.29)$$

Substitution of the values of $\mathbf{J}$ and $\mathbf{K}$ from eqs.(6.28) and (6.29) into eqs.(6.2a) and (6.2b), respectively, upon simplification yields

$$\begin{bmatrix} s_1^2/l_1^2 & 0 & 0 & -p_1 s_1/l_1^2 & 0 & 0 \\ 0 & s_2^2/l_2^2 & 0 & 0 & -p_2 s_2/l_2^2 & 0 \\ 0 & 0 & s_3^2/l_3^2 & 0 & 0 & -p_3 s_3/l_3^2 \\ -p_1 s_1/l_1^2 & 0 & 0 & s_1^2/r_1^2 + p_1^2/l_1^2 & 0 & 0 \\ 0 & -p_2 s_2/l_2^2 & 0 & 0 & s_2^2/r_2^2 + p_2^2/l_2^2 & 0 \\ 0 & 0 & -p_3 s_3/l_3^2 & 0 & 0 & s_3^2/r_3^2 + p_3^2/l_3^2 \end{bmatrix} = \sigma^2 \mathbf{1} \quad (6.30a)$$

and

$$\begin{bmatrix} \mathbf{m}_1^T \mathbf{m}_1 / L^2 & \mathbf{m}_1^T \mathbf{m}_2 / L^2 & \mathbf{m}_1^T \mathbf{m}_3 / L^2 & \mathbf{m}_1^T \mathbf{q}_1 / L^2 & \mathbf{m}_1^T \mathbf{q}_2 / L^2 & \mathbf{m}_1^T \mathbf{q}_3 / L^2 \\ \mathbf{m}_2^T \mathbf{m}_1 / L^2 & \mathbf{m}_2^T \mathbf{m}_2 / L^2 & \mathbf{m}_2^T \mathbf{m}_3 / L^2 & \mathbf{m}_2^T \mathbf{q}_1 / L^2 & \mathbf{m}_2^T \mathbf{q}_2 / L^2 & \mathbf{m}_2^T \mathbf{q}_3 / L^2 \\ \mathbf{m}_3^T \mathbf{m}_1 / L^2 & \mathbf{m}_3^T \mathbf{m}_2 / L^2 & \mathbf{m}_3^T \mathbf{m}_3 / L^2 & \mathbf{m}_3^T \mathbf{q}_1 / L^2 & \mathbf{m}_3^T \mathbf{q}_2 / L^2 & \mathbf{m}_3^T \mathbf{q}_3 / L^2 \\ \mathbf{q}_1^T \mathbf{m}_1 / L^2 & \mathbf{q}_1^T \mathbf{m}_2 / L^2 & \mathbf{q}_1^T \mathbf{m}_3 / L^2 & s_{11} & s_{12} & s_{13} \\ \mathbf{q}_2^T \mathbf{m}_1 / L^2 & \mathbf{q}_2^T \mathbf{m}_2 / L^2 & \mathbf{q}_2^T \mathbf{m}_3 / L^2 & s_{21} & s_{22} & s_{23} \\ \mathbf{q}_3^T \mathbf{m}_1 / L^2 & \mathbf{q}_3^T \mathbf{m}_2 / L^2 & \mathbf{q}_3^T \mathbf{m}_3 / L^2 & s_{31} & s_{32} & s_{33} \end{bmatrix} = \tau^2 \mathbf{1} \quad (6.30b)$$

where

$$s_i = \mathbf{a}_i^T \mathbf{m}_i \quad (6.30c)$$

$$s_{ij} = \mathbf{q}_i^T \mathbf{q}_j / L^2 + \mathbf{m}_i^T \mathbf{m}_j / (r_i r_j) \quad (6.30d)$$

Equations (6.30a) and (6.30b) lead to the conditions for isotropy, i.e., for $i, j = 1, 2, 3$, we have

$$s_i^2 / l_i^2 = \sigma^2 \quad (6.31a)$$

$$p_i s_i = 0 \quad (6.31b)$$

$$s_i^2 / r_i^2 + p_i^2 / l_i^2 = \sigma^2 \quad (6.31c)$$

$$\mathbf{m}_i^T \mathbf{m}_j / L^2 = \begin{cases} \tau^2, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \quad (6.31d)$$

$$\mathbf{q}_i^T \mathbf{m}_j = 0 \quad (6.31e)$$

$$s_{ij} = \begin{cases} \tau^2, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \quad (6.31f)$$

and hence,

$$l_i = r_i \quad (6.32a)$$

$$p_i = 0 \quad (6.32b)$$

$$\mathbf{q}_i^T \mathbf{q}_i = L^2 (1 - L^2 / l_i^2) \tau^2 \quad (6.32c)$$

$$b_1 = b_2 = b_3 \quad (6.32d)$$

$$(\mathbf{c}_1 \times \mathbf{m}_1)^T (\mathbf{c}_1 \times \mathbf{m}_1) = (\mathbf{c}_2 \times \mathbf{m}_2)^T (\mathbf{c}_2 \times \mathbf{m}_2) = (\mathbf{c}_3 \times \mathbf{m}_3)^T (\mathbf{c}_3 \times \mathbf{m}_3) \quad (6.32e)$$

Moreover, eqs.(6.31a–f), the isotropy conditions, produce manipulators with the following characteristics:

- 1) The three planes containing vectors $\mathbf{r}_i$ and $\mathbf{b}_i$, for $i = 1, 2, 3$, are orthogonal;
- 2) the set $\{\mathbf{r}_i\}_1^3$ is orthogonal;
- 3) the set $\{\mathbf{b}_i\}_1^3$ is orthogonal;
- 4) $\mathbf{a}_i$, for $i = 1, 2, 3$, is perpendicular to the plane of vectors $\mathbf{r}_i$ and $\mathbf{b}_i$;
- 5) the distances between $\mathbf{a}_i$ and $\mathbf{b}_i$, for $i = 1, 2, 3$, are equal.

The foregoing characteristics lead to a one-parameter continuum for isotropic designs of the manipulator. The one-dimensional design parameter r is defined as follows:

$$r = r_1 = r_2 = r_3 \quad (6.33)$$

The continuum of isotropic design is given as

$$0 < r < \infty \quad (6.34)$$

A typical isotropic design of the manipulator is depicted in Fig. (6.5).

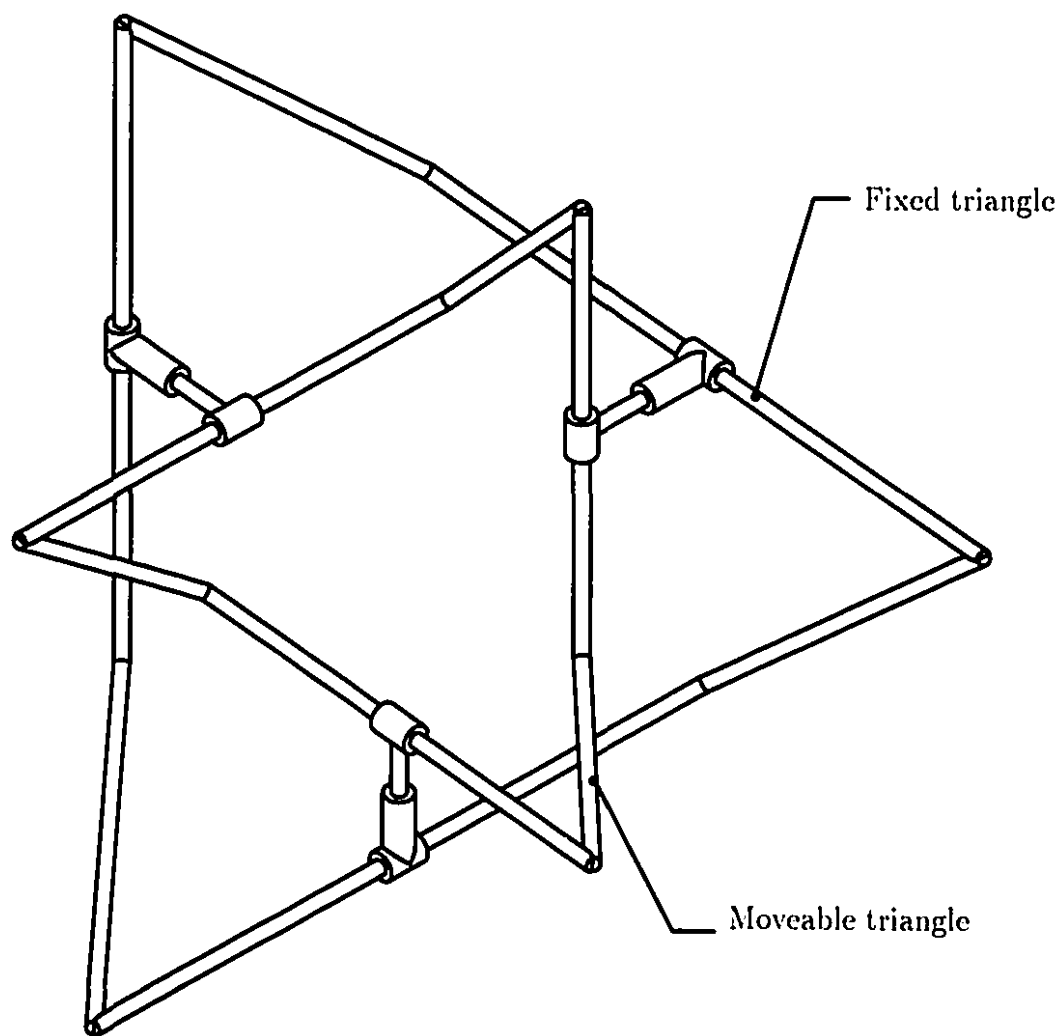


Figure 6.5: An isotropic design of a spatial 6-dof, DT manipulator

Chapter 7

Concluding Remarks

7.1 Conclusions

Research interest in parallel manipulators was prompted by the realization that serial-manipulator performance is deficient. The source of deficiencies in these manipulators is their cantilever type of link loading. Therefore, the obvious alternative is a parallel architecture, in which the end-effector is supported with a multiple point support. However, long slender legs, which are the source of flexibility, are present in parallel manipulators. A novel class of parallel device, namely, double-triangular (DT) manipulators, in three versions, was introduced to alleviate this problem. Minimum leg lengths occur as a common feature of this new class of manipulators.

Solutions to the direct kinematics (DK) of planar, spherical and spatial DT manipulators were attempted and obtained. Using planar trigonometry, we found a quadratic equation solution for the planar DT manipulator. The DK of the spherical DT device was solved, as a 16th order polynomial, by means of spherical trigonometry. These results inductively led us to invoke methods of spatial trigonometry to treat skew lines. Spatial trigonometric relationships, in turn, were expressed in dual-number algebra, while the 6 real components of a unit dual vector are the Plücker coordinates of a line. With the aid of spatial trigonometry, we formulated and solved

the direct kinematic problem of several versions of spatial DT manipulators. These results revealed that screw operators and the Plücker coordinates of a line expressed by unit dual quaternions and unit dual vectors, respectively, provide the most efficient means to formulate, manipulate and solve the kinematics of complicated problems where motion is constrained by line contact.

A dual 3×3 matrix representing a screw motion was derived in an invariant form. It was shown that the linear invariants of this matrix provide a convenient way to compute the screw axis, the angle of rotation and the displacement along the screw axis of a general motion. These invariants have been traditionally computed by equation solving, which should be avoided in real-time applications. Therefore, obtaining these parameters from the linear invariants of the dual matrix reduces the computational burden to simple sums and differences.

It is customary to express Jacobian matrices of parallel manipulators component-wise. Indeed, this practice is frequently encountered in the robotics literature. This practice leads to equations that are frame-dependent and cumbersome to interpret. The component-wise expressions in such Jacobian matrices are lengthy and do not give much geometric insight into the behaviour of the manipulator. As an alternative, we derived the Jacobian matrices of certain large classes of parallel manipulators in an invariant form. The Jacobian matrices found in this way are compact, give direct geometric interpretations of the manipulator behaviour, are frame-independent, and are algebraically simpler. Moreover, we proposed a general method to derive the Jacobian matrices of parallel manipulators at large. For example, we unified the Jacobian matrices for a large class containing 20 manipulator types into a single formula.

A general classification of singularities encountered in parallel manipulators was introduced and categorized in three singularity types. This classification scheme relies on the conditioning of the Jacobian matrices. Having derived the Jacobian matrices in an invariant form, allowed us to detect all singularities within the workspaces

of the manipulators under study. Moreover, we showed that, contrary to earlier claims, the third type of singularity is not necessarily architecture-dependent.

An important property of robotic manipulators is their dexterity. We regarded dexterity in the context of local kinetostatic accuracy. Among various measures proposed for quantification of dexterity, we adopted the condition number. This measure does not share the drawbacks suffered by other measures like the determinant, the manipulability and the minimum singular value. A manipulator design with optimally conditioned Jacobian matrices was called *isotropic*. Having formulated the Jacobian matrices of the manipulators at hand, in an invariant form, we found the conditions leading to isotropic designs. For several manipulators, we were able to find the complete set of isotropic designs. The isotropic design parameter spaces of these manipulators turn out to be a continuum of at least one dimension. This provides a substantial domain of dextrous design choice to fit many situations, which should admit design criteria other than isotropy, e.g., workspace volume, global dexterity.

7.2 Consideration for Future Work

A few recommended topics for future research are listed below:

1. A polynomial of degree 16 was found for the direct kinematic problem of spherical DT manipulator. The result implies that the polynomial has at most 16 solutions, among which only eight would be real positive. Since only the real positive solutions are acceptable, the direct kinematic problem of this device has at most eight solutions. However, in tests we ran, we could find no instance with more than two geometrically distinct solutions. Therefore, the polynomial is not minimal. Finding the minimal polynomial would be a topic of further research.
2. The direct kinematic problem of all versions of spatial DT manipulators was formulated in this thesis, without closed form solutions. This issue remains as

a challenging research problem.

3. It has been shown that dual quaternion algebra is an excellent tool to handle the kinematics of line-contact constrained mechanisms. The kinematic study of other mechanisms of this type, such as the double-tetrahedral mechanism, by this means, constitutes another possible extension to our work.
4. The main reason why dual numbers, quaternions and dual quaternions are not popular is that they are difficult to work with. To overcome this obstacle the author implemented some user-defined functions in MATHEMATICA to handle some dual number algebraic operators. It is suggested that a computational algebraic code be developed to make these computations as convenient as those currently available for complex, vector and matrix algebras.
5. It was shown that expressing the Jacobian matrices in an invariant form makes it easier to effectively handle the issues of isotropy and singularity. Another challenging and fruitful problem is to find the continuum of isotropic designs and singular configurations of the most common parallel manipulator, i.e., the Stewart-Gough platform, with invariant forms of its Jacobian matrices.
6. Multi-parameter continua of isotropic designs for some manipulators were found. This allows one to incorporate design criteria other than isotropy. Clearly, designing manipulators with multi-variate objective functions, including isotropy, is a topic worthy of further research.

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Appendix A

Bezout's Method

Given k homogeneous equations in k variables, or k non-homogeneous equations in $k - 1$ variables, it is always possible to combine the equations so as to obtain from them a single monovariate equation $\Delta = 0$. Δ being called the *eliminant* of the system of equations.

There are several methods to do this. A method, known as *Bezout's method*, is faster than others (Salmon, 1964). It is demonstrated with an example here where two homogeneous quartic equations in two variables are reduced to a univariate polynomial. Consider the two equations

$$a_0x^4 + a_1x^3y + a_2x^2y^2 + a_3xy^3 + a_4y^4 = 0$$

$$b_0x^4 + b_1x^3y + b_2x^2y^2 + b_3xy^3 + b_4y^4 = 0$$

Multiplying the first equation by b_0 , and the second by a_0 , and subtracting, then dividing the result by y , gives

$$a_0b_1x^3 + a_0b_2x^2y + a_0b_3xy^2 + a_0b_4y^3 = 0 \quad (\text{A.1})$$

Again, using the same procedure with respective multipliers $b_0x + b_1y$ and $a_0x + a_1y$, and the divisor y^2 , gives

$$a_0b_2x^3 + (a_0b_3 + a_1b_2)x^2y + (a_0b_4 + a_1b_3)xy^2 + a_1b_4y^3 = 0 \quad (\text{A.2})$$

Now, repeating the procedure for the third time, with respective multipliers $b_0x^2 + b_1xy + b_2y^2$ and $a_0x^2 + a_1xy + a_2y^2$, and divisor y^3 , produces

$$a_0b_3x^3 + (a_0b_4 + a_1b_3)x^2y + (a_1b_4 + a_2b_3)xy^2 + a_2b_4y^3 = 0 \quad (\text{A.3})$$

Finally, the fourth equation is derived with respective multipliers $b_0x^3 + b_1x^2y + b_2xy^2 + b_3y^3$ and $a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3$, and divisor y^4 , namely,

$$a_0b_4x^3 + a_1b_4x^2y + a_2b_4xy^2 + a_3b_4y^3 = 0 \quad (\text{A.4})$$

From the four eqs.(A.1–A.4), we can eliminate linearly the four quantities, x^3, x^2y, xy^2 and y^3 , and obtain the eliminant

$$\Delta = \det \begin{bmatrix} a_0b_1 & a_0b_2 & a_0b_3 & a_0b_4 \\ a_0b_2 & a_0b_3 + a_1b_2 & a_0b_4 + a_1b_3 & a_1b_4 \\ a_0b_3 & a_0b_4 + a_1b_3 & a_1b_4 + a_2b_3 & a_2b_4 \\ a_0b_4 & a_1b_4 & a_2b_4 & a_3b_4 \end{bmatrix} \quad (\text{A.5})$$

In a similar manner we derive the eliminant of higher-orders equations.

Appendix B

Coefficients of Equation (4.19b)

In this Appendix we tabulate the coefficients of eq.(4.19b) which were obtained with MATHEMATICA, a software package for symbolic computations.

$$\begin{aligned} A_{10} &= 4(cQ_1^2cD_2^2 + 2cQ_1cQ_3cD_2cE_2 + cQ_3^2cE_2^2 + cQ_1^2cf^2sD_2^2 - cD_2^2sE_2^2 - \\ &\quad 2cQ_1cQ_3cfsD_2sE_2 - 2cfcD_2cE_2sD_2sE_2 + cQ_3^2sE_2^2 - cf^2cE_2^2sD_2^2) \\ A_{11} &= 16cd(cfcE_2sD_2 + cD_2sE_2)(cQ_1cQ_3 + cD_2cE_2 - cfsD_2sE_2) \\ A_{12} &= 8(cQ_1^2cD_2^2 - cQ_3^2cE_2^2 + 2cQ_3^2cd^2cE_2^2 - cQ_3^2sE_2^2 + cQ_1^2cf^2sD_2^2 + \\ &\quad 4cfcD_2cE_2sD_2sE_2 - 2cf^2cd^2sD_2^2sE_2^2 + cf^2cE_2^2sD_2^2 - 2cd^2cD_2^2cE_2^2 + \\ &\quad 2cfcD_2cE_2sD_2sE_2cD_2^2sE_2^2 + 2cQ_3^2cd^2sE_2^2) \\ A_{13} &= 16cd(cfcE_2sD_2 + cD_2sE_2)(cQ_1cQ_3 - cD_2cE_2 + cfsD_2sE_2) \\ A_{14} &= 4(cQ_1^2cD_2^2 + cQ_3^2cE_2^2 + cQ_3^2sE_2^2 - cQ_1^2cf^2sD_2^2 - cf^2cE_2^2sD_2^2 + \\ &\quad 2cQ_1cQ_3cfsD_2sE_2 + cD_2^2sE_2^2 - 2cfcD_2cE_2sD_2sE_2 - 2cQ_1cQ_3cD_2cE_2) \\ A_{20} &= 16(cf^2cD_2cE_2^2sD_2 + cQ_1cQ_3cfcD_2sE_2 + cQ_1^2cD_2sD_2 - cQ_1^2cf^2cD_2sD_2 - \\ &\quad cfcE_2sD_2^2sE_2 + cfcD_2^2cE_2sE_2 + cQ_1cQ_3cE_2sD_2 - cD_2sD_2sE_2^2) \\ A_{21} &= 32cd(cQ_1cQ_3sD_2sE_2 + 2cD_2cE_2sD_2sE_2 + cfcD_2^2sE_2^2 - cfsD_2^2sE_2^2 + \\ &\quad 2cf^2cD_2cE_2sD_2sE_2 + cfcE_2^2sD_2^2 - cQ_1cQ_3cfcD_2cE_2 - cfcD_2^2cE_2^2) \end{aligned}$$

$$A_{22} = 32(-2cd^2cD_2cE_2^2sD_2 - cfcD_2^2cE_2sE_2 + cQ_1^2cD_2sD_2 - cQ_1^2cf^2cD_2sD_2 - \\ cf^2cD_2cE_2^2sD_2 + 2cf^2cd^2cD_2sD_2sE_2^2 - 2cfc d^2cD_2^2cE_2sE_2 + cfcE_2sD_2^2sE_2 \\ + 2cfc d^2cE_2sD_2^2sE_2 + cD_2sD_2sE_2^2)$$

$$A_{23} = 32cd(cQ_1cQ_3sD_2sE_2 - 2cD_2cE_2sD_2sE_2 - cfcD_2^2sE_2^2 + cfsD_2^2sE_2^2 - \\ 2cf^2cD_2cE_2sD_2sE_2 + cfcD_2^2cE_2^2 - cQ_1cQ_3cfcD_2cE_2 - cfcE_2^2sD_2^2)$$

$$A_{24} = 16(-cQ_1cQ_3cfcD_2sE_2 - cfcE_2sD_2^2sE_2 + cf^2cD_2cE_2^2sD_2 + cfcD_2^2cE_2sE_2 - \\ cD_2sD_2sE_2^2 + cQ_1^2cD_2sD_2 - cQ_1^2cf^2cD_2sD_2 - cQ_1cQ_3cE_2sD_2)$$

$$A_{30} = 8(-cQ_1^2cD_2^2 + 2cQ_1^2cf^2cD_2^2 + cQ_3^2cE_2^2 - 2cf^2cD_2^2cE_2^2 + 2cQ_1^2sD_2^2 - \\ cQ_1^2cf^2sD_2^2 + cf^2cE_2^2sD_2^2 + 6cfcD_2cE_2sD_2sE_2 + cQ_3^2sE_2^2 + cD_2^2sE_2^2 - \\ 2sD_2^2sE_2^2)$$

$$A_{31} = 32cd(2cE_2sD_2^2sE_2 - 3cfcD_2cE_2^2sD_2 - cD_2^2cE_2sE_2 - 2cf^2cD_2^2cE_2sE_2 + \\ cf^2cE_2sD_2^2sE_2 + 3cfcD_2sD_2sE_2^2)$$

$$A_{32} = 16(-cQ_1^2cD_2^2 + 2cQ_1^2cf^2cD_2^2 - cQ_3^2cE_2^2 + 2cQ_3^2cd^2cE_2^2 + 2cf^2cD_2^2cE_2^2 + \\ 2cd^2cD_2^2cE_2^2 + 2cQ_1^2sD_2^2 - cQ_1^2cf^2sD_2^2 - 12cfc d^2cD_2cE_2sD_2 - 4cd^2cE_2^2sD_2^2 - \\ 6cfcD_2cE_2sD_2sE_2 - cD_2^2sE_2^2 - cf^2cE_2^2sD_2^2sE_2 - cQ_3^2sE_2^2 + 2cQ_3^2cd^2sE_2^2 - \\ 4cf^2cd^2cD_2^2sE_2^2 + 2sD_2^2sE_2^2 + 2cf^2cd^2sD_2^2sE_2^2)$$

and

$$A_{rs} = (-1)^{9-r-s} A_{6-r,4-s}$$

for $(r, s) = (3, 3), (3, 4)$ and $r = 4, 5, s = 0, \dots, 4$, where $c(\cdot) = \cos(\cdot)$ and $s(\cdot) = \sin(\cdot)$.

Appendix C

Coefficients of Equation (4.38)

Here we tabulate q_i , for $i = 1, \dots, 8$, of eq.(4.38) which were obtained with MATHEMATICA, a software package for symbolic computations.

$$\begin{aligned} q_1 &= 0.1589146384409058 \cos \psi_1 + 0.284268244827663 \sin \psi_3 \\ &\quad - 0.6356385402223707 \sin \psi_3 \sin \psi_1 - 0.4264022119870481 \sin \psi_2 \\ q_2 &= -0.4020142020702365 \cos \psi_3 - 0.6356385402246087 \cos \psi_1 \sin \psi_3 - \\ &\quad 0.1589146384403463 \sin \psi_1 + 0.6396018356438479 \sin \psi_2 \\ q_3 &= -0.898932680742382 \cos \psi_3 \cos \psi_1 - 0.7687052862670159 \cos \psi_2 + \\ &\quad 0.2842656920009319 \sin \psi_3 + 0.6356442485266173 \sin \psi_3 \sin \psi_1 \\ q_4 &= 0.284268244827663 r_1 \cos \psi_3 + 0.0842725261027617 \cos \psi_1 - \\ &\quad 0.898932680742382 \cos \psi_3 \cos \psi_1 - 0.4264022119870481 r_3 \cos \psi_2 + \\ &\quad .3017665551147022 \sin \psi_3 + 0.6356385402246087 \cos \psi_1 \sin \psi_3 - \\ &\quad 0.6356385402223707 r_2 \cos \psi_1 \sin \psi_3 + 0.1589142167460837 \sin \psi_1 - \\ &\quad 0.1589146384409058 r_2 \sin \psi_1 + 0.4494667898948425 \cos \psi_3 \sin \psi_1 - \\ &\quad 0.6356385402223707 r_1 \cos \psi_3 \sin \psi_1 + 0.2786954677580153 \sin \psi_3 \sin \psi_1 \\ &\quad - 0.4215550405217529 \sin \psi_2 \end{aligned}$$

$$\begin{aligned}
q_5 = & -0.3262577907206018 \cos \psi_3 + 0.1589146384409058 \cos \psi_1 - \\
& 0.1589146384403463 r_2 \cos \psi_1 - 0.6356385402246087 r_1 \cos \psi_3 \cos \psi_1 - \\
& 0.7687062862670159 \cos \psi_2 + 0.6396018356438479 r_3 \cos \psi_2 + \\
& 0.4020142020702365 r_1 \sin \psi_3 - 0.3569479374078874 \cos \psi_1 \sin \psi_3 - \\
& 0.0842727369495963 \sin \psi_1 - 0.3178148330523365 \sin \psi_3 \sin \psi_1 + \\
& 0.6356385402246087 r_2 \sin \psi_3 \sin \psi_1 + 0.2858790240055581 \sin \psi_2 \\
q_6 = & 0.2842656920009319 r_1 \cos \psi_3 - 0.1589146384409058 \cos \psi_1 + \\
& 0.1694026843444782 \cos \psi_3 \cos \psi_1 - 0.471702595330201 \cos \psi_2 + \\
& 0.1596309991409217 \sin \psi_3 - 0.6356442485288552 \cos \psi_1 \sin \psi_3 + \\
& 0.898932680742382 r_1 \cos \psi_1 \sin \psi_3 + 0.6356442485266173 r_2 \cos \psi_1 \sin \psi_3 + \\
& 0.07945709559545802 \sin \psi_1 - 0.89893029534063 \cos \psi_3 \sin \psi_1 + \\
& 0.6356442485266173 r_1 \cos \psi_3 \sin \psi_1 + 0.898932680742382 r_2 \cos \psi_3 \sin \psi_1 + \\
& 0.03912415371435915 \sin \psi_3 \sin \psi_1 - 0.6396018356438479 \sin \psi_2 + \\
& 0.7687062862670159 r_3 \sin \psi_2 \\
q_7 = & -0.5685332823940687 - 0.6356442485288552 \cos \psi_1 \sin \psi_3 - \\
& 0.898932680739217 \cos \psi_3 \sin \psi_1 \\
q_8 = & 0.6756731531896393 + 0.898932680742382 \cos \psi_3 \cos \psi_1 - \\
& 0.6356442485288552 r_1 \cos \psi_3 \cos \psi_1 - 0.898932680739217 r_2 \cos \psi_3 \cos \psi_1 + \\
& 0.5965152298796568 \cos \psi_1 \sin \psi_3 + 0.1694014916445885 \cos \psi_3 \sin \psi_1 \\
& -0.953462571444399 \sin \psi_3 \sin \psi_1 + 0.898932680739217 r_1 \sin \psi_3 \sin \psi_1 + \\
& 0.6356442485288552 r_2 \sin \psi_3 \sin \psi_1
\end{aligned}$$

Appendix D

Coefficients of Equation (4.41)

Here we tabulate q_i , for $i = 1, \dots, 8$, of eq.(4.41) which were obtained with MATHEMATICA, a software package for symbolic computations.

$$\begin{aligned} q_1 &= -0.898932680742382 \cos \mu_3 \cos \mu_1 \sin \psi_3 \sin \psi_1 - \\ &\quad 0.4020142020702365 \sin \mu_3 \sin \psi_3 - 0.898932680742382 \cos \psi_3 \sin \mu_1 \sin \psi_1 \\ &\quad + 0.1589146384409058 \cos \psi_1 + 0.7687062862670159 \cos \mu_2 \sin \psi_2 \\ q_2 &= -0.4020142020702365 \cos \psi_3 + 0.898932680742382 \cos \mu_3 \cos \psi_1 \sin \psi_3 + \\ &\quad 0.1589146384409058 \cos \mu_1 \sin \psi_1 - 0.7687062862670159 \sin \mu_2 \sin \psi_2 + \\ &\quad 0.898932680742382 \sin \mu_3 \sin \mu_1 \sin \psi_3 \sin \psi_1 0.7687062862670159 \sin \mu_2 \sin \psi_2 \\ q_3 &= -0.898932680742382 \cos \psi_3 \cos \psi_1 - 0.7687062862670159 \cos \psi_2 - \\ &\quad 0.4020142020702365 \cos \mu_3 \sin \psi_3 - 0.1589146384409058 \sin \mu_1 \sin \psi_1 + \\ &\quad 0.898932680742382 \cos \mu_1 \sin \mu_3 \sin \psi_3 \sin \psi_1 \\ q_4 &= 0.0842725261027617 \cos \psi_1 - 0.898932680742382 \cos \psi_3 \cos \psi_1 + \\ &\quad 0.3198002640379491 \cos \mu_2 \cos \psi_2 - 0.1421333271845383 \cos \psi_3 \sin \mu_3 - \\ &\quad 0.4494645425058293 \cos \psi_3 \cos \psi_1 \sin \mu_1 - 0.1005035505175591 \cos \mu_3 \sin \psi_3 \\ &\quad - 0.898932680742382 \cos \mu_3 \cos \psi_1 \sin \psi_3 - \end{aligned}$$

$$\begin{aligned}
& 0.4494645425058293 \cos \mu_3 \cos \mu_1 \cos \psi_1 \sin \psi_3 \\
& -0.3262577907206018 \sin \mu_3 \sin \psi_3 - 0.07945700139117601 \sin \psi_1 - \\
& 0.1589146384409058 \cos \mu_1 \sin \psi_1 - 0.4494663403711908 \cos \mu_1 \cos \psi_3 \sin \psi_1 \\
& -0.3178203460745112 \cos \mu_3 \cos \mu_1 \cos \psi_3 \sin \psi_1 - \\
& 0.1589146384409058 \sin \mu_1 \sin \psi_1 + 0.471702595330201 \cos \mu_2 \sin \psi_2 + \\
& 0.4494663403711908 \cos \mu_3 \sin \mu_1 \sin \psi_3 \sin \psi_1 + \\
& 0.2247331701855954 \cos \mu_1 \sin \mu_3 \sin \psi_3 \sin \psi_1 + \\
& 0.3178203460745112 \sin \mu_1 \sin \psi_3 \sin \psi_1 \\
& +0.1694026843444782 \cos \psi_3 \sin \mu_1 \sin \psi_1 \\
& +0.1694026843444782 \cos \mu_3 \cos \mu_1 \sin \psi_3 \sin \psi_1 \\
& +0.1921773402730402 \sin \mu_2 \sin \psi_2 \\
q_5 = & -0.3262577907206018 \cos \psi_3 + 0.1589146384409058 \cos \psi_1 + \\
& 0.07945700139117601 \cos \mu_1 \cos \psi_1 + 0.3178203460745112 \cos \mu_3 \cos \psi_3 \cos \psi_1 \\
& -0.7687062862670159 \cos \psi_2 - 0.3198002640379491 \cos \psi_2 \sin \mu_2 + \\
& 0.1421333271845383 \sin \psi_3 - 0.1694026843444782 \cos \mu_3 \cos \psi_1 \sin \psi_3 + \\
& 0.6741995105567862 \cos \psi_1 \sin \mu_3 \sin \psi_3 + \\
& 0.4494645425058293 \cos \psi_1 \sin \mu_3 \sin \mu_1 \sin \psi_3 \\
& +0.0842725261027617 \cos \mu_1 \sin \psi_1 - 0.0794573192204529 \sin \mu_1 \sin \psi_1 + \\
& 0.3178203460745112 \cos \psi_3 \sin \mu_3 \sin \mu_1 \sin \psi_1 - \\
& 0.4494645425058293 \cos \mu_3 \sin \psi_3 \sin \psi_1 \\
& -0.898932680742382 \cos \mu_3 \cos \mu_1 \sin \psi_3 \sin \psi_1 + \\
& 0.4494663403711908 \cos \mu_1 \sin \mu_3 \sin \psi_3 \sin \psi_1 - \\
& 0.6741995105567862 \cos \mu_3 \sin \mu_1 \sin \psi_3 \sin \psi_1 - \\
& 0.1694026843444782 \sin \mu_3 \sin \mu_1 \sin \psi_3 \sin \psi_1 + \\
& 0.1921773402730402 \cos \mu_2 \sin \psi_2 - 0.471702595330201 \sin \mu_2 \sin \psi_2
\end{aligned}$$

$$\begin{aligned}
q_6 = & -0.1421333271845383 \cos \mu_3 \cos \psi_3 - 0.1589146384409058 \cos \psi_1 + \\
& 0.1694026843444782 \cos \psi_3 \cos \psi_1 - 0.471702595330201 \cos \psi_2 - \\
& 0.07945700139117601 \cos \psi_1 \sin \mu_1 - 0.3262577907206018 \cos \mu_3 \sin \psi_3 + \\
& 0.3178203460745112 \cos \psi_1 \sin \psi_3 + 0.1005035505175591 \sin \mu_3 \sin \psi_3 + \\
& 0.898932680742382 \cos \psi_1 \sin \mu_3 \sin \psi_3 + \\
& 0.4494645425058293 \cos \mu_1 \cos \psi_1 \sin \mu_3 \sin \psi_3 - \\
& 0.0794573192204529 \cos \mu_1 \sin \psi_1 + 0.4494645425058293 \cos \psi_3 \sin \psi_1 + \\
& 0.898932680742382 \cos \mu_1 \cos \psi_3 \sin \psi_1 + \\
& 0.3178203460745112 \cos \mu_1 \cos \psi_3 \sin \mu_3 \sin \psi_1 - \\
& 0.0842725261027617 \sin \mu_1 \sin \psi_1 + 0.898932680742382 \cos \psi_3 \sin \mu_1 \sin \psi_1 \\
& + 0.2247331701855954 \cos \mu_3 \cos \mu_1 \sin \psi_3 \sin \psi_1 - \\
& 0.1694026843444782 \cos \mu_1 \sin \mu_3 \sin \psi_3 \sin \psi_1 - \\
& 0.4494663403711908 \sin \mu_3 \sin \mu_1 \sin \psi_3 \sin \psi_1 + 0.3198002640379491 \sin \psi_2 \\
& + 0.7687062862670159 \sin \mu_2 \sin \psi_2 \\
q_7 = & -0.5685332823940687 + 0.898932680742382 \cos \psi_1 \sin \mu_3 \sin \psi_3 + \\
& 0.898932680742382 \cos \mu_1 \cos \psi_3 \sin \psi_1 - \\
& 0.898932680742382 \cos \mu_3 \sin \mu_1 \sin \psi_3 \sin \psi_1 \\
q_8 = & 0.6756731531896393 + 0.898932680742382 \cos \psi_3 \cos \psi_1 + \\
& 0.4494645425058293 \cos \mu_1 \cos \psi_3 \cos \psi_1 + \\
& 0.3178203460745112 \cos \psi_3 \cos \psi_1 \sin \mu_3 - \\
& 0.6741995105567862 \cos \mu_3 \cos \psi_1 \sin \psi_3 - \\
& 0.1694026843444782 \cos \psi_1 \sin \mu_3 \sin \psi_3 - \\
& 0.4494645425058293 \cos \mu_3 \cos \psi_1 \sin \mu_1 \sin \psi_3 - \\
& 0.1694026843444782 \cos \mu_1 \cos \psi_3 \sin \psi_1 - \\
& 0.4494663403711908 \cos \psi_3 \sin \mu_1 \sin \psi_1
\end{aligned}$$

$$\begin{aligned}
& -0.3178203460745112 \cos \mu_3 \cos \psi_3 \sin \mu_1 \sin \psi_1 - \\
& 0.3178203460745112 \cos \mu_1 \sin \psi_3 \sin \psi_1 - \\
& 0.4494645425058293 \sin \mu_3 \sin \psi_3 \sin \psi_1 \\
& -0.4494663403711908 \cos \mu_3 \cos \mu_1 \sin \psi_3 \sin \psi_1 \\
& -0.898932680742382 \cos \mu_1 \sin \mu_3 \sin \psi_3 \sin \psi_1 + \\
& 0.1694026843444782 \cos \mu_3 \sin \mu_1 \sin \psi_3 \sin \psi_1 - \\
& 0.6741995105567862 \sin \mu_3 \sin \mu_1 \sin \psi_3 \sin \psi_1
\end{aligned}$$

Appendix E

Mechanical Designs of Planar and Spherical DT Manipulators

Typical designs of the planar and spherical DT manipulators are depicted in Figs. (E.1) and (E.2), respectively.

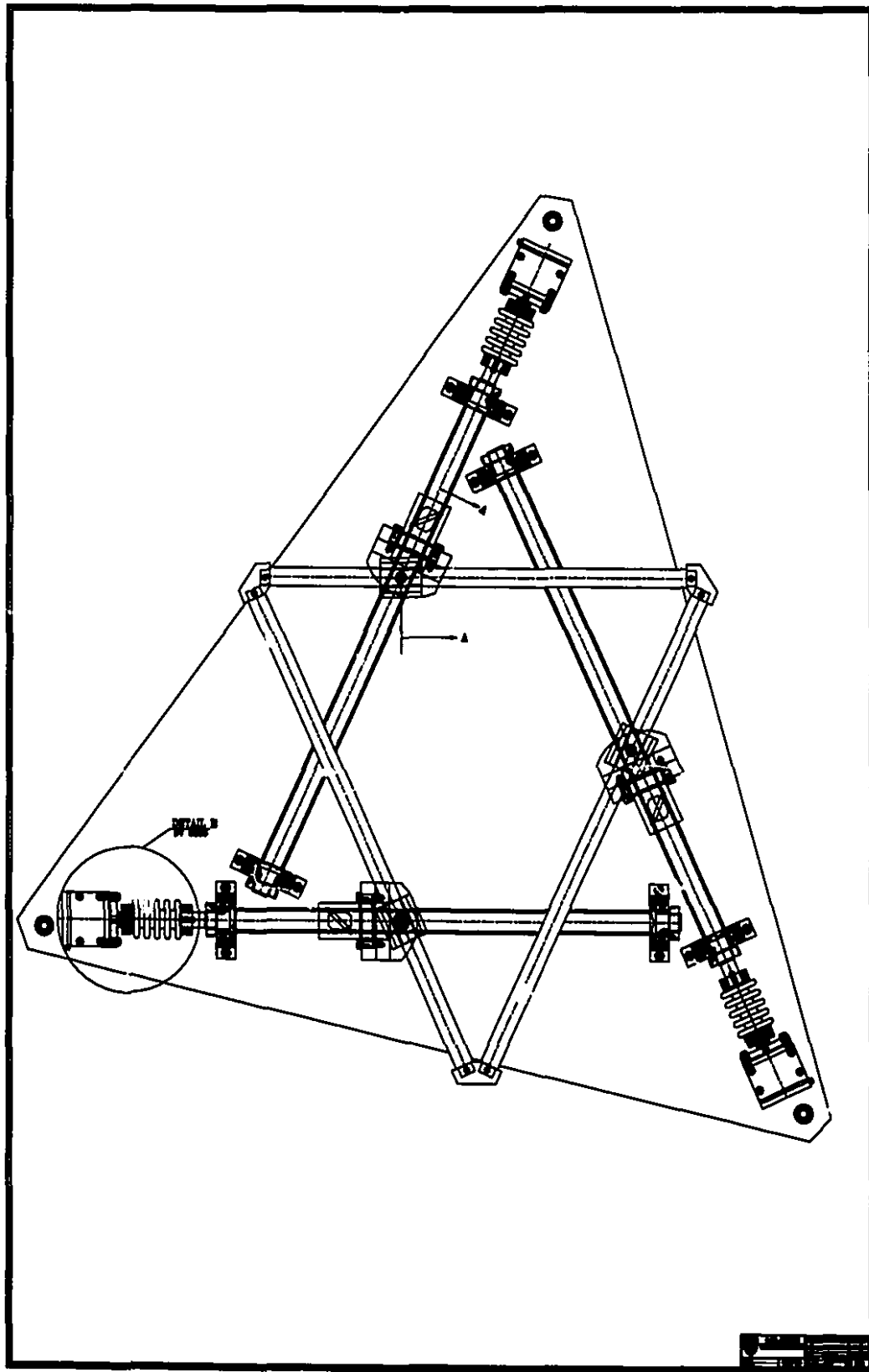
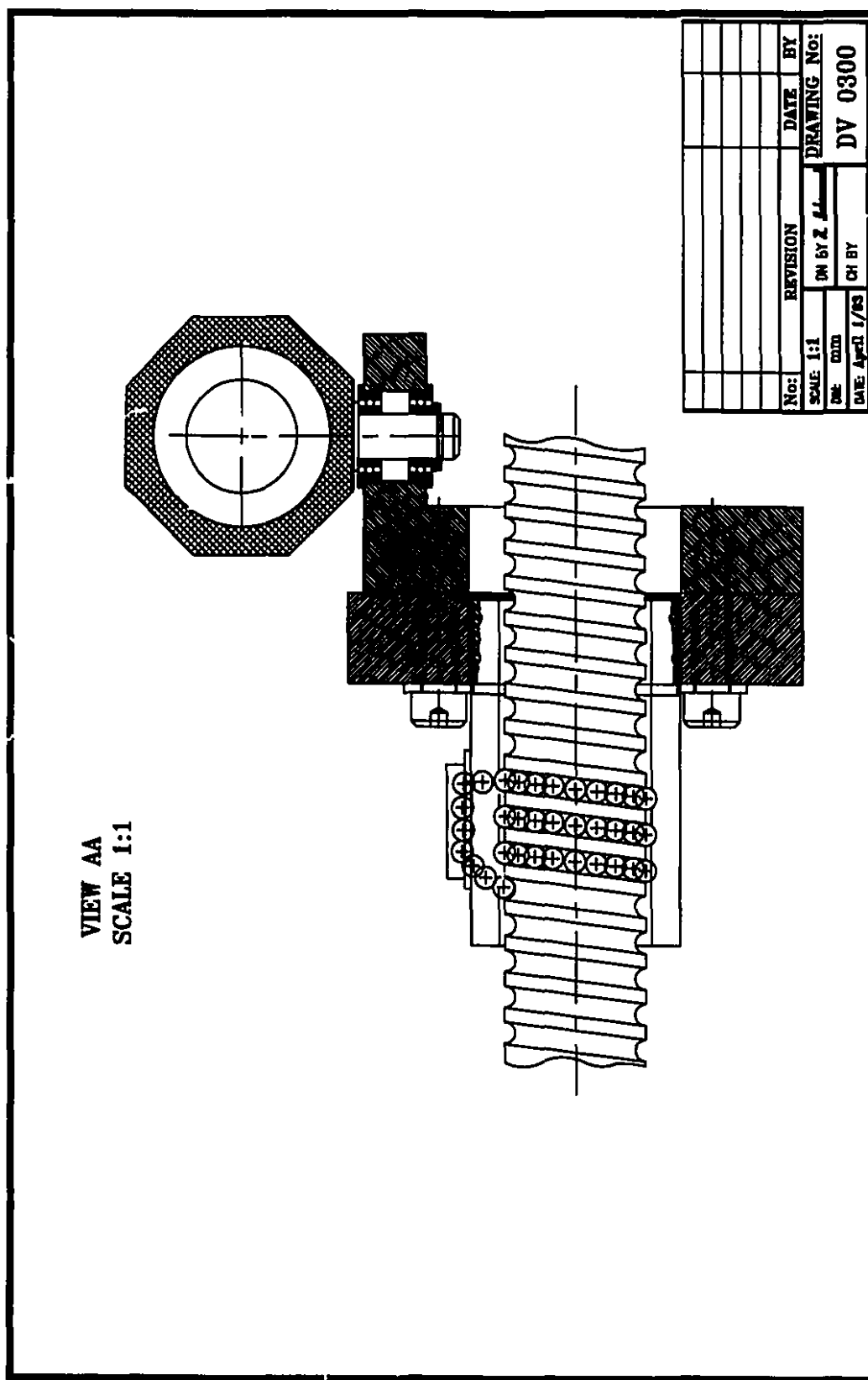
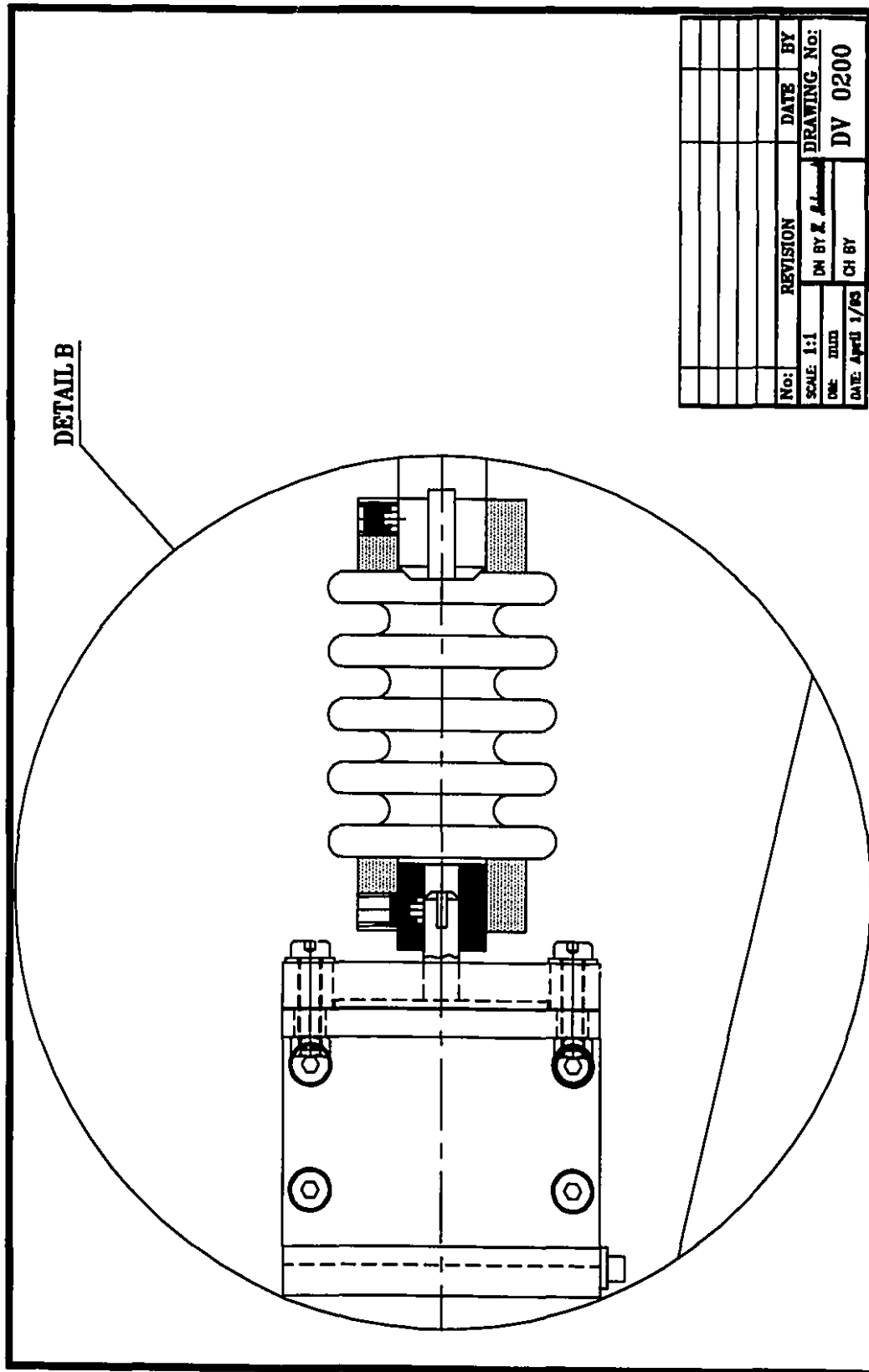


Figure E.1: A typical design of planar DT manipulators





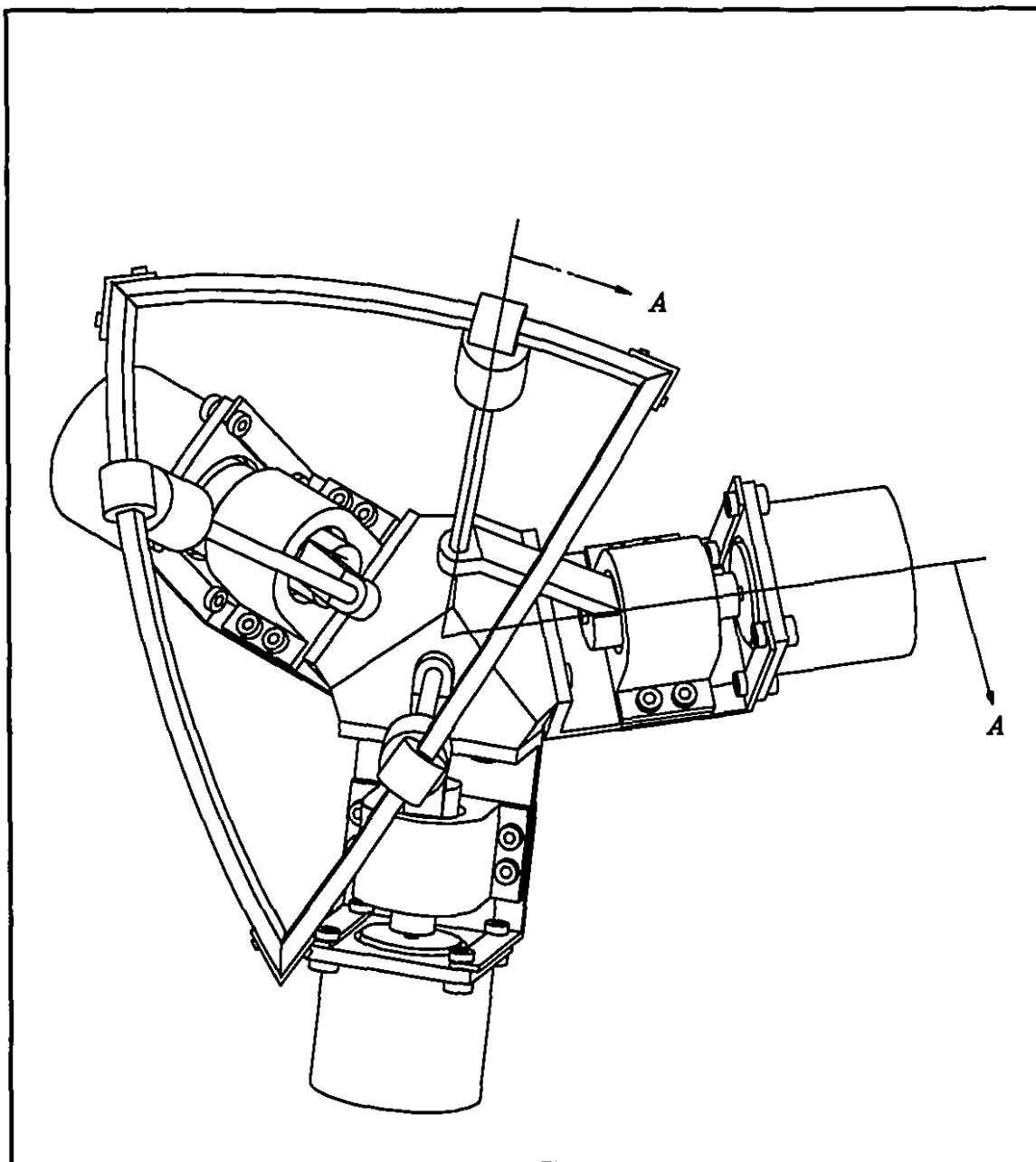


Figure E.2: A typical design of spherical DT manipulators

