ALMOST PERIODIC FUNCTIONS
ON THE ROTATION GROUP

by

James Henniger, B.Sc.

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfilment of the requirements for the degree of Master of Science.

Department of Mathematics,
McGill University,
Montreal.

April 1962.

PREFACE

The numbers in parentheses appearing in the footnotes refer to the corresponding sources of reference listed in the bibliography.

I wish to express my appreciation to Professor Schwerdtfeger for his directing my interest to the subject of almost periodic functions and for his most helpful instruction in the development of the thesis.

TABLE OF CONTENTS

Almost Periodic Functions on the Rotation Group

Introduction

The aim of this thesis is to exhibit the bounded representations of the rotation group, Ω_n, by considering the almost periodic functions on Ω_n. By Ω_n is meant the rotation group in R_n, or more explicitly the group of all proper orthogonal n-matrices where $n \geq 3$. The case $n = 3$ will be given special consideration.

The presentation can be divided roughly into three parts. The first part is a brief description of the theory of almost periodic functions on an arbitrary group, as first developed by von Neumann. This section serves only to display the properties of almost periodic functions and their relations to group representations which will be needed for the discussion of the rotation group. The theorems and properties are only stated in this part and no attempt is made to be complete or rigorous. For those less familiar with the theory, the exposition is illustrated by the familiar example of the continuous periodic functions.

The second part deals with the almost periodic functions on Ω_n, and, corresponding to these, the almost periodic functions on the sphere S^n. The properties of spherical harmonics are investigated and it is shown that the irreducible modules of almost periodic functions on S^n are precisely the sets of harmonic functions of degree s on S^n, for $s = 1, 2, 3, \ldots$.

In the last part, the irreducible representations of Ω_n are exhibited using the modules of almost periodic functions on Ω_n found in the preceding section. A slightly different approach is adopted for Ω_3, using the representations of Ω_3 by quaternions. This results in the finding of representations of all dimensions, the coefficients of the matrices being expressed as spherical harmonics on the unit sphere in R_4.

I Almost Periodic Functions*

1. Definition

Let us begin by considering an arbitrary group G with elements $a, b, \ldots, x, y, \ldots,$ and complex-valued functions $f(x), g(x), \ldots,$ on these elements. A function $f(x)$ is called almost periodic (a.p.) on G if for every $\varepsilon > 0$ there exist finitely many subsets $A_1, \ldots, A_n$ of G such that $A_1 \cup A_2 \cup \ldots \cup A_n = G$, and $|f(axb) - f(ayb)| < \varepsilon$ for any x, y belonging to any one of the subsets A_i, $i = 1, 2, \ldots, n$, and for any a, b in the group G. The set of subsets $A_1, \ldots, A_n$ will be called a partition of G for $f(x)$ and ε.

As an example of almost periodic functions, we may consider the continuous periodic functions on the real line with period 2π. In this case the group will be the additive group of real numbers mod 2π, and the functions on the group will be the continuous functions on $[0, 2\pi]$ with $f(x + 2\pi) = f(x)$. Since the functions are uniformly continuous on the closed interval $[0, 2\pi]$, for any $f(x)$ we may choose a δ so that $|f(x) - f(y)| < \varepsilon$ for all x, y such that $|x - y| < \delta$ (mod 2π). Hence for a partition for $f(x)$ and ε, we may take any division of the segment $[0, 2\pi]$ into seg-

* (1): pp 24 - 65

ments of length less than δ. Then if x, y belong to any one of these segments, $|f(x + a) - f(y + a)| < \varepsilon$ for all real a.

Returning to the general case, we mention only that almost periodicity implies the following properties:

1) if $f(x)$ is a.p. on G, then $f(x)$ is bounded on G;

2) if $f(x)$ and $g(x)$ are a.p. on G, then $f(x) + g(x)$ and $f(x)\, g(x)$ are also;

3) if $f(x)$ is a.p., and φ is a continuous complex valued function defined on complex numbers, then $\varphi[f(x)]$ is a.p. on $x \in G$

- in particular $\overline{f(x)}$, $|f(x)|$, $\alpha f(x)$, and $f(x^{-1})$ are a.p. if $f(x)$ is;

4) if a sequence $\{f_i(x)\}$ of a.p. functions converges uniformly to $f(x)$, then $f(x)$ is a.p. on $x \in G$ also.

2. <u>The Mean Value</u>

Because of the restricted range of values of $f(x)$ over each subset A_i of a partition, one might expect to find a "mean value" of $f(x)$ by forming $\frac{1}{n} \sum_{i=1}^{n} f(a_i)$, where $a_i \in A_i$ of some partition for $f(x)$ and ε, and then letting $\varepsilon \to 0$. A unique limit in the above process may be obtained by considering <u>minimal partitions</u> - that is, a partition for $f(x)$

and ε, $A_1, \ldots, A_M$ where M is the smallest number of subsets a partition for $f(x)$ and ε may have. The existence of the above mean value is asserted by the Mean Value Theorem:

For each a.p. function $f(x)$, there exists a number $M_x\{f(x)\}$, called its mean value, such that for each $\varepsilon > 0$, there can be found elements of the group, $a_1, \ldots, a_n$, such that uniformly for c and $d \in G$ $|M_x\{f(x)\} - \frac{1}{n} \sum_{i=1}^{n} f(ca_i d)| < 2\varepsilon$.

The four important properties of the mean value are as follows:

1) $M_x\{\alpha f(x) + \beta g(x)\} = \alpha M_x\{f(x)\} + \beta M_x\{g(x)\}$

2) $M_x\{f(xa)\} = M_x\{f(x)\}$ for any $a \in G$

3) $M_x\{f(x)\} \leq M_x\{g(x)\}$ if f, g are real and $f(x) \leq g(x)$ for all x

4) $M_x\{1\} = 1$

In fact these four properties uniquely determine the mean value: that is, if $M'_x\{f(x)\}$ is a number associated with $f(x)$ having the above four properties, then $M'_x\{f(x)\} = M_x\{f(x)\}$. A fifth important property of the mean value is this:

5) if $\{f_i(x)\}$ is a sequence of a.p. functions converging uniformly to $f(x)$, i.e. $\lim_{i\to\infty} f_i(x) = f(x)$ then $\lim_{i\to\infty} M\{f_i(x)\} = M\{f(x)\}$.

The mean value of an a.p. function enables us to define a scalar product of two a.p. functions, since the product of a.p. functions is also almost periodic. If $f(x)$, $g(x)$ are a.p. on G, then the scalar product of f and g is defined as

$$(f, g) = M_x\{f(x)\ \overline{g(x)}\}.$$

When one considers the properties of the mean value, the following properties of the scalar product become evident:

1) (Hermitian) $(f, g) = \overline{(g, f)}$

2) (Linearity) $(\alpha f + \beta g, h) = \alpha(f, h) + \beta(g, h)$

3) (Invariance) $(f(axb), g(axb)) = (f(x), g(x))$

4) (Continuity) if $f_i(x)$ converges uniformly to $f(x)$, then $\lim_{i\to\infty} (f_i, g) = (f,g)$.

With the help of the scalar product, we may define the norm of an a.p. function:

$$N(f) = (f, f);$$

and the distance between two a.p. functions:

$$D(f, g) = \sqrt{N(f - g)}$$

One may show, using the properties of the mean value, that the usual properties of a norm and distance function are satisfied by the norm and distance given in the above definitions. Two functions f and g will be said to be orthogonal to each other if $(f, g) = 0$.

Referring back to our example of a.p. functions, i.e. the continuous periodic functions on the real line, one can easily see that the mean value M_x in this case is the mean value in the usual sense - that is

$$M_x\{f(x)\} = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx$$

3. <u>Modules of Almost Periodic Functions</u>

A non-empty set R of almost periodic functions on a group G is called a right-invariant module (more briefly invariant module) when:

1) from $f, g \in R$ it follows that $\alpha f + \beta g \in R$ with complex coefficients α, β, and

2) from $f(x) \in R$ it follows that $f(xc) \in R$ where c is any element of G.

The module is said to be closed if any uniformly convergent sequence of a.p. functions in R has its limit in R; i.e. if $f_i(x) \in R$ and $\lim_{i\to\infty} f_i(x) = f(x)$ uniformly in x then $f(x) \in R$. If a module R contains n functions $f_1, \ldots, f_n$ such that for any $f(x) \in R$, $f(x) = \alpha_1 f_1(x) + \ldots + \alpha_n f_n(x)$ for suitable complex coefficients $\alpha_1, \ldots, \alpha_n$, then the module is said to be finite. If the n functions are linearly independent, then the $f_1, \ldots, f_n$ are said to form a basis of R, and the dimension of R is n.

If R and R_1 are both invariant modules where $R_1 \subset R$, then R_1 is said to be a proper invariant submodule of R if there exists an $f(x) \in R$ such that $f(x) \notin R_1$, and $R_1 \not\equiv \{0\}$ where $\{0\}$ is the module consisting only of the a.p. function $g(x) \equiv 0$. A closed, invariant module $R \neq \{0\}$ is said to be irreducible when it contains no proper closed invariant submodule. Note that if R is irreducible, and $f(x)$ is any function of R, then the set of functions $f_{a_i}(x) = f(xa_i)$, where $a_i \in G$, for suitable a_i, span R. Otherwise the module spanned by the $f_{a_i}(x)$ is an invariant submodule of R.

We shall make one final definition on modules of a.p. functions. Suppose $\{R_\alpha\}$, $\alpha \in A$ is a set (not necessarily countable) of closed invariant modules of a.p. functions on a group G. Consider the set of all functions of the form $f(x) = \sum_{i=1}^{n} f_{\alpha_i}$, where each $f_{\alpha_i} \in R_{\alpha_i}$, $\alpha_i \in A$, and n takes on all finite positive integral values. Call this set $\mathfrak{M}$. Now form the closure of $\mathfrak{M}$, R, by adding to $\mathfrak{M}$ the limit functions of all uniformly convergent sequences in $\mathfrak{M}$. Clearly, R is the smallest closed invariant module containing all R_α. R is said to be the sum of the modules R_α, and we shall write this as $R = \sum_{\alpha \in A} R_\alpha$.

In the following sections, we shall see that the finite irreducible invariant modules of a.p. functions play an important role in the representations of groups. With this and the above definitions in mind, we state the principal theorem in the theory of a.p. functions on groups:

<u>THEOREM</u>: Every closed invariant module R of a.p. functions on a group G is the sum of finite, irreducible, invariant modules R_ν: $R = \sum_\nu R_\nu$.

In our example of the continuous periodic functions, it turns out that the irreducible invariant modules are one dimensional and have as bases the functions $e^{i\nu x}$, $\nu = \pm 1, \pm 2, \ldots, \pm n, \ldots$. The module R_ν spanned by $e^{i\nu x}$ is the set of functions $\alpha e^{i\nu x}$, α complex. It is invariant since $\alpha e^{i\nu(x+a)} = \alpha' e^{i\nu x}$ where $\alpha' = \alpha e^{i\nu a}$, and since it is finite dimensional it is necessarily closed.

4. <u>The Representations of Groups</u>*

As before, we let $x, y, \ldots$ be elements of an arbitrary group G. One speaks of a representation of G when to each element $x \in G$ there exists a non-singular s-rowed square matrix $D(x)$ such that $D(x)\,D(y) = D(xy)$. The coefficients $D_{\rho\sigma}(x)$ of the matrix are complex valued functions of the group elements.

* (1): pp 9 - 18; pp 46; pp 119 - 128.

Two representations $D(x)$ and $D'(x)$ will be considered as being equivalent if there exists a constant, non-singular matrix A such that $D(x) = A\,D'(x)\,A^{-1}$ for all $x \in G$.

A representation of the group is called reducible if there exists a representation of the form

$$\begin{pmatrix} D_1(x) & & & 0 \\ & D_2(x) & & \\ & & \ddots & \\ 0 & & & D_r(x) \end{pmatrix}$$

to which it is equivalent. The matrices $D_i(x)$ along the diagonal are s_i-rowed representations of G, and there are zeros elsewhere. If a representation of the above form to which $D(x)$ is equivalent <u>cannot</u> be found, then $D(x)$ is said to be <u>irreducible</u>.

A representation $D(x) = (D_{\rho\sigma}(x))$ is <u>unitary</u> if for every $x \in G$, $D(x)\,D^*(x) = E$ where $D^*(x) = (\overline{D_{\sigma\rho}(x)})$ and E is the unit matrix. $D(x)$ is <u>normal</u> if it is equivalent to a unitary representation.

$D(x)$ is said to be <u>bounded</u> if each of its coefficients $D_{\rho\sigma}(x)$, considered as functions on G, are bounded: i.e. $|D_{\rho\sigma}(x)| < M$ for all $x \in G$ and all ρ and σ.

Now bounded representations are important for our purposes for the following reasons. Firstly, the coefficients $D_{\rho\sigma}(x)$ of a bounded representation of G are almost periodic functions on G. Secondly, using the existence of a mean value of a.p. functions, one can prove that a representation is normal if and only if it is bounded.

To each bounded irreducible representation D(x), then, corresponds an equivalent irreducible unitary representation $D^{(\nu)}(x)$ (since it is normal). The relation $D(x) \sim D'(x)$ iff $D(x) = A\, D'(x)\, A^{-1}$ is well known to be an equivalence relation, and hence we may assume $D^{(\nu)}(x)$ to be the representative of its equivalence class. Now if we consider all bounded irreducible representations on G, and the system of all representatives of their equivalence classes $\{D^{(\nu)}(x)\}$, then this system is a complete system of bounded unitary representations of G in the sense that to any bounded irreducible representation of G corresponds a $D^{(\nu)}(x)$, in the system to which it is equivalent; and only one $D^{(\nu)}(x)$, since the $D^{(\nu)}(x)$ are all inequivalent. Thus, any bounded representation of G may be written as $A\, C(x)\, A^{-1}$ where

$$C(x) = \begin{pmatrix} C_1(x) & & & 0 \\ & C_2(x) & & \\ & & \ddots & \\ 0 & & & C_r(x) \end{pmatrix}$$

in which each $C_i(x)$ is equivalent to some $D^{(\nu)}(x)$ belonging to the complete system of unitary representations $\{D^{(\nu)}(x)\}$.

To sum up then, from the representations in the complete system of bounded irreducible inequivalent representations, one is able to obtain a survey, up to equivalence, of all bounded representations of G. Throughout this thesis we shall consider only bounded representations.

5. Modules of A.P. Functions as Representation Modules

We are now in a position to show the correspondence between the finite, irreducible, right-invariant modules of a.p. functions, and the irreducible representations on the same group G.

Let $\mathfrak{M}$ be a finite invariant (not necessarily irreducible) module of a.p. functions on G, and $f_1(x), \ldots, f_n(x)$ be a basis of $\mathfrak{M}$. Since $\mathfrak{M}$ is right-invariant, $f_i(xc) \in G$, where c is any element of G and $f_i(x)$ is any one of the basis functions. Therefore we may write $f_i(xc)$ as a linear combination of $f_1(x), \ldots, f_n(x)$ - i.e.

$$f_i(xc) = \sum_{j=1}^{n} D_{ij}(c)\, f_j(x),$$

and similarly $f_i(xcd) = \sum_{j=1}^{n} D_{ij}(cd)\, f_j(x).$ (1)

$$\text{But}\quad f_i(xcd) = \sum_{j=1}^{n} D_{ij}(c)\, f_j(xd)$$

$$= \sum_{j=1}^{n} D_{ij}(c)\, \Big(\sum_{k=1}^{n} D_{jk}(d)\, f_k(x)\Big)$$

$$= \sum_{k=1}^{n} \Big(\sum_{j=1}^{n} D_{ij}(c)\, D_{jk}(d)\Big)\, f_k(x). \quad (2)$$

Letting $D(c) = (D_{ij}(c))$ represent the matrix represented by the transformation $f_i(x) \rightarrow f_i(xc)$, then $D_{ik}(cd) = \sum_{j=1}^{n} D(c)_{ij} D(d)_{jk}$ from (1) and (2) or $D(cd) = D(c) D(d)$. That is, the transformation matrix $D(c)$ associated with the transformation $f(x) \longrightarrow f(xc)$ in the above manner is a representation of G.

If A is any non-singular $n \times n$ matrix, then the equivalent representation $AD(x)A^{-1}$ is the transformation matrix associated with above transformation for the new basis $g_i(x) = \sum_{j=1}^{n} A_{ij} f_j(x)$, $i = 1, \ldots, n$. That is, $g_i(xc) = \sum_{k=1}^{n} (AD(c)A^{-1})_{ik} g_k(x)$. Conversely for any new basis $g_1, \ldots, g_n$, we may determine an $n \times n$ matrix A such that $g_i(x) = \sum_{j=1}^{n} A_{ij} f_j(x)$, so that if $D(c)$ is the matrix associated with the above transformation with respect to the basis $f_1(x), \ldots, f_n(x)$, then $A\, D(c)\, A^{-1}$ is the matrix associated with the same transformation with respect to the basis $g_1(x), \ldots, g_n(x)$. Thus the representations resulting from a given invariant module $\mathfrak{M}$ are equivalent; in fact the module yields all the equivalent representations, each representation corresponding to a particular basis of $\mathfrak{M}$.

Clearly, irreducible modules give rise to irreducible representations, and conversely, if a representation associated with a module $\mathfrak{M}$ is irreducible, then $\mathfrak{M}$ is irreducible. For if $D(x)$ is reducible, and associated with a module $\mathfrak{M}$, then it is equivalent to a representation of the form $C(x) = \begin{pmatrix} C_1(x) & 0 \\ 0 & C_2(x) \end{pmatrix}$

where $C_1(x)$ has r rows and columns. Then if $f_1(x), \ldots, f_r(x), \ldots, f_n(x)$ is the basis associated with $C(x)$, one can see that $f_i(xc)$, $i \leq r$, is a linear combination of $f_i(x)$, $i = 1, \ldots, r$, for all c so that the module $\mathfrak{M}_1$ spanned by $f_1(x), \ldots, f_r(x)$ is an invariant submodule of $\mathfrak{M}$. Conversely, if $\mathfrak{M}$ is reducible, it is easy to show in a like manner that $\mathfrak{M}$ gives rise to reducible representations.

If the basis $f_1, \ldots, f_n$ of the invariant module $\mathfrak{M}$ is taken to be orthonormal - i.e. $(f_i, f_j) = \delta_{ij}$, then the representation associated with this basis will be unitary.

Now the functions $D_{\rho\sigma}(x)$ of a representation $D(x) = (D_{\rho\sigma}(x))$ associated with a module $\mathfrak{M}$ form a set of functions which span the module $\mathfrak{M}$. One simply writes $f_i(xc) = \sum_{j=1}^{n} D_{ij}(c)\, f_j(x)$, and putting $x = 1$, obtains

$$f_i(c) = \sum_{j=1}^{n} f_j(1)\, D_{ij}(c) \quad \text{for all } c \in G.$$

The basis functions are thus linear combinations of the $D_{ij}(x)$, and since any $f(x) \in \mathfrak{M}$ is a linear combination of the basis functions $f_1(x), \ldots, f_r(x)$, it is therefore a linear combination of the $D_{ij}(x)$.

It is evident from the above relations that the functions $D_{ij}(x)$ are bounded, since the a.p. functions are bounded, and hence the representations resulting from a module $\mathfrak{M}$ are bounded representations. If we recall now the complete system of inequivalent irreducible bounded representations $\{D^{(\nu)}(x)\}$, then the representations to which $\mathfrak{M}$ gives rise are represented in this system, and we may choose a basis of $\mathfrak{M}$ which gives rise to the $D^{(\nu)}(x)$ which represents this set of equivalent representations. Restricting ourselves to these unitary representations $D^{(\nu)}(x)$, we now state the following theorem, one of the most important theorems on a.p. functions:

<u>THEOREM</u>: The functions $D^{(\nu)}_{\rho\sigma}(x)$, coefficients of the inequivalent, irreducible, bounded, unitary representations in the above complete system of representations of G, form, in the space of all a.p. functions on G an orthogonal, and in a certain sense, normal system of functions:

i.e. $$(D^{(\nu)}_{\rho\sigma}(x), D^{(\mu)}_{\tau\omega}(x)) = \begin{cases} 1/S^{(\nu)} & \text{when } \sigma = \omega, \ \rho = \tau, \ \mu = \nu \\ 0 & \text{otherwise} \end{cases}$$

$S^{(\nu)}$ is the number of rows in the matrix $D^{(\nu)}(x)$.

Now the set of all a.p. functions on a group G is a closed invariant module, R, and is therefore the sum of irreducible finite invariant modules R_ν, i.e. $R = \sum R_\nu$. By the definition of the sum of modules, this means that any $f \in R$ can be uniformly approximated by finite sums $\sum_{\text{finite}} {}_i f_{\nu_i}$ where f_{ν_i} belongs to some R_ν. This f_{ν_i} in turn can be written as $f_{\nu_i}(x) = \sum_{\rho,\sigma}^{s^{(\nu_i)}} a_{\rho\sigma}^{(\nu_i)i} D_{\rho\sigma}^{(\nu_i)}(x)$, where $D^{(\nu_i)}(x)$ is the representation in $\{D^{(\nu)}(x)\}$ corresponding to the module R_{ν_i}. Thus we have the <u>Approximation Theorem</u>:

Each a.p. function $f(x)$ can be uniformly approximated by finite series of the following form:

$$\sum_{i=1}^{n} s^{(\nu_i)} \sum_{\rho,\sigma=1}^{(s^{(\nu_i)})} a_{\rho\sigma}^{(\nu_i)} D_{\rho\sigma}^{(\nu_i)}(x),$$

where the $D_{\rho\sigma}^{(\nu_i)}(x)$ are coefficients of the unitary irreducible representations of a complete system $\{D^{(\nu)}(x)\}$ of representations.

We may illustrate these developments by referring to our example of the continuous periodic functions. It turned out that all the irreducible modules in this case were one-dimensional, and spanned by the functions $e^{i\nu x}$ $\nu = \pm 1, \pm 2, \ldots$. The complete set of irreducible representations of the real numbers mod 2π are given by $D^{(\nu)}(a) = e^{i\nu a}$ since $e^{i\nu(x+a)} = (e^{i\nu a})(e^{i\nu x})$. These coefficients are certainly orthonormal since $(e^{i\nu x}, e^{i\mu x}) = \delta_{\nu\mu}$. Further, the general approximation theorem proved above reduces in this case to the familiar Weierstrass Approximation Theorem:

Any continuous function on the interval $[0, 2\pi]$ with $f(0) = f(2\pi)$ can be uniformly approximated by finite sums of the form $\sum_{\nu=1}^{n} a_\nu e^{ix\nu}$.

Now if we consider any periodic continuous function $f(x)$, we may form its Fourier Coefficients $\alpha_\nu = (f(x), e^{i\nu x})$, and associate with $f(x)$ a Fourier Series $f(x) \sim \sum_{\nu=-\infty}^{\infty} \alpha_\nu e^{i\nu x}$. One can then prove the Bessel Inequality:

$$\sum_{\nu=-\infty}^{\infty} |\alpha_\nu|^2 \leq (f, f) ,$$

and using the Weierstrass Theorem, one can show that the equality holds

$$\text{i.e. } \sum_{\nu=-\infty}^{\infty} |\alpha_\nu|^2 = (f, f).$$

This relation is the well-known Parseval Equation or Completeness Relation. From it follows that $f(x)$ can be approximated <u>in the mean</u> by its Fourier Series -

$$\text{i.e. } N(f - \sum_{\nu=-n}^{n} \alpha_\nu e^{i\nu x}) \to 0 \quad \text{as} \quad n \to \infty.$$

Now these properties of Fourier Series may be extended to similar results for a.p. functions in general. Proceeding in a parallel fashion, we call $\alpha_{\rho\sigma}^{(\nu_i)} = (f(x), D_{\rho\sigma}^{(\nu_i)}(x))$ the Fourier coefficient of $f(x)$ with respect to $D_{\rho\sigma}^{(\nu_i)}$, and associate with $f(x)$ a Fourier Series:

$$f(x) \sim \sum_{i=1}^{\infty} S_{(\nu_i)} \sum_{\rho,\sigma=1}^{S(\nu_i)} \alpha_{\rho\sigma}^{(\nu_i)} D_{\rho\sigma}^{(\nu_i)}(x) .$$

It can be shown that at most a countable number of the $\alpha_{\rho\sigma}^{(\nu_i)}$'s are different from 0, so that the countable sum in the Fourier Series is justified. As for the previous case, we can state the Parseval Equation:

$\sum_{\nu} s_{(\nu)} \sum_{\rho,\sigma=1}^{s_{(\nu)}} |\alpha_{\rho\sigma}^{(\nu)}|^2 = (f, f)$ in its generalized form, where the $\alpha_{\rho\sigma}^{(\nu)}$ are the Fourier coefficients with respect to a <u>complete</u> system of inequivalent unitary representations of G. And from this follows the completeness of the system $\{D^{(\nu)}(x)\}$ - i.e. that every a.p. function can be <u>approximated</u> <u>in</u> <u>the</u> <u>mean</u> by its Fourier Series:

$$N(f(x) - \sum_{i=1}^{n} s_{(\nu_i)} \sum_{\rho,\sigma=1}^{s_{(\nu_i)}} \alpha_{\rho\sigma}^{(\nu_i)} D_{\rho\sigma}^{(\nu_i)}(x)) \to 0 \quad \text{as } n \to \infty.$$

The Parseval relation is equivalent to the completeness of the system of functions $D_{\rho\sigma}^{(\nu)}(x)$ in the following sense: if $f(x)$ is any a.p. function such that $(f(x), D_{\rho\sigma}^{(\nu)}(x)) = 0$ for all ρ, σ, and ν, then $f(x) \equiv 0$. Thus to show that a system of representations is complete (or equivalently that a system of irreducible modules of a.p. functions is complete - i.e. exhausts the set of all a.p. functions on the group) then one need only show that if $(f(x), D_{\rho\sigma}^{(\nu)}(x))$ is zero for all $f(x)$, a.p. on G, and for all ρ, σ, ν, then $f(x) \equiv 0$.

We shall make use of one more property of a.p. functions. Using topological considerations on the space of a.p. functions, one can show that the definition of a.p. functions can be modified slightly. One can prove the following theorem:

If $f(x)$ is a function of the elements x of a group G, and to every $\varepsilon > 0$ there exists a covering $A_1, \ldots, A_n$ of G - i.e. $A_1 \cup A_2 \cup \ldots \cup A_n = G$ such that uniformly in $d \in G$

$$|f(xd) - f(yd)| < \varepsilon \quad \text{for} \quad x, y \in A_i$$

then $f(x)$ is almost periodic.

This means that instead of requiring invariance of the inequality under both left and right translation of the elements x and y, it is sufficient to consider only right translations.

During the remainder of this thesis we shall deal with the a.p. functions and representations of the rotation group in R^n, and in particular in R^3. By the rotation group we mean the group of distance preserving transformations of the unit sphere, S^n, into itself - i.e. the group of proper orthogonal $n \times n$ matrices.

Before we consider the a.p. functions on the rotation group, we shall consider a slightly different concept of almost periodicity, that of almost periodicity on the sphere.

II

1. Almost Periodic Functions on the Sphere S^n

We denote by S^n, $n \geq 3$, the sphere of radius ρ in R^n and by P, Q, ... points on S^n - i.e. if $P = (x_1, \ldots, x_n)$ and $P \in S^n$, then $\sum_{i=1}^n x_i^2 = \rho^2$. Ω_n will denote the rotation group in R^n - i.e. the group of proper orthogonal n-matrices, $x, y, \ldots, \nu, \mu, \ldots$ denoting its elements. We shall denote by $P.Q$ the scalar product of the two points $P = (x_1, \ldots, x_n)$, $Q = (y_1, \ldots, y_n)$: $P.Q = \sum_{i=1}^n x_i y_i$. We shall have occasion to use the fact that if $w(P_1, P_2)$ is a function on S^n, $n \geq 3$, such that $w(P_1\nu, P_2\nu) = w(P_1, P_2)$ for all $\nu \in \Omega_n$, then w is a function only of $P_1.P_2$. Notice that due to the importance of right-invariance in modules of a.p. functions, we are using matrix multiplication on the right side of the point P.

Now S^n is compact. If f(P) is a continuous complex function on S^n, then to every ε there corresponds a δ such that if $d(P, Q) < \delta$, where d( ) is the metric on S^n, then $|f(P) - f(Q)| < \varepsilon$. Thus we may choose finitely many δ-spheres $A_1, \ldots, A_n$ such that $A_1 \cup A_2 \cup \ldots \cup A_n = S^n$, and $|f(P) - f(Q)| < \varepsilon$ when $P, Q \in A_i$ for any i. Since a δ-sphere is transformed into another δ-sphere under a rotation of S^n, we have that $|f(P\nu) - f(Q\nu)| < \varepsilon$ for $P, Q \in A_i$ and any $\nu \in \Omega_n$. Such a function f is called almost periodic on S^n.

Now the continuous a.p. functions on S^n form a closed right-invariant module $\mathfrak{M}$ in the following sense:

1) if $f(P), g(P) \in \mathfrak{M}$, $\alpha f(P) + \beta g(P) \in \mathfrak{M}$

2) if $f(P) \in \mathfrak{M}$, $f(P\nu) \in \mathfrak{M}$ for any $\nu \in \Omega_n$

3) if $f_i(P)$ is a uniformly convergent sequence of functions $f_i \in \mathfrak{M}$, then $\lim_{i\to\infty} f_i = f(P)$ also belongs to $\mathfrak{M}$. The above properties follow from the continuity of the functions $\alpha f + \beta g$, $f(P\nu)$, and $\lim_{i\to\infty} f_i(P)$.

To each of the above a.p. functions on S^n we may associate an a.p. function on Ω_n in the following obvious way. Let P_o be a particular point of S^n, and define $F(x)$ on Ω_n by the equation $F(x) = f(P_o x) = f(P)$ where $P_o x = P$. Ω_n may be divided into subsets $B_1, \ldots, B_n$ in the following manner: $\nu \in B_i$ iff $P_o \nu = P \in A_i$ where A_i belongs to the set of δ-spheres for $f(P)$ mentioned previously. Then clearly $\{B_i\}$ is a division of Ω_n for $F(x)$ and ε, since $|F(x\nu) - F(y\nu)| = |f(P_o x\nu) - f(P_o y\nu)|$, so that if $x, y \in B_i$, then $P_o x, P_o y \in A_i$ and therefore $|f(P_o x\nu) - f(P_o y\nu)| < \varepsilon$ for any $\nu \in \Omega_n$. Hence $F(x)$ is an a.p. function on Ω_n.

The set R of all such functions $F(x) = f(P_o x)$ for a particular P_o forms a closed right-invariant module of a.p. functions on Ω_n, as may be seen by considering the corresponding module $\mathcal{M}$ of a.p. functions on S^n. For example,

$$\begin{aligned} F(x) \in R &\Rightarrow f(P_o x) \in \mathcal{M} \\ &\Rightarrow f(P_o x \nu) \in \mathcal{M} \text{ for any } \nu \in \Omega_n \\ &\Rightarrow F(x\nu) \in R \text{ for any } \nu \in \Omega_n \end{aligned}$$

The other requirements follow in a similar manner.

Since R is a right-invariant closed module of a.p. functions, the principal theorem on modules of a.p. functions asserts that it is the sum of finite-dimensional invariant irreducible modules R_α, $\alpha \in A$. By the correspondence $F(x) \leftrightarrow f(P_o x) = f(P)$ where $F(x) \in R$ and $f(P) \in \mathcal{M}$, one can see that $\mathcal{M}$ is the sum of finite-dimensional invariant irreducible modules of a.p. functions on S^n. The modules $\mathcal{M}_\alpha$ are defined by the relation $\{f(P) \in \mathcal{M}_\alpha \text{ iff } F(x) \in R_\alpha\}$. $\mathcal{M} = \sum_{\alpha \in A} \mathcal{M}_\alpha$ in the sense that for every ε, and any $f(P) \in \mathcal{M}$, there exist finitely many α_i's, such that $|f(P) - \sum_1^n f_{\alpha_i}(P)| < \varepsilon$ where $f_{\alpha_i}(P) \in \mathcal{M}_{\alpha_i}$. $\mathcal{M}_\alpha$ is invariant in the sense that if $f_\alpha(P) \in \mathcal{M}_\alpha$, then $f_\alpha(P\nu) \in \mathcal{M}_\alpha$ for any $\nu \in \Omega_n$. And $\mathcal{M}_\alpha$ is irreducible in the sense that $\{0\}$ is the only invariant submodule of $\mathcal{M}_\alpha$. That is, if $f_\alpha(P)$ is any function of $\mathcal{M}_\alpha$, then there exist $\nu_1, \ldots, \nu_n$ such that

$f_\alpha(P\nu_1), \ldots, f_\alpha(P\nu_n)$ form a basis of $\mathfrak{M}_\alpha$.

Now the mean value of a function $F(x)$ belonging to R reduces to the integral of the corresponding function $f(P)$ belonging to $\mathfrak{M}$ over S^n. That is, if $d\,S^n(P)$ is the element of volume of S^n at P, and w is the "volume" of S^n, then

$$M_x\{F(x)\} = \frac{1}{w}\int_{S^n} f(P)\,d\,S^n(P),$$

where $F(x) = f(P_o x) = f(P)$. That this is so can be seen by considering the following properties, which uniquely determine the mean value M_x:

1) $$\frac{1}{w}\int_{S^n}[\alpha f(P) + \beta g(P)]\,d\,S^n(P) = \frac{\alpha}{w}\int_{S^n} f(P)\,d\,S^n(P) + \frac{\beta}{w}\int_{S^n} g(P)\,d\,S^n(P)$$

2) $$\frac{1}{w}\int_{S^n} f(P\nu)\,d\,S^n(P) = \frac{1}{w}\int_{S^n} f(P)\,d\,S^n(P\nu^{-1}) = \frac{1}{w}\int_{S^n} f(P)\,d\,S^n(P)$$

since $d\,S^n(P) = d\,S^n(P\mu)$ for any $\mu \in \Omega_n$.

3) $$\frac{1}{w}\int_{S^n} f(P)\,d\,S^n(P) \leq \frac{1}{w}\int_{S^n} g(P)\,d\,S^n(P)$$

if f, g are real and $f(P) \leq g(P)$ for all P.

4) $$\frac{1}{w}\int_{S^n} 1\,d\,S^n(P) = 1$$

Then if $M'_x\{F(x)\} = \frac{1}{w}\int_{S^n} f(P)\,d\,S^n(P)$, M'_x satisfies the four conditions that determine M_x, so that $M'_x = M_x$.

It is clear from the preceding discussion that one may form a mean value over Ω_n of an a.p. function on the sphere - i.e. of $f(P) \in \mathfrak{M}$. For if $F(x) = f(P_1 x) = f(P)$ for some particular P_1, then $F(x)$ is a.p. on Ω_n, so that we may put $\underset{x \in \Omega_n}{M_x} \{f(P)\} = M_x\{f(P_1 x)\} = M_x\{F(x)\}$. One may extend this notion to the mean value of a product of functions of $\mathfrak{M}$, say $f_1(P)$, $f_2(P')$. Then $F_1(x) = f_1(P_1 x) = f_1(P)$ and $F_2(x) = f_2(P_2 x) = f_2(P')$, where $P_1 x = P$, and $P_2 x = P'$, are a.p. functions on Ω_n, so that their product is a.p. on Ω_n. Let $H(x) = F_1(x)\, F_2(x)$. Then we can find the mean value of $H(x)$ and call this the mean value over Ω_n of $f_1(P_1 x)\, f_2(P_2 x)$. Put $w(P_1, P_2) = M_x\{f_1(P_1 x)\, f_2(P_2 x)\} = M_x\{H(x)\}$. The two functions $f_1(P)$, $f_2(P) \in \mathfrak{M}$ are said to be orthogonal over Ω_n if $w(P, P) = M_x\{f_1(Px)\, \overline{f_2(Px)}\} = 0$. We investigate now the properties of this function $w(P, Q) = M_x\{f_1(Px)\, f_2(Qx)\}$.

Firstly $w(P, Q)$, as a function of P, keeping Q fixed, is continuous. For, $f_1(P)$ being uniformly continuous on S^n, there exists a $\delta = \delta(\varepsilon)$ for every ε such that $|f_1(P_1) - f_1(P_2)| < \varepsilon/M$ when $d(P_1, P_2) < \delta$, $d(P_1, P_2)$ being the metric on S^n, and $M = \underset{Q \in S^n}{\text{l u b}} |f_2(Q)|$.

Then $|w(P_1, Q) - w(P_2, Q)|$

$$= |M_\nu\{f_1(P_1\nu)\, f_2(Q\nu) - f_1(P_2\nu)\, f_2(Q\nu)\}|$$
$$\leq M_\nu\{\, |f_1(P_1\nu) - f_1(P_2\nu)|\ \ |f_2(Q\nu)|\}$$
$$< \varepsilon \text{ if } d(P_1, P_2) < \delta.$$

Similarly $w(P, Q)$ is a continuous function of Q, keeping P fixed. Further, $w(P, Q)$ may be written as the limit of the sequence $h_n(P, Q) = \frac{1}{n}\sum_{i=1}^{n} f_1(P\nu_i)\, f_2(Q\nu_i)$ where the ν_i are the elements used in the corresponding sequence $\{H(\nu_i)\}$ used to find the mean value $M_\nu\{H(\nu)\}$. Each term of the sequence $\{h_n\}$, considered as a function of P, is continuous on S^n. In fact the sequence $h_n(P, Q)$ is equicontinuous. For if δ and ε are defined as above, then $|f_1(P_1\nu_i)\, f_2(Q\nu_i) - f_1(P_2\nu_i)\, f_2(Q\nu_i)| < \varepsilon$ for all i when $d(P_1, P_2) < \delta$. Therefore

$|H_n(P_1, Q) - H_n(P_2, Q)| =$

$$\left|\frac{1}{n}\sum_{i=1}^{n} f_1(P_1\nu_i)\, f_2(Q\nu_i) - \frac{1}{n}\sum_{i=1}^{n} f_1(P_2\nu_i)\, f_2(Q\nu_i)\right|$$

$< \varepsilon$ for all n. Hence $w(P, Q)$ is the continuous limit of an equicontinuous sequence of functions on the compact space S^n, and the sequence is therefore uniformly convergent.

Now if $f_1(P)$ belongs to an invariant (not necessarily irreducible) closed module $\mathfrak{M}_1$, and f_2 to another such module $\mathfrak{M}_2$, then each of the terms $\frac{1}{n}\sum_{i=1}^{n} f_1(P\nu_i)\, f_2(Q\nu_i)$ belongs to $\mathfrak{M}_1$, if Q is held constant, and since $w(P, Q)$ is the limit to which the sequence converges uniformly in P, w(P, Q) is an a.p. function of P in $\mathfrak{M}_1$, since $\mathfrak{M}_1$ is closed. Similarly, if P is held constant w(P, Q) is an a.p. function of Q, in $\mathfrak{M}_2$.

w(P, Q) is invariant under Ω_n; i.e. $w(P\mu, Q\mu) = w(P, Q)$. This is a consequence of the properties of the mean value. It follows that w(P, Q) is a function only of P.Q, as mentioned at the beginning of this section. And therefore $w(P, Q) = w(Q, P)$ since $P.Q = Q.P$.

Now $w(P, Q_0)$, for some particular value of $Q = Q_0$, is a function of the invariant module $\mathfrak{M}_1$, since $f_1(P) \epsilon \mathfrak{M}_1$. But $w(P, Q_0) = w(Q_0, P)$, and the latter is a function of the invariant module $\mathfrak{M}_2$ since $f_2(P) \epsilon \mathfrak{M}_2$. We may conclude then, that w(P, Q) is a function of both $\mathfrak{M}_1$ and $\mathfrak{M}_2$, considered as a function of P. This same property holds if w(P, Q) is considered as a function of Q. Hence if $\mathfrak{M}_1$ and $\mathfrak{M}_2$ have only the function 0 in common, $w(P, Q) \equiv 0$. In particular, if $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are irreducible invariant <u>distinct</u> modules of $\mathfrak{M}$, they can have only the function 0 as their common element so that $M_\nu\{f_1(P\nu)\, f_2(Q\nu)\} = 0$ for all $f_1 \epsilon \mathfrak{M}_1$ and $f_2 \epsilon \mathfrak{M}_2$.

As an example of an invariant module of continuous functions on S^n, consider the set of all homogeneous polynomials of $P \in S^n$ of degree m. These are functions of the form $\sum_{r_1,\dots,r_n} a_{r_1 \dots r_n} x_1^{r_1} x_2^{r_2} \dots x_n^{r_n}$ where $\sum_{i=1}^{n} x_i^2 = \rho^2$ and $r_1 + r_2 + \dots + r_n = m$, $a_{r_1 \dots r_n}$ complex. Being continuous, these functions belong to $\mathfrak{M}$, and the set of all such functions for some particular m is certainly invariant under Ω_n. This invariant module must contain at least one of the irreducible invariant modules of $\mathfrak{M}$. That each of the irreducible modules consists of homogeneous polynomials of some particular degree m will be shown in the next section. In fact we will show that the complete set of irreducible modules of $\mathfrak{M}$ is the set of systems of harmonic functions on S^n of degree $s = 0, 1, 2, \dots$, or if $\rho = 1$, the systems of spherical harmonics of degree s.

2. Spherical Harmonics in n - dimensions

We shall investigate the system of continuous solutions of the equation (1) $\Delta f = 0$, where $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ and $f = f(P) = f(x_1, \dots, x_n)$, $P \in S^n$.

The operator is invariant under orthogonal transformations of coordinates. That is

$$\sum_{i=1}^{n} \frac{\partial^2 f(P)}{\partial x_i^2} = \sum_{i=1}^{n} \frac{\partial^2 f(P)}{\partial x_i'^2}$$

where the transformation from the coordinates $(x_1, \ldots, x_n)$ to $(x_1', \ldots, x_n')$ is accomplished by an orthogonal matrix, as can easily be verified by making the substitution of new coordinates and simplifying using the orthogonality relations. This implies that if $\Delta f(P) = 0$, then $\Delta f(P\nu) = 0$ also, so that a system of solutions of $\Delta f = 0$ is invariant under Ω_n. Also, if f, g are solutions of the equation, $\alpha f + \beta g$ are also solutions. Hence if we restrict the solutions to continuous functions on S^n, the resulting system of solutions is an invariant submodule of $\mathfrak{M}$.

If we now require that a system of solutions be homogeneous polynomials of degree s, then the resulting module will be an invariant submodule of the module of all homogeneous polynomials of $P \epsilon S^n$ of degree s. To show that there are such solutions, we actually find a solution of the form

$$f(x_1, \ldots, x_n) = \sum_{r=0}^{s/2} a_r x_n^{s-2r} (x_1^2 + \ldots + x_{n-1}^2)^r$$

$$= a_0 x^s + a_1 x^{s-2}(x_1^2 + \ldots + x_{n-1}^2) + \ldots \quad .$$

By straightforward calculations, one finds that $\Delta f =$
$\sum_0^{s/2-1} \left[(s-2r)(s-2r-1)a_r + 2(r+1)(n+2r-1)a_{r+1}\right] x_n^{s-2r-2}(x_1^2 + \ldots + x_{n-1}^2)^r$
and putting $\Delta f = 0$, the above equation yields the recursion formula

$$a_{r+1} = \frac{-(s-2r)(s-2r-1)}{2(r+1)(n+2r-1)} a_r$$

Putting $a_0 = f(0, 0, \ldots, 1) = 1$, we call the resulting function $v_s(P) = v_s(x_1, \ldots, x_n)$.

A homogeneous polynomial of degree s on S^n which is a solution of (1) is called a harmonic function on S^n and denoted by $f(\rho, P)$ where ρ is the radius of S^n. In particular the above function v_s is a harmonic function and

$$v_s(\rho, P) = \sum_{r=0}^{s/2} a_r x_n^{s-2r} (x_1^2 + \ldots + x_{n-1}^2)^r.$$

To each harmonic function $f(\rho, P)$ there corresponds a function on the unit sphere w_n (i.e. $\rho = 1$): $f(1, P) = \frac{f(\rho, P)}{\rho^s}$ where s is the degree of $f(\rho, P)$ and $\rho^s = (\sum_1^n x_i^2)^{s/2}$. Putting $\xi_i = x_i/\rho$, then $\sum_{i=1}^n \xi_i^2 = 1$, and $\frac{f(\rho, P)}{\rho^s} = f(\frac{x_1}{\rho}, \ldots, \frac{x_n}{\rho}) = f(\xi_1, \ldots, \xi_n)$ if $f(\rho, P) = f(x_1, \ldots, x_n)$, because of the homogeneity of f. Thus $f(1, P)$ may be regarded as a function on w_n. The functions $f(1, P)$ are called spherical harmonics of degree s. It is clear that the spherical harmonics of degree s form an invariant finite dimensional module $\subset \mathfrak{M}'$, where $\mathfrak{M}'$ is the module of continuous functions on w_n - i.e. on S^n for $\rho = 1$, since the set of corresponding harmonic functions $f(\rho, P)$ form an invariant submodule of the module of all homogeneous polynomials of degree s on S^n, which is certainly finite. This module of spherical harmonics we call (Y_s), and we denote its dimension by N_s. In particular $v_s(1, P) = V_s(P)$ is a spherical harmonic, and

$$V_s(P) = \sum_0^{s/2} a_r \xi_n^{s-2r} (\xi_1^2 + \dots + \xi_{n-1}^2)^r$$
$$= \sum_0^{s/2} a_r \xi_n^{s-2r} (1 - \xi_n^2)^r$$
$$= \xi_n^s + a_1 \xi_n^{s-2}(1 - \xi_n^2) + \dots$$

$V_s(P)$ is thus reduced to a polynomial in ξ_n which we denote by $\pi_s(\xi_n)$. It could be that $1-a_1+a_2+\dots = 0$ in which case the coefficient of ξ_n^s in $\pi_s(\xi_n)$ would be zero, so that $\pi_s(\xi_n)$ would have degree $< s$. That this cannot be so follows from the next

<u>THEOREM</u>: Two modules of spherical harmonics of different degrees $s < s'$ have no functions in common.

<u>Proof</u>: Let $f(P)$ be a function of both (Y_s) and $(Y_{s'})$ where $s' = s + h$, $h > 0$. This implies that $\rho^s f$ is a harmonic function of degree s on S^n and that $\rho^{s'} f$ is also a harmonic function of degree s'. That is, if $g(\rho, P) = \rho^s f$, then $g(\rho, P)$ is a solution of (1) as is $\rho^h g(\rho, P) = (\sum_1^n x_i^2)^{h/2} g(\rho, P)$. Now if $\Delta g = 0$ then $\Delta \rho^h g \neq 0$ for $h > 0$ since

$$\Delta(\rho^h g) = \rho^h \Delta g + 2h\rho^{h-2} \sum x_i \frac{\partial g}{\partial x_i} + h(n+h-2)\rho^{h-2} g .$$

But $\Delta g = 0$ and $\sum x_i \frac{\partial g}{\partial x_i} = sg$ because of the homogeneity of g, so that $\Delta(\rho^h g) = h(n + h + 2s - 2)g$. Then since $h > 0$, $\Delta \rho^h g \neq 0$ so that $\rho^h g$ is not a harmonic function. Hence it follows that if f belongs to both (Y_s) and $(Y_{s'})$, $f \equiv 0$.

<u>Corollary</u>: If $f(P) \in (Y_s)$ and $g(P) \in (Y_{s'})$, then $M_\nu\{f(P\nu)\ \overline{g(Q\nu)}\} = w(P, Q) \equiv 0$.
Let $f(\rho, P) = \rho^s f(P)$ and $g(\rho, Q) = \rho^{s'} g(Q)$. Then $f(\rho, P)$ and $g(\rho, Q)$ are harmonic functions of degree s and s' respectively, and further, $\overline{g(\rho, Q)}$ belongs to the same module as does $g(\rho, Q)$. But since (Y_s) and $(Y_{s'})$ have no spherical function in common, the modules containing $f(\rho, P)$ and $g(\rho, Q)$ have no element in common, so $0 \equiv M_\nu\{f(\rho, P\nu)\ \overline{g(\rho, Q\nu)}\} = \rho^{s+s'} M_\nu\{f(P\nu)\ \overline{g(P\nu)}\}$.
In particular, if $P = Q$,

$$M_\nu\{f(P\nu)\ \overline{g(P\nu)}\} = \frac{1}{w_n} \int_{w_n} f(P)\ \overline{g(P)}\ d\ w_n(P) = 0$$

where the integral is taken over the unit sphere. In other words, two spherical harmonics of different degrees are orthogonal over the unit sphere.

It follows that the $V_s(P)$, $s = 0, 1, 2, \ldots$, belonging to (Y_s) are orthogonal. That is, $\pi_s(\xi_n)$ is orthogonal to all $\pi_{s'}(\xi_n)$, $s' < s$. Hence the degree of $\pi_s(\xi_n)$ is greater than the degree of all $\pi_{s'}(\xi_n)$, $s' < s$. Since the degree of $\pi_1(\xi)$ is one, it follows by induction on s that the degree of $\pi_s(\xi_n)$, which we knew to be $\leq s$ is actually s.

We may write $\xi_n = P.X_n$ where $X_n = (0, 0, \ldots, 1)$|
Then $V_s(P) = \pi_s(P.X_n)$, and we let this function be $V_s(P, X_n)$ henceforth. $V_s(P, X_n)$ is the only spherical function of degree s involving only ξ_n, apart from

a constant factor, as may be seen by considering the corresponding harmonic function. $\pi_s(P.X_n) = \pi_s(P\nu.X_n\nu)$ so that $V_s(P, X_n) = V_s(P\nu, X_n\nu)$. $V_s(P\nu, X_n)$ also belongs to (Y_s) (because of the invariance of Δ under Ω_n), and we may put $V_s(P\nu, X_n) = V_s(P, X_n\nu^{-1})$ $= V_s(P, Q)$ where $Q\nu = X_n$, $\nu \in \Omega_n$. Then $V_s(P, Q) =$ $\pi_s(P.Q)$, so that $V_s(P, Q) = V_s(Q, P)$, $V_s(P\nu, Q\nu) =$ $V_s(P, Q)$, $V_s(P, P) = \pi_s(1) = 1$. From these properties we may conclude that $V_s(P, Q)$ is the same function considered either as a function of P, or of Q, and is the only spherical harmonic of degree s which is a function only of P.Q .

We are now in a position to show that (Y_s), considered as a module of a.p. functions on the unit sphere, is irreducible. Let $y_s(P) \in (Y_s)$. Then

$$w(P, Q) = M_\nu\{y_s(P\nu)\ \overline{y_s(Q\nu)}\}$$

is such that $w(P, Q)$ is a function only of P.Q, and and belongs to (Y_s) (cf. pg. 23). Therefore $w(P, Q)$ $= c\ V_s(P, Q)$ where c is a constant. Since $w(P, Q) =$ $M_\nu\{|y_s(P\nu)|^2\}$ is > 0, and $V_s(P, P) = 1$, we see that $c > 0$. But $y_s(P)$ was any function in (Y_s), so that $V_s(P, Q) = (1/c)\ w(P, Q)$ belongs to any invariant submodule of (Y_s), since if $y_s(P)$ belongs to any such submodule, $w(P, Q)$ (consdiered as a function of P or Q) also belongs to that submodule. Therefore $V_s(P, Q)$

generates all the irreducible invariant modules belonging to (Y_s), so that they are identical, and coincide with (Y_s). In other words (Y_s) is irreducible.

The irreducible modules (Y_s) for $s = 0, 1, 2, \ldots$ are the only irreducible modules of continuous functions on w_n. We may prove this by showing that the system of functions Y_s, $Y_s \in (Y_s)$, $s = 0, 1, 2, \ldots$ form a complete system of functions with respect to w_n. That is, if $f(P)$ is a continuous function on w_n, then if $M_\nu\{f(P\nu)\,\overline{Y_s(P\nu)}\} = 0$ for all $Y_s \in (Y_s)$, and all $s = 0, 1, 2, \ldots$ then $f \equiv 0$. Here $M_\nu\{f(P\nu)\,\overline{Y_s(P\nu)}\} = \frac{1}{w_n}\int_{w_n} f(P)\,\overline{Y_s(P)}\,d\,w_n(P)$ where $d\,w_n(P)$ is the element of volume of w_n at the point P.

Now the element of volume $d\,w_n(P)$ may be reduced in the following way. Write $P = (\xi_n', \eta_1, \ldots, \eta_{n-1})$ where $\xi_n', \eta_1, \ldots, \eta_{n-1}$ are new coordinates, and $\xi_n = \xi_n' = \eta_n$, $\eta_i = \xi_i/\sqrt{1-\xi_i^2}$ for $i = 1, 2, \ldots, n-1$. Then $\sum_{i=1}^{n-1} \eta_i^2 = 1$ so that $S = (\eta_1, \ldots, \eta_{n-1})$ is a point in the subsphere of w_n, which we call w_{n-1}, defined by $\xi_n = 0$. Then each point P of w_n may be written as (ξ_n, S). If one assumes a system of curvilinear coordinates in w_{n-1}, and one applies known formulae to the element of volume $d\,w_n(P)$, one can obtain the formula $d\,w_n(P) = (1 - \xi_n^2)^{(n-3)/2}\,d\,\xi_n\;d\,w_{n-1}(S)$ where $d\,w_{n-1}(S)$ is the element of volume of w_{n-1} at the point S.*

* (2): pp 233

THEOREM: The system of functions (Y_s), $s = 0, 1, 2, \ldots$ is complete.

Proof: Suppose there exists a continuous function $f(P)$ on w_n, orthogonal to all $Y_s \in (Y_s)$, for all s. That is,

$$\int_{w_n} f(P)\, \overline{Y_s(P)}\, d\, w_n(P) = 0$$

We may write $f(P) = f(\xi_n, S)$, $S \in w_{n-1}$, and we may take, in particular $\pi_s(\xi_n) \in (Y_s)$. Then the above equation becomes

$$\int_{-1}^{1} d\,\xi_n\, (1-\xi_n^2)^{(n-3)/2}\, \pi_s(\xi_n) \int_{w_{n-1}} f(\xi_n, S)\, d\, w_n(S) = 0$$

Putting $\varphi(\xi_n) = \int_{w_{n-1}} f(\xi_n, S)\, d\, w_n(S)$, we have that

$$\int_{-1}^{1} \varphi(\xi_n)\, (1 - \xi_n^2)^{(n-3)/2}\, \pi_s(\xi_n) = 0 \quad \text{for all}$$

$s = 0, 1, 2, \ldots$. But $\pi_s(\xi_n)$ was shown to be a polynomial in ξ_n of degree s. A system of these polynomials for $s = 0, 1, 2, \ldots$ is complete on the interval $[-1, 1]$, because of the completeness on $[a, b]$ of $1, x, x^2, \ldots$. Since $\varphi(\xi_n)\, (1 - \xi_n^2)^{(n-3)/2}$ is orthogonal over $[-1, 1]$ to all $\pi_s(\xi_n)$, it must be identically zero. Therefore $\varphi(\xi_n) = 0$ except perhaps at $\xi_n = \pm 1$. But since $\varphi(\xi_n)$ is continuous, $\varphi(\xi_n) \equiv 0$; i.e. $\int_{w_{n-1}} f(\xi_n, S)\, d\, w_n(S) \equiv 0$. If we put $\xi_n = 1$, that is, $P = (0, 0, \ldots, 1) = X_n$, then $S = (0, 0, \ldots, 0)$, so that $\int_{w_{n-1}} f(X_n)\, d\, w_n(S) = w_{n-1}\, f(X_n) = 0$. That is, $f(X_n) = 0$.

Now $Y_s(P\nu^{-1})$, $\nu \in \Omega_n$, also belongs to (Y_s), so that $f(P)$ is orthogonal to all $Y_s(P\nu^{-1})$, or in other words $f(P\nu)$ is orthogonal to all $Y_s(P)$. From the previous paragraph, we may conclude $f(X_n\nu) = 0$ for any $\nu \in \Omega_n$. But $X_n\nu = P$ for any $P \in w_n$, for a suitable ν, so that $f(P) \equiv 0$. Hence the system (Y_s), $s = 0, 1, 2, \ldots$ is complete.

In each (Y_s) we may choose N_s functions $Y_s^1, Y_s^2, \ldots, Y_s^{N_s}(P)$ which constitute a basis of (Y_s). Then the completeness theorem implies that every $f(P)$ may be approximated in the mean by a series of the form $\sum_{s=0}^{N} \sum_{i=1}^{N_s} a_s^i Y_s^i(P)$. And this implies that every irreducible module of the closed invariant module of all continuous functions on w_n is to be found among the (Y_s), for some s (see pg. 16). But this module is the sum of its irreducible modules, so that $|f - \sum_{s=0}^{N} y_s(P)| < \varepsilon$ for any ε and suitable $y_s(P) \in (Y_s)$, for any f continuous on w_n. Therefore, for any continuous $f(P)$, $P \in w_n$, we have $|f(P) - \sum_{s=0}^{N} \sum_{i=1}^{N_s} a_s^i Y_s^i(P)| < \varepsilon$ for any ε. That is, $f(P)$ may be uniformly approximated by spherical functions.

Similar results may be obtained for a function $f(\rho, P)$ on S^n, by considering the complete system of functions $(\rho^s Y_s)$ for $s = 0, 1, 2, \ldots$.

III

1. Modules of Spherical Harmonics as Representation Modules

Let (Y_s) be an irreducible modules of spherical harmonics of degree s, of dimension N_s, with an orthonormal basis $Y_s^1(P), \ldots, Y_s^{N_s}(P)$, $P \in w_n$. To each $Y_s(P)$ we may associate a function $y_s(\nu)$ on Ω_n by the relation $Y_s(P) = Y_s(P_o\nu) = y_s(\nu)$, for some fixed P_o. Then, as stated before, $y_s(\nu)$ is an a.p. function on Ω_n and the set of all such functions, R_s, for a given (Y_s) and a certain P_o, forms an irreducible right-invariant finite module of a.p. functions, with the orthonormal basis $y_s^i(x)$ where $y_s^i(x) = Y_s^i(P_o x)$. The $y_s^i(x)$ are orthonormal since

$$\begin{aligned}(y_s^i(x), y_s^j(x)) &= M_x\{y_s^i(x)\,\overline{y_s^j(x)}\} \\ &= M_x\{Y_s^i(P_o x)\,\overline{Y_s^j(P_o x)}\} \\ &= \frac{1}{w_n}\int_{w_n} Y_s^i(P)\,\overline{Y_s^j(P)}\,d\,w_n(P) \\ &= \delta_{ij}\end{aligned}$$

R_s is not left invariant since if $f(x) \in R_s$, then $f(x) = F(P_o x)$ where $F(P_o x) \in (Y_s)$, but $f(\nu x) = F(P_o \nu x) = F(Q_o x)$ to which there need not correspond a function $f'(x) = f(\nu x)$ in R_s.

Since R_s is invariant and irreducible, the transformation $y_s^i(x\nu) = \sum_{j=1}^{N_s} D_{ij}^{(s)}(\nu)\, y_s^j(x)$ yields an irreducible representation of Ω_n through the matrices $D^{(s)}(\nu) = (D^{(s)}_{ij}(\nu))$. Since the basis $y_s{}^i$ is orthonormal, the representation $D^{(s)}(x)$ is unitary and may be taken as the representation of its equivalence class in the complete system $\{D^{(\nu)}(x)\}$ of irreducible inequivalent representations. Thus the functions $D^{(s)}_{ij}$ are orthonormal in the sense that

$$(D^{(s)}_{ij}, D^{(s)}_{kl}) = \begin{cases} 1/s & \text{if } i = k, \ j = l \\ 0 & \text{otherwise} \end{cases}$$

where $(D^{(s)}_{ij}, D^{(s)}_{kl}) = \underset{x \in \Omega_n}{M_x} \{D^{(s)}_{ij}(x) \overline{D^{(s)}_{kl}(x)}\}$.

Since the $y_s{}^i(x)$ are orthonormal, the $D^{(s)}_{ij}$ may be given by the following relations

$$D^{(s)}_{ij}(\nu) = (y_s{}^i(x\nu), y_s{}^j(x)).$$

From this it is clear that the representations are independent of the point P_o used to define R_s. For suppose we have another module $R_s{}'$ of a.p. functions on Ω_n derived from (Y_s) by the relation $y_s{}^{i\prime}(x) = Y_s{}^i(P_o{}'x)$ where $P_o{}'x = P \in S^n$. Then if $y_s{}^{i\prime}(x\nu) = \sum_{j=1}^{N_s} D^{(s)\prime}_{ij}(\nu)\, y_s{}^{j\prime}(x)$ then

$$\begin{aligned} D^{(s)\prime}_{ij}(\nu) &= (y_s{}^{i\prime}(x\nu), y_s{}^{j\prime}(x)) \\ &= (y_s{}^{i\prime}(\mu x\nu), y_s{}^{j\prime}(\mu x)) \\ &= (y_s{}^i(x\nu), y_s{}^j(x)) \\ &= D^{(s)}_{ij}(\nu) \end{aligned}$$

where $P_o{}'\mu = P_o$ so that $y_s{}^{i\prime}(\mu x) = Y_s{}^i(P_o{}'\mu x) = Y_s{}^i(P_o x)$.

The functions $D_{ij}^{(s)}(\nu)$ may be expressed in terms of the $y_s{}^i(\nu)$, in fact as a linear combination of $y_s{}^i(x_k\nu)$ for suitable x_k. We may choose N_s elements of Ω_n, x_k, $k = 1, \ldots, N_s$ such that the determinant $\det(y_s{}^j(x_k))$ $j = 1, \ldots, N_s$, $k = 1, \ldots, N_s$ is not zero. This follows from the linear independence of the $y_s{}^i(x)$. Then the N_s linear equations in the N_s unknowns, $D_{ij}^{(s)}(\nu)$, $j = 1, \ldots, N_s$,

$$y_s{}^i(x_k\nu) = \sum_{j=1}^{N_s} D_{ij}^{(s)}(\nu)\, y_s{}^i(x_k)$$

may be solved for the $D_{ij}^{(s)}$ in terms of the $y_s{}^i(x_k)$; i.e. $D_{ij}^{(s)}(\nu) = \sum_{k=1}^{N_s} a_k\, y_s{}^i(x_k\nu)$ for suitable constants a_k.

It has been shown that the dimension of (Y_s) is given by $N_s = \frac{(s+1)(s+2)\ldots(s+n-3)}{(n-2)!}(2s+n-2)$.*

For $s = 1$, this becomes $N_1 = n$. Hence the module (Y_1) has dimension n, and therefore the representation of Ω_n for $s = 1$ consists of n-matrices. In fact, if the basis of (Y_1) is taken as $Y_1(P) = x_1, \ldots, Y_n(P) = x_n$, then it is easy to see that the representation given in this case is the group of proper orthogonal matrices with which one started.

From the completeness of the modules of spherical harmonics, (Y_s), with respect to the continuous

* (2): pp. 222

functions on w_n follows the completeness of the corresponding modules of a.p. functions $\{y_s(\nu)\}$ with respect to the continuous a.p. functions on Ω_n. Thus the representations above may be regarded as the complete system of bounded irreducible "continuous" representations of Ω_n.

For $n = 3$, $N_s = 2s+1$. Thus the irreducible, bounded, continuous representations of Ω_3 are of dimension $1, 3, \ldots, 2s+1, \ldots$.

It is well known that the special unitary group given by matrices of the form

$$\begin{pmatrix} x_1+ix_4 , & -x_2+ix_3 \\ x_2+ix_3 , & x_1-ix_4 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \delta & \gamma \end{pmatrix}$$

where $\alpha, \beta, \gamma, \delta$ are the so-called "Cayley-Klein Parameters" is a two valued representation of the rotation group Ω_3. It is not included in the above set of representations since it is of dimension 2. - i.e. consists of 2-matrices. In order to find other representations not given by the above set we shall adopt a new approach for the rotation group Ω_3, which is developed in the next section.

2. Rotation Group in R_3

Closely allied with the two valued representation of Ω_3 by the special unitary group is the representation of Ω_3 by the quaternion group of quaternions of norm 1. *

(3): pp. 210 - 211; 242.

We can represent this group by matrices of the form

$$q = \begin{pmatrix} q_1 & -q_2 & -q_3 & -q_4 \\ q_2 & q_1 & -q_4 & q_3 \\ q_3 & q_4 & q_1 & -q_2 \\ q_4 & -q_3 & q_2 & q_1 \end{pmatrix}$$

where $\sum_{i=1}^{4} q_i^2 = 1$. These matrices are proper orthogonal, and from a group - i.e. the product of two of them is another matrix of the same form. Hence they are a proper subgroup of Ω_4. They form a two-valued representation of Ω_3 in the sense that to each element μ of Ω_3 there exist the two quaternions $\pm m$, such that if $\pm m$, $\pm n \in Q$ and correspond to μ, ν of Ω_3 then to $\mu\nu$ corresponds $\pm(mn)$.

Clearly the quaternion

$$x = \begin{pmatrix} x_1 & -x_2 & -x_3 & -x_4 \\ x_2 & x_1 & -x_4 & x_3 \\ x_3 & x_4 & x_1 & -x_2 \\ x_4 & -x_3 & x_2 & x_1 \end{pmatrix}$$

is uniquely determined by its first row, and therefore to x we may associate the point $X = (x_1, -x_2, -x_3, -x_4)$ on w_4 since $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. This correspondence is 1 - 1. To xu corresponds the point given by its first row, which is the point

$$Xu = (x_1 \quad -x_2 \quad -x_3 \quad -x_4) \begin{pmatrix} u_1 & -u_2 & -u_3 & -u_4 \\ u_2 & u_1 & -u_4 & u_3 \\ u_3 & u_4 & u_1 & -u_2 \\ u_4 & -u_3 & u_2 & u_1 \end{pmatrix}$$

We may also multiply x on the left by u, and to ux corresponds the point Ux where $U = (u_1, -u_2, -u_3, -u_4)$. This, however, may be written as right multiplication of X by a matrix $\tilde{u}$ - i.e. $X\tilde{u}$ where

$$\tilde{u} = \begin{pmatrix} u_1 & -u_2 & -u_3 & -u_4 \\ u_2 & u_1 & u_4 & -u_3 \\ u_3 & -u_4 & u_1 & u_2 \\ u_4 & u_3 & -u_2 & u_1 \end{pmatrix} .$$

$\tilde{u}$, by inspection is orthogonal, and its determinant is $u_1^2 + u_2^2 + u_3^2 + u_4^2$ which is 1, and hence $\tilde{u} \in \Omega_4$.

Each function $f(X)$, $X = (x_1, -x_2, -x_3, -x_4)$, on w_4 then, may be considered as a function $f(x)$ on the quaternion determined by X. The converse of this statement is true also. In particular the a.p. functions on the sphere w_4 may be taken as the a.p. functions on Q. The modules (Y_s) of spherical functions on w_n are invariant under Ω_4, and therefore invariant under Q; i.e. if $f(X)$ (or $f(x)$) $\in (Y_s)$, then $f(Xq) \in (Y_s)$, $q \in Q$, where X is the point determined by x. In fact if $f(x) \in (Y_s)$ then $f(x\nu) \in (Y_s)$ where $\nu \in \Omega_4$, and so $f(ux) \in (Y_s)$ if $u \in Q$, since $f(ux) = f(x\tilde{u})$ where $\tilde{u}$ is the matrix of Ω_4 discussed in the last paragraph. The module (Y_s) while irreducible under right multiplication by Ω_4, is not necessarily irreducible under right multiplication by Q.

(Y_s) is in fact reducible under Q. For let $f_1(x), \ldots, f_n(x)$, $n \le N_s$, be a basis of an irreducible right-invariant submodule of (Y_s). Then we can write

$$f_i(xu) = \sum_{j=1}^{n} D_{ij}(u)\, f_j(x).$$

Since the $f_j(x)$ are linearly independent, we may choose x_k, $k = 1, 2, \ldots, n$ so that the determinant $\det (f_j(x_k)) \neq 0$. Then, the $D_{ij}(u)$ may be found in terms of $f_i(x_k u)$, $k = 1, 2, \ldots, n$ - i.e. $D_{ij}(u) = \sum_{k=1}^{n} a_k\, f_i(x_k u)$ for suitable a_k's. Since $f(x_k u)$ as a function of u belongs to (Y_s), $D_{ij}(u)$ is also a function of (Y_s). The functions $D_{ij}(u)$ may be taken to be orthogonal, so that (Y_s) contains the n^2 orthogonal (and hence linearly independent) functions $D_{ij}(x)$, $i, j = 1, \ldots, n$, so that $N_s \ge n^2$. Therefore (Y_s) is necessarily reducible.

It should be noted that the orthogonality referred to is orthogonality over w_4 as well as that over Q.

The mean value over Q can be taken to be the integral over w_4, the points of w_4 representing the elements of Q.

$$\underset{x \in Q}{M_x}\{f(x)\} = \frac{1}{w_4} \int_{w_4} f(X)\, d\, w_4(X)$$

where X is the point $(x_1, -x_2, -x_3, -x_4)$ corresponding to the quaternion x. That this is actually the mean value can be seen by considering the four determining properties of $M_x\{\ \}$ as was done in the similar case of the almost periodic functions on the sphere. However, because of the 1 - 1 relation $x \longleftrightarrow X$, this mean value is the mean value of the function on the group as well as the function on the sphere. Whereas in the case of a.p. functions $f(P)$ on the sphere and the corresponding function $F(x) = f(P_o x)$ for some particular $P_o = Px^{-1}$, the mean value of $F(x)$ over G happened to have the same value as the mean value of the a.p. function $f(P)$ on S^n, for some particular P_o.

We will now show that there is only one irreducible inequivalent representation of Q associated with each module (Y_s). That is, all irreducible submodules of (Y_s) are of dimension n, each yielding equivalent representations of Q, so that the $D_{ij}(u)$ mentioned in the preceding paragraph span (Y_s), and therefore $N_s = n^2$.

First of all let us note that if $f_1, \ldots, f_{N_s}$ is a basis of (Y_s), then we can find a function $f(xA)$, where A is a variable element of Ω_4, such that $f(xA_i) = f_i(x)$ for suitable A_i. For of the 16 coefficients of the matrices of Ω_4, it is well known that 6 of them are independent - that is, each $A_i \in \Omega_4$ may be completely determined by six of its coefficients, which we shall call its parameters, labelled by $a_1{}^i$, $\ldots$, $a_6{}^i$. If $A \in Q$ then it is completely determined by three of these which we may take to be $a_1{}^i, a_2{}^i, a_3{}^i$.

If $a_1, \ldots, a_6$ are the parameters of a variable matrix A, we put $(a^i - a) = (a_1{}^i - a_1)(a_2{}^i - a_2) \ldots (a_6{}^i - a_6)$. We can find N_s matrices A_j, $j = 1, \ldots, N_s$, such that $a_i{}^j \neq a_i{}^k$ for $j \neq k$. Then the function

$$f(xA) = \sum_{j=1}^{N_s} \left(\prod_{\substack{i=1 \\ i \neq j}}^{N_s} \frac{(a^i - a)}{(a^i - a^j)} \right) f_j(x)$$

is such that $f(xA_j) = f_j(x)$.

Now let $\mathfrak{M}_1, \ldots, \mathfrak{M}_m$ be the modules of (Y_s) which are irreducible and invariant under Q. Then there exists a function $f(x) \epsilon (Y_s)$ such that $f(xA_i) \epsilon \mathfrak{M}_i$ for suitable $A_i \epsilon \Omega_4$. If n_i is the dimension of $\mathfrak{M}_i$, there exist n_i elements of Q, $u_1, \ldots, u_{n_i}$, such that $f(xA_iu_j)$, $j = 1, \ldots, n_i$, form a basis of $\mathfrak{M}_i$. From the considerations of the last paragraph, one can see that the $f(xA_iu_j)$ may be taken to be orthogonal: i.e. $(f(xA_iu_j), f(xA_iu_k)) = \delta_{jk}$.

However, $(f(xA_iu_j), f(xA_iu_k)) = (f(xu_j), f(xu_k))$

$$= (f(xA_lu_j), f(xA_lu_k)),$$

for any $l = 1, \ldots, m$. One can easily see from this that $f(xA_lu_j)$, $j = 1, \ldots, n_l$, form an orthonormal basis of $\mathfrak{M}_l$ for any l so that $n_i = n$ for all i. If $D^{(l)}_{ij}(u)$ are the coefficients of the representations corresponding to $\mathfrak{M}_l$, then

$$f(xuA_lu_i) = \sum_{j=1}^{n} D^{(l)}_{ij}(u)\, f(xA_lu_j)$$

so $D^{(l)}_{ij}(u) = (f(xuA_lu_i), f(xA_lu_j))$.

But $(f(xuA_l u_i), f(xA_l u_j)) = (f(xuA_k u_i), f(xA_k u_i))$ for all l, k, so that $D_{ij}^{(l)}(u) = D_{ij}^{(k)}(u)$, and consequently all representations $D^{(l)}(u)$, $l = 1, \ldots, m$ are equivalent.

But (Y_s) is a closed invariant module of a.p. functions on Q, so that it is the sum of its irreducible modules. All these modules are of dimension n and yield equivalent representations of Q, so that the n^2 functions $D_{ij}(u)$ of the unitary representative of these representations form a basis of (Y_s). Hence $N_s = n^2$ for some n. But $N_s = (s+1)^2$ when the sphere w_4 is considered, where s is the degree of the homogeneous polynomials comprising (Y_s). Hence the irreducible modules belonging to (Y_s) are of dimension $s+1$, so that there are irreducible representations of Q of all dimensions $1, 2, \ldots, s+1, \ldots$.

The irreducible modules of (Y_s) may be found in the following manner. First of all we find an orthogonal normal basis $f_1, \ldots, f_{(s+1)^2}$ of (Y_s). ((Y_s) is generated by the functions $V_s(P, Q_k)$ for suitable Q_k.) Then we find the $(s+1)^4$ functions $\varphi_{ij}(u)$ associated with the transformations $f_i(x) \longrightarrow f_i(xu)$

$$f_i(xu) = \sum_{j=1}^{(s+1)^2} \varphi_{ij}(u)\, f_j(x)$$

so that
$$\varphi_{ij}(u) = (f_i(xu), f_j(x))$$
$$= \frac{1}{w_4} \int_{w_4} f_i(Xu)\, \overline{f_j(X)}\, d\, w_4(X).$$

Now we shall take s+1 functions of (Y_s) of the form $g_i(x) = \sum_{j=1}^{(s+1)^2} a_{ij} f_j(x)$, where the a_{ij} are as yet undetermined, and apply the condition that $g_i(x) \in \mathfrak{M}_i$, where $\mathfrak{M}_i$, $i = 1, \ldots, s+1$, are the irreducible modules of (Y_s). This is equivalent to saying that $(g_i(xu), g_j(x)) = 0$ for all $u \in Q$ and $i \neq j$. Applying these conditions to determine the a_{ij}, we find

$$\begin{aligned}
(g_i(xu), g_j(x)) &= \left(\sum_{k=1}^{(s+1)^2} a_{ik} f_k(xu), \sum_{l=1}^{(s+1)^2} a_{jl} f_l(x)\right) \\
&= \sum_k \sum_l a_{ik} \bar{a}_{jl} (f_k(xu), f_l(x)) \\
&= \sum_{k,l} a_{ik} \bar{a}_{jl} \sum_{m=1}^{(s+1)^2} \varphi_{km}(u) (f_m(x), f_l(x)) \\
&= \sum_{k,l} a_{ik} \bar{a}_{jl} \sum_m \varphi_{km}(u) \delta_{ml} \\
&= \sum_{k,l} a_{ik} \bar{a}_{jl} \varphi_{kl}(u).
\end{aligned}$$

Thus the problem of finding the functions $g_i(x)$ is reduced to the problem of finding $(s+1)^3$ coefficients a_{kl}, $k = 1, \ldots, s+1$, $l = 1, \ldots, (s+1)^2$, such that $\sum_{k,l=1}^{(s+1)^2} a_{ik} a_{jl} \varphi_{kl}(u) = 0$ for all $u \in Q$ if $i \neq j$.

Now it will be remembered that (Y_s) consists of homogeneous polynomials in x_1, x_2, x_3, x_4 of degree s. Thus the coefficients of the representations $D_{ij}(x)$ may be explicitly represented as homogeneous polynomials in $x_1, \ldots, x_4$. The representations of Q are thus at most one valued, and the corresponding representations of Ω_3 are at most two-valued. In fact, since for even s, the two points (or quaternions) $\pm x$ yield the same values of $D_{ij}(x)$, the corresponding odd dimensional representations of Ω_3 consisting of (s+1) - matrices, are at most one valued.

In particular, for s = 1, we obtain the special unitary group as a two-valued representation of Ω_3:

$$\begin{pmatrix} u_1 + iu_3 , & -u_2 + iu_4 \\ u_2 + iu_4 , & u_1 - iu_3 \end{pmatrix} .$$

This gives us nothing new, since the quaternion group was in fact a representation of this group. The representations by (2s)-matrices for s = 2, 3, ... are probably two-valued also.

For s = 2, we obtain the usual faithful representation of Ω_3 by 3-matrices given by

$$\begin{pmatrix} u_1{}^2+u_2{}^2-u_3{}^2-u_4{}^2 , & -2(u_1u_4 - u_2u_3) , & 2(u_1u_3 + u_2u_4) \\ -2(u_1u_4 + u_1u_2) , & u_1{}^2-u_2{}^2+u_3{}^2-u_4{}^2 , & -2(u_1u_2 - u_3u_4) \\ -2(u_1u_3 - u_2u_4) , & 2(u_1u_2 + u_3u_4) , & u_1{}^2+u_2{}^2-u_3{}^2+u_4{}^2 \end{pmatrix}$$

the coefficients of which may be expressed in terms of the Cayley - Klein Parameters.

References

(1) W. Maak: Fastperiodische Funktionen
(Springer - Verlag, 1950)

(2) Guido Ascoli: Sopra I Systemi Lineari Isotropi e le loro Proprietà Integrali
(<u>Commentationes</u>; Pontificia Academia Scientiarum: Anno VII, Vol. VII, N. 10)

(3) H. Schwerdtfeger: Introduction to Linear Algebra and the Theory of Matrices
(Noordhoff, 1950)