THE WEIERSTRASS E-FUNCTION

by

John F. Shaw

Submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Master of Science.

McGill University

April, 1966.

ACKNOWLEDGEMENTS

I am very grateful to Professor Fox for suggesting the topic of this thesis, and for his advice and assistance during its preparation.

Also, I express my sincere thanks to Dr. Rosenthall, whose patient encouragement and goodwill helped to alleviate many of my personal burdens.

For his many acts of generosity over the years, I am deeply indebted to Dr. F.R. Terroux, without whose help this work would not have been possible.

TABLE OF CONTENTS

Table of Contents.

1. Introduction - historical note.

The basic problem of the Calculus of Variations is that of finding a point in function space at which a given integral attains a maximum or a minimum value. The type of problem that can be solved by means of this calculus is referred to as a variational problem, and the extreme value of the integral is called an extremum.

The simplest case involves a line integral and a function space whose elements are of the form $y = f(x)$. The most elementary problem of this kind is the determination of the shortest distance between two points in the plane. Because of its extreme simplicity, it is probably the oldest variational problem known to man, going back as far as the ability of the human mind to grasp the relevant ideas. This, however, is merely a conjecture; the earliest records of the subject belong to classical times, when variational problems were well-known.

Vergil (Aeneid: i, 368) relates that Dido, seeking a plot of land for the building of Carthage, needed to know the shape of a fixed length of cord that would enclose the largest area. The Greeks knew that the answer was a circle, and in the third century B.C. Archimedes had considered the analogous situation in three dimensions. These problems are now referred to as <u>isoperimetric problems</u>.

It was also recognised by the ancients that nature often behaves according to some maximum or minimum principle. Heron of Alexandria showed in the first century A.D. that a light ray reflected in a plane mirror follows the shortest possible path - an important discovery in the theory of optics that Fermat was to generalise sixteen hundred years later.

These and other results of a similar kind were obtained by early natural philosophers largely through intuition and experiment. It was not until the seventeenth century that general methods of solution for such problems were developed and an adequate mathematical theory was evolved to substantiate them.

The earliest hint of the subject after the classical period was given in 1630, when the problem of the curve of quickest descent, later reformulated by Johann Bernoulli, appeared somewhat vaguely in the writings of Galileo. Again in 1638, Galileo considered the problem of finding the shape assumed by a chain suspended freely from two points (the catenary). In 1686 [1], Isaac Newton determined the characteristic property of a curve generating a surface of revolution which cleaves through a fluid with minimum resistance.

For a long time, however, it was not recognised that these examples had anything in common, and only the last attracted more than a passing interest, mainly because the result proved to be theoretically unsound. In fact, by substituting zig-zag lines for the meridian curve, Legendre transformed Newton's surface into one whose resistance was even smaller.

Actually, Legendre's method contains the germ of the idea which led to the construction of the E-function, although more subtle techniques than the ones of his day were needed to implement it. The 'paradox' of Newton's problem is one of the many applications within the scope of the E-function test.

It was not until 1696 that the calculus of variations slowly began to emerge as a new branch of mathematics. Writing in the Acta Eruditorum that year, Johann Bernoulli

---

[1] The year of the presentation of the Principia to the Royal Society. The date 1687, given by some authors, is the year of publication.

challenged the mathematicians of Europe with a new version of Galileo's problem: to find the path between two points in a vertical plane along which a particle would slide under gravity in the shortest time. The brachistochrone problem (brachistos = shortest, chronos = time), as it came to be known, was soon widely recognised as a new type of problem whose solution was not accessible by any method generally known at the time. Bernoulli himself found the answer by an ingenious and quite non-rigorous adaptation of Fermat's Principle, and other leading mathematicians of the day quickly obtained solutions by their own methods. The foundations of a general theory were laid by James Bernoulli in 1697.

The history of the calculus of variations in its modern form begins with the work of L. Euler (1707-83) and J.L. Lagrange (1736-1813). Euler saw that the methods of James Bernoulli were widely applicable, and he deduced in 1744 the first necessary condition for the existence of an extremum in the form of a differential equation, now referred to as Euler's equation. A solution of this equation is known as a characteristic curve or extremal.

In 1760-62, Lagrange devised the δ-notation for variations which have given the theory its name, and greatly simplified and extended the results of Euler. In order to compare the value of a given integral J(C) along a path C between fixed end-points with its value along a neighboring path between the same points, Lagrange replaced C by a similar path whose slope and ordinates were arbitrarily close to those of C. Lagrange called this process a <u>variation</u> of the path C, later known as a <u>weak</u> variation because of the restriction concerning the slope. The new path of integration was denoted by $C+\delta C$.

The symbols δJ, $\delta^2 J, \ldots,$ were used by Lagrange to represent integrals of homogeneous polynomials in small quantities of the first, second and higher orders occurring in successive terms of the Taylor series expansion of the difference $J(C+\delta C) - J(C)$. These expressions are called the <u>first</u>, <u>second</u> and higher <u>variations</u> of the integral J, and are of fundamental importance in the theory of the first three necessary conditions for an extremum. In fact, their role is similar to that of the corresponding derivatives in the elementary theory of the differential calculus. In particular, the vanishing of the first variation is a necessary condition which leads directly to Euler's equation.

Following the work of Lagrange, interest developed in the problem of finding a sufficient condition which would guarantee the existence of an extremum for the given integral. Since the vanishing of the first variation is necessary, attention turned for the first time to the second variation. In 1786 Adrien Marie Legendre (1752-1833) found a criterion for distinguishing between maxima and minima, and published a second necessary condition which an extremal arc must satisfy. But the condition thus discovered was not also a sufficient one.

Furthermore, in his reduction of the second variation Legendre used a transformation which could not be justified in all cases. The difficulty was analysed in 1838 by C.G.Jacobi (1804-1851) who in doing so discovered the existence of the point, known as the <u>critical point</u>, beyond which the minimising[1] properties of an arc will fail. Location of this critical point is the essence of a third necessary condition, known as the <u>Jacobi criterion</u>. Jacobi's discovery marks the end of what

---

[1]To avoid tedious repetition of the phrase 'maximising or minimising' and similar expressions, the convention whereby both cases are referred to as a "minimum" will be adopted from now on whenever it is convenient. There is no loss of generality in doing so, because every function which minimises the integral J, maximises -J. The word 'extremising' has been avoided for obvious reasons.

may be called the classical period in the calculus of variations. For several decades there was no further progress of any importance, and interest in the subject remained at a low ebb. The next phase of development begins in 1879, and is dominated by the name of Karl Weierstrass. A résumé of his work is given in the next section.

2. Weierstrass and the E-function.

A distinguishing feature of the mathematics of the late nineteenth century is to be found in the high standard of rigor which governs every aspect of analysis from the statement of definitions to the construction of proofs. This feature seems to have been particularly useful in the re-appraisal of earlier results. One of the leading exponents of the new trend was Weierstrass (1815-1897), who, after a lifetime devoted to other topics, appropriately turned his attention to the calculus of variations around 1869. At the time, this subject, as other branches of analysis, was much in need of greater precision in the formulation of problems, and in the methods and reasoning applied to their solution. Furthermore, the problem of the sufficiency condition, still unsolved, represented a serious gap in the theory.

Over the following decade, Weierstrass undertook a complete re-examination of the first and second variations and the criteria of his predecessors. The crowning achievement of those years was the establishment in terms of the E-function of a workable sufficiency criterion for the very first time.

Weierstrass started at the roots of classical theory by analysing the method of weak variations, and he found it unsatisfactory. He noticed that the zig-zag lines which occurred in some of the counter-examples lay outside the scope of this type of variation on account of their fluctuating slope, and concluded that this technique must also be inadequate for the treatment of smooth curves of similar shape. Hence, this method could never lead to anything more than an indefinite search for the properties of minimising arcs.

To overcome these limitations, Weierstrass introduced the strong variation, in which the slope of the alternative path was permitted to assume any finite value. Conceptually, the step from weak to strong variations was a simple one, but its application required formidable ingenuity and manipulative skill of the highest order. In the hands of anyone but Weierstrass, the idea probably would have been useless.

Starting from this point, Weierstrass probed more deeply for the characteristics which guarantee the minimising properties of an extremal arc. In order to manipulate the strong variation analytically, he introduced the system of integral curves corresponding to Euler's equation, and embedded the extremal in it. The conditions under which this procedure is possible are related to the Jacobi criterion, and the underlying theory, now considerably refined, is known as the *theory of fields*. The system of integral curves is called a *field of extremals*.

Using the strong variation, Weierstrass began, as Lagrange had done before him, by considering a simple expression of the form

$$J(\tilde{C}) - J(\bar{C}),$$

where $J(\bar{C})$ represents the integral of the given function taken along the extremal path $\bar{C}$, and $J(\tilde{C})$ represents the same integral evaluated along the arbitrary alternative path $\tilde{C}$ defined by the strong variation. Thus, if $J(\bar{C})$ were a minimum, then the difference

$$J(\tilde{C}) - J(\bar{C})$$

would be non-negative for all paths $\tilde{C}$, and conversely.

The difficulty clearly arose in the manipulation of the term $J(\tilde{C})$ which, being arbitrary, was not related to any known quantity, whereas a simple test was needed that could be applied directly to the function whose integral was to be minimised. Aided by the theory of fields, and using only the techniques of classical analysis, Weierstrass devised what is now called the E-function test. A detailed derivation of Weierstrass's result will be given later. For the purposes of this introduction, the following outline of his method will be sufficient.

Weierstrass assumed that a unique integral curve passed through every point of $\tilde{C}$, and considered paths of integration partly along an integral curve and partly along $\tilde{C}$. The abscissa t of the point at which these paths joined played the role of a parameter upon which the value of the entire integral depended. Thus, the integral between the two fixed end-points along the integral curve and $\tilde{C}$ was a function of t, denoted by S(t). From the definition of S(t) it follows that $S(a)=J(\tilde{C})$, and $S(b)=J(\bar{C})$, where a and b are the abscissas of the left and right end-points respectively. Consequently the difference $\Delta J = J(\tilde{C}) - J(\bar{C})$ may be written $S(a) - S(b)$. Apart from the latter expression, the value of S(t) is limited, and it is merely another symbol. On the other hand, the derivative S'(t) turns out to be most useful. Weierstrass calculated this derivative from first principles, using the definition

$$\lim_{h \to 0} \frac{S(t+h) - S(t)}{h}$$

and by means of an ingenious lemma succeeded in eliminating from the result all the terms which arose from the unknown quantity $J(\tilde{C})$, with the exception of the slope q of $\tilde{C}$ at the point x = t. The other terms could be obtained directly from the integral to be minimised.

From what has been said, it is clear that the integral of -S'(t) from a to b is equal to ΔJ. Thus, -S'(t) may be regarded as a measure of the amount by which $J(\tilde{C})$ exceeds, or is exceeded by, $J(\bar{C})$. In view of this property, and because of the number of terms involved, Weierstrass gave the integrand -S'(t) a special name - the excess function or E-function as it is now known, and denoted it by the symbol $E(x,y,\frac{dy}{dx},q)$. It is continuous, and in the two-dimensional space that Weierstrass considered[1], it is a function of four variables.

---

[1] The treatment given here and later on is a modification of the discussion given by Weierstrass in his lectures (1879) for the case of parameter representation.

In more complicated spaces, the E-function is correspondingly more elaborate. Obviously a minimum is ensured for $J(\bar{C})$ if ΔJ is non-negative for all paths $\tilde{C}$, which is true, provided that the integrand E has the same characteristic. Essentially, this is the sufficiency condition that Weierstrass established.

Very little of this work was revealed until 1879 when Weierstrass gave a series of lectures on the calculus of variations at the University of Berlin in the summer of that year.[1] With this episode, the formative years of the subject came to a brilliant close.

The sufficiency criterion, based on the formula $\Delta J = \int_a^b E.dx$ will be referred to as <u>Weierstrass's theorem</u>. Alternative derivations of this important formula were published around 1900. The most direct method, due to Kneser and Osgood[2], consists of differentiation of the expression for $J(\tilde{C}) - J(\bar{C})$, and is especially suited to the derivation of Weierstrass's fourth necessary condition. The most elegant proof of Weierstrass's theorem is due to Hilbert, although the simplicity

---

[1] None of Weierstrass's discoveries was ever published directly by the author. The main sources of material for this subject are Kneser's Lehrbuch, and privately circulated lecture notes.

[2] Kneser: Lehrbuch der Variationsrechnung (1900), Section 20.
Osgood: Transactions of the American Mathematical Society, vol.II, (p.116.

of the description belies the ingenuity of the method.

Hilbert discovered an integral, which may be denoted by H, with the following properties.

(a) H has the value $J(\bar{C})$ when evaluated along an extremal $\bar{C}$ of the field.

(b) The value of the integral taken along any path lying inside the field of extremals depends only on the end-points. In other words, in the interior of a field of extremals, the Hilbert integral is exact.

The main steps of Hilbert's proof are as follows. According to (a), $J(\bar{C}) = H(\bar{C})$. Secondly, $H(\bar{C}) = H(\tilde{C})$, by property (b). Therefore $\Delta J = J(\tilde{C}) - J(\bar{C})$ may be written $\Delta J = J(\tilde{C}) - H(\tilde{C})$. Since both integrals now have the same path of integration $\tilde{C}$, the integral sign may be dropped, and the sign of the integrand becomes the determinant characteristic. This integrand is the E-function. Hilbert's theorem is ideally suited to the proof of the Weierstrass theorem in (n+1)-dimensional Euclidean space, whereas Weierstrass's original derivation is not readily adaptable to this situation because of its rather cumbersome notation. A full account of all three methods will be given in due course, each in its appropriate context. The E-function, of course, figures predominantly in all of them.

By means of the E-function test, the difficulties with Newton's Solid and similar problems can be very simply resolved. In modern language, Newton's solution furnishes a weak but not a strong minimum with respect to the given functional. Apart from being the key to the Sufficiency Criterion, however, the E-function also plays a number of other important roles, as the following examples show. With slight modifications, for instance, the E-function test can be used as the basis of a fourth necessary condition, discovered by Weierstrass in the early stages of his research. This fourth condition, in turn, can be used to prove Legendre's Necessary Condition for a weak minimum. The vanishing of the E-function at a corner[1] of a so-called broken extremal is equivalent to one of the classical corner conditions. Also, the E-function is closely related to the Hilbert integral, whose integrand is part of the E-function for the same functional.

Since the E-function is so deeply embedded in the fabric

---

[1] A corner is a point on the extremal where the derivative has a discontinuity of the first kind.

of the theory [1], it is impossible to discuss it coherently without careful and detailed preliminaries depending on the language and ideas of older results. For this reason, and to bring the importance of the E-function into the sharpest possible focus, it will be necessary to start at the beginning of the subject by giving a derivation of Euler's equation.

Obviously, the fundamental problem can be presented in many different forms. The integrand may be a function of any number of independent and dependent variables and their derivatives up to any order. Also, the path of integration may be subject to different types of constraint. It is beyond the scope of this paper to give an exhaustive treatment covering all possible ramifications. Instead, a thorough treatment of one fairly broad case will suffice. Unless it is otherwise stated, the discussion will be confined to the fixed end-point problem in a Euclidean space of n+1 dimensions.

The analogies in (n+1)-dimensional space to the classical theorems in two dimensions are, in most cases,

---

[1] It is possible to identify the E-function with a directional derivative of the difference of two integrals (see W.S. Kimball, Calculus of Variations). In this writer's opinion, however, this interpretation is rather far-fetched, and its construction is much too artificial to be included here.

interesting enough to justify this choice, especially since they are not merely notational extensions of the simpler theorems, but have, in general, more complicated forms. At the same time, all the classical results are included as a special case, and where such generalisations are trivial, or notational difficulties arise, the number of dimensions may be reduced, and the theory discussed in its original form. Such transitions will be made from time to time for convenience.

3. Definitions, Terminology and Notation.

The following definitions will be found useful.

Definition 1. A function f of n variables is said to be of class C^m, $m \geq 0$, in those variables on a domain S if and only if f is continuous on S, together with all its partial derivatives up to and including those of the mth order.

Definition 2. A function y(x) of a single variable x is said to be of class C^m, $m \geq 0$, on an interval $[a,b]$, if y(x) is continuous on the interval, together with all its derivatives up to and including those of the mth order. The class C^0 will be denoted by C with no superscript.

Definition 3. A function y(x) of a single variable x is said to be of class D^m, $m > 0$, on an interval $[a,b]$ if y(x) is continuous on the interval, and if the interval can be divided into a finite set of sub-intervals on the closure of each of which y(x) is of class C^m.

<u>Definition 4.</u> A function $y(x)$ is said to be of class D^0 on the interval $[a,b]$ if this interval can be divided into a finite set of sub-intervals on the interior of each of which $y(x)$ is of class C, and at the ends of which $y(x)$ possesses finite left- and right-hand limits.

<u>Definition 5.</u> Let E_k denote the real Euclidean space of k dimensions. A continuous mapping g of an interval $[a,b]$ into E_k is called a <u>curve</u> in E_k, and is said to join the points $(a, g(a))$ and $(b, g(b))$ in E_{k+1}. If g is one-one on $[a,b]$, the curve is called an <u>arc</u>. A curve is said to be of class $C^m(D^m)$, etc., if the function $g(x)$ is of class $C^m(D^m)$.

<u>Terminology.</u>

Functions of class D^0 will be called <u>piecewise-continuous</u>.
Functions of class D^1 are called <u>piecewise-smooth</u>.
Functions of class C are called <u>smooth</u>.
A simply-connected set of points of E_n which includes all or part of its boundary will be referred to as a <u>region</u> in E_n.

<u>Notation.</u>

Partial derivatives are denoted by the usual symbols. For instance, the partial derivative of $f(x, y_1, \ldots, y_n; z_1, \ldots, z_n)$ with respect to z_i will be denoted by f_{z_i} or by $\frac{\partial f}{\partial z_i}$. When the z_i are clearly identified with the derivatives $y_i'(x)$ $(i = 1, \ldots, n)$, the notation $f_{y_i'}$ or $\frac{\partial f}{\partial y_i'}$ will be used instead.

4. The fundamental problem.

Let

$$(x, y_1,\dots,y_n) = (x,y)$$

be the Cartesian coordinates of a point (x,y) in the Euclidean space E_{n+1}, and let G be an open set in E_{n+1}. Let M be the function space consisting of all continuous vector functions which map the interval $[a,b]$ on the real axis into E_n. These functions are representable in the form

(1) $$y(x) = \{y_1(x),\dots,y_n(x)\} \qquad x \in [a,b]$$

with piecewise continuous derivatives

(2) $$y'(x) = \{y_1'(x),\dots,y_n'(x)\} \qquad x \in [a,b]$$

whose graphs lie in G, and which coincide at $x=a$ and $x = b$. In other words, M consists of functions of class D^1 on $[a,b]$ whose curves join two fixed distinct points of G. The curves determined by these functions will be referred to as <u>admissible curves</u>.

Now let

(3) $$f(x,y_1,\dots,y_n;z_1,\dots,z_n) = f(x,y,z)$$

be a real-valued function defined at every point $(x,y_1,\dots,y_n)$

of G, and for arbitrary finite values $z_1,\ldots,z_n$, where y and z are abbreviations representing n-dimensional vectors. In addition, it is assumed that f and its partial derivatives f_{y_i}, f_{z_i} $(i = 1,\ldots,n)$ are of class C^2 for $(x,y) \in G$ and finite z. On these assumptions the integral

$$J(y) = \int_a^b f(x,y,y')dx \qquad y \in M, \tag{4}$$

has a well-defined value.[1] Thus the integral (4) defines a real-valued functional J(y) on M. The right member of (4) will be referred to as the functional in ordinary form, and in this case the basic problem is that of finding the elements y belonging to M for which this functional attains a maximum or a minimum value.

5. Absolute and relative extrema.

As in the ordinary differential calculus, it is convenient here to distinguish between absolute and relative extrema, although the first kind can generally be reduced to the second. To make this distinction precise, it is necessary to introduce the concept of neighborhoods.

---

[1] provided that, in the case of a curve with corners, the integral is defined as a sum of integrals taken between successive corners.

Definition 6. A zero-order e-neighborhood of a continuous vector function $\bar{y}(x)$ is the collection of all continuous vector functions $y(x)$ for which the inequality

$$\sum_{i=1}^{n} \left| y_i(x) - \bar{y}_i(x) \right| < e \tag{5}$$

is satisfied for $a \leq x \leq b$. A neighborhood of zero order will be referred to as a strong neighborhood. The expression $\| y(x) - \bar{y}(x) \|_0 < e$ denotes that $y(x)$ belongs to a strong neighborhood of $y(x)$.

Definition 7. A first-order e-neighborhood of a piecewise-smooth vector function $\bar{y}(x)$ is the collection of all piecewise-smooth vector functions $y(x)$ that satisfy (5) and also satisfy the inequality

$$\sum_{i=1}^{n} \left| y_i'(x) - \bar{y}_i'(x) \right| < e \tag{6}$$

for every point x belonging to $[a,b]$ at which $y_i'(x)$ and $\bar{y}_i'(x)$ are defined.[1] A neighborhood of the first order is called a weak neighborhood, and the notation

$$\| y(x) - \bar{y}(x) \|_1 < e \tag{7}$$

will be used to denote that $y(x)$ belongs to a weak neighborhood of $\bar{y}(x)$.

---

[1]It follows from the definition of piecewise-smooth that at points t where one or both of the derivatives in (6) fail to exist, the left- and right-hand derivatives must satisfy the inequality (6) at x = t.

Remarks.

1. The neighborhoods defined above obviously have the familiar properties of a metric. The triangular inequality, however, is the only one that will be needed.

2. It follows from the definition that every strong e-neighborhood of a given function $\bar{y}(x)$ contains the weak neighborhood for the same value of e.

3. The variations (q.v., p.5) of a given path of integration C are called strong or weak according as the neighborhood of C in which they are situated is strong or weak.

4. The definition of weak neighborhood is inapplicable in cases such as the cycloid (brachistochrone problem) where the derivatives at the cusps are infinite.[1]

Definition 8. The functional J(y) is said to be continuous with respect to weak neighborhoods at the function $\bar{y}(x)$ in its domain if and only if for every $e > 0$ there is a $d > 0$ such that $|J(y) - J(\bar{y})| < e$ whenever $\|y - \bar{y}\|_1 < d$. Continuity with respect to strong neighborhoods is defined similarly.

Definition 9. The functional J(y) is said to have an absolute minimum with respect to the collection M for $y = \bar{y}$ if and only if

$$(8) \qquad J(y) \geq J(\bar{y})$$

for all y belonging to M. The absolute maximum of J(y) is

[1] The resulting difficulties, however, are not serious (cf.p.117).

defined similarly.

Definition 10. The functional $J(y)$ is said to have a relative minimum (or maximum) for $y = \bar{y}$ if there exists a neighborhood $N \subset M$ such that $J(y)$ has an absolute minimum (or maximum) for $y = \bar{y}$ with respect to N. In this case the extremum is designated as 'strong' or 'weak' according as the corresponding neighborhood is strong or weak.

Definition 11. The relative minimum (or maximum) will be called proper if there exists a neighborhood N such that in (8) the sign $>$ (or $<$) holds for all $y \neq \bar{y}$, and improper if, however the neighborhood N may be chosen, there exists some function $y \neq \bar{y}$ for which the equality sign has to be taken.

It is clear from Remark 2 on page 21 that every strong extremum is simultaneously a weak extremum, for if $J(\bar{y})$ is an extremum with respect to all y such that $\|y - \bar{y}\|_0 < e$, then $J(y)$ is an extremum with respect to all y such that $\|y - \bar{y}\|_1 < e$.

On the other hand, a curve that furnishes a weak extremum may not yield a strong extremum, and in general, a curve that furnishes an extremum in a given class of admissible curves, may not do so if the class is enlarged. An important exception, however, is the enlargement from C^1 to D^1. A well-known lemma on rounding-off corners shows that no new solutions are introduced.*
Therefore it may be assumed that weak extrema are smooth curves

*Carathéodory, Variationsrechnung, p.191.
Hadamard, Calcul des Variations, p.51.

6. The First Necessary Condition - Euler's equation.

To clarify the remarks that follow, the meaning of the term 'variation' must now be made more precise.

Definition 12. Let $J(y)$ be a functional defined on M (q.v., p. 18), and let $y(x)$ be a vector function of class D^1 belonging to M. Let $s(x) = \{s_1(x),\ldots,s_n(x)\}$ be a vector function of class D^1 that satisfies the condition $s_i(a) = s_i(b) = 0$ $(i = 1,\ldots,n)$. The functional $J(y+es)$ may then be considered a function of the parameter e. The nth variation of the functional $J(y)$ at y is defined to be

$$\left.\frac{d^nJ(y+es)}{de^k}\right|_{e=0} . e^n \qquad n = 1,2,\ldots .$$

provided the derivative exists, and it is denoted by $\delta^n J$. It is easy to see that the nth variation is identically equal to the term of order n (if it exists) multiplied by n! in the Taylor's expansion of $J(y+es)$. Definition 12 includes the zero-order variation, viz., the functional $J(y)$ itself.

The classic technique in the calculus of variations, patterned directly on the techniques of finite-dimensional calculus, depends upon the concept of a function yielding an extremum as a point in function space, and the characterisation of that point by various necessary conditions. The most important of these conditions is expressed by Euler's equation for the derivation of which only the first variation is needed.

As the following analysis will show, the first variation is one of the more striking examples of the analogy between the variations δJ, $\delta^2 J$,..and the differentials df, d^2f,...of a function $f(x)$. Unfortunately, these analogies are of limited application. Bliss[1] sums up the situation very neatly: "In the years between 1800 and 1850 these analogies were much studied, and the literature of that period indicates a strong desire to unify the methods of the differential calculus and the calculus of variations. Analysis of the type elaborately developed at that time seems, however, to have been relatively unproductive of important results."

In deriving necessary conditions for an extremum, the situation may be specialised in any convenient manner. In particular, weak variations may be used to derive Euler's equations. Since every strong extremum is simultaneously a weak extremum, there is no loss of generality in doing so, although there are two distinct advantages. The weak variation is easier to manipulate than the strong one, and secondly,

---

[1] Lectures on the Calculus of Variations, pp.6-7.

the functionals encountered in the calculus of variations are generally continuous with respect to weak neighborhoods, but not with respect to strong ones.[1] Therefore, finding a weak extremum is generally easier than finding a strong one.

The following theorem is a formal statement of the first necessary condition.

<u>Theorem 1.</u> Let $J(y)$ be a functional defined on the function space M, and suppose that J has a relative minimum or maximum at $y = y(\bar{x})$. Then

$$f_{z_i}(x,\bar{y}(x),\bar{y}'(x)) = \int_a^x f_{y_i}(x,\bar{y}(x),\bar{y}'(x))dx + c_i$$

$(i = 1,\ldots,n)$ for x in $[a,b]$ and for suitable constants c_i.

<u>Proof.</u> Let $y_i = y_i(x,e)$ denote a family of admissible curves of the form[2]

(9) $$y_i(x,e) = \bar{y}_i(x) + es_i(x) \qquad (i = 1,\ldots,n)$$

where e is a parameter near $e = 0$, $s(x) = (s_1(x),\ldots,s_n(x))$ is

---

[1] Arc length is a typical example of such a functional. Given any rectifiable curve, it is possible, for $e > 0$, to find a second curve in a strong e-neighborhood of the first whose length differs from that of the first curve by an arbitrarily large factor.

[2] The functions defined by (9) are clearly weak variations of $\bar{y}(x)$. Theorem 1 can also be proved by using strong variations (see Bolza, Lectures on the Calculus of Variations, Dover (1961), p.19.).

a piecewise-smooth function that satisfies the conditions

$$(10) \qquad s_i(a) = s_i(b) = 0 \qquad (i = 1,\ldots,n).$$

If $|e|$ is sufficiently small, the functions $y(x,e) = \bar{y}(x)+es(x)$ lie in a weak neighborhood of the function $y(x)$. Equation (10) provides that the end-point conditions are also satisfied. Hence the functional $J(y)$, considered over the collection of functions defined by (9), has an extremum for $\bar{y}$ when $e = 0$. But $J(y + es) = \phi(e)$. Consequently

$$(11) \qquad \phi'(0) = 0.$$

Differentiation of $\phi(e)$ with respect to the parameter e gives

$$\phi'(e) = \int_a^b \sum_{i=1}^n \left\{ f_{y_i}(x,y,y')s_i(x) + f_{y_i'}(x,y,y')s_i(x) \right\} dx$$

Let $f_{y_i}(x,\bar{y},\bar{y}') = \bar{f}_{y_i}$ and $f_{y_i'}(x,\bar{y},\bar{y}') = \bar{f}_{y_i'}$. Then it follows that

$$\phi'(0) = \int_a^b \sum_{i=1}^n \left\{ \bar{f}_{y_i} s_i(x) + \bar{f}_{y_i'} s_i'(x) \right\} dx$$

Thus, condition (11) becomes

$$(12) \qquad \int_a^b \sum_{i=1}^n \left\{ \bar{f}_{y_i} s_i(x) + \bar{f}_{y_i'} s_i'(x) \right\} dx = 0.$$

The left-hand side of (12) is proportional to the first variation of J along $\bar{y}$. This variation is completely determined when $\bar{y}$ and the function $s(x)$ are given. Equation (12) can also be obtained by a method based upon Taylor's formula, although this is not as elegant as the method of differentiation with respect to e.[1] The terms $s_i \bar{f}_{y_i}$ in (12) can be integrated by parts,[2] giving the result

$$(13) \qquad J'(0) = \int_a^b \sum_{i=1}^{n} s_i' \left[\bar{f}_{z_i} - \int_a^x \bar{f}_{y_i} dx \right] dx = 0,$$

since the integrated terms vanish on account of (10). The result will follow from (13) and the following lemma.

Lemma 1. If $g(x)$ is of class D^0 on $[a,b]$ and

$$(14) \qquad \int_a^b s'(x) . g(x) dx = 0$$

for all functions $s(x)$ of class D^1 which vanish at a and b, then $g(x)$ is constant on $[a,b]$.

Proof. Let c be a constant such that

$$\int_a^b (g(x) - c) dx = 0.$$

The function $s(x) = \int_a^x (g(x) - c) dx$ is then a function $s(x)$ of

---

[1] Neither method leads to sufficient conditions. This is one case where the analogies with differential calculus are not helpful.

[2] The integration by parts of the second term in (12), which is the procedure usually followed in elementary treatments, presupposes the existence of $\bar{y}''(x)$. This was first noticed by Du Bois-Reymond (1879), to whom the above lemma is ascribed.

the type admitted in the lemma. For this function s(x), (14) takes the form

$$0 = \int_a^b (g(x) - c)g(x)dx = \int_a^b (g(x) - c)^2 dx$$

from which it follows that g(x) = c. Q.E.D.

Now in (13), let all the functions $s_i(x)$ be taken identically zero except one, say $s_k(x)$. According to Lemma 1, the coefficient of s'_k in the integrand in (13) must be constant. This statement is true for k = 1,...,n. The theorem is thereby proved. Q.E.D.

The equations

$$f_{z_i} - \int_a^x f_{y_i} = c_i \qquad (i = 1,\ldots,n) \tag{15}$$

are called the <u>Euler equations in integrated form.</u>

<u>Theorem 2.</u> Let $\overline{y}(x)$ be a piecewise-smooth vector function which lies in the region G and provides a weak relative extremum for the functional J(y). Then $\overline{y}(x)$ satisfies the equation

$$f - \sum_{i=1}^{n} y'_i f_{y'_i} - \int_a^x f_x dx = C \tag{16}$$

where C is some constant.

Proof. Let a transformation from the coordinates $(x, y_1, \ldots, y_n)$ be made to other rectilinear coordinates $(u, v_1, \ldots v_n)$ in which the new axes make an arbitrarily small angle with the original axes. The equation

$$(17) \qquad y = \bar{y}(x) \qquad (a \leq x \leq b)$$

becomes

$$(18) \qquad v = \bar{v}(u) \qquad (c \leq u \leq d)$$

where c and d are the new abscissas of the points a and b respectively. Every curve $v = v(u)$ in a sufficiently small neighborhood of the curve (18) is the transform of a curve $y = y(x)$ that lies in a weak neighborhood of the curve (17). When the functional $J(y)$ has been written in terms of the new variables, it is clear that $\bar{v}(u)$ must satisfy the Euler equations in integral form for the transformed functional. Expression of these equations in terms of the original coordinates then gives a new set of equations that the function $\bar{y}(x)$ must satisfy, namely, equations (16).

The equations of transformation are

$$x = u + tv_1, \; y_i = v_i \qquad (i = 1, \ldots, n)$$

in which t is a constant arbitrarily small in absolute value.

The functional

$$\int_a^b f(x, y, y')dx$$

is transformed to

$$\int_c^d F(u,v,\dot{v})du = \int_c^d f\left(u + tv_1,\ v,\ \frac{\dot{v}}{1 + t\dot{v}_1}\right)(1 + t\dot{v}_1)du$$

in which differentiation with respect to u is indicated by a dot. The first of equations (15)

$$F_{\dot{v}_1} - \int_c^u F_{v_1} du = C$$

may be written

$$tf + f_{z_1} - \frac{t}{1 + t\dot{v}_1}\sum_{i=1}^{n} \dot{v}_i f_{z_i} - \int_c^u (tf_x + f_{y_1})(1 + t\dot{v}_1)du = C.$$

In terms of the old coordinates, the last equation is

$$(19) \qquad tf + f_{z_1} - t\sum_{i=1}^{n} y_i' f_{z_i} - \int_a^x (tf_x + f_{y_1})dx = \text{const.},$$

since $\dot{v}_i = \frac{dy_i}{dx}\frac{dx}{du}$ $(i = 1,\ldots,n)$. The function $\bar{y}(x)$ must satisfy condition (19). Finally, if (19) is combined with the first equation of (15), the result is (16). Q.E.D.

The following corollaries may now be deduced from Theorems 1 and 2 respectively.

<u>Corollary 1.1</u> Each segment of class C^1 of a minimising arc $y = \bar{y}(x)$ must satisfy the Euler equations in differentiated form

$$(20) \qquad \frac{d}{dx}(f_{y_i'}) - f_{y_i} = 0 \qquad (i = 1,\ldots,n).$$

<u>Proof.</u> If $\bar{y}'(x)$ is continuous in an interval, then the function $\bar{f}_{y_i}$ is continuous in the same interval. Hence the function

$$\int_a^x \bar{f}_{y_i} dx,$$

where $\bar{f}_{y_i}$ means $f_{y_i}(x,\bar{y}(x),\bar{y}'(x))$, has a continuous derivative of the first order in the interval. Consequently, equations (15) can be differentiated, after replacing y by $\bar{y}(x)$. Q.E.D.

From now on, the expression 'Euler's equations' will be understood to mean equations (20). It should be emphasised that the existence of the derivatives which occur in (20) was not assumed in advance, but follows as part of the result. A curve of class C^1 on some open interval containing $[a,b]$ which satisfies the equations (20) will be called an <u>extremal</u>.

<u>Corollary 2.1</u> Each segment of class C^1 of a minimising arc $y = \bar{y}(x)$ must satisfy the equation

$$(21) \qquad \frac{d}{dx}\Big(f - \sum_{i=1}^{n} y_i' f_{z_i}\Big) - f_x = 0,$$

where $z_i = \bar{y}_i'$.

<u>Proof.</u> If $\bar{y}'(x)$ is continuous in an interval, then the function

$$\int_a^x \bar{f}_x dx$$

has a continuous derivative with respect to x in that interval. Therefore, equation (16) can be differentiated, after replacing y by $\bar{y}(x)$. The result is (21) Q.E.D.

<u>Corollary 2.2</u>

(a) If the integrand function $f(x,y,y')$ does not contain the components y_i of the vector y, but does contain y_i', the Euler equations have a first integral

$$f_{y_i'} = c_i \qquad (i = 1,\dots,n).$$

(b) If the functional has the form $\int_a^b f(y,y')dx$, then a vector function $\bar{y}(x)$ that affords it a weak relative extremum satisfies the equation

$$(22) \qquad f - \sum_{i=1}^{n} y_i' f_{y_i'} = C$$

where C is a constant. Corollaries 2.2 have special significance for the case n = 1 in which the extremal is defined by one equation.

<u>Proof.</u> A proof is needed only for the second assertion. By hypothesis, it is clear that (22) is a first integral of (21). Q.E.D.

<u>Corollary 3.</u> At each corner x = c on an extremum curve $\bar{y}$, the following conditions hold.

$$\bar{f}_{y_i'}\Big|_{c-0} = \bar{f}_{y_i'}\Big|_{c+0} \qquad (i = 1,\dots,n), \tag{23}$$

$$\bar{f} - \sum_{i=1}^{n} \bar{y}_i'\bar{f}_{y_i'}\Big|_{c-0} = \bar{f} - \sum_{i=1}^{n} \bar{y}_i'\bar{f}_{y_i'}\Big|_{c+0}. \tag{24}$$

Equations (23) and (24) are known as the Weierstrass-Erdmann corner conditions. A curve of class D^1 which satisfies Euler's equations is called a <u>discontinuous solution</u> if it actually possesses a corner. According to the lemma mentioned on page 22, in any discussion of weak relative extrema, curves of class D^1 may be replaced by curves of class C^1 whenever it is convenient to do so.

<u>Proof.</u> Equations (23) follow from the continuity

of the function

$$\int_a^x \bar{f}_{y_i}\, dx$$

at each corner point. Therefore, by (15), $\bar{f}_{y_i'}$ is also continuous. Similarly (24) follows from (16) and the continuity of

$$\int_a^x \bar{f}_x dx \qquad \text{Q.E.D.}$$

Equation (24) can also be obtained by means of the E-function, as will be shown later. Apart from this, there will be no further occasion to discuss discontinuous solutions. On the other hand, piecewise-smooth curves are very useful in the construction of counter-examples, as the discussion of the next section will show.

7. Insufficiency of Euler's equations for a strong extremum. Weierstrass's Lemma and the E-function.

From the remarks in the introduction, it follows that there is no reason to expect that Euler's equations (20) are sufficient for a strong minimum, and in fact, it is easy to construct counter-examples which show that they are not. The first example of this kind was Newton's Solid of Least Resistance. Other problems of this type, but not so complicated,[1] were discovered later. The two examples which follow are among the simpler ones, and will suffice to illustrate the point.

<u>Example 1.</u> Consider the case n = 1 and the functional

$$\int_0^6 y'^2(1 - y')^2 dx$$

with boundary conditions y(0) = 0, y(6) = 1. The points with Cartesian coordinates (0,0) and (6,1) are denoted by A and B respectively.

Since f is a function of y' alone, (20) reduces in this case to

$$f_{y'} = c$$

by Corollary 2.2. Therefore the extremals are solutions of the equation y' = constant, and are straight lines. If the boundary conditions are taken into consideration, the solution is easily seen to be

$$\bar{C}: \bar{y} = \frac{x}{6}$$

[1] thus indicating a theoretical defect quite apart from any difficulties arising from the unrealistic physical assumptions of the problem.

Nevertheless, if the slope m of the line AB is such that $0 < m < 1$, a minimum does not take place. For in this case, in any e-neighborhood of $\bar{C}$, a broken line joining A and B may be drawn, consisting of straight line segments whose slopes are alternately 0 and 1. For such a broken line, J(y) is obviously zero, whereas $J(\bar{C}) = 25/216$ (cf. Fig. 1). Therefore $\bar{C}$ does not furnish J(y) with a strong minimum.

That $\bar{C}$ does not furnish J(y) with a strong maximum can be shown by considering the broken line whose equation is

$$\tilde{y} = \begin{cases} \dfrac{1+e}{6-e} \cdot x, & 0 \leq x \leq 6-e \\ 7-x, & 6-e < x \leq 6, \end{cases}$$

where $e > 0$ is arbitrarily small. Since the function $f(x) = x^2(1-x)^2$ increases in $(0,\frac{1}{2})$, it is clear from the relation $\tilde{y}' > \bar{y}'$ that

$$\int_0^{6-e} \tilde{y}'^2 (1-\tilde{y}')^2 dx > \int_0^{6-e} \bar{y}'^2 (1-\bar{y}')^2 dx$$

Also

$$\int_{6-e}^{6} 4dx \qquad \int_{6-e}^{6} 25/1296$$

Therefore $J(\tilde{y}) > J(\bar{y})$ and the statement is proved (cf. Fig. 2).

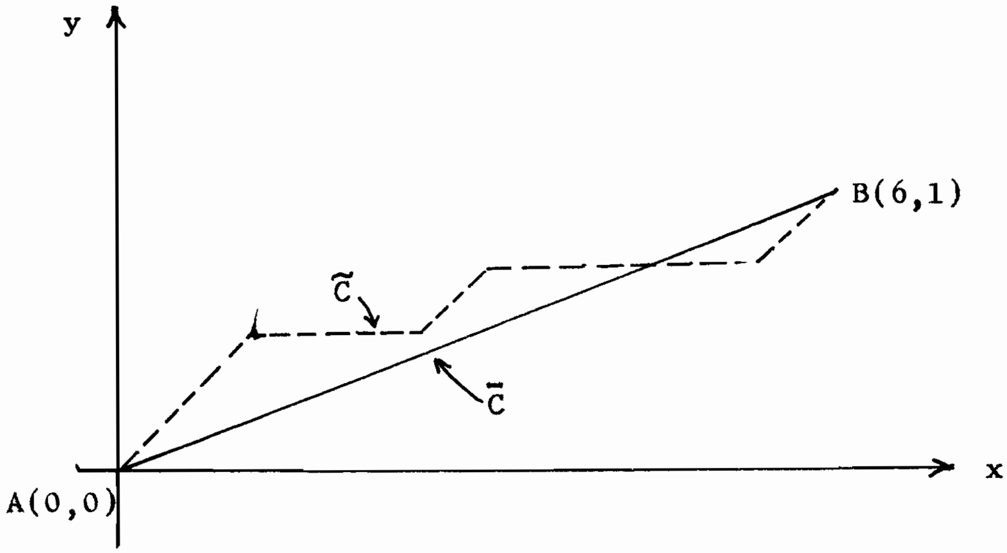


Fig. 1. Example illustrating insufficiency of Euler's equation to guarantee a strong minimum. $J(\tilde{C}) = 0$; $J(\bar{C}) = 25/216$. The scale has been exaggerated for the sake of clarity.

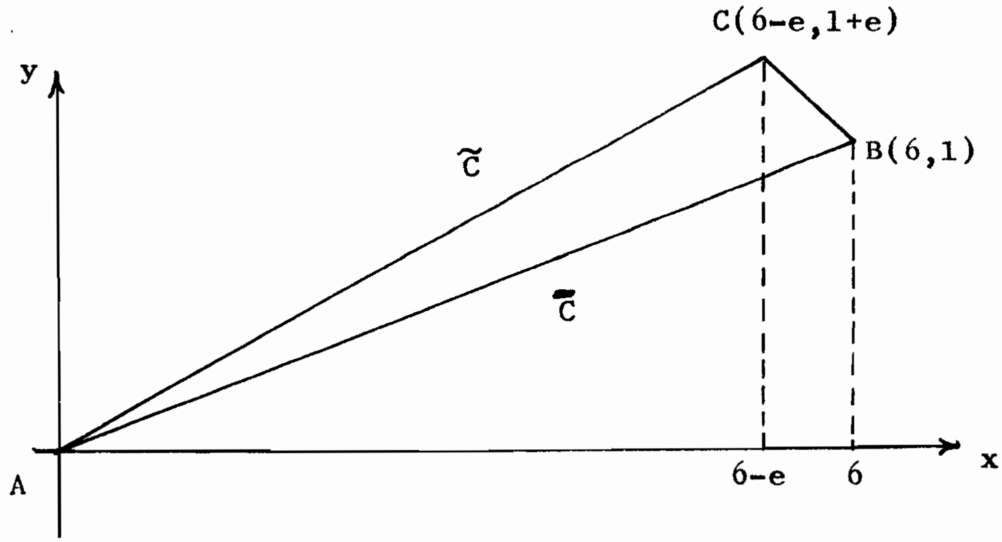


Fig. 2. Construction for proving that $\bar{C}$ as defined in example 1 does not furnish a maximum for the given functional. $J(\bar{C}) = 25/216$; $J(\tilde{C}) > J(\bar{C})$.

<u>Example 2.</u> Let $J(y) = \int_0^1 (y'^2 + y'^3)dx$

with boundary conditions $y(0) = 0$, $y(1) = 0$. As in the previous example, the end-points are denoted by A and B respectively. As before, application of Corollary 2.2 shows that the extremals are straight lines, $\bar{C}$ is the segment $[0,1]$ of the x-axis, and $J(\bar{C}) = 0$. By proper choice of the path of integration, however, $J(y)$ can be made both positive and negative. For if p, q are chosen so that $0 < p < 1$ and $q > 0$, and y is defined to be the broken line APB, where P has coordinates $(1-p, q)$, then

$$J(y) = \frac{q^2}{p(1-p)}\left(1 + \frac{q}{1-p} - \frac{q}{p}\right).$$

Any e-neighborhood of $\bar{C}$ being given, it is possible to choose $q < e$. Then p can always be taken so small that $J(\tilde{y}) < 0$ (cf. Fig. 3).

Similarly, by choice of p,q so that $0 < p < 1$, $q < 0$, a broken line APB with equation $y = \tilde{y}(x)$ can be constructed so that $J(\tilde{y}) > 0$. In this case the coordinates of P are $(1-p, q)$ as before.

The situation here is analogous to that of the extremum problem in differential calculus. The required curve is to be found among the solutions to Euler's equation, i.e., among the

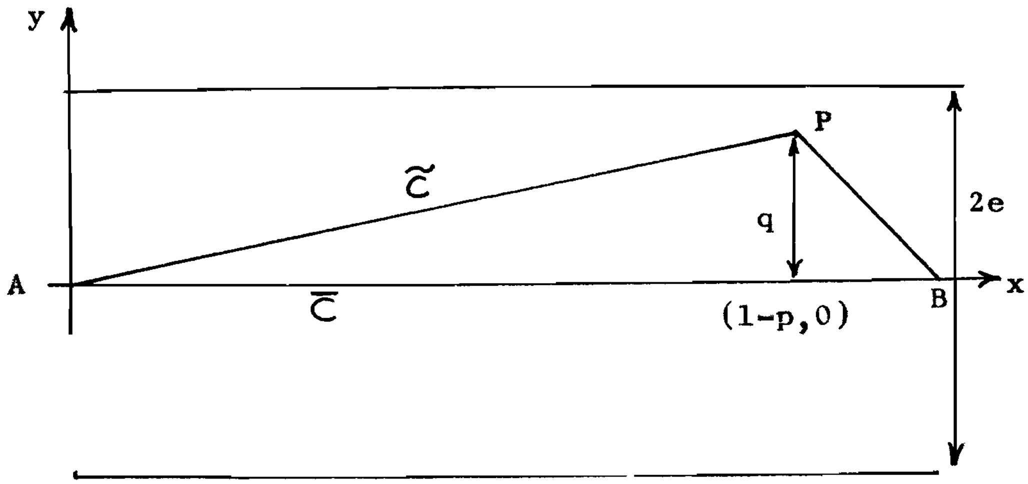


Fig. 3. Construction of broken path APB showing that the segment AB is not a strong minimum for $J(y) = \int_0^1 (y'^2 + y'^3)dx$. (see Example 2).

extremals, but as the above examples show, not all extremals are extreme points for the given functional. In order to distinguish extreme points, an additional criterion is needed. The derivation of such a criterion begins with a lemma due to Weierstrass. The assumption n = 1 still holds.

Weierstrass's Lemma and the E-function.

Suppose that there are given in the region G

(i) an extremal $\bar{C}$ of class C^2: $y = \bar{y}(x)$,

(ii) a curve $\tilde{C}$ of class C^1: $y = \tilde{y}(x)$, meeting $\bar{C}$ at a point $B(b,\bar{y}(b))$, and

(iii) a point $A(a,\bar{y}(a))$ on $\bar{C}$ such that $a < b$.

Let D be the point of $\tilde{C}$ whose abscissa is b+h, where h is an arbitrarily small positive number. Now select an arbitrary function t(x) of class C^1 in G satisfying the conditions

$$t_1 = t(a) = 0$$

$$t_2 = t(b) \neq 0.$$

Then a number e can be so determined that the curve

$$C: \; y = \bar{y}(x) + et(x)$$

which necessarily passes through the point A, also passes through D (cf. Fig. 4). To find e, it is necessary to solve the equation

(25) $$\tilde{y}(b+h) = \bar{y}(b+h) + et(b+h).$$

Since t(x) is continuous, it is possible to write

$$et(b+h) = et(b) + eo(1)$$

where $o(1) \rightarrow 0$ as $h \rightarrow 0$ (the symbol 'o' is the small Landau o-symbol). Also, because $\tilde{y}$ and $\bar{y}$ are at least of class C^1, it

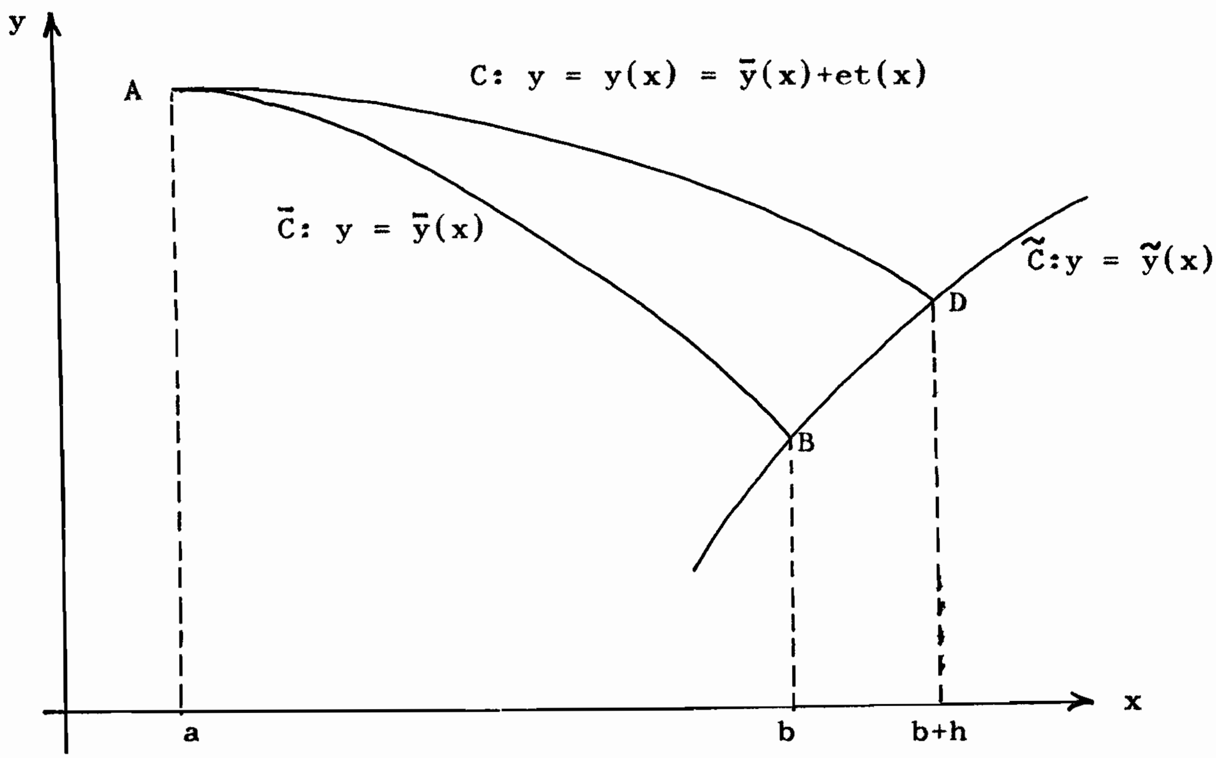


Fig. 4. Construction for proof of Weierstrass's Lemma.

follows that

(26) $$\tilde{y}(b+h) - \bar{y}(b+h) = (\tilde{y}'(b) - \bar{y}'(b))h + o(h).$$

Therefore, from (25) and (26)

(27) $$e = \frac{\left[\tilde{y}'(b) - \bar{y}'(b)\right] h+o(h)}{t(b) + o(1)} = h\left[\frac{\tilde{y}'(b) - \bar{y}'(b)+o(1)}{t(b) + o(1)}\right].$$

By the definition of o(1), (27) can be written

(28) $$e = h\left[\frac{\tilde{y}'(b) - \bar{y}'(b)}{t(b)}\right] + o(1).$$

Therefore $e \to 0$ as $h \to o$, and $e = o(1)$. The difference

$$\Delta J = J_{AD}(C) - \left[J_{AB}(\bar{C}) + J_{BD}(\tilde{C})\right]$$

will now be computed. Here the integrals J are taken along the indicated path from the point represented by the first subscript to the point represented by the second.

Let $f, \bar{f}, \tilde{f}$ or $f(x), \bar{f}(x), \tilde{f}(x)$ denote $f(x,y(x),y'(x))$, $f(x,\bar{y}(x),\bar{y}'(x))$, $f(x,\tilde{y}(x),\tilde{y}'(x))$ respectively. Then

$$\Delta J = \int_a^{b+h} f - \int_a^b \bar{f} - \int_b^{b+h} \tilde{f}$$

$$= \int_a^b f + \int_b^{b+h} f - \int_a^b \bar{f} - \int_b^{b+h} \tilde{f}.$$ Hence

(29) $$\Delta J = \int_a^b (f - \bar{f})dx + \int_b^{b+h} (f - \tilde{f})dx$$

Application of Taylor's theorem to the first integral and an integration by parts gives

$$\int_a^b (f - \bar{f})dx = \int_a^b f(x,\bar{y} + et,\bar{y}' + et')dx - \int_a^b f(x,\bar{y},\bar{y}')dx$$

$$= e\int_a^b (t\bar{f}_y + t'\bar{f}_{y'})dx + R_2$$

$$= e\int_a^b (t\bar{f}_y - t\frac{d}{dx}\bar{f}_{y'}) + et(b)\bar{f}_{y'}(b) + R_2$$

where R_2 is a term involving e^2 which can be written o(h). Since $\bar{y}$ is an extremal, the first term vanishes. Hence, by (28) there results

$$(30) \qquad \int_a^b (f - \bar{f})dx = h\Big[(\tilde{y}'(b) - \bar{y}'(b))\bar{f}_{y'}(b) + o(1)\Big]$$

where $\bar{f}_{y'}(b)$ means $\frac{\partial}{\partial z}f(b,\bar{y}(b),\bar{y}'(b))$. By the Mean Value Theorem for integrals, the second integral in (29) may be written

$$(31) \quad \int_b^{b+h} (f-\tilde{f})dx = h\Big\{f(b+\theta h,y(b+\theta h),y'(b+\theta h)) - \tilde{f}(b+\theta h)\Big\}$$

where $0 < \theta < 1$. Since f(x) and $\tilde{f}(x)$ are continuous, (31) can be expressed

$$\int_a^{b+h} (f-\tilde{f})dx = h\left[f(b,\bar{y}(b)+et(b),\bar{y}'(b)+et'(b)) - \tilde{f}(b) + o(1)\right]$$

Using the continuity of f a second time, and the fact that e = o(1) gives

$$(32) \qquad \int_b^{b+h} (f-\tilde{f})dx = h\left[\bar{f}(b) - \tilde{f}(b) + o(1)\right]$$

Therefore by (29), (30) and (32)

$$(33) \qquad \Delta J = h\left\{(\tilde{y}'(b)-\bar{y}'(b))\bar{f}_{y'}(b) + \bar{f}(b) - \tilde{f}(b) + o(1)\right\}$$

Similarly, if K is that point of C whose abscissa is b-h, then e can be so determined that the curve

$$C: \; y = \bar{y}(x) + et(x)$$

passes through K (the other symbols have the same meaning as before). Then, by a similar process, the following result is obtained

$$\Delta J = J_{AK}(C) + J_{KB}(\tilde{C}) - J_{AB}(\bar{C})$$

$$= -h\left[(\tilde{y}'(b) - \bar{y}'(b))\bar{f}_{y'}(b) + \bar{f}(b) - \tilde{f}(b) + o(1)\right]$$

Now define

$$(34)\qquad E(x,y,z,q) = f(x,y,q) - f(x,y,z) - (q-z)f_z(x,y,z)$$

Then by (34) the preceding results may be written

$$(35a)\qquad J_{AD}(C) - \left[(J_{AB}(\bar{C}) + J_{BD}(\tilde{C}))\right]$$
$$= -h\left\{E(b,\bar{y}(b),\bar{y}'(b),\tilde{y}'(b)) + o(1)\right\}$$

$$(35b)\qquad J_{AK}(C) + J_{KB}(\tilde{C}) - J_{AB}(\bar{C})$$
$$= h\left\{E(b,\bar{y}(b),\bar{y}'(b),\tilde{y}'(b)) + o(1)\right\}$$

These two[1] formulas will be referred to as Weierstrass's Lemma. The function E(x,y,z,q) defined by (34) is called the Weierstrass E-function. With obvious modifications, this lemma may be extended to a space of n+1 dimensions, although in view of the notational difficulties to do so here is hardly worthwhile. As mentioned earlier, the E-function for spaces of dimension higher than two can be obtained more elegantly by other methods. According to the assumptions made earlier regarding the function f[2], the E-function is a real-valued function of class C^2 defined for $(x,y) \varepsilon G$,[3] and for real finite values z,q. E(x,y,z,q) = 0 if z = q.

---

[1] Unfortunately these formulas cannot be condensed into a single equation because the differences defined by the left-hand sides are not symmetrical with respect to $\bar{C}$.

[2] cf.pp.18-19.

[3] in this case an open, bounded and simply-connected region.

8. The E-function in two dimensions.

8.1 The E-function and the Fourth Necessary Condition.

From the results of the preceding section, Weierstrass's Fourth Condition for the case n = 1 follows immediately. Through an arbitrary point $B(b,\bar{y}(b))$ of an extremal $\bar{C}$, let there be drawn arbitrarily a curve $\tilde{C}$: $y = \tilde{y}(x)$ of class C^1 (cf. Fig. 5).

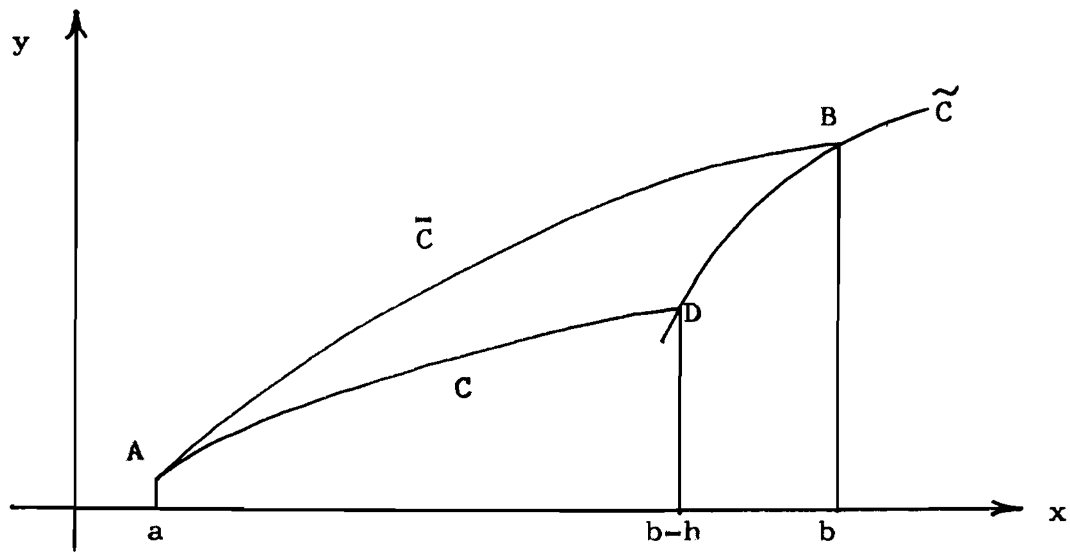


Fig. 5 Construction for Weierstrass's Fourth Condition in two dimensions.

Let D denote that point of $\tilde{C}$ whose abscissa is b-h, h being a small positive quantity. As in the proof of Weierstrass's Lemma, construct a curve C: $y = \bar{y} + et$ of class C^1 from A to D and replace the arc of integration AB of $\bar{C}$ by the curve ADB. By taking h sufficiently small, the curve ADB can be made to lie in an arbitrarily small strong neighborhood of $\bar{C}$. For this variation of $\bar{C}$, ΔJ is given by

$$(36) \qquad \Delta J = J_{AD} + \tilde{J}_{DB} - \bar{J}_{AB}$$

in the notation of the previous section. But according to (35b), (36) can be written

$$(37) \qquad \Delta J = h\, E(b,\bar{y}(b),\bar{y}'(b),\tilde{y}'(b)) + o(h).$$

The following theorem for the case n = 1 is thereby proved.

<u>Theorem 3.</u> For an arbitrary finite number q, the inequality

$$(38) \qquad E(x,y,y',q) \geq 0$$

must be satisfied at every point of an extremal $y = \bar{y}(x)$, $a \leq x \leq b$, if the extremal furnishes the functional

$$\int_a^b f(x,y,y')dx$$

with a <u>strong</u> minimum.[1]

---

[1]Unless otherwise stated, every theorem concerning a minimum condition involving an inequality, applies also to a maximum with the inequality reversed (see footnote, p.6).

Application of Taylor's formula to the difference

$$f(x,y,z) - f(x,y,q)$$

yields the following relation between the E-function and f_{zz}

$$(39) \qquad E(x,y,z,q) = \frac{(q-z)^2}{2} f_{zz}(x,y,z^*)$$

where $z^* = z + \theta(q-z)$, $0 < \theta < 1$. This proves

<u>Corollary 3.1</u> Weierstrass's Necessary Condition is satisfied if for every point (x,y) on $\tilde{C}$, and for every finite number q

$$(40) \qquad f_{zz}(x,y,q) \geq 0.$$

8.2 Geometrical Interpretation of the E-function.

For $n = 1$, the E-function has a simple geometrical interpretation[1], since if $f(x,y,z)$ is regarded as a function of z alone, x and y being fixed, then

$$E(x,y,z_o,q) = f(x,y,q)-f(x,y,z_o) - (q-z_o)f_z(x,y,z_o)$$

is the vertical distance at q from the curve C representing $f(x,y,z)$ to the tangent to C drawn through a fixed point $z = z_o$ of C (cf. Fig. 6).

---

[1] Due to Zermelo, Dissertation, 1894.

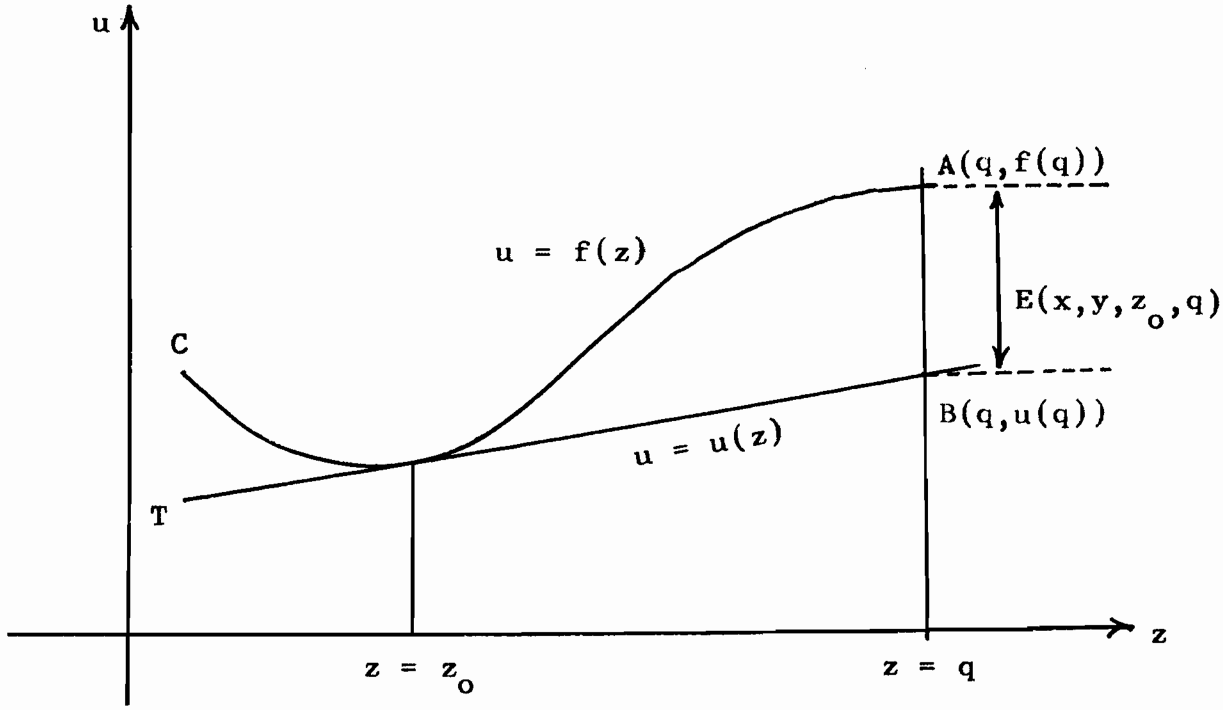


Fig. 6 Geometrical interpretation of the E-function for the case n = 1.

If f(z) denotes f(x,y,z) with x and y fixed, and u represents the values of f(z), then the tangent T to f(z) at z_o has equation

$$T: u(z) = f(z_o) + (z - z_o)f'(z_o)$$

and the ordinate of the point whose abscissa is z = q is given by

$$u(q) = f(z_o) + (q - z_o)f'(z_o).$$

Therefore AB = $f(q) - f(z_o) - (q - z_o)f'(z_o)$. In the original notation, the latter is equivalent to

$$AB = E(x,y,z_o,q)$$

Now identify y, z_o with $\bar{y}(x)$, $\bar{y}'(x)$ respectively. Let x range over the interval $[a,b]$ and let Z denote the range of $\bar{y}'(x)$. Then the condition expressed by (38)

$$E(x,\bar{y},\bar{y}', q) \geq 0 \qquad (a \leq x \leq b)$$

means geometrically that the curve C: $u = f(z)$ cannot lie below any of its tangents determined by points z belonging to Z. Therefore, in order that (38) may hold, it is sufficient that the curve C shall be everywhere concave up, i.e.,

$$f''(z) \geq 0$$

for all finite z. The latter condition is equivalent to condition (40) of Corollary 3.1.

8.3 The E-function and Legendre's Weak Condition.

If in the proof of Theorem 3 the curve ADB is restricted to weak variations, then obviously similar reasoning leads to the inequality (38) in weakened form. More precisely, for a weak minimum, it is necessary that (38) hold for $(x,y,y') = (x,\bar{y},\bar{y}')$ and q sufficiently near $\bar{y}'$. This modification leads to the following result.

<u>Theorem 4.</u> If the extremal $y = \bar{y}(x)$ furnishes a weak minimum for the functional

$$\int_a^b f(x,y,y')dx$$

then the inequality

$$f_{zz}(x,\bar{y}(x),\bar{y}'(x)) \geq 0 \tag{41}$$

is satisfied for all x in the interval $[a,b]$.

<u>Proof.</u> Define $g(e) = E(x,\bar{y},\ \bar{y}',\ \bar{y}' + ez)$, where z is an arbitrary finite variable. It follows that

$$g'(e) = zf_z(x,\bar{y},\bar{y}' + ez) - zf_z(x,\bar{y},\bar{y}'),$$
$$g''(e) = z^2 f_{zz}(x,\bar{y},\bar{y}' + ez),$$

and hence

$$g''(0) = z^2 f_{zz}(x,\bar{y},\bar{y}')$$

From the Weierstrass Condition in its weak form, it follows that g(e) has a relative minimum when e = 0, and hence $g''(0) \geq 0$. Q.E.D.

Inequality (41) in Theorem 4 is known as Legendre's Necessary Condition for a weak minimum, and was discovered in 1786. Since a strong minimum is a fortiori also a weak minimum, Legendre's Condition is also necessary for a strong minimum, and may be referred to without ambiguity simply as Legendre's Condition. As the preceding result shows, Legendre's Condition is contained in the Weierstrass Condition.

8.4 The E-function and the Second Corner Condition.

As will now be shown, the E-function vanishes at each corner of a discontinuous solution curve, and in consequence of this, it is possible to prove the second corner condition (24) if the first one is assumed. In Fig. 7, let ACB be a minimising extremal $\bar{C}$: $y = \bar{y}(x)$, and consider successively the variations ADB and AKC. The corresponding values of the total variations ΔJ are, by (35a) and (35b)

$$-h\left\{E(c,\bar{y}(c),\bar{y}'(c-0),\ \bar{y}'(c+0)\right\} + o(h)$$

and

$$h\left\{E(c,\bar{y}(c),\bar{y}'(c-0),\ \bar{y}'(c+0)\right\} + o(h)$$

where $o(h) \to 0$ as $h \to 0$, and $y'(c-0), y'(c+0)$ are the left- and right-hand derivatives respectively of $\bar{y}(x)$ at $x = c$. Since $\Delta J \geq 0$, it follows that $E(c,\bar{y}(c),\bar{y}'(c-0),\bar{y}'(c+0)) = 0$, which, on account of (23) is equivalent to (24) Q.E.D.

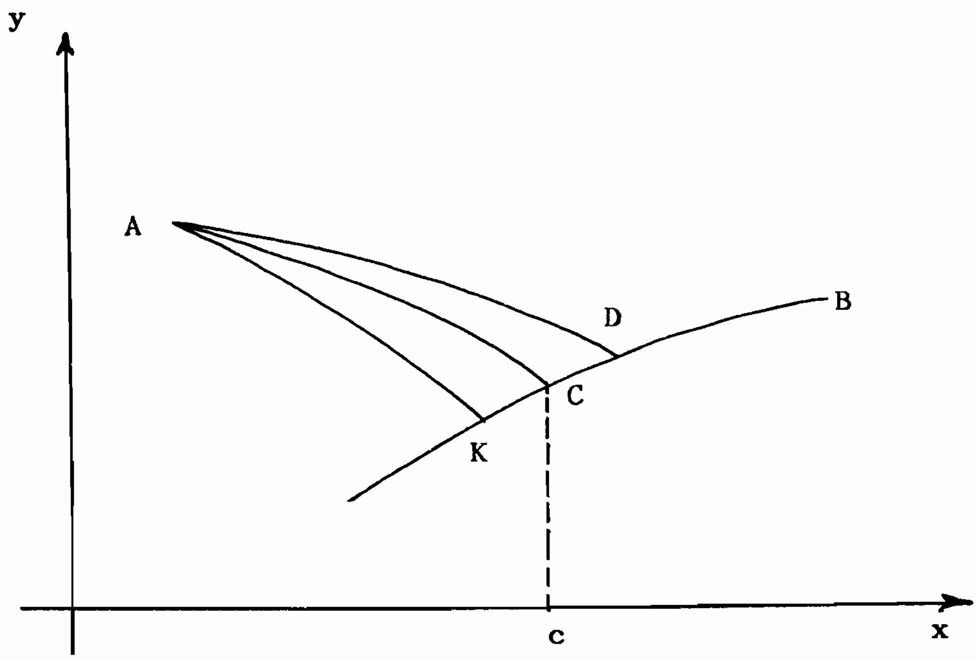


Fig. 7 Construction for proving that the E-function vanishes at a corner.

8.5 Applications of the E-function condition.

It is now possible to show by means of Theorem 3 that the solutions to the two problems examined on pages 35-38 are not strong extrema for the corresponding functionals.

Example 1. $$J(y) = \int_0^6 y'^2(1-y')^2 dx, \quad y(0) = 0, \quad y(6) = 1.$$

In this case, the E-function is

$$E(x,y,z,q) = (q-z)^2\left[q^2 + q(2z - 2) + 3z^2 - 4z + 1\right].$$

The quadratic in q

$$q^2 + q(2z - 2) + 3z^2 - 4z + 1$$

is always non-negative if $z(z-1) \geq 0$; it can change sign if $z(z-1) < 0$. Hence, if $z \leq 0$ or if $z \geq 1$, the condition of Theorem 3 is satisfied. If $0 < z < 1$, that condition is not satisfied, and therefore the line $y = \frac{x}{6}$ does not furnish a strong extremum. This agrees with the previous result.

<u>Example 2.</u> $J(y) = \int_0^1 (y'^2 + y'^3)dx, \quad y(0) = 0, \; y(1) = 0.$

For this problem, the solution of Euler's equation is $\bar{y}(x) = 0$, i.e., the segment $[0,1]$ of the x-axis. Hence both $\bar{y}$ and $\bar{y}'$ are zero identically. The E-function is

$$E(x,y,z,q) = (q-z)\left[q^2 + (z+1)q - (2z^2+z)\right]$$

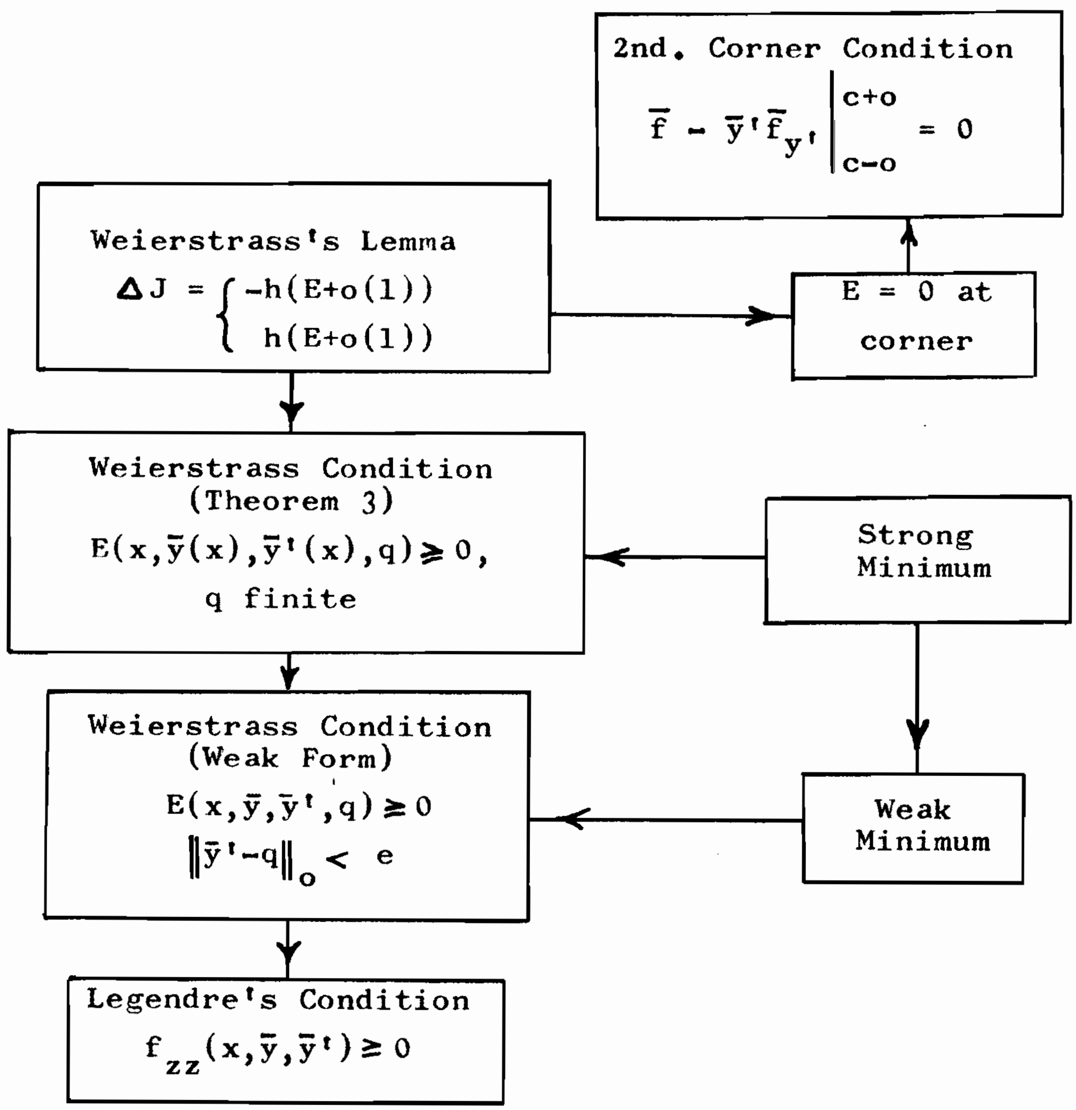
and

$$E(x,\bar{y},\bar{y}',q) = E(x,0,0,q) = q^2(1+q)$$

In this case, E can be made positive or negative by choice of q, and so the condition of Theorem 3 is not satisfied. Hence the curve y = 0 does not furnish a strong extremum, in agreement with the earlier result.

8.6 Summary.

The relations between the various conditions in which the E-function occurs are illustrated in the following diagram. The minimising curve is denoted by $\bar{y}$ or by $\bar{y}(x)$, and the arrows denote implication.



The Weierstrass Condition, although necessary for a strong minimum, is not also sufficient, as the following example shows. Consider the functional

$$J(y) = \int_0^1 (ay'^2 - 4byy'^3 + 2bxy'^4)dx, \quad y(0) = y(1) = 0,$$

where $a > 0$, $b > 0$, and the extremal $y = \bar{y}(x) = 0$. In view of the choice of $\bar{y}$, the E-function reduces to

$$E(x,\bar{y},\bar{y}',q) = E(x,o,o,q) = q^2(a + 2bxq^2) \geq 0,$$

and the Weierstrass Condition is satisfied. Nevertheless, the curve $\bar{y} = 0$ does not furnish the functional $J(y)$ with a strong minimum. For consider

$$\tilde{y}(x) = \begin{cases} \dfrac{kx}{h}, & 0 \leq x \leq h, \\[2ex] \dfrac{k(1-x)}{1-h}, & h \leq x \leq 1. \end{cases}$$

In this case, $\Delta J = J(\tilde{y}) - J(\bar{y})$

$$= k^2\left[-\frac{bk^2}{h^2} + \frac{a}{h} + a + 3bk^2\right] + o(1)$$

where $o(1) \to 0$ as $h \to 0$. Obviously $\Delta J < 0$ for all sufficiently small values of h. Hence the extremal $\bar{y} = 0$ is not a strong minimum for the given functional, and the Weierstrass Condition is not sufficient. The E-function Sufficient Condition is derived in the next section.

9. Weierstrass's Theorem and the E-function Sufficient Condition.

After adding a fourth necessary condition to those of Euler, Legendre and Jacobi, Weierstrass showed that, when suitably strengthened, these four conditions will actually guarantee the minimising property.

In particular, the E-function Sufficient Condition is equivalent to a generalisation of the converse of Theorem 3. The derivation of this condition given below is based on the original proof of Weierstrass for problems in the plane, and depends on the idea of families of extremals rather than individual curves. Weierstrass called such a family a <u>field of extremals</u>. Since Weierstrass's time, these families have been the subject of extensive study, resulting in an elaborate theory now referred to as the <u>theory of fields</u>.

In Section 9 below, there is given a formal definition, in terms of the Hilbert integral, of a field more general than the one used by Weierstrass, and applicable either in the plane or in higher dimensions. This definition will be needed for the proof of Weierstrass's theorem in n+1 dimensions. At the moment, however, there is nothing to be gained from such a definition, since Weierstrass's proof does not make use of the Hilbert integral. Therefore, the outline which follows will be sufficient for the case n = 1.

Let $\bar{C}$ be an extremal for the functional $\int_a^b f(x,y,y')dx$, where $\bar{C}$ is defined on $[A,B]$, $A < a$, $b < B$, and suppose that along $\bar{C}$ the condition

$$(42) \qquad f_{zz}(x,y,y') \neq 0$$

holds. According to the theory of differential equations, there exists an open region D containing the curve $\bar{C}$ such that only one integral curve

$$(43) \qquad y = y(x,c_1,c_2) \qquad (A \leq x \leq B)$$

of Euler's equation of class C^2 passes through each point in the interior of D with given slope m $(m \neq \pm \pi/2)$.

Now let V be the point $(x_o,\bar{y}(x_o))$, where $y = \bar{y}(x)$ is the equation of $\bar{C}$, and $A < x_o < a$. From the extremals (43) select the one-parameter family consisting of the pencil of extremals which pass through the point V, using the slope m of each extremal at V as the family parameter. The pencil can thus be represented in the form

$$(44) \qquad y = g(x,m).$$

If $\bar{m}$ is the slope of $\bar{C}$ at V, then the pencil (44) contains $\bar{C}$ for $m = \bar{m}$. A theorem on the dependence of a solution of a differential equation on parameters provides that both $g(x,m)$ and $g_x(x,m)$ are continuously differentiable with respect to m. Hence, in some region

$$A \leq x \leq B, \qquad |m - \bar{m}| \leq d$$

where $d > 0$, $g(x,m)$ is of class C^2 with respect to its arguments. Now let k denote a positive quantity less than d, and denote by S_k the set of points (x,y) determined by (44) as m, x range over the intervals $[\bar{m}-k, \bar{m}+k]$ and $[a,b]$ respectively. In the discussion that follows, it will also be necessary to assume that

$$\text{(45)} \qquad g\ (x,\bar{m}) \neq 0 \qquad a \leq x \leq b.$$

This assumption guarantees the existence of a $k > 0$ such that through every point (x,y) of S_k there passes a unique extremal of the set (44) for which $|m-\bar{m}| \leq k$. Analytically, this means that there exists a single-valued function

$$m = m(x,y)$$

such that

$$\text{(46)} \qquad y = g(x,m(x,y))$$

identically, and $|m(x,y) - \bar{m}| \leq k$ for every (x,y) in S_k. The existence and continuity of the first partial derivatives of $m(x,y)$ follows from the theorem on implicit functions. In fact, the values of these partial derivatives, obtained from (46) by the ordinary rules for the differentiation of implicit functions, are

$$m_x = -\frac{g_x}{g_m},$$

and

$$m_y = \frac{1}{g_y}.$$

Now let P, Q be the points $(a,\bar{y}(a))$ and $(b,\bar{y}(b))$ respectively and let $\tilde{C}$: $y = \tilde{y}(x)$ denote any curve of class C^1 joining the points P and Q, and lying in some region S_k with the properties described above. Denote by T any point $(t, \tilde{y}(t))$ on $\tilde{C}$. By assumption, there passes through T a unique extremal of the set (44) which may be denoted by

$$\bar{C}_t: \; y = \bar{y}(x,m_t).$$

The situation is illustrated in Fig. 8. For the sake of clarity, no attempt has been made to include the region S_k. Now consider the integral J taken from V to T along $\bar{C}_t$, and from T to Q along $\tilde{C}$. Denote its value by $S(t)$:

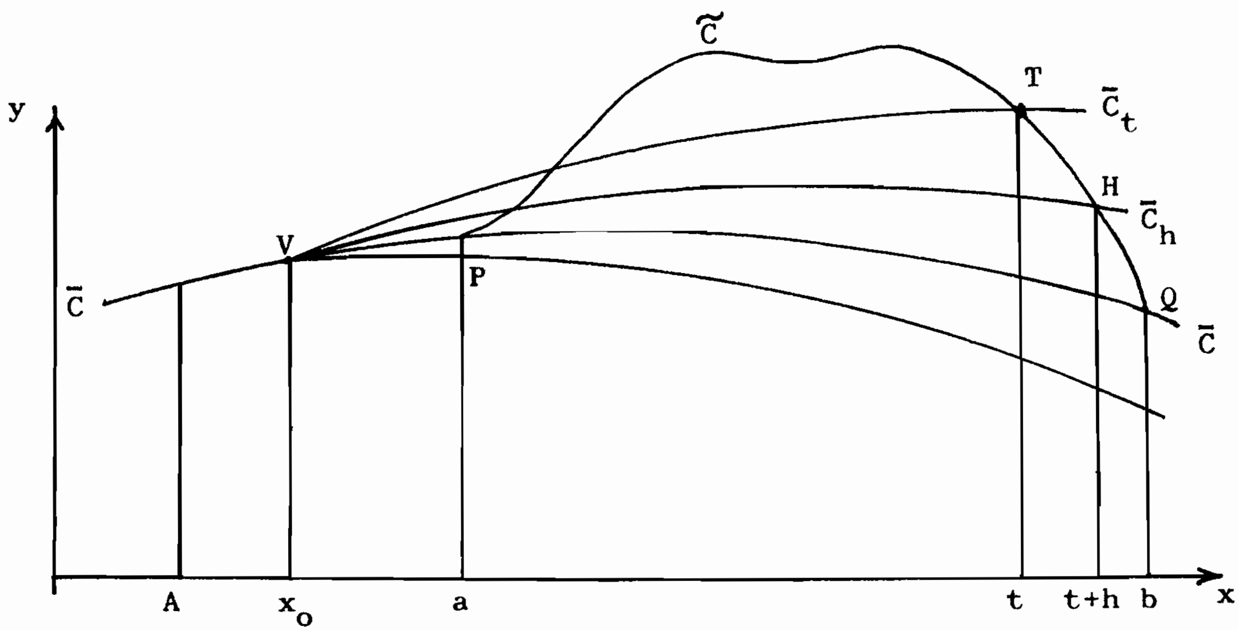


Fig. 8 Pencil of extremals with vertex V used in the proof of the E-function Sufficient Condition.

$$S(t) = J_{VT}(\bar{C}_t) + J_{TQ}(\tilde{C})$$

$$= \int_{x_o}^{t} f(x,\bar{y}(x,m\ ),\bar{y}'(x,m\ ))dx + \int_{t}^{b} f(x,\tilde{y},\tilde{y}')dx$$

For $t = a$, $t = b$, $S(t)$ takes the values

$$S(a) = J_{VP}(\bar{C}) + J_{PQ}(\tilde{C}),$$

$$S(b) = J_{VQ}(\bar{C}).$$

Therefore $J_{PQ}(\tilde{C}) - J_{PQ}(\bar{C}) = \Delta J = S(a) - S(b)$.

It will now be shown that the function S(t) is continuous and differentiable in (a,b) and that

$$S'(t) = -E(t,\bar{y}(t,m_t),\ \bar{y}'(t,m_t),\ \tilde{y}'(t)) \tag{47}$$

where the primes denote differentiation with respect to x. Let H be the point of C whose abscissa is t+h, h being a small positive quantity, and let

$$\bar{C}_h:\ y = \bar{y}(x,m_h)$$

be the unique extremal of the set (44) which passes through the point H. Then it follows that

$$\begin{aligned} S(t+h) - S(t) &= J_{VH}(\bar{C}_h)+J_{HQ}(\tilde{C})-J_{VT}(\bar{C}_t)-J_{TQ}(\tilde{C}) \\ &= J_{VH}(\bar{C}_h)-J_{VT}(\bar{C}_t)+J_{HT}(\tilde{C})+J_{TQ}(\tilde{C})-J_{TQ}(\tilde{C}) \\ &= J_{VH}(\bar{C}_h) - J_{VT}(\bar{C}_t) - J_{TH}(\tilde{C}) \\ &= J_{VH}(\bar{C}_h) - \left[J_{VT}(\bar{C}_t) + J_{TH}(\tilde{C})\right] \end{aligned}$$

By Weierstrass's Lemma, the last expression can be written

$$S(t+h)-S(t) = -h\left[E(t,\bar{y}_t,\bar{y}'_t,\tilde{y}'(t)) + o(1)\right] \tag{48}$$

where $\bar{y}_t, \bar{y}'_t$ denote $\bar{y}(t,m_t), \bar{y}'(t,m_t)$ respectively, and $o(1) \rightarrow 0$ as $h \rightarrow 0$. Similarly, if H is that point of $\tilde{C}$ whose abscissa is t-h, it can be shown that

$$S(t-h)-S(t) = h\left[E(t,\bar{y}_t,\bar{y}'_t,\tilde{y}'(t)) + o(1)\right]. \tag{49}$$

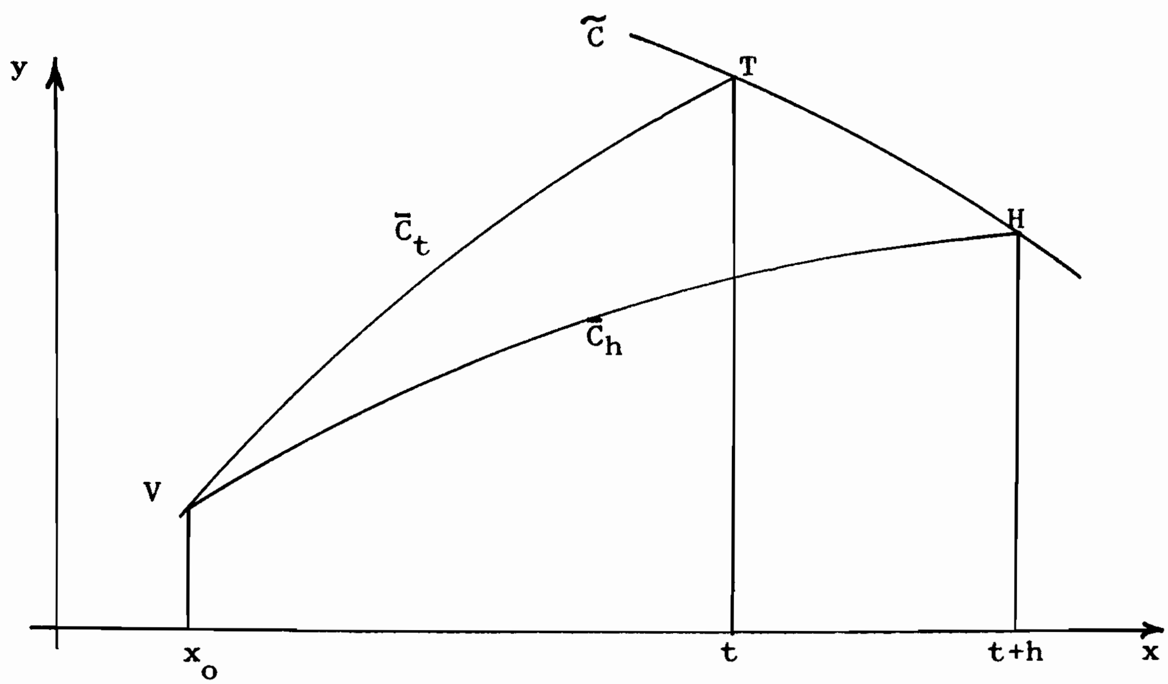


Fig. 9 Construction for the application of Weierstrass's Lemma to the proof of the E-function Sufficient Condition.

From (48) and (49), for positive or negative h,

$$S'(t) = \lim_{h \to 0} \left[\frac{S(t+h) - S(t)}{h} \right] = -E(t, \bar{y}_t, \bar{y}'_t, \tilde{y}'(t))$$

and (47) is hereby proved. Integration of (47) between the limits a and b gives, in view of the fact that $\Delta J = S(a)-S(b)$,

$$(50) \qquad \Delta J = \int_a^b E(t, \bar{y}_t, \bar{y}'_t, \tilde{y}'(t))dt.$$

Formula (50) will be referred to as <u>Weierstrass's Theorem</u>.

This formula is true also in the case where $\tilde{C}$ is of class D^1. For if $\tilde{C}$ has a corner at T, then (48) and (49) still hold provided that $\tilde{y}'(t)$ is understood to mean the right- and left-hand derivative of $\tilde{y}(t)$ respectively. Weierstrass's theorem can also be proved by the process of differentiation of a definite integral with respect to a parameter.

<u>Definition 13.</u> A region D in the plane is said to be <u>simply covered</u> by a family of extremals if through each point of D there passes a unique extremal of the family.

Weierstrass's Theorem now leads immediately to the E-function Sufficient Condition.

<u>Theorem 5.</u> Let $y = \bar{y}(x)$ be an extremal for the functional in ordinary form for the case $n = 1$. Suppose that the extremal is contained in a pencil P: $y = g(x,m)$ for $m = \bar{m}$, and that there exists a region S containing the extremal for $a \leq x \leq b$, which is simply covered by P. Let $p(x,y)$ denote the slope at (x,y) of the member of the pencil P which passes through (x,y). Then if

$$E(x,y,p(x,y),q) \geq 0 \qquad (51)$$

for every finite q and for all (x,y) in S, the curve $y = \bar{y}(x)$ furnishes the given functional with a <u>strong minimum</u>.

According to (39) it is possible to write

$$E(x,y,p(x,y),q) = \frac{[q-p(x,y)]^2}{2} f_{zz}(x,y,p^*)$$

where $p^* = p(x,y) + \theta(q-p(x,y))$, $0 < \theta < 1$. Therefore the inequality $E(x,y,p,q) \geq 0$ is ensured provided that $f_{zz}(x,y,z) \geq 0$ at every point of S for finite z. In certain cases, the latter inequality is simpler to use than (51). This statement applies to many well-known problems.

Suppose that $J(y) = \int_a^b h(x,y)ds = \int_a^b h(x,y)\sqrt{1+y'^2}\,dx$,

where the positive value of the root is taken, and s denotes the arc length measured along the path of integration. In this case

$$f_{y'y'} = \frac{h(x,y)}{(1+y'^2)^{3/2}}$$

so that J(y) has a strong minimum at $y = \bar{y}$ if $h(x,y) \geq 0$ at all points of some region containing $\bar{y}$.[1] This result includes the following special cases in which the condition (51) for a strong minimum is satisfied:

1. The shortest distance between two points in the plane. Here $J = \int_a^b ds$, $h(x,y) = 1$ and the condition for a strong minimum is realised.

---

[1]The case where h(x,y) is identically zero along $\bar{y}$ is excluded.

2. Problems involving catenaries and minimal surfaces, where $J(y) = \int_a^b y.ds$. If the arc of the catenary lies above the x-axis, $y \geqslant 0$, and the condition for a strong minimum is satisfied.

3. The Principle of Least Action for a particle in a conservative field of force. The integral is $\int_a^b v.ds$ where v is the speed. In this case, v must be positive at all points of the dynamical path.

4. Fermat's Principle of Least Time, optical paths and the brachistochrone. The integral to be minimised is $\int_a^b \frac{ds}{v}$, where v is the speed. As in case 3, v must be positive at all points of the optical or dynamical path.

Of course, before any firm conclusions can be reached concerning the existence of a strong minimum, the other hypotheses of Theorem 5 must be checked also. Application of this theorem to some of the above problems will be considered in greater detail in Section 13. For the moment, the following modification of Example 1 (see pp.35, 53) will suffice to illustrate the use of Theorem 5.

<u>Example.</u> $$J(y) = \int_0^1 y'^2(1-y')^2 dx, \quad y(0) = 0,\ y(1) = 2.$$

The two-parameter family of solutions of Euler's equation is

$$y = mx + b,$$

and the extremal satisfying the boundary conditions is

$$\bar{C}:\ y = 2x.$$

For arbitrary $e > 0$, a pencil of extremals passing through the vertex at $(-e,-2e)$ on $\bar{C}$ is given by

(52) $$y = mx + e(m - 2),$$

which contains $\bar{C}$ for $m = 2$. This pencil is obviously a simple covering for any sufficiently small region containing the curve $\bar{C}$ for x in the interval $[0,1]$. The slope function for the pencil (52) is

$$p(x,y) = \frac{y + 2e}{x + e}.$$

It was shown earlier (see page 53) that

$$E(x,y,z,q) = (q - z)^2\left[q^2 + q(2z - 2) + 3z^2 - 4z + 1\right].$$

and the quadratic factor is non-negative for all finite values of q provided that $z(z - 1) \geq 0$, i.e., in particular, if $z \geq 1$. Therefore, if $p(x,y) \geq 1$ in a region which contains

$\bar{C}$: $y = 2x$, $0 \leq x \leq 1$, and is simply covered by the pencil (52), the hypothesis of Theorem 5 is satisfied, and $J(y)$ has a strong minimum for $y = 2x$. The existence of such a region is obvious from the following considerations. For arbitrarily small $d > 0$, let

$$\bar{C}_1: y = (2 - d/e)x - d$$
$$\bar{C}_2: y = (2 + d/e)x + d$$

be extremals of the pencil (52). If d is small enough, then $2 - (d/e) > 1$ and all the extremals lying in the cone K with vertex V determined by $\bar{C}_1$ and $\bar{C}_2$ have slope larger than 1. Any region R containing the segment

$$S = \left\{ (x,y)/y = 2x,\ 0 \leq x \leq 1 \right\}$$

and lying in K satisfies the requirements of Theorem 5 (cf.Fig.10).

The more general discussion of fields and the resulting theorems for the (n+1)-dimensional case which will be given in the next section, show that in the previous example it is possible (and much more convenient) to use the family of extremals $y = 2x+b$ parallel to $\bar{C}$, with $p(x,y) = 2$ everywhere. The use of a pencil in the above example was merely a restriction imposed by the hypotheses of Theorem 5. This restriction will be dropped in Section 10.

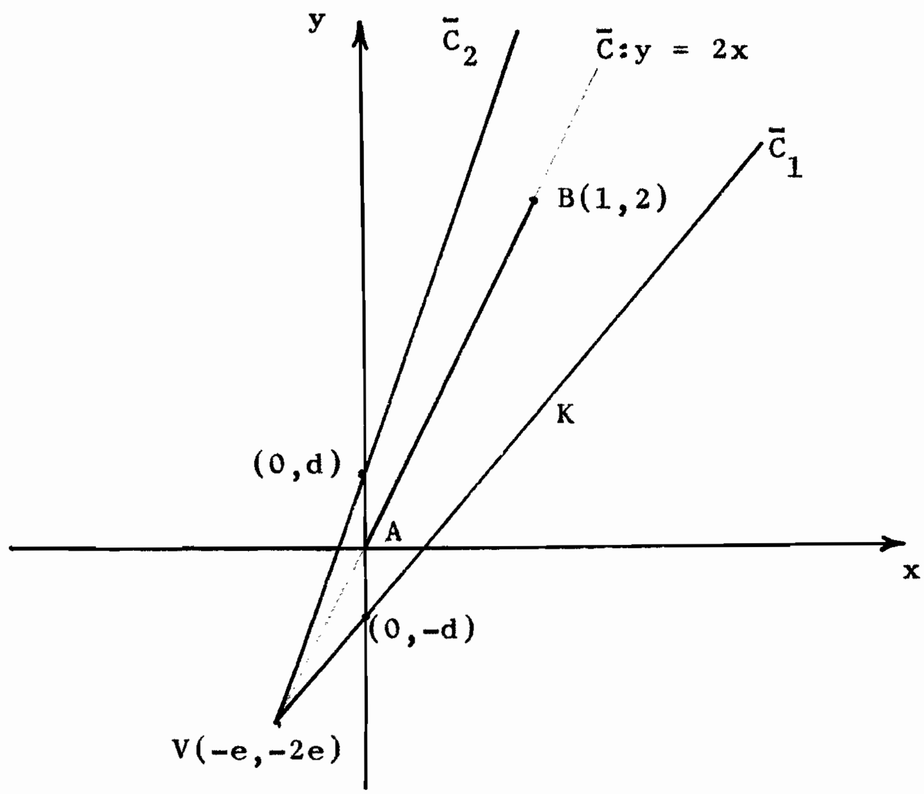


Fig. 10 Construction of pencil of extremals (cf.p.67).

The cone of extremals K with vertex V furnishes a pencil for the application of the E-function Sufficient Condition (Theorem 5). K covers any sufficiently small region containing the segment AB of the extremal $\bar{C}$: $y = 2x$, $0 \leq x \leq 1$. The equations of the boundaries of K are

$$\bar{C}_1: \; y = (2 - d/e)x - d$$

$$\bar{C}_2: \; y = (2 + d/e)x + d$$

where $d > 0$ is arbitrarily small.

10. The theory of fields for the (n+1)-dimensional problem.

As the last example shows, the pencil of extremals is not necessarily the most convenient family to use in applying Theorem 5, and the question naturally arises whether other families may be used instead. It will be shown later that in the application of Theorem 5, any family of extremals may be used which simply covers a region containing the extremum curve. A fundamental notion in the proof of this statement is the <u>field of extremals</u>.

<u>Definition 14.</u> A field for the functional $\int_a^b f(x,y,y')dx$ is any region $D \subset G^1$ with a vector function

$$(53) \qquad p(x,y) = \left\{p_1(x,y),\cdots, p_n(x,y)\right\}$$

where (x,y) denotes $(x,y_1,\cdots,y_n)$, satisfying the conditions

(53a) the functions $p_i(x,y)$ are of class C^1 in D $(i = 1,\cdots,n)$,

(53b) the integral

$$(54) \qquad \iint\left\{\left[f(x,y,p) - \sum_{i=1}^{n} p_i f_{z_i}(x,y,p)\right]dx + \sum_{i=1}^{n} f_{z_i}(x,y,p)dy_i\right\}$$

depends only on the end-points of the path of integration. The integral (54) is known as the <u>Hilbert invariant integral</u>.

[1]The notation and assumptions are the same as in Section 4.

The region D is said to be covered by a field, and the vector function $p(x,y)$ is called the slope function of the field.

Definition 15. The solutions of the system of first-order differential equations

$$(55) \qquad \frac{dy_i}{dx} = p_i(x,y_1,\cdots,y_n) \qquad (i = 1,\cdots,n)$$

are called the trajectories of the field. An existence theorem for ordinary differential equations provides that through each point of the region D there passes a unique trajectory, and that the collection of all such trajectories is an n-parameter family

$$(56) \qquad y_i = \phi_i(x,b\ ,\cdots,b\ ) \qquad (i = 1,\cdots,n)$$

Equations (56) may be written more concisely in the form

$$(57) \qquad y = \phi(x,b)$$

in which ϕ represents a vector function of class[1] C^2 in some region R of the space of the variables $(x,b_1,\cdots,b_n)$. Since the Hilbert integral is independent of the path, the expression under the integral sign in (54) is an exact differential. Application of the usual necessary condition

---

[1]The statement follows from (55) and condition (53a).

for exactness shows, after some manipulation, that the trajectories of the field are solutions of Euler's equation, i.e., they are extremals. Note that along each trajectory arc (extremal) the Hilbert integral (54) reduces to the functional $\int f(x,y,y')dx$ evaluated along this arc. The conditions under which the converse statement is true, i.e., the conditions under which an extremal is a trajectory of the field will be given below.

<u>Construction of a field for the (n+1)-dimensional problem.</u>

<u>Definition 16.</u> A region D in the space of the variables (x,y) is said to be <u>simply covered</u> by an n-parameter family of extremals

$$y_i = \phi_i(x,b_1,\cdots,b_n) \qquad (i = 1,\cdots,n), \tag{58}$$

if through each point of D there passes a unique extremal of the family.

Suppose that there is given an n-parameter family of extremals (58) which simply covers a region D in the space of the variables (x,y). This means analytically that there exist single-valued functions

$$b_i = b_i(x,y_1,\cdots,y_n) \qquad (i = 1,\cdots,n),$$

on D which define a point set R in the space of the variables (x,b) and which the equations (58) transform into the region D. Let the following assumptions be made.

(a) R is a simply-connected region.

(59)[1] (b) The Jacobian $\dfrac{\partial(\phi_1,\dots,\phi_n)}{\partial(b_1,\dots,b_n)} \neq 0$ everywhere in R.

(c) The ϕ_i are of class C^2 in R.

Now set $v_i = f_{z_i}(x,y,y')$, $(i = 1,\dots,n)$, where the v_i can be expressed as functions of x and the b_i by means of (58). With these assumptions, it can be shown that an n-parameter family of extremals associated with the functional $\int_a^b f(x,y,y')dx$ is the family of all trajectories of the field for this functional if and only if all the quantities

$$\sum_{i=1}^{n}\left(\frac{\partial y_i}{\partial b_t}\frac{\partial v_i}{\partial b_s} - \frac{\partial y_i}{\partial b_s}\frac{\partial v_i}{\partial b_t}\right) \qquad (s,t = 1,\cdots,n) \tag{60}$$

expressed in terms of y_i and v_i vanish identically. The quantities in (60) are known as the <u>Lagrange brackets</u>.

---

[1]Condition (a) ensures that the well-known necessary condition for an integral to be independent of the path is also sufficient. Conditions (b) and (c) imply, according to the well-known theorem on implicit functions, that the solutions b_i $(i = 1,\dots,n)$ of (58) are of class C^2. The latter, in turn, guarantees condition (53a) in the definition of a field (Definition 14).

An n-parameter family of extremals (58) for which every Lagrange bracket vanishes identically is called a <u>Mayer family</u>. Thus, an n-parameter family that satisfies the conditions (59) generates a field if and only if it is a Mayer family. For $n = 1$, every one-parameter family of extremals is a Mayer family. It can be shown that the Lagrange brackets expressed in the form (60) are independent of x, so that in order to determine whether a given family is a Mayer family, it is sufficient to verify that the Lagrange bracket vanishes at only one point. An important application of this principle leads to the idea of <u>central fields</u>.

Let (x,y) be an arbitrary point of $D \subset G$, and let $C = (c, c_1, \ldots, c_n)$ be a point lying outside D. Suppose that all the extremals $y = \phi(x,b)$ of the functional pass through C, but that a unique extremal passes through (x,y). Then from (58) it follows that

(61) $$c_i = \phi_i(c,b) \qquad (i = 1,\ldots,n),$$

and so the right members of (61) do not contain the parameters $b_1,\ldots,b_n$ explicitly. Therefore

(62) $$\left.\frac{\partial y_i}{\partial b_k}\right|_{x=c} = \frac{\partial \phi_i(c,b)}{\partial b_k} = 0 \qquad (i,k = 1,\ldots,n),$$

which implies that all the Lagrange brackets vanish at $x = c$. Since these brackets are independent of x, (62) implies that they vanish for all x. Hence this family of extremals is a Mayer family. This result leads to the following definition.

Definition 17. Let a unique extremal of the family $y = \phi(x,b)$ pass through an arbitrary point of D and suppose that all extremals of the family intersect at $(c,c_1,\cdots,c_n)$. Then D is called a central field,[1] and $(c,c_1,\ldots,c_n)$ is called the vertex.

Central fields are sometimes referred to as improper fields to distinguish them from fields which have no vertex, known as proper fields. Evidently the pencils of extremals considered earlier are examples of central fields.[2]

In the next section it will be shown that any family of extremals which generates a field for the given functional may be used in the hypothesis of Theorem 5.

---

[1] Here, as with proper fields (q.v.), it is often convenient to apply the word 'field' to the collection of extremal curves rather than to the region D.

[2] The correct designation, of course, may depend on the region D covered by the family. For example, the family $y = C.\sin x$ in the plane, where $0 < C < \infty$, generates a proper field for $0 < x < \pi$, and an improper field for $0 \leq x < \pi$.

11. The E-function in (n+1) dimensions.

Definition 18. The E-function for the functional

$$\int_a^b f(x,y,y')dx = \int_a^b f(x,y_1,\dots,y_n;y_1',\dots,y_n')dx$$

is defined as the following function of (3n+1) variables.

$$(63)\quad E(x,y,z,q) = f(x,y,q) - f(x,y,z) - \sum_{i=1}^{n} (q_i-z_i)f_{z_i}(x,y,z).$$

According to the assumptions made earlier regarding the integrand f,[1] the E-function is a real-valued function of class C^2 defined on the direct product G X E_{2n}. Note that E = 0 when z = q. If f is regarded as a function of its last vector argument, then E is the difference between the value of f at the point q and the linear part of its Taylor expansion about the point z. Hence, an application of the mean-value theorem gives

$$(64)\qquad E(x,y,z,q) = \tfrac{1}{2}\sum_{i,k=1}^{n} (q_i-z_i)(q_k-z_k)f_{z_i z_k}(x,y,z+\theta(q-z))$$

where $0 < \theta < 1$.

Definition 19. If D is a field covering a region R which contains the extremal $y = \bar{y}(x)$, and has this extremal as one of its trajectories, then $\bar{y}$ is said to be embedded in D.

[1]See pp.18-19.

Now let $A(a,a_1,\ldots,a_n)$ and $B(b,b_1,\ldots,b_n)$ be two points in G and suppose that $y = \bar{y}(x)$ is an extremal for the given functional which joins A and B. The following theorem will be proved.

<u>Theorem 6.</u> Let $\bar{C}$: $y = \bar{y}(x)$ be an extremal which can be embedded in a field, and let R be a region containing $\bar{C}$ which is simply covered by the field. Suppose that throughout R the inequality

$$E(x,y,p(x,y),\ q) \geq 0 \qquad (a \leq x \leq b) \tag{65}$$

holds for arbitrary finite values of the vector q and for the slope function of the field p(x,y). Then the functional

$$\int_a^b f(x,y,y')dx$$

has a <u>strong minimum</u> at $y = \bar{y}(x)$. Obviously, if $E > 0$ for all $p(x,y) \neq q$, then $\bar{C}$ furnishes a proper minimum.

<u>Proof.</u> Let $\tilde{C}$: $y = \tilde{y}(x)$ be a piecewise-smooth curve lying in R, and let $\bar{C}$: $y = \bar{y}(x)$ denote an extremal. It is assumed that both of these curves satisfy the boundary conditions. Since the Hilbert integral taken along an extremal arc reduces to the given functional evaluated along that arc, it follows that

$$J(\bar{y}) = \int_{\bar{C}} \left[f(x,y,p) - \sum_{i=1}^{n} p_i f_{z_i}(x,y,p)\right]dx + \int_{\bar{C}} \sum_{i=1}^{n} f_{z_i}(x,y,p)dy_i \tag{66}$$

Because the value of the Hilbert integral is independent of the path of integration, the right-hand side of (66) may be evaluated along the path C. Therefore

$$(67)\qquad J(\bar{y}) = \int_a^b \Big[f(x,\tilde{y},p(x,\tilde{y})) - \sum_{i=1}^{n} \tilde{p}_i f_{z_i}(x,\tilde{y},p(x,\tilde{y})) \, dx + \int_a^b \sum_{i=1}^{n} f_{z_i}(x,\tilde{y},p(x,\tilde{y})) d\tilde{y}_i,$$

where $\tilde{p}_i = p_i(x,\tilde{y})$. Since $d\tilde{y}_i = \tilde{y}_i' dx$, (67) can be written

$$(68)\qquad J(\bar{y}) = \int_a^b \Big\{ f(x,\tilde{y},p(x,\tilde{y})) - \sum_{i=1}^{n} \big[p_i(x,y) - y_i' \big] f_{z_i}(x,y,p) \Big\} dx.$$

Also,

$$(69)\qquad J(\tilde{y}) = \int_a^b f(x,\tilde{y},\tilde{y}')dx$$

If (68) is subtracted from (69), the result is

$$(70)\qquad J(\tilde{y}) - J(\bar{y}) = \int_a^b \Big\{ f(x,\tilde{y},\tilde{y}') - f(x,\tilde{y},p(x,\tilde{y})) + \sum_{i=1}^{n} \big[p_i(x,\tilde{y}) - \tilde{y}_i' \big] f_{z_i}(x,\tilde{y},p(x,\tilde{y})) \Big\} dx,$$

i.e.,

$$(71)\qquad \Delta J = \int_a^b E(x,\tilde{y},p(x,\tilde{y}),\tilde{y}')dx.$$

According to the hypothesis (65), the value of the integral in (71) is non-negative, and hence $J(\tilde{y}) \geq J(\bar{y})$. Q.E.D.

Formula (71) is the (n+1)-dimensional extension of formula (50) (p.63), although these formulas are formally identical, since in the latter, $\bar{y}_t = \tilde{y}(t)$ and $\bar{y}_t'$ has the same meaning as $p(x,\tilde{y})$ in (71). Both formulas will be referred to as 'Weierstrass's theorem' when there is no ambiguity involved.

Theorem 6 shows that if the extremal arc $\bar{C}$ can be embedded in a field in which the E-function has the property (65), then its minimising properties are assured within the field, and no further tests need be applied.

Since, according to (64), the E-function can be written in the form

$$(72)\quad E(x,y,p,q) = \tfrac{1}{2}\sum_{i,k=1}^{n}(q_i-p_i)(q_k-p_k)f_{z_i z_k}(x,y,p+\theta(q-p)),$$

$(0 < \theta < 1)$, condition (65) may be replaced by the condition that at every point of some region R containing the extremal $y = \bar{y}(x)$, the matrix $\left[f_{z_i z_k}(x,y,z)\right]$ be non-negative definite for every finite z.[1]

[1]This is the analogue of the condition $f_{zz}(x,y,z) \geq 0$ (see p.65).

The Legendre Strong Condition.

One of the consequences of Weierstrass's theorem is the so-called Legendre Strong Condition. The precise statement is as follows.

Theorem 7. If an extremal can be embedded in a field, and if

$$\sum_{i,k=1}^{n} f_{y_i' y_k'}(x,\bar{y},\bar{y}')t_i t_k > 0 \qquad (a \leq x \leq b) \tag{73}$$

for arbitrary values of t_i that satisfy the condition

$$\sum_{i=1}^{n} t_i^2 \neq 0,$$

then the extremal furnishes a weak minimum for the functional $\int_a^b f(x,y,y')dx$.

In 1786 Legendre analysed the second variation from first principles, and used it to derive condition (73) for $n = 1$. He concluded that (73) alone was sufficient to guarantee a weak minimum, but his result was only partially correct. The hypothesis that $\bar{y}(x)$ can be embedded in a field is essential here.

Proof of Theorem 7. Let $y = \tilde{y}(x)$, $a \leq x \leq b$, be an arbitrary smooth curve that lies in a weak neighborhood of the curve

$y = \bar{y}(x)$ and satisfies the boundary conditions. The value of e which characterises the neighborhood is to be determined later. From (71) it follows that

$$(74) \qquad \Delta J = J(\tilde{y}) - J(\bar{y}) = \int_a^b E(x,\tilde{y},p(x,\tilde{y}),\tilde{y}')dx$$

and by (72)

$$J = \tfrac{1}{2}\int_a^b \sum_{i,k=1}^{n} (\tilde{y}_i' - p_i(x,\tilde{y}))(\tilde{y}_k' - p_k(x,\tilde{y}))f_{y_i' y_k'}(x,\tilde{y}(x),s(x))$$

where $s(x) = p(x,\tilde{y}) + \theta(\tilde{y}'(x) - p(x,\tilde{y}))$, $0 < \theta < 1$. Therefore, by a well-known property of quadratic forms

$$(75) \qquad \Delta J \geq \tfrac{1}{2}\int_a^b \sum_{i=1}^{n} (\tilde{y}_i' - p_i(x,\tilde{y}))^2 \lambda(x,\tilde{y}(x), s(x)),$$

where $\lambda(x,y,z)$ is the minimum on the hypersphere $\sum_{i=1}^{n} t_i^2 = 1$ of the real quadratic form

$$\sum_{i,k=1}^{n} f_{y_i' y_k'}(x,y,z)t_i t_k.$$

The function $\lambda(x,y,z)$ is continuous in all its arguments, and by hypothesis $\lambda(x,\bar{y},\bar{y}') \geq c > 0$, $a \leq x \leq b$, where c is a constant. Therefore, for every positive number in the interval (0,c), in particular for $\frac{1}{2}c$ say, there is a $d > 0$ such that $\lambda(x,\tilde{y}(x), s(x)) \geq \frac{1}{2}c$, $a \leq x \leq b$, whenever

$$\|\tilde{y}(x) - \bar{y}(x)\|_0 \leq d$$

$$\|s(x) - \bar{y}'(x)\|_0 \leq d \qquad (a \leq x \leq b).$$

Also, by the continuity of p(x,y), and the fact that $p(x,\bar{y}) = \bar{y}'(x)$, it follows that for the above d there exists a number $e: 0 < e \leq d$, such that

$$\| p(x,\tilde{y}) - \bar{y}'(x) \|_o \leq d$$

whenever

$$\| \tilde{y} - \bar{y} \|_o \leq e.$$

Let this e determine the weak neighborhood of $\bar{y}$ in which $\tilde{y}$ lies. Then it follows that

$$\| s(x) - \bar{y}'(x) \|_o = \| p(x,\tilde{y}) - \bar{y}'(x) + \theta(\tilde{y}' - p(x,\tilde{y}) \|_o$$

$$= \| p(x,\tilde{y}) - \bar{y}'(x) + \theta\bar{y}' - \theta\bar{y}' + \theta(\tilde{y}' - p(x,\tilde{y}) \|_o$$

$$= \| \tilde{p} - \bar{y}' - \theta\tilde{p} + \theta\bar{y}' + \theta\tilde{y}' - \theta\bar{y}' \|_o$$

$$= \| (1 - \theta)(\tilde{p} - \bar{y}') + \theta(\tilde{y}' - \bar{y}') \|_o$$

$$(1 - \theta)\| \tilde{p} - \bar{y}' \|_o + \theta \| \tilde{y}' - \bar{y}' \|_o$$

(by the triangular inequality)

$$\leq (1 - \theta)d + \theta e$$

$$= d - (d - e)\theta \leq d.$$

Therefore $\lambda(x, \tilde{y}(x), s(x)) \geq \frac{1}{2}c \qquad (a \leq x \leq b),$

and by (75)

$$\Delta J \geq c \int_a^b \sum_{i=1}^{n} (\tilde{y}_i' - p_i(x,\tilde{y}))^2 \geq 0 \qquad (a \leq x \leq b).$$

Hence

$$J(\tilde{y}) \geq J(\bar{y}). \qquad \text{Q.E.D.}$$

In contrast to (73), the inequality

$$\sum_{i,k=1}^{n} f_{y_i' y_k'}(x,\bar{y},\bar{y}')t_i t_k \geq 0 \qquad (a \leq x \leq b)$$

is called the Legendre weak condition, and is necessary for a weak minimum, being the (n+1)-dimensional extension of (41) in Theorem 4 (p.51). Its relation to the E-function will be discussed in the next section.

12. Further applications of the E-function.

It will now be shown that the conditions expressed by the inequalities (65) and (73) are necessary, in a weakened form, for minimising the functional $\int_a^b f(x,y,y')dx$. The first result is an extension of Theorem 3 (p.47) to (n+1) dimensions.

12.1 The E-function Necessary Condition in (n+1)-space.

Theorem 8. For an arbitrary finite vector q, the inequality

$$E(x,y,y',q) \geq 0 \tag{76}$$

must be satisfied at every point of an extremal

$$\bar{C}: \; y = \bar{y}(x) \qquad (a \leq x \leq b)$$

which furnishes the functional $\int_a^b f(x,y,y')dx$ with a strong minimum (in contrast to the hypothesis of Theorem 6, $E \geq 0$ only along the extremal, and not throughout a region containing it.).

Proof. Let $\bar{C}$ be represented by $y = \bar{y}(x)$. Let $(x_1,\bar{y}_1) = (x_1,\bar{y}(x_1))$ be any point of $\bar{C}$. It is assumed that $x_1 > a$. The case $x_1 = a$ will be considered separately.

Let $\tilde{C}$: $y = \tilde{y}(x)$ be a smooth curve defined on some open interval I containing x_1, which passes through $(x_1, \bar{y}_1)$, with arbitrary finite derivative $q = \tilde{y}'(x_1)$ at that point. The situation for $n = 1$ (or for one of the x-y_i planes) is illustrated in Fig. 11.

Now let t $(t \leq x_1)$[1] be an arbitrary parameter in I. A one-parameter family of curves C_t: $y = y(x,t)$ will be constructed, joining the initial point $A(a, \bar{y}(a))$ of $\bar{C}$ to the point T: $(t, \tilde{y}(t))$ and which reduces to $\bar{C}$ when $t = x_1$. If $\bar{C}$ furnishes a relative minimum or maximum, then it is assumed that t is sufficiently close to x_1 to ensure that C_t lies inside an arbitrarily small neighborhood of $\bar{C}$.

The family $y(x,t)$ can be constructed in a number of ways. One of the simplest is to define $y_i(x,t)$ for any x, $a \leq x \leq t$, according to the relation

$$\bar{y}_i(x) - y_i(x,t) = k(x - a) \qquad (i = 1,\ldots,n)$$

where k is a constant. For $x = t$, this definition gives

$$\bar{y}_i(t) - \tilde{y}_i(t) = k(t-a)$$

Therefore $k = \dfrac{\bar{y}_i(t) - \tilde{y}_i(t)}{t - a}$, and $y_i(x,t)$ can be written

[1]The case $t > x_1$ does not correspond to an admissible variation.

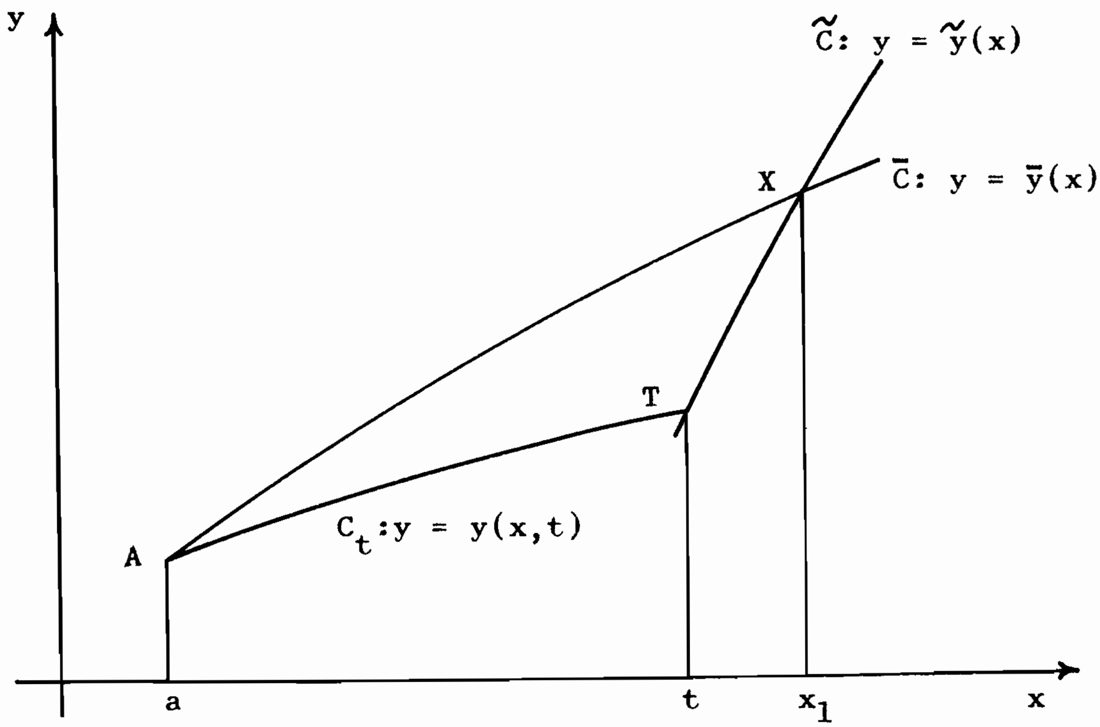


Fig. 11 Construction for the E-function Necessary Condition for the case $n = 1$ $(t \leq x_1)$.

$$(77) \qquad y_i(x,t) = \frac{x-a}{t-a}\left[\tilde{y}_i(t) - \bar{y}_i(t)\right] + \bar{y}_i(x) \qquad (i = 1,\ldots,n).$$

The assumption $t > x_1$, which will be needed later, leads formally to the same result. Now introduce the function

$$(78) \qquad J(t) = \int_a^t f(x,y,y_x)dx + \int_t^{x_1} f(x,\tilde{y},\tilde{y}')dx - \int_a^{x_1} f(x,\bar{y},\bar{y}')dx,$$

i.e., $J(t)$ is the value of the integral of f taken along the varied path minus its value taken along the extremal (in Fig.11 these paths are ATX and AX respectively). Since $\bar{C}$ is a minimising arc, it follows that $J'(x_1) \leq 0$.[1] Therefore it is sufficient for the proof of the theorem to show that

(79) $$J'(x_1) = -E(x_1, \bar{y}(x_1), \bar{y}'(x_1), q).$$

Now $y_i(x,t)$ is of class C^1, $a \leq x \leq t$, $a \leq t \leq x_1$, $i = 1,\ldots,n$. From the assumptions concerning f and the curves $\bar{C}$ and $\tilde{C}$, it follows that

$$\frac{\partial f(x,y(x,t),y'(x,t))}{\partial t} \qquad \begin{array}{l} a \leq x \leq t, \\ a \leq t \leq x_1 \end{array}$$

is a continuous function of x and t. The rule for differentiation of a definite integral with respect to a parameter may now be applied. Differentiation of (78) with respect to t gives

(80) $$J'(t) = f(t,y(t,t),y_x(t,t)) - f(t,\tilde{y}(t),\tilde{y}'(t))$$

$$+ \int_a^t \sum_{i=1}^n \left[f_{y_i}(x,t) y_{it}(x,t) + f_{y_i'}(x,t) y_{itx}(x,t) \right] dx,$$

where $f_{y_i}(x,t)$, $f_{y_i'}(x,t)$ denote $\frac{\partial}{\partial y_i} f(x,y(x,t),y_x(x,t))$,

---

[1]It does not follow, however, that J has a minimum for $t = x_1$. If the functional corresponds to arc length, for example, $J(t)$ will continue to decrease as t passes to the right of x_1. Therefore it cannot be asserted that $J'(x_1) = 0$.

$\dfrac{\partial}{\partial y_{ix}} f(x,y(x,t),y\ (x,t))$ respectively. Therefore

$$J'(x_1) = f(x_1,y(x_1,x_1),y_x(x_1,x_1)) - f(x_1,\tilde{y}(x_1),\tilde{y}'(x_1))$$
$$+ \int_a^{x_1} \sum_{i=1}^{n} y_{it}(x,x_1)f_{y_i}(x,x_1)dx$$

$$+ \int_a^{x_1} \sum_{i=1}^{n} y_{itx}(x,x_1)f_{y_i'}(x,x_1)dx$$

Since $y_i(x_1,x_1) = \bar{y}_i(x_1)$ and $y_{ix}(x_1,x_1) = \bar{y}_i'(x_1)$, the first term on the right equals $f(x_1,\bar{y}(x_1),\bar{y}'(x_1))$. The term involving y_{itx} may be integrated by parts. In view of the fact that $\bar{y}(x)$ satisfies Euler's equations, the result is

$$(81)\quad J'(x_1) = f(x_1,\bar{y}_1,\bar{y}'(x_1)) - f(x_1,\bar{y}_1,\tilde{y}'(x_1))$$
$$+ \sum_{i=1}^{n} y_{it}(x,x\ )f_{y_i'}(x,x\ )\Big|_a^{x_1}$$

since $\tilde{y}(x_1) = \bar{y}(x_1)$. From the identities

$$y_i(a,t) = \bar{y}_i(a),$$
$$y_i(t,t) = \tilde{y}_i(t) \qquad (i = 1,\ldots,n),$$

it follows, upon differentiating with respect to t and putting $t = x_1$ that

$$y_{it}(a,x_1) = 0, \qquad p_i + y_{it}(x_1,x_1) = q_i,$$

where $p_i = y_{ix}(x_1,x_1) = \bar{y}_i'(x_1)$ and $q_i = \tilde{y}_i'(x_1)$.

Application of these results to (81) gives

$$(82)\quad J'(x_1) = f(x_1,\bar{y}_1,p) - f(x_1,\bar{y}_1,q) + \sum_{i=1}^{n} (q_i-p_i) f_{y_i'}(x_1,\bar{y}_1,p)$$

$$= - E(x_1,\bar{y}_1,p,q).$$

The theorem now follows from the condition $J'(x_1) \leq 0$. Q.E.D.

The case $x_1 = a$ mentioned earlier (p.84) requires only a few simple changes. Let $\tilde{C}$: $y = \tilde{y}(x)$ be a smooth curve defined on some open interval I containing the number $x = a$, which passes through $A(a,\bar{y}(a))$ with arbitrary finite derivative $q = \tilde{y}'(a)$ at that point. Now let x_1^* be the abscissa of any point on $\bar{C}$ distinct from A, and let t be an arbitrary parameter in I ($t > a$). As in the previous case, a one-parameter family of curves C_t: $y = y(x,t)$ may be constructed which join the point $(x_1,\bar{y}(x_1))$ to the point $(t,\tilde{y}(t))$, and reduce to $\bar{C}$ when $t = a$. The situation for $n = 1$ is shown in Fig. 12. The geometrical condition which determines the

*This symbol has a different meaning here from the one on p.84.

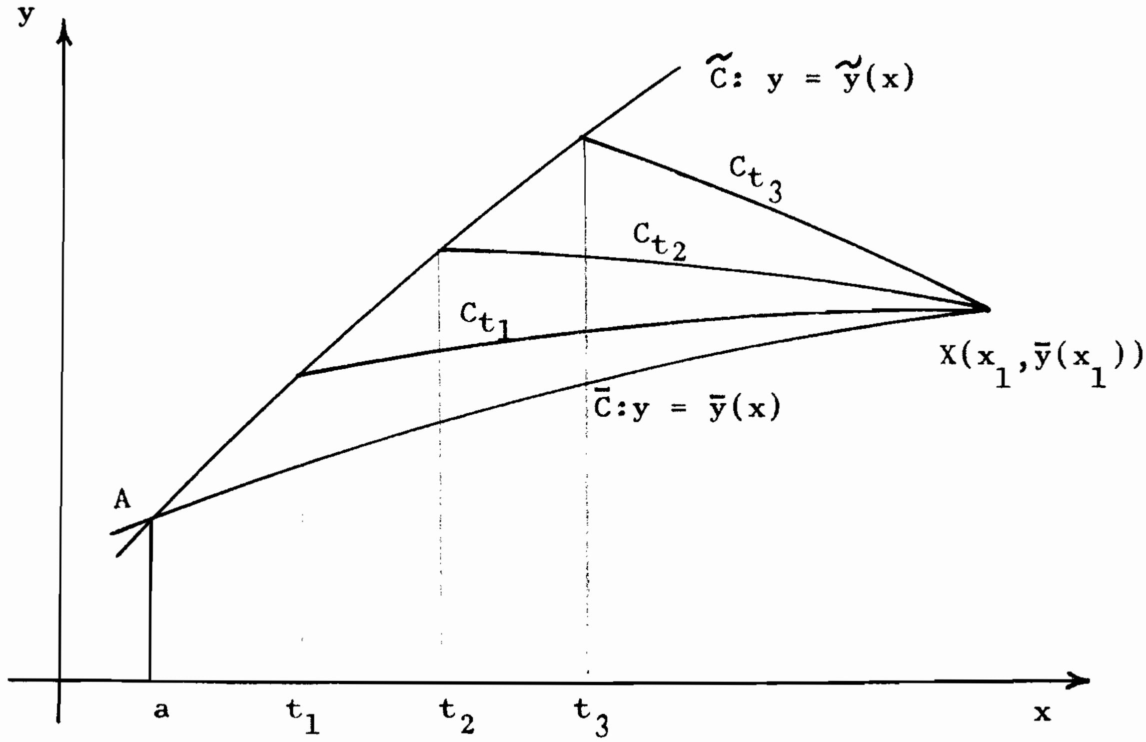


Fig. 12 Construction for the proof of the E-function Necessary Condition to include the point A on the extremal $\bar{C}$. The numbers t_1, t_2, t_3 are values of the parameter t, and the curves marked $C_{t_1}, C_{t_2}, C_{t_3}$ are typical members of the family $y = y(x,t)$.

family C_t: $y = y(x,t)$ is the same as in the previous case, and so the equation of the family is formally identical to (77). Let T be the point of $\tilde{C}$ with coordinates $(t, \tilde{y}(t))$. Then the difference $J_{ATX} - J_{AX}$ is given by

$$(83)\quad J(t) = \int_a^t f(x, \tilde{y}(x), \tilde{y}'(x))dx + \int_t^{x_1} f(x, y(x,t), y_x(x,t))dx - \int_a^{x_1} \bar{f}dx$$

Since $\bar{C}$ furnishes a minimum, it is obvious from the construction of the variation ATX that J(t) increases as t increases. Consequently $J'(a) \geq 0$, and it is sufficient to show that $J'(a) = E(a,\bar{y}(a),p,q)$, where $p = \bar{y}'(a)$, $q = \tilde{y}'(a)$. Differentiation of (83) gives

$$J'(t) = f(t,\tilde{y}(t),\tilde{y}'(t)) - f(t,y(t,t),y_x(t,t))$$

$$+ \int_t^{x_1} \sum_{i=1}^{n} f_{y_i}(x,y(x,t),y_x(x,t)).y_{it}(x,t)dx$$

$$+ \int_t^{x_1} \sum_{i=1}^{n} f_{y_i'}(x,y(x,t),y_x(x,t)).y_{itx}(x,t)dx.$$

Therefore

$$(84)\quad J'(a) = f(a,\bar{y}(a),q) - f(a,\bar{y}(a),p)$$

$$+ \sum_{i=1}^{n} f_{y_i'}(x_1,y(x_1,a),y_x(x_1,a)).y_{it}(x_1,a)$$

$$- \sum_{i=1}^{n} f_{y_i'}(a,y(a,a),y_x(a,a)).y_{it}(a,a)$$

since, when $t = a$, $C_a = \bar{C}$ which satisfies Euler's equations.

As a result, the terms in the expression for $J'(t)$ which involve integrals vanish after integration by parts as in the previous case. From the identities

$$y_i(x_1,t) = \bar{y}_i(x_1),$$

$$y_i(t,t) = \tilde{y}_i(t),$$

it follows, upon differentiating with respect to t that

$$y_{it}(x_1,t) = 0$$

and

$$y_{ix}(t,t) + y_{it}(t,t) = \tilde{y}_i'(t).$$

For $t = a$, this yields

$$y_{it}(x_1,a) = 0$$

and

$$y_{it}(a,a) = \tilde{y}_i'(a) - y_{ix}(a,a)$$

$$= q_i - p_i.$$

Since $y(a,a) = \bar{y}(a)$, the summation in (84) reduces to

$$-\sum_{i=1}^{n} (q_i - p_i) f_{y_i'}(a,\bar{y}(a),p).$$

Combination of these results shows that

$$J'(a) = f(a,\bar{y}(a),q) - f(a,\bar{y}(a),p) - \sum_{i=1}^{n} (q_i - p_i) f_{y_i'}(a,\bar{y}(a),p).$$

Therefore $J'(a) = E(a,\bar{y}(a),p,q)$. Q.E.D.

12.2 The Legendre Weak Condition.

If the hypothesis of Theorem 8 is modified by allowing $\bar{y}(x)$ to afford a weak minimum to the given functional, then the conclusion must hold in its weak form, i.e., $E(x,y,y',q) \geq 0$ for (x,y,y') on $\bar{C}$: $y = \bar{y}(x)$ and for q within a suitable strong neighborhood of $\bar{y}'(x)$. As a consequence of this weak condition, it is possible to derive the Legendre Weak Condition given by the following theorem.

Theorem 9. Let the functional $\int_a^b f(x,y,y')dx$ have a weak minimum at $y = \bar{y}(x)$, $a \leq x \leq b$. Then the inequality

$$(85) \qquad \sum_{i,k=1}^{n} f_{y_i' y_k'}(x,y,y') z_i z_k \geq 0$$

is satisfied at all points of the extremal $y = \bar{y}(x)$ for arbitrary finite values of z_i $(i = 1,\ldots,n)$.

Proof. Define $\phi(e) = E(x,\bar{y},\bar{y}',\bar{y}'+ez)$, where the fourth argument has components $\bar{y}_i(x) + ez_i$ for arbitrary finite values z_i $(i = 1,\ldots,n)$ and $e > 0$ is arbitrarily small. Note that $\phi(0) = 0$. Now

$$\phi'(e) = \sum_{i=1}^{n} z_i f_{y_i'}(x,\bar{y},\bar{y}'+ez) - \sum_{i=1}^{n} z_i f_{y_i}(x,\bar{y},\bar{y}')$$

and $\phi''(e) = \sum_{i,k=1}^{n} f_{y_i'y_k'}(x,\bar{y},\bar{y}'+ez)z_i z_k$. Obviously $\phi''(0)$ is equal to the left member of (85). From the E-function condition in its weak form it follows that $\phi(e)$ has a relative minimum when $e = 0$. Therefore $\phi''(0) \geq 0$. Q.E.D.

12.3 The Second Corner Condition.

It was shown earlier (p.52) that the E-function vanishes at the corner of an extremal arc in two dimensions, and as a consequence the second corner condition (24) was proved for the case $n = 1$. By means of the E-function and a similar argument, it is now possible to prove the relation (24) for all n. First, it is necessary to reconstruct Fig. 11 and take $t > x_1$. The result is shown in Fig. 13.

Now consider

$$(86) \qquad J(t) = \int_{C_t} f.dx - \left(\int_{\bar{C}} f.dx + \int_{\tilde{C}} f.dx\right).$$

Formally, this leads as before to the result $J'(x_1) = -E$. Now let $y = \bar{y}_1(x)$ and $y = \bar{y}_2(x)$ be smooth curves in G which are defined on $[a,b]$. Suppose that $y = \bar{y}(x)$ is an extremal with a corner at $x = c$ which coincides with $\bar{y}_1$ on $[a,c]$

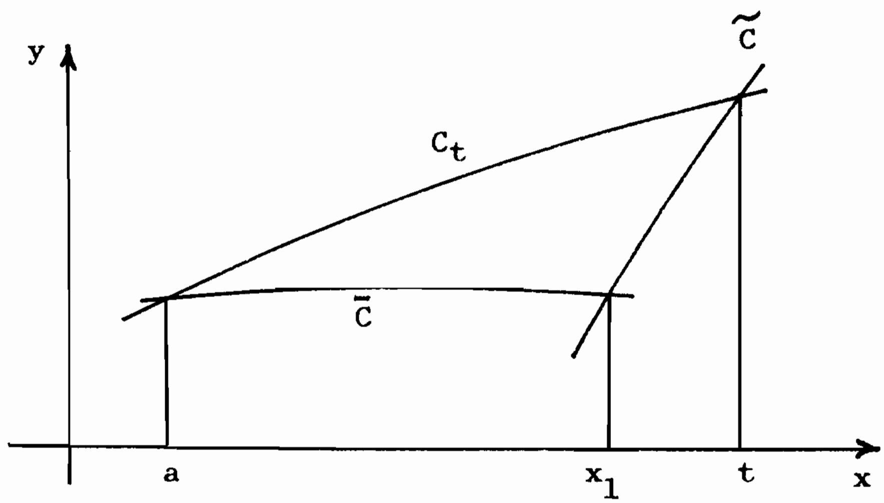


Fig. 13 Construction used in the proof of the second corner condition (24).

and with $\bar{y}_2$ on $[c,b]$. It is assumed that $\bar{y}(x)$ is a minimising function. Consider the variations

$$(87) \qquad y(x,t) = \frac{x-a}{t-a}\left[\bar{y}_2(t) - \bar{y}_1(t)\right] + \bar{y}_1(x),$$

where t is a parameter which may be taken arbitrarily close to c. In contrast to the situation encountered in the proof of Theorem 8, both $t \geq c$ and $t < c$ represent admissible variations.[1] The usual two-dimensional illustration is given in Fig. 14. By hypothesis, there exists a neighborhood N

---

[1] cf. footnote, p.85.

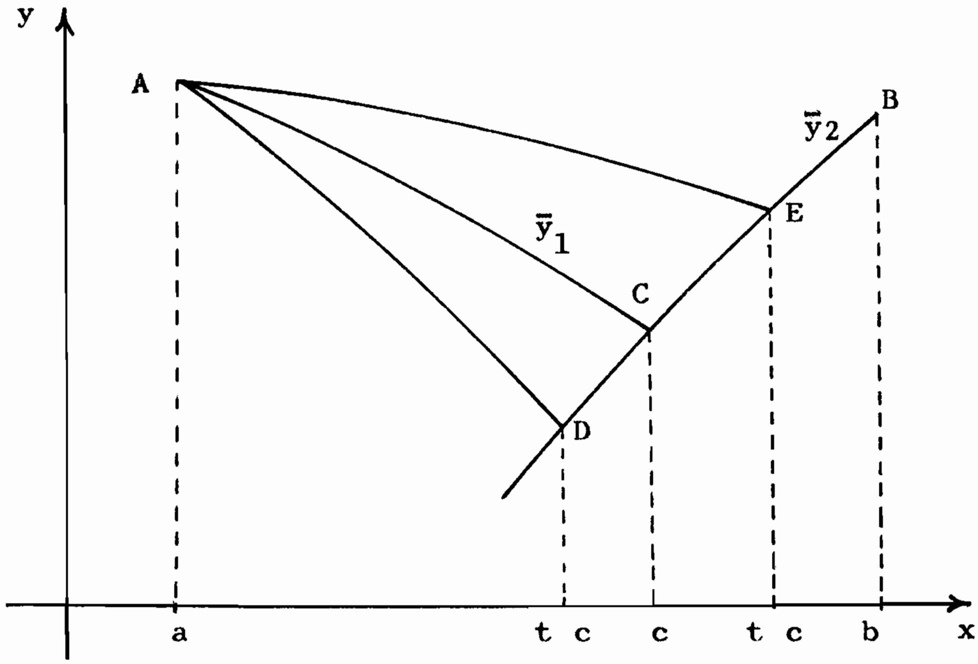


Fig. 14 Variations of broken extremal ACB used in the proof of the second corner condition.

of $\bar{y}(x)$ such that the given functional has an absolute minimum for $y = \bar{y}(x)$ with respect to N.[1] Select $e > 0$ so that the interval $(c-e, c+e)$ is small enough to ensure that the variations (87) lie inside N. If the integration along $\tilde{C}$ in (86) is taken with lower limit = x_1 and upper

---

[1] cf. Definition 10, p.22.

limit = t, then J(t) so defined represents, for all values of t, the increase in the value of the functional when the path ACB is varied according to (87). It then follows that

$$J(t) \text{ is non-increasing in } (c-e,c],$$
$$J(t) \text{ is non-decreasing in } [c,c+e),$$

as t increases. The last two statements imply $J'(c) \leq 0$ and $J'(c) \geq 0$ respectively. Hence $J'(c) = 0$. Therefore, by (79)

$$E(c,\bar{y}_1(c),\bar{y}_1'(c),\bar{y}_2'(c)) = 0.$$

By definition of $\bar{y}_1$ and $\bar{y}_2$, the last equation can be written using left- and right-hand limits. The result is

$$E(c,\bar{y}(c), \bar{y}'(c-0),\bar{y}'(c+0)) = 0$$

which on account of (23) is equivalent to (24).

13. Applications of the E-function Sufficient Condition.

To illustrate the foregoing theory, it is appropriate to consider in greater detail some of the classical problems that became famous during the early years of the subject. By means of the E-function criterion of Theorem 6, it can be shown that all of them except Newton's problem have strong solutions.

Although these problems can generally be stated in terms of functionals in ordinary form, the solution of Euler's equation in a number of cases requires the use of parameters. The parametric form of the problem has an elaborate theory of its own, due mainly to Weierstrass, which cannot be described here. In the two problems where parameters intrude, however, the equations can be reduced to ordinary form by one means or another. The brachistochrone is the one instance where difficulties arise, particularly in the verification of conditions (59b) and (59c). Fortunately, the essential properties in this case can be deduced by arguments which avoid the use of those conditions. The other examples present no difficulty in this respect.

Since there are no multiply-connected regions involved, it will not be necessary to check (59a). Also, condition (59c) holds in all cases except the brachistochrone. In establishing the existence of a field, therefore, (59b) is the only condition that needs direct verification. Secondly, every one-parameter family in two-dimensions is a Mayer family. Therefore it is necessary only in the first example to show that the Lagrange bracket (60) vanishes.

The examples to be discussed are as follows:

1. The shortest distance between two points.
2. The Isoperimetric Problem.
3. The Brachistochrone.
4. $J(y) = \int_0^a (6y'^2 - y'^4 + yy')dx$, $y(0) = 0$, $y(a) = b$, $a > 0$.

Although the first example, in some shape or form, is probably as old as mankind, a rigorous statement of the problem dates from comparatively recent times. Moreover, the arc-length integral suggests an elegant generalisation which neatly illustrates the theory for (n+1) dimensions.

13.1 The shortest distance between two points.

Given two points $A(a,a_1,\ldots,a_n)$ and $B(b,b_1,\ldots,b_n)$ in (n+1)-dimensional Euclidean space, find the curve $y = \bar{y}(x)$ such that the functional

$$\int_a^b \left[1 + \sum_{i=1}^{n} y_i'^2\right]^{\frac{1}{2}} dx \qquad \left(y_i' = \frac{dy_i}{dx}\right)$$

is a minimum.

Solution. Since f does not contain y_i explicitly ($i = 1,\dots,n$), the Euler equations (20) reduce to

$$\frac{d}{dx}(f_{y_i'}) = 0 \qquad (i = 1,\dots,n),$$

or

$$f_{y_i'} = w_i \qquad (i = 1,\dots,n),$$

where the w_i are constants. More precisely,

$$y_i' = w_i \left[1 + \sum_{i=1}^{n} y_i'^2\right]^{\frac{1}{2}} \qquad (i = 1,\dots,n). \tag{88}$$

Squaring both sides of (88) gives

$$\sum_{k=1}^{n} (\delta_{ik} - w_i^2).y_k'^2 = w_i^2 \qquad (i = 1,\dots,n), \tag{89}$$

where δ_{ki} is the Kronecker delta. Equation (89) represents n equations in the n unknowns $y_k'^2$. The matrix (a_{ij}) of coefficients is

$$A = \begin{bmatrix} 1-w_1^2 & -w_1^2 & -w_1^2 & \cdot & \cdot & \cdot & -w_1^2 \\ -w_2^2 & 1-w_2^2 & -w_2^2 & \cdot & \cdot & \cdot & -w_2^2 \\ -w_3^2 & -w_3^2 & 1-w_3^2 & \cdot & \cdot & \cdot & -w_3^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -w_n^2 & -w_n^2 & -w_n^2 & & & & 1-w_n^2 \end{bmatrix}$$

The matrix B obtained from A by adding to the first row all the other rows is

$$B = \begin{bmatrix} 1-S & 1-S & \cdot & \cdot & \cdot & \cdot & 1-S \\ -w_2^2 & 1-w_2^2 & \cdot & \cdot & \cdot & \cdot & -w_2^2 \\ -w_3^2 & -w_3^2 & \cdot & \cdot & \cdot & \cdot & -w_3^2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -w_n^2 & -w_n^2 & \cdot & \cdot & \cdot & \cdot & 1-w_n^2 \end{bmatrix}$$

where $S = \sum_{i=1}^{n} w_i^2$ and $|B| = |A|$. The matrix C obtained from B by subtracting the first column from all the other columns is

$$C = \begin{bmatrix} 1-S & 0 & 0 & \cdot & \cdot & \cdot & 0 \\ -w_2^2 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ -w_3^2 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -w_n^2 & 0 & 0 & \cdot & \cdot & \cdot & 1 \end{bmatrix}$$

where
$$c_{11} = 1 - \sum_{i=1}^{n} w_i^2$$

$$c_{1j} = 0, \ j = 2,\ldots,n,$$

$$c_{i1} = -w_i^2, \ i = 2,\ldots,n,$$

$$c_{ij} = \delta_{ij}, i,j = 2,\ldots,n,$$

and $|C| = |B| = |A|$. Expansion of the determinant $|C|$ by the elements and cofactors of the first row shows that

$$|C| = c_{11} = 1 - \sum_{i=1}^{n} w_i^2 = |A|.$$

From (88) it follows that

$$\sum_{i=1}^{n} w_i^2 = \frac{\sum_{i=1}^{n} y_i'^2}{\sqrt{1 + \sum_{i=1}^{n} y_i'^2}} < 1.$$

Therefore $|A| > 0$ and the system of equations (89) has a unique solution. The inverse matrix A^{-1} has elements a_{ij}^{-1} where

$$a_{ij}^{-1} = \frac{w_i^2 + \delta_{ij}(1-S)}{1 - S}.$$

The last equation may easily be verified by direct evaluation of

$$\sum_{k=1}^{n} a^{-1}_{ik} a_{kj} .$$

Multiplication of (89) by A^{-1} yields

$$y_i'^2 = \frac{w_i^2}{1 - S} \qquad (i = 1, \ldots, n)$$

or, taking the positive square root on both sides

(90) $$y_i' = \frac{w_i}{\sqrt{1 - S}} \qquad (i = 1, \ldots, n)$$

The primitive of (90) may be written in the conventional form

(91) $$y_i = c_i x + d_i \qquad (i = 1, \ldots, n)$$

where $c_i = w_i(1-S)^{-\frac{1}{2}}$, $S = \sum_{i=1}^{n} w_i^2$. The 2n constants c_i, d_i $(i = 1, \ldots, n)$ are determined by the requirement that the extremal shall pass through $A(a, a_1, \ldots, a_n)$ and $B(b, b_1, \ldots, b_n)$. The complete solution is

$$\bar{C}: \bar{y}_i = \frac{b_i - a_i}{b - a} x + \frac{a_i b - a b_i}{b - a} .$$

The family of extremals

$$y_i = m_i x + d_i = y_i(x, d_i) \tag{92}$$

where m_i denotes $(b_i - a_i)/(b-a)$ contains $\bar{C}$ for $d_i = \dfrac{a_i b - ab_i}{b - a}$.

To show that this family generates a field for the given functional, it is necessary to verify that condition (59b) is satisfied and the Lagrange bracket (60) is identically zero. The first requirement is met because

$$\frac{\partial(y_1, \ldots, y_n)}{\partial(d_1, \ldots, d_n)} = \left[\delta_{ij}\right] \neq 0.$$

To verify the second condition, note that $\dfrac{\partial v_i}{\partial d_j} = 0$ for all i,j.

Therefore (60) reduces to zero, and the family (92) is a Mayer family which generates a field for the given functional. The components p_i of the slope function of the field are given by

$$p_i(x,y) = m_i = \frac{b_i - a_i}{b - a} \qquad (i = 1, \ldots, n).$$

<u>The E-function test.</u>

According to (63)

$$E(x,y,p,q) = \left[1 + \sum_{i=1}^{n} q_i^2\right]^{\frac{1}{2}} - \left[1 + \sum_{i=1}^{n} m_i^2\right]^{\frac{1}{2}} - \frac{\sum_{i=1}^{n} q_i - m_i) m_i}{\left[1 + \sum_{i=1}^{n} m_i^2\right]^{\frac{1}{2}}}$$

$$= \frac{\left[1+\sum_{i=1}^{n} q_i^2\right]^{\frac{1}{2}}\left[1+\sum_{i=1}^{n} m_i^2\right]^{\frac{1}{2}} - 1 - \sum_{i=1}^{n} m_i^2 + \sum_{i=1}^{n} m_i^2 - \sum_{i=1}^{n} m_i q_i}{\left[1+\sum_{i=1}^{n} m_i^2\right]^{\frac{1}{2}}}$$

$$\frac{\left[1+\sum_{i=1}^{n} q_i^2\right]^{\frac{1}{2}}\left[1+\sum_{i=1}^{n} m_i^2\right]^{\frac{1}{2}} - \left[1+\sum_{i=1}^{n} m_i q_i\right]}{\left[1+\sum_{i=1}^{n} m_i^2\right]^{\frac{1}{2}}}.$$

By the Cauchy-Schwartz inequality applied to the sets of numbers $(1,q_1,\ldots,q_n)$ and $(1,m_1,\ldots,m_n)$ it follows that

$$E(x,y,p,q) \geq 0$$

and hence (65) is satisfied. Therefore the curve $\bar{C}$ whose equation is given above furnishes a strong minimum. For $n = 1, 2$, this result yields the familiar equations of a straight line, the numbers $(1,m_1,m_2)$ being in the latter case the direction numbers of the line.

13.2 The Isoperimetric Problem.

In its narrow sense, the isoperimetric problem may be stated as follows: among all closed curves of a given length L, find the one which encloses a maximum area. The answer to this problem and its three-dimensional analogue was known in classical times, but only modern methods of analysis have yielded a rigorous proof.

In modern terminology, the problem is that of finding the maximum of the functional in parametric form

$$\frac{1}{2}\int_{t_1}^{t_2}(x\dot{y} - y\dot{x})dt$$

subject to the side condition

$$\int_{t_1}^{t_2}\sqrt{\dot{x}^2 + \dot{y}^2}\,dt = L$$

where x(t) and y(t) are of class C^1 (or D^1), and $x(t_1) = x(t_2)$, $y(t_1) = y(t_2)$. In order to apply the theory of functionals in ordinary form, the problem must be reduced to one which has no side condition.

First, the length of a closed curve will not vary if each arc of it is replaced by another arc symmetric to it with respect to the common chord. Therefore, the desired curve must be convex, and if two points divide it into two arcs of equal length[1], the areas of the figures bounded by each of these arcs and their common chord must be equal.

Thus, the isoperimetric problem reduces to this: of all arcs of length $\frac{1}{2}L$ that lie in the upper half-plane and have their end-points A and B on the x-axis, find the one that encloses the greatest area with the x-axis if the end-point A is given (say $x = 0$), and B lies to the right of A.

Let s denote the arc length measured along the curve from the point $x = 0$, and consider admissible curves given by the equations

$$x = x(s), \quad y = y(s), \qquad (0 \leq s \leq \tfrac{1}{2}L)$$

in which the functions $x(s)$ and $y(s)$ are connected by the relation

(93) $$x'^2 + y'^2 = 1$$

and satisfy the conditions

(94) $$x(0) = 0, \; y(0) = 0, \; y(\tfrac{1}{2}L) = 0.$$

---

[1] It follows from the assumptions that the curve is rectifiable.

The functional to be maximised is

(95) $$J = \int_0^{\frac{1}{2}L} y(s)x'(s)ds, \qquad y \geq 0.$$

According to (93), it is possible to write (95) in the form

(96) $$J(y) = \int_0^{\frac{1}{2}L} y(s)\sqrt{1 - y'^2(s)}\, ds$$

with boundary conditions

(97) $$y(0) = y(\tfrac{1}{2}L) = 0.$$

Clearly (96) is the equation of a functional in ordinary form. Since x does not occur explicitly in the integrand, it follows from Corollary 2.2b (p.32) that Euler's equation has a first integral

(98) $$f - y'f_{y'} = C \qquad (y' = \frac{dy}{ds}),$$

where C is a constant. For this problem (98) implies

$$y = C\sqrt{1 - y'^2}$$

which reduces to the elementary separable equation

$$\frac{dy}{\sqrt{C^2 - y^2}} = \frac{ds}{C} \qquad (C \neq 0),$$

whose solution is

$$y = C \sin \frac{s-s_o}{C} \qquad C \neq 0)$$

where s_o is a constant of integration. The equation of the pencil of extremals passing through the origin is

$$y = C \sin \frac{s}{C}$$

(see Fig. 15). For $C = L/2\pi$ this pencil includes the extremal $\bar{C}$ satisfying the boundary conditions:

$$(99) \qquad \bar{C}\colon\ y = \frac{L}{2\pi} \sin \frac{2\pi s}{L}$$

The family of curves

$$(100) \qquad y = C \cos \left(\frac{s}{C} - \frac{L}{4C}\right) \qquad |x-\tfrac{1}{4}L| \leq \frac{\pi C}{2}$$

forms a simple covering of any finite region D which contains the extremal $\bar{C}$ and is bounded below by the x-axis, provided the parameter C is taken large enough. Since n = 1, the family (100) is a Mayer family. There remains only the verification of (59b). Partial differentiation of (100) with respect to C gives

$$y_C = \cos \theta + \theta \sin \theta$$

where $\theta = (s - \tfrac{1}{4}L)/C$. In view of the range of s, it is sufficient

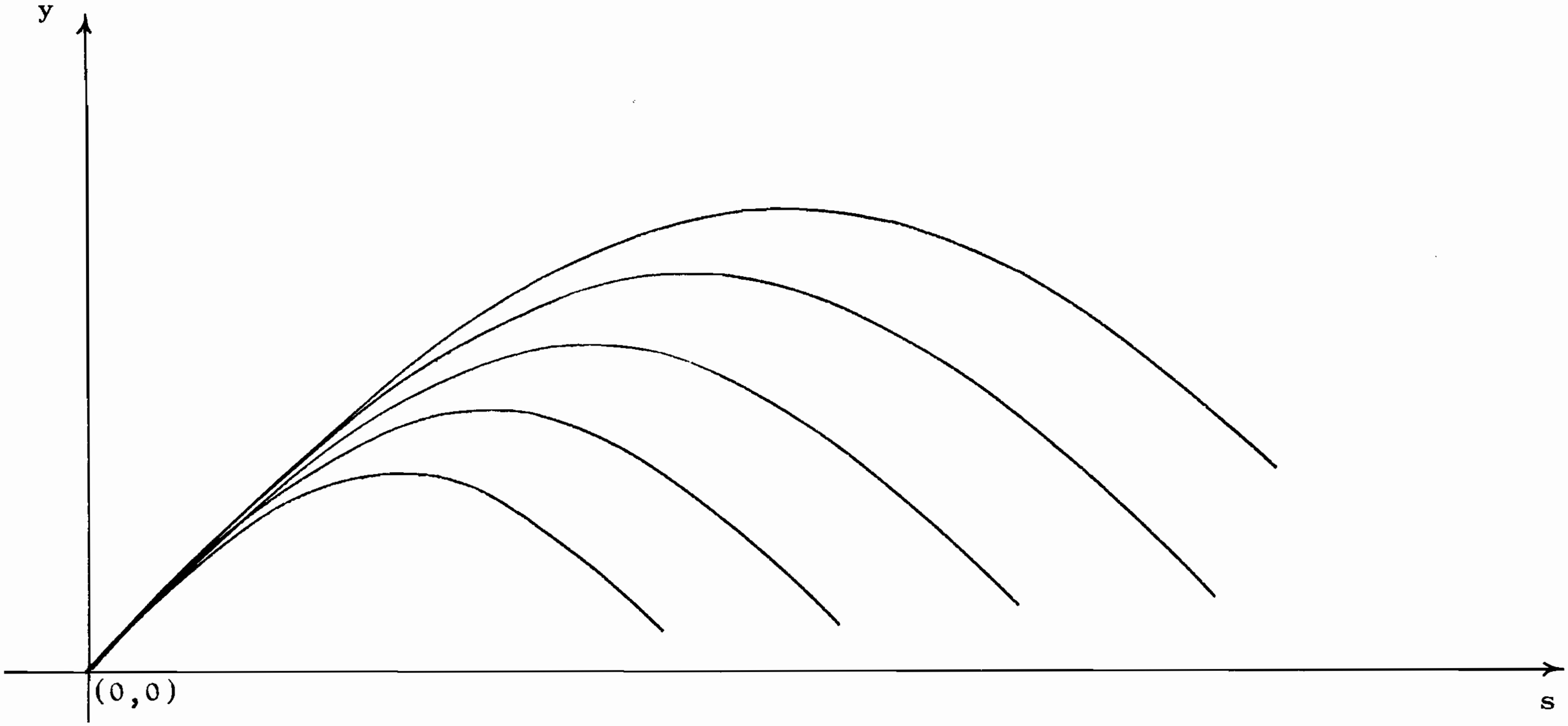


Fig. 15 Pencil of extremals $y = C \sin (s/C)$ for the isoperimetric problem containing the solution curve for $C = L/2\pi$.

to note that the function $f(\theta) = \cos\theta + \theta\sin\theta \neq 0$ in the interval $[-\pi/2, \pi/2]$. Therefore (59b) holds and the family (100) forms a proper field for the functional (96) (Fig.16).

Existence of the Hilbert integral.

The Hilbert integral H for the functional (96) is

(101) $$H = \int \frac{ydx - pydy}{\sqrt{1 - p^2}} dx$$

where $p = p(x,y)$ is the slope function of the field. According to (100), $p = \pm 1$ at the end-points of the extremals, and therefore (101) is an improper integral. Clearly, if H diverges, the E-function test is meaningless and Theorem 6 cannot be applied. Formally, H is exact, and if it exists, has the value $J(\bar{y})$ when the path of integration joins the points A and B (q.v.). Exactness, however, is not sufficient for integrability.[1] In order to show that H exists, new coordinates will be introduced to remove the singularity at the end-points.

To this end, let a new y-axis be introduced for the family (100) at $s = \frac{1}{4}L$. Referred to this new system, equations (100) become

(102) $$y = C\cos(s/C) \qquad |s| \leq \tfrac{1}{2}\pi C.$$

[1]The difficulty arises because the integrand f is not of class C^2

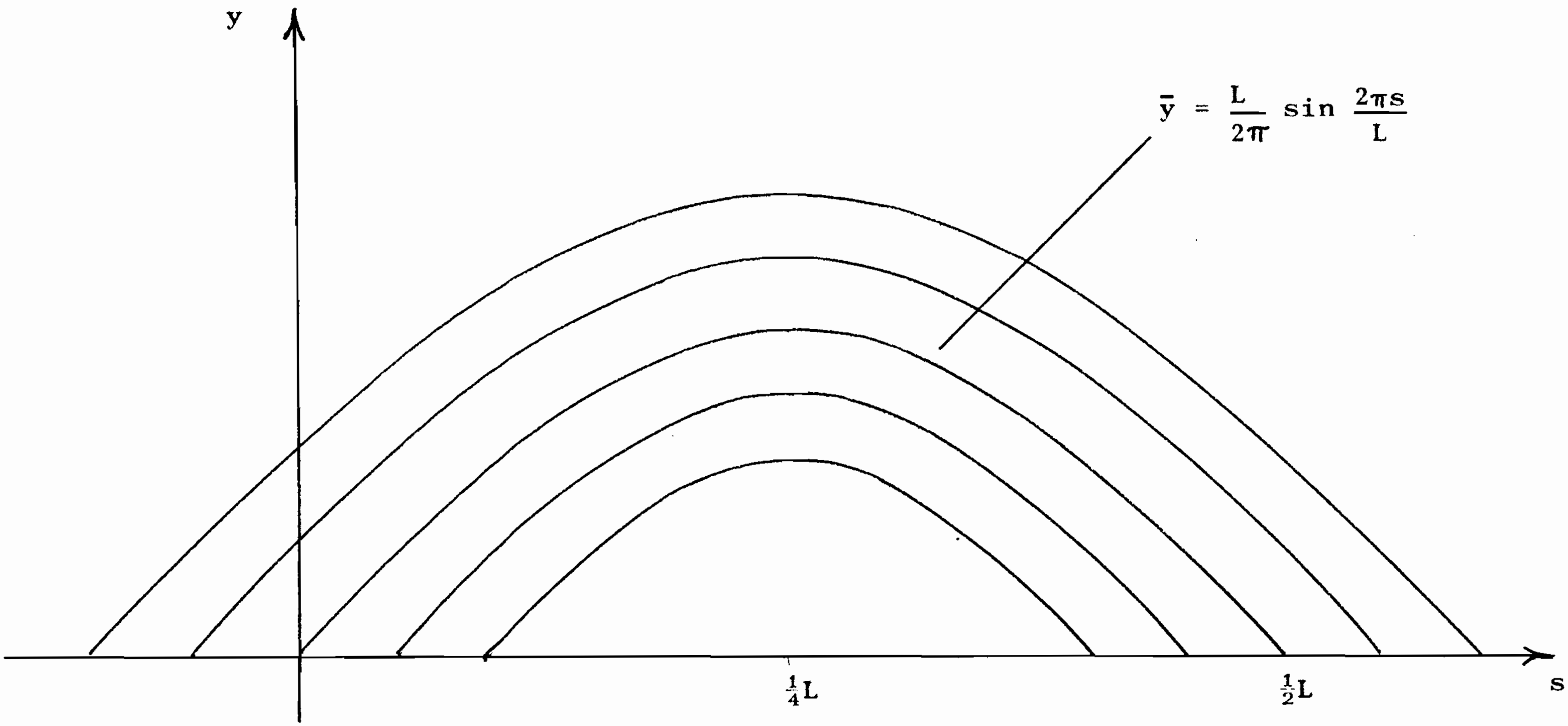


Fig.16 Portion of proper field for the isoperimetric problem. The field equation is $y = C \cos (s/C - L/4C)$ where C is the field parameter, $0 < C < \infty$, $|s-\frac{1}{4}L| \leq \frac{1}{2}\pi C$. The field contains the required extremal for $C = L/2\pi$.

Now define coordinates (r,t) by the relations

$$(103) \qquad y = r \cos t \qquad |t| \leq \tfrac{1}{2}\pi$$
$$s = rt \qquad 0 < r < \infty .$$

Comparison of (102) and (103) shows that the lines r = constant are the extremals of the field, while from (103) the lines t = constant are straight lines y = ms radiating from the origin. Thus every point in the field is completely determined by the intersection of a unique extremal r = constant and a unique radial line t = constant (Fig.17). From (103)

$$(104) \qquad dy = dr \cos t - (r \sin t)dt$$
$$ds = t.dr + r.dt$$
$$p(t) = dy/ds = - \sin \frac{s}{C} = - \sin \frac{rt}{r}$$
$$= - \sin t.$$

Substitution from (104) into (101) yields

$$(105) \qquad H = \int \frac{r \cos t\ (tdr+rdt) + r \cos t.\sin t(dr.\cos t - rdt.\sin t)}{\cos t}$$
$$= \int \frac{r \cos t.(t + \cos t.\sin t).dr + r^2 \cos t(1-\sin^2 t).dt}{\cos t}$$
$$= \int r(t + \sin t.\cos t).dr + r^2(1 - \sin^2 t).dt$$
$$= \int d(\tfrac{1}{2}r^2(t + \sin t.\cos t)$$

which contains no singularity. Therefore H exists and the

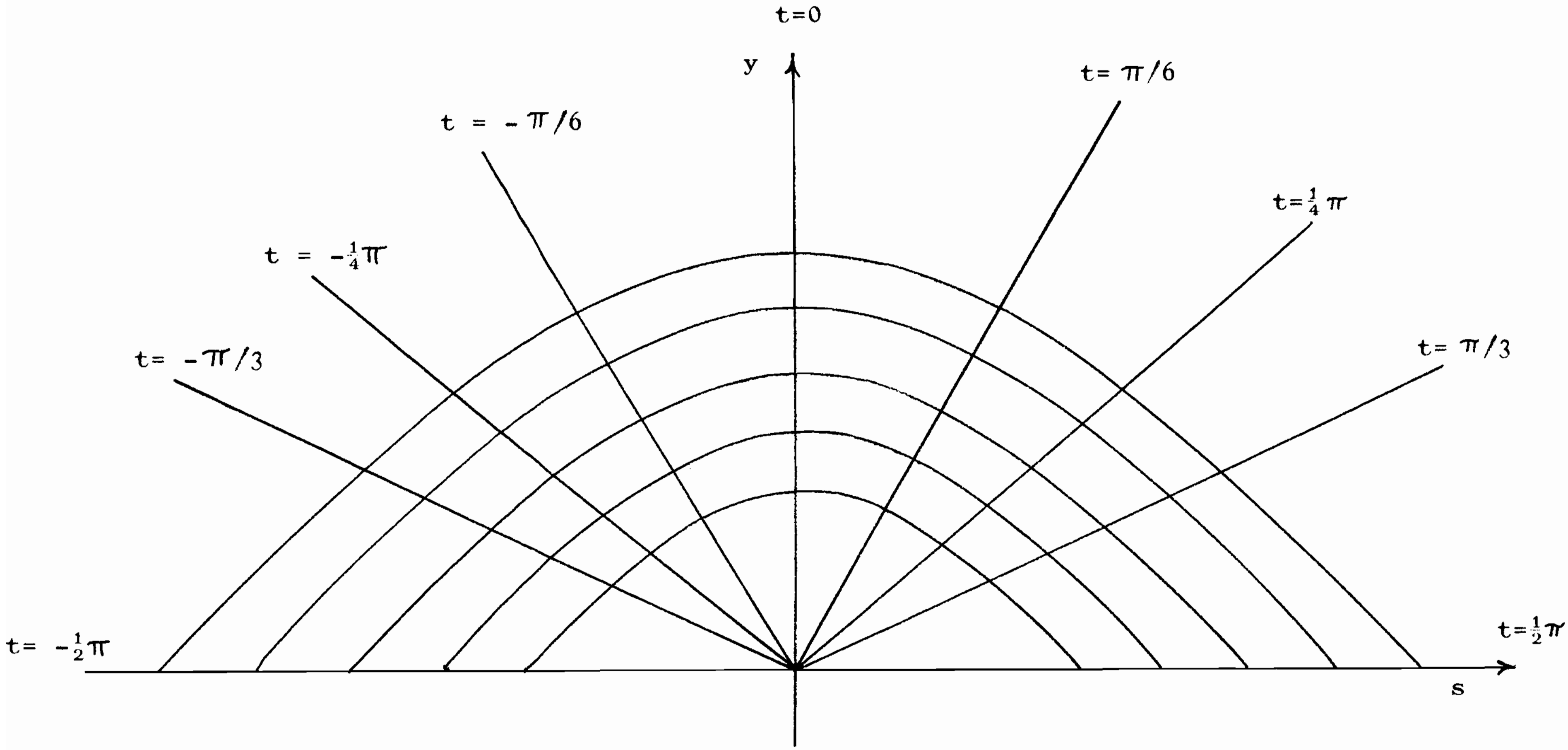


Fig. 17 Coordinate system (r,t) used for the evaluation of the Hilbert integral in the isoperimetric problem.

maximum area may now be evaluated by using the fact that $H(\bar{y}) = J(\bar{y})$. In the (r,t) coordinate system, the end-points of the extremal (99) have coordinates $(L/2\pi, -\frac{1}{2}\pi)$ and $(L/2\quad, \frac{1}{2}\quad)$. Substitution into the right member of (105) gives

$$H = (L^2/8\pi^2)\ (\tfrac{1}{2}\pi + 0) - (-\tfrac{1}{2}\pi + 0)$$

$$= L^2/8\pi\ .$$

<u>The E-function test.</u>

The E-function for the functional (96) is given by

$$(106)\quad E(x,y,p,q) = \frac{y}{\sqrt{1-p^2}}\left(\sqrt{1-q^2}\sqrt{1-p^2} - (1-pq)\right)$$

where p is the slope function of the field and q is an arbitrary finite number. The indicated square roots are positive. Since $(p-q)^2 \geq 0$, it follows that

$$-2pq \geq -p^2 - q^2$$

i.e.,

$$1-2pq+p^2q^2 \geq 1-p^2-q^2+p^2q^2$$

or

$$(1-pq)^2 \geq (1-p^2)(1-q^2)$$

and hence

$$(1-pq) \geq \sqrt{1-p^2}\sqrt{1-q^2}$$

Since $y \geq 0$, the last result implies $E(x,y,p,q) \leq 0$, the condition for a strong maximum. The foregoing results imply that the extremals

(107) $$\bar{y} = (L/2\pi) \sin (2\pi s/L), \qquad \bar{y}(s) \geq 0$$

afford a strong maximum with respect to the functional (96). From (93) it follows that

(108) $$\bar{x}' = \pm \sin 2\pi s/L$$

and since B lies to the right of A, the minus sign may be excluded. Integration of (108) yields

$$\bar{x} = (L/2\pi)(K - \cos 2\pi s/L)$$

where K is a constant of integration. Since x(0) = 0, it follows that K = 1. Hence

(109) $$\bar{x} = (L/2\pi)\ 1 - \cos 2\pi s/L$$

Elimination of s from (107) and (109) gives

$$(\bar{x} - L/2\pi)^2 + \bar{y}^2 = (L/2\pi)^2, \quad y \geq 0.$$

Therefore the semi-circle is the curve of fixed length that encloses maximum area with the x-axis, and that area is $L^2/8\pi$ in agreement with the previous calculation. It then follows that the circle is a strong maximum for the isoperimetric problem.

13.3 The Brachistochrone.

Given two points A and B which are not in the same horizontal plane, find the vertical curve joining them such that a particle with zero initial velocity at A will slide without friction along the curve to the point B in the shortest possible time.

Solution. Let $A(a,a_1)$ and $B(b,b_1)$ be the two points $(a_1 \neq b_1)$. The functional to be minimised is

$$(110) \qquad \int_a^b \frac{\sqrt{1+y'^2}}{\sqrt{y-a_1}}\,.dx$$

where the positive y-axis is taken vertically downward. Since the integrand does not contain x explicitly, Euler's equation[1] admits an immediate first integral, viz.,

$$\frac{1}{\sqrt{1+y'^2}\sqrt{y-a_1}} = \text{constant}$$

---

[1]Since the derivatives of cycloid arcs are infinite at the cusps, weak neighborhoods of such curves do not exist. Fortunately, Euler's equation may be derived by means of strong neighborhoods, which are well-defined for this case (see Remark 4, p.21 and footnote 2, p.25).

or

$$y' = \sqrt{\frac{2c - (y - a_1)}{y - a_1}}$$

where c is a constant. By setting $y-a_1 = 2c \sin^2 \frac{1}{2}t$, where t is a parameter, the solution may be obtained in the form

$$\text{(111)} \qquad \begin{aligned} x &= a + c(t - \sin t) \\ y &= a_1 + c(1 - \cos t) \qquad (0 \leq t < 2\pi). \end{aligned}$$

The equations (111) represent a pencil of cycloids with vertex at (a, a_1) generated by a circle of radius c rolling along the line $y = a_1$ (Fig.18).

The value of c in (111) is determined by the requirement that the curve shall pass through the point $B(b, b_1)$. The complete solution leads to a transcendental equation which must be solved with the help of tables or by approximation. Let the parameter c be denoted by r and let r_o be the value of r for the curve which passes through B. The solution may then be represented in the form

$$\text{(112)} \qquad \bar{C}: \begin{aligned} \bar{x} &= a + r_o(t - \sin t) \\ \bar{y} &= a_1 + r_o(1 - \cos t) \qquad (0 \leq t < 2\pi). \end{aligned}$$

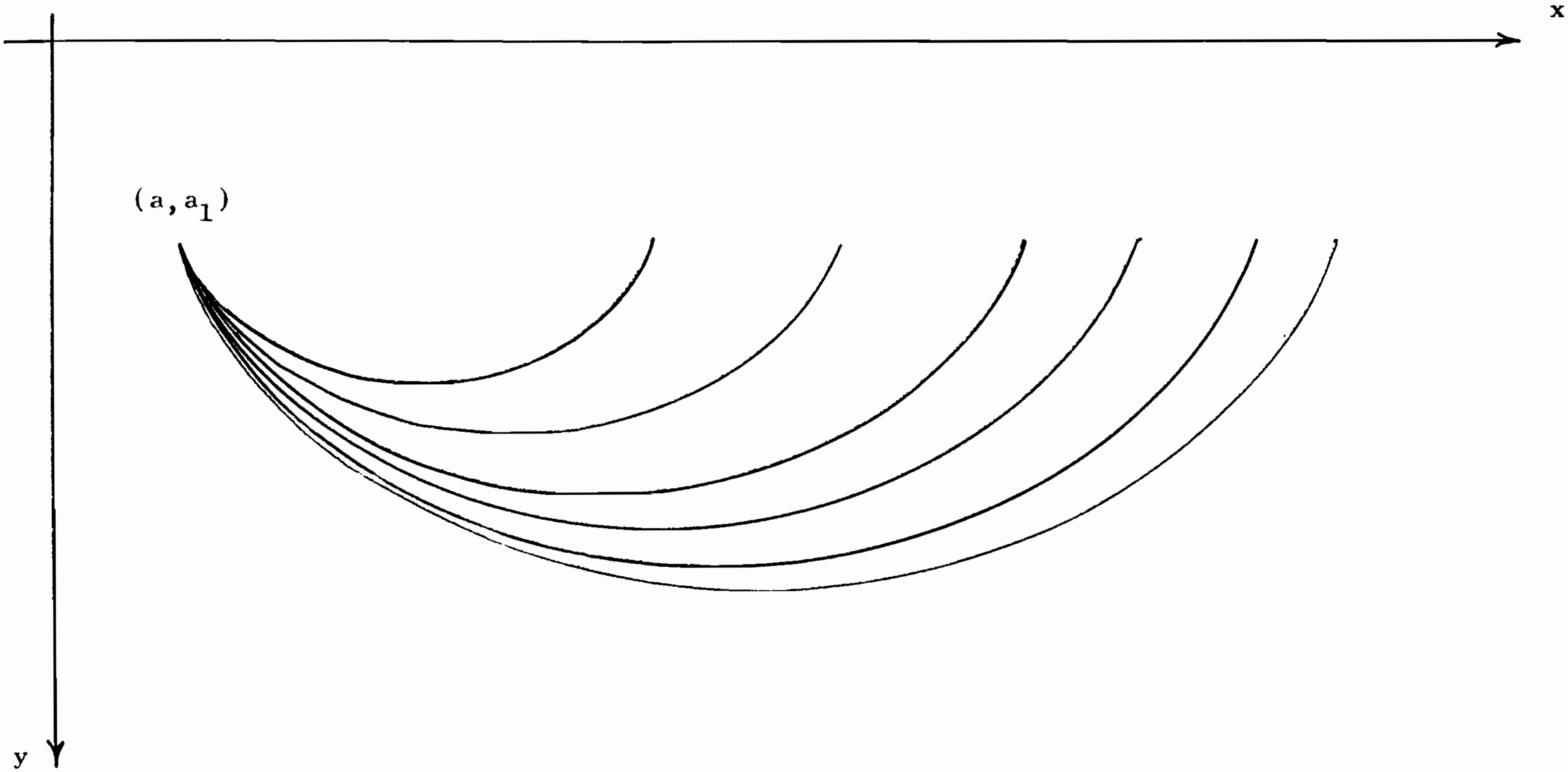


Fig. 18 The Brachistochrone Problem. A pencil of cycloids through the initial point.

The family of concentric cycloids[1]

$$x = a + r(t - \sin t) + \pi(r_o - r) \tag{113}$$
$$y = a_1 + r(1 - \cos t)$$

where r assumes any positive value, contains $\bar{C}$ for $r = r_o$ and forms a simple covering of any region containing $\bar{C}$ and bounded above by the line $y = a_1$. For instance, if r_1 and r_2 are values of r such that $r_1 < r_o < r_2$, then the region D of Definition 16 (p.72) may be taken as the area bounded by the cycloids whose parameters are r_1 and r_2 and by the line $y = a_1$. Analytically, this means that there exists a function $r = r(x,y)$ defined on D.

Beyond this point, the theory of the functional in ordinary form, particularly that of Section 10, is inapplicable for several reasons. First of all, the curves (113) are not extremals according to the definition on page 31 because they are not of class C^1. Secondly, condition (59b) fails along the line $y = a_1$. Also, the Hilbert integral contains singularities, i.e., it is improper.

The proof of Theorem 6, however, shows that it is sufficient that any admissible variation of (111) be covered by the family (113) and that the Hilbert integral shall be

---

[1]The centre, for a single arc, is the mid-point of the line traced out by the generating circle in one complete rotation.

independent of the path in any region simply covered by this family. In other words, the brachistochrone is one case where the results of Section 10 and the proof of Theorem 6 can be obtained with weaker hypotheses. A proof that the Hilbert integral exists and is independent of the path is given below. In anticipation of this result, the curves (112) and (113) will be referred to as 'extremals' and the family (113) will be referred to as a 'field.' (Fig.19).

The E-function test.

Formally, the E-function integral is

$$(114)\quad \int_a^b E(x,y,p,q) = \int_a^b \frac{\sqrt{1+q^2}\sqrt{1+p^2} - (1+pq)}{\sqrt{y-a_1}\sqrt{1+p^2}}\,dx$$

$$\int_a^b \frac{\sqrt{1+q^2}}{\sqrt{y-a_1}}\,dx - \int_a^b \frac{1+pq}{\sqrt{y-a_1}\sqrt{1+p^2}}\,dx$$

The first term on the right of (114) is clearly the time of descent along an arbitrary path[1] of slope $q(x)$ joining the points A and B. The existence of the second integral, however, is dubious on account of the singularity at each end ($y = a_1$ and $p = \pm\infty$). Hence, a closer examination of the Hilbert

[1]The physical interpretation, of course, is meaningful only for smooth paths, and therefore all strong variations must be of this class.

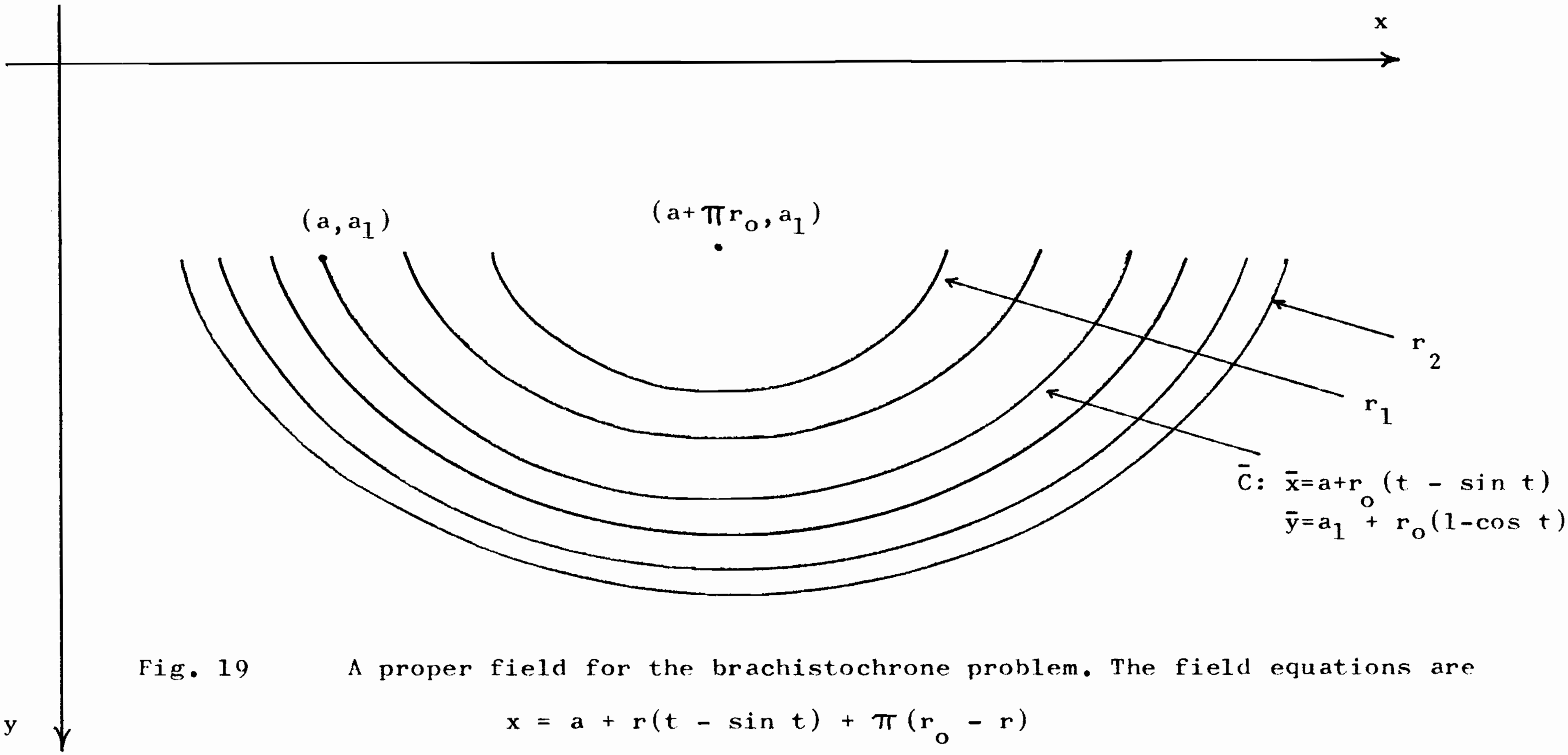


Fig. 19 A proper field for the brachistochrone problem. The field equations are

$$x = a + r(t - \sin t) + \pi(r_o - r)$$

$$y = a_1 + r(1 - \cos t).$$

The field contains the extremal $\bar{C}$ for $r = r_o$. The common centre of this family of concentric cycloids is located at $(a+\pi r_o, a_1)$.

integral is required. By a method similar to the one used in the previous example, the singularity in (114) may be removed.

The first step is the introduction of new axes with origin at the point $(a + \pi r_o, a_1)$, the centre of the family (113). Referred to this new system, the equations of the family are

$$(115) \qquad x = r(t - \sin t) - \pi r$$
$$y = r(1 - \cos t),$$

and the functional (110) assumes the form

$$\int_{a'}^{b'} \frac{\sqrt{1 + y'^2}}{\sqrt{y}}\, dx \qquad a' = -\pi r_o, \quad b' = b - (a + \pi r_o).$$

The field of extremals (115) is symmetrical with respect to the y-axis, and these equations may be used to define curvilinear coordinates (r,t). The curves r = constant are the cycloid arcs, and the curves t = constant are straight lines passing through the origin (Fig. 20). It is now a simple matter to obtain an expression for the Hilbert integral in terms of the coordinates (r,t). The Hilbert integral referred to the new Cartesian axes is given by

$$(116) \qquad H = \int \frac{dx + p.dy}{\sqrt{y}\sqrt{1+p^2}}$$

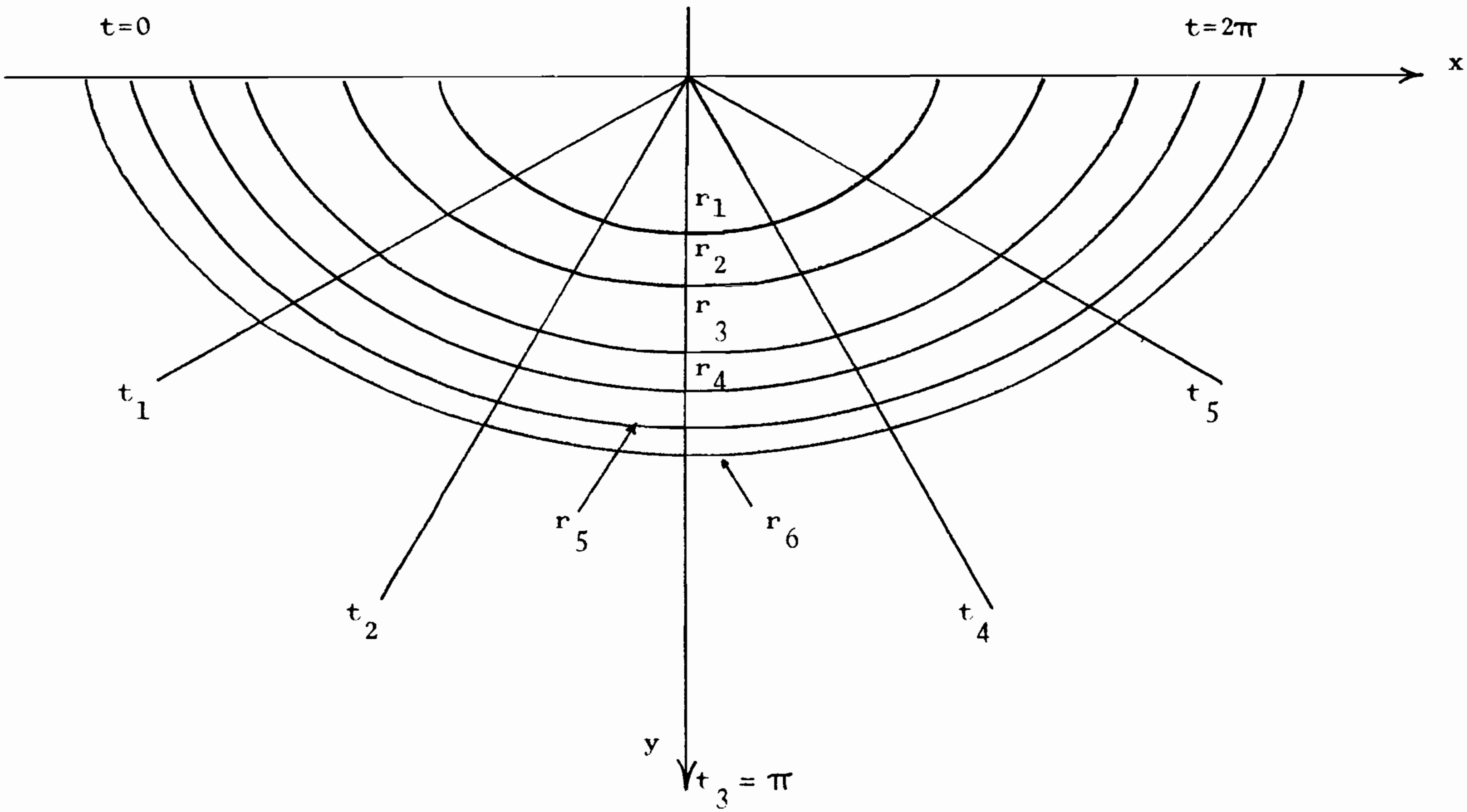


Fig. 20 Symmetric proper field for the brachistochrone used to define a system of curvilinear coordinates (r,t). The lines t = constant are radial lines and the curves r = constant are extremals.

where p is the slope function of the field. From (115) it follows that

(117)
$$dx = dr(t-\pi-\sin t) + rdt(1-\cos t)$$
$$dy = dr(1-\cos t) + r.dt.\sin t$$
$$y' = \left(\frac{dy}{dt}\right)\Big/\frac{dx}{dt} = \frac{\sin t}{1-\cos t} = p(t).$$

The last equation follows from the fact that along an extremal the value of r is constant. The fact that the slope at any point in the field depends only on t follows, of course, from (115).

Substitution from (115) and (117) into (116) gives

$$H = \int \frac{dr(t-\pi-\sin t)+rdt(1-\cos t) + dr\sin t + \dfrac{r\sin^2 t dt}{1-\cos t}}{\sqrt{r}\,\sqrt{1-\cos t}\,\sqrt{\dfrac{2}{1-\cos t}}}$$

$$= \int \frac{dr(t-\pi-\sin t) + r.dt(1-\cos t) + dr.\sin t + \dfrac{r\sin^2 t.dt}{1-\cos t}}{\sqrt{2r}}$$

$$= \int \frac{(t-\pi)dr}{\sqrt{2r}} + \int \frac{r.dt}{\sqrt{2r}}\left[1-\cos t + \frac{\sin^2 t}{1-\cos t}\right]$$

$$= \int \frac{(t-\pi)}{\sqrt{2r}}dr + \int \sqrt{2r}.dt$$

$$= \int d(\sqrt{2r}(t-\pi)).$$

The last integral is exact and has no singularities. The minimum time T may now be calculated readily, since $H(\bar{y})=J(\bar{y})$. If B lies on the line $t=t_o$, then the coordinates of B are (r_o,t_o). Therefore

$$T = \sqrt{2r_o}\left[(t_o-\pi) - (0-\pi)\right] = \sqrt{2r_o}\,t_o.$$

<u>The E-function test.</u>

From (114), the sign of the E-function depends on the sign of

$$\sqrt{1+q^2}\sqrt{1+p^2} - (1+pq)$$

where the indicated square roots are positive, and p is the slope function of the field. Since $1+pq \leq |1+pq| \leq 1 + |pq|$, it follows that

$$1+pq \leq \sqrt{1+p^2}\sqrt{1+q^2}$$

by the Cauchy-Schwartz inequality. Therefore $E(x,y,p,q) \geq 0$ and Theorem 6 shows that the cycloid furnishes a strong minimum for the brachistochrone problem.

13.4 Classification of extrema.

The next example shows how the E-function may be used to separate the extremals into classes according to the type of extremum.

Examine the extrema of class C^1 of the functional

(118) $$\int_0^a 6y'^2 - y'^4 + yy')dx, \quad y(0) = 0, \quad y(a)=b, \quad a > 0.$$

Solution According to Corollary 2.2b (p.32), Euler's equation has a first integral

(119) $$f - y'f_{y'} = C$$

where C is a constant. In terms of (118), this integral is

$$3y'^4 - 6y'^2 = C$$

or

$$y' = \text{constant}.$$

Therefore the extremals are straight lines. The pencil of extremals passing through the initial point (0,0) is given by

(120) $$y = mx$$

with parameter m, which, for m = b/a includes the extremal $\bar{C}$ satisfying the boundary conditions (Fig.21a). The family of extremals

(121) $$y = (b/a)x + c$$

with parameter c is obviously a field[1] which contains $\bar{C}$ for $c = 0$ (Fig.21b). Differentiation of the integrand in (118) gives

$$f_{y'y'}(x,\bar{y},\bar{y}') = 12(1 - \frac{b^2}{a^2}).$$

Therefore, it follows by Theorem 7 (p.80) that

(i) if $|b| < a$, then $\bar{C}: y = \frac{b}{a}x$ affords at least a weak minimum,

(ii) if $|b| > a$, then $\bar{C}$ affords at least a weak maximum.

The regions containing the weak extrema are shown in Fig.22.

The regions corresponding to strong extrema, if they exist, may be identified as subsets of the regions shown in Fig.22 by means of the E-function. From (118)

$$E(x,y,p,q) = 6q^2-q^4+yq-6p^2+p^4-yp-(q-p)(12p-4p^3+y)$$
$$= -(q-p)^2(q^2 + 2pq - (6 - 3p^2)).$$

The sign of E is opposite to the sign of

$$q^2 + 2pq - (6 - 3p^2) \tag{122}$$

and therefore, if the discriminant $4(6-2p^2)$ of the quadratic expression (122) is not greater than zero, i.e., if $p \geq \sqrt{3}$ then for arbitrary q that expression has non-negative values. On the other hand, if $p < \sqrt{3}$, then (122) changes sign.

---

[1] Every one-parameter family is a Mayer family and $\frac{\partial y}{\partial c} = 1$.

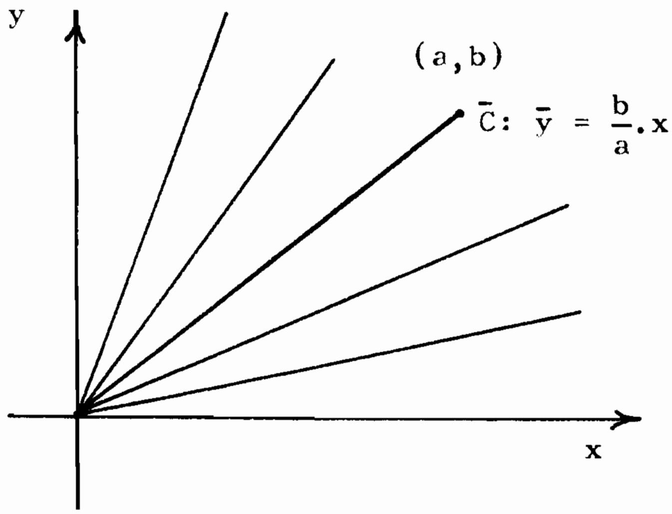


Fig.21a Central field afforded by pencil of extremals for Example 4 with $b > 0$.

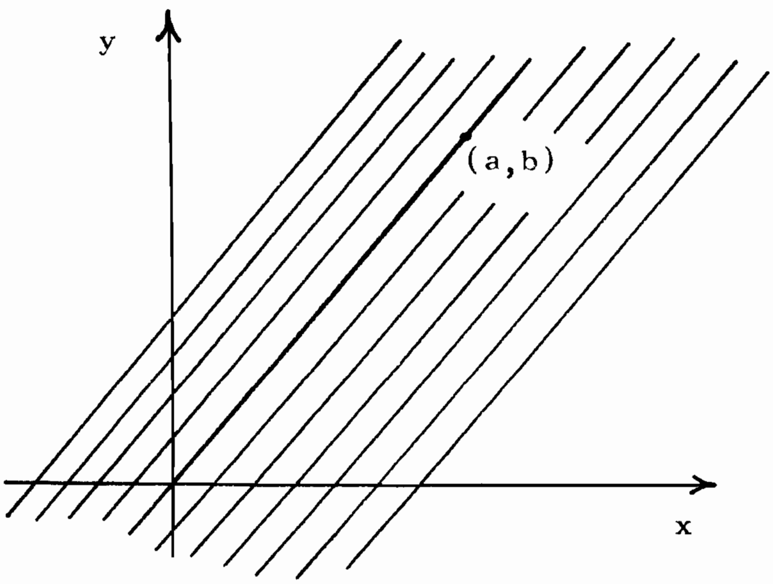


Fig.21b Proper field for Example 4 with $b > 0$. The extremal $\bar{C}$ is represented by the heavy line joining the origin to the point (a,b).

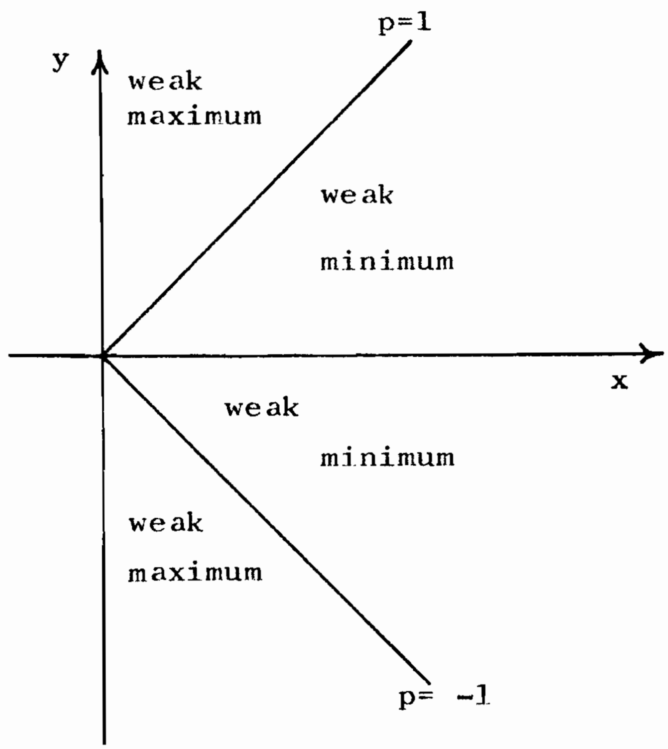


Fig.22 Regions of weak extrema for Example 4.

Therefore

(i) if $|p| \geq \sqrt{3}$, $E(x,y,p,q) \leq 0$ for all values of its arguments, and

(ii) if $|p| < \sqrt{3}$, $E(x,y,p,q)$ may change sign for appropriate values of q.[1]

By Theorem 6 (p.77), the first statement implies that any straight line $y = mx$ for which $|m| \geq \sqrt{3}$ and which satisfies the boundary conditions, furnishes the functional (118) with a strong maximum. On the other hand, by Theorem 8

[1]If p=1,(122) is positive for q=2, negative for q=0.

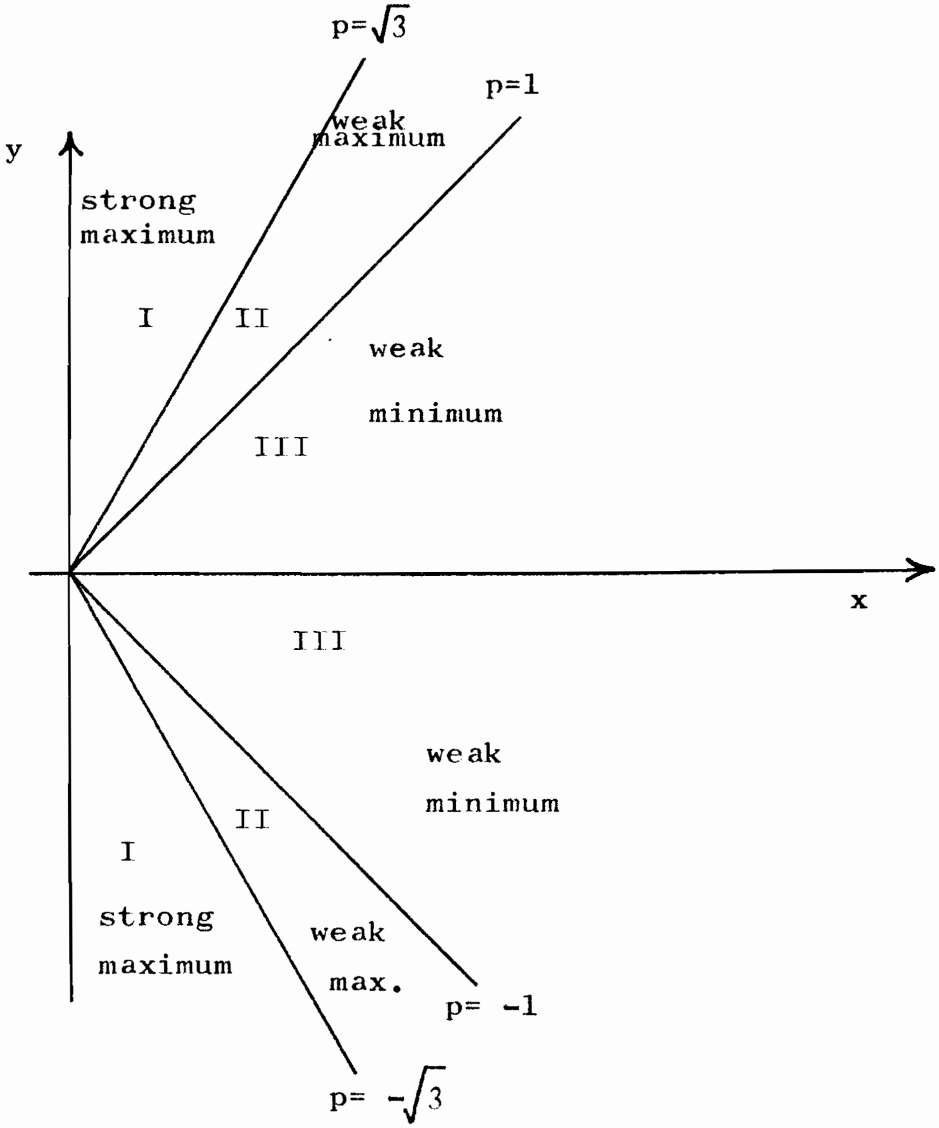


Fig.23 Regions of extrema for Example 4.

(p.84), if $|m| < \sqrt{3}$, the line furnishes neither a strong maximum nor a strong minimum. These results are illustrated in Fig.23.

None of the regions illustrated includes both its boundaries. Obviously $J(y)$ is not defined for the y-axis, so the latter does not belong to region I. The lines $y = \pm\sqrt{3}x$ belong to both regions I and II, since a strong extremum is, a fortiori, a weak extremum.

The theory gives no information concerning the lines $y = \pm x$ which separate regions II and III. For in this case $p^2 = 1$, and $f_{y'y'} = 0$ along the extremal. By Theorem 7 (p.80) and Theorem 9 (p.93) respectively, it follows that the Legendre Sufficient Condition for a weak minimum does not hold, and the Legendre Necessary Condition for a weak (and hence for a strong) minimum does not fail. But since the last term in the integrand of (118) is exact, and since the quantity $6y'^2 - y'^4$ is strictly increasing near $y' = 1$, it follows that the lines $y = \pm x$ belong neither to region II nor to region III.

14. Application of the E-function to Newton's problem.

Two illustrations of the use of the E-function Necessary Condition have already been given (pp.53,54). As a final example, to round out the remarks on p.97, and to implement the objectives outlined in the introduction, this condition will now be applied to Newton's problem.

Although it is not the best, nor the simplest illustration of this use of the E-function, this example is the earliest case of a variational problem that has no strong solution, and as such, it retains a historical interest that has outlasted its practical value. It therefore provides an appropriate topic for the conclusion of this discussion.

It was Legendre who showed in 1786, one hundred years after the Principia, that the apparent solution to Newton's problem, which satisfied all the known conditions for a weak minimum, could be made arbitrarily small by the use of zig-zag lines.[1] Of course, the ideas behind the zig-zag line and the strong variation are very close, but the connection did not become clear until 1879 when Weierstrass's work was made public. As will be shown, the E-function test of Theorem 8 disposes of Newton's 'paradox' in a few lines.

---

1 "Mémoire sur la manière de distinguer les maxima des minima dans le calcul des variations.", Mémoires de l'Académie des Sciences (1786). Even with a sound theoretical basis,however, the problem still bristles with difficulties that are non-mathematical. The assumption of zero tangential resistance, for example, is unrealistic, but persisted for many years contrary to well-authenticated observations of Naval vessels. Forsyth (Calculus of Variations, p.344) offers some rather quaint statistics on this subject from old Navy records.

Newton's problem is the most complex of all the classical examples, and a complete examination of it is beyond the scope of this work.[1] The material that follows will be confined to a demonstration that the extremals given by Euler's equation furnish a weak but not a strong solution.

Newton's Solid of Revolution of Minimum Resistance.

A solid of revolution moves with constant velocity in the direction of its axis in a perfect incompressible fluid. If the resisting pressure at any point is proportional to the square of the normal component of the velocity, find the shape of the solid in order that the total resistance shall be a minimum.

Solution. Let the y-axis be the axis of revolution, the positive direction being the direction of motion, and let v denote the constant velocity (Fig.24). The normal component of velocity is $v.\cos\psi$ and hence the total force in the normal direction upon an element of area $(x.d\phi)ds$ is proportional to $x(v\cos\psi)^2 d\phi ds$ where ds is an element of arc

[1]The most thorough analysis of Newton's problem in the literature is due to W.S. Kimball (op.cit., Chap.10). In particular, a modification of the orthodox solution to give closed surfaces is discussed in detail.

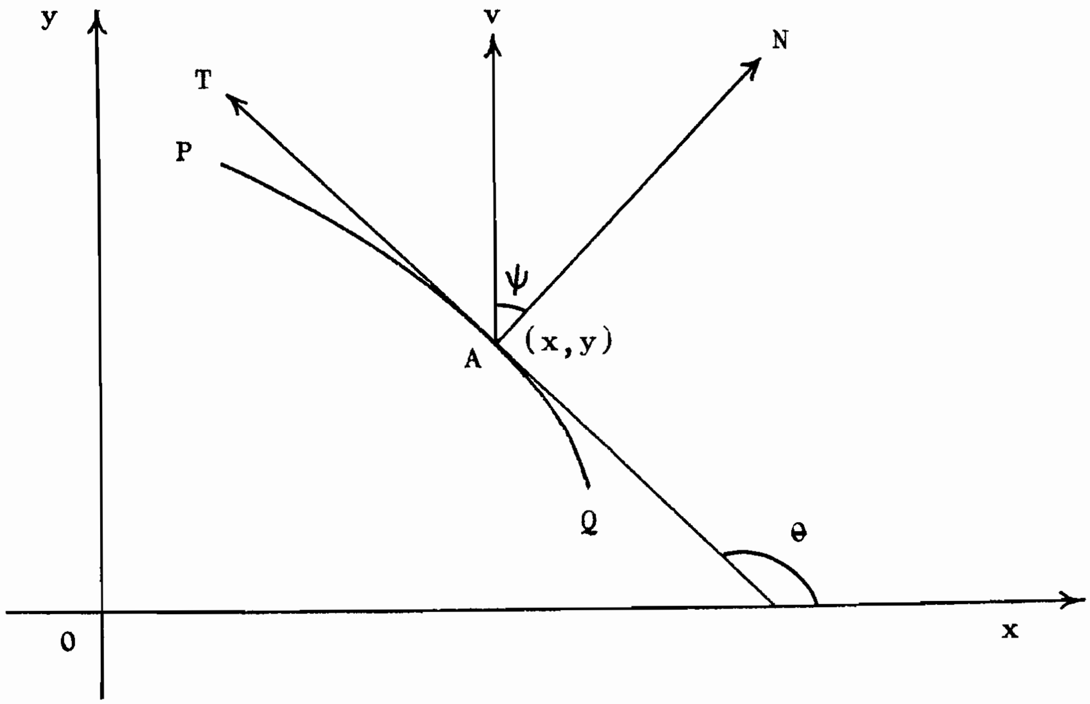


Fig.24 Portion PQ of meridian curve for Newton's Problem.

T is the positive tangent to PQ at A,

N is the outward-drawn normal at A,

ψ is the angle between N and the positive y-axis,

θ is the angle between T and the positive x-axis.

of the segment PQ, and $d\phi$ is a small angle of rotation of PQ about the y-axis. The total resisting force opposing the motion is therefore proportional to

$$(123) \qquad v^2 \int_{s(a)}^{s(b)} x.\cos^3\psi \int_0^{2\pi} d\phi.ds \;=\; 2\pi\, v^2 \int_{s(a)}^{s(b)} x.\cos^3\psi\, ds$$

where a and b are the abscissas of the end-points of the meridian curve. Since $\cos\theta = dx/ds$, $\psi = \pi - \theta$, it follows that

$$
\begin{aligned}
(dx/ds)^2 &= 1/(1 + y'^2),\\
\cos\psi\, ds &= -\, dx, \qquad (124)\\
\cos^2\psi &= 1/(1 + y'^2).
\end{aligned}
$$

Therefore (123) may be written

$$(125) \qquad -2\pi v^2 \int_a^b \frac{x\,dx}{1 + y'^2}$$

i.e., the resisting force is proportional to

$$(126) \qquad J(y) = \int_a^b \frac{x\,dx}{1 + y'^2}$$

which is the functional to be minimised. Since the integrand in (126) does not contain y explicitly, it follows from Corollary 2.2a (p.32) that the Euler equation has a first integral

$$(127) \qquad f_{y'} = c,$$

i.e.,

$$(128) \qquad \frac{xy'}{(1 + y'^2)^2} = c,$$

where c is a constant. In anticipation of later results, it is assumed that $c < 0$ in (128), since only the negative values of c are meaningful.[1] Let t denote $\frac{dy}{dx}$. Then the equations of the meridian curve may be obtained in parametric form from (128):

$$x = \frac{c}{t}(1 + t^2)^2 \tag{129}$$

$$y = y_o + c\left(\frac{3t^4}{4} + t^2 - \log|t|\right)^*$$

The graph of (129) has two branches and a cusp corresponding to $t = -(1/\sqrt{3})$. According to Theorem 9 (p.93) (Legendre's Weak Condition), it is necessary that $f_{y'y'}(x,\bar{y},\bar{y}') \geq 0$ for all x in $[a,b]$. Therefore

$$\frac{2x(3y'^2 - 1)}{(1 + y'^2)^3} \geq 0 \tag{130}$$

and hence $y' \leq -(1/\sqrt{3})$ or $y' \geq (1/\sqrt{3})$. The graph of (129) may be simplified by taking

$$y_o = -c(5 + 6 \log 3)/12 = -(0.966)c$$

in which case $y(-\frac{1}{\sqrt{3}}) = 0$, i.e., the cusp lies on the x-axis.

---

[1] If c=0, the solution is a weak maximum, and the case $c > 0$ leads to curves that are concave in the direction of motion, and to which Newton's law of resistance does not apply.

*All logarithms shown are to base e.

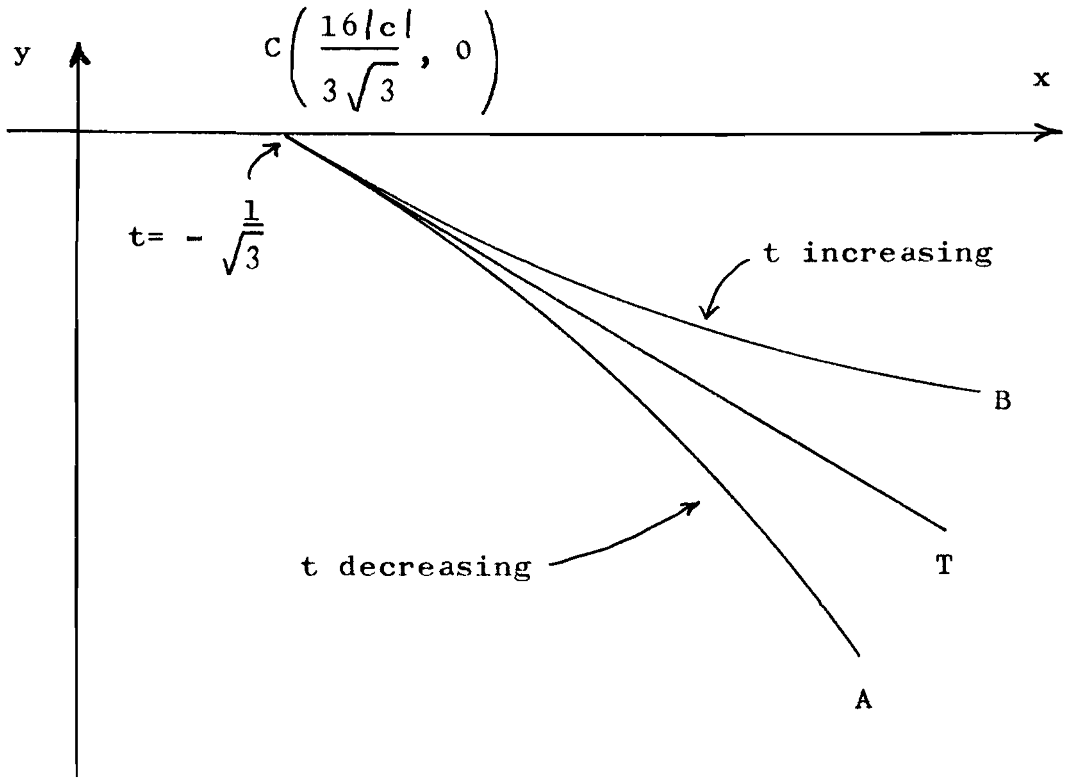


Fig.25 Meridian curves for Newton's problem given by

$$x = \frac{c}{t}(1 + t^2)^2$$

$$y = c\left(\frac{3t^4}{4} + t^2 - \log|t| - 0.966\right), \qquad c < 0.$$

The branch CB is concave up[1] and Newton's law of resistance is not applicable to the corresponding solid of revolution. The branch CA (excluding C) furnishes a weak minimum.

---

[1] $y'' = \dfrac{t^2}{c(3t^2-1)(t^2+1)} > 0$, since $c < 0$.

The result is illustrated in Fig.25. From (129) it follows that

$$\frac{\partial x}{\partial c} = \frac{(1 + t^2)^2}{t},$$

$$\frac{\partial x}{\partial t} = \frac{c(3t^2-1)(1+t^2)}{t^2},$$

$$\frac{\partial y}{\partial c} = (\frac{3t^4}{4} + t \; - \log|t| - 0.966),$$

$$\frac{\partial y}{\partial t} = \frac{c(3t^2 - 1)(t^2 + 1)}{t}.$$

Therefore

$$\frac{\partial(x,y)}{\partial(c,t)} = \frac{c(3t^2-1)(t^2+1)(\frac{t^4}{4}+t^2+ \log|t| + 1.966)}{t^2} \neq 0$$

provided $t < -\frac{1}{\sqrt{3}}$. Therefore, to each point (x,y) in the interior of quadrant IV, there corresponds a unique ordered pair of values (c,t) satisfying (129), i.e., there exist well-defined functions

$$c = c(x,y),$$
$$t = t(x,y)$$

in a neighborhood of each point (x,y), which satisfy (129) identically. It follows that through each point (x,y) in the interior of quadrant IV, there passes a unique extremal of the family (129), and hence quadrant IV is simply-covered by that family.[1]

Also, from the theory of Jacobians, it is easily seen that if y = f(x,c) is the one-parameter representation corresponding to (129) then

$$\left.\frac{\partial f}{\partial c}\right|_{x\ \text{constant}} = - \frac{\dfrac{\partial(x,y)}{\partial(c,t)}}{\dfrac{\partial x}{\partial t}} .$$

Therefore

$$\frac{\partial f}{\partial c} = -\left(\frac{t^4}{4} + t^2 + \log|t| + 1.966\right) \neq 0$$

Hence the conditions for a field are satisfied and the extremals (129) furnish a proper field for the functional (126).

In a practical case, the ratio of height to base radius completely determines the shape of the surface of revolution; the constant c determines the size or scale.

Application of Theorem 7 (p.80) to relation (130) shows that the branch of the curve (129) corresponding to $t < -(1/\sqrt{3})$ furnishes a weak minimum for the functional (126).

[1] including the points on the positive x-axis.

In Fig.25 such a branch is represented by the segment CA, excluding the point C.

The E-function test.

Application of Theorem 8 (p.84), the E-function Necessary Condition, shows that these curves furnish neither a strong minimum nor a strong maximum. From (126) and the definition of the E-function:

$$E(x,y,y',q) = \frac{x(q - y')^2(2yq + y'^2 - 1)}{(1 + q^2)(1 + y'^2)^2}$$

Obviously there exist finite values of q such that $E(x,y,y',q)$ can be made both positive and negative along any extremal curve. Therefore a curve of the family (129) affords a weak but not a strong minimum with respect to the functional (126).

According to the theory of strong variations leading to Theorem 3 (p.47), the last result implies that it is possible to construct solids of revolution which offer less resistance to fluids than those obtained by rotating the meridian curves (129). The construction used in the proof of Theorem 3 suggests that the meridian curves for these less resistant solids are of class D^1 rather than C^1. As mentioned earlier, this is precisely Legendre's result of 1786.

15. Summary and Conclusion.

Beginning in 1696 with the brachistochrone problem of Johann Bernoulli, the modern development of the calculus of variations occupied roughly the following two hundred years. Euler, Legendre and Jacobi investigated the necessary properties of a minimising arc for a functional in ordinary form for the case $n = 1$, and the three conditions associated with their names appeared in 1744, 1786 and 1837 respectively.

These conditions, however, were not sufficient to guarantee the existence of a minimising curve. In a number of well-known cases, for instance, the extremal satisfied the three conditions for a weak minimum, and yet a second curve could be found for which the value of the functional was still smaller. The most famous counter-example of this kind was Newton's problem of the Solid of Least Resistance, first analysed by Legendre. The problem of a sufficiency criterion which thus arose was not settled for many decades.

Finally, in 1879, Weierstrass added a fourth necessary condition and showed that the four conditions then known, when suitably strengthened, would actually guarantee the minimising property.

Realizing the limitations of the weak variation used by his predecessors, Weierstrass introduced a more general type of comparison path, known as the strong variation, which is the basis of all his results.

In order to deal with the strong variations analytically, Weierstrass introduced fields of extremals, thus laying the foundation of the theory of fields. To characterise the property of an arc which guarantees a minimum for the given functional, Weierstrass introduced the function $E(x,y,z,q)$, known as the E-function. By means of it, a sufficiency condition was finally established, and the difficulties associated with Newton's and similar problems were resolved.

Far from being a notational convenience, the E-function plays a central role in the theory of the calculus of variations. Not only is it the basis of the sufficiency criterion, but it yields a fourth necessary condition and appears in other important contexts. It can be used to prove the Second Corner Condition, the Legendre Weak Condition (necessary for a strong minimum) and the Legendre Strong Condition (sufficient for a weak minimum). In the last two cases, much of the cumbersome theory of the second variation developed by writers before the time of Weierstrass can be avoided. These remarks are illustrated in Fig.26 which shows the relation between the E-function and the other major conditions.

The arrows shown in the figure do not denote implication in the strict sense of the word, but rather that the condition to the left of the arrow can be used, possibly in combination with other assumptions obvious from the text, to prove the condition shown to the right. A similar statement holds for arrows which are vertical. This type of 'implication' , of course, is not necessarily transitive. The meaning of the symbols is given in the following index.

$\bar{C}$: $y = \bar{y}(x)$ is an extremal which can be embedded in a field with slope function $p(x,y)$.

$\tilde{C}$: $y = \tilde{y}(x)$ is a piecewise-smooth curve lying in R (q.v., p.77).

$\bar{f}, \bar{f}_{y_i'}$ denote $f(x,\bar{y},\bar{y}')$, $f_{y_i'}(x,\bar{y},\bar{y}')$ respectively.

$q = (q_1,\dots,q_n)$ is an arbitrary finite vector in E_n.

$t_1,\dots,t_n$ are arbitrary finite numbers which are not all zero.

$J(t)$ is defined on p.86.

$\bar{p}, p$ represent $p(x,\bar{y})$, $p(x,y)$ respectively.

$z_1,\dots,z_n$ are arbitrary finite numbers.

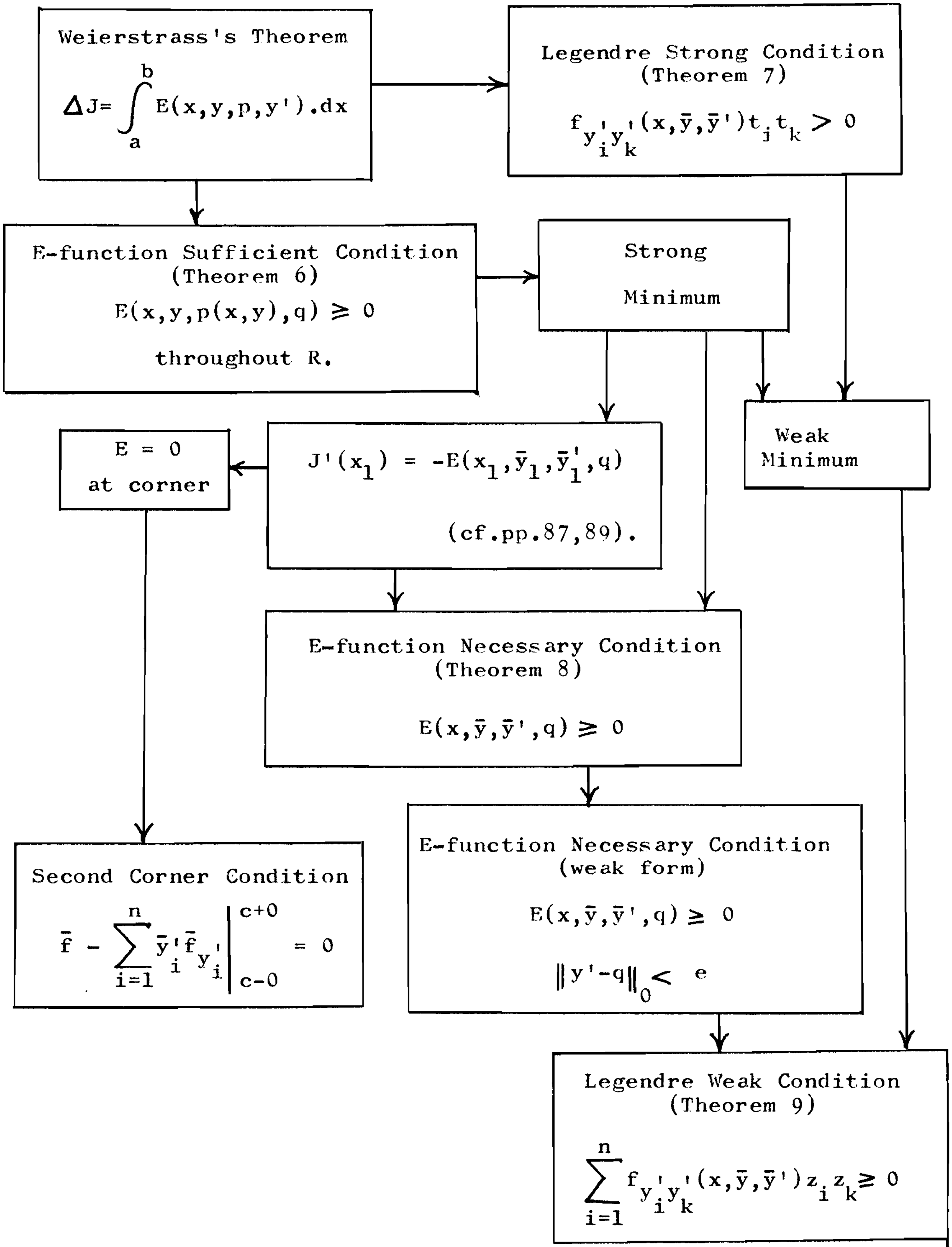


Fig.26 The relation between the E-function and some of the major theorems in the Calculus of Variations.

BIBLIOGRAPHY

1. Akhiezer, N.I., The Calculus of Variations, New York, Blaisdell Publishing Co., 1962.

2. Bliss, G.A., Lectures on the Calculus of Variations, Chicago, University of Chicago Press, 1946.

3. Bliss, G.A., The Weierstrass E-function for problems of the calculus of variations in space, Trans. Am. Math. Soc. 15 (1914), 369-378.

4. Bolza, O., Lectures on the Calculus of Variations, New York, Dover Publications, 1961.

5. Bolza, O., Some Instructive Examples in the Calculus of Variations, Bull. Am. Math. Soc. (2), Vol. IX (1902), p.3.

6. Carathéodory, C., Variationsrechnung und partielle Differentialgleichungen erste Ordnung, Leipzig, B.G. Teubner, 1935.

7. Elsgolc, L.E., Calculus of Variations, New York, Pergamon Press, 1961.

8. Forsyth, A.R., Calculus of Variations, New York, Dover Publications, 1960.

9. Fox, C., An Introduction to the Calculus of Variations, London, Oxford University Press, 1950.

10. Funk, P., Variationsrechnung und ihre Anwendung in Physik und Technik, Berlin, Springer-Verlag, 1962.

11. Gelfand, I.M., Fomin, S.V., Calculus of Variations, Englewood Cliffs, Prentice Hall, 1963.

12. Hadamard, J.S., Leçons sur le calcul des variations, Paris, A. Hermann & Fils, 1910.

13. Kimball, W.S., Calculus of Variations by Parallel Displacement, London, Butterworth's Scientific Publications, 1952.

14. Kneser, A., Lehrbuch der Variationsrechnung, Braunschweig, F. Vieweg & Sohn, 1900.

15. Morse, M., The Calculus of Variations in the Large, New York, American Mathematical Society, 1934.

16. Weinstock, R., Calculus of Variations with Applications to Physics and Engineering, New York, McGraw Hill, 1952.

INDEX

Index.

Index.