THREE ESSAYS ON THE PRICING OF FIXED INCOME SECURITIES WITH CREDIT RISK

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DEDICATED TO MY PARENTS AND MY SISTER

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Abstract

This thesis studies the impacts of credit risk, or the risk of default, on the pricing of fixed income securities. It consists of three essays. The first essay extends the classical corporate debt pricing model in Merton (1974) to incorporate stochastic volatility (SV) in the underlying firm asset value and derive a closed-form solution for the price of corporate bond. Simulation results show that the SV specification for firm asset value greatly increases the resulting credit spread levels. Therefore, the SV model addresses one major deficiency of the Merton-type models: namely, at short maturities the Merton model is unable to generate credit spreads high enough to be compatible with those observed in the market. In the second essay, we develop a two-factor affine model for the credit spreads on corporate bonds. The first factor can be interpreted as the level of the spread, and the second factor is the volatility of the spread. Our empirical results show that the model is successful at fitting actual corporate bond credit spreads. In addition, key properties of actual credit spreads are better captured by the model. Finally, the third essay proposes a model of interest rate swap spreads. The model accommodates both the default risk inherent in swap contracts and the liquidity difference between the swap and Treasury markets. The default risk and liquidity components of swap spreads are found to behave very differently: first, the default risk component is positively related to the riskless interest rate, whereas the liquidity component is negatively correlated with the riskless interest rate; second, although default risk accounts for the largest share of the levels of swap spreads, the liquidity component is much more volatile; and finally, while the default risk component has been historically positive, the liquidity component was negative for much of the 1990s and has become positive since the financial market turmoil in 1998.

Résumé

Cette thèse étudie les impacts du risqué de crédit ou le risqué de défaut sur l'évaluation des titres à revenus fixes. Elle consiste en trois essais. Le premier essai étend le modèle classique de l'évaluation de la dette corporative selon Merton (1974) pour inclure le cas de la volatilité stochastique (SV) sur la valeur de l'actif sous-jacent et dériver une solution fermée pour le prix des obligations corporatives. Les résultats de simulation montrent que la spécification SV pour la valeur de l'actif de la firme augmente grandement les niveaux des spreads résultants. Ainsi, le modèle SV soulève une déficience majeure des modèles du type Merton : Plus particulièrement, pour les courtes maturités, le modèle de Merton est incapable de générer des marges de crédit assez élevées pour être compatibles avec les données observées sur le marché. Dans le second essai, nous développons un modèle affiné à deux facteurs pour la marge de crédit des obligations corporatives. Le premier facteur peut être interprété comme le niveau du spread alors que le second, correspond à sa volatilité. Nos résultats empiriques montrent que le modèle explique les marges de crédit sur les obligations corporatives observées. De plus, les propriétés essentielles des marges de crédit sont mieux capturées par le modèle. Finalement, le troisième essai un modèle de marge de swap de taux d'intérêt. Il accommode le risque de défaut inhérent aux contrats swap et le différentiel de liquidité entre le marché des swaps et celui des bons du trésor. Les deux composantes que sont le risque de défaut et la liquidité des marges sur swap se comportent différemment. D'abord, le risque de défaut est relié positivement au taux d'intérêt sans risque, alors que la liquidité est corrélée négativement avec ce même taux sans risque. Ensuite, bien que le risque de défaut tienne compte de la grande part des niveaux des marges swap, la liquidité est beaucoup plus volatile; et finalement, alors que le risque de défaut a été historiquement positif, la composante liquidité a été négative pour la plus grande partie des années 1990 et est devenue positive depuis la crise financière de 1998.

Contributions of Authors

Among the three essays that consist of this thesis, the second essay, "Modeling the Dynamics of Credit Spreads with Stochastic Volatility," is a joint paper of me and Prof. Kris Jacobs, chair of my thesis committee. Prof. Jacobs and I have made equally substantial contributions to this paper and are therefore equally responsible for any remaining errors in this paper, if there are any.

Table of Contents

Introduction

Credit risk or default risk refers to the risk of the possible default of financial securities. Almost every financial contract and security is affected by certain types of credit risk. Therefore, both academics and practitioners have a keen interest in accurately measuring the credit risk embedded in financial securities. This thesis makes some contributions to the accurate measurement of credit risk. Specifically, we investigate the impacts of credit risk on the pricing of fixed income securities (e.g. corporate bonds and interest rate swaps etc.). The thesis concentrates on the following three fundamental questions in modeling credit risk:

1) What are the implications of credit risk for the pricing of fixed income securities?

2) Can we develop more satisfactory credit risk models that better capture the observed credit spreads on fixed income securities?

3) Are the observed credit spreads on fixed income securities solely attributed to credit risk? If not, what are the non-credit (default) components?

Currently there are two broadly specified approaches to modeling credit risk: the structural approach and the reduced form approach. The first approach is based on the value of the firm. This approach specifies a default threshold and models the allocation of the firm's residual value upon default. The structural approach was initiated by Merton (1974) and was subsequently extended by Longstaff and Schwartz (1995) and others. On the other hand, the reduced form approach treats default as an exogenous event and usually uses a Poisson process to modeling the occurrence of default. This approach was first introduced by Madan and Unal (1994) and Jarrow and Turnbull (1995).

This thesis uses both structural and reduced form approaches. It proposes better models of credit risk and studies the individual components of the credit spreads. The models developed in this thesis are tractable and can be easily implemented in practice. In addition, the empirical results obtained in this thesis have important implications for not only the pricing of various types of fixed income securities, but also for the management of fixed income portfolios.

The thesis consists of three essays. The first and second essays address both the first and second aforementioned questions, but using different modeling approaches. In particular, in the first essay, "A Structural Model of Corporate Debt with Stochastic Volatility," we extend the structural corporate debt pricing model in Merton (1974) to incorporate stochastic volatility (SV) in the underlying firm asset value and derive a closed-form solution for the price of corporate bond. Simulation results show that for realistic parameter values, the SV specification for firm asset value greatly increases the resulting credit spread levels. Therefore, the SV model addresses one major deficiency of the Merton-type models: namely, at short maturities the Merton model is unable to generate credit spreads high enough to be compatible with those observed in the market.

In the second essay, "Modeling the Dynamics of Credit Spreads with Stochastic Volatility" (co-authored with Prof. Kris Jacobs), we develop a two-factor reduced form model for the credit spreads on corporate bonds. The first factor can be interpreted as the level of the spread, and the second factor is the volatility of the spread. The riskless interest rate is modeled using a standard two-factor affine model, thus leading to a four-factor model for corporate yields. This approach allows us to model the volatility of corporate credit spreads as stochastic, and also allows us to capture higher moments of credit spreads. We use an extended Kalman filter approach to estimate our model on corporate bond prices for 108 firms. The model is found to be successful at fitting actual corporate bond credit spreads, resulting in a significantly lower root mean square error (RMSE) than a standard alternative model in both in-sample and out-of-sample analyses. In addition, key properties of actual credit spreads are better captured by the model.

Finally, the third essay, "Decomposing the Default Risk and Liquidity Components of Interest Rate Swap Spreads," addresses the third question mentioned earlier. To this end, we study the individual components of one of the most important credit spreads in the financial markets: interest rate swap spreads. We propose a reduced form model of interest rate swap spreads. The model accommodates both the default risk inherent in the swap contracts and the liquidity difference between the swap and Treasury markets. Empirical results show that the default risk and liquidity components of swap spreads behave very differently: first, the default risk component is positively related to the riskless interest rate, whereas the liquidity component is negatively correlated with the riskless interest rate; second, although default risk accounts for the largest shares of the levels of swap spreads, the liquidity component is much more volatile; and finally, while the default risk component has been historically positive, the liquidity component was negative for much of the 1990s and has become positive since the financial market turmoil in 1998.

Literature Review

Credit risk, or the risk of default, has received a lot of attention in the finance literature since the early 1990s. Saunders and Allen (2002) identify at least seven reasons for this surge in interest. First, there has been a significant increase in bankruptcies worldwide. Second, the average credit quality of borrowers in capital markets has declined as more and more small and mid-sized firms have gained access to bank loans. Third, profit margins have become very thin. Therefore, the risk-return trade-off from lending has worsened. Fourth, real asset values in many markets have decreased, thus eroding the value of collateral. Fifth, the explosive expansion of off-balance sheet derivative markets has introduced a significant credit exposure. Sixth, the advances in technology have made accurate measurement of credit risk feasible. Finally, because of increasing scrutiny from regulators, especially the Bank for International Settlements (BIS), banks and financial institutions are under growing pressure to measure credit risk accurately. In response to this changing environment, academics and practitioners have been working together to develop newer and better credit risk models since the last decade.

There are two types of credit risk. First, the issuer of a security could default, even if the underlying security itself is default-free. A typical example of this type of credit risk will be an over-the-counter (OTC) call option written on Treasury bills. Second, the assets underlying a derivative security may be subject to default, paying less than what has been promised. This is the case, for instance, with the imbedded option in corporate bond. Models for accurate measurement of both types of credit risk are being actively developed.

There are three key issues that credit risk models have to address. First, credit risk models are primarily used to price bank loans and corporate bonds. These types of loans serve as basic building blocks for modeling credit risk in much the same way that zero coupon bonds are used to set up a term structure of riskless interest rates from which various types of interest rate derivatives can be priced. At a more fundamental level, the valuation of corporate bonds is intimately related to the optimal capital structure choice of a firm. Second, credit risk models are employed to manage credit risky portfolios. The difficult question here is not only describing the stochastic behavior of prices but also measuring correlation between price movements. Third, as the credit derivative markets grow in both depth and breadth, using credit risk models to price credit derivatives has become an indispensable part of banks and financial institutions' risk management practice.

From a modeling perspective, the two quantities that determine the prices of credit risky securities and derivatives are default probability and recovery rate. The latter quantity refers to the fraction of the promised payments which the defaulting entities are able to pay. Accordingly, any credit risk model has to take three sources of risk into account: the riskless interest rate risk, default risk, and recovery rate risk. Different models are mainly characterized by different specifications for the latter two types of risk.

Currently, there are two broadly specified approaches to modeling credit risk: models based on the value of the firm (where "firm" is used as a generic term for the issuer of the bond) and models based on the default intensity. The models used in these two approaches are called *structural models* and *reduced form models*, respectively. In structural models, default is assumed to occur when the underlying firm value process first passes a certain threshold. Therefore in this type of models default is endogenously determined. In contrast, in reduced form models default is taken as exogenous and is assumed to occur when an exogenous Poisson process jumps, capturing the idea that default time takes bondholders by surprise.

Merton (1974) developed the first structural model. His model was subsequently extended by a number of authors. For example, Leland (1994), Anderson and Sundaresan (1996), Leland and Toft (1996), and Mella-Barral and Perraudin (1997) incorporate bankruptcy costs, taxes, and violations of the absolute priority rule in situations of financial distress. Likewise, Kim, Ramaswamy, and Sundaresan (1993), Shimko, Tejima, and Van Deventer (1993), Longstafff and Schwartz (1995), and Wang (1998) have modified the original Merton model to include a stochastic interest rate. Finally, Zhou (1997) substitutes a jump-diffusion process for the geometric Brownian motion process assumed in Merton (1974) to describe the dynamics of firm asset value. Reduced form models were first introduced by Madan and Unal (1994) and Jarrow and Turnbull (1995).

Much of the recent work on credit risk modeling is in this category, see, among others, Jarrow, Lando, and Turnbull (1997), Duffee (1999), and Duffie and Singleton (1999).

Structural models provide very helpful insights into the qualitative aspects of credit risk modeling, whereas reduced form models can not. On the other hand, the structural approach has proven difficult to use for practical applications such as valuing individual corporate debt securities since firm value, which is the key modeling variable of this approach, is rarely observable. In contrast, the reduced form approach is easier to implement and has the advantage of being able to match the levels of credit spreads observed in the market.

It should be noted that the distinction between the structural and reduced form models is not as clear-cut as it appears to be. For instance, a firm-value-based model can result in a default intensity that is similar to that in reduced form models if bondholders can not observe the firm's asset value directly and have to rely on equityholders' periodic and imperfect accounting reports (Duffie and Lando (2001)). In addition, structural models could be easily transformed into reduced form models if we model the value of the firm using a jump process; on the other hand, reduced form models can easily incorporate firm value by using it as a variable determining the default intensity.[1]

Roughly speaking, recovery rates can be modeled in two ways: we may either assume them to be a function of the firm value using the option features of corporate debt or model them as an essentially exogenous quantity that must be estimated from historical data or implied out from observed prices. As a rule of thumb, structural models use one or both of these approaches to model recovery rates, whereas reduced form models normally specify recovery rates as exogenous.

In recent years, important progresses have also been made in modeling credit risk at a portfolio level. For example, currently there are three standard credit risk management approaches that are being widely used by the finance industry worldwide. They are: the KMV and Moody's, CreditMetrics, and Credit Risk Plus approaches. Among these three approaches, the KMV and Moody's method is based on the structural

[1] Jarrow (2001) makes the interesting point that, prior to his work, structural models used only equity prices, avoiding debt prices as too noisy, whereas reduced form models only relied on debt prices, eschewing equity prices altogether. The reduced form model in Jarrow (2001) is claimed to be the first model that uses both debt and equity prices to measure credit risk exposure.

model in Merton (1974). It determines the empirical expected default frequency (EDF) by using KMV and Moody's extensive credit history database. The primary advantage of this approach is that it utilizes stock price data that are highly indicative of the firm's changing financial conditions. The CreditMetrics method was introduced by J.P. Morgan and is also based on the structural approach. This method utilizes credit rating transition matrices, which are publicly available, to compute the firm's credit value at risk (VaR). This approach is characterized by: (1) it considers both an upside and downside to loan values; and (2) it takes into account the actual distribution of estimated future loan values when calculating a capital requirement on a loan. Lastly, the Credit Risk Plus approach was proposed by Credit Suisse Financial Products and is an actuarial method similar to those found in the property insurance literature. The major advantage of this approach is that its data input requirement is minimal, relative to e.g. the CreditMetrics approach. Its major disadvantage is that unlike the CreditMetrics approach, it only considers loss rates rather than loan value changes. Active research is also being conducted to refine the three aforementioned methods and to develop newer and finer methods to manage portfolio credit risk.

There are several particularly interesting areas for future research. First, more work is needed on the implementation of credit risk models, especially on the implementation of structural models since the fundamental variable in these models, firm asset value, is not observable. As a compromise, in order to implement these models, we need to use equity values as a proxy, since the equity of a levered firm can be thought of as a call option written the underlying firm value. This practice will lead to multiple layers of measurement errors in estimation.

Second, we have to develop models that explicitly capture the correlation between market risk and credit risk. Economic theories tell us that market risk and credit risk are not separable but related. Yet the present credit risk models usually treat these two sources of risk as if they are independent of each other. Therefore, a promising avenue for future research is to explicitly capture the correlation between market and credit risk. There are at least two reasons for so doing. First, risk management at financial institutions requires a unified framework that combines both types of risk "under one umbrella," thus facilitating the calculation of the overall VaR for financial institutions.

Second, recent empirical evidence has shown that traditional credit risk models have great difficulties in fitting the changes of credit spreads on corporate bonds (see e.g. Collin-Dufresne, Goldstein, and Martin (2001)). A possible explanation for this failure of credit risk models is that the stock and bond markets might be segmented. This emphasizes the need for further research into the interaction between market risk and credit risk. The recent work of Jarrow and Turnbull (2000) contains such an attempt.

Third, credit derivatives were one of the most important financial innovations in the past decade. Credit derivatives can be defined as contracts whose payoffs are contingent on certain underlying credit events or measures of credit risk quality such as credit spreads. For those investors whose portfolio values are highly sensitive to shifts in the credit spreads, these derivative contracts offer them a very valuable new tool for managing and hedging this type of risk. These derivative securities allow them to manage and transfer credit risk efficiently. Key credit derivatives include credit default swaps, total-return swaps, and credit spread options etc. Credit default swaps pay the buyer of the credit protection a contracted contingent amount when a stipulated credit event occurs, such as default. The contingent amount is usually the difference between the face value of the corporate bond and its market price at the time of the credit event. In return, the credit protection seller receives periodic premium payments from the protection buyer until the time of the credit event, or the maturity date of the default swap, whichever comes first. Total-return swaps pay the net return of one asset class over another. If the two asset classes differ mainly in terms of credit risk, such as a Treasury bond vs. a corporate bond with matched maturity, then the total-return swap can be considered as a credit derivative. Finally, credit spread options convey the right to trade bonds at given spreads over a reference yield, such as the Treasury yield curve. As our understanding of credit risk deepens, we need to know more about the pricing of credit derivatives.

Finally, there is this issue of measuring the credit risk of off-balance sheet derivative securities. The current practice for so doing has two major problems. First, the present method uses models of the stochastic behavior of financial variables while ignoring their inherent oversimplifications and the uncertainty in the model parameters. Second, the current approach ignores the correlation between the exposure to different derivative instruments and the probability of counterparty default. Because of these two problems, the present approach may cause large errors in the estimation of distributions of both future credit exposure and future credit losses.

The following essay develops a structural model of corporate debt with stochastic volatility. In contrast, the classical Merton's (1974) model and its existing extensions all assume constant volatility of the firm value. Simulation results show that for realistic parameter values, the stochastic volatility specification for firm asset value greatly increases the resulting credit spread levels.

A Structural Model of Corporate Debt with Stochastic Volatility

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Abstract

This paper extends the corporate debt pricing model in Merton (1974) to incorporate stochastic volatility (SV) in the underlying firm asset value and derives a closed-form solution for the price of corporate debt. Simulation results show that for realistic parameter values, the SV specification for firm asset value greatly increases the resulting credit spread levels. In particular, for debt maturities of less than or equal to five years, the average increase in credit spread levels is 33 basis points (or equivalently, 32.35%) for a typical firm. Therefore, the SV model addresses one major deficiency of the Merton-type models: namely, at short maturities the Merton model is unable to generate credit spreads high enough to be compatible with those observed in the market.

JEL Classifications: G12, G13, G33

Keywords: credit risk; credit spreads; stochastic volatility; structural models.

1 Introduction

The corporate debt market is among the largest financial markets. For example, according to a recent estimate of the Bond Market Association, the total amount of U.S. corporate debt outstanding is more than US$ 3 trillion, and the U.S. corporate debt market has surpassed the U.S. Treasury market as the largest segment of the U.S. fixed income market (Ericsson and Reneby (2001)). Most corporate debt commands a sizeable spread over the yield of riskless debt (e.g. U.S. Treasury bills, notes etc.). This spread is called the *yield spread.* The components of this yield spread have long been a major research interest of financial economists. Academics generally agree that one of the main components of yield spreads is a credit spread that compensates for credit risk, i.e. the possible default and credit downgrade of corporate debt. Given the size of corporate debt market, the potential losses from corporate bond defaults are large. Therefore, both academics and practitioners have a keen interest in accurately measuring the credit risk embedded in corporate debts.

This paper contributes to the fast-growing literature on modeling the credit spread of corporate debt. Currently there are two broadly specified approaches to modeling credit risk. The first approach is based on the value of the firm, where "firm" should be considered as a generic term for the issuer of the bond. This approach specifies a default threshold and models the allocation of residual value upon default exogenously. The models used in this approach are called *structural* models. Merton (1974) first developed structural models, and they were subsequently extended in Longstaff and Schwartz (1995) and others.

The second approach considers default as exogenous and assumes that default will occur when an exogenous Poisson process jumps. The models used in this approach are called *reduced form* models. Much of the recent work on credit risk modeling follows this approach (see e.g. Madan and Unal (1994) and Duffie and Singleton (1999)).[1]

Structural models provide very helpful insights into the qualitative aspects of credit risk

[1]Sundaresan (2000) reviews the existing structural and reduced form models.

modeling, whereas reduced form models can not. On the other hand, the structural approach has proven to be quite difficult to use for practical applications such as valuing individual corporate debt securities. In addition, it is often criticized for its inability to generate credit spreads high enough to be compatible with those observed in the market. In contrast, the reduced form approach has the advantage of being able to match the levels of credit spreads prevailing in the market.

This paper presents a structural model of credit risk. It extends Merton's (1974) basic model to allow the volatility of the firm's asset value to be stochastic. In contrast, all existing structural models assume constant volatility of the firm value (see the extensions of the Merton model discussed in Section 2.1). Academics have ignored the presence of stochastic volatility (SV) in the underlying firm asset value and its potential impact on credit spread levels. However, the constant volatility of firm value assumption is clearly counterfactual: in reality, the volatility of firm asset value–measured as the sum of book value of firm debt and market value of firm equity–changes over time.[2] Accordingly, in the present paper we directly model the volatility of firm value as stochastic using the setup developed by Merton (1974). This exercise is analogous to the approaches in the option pricing literature that modify the Black-Scholes model by incorporating SV in the underlying stock price to correct for the Black-Scholes model's wide-documented pricing biases.

The discussion of the model's performance is organized around the following two questions: first, does the incorporation of SV in firm value increase the levels of credit spreads generated by the Merton model? If so, to what extent? Second, does the inclusion of SV of firm value alter the shape of the credit spread curve in the Merton model, especially at short maturities? (At short maturities, credit spreads in the Merton model are close to zero, which is contrary to reality).

[2]It is worth pointing out that since firm value is unobservable, it is difficult, if not impossible, to measure the volatility of firm value precisely in practice. However, an indication of the time-varying volatility of firm value is the well- established fact that the volatility of market equity price is stochastic.

We demonstrate that for realistic parameter values the addition of SV to the process for firm value can greatly increase credit spread levels in the Merton model, on average by 33 basis points (bps), or equivalently 32.35%, in the base case for debt maturities of less than or equal to five years. This finding is especially encouraging because it is precisely at short maturities that the Merton-type models are most vulnerable to the criticism that their resulting credit spread levels are too low to be realistic. Moreover, the framework developed is both flexible and practical in that it can be extended to allow for possible early default prior to debt maturity. It can also be applied to valuing various types of corporate debt securities and credit derivatives.

The rest of this paper is organized as follows: we review the Merton model in Section 2, together with the empirical evidence on the model; the SV model and a closed-form solution for corporate discount bond value are given in Section 3; Section 4 reports and discusses the simulation results; Section 5 examines the effect of the relative location of the current variance of firm value; and finally, Section 6 concludes.

2 A review of Merton's (1974) model

In this section, we briefly review the Merton model (in Section 2.1) and discuss its empirical performance (in Section 2.2).

2.1 The theoretical model

In the seminal work of Black and Scholes (1973), it is pointed out that corporate securities can be regarded as contingent claims written on the firm's assets. The Merton model uses this insight to pricing corporate debt and serves as a classic example of contingent claim analysis (CCA). The Merton model remains an indispensable tool in credit risk modeling, and its framework has been applied to measure default risk in the swap markets (see Cooper and Mello (1991)).

Merton (1974) makes the following assumptions

Assumption 1: Under the risk-neutral probability measure, the firm asset value at time t, V_t, is assumed to follow the diffusion

$$dV_t = rV_t dt + \sigma V_t dz_t, \tag{1}$$

where r is the instantaneous risk-free interest rate (see Assumption 4 below), σ^2 is the instantaneous variance of the return on the firm asset value per unit of time, and z_t is a standard Wiener process.

Assumption 2: A "perfect and frictionless" market. Trading takes place continuously in time. Arbitrage opportunities are precluded.

Assumption 3: The Modigliani-Miller theorem holds so that the firm value is invariant to the firm's capital structure choice.

Assumption 4: A constant instantaneous riskless interest rate, r, for both borrowing and lending.

Assumption 5: The firm has both a single issue of zero-coupon (i.e. pure discount) bond with face value of D and maturity date T, and a non-dividend-paying equity. In the event that the promised payment D is not made at date T, the bondholders immediately take over the firm and the shareholders receive nothing.

The geometric Brownian motion process in equation (1) is similar to that assumed for the stock price in Black and Scholes (1973). The absence of arbitrage opportunities in Assumption 2 is equivalent to the existence of an equivalent martingale measure (see Harrison and Kreps (1979)). This implies that we can readily apply standard derivative pricing approaches to value corporate debt in the present context. Assumption 3 is a standard assumption made in the literature and is actually proved as part of the analysis in Merton (1974). The constant riskless interest rate assumption is made to clearly distinguish the effect of credit risk on corporate debt value from the term structure of interest rate effect.

Finally, Assumption 5 implies that the equity functions as the residual claim on the firm's assets upon maturity of the debt. At the maturity date of the debt, the payoff to the equity is equivalent to that of a European call option written on the firm's asset value with a strike price equal to the face value of the debt and the time to maturity coinciding with that of debt. Using a no-arbitrage argument, the equity must be of the same value as that of the call option at the initiation date of the debt. It follows that the debt value is the difference between the firm value and the call option value that can be calculated using the Black-Scholes formula.

Many of the above assumptions have subsequently been relaxed in an attempt to bring the Merton model more in line with reality. For example, the "perfect market" assumption is relaxed by Leland (1994), Anderson and Sundaresan (1996), Leland and Toft (1996), and Mella-Barral and Perraudin (1997) to incorporate bankruptcy costs, taxes, and violations of the absolute priority rule in situations of financial distress. Likewise, Kim, Ramaswamy, and Sundaresan (1993), Shimko, Tejima, and Van Deventer (1993), Longstafff and Schwartz (1995), and Wang (1998) have modified the basic model to include a stochastic interest rate. Finally, Zhou (1997) substitutes a jump-diffusion process for the geometric Brownian motion process assumed in equation (1).[3] These efforts to modify the Merton model have

[3]Zhou (1997) maintains the constant volatility of firm value assumption. His results show that including jumps in firm asset value increases the credit spread levels in the Merton model, particularly at short maturities. Allowing for jumps in firm asset value is attractive in that it addresses one deficiency of the Merton-type models: namely, all these models assume that firm value follows a diffusion process. Under a diffusion process, default can only be triggered by a gradual decline in firm asset value over a longer time horizon, therefore in these models default can only occur *expectedly*. On the other hand, by incorporating jumps in firm asset value, Zhou (1997) can accommodate the empirical fact that in reality firms do sometimes default *unexpectedly* due to a sudden drop in firm value over a short period of time. However, it is important to realize that the SV and jump models are simply two different modeling approaches. They are not conflicting but complementary. When compared to jumps, a SV process is easier to estimate. Also SV can be linked more fundamentally to economic factors (e.g. leverage ratios) that drive the firm asset value process.

achieved some success empirically as evidenced in Anderson and Sundaresan (2000). These two authors test the models of Merton (1974), Leland (1994), and Anderson and Sundaresan (1996) on yield indices of U.S. investment grade corporate bonds and find that the latter two models outperform that of Merton since they better fit the observed yields and result in more plausible parameter estimates.

The Merton model is used as a benchmark in this study. In this model the formulas for equity value and discount bond value are the following

$$E(V,\tau) = V\Phi(x_1) - De^{-r\tau}\Phi(x_2), \tag{2}$$

where $E(\cdot)$ denotes the value of equity, $\Phi(\cdot)$ is the cumulative standard normal distribution function, $\tau = T - t$ is the time remaining until debt matures, and $x_1 \equiv \frac{\ln(V/D)+(r+\frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$, $x_2 \equiv x_1 - \sigma\sqrt{\tau}$. Also recall that V is the firm asset value, D is the face value of the bond, and σ is the volatility of the rate of return on firm asset value.

The debt value can be written as

$$\begin{aligned} B(V,\tau) &= V - E(V,\tau) \\ &= De^{-r\tau}\{\Phi[h_2(d,\sigma^2\tau)] + \frac{1}{d}\Phi[h_1(d,\sigma^2\tau)]\}, \end{aligned} \tag{3}$$

where $B(\cdot)$ denotes the value of debt, $h_1(d,\sigma^2\tau) \equiv \frac{-[\frac{1}{2}\sigma^2\tau-\ln(d)]}{\sigma\sqrt{\tau}}$, $h_2(d,\sigma^2\tau) \equiv \frac{-[\frac{1}{2}\sigma^2\tau+\ln(d)]}{\sigma\sqrt{\tau}}$, $d \equiv \frac{De^{-r\tau}}{V}$. The "d" parameter is the so-called "quasi" debt-firm value ratio (see Merton (1974, p. 455)). It is typically higher than the firm's market leverage ratio and lower than the firm's book debt-to-capitalization ratio.

Equivalently, we may rewrite equation (3) in terms of the yield spread as

$$R(\tau) - r = -\frac{1}{\tau}\ln\{\Phi[h_2(d,\sigma^2\tau)] + \frac{1}{d}\Phi[h_1(d,\sigma^2\tau)]\}, \tag{4}$$

where $R(\tau)$ is the yield-to-maturity on the risky debt provided that the firm does not default. Therefore, $R(\tau)-r$ defines a risk premium, and, overall, equation (4) defines the *risk structure of interest rates* (see Merton (1974, p. 454)).

2.2 The empirical performance of the Merton model

Generally speaking, the Merton model has not done well in empirical tests. On the positive side, Sarig and Warga (1989) find that the term structure of credit spreads predicted by the Merton model is consistent with what they have observed in the data. On the negative side, numerous studies of the Merton model document that the model tends to underestimate credit spread levels, particularly at short maturities.[4]

Perhaps the most widely cited empirical study of the Merton model to date is that of Jones, Mason, and Rosenfeld (1984). The model considered in that study is the Merton model for a single issue of non-convertible callable coupon bond with a sinking-fund feature. This study investigates a sample of twenty-seven firms that were found to have a simple capital structure (i.e. one class of stock, no convertible bonds, a small number of debt issues, and no preferred stock) in January 1975. The sample consists of monthly data between January 1975 and January 1981. When comparing prices predicted by the model with market prices, Jones et al. find that for the entire sample the average pricing error is 4.5 percent and the absolute values of the errors average 8.5 percent. More importantly, they find that for investment-grade bonds the Merton model performs no better than a naive model that assumes no credit risk. However, the Merton model does exhibit some improvement over the naive model when applied to speculative-grade bonds. Overall, Jones et al. conclude that the Merton model overprices corporate debt, or equivalently, underestimates credit spread levels.

[4]The results in Elton, Gruber, Agrawal, and Mann (1999) indicate that state taxes are an important component of credit spreads. The reason is that although coupon payments on corporate bonds are subject to state taxes, coupon payments on government bonds are not. Investors of corporate bonds thus require higher yields on corporate bonds to compensate them for the difference in state taxes. However, state taxes have been absent from almost all existing models of credit risk (and credit spreads), including the Merton model. It then follows that the inability of the Merton model to generate high enough credit spreads may be partly due to the model's failure to account for the impact of state taxes.

The finding in Ogden (1987) echoes that of Jones et al. (1984). He examines a sample of fifty-seven bond offerings and their transaction prices during the years 1973-1985. In contrast to Jones et al. (1984), Ogden analyzes spreads, not bond prices. He reports that the average pricing error of the Merton model is minus 104 bps. Thus, the Merton model appears to undervalue credit spreads. Similarly, Franks and Torous (1989), in their empirical study of U.S. firms in reorganization, document that for typical parameter values the Merton model underestimates credit spread levels by 109 bps–however, their finding is not very surprising since they examine firms in financial distress.

Similar evidence, for instance, can also be found in Kim et al. (1993). This study relies on numerical simulations, showing that even with excessive values for the "quasi" leverage ratio and firm value volatility, the maximum credit spread level produced by the Merton model for a ten-year corporate bond with an annual coupon rate of 9 percent is no higher than 120 bps. In contrast, over the 1926-1986 period, the yield spreads on AAA-rated bonds had a range of 15 to 215 bps with an average of 77 bps, while the yield spreads on BAA-rated bonds (also investment-graded) ranged from 51 to 787 bps and had a mean of 198 bps (see Kim et al.(1993)).

More recent studies that use better quality bond data than the earlier ones reach the same conclusion. For instance, Eom, Helwege, and Huang (2000) find that the spreads predicted by the Merton model are significantly lower than the observed spreads, particularly for bonds of high rating and shorter maturity, and bonds issued by firms with low volatility. According to the monthly Lehman Brothers Bond Index Data from 1973 to 1993, historically the average yield spreads of ten-year corporate bonds of various credit ratings over U.S. Treasury securities of similar maturities are: Aaa: 63 bps; Aa: 91 bps; A: 123 bps; Baa: 194 bps; Ba: 299 bps; and B: 408 bps. Using numerical simulations, Huang and Huang (2000) report that for the Merton model, the portion of spreads that is attributable to credit risk is: Aaa: 8 bps; Aa: 10 bps; A: 14.3 bps; Baa: 32 bps; Ba: 137.9 bps; and B: 363.3 bps. Since credit risk is assumed to be the only source of yield spread in the Merton model, their simulation results serve as indirect evidence that the Merton model underestimates spread levels, especially for high-rated bonds. However, it is apparent from Huang and Huang's (2000) findings that the Merton model does reasonably well for low-rated bonds.

3 A stochastic volatility model of corporate debt

In this section, we introduce a new structural model of credit risk with SV in firm asset value and present a closed-form solution for corporate debt value. Similar to all the existing extensions to the Merton model (see Section 2.1), we relax one of the basic assumptions made in Merton (1974). In particular, we maintain all the assumptions of Merton (1974) except allowing for SV in firm value. That is, we modify Assumption 1 of Merton (1974) to

Assumption 1': Under the risk-neutral probability measure, firm asset value at time t, V_t, is assumed to follow the diffusion

$$dV_t = rV_t dt + \sqrt{\xi_t} V_t dz_{1t}, \tag{5}$$

where z_{1t} is a standard Wiener process. The instantaneous variance of the return on firm value ξ_t is given by a familiar Cox, Ingersoll, and Ross (1985)-type mean-reverting square-root process

$$d\xi_t = \kappa(\theta - \xi_t)dt + \eta\sqrt{\xi_t} dz_{2t}, \tag{6}$$

where κ is the mean-reversion speed of the instantaneous variance, θ is the long-run mean level of variance, η is the "volatility of volatility" parameter, and z_{2t} is another standard Wiener process which has an instantaneous correlation coefficient ρ with z_{1t}. The stochastic volatility model in equations (5) and (6) is inspired by Heston (1993), who uses similar dynamics to model stock price and its volatility.

The present model can be regarded as a two-factor model of corporate debt, where the two factors are firm asset value and its instantaneous SV, respectively. In contrast, the

Merton model is a one-factor model where the only factor underpinning the credit risk is firm value. The SV model nests the Merton model for η equal to zero (see also Section 4.3).

It can then be shown (see Scott (1987)) that the value of any contingent claim $U(V, \xi, t)$ must satisfy the following partial differential equation (PDE)

$$\frac{1}{2}\xi V^2 \frac{\partial^2 U}{\partial V^2} + \rho\eta\xi V \frac{\partial^2 U}{\partial V \partial \xi} + \frac{1}{2}\eta^2 \xi \frac{\partial^2 U}{\partial \xi^2} + rV \frac{\partial U}{\partial V} + \kappa(\theta - \xi)\frac{\partial U}{\partial \xi} - rU + \frac{\partial U}{\partial t} = 0. \tag{7}$$

A European call option with strike price D and maturing at time T satisfies the above PDE subject to the following boundary conditions

$$\begin{aligned}
E(V, \xi, T) &= \max(0, V - D), \\
E(0, \xi, t) &= 0, \\
\frac{\partial E(\infty, \xi, t)}{\partial V} &= 1, \\
rV \frac{\partial E(V, 0, t)}{\partial V} + \kappa\theta \frac{\partial E(V, 0, t)}{\partial \xi} - rE(V, 0, t) + \frac{\partial E(V, 0, t)}{\partial t} &= 0, \\
E(V, \infty, t) &= V.
\end{aligned} \tag{8}$$

By analogy with the Black-Scholes formula, we guess a solution of the form

$$E(V, \xi, t) = VP_1 - De^{-r(T-t)}P_2, \tag{9}$$

where P_1 and P_2 can be interpreted as risk-neutral probabilities. In particular, P_2 corresponds to the risk-neutral conditional probability that at maturity date T, firm asset value V_T is no less than the face value of the debt D, and firm is thus able to make its debt payment (i.e. this European call option expires in-the-money). In terms of the logarithm of the spot firm asset value, $x = ln[V]$, the *characteristic functions* corresponding to probabilities $P_j (j = 1, 2)$ are given by

$$f_j(x, \xi, t; u) = e^{A(T-t;u) + B(T-t;u)\xi + iux}, \tag{10}$$

where

$$A(\tau;u) = rui\tau + \frac{c}{\eta^2}\{(d_j - \rho\eta ui + n)\tau - 2\ln[\frac{1-me^{n\tau}}{1-m}]\},$$
$$B(\tau;u) = \frac{d_j - \rho\eta ui + n}{\eta^2}[\frac{1-e^{n\tau}}{1-me^{n\tau}}],$$
$$m = \frac{d_j - \rho\eta ui + n}{d_j - \rho\eta ui - n},$$
$$n = \sqrt{(\rho\eta ui - d_j)^2 - \eta^2(2\phi_j ui - u^2)},$$
$$\phi_1 = \frac{1}{2}, \phi_2 = -\frac{1}{2}, c = \kappa\theta, d_1 = \kappa - \rho\eta, d_2 = \kappa, \tau = T - t.$$

We can then obtain the risk-neutral probabilities P_j, $j = 1,2$, by inverting the characteristic functions[5]

$$P_j(x,\xi,T;\ln[D]) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}[\frac{e^{-iu\ln[D]}f_j(x,\xi,T;u)}{iu}]du. \tag{11}$$

Together, equations (9), (10), and (11) give a closed-form solution for a European call option price.

In the present context, we know from Merton's (1974) analysis that equations (9), (10), and (11) also give the value of the firm's equity. By design, the value of the firm's debt is given by

$$B(V,\xi,t) = V - E(V,\xi,t), \tag{12}$$

and the risk structure of interest rates can be expressed as

$$R(\tau) - r = \frac{1}{\tau}(\ln D - \ln B(V,\xi,t)) - r. \tag{13}$$

4 Simulation results

This section discusses the simulation results. The results are further divided into three groups: results obtained in the base case, results on default probability and recovery rate,

[5]Note that characteristic functions always exist, and in Kendall and Stuart (1977) it is demonstrated that the integral in equation (11) converges.

and those of the comparative statics.

4.1 Results for the base case

Table I contains the base case parameter values that used in the numerical simulations. The current annual variance level of firm value is set at 0.1, which corresponds to an annualized volatility (standard variation) of 0.316. This is consistent with Jones et al. (1984), who estimate that the individual firms in their sample have an annualized volatility of firm value ranging from 0.052 to 0.628. For the base case the current variance is set at its long-run mean level, because of the assumed mean-reversion in firm variance level (we will examine the robustness of this assumption in Section 5). An annualized riskless interest rate of 6 percent is used, the same as in Longstaff and Schwartz (1995) and Zhou (1997). Finally, a negative correlation coefficient (ρ = -0.5) is chosen between the value of the firm and the SV of firm value. That is, we conjecture that firm asset value and its volatility are negatively correlated. It must be noted that the validity of this conjecture and the relationship between firm value movement and its volatility are an open empirical question. This paucity of evidence is partially due to the difficulty in measuring firm value accurately, since it is unobservable and in reality firms have very complex capital structures. The usual practice of measuring firm value as the sum of a firm's book-value of debt and market-value of equity is only an approximation. In contrast, the "leverage effect" in stock price volatility has been widely documented in the empirical literature (see among others, Black (1976) and Christie (1982)). We will examine the impact of alternative choices for ρ in Section 5.

Table II presents, for the base case parameters, the yield spreads generated by the SV model, those produced by the Merton model, and the difference between the two when d, the "quasi" debt-firm value ratio (or leverage ratio), is set at 0.2 or 0.5. The values of d are identical to those used in Table 1 in Merton (1974, p. 457). In addition, we consider six debt maturities: 0.5, 1, 2, 5, 10, and 15 years. Figure 1 gives a graphic illustration of the difference in yield spreads.

Several findings are noteworthy. First, it is evident from both the table and the figure that the maximal increase in yield spread levels resulted from incorporating SV of firm value into the Merton model occurs at maturities of less than or equal to five years. At these short maturities, the SV specification for firm asset value increases the resulting credit spread levels by an average of 10 bps when $d = 0.2$ and 33 bps for $d = 0.5$. The significance of these increases is best illustrated in relative terms. For debts with maturities less than or equal to five years, the average yield spread generated by the Merton model is only 3.4 bps when $d = 0.2$ and is 102 bps when $d = 0.5$. Therefore, by incorporating SV of firm value, the yield spread levels increase compared to the Merton model on average by 294% for $d = 0.2$ and 32.35% for $d = 0.5$! In addition, when $d = 0.5$ the yield spread curve peaks at around 2.5 years. These results are particularly encouraging because, as suggested by the empirical evidence in Section 2.2, it is precisely at short maturities that the Merton-type models have greatest difficulty in matching the observed credit spread levels. This result also shows that the SV specification for firm value can substantially alter the shape of the credit spread curve predicted by the Merton model, especially at short maturities.

Second, by incorporating SV of firm value into the Merton model, we can increase the resulting credit spread levels by a sizeable amount across all maturities: on average 16 bps when $d = 0.2$ and 12 bps for $d = 0.5$; or in relative terms, on average 48.48% for $d = 0.2$ and 6.9% for $d = 0.5$. For instance, when d is set at 0.2, for a debt with maturity of 5 years, the Merton model generates a credit spread of merely 12 bps, whereas the SV model generates 35 bps, almost triple that of Merton. Likewise, at 1 year of maturity, for a firm with d of 0.5, the SV model's credit spread (55 bps) is far more than twice as large as in the Merton model (22 bps). Recall that d is an upward biased estimate of the actual leverage ratios. Thus when we set d equal to 0.2 and 0.5, we are essentially examining firms with low and medium leverage ratios. Roughly speaking, bonds of these firms are of investment-grade: BBB or higher according to Standard & Poor's, or Baa or higher according to Moody's. (In fact, it is reasonable to say that firms with d of 0.2 have bonds of highest ratings–for instance, AAA according to Standard & Poor's or Aaa according to Moody's–because the Aaa-rated firms had a historical average leverage ratio of 13.08% (Standard & Poor's (1999)).) Since it is clear from the empirical evidence in Section 2.2 that the Merton model tends to underestimate spread levels for high-rated bonds, our study shows that the specification for SV in firm value can help significantly increase Merton's yield spreads for firms with high-rated bonds.[6]

Third, as time to maturity lengthens, the effect of SV in firm value diminishes and the yield spreads of the SV model converge with those of the Merton model. This is as expected since SV is mean reverting, and thus will be pulled back to its long-run mean level, θ, as time to maturity increases. (In the base case, the firm value volatility is assumed to have a mean-reversion speed parameter, κ, equal to 0.5, which implies a half-life of mean-reversion of about 1.4 years.) It then follows that in the long run, the impact of SV of firm value on credit spread levels will diminish.[7]

Finally, as Panel A of Table II indicates, for firms with very low leverage ratios (d = 0.2 in the table), and for very short maturities (maturities of 0.5 and 1 year in the table), the SV model may generate credit spreads lower than those of the Merton model, which are essentially zero. The explanation for this finding is the following. At very short maturities, a marginally levered firm can be thought of as being well above its default boundary and the likelihood of it defaulting during the time remaining to maturity is very low, because of the

[6]In this paper we focus on investment-grade bonds. The properties of credit spreads for speculative-grade bonds are studied in e.g. Sarig and Warga (1989), Fons (1994), and Helwege and Turner (1999). In general, the evidence on the credit spreads for speculative-grade bonds is inconclusive.

[7]The Merton model generates counterfactual near-zero credit spreads at sufficiently long maturities. One possible explanation for this problem concerns the fact that in the Merton model the firm leverage ratios are non-stationary (Collin-Dufresne and Goldstein (2001)). Developing a structural model of corporate debt in which the firm's leverage ratio is assumed to be stationary (as in Collin-Dufresne and Goldstein (2001)), while at the same time allowing firm value volatility to be stochastic, may be a fruitful path for future research.

assumed diffusion process for the firm asset value (see also Footnote 3 in Section 2.1). This explains the almost zero credit spreads generated by the Merton model, since in the Merton model default risk is assumed to be the only source of credit spreads. On the other hand, the effects of SV on credit spreads take time to "materialize," due to the assumed stochastic process for the volatility. It follows that the SV model may not be able to improve on the Merton's credit spread levels in this case.[8] In contrast, for firms with medium leverage ratios ($d = 0.5$ in Panel B of Table II), the SV specification for firm asset value largely increases the Merton's credit spreads at every debt maturity except the longest two (10 and 15 years in the table).

To verify whether the parameter values used in our simulations are realistic, we calculate in Table III the stock price volatility in the Merton model that is implied by the chosen parameter values. This volatility is calculated as follows

$$\sigma_E = \sigma_V \frac{V \frac{\partial E}{\partial V}}{E}, \tag{14}$$

where σ_E and σ_V refer to the volatilities of stock price and firm value, respectively. The majority of stock price volatility obtained in Table III are in the range of 38% to 60%, slightly higher than the value for the annualized actual stock price volatility of 20% to 40% documented in Hull (2000). Therefore, we conclude that our parameter values are reasonable. Notice from Table III that for a given value of d, the implied stock price volatilities also change as we vary debt maturities. This is due to the fact that in formula (14), both equity value E (equal to a call option price in the present context) and $\frac{\partial E}{\partial V}$ ($= \Phi[x_1]$) change as time to maturity changes.

[8]In Panel A of Table II, when $d = 0.2$, for maturities of 0.5 and 1 year the SV model generates negative credit spreads. This result is counter-intuitive since it implies that default risky debts are more valuable than the corresponding default-free bonds. We believe that this result is due to the slight inaccuracy of the numerical algorithm used, which is unavoidable. However, the main conclusions of this paper still hold.

4.2 Results on default probabilities and recovery rates

Given the results in Section 4.1, we only report the computational results obtained when the value of d is set at 0.5 throughout the rest of this paper to save space, unless otherwise stated.[9] Since, by definition, d is always higher than market leverage ratios, a value of 0.5 for d is consistent with the evidence in Huang and Huang (2000) that the historical average leverage ratio of Baa-rated firms was 43.28% (Standard & Poor's (1999)).

Figure 2 reports the risk-neutral conditional cumulative default probability calculated for both the Merton and SV models. Three findings are noteworthy. First, the SV model generates a higher default probability than the Merton model for debts with maturities less than or equal to three years; however, the difference between the two models' default probabilities is small (on average about 2% across maturities). (In the results not reported here, when d is set at 0.2, the cumulative default probability in the SV model is higher than that in the Merton model at almost every maturity, with the largest probability difference–around 2.6%–occurring at a maturity of about 5 years.) On the other hand, we see from Panel B in Table II that the SV model produces significantly higher yield spreads than the Merton model for debts with maturities less than or equal to five years (on average 33 bps). Because this is an important result, we now offer the intuition for it.

In both the SV and Merton models, credit risk affects yield spread in two forms: through the default probability and through the recovery rate of debt when default occurs. Obviously, when calculating credit spreads in reality, the recovery rate is as important as the default probability: a higher (resp. lower) default probability, coupled with a lower (resp. higher) recovery rate, naturally leads to a lower (resp. higher) debt price (and a higher (resp. lower) yield spread), all else being equal; on the other hand, a higher default probability, but together with a sufficiently higher recovery rate, may actually result in a higher debt price (and a lower yield spread) when compared to a case where both the default probability and

[9]The results for the case where $d = 0.2$ are available upon request.

the recovery rate are sufficiently lower, all else being the same. In both the Merton and SV models, the recovery rate upon default depends on the firm's remaining asset value at that time. Since both models preclude early default prior to debt maturity (i.e. default can only occur when debt matures), the firm's asset value upon default is stochastic. It then follows that in both models the recovery rate of debt is also stochastic. As mentioned in Section 3, the SV model has one more factor underlying credit risk–the SV of firm asset value–when compared to the Merton model, which has only one factor–firm value. Therefore, it is reasonable to believe that the SV model can add more variations to credit risk, resulting not only in a more variable default probability, but also in a more volatile recovery rate compared to the Merton model.

To verify the above intuition, we run some Monte Carlo simulations (with 1,000 replications) to compute the mean recovery rate in the SV and Merton models.[10] The results are reproduced graphically in Figure 3. Figure 4 contains the results obtained for debts with maturities less than or equal to fives years. Consistent with our intuition, both Figure 3 and Figure 4 show that the SV model generates a lower mean recovery rate than that in the Merton model for maturities less than or equal to five years; for maturities over five years, these two models produce almost indistinguishable mean recovery rates. In a parallel Monte Carlo simulation exercise, we find that the volatility of the recovery rate in the SV model is generally higher than that in the Merton model, again consistent with our intuition. (The results of the latter Monte Carlo exercise are available upon request.) Therefore a small difference in default probabilities (in Figure 2) can lead to a much larger difference in credit spread levels (in Panel B of Table II). A numerical example helps to further illustrate this point. In the base case, a bond with a maturity of 2 years has a risk-neutral conditional cumulative default probability of 10.57% in the SV model and a probability of 9.24% in the Merton model. The difference in probabilities is 1.33%. Monte Carlo simulations show that

[10]The results obtained after 5,000 replications of Monte Carlo simulations (not reported) are virtually indistinguishable.

this bond has a recovery rate of 39.6% of par in the SV model and a recovery rate of 41.6% of par in the Merton model.[11] The difference in recovery rates is therefore -2%. Under the risk-neutral measure, we calculate the bond price as the expected payoff of the bond (in the event of default and in the event of no default) discounted at the risk free interest rate for the time to maturity. The corresponding yield spreads in the two models can then be readily computed. The difference in yield spreads is found to be 53 bps, only slightly higher than that reported in Panel B of Table II, which is 47 bps.

Second, although the magnitudes of the reported default probabilities appear to be large, they are indeed consistent with those found in Jarrow, Lando, and Turnbull (1997). The estimates also correspond approximately to the Moody's weighted-average cumulative default rates (1970-1993) for Ba- and B-rated firms reported in Fons (1994). This result provides additional evidence that the parameter values used to generate Table II and Figure 1 are reasonable. Also notice that the default probabilities reported in Figure 2 are the risk-neutral ones, which are found to be much larger than the actual ones (see Duffee (1999)).

Finally, notice that in Figure 2 over longer period of time (at maturities of more than five years), the Merton model actually produces a higher default probability than the SV model, though the difference is again not large. On the other hand, Figure 3 provides weak evidence that at maturities longer than eight years, the SV model generates a higher mean recovery rate than the Merton model. Together, these two facts partially explain the observations in Table II and Figure 1 that the impact of SV on credit spreads diminishes at long maturities.

4.3 Comparative statics

In Figure 5, we experiment with different values of κ, the mean-reversion speed of the stochastic variance of firm value. A lower value of κ means that a longer time is needed for

[11] According to Moody's estimate, senior unsecured debts have an average recovery rate of 44% of par in the event of default.

the current variance to return to its long-run mean level once it deviates from the latter. Thus, we will expect the SV to have a larger impact on credit spread levels when κ is low. The opposite reasoning holds when κ is high. Figure 5 indicates that for maturities less than or equal to eight years the results confirm this reasoning. Figure 6 reports the results for $d = 0.2$. Across maturities the results obtained are consistent with our intuition.

Next, we study the effect of the volatility of volatility parameter η in Figure 7 (where $d = 0.5$) and Figure 8 (where $d = 0.2$). If η is zero, the volatility is deterministic, and the continuously compounded return on firm asset value is normally distributed, and the SV model nests that of Merton. Otherwise, an increase in η increases the kurtosis (or the fourth moment) of the underlying distribution of firm asset value. A higher value of η thus implies a more volatile SV itself, creating a larger swing around the upward trend in firm value, $rV_t dt$ (provided that $r > 0$, which is generally true since r denotes the nominal risk free interest rate). Intuitively, a higher value of η therefore leads to a larger increase in credit spread levels brought about by SV in firm value, all else being the same. The pattern shown in Figure 7 and Figure 8 confirms this intuition.

Finally, Figure 9 demonstrates the effect of the correlation coefficient ρ between SV and firm value. The correlation parameter ρ is positively correlated with the skewness (or the third moment) of the distribution of the returns on firm asset value. Specifically, a positive ρ (= 0.5) may render a SV credit spread lower than that of the Merton model, and the resulting term structure can be inverted humped-shaped. The opposite effect holds true for a negative ρ (= -0.5), i.e., when ρ is negative, SV credit spreads are in general higher than those of the Merton model, and the resulting term structure is of humped-shape. When ρ is equal to zero, the resulting pattern will be a mixture of the two extremes just mentioned, but bears more "resemblance" to the pattern associated with a negative ρ. We can explain this finding using option pricing theory. Using the put-call parity for European options, we may alternatively write the corporate debt value as the difference between the face value of debt (discounted at the risk free interest rate) and a European put option written on firm value that shares all the characteristics of the call option–the equity. When firm value goes down, a negative ρ will increase firm value volatility. Both of these effects will tend to raise the put price, driving down debt value, and increasing the credit spread. On the other hand, when ρ is positive, a decline in firm value also brings down volatility. The first force increases put price, while the second force works to depress put value. These two forces will tend to offset each other in such a way that the overall impact on put price (and thus on debt value and credit spread) becomes ambiguous. However, judging from the graph, it seems that in the latter case the decrease in volatility has a greater impact on put option value and subsequently on credit spread levels.

The analysis above shows that we can conveniently explain the properties of credit spreads in terms of the first four moments–mean, variance, skewness, and kurtosis–of the underlying distribution of firm asset value under the risk-neutral probability measure.

5 The location of current variance relative to its long-run mean

In this section, we report an additional implication of SV in firm asset value. This implication is in a sense more subtle than that reported previously. In Figures 10, 11, and 12, we investigate the influence of the location of the current variance ξ_t relative to its long run mean θ on credit spread levels for different values of ρ. In particular, we set ρ equal to -0.5, 0, and 0.5 in Figures 10, 11, and 12, respectively. We then conduct the following experiment: for each value of ρ, we set current variances evenly on both sides of its long run mean level. We know from the analysis in Merton (1974) that credit spreads monotonically increase in volatility for a fixed debt maturity. Therefore, if the effect of SV is not substantive, we will expect to observe that at a given maturity when ξ_t is higher than θ, SV increases credit spreads by an amount (almost) identical to the decrease in credit spreads when ξ_t is lower than θ. On the other hand, if the effect of SV is substantive, we will expect to see that SV increases credit spreads in the case where ξ_t is higher than θ by an amount (much) larger than the decrease in credit spreads when ξ_t is lower than θ.

A number of observations are important. First, we see from all these three figures that for a given correlation coefficient ρ, when current variance is lower than its long run mean, SV predicts a lower spread than that of Merton, and when current variance is above its long run mean, SV increases credit spread levels. This pattern in credit spreads is consistent with our intuition. Second, for comparison purpose we have defined the net increase in credit spreads as the increase in spreads minus the corresponding decrease (in absolute terms) in spreads. Notice from Figure 10 that when ρ is negative, the net increase in the generated credit spreads is both positive and significant, particularly for maturities of less than two years: for instance the mean (averaged across all maturities) net increase in credit spreads in case when $\theta = .1$ and $\xi_t = .15$ over those in case where $\theta = .1$ and $\xi_t = .05$ is 25 bps. Similarly, in Figure 11 where ρ is set at zero the net increase in credit spreads is also positive for the most part. We assume a positive value for ρ in Figure 12. The same pattern in credit spreads exhibits in this figure: SV predicts a higher spread than the Merton model when the current variance is above its long run mean, and the opposite is true when the current variance is lower than its long run mean. However, in this case the net increase in credit spreads brought out by SV is negative. Overall, it is fair to say that the inclusion of SV of firm asset value can substantially increase the resulting credit spread levels, although the net effect of SV depends critically on the sign of the correlation coefficient ρ.

6 Conclusion

This paper extends the seminal work of Merton (1974) by developing a structural model of corporate debt that incorporates the stochastic volatility of firm asset value. We derive a closed-form solution for the resulting corporate debt price. Computational experiments show that for realistic parameter values incorporating SV can substantially increase the generated credit spread levels in both absolute and relative terms. The largest increase in spreads occurs for debts with maturities less than or equal to five years, even for firms with a low or medium leverage ratio: for a typical firm an increase on average of 33 bps, or equivalently, 32.35% of the original credit spread levels in the Merton model. This result is encouraging since it is well known that contrary to observed credit spreads, the credit spreads predicted by the Merton model are not much higher than zero for short-maturity debts, especially for high-grade ones.

When we implement the SV model in practice, we need to estimate the parameters of the stochastic processes for firm asset value V and its variance ξ. None of these two quantities are observable. One promising way to overcome this problem is the maximum likelihood estimation method suggested in Duan, Gauthier, Simonato, and Zaanoun (2002). Their approach utilizes the fact that in structural models (including the SV model) equity can be regarded as an option written on firm value. Since equity prices and their volatility are observable, we can apply a transformed data technique to write down the likelihood function in explicit forms. We refer the interested reader to Duan et al. (2002) for further details.

Table I:
Base case parameter values

Under the risk-neutral probability measure, the firm asset value is assumed to follow the joint diffusion process below

$$\begin{aligned} dV_t &= rV_t dt + \sqrt{\xi_t} V_t dz_{1t}, \\ d\xi_t &= \kappa(\theta - \xi_t)dt + \eta\sqrt{\xi_t} dz_{2t}, \end{aligned}$$

where $dz_{1t}dz_{2t} = \rho dt$.

Parameter	*Value*
mean reversion speed	$\kappa = 0.5$
long-run mean of variance	$\theta = 0.1$
current variance	$\xi_t = 0.1$
correlation between z_{1t} and z_{2t}	$\rho = -0.5$
volatility of volatility	$\eta = 0.225$
annual riskless interest rate	$r = 0.06$
current firm asset value	$V = 100$

Table II: Base case results

The following two tables contain the results obtained using the base case parameter values (given in Table I) where d, the "quasi" debt-firm value ratio, is set at 0.2 and 0.5, respectively. The *SV's YS* is the yield spread generated by the SV model. The *Merton's YS* is the yield spread generated by the Merton model. The *YS difference* is the yield spread of the SV model minus that of the Merton model. All the numbers have been rounded to the nearest decimal point.

Panel A: $d = 0.2$

Debt maturity (in years)	*SV's YS* (in basis points)	*Merton's YS* (in basis points)	*YS difference* (in basis points)
0.5	-4	0	-4
1	-2	0	-2
2	4	0	4
5	35	12	23
10	68	47	21
15	86	75	11

(Panel B is continued on the next page.)

Panel B: $d = 0.5$

Debt maturity (in years)	*SV's YS* (in basis points)	*Merton's YS* (in basis points)	*YS difference* (in basis points)
0.5	9	2	7
1	55	22	33
2	129	82	47
5	196	174	22
10	211	211	0
15	211	219	-8

Table III: Implied Merton volatilities

The *implied Merton volatility* is the stock price volatility in the Merton model that is implied by the parameter values chosen for the base case. All the numbers have been rounded to the nearest decimal point.

Debt maturity (in years)	*Implied Merton volatility* (d = *0.2*) (in percentage points)	*Implied Merton volatility* (d = *0.5*) (in percentage points)
0.5	40	63
1	40	63
2	40	60
5	39	53
10	38	47
15	37	44

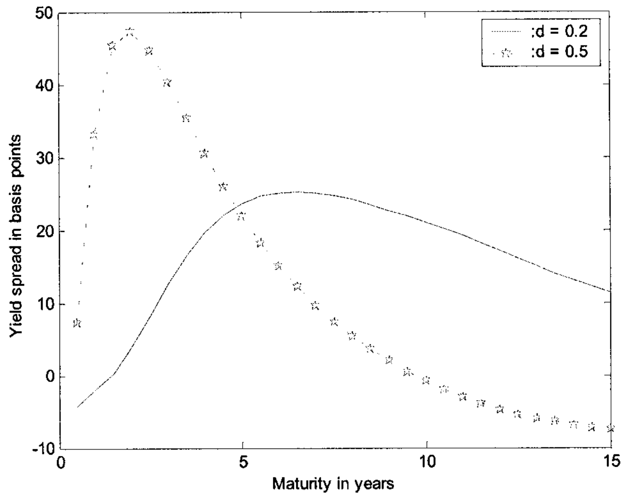

Figure 1:

Yield spread of the SV model minus that of the Merton model where d, the "quasi" debt-firm value ratio, is set at 0.2 and 0.5, respectively. Base case parameters: $\kappa = 0.5$, $\xi_t = \theta = 0.1$, $\eta = 0.225$, $\rho =$ -0.5, $r = 0.06$, V =100.

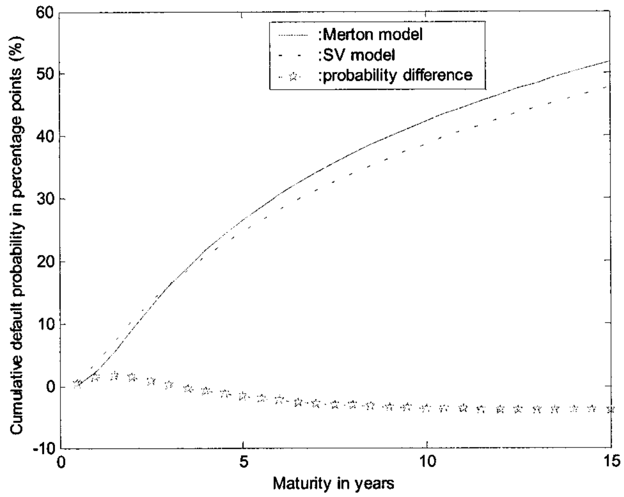

Figure 2:

Risk-neutral conditional cumulative default probability of the SV model and the Merton model. The *probability difference* is the probability of the SV model minus that of the Merton model. Base case parameters: $\kappa = 0.5$, $\xi_t = \theta = 0.1$, $\eta = 0.225$, $\rho =$ -0.5, $r = 0.06$, $d = 0.5$, V =100.

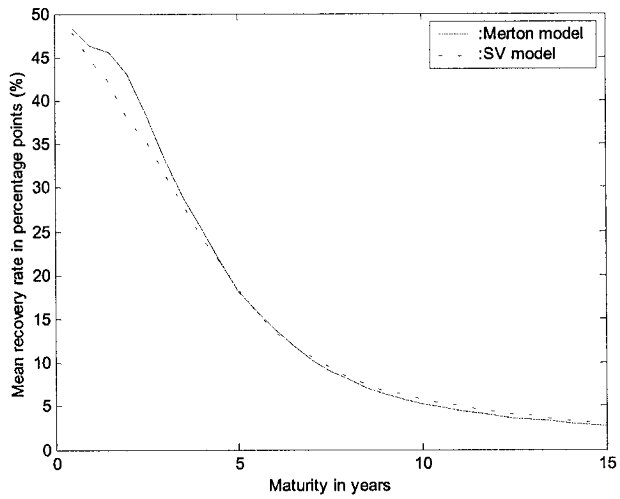

Figure 3:

Mean recovery rate in the SV and Merton models, which is calculated using Monte Carlo simulations (with 1,000 replications). Base case parameters: $\kappa = 0.5$, $\xi_t = \theta = 0.1$, $\eta = 0.225$, $\rho =$ -0.5, $r = 0.06$, $d = 0.5$, $V = 100$.

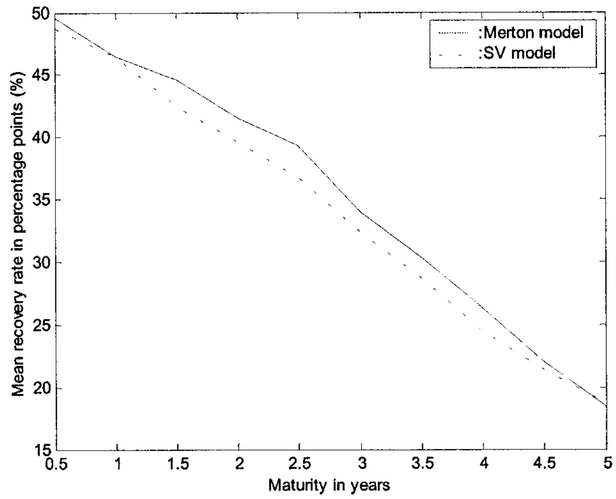

Figure 4:

Mean recovery rate in the SV and Merton models for debts with maturities less than or equal to five years. The recovery rate is calculated using Monte Carlo simulations (with 1,000 replications). Parameter values are the same as those used in Figure 3.

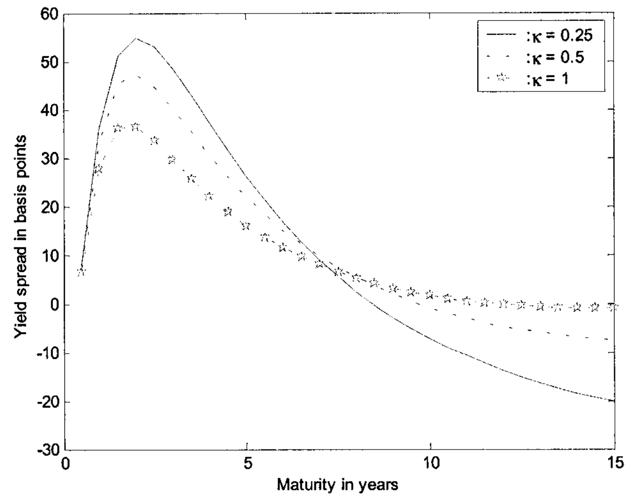

Figure 5:

Yield spread of the SV model minus that of the Merton model for various values of κ, the mean-reversion speed of firm value variance. Other parameter values: $\xi_t = \theta = 0.1$, $\eta =$ 0.225, $\rho =$ -0.5, $r = 0.06$, $d = 0.5$, $V = 100$.

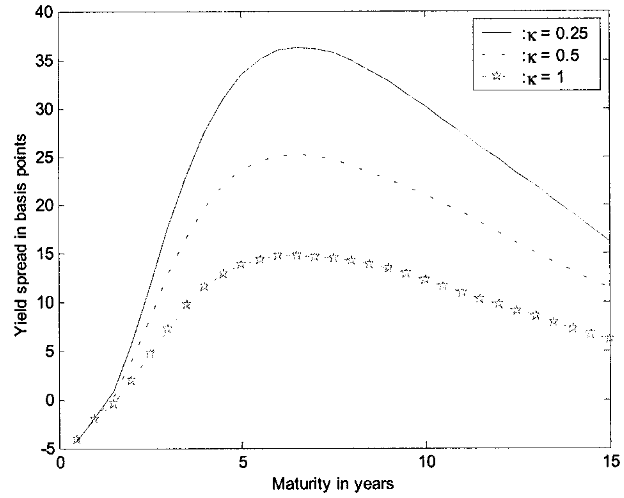

Figure 6:

Yield spread of the SV model minus that of the Merton model for various values of κ. Other parameter values are the same as those in Figure 5 except that $d = 0.2$.

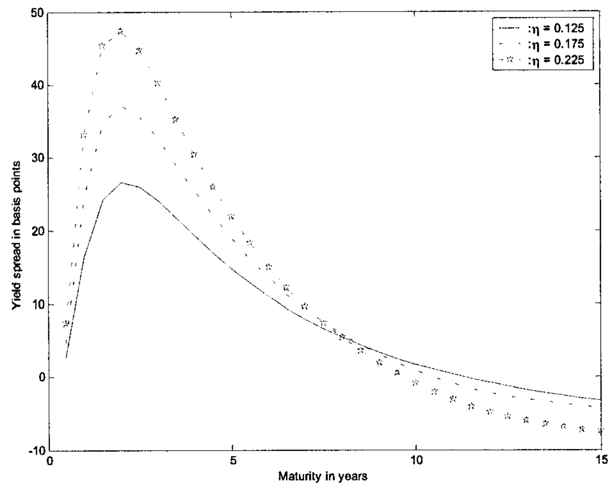

Figure 7:

Yield spread of the SV model minus that of the Merton model for various values of η, the volatility of firm value volatility. Other parameter values: $\kappa = 0.5$, $\xi_t = \theta = 0.1$, $\rho =$ -0.5, $r = 0.06$, $d = 0.5$, $V = 100$.

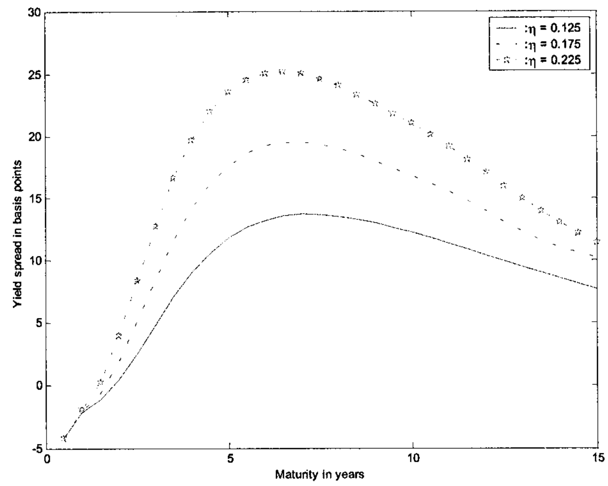

Figure 8:

Yield spread of the SV model minus that of the Merton model for various values of η. Other parameter values are the same as those in Figure 7 except that $d = 0.2$.

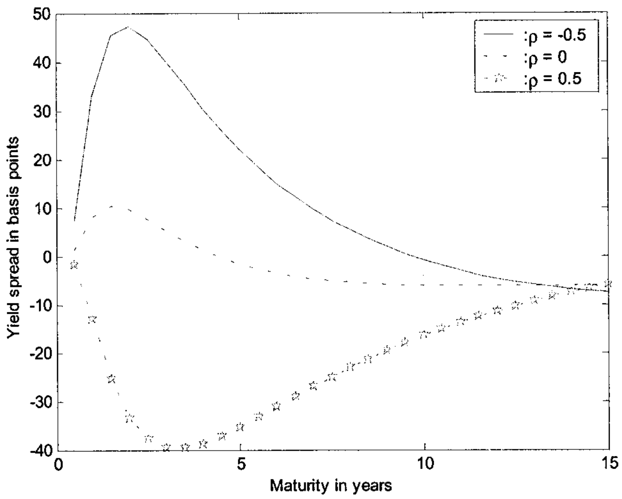

Figure 9:

Yield spread of the SV model minus that of the Merton model for various values of ρ, the correlation coefficient between firm value and its volatility. Other parameter values: κ = 0.5, $\xi_t = \theta = 0.1$, $\eta = 0.225$, $r = 0.06$, $d = 0.5$, $V = 100$.

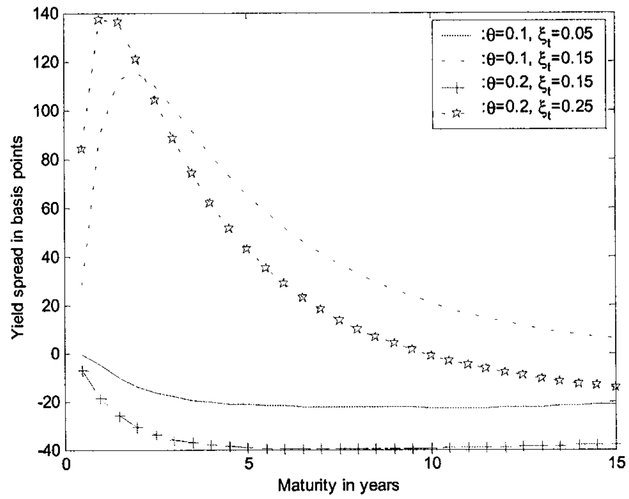

Figure 10:

Yield spread of the SV model minus that of the Merton model for various values of long run mean level of variance, θ, and current variance level, ξ_t. The correlation coefficient between firm value and its stochastic volatility, ρ, is set at -0.5. Other parameter values: $\kappa = 0.5$, $\eta = 0.225$, $r = 0.06$, $d = 0.5$, $V = 100$.

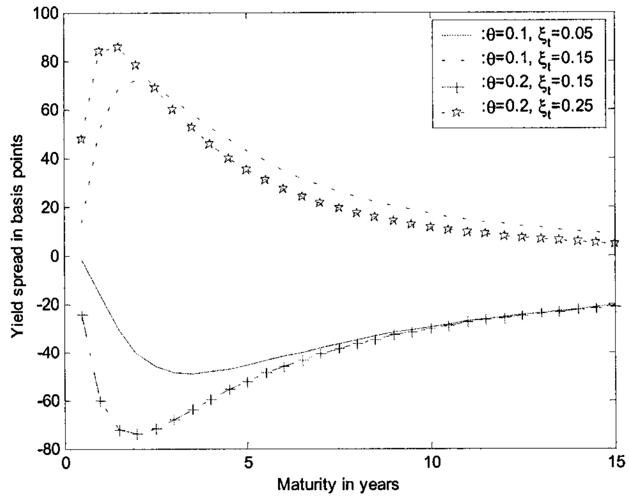

Figure 11:

Yield spread of the SV model minus that of the Merton model for various values of long run mean level of variance, θ, and current variance level, ξ_t. The correlation coefficient between firm value and its stochastic volatility, ρ, is set at 0. Other parameter values: $\kappa = 0.5$, $\eta = 0.225$, $r = 0.06$, $d = 0.5$, $V = 100$.

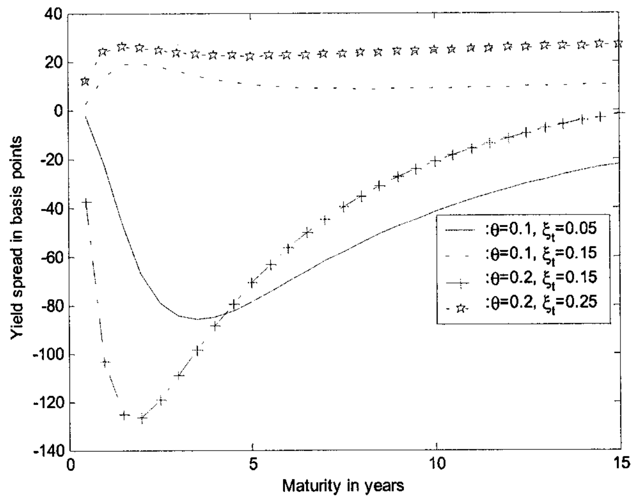

Figure 12:

Yield spread of the SV model minus that of the Merton model for various values of long run mean level of variance, θ, and current variance level, ξ_t. The correlation coefficient between firm value and its stochastic volatility, ρ, is set at 0.5. Other parameter values: $\kappa = 0.5$, $\eta = 0.225$, $r = 0.06$, $d = 0.5$, $V = 100$.

The following essay develops a reduced form model of corporate credit spreads with stochastic volatility. In contrast, the immediately preceding essay uses the structural approach. The first factor of the model can be interpreted as the level of the spread, and the second factor of the model is the volatility of the spread. Empirical results show that the model is successful at fitting actual corporate bond credit spreads. In addition, key properties of actual credit spreads are better captured by the model.

Modeling the Dynamics of Credit Spreads with Stochastic Volatility

Kris Jacobs and Xiaofei Li

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Abstract

This paper investigates a two-factor affine model for the credit spreads on corporate bonds. The first factor can be interpreted as the level of the spread, and the second factor is the volatility of the spread. The riskless interest rate is modeled using a standard two-factor affine model, thus leading to a four-factor model for corporate yields. This approach allows us to model the volatility of corporate credit spreads as stochastic, and also allows us to capture higher moments of credit spreads. We use an extended Kalman filter approach to estimate our model on corporate bond prices for 108 firms. The model is found to be successful at fitting actual corporate bond credit spreads, resulting in a significantly lower root mean square error (RMSE) than a standard alternative model in both in-sample and out-of-sample analyses. In addition, key properties of actual credit spreads are better captured by the model.

JEL Classification: G12

Keywords: credit risk; credit spreads; reduced form models; stochastic volatility.

1 Introduction

Understanding the pricing of default risk or credit risk is of critical importance since almost every financial security is affected by certain types of credit risk. Over the last decade, the financial industry has come under increasing pressure to understand and quantify this risk, because of the growth in financial markets that trade credit sensitive products, and because of increasing scrutiny from regulators. As a result, the academic literature on modeling credit risk has been growing fast. Currently, there are two main approaches to modeling credit risk. The first approach is called the *structural* approach, first developed in Merton (1974). The models used in the structural approach are based on the value of the firm. In these models default is regarded as an endogenous decision usually made by the firm's equityholders. The second approach is called the *reduced form* approach. It considers default as both a surprise and an exogenous event. This approach uses stochastic processes similar to those used in the modeling of the riskless term structure to model the default probability (see e.g. Madan and Unal (1994) and Duffie and Singleton (1999)). The models used in this approach have proven to be relatively easier to use for practical applications. This paper presents a reduced form model of credit risk. In particular, we use a two-factor affine model to describe the joint dynamics of the instantaneous default probability and the volatility of the default probability.

There are compelling reasons for incorporating stochastic volatility into models of default probability and credit spreads. According to the contingent claim analysis in Merton (1974), the corporate bond price is equal to the price difference between a riskless bond and a put option written on the assets of the firm. Since volatility plays a crucial role in pricing options, it will also have a major impact on corporate bond prices and yields. It is therefore critically important to model volatility correctly. At the empirical level, academics and practitioners have long noted that the volatility of credit spreads changes through time. To illustrate this feature of credit spread data, consider Figures 1 and 2, which plot different time series of credit spread indices for different ratings from Moody's and Standard & Poor's, respectively (these data are described in more detail below). Figure 3 also plots the relative changes for the credit spreads from Moody's. These figures clearly show that the volatility of credit spreads is time-varying. Relative changes in credit spreads were significantly larger during the first halves of the sample periods. In his analysis of credit spreads, Duffee (1999, p.198) finds "persistent fluctuations in the volatilities of yields (GARCH-like effects) that are not captured by the model." Miu (2001) reaches a similar conclusion and argues for introducing volatility as a second factor in the default probability process. Finally, the market for credit derivatives has been expanding dramatically since 1990s. Since volatility is fundamental to pricing derivative securities, a stochastic volatility model of credit spreads will help us to more accurately value credit derivatives such as credit spread options, and to more properly manage the credit risk of fixed-income portfolios.

A stochastic volatility model can also capture the skewness naturally embedded in credit data. Because one component of the risky bond price is a put option, the Merton (1974) model also implies that its distribution is negatively skewed. Therefore we expect to find positive skewness for the distributions of risky yields and spreads. Tables 1 and 2 present summary statistics for the Moody's and Standard & Poor's credit spread indices. Table 1 provides strong evidence of positive skewness in credit spreads. The evidence in Table 2 is more mixed, with some series displaying (small) negative skewness and others displaying (relatively larger) positive skewness. The credit spread model in this paper can capture this skewness.

The objectives of this paper are two-fold. First, we propose a new reduced form model of credit risk that explicitly accounts for stochastic volatility in default probability and the correlation between the riskless interest rate and default probability, and we estimate the model using prices of U.S. Treasury bonds and corporate bonds. Second, we compare the performance of our model to an alternative model that is developed in Duffee (1999), in which stochastic volatility in default probability is not explicitly taken into account. We estimate our model and the alternative model on data of month-end corporate bond prices from January 1985 to March 1998. In total, the data consist of more than 44,000 bond-price observations across 108 firms. These bonds are primarily investment-grade. For every firm, we divide the available data into two sub-periods: the last 12 months are used for out-of-sample forecasting, and the rest are used for in-sample estimation.

Overall, the stochastic volatility model fits corporate yields much better than the alternative model. At the aggregate level, the median in-sample RMSE of our model is 9.30 basis points (bps), versus 11.99 bps for the alternative model; in the out-of-sample analysis, the stochastic volatility model achieves an median RMSE of 12.05 bps, relative to 13.89 bps for the alternative model. More importantly, at the individual firm level, the stochastic volatility model realizes a lower in-sample RMSE than the competing model for every single firm in the sample and a lower out-of-sample RMSE for slightly more than two-thirds of firms in the sample. The parameter estimates indicate that the probability of default is mean-reverting under the physical measure and can also be mean-reverting under the risk-neutral measure in the stochastic volatility model, whereas the estimates of the benchmark model indicate that the probability of default is mean-reverting under the physical measure but mean-averting under the risk-neutral measure. For the vast majority of the firms, the default probability and credit spreads are found to be strongly negatively correlated with the default-free interest rate. This finding is consistent with many previous empirical studies. To investigate the robustness of our findings, we also repeat the estimation using credit spread indices. These estimation results confirm our findings obtained using individual firm data.

This paper is part of a growing list of articles that empirically test reduced form models of credit risk.[1] Madan and Unal (1994) examine yields on certificates of deposit issued by thrift institutions. Duffie and Singleton (1997), Liu, Longstaff, and Mandell (2000), Collin-

[1]Recent empirical studies that estimate structural models of credit risk include Eom, Helwege, and Huang (2000) and Ericsson and Reneby (2001). In addition, using numerical simulations, Huang and Huang (2000) compare the performances of several well-known structural models of corporate debt.

Dufresne and Solnik (2001), and Grinblatt (2002) study interest rate swap yields. Nielsen and Ronn (1997) use data on both corporate bonds and interest rate swap yields. Duffee (1999), Bakshi, Madan, and Zhang (2001), and Miu (2001) analyze corporate bond prices of individual firms. All of these applications, including ours, are special cases of the affine family of term structure models. However, our model specification is new in the context of credit spreads, and it is easy to implement.

The rest of the paper is organized as follows. Section 2 introduces the two models used in the empirical analysis. Section 3 discusses the data. The estimation method is discussed in Section 4. Section 5 reports the basic results. Section 6 further explores the empirical results. Finally, Section 7 concludes the article. All technical details can be found in the Appendix B.

2 Models of corporate bond prices

This section consists of three parts. Section 2.1 describes the proposed stochastic volatility model of credit spreads. Section 2.2 briefly summarizes Duffee's (1999) model, which is used as a benchmark. In this model, stochastic volatility of credit spreads is not explicitly accounted for. In Section 2.3, we discuss the stochastic volatility model in more detail, including the parameterization of the model, assumptions on the recovery rate, and the pricing of coupon bonds.

2.1 A model of credit spreads with stochastic volatility

First consider the default-free interest rate. Under the physical (or actual) probability measure P, the instantaneous nominal riskless interest rate is denoted by i_t, and is assumed to be equal to the sum of a constant, c, and two factors that follow independent square-root diffusion processes

$$i_t = c + f_{1t} + f_{2t}, \tag{1}$$

$$df_{it} = \phi_i(\mu_i - f_{it})dt + \sigma_i\sqrt{f_{it}}dw_{it}, \tag{2}$$

for $i = 1, 2$. In equation (2), ϕ_i denotes the speed of mean reversion of factor f_{it}, and μ_i can be interpreted as the long-run mean of f_{it}, $i = 1, 2$. The two standard Brownian motions w_{1t} and w_{2t} are assumed to be independent. The specification in equations (1) and (2) has appeared in Cox, Ingersoll, and Ross (1985) and Pearson and Sun (1994), and the latter calls it a *translated* square-root model, due to the presence of the constant term c. This model belongs to the class of *exponential affine* (or simply *affine*) term structure of interest rate models.[2]

Following Cox, Ingersoll, and Ross (1985), we write the process in (2) under the risk-neutral probability measure Q as

$$df_{it} = (\phi_i\mu_i - (\phi_i + \pi_i)f_{it})dt + \sigma_i\sqrt{f_{it}}d\widehat{w}_{it}, \tag{3}$$

where π_i $(i = 1, 2)$ are the risk premiums and $\widehat{w}_{1t}$ and $\widehat{w}_{2t}$ are two independent Brownian motions under the measure Q. Notice that in this model, $\pi_i < 0$ $(i = 1, 2)$ implies that investors demand positive compensation for bearing interest rate risk.

The choice of the above riskless interest rate model was driven by three considerations. First, it is well known that the dynamics of riskless yield curves can not be adequately captured by a single factor. Instead, at least a two-factor model should be used. According to Litterman and Scheinkman (1991), the first two factors account for nearly 96% of variations in yield curves. Second, the above specification permits a well-known and simple closed-form solution for zero-coupon bond prices (see e.g. Pearson and Sun (1994)), which greatly

[2]Duffie and Kan (1996) characterize affine term structure of interest rate models. Dai and Singleton (2000) conduct a thorough specification analysis of various affine interest rate models.

facilitates estimation. Third, this interest rate model results in a reasonably good fit for the risk free term structure.[3]

As is common in all reduced form models, default is assumed to occur when an exogenous Poisson process with intensity λ jumps (see Lando (1997)). Under the equivalent martingale measure Q, the intensity of this Poisson process for firm j at time t is denoted by λ_{jt}, which can be interpreted as follows. Consider a firm j that is not in default at time t. Under the measure Q, the probability that this firm defaults during a subsequent sufficiently small time interval $(t, t+dt)$, conditioning on the information available at time t, is $\lambda_{jt}dt$. Consequently, we may interpret λ_{jt} as the instantaneous default probability. Different reduced form models are mainly characterized by different specifications for λ_{jt}.

Next, consider a zero-coupon bond issued by firm j, which promises to pay one dollar at the maturity date of T, unless the firm defaults before T. If default occurs, the bondholders recover nothing. We denote by $B_j(t, T, 0, 0)$ the price of this bond at time t, where the third argument in $B_j(t, T, 0, 0)$ refers to the coupon rate, and the fourth argument denotes the recovery rate. We will relax the zero-recovery assumption and examine coupon bonds in Section 2.3. Under certain regularity conditions (see Duffie and Singleton (1999)), this corporate bond price is given as

$$B_j(t, T, 0, 0) = E_t^Q\{\exp[-\int_t^T (i_u + \lambda_{ju})du]\}, \tag{4}$$

where $i_t + \lambda_{jt}$ is the so-called "adjusted discount rate" for firm j at time t, and $E_t^Q(\cdot)$ denotes the conditional expectation under the risk-neutral probability measure Q, relying on all the information available at time t.

It is clear from equation (4) that for a near-maturity, zero-coupon, and zero-recovery corporate bond, the instantaneous default probability, λ_{jt}, can also be interpreted as the

[3]This riskless interest rate model has one weakness: namely, in this model the nominal riskless interest rate, i_t, can become negative, which is contrary to reality. This model allows i_t to become negative since the constant term c may take negative values.

instantaneous credit spread on this bond. Consequently, we will use the terms "instantaneous default probability" and "credit spreads" interchangeably in this article; they both refer to λ_{jt}. It should be pointed out that λ_{jt}, the default probability, may be interpreted in a broader sense, since the spread between risky and riskless yields also results from factors other than default risk, such as the liquidity premium or state tax differences.

In this paper, we model default probability as a two-factor affine process under the physical measure P

$$\begin{aligned} \lambda_{jt} &= c_j + \lambda^*_{jt} + \delta_{1j}(f_{1t} - \overline{f_{1t}}) + \delta_{2j}(f_{2t} - \overline{f_{2t}}), \\ d\lambda^*_{jt} &= \alpha(\overline{\lambda} - \lambda^*_{jt})dt + \sqrt{v_{jt}}dz_{1j,t}, \\ dv_{jt} &= \gamma(\overline{v} - v_{jt})dt + \xi\sqrt{v_{jt}}dz_{2j,t}, \end{aligned} \tag{5}$$

where v_{jt} is the instantaneous variance of the default probability of firm j at time t; $\overline{\lambda}$ and $\overline{v}$ are the unconditional (or long-run) means of λ^*_{jt} and v_{jt}, respectively; α and γ capture the mean-reversion of λ^*_{jt} and v_{jt}, respectively; and ξ is the "volatility of volatility" parameter, which determines the kurtosis of λ^*_{jt}.

The two standard Brownian motions $z_{1j,t}$ and $z_{2j,t}$ are correlated with coefficient ρ, which is positively correlated with the skewness of credit spreads. This feature of the model enables us to capture the skewness exhibited by the distribution of credit spreads. Furthermore, we assume that $z_{1j,t}$ and $z_{2j,t}$ are independent of the two Brownian motions w_{1t} and w_{2t} in the default-free interest rate process. Finally, $\overline{f_{1t}}$ and $\overline{f_{2t}}$ are the means of the smoothed estimates of the two riskless interest rate factors, f_{1t} and f_{2t}, over the sample period of corporate bonds.

Again using a standard assumption of prices of risk, the stochastic processes followed by λ^*_{jt} and v_{jt} under the risk-neutral measure Q are given by

$$\begin{aligned} d\lambda^*_{jt} &= (\alpha\overline{\lambda} - \alpha\lambda^*_{jt} + \eta_1 v_{jt})dt + \sqrt{v_{jt}}d\widehat{z}_{1j,t}, \\ dv_{jt} &= (\gamma\overline{v} - (\gamma + \xi\eta_2)v_{jt})dt + \xi\sqrt{v_{jt}}d\widehat{z}_{2j,t}, \end{aligned} \tag{6}$$

where again the two standard Brownian motions $\widehat{z}_{1j,t}$ and $\widehat{z}_{2j,t}$ have a correlation coefficient of ρ and they are independent of the Brownian motions $\widehat{w}_{1t}$ and $\widehat{w}_{2t}$ in the default-free interest rate process. Also, η_i ($i = 1, 2$) are the risk premium parameters. Note that in this model, $\eta_1 > 0$ and $\eta_2 < 0$ indicate a positive risk premium in λ^*_{jt} and v_{jt}, respectively.

The model for λ^*_{jt} and v_{jt} in equation (5) is inspired by Fong and Vasicek (1991), who use similar dynamics to model the default-free interest rate. The model is a member of the affine family of interest rate models. Although the model has a closed-form solution for zero-coupon bond prices, the solution is quite complicated and involves complex algebra. We therefore use a series solution method suggested in Selby and Strickland (1995) to compute bond prices in this paper. The Selby and Strickland method has proven to be both accurate and fast. Appendix B.1.1 contains a brief introduction to their method.

The specification in (5) and (6) captures, in a tractable way, three prominent empirical features of actual credit spreads. First, credit spreads (and default probability) appear to vary stochastically over time. The stochastic λ^*_{jt} term within λ_{jt} accounts for this feature. Second, the volatility of credit spreads is itself stochastic. This fact is modeled parsimoniously by a stochastic v_{jt}. Third, credit spreads on corporate bonds non-trivially depend on the movements of default-free interest rates. In the model, this dependence is solely captured by the δ_{1j} and δ_{2j} coefficients in λ_{jt}, due to the assumed independence of the Brownian motions driving the riskless interest rate and the default probability.

2.2 A benchmark model

To evaluate the empirical performance of the stochastic volatility credit spread model, we also estimate a benchmark model. The benchmark model that we use is the model developed in Duffee (1999). The default-free interest rate component of his model is identical to the specification in equations (1), (2), and (3).

Under the P measure, Duffee (1999) models the default probability using a one-factor translated square-root process with two components linked to the riskless term structure

$$\lambda_{jt} = c_j + \lambda^*_{jt} + \delta_{1j}(f_{1t} - \overline{f_{1t}}) + \delta_{2j}(f_{2t} - \overline{f_{2t}}), \tag{7}$$
$$d\lambda^*_{jt} = \kappa_j(\theta_j - \lambda^*_{jt})dt + \sigma_j\sqrt{\lambda^*_{jt}}du_{jt},$$

where the standard Brownian motion u_{jt} is independent of the two Brownian motions w_{1t} and w_{2t} in the riskless interest rate process.

Under the risk-neutral measure, the process for λ^*_{jt} is given by

$$d\lambda^*_{jt} = (\kappa_j\theta_j - (\kappa_j + \pi_j)\lambda^*_{jt})dt + \sigma_j\sqrt{\lambda^*_{jt}}d\widehat{u}_{jt}, \tag{8}$$

in which π_j (< 0) is the risk premium, and the Brownian motion $\widehat{u}_{jt}$ is independent of the Brownian motions $\widehat{w}_{1t}$ and $\widehat{w}_{2t}$ in the riskless interest rate specification.

Notice that in the Duffee model, the volatility of the credit spreads, $\sigma_j\sqrt{\lambda^*_{jt}}$, is not constant but time-varying due to the stochastic nature of λ^*_{jt}. However, instead of modeling the credit spread volatility as a separate diffusion process, in his model the volatility of the credit spreads is proportional to the level of the spreads itself and can not move independently of λ^*_{jt}.

Together, equations (1) through (3) and (7) through (8) constitute a three-factor affine model of corporate bond yields, and equations (7) and (8) alone result in a one-factor affine model of credit spreads. Henceforth we will refer to this model as the Duffee model or the benchmark model. This model leads to well-known closed-form solutions for corporate zero-coupon bond price. In contrast, the four-factor affine model of corporate yields in equations (1) through (3) and (5) through (6) has an additional factor that captures the stylized fact that the volatility of credit spreads is stochastic. Henceforth we will refer to this model as the stochastic volatility model of credit spreads, or for short the stochastic volatility model, even though the volatility of the riskless component of the risky yield is not formally modeled.[4] Notice that the four-factor model does not nest the three-factor benchmark model since in

[4]It is worth pointing out that our model and the Duffee model share a common weakness: in both settings,

the special case where the v_{jt} process in equation (5) reduces to a constant, the process for λ^*_{jt} in equation (5) becomes a so-called Ornstein-Uhlenbeck process, similar to the one used in Vasicek (1977), which is different from the square-root process for λ^*_{jt} in equation (7).[5]

2.3 Further discussion of the stochastic volatility model

Denote the "adjusted discount rate" for firm j at time t by $R_{jt} \equiv i_t + \lambda_{jt} L_{jt}$, where L_{jt} is the expected loss rate of firm j's defaultable bond's value if default were to occur at time t, and the product term $\lambda_{jt} L_{jt}$ is called the mean loss rate (Duffie and Singleton (1999)). It follows that the recovery rate of firm j at time t is equal to $1 - L_{jt}$. For simplicity, in our discussion in Section 2.1, we have assumed no recovery upon default, i.e., $L_{jt} = 1$. We now relax this no-recovery assumption. When the recovery rate is non-zero, i.e., when $0 \leq L_{jt} < 1$, the price of a zero-coupon corporate bond with a recovery rate of $1 - L_{jt}$, $B_j(t, T, 0, 1 - L_{jt})$, is given by

$$\begin{aligned} B_j(t,T,0,1-L_{jt}) &= E_t^Q\{\exp[-\int_t^T (i_u + \lambda_{ju} L_{ju}) du]\} \qquad (9) \\ &= E_t^Q\{\exp[-\int_t^T R_{ju} du]\}. \end{aligned}$$

When using the above pricing relationship, we may either parameterize R_{jt} directly, or parameterize the components of R_{jt}, namely, i_t, λ_{jt}, and L_{jt}, individually. Duffie and Singleton (1997) and Dai and Singleton (2000) take the former approach to modeling the term structure of interest-rate swap yields. The latter approach is adopted by Duffee (1999)

the default probability, λ_{jt}, can become negative, which is conceptually odd. In our model, λ_{jt} may become negative because the stochastic process followed by λ^*_{jt} allows λ^*_{jt} to take negative values. On the other hand, in the Duffee model, λ_{jt} may fall below zero if either c_j or δ_{ij} ($i = 1, 2$) is negative.

[5]We also estimated the nested three-factor affine model of corporate yields in which the process for λ^*_{jt} is given by an Ornstein-Uhlenbeck process. The performance of this nested model is worse than that of the benchmark model.

and Collin-Dufresne and Solnik (2001), where they parameterize i_t and the product term $\lambda_{jt}L_{jt}$ separately. In this paper, we choose to parameterize i_t and $\lambda_{jt}L_{jt}$ separately in order to extract information about the mean loss rate $\lambda_{jt}L_{jt}$ from historical defaultable bond prices. This information can be used to value other defaultable claims, such as credit default swaps.

In this paper, we assume that if default were to occur at τ_d, $t < \tau_d \leq T$, the bondholders would receive, upon default, a fixed fraction of $(1 - L)$ of the face value of the original corporate bond. In other words, in the event of default, the corporate bondholders are assumed to receive a fixed $(1 - L)$ unit of an otherwise identical default-free bond for every one dollar of the face value of the original default-risky bond. Jarrow, Lando, and Turnbull (1997) and Duffee (1999) make a similar assumption on the recovery rate.

The assumption of a constant recovery rate $(1-L)$ can be justified by two findings. First, a recent empirical study in Skinner and Diaz (2001) discovers that for the purpose of accurately pricing defaultable bonds, a stochastic recovery rate is of only second-order importance, relative to e.g. a correct parameterization of default probability. Second, as pointed out in Duffie and Singleton (1999), since λ_{jt} and L_{jt} only appear in the pricing relationship (9) as a cross product term, it is impossible to identify them separately by using data on corporate bonds alone. Instead, data on credit derivatives, of which payoffs are nonlinearly dependent on λ_{jt} and L_{jt}, are required. Since the main focus of this paper is on modeling default probability λ_{jt}, a constant recovery rate assumption makes identification of λ_{jt} possible. It also permits a closed-form solution of bond prices, which significantly facilitates estimation. However, it should be kept in mind that the framework can be extended to accommodate a stochastic recovery rate without additional conceptual difficulty.

We set the recovery rate at 44% of par in our empirical work. This is consistent with Moody's finding that the average recovery rate of senior unsecured bonds is approximately 44% of the par value (of the original bond) if default occurs. A similar assumption is made in Duffee (1999), who fixes the recovery rate at 44% of par, and also in Duffie and Lando (2001), who assume a constant recovery rate of 43.3%.

Applying a standard no-arbitrage argument, we can write the before-default price of a zero-coupon corporate bond with a constant recovery rate of $(1 - L)$ as

$$B_j(t, T, 0, 1 - L) = (1 - L)G(t, T, 0) + LB_j(t, T, 0, 0), \tag{10}$$

where $G(t, T, 0)$ denotes the price at time t of a default-free zero-coupon bond that matures at time T.

The bulk of the corporate bond data consist of coupon bonds. We use the so-called "portfolio of zeros" approach to valuing corporate coupon bonds. In Appendix B.1.2, we define this approach and provide theoretical justifications for its use in the present context. Appendix B.1.2 also contains the formulas for coupon bonds.

3 Data

3.1 Data on the risk free interest rate

Month-end US Treasury prices (the averages of the reported bid and ask prices) from January 1985 to March 1998 are obtained from the CRSP US Treasury Cross-Sectional File. In each month, the most recently issued (or on-the-run) noncallable Treasury bills, notes or bonds with maturities closest to 3 and 6 months and 1, 2, 3, 5, 10 and 30 years are selected.[6]

[6] Duffee (1999) uses data on the second most recently issued (or off-the-run) U.S. Treasury securities in order to avoid capturing any special liquidity premium associated with the on-the-run Treasury securities. He reports no material difference between on-the-runs and off-the-runs. As a robustness check, we also estimate our riskless interest rate model using off-the-run data and the results are very similar to those obtained using on-the-runs. We therefore only report results on on-the-runs. Several other empirical studies, e.g. Duffie and Singleton (1997) and Miu (2001), also use the on-the-run U.S. Treasury data.

3.2 Data on corporate bonds

Month-end corporate bond bid prices are collected from the Fixed Income Securities Database (also known as the Lehman Brothers Fixed Income Database or the Warga Database) over the period beginning January 1985 and ending March 1998, encompassing a sample period of 159 months. Before 1985, firms rarely issued non-callable bonds. Hence we only use a sample period starting in January 1985. All price observations included in the sample are indicative trader-quoted prices. That is, prices that were calculated using a matrix algorithm are dropped. Only investment-grade, non-callable, non-putable, senior unsecured straight bonds with semi-annual coupons and no sinking fund provisions, having remaining maturities no longer than 35 years and no shorter than 1 year, are selected. Only those firms for which there are at least three bonds (not necessarily the same bonds) outstanding in a given month for at least 48 months (not necessarily consecutively) are considered. Finally, we only include bonds in the sample that make up the Lehman Brothers bond index or are about to enter the index. There are 108 firms that satisfy all of the above requirements. Among these firms, 65 are industrial firms, 28 are financial firms, and 15 are utility firms. The final data set consists of a total of 44,298 qualified bond price observations. Appendix A contains a complete listing of the corporations included in the dataset.

In some instances, we compare estimation results across credit ratings. To do so, we use the bond rating supplied by Moody's, which defines a firm's credit rating as the rating on its senior unsecured bonds. The credit rating assigned to a firm in the sample is the mean of the ratings of the firm's bonds used in estimation. This procedure results in 12 Aa-rated firms, 60 A-rated firms, and 36 Baa-rated firms. That is, the sample is dominated by A- and Baa-rated firms.[7]

[7] In our original sample, there were also 3 Aaa-rated firms. We have excluded them from our subsequent analyses for two reasons. First, the limited sample size makes any inference from the results on these firms inconclusive. Second, in our sample, the average actual yield on these 3 Aaa-rated firms is higher than those on the Aa-, A-, and Baa-rated firms, and the mean actual credit spread for these 3 firms is higher than that

Table 3 contains summary statistics for the corporate bond data. Panel A reports that the median firm has 73 months of valid bond price observations, while none of the firms in the sample has valid observations in every month. The second row of Panel A in Table 3 reports the mean number of fitted bonds per month, which is calculated over those months in which a firm has at least three qualified bonds outstanding. The median number of fitted bonds is 4.4, and the maximum number is 12.36. Therefore, although across all the firms and the entire sample period this article uses 44,298 bond price observations, the credit spreads for the median firm are estimated using just 4 bonds per month. For some firms, certain parameters are therefore estimated imprecisely. The third, fourth, and fifth rows of Panel A report the remaining years to maturity of the bonds used in estimation. The median firm has a minimum maturity of 1.06 years, a mean maturity of 7.92 years, and a maximum maturity of 20.25 years. Finally, according to the last row of Panel A, the median firm has a mean annual coupon rate of 8.37%. For one firm (Allied Corp.), the data set exclusively contains zero-coupon bonds.

Panels B and C present means (in bps), standard deviations (in %), skewness and kurtosis for yields and credit spreads in this sample. The credit spreads in Panel C are calculated as the differences between the yields (used in Panel B) and the riskless interest rates implied by the default-free term structure model. The median firm has a mean yield of 709.56 bps and a mean credit spread of 246.07 bps. Similar to the findings in Table 1 and Table 2, yields and credit spreads in this sample exhibit significant positive skewness. There is not much evidence of excess kurtosis in either the yield data or the credit spread data.

To demonstrate the robustness of our results, we repeat the empirical exercise using credit spread indices. The advantage of this type of data is that they span a much longer time horizon than the individual firm data. We obtain Moody's 10-year and 30-year Aaa and Baa monthly credit spread indices from the Federal Reserve Board's G.13 release. The

for the Aa-rated firms. This is an anomaly likely caused by the very limited sample size of the Aaa-rated firms.

sample periods are January 1960 to April 2003 for 10-year maturity spreads and February 1977 to February 2002 for 30-year maturity spreads. We also obtain weekly credit spread index data for various maturities from Standard & Poor's for the sample period August 6, 1996 to September 11, 2001. The time series plots of a subset of these data are in Figure 1 and Figure 2, respectively, and Figure 3 presents the relative changes in the spreads for the Moody's data. Tables 1 and 2 present descriptive statistics for these data. As mentioned before, the Moody's data display positive skewness, while the Standard & Poor's data in some cases display small negative skewness, and large positive skewness in other cases.

4 Estimation methodology

These two models can be estimated using a number of different methods. We adopt the extended Kalman filter (EKF) approach to estimate the riskless interest rate and corporate bond models. This approach has been successfully used in, among others, Claessens and Pennacchi (1996), Babbs and Nowman (1999), de Jong (2000), Duan and Simonato (1999), Duffee (1999), Geyer and Pichler (1999), and Miu (2001). There are at least three major advantages associated with the EKF approach. First, it allows us to use both cross-sectional and time-series information contained in the riskless bond prices and corporate bond prices. Second, this approach correctly treats the underlying state variables (or factors) as unobservable, which is consistent with the theoretical models. Third, as a by-product of the EKF, estimates of the state variables are also generated, which is useful for our analysis in Section 6.[8] Appendix B.2.1 contains a brief summary of the EKF approach.

[8]For nonlinear and non-Gaussian models, such as the two models considered here, the parameter estimates obtained from the EKF may be inconsistent. However, Monte Carlo evidence in Lund (1997), Duan and Simonato (1999), and de Jong (2000) suggests that this inconsistency is of fairly limited importance for the typical sample size encountered in term structure modeling. Duffee and Stanton (2004) advocate the use of the EKF instead of a more sophisticated method in a similar context.

Because of the assumed independence between the Brownian motions driving the riskless interest rate and those underpinning the default probability, we can follow the two-step estimation procedure proposed in Duffee (1999). In the first step, we estimate the default-free term structure using U.S. Treasury prices alone. In the second step, we assume that the parameter estimates of the riskless interest rate obtained from stage one are the true parameters and use them to estimate the parameters of the default probability of each individual firm in the sample. That is, we run an EKF to estimate the riskless interest rate in the first step, and in step two we run an EKF for each individual firm to estimate its credit spread process.

4.1 Estimation of the default-free interest rate

At time t, we observe a cross-section of U.S. Treasury bond prices $G_t = (G_{1,t}, ..., G_{8,t})'$. We collect the two unobservable state variables in the vector F_t. For notational simplicity, we suppress the dependence of the model on the parameters to be estimated and write down the measurement equation and the transition equation of the Kalman filter as

$$G_t = m(F_t) + \epsilon_t, \ \ E_{t-1}(\epsilon_t \epsilon_t') = \Lambda, \tag{11}$$

$$F_t = a + bF_{t-1} + \varsigma_t, \ \ E_{t-1}(\varsigma_t \varsigma_t') = V(F_{t-1}). \tag{12}$$

In the measurement equation (11), the function $m(F_t)$ maps the two state variables in F_t to bond prices, and we know this mapping in closed-form; ϵ_t is the white noise measurement error at time t and has a constant conditional variance-covariance matrix given by Λ. In the transition equation (12), ς_t is also a white noise error term, of which the conditional variance-covariance matrix is $V(F_{t-1})$. This matrix depends on the values of the state variables at time $t-1$. The explicit forms of components a, b, and $V(F_{t-1})$ are presented in Appendix B.2.2.

We assume that the default-free interest rate process is stationary. Therefore, we can use the unconditional means of factors f_1 and f_2, μ_1 and μ_2, respectively, to initiate iterations on the riskless interest rate. We write the measurement equation in terms of bond prices instead of yields for two reasons. First, for coupon bonds there is no linear mapping between the state variables of the model and the bond yields. As a result, we have to numerically solve for the yields. We conjecture that the error occurred in this yield extraction process may be carried over to the subsequent estimation process. We emphasize that this is only a conjecture since we are unaware of any empirical study that addresses this issue. Second, writing the measurement equation in terms of prices allows us to analytically calculate the derivatives of the function $m(\cdot)$ with respect to parameters of the model, which facilitates estimation.

Finally, the nonlinear mapping $m(\cdot)$ between the coupon bond prices and the state variables makes identification of all the risk premium parameters of the model possible. This point is made in Dai and Singleton (2000).

4.2 Estimation of the default probability

When estimating the default probability of an individual firm in the second step, we consider the parameter estimates obtained from phase one as the true parameters of the model. We also use the smoothed estimates (i.e., estimates based on information through the entire sample) of the two unobserved riskless factors from phase one. These smoothed estimates are produced by the Kalman filter and we denote them by $\widehat{f_{it}}$ $(i = 1, 2)$. We take the means of these estimates over those months in which a firm has valid corporate bond price observations. These means are denoted by $\overline{f_{it}}$ $(i = 1, 2)$ and they appear in equations (5) and (7).

Consider a given firm j. In month t, we observe a cross-section of U_{jt} corporate bond prices issued by this firm. We stack these bond prices into a vector $B_{jt} = (B_{j,1,t}, ..., B_{j,U_{jt},t})'$. The last time that firm j's bond prices were observed was in month $t-\tau$, where due to missing observations, τ is not necessarily equal to one. The measurement and transition equations are (we again ignore the dependence of the model on the parameters to be estimated)

$$B_{jt} = m_j(\Sigma_{jt}, F_t) + \epsilon_{jt}, \;\; E_{t-\tau}(\epsilon_{jt}\epsilon_{jt}') = \Lambda_{jt}, \tag{13}$$

$$\Sigma_{jt} = a_j + b_j\Sigma_{j,t-\tau} + \varsigma_{jt}, \;\; E_{t-\tau}(\varsigma_{jt}\varsigma_{jt}') = \Gamma(\Sigma_{j,t-\tau}). \tag{14}$$

In the measurement equation (13), vector $F_t = (\widehat{f_{1t}}, \widehat{f_{2t}})'$. The function $m_j(\cdot)$ maps the default-risky state variables stored in Σ_{jt} and the smoothed estimates of the riskless factors in F_t into corporate bond prices B_{jt}. (For the stochastic volatility model, $\Sigma_{jt} = (\lambda_{jt}^*, v_{jt})'$; for the benchmark model, $\Sigma_{jt} = \lambda_{jt}^*$.) The terms ϵ_{jt}, ς_{jt}, and $\Gamma(\Sigma_{j,t-\tau})$ can be interpreted analogously to their counterparts in equations (11) and (12). The Λ_{jt} matrix is a $U_{jt} \times U_{jt}$ diagonal matrix with diagonal entry S_j, the common measurement error variance of the bond prices of firm j. We assume a common error variance since for a given firm, the number of bonds and the maturities of the bonds are time-varying.

The functional forms of a_j, b_j, and $\Gamma(\Sigma_{j,t-\tau})$ in the transition equation (14) are given in Appendix B.2.2. In addition, in Appendix B.3 we give, in closed-form, the first two conditional moments of the two state variables of the stochastic volatility model of credit spreads. These moments are used in the empirical work and to the best of our knowledge, they have not been presented anywhere before.

Finally, unlike the assumption of stationarity made for the risk-free interest rate, we do not assume that the default probability of an individual firm is stationary. As a result, we can not use the unconditional means of the risky state variables in Σ_{jt} as starting points for estimating the default probability. Instead, we filter an estimate of the initial values of Σ_{jt} and the variance-covariance matrix associated with this estimate out of firm j's first month bond data. We refer the interested reader to Duffee (1999, p. 208) for further details.

5 Empirical results

This section is divided into four parts. Section 5.1 summarizes the estimation results on the default-free interest rate. Section 5.2 discusses the in-sample estimation results for the credit spread models. Section 5.3 reports the out-of-sample results. Section 5.4 presents the estimation results on credit spread indices.

5.1 Results on the riskless interest rate

Table 4 reports the estimation results on the default-free interest rate model. The robust standard errors for the parameter estimates are calculated using the formulas in White (1982). The standard errors are generally very small, indicating that the riskless model parameters are estimated quite precisely. The parameter estimates imply that the first factor exhibits strong mean-reversion with a half-life of around 1.25 years. In contrast, the second factor exhibits little mean-reversion, with a half-life of more than 34 years. The estimates of the risk premia are both negative, which is consistent with the theoretical model, although the estimate of the second risk premium, π_2, and the associated standard error indicate that π_2 is of little economic significance and is not statistically different from zero.

The first factor of the riskless model may be interpreted as the negative of the slope of the riskless term structure. The correlation between the smoothed estimates of this factor and the slope of the Treasury yield curve (defined as the difference between the 30-year Treasury bond yield and the 3-month Treasury bill yield) is -0.95, and the correlation between the first differences of these two series is -0.82. On the other hand, the second riskless factor moves closely with long-term Treasury bond yields since the correlation between the smoothed estimates of this factor and the yields on 30-year Treasury bonds is 0.98, and the first differences of these two series are strongly correlated with correlation coefficient 0.93. It is common practice in modeling the riskless term structure to interpret one risk-free factor as the slope of the yield curve, while another factor as the level of the yield curve.

The parameter estimates for the riskless term structure are generally consistent with those of, among others, Duan and Simonato (1999), Duffee (1999), and Geyer and Pichler (1999), although the sample periods in these studies are different. Our results are also similar to those in Pearson and Sun (1994), although both their estimation methodology and sample period differ. The fit of this two-factor model to the Treasury yield curves is overall comparable to the results in Duffee (1999). We achieve a much better fit for the short maturities but a slightly inferior fit for long-maturity bonds. Unlike Duffee (1999), who arbitrarily fixes the constant term c at -1, we estimate this constant along with other parameters of the model. An estimate of -0.48 for c, coupled with the estimates of μ_1 and μ_2, implies a long-run mean of the riskless short rate of around 9%, which is reasonable. Moreover, Duffee (1999) argues that in order for this riskless interest rate model to fit both a low, flat term structure and a high, steep term structure, while at the same time without incurring unrealistically high volatility, we need a negative estimate of the constant term. Our results support his claim.

5.2 In-sample results on credit spread models

Table 5 summarizes the in-sample RMSE fit of the stochastic volatility model, and Table 6 presents the corresponding RMSE for the benchmark model. Tables 5 and 6 also report the median and interquartile ranges for the parameter estimates. In Appendix A, we break down, firm by firm, the RMSE results for the two models. Notice that here the RMSE is calculated based on the contemporaneous predictions of the state variables in Σ_{jt} (i.e., the estimates of Σ_{jt} using information available through time t). This is in contrast to Table 4, where the RMSE is computed using the smoothed estimates (based on information through the entire sample) of the state variables. Duffee (1999) computes the RMSEs similarly.

Since for any given firm, its term structure of credit spreads is estimated using a relatively small number of bonds, the resulting parameter estimates are sometimes not very precise. Also, there are substantial interquartile variations in the parameter estimates, as reported in Table 5 and Table 6. Consequently, we will concentrate on the median parameter estimates and the median RMSEs in the remainder of this section.

Table 5 indicates that in the stochastic volatility model, both the instantaneous default probability (and credit spread) and its volatility are mean-reverting under the physical measure, because both estimates of α and γ are positive (see equation (5)). A related study by Prigent, Renault, and Scaillet (2001) also find strong mean-reversion in Moody's credit spread series. Under the risk-neutral measure, we can rewrite equation (6) in a slightly different form as

$$\begin{aligned} d\lambda^*_{jt} &= \alpha(\overline{\lambda} - \lambda^*_{jt} + \frac{\eta_1 v_{jt}}{\alpha})dt + \sqrt{v_{jt}}d\widehat{z}_{1j,t}, \\ dv_{jt} &= (\gamma + \xi\eta_2)(\frac{\gamma\overline{v}}{\gamma + \xi\eta_2} - v_{jt})dt + \xi\sqrt{v_{jt}}d\widehat{z}_{2j,t}. \end{aligned} \quad (15)$$

Notice that the variance of credit spreads, v_{jt}, now appears in the drift term of λ^*_{jt}. Therefore, α is not the mean-reversion parameter for λ^*_{jt} under the Q measure. The parameter estimates in Table 5 suggest that the term $\frac{\eta_1 v_{jt}}{\alpha}$ is often positive. It then follows from equation (15) that when λ^*_{jt} is below its unconditional mean $\overline{\lambda}$ so that $(\overline{\lambda} - \lambda^*_{jt})$ is positive (and λ^*_{jt} will increase towards its mean level of $\overline{\lambda}$), the variance of credit spreads v_{jt} may increase the mean-reversion of credit spreads by making the drift of λ^*_{jt} bigger. The opposite effect holds when λ^*_{jt} is above its unconditional mean $\overline{\lambda}$ so that $(\overline{\lambda} - \lambda^*_{jt})$ is negative (and λ^*_{jt} will decrease towards $\overline{\lambda}$). We conclude that in the stochastic volatility model, the default risk λ^*_{jt} can exhibit mean-reversion under the risk-neutral measure for reasonable combinations of parameter values. From Table 6 we see that although the default probability exhibits mean-reversion under the physical measure in the Duffee model, it displays mean-aversion ($\kappa_j + \pi_j < 0$, see equation (8)) under the Q measure. This finding is consistent with that in Duffee (1999). Finally, the mean-reversion parameter for v_{jt} is $\gamma + \xi\eta_2$. The results in Table 5 then show that the volatility of default probability is mean-averting (i.e. non-stationary) for about 50% of the firms.

In both models, parameters δ_{1j} and δ_{2j} capture the correlation between credit spreads and the default-free interest rate. Table 5 and Table 6 indicate that the estimates of δ_{1j} and δ_{2j} are primarily negative and are larger (in absolute terms) than the estimates in Duffee (1999). To appreciate the economic significance of the estimates of δ_{1j} and δ_{2j}, suppose that the first riskless factor f_{1t} drops by 100 bps. This increases λ_{jt} by 0.00475 in the stochastic volatility model and by 0.00242 in the benchmark model, according to the median estimates of δ_{1j} in Table 5 and Table 6, respectively. Given a recovery rate of 44%, this increase in λ_{jt} translates into an increase of 26.6 bps and 13.6 bps in the credit spreads on a near-maturity zero-coupon corporate bond in the stochastic volatility model and in the Duffee model, respectively. Similarly, the median estimates of δ_{2j} reported in Table 5 and Table 6 imply that a 100 bps decline in the second riskless factor f_{2t} corresponds to an increase of 7.5 bps and 3.7 bps in the credit spreads on a near-maturity zero-coupon corporate bond in the stochastic volatility model and in the Duffee model, respectively. A negative relationship between the riskless interest rate and credit spreads is consistent with the structural model in e.g. Longstaff and Schwartz (1995): an increase in the risk-free interest rate increases the drift of the process for firm asset value under the measure Q. Other things being equal, this increase in firm value will pull the firm further away from the default threshold, increasing the bond prices of the firm, thus lowering the bond credit spreads. This finding of a negative relationship also confirms the results of many previous empirical studies, such as Duffee (1998) and Collin-Dufresne, Goldstein, and Martin (2001). On the other hand, Neal, Rolph, and Morris (2000) and David (2002) suggest that the relationship between credit spreads and the riskless interest rate is not constant, but depends on factors such as maturity and the state of the business cycle.

In Table 5, the estimates of the first risk premium parameter, η_1, are positive, while the estimates of the second risk premium parameter, η_2, are negative. In Table 6 the estimates of the risk premium parameter, π_j, are negative. These results are consistent with the theoretical models developed in Section 2 and imply that investors demand compensation for bearing not only the time-varying default risk but also the risk associated with the stochastic volatility of credit spreads. Finally, note that in Table 5 the estimates of η_1 and η_2 are large numbers (in absolute value). This is due to the fact that in equation (15) η_1 and η_2 appear in the product terms of $\eta_1 v_{jt}$ and $\xi\eta_2$, respectively, and the estimates of v_{jt} and ξ are relatively small.

Table 5 and Table 6 also report the interquartile ranges of the mean fitted values of the state variables of the two models. These fitted values are based on the contemporaneous predictions of the state variables of the models, consistent with the way in which the in-sample RMSE is computed. Table 5 shows that in the stochastic volatility model, the median firm has a mean instantaneous default probability λ_{jt} of 1.8% per annum, while the corresponding value for the benchmark model in Table 6 is 1.4% per annum. Both estimates are fairly close to the estimate in Duffee (1999), which is 1.36% per annum, but of course the default probability in the data is much lower. Also, it is worth noticing that although a negative estimate of c_j in the stochastic volatility model might raise the prospect of a negative default probability λ_{jt} (see equation (5)), which is theoretically inconsistent, the mean estimates of λ_{jt} are always positive, as reported in Table 5. Table 5 also shows that the median value of the mean instantaneous variance of credit spreads, v_{jt}, is 0.0000581, which translates into an instantaneous volatility of credit spreads of 0.0076. For the benchmark model, Table 6 reports that the median firm has a mean instantaneous volatility of credit spreads, $\sigma_j\sqrt{\lambda^*_{jt}}$, of 0.002, which is lower than the estimate in the stochastic volatility model.

The stochastic volatility model fits the corporate bond prices better than the benchmark model. Table 5 and Table 6 show that the median in-sample RMSE of the stochastic volatility model is 9.30 bps, versus 11.99 bps for the benchmark model. In addition, according to Appendix A, the stochastic volatility model generates a lower in-sample RMSE than the benchmark model for every single firm in the sample. As another indication of the better fit achieved by the stochastic volatility model, the median estimate of the measurement error volatility, $\sqrt{S_j}$ (see equation (13)), in the stochastic volatility model is 0.369 dollars, while it is 0.611 dollars for the benchmark model (all corporate bonds in the sample have a face value of 100 dollars). The better in-sample fit of the stochastic volatility model should come as no great surprise because it has one more factor than the benchmark model. However, it must be noted that the stochastic volatility model does not nest the benchmark model (see Section 2.2). Therefore, the better in-sample fit achieved by the stochastic volatility model is encouraging.

5.3 Out-of-sample results on credit spread models

To evaluate a model's performance, a model's out-of-sample pricing performance is more important. A more richly parameterized model is expected to perform better in-sample than a more sparsely parameterized alternative model, but this may not be the case out-of-sample. The reason is that models with extra parameters may be penalized in an out-of-sample analysis because of the difficulty in identifying those extra parameters, given the limited sample size of available data.

In this paper, we use as the out-of-sample period the last 12 months in which a firm has valid bond price observations. We conduct the out-of-sample test as follows. We use the in-sample parameter estimates from Table 5 and Table 6, together with the smoothed estimates of the riskless factors in the out-of-sample period and the risk-free model parameters (estimated over the entire sample), to generate a sequence of estimates of the default risky state variables in each of the 12 months in the out-of-sample period. We then use these estimates, as well as the in-sample parameter estimates and information on the riskless term structure, to price corporate bonds in the out-of-sample period. Finally, we compute the corresponding RMSE to gauge the out-of-sample performance of the two models.

The out-of-sample results on the stochastic volatility model are presented in the last row in Table 5. The bottom row of Table 6 reports the results on the benchmark model.

Again, Appendix A presents the out-of-sample performance of the models on a firm by firm basis. It is clear from Table 5 and Table 6 that the stochastic volatility model compares favorably with the benchmark model in out-of-sample forecasting: the median RMSE in the stochastic volatility model is 12.05 bps, down from 13.89 bps for the Duffee model. Appendix A also shows that in slightly more than two-thirds of the cases (for 75 firms out of the total 108 firms), the stochastic volatility model leads to a lower out-of-sample RMSE than the benchmark model.

5.4 Results on credit spread indices

To ensure that the estimation results are robust, we also estimate the stochastic volatility model and the benchmark model on credit spread index data from Standard and Poor's. The estimation results are reported in Table 7A and Table 7B for the stochastic volatility model and the benchmark model, respectively. The stochastic volatility model achieves a lower RMSE than the benchmark model for every credit rating group. As to parameter estimates, they are generally consistent with those reported in Table 5 and Table 6. It is interesting to note that the estimates of the constant term c_j are negative for the majority of credit spread indices in the Duffee model, in contrast to the estimates presented in Table 6, where they are universally positive.

6 Exploration of the empirical results

In this section, we further explore the empirical results. This section is further divided into three subsections. In Section 6.1, we discuss the estimation results for the two credit spread models by credit rating. In Section 6.2, we examine the fit of the two models by credit rating and bond maturity. In Section 6.3, we analyze the role played by the constant term c_j in the two models.

6.1 Estimation results for credit spread models by credit rating

Table 8 reports the median parameter estimates and the median mean fitted values of the state variables for the stochastic volatility model of credit spreads for firms rated Aa, A, and Baa in the sample. In Table 9 we conduct a similar analysis for the benchmark model. The first thing to notice from Table 8 and Table 9 is that there is substantial variation in the estimates across rating classes. While this could be a genuine feature of the data, it is also possible that this finding is due to the lack of precision in the estimates, which is caused by the relatively small number of bond price observations available to estimate an individual firm's term structure of credit spreads.

Table 8 indicates a modest positive relationship between credit spreads and their volatility since the median estimate of ρ for the A-rated group is 0.009, while the corresponding estimate for the Baa-rated group is 0.018. For the relatively small sample of Aa bonds, the estimate of ρ is much larger. Table 1 and Table 2 also report a positive relationship between credit spreads and their volatility. There we observe that as credit ratings drop, credit spread levels go up and at the same time the standard deviations of credit spreads generally increase, thus resulting in a positive relationship between credit spreads and their volatility. A positive median estimate of ρ for all the three rating groups also confirms the evidence of positive skewness of credit spreads reported in Table 1 and Table 2, since in the stochastic volatility model ρ captures the skewness of credit spreads. The "volatility of volatility" parameter ξ also appears to increase as credit rating declines: the median estimates of ξ for the A and Baa rated groups are higher than that for the Aa-rated group.

In both Table 8 and Table 9, the estimates of δ_{1j} and δ_{2j} generally are more negative for lower credit ratings, even if the pattern is clearer in Table 8. For example, in Table 8 the median estimate of δ_{1j} for the Aa-rated group is -0.365, which declines to -0.459 for the A-rated group, then further decreases to -0.558 for the Baa-rated group. Table 8 also reports that the median estimates of δ_{2j} are -0.097 for the Aa-rated firms and -0.141 for the A-rated firms. This relationship can be explained intuitively using the Merton (1974) model: other things being equal, a lower-rated firm is closer to the default boundary. The firm value process of such a firm is more sensitive to changes in the riskless interest rate since an increase in the default-free interest rate translates into an increased drift of the firm value process under the risk-neutral measure, which pulls the firm away from the default boundary, increases the firm's bond prices, and decreases its credit spreads (see also Section 5.2).

Table 8 and Table 9 also show that the risk premium parameter estimates for both models generally increase in (absolute) magnitude as firms' credit rating worsens. This pattern is consistent with intuition: it implies that investors require higher compensation for bearing the default risk and the volatility risk of credit spreads as firms' credit rating declines.

Table 8 reports the median mean fitted values of λ_{jt}, λ^*_{jt}, and v_{jt}, and Table 9 reports the median mean fitted values of λ_{jt}, λ^*_{jt}, and $\sigma_j\sqrt{\lambda^*_{jt}}$ (which measures the volatility of credit spreads in the benchmark model) for the three rating groups. These fitted values are based on the contemporaneous predictions of the state variables in the models, which is consistent with Table 5 and Table 6.

The median estimates of v_{jt} and $\sigma_j\sqrt{\lambda^*_{jt}}$ reveal that the credit spreads of lower-rated firms are generally more volatile than the credit spreads of higher-rated firms. In Table 8, the median value of v_{jt}, the credit spread variance, is 0.0000294 for the Aa-rated firms, and increases to 0.0000715 for the A-rated firms. Although for the Baa-rated firms the median estimate of v_{jt} is lower than that for the A-rated firms, in the results not reported, the 25% and 75% inter-quartile values of v_{jt} for the Baa-rated firms are higher than the corresponding values for the A-rated firms. These results also echo the evidence presented in Table 1 and Table 2, where we observe that the standard deviations of lower-rated credit spreads are in general higher than those of their higher-rated counterparts. The higher volatility associated with the credit spreads on lower-rated bonds implies that managers of bond portfolios consisting of mainly lower-rated bonds should pay more attention to hedging their exposure to the volatility risk.

According to the median estimates of λ_{jt} reported in Table 8 and Table 9, the lower-rated firms have a higher default probability. For example, in Table 8 the median estimate of λ_{jt}, the default probability, is 1.2% for the Aa-rated firms, which rises to 2.8% for the Baa-rated firms. These findings support the claim that the commonly used credit ratings are a good first indicator of a firm's creditworthiness.

6.2 RMSEs of credit spread models by credit rating and maturity

The last two rows in Table 8 and Table 9 tabulate the median in- and out-of-sample RMSEs for the stochastic volatility model and the benchmark model, respectively. These two tables reveal that the stochastic volatility model has a better fit than the benchmark model in both in- and out-of-sample analyses, resulting in a lower RMSE in both cases for all three rating groups. Also, it is interesting to note that the fit of both models worsens as credit ratings fall; the in- and out-of-sample RMSEs for both models become bigger as credit ratings get worse.

We now examine the fit of the two models from the perspective of individual bonds. In Panel A of Table 10A, we first divide all the qualified bond price observations in the in-sample periods into different maturity groups. We then report and compare the in-sample RMSEs for the stochastic volatility and benchmark models for these maturity groups. Again, the stochastic volatility model produces a lower in-sample RMSE than the benchmark model for bonds in every maturity group. Similarly, Panel A of Table 10B reports and compares the out-of-sample RMSEs for both models for bonds in various maturity groups. The stochastic volatility model achieves a lower out-of-sample RMSE than the benchmark model for every maturity group except for bonds with maturities ranging from 20 years to 25 years. In Panel B of Table 10A, we first divide all the valid bond price observations in the in-sample periods into different credit rating classes. We then calculate and report the in-sample RMSEs for both models for bonds within each credit rating class. In Panel B of Table 10B, we conduct a similar analysis on bonds in the out-of-sample periods. We observe from both these two panels that the stochastic volatility model performs better than the benchmark model in not only the in-sample but also the out-of-sample tests. Again, the fit of both models gets worse as bond ratings decline, since both the in-sample and out-of-sample RMSEs of the two models rise as bonds become less creditworthy.

6.3 Role of the constant term c_j

In Figure 4 and Figure 5, we plot the average credit spreads of the Aa-, A-, and Baa-rated firms for the stochastic volatility model and the benchmark model, respectively. These figures are generated as follows. For every firm in each rating group, we take its parameter estimates and its mean fitted values of λ^*_{jt} and v_{jt}. The parameter estimates for the riskless term structure model are taken from Table 4, and for simplicity, we have set the two riskless factors, f_{1t} and f_{2t}, to their sample means over the in-sample period used for estimation of the firm's credit spread term structure. Using all these information, we calculate the credit spreads corresponding to the firm's parameter estimates. We then average across the credit spreads corresponding to all the firms in each rating group and plot the resulting credit spread curves.

In both Figure 4 and Figure 5, the credit spread curves of the lower-rated firms lie above those of the higher-rated firms. In addition, both figures show that the credit spread curves of the lower-rated firms are steeper than those of the higher-rated firms. The pattern of the credit spread term structures exhibited in Figure 4 and Figure 5 is consistent with the stylized facts about the investment-grade credit spreads (see e.g. Litterman and Iben (1991) and Fons (1994)).

The median estimate of the constant term c_j in the stochastic volatility model is negative, while it is positive in the benchmark model. Duffee (1999) argues that the combination of

$c_j > 0$ and $\kappa_j + \pi_j < 0$ is required for his model to fit both the level and slope of the credit spread curves in Figure 5. His reasoning can be briefly summarized as follows. Because $\kappa_j + \pi_j < 0$ under measure Q, investors price the corporate bonds as if the embedded default risk is explosive. That is, they expect λ^*_{jt} to rise through time. For a fixed value of $\kappa_j + \pi_j$, a rising λ^*_{jt} implies a larger drift term in equation (8) and a upward-sloping credit spread curve. However, the slope of the resulting credit spread term structure for higher-rated firms may be too steep to match the slope of the credit spread curve. Therefore, a positive c_j is required in order to depress the overall steepness of the yield spread curves. Without the c_j parameter, for highly rated firms, the yield spread curves generated by the Duffee model would be too steep to be consistent with those found in the data.

Similarly, the pair of $c_j < 0$ and $\alpha > 0$ is necessary for the stochastic volatility model to fit both the level and slope of the credit spread curves in Figure 4. A positive estimate of α implies that the default risk is mean-reverting (i.e. stationary) under the physical measure and can be mean-reverting as well under the risk-neutral measure (see Section 5.2). That is, investors do not expect λ^*_{jt} to rise through time. Instead they expect λ^*_{jt} to likely return to its long-run mean level after a sufficient length of time. (The mean-reversion of λ^*_{jt} in the stochastic volatility model is moderate since e.g. the median estimate of α for the A-rated firms is 0.063, which implies a half-life of about 11 years.) As a result, the stochastic volatility model will not imply an overly steep credit spread curve for highly rated firms. Consequently, the role played by the constant term c_j in the stochastic volatility model is mainly to dilute the effect of λ^*_{jt} through the sum of c_j and λ^*_{jt} in equation (5), since the estimates of λ^*_{jt} are comparatively large for the stochastic volatility model, in order to fit the levels of credit spreads.

7 Conclusion

This paper presents a two-factor affine model of default probability and credit spreads. The first factor can be interpreted as the level of credit spreads, and the second factor is the volatility of credit spreads. This default risk model also allows for a close relationship between credit spreads and the riskless interest rate, a characteristic supported by the empirical findings.

Using a large sample of corporate bond price data, we compare the stochastic volatility model to a benchmark model in which the volatility of credit spreads is not recognized as a distinct state variable. The stochastic volatility model performs better than the benchmark model, resulting in a lower RMSE (in bps) in both in-sample and out-of-sample tests. The properties of actual credit spreads are better captured by the stochastic volatility model. Therefore, the empirical results demonstrate the importance of including the volatility of credit spreads as a second factor in default risk models.

These results question the ability of a single-factor diffusion process to model adequately both the dynamics of credit spreads and the dynamics of credit spread volatility. We propose a multi-factor reduced form model instead. The model is tractable as well as flexible, and the empirical results show that it fits corporate yield curves reasonably well.

In future work, it will be interesting to use the model to value various types of credit derivatives. The use of credit derivatives has been growing at a tremendous pace, reflecting an increase in both transaction volumes and market participants (J.P. Morgan (1999)). All major types of credit derivatives (such as credit default swaps, total return swaps and credit spread options) are significantly affected by changes in credit spreads and default probabilities (Das (1999)). Since the stochastic volatility model approximates the dynamics of credit spreads and default probabilities more realistically and more satisfactorily, it may lead to more accurate pricing of credit derivatives.

Appendix A: Empirical results (in basis points) firm by firm

Firm number	Firm name	In-sample Duffee's model	Stoch. Volati. model	Out-of-sample Duffee's model	Stoch. Volati. model
1	ALLIED - SIGNAL INC	8.308	6.608	12.194	12.562
2	ALLIED CORP	12.180	8.876	13.053	11.412
3	AMERICAN BRANDS	13.401	12.163	16.451	13.047
4	AMERICAN EXPRESS CREDIT	12.924	7.675	8.744	7.602
5	AMERICAN GENERAL FIN CORP	17.617	15.544	8.590	9.320
6	AMR CORPORATION	21.442	20.251	18.440	21.722
7	AON CORP	4.709	3.419	8.767	8.122
8	ARCHER-DANIELS-MIDLAND	11.484	10.997	12.237	12.450
9	ARCO CHEMICAL CO	14.575	12.740	11.742	13.081
10	ARISTAR INC	9.725	7.508	18.517	14.254
11	ATLANTIC RICHFIELD	15.834	14.169	18.007	17.407
12	AVCO FINANCIAL SERVICES	9.787	7.761	12.190	11.161
13	BAXTER INTERNATIONAL INC	11.678	8.668	7.919	7.158
14	BEAR STEARNS CO, INC	5.591	5.009	12.831	8.884
15	BELL TEL OF PENN	4.374	2.214	6.315	6.600
16	BENEFICIAL CORP	13.408	10.804	31.981	17.600
17	BOEING CO	8.286	8.012	11.949	11.957
18	BOSTON EDISON	9.506	9.252	11.617	11.063
19	BOWATER	16.152	15.469	17.457	15.474
20	BP AMERICA INC	5.483	3.100	8.796	8.617
21	BURLINGTON RESOURCES INC	13.525	12.283	15.075	15.067
22	CATERPILLAR INC	22.091	15.186	12.403	15.581
23	CHRYSLER FINANCIAL	15.707	14.905	26.102	5.275
24	CIGNA CORPORATION	8.837	6.699	19.790	12.935
25	CIT GROUP HOLDINGS	12.332	10.610	6.137	6.620
26	CITICORP	9.046	6.874	26.388	27.842
27	COASTAL CORPORATION	12.007	10.213	16.492	9.154
28	COCA - COLA ENTERPRISES INC.	16.922	15.078	15.533	16.210
29	COCA-COLA CO	5.131	3.472	5.972	6.600
30	COMMERCIAL CREDIT	12.582	8.225	12.645	7.604
31	CONSOLIDATED ED OF NY	7.139	4.775	8.271	6.765
32	CONSOLIDATED NATURAL GAS	10.325	8.015	11.310	7.671
33	CSX CORP	10.857	10.573	14.396	13.994
34	DAYTON HUDSON CORP	19.319	17.371	15.335	14.698
35	DELTA AIRLINES, INC.	17.991	17.383	17.105	15.273
36	DILLARD DEPARTMENT STORES	13.566	13.167	14.395	21.448
37	DOLE FOOD CO	10.610	3.716	11.779	9.319
38	DOW CHEMICAL	15.694	14.549	16.077	17.043
39	DOW CHEMICAL B.V.	8.557	7.432	10.971	8.971
40	EATON CORP	12.535	11.147	14.780	14.192
41	ENRON CORP	11.983	10.556	15.057	11.610
42	FEDERAL EXPRESS CORP	14.949	7.032	24.252	8.513
43	FIRST INTERSTATE BANCORP	13.512	10.584	7.670	9.032
44	FORD CAPITAL B.V.	10.603	8.339	10.566	8.291
45	FORD MOTOR	15.330	13.085	12.806	14.413
46	GENERAL MOTORS	14.286	10.304	17.723	15.528
47	GENERAL MOTORS ACPT CORP	12.365	9.081	7.936	10.114
48	GEORGIA PACIFIC	22.804	17.164	18.000	15.724
49	GREAT WESTERN FIN CORP	4.376	2.701	12.371	5.169
50	GTE CORP	10.818	8.257	16.667	14.151
51	HELLER FINANCIAL, INC	9.833	7.584	11.518	10.079
52	HERTZ CORP	11.891	10.619	13.939	11.088
53	HOUSEHOLD FINANCE	17.371	16.294	9.717	8.245
54	INTERNATIONAL LEASE FINANCE	17.403	16.158	13.989	8.797
55	INTERNATIONAL PAPER	17.729	12.896	13.036	14.940
56	INTL BUSINESS MACHINES	14.617	14.131	13.089	14.310
57	JAMES RIVER CORP	6.655	3.475	17.910	17.124
58	LEHMAN BROTHERS HOLDINGS INC	5.784	5.678	16.706	12.493
59	LIMITED, INC	8.002	4.624	13.133	8.236
60	LORAL CORPORATION	16.406	15.514	17.253	17.754
61	LOUISIANA LAND & EXPLORATION	12.912	10.940	11.294	12.841
62	MARRIOTT CORPORATION	21.433	20.355	45.492	39.707

63	MARTIN MARIETTA	9.402	9.085	16.809	22.294
64	MASCO CORP	10.832	7.479	7.861	6.707
65	MAY DEPARTMENT STORES	16.221	14.658	14.948	12.438
66	MAYTAG CORPORATION	7.352	4.245	11.687	11.230
67	MERRILL LYNCH & CO.	11.160	9.508	14.765	7.428
68	MOBIL CORP	12.053	8.841	15.289	9.321
69	MORGAN STANLEY GROUP INC	13.862	10.764	17.366	7.692
70	NEW ENGLAND TEL + TEL	9.509	8.038	11.737	13.298
71	NEW YORK TELEPHONE	10.705	10.559	11.531	9.620
72	NORWEST FINANCIAL INC.	9.290	7.636	17.681	8.399
73	OCCIDENTAL PETROLEUM	21.950	15.061	7.195	13.976
74	PACIFIC BELL	7.331	5.624	11.095	9.597
75	PAINE WEBBER INC	10.829	7.982	19.399	12.668
76	PEPSICO INC	9.009	6.711	11.952	7.841
77	PHILIP MORRIS COS. INC	8.003	7.951	15.983	9.632
78	PROCTER + GAMBLE CO	9.516	9.341	8.495	5.824
79	RALSTON PURINA CO	16.139	15.898	17.964	18.211
80	ROCKWELL INTERNATIONAL	4.922	4.101	8.249	9.112
81	SALOMON INC	6.814	4.605	13.530	9.948
82	SCOTT PAPER	10.828	7.306	16.037	18.292
83	SEAGRAM JOSEPH E + SONS	9.644	8.290	11.210	10.601
84	SEARS ROEBUCK + CO	16.021	9.377	9.149	7.033
85	SECURITY PACIFIC CORP	12.086	10.218	33.141	31.708
86	SHOPKO STORES, INC	22.672	16.318	29.509	27.990
87	SOUTHERN CALIF EDISON	4.125	3.776	6.954	5.638
88	SOUTHWEST AIRLINES CO.	6.858	6.267	8.611	8.815
89	SUNAMERICA INC	14.155	11.902	22.523	23.709
90	TELE-COMMUNICATIONS	17.791	16.832	33.558	37.417
91	TENNECO CREDIT CORP	11.260	7.154	17.046	15.091
92	TENNECO INC	12.219	8.375	17.154	12.218
93	TENNESSEE GAS PIPELINE CO	9.294	6.460	20.404	19.680
94	TEXACO CAPITAL INC.	14.811	13.938	13.444	12.696
95	TEXAS EASTERN TRANSMISSN	3.957	3.204	6.921	6.781
96	TEXAS INSTRUMENTS	7.520	3.417	13.072	7.307
97	TIME WARNER ENT	17.447	16.134	13.845	12.997
98	TRANSAMERICA FINANCIAL	14.577	11.218	18.815	7.912
99	UNION OIL OF CALIFORNIA	9.195	8.748	5.380	6.831
100	UNITED AIR LINES INC	19.939	18.421	22.845	13.736
101	USX CORP	16.598	15.883	19.262	16.754
102	WAL-MART STORES, INC	8.990	8.689	10.890	12.140
103	WEYERHAEUSER CO	13.723	12.381	15.281	14.705
104	WHIRLPOOL CORP	10.116	8.134	15.698	14.365
105	WILLAMETTE IND	10.071	7.007	18.001	17.152
106	WILLIAMS COS	17.444	16.007	26.410	20.446
107	XEROX CORP	13.275	9.442	6.777	8.546
108	XEROX CREDIT CORP	16.672	12.971	24.225	10.964
25%		9.266	7.268	11.273	8.538
Median		11.995	9.297	13.892	12.048
75%		15.421	13.105	17.389	15.073

Appendix B: Technical appendices

B.1 Bond pricing formulas

B.1.1 Solving the Riccati equation using the method of Selby and Strickland (1995)

The class of exponential-affine (or simply affine) term structure models is a class of models in which the yields to maturity are affine functions of some unobservable state variables X_t, thedynamics of which are assumed to be

$$dX_t = \Pi(X_t; \Phi)dt + \Psi(X_t; \Phi)dZ_t, \tag{B.1}$$

where Z_t is a vector of independent Brownian motions and Φ is a vector of the model parameters. The generic form of bond pricing formula for this class of models is

$$G(t, T, 0) = \exp(A(\Phi, T-t) + B(\Phi, T-t)X_t), \tag{B.2}$$

where $G(t, T, 0)$ denotes the time t price of a riskless zero-coupon bond that matures at time T. Let $Y_t(X_t; \Phi, T-t)$ denote the time t continuously compounded yield to maturity on this bond, then the formula of this yield is given by

$$Y_t(X_t; \Phi, T-t) = -\frac{1}{T-t}A(\Phi, T-t) - \frac{1}{T-t}B(\Phi, T-t)X_t, \tag{B.3}$$

which is affine in the state variables X_t.

In the present context, the stochastic volatility model for λ_{jt} in equation (5) leads to a closed-form solution to the price of a default-risky zero-coupon no-recovery bond issued by firm j with a face value of one dollar as

$$\begin{aligned} B_j(t, T, 0, 0) &= \exp[-\tau(c + c_j - \delta_{1j}\overline{f_{1t}} - \delta_{2j}\overline{f_{2t}})] \exp[-\lambda^*_{jt}D(\tau) + v_{jt}F(\tau) + K(\tau)] \\ &\cdot E^Q_t[\exp(-\int_t^T f^*_{1u}du)] \cdot E^Q_t[\exp(-\int_t^T f^*_{2u}du)], \end{aligned} \tag{B.4}$$

where $f^*_{it} \equiv f_{it}(1 + \delta_{ij})$, $i = 1, 2$, and $\tau \equiv T - t$. The first exponential component of the solution in equation (B.4) is a constant, and the two conditional expectation terms in (B.4) can be solved in simple closed-form (see e.g. Pearson and Sun (1994)).

In equation (B.4), the three functions D, F, and K have the time to maturity, τ, as their only variable. They are the solutions to the following system of ordinary differential equations (ODEs)

$$D' + \alpha D - 1 = 0, \quad D(0) = 0; \tag{B.5}$$

$$F' = \frac{1}{2}\xi^2 F^2 - (\gamma + \xi\eta_2)F - \rho\xi DF - \eta_1 D + \frac{1}{2}D^2, \quad F(0) = 0; \tag{B.6}$$

$$K' = -\alpha\overline{\lambda}D + \gamma\overline{v}F, \quad K(0) = 0. \tag{B.7}$$

In the above system of ODEs, D' denotes $\frac{\partial D}{\partial \tau}$, F' and K' are defined analogously; and $D(0) = 0$, $F(0) = 0$, and $G(0) = 0$ are the initial conditions.

The ODE in (B.5), which is for function D, can be solved in simple closed-form as $D(\tau) = \frac{1}{\alpha}(1 - e^{-\alpha\tau})$, and function K can be found by direct integration once we know both D and F. The difficult part lies in finding the solution to the ODE for F in (B.6), which is a Riccati equation. A Riccati equation is one type of *nonlinear* first-order ODE. It is nonlinear due to the presence of quadratic terms in it, e.g. the F^2 component in (B.6). In the current case, although function F can be found in closed-form, the solution is fairly complicated and contains complex algebra.

To overcome this difficulty, Selby and Strickland (1995) make a simple substitution

$$H(s) = \exp[-\frac{1}{2}\xi^2 \int_t^s F(u)du]. \tag{B.8}$$

This substitution transforms the nonlinear ODE in (B.6) into an equivalent *linear* second-order ODE for H. Under this substitution, functions F and K can be rewritten as

$$F(\tau) = -\frac{2}{\xi^2}\frac{H'(\tau)}{H(\tau)}, \tag{B.9}$$

$$K(\tau) = \overline{\lambda}(D(\tau) - \tau) - \frac{2\gamma\overline{v}}{\xi^2}\ln H(\tau). \tag{B.10}$$

Therefore the solution to the bond pricing formula in (B.4) amounts to evaluating $H(\tau)$ and

$H'(\tau)$. A further substitution

$$\tau = -\frac{1}{\alpha}\ln(x), \quad 0 \leq x \leq 1;$$
$$H(\tau) = x^{\beta}Q(x) \tag{B.11}$$

reduces the ODE for H to a homogeneous linear ODE of second order for Q, which can be solved by using a standard series solution method. Once we obtain the solution to Q, we can retrace, substituting Q back into (B.11) for H, and then substituting H back into (B.9) and (B.10) for F and K, respectively. For computational details, please refer to Selby and Strickland (1995).

B.1.2 The coupon bond pricing formula with non-zero recovery rate

Using a no-arbitrage argument, we know that the price of a default-free coupon bond is the sum of the values of individual claims to its remaining coupon payments and its principal, where each of these claims can be regarded as a zero-coupon bond. This is the so-called "portfolio of zeros" approach to valuing a riskless coupon bond. In contrast, when default is a factor, this "portfolio of zeros" approach needs some reconsideration since all remaining coupons now share the same default time if default occurs at or before time T. However, this approach is theoretically justified in Duffie and Singleton (1999) provided that λ_{jt} and L_{jt} are "exogenous," in that they do not depend on the value of the defaultable claim itself, as is the case when pricing corporate bonds. This exogenous assumption is valid in our framework since we are pricing corporate bonds and the recovery rate is assumed to be fixed.

We denote by $G(t, T, cp)$ the price of a default-free coupon bond that pays cp dollars at date T and every six months (i.e. 0.5 years, see equation (B.12)) below) before T. In addition, at the maturity date T, the bond also pays its holder the principal value of one dollar. Similarly, we use $B_j(t, T, cp, 0)$ to denote an otherwise equivalent corporate coupon bond with zero recovery in the event of default. No-arbitrage arguments tell us that we can apply the "portfolio of zeros" approach to value these two coupon bonds as

$$G(t,T,cp) = cp\sum_{k=1}^{N} G(t,t+0.5k,0) + G(t,T,0), \tag{B.12}$$

$$B_j(t,T,cp,0) = cp\sum_{k=1}^{N} B_j(t,t+0.5k,0,0) + B_j(t,T,0,0), \tag{B.13}$$

where in both equations (B.12) and (B.13), N is the total number of coupon payments and is equal to $2(T-t)$. The riskless zero-coupon bond prices that comprise equation (B.12) are given by a well-known closed-form formula (see, for example, Pearson and Sun (1994)). The zero-coupon corporate bond prices in (B.13), such as $B_j(t,T,0,0)$, are also in analytical form, and can be derived using equation (4) and the Selby and Strickland method.

Finally, we denote by $B_j(t,T,cp,1-L)$ the time t price of a corporate coupon bond with a constant recovery rate of $1-L$. The assumption made on recovery rate in this paper implies that upon default at time τ_d ($t < \tau_d \leq T$), the bondholder essentially receives a $1-L$ fraction of an otherwise equivalent riskless bond. As a result, beginning at τ_d, the bondholder is going to receive $cp(1-L)$ dollars of coupon payment every six months prior to T. At time T, she is going to receive a total of $(1-L)(1+cp)$ dollars of final coupon payment plus principal. A modification of equation (10) shows that the value of $B_j(t,T,cp,1-L)$ is equal to

$$B_j(t,T,cp,1-L) = (1-L)G(t,T,cp) + LB_j(t,T,cp,0). \tag{B.14}$$

B.2 The Kalman filter

B.2.1 A brief summary of the Kalman filter

Consider an $n \times 1$ vector of variables observed at time t, y_t, and a $r \times 1$ unobservable state vector, ξ_t. The *state-space* representation of the dynamics of y_t is given by

$$\xi_{t+1} = F\xi_t + \epsilon_{t+1}, \tag{B.15}$$

$$y_t = H\xi_t + \omega_t, \tag{B.16}$$

where F and H are $r \times r$ and $n \times r$ matrices, respectively. The $r \times 1$ vector ϵ_t and $n \times 1$ vector ω_t are vectors of white noise, $E(\epsilon_t \epsilon_\tau') = Q$ and $E(\omega_t \omega_\tau') = R$, for $t \neq \tau$. Here, Q and R are of dimensions $r \times r$ and $n \times n$, respectively. In addition, the disturbances ϵ_t and ω_t are assumed to be uncorrelated at all lags. In the Kalman filter setup, equation (B.15) is called the *transition equation* (or the *state equation*), and equation (B.16) is called the *measurement* equation (or the *observation* equation).

The Kalman filter is an algorithm for calculating linear least squares forecasts of the state vector on the basis of data observed through date t, $\xi_{t+1|t} \equiv E(\xi_{t+1}|Y_t)$, where $Y_t \equiv (y_t', ..., y_1')$ and $E(\xi_{t+1}|Y_t)$ denotes the linear projection of ξ_{t+1} on Y_t. It calculates these forecasts recursively, generating $\xi_{1|0}$, $\xi_{2|1}$, ..., $\xi_{T|T-1}$ in succession. Associated with each of these forecasts is a variance-covariance matrix based on the one-step-ahead prediction error

$$P_{t+1|t} \equiv E[(\xi_{t+1} - \xi_{t+1|t})(\xi_{t+1} - \xi_{t+1|t})'],$$

which will be used to evaluate the likelihood function.

The key equations for the Kalman filter are

$$\xi_{t|t} = \xi_{t|t-1} + P_{t|t-1}H'(HP_{t|t-1}H' + R)^{-1}(y_t - H\xi_{t|t-1}), \tag{B.17}$$

$$\xi_{t+1|t} = F\xi_{t|t}, \tag{B.18}$$

$$P_{t|t} = P_{t|t-1} - P_{t|t-1}H'(HP_{t|t-1}H' + R)^{-1}HP_{t|t-1}, \tag{B.19}$$

$$P_{t+1|t} = FP_{t|t}F' + Q. \tag{B.20}$$

Among these four equations, equations (B.18) and (B.20) belong to the (one-step-ahead) prediction stage; equations (B.17) and (B.19) are for the updating stage, i.e., using the information available through date t to update the previous (one-step-ahead) estimate of ξ_t.

For a linear and Gaussian model, the method of maximum likelihood estimation (MLE) can be used to estimate parameters of the model. The sample likelihood is given by

$$\sum_{t=1}^{T} \ln f(y_t|Y_{t-1}), \tag{B.21}$$

where we have

$$f(y_t|Y_{t-1}) =$$
$$(2\pi)^{-\frac{n}{2}}|HP_{t|t-1}H' + R|^{-\frac{1}{2}} \exp[-\frac{1}{2}(y_t - H\xi_{t|t-1})'(HP_{t|t-1}H' + R)^{-1}(y_t - H\xi_{t|t-1})]. \tag{B.22}$$

When the measurement equation is nonlinear

$$y_t = H(\xi_t) + \omega_t, \tag{B.23}$$

where $H(\cdot)$ is a nonlinear function, the *extended Kalman filter* (EKF) can be used to obtain an approximate filter. In particular, we replace the $H(\xi_t)$ function in (B.23) with its first-order Taylor's approximation around $\xi_t = \xi_{t|t-1}$ so that

$$y_t = H(\xi_{t|t-1}) + \frac{\partial H(\xi_t)}{\partial \xi_t'}|_{\xi_t=\xi_{t|t-1}}(\xi_t - \xi_{t|t-1}) + \omega_t. \tag{B.24}$$

The idea behind the EKF is that to treat equation (B.24), together with (B.15), as if they were the true model. It follows that equations (B.17) through (B.20) will have to be modified accordingly. For details, please refer to Hamilton (1994) and Harvey (1990). Also notice that the parameter estimates obtained from the EKF will be quasi-maximum likelihood estimates (QMLE), rather than MLE as in the linear model case. See also Footnote 8.

B.2.2 Details of the transition equations used in estimation

For estimation of the default-free interest rate model, the components a and b of the transition equation in (12) are given by

$$\begin{aligned} a &= \begin{bmatrix} \mu_1(1 - e^{-\phi_1/12}) \\ \mu_2(1 - e^{-\phi_2/12}) \end{bmatrix}, \\ b &= \begin{bmatrix} e^{-\phi_1/12} & 0 \\ 0 & e^{-\phi_2/12} \end{bmatrix}, \end{aligned}$$

and $V(F_{t-1})$ is a 2×2 diagonal matrix with elements

$$V_{i,i}(F_{t-1}) = \phi_i^{-1}\sigma_i^2[f_{i,t-1}(e^{-\phi_i/12} - e^{-2\phi_i/12}) + \frac{\mu_i}{2}(1 - e^{-\phi_i/12})^2], \text{ for } i = 1, 2.$$

For estimation of the stochastic volatility model of credit spreads, the components a_j and b_j of the transition equation in (14) are

$$\begin{aligned} a_j &= \begin{bmatrix} \overline{\lambda}(1 - e^{-\tau\alpha/12}) \\ \overline{v}(1 - e^{-\tau\gamma/12}) \end{bmatrix}, \\ b_j &= \begin{bmatrix} e^{-\tau\alpha/12} & 0 \\ 0 & e^{-\tau\gamma/12} \end{bmatrix}, \end{aligned}$$

and $\Gamma(\Sigma_{j,t-\tau})$ is a 2×2 matrix with the two diagonal elements given by $(\frac{v_{j,t-\tau}-\overline{v}}{2\alpha-\gamma})(e^{-\tau\gamma/12} - e^{-2\tau\alpha/12}) + \frac{\overline{v}}{2\alpha}(1 - e^{-2\tau\alpha/12})$ and $v_{j,t-\tau}\frac{\xi^2}{\gamma}(e^{-\tau\gamma/12} - e^{-2\tau\gamma/12}) + \frac{\xi^2}{2\gamma}\overline{v}(1 - e^{-\tau\gamma/12})^2$, respectively, and the off-diagonal element is $\rho\xi[\frac{\overline{v}}{\alpha+\gamma}(1 - e^{-\tau(\alpha+\gamma)/12}) + \frac{(v_{j,t-\tau}-\overline{v})}{\alpha}(e^{-\tau\gamma/12} - e^{-\tau(\alpha+\gamma)/12})]$, where we recall that τ indicates the number of months elapsed between successive observations of corporate bond prices of firm j.

B.3 The conditional moments of the stochastic volatility model of credit spreads

Assume that an $n \times 1$ vector X_t follows the stochastic differential equation (SDE)

$$dX_t = U(X_t; \Psi)dt + \Sigma(X_t; \Psi)dW_t, \tag{B.25}$$

where W_t is an $n \times 1$ vector of independent standard Brownian motions. If $U(X_t; \Psi)$ and $\Sigma(X_t; \Psi)\Sigma(X_t; \Psi)'$ are affine functions of X_t so that $U(X_t; \Psi)$ can be written in the form of $G + KX_t$, where G and K are matrices of dimension $n \times 1$ and $n \times n$, respectively, then the mean and variance of X_{t+h}, conditional on X_t, are also affine functions of X_t as long as K is diagonable (i.e., all of the eigenvalues of K are distinct). Here h denotes a sufficiently small length of time.

Denote the eigenvalue decomposition of K by QkQ^{-1}, where $Q = [Q_1\ Q_2\ \dots\ Q_n]$ and Q_i, $i = 1, 2, \dots, n$, are the n linearly independent eigenvectors of K, and k is a square diagonal matrix with elements along its main diagonal being the n distinct eigenvalues of K. Then Duan and Simonato (1999) show that the conditional mean of X_{t+h}, $E(X_{t+h}|X_t)$, is given by

$$E(X_{t+h}|X_t) = Qe^{kh}Q^{-1}X_t + Q(e^{kh} - I)k^{-1}Q^{-1}G, \tag{B.26}$$

which is clearly affine in X_t. Similarly, we can compute the conditional variance of X_{t+h}, $Var(X_{t+h}|X_t)$. The required formulas for which can be found in Appendix B in Duan and Simonato (1999).

We can use the above result to derive the conditional moments of the stochastic volatility model of credit spreads. The model is

$$\begin{aligned} d\lambda^*_{jt} &= \alpha(\overline{\lambda} - \lambda^*_{jt})dt + \sqrt{v_{jt}}dz_{1j,t}, \\ dv_{jt} &= \gamma(\overline{v} - v_{jt})dt + \xi\sqrt{v_{jt}}dz_{2j,t}, \end{aligned} \tag{B.27}$$

where $corr(z_{1j,t},\ z_{2j,t}) = \rho$. Using a change of variable technique, we can rewrite the above model as

$$\begin{gathered} d\lambda^*_{jt} = \alpha(\overline{\lambda} - \lambda^*_{jt})dt + \sigma\eta\sqrt{u_{jt}}dz_{2j,t} + \sqrt{u_{jt}}dz_{3j,t}, \\ du_{jt} = \gamma(\overline{u} - u_{jt})dt + \eta\sqrt{u_{jt}}dz_{2j,t}, \end{gathered} \tag{B.28}$$

where $u_{jt} \equiv (1-\rho^2)v_{jt}$, $\overline{u} \equiv (1-\rho^2)\overline{v}$, $\eta \equiv \sqrt{1-\rho^2}\xi$, and $\sigma \equiv \frac{\rho}{\eta\sqrt{1-\rho^2}}$. The two Brownian motions $z_{2j,t}$ and $z_{3j,t}$ are now independent. We can rewrite the model in (B.28) in matrix form similar to equation (B.25) as

$$\begin{aligned} G &= \begin{bmatrix} \alpha\overline{\lambda} \\ \gamma\overline{u} \end{bmatrix},\ K = \begin{bmatrix} -\alpha & 0 \\ 0 & -\gamma \end{bmatrix}, \\ X_t &= \begin{bmatrix} \lambda^*_{jt} \\ u_{jt} \end{bmatrix},\ dW_t = \begin{bmatrix} dz_{2j,t} \\ dz_{3j,t} \end{bmatrix}, \end{aligned}$$

and

$$\Sigma(X_t;\Psi) = \begin{bmatrix} \sigma\eta\sqrt{u_{jt}} & \sqrt{u_{jt}} \\ \eta\sqrt{u_{jt}} & 0 \end{bmatrix}.$$

Notice that in the present model, the matrix K is diagonable if and only if $\alpha \neq \gamma$.

Substituting the above matrices into the formulas in Duan and Simonato (1999), one obtains after some manipulation the conditional moments of this model in terms of the original state variables λ_{jt}^* and v_{jt} and the model parameters. The conditional means are

$$\begin{aligned} E(\lambda_{js}^*|\lambda_{jt}^*) &= \lambda_{jt}^* e^{-\alpha(s-t)} + \overline{\lambda}(1 - e^{-\alpha(s-t)}), \\ E(v_{js}|v_{jt}) &= v_{jt}e^{-\gamma(s-t)} + \overline{v}(1 - e^{-\gamma(s-t)}), \text{ for } s \geq t; \end{aligned} \tag{B.29}$$

and the conditional variances and covariances are

$$\begin{aligned} Var(\lambda_{js}^*|\lambda_{jt}^*) &= (\frac{v_{jt} - \overline{v}}{2\alpha - \gamma})(e^{-\gamma(s-t)} - e^{-2\alpha(s-t)}) \\ &\quad + \frac{\overline{v}}{2\alpha}(1 - e^{-2\alpha(s-t)}), \\ Cov(\lambda_{js}^*, v_{js}|\lambda_{jt}^*, v_{jt}) &= \rho\xi[\frac{\overline{v}}{\alpha+\gamma}(1 - e^{-(\alpha+\gamma)(s-t)}) + \frac{(v_{jt} - \overline{v})}{\alpha}(e^{-\gamma(s-t)} \\ &\quad - e^{-(\alpha+\gamma)(s-t)})], \\ Var(v_{js}|v_{jt}) &= v_{jt}\frac{\xi^2}{\gamma}(e^{-\gamma(s-t)} - e^{-2\gamma(s-t)}) + \frac{\xi^2}{2\gamma}\overline{v}(1 - e^{-\gamma(s-t)})^2, \text{ for } s \geq t. \end{aligned} \tag{B.30}$$

As $s \to \infty$, the processes for λ_{jt}^* and v_{jt} have a steady-state (unconditional) distribution with mean given by $\overline{\lambda}$ and $\overline{v}$, and variance equal to $\frac{\overline{v}}{2\alpha}$ and $\frac{\xi^2\overline{v}}{2\gamma}$, respectively. In addition, the steady-state covariance between λ_{js}^* and v_{js} is $\overline{v}\frac{\rho\xi}{\alpha+\gamma}$. Finally, the above conditional moments can be derived alternatively using the method of Fisher and Gilles (1996).

Table 1

Summary statistics for Moody's credit spread indices

Moody's 10-year and 30-year Aaa and Baa credit spreads (in basis points). The data are monthly for January 1960 - April 2003 (for 10-year spreads) and February 1977 - February 2002 (for 30-year spreads). Source of data: Federal Reserve Board's G.13 release.

	Mean (in bps)	Standard deviation (in %)	Skewness	Kurtosis
10-year Aaa	80.44	0.50	0.72	3.37
10-year Baa	179.73	0.70	0.37	2.71
30-year Aaa	76.62	0.38	0.90	3.72
30-year Baa	184.53	0.60	0.81	3.07

Table 2

Summary statistics for Standard & Poor's credit spread indices

Standard & Poor's AAA, AA, A, and BBB credit spreads (in basis points) for various maturities. The data are weekly and cover the sample period August 6, 1996 - September 11, 2001. Source of data: Standard and Poor's.

Panel A. AAA	1-year	5-year	10-year	15-year	20-year	25-year
Mean (in bps)	36.97	69.31	83.01	97.76	96.50	109.28
Standard deviation (in %)	0.24	0.28	0.32	0.40	0.35	0.38
Skewness	0.42	-0.02	-0.01	0.22×10^{-2}	0.07	0.04
Kurtosis	2.28	1.55	1.57	1.56	1.67	1.66
Panel B: AA						
Mean (in bps)	43.45	84.18	101.39	118.17	118.35	132.29
Standard deviation (in %)	0.25	0.33	0.39	0.48	0.44	0.47
Skewness	0.27	-0.04	-0.06	-0.10	-0.04	-0.05
Kurtosis	1.90	1.63	1.62	1.60	1.65	1.63
Panel C: A						
Mean (in bps)	67.66	111.07	129.49	146.94	147.59	161.91
Standard deviation (in %)	0.36	0.44	0.49	0.58	0.53	0.56
Skewness	0.07	0.03	0.09	0.09	0.16	0.13
Kurtosis	1.78	1.61	1.66	1.67	1.72	1.68
Panel D: BBB						
Mean (in bps)	111.06	157.54	177.26	195.54	196.74	NA
Standard deviation (in %)	0.58	0.62	0.65	0.73	0.68	NA
Skewness	0.11	-0.05	-0.05	-0.04	-0.03	NA
Kurtosis	1.79	1.52	1.51	1.53	1.50	NA

Table 3
Summary statistics for corporate bond data

Corporate bond data are extracted from the Lehman Brothers Fixed Income Database. Every firm included in the sample must have at least 48 months in which at least 3 qualified bond price observations are available. There are in total 108 such firms over the sample period beginning January 1985 and ending March 1998. In Panel B, *yields* denote actual yields to maturity on firms' outstanding bonds. In Panel C, *credit spreads* are defined as the spreads of firms' actual yields to maturity over the riskless interest rates implied by the default-free interest rate model.

	Across 108 firms				
Panel A: corporate bonds	Minimum	1^{st} quartile	Median	3^{rd} quartile	Maximum
Months of data	48	60	73	88	157
Mean number of bonds	3	3.55	4.40	5.77	12.36
Mean years to maturity	2.46	5.42	7.92	15.12	27.79
Minimum years to maturity	1.02	1.02	1.06	1.59	24.19
Maximum years to maturity	5.03	12.02	20.25	30.39	33.44
Mean annual coupon rate	0	7.69	8.37	9.03	11.74
Panel B: yields					
Mean (in bps)	614.43	688.15	709.56	747.54	922.71
Standard deviation (in %)	0.52	0.65	0.77	1.04	1.99
Skewness	-0.69	0.21	0.41	0.59	1.69
Kurtosis	1.45	2.07	2.43	2.75	7.30
Panel C: credit spreads					
Mean (in bps)	136.85	202.45	246.07	276.57	390.20
Standard deviation (in %)	0.54	0.98	1.14	1.37	2
Skewness	-0.65	0.25	0.39	0.68	1.89
Kurtosis	1.44	1.66	1.83	2.24	8.40

Table 4

Estimation results for the default-free term structure

The instantaneous default-free interest rate, i_t, is modeled as

$$i_t = c + f_{1t} + f_{2t},$$
$$df_{it} = \phi_i(\mu_i - f_{it})dt + \sigma_i\sqrt{f_{it}}dw_{it}, \text{ (under the } P \text{ measure)}$$
$$df_{it} = (\phi_i\mu_i - (\phi_i + \pi_i)f_{it})dt + \sigma_i\sqrt{f_{it}}d\widehat{w}_{it}, \text{ (under the } Q \text{ measure)}$$

for $i = 1, 2$. We use an extended Kalman filter approach to estimate the above riskless interest rate model. The data are selected from the CRSP and include month-end price observations of the most recently issued Treasury bonds with maturities closest to 3 and 6 months and 1, 2, 3, 5, 10, and 30 years. The robust standard errors for the parameter estimates are calculated following White (1982) and are presented in parentheses.

i	ϕ_i	μ_i	σ_i	π_i	$\phi_i + \pi_i$	c
1	0.56	0.47	0.02	-0.03	0.53	-0.48
	(0.0002)	(0.00005)	(0.00006)	(0.00003)		(0.00028)
2	0.02	0.10	0.05	-0.00008	0.02	-
	(0.0005)	(0.00011)	(0.017)	(0.059)		

Bond maturity	RMSE (in bps)
3 months	30.77
6 months	17.15
1 year	4.55
2 year	10.50
3 years	8.18
5 years	4.77
10 years	10.40
30 years	18.40

Table 5

Estimation results for the stochastic volatility model of credit spreads

Under the physical measure P, firm j' s instantaneous default probability at time t, λ_{jt}, is assumed to follow the dynamics

$$\lambda_{jt} = c_j + \lambda^*_{jt} + \delta_{1j}(f_{1t} - \overline{f_{1t}}) + \delta_{2j}(f_{2t} - \overline{f_{2t}}),$$
$$d\lambda^*_{jt} = \alpha(\overline{\lambda} - \lambda^*_{jt})dt + \sqrt{v_{jt}}dz_{1j,t},$$
$$dv_{jt} = \gamma(\overline{v} - v_{jt})dt + \xi\sqrt{v_{jt}}dz_{2j,t},$$

where $corr(z_{1j,t},\, z_{2j,t}) = \rho$, and f_{1t} and f_{2t} are the two riskless factors of the default-free interest rate model. Under the risk-neutral measure Q, the processes for λ^*_{jt} and v_{jt} are

$$\begin{aligned} d\lambda^*_{jt} &= (\alpha\overline{\lambda} - \alpha\lambda^*_{jt} + \eta_1 v_{jt})dt + \sqrt{v_{jt}}d\widehat{z}_{1j,t}, \\ dv_{jt} &= (\gamma\overline{v} - (\gamma + \xi\eta_2)v_{jt})dt + \xi\sqrt{v_{jt}}d\widehat{z}_{2j,t}, \end{aligned}$$

where $corr(\widehat{z}_{1j,t},\, \widehat{z}_{2j,t}) = \rho$. An extended Kalman filter approach is adopted to estimate the above stochastic volatility model of credit spreads. The data consist of month-end corporate coupon bond prices, which are assumed to be observed with measurement errors that are normally distributed with mean zero and variance S_j.

(Table 5 is continued on the next page.)

Table 5 (continued)

Variable	1^{st} quartile	Median	3^{rd} quartile
c_j	-0.437	-0.061	-0.054
α	0.035	0.056	0.093
$\overline{\lambda}$	0.063	0.079	0.480
γ	0.006	0.077	0.179
$\overline{v}$	0.721×10^{-8}	0.531×10^{-6}	0.886×10^{-6}
ξ	0.004	0.006	0.007
ρ	-0.044	0.011	0.600
$\delta_{1,j}$	-0.766	-0.475	0.026
$\delta_{2,j}$	-0.202	-0.134	-0.026
$\sqrt{S_j}$	0.271	0.369	0.727
η_1	6.379	9.956	15.770
η_2	-68.046	-19.365	-10.425
$\gamma + \xi\eta_2$	-0.310	-0.009	0.106
Mean fitted λ_{jt}	0.012	0.018	0.032
Mean fitted λ^*_{jt}	0.069	0.085	0.475
Mean fitted v_{jt}	0.241×10^{-4}	0.581×10^{-4}	0.121×10^{-3}
In-sample RMSE (in bps)	7.27	9.30	13.11
Out-of-sample RMSE (in bps)	8.54	12.05	15.07

Table 6

Estimation results for the benchmark model of credit spreads

In the benchmark model, under the physical measure P, firm j's instantaneous default probability at time t, λ_{jt}, is given by

$$\lambda_{jt} = c_j + \lambda^*_{jt} + \delta_{1j}(f_{1t} - \overline{f_{1t}}) + \delta_{2j}(f_{2t} - \overline{f_{2t}}),$$
$$d\lambda^*_{jt} = \kappa_j(\theta_j - \lambda^*_{jt})dt + \sigma_j\sqrt{\lambda^*_{jt}}du_{jt},$$

where f_{1t} and f_{2t} are the two riskless factors of the default-free interest rate model. Under the risk-neutral measure Q, the process for λ^*_{jt} becomes

$$d\lambda^*_{jt} = (\kappa_j\theta_j - (\kappa_j + \pi_j)\lambda^*_{jt})dt + \sigma_j\sqrt{\lambda^*_{jt}}d\widehat{u}_{jt}.$$

We use an extended Kalman filter approach to estimate the benchmark model. The data consist of month-end corporate coupon bond prices, which are assumed to be observed with measurement errors that are normally distributed with mean zero and variance S_j. (Table 6 is continued on the next page.)

Table 6 (continued)

Variable	1^{st} quartile	Median	3^{rd} quartile
c_j	0.006	0.011	0.015
κ_j	0.301×10^{-7}	0.026	0.376
θ_j	0.905×10^{-7}	0.242×10^{-3}	0.002
σ_j	0.024	0.045	0.064
δ_{1j}	-0.480	-0.242	-0.088
δ_{2j}	-0.183	-0.066	0.037
$\sqrt{S_j}$	0.372	0.611	1.314
π_j	-0.567	-0.326	-0.191
$\kappa_j + \pi_j$	-0.363	-0.223	-0.130
Mean fitted λ_{jt}	0.010	0.014	0.019
Mean fitted λ^*_{jt}	0.493×10^{-3}	0.003	0.006
Mean fitted $\sigma_j\sqrt{\lambda^*_{jt}}$	0.615×10^{-3}	0.002	0.005
In-sample RMSE (in bps)	9.27	11.99	15.42
Out-of-sample RMSE (in bps)	11.27	13.89	17.39

Table 7A

Estimation results for the stochastic volatility model of credit spreads using Standard & Poor's credit spread indices

Data used are described in Table 2. Parameters are defined in Table 5.

Variable	AAA	AA	A	BBB
c_j	-0.059	-0.073	-0.159	-0.060
α	0.031	0.043	0.037	0.040
$\overline{\lambda}$	0.065	0.087	0.170	0.075
γ	0.260	0.187	0.126	0.128
$\overline{v}$	0.484×10^{-7}	0.197×10^{-5}	0.373×10^{-5}	0.400×10^{-5}
ξ	0.005	0.013	0.004	0.005
ρ	0.367	0.392	0.549	0.374
δ_{1j}	-0.303	-0.113	-0.190	-0.370
δ_{2j}	-0.163	0.072	-0.121	-0.100
$\sqrt{S_j}$	5.889×10^{-4}	5.515×10^{-4}	6.151×10^{-4}	6.449×10^{-4}
η_1	20.493	17.974	11.430	9.748
η_2	-35.003	-0.761	-30.454	-18.518
$\gamma+\xi\eta_2$	0.085	0.177	0.004	0.035
RMSE (in bps)	5.09	4.76	5.47	5.71

Table 7B

Estimation results for the benchmark model of credit spreads using Standard & Poor's credit spread indices

Data used are described in Table 2. Parameters are defined in Table 6.

Variable	AAA	AA	A	BBB
c_j	-0.297×10^{-3}	-0.476×10^{-3}	-0.185×10^{-3}	0.292×10^{-5}
κ_j	0.170	0.157	0.080	0.019
θ_j	0.182×10^{-2}	0.226×10^{-2}	0.585×10^{-2}	0.033
σ_j	0.070	0.072	0.065	0.064
δ_{1j}	-0.085	-0.062	-0.080	-0.164
δ_{2j}	0.110	0.153	0.200	0.287
$\sqrt{S_j}$	8.784×10^{-4}	9.243×10^{-4}	8.433×10^{-4}	7.916×10^{-4}
π_j	-0.266	-0.263	-0.158	-0.070
$\kappa_j + \pi_j$	-0.096	-0.106	-0.078	-0.051
RMSE (in bps)	8.23	8.69	7.90	7.28

Table 8

Median estimation results for the stochastic volatility model of credit spreads sorted by credit rating

Parameters are defined in Table 5. A firm's credit rating is defined as the mean of the Moody's ratings on the firm's qualified bonds over the sample period used in the estimation of its term structure of credit spreads.

Variable	Aa	A	Baa
Number of firms	12	60	36
c_j	-0.614$\times 10^{-1}$	-0.608$\times 10^{-1}$	-0.612$\times 10^{-1}$
α	0.033	0.063	0.053
$\overline{\lambda}$	0.075	0.077	0.080
γ	0.144	0.100	0.034
$\overline{v}$	0.498$\times 10^{-6}$	0.531$\times 10^{-6}$	0.510$\times 10^{-6}$
ξ	0.456$\times 10^{-2}$	0.578$\times 10^{-2}$	0.558$\times 10^{-2}$
ρ	0.375	0.009	0.018
δ_{1j}	-0.365	-0.459	-0.558
δ_{2j}	-0.097	-0.141	-0.134
η_1	10.059	9.385	10.870
η_2	-18.146	-15.185	-42.839
$\gamma + \xi\eta_2$	0.024	0.059	-0.195
Mean fitted λ_{jt}	0.012	0.016	0.028
Mean fitted λ^*_{jt}	0.073	0.081	0.098
Mean fitted v_{jt}	0.294$\times 10^{-4}$	0.715$\times 10^{-4}$	0.637$\times 10^{-4}$
In-sample RMSE (in bps)	7.66	9.41	10.69
Out-of-sample RMSE (in bps)	8.76	11.12	13.98

Table 9

Median estimation results for the benchmark model of credit spreads sorted by credit rating

Parameters are defined in Table 6. A firm's credit rating is defined as the mean of the Moody's ratings on the firm's qualified bonds over the sample period used in the estimation of its term structure of credit spreads.

Variable	Aa	A	Baa
Number of firms	12	60	36
c_j	0.006	0.009	0.014
κ_j	0.590×10^{-6}	0.030	0.046
θ_j	0.512×10^{-5}	0.370×10^{-3}	0.508×10^{-3}
σ_j	0.030	0.045	0.047
δ_{1j}	-0.217	-0.191	-0.443
δ_{2j}	-0.068	-0.040	-0.092
π_j	-0.274	-0.331	-0.326
$\kappa_j + \pi_j$	-0.274	-0.226	-0.208
Mean fitted λ_{jt}	0.007	0.013	0.021
Mean fitted λ^*_{jt}	0.001	0.002	0.006
Mean fitted $\sigma_{jt}\sqrt{\lambda^*_{jt}}$	0.152×10^{-2}	0.189×10^{-2}	0.004
In-sample RMSE (in bps)	9.14	12.13	13.16
Out-of-sample RMSE (in bps)	9.82	13.08	17.35

Table 10A

In-sample RMSEs by maturity and credit rating

Across all 108 firms in the sample, there are in total 38,197 observations on bond prices in the in-sample periods. We report in-sample RMSEs (in bps) for these bonds by maturity and credit rating. The maturity of the bond (**M**) is measured in years. The credit rating classese are defined according to Moody's ratings on firms' individual bonds.

Panel A: maturity	Number of bonds	SV model	Benchmark model
$1 < M \leq 5$	16,412	12.34	15.39
$5 < M \leq 10$	10,941	8.12	10.28
$10 < M \leq 15$	3,038	8.18	9.79
$15 < M \leq 20$	3,069	15.14	15.56
$20 < M \leq 25$	674	13.22	13.57
$25 < M \leq 30$	3,660	14.79	15.15
$30 < M \leq 35$	403	13.03	14.22
Panel B: credit rating			
Aa	6,023	8.30	10.77
A	21,517	11.29	13.19
Baa	10,561	13.53	15.85
Below Baa	96	15.17	17.07

Table 10B

Out-of-sample RMSEs by maturity and credit rating

Across all 108 firms in the sample, there are in total 6,101 observations on bond prices in the out-of-sample periods. We report out-of-sample RMSEs (in bps) for these bonds by maturity and credit rating. The maturity of the bond (**M**) is measured in years. The credit rating classes are defined according to Moody's ratings on firms' individual bonds.

Panel A: maturity	Number of bonds	SV model	Benchmark model
$1 < M \leq 5$	1,858	14.38	17.39
$5 < M \leq 10$	2,035	14.82	17.41
$10 < M \leq 15$	734	12.70	13.88
$15 < M \leq 20$	565	15.05	16.15
$20 < M \leq 25$	119	15.93	15.02
$25 < M \leq 30$	593	18.48	19.05
$30 < M \leq 35$	197	15.66	16.40
Panel B: credit rating			
Aa	937	11.81	13.03
A	3,561	14.52	16.94
Baa	1,588	17.31	19.09
Below Baa	15	8.74	13.50

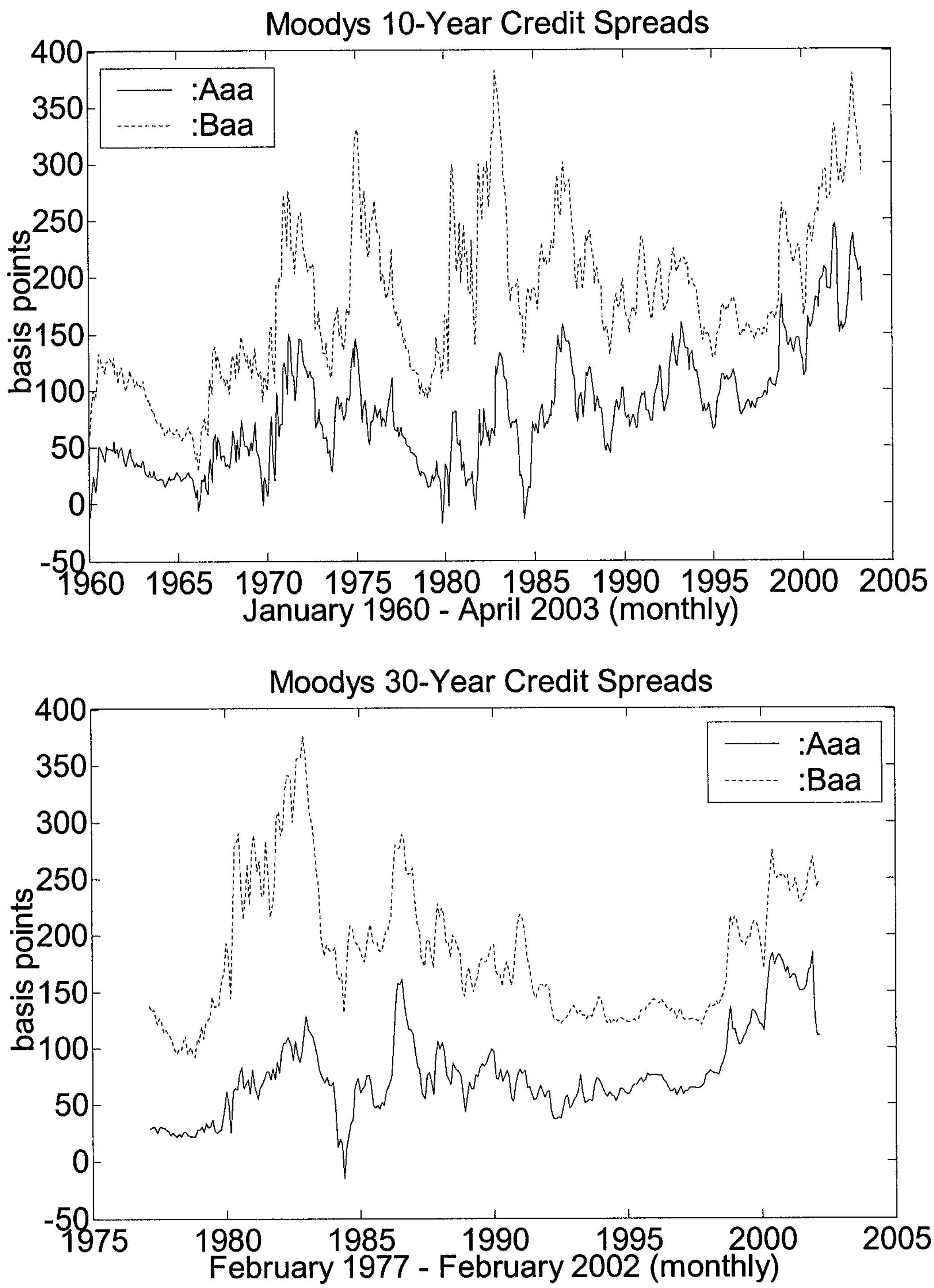

Figure 1: Moody's 10-year and 30-year Aaa and Baa credit spreads (in bps). Data used are described in Table 1.

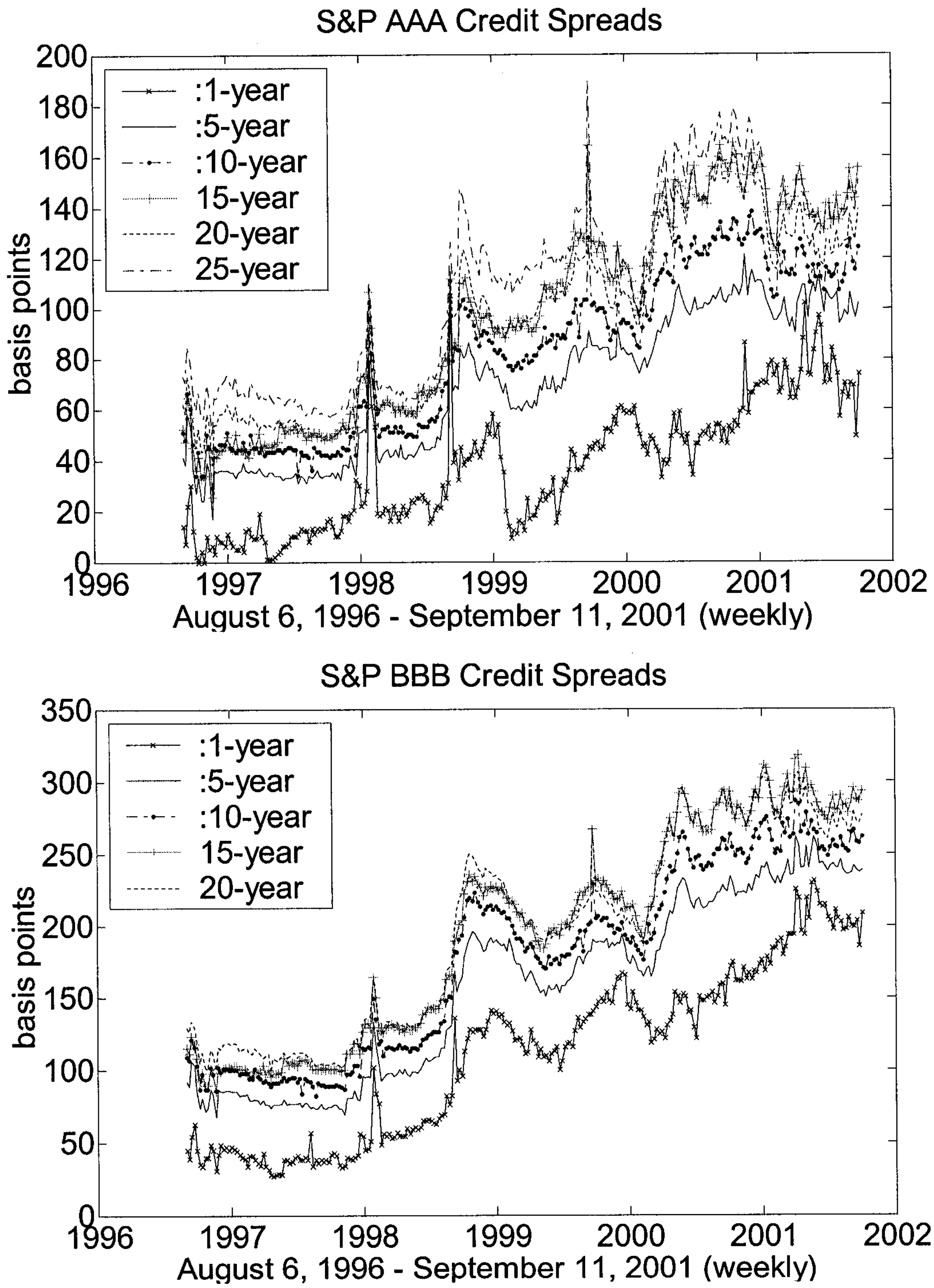

Figure 2: Standard and Poor's AAA and BBB credit spreads (in bps) for various maturities. Data used are described in Table 2.

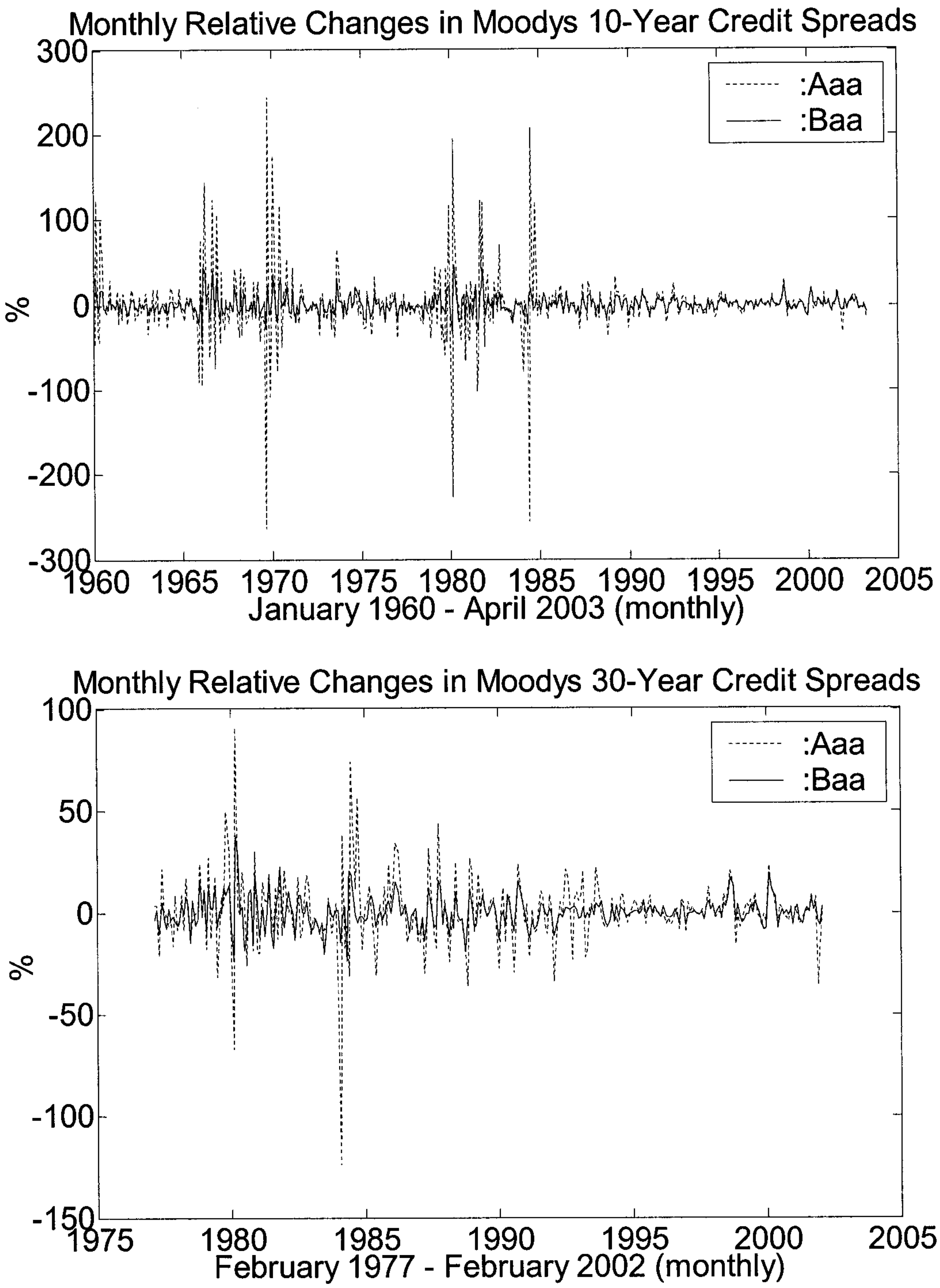

Figure 3: Monthly relative changes in Moody's 10-year and 30-year Aaa and Baa credit spreads. Data used are described in Table 1. Relative change in credit spreads from month t to month t-1 is defined as $\ln(CS_t/CS_{t-1})$, where CS_t denotes credit spread in month t.

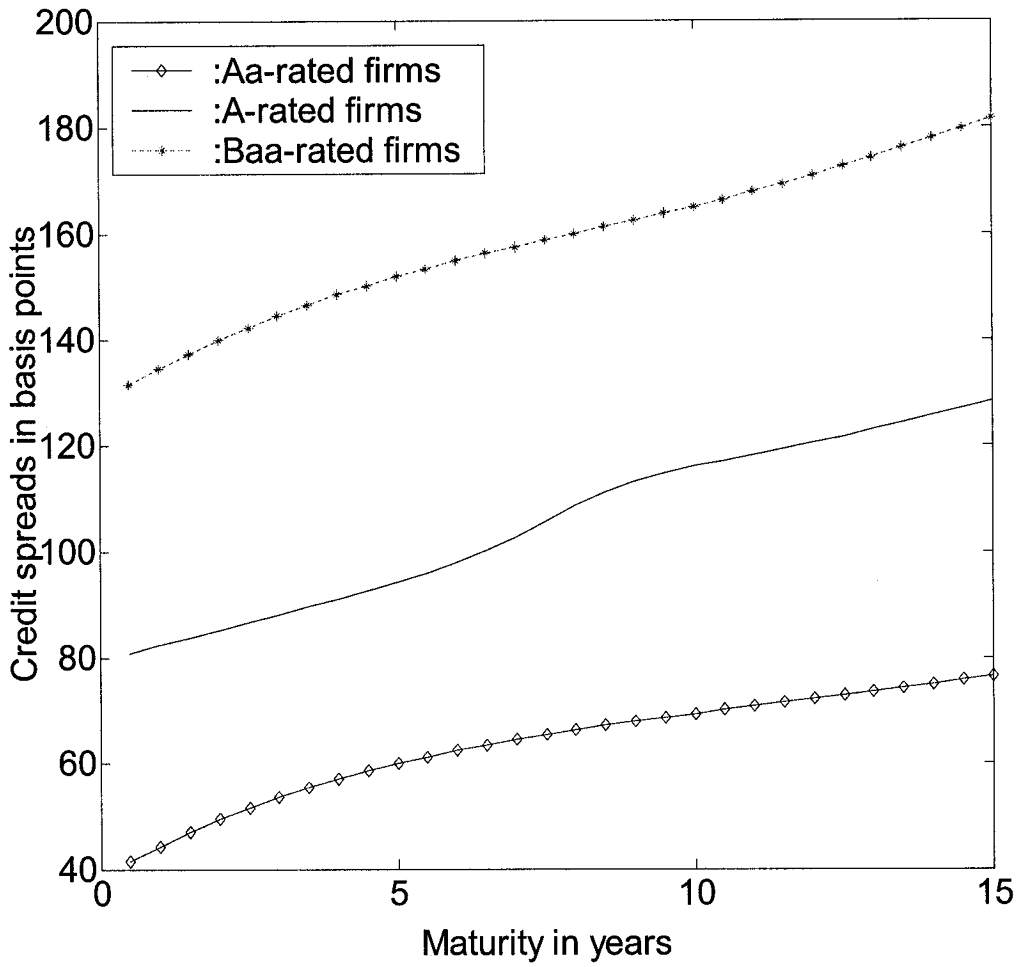

Figure 4: The average credit spreads of the Aa-, A-, and Baa-rated firms implied by the stochastic volatility model of credit spreads.

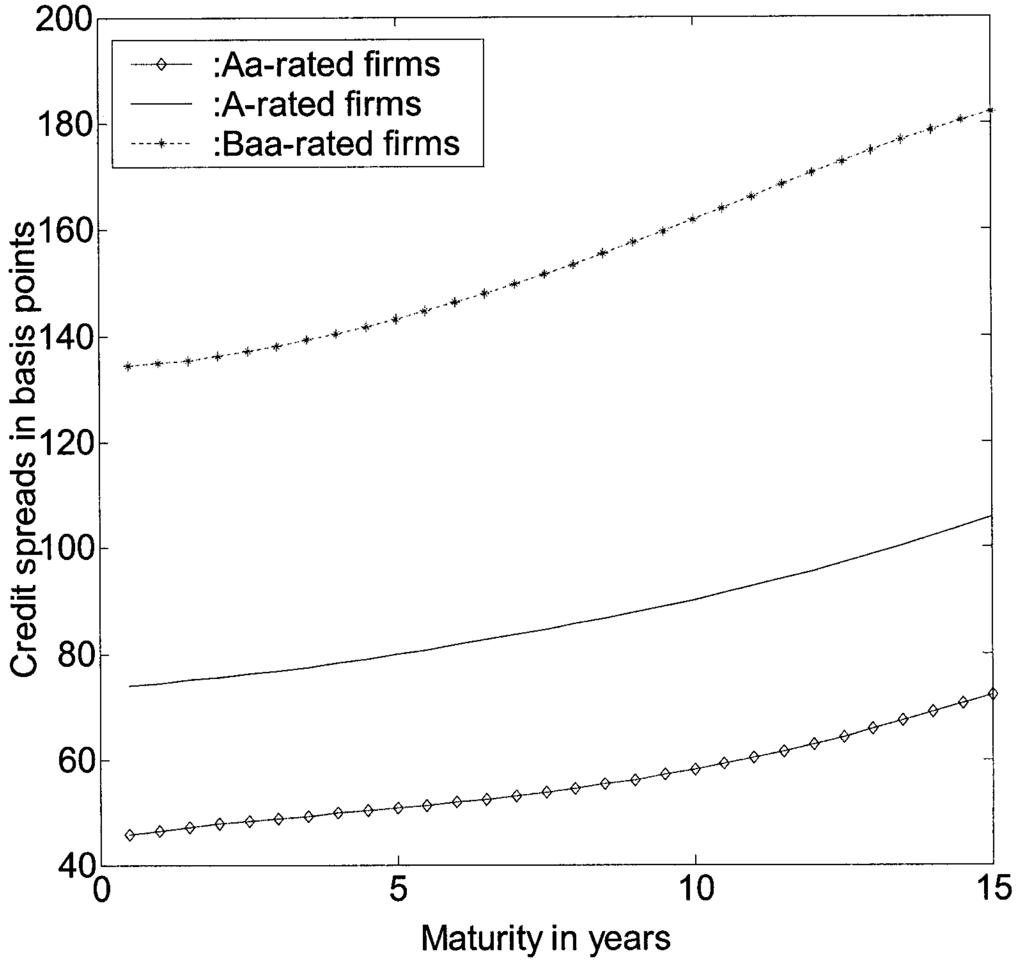

Figure 5: The average credit spreads of the Aa-, A-, and Baa-rated firms implied by the benchmark model of credit spreads.

The following essay proposes a reduced form model of interest rate swap spreads and studies the individual components of swap spreads. The model accommodates both the default risk inherent in swap contracts and the liquidity difference between the swap and Treasury markets. The default risk and liquidity components of swap spreads are found to behave very differently. The immediately preceding essay also uses the reduced form approach; on the other hand, the first essay uses the structural approach.

Decomposing the Default Risk and Liquidity Components of Interest Rate Swap Spreads

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Abstract

This paper develops a reduced form model of interest rate swap spreads. The model accommodates both the default risk inherent in swap contracts and the liquidity difference between the swap and Treasury markets. We use an extended Kalman filter approach to estimate the model parameters. The model fits the swap rates well. We then solve for the implied general collateral repo rates and use them to decompose the swap spreads into their default risk and liquidity components. This exercise shows that the default risk and liquidity components of swap spreads behave very differently: although default risk accounts for the largest share of the levels of swap spreads, the liquidity component is much more volatile. In addition, while the default risk component has been historically positive, the liquidity component was negative for much of the 1990s and has become positive since the financial market turmoils in 1998.

JEL Classification: G12

Keywords: default risk; general collateral repo rate; liquidity; reduced form models; swap spread.

1 Introduction

A swap is an over-the-counter (OTC) derivative contract in which two parties agree to exchange cash flows in the future. The simplest example of a swap is a forward contract. The two most popular types of swaps are *interest rate swaps* and *currency swaps.* This paper focuses on interest rate swaps since they are the largest segment of the global swap markets. (Henceforth we will use the terms "interest rate swap(s)" and "swap(s)" interchangeably.) Normally there are two counterparties in a "plain-vanilla" fixed for floating interest rate swap, counterparty A and counterparty B. The swap contract stipulates that counterparty A pays counterparty B, usually through a swap dealer, cash flows equal to interest at a fixed rate decided *a priori* on a principal for the *tenor* (maturity) of the swap. At the same time, counterparty B pays counterparty A, again through a swap dealer, cash flows equal to interest at a floating rate on the same principal for the same period of time. The floating rates in swaps are most often set at the London Interbank Offer Rates (LIBOR). The fixed rate exchanged for floating rate in an interest rate swap is referred to as the *swap rate.* The swap rate is usually chosen so that the swap is worth (approximately) zero at its initiation date. The *swap spread* is defined as the difference between the swap rate and the risk-free interest rate for the same maturity as the swap. Finally, note that the principal used to calculate interest payments in an interest rate swap is never exchanged. Therefore it is called *notional principal.*

The first swap contract was initiated in August 1981 (Litzenberger (1992)). Since then the interest rate swap markets have been growing tremendously and have become one of the most active and important derivative markets worldwide. A recent study by the Bank for International Settlements (BIS) estimates that the notional amount of outstanding interest rate swaps was US$ 79.161 trillion by the end of 2002, far exceeding the size of the U.S. Treasury debt market.[1] The interest rate swap markets are also highly liquid: the average

[1]It should be noted that since market participants tend to close out a swap position by entering into a

bid-ask spreads in swap rates are merely 3-4 basis points (bps) (Hull (2000)). Besides the sheer size of the swap markets, swaps provide data that are of great importance. Many financial institutions use the swap rates to generate the benchmark rates for their derivative models (Grinblatt (2002)). And in the marketplace there is a growing trend that the Treasury yields are replaced by the swap rates as the reference term structure (Duffie, Pedersen, and Singleton (2000) and Houweling and Vorst (2001)). An additional advantage of swaps is that the swap rates are highly correlated with yields on other fixed-income securities, even during turbulent times, thus making swaps a better hedging instrument than government bonds. Finally, the swap options (or *swaptions*), which are options to initiate new swaps, or to terminate existing swaps, are one of the most-heavily-traded interest rate derivatives (Longstaff, Santa-Clara, and Schwartz (2000)).

As the swap markets evolve and gain prominence, the academic literature on modeling swap spreads has also been growing fast. Since the largest component of swap spreads is due to the default risk embedded in swap contract (as suggested by the empirical results of this paper), the literature on modeling swap spreads can be regarded as part of the academic studies of modeling default risk. Currently, there are two main approaches to modeling default risk: the structural approach, which is first developed in Merton (1974); and the reduced form approach, which is advocated in Duffie and Singleton (1999). (Lando (1997) contains an introduction to these two approaches.) This paper develops a reduced form model of swap spreads. In particular, we use a two-factor affine model to capture the dynamics of swap spreads. The two factors can be interpreted as the default risk and liquidity components of swap spreads, respectively. A notable feature of the model is that it allows us to study the *individual* relationships between the default risk and liquidity components of swap spreads and the risk-free interest rates. In contrast, previous models of swap spreads have all examined the relationship between the riskless interest rate and the swap spread as

new offsetting swap contract (and therefore a swap position may still remain in the trading book even after it is terminated), the notional amount of swaps outstanding may be exaggerated to a large extent (He (2000)).

a whole.

We estimate the model on weekly data of the 6-month LIBOR rates and swap rates for a wide array of maturities for May 22, 1991 to April 30, 2003. We also supplement the swap rate data with data on the default risk and liquidity components of the implied 3-month LIBOR spread (these data are described in Section 5.2). The model fits swap rates well, resulting in an average root mean squared error (RMSE) of 5.1 basis points (bps) across the maturities of swap rates. The parameter estimates indicate that the default risk component of swap spreads is positively related to the risk-free interest rate, whereas the liquidity component of swap spreads is negatively related to the risk-free rate. Moreover, the default risk component exhibits much stronger mean reversion than the liquidity component. To further investigate the individual components of swap spreads, we solve for the implied general collateral (GC) repo rates and use them to decompose the model-implied swap spreads into their default risk and liquidity components. This decomposition exercise serves the main objective of this paper. This exercise shows that the default risk component and the liquidity component of swap spreads behave quite differently: although the default risk component accounts for the largest share of the levels of swap spreads, the liquidity component adds disproportionately large variations to the movement of swap spreads. Recent events in the U.S. swap markets help to further illustrate this point. For example, in the fall of 1998, in the aftermath of Russian default and the collapse of LTCM, swap spreads widened substantially. Default risk could not explain this sudden increase in swap spreads since in the U.S. government bond markets, where default risk was not an issue, the slope of the Treasury yield curve also steepened dramatically. Instead, the event of 1998 can be mainly characterized as a liquidity event due to "flight to quality" and concern of systematic failure in the financial industry. The decomposition exercise also reveals that although the default risk component of swap spreads has been historically positive and relatively stable, the liquidity component of swap spreads varied substantially through time and seemed to display two regimes: before 1998, the liquidity component was mainly negative and had a low volatility; after 1998, the liquidity component has become both positive and more volatile.

This paper makes two distinct contributions to the literature on modeling swap spreads. First, the model specification in this paper is new and consistent with many empirical features of swap spreads. The model is also tractable and easy to implement. Second, the decomposition exercise enables us to "peek behind the curtain," i.e. to study the default risk and liquidity components of swap spreads, which are otherwise unobservable.

The rest of the paper proceeds as follows. Section 2 briefly reviews the literature on swap spreads. Section 3 presents the swap spread model. Section 4 describes the data. Section 5 discusses the estimation methodology. Section 6 reports and discusses the estimation results. Finally, Section 7 offers concluding remarks and directions for future research. All technical details can be found in Appendix A.

2 The literature on interest rate swaps

This section consists of two parts. Section 2.1 reviews the literature on valuing interest rate swaps. Section 2.2 examines the default risk and liquidity components of swap spreads.

2.1 The pricing of interest rate swaps

Most of the earlier theories on interest rate swaps focus on the economic benefits offered by swaps. Bicksler and Chen (1986) provide an argument suggesting that counterparties with asymmetric credit ratings could both benefit from an interest rate swap if one of them enjoys a comparative advantage in the fixed-rate debt market, whereas the other has a comparative advantage in the floating-rate loan market. The papers of Arak, Estrella, Goodman, and Silver (1988), Wall and Pringle (1988), Titman (1992), and Li and Mao (2003) demonstrate that firms with private information about their future creditworthiness can benefit in terms of a reduction in borrowing cost and/or a mitigation in agency cost from rolling over short-term floating-rate loans. Although borrowing short-term floating-rate debts subjects firms to considerable interest rate risk, firms can hedge this risk by entering into a floating for fixed interest rate swap. Finally, Smith, Smithson, and Wakeman (1988) suggest several alternative reasons for the popularity of interest rate swaps. Their arguments are based on tax and regulatory arbitrage and market completion.

The earlier empirical studies of interest rate swaps, such as those of Sun, Sundaresan, and Wang (1993), Brown, Harlow, and Smith (1994), Cossin and Pirotte (1997), Minton (1997), and Lang, Litzenberger, and Liu (1998), run linear regressions of swap rates or swap spreads on various economic variables, including the riskless interest rates and the proxies for default risk and hedging costs. These studies generally find that default risk is priced in swap spreads. However, other factors, such as hedging costs, can also exert significant influence on swap spreads. None of these studies develops a dynamic swap-pricing model.

Recently developed models of swap spreads can be divided into two groups: the structural models and the reduced form models. Cooper and Mello (1991) and Li (1998) develop structural models using the insight of Merton (1974) to pricing interest rate and currency swaps. Reduced form models of swap spreads have been developed in, among others, Sundaresan (1991), Duffie and Huang (1996), Duffie and Singleton (1997), He (2000), Liu, Longstaff, and Mandell (2000), Collin-Dufresne and Solnik (2001), and Grinblatt (2002). Finally, Dai and Singleton (2000) and Jagannathan, Kaplin, and Sun (2003) evaluate several well-known affine term structure models on data of LIBOR and swap rates.

In Sundaresan (1991), swap spread is assumed to result entirely from default risk. In contrast, Grinblatt (2002) assumes that swap contract is free of default risk and swap spreads exist due to the liquidity difference between the Treasury and swap markets. Duffie and Singleton (1997) present a model of swap rates. They find that both liquidity and default risk are required in order to explain the variations in swap spreads. However, the effects of the liquidity factor do not last long, whereas the impacts of the default risk factor become more important over longer time horizons. He (2000) and Liu, Longstaff, and Mandell (2000) both conclude that default risk alone can not explain the volatility observed in recent swap markets. Instead, it is the liquidity component of swap spreads that adds most variations to swap spreads. Their finding is consistent with the results of the decomposition exercise in Section 6.3. Finally, Collin-Dufresne and Solnik (2001) examine the relationship between corporate bond yields and swap rates (the LIBOR-swap spread).

2.2 The default risk and liquidity components of swap spreads

A swap contract generally involves two types of default risk. First, the two counterparties to a swap may default on their future obligations. This is called *counterparty default risk*. Second, because the underlying floating rate in a swap contract is usually set at the LIBOR rate, which is a default-risky interest rate, an interest rate swap is subject to default risk even if the two counterparties to the swap do not default.

It can be shown that compared to a corporate bond with similar credit rating, the counterparty default risk in a swap contract is much smaller. The reasons are the following. First, due to the nature of the swap contract, the counterparty default risk matters only when the swap has a positive value to the non-defaulting party and a negative value to the defaulting party. As a result, the default risk of swaps is reduced substantially. Second, unlike in the case of corporate bonds, the notional principal of a swap is never exchanged, thus the amount "at stake" in a swap is only the net interest payment, which is much lower than the notional principal of the swap. Third, the unusual treatment of swaps under default events also alleviates the losses from defaults to swap counterparties (Litzenberger (1992)). Fourth, the current industry practices of netting, imposing collateral, and marking to market to a great extent eliminate the risk of default by either counterparty (He (2000)). Moreover, parties who are less creditworthy are either not allowed to participate in the swap markets or required to place additional collateral. As a result, several studies including He (2000), Collin-Dufresne and Solnik (2001), and Grinblatt (2002) assume that swaps are free of counterparty default risk. Fifth, results in e.g. Duffie and Huang (1996), Li (1998), and Huge and Lando (1999) show that swap spread is very insensitive to credit rating differences between counterparties. Lastly, Bomfim (2002) shows that counterparty default risk had minimal impact on swap spreads even during market turbulent times, such as the Russian default crisis of 1998. However, it is important to note that although the counterparty default risk inherent in swaps may have been minimized, by no means this implies that swaps are free of default risk. On the contrary, swaps are still default-risky. After all, the reason that all the above-mentioned mechanisms exist is to mitigate the counterparty default risk of swaps. Accordingly, in this paper we explicitly take the default risk inherent in swaps into account, without being precise *a priori* about the sources of this default risk.

We now turn to the liquidity component of swap spreads. Academics generally agree that asset prices contain liquidity premium (Amihud and Mendelson (1988, 1991), Kamara (1994), Longstaff (1995a, 1995b, and 2001), and Ericsson and Renault (2001)). The liquidity component of swap spreads stems from the liquidity difference between the Treasury and swap markets. The Treasury market is considered more liquid than the swap market, not so much because it has lower bid-ask spreads as because Treasury bonds, especially the most recently issued (or *on-the-run*) ones, are preferred collateral in the overnight repurchase agreement (repo) markets, whereas swaps are not. As a result, owners of on-the-run Treasury bonds can borrow at a special repo rate that is lower than the prevailing general collateral repo rate.[2] In effect, there are daily cash flows (similar to some kinds of "dividends") to owners of Treasury bonds, equal to the difference between the special and general collateral repo rates.[3]

[2]On some days, the special repo rates can be 500 bps lower than the corresponding general collateral repo rates (Sundaresan (1997, pp. 59-60)).

[3]Duffie (1996) develops a theoretical model to explain the causes and effects of special repo rates. His analysis predicts that the liquidity premium associated with on-the-run issues is due to special repo rates. Jordan and Jordan (1997) confirm this prediction.

In this paper, we model the above liquidity advantage of Treasury securities as a convenience yield that accrues to holders of Treasury securities but is lost to investors of swaps. A similar modeling approach is also adopted by He (2000), Cherian, Jacquier and Jarrow (2001), Janosi, Jarrow, and Yildirim (2001), Jarrow (2001), and Grinblatt (2002) in related contexts.

3 Model of the swap rates

Since the swap spread is defined as the difference between the swap rate and the corresponding riskless interest rate, a model of swap rate (or equivalently swap spread) naturally begins with modeling the riskless interest rate. In Section 3.1, we first introduce a standard two-factor affine model of the riskless interest rate. Then in Section 3.2, we present a two-factor affine model of swap spreads. Together, the models in Section 3.1 and Section 3.2 lead to a four-factor affine model for swap rates.

3.1 Model of the default-free interest rate

Under the physical probability measure P, the instantaneous nominal default-free interest rate, r_t, is modeled as the sum of a constant, α_r, and two independent factors f_{1t} and f_{2t} that follow square-root diffusion processes

$$r_t = \alpha_r + f_{1t} + f_{2t}, \tag{1}$$

$$df_{it} = \phi_i(\mu_i - f_{it})dt + \sigma_i\sqrt{f_{it}}dz_{it}, \quad \text{for } i = 1, 2, \tag{2}$$

where in equation (2), the parameters ϕ_i and μ_i can be interpreted as the mean reversion speeds and the unconditional means of factors f_{it}, i = 1, 2, respectively. The two standard Brownian motions z_{1t} and z_{2t} are assumed to be independent. The model specified in equations (1) and (2) is called a *translated square-root model* and has appeared in Cox, Ingersoll, and Ross (1985) and Pearson and Sun (1994). This model is a member of the family of *exponential affine* (or *affine* for short) models of the riskless interest rate.

Making the same assumption of prices of risk as that in Cox, Ingersoll, and Ross (1985), we write the stochastic process in equation (2) as

$$df_{it} = (\phi_i \mu_i - (\phi_i + \pi_i) f_{it})dt + \sigma_i \sqrt{f_{it}} d\widehat{z}_{it}, \quad \text{for } i = 1, 2, \tag{3}$$

under the risk-neutral probability measure Q. In equation (3), $\widehat{z}_{1t}$ and $\widehat{z}_{2t}$ are two independent standard Brownian motions under the Q measure, and π_i (< 0), $i = 1, 2$, are the risk premiums for bearing interest rate risk.

The riskless interest rate model presented in equations (1) through (3) conforms to the common practice that uses at least two factors to capture the dynamics of the riskless term structure (Litterman and Scheinkman (1991)). The time t price of a default-free zero-coupon bond that matures at time T, $G(t,T)$, is given by

$$G(t,T) = E_t^Q[\exp(-\int_t^T r_s ds)], \tag{4}$$

where $E_t^Q[\cdot]$ denotes the conditional expectation taken under the measure Q, utilizing all the information known at time t. Under this model, the conditional expectation in equation (4) can be solved in simple closed-form, the formulas for which are given in Appendix A.1.1.

Finally, the constant maturity Treasury (CMT) rates used in estimation of the riskless interest rate model are yields on par Treasury bonds. The formula for the CMT rate with a remaining maturity of $T - t$ years is given as

$$CMT_{t,T} = 2[\frac{1 - G(t,T)}{\sum_{j=1}^{2T} G(t, j/2)}], \tag{5}$$

where $CMT_{t,T}$ denotes the time t value of a CMT rate for a maturity of $T - t$ years. (An informal proof of the above formula is in Hull (2000, pp. 89-90).)

3.2 Model of the swap spreads

The instantaneous credit spread, s_t, is modeled as

$$s_t = h_t + l_t. \tag{6}$$

(Here the term "credit spread" should not be interpreted as a spread that is *solely* due to default risk. In fact, as the subsequent arguments will make clear, non-default features, such as liquidity, may also contribute to the spread s_t.) In equation (6), the state variable h_t can be interpreted as the default risk component of the credit spread s_t, whereas the state variable l_t can be considered as the liquidity component (or more generally, the non-default-risk component) of the spread s_t. As has been pointed out in Section 2.2, we model the factor l_t as a convenience yield resulting from the liquidity advantage of on-the-run Treasury securities. Both h_t and l_t are unobservable. The main objective of this paper is to use the default risk and liquidity components of the implied 3-month LIBOR spread, which can be observed, to identify the factors h_t and l_t separately and to simulate the default risk and liquidity components of the swap spreads for longer maturities.

To facilitate the main task of this study, which is to examine the empirical properties of swap spreads, we adopt a particularly tractable specification for the factors h_t and l_t. More elaborate model specifications can also be used without additional conceptual difficulty, although in that case the computations will become more involved. Under the physical measure P, the dynamics of the factors h_t and l_t are assumed to be

$$\begin{aligned} h_t &= \beta_h r_t + h_t^*, \\ dh_t^* &= \kappa_h(\theta_h - h_t^*)dt + \sigma_h \sqrt{h_t^*} dw_{1t}; \end{aligned} \tag{7}$$

$$\begin{aligned} l_t &= \beta_L r_t + l_t^*, \\ dl_t^* &= \kappa_L(\theta_L - l_t^*)dt + \sigma_L dw_{2t}; \end{aligned} \tag{8}$$

where the two standard Brownian motions w_{1t} and w_{2t} are independent, and both of them are independent of the Brownian motions z_{1t} and z_{2t} in the riskless interest rate process in equation (2). In equation (7), the parameters κ_h and θ_h can be interpreted analogously to the parameters ϕ_i and μ_i in equation (2), respectively. Likewise for the parameters κ_L and θ_L in equation (8).

Adopting a standard assumption of prices of risk, we write the dynamics of h_t and l_t under the risk-neutral measure Q as

$$
\begin{aligned}
h_t &= \beta_h r_t + h_t^*, \\
dh_t^* &= (\kappa_h \theta_h - (\kappa_h + \lambda_h) h_t^*)dt + \sigma_h \sqrt{h_t^*} d\widehat{w}_{1t}; \qquad (9) \\
l_t &= \beta_L r_t + l_t^*, \\
dl_t^* &= (\kappa_L \theta_L - \kappa_L l_t^* - \lambda_L \sigma_L)dt + \sigma_L d\widehat{w}_{2t}. \qquad (10)
\end{aligned}
$$

Again, in equations (9) and (10) the two standard Brownian motions $\widehat{w}_{1t}$ and $\widehat{w}_{2t}$ are independent, and both of them are assumed to be independent of the Brownian motions $\widehat{z}_{1t}$ and $\widehat{z}_{2t}$ in equation (3). Furthermore, $\lambda_h(< 0)$ and $\lambda_L(< 0)$ are the risk premiums for bearing default risk and liquidity risk, respectively. Together, the riskless interest rate model in equations (1) through (3) and the swap spread model in equations (6) through (10) consist of a four-factor affine model for swap rates.

Our choice of a mean-reverting stochastic process for the liquidity-induced convenience yield l_t is justified by the theoretical analysis in Brennan (1991) and the empirical evidence in Gibson and Schwartz (1990) and Schwartz (1997) on copper, oil, and gold futures contracts. Convenience yield is caused by the scarcity of the underlying commodities. As a result, arbitrage activities will make convenience yield to revert to its long-run mean level once it deviates from the latter. In the present example, this scarce "commodity" is the on-the-run Treasury securities that can be placed as collateral in the special repo market.

The swap spread model in equations (6) through (10) captures four prominent properties of actual swap spreads. First, swap spreads are stochastic. The h_t^* and l_t^* terms in the above model account for this feature. Second, swap spreads are correlated with the riskless interest rates. An interesting feature of the present model is that it allows us to study the correlation between the default risk factor h_t and the riskless interest rate (as captured by the parameter β_h) *separately* from the correlation between the liquidity factor l_t and the riskless rate (as captured by the parameter β_L). In contrast, previous models of swap spreads have all studied the relationship between the riskless interest rate and the swap spread as a whole. Third, this specification allows the credit spread s_t to become negative since the processes assumed for l_t in equations (8) and (10), which are Gaussian processes, allow l_t to become negative. This feature of the model can be justified by the fact that although the swap spreads in U.S. have been historically positive, negative swap spreads did occur in other markets, such as in Japan (Eom, Subrahmanyam, and Uno (2000)). Finally, equation (6) indicates that in the extreme case where there is neither default risk in swaps nor liquidity difference between the Treasury and swap markets, the swap spread will decrease to zero, as we will expect.

The swap rates used in this paper are valued as yields on par credit-risky bonds. That is,

$$swap_{t,T} = 2[\frac{1 - P(t,T)}{\sum_{j=1}^{2T} P(t, j/2)}], \tag{11}$$

where $swap_{t,T}$ denotes the time t swap rate for a maturity of $T - t$ years. (A proof of the above formula is in Duffie and Singleton (1997, p. 1290).) In equation (11), $P(t,T)$ denotes the time t price of a credit-risky zero-coupon bond that matures at time T and is valued as

$$P(t,T) = E_t^Q[\exp(-\int_t^T (r_s + s_s)ds)]. \tag{12}$$

In this model, the conditional expectation in equation (12) can be solved in simple closed-form, as presented in Appendix A.1.2. Finally, by definition, the time t swap spread for a maturity of $T - t$ years is valued as $(swap_{t,T} - CMT_{t,T})$.

Finallly, the formula for the time t LIBOR rate for a maturity of $\tau(< 1)$ year, $LIBOR_{t,t+\tau}$, is

$$LIBOR_{t,t+\tau} = \frac{360}{AD}[\frac{1}{P(t,t+\tau)} - 1], \tag{13}$$

where in equation (13), AD denotes the actual number of days in the τ-year maturity. (A proof of equation (13) can be found in Sundaresan (1997, p. 526).)

In writing equations (11) and (13), several important assumptions have implicitly been made. These assumptions include that the default risk in the swap contract is exogenous, that the credit qualities of the counterparties to swaps are periodically refreshed, that the counterparties to swaps have symmetric creditworthiness, and the LIBOR and swap markets are homogenous in terms of their credit qualities (Duffie and Singleton (1997, pp. 1292-1294)). Duffie and Huang (1996) and Li (1998) provide some support for the validity of these assumptions.

4 Data

The main source of data for this paper is Datastream. The sample period is from May 22, 1991 to April 30, 2003.[4] For every week in this time period, the Wednesday observations of CMT rates for maturities of 3 and 6 months, 2, 3, 5, 7, and 10 years, the 3- and 6-month LIBOR rates, and swap rates for maturities of 2, 3, 5, 7, and 10 years are collected from Datastream. The data are then scrutinized to eliminate suspicious quotes. The final data set contains 607 weekly cross-sections of valid CMT, LIBOR, and swap rates.

We use Wednesday quotes since holidays (and the resulting market close) are least likely to occur on Wednesdays, thus ensuring the continuity of the time series. The CMT rates are based on the yields on the on-the-run Treasury securities that are most likely to be placed as collateral in the special repo market and earn their owners special repo rates. Therefore, the CMT rates provide an accurate estimate of the liquidity advantage of on-the-run Treasury securities. The maturities of the swap rates selected are the most liquid and heavily-traded

[4]The choice of this sample period was made to accommodate the availability of data on the 3-month GC repo rates, since the earliest date for which repo rates are available from Bloomberg is May 21, 1991. Data on the CMT rates, LIBOR rates, and swap rates over longer sample periods are certainly available.

maturities in the swap markets. We also take into account the difference in day count conventions between LIBOR and swap rates: LIBOR uses an actual/360 day count basis, whereas swap rates are quoted on a 30/360 basis. In counting the actual numbers of days in the maturities of LIBOR contracts, we follow the market practice and use the *modified following business day convention* (Hull (2000, p. 128)).

In order to examine in detail the components of swap spreads, we also obtain weekly (Wednesday) data on the 3-month GC repo rate from Bloomberg for the same time period. Longstaff (2000) advocates the use of GC repo rates as an alternative measure of the riskless term structure. His main arguments can be summarized as the following. First, the repo rate is essentially a default-free rate due to the design of the repo contract. Second, unlike the yields on on-the-run Treasury securities, repo rates do not contain liquidity premiums.

Table 1 presents the summary statistics for the CMT rate, LIBOR rate, swap rate, and GC repo rate data used in this paper and their respective first differences. In Table 1, the LIBOR and swap spreads are defined as the differences between the LIBOR and swap rates and the CMT rates with corresponding maturities. Several interesting findings are apparent. First, the LIBOR and swap rates are strongly negatively skewed, whereas the LIBOR and swap spreads are strongly positively skewed. In addition, there is only modest evidence of excess kurtosis in both series. Second, the first differences of LIBOR and swap rates and the first differences of LIBOR and swap spreads exhibit significant excess kurtosis. Third, both the 3-month GC repo rate and its first differences are strongly negatively skewed, and the first difference series also displays strong excess kurtosis. Fourth, the standard deviations of the CMT rates appear to decrease with maturities. The same thing can also be said about the standard deviations of the LIBOR and swap rates. Lastly, Figure 1 depicts the actual 2- and 10-year swap rates in our sample. Plots of the swap rates of other maturities are very similar. It is clear from Figure 1 that swap rates have been declining since 2001.

5 Estimation methodology

We use an extended Kalman filter (EKF) approach to estimate the riskless interest rate model and the swap spread model developed in Section 3.[5] In the present context, using the EKF approach has at least three advantages. First, the EKF approach utilizes both the cross-sectional and time-series information contained in the CMT, LIBOR, and swap yield curves, thus increasing the efficiency of estimation. Second, consistent with the theoretical models, this approach correctly treats the underlying state variables as unobservable. Third, estimates of the state variables are generated in an iterative and efficient manner, which facilitates the analysis in Section 6. The EKF approach also compares favorably to the other estimation methods that have been used in the existing literature on swap spreads. For example, the studies of Duffie and Singleton (1997), Liu, Longstaff, and Mandell (2000), Collin-Dufresne and Solnik (2001), and Jagannathan, Kaplin, and Sun (2003) all adopt the maximum likelihood estimation method suggested in Chen and Scott (1993) and Pearson and Sun (1994). Although this method has some advantages, it is very time-consuming since an optimization routine has to be used to recover the values of the model state variables at every trial of parameter values. In contrast, the EKF approach is much easier to use.

We follow the two-stage estimation procedure suggested in Duffee (1999) to estimate the riskless interest rate and swap spread models separately. In the first stage, we estimate the riskless term structure using data on the CMT rates alone. The parameter estimates obtained after the first stage are assumed to be the true parameters of the riskless model and are used in the second stage to estimate the parameters of the swap spread model. This two-stage estimation procedure is adopted for the following three reasons. First, this procedure helps the numerical algorithm to identify model parameters since the total number of the parameters for the riskless interest rate and swap spread models is quite large relative to data available. Second, this estimation procedure leads to reasonably good fit for both

[5]Please see Harvey (1990) and Hamilton (1994) for an introduction to the EKF estimation method.

the riskless and swap yield curves. Third, the assumed independence between the Brownian motions driving the process for the riskless interest rate and those driving the processes for the LIBOR and swap spreads makes this approach feasible.

The rest of this section is organized as follows. Section 5.1 presents the measurement and transition equations used in estimation of the riskless interest rate model. Section 5.2 explains in detail how we estimate the swap spread model.

5.1 Estimation of the default-free term structure

During every week t in the sample period, a cross-section of seven CMT rates is observed. These CMT rates are for maturities of 3 and 6 months, and 2, 3, 5, 7 and 10 years and are collected in the vector R_{Gt}. The measurement and transition equations of the Kalman filter are given by

$$R_{Gt} = m(\Theta_t) + \varepsilon_t, \quad E_{t-1}(\varepsilon_t \varepsilon_t^{'}) = \Lambda, \tag{14}$$

$$\Theta_t = a + b\Theta_{t-1} + \varkappa_t, \quad E_{t-1}(\varkappa_t \varkappa_t^{'}) = V(\Theta_{t-1}). \tag{15}$$

In the measurement equation (14), the vector Θ_t stores the two state variables in the riskless model, f_{1t} and f_{2t}. The function $m(\Theta_t)$ maps the state variables in Θ_t to the observed CMT rates in R_{Gt}. This mapping is known in closed-form. The measurement error in week t, ε_t, is assumed to be normal and independent both serially and cross-sectionally with a constant conditional variance-covariance matrix of Λ. In the transition equation (15), the white noise measurement error at time t, $\varkappa_t$, has a conditional variance-covariance matrix given by $V(\Theta_{t-1})$ that is time-varying. The details of the transition equation (15) are given in Appendix A.2.1. Finally, when estimating the riskless term structure, we assume that the process for the default-free interest rate is stationary. This assumption allows us to use the unconditional moments of the state variables of the riskless model to initiate iterations on the riskless interest rates.

5.2 Estimation of the swap spread model

Similarly, the measurement and transition equations of the Kalman filter used in estimation of the swap spread model can be written as

$$R_{st} = n(\Sigma_t, \widehat{\Theta}_t) + e_t, \quad E_{t-1}(e_t e_t^{'}) = \Xi, \tag{16}$$

$$\Sigma_t = p + q\Sigma_{t-1} + \varphi_t, \quad E_{t-1}(\varphi_t \varphi_t^{'}) = \Gamma(\Sigma_{t-1}). \tag{17}$$

In the measurement equation (16), the vector R_{st} contains the data on the default risk and liquidity components of the implied 3-month LIBOR spread in week t and the observations of the 6-month LIBOR rate and 2-, 3-, 5-, 7-, and 10-year swap rates in week t. The implied 3-month LIBOR spread is defined as the spread of the actual 3-month LIBOR rate over the implied 3-month riskless interest rate, which is computed using the estimation results for the risk-free interest rate model from phase one. The 3-month LIBOR spread is decomposed as follows. The default risk component of the spread is calculated as the difference between the 3-month LIBOR rate and the actual 3-month GC repo rate. The liquidity component of the spread is computed as the difference between the 3-month GC repo rate and the 3-month riskless interest rate. This decomposition approach can be justified by the facts that: first, the GC repo rate is essentially a default-free rate and contains no liquidity premium; and second, the CMT rates used in estimation of the riskless interest rate model depend heavily on the on-the-run Treasury issues that are most likely to be placed as collateral in the special repo markets. Therefore the CMT rates contain substantial liquidity premium. It follows that the difference between the LIBOR and GC repo rates can be considered as a pure default-risk component of the LIBOR spread, whereas the difference between the GC repo rate and the implied riskless interest rate can be regarded as a pure liquidity component of the LIBOR spread. A similar decomposition approach is also used in Liu, Longstaff, and Mandell (2000). Table 3 reports the summary statistics for the implied 3-month LIBOR spread and its default and liquidity components. A graphical illustration of Table 3 is in Figure 2. According to Table 3, the 3-month LIBOR spread has a mean of 30.155 bps, of which the vast majority is accounted for by the default risk component (26.538 bps out of 30.155 bps), whereas the liquidity component only accounts for a small part of the spread (3.616 bps out of 30.155 bps). However, the variations of the LIBOR spread seem to mainly stem from the liquidity component: the liquidity component has a standard deviation of 0.471%, slightly higher than that of the spread, which is 0.438%, and is nearly four times that of the default risk component, which is 0.12%. Figure 2 supports this finding. In this figure, the default risk component appears to be more stable and is always positive. The liquidity component and the implied spread, on the other hand, are much more volatile and can become negative. As we will see in Section 6.3, the default risk and liquidity components of swap spreads for longer maturities behave similarly.

In equation (16), the vector $\widehat{\Theta}_t$ stores the *smoothed estimates* (i.e. estimates based on information through the entire sample) of the two riskless factors f_{1t} and f_{2t} in week t. These smoothed estimates are assumed to be equal to the true values of the riskless factors for the estimation of the swap spread model. In addition, the parameter estimates for the riskless model obtained from the first stage are assumed to be the true parameters of the riskless interest rate model. The vector Σ_t collects the two state variables of the swap spread model: h_t and l_t. Finally, the Ξ matrix is a 8-by-8 diagonal matrix with elements S_1, S_2, S_3, S_4, ..., S_4, where S_1, S_2, and S_3 denote the measurement error volatilities associated with the default risk and liquidity components of the 3-month LIBOR spread and the 6-month LIBOR rate, respectively, and S_4 indicates the common measurement error volatility for swap rates of stated maturities. We assume a common measurement error volatility for swap rates for two reasons. First, doing so further reduces the number of parameters to be identified, thus increasing the efficiency of estimation. Second, in the unreported estimation results where swap rates are allowed to have different measurement error volatilities, the resulted estimates of swap rate volatilities are very similar and are significantly lower than the measurement error volatility for the 6-month LIBOR rate.

The rest of the terms in the measurement equation (16) and the transition equation (17) can be interpreted analogously to their counterparts in equations (14) and (15), respectively. The details of equation (17) are presented in Appendix A.2.2. Lastly, similar to the assumption of stationarity made for the default-free interest rates, we assume that the stochastic processes in the swap spread model are stationary. The estimation results, however, are not very sensitive to this assumption.

6 Empirical results

This section reports the estimation results for both the riskless interest rate model and the swap spread model. The section is divided into three parts. Section 6.1 and Section 6.2 discuss the results on the default-free model and the swap spread model, respectively. Section 6.3 further explores the components of swap spreads.

6.1 Estimation results on the riskless interest rates

Panel A in Table 2 presents the parameter estimates and the associated robust standard errors that are calculated following White (1982) for the riskless interest rate model. The standard errors are generally much smaller than their corresponding parameter estimates, indicating that the parameters have been estimated very precisely. The first state variable of the model, f_{1t}, is found to exhibit much stronger mean reversion than the second riskless factor, f_{2t}, which has almost no mean reversion and is therefore close to a martingale. The two risk premium parameters, π_1 and π_2, are both of the desired signs, implying that investors demand positive compensations for bearing interest rate risk. However, the estimate of the second risk premium π_2 and its associated standard error indicate that π_2 is neither economically nor statistically significant. The constant term α_r is found to be negative and is statistically very significant. A negative constant term is required for the present riskless model to fit both a low, flat yield curve and a high, steep yield curve with realistic volatility levels (Duffee (1999)). Overall, the parameter estimates for the riskless model in this paper are generally consistent with the results in other recent studies such as Pearson and Sun (1994), Duan and Simonato (1999), and Duffee (1999).

Panel B in Table 2 reports the fit of this riskless model for the CMT rates. As shown, although the model does not fit well for the 3- and 6-month CMT rates, it is quite successful in fitting the CMT rates with longer maturities.[6] Using different data, Duffee (1999) achieves a similar fit for the riskless term structure.

6.2 Estimation results on the swap spreads

In any week t in our sample period, we observe the default risk and liquidity components of the 3-month LIBOR spread, the 6-month LIBOR rate, and the swap rates for maturities of 2, 3, 5, 7, and 10 years. We use all these data to estimate the swap spread model. We supplement the LIBOR and swap rate data with data on the individual components of the LIBOR spread because the default risk and liquidity factors of the swap spread model are unobservable and are thus difficult to differentiate in practice. The default risk and liquidity components of the 3-month LIBOR spread, however, are observable (see Section 5.2) and are known to have different default or liquidity properties. Using these components, together with the LIBOR and swap rates, to estimate the swap spread model will therefore help us identify the default risk and liquidity components of the 6-month LIBOR rate and swap rates for longer maturities.

For estimation of the swap spread model, we need the pricing formula for the GC repo rate for maturity less than 1 year. In this paper, we value the GC repo rate in a way very

[6]The RMSE results presented in Table 2 and Table 4 are based on the *contemporaneous estimates* of the model state variables (i.e. estimates contingent on information available concurrently). We have also computed the root mean squared errors using the smoothed estimates of the state variables. The results are very similar and are therefore not reported.

similar to the LIBOR rate with the same maturity. That is,

$$Repo_{t,t+\tau} = \frac{360}{AD}[\frac{1}{GC(t,t+\tau)} - 1], \tag{18}$$

where $Repo_{t,t+\tau}$ denotes the time t value of a GC repo rate with a maturity of $\tau(< 1)$ year, AD denotes the actual number of days between time t and time $t+\tau$, and

$$GC(t,t+\tau) = E_t^Q[\exp(-\int_t^T (r_s + l_s)ds)] \tag{19}$$

denotes the price of a "hypothetical" zero-coupon bond that is default-free but does not have the liquidity advantage of on-the-run Treasury securities. This zero-coupon bond is constructed to capture the notion that the GC repo rate is a default-free rate and does not contain liquidity premium. The conditional expectation in equation (19) can be solved in simple closed-form, the formula for which is given in Appendix A.1.3. Finally, the pricing formula for the GC repo rate in equation (18) can be justified by the fact that both the LIBOR and repo contracts are money market instruments and use an actual/360 day count basis and simple interest calculation.

For the decomposition exercise in Section 6.3, we also need the pricing formula for the GC repo rates with maturities longer than 1 year. GC repo contracts with maturities longer than 1 year rarely exist. However, the pricing formula for them is necessary for our decomposition exercise. In this paper, we price the GC repo rate as the yield on a "hypothetical" par bond with the same maturity. That is,

$$Repo_{t,T} = 2[\frac{1 - GC(t,T)}{\sum_{j=1}^{2T} GC(t,j/2)}]. \tag{20}$$

The pricing formula in equation (20) can be justified by the following. The borrower in the repo market, who is also the owner of the underlying collateral security, essentially issues a coupon bond that is priced at par to the lender. The repo rate is the coupon rate of this bond, which is also the yield to maturity on this bond since the bond sells at par. The bond is a par bond because the face value of the bond is the amount borrowed. The bond is a coupon bond since the maturity of the repo contract goes beyond 1 year and the bond thereby has to pay coupons. (Duffie (1996) and Jordan and Jordan (1997) contain a description of the repo rates and the repo markets.) The pricing formula in equation (20) is also very similar to the formulas developed in Sun, Sundaresan, and Wang (1993) and Collin-Dufresne and Solnik (2001) for the yields on the LIBOR par bonds.

The estimation results for the swap spread model are reported in Panel A in Table 4. Several findings are noteworthy. First, the parameter β_h, which captures the correlation between the riskless interest rate and the default risk factor h_t, is positive and significant. On the other hand, the parameter β_L, which captures the correlation between the riskless rate and the liquidity factor l_t, is negative and significant. Second, the default risk factor, h_t, displays much stronger mean reversion than the liquidity factor, l_t. As we will see in Section 6.3, this difference in mean reversion may partially explain the relatively stable behavior of the default risk component of swap spread, when compared to the liquidity component of swap spread. Third, the estimates of the risk premiums, λ_h and λ_L, are both of the right signs, indicating that investors in swap markets demand compensations for bearing not only default risk but also the liquidity difference between the swap and Treasury markets.

A positive correlation between the default risk component of swap spread and the riskless interest rate may appear to be surprising at first glance since many previous studies have documented a negative relationship between credit spread and the riskless interest rate. However, it should be noted that all these studies have treated credit spread as a whole and have not distinguished the *individual* components of credit spread and their respective relations with the riskless interest rate. Besides default risk, credit spreads usually contain sizeable non-default-risk components, due to e.g. liquidity. These non-default-risk components can be negatively related to the riskless interest rate, as has been documented in this paper. It follows that a positive correlation between the default risk component of credit spread and the riskless interest rate does not necessarily contradict the possibly negative relationship between the credit spread as a whole and the risk-free interest rate. Furthermore, the negative correlation between credit spread and the riskless interest rate found in many previous studies is not a foregone conclusion. Neal, Rolph, and Morris (2000), for instance, find that the relationship between credit spread and the riskless interest rate is not constant but depends on the time horizon: although in the short-run higher riskless interest rates will cause credit spreads to decrease, the opposite effect will occur over the long-run and higher riskless rates will actually cause credit spreads to increase. Clearly, more research is needed on the exact relationship between credit spread and the risk-free interest rate.

Finally, a negative relationship between the liquidity component of swap spread and the riskless interest rate can be best explained using the recent experience in the U.S. swap market. In early 2000, the U.S. Treasury Department announced buying back US$ 30 billion worth of U.S. government bonds. Consequently the swap spreads in the U.S. market exploded to their highest levels ever (He (2000)). A reduction in the supply of Treasury bonds caused Treasury bond prices to rise and the prevailing riskless interest rates to decline. (According to Schinasi, Kramer, and Smith (2001), the levels of U.S. Treasury yield curves did decline in 2000.) At the same time, there were less securities available for being used as collateral in the special repo market. As a result, the liquidity benefits that accrue to owners of such securities increased and the value of the liquidity component of swap spread therefore increased.[7]

Panel B in Table 4 presents the fit of this swap spread model for the LIBOR and swap rates. This model achieves a good fit for swap rates with a resulting average RMSE of 5.1 bps, which is fairly comparable to the prevailing average bid-ask spreads of 3-4 bps in the swap markets. On the other hand, the fit for the 6-month LIBOR rate is relatively inferior.

[7]The repurchase of U.S. Treasury securities may have reduced liquidity in the U.S. Treasury markets, since the bid-ask spreads and the spread between yields on off-the-run and on-the-run Treasury securities in the U.S. Treasury markets had increased sharply in 2000 (Schinasi et al. (2001)). However, our conclusion should remain unchanged because as the supply of Treasuries declined, the scarcity of acceptable collateral in repo markets led to a widening spread between the GC repo rate and special repo rate.

The relatively poor fit for the LIBOR rate is also indicated by the estimates of S_3 and S_4, which are the measurement error volatilities associated with the LIBOR and swap rates, respectively: the estimate of S_3 is substantially larger than the estimate of S_4.

6.3 The components of swap spreads

Here we decompose the LIBOR and swap spreads into their respective default risk and liquidity components. The LIBOR and swap spreads are defined as the spreads of the LIBOR and swap rates over the corresponding CMT rates for the same maturities, which are implied by the parameter estimates for the riskless model and the contemporaneous estimates of the riskless model state variables. (The results of the decomposition exercise are very similar when we use the actual CMT rates instead.) The decomposition exercise is proceeded as follows. First, we simulate the GC repo rates for maturities of 6 months and 2, 3, 5, 7, and 10 years using the pricing formulas in equations (18) and (20) and the estimation results for the swap spread model. The GC repo rates for these maturities have to be simulated since data on them are not available. Second, we compute the differences between the LIBOR and swap rates for stated maturities and the simulated GC repo rates for the same maturities. These differences can be interpreted as the default risk components of the LIBOR and swap spreads since the GC repo rate is a default-free rate and has no liquidity premium. Third, we calculate the differences between the simulated GC repo rates and the implied CMT rates for the same maturities. These differences can be interpreted as the liquidity components of the LIBOR and swap spreads (see also Section 5.2).

We report the summary statistics for the resulted default risk and liquidity components in Table 5. Table 5 implies an upward-sloping term structure of swap spreads: as maturity goes up, the level of swap spread also increases. The means and standard deviations of the swap spreads and their default risk and liquidity components reported in Table 5 indicates that across maturities, default risk is the main source of the levels of swap spreads, whereas liquidity is the predominant source of the volatilities of swap spreads. (The only exception is the 6-month LIBOR spread, of which the default risk component has a larger standard deviation than the liquidity component.) However, the impacts of the liquidity component on the levels of swap spreads become larger as maturities increase. For example, at 2 years of maturity, the mean liquidity component is only 4.96 bps, or 14.16% of the mean swap spread of 35.02 bps; at 10 years of maturity, however, the mean liquidity component increases to 21.92 bps, accounting for 39.74% of the mean swap spread of 55.16 bps. The impacts of the default risk component, on the other hand, exhibit a humped-shape, though a modest one: the mean level of the default risk component reaches a peak of 35.26 bps at 5 years of maturity, subsequently decreases to 33.72 bps at 7 years of maturity, and further decreases to 33.23 bps at 10 years of maturity. This finding is in contrast to the results in Duffie and Singleton (1997), where they find that the effects of liquidity diminish as maturities increase, whereas the impacts of default risk become more important over longer time horizons.

The results in Table 5 are also graphically illustrated in Figure 3 through Figure 8, for maturities of 6 months, 2, 3, 5, 7, and 10 years, respectively. The figures are very similar. They show that the default risk component is more stable and stays within a narrower range, whereas the liquidity component and the swap spreads are much more volatile. The default risk component of swap spread is always positive, which is consistent with the stochastic process assumed for the default risk factor h_t in equation (7). On the other hand, consistent with the dynamics assumed for the liquidity factor l_t in equation (8), the liquidity component of swap spread can become negative. A closer look at the figures reveals that both the swap spread and its liquidity component seem to display two distinct regimes. In particular, during the time period of 1992 to 1998, both swap spread and its liquidity component were low in both levels and volatilities (this was the period in which the liquidity component was mainly negative). On the contrary, during the later time period of 1998-2002, both the swap spread and the liquidity component were high in both levels and volatilities (this was the period in which the liquidity component was predominantly positive). In contrast, the default risk component of swap spread has been quite stable throughout the entire sample period and did not exhibit changes in regimes. To further verify this finding, we divide the sample period into two sub-periods: from May 22, 1991 to December 31, 1997 and from January 7, 1998 to April 30, 2003. We then compute the summary statistics for the LIBOR and swap spreads and their components and report them in Table 6A for the first sub-period and in Table 6B for the second sub-period. Table 6A and Table 6B clearly show that as we move from the first sub-period to the second sub-period, both the levels and standard deviations of the swap spread and its liquidity component significantly increase, whereas there are only marginal changes in the level and standard deviation of the default risk component. Therefore, an interesting topic for future research is to model the swap spread within a regime-switching framework.[8] Finally, the decomposition exercise is able to capture the recent increases in the swap spread and its liquidity component after the fall 1998 financial market turmoils, during the second half of 1999 due to concerns over Y2K, and in 2000 because of the Treasury buyback mentioned earlier.

7 Conclusion

This paper develops a two-factor affine model of swap spreads. The two factors can be interpreted as the default risk and liquidity components inherent in swap spreads, respectively. One notable feature of the model is that in contrast to the previous literature, it allows us to consider the relationships between the individual components of swap spreads and the riskless interest rates separately.

This model fits the swap rate data well, resulting in an average RMSE that is comparable to the average bid-ask spreads in the swap markets. Parameter estimates indicate that the default risk and liquidity components of swap spreads are related to the riskless interest rates

[8]Chapman and Pearson (2001) and Dai and Singleton (2003) review the literature on modeling the term structure of interest rates with regime shift.

in a differing way: while default risk is positively related to the riskless interest rates, liquidity is found to be negatively related to the riskless rate. A further study of the components of swap spreads reveals that although default risk accounts for the "lion's share" of the levels of swap spreads, the volatility of swap spreads is mainly attributed to changes in liquidity. One implication of this finding is that if swap spreads contain risk premiums, these risk premiums are more likely compensations for liquidity risk. Therefore, a fruitful direction for future research is to study the pricing of liquidity premium by swap market participants. Finally, the fact that the volatility of swap spreads is time-varying suggest that we need a stochastic volatility model to better capture the dynamics of swap spreads. The results of this paper further suggest that the stochastic volatility of swap spreads mainly stem from the liquidity component of swap spreads. Developing such a model is another interesting avenue for future research.

Appendix A: Technical Appendices

A.1 Zero-coupon bond pricing formulas

A.1.1 The risk-free interest rate model

The time t price of a default-free zero-coupon bond that matures at time T, $G(t,T)$, is given by

$$G(t,T) = A_1(\tau)A_2(\tau)\exp[-\alpha_r\tau - B_1(\tau)f_{1t} - B_2(\tau)f_{2t}], \tag{A.1}$$

where f_{1t} and f_{2t} denote the two risk-free factors at time t and $\tau \equiv T - t$. The functions $A_i(\tau)$ and $B_i(\tau)$, $i = 1, 2$, are given as

$$\begin{aligned} A_i(\tau) &= \{\frac{2\varsigma_i \exp[(\phi_i + \pi_i + \varsigma_i)\tau/2]}{(\phi_i + \pi_i + \varsigma_i)[\exp(\varsigma_i\tau) - 1] + 2\varsigma_i}\}^{2\phi_i\mu_i/\sigma_i^2}, \\ B_i(\tau) &= \{\frac{2[\exp(\varsigma_i\tau) - 1]}{(\phi_i + \pi_i + \varsigma_i)[\exp(\varsigma_i\tau) - 1] + 2\varsigma_i}\}, \\ \text{with} \quad \varsigma_i &\equiv \sqrt{(\phi_i + \pi_i)^2 + 2\sigma_i^2}. \end{aligned} \tag{A.2}$$

A.1.2 The swap spread model

The time t price of a credit-risky zero-coupon bond that matures at time T, $P(t,T)$, is given by

$$P(t,T) = A_1^*(\tau)A_2^*(\tau)A_h(\tau)A_L(\tau)\exp[-(1+\beta_h+\beta_L)\alpha_r\tau - B_1^*(\tau)f_{1t}^* - B_2^*(\tau)f_{2t}^* \tag{A.3}$$

$$-B_h(\tau)h_t^* - B_L(\tau)l_t^*],$$

where $f_{it}^* \equiv (1 + \beta_h + \beta_L)f_{it}$ for $i = 1, 2$ and $\tau \equiv T - t$. In equation (A.3), the functions $A_i^*(\tau)$ and $B_i^*(\tau)$, $i = 1, 2$, are the same as the functions $A_i(\tau)$ and $B_i(\tau)$, $i = 1, 2$, in equation (A.2) except that μ_i is replaced with $\mu_i^* \equiv (1 + \beta_h + \beta_L)\mu_i$ and σ_i is replaced with $\sigma_i^* \equiv \sqrt{(1 + \beta_h + \beta_L)}\sigma_i$. On the other hand, the functions $A_h(\tau)$ and $B_h(\tau)$ are defined analogously to the functions $A_i(\tau)$ and $B_i(\tau)$, $i = 1, 2$, in equation (A.2) with ϕ_i, μ_i, σ_i, and π_i replaced with κ_h, θ_h, σ_h, and λ_h, respectively. Finally, the functions $A_L(\tau)$ and $B_L(\tau)$ are

$$\begin{aligned} A_L(\tau) &= \exp\{\gamma[B_L(\tau) - \tau] - \frac{\sigma_L^2 B_L^2(\tau)}{4\kappa_L}\}, \\ B_L(\tau) &= \frac{1}{\kappa_L}[1 - \exp(-\kappa_L \tau)], \\ \text{with} \quad \gamma &\equiv \theta_L - \frac{\sigma_L \lambda_L}{\kappa_L} - \frac{\sigma_L^2}{2\kappa_L^2}. \end{aligned} \tag{A.4}$$

A.1.3 The GC repo rate

The pricing formula for $GC(t,T)$ is

$$GC(t,T) = A_1^{GC}(\tau)A_2^{GC}(\tau)A_L(\tau)\exp[-(1+\beta_L)\alpha_r\tau - B_1^{GC}(\tau)f_{1t}^{GC} - B_2^{GC}(\tau)f_{2t}^{GC} - B_L(\tau)l_t^*], \tag{A.5}$$

where $f_{it}^{GC} \equiv (1+\beta_L)f_{it}$ for $i = 1, 2$ and $\tau \equiv T - t$. In equation (A.5), the functions $A_i^{GC}(\tau)$ and $B_i^{GC}(\tau)$, $i = 1, 2$, are the same as the functions $A_i(\tau)$ and $B_i(\tau)$, $i = 1, 2$, in equation (A.2) except that μ_i is replaced with $\mu_i^{GC} \equiv (1+\beta_L)\mu_i$ and σ_i is replaced with $\sigma_i^{GC} \equiv \sqrt{(1+\beta_L)}\sigma_i$. In addition, the formulas for functions $A_L(\tau)$ and $B_L(\tau)$ have been given in equation (A.4).

A.2. Details of the transition equations

A.2.1 The risk-free interest rate model

The components a and b of the transition equation in equation (15) are

$$a = \begin{bmatrix} \mu_1[1 - \exp(-\frac{\phi_1}{52})] \\ \mu_2[1 - \exp(-\frac{\phi_2}{52})] \end{bmatrix}, \tag{A.6}$$

and

$$b = \begin{bmatrix} \exp(-\frac{\phi_1}{52}) & 0 \\ 0 & \exp(-\frac{\phi_2}{52}) \end{bmatrix}. \tag{A.7}$$

Finally, the term $V(\Theta_{t-1})$ in equation (15) is a 2×2 diagonal matrix with elements

$$V_{ii}(\Theta_{t-1}) = \frac{\sigma_i^2}{\phi_i}[f_{i,t-1}(e^{-\frac{\phi_i}{52}} - e^{-\frac{2\phi_i}{52}}) + \frac{\mu_i}{2}(1 - e^{-\frac{\phi_i}{52}})^2], \quad \text{for} \quad i = 1, 2. \tag{A.8}$$

A.2.2 The swap spread model

The components p and q of the transition equation in equation (17) are

$$p = \begin{bmatrix} \theta_h[1 - \exp(-\frac{\kappa_h}{52})] \\ \theta_L[1 - \exp(-\frac{\kappa_L}{52})] \end{bmatrix}, \tag{A.9}$$

and

$$q = \begin{bmatrix} \exp(-\frac{\kappa_h}{52}) & 0 \\ 0 & \exp(-\frac{\kappa_L}{52}) \end{bmatrix}. \tag{A.10}$$

Finally, the term $\Gamma(\Sigma_{t-1})$ in equation (17) is a 2×2 diagonal matrix with elements

$$\Gamma_{11}(\Sigma_{t-1}) = \frac{\sigma_h^2}{\kappa_h}[h_{t-1}^*(e^{-\frac{\kappa_h}{52}} - e^{-\frac{2\kappa_h}{52}}) + \frac{\theta_h}{2}(1 - e^{-\frac{\kappa_h}{52}})^2], \quad \text{and} \tag{A.11}$$
$$\Gamma_{22}(\Sigma_{t-1}) = \frac{\sigma_L^2}{2\kappa_L}(1 - e^{-\frac{2\kappa_L}{52}}).$$

Table 1
Summary statistics for the CMT rate, LIBOR rate, swap rate, and GC repo rate data

Data are weekly (Wednesday) observations. The sample period is from 5/22/1991 to 4/30/2003. The CMT rate, LIBOR rate, and swap rate data are from Datastream. In the Table, *CMT* indicates the constant maturity Treasury rate; *LIBOR spread* and *swap spread* are defined as the spreads of the LIBOR and swap rates over the corresponding CMT rates, respectively; and *GC repo* denotes the 3-month general collateral repo rate obtained from Bloomberg.

	Levels				First differences			
	Mean (in bps)	Standard deviation (in %)	Skewness	Kurtosis	Mean (in bps)	Standard deviation (in %)	Skewness	Kurtosis
3-M CMT	430.80	1.39	-0.78	2.55	-0.74	0.10	-1.72	18.94
6-M CMT	446.44	1.44	-0.78	2.59	-0.79	0.10	-1.86	20.52
2-Y CMT	506.32	1.35	-0.80	3.13	-0.87	0.14	-0.43	6.99
3-Y CMT	528.91	1.26	-0.76	3.30	-0.85	0.14	-0.16	5.52
5-Y CMT	564.48	1.12	-0.54	3.21	-0.80	0.14	0.04	4.25
7-Y CMT	589.81	1.03	-0.37	3.02	-0.75	0.14	0.26	3.57
10-Y CMT	602.08	1.00	-0.08	2.65	-0.69	0.13	0.23	3.44
3-M LIBOR	471.33	1.51	-0.80	2.50	-0.78	0.10	-1.78	22.71
6-M LIBOR	480.59	1.53	-0.79	2.56	-0.81	0.11	-1.86	18.79
2-Y swap	541.35	1.39	-0.72	3.08	-0.91	0.15	-0.22	5.39
3-Y swap	571.73	1.26	-0.69	3.31	-0.89	0.15	-0.08	4.60
5-Y swap	611.73	1.11	-0.54	3.36	-0.85	0.15	0.00	4.09
7-Y swap	635.98	1.04	-0.36	3.20	-0.80	0.14	0.03	3.86
10-Y swap	659.20	0.98	-0.20	2.96	-0.74	0.14	0.08	3.81
GC repo	444.79	1.48	-0.77	2.44	-0.77	0.11	-1.66	15.88
3-M LIBOR spread	40.53	0.21	1.20	5.21	-0.04	0.09	0.39	7.83
6-M LIBOR spread	34.15	0.17	1.18	4.48	-0.02	0.07	-0.01	7.00
2-Y swap spread	35.03	0.18	0.76	2.78	-0.04	0.06	-0.30	7.60
3-Y swap spread	42.82	0.20	0.58	2.31	-0.04	0.07	-0.33	7.93
5-Y swap spread	47.26	0.23	0.83	2.67	-0.05	0.06	-0.01	4.80
7-Y swap spread	46.16	0.21	1.09	3.58	-0.05	0.06	-0.18	4.53
10-Y swap spread	57.12	0.26	1.23	3.70	-0.05	0.07	-0.49	8.40

Table 2
Estimation results for the risk-free interest rate model

The instantaneous risk-free interest rate, r_t, is modeled as

$$r_t = \alpha_r + f_{1t} + f_{2t},$$

where α_r is a constant and the processes for the riskless factors f_{1t} and f_{2t} are given in Section 3.1. We use an extended Kalman filter approach to estimate the above risk-free interest rate model on weekly data of the constant Maturity Treasury (CMT) rates for maturities of 3 and 6 months, and 2, 3, 5, 7, and 10 years. The data used are described in Table 1. The robust standard errors for the parameter estimates are calculated following White (1982). The RMSE reported in Panel B is calculated using the contemporaneous estimates of the state variables of the model.

Panel A: Variable		**Parameter estimates**			**Robust standard errors**
α_r		-0.654			0.027
ϕ_1		0.346			0.007
μ_1		0.645			0.025
σ_1		0.013			0.003
π_1		-0.022			0.003
ϕ_2		0.009			0.007
μ_2		0.109			0.006
σ_2		0.076			0.018
π_2		-7.200E-07			0.008
Panel B:		3-M CMT			6-M CMT
RMSE (in basis points)		47.337			31.647
	2-Y CMT	3-Y CMT	5-Y CMT	7-Y CMT	10-Y CMT
RMSE (in basis points)	0.198	4.563	4.171	6.631	7.188

Table 3
Summary statistics for the implied 3-month LIBOR spread and its components

The *implied 3-month LIBOR spread* is calculated as the spread of the actual 3-month LIBOR rate over the 3-month riskless interest rate implied by the risk-free interest rate model. The *default risk component* of this implied spread is computed as the difference between the actual 3-month LIBOR rate and the actual 3-month general collateral repo rate. The *liquidity component* of this implied spread is defined as the difference between the actual 3-month general collateral repo rate and the 3-month riskless interest rate implied by the risk-free interest rate model.

	Mean (in bps)	Standard deviation (in %)	Minimum (in bps)	Median (in bps)	Maximum (in bps)
Implied 3-month LIBOR spread	30.155	0.438	-92.940	29.438	150.291
Default risk component	26.538	0.120	7.000	24.250	98.250
Liquidity component	3.616	0.471	-170.984	3.971	128.385

Table 4
Estimation results for the swap spread model

The swap spread model is given in Section 3. We adopt an extended Kalman filter approach to estimate the swap spread model on weekly data of the default risk and liquidity components of the implied 3-month LIBOR spread, defined in Table 3, and weekly data of the 6-month LIBOR and 2-, 3-, 5-, 7-, and 10-year swap rates, described in Table 1. We assume that the default risk and liquidity components (denoted *3-M default* and *3-M liquidity*, respectively, in Panel B) and the LIBOR and swap rates are observed with measurement errors that are normally distributed with mean zero and volatilities of S_1, S_2, S_3, and S_4, respectively. The robust standard errors for the parameter estimates are calculated following White (1982). The RMSE reported in Panel B is calculated using the contemporaneous estimates of the state variables of the model.

Panel A: Variable	**Parameter estimates**	**Robust standard errors**
β_h	0.018	6.121E-04
κ_h	5.575	0.002
θ_h	0.002	0.001
σ_h	0.157	0.001
λ_h	-0.130	0.002
β_L	-0.033	3.713E-04
κ_L	3.106E-04	0.002
θ_L	0.017	5.899E-04
σ_L	0.004	0.002
λ_L	-0.154	0.003
S_1	5.965E-04	0.002
S_2	0.004	0.003
S_3	0.003	0.001
S_4	5.804E-04	0.002

Table 4 (continued)
Estimation results for the swap spread model

Panel B:	3-M default			3-M liquidity		
RMSE (in basis points)	4.514			43.887		
	6-M LIBOR	2-Y swap	3-Y swap	5-Y swap	7-Y swap	10-Y swap
RMSE (in basis points)	31.045	6.255	4.806	4.048	3.692	6.670

Table 5
Summary statistics for the implied LIBOR and swap spreads and their respective default risk and liquidity components

Data used are described in Table 1. Details of the decomposition exercise are given in Section 6.3. In the Table, *Implied spread* refers to the LIBOR or swap spread for a given maturity, and *Default risk* and *Liquidity* indicate the default risk and liquidity components of the LIBOR or swap spread, respectively.

	6-M LIBOR			2-Y swap		
	Implied spread	**Default risk**	**Liquidity**	**Implied spread**	**Default risk**	**Liquidity**
Mean (in bps)	28.93	27.79	1.14	35.02	30.06	4.96
Median (in bps)	29.78	31.27	-2.59	31.00	29.82	-0.71
Standard deviation (in %)	0.31	0.29	0.20	0.18	0.07	0.21
Minimum (in bps)	-52.21	-57.00	-31.20	1.00	10.51	-24.18
Maximum (in bps)	122.68	93.03	60.27	91.50	57.55	61.84
	3-Y swap			**5-Y swap**		
	Implied spread	**Default risk**	**Liquidity**	**Implied spread**	**Default risk**	**Liquidity**
Mean (in bps)	41.08	34.05	7.03	46.60	35.26	11.34
Median (in bps)	35.59	33.97	1.32	39.15	34.37	5.54
Standard deviation (in %)	0.21	0.06	0.21	0.23	0.04	0.21
Minimum (in bps)	2.75	20.39	-21.77	11.26	12.40	-16.93
Maximum (in bps)	97.68	58.01	64.19	107.50	47.26	68.96
	7-Y swap			**10-Y swap**		
	Implied spread	**Default risk**	**Liquidity**	**Implied spread**	**Default risk**	**Liquidity**
Mean (in bps)	49.41	33.72	15.69	55.16	33.23	21.92
Median (in bps)	40.80	33.45	9.85	45.93	33.51	16.19
Standard deviation (in %)	0.24	0.04	0.22	0.24	0.07	0.22
Minimum (in bps)	12.70	14.66	-12.62	17.94	11.64	-7.37
Maximum (in bps)	115.37	46.81	73.67	127.92	53.90	80.39

Table 6A

Summary statistics for the implied LIBOR and swap spreads and their respective default risk and liquidity components: 5/22/1991 – 12/31/1997

Data used are described in Table 1. Details of the decomposition exercise are given in Section 6.3. In the Table, *Implied spread* refers to the LIBOR or swap spread for a given maturity, and *Default risk* and *Liquidity* indicate the default risk and liquidity components of the LIBOR or swap spread, respectively.

	6-M LIBOR			2-Y swap		
	Implied spread	**Default risk**	**Liquidity**	**Implied spread**	**Default risk**	**Liquidity**
Mean (in bps)	19.90	32.52	-12.62	23.69	33.19	-9.50
Median (in bps)	20.96	33.46	-15.65	21.00	32.92	-12.07
Standard deviation (in %)	0.26	0.26	0.10	0.11	0.07	0.09
Minimum (in bps)	-42.80	-40.14	-31.20	1.00	11.32	-24.18
Maximum (in bps)	76.05	87.39	20.92	69.00	57.55	23.45

	3-Y swap			5-Y swap		
	Implied spread	**Default risk**	**Liquidity**	**Implied spread**	**Default risk**	**Liquidity**
Mean (in bps)	27.69	35.33	-7.65	31.95	35.54	-3.59
Median (in bps)	25.33	35.06	-9.87	28.77	34.19	-5.94
Standard deviation (in %)	0.12	0.06	0.09	0.11	0.04	0.08
Minimum (in bps)	2.75	22.55	-21.77	11.26	27.48	-16.93
Maximum (in bps)	70.82	53.28	24.93	69.69	47.26	28.36

	7-Y swap			10-Y swap		
	Implied spread	**Default risk**	**Liquidity**	**Implied spread**	**Default risk**	**Liquidity**
Mean (in bps)	33.78	33.16	0.62	40.44	33.66	6.78
Median (in bps)	31.96	33.08	-1.40	39.44	33.82	5.00
Standard deviation (in %)	0.10	0.03	0.08	0.09	0.06	0.08
Minimum (in bps)	12.70	23.81	-12.62	17.94	18.20	-7.37
Maximum (in bps)	74.60	44.38	32.05	78.30	53.90	37.56

Table 6B
Summary statistics for the implied LIBOR and swap spreads and their respective default risk and liquidity components: 1/7/1998 – 4/30/2003

Data used are described in Table 1. Details of the decomposition exercise are given in Section 6.3. In the Table, *Implied spread* refers to the LIBOR or swap spread for a given maturity, and *Default risk* and *Liquidity* indicate the default risk and liquidity components of the LIBOR or swap spread, respectively.

		6-M LIBOR			2-Y swap	
	Implied spread	**Default risk**	**Liquidity**	**Implied spread**	**Default risk**	**Liquidity**
Mean (in bps)	40.12	21.92	18.20	49.07	26.17	22.89
Median (in bps)	41.53	25.33	17.14	47.50	25.51	21.70
Standard deviation (in %)	0.33	0.31	0.17	0.16	0.06	0.17
Minimum (in bps)	-52.21	-57.00	-16.90	20.00	10.51	-5.85
Maximum (in bps)	122.68	93.03	60.27	91.50	45.97	61.84
		3-Y swap			**5-Y swap**	
	Implied spread	**Default risk**	**Liquidity**	**Implied spread**	**Default risk**	**Liquidity**
Mean (in bps)	57.68	32.47	25.22	64.77	34.91	29.86
Median (in bps)	57.32	32.88	24.05	64.30	34.59	28.45
Standard deviation (in %)	0.17	0.05	0.17	0.19	0.04	0.18
Minimum (in bps)	22.16	20.39	-3.06	29.97	12.40	2.11
Maximum (in bps)	97.68	58.01	64.19	107.50	46.92	68.96
		7-Y swap			**10-Y swap**	
	Implied spread	**Default risk**	**Liquidity**	**Implied spread**	**Default risk**	**Liquidity**
Mean (in bps)	68.78	34.41	34.37	73.40	32.70	40.70
Median (in bps)	68.44	34.19	32.63	73.05	32.83	38.65
Standard deviation (in %)	0.22	0.04	0.19	0.25	0.08	0.19
Minimum (in bps)	34.13	14.66	5.51	31.71	11.64	10.45
Maximum (in bps)	115.37	46.81	73.67	127.92	53.32	80.39

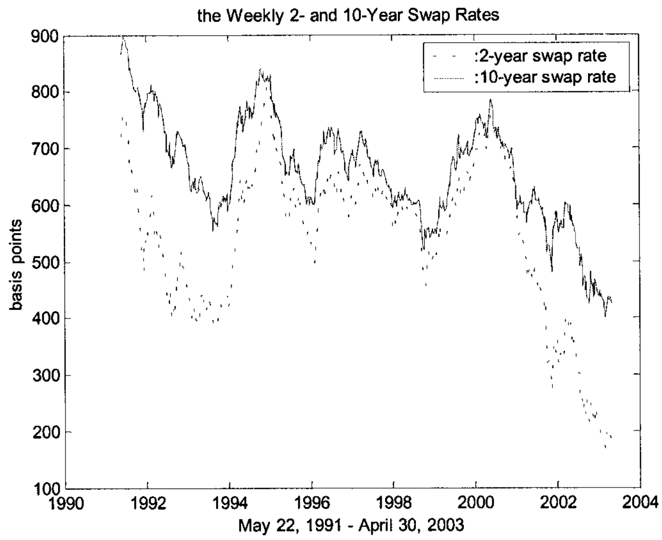

Figure 1:

The actual 2- and 10-year swap rates. Data used are weekly and are described in Table 1.

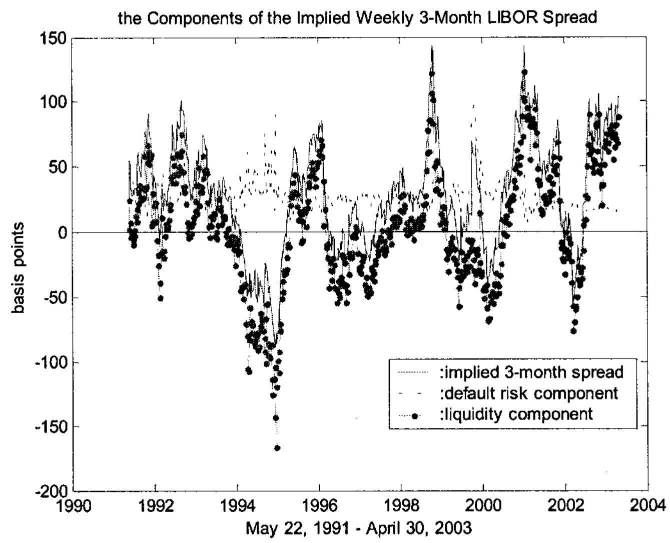

Figure 2:

The implied 3-month LIBOR spread and its default risk and liquidity components. Data used are weekly and are described in Table 1 and Table 3.

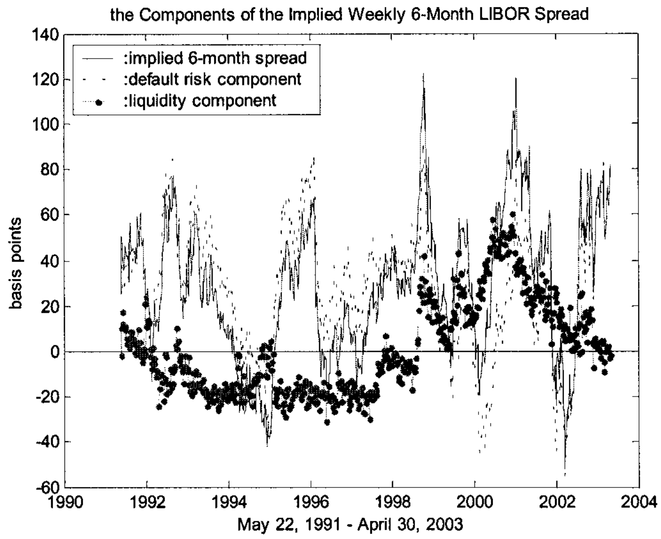

Figure 3:

The implied 6-month LIBOR spread and its default risk and liquidity components. Data used are weekly and are described in Table 1.

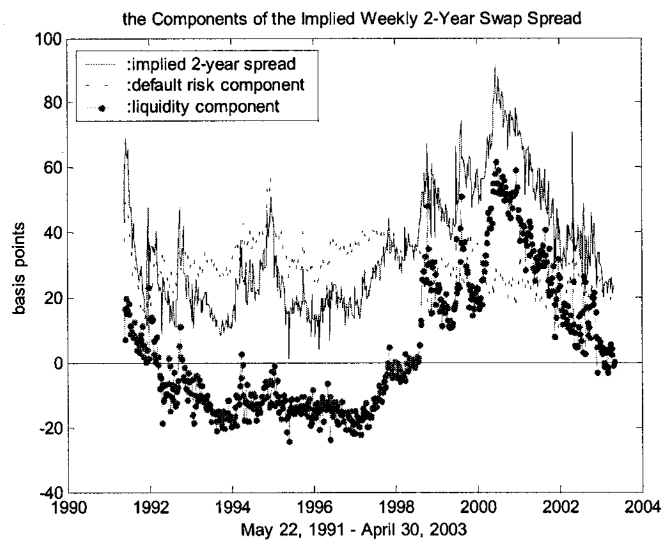

Figure 4:

The implied 2-year swap spread and its default risk and liquidity components. Data used are weekly and are described in Table 1.

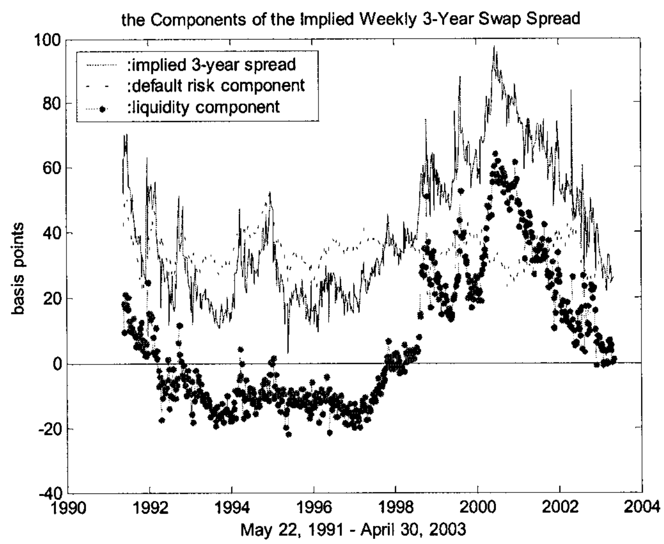

Figure 5:

The implied 3-year swap spread and its default risk and liquidity components. Data used are weekly and are described in Table 1.

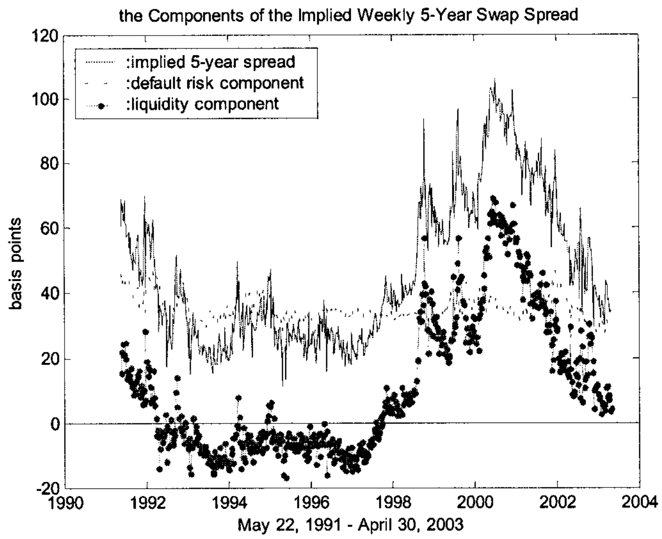

Figure 6:

The implied 5-year swap spread and its default risk and liquidity components. Data used are weekly and are described in Table 1.

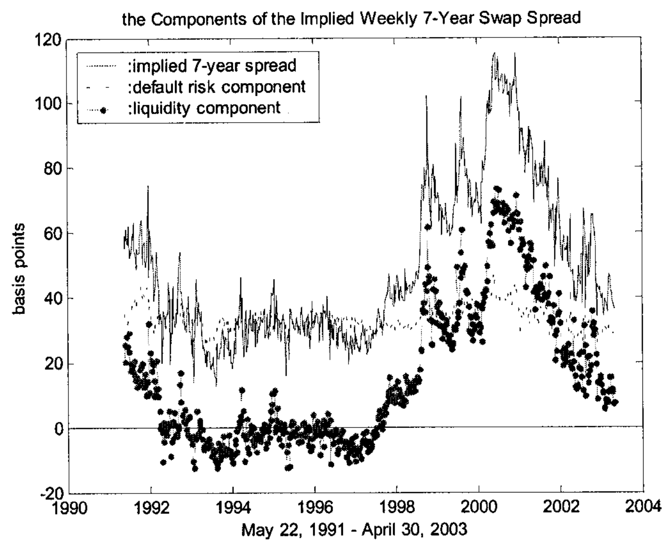

Figure 7:

The implied 7-year swap spread and its default risk and liquidity components. Data used are weekly and are described in Table 1.

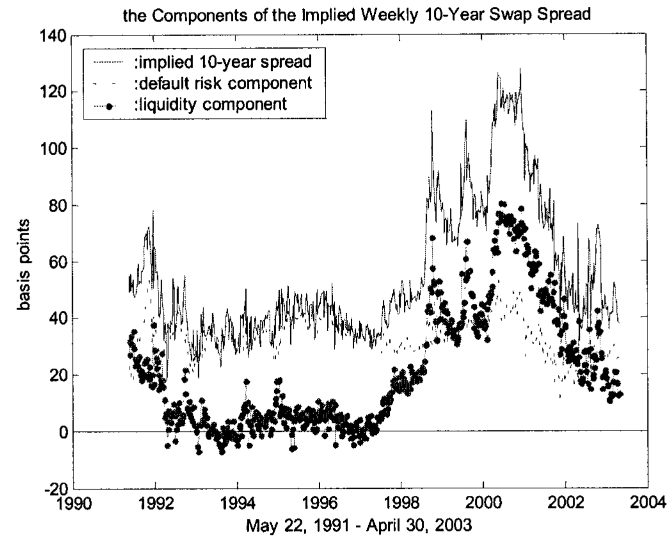

Figure 8:

The implied 10-year swap spread and its default risk and liquidity components. Data used are weekly and are described in Table 1.

Summary and Conclusion

This thesis contributes to the fast growing literature on modeling credit risk, or the risk of default. In particular, it studies the impacts of credit risk on the pricing of fixed income securities. It concentrates on the following three fundamental questions in modeling credit risk:

1) What are the implications of credit risk for the pricing of fixed income securities?

2) Can we develop more satisfactory credit risk models that better capture the observed credit spreads on fixed income securities?

3) Are the observed credit spreads on fixed income securities solely due to credit risk? If not, what are the non-credit (default) components?

The first and second questions are addressed in both the first and second essays of this thesis. In both essays, we incorporate stochastic volatility into models for corporate bond prices. The first essay uses the structural approach, whereas the second essay relies on the reduced form approach. In both papers, the inclusion of stochastic volatility is shown to have significant impacts on credit spread levels and to improve on previous credit risk models in a number of aspects. Furthermore, properties of actual credit spreads are better captured by the models developed in these two essays. Finally, the third essay of this thesis addresses the third question mentioned above. Here we propose a reduced form model of interest rate swap spreads and uses this model to decompose the swap spreads into their default risk and liquidity components, which are otherwise unobservable. This decomposition exercise sheds new light on the composition of swap spreads, which are one of the most important credit risk spreads, and reveals that the default risk and liquidity components of swap spreads behave quite differently.

The results presented in this thesis point to several interesting avenues for future research. First, the models developed in this thesis can be used to price various types of credit derivatives. Credit derivatives have been one of the most important financial innovations in the past decade and their use has been growing at a rapid pace. Since the models proposed in this thesis can approximate the dynamics of credit spreads satisfactorily, they will lead to more accurate valuation of credit derivatives.

Second, the finding of the third essay that although default risk accounts for the largest shares of the levels of interest rate swap spreads, the liquidity component of swap spreads is much more volatile suggests a need to study the pricing of liquidity premium in the swap markets. More specifically, if swap spreads contain risk premiums, these risk premiums are more likely compensations for liquidity risk.

Third, the results of the first two essays question the ability of a single-factor diffusion process to model adequately the dynamics of credit spreads. We develop two multi-factor models instead. These models can fit the observed credit spreads reasonably well. However, an important question still remains: how many factors are necessary to *fully* capture the dynamics of corporate yield curves? If there are still factors to consider, what are they? And how to incorporate them into credit risk models in a tractable way? This and other questions are left for our future research.

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