Essays on Asset Pricing with Heterogeneous Beliefs and Bounded Rational Investor

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Abstract

The thesis includes two essays on asset pricing. In the first essay, "*Asset Pricing in a Monetary Economy with Heterogeneous Beliefs*", we shed new light on the role of monetary policy in asset pricing by focusing on the case where investors have heterogeneous expectations about future monetary policy. Under heterogeneity in beliefs, investors place bets against each other on the evolution of money supply, and as a result, the sharing of wealth in the economy evolves stochastically over time, making money non-neutral. Employing a continuous-time, general equilibrium model, we establish these fluctuations to be rich in implications, in that they majorly affect the equilibrium prices of all assets, as well as inflation. In particular, we find that the stock market volatility may be significantly increased by the heterogeneity in beliefs, a conclusion supported by our empirical analysis. The second essay is titled with "*Asset Pricing and Welfare Analysis with Bounded Rational Investors*". Motivated by the fact that investors have limited ability and insufficient knowledge to process information, I model investors' bounded-rational behavior in processing information and study its implications on asset pricing. Bounded rational investors perceive "correlated" information (which consists of news that is correlated with fundamentals, but provides no information on them) as "fundamental" information. This generates "bounded rational risk". Asset prices and volatilities of asset returns are derived. Specially, the equity premium and the stock volatility are raised under some conditions. I also analyze the welfare impact of bounded rationality.

Résumé

Cette thèse comprend deux essais sur l'évaluation des actifs. Dans le premier essai, « Evaluation des Actifs dans une Economie Monétaire avec Croyances Hétérogènes », nous apportons un éclairage nouveau sur le rôle de la politique monétaire dans l'évaluation des actifs an considérant le cas où les investisseurs ont des anticipations hétérogènes quant à la politique monétaire future. En présence de croyances hétérogènes, les investisseurs parient les uns contre les autres relativement à l'évolution de la masse monétaire, si bien que la répartition de la richesse dans l'économie fluctue de manière aléatoire, ce qui fait que la monnaie a un effet sur l'économie réelle. Dans un modèle d'équilibre général en temps continu, nous montrons que ces fluctuations sont riches en implications, car elles impactent de façon importante les prix de tous les actifs, ainsi que l'inflation. En particulier, nous montrons que la volatilité des actions peut être significativement augmentée par l'hétérogénéité des croyances, une conclusion confirmée par notre analyse empirique. Le deuxième essai est intitulé « Evaluation des Actifs et Analyse du Bien-être avec des Investisseurs à Rationalité Limitée ». Motivé par le fait que les investisseurs disposent de capacités et de connaissances insuffisantes pour analyser l'information, je modèle le comportement de rationalité limitée des investisseurs dans l'analyse de l'information et étudie ses implications sur les prix. Les investisseurs à rationalité limitée perçoivent l'information corrélée (qui se compose de nouvelles corrélées avec les fondamentaux, mais qui ne fournissent pas d'information sur ceux-ci) comme étant de l'information sir les fondamentaux. Ceci engendre du risque de rationalité limitée. Je calcule les prix et les volatilités des actifs. En particulier, la prime de risque et la volatilité des actions sont augmentées sous certaines conditions. J'analyse aussi l'impact de la rationalité limitée sur le bien-être.

To my parents, my wife, and my brothers and sister

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Contribution of Authors

Among the two essays of this thesis, I coauthored the first essay, entitled "Asset Pricing in a Monetary Economy with Heterogeneous Beliefs", with Professor Benjamin Croitoru. All authors have made equal contributions to this paper.

I am solely responsible for the second paper, titled "Asset Pricing and Welfare Analysis with Bounded Rational Investor".

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Chapter 1

INTRODUCTION

The heterogeneous beliefs and irrational (or bounded rational) behavior of investors has been an important research topic in asset pricing literature. When investors have incomplete information about economic variables, it is natural to expect (based on intuition, casual empiricism, formal surveys and experiments) that they have different opinions and that their behavior deviates from rationality. The objective of this thesis is to improve the understanding of the impact of heterogeneous beliefs and investors' bounded rational behavior on asset pricing. Chapter II reviews the related literature.

The existing literature suggests that asset prices and their dynamics (e.g., asset volatility and risk premium) are affected by the presence of heterogeneous beliefs. However, despite the impact of monetary policy on asset prices and investors' differing opinions about future monetary policy, this issue has not been explored in depth so far. Motivated by these facts, the first essay of this thesis, Chapter III, attempts to better understand the effects of monetary policy on asset prices by developing an equilibrium model in which investors disagree on expected future monetary policy.

In a continuous-time general equilibrium model, there are two investors who observe the changes in money supply but disagree on its expected growth rate. Investors' utility is assumed to be a function of both consumption and money holdings. There are two sources of uncertainty: one associated with money supply and the other associated with aggregate consumption. We use the martingale representation technology (Cox and Huang (1989), Karatzas, Lehoczky and Shreve (1987)) to solve the investors' optimization, and construct a representative agent with stochastic weights (Cuoco and He (1994), Basak and Cuoco (1998)) to solve for the equilibrium. Under heterogeneous beliefs, investors bet against each other on the future money supply and so shocks in money supply affect the distribution of wealth across investors and investors' consumption: heterogeneity in beliefs makes money non-neutral. Moreover, using economists' forecasts as provided by the Survey of Professional Forecasters and S&P 500 returns, we empirically investigate the relationship between difference in beliefs on future money growth and stock market volatility. This may help explain high level and idiosyncratic, time-varying behavior of stock market volatility, which are some of the key puzzles facing financial economists (e.g., Schwert (1989)).

The second essay, Chapter IV, motivated by the fact that investors have limited ability and insufficient knowledge to process the wealth of available information, investigates the implications of investors' bounded rational behavior on asset pricing. In financial markets, investors face two types of information: "fundamental" information provides investors information about expected changes in fundamentals (e.g., dividends, earnings, and cash flows), while "correlated" information (e.g., the unemployment rate) is only correlated with fundamentals and provides no information on these; it may, however, provide investors with information about shocks to fundamentals.

There are two types of investor: the rational investor can distinguish between these two types of information, while the bounded rational investor cannot and incorrectly assumes that both fundamental and correlated information provide information on fundamentals. They estimate the unobserved dividend growth by observing the realizations of dividends and "correlated" information. The rational investor updates his beliefs accurately, which generates "learning risk" (Brennan (1998)), while the bounded rational investor's incorrect updating of his beliefs generates "bounded rational risk". Using methodologies similar to those in Chapter III, I analyze the implications of investor's bounded rational behavior on asset pricing (e.g., stock price, stock volatility and equity premium) and rational investor's welfare changes relative to a benchmark economy in which all investors are rational. The economic setting, as well as theoretical and simulation results are presented in Chapter IV.

Chapter V summarizes the main findings of the thesis and suggests some related topics for future studies.

Chapter 2

LITERATURE REVIEW

2.1 Introduction

The difference in opinions across investors and the irrational behavior of investors are two important research topics in asset pricing. In this chapter, I review the literature related to these two topics. The chapter is organized as follows: Section 2.2 reviews the literature on asset pricing with heterogeneous beliefs and supporting empirical evidence. Section 2.3 reviews the literature on asset pricing with irrational (or bounded rational) investors, and Section 2.4 reviews recent works which link these two topics and thus have richer implications for asset pricing.

2.2 Asset Pricing with Heterogeneous Beliefs

An important assumption of standard asset pricing is that investors are homogeneous, an immediate implication of which is that there is no trade in financial assets. As pointed out by Abel (1989), "*In order to generate trade among investors, it is necessary to assume that investors have different subjective beliefs, different utility functions and/or different opportunity sets.*"

Pioneered by Miller (1977) and Williams (1977), a growing theoretical literature investigates the effect of heterogeneous beliefs on asset prices, their dynamics and investors' portfolio allocation. The earlier theoretical models include Miller (1977), Williams (1977), Harrison and Kreps (1978), Varian (1989), Abel (1989), Harris and Raviv (1993), Detemple and Murthy (1994), Zapatero (1998) and Basak (2000). Recently, Kyle and Lin (2003), Scheinkman and Xiong (2003), Basak (2005), Dumas, Kurshev and Uppal (2005), Berrada (2006a), Buraschi and Jiltsov (2006), David (2006), Gallmeyer and Hollifield (2006), Hong, Scheinkman and Xiong (2006), Xiong and Yan (2006), and Li (2007) investigate the impacts of heterogeneous beliefs from various perspectives.

Miller (1977) suggests that if investors have heterogeneous beliefs about economic variables and short sales are not allowed, the stock price will reflect the opinion of the more optimistic investor. In this setting, the stock price can never exceed the valuation of the more optimistic investor. The Miller (1977) model is static and so it is silent on the dynamics of trading. Harris and Raviv (1993), Kandel and Pearson (1995), and Kyle and Lin (2003) study models in which trading is generated by heterogeneous beliefs. Harrison and Kreps (1978) present a multi-period discrete model which investigates the joint behavior of volume and overpricing. The theoretical developments of Harrison and Kreps (1978) include Scheinkman and Xiong (2003) and Hong, Scheinkman and Xiong (2006). Moreover, Chen, Hong and Stein (2002) and Diether, Malloy and Scherbina (2002) present evidence that supports the prediction of Miller's model. Hong and Stein (2006) review the related literature. Here, we focus on the implications of heterogeneous beliefs for asset pricing rather than for trading volume.

2.2.1 The Asset Pricing Model of Williams (1977)

Williams presents a continuous-time asset pricing model, in which asset prices follow a geometric Brownian motion and the expected returns are unknown constants. Investors can correctly estimate the variance-covariance matrix of returns by observing the prior realizations of asset prices. However, it is hard for them to correctly estimate expected returns conditional on available information, and thus may have different estimates.

Each investor maximizes his utility function like the agent in Merton (1971, 1973). By aggregating the wealth across investors, in equilibrium, the usual Security Market Line (SML) is replaced by a new SML, which specifies a linear trade-off between expected average return and investors' perceived risk. This has important implications for existing empirical tests of the capital asset pricing model. In contrast to the usual model with homogeneous beliefs, variances and covariances of residual returns have an important impact on assets' expected returns.

2.2.2 The Asset Pricing Models of Detemple and Murthy (1994), Zapatero (1998) and Basak (2000)

In a production economy, Abel (1989) considers a two-period version of the Lucas (1978) asset pricing model, and suggests that if the riskless rate of return is determined endogenously, increased heterogeneity increases the riskless rate of return and equity premium, and reduces stock price. The results presented by Abel (1989) are quite interesting and can help us understand "equity premium puzzle" (Mehra and Prescott, 1985).

Detemple and Murthy (1994), Zapatero (1998), and Basak (2000) extend Williams (1977) and Abel (1989) to a continuous-time framework. They analyze the role of heterogeneous beliefs in equilibrium variables from different perspectives. Detemple and Murthy (1994) focus on increased real interest rates, Zapatero (1998) investigates increased volatility of real interest rates, and Basak (2000) studies increased volatility of investors' consumption growth and perceived state-price.

Detemple and Murthy (1994)

Detemple and Murthy consider a continuous-time production economy of Cox, Ingersoll and Ross (1985) with a Brownian uncertainty structure, in which investors disagree on expected production growth.

The cumulative rate of production growth satisfies a Brownian motion process, and the drift of production growth is unobservable since investors have incomplete information. Investors observe the realizations of production growth as well as output, but have heterogeneous beliefs about the drift of production growth. The mechanism is as follows: investors interpret the content of information relative to their beliefs, and so their posterior beliefs differ.

Each investor maximizes his utility to derive perceived real interest rate. Detemple and Murthy find that the real interest rate is a wealth-weight average of investors' perceived real interest rate. In equilibrium, the wealth share of each investor follows a stochastic process. Moreover, the asset price is a wealth-weighted average of the prices that would prevail in economies populated by homogeneous agents. They also find that the optimistic investor takes more aggressive positions in risky technology and benefits from positive innovations. Moreover, they consider a case in which the asset market is incomplete and investors have incentive to create new assets to increase the space of potential consumption allocation, and suggest that financial innovation affects the dynamics of asset allocations and prices.

Although Detemple and Murthy derive stock price explicitly, their model is limited to the analysis of the dynamics of asset prices; e.g., volatility and equity premium. Investors are assumed to have logarithmic preferences, and so stock volatility is not affected by heterogeneity in beliefs.

Zapatero (1998)

Zapatero uses a pure exchange economy version of the model in Detemple and Murthy (1994) to investigate the effects of financial innovation and the changes in information structure on volatility of real interest rates.

The exogenous consumption good follows the dynamics with unobserved constant growth μ given by

$$dp\,(t)\,/p\,(t) = \mu dt + \rho_1 d\omega_1\,(t) + \rho_2 d\omega_2\,(t)\,. \tag{2.1}$$

where $\omega \equiv (\omega_1, \omega_2)$ is a standard 2-dimensional Brownian motion. Besides the above endowment process, investors observe a public index with the dynamics

$$ds\,(t)\,/s\,(t) = \varepsilon dt + \sigma_{21} d\omega_1\,(t) + \sigma_{22} d\omega_2\,(t)\,. \tag{2.2}$$

equation (2.2) means that the public index provides information about Brownian motion process ω rather than consumption growth μ. Investor i $(i = 1, 2)$ updates his estimate of μ dynamically by observing the realization of consumption and index via the Bayes rule.

Zapatero analyzes two cases. In the first case, there are two assets: a riskless bond and a risky asset (e.g., stock). In this situation, there is no public index, so the financial market is incomplete. In the second case, he introduces the public index and a second risky asset (e.g., contingent claim) to complete the market. By solving the investor's optimal problem, the real interest rates are derived for the above two cases. Zapatero finds that, by introducing a second risky asset to complete the market, the volatility of the real interest rate rises, which means financial innovation makes the real interest rate more volatile.

Zapatero acknowledges that his model leaves some questions unanswered. First, investors are myopic when they have logarithmic preferences, and so he cannot analyze investors' hedging policy. Second, the effect of financial innovation on volatility of the real interest rate will disappear as investors update their beliefs dynamically and gradually converge to the real rate in the long run. The recent work by Dumas, Kurshev and Uppal (2005) analyzes investors' hedging policy when they have CRRA preference. The literature on the survival of irrational investors partially investigates the second issue, and finds that this convergence process possibly takes hundreds of years (Yan (2006)).

Basak (2000)

Basak investigates the existence of a sunspot equilibrium in which an extraneous process that is unrelated with market fundamentals affects asset prices when agents have different beliefs. In a pure exchange economy, investors observe two exogenous processes: the aggregate endowment process and an extraneous process. They have full information on the endowment process, but incomplete information on the extraneous process from which they make different inferences about its growth μ_z. Investors update their beliefs about μ_z in a Bayesian fashion.

By converting the investors' dynamic optimization problem into the static variational problem, he derives each investor's optimal consumption $c_i(t)$ $(i = 1, 2)$ and its dynamic process. To solve the equilibrium, he constructs a representative agent with utility function given by

$$U(c(t); \Lambda(t)) \equiv \max_{c_1+c_2=c} \lambda_1(t) u_1(c_1(t)) + \lambda_2(t) u_2(c_2(t)). \tag{2.3}$$

where u_i is the utility function of investor i. In equilibrium, he derives the state-price density perceived by each investor and demonstrates that the weighting $\eta(t) \equiv \lambda_2(t)/\lambda_1(t)$ is stochastic due to investors' facing differing state price densities.

When investors are assumed to have CRRA preferences, he finds that volatilities of state price and individual consumption process are unambiguously increased, and the real interest rate and the perceived consumption growths of the two investors are increased in the presence of extraneous uncertainty if investors are more risk averse than log utility. I refer readers to the survey of Basak (2005), which provides additional interesting future research topics on heterogeneous beliefs.

2.2.3 Recent Developments

Scheinkman and Xiong (2003) present a model in which overconfidence generates disagreements among investors, where overconfidence indicates that investors overestimate the correlation between innovations in the signal and innovations in the unobserved variables. With short-selling constraints, they analyze links between asset prices, trading volume, and price volatility. Dumas, Kurshev and Uppal (2005) extend this work to provide a model in which all investors are risk averse and are allowed to short sell. There are two types of investor: one has correct belief about signal and the other is overconfident in it. Dumas, Kurshev and Uppal (2005) analyze the implications of the irrational investor on asset pricing and hedging strategies, and find that the overconfident investor adds "noise" for which all investors require a risk premium; consequently, the price levels of assets are reduced, and the volatility and the risk premium of assets increase. Moreover, the rational investor reduces the proportion of wealth invested into equity except when he is extremely optimistic about future dividend growth. I further review this paper in Section 2.3. In Gallmeyer and Hollifield (2006), all investors are rational and the short-sale constraint can have large effects on stock volatility.

Berrada (2006a) extends Detemple and Murthy (1994) to a pure exchange economy where the growth rate of aggregate dividend is unobservable and follows a mean-reverting process. Investors not only have heterogeneous beliefs about the unobserved dividend growth but their utility has different levels of relative risk aversion. Berrada finds that the market price of risk and the stock volatility are much higher, and the riskless rate of return lower, than in an equivalent full information economy.

Li (2007) considers a model in which investors have heterogeneous beliefs about the structure of a dividend process, and have logarithmic preferences (or they are risk neutral). The heterogeneity of beliefs generates several empirical effects – excessive volatility, leverage effects, and positive relationships between price and trading volume and between volatility and volume.

Recent papers complete the earlier models integrating heterogeneous beliefs into bond and option markets. Buraschi and Jiltsov (2006) provide option pricing and volume implications for an economy with heterogeneous agents who face model uncertainty and have different beliefs on expected returns. They use survey data to build an Index of Dispersion in Beliefs and find that a model that takes information heterogeneity into account can explain the dynamics of option volume and the smile better than can reduced form models with stochastic volatility. Xiong and Yan (2006) present a dynamic equilibrium model of bond markets, in which two groups of investors hold heterogeneous expectations about future economic conditions. They demonstrate that the relative wealth fluctuation between the two groups of investors caused by their speculative positions amplifies bond yield volatility and generates time-varying risk premia, thus providing an explanation for both the "excessive volatility puzzle" of bond yields and the failure of the expectation hypothesis.

David (2006) extends the analysis of continuous time models of heterogeneous beliefs to the case of recurrent jumps in the underlying drift of the diffusion process. He claims that the dispersion process in these models declines monotonically over time and asymptotes to zero. Therefore, the dispersion of beliefs across investors has a temporary effect on conditional risk premium, but is unable to match the large risk premium in long samples of data. In his analysis, however, investors have different underlying models of the data generating process as opposed to differing initial priors as in the above papers, and the dispersion process recurrently fluctuates and leads to a large equity premium over long horizons.

2.2.4 Empirical Evidence

At the empirical level, Kandel and Pearson (1995) provide evidence on the relationship between trading volume and stock returns around public announcements, and argue that the evidence is consistent with the assumption that agents interpret public information differently. Welch (2000) reports high disparity, with views ranging from 2% for the pessimists to 13% for the optimists. Mankiw, Reis and Wolfers (2003) analyze 50 years of inflation expectations data from several sources, and document substantial disagreement among both consumers and professional economists about expected future inflation. Anderson, Ghysels and Juergens (2005) show that there exists significant disagreement among stock analysts about expected earnings, and that this heterogeneity in beliefs has an impact on asset pricing. Pavlova and Rigobon (2006) provide empirical support for a model of international stock prices and exchange rates with heterogeneity in beliefs, and Buraschi and Jiltsov (2006) construct an index of difference in beliefs on fundamentals and find that differences of opinion can help explain the dynamics of option trading volume.

2.3 Asset Pricing with Bounded Rational Investors

Most neoclassical asset pricing models rely on the assumption that investors are rational in making investment decisions. Barberis and Thaler (2003) claim that "*...rationality means two things. First, when they receive new information, agents update their beliefs correctly, in the manner described by Bayes' law. Second, given their beliefs, agents make choices that are normatively acceptable, in the sense that they are consistent with Savage's notion of Subjective*

Expected Utility (SEU)." However, there is a growing literature that takes into account investors' irrational or bounded rational behavior and considers its implications on asset pricing. Simon (1955, 1987) formally defines bounded rationality as "*...rational choice that takes into account the cognitive limitations of the decision-maker – limitations of both knowledge and computational capacity.*"

2.3.1 The Asset Pricing Model of De Long, Shleifer, Summers and Waldmann (1990)

The literature on the effect of irrational investors on asset prices dates back to Friedman (1953), who argues that irrational investors with erroneous beliefs about fundamentals have no impact on asset prices in the long run. De Long, Shleifer, Summers and Waldmann (1990) present a partial general equilibrium model in which the unpredictability of noise traders' beliefs creates a risk in asset prices that deters rational arbitrageurs from aggressively betting against them. This makes assets less attractive to risk-averse arbitrageurs and so drives down prices.

In their setting, there are two types of investor: the sophisticated (or rational) investor and the noise trader (or irrational investor) who trade on two assets (e.g., a riskless bond and a stock). The sophisticated investor accurately perceives the distribution of stock return, and the noise trader misperceives expected stock price by an *i.i.d.* normal distribution variable. Investors live in two periods: they choose the portfolio in period t and consume the goods (e.g., dividends) in period $t+1$. Each investor maximizes his utility which is a constant absolute risk aversion function of wealth.

Their model has several implications for asset price behavior. First, since noise trader risk limits the effectiveness of arbitrage, stock price becomes more volatile. Second, the underpricing of assets relative to fundamental values caused by noise trader risk can help us explain the underpricing of close-end mutual funds and the "equity premium puzzle". Third, their model can be used to analyze the optimal investment strategy of rational investors[1]. Finally, the irrational investors can survive and affect asset prices in the long run.

2.3.2 Recent Developments

Based on De Long, Shleifer, Summers, and Waldmann (1990) and the psychological evidence (Kahneman and Tversky (1974)), several behavioral models analyze the cross-sectional properties of asset returns: the under- and overreaction of stock prices to news. Barberis, Shleifer and Vishny (1998) consider a model in which the true earnings process is a random walk, but investors believe that earnings either follow a steady growth path in the long-run or are mean-reverting. When a positive earnings surprise is followed by another positive surprise, the investor raises the likelihood that he is in the growth regime, whereas when a positive surprise is followed by a negative surprise,

[1] Dumas, Kurshev, and Uppal (2005) investigate investors' hedging policy in a general equilibrium framework.

the investor raises the likelihood that he is in the mean-reverting regime. The model is able to replicate the empirical observations of continuation and reversal of stock returns. Daniel, Hirshleifer and Subrahmanyam (1998) assume that investors are overconfident about the precision of private signal about the asset payoff. The overconfidence increases if the private signal is confirmed by public information, but decreases slowly if the private signal contrasts with public information. In Hong and Stein (1999), investors are only able to process a subset of available information. There are two types of investor: one investor makes forecasts based on private signals about future fundamentals, and so does not condition on current or past prices. The other investor, in contrast, conditions on past price changes. However, his forecasts must be "simple" (i.e., univariate) functions of the history of past prices. Hirshleifer (2001) and Barberis and Thaler (2003) survey the literature about the impacts of investor psychology on asset pricing and recent developments in behavioral finance, respectively.

Boswijk, Hommes and Manzan (2006) present an asset pricing model, in which investors have homogeneous beliefs about future cash flows, but disagree on the speed of reversion of stock prices towards the intrinsic value. Given that the overvaluation of the stock is common knowledge, fundamentalists believe that stock price will move back towards its fundamental value, while trend followers expect that the price trend will continue in the short run. They calibrate the model to annual U.S. stock price data between 1871 and 2003 and offer an explanation for the recent stock price run-up.

Motivated by Friedman (1953) and De Long, Shleifer, Summers, and Waldmann (1990), recent works further investigate whether irrational investors can survive and have an impact on asset prices in the long run. In Kogan, Ross, Wang, and Westerfield (2006), there are two investors with the same constant relative risk aversion utility over terminal wealth. One investor has perfect knowledge of all parameters of the underlying stochastic process. The other, the irrational investor, uses parameters with a constant deviation from the true value. They show that the price impact of the irrational investor does not depend on his long-run survival, and he can have a significant impact on asset prices even when his wealth becomes negligible. Yan (2006) presents a dynamic general equilibrium model with two investors having identical preferences. One investor has rational expectations while the other has an incorrect belief concerning the mean growth rate of the economy. He suggests that the selection process is excessively slow, although the investor with an incorrect belief cannot survive in the long run. Since this review focuses on the implications of irrational behavior for asset pricing, I will next review the most relevant papers on the subject.

Dumas, Kurshev, and Uppal (2005)

The main objective of their paper is to understand the strategy of rational investors and the impact of this strategy on the irrational investors responsible for excess volatility. Dumas, Kurshev and Uppal (2005) consider an economic framework similar to that in Scheinkman and Xiong (2003) except that investors are risk averse and can short sell. Moreover, there are two types of investor with one that has correct belief (they call him "rational investor A") regarding signal and the other one (they call him "overconfident investor B") who is overconfident in it. The aggregate dividend process δ follows a Brownian motion process, and the expected dividend growth is unobservable. Investors have to estimate it by observing the realizations of dividends and a public signal. The signal provides information about dividend growth rather than about current shock to dividend growth. There are two types of investor: rational investor A has correct perception of the role of the public signal, while overconfident investor B incorrectly believes that the public signal is positively correlated with dividend growth.

Each investor, who has CRRA preference, chooses his optimal consumption and portfolio to maximize his utility, and the equilibrium is solved by constructing a representative investor. In equilibrium, Dumas, Kurshev and Uppal (2005) find the following results. First, the price levels of stock and bond are reduced by the presence of the irrational investor since the irrational investor adds "noise" for which all investors require a risk premium. Second, the volatility of both bond and stock return as well as the correlation between them are increased. Third, the total holding of equity in the case of irrationality is reduced relative to the case of rationality when there is agreement, because risk averse investors are deterred by the presence of the irrational investor. Finally, the fact that the irrational investor is not eliminated from the population instantly implies that the phenomenon of excess volatility will not disappear quickly.

Berrada (2006b)

Motivated by the fact that investors show cognitive bias in making decisions, Berrada investigates, by simulation, whether the under- (over-) reactive investor can survive in the long run and has an impact on stock price and its volatility. He considers a model in which the expected changes in fundamentals is unobservable and some investors display learning bias and over- and under-react to the arrival of new information. In a pure-exchange economy, the exogenous dividend follows a stochastic process and the dividend growth is unobservable. There are different types of investor: some are Bayesian learners and some display learning bias, and they must estimate dividend growth by observing the realization of dividends. The Bayesian learners correctly update their estimates via filtering theory (Lipster and Shiryaev (2001)). However, the over-reactive (under-reactive) investor assigns a high (low) weight to his updated belief.

Each investor chooses the optimal consumption and the portfolio to maximize his utility. Using the Clark-Ocone formula and the martingale representative theorem, Berrada derives a quasi-analytical solution for stock volatility, and finds that the presence of the under/over-reactive investor decreases/increases stock volatility. Moreover, he suggests that over a reasonable horizon, under- or over-reaction has little impact on investors' consumption share, which is similar to the findings in Dumas, Kurshev and Uppal (2005) and Yan (2006).

2.4 Asset Pricing with Heterogeneous Beliefs and Bounded Rational Investors

Recent developments link investors' irrationality to their heterogeneity, and find additional interesting implications for asset pricing; for example, De Long, Shleifer, Summers, and Waldmann (1990), Hong and Stein (1999), Dumas, Kurshev and Uppal (2005), Berrada (2006b), Boswijk, Hommes and Manzan (2006) and Yan (2006) present theoretical models in which there are two types of investor: one is rational in processing information and the other is irrational or overconfident. In this framework, they analyze the price impact of irrational investors on asset prices and their survival in the long run.

Chapter 3

ASSET PRICING IN A MONETARY ECONOMY WITH HETEROGENEOUS BELIEFS

Abstract

In this chapter, we shed new light on the role of monetary policy in asset pricing by focusing on the case where investors have heterogeneous expectations about future monetary policy. This case is realistic, because central banks are typically less than perfectly open on their intentions. Accordingly, surveys of economists in the press reveal that they frequently disagree in their expectations. Under heterogeneity in beliefs, investors place bets against each other on the evolution of the money supply, and as a result, the sharing of wealth in the economy evolves stochastically over time, making money non-neutral. Employing a continuous-time, general equilibrium model, we establish these fluctuations to be rich in implications, in that they majorly affect the equilibrium prices of all assets, as well as inflation. In some specific cases, we are able to derive explicit formulas for important economic quantities. In particular, we find that stock market volatility may be significantly increased by the heterogeneity in beliefs, a conclusion supported by our empirical analysis. In addition to generating interesting behavior on the part of asset prices, our model provides a natural framework in which to assess the impact of the transparency of monetary policy, a topical and controversial issue. Our model is particularly appropriate for the study of the effects of transparency, because it is intuitive that one of the key effects of increased transparency should be a drop in the amount of heterogeneity in beliefs.

3.1 Introduction

The impact of monetary policy on financial asset prices can hardly be overstated. Its impact on the value of money itself (in other words, inflation) is obvious, but it is also possibly the single most important factor affecting stock market returns. While the academic literature provides many studies documenting this (see, e.g., Thorbecke (1997)), it is not necessary to go that far: by simply browsing through financial pages in the press, it is clear that participants in financial markets attach tremendous importance to the actions of central banks. Any upcoming decision by the Federal Reserve is awaited with great anticipation and, as a news article on the BBC website[1] put it, "A mere word from Mr. Greenspan can cause the stock market and the dollar to rise and fall." The mere anticipation may suffice to cause stock market moves: for example, CNN stated on September 8, 2004:[2] "cautiousness ahead of a key address by Fed Chairman Alan Greenspan before a Congressional panel could take stocks lower Wednesday."

One key feature of central banks' policies is that they are typically hard to anticipate. As the same BBC article puts it, Alan Greenspan "is also famous for his ability to keep the markets and the politicians guessing." Part of the monetary economics literature suggests that transparency in monetary policy may compromise its effectiveness, and this school of thought seems to still exert some influence, even though there has been a trend toward more transparency.

[1]http://news.bbc.co.uk/1/hi/business/business basics/178569.stm
[2]http://money.cnn.com/2004/09/08/markets/stockswatch/

(A survey by Carpenter (2004) points that "there is a lack of consensus on whether central bank transparency is beneficial." It is interesting to note that the author of the survey is an economist at the Fed.) While many central banks have switched to following more formal rules in the recent past, by far the most important central bank in the world, the U.S. Fed has not adopted such rules and remains difficult to read. As economist Robert Barro put it,[3] "Beginning with New Zealand in 1989, a number of central banks successfully designed and followed formal rules for inflation targeting. Examples include Australia, Britain, Canada and Sweden. (...) Despite increases in transparency, the U.S. system remains opaque and relies more on the judgment and credibility of its chair."

The fact that there is imperfect information on monetary policy makes it likely that market participants hold heterogeneous beliefs on its future evolution. The presence of disagreement among traders, and its importance for understanding prices, is well established in other areas of finance. For example, a recent study by Anderson, Ghysels and Juergens (2005) shows that there exists significant disagreement among stock analysts about expected earnings, and that this heterogeneity in beliefs matters for asset pricing. It is not necessary to invoke the academic literature to verify that such heterogeneous beliefs are also present regarding monetary policy. Examples of disagreement abound in the popular press. For example, on September 24, 2004, following an interest rate hike by the Fed, CNBC surveyed several economists on their take

[3] Business Week, November 7, 2005.

on the near future of US monetary policy:[4] "While most analysts still expect a rate hike November 10, some disagree as economists debate just when and whether the Fed will pause in its tightening cycle. Wachovia economist Mark Vitner said he expects the Fed to keep rates unchanged in November." We are not aware of any academic study directly documenting heterogeneous beliefs on monetary policy, but Mankiw, Reis and Wolfers (2003) establish substantial disagreement (both among professional economists and consumers) in expectations on inflation, a quantity closely related to monetary policy.

Given the impact of monetary policy on asset prices, it is surprising how little academic work has been done to incorporate money into an asset pricing framework. In almost all models of financial asset pricing, the value of financial securities and their payoffs are denominated in units of consumption goods, and money is not present. There exists a small intersection between monetary economics and asset pricing (see Bakshi and Chen (1996) and the references therein), but it does not incorporate heterogeneity in beliefs on monetary policy.

In this paper, we attempt to better understand asset prices and the effects of monetary policy by building an equilibrium model that incorporates investors' disagreement on monetary policy. To our knowledge, this analysis has not been performed in a framework that allows for a realistic modeling of asset prices (even though Basak (2005) suggests that such an analysis could be performed, and sketches a model to do so). We show that this heterogeneity

[4] http://www.msnbc.msn.com/id/6064674/

in beliefs has important implications on equilibrium economic quantities (inflation, interest rates, stock returns, sharing of wealth across investors), and explore the direction and the magnitude of these effects. In particular, the volatility of the stock market is considerably affected.

We employ a general equilibrium model in continuous time. We chose to employ a continuous time model for tractability (it makes it possible to obtain explicit formulas in many cases) and also because of the availability of well-understood benchmark models, with which we will be able to compare our results. Our model builds on the work of Bakshi and Chen (1996), who start with a discrete time framework and then take the continuous-time limit. This model is extended by Basak and Gallmeyer (1999) to an international context with two currencies, and by Lioui and Poncet (2004) and Buraschi and Jiltsov (2005) to a production economy. (None of these studies include heterogeneity in beliefs, and so there is no redundancy with this paper).

The main features of our model are as follows. Money plays a role because investors' utilities are assumed to be a function of money holdings in addition to consumption. "Money-in-the-utility-function" (MIUF) is one of the modeling strategies that are most popular with monetary economists. It can be interpreted as capturing in reduced form the fact that money renders transaction services, and that it is necessary for investors to hold cash balances (even though they forfeit interest income by doing so) in order to be able to consume. Feenstra (1986) has shown that MIUF is approximately equivalent to assuming a "cash-in-advance" or "Clower" constraint (the other modeling strategy commonly employed to generate an economic role for money, where investors are constrained to hold cash balances proportional to the level of their consumption; see, e.g., Blanchard and Fischer (1989)). The main advantage of the MIUF formulation is that it is quite tractable and often allows for explicit computations. Money enters the economy by being endowed to the investors by the central bank. The money supply is assumed to summarize the central bank's monetary policy, a standard simplification in monetary economics. The money supply is assumed to follow an Ito process, whose drift is chosen by the central bank. The presence of a random component in the money supply is quite realistic, as it is clear that central banks only imperfectly control the money supply (due in particular to their imperfect control of the decisions of the banking sector). There are two investors in our model (both with MIUF), who observe the changes in the money supply, but have incomplete information on its dynamics. We assume that the two investors have heterogeneous beliefs so that, even though they have symmetric information, they disagree on the estimated expected money supply growth. There are two sources of uncertainty, one associated with money supply, and the other associated with aggregate consumption, which are allowed to be correlated.

Three securities are available for trading: a risky stock (equity) representing a claim on aggregate consumption, a real riskless bond (whose payments are indexed on inflation) and a nominal riskless bond (whose payments are denominated in nominal terms and not protected against inflation; thus, the nominal bond is risky in real terms). Since there are three securities for two sources of uncertainty, markets are complete. We use martingale representation technology (Cox and Huang (1989), Karatzas, Lehoczky and Shreve (1987)) to solve the investors' optimization, and construct a representative agent with stochastic weights (Cuoco and He (1994), Basak and Cuoco (1998)) to solve for the equilibrium.

In a general framework, without making particular assumptions on the dynamics of aggregate consumption and money supply and investors' preferences, we find that the equilibrium economic quantities are all affected by the presence of heterogeneous beliefs. The most important feature of the model is that, under heterogeneous beliefs, agents place bets against each other on the money supply and so shocks in the money supply affect the distribution of wealth in the economy. Thus, money supply shocks affect consumptions: heterogeneity in beliefs makes money non-neutral. The extra risk for the investors (a type of risk often referred to as "trading risk" in finance) leads to an extra factor in the pricing of assets, that subsists even when preferences are separable. One interesting result is that, when the investors' utilities are additively separable, financial assets' risk premia are only affected by their exposure to monetary risk in the presence of heterogeneous beliefs. This shows that the heterogeneity in beliefs has a profound effect on the equilibrium, because the structure of the expressions (not just the numerical values) is affected. This result is independent of the specific shape of the investors' utilities and implies that it is possible to obtain novel, interesting implications in a tractable setup, with separable preferences (models with non-separable preferences are typically very intractable). To derive sharper implications, we move to such a setup, and analyze two examples in depth, where simplifying assumptions are made on the dynamic behavior of aggregate consumption and money supply, and on investor preferences.

In the first example, we assume that investors exhibit separable, logarithmic expected utility preferences, which considerably simplifies the computations. The main advantage of this case is that an explicit computation of the price of money (expressed in units of the consumption good), expected inflation and nominal interest rates is possible. (In general, these quantities are characterized by a backward stochastic differential equation whose solution requires complex numerical techniques, such as those employed by Basak and Gallmeyer (1999)). The term structure of nominal interest rates can also be computed explicitly. In spite of the relative simplicity of this case, the implications are rich. Nominal interest rates are driven by investors' expectations of future monetary policy – including the short rate, which is surprising under logarithmic utility (which implies myopic behavior). The most robust implication is that the heterogeneity in beliefs increases the volatility of inflation: when a positive shock to the money supply occurs, not only does the extra amount of money cause inflation, but those investors who expect higher money supply growth and higher future inflation "win their bet" and so their weight in the economy increases, which generates extra inflation. Real asset prices, on the other hand, are unaffected in the logarithmic case. This provides the main motivation for our second example, in which heterogeneity in beliefs affects real asset prices.

In our second main example, we examine the case where investors have separable, constant relative risk aversion (CRRA) utility functions. In this case, it is not possible to solve explicitly for the price of money and the nominal interest rate, but it is possible to provide an explicit formula for the stock price and its volatility. It is rare to be able to explicitly compute a stock price that is not "trivial" (for example, equal to the amount of aggregate dividends) and exhibits interesting properties, as is the case here. In particular, heterogeneity in beliefs on monetary policy generates a much higher volatility for the stock. Real interest rates are also increased. The implications on the equity premium are ambiguous; nonetheless, for some plausible parameter values, the equity premium is much increased over a standard model.

The high level and the idiosyncratic, time-varying behavior of stock market volatility are one of the key puzzles facing financial economists (see, e.g., Schwert (1989)). As Engle and Rangel (2005) put it, "The number of models that have been developed to predict volatility based on time series information is astronomical, but models that incorporate economic variables are hard to find." Thus, highlighting the significant impact of heterogeneous beliefs on monetary policy on volatility may be the key empirical implication of our paper, and we perform an empirical analysis of this relationship. Using economists' forecasts as provided by the Survey of Professional Forecasters and S&P 500 returns, this prediction of our theoretical model is supported by the data, with a significant positive relationship between heterogeneity in beliefs and volatility. One interesting fact is that heterogeneity in beliefs on real output does not appear to have a significant impact on volatility; this suggests that our empirical results are not simply due to a higher level of uncertainty in the economy generating volatility, but that disagreement on the monetary side of the economy plays a specific, important role, as predicted by our model.

In addition to furthering our understanding of asset prices, our model provides a natural framework in which to assess the impact of the transparency of monetary policy, and how much transparency and openness on the part of central banks are optimal – a topical and controversial issue. Our model is particularly appropriate for the study of the effects of transparency, because it is intuitive that one of the key effects of increased transparency should be a drop in the amount of heterogeneity in beliefs: the more transparent monetary policy is, the less subjective prior beliefs enter forecasts, and the more similar individual forecasts should be. Hence, our model suggests that increased transparency could potentially, among other implications, reduce the volatility of inflation and stock prices, as well as the level of real interest rates.

The rest of the chapter is organized as follows. Section 3.2 describes our model, and Section 3.3 characterizes the equilibrium in a general setup. Section 3.4 and 3.5 are devoted to two special cases, where investors have, respectively, separable logarithmic and separable CRRA preferences. Section 3.6 provides our empirical analysis, Section 3.7 concludes, and the Appendix provides all proofs and empirical results.

3.2 The Economic Setup

We consider a continuous-time, pure exchange, finite horizon ($[0, T]$), economy populated with two investors $i = 1, 2$, possibly heterogeneous in their beliefs, preferences and endowments. The uncertainty is generated by a two-dimensional Brownian motion, $\omega = (\omega_\varepsilon, \omega_M)^T$. There is a single consumption good that serves as the numeraire. In addition to their consumption, agents derive utility from holding money balances, which captures in reduced form the transaction services rendered by money. We assume sufficient regularity for all stochastic differential equations and investors' optimization problems to have a solution.

3.2.1 The Aggregate Consumption, Money Supply and Information Structure

The aggregate endowment of consumption ε and the money supply M are assumed to be positive and to follow Ito processes:

$$d\varepsilon(t) = \varepsilon(t)\left[\mu_\varepsilon(t)\,dt + \sigma_\varepsilon(t)\,d\omega_\varepsilon(t)\right], \tag{3.1}$$

$$dM(t) = M(t)\left[\mu_M(t)\,dt + \sigma_{M\varepsilon}(t)\,d\omega_\varepsilon(t) + \sigma_{MM}(t)\,d\omega_M(t)\right], \tag{3.2}$$

where $\omega = (\omega_\varepsilon, \omega_M)^T$ is a standard 2-dimensional Brownian motion. While ω_ε (the consumption risk) and ω_M (the "pure" monetary risk) are independent, this formulation allows for correlation between aggregate consumption and money supply. The (instantaneous) correlation is given by $\rho(t) = \sigma_{M\varepsilon}(t) / \left(\sigma_{M\varepsilon}(t)^2 + \sigma_{MM}(t)^2\right)$

For simplicity, we sometimes denote the total volatility of the money supply by $\sigma_M(t) = \left(\sigma_{M\varepsilon}(t)^2 + \sigma_{MM}(t)^2\right)^{1/2}$.

The investors commonly observe the processes ε and M, but have incomplete information and heterogeneous beliefs on the dynamics of the money supply.[5] They observe the volatility coefficients $\sigma_{M\varepsilon}$, σ_{MM} (from the quadratic variation and covariation with ε), but must estimate μ_M. We denote agent i's estimate of μ_M by μ_M^i. Even though investors have symmetric information on μ_M, due to their heterogeneous prior beliefs they may disagree in their estimates of μ_M. We denote the (normalized) difference in their estimates by $\bar{\mu_M}(t) = \left(\mu_M^1(t) - \mu_M^2(t)\right)/\sigma_{MM}(t)$. Agent i's innovation process (or estimate for the monetary risk factor ω_M) is given by:

$$\omega_M^i(t) = \int_0^t \frac{1}{\sigma_{MM}(s)}\left(\frac{dM(s)}{M(s)} - \mu_M^i(s)\,ds - \sigma_{M\varepsilon}(s)\,d\omega_\varepsilon(s)\right), \tag{3.3}$$

the estimate that reconciles the investor's estimate of μ_M with his observation of the money supply. The investors' innovation processes are related by

$$d\omega_M^1(t) = d\omega_M^2(t) - \bar{\mu_M}(t)\,dt. \tag{3.4}$$

By Girsanov's theorem, ω_M^i is a Brownian motion under agent i's subjective beliefs, and his perceived dynamics for the money supply are as follows

$$dM(t) = M(t)\left[\mu_M^i(t)\,dt + \sigma_{M\varepsilon}(t)\,d\omega_\varepsilon(t) + \sigma_{MM}(t)\,d\omega_M^i(t)\right]. \tag{3.5}$$

[5]It would be easy to additionally incorporate heterogeneity in beliefs on the aggregate endowment growth. Here, we choose to focus on heterogeneity in beliefs on the money supply growth, a novelty of this work, and so, in order to not obscure its implications, we assume homogeneous beliefs on consumption growth. The effects of heterogeneous beliefs on aggregate consumption growth are well-understood (e.g., Basak (2005)).

It will be verified that, in equilibrium, the price of money p, expressed in units of the consumption good (the numeraire), follows an Ito process with dynamics ($i = 1, 2$):

$$\begin{aligned} dp(t) &= p(t)\left[\mu_p(t)\,dt + \sigma_{p\varepsilon}(t)\,d\omega_\varepsilon(t) + \sigma_{pM}(t)\,d\omega_M(t)\right], \\ &= p(t)\left[\mu_p^i(t)\,dt + \sigma_{p\varepsilon}(t)\,d\omega_\varepsilon(t) + \sigma_{pM}(t)\,d\omega_M^i(t)\right], \end{aligned} \tag{3.6}$$

under the "objective" probability and as perceived by the investors, respectively. The expected inflation, as perceived by investor i, is denoted by $\pi^i(t) = -\mu_p^i(t)$, consistent with the past literature.

3.2.2 Investment Opportunities

There are three securities available for continuous trading. The first is a zero-net supply, riskless (in real terms) bond paying-off the real interest rate r. Its price follows

$$dB(t) = B(t)\,r(t)\,dt. \tag{3.7}$$

There is also a zero-net supply, nominally riskless bond paying-off the nominal interest rate R. An application of Ito's lemma shows that the real price of the nominal bond, B_m, has dynamics ($i = 1, 2$):

$$\begin{aligned} dB_m(t) &= B_m(t)\begin{bmatrix} \left(\mu_p(t) + R(t)\right)dt + \sigma_{p\varepsilon}(t)\,d\omega_\varepsilon(t) \\ +\sigma_{pM}(t)\,d\omega_M(t) \end{bmatrix} \\ &= B_m(t)\begin{bmatrix} \left(\mu_p^i(t) + R(t)\right)dt + \sigma_{p\varepsilon}(t)\,d\omega_\varepsilon(t) \\ +\sigma_{pM}(t)\,d\omega_M^i(t) \end{bmatrix}. \end{aligned} \tag{3.8}$$

Thus, the nominal bond is risky in real terms. Finally, there is a risky stock representing a claim on the aggregate consumption ε, with a total supply of one share, whose price S has dynamics ($i = 1, 2$):

$$\begin{aligned} dS(t) + \varepsilon(t)\,dt &= S(t)\begin{bmatrix} \mu_S(t)\,dt + \sigma_{S\varepsilon}(t)\,d\omega_\varepsilon(t) \\ +\sigma_{SM}(t)\,d\omega_M(t) \end{bmatrix} \\ &= S(t)\begin{bmatrix} \mu_S^i(t)\,dt + \sigma_{S\varepsilon}(t)\,d\omega_\varepsilon(t) \\ +\sigma_{SM}(t)\,d\omega_M^i(t) \end{bmatrix}. \end{aligned} \tag{3.9}$$

Agents facing the same price processes S, B_m and equation (3.4) imply that the perceived expected returns are related by

$$\begin{aligned} \mu_p^1(t) - \mu_p^2(t) &= \sigma_{pM}(t)\,\bar{\mu_M}(t)\,, \\ \mu_S^1(t) - \mu_S^2(t) &= \sigma_{SM}(t)\,\bar{\mu_M}(t)\,. \end{aligned} \tag{3.10}$$

Assuming that the (endogenous) volatility coefficients $\sigma_{p\varepsilon}$, σ_{pM}, $\sigma_{S\varepsilon}$, σ_{SM} are nonzero, the market is complete and investor i faces unique market prices of risk, θ_ε^i and θ_M^i , associated with aggregate consumption and pure monetary uncertainty, respectively. θ_ε^i and θ_M^i solve:

$$\begin{pmatrix} \sigma_{p\varepsilon}(t) & \sigma_{pM}(t) \\ \sigma_{S\varepsilon}(t) & \sigma_{SM}(t) \end{pmatrix}\begin{pmatrix} \theta_\varepsilon^i(t) \\ \theta_M^i(t) \end{pmatrix} = \begin{pmatrix} \mu_p^i(t) + R(t) - r(t) \\ \mu_S^i(t) - r(t) \end{pmatrix}, \tag{3.11}$$

(3.11) implies that the investors only disagree on the market price of pure monetary risk, and face the same market price of consumption risk, $\theta_\varepsilon^1 = \theta_\varepsilon^2 \equiv \theta_\varepsilon$. Agent i's perceived state-price density is given by

$$d\xi^i(t) = -\xi^i(t)\left[r(t)\,dt + \theta_\varepsilon(t)\,d\omega_\varepsilon(t) + \theta_M^i(t)\,d\omega_M^i(t)\right]. \tag{3.12}$$

No-arbitrage implies that, under standard regularity conditions, the price of any asset is given by a present value formula. In particular, the stock and money prices are given by $(i = 1, 2)$

$$S(t) = \frac{1}{\xi^i(t)} E_t^i \left[\int_t^T \xi^i(s) \varepsilon(s) \, ds \right], \tag{3.13}$$

$$p(t) = \frac{1}{\xi^i(t)} E_t^i \left[\int_t^T \xi^i(s) R(s) p(s) \, ds \right], \tag{3.14}$$

where E_t^i denotes the time t conditional expectation under agent i's beliefs. The first equation is standard. The second is intuitive, if one thinks of one unit of money as an asset worth p, and paying-off a continuous dividend equal to the nominal interest rate, R (worth Rp in real terms).

3.2.3 Investors' Optimization

Investor i is endowed with $a^i > 0$ share of the stock, and $b^i > 0$ share of the money supply (with $a^1 + a^2 = b^1 + b^2 = 1$), so that his initial wealth is given by $W^i(0) = a^i S(0) + b^i p(0) M(0)$. He then chooses his consumption rate $c^i \geq 0$, money balance $m^i \geq 0$ and portfolio policy $\pi^i = (\pi_0^i, \pi_B^i, \pi_S^i)$, where π_0^i, π_B^i, and π_S^i denote the amounts of the numeraire invested by agent i in the real riskless bond, the nominal bond and the stock, respectively, so as to maximize his cumulated lifetime expected utility $E^i \left[\int_0^T u^i \left(c^i(t), p(t) m^i(t) \right) dt \right]$. Agent i's utility u^i of consumption and real money holding is assumed to be strictly increasing, strictly concave and three times continuously differentiable in both of its arguments, and to satisfy standard Inada conditions.[6] We define the

[6] That is, denoting the first derivatives of ui by u^i by $u_c^i \equiv \frac{\partial u^i}{\partial c^i}$, $u_m^i \equiv \frac{\partial u^i}{\partial m^i}$, $u_c^i(0) = u_m^i(0) = \infty$ and $u_c^i(\infty) = u_m^i(\infty) = 0$.

gradient by $Du^i \equiv \left(\frac{\partial u^i}{\partial c^i}, \frac{\partial u^i}{\partial m^i}\right)$; under our (standard) assumptions on u^i, it has an inverse, denoted by $J^i(.,.)$. An admissible policy (c^i, m^i, π^i) is one such that the resulting wealth process $W_i(t) = p(t)m^i(t) + \pi_0^i(t) + \pi_B^i(t) + \pi_S^i(t)$ is bounded from below, satisfies $W_i(T) \geq 0$ and obeys the dynamic budget constraint:

$$\begin{aligned} dW^i(t) &= \left[W^i(t) r(t) - c^i(t)\right] dt \qquad (3.15) \\ &+ \pi_B^i(t) \begin{bmatrix} \left(\mu_p^i(t) + R(t) - r(t)\right) dt \\ +\sigma_{p\varepsilon}(t) d\omega_\varepsilon(t) + \sigma_{pM}(t) d\omega_M^i(t) \end{bmatrix} \\ &+ \pi_S^i(t) \begin{bmatrix} \left(\mu_S^i(t) - r(t)\right) dt \\ +\sigma_{S\varepsilon}(t) d\omega_\varepsilon(t) + \sigma_{SM}(t) d\omega_M^i(t) \end{bmatrix} \\ &+ p(t) m^i(t) \begin{bmatrix} \left(\mu_p^i(t) - r(t)\right) dt \\ +\sigma_{p\varepsilon}(t) d\omega_\varepsilon(t) + \sigma_{pM}(t) d\omega_M^i(t) \end{bmatrix} \\ &+ p(t) M^i(t) \begin{bmatrix} \left(\mu_M^i(t) + \sigma_{pM}(t)\sigma_{MM}(t) + \sigma_{p\varepsilon}(t)\sigma_{M\varepsilon}(t)\right) dt \\ +\sigma_{M\varepsilon}(t) d\omega_\varepsilon(t) + \sigma_{MM\varepsilon}(t) d\omega_M^i(t) \end{bmatrix}. \end{aligned}$$

The first three lines of the dynamic budget constraint are standard, capturing the impact of consumption and gains from trading in securities. The fourth line is equal to $m^i(t)(dp(t) - p(t)r(t)dt)$ and accounts for the changes in the value of investor i's money holding due to inflation. The last line (equal to $b^i(t)(p(t) + dp(t)dM(t)))$ captures investor i's endowment of the money supply.

Standard martingale techniques (Cox and Huang (1989), Karatzas, Lehoczky and Shreve (1987)) imply that investor i's problem is equivalent to the following static problem:

$$\max_{c^i,m^i} E^i\left[\int_0^T u^i\left(c^i(t), p(t)\, m^i(t)\right) dt\right], \tag{3.16}$$

$$\begin{aligned} s.t.\ \ & E^i\left[\int_0^T \xi^i(t)\left(c^i(t) + R(t)\, p(t)\, m^i(t)\right) dt\right] \\ \leq\ & \xi^i(0)\, a^i S(0) + b^i E^i\left[\int_0^T \xi^i(t)\, R(t)\, p(t)\, M(t)\, dt\right]. \end{aligned} \tag{3.17}$$

The static budget constraint (3.17) takes into account, in the left hand side, the present value of the investor's lifetime consumption, as well as the cost of holding a money balance in the amount given by m^i; this cost is given by the interest that would be paid to borrow that amount (or, equivalently, the interest that is lost by holding a money balance instead of investing), i.e., Rm^i in units of money, or Rpm^i in units of consumption good. Similarly, receiving the endowment of money b^iM is equivalent to receiving a stochastic endowment at a rate of b^iRpM, the income that would be obtained by investing the endowment, and this stochastic endowment appears in the right hand side, together with the value of the stock that agent i is endowed with.

Proposition 1 characterizes investor i's optimal policy.

Proposition 1 *Investor i's optimal consumption $\hat{c}^i$ and money holding $\hat{m}^i$ solve*

$$u_c^i\left(\hat{c}^i(t),\ p(t)\,\hat{m}^i(t)\right) = y^i\xi^i(t), \tag{3.18}$$

$$u_m^i\left(\hat{c}^i(t),\ p(t)\,\hat{m}^i(t)\right) = y^i\xi^i(t)\, R(t), \tag{3.19}$$

where y^i is such that the investor's static budget constraint holds with equality, *i.e.,*

$$
\begin{aligned}
& E^i \left[\int_0^T \xi^i(t) \left(c^i(t) + R(t) p(t) m^i(t) \right) dt \right] \\
= \; & \xi^i(0) a^i S(0) + b^i E^i \left[\int_0^T \xi^i(t) R(t) p(t) M(t) dt \right]. \qquad (3.20)
\end{aligned}
$$

In addition to characterizing agent i's optimal consumption, equations (3.18)-(3.19) reveal that the nominal short rate is given by the ratio of the marginal utility of holding money to the marginal utility of consumption. This is intuitive: by holding the money balance m^i, agent i gives up investing this amount of money and receiving interest on it, at a rate of R, which is thus the implicit cost of holding money. At the optimum, this cost should be proportional to the marginal benefit of holding money, or the value of the services rendered by holding money.

The case of separable preferences. For tractability and clarity of our intuitions, we often make the assumption that both investors' preferences are separable with respect to consumption and money holdings, i.e.,

$$
u^i \left(c^i, pm^i \right) = v^{ic}(c) + v^{im}(pm), \qquad (3.21)
$$

and refer to this case as the "separable case".

3.3 Equilibrium

Definition 1 *(Competitive Equilibrium). An equilibrium is a price system* (r, R, S, p) *and admissible policies* (c^i, m^i, π^i), $i = 1, 2$ *such that: (i) agents* *choose their optimal policies given their beliefs; and (ii) markets for consumption, money and securities clear, i.e.,*

$$c^1(t) + c^2(t) = \varepsilon(t), \tag{3.22}$$

$$m^1(t) + m^2(t) = M(t), \tag{3.23}$$

$$\begin{aligned} \pi_B^1(t) + \pi_B^2(t) &= 0, \\ \pi_S^1(t) + \pi_S^2(t) &= S(t), \\ W^1(t) + W^2(t) &= S(t) + p(t) M(t). \end{aligned} \tag{3.24}$$

For convenience in the exposition of our results, we introduce a representative investor with utility defined by

$$U(c, pm; \lambda) \equiv \max_{c^1+c^2=c;\; m^1+m^2=m} u^1\left(c^1, pm^1\right) + \lambda u^2\left(c^2, pm^2\right), \tag{3.25}$$

where the relative weight $\lambda > 0$ is allowed to be stochastic. We denote the partial derivatives of $U_c \equiv \frac{\partial U}{\partial c}$, $U_{cc} \equiv \frac{\partial^2 U}{\partial c^2}$, etc.

Proposition 2 provides a characterization of equilibrium asset prices in our economy.

Proposition 2 *Assume that an equilibrium exists. Then, the investors' state-price densities are given by*

$$\begin{aligned} \xi^1(t) &= U_c(\varepsilon(t), p(t) M(t); \lambda(t)) \\ \xi^2(t) &= \frac{\lambda(0)}{\lambda(t)} \xi^1(t), \end{aligned} \tag{3.26}$$

and their consumptions and real money holdings by

$$\begin{aligned} \left(c^1(t), p(t)\, m^1(t)\right) &= J^1\left(U_c(.), U_m(.)\right), \\ \left(c^2(t), p(t)\, m^2(t)\right) &= J^1\left(\frac{U_c(.)}{\lambda(t)}, \frac{U_m(.)}{\lambda(t)}\right), \end{aligned} \tag{3.27}$$

where $U_c(.) \equiv U_c(\varepsilon(t), p(t) M(t);\ \lambda(t))$; $U_m(.) \equiv U_m(\varepsilon(t), p(t) M(t);\ \lambda(t))$ *and* λ *has dynamics*

$$\begin{aligned} \frac{d\lambda(t)}{\lambda(t)} &= -\bar{\mu_M}(t)\, d\omega_M^1(t) \\ &= -\bar{\mu_M}(t)\left(\frac{\mu_M(t) - \mu_M^1(t)}{\sigma_{MM}(t)} dt + d\omega_M(t)\right), \end{aligned} \tag{3.28}$$

and $\lambda(0) = 1/y^2$*, where* y^2 *satisfies investor 2's static budget constraint with equality (equation (3.17)).*[7]

The nominal interest rate and money price are given by

$$R(t) = \frac{U_m(\varepsilon(t), p(t) M(t); \lambda(t))}{U_c(\varepsilon(t), p(t) M(t); \lambda(t))}, \tag{3.29}$$

$$p(t) = E_t^i\left[\frac{\int_t^T U_m(\varepsilon(s), p(s) M(s); \lambda(s))\, p(s)\, ds}{U_c(\varepsilon(t), p(t) M(t); \lambda(t))}\right]. \tag{3.30}$$

The main difference with a standard model with homogeneous beliefs is that the two investors effectively face different state-prices densities; this is not surprising if one recalls the interpretation of a state-price density as a state-price per unit of probability, and that under heterogeneous beliefs the investors assign different probabilities to the possible states. Thus, the equilibrium allocation is not Pareto efficient, and only solves the representative agent's

[7]The investors' budget constraints are equivalent, and only determine the ratio y^1/y^2. Without loss of generality, we set $y^1 = 1$.

problem if the agents' relative weight λ is allowed to be stochastic (making the representative agent's preferences state-dependent) (Cuoco and He (1994)).

The relative weight of the two agents in the economy is driven by the heterogeneity in beliefs $\bar{\mu}_M$. The intuition for this is simple: the more optimistic investor (with the higher estimate for μ_M) invests in a portfolio that is more positively correlated with the money supply than the more pessimistic investor. Effectively, he is betting (against the pessimistic investor) that the money supply will grow by a lot. If the realization of the Brownian motion representing pure monetary risk is high, he wins his bet, and his weight in the economy increases, at the expense of the other investor. In this case, not only does he hold more money, he also consumes more, even if preferences are separable. This is in contrast with a model without heterogeneity in beliefs, where pure monetary risk would affect consumptions only indirectly, via the effect of money holdings on marginal utility of consumption (an effect that disappears in the separable case).

The nominal interest rate provides a measure of the value of the services rendered by holding money (relative to the marginal utility of consuming), and is driven primarily by the quantity of money in the economy, relative to aggregate consumption. While a similar interpretation holds for the price of money itself, its value takes into account the future "dividends" from holding money (i.e., the future transaction services rendered by money), and as a result its price follows a backward stochastic differential equation (3.30) that is difficult to analyze.

Proposition 3 provides more explicit characterization of the real asset prices.

Proposition 3 *In equilibrium, the market prices of aggregate consumption risk and pure monetary risk are given by*

$$\theta_{\varepsilon}(t) = A_c(t)\,\varepsilon(t)\,\sigma_{\varepsilon}(t) + A_m(t)\,p(t)\,M(t)\left(\sigma_{p\varepsilon}(t) + \sigma_{M\varepsilon}(t)\right), \tag{3.31}$$

$$\begin{aligned} \theta_M^1(t) &= A_m(t)\,p(t)\,M(t)\left(\sigma_{pM}(t) + \sigma_{MM}(t)\right) - A_{\lambda}(t)\,\lambda(t)\,\bar{\mu_M}(t), \\ \theta_M^2(t) &= \theta_M^1(t) - \bar{\mu_M}(t), \end{aligned} \tag{3.32}$$

where $A_j(t) \equiv -U_{cj}(\varepsilon(t), p(s)\,M(t); \lambda(t))/U_c(\varepsilon(t), p(t)\,M(t); \lambda(t))$, $j \in \{c, m, \lambda\}$.

The real interest rate is given by

$$\begin{aligned} r(t) &= -A_c(t)\,\varepsilon(t)\,\mu_{\varepsilon}(t) + A_m(t)\,p(t)\,M(t)\begin{pmatrix} \mu_M^1(t) + \sigma_{MM}(t)\,\sigma_{pM}(t) \\ +\mu_p^1(t) + \sigma_{p\varepsilon}(t)\,\sigma_{M\varepsilon}(t) \end{pmatrix} \\ &\quad + \frac{1}{2}B_{cc}\varepsilon(t)^2\,\sigma_{\varepsilon}(t)^2 + \frac{1}{2}B_{mm}\left(p(t)\,M(t)\right)^2\begin{pmatrix} \left(\sigma_{MM}(t) + \sigma_{pM}(t)\right)^2 \\ + \left(\sigma_{p\varepsilon}(t) + \sigma_{M\varepsilon}(t)\right)^2 \end{pmatrix} \\ &\quad + \frac{1}{2}B_{cm}\varepsilon(t)\,p(t)\,M(t)\,\sigma_{\varepsilon}(t)\left(\sigma_{p\varepsilon}(t) + \sigma_{M\varepsilon}(t)\right) \\ &\quad + \frac{1}{2}B_{\lambda\lambda}\left(\lambda(t)\,\bar{\mu_M}(t)\right)^2 - B_{m\lambda}\lambda(t)\,p(t)\,M(t)\left(\sigma_{MM}(t) + \sigma_{pM}(t)\right)\bar{\mu_M}(t), \end{aligned} \tag{3.33}$$

where $B_{jk}(t) \equiv -U_{cjk}(\varepsilon(t), p(t)\,M(t); \lambda(t))/U_c(\varepsilon(t), p(t)\,M(t); \lambda(t))$, $j \in \{c, m, \lambda\}$.

While the market price of aggregate consumption risk is only indirectly affected by the disagreement (via its effect on λ), the market prices of pure monetary risk are directly affected.

If for ease of exposition we assume that both investors' preferences are separable, implying that all the cross-partial derivatives U_{cm}, U_{cmm},U_{ccm}, and $U_{cm\lambda}$ are zero (as well as A_m), we note that the market price of pure monetary risk is only non-zero in the presence of heterogeneity in beliefs. This is intuitive: under homogeneous beliefs, a shock affecting only the money supply only has an impact on the investors' money balances, not on their consumptions, and so is not correlated with the marginal utility of consumption. Thus, exposure to such risks is not rewarded by a higher expected return. In the presence of heterogeneous beliefs, however, investors effectively place opposite bets on the evolution of the money supply. The investor who wins the bet receives wealth from the other investor, and his share of consumption increases. Thus, in this case pure monetary shocks impact agents' consumptions and so exposure to this type of risk is priced. The more optimistic agent facing a higher market price of monetary risk is enticed to invest in a portfolio that is more positively correlated with monetary uncertainty, which is the mechanism he uses to "bet" that the money supply will grow a lot. Under separable preferences, algebraic manipulation shows that $A_\lambda(t) = -A^1(t)/(\lambda(t)(A^1(t) + A^2(t)))$, where $A^i = -u^i_{cc}/u^i_c$ is agent i's absolute risk aversion, and so we have: $\theta^1_M(t) = \bar{\mu_M}(t)\, A^1(t)/((A^1(t) + A^2(t)))$, $\theta^2_M(t) = -\bar{\mu_M}(t)\, A^2(t)/((A^1(t) + A^2(t)))$. Under heterogeneity in beliefs, investors must take the opposite sides of the same bet, and so it is necessary to reward monetary risk-taking in proportion with their risk-aversions (to take a bet of the same size, the more risk-averse agent needs to be rewarded more).

When preferences are not separable, this interpretation remains essentially true, but there is an additional effect: independently of heterogeneity in beliefs, investors' marginal utilities of consumptions are affected by monetary shocks. This indirect effect of monetary shocks is captured by the terms in $A_m = -U_{cm}/U_c$: investors receive extra return to make up for their utility of consumption being affected by monetary shocks.

The real interest rate is also affected by the disagreement about expected monetary policy, which holds whether or not preferences are separable. If one recalls the familiar interpretation of the real interest rate as minus the instantaneous expected rate of change of marginal utility, this is not surprising. The heterogeneity in beliefs, via its impact on λ, increases the variability of consumption, because on top of the fundamental uncertainty, agents' consumptions are affected by their betting against each other. In the expression for the interest rate (3.33), the first three lines are almost as in a standard model (relative risk aversion times consumption growth plus one half times consumption variance times prudence), but made more complicated by the inclusion of monetary risk in addition to consumption risk. The terms in the last line, however, arise only under heterogeneity in beliefs, reflecting the impact of disagreement risk on the investors' precautionary savings motive: in general, the extra risk under heterogeneity in beliefs entices investors to save more, and the interest rate must be lowered to counteract this tendency (so that markets clear).

These results show that, under heterogeneous beliefs, there is a "spillover" effect of the monetary sphere into the real side of the economy, while models assuming homogeneous beliefs might erroneously conclude that there is no effect (with money being neutral under separable preferences) or only a limited, indirect effect. In later sections, we will be able to assess the size of this effect and its importance for asset pricing.

In general, it is not possible to solve explicitly for the price of money and the price of the stock (although this is possible in some setups that we will investigate in the next sections). It is possible, however, to provide meaningful expressions for the stock risk premium and the expected inflation. An alternative way to state the first result in Proposition 3 is an appropriately modified version of the familiar CCAPM:

$$\begin{aligned}
\mu_S^1(t) - r(t) &= A_c(t)\,\varepsilon(t)\,\sigma_\varepsilon(t)\,\sigma_{S\varepsilon}(t) \\
&\quad + A_m(t)\,p(t)\,M(t)\begin{pmatrix} \sigma_{S\varepsilon}(t)\left(\sigma_{p\varepsilon}(t) + \sigma_{M\varepsilon}(t)\right) \\ +\sigma_{SM}(t)\left(\sigma_{pM}(t) + \sigma_{MM}(t)\right) \end{pmatrix} \\
&\quad - A_\lambda(t)\,\lambda(t)\,\bar{\mu_M}(t)\,\sigma_{SM}(t)\,. \qquad (3.34)
\end{aligned}$$

The first term is standard, the second arises from correlations between the stock and the real value of money balances and disappears in the separable case. The last term reflects heterogeneity in beliefs: the more bullish agent 1 is about money supply growth relative to agent 2, the higher his perceived expected return for the stock (assuming positive correlation between stock and money supply). This is intuitive: the higher stock expected return entices him to invest more in it, making his portfolio more positively correlated with the money supply; this is how the more bullish investor "places his bet" that the money supply is going to grow a lot. This makes it clear that heterogeneity in beliefs on the money supply appears as an additional factor in the pricing of financial assets, even in the separable case. More interestingly, although it may not be obvious, a similar result applies to inflation. When its value is measured in real terms, the nominal riskless bond is a risky asset and its expected return must also be consistent with the CCAPM, which implies:

$$\begin{aligned}\pi^1(t) &= R(t) - r(t) - A_c(t)\,\varepsilon(t)\,\sigma_\varepsilon(t)\,\sigma_{p\varepsilon}(t) \\ &\quad -A_m(t)\,p(t)\,M(t)\begin{pmatrix}\sigma_{p\varepsilon}(t)\left(\sigma_{p\varepsilon}(t)+\sigma_{M\varepsilon}(t)\right)\\ +\sigma_{pM}(t)\left(\sigma_{pM}(t)+\sigma_{MM}(t)\right)\end{pmatrix} \\ &\quad +A_\lambda(t)\,\lambda(t)\,\bar{\mu_M}(t)\,\sigma_{pM}(t). \end{aligned} \tag{3.35}$$

This expression emphasizes that, due to the inflation risk that is borne by nominal bondholders, the nominal interest rate differs from the real riskless rate by more than expected inflation; it also includes a number of risk premia. The intuition on the effect of heterogeneity in beliefs on the stock return still applies here. It is, however, impossible to make an unambiguous prediction on the effect of heterogeneity in beliefs on inflation, because of its ambiguous effect on the real interest rate.

The most important insight of this section is that, under heterogeneous beliefs on monetary policy, agents place bets against each other on the money supply and so shocks in the money supply affect the distribution of wealth in the economy (proxied for by λ). Relative to the homogeneous beliefs case where the distribution of wealth is fixed, this generates extra risk for the investors (a particular kind of "trading risk"), leading to an extra factor in the pricing of assets, that subsists even when preferences are separable.

3.4 The Case of Separable Logarithmic Preferences

In this section, we make a couple of simplifying assumptions, Conditions 1 and 2 below. This allows us to explicitly solve all quantities in our economy.

Condition 1 *Both investors have separable, logarithmic preferences, i.e.* ($i = 1, 2$)

$$u^i\left(c^i, pm^i\right) = \phi \log\left(c^i\right) + (1-\phi)\log\left(pm^i\right), \tag{3.36}$$

where $\phi \in (0,1)$.

Condition 2 *All dynamics coefficients for the aggregate consumption and the money supply, as well as the two investors' estimates for these, are constant, i.e.,* μ_ε, σ_ε, μ_M, σ_{MM}, $\sigma_{M\varepsilon}$, μ_M^1, *and* μ_M^2 *are constant.*

Condition 1 implies that the representative agent utility function can be computed explicitly. We have:

$$\begin{aligned} U(c, pm; \lambda) \;\equiv\;& \phi\left[\log\left(\frac{c}{1+\lambda}\right) + \lambda\log\left(\frac{\lambda c}{1+\lambda}\right)\right] \\ &+ (1-\phi)\left[\log\left(\frac{pm}{1+\lambda}\right) + \lambda\log\left(\frac{\lambda pm}{1+\lambda}\right)\right]. \end{aligned} \tag{3.37}$$

Condition 2 implies that both the aggregate consumption and the money supply follow geometric Brownian motions, both under the objective probability and under each investor's beliefs. It also implies that the disagreement process, $\bar{\mu_M}$, is a constant as well. The fact that μ_M^1 and μ_M^2 are constant means that the investors do not update their beliefs as more observations of the money supply become available. This is not an appealing assumption, but adds considerable tractability and allows us to derive many explicit results. In addition, in real-life it is debatable whether most investors update their beliefs at all, and how fast they do it. Models with rational Bayesian updating in continuous-time seem to overstate the speed at which investors update their beliefs. Our case could be viewed as an approximation of a case where agents update slowly (or of an overlapping generations model where new, inexperienced investors continuously enter the market, leading to constant aggregate heterogeneity in beliefs in the economy). In fact, we can solve the model with Bayesian updating in the case where agents have normally distributed prior beliefs on μ_M (with an identical variance across agents, implying that $\bar{\mu_M}$ is deterministic); equilibrium expressions are affected and made more complicated, but our qualitative insights remain unchanged.[8]

[8]The model remains tractable because, under deterministic dynamics for $\bar{\mu_M}$, the stochastic weighting λ is lognormally distributed, as it is under Condition 2, and prices can still be computed explicitly.

3.4.1 Characterization of Equilibrium

Proposition 4 reports equilibrium allocations and prices under Conditions 1 and 2.

Proposition 4 *In equilibrium, the consumption and money holdings, state-price densities, money price and nominal interest rate are as follows:*

$$c^1(t) = \frac{\varepsilon(t)}{1+\lambda(t)}, \quad c^2(t) = \frac{\lambda(t)\varepsilon(t)}{1+\lambda(t)}, \tag{3.38}$$

$$m^1(t) = \frac{M(t)}{1+\lambda(t)}, \quad m^2(t) = \frac{\lambda(t) M(t)}{1+\lambda(t)}, \tag{3.39}$$

$$\xi^1(t) = \phi\frac{1+\lambda(t)}{\varepsilon(t)}, \quad \xi^2(t) = \phi\frac{\phi a^2 + (1-\phi) b^2}{\phi a^1 + (1-\phi) b^1}\frac{1+\lambda(t)}{\lambda(t)\varepsilon(t)}, \tag{3.40}$$

$$p(t) = \frac{1-\phi}{\phi}\frac{F^1(t) + \lambda(t) F^2(t)}{1+\lambda(t)}\frac{\varepsilon(t)}{M(t)}, \tag{3.41}$$

$$R(t) = \frac{1+\lambda(t)}{F^1(t) + \lambda(t) F^2(t)}, \tag{3.42}$$

where $(i = 1, 2)$

$$F^i(t) = \frac{\exp\left[\left(\sigma_M^2 - \mu_M^i\right)(T-t)\right] - 1}{\sigma_M^2 - \mu_M^i}, \tag{3.43}$$

and the weighting process is given by

$$\lambda(0) = \frac{\phi a^2 + (1-\phi) b^2}{\phi a^1 + (1-\phi) b^1}, \quad \frac{d\lambda(t)}{\lambda(t)} = -\bar{\mu}_M(t)\, d\omega_M^1(t). \tag{3.44}$$

Under logarithmic preferences, both investors' consumptions and money holdings are proportional to their shares of the wealth in the economy $(1/(1+\lambda(t))$ for agent 1 and $\lambda(t)/(1+\lambda(t))$ for agent 2). At time 0, the agents' weights in the economy are proportional to their initial endowments but, under heterogeneous beliefs, they evolve stochastically as agents place bets on the evolution of the money supply against each other, and the winning investor becomes wealthier.

One of the main difficulties in our model is that, in general, the price of money solves a backward stochastic differential equation (3.30) involving the future values of p. Such an equation typically is very difficult, if not impossible, to solve explicitly. The key simplification that makes the logarithmic case tractable is that, due to the properties of the logarithmic function, the price of money "separates out" of the utility of real money holdings. The marginal utility of real money holdings is inversely proportional to the money price p, and so the product $Rp = (U_m/U_c)/p$ does not depend on p. As a result, the future values of p disappear from the equation for p, and the computation can be performed explicitly.

Both money price and nominal interest rate are equal to weighted averages (more specifically, in the case of the nominal interest rate, a harmonic mean) of their values in an (otherwise identical) homogeneous beliefs economy populated only with agents of each type, 1 and 2; unsurprisingly, the weights used in this averaging are the agents' wealth shares $(1/(1+\lambda(t))$ and $\lambda(t)/(1+\lambda(t)))$. In the homogeneous beliefs economy, the price of money – or the present value of future services rendered by money, measured by the marginal utility of money holdings – is given by $p(t) = ((1-\phi)/\phi)\,\varepsilon(t)\,F^i(t)/M(t)$, where $F^i(t)$ is as in equation (3.43) or, equivalently, $F^i(t) = E^i[\int_t^T (1/M(s))ds]/(1/M(t))$. $F^i(t)$ can be viewed as a measure of the expected future marginal utilities from money holdings (relative to the current marginal utility). Unsurprisingly, the relative price of money is decreasing in the current quantity of money (and increasing in the aggregate endowment), but it is also decreasing in μ_M^i: the more the money supply is expected to grow in the future, the less valuable money is now.

The nominal short rate is the payoff an investor is willing to give up in order to hold a money balance, which equals, at the agent's optimum, the marginal utility of holding money relative to marginal utility of consumption. In the logarithmic case, we have $P(t) = ((1-\phi)/\phi)\,\varepsilon(t)/(p(t)\,M(t))$. Substituting $p(t)$, we see that, in the homogeneous belief economy with agents of type i, $R(t) = 1/F^i(t)$: the nominal interest rate is purely driven by expectations on the future growth of money, and is deterministic. While this may appear surprising, the real interest flow to a nominally riskless bond holder is given by $R(t)p(t) = ((1-\phi)/\phi)\,\varepsilon(t)/M(t)$; the nominal short rate makes up for fluctuations in the price of money that are due to future expectations of money growth and, as a result, the real return to nominal bondholders is independent of future expectations. In our heterogeneous beliefs economy, the nominal short rate is additionally affected by the distribution of wealth in the economy, which fluctuates in response to monetary shocks. Interestingly, despite the myopic behavior associated with logarithmic utility, future expectations of money supply growth play a key role in the pricing of monetary assets.

Real prices, however, are largely unaffected. The real short rate, stock price and stock dynamics are as in a standard economy:

$$\begin{aligned} r(t) &= \mu_\varepsilon - \sigma_\varepsilon^2, \quad S(t) = (T-t)\,\varepsilon(t), \\ \mu_S(t) &= \mu_\varepsilon, \quad \sigma_{S\varepsilon}(t) = \sigma_\varepsilon, \quad \sigma_{SM}(t) = 0. \end{aligned} \tag{3.45}$$

The market prices of risk, in contrast, are affected by heterogeneity in beliefs, and are as provided by Proposition 5.

Proposition 5 *In equilibrium, the market prices of risk are as follows:*

$$\begin{aligned} \theta_\varepsilon(t) &= \sigma_\varepsilon, \\ \theta_M^1(t) &= \frac{\lambda(t)}{1+\lambda(t)}\bar{\mu}_M, \quad \theta_M^2(t) = -\frac{1}{1+\lambda(t)}\bar{\mu}_M. \end{aligned} \tag{3.46}$$

While the market price of consumption risk remains as in a standard economy, due to the heterogeneity in beliefs, the market prices of monetary risk are individual-specific. The investor who expects a higher money supply growth rate faces a higher market price of monetary risk, as he expects any asset with a return positively correlated with the money supply to have a higher return than the other investor does. From the viewpoint of consumption smoothing, such an asset is not very valuable to him, because if the money supply increases by a lot, he will become wealthier and so he is not willing to pay a lot for this asset. Thus, he faces a high expected return for the asset. In the case of homogeneous beliefs, agents' shares of consumption are fixed, and pure monetary risk is not correlated with agents' consumptions. Thus, the market price of monetary risk is zero.

3.4.2 The Price of Money and Inflation

The dynamics of the price of money are as follows. The expected inflation is:

$$\begin{aligned}\pi^1(t) &= \mu_M^1 - \mu_\varepsilon - \sigma_M^2 + \rho\sigma_\varepsilon\sigma_M + \frac{1}{F^1(t) + \lambda(t)F^2(t)} * \\ &\quad \left(\begin{array}{c} 1+\lambda(t) \\ +\frac{F^2(t)-F^1(t)}{1+\lambda(t)}\lambda(t)\bar{\mu}_M\left(\frac{\lambda(t)}{1+\lambda(t)}\bar{\mu}_M - \sigma_M\right) \\ -\left(\mu_M^1 - \sigma_M^2\right)F^1(t) - \left(\mu_M^2 - \sigma_M^2\right)F^2(t)\lambda(t)\end{array}\right), \end{aligned} \tag{3.47}$$

and the diffusion coefficients are:

$$\begin{aligned}\sigma_{p\varepsilon}(t) &= \sigma_\varepsilon - \sigma_{M\varepsilon}, \\ \sigma_{pM}(t) &= -\left(\sigma_{MM} + \frac{F^1(t) - F^2(t)}{1+\lambda(t)}\frac{\lambda(t)}{1+\lambda(t)}\bar{\mu}_M\right).\end{aligned} \tag{3.48}$$

While its effect on expected inflation is ambiguous, the heterogeneity in beliefs is revealed to increase the volatility of the price of money. Assuming a positive correlation between money supply and economic activity ($\rho \geq 0$), the higher the heterogeneity in beliefs, the more the price of money drops in response to unexpected increases in the money supply (as the product $\left(F^1(t) - F^2(t)\right)\bar{\mu}_M(t)$ is positive, and increases with heterogeneity in beliefs). This is intuitive: when such a shock occurs, not only does the quantity of money increases, but the investor who expects higher money growth wins his bet, and so his weight in the economy increases; since he is also the one who places a lower value on the future services rendered by money (and thus expects higher inflation), the price of money decreases further. As is the case for most quantities in this economy, the price of money is essentially a weighted average of the two investors' private valuations; when a positive shock to the money supply occurs, both agents' valuations of money drop, and the weight of the lower of the two valuations increases. These two effects reinforce each other, leading to more volatile inflation.

3.4.3 Nominal Interest Rates

The dynamic behavior of the nominal short rate is dramatically affected by heterogeneity in beliefs as, in this setup, it would be deterministic under homogeneous beliefs. Its dynamics follow the Ito process: $dR(t) = \mu_R(t)\,dt + \sigma_R(t)\,dw_M^1(t)$, where

$$\begin{aligned}
\mu_R(t) &= R(t)^2 \frac{F^2(t)\left(F^2(t) - F^1(t)\right)}{\left(1+\lambda(t)\right)^3} \lambda(t)^2 \bar{\mu}_M^{\,2} \\
&\quad + \frac{R(t)^2}{1+\lambda(t)} \left\{ \begin{array}{c} \exp\left[\left(\sigma_M^2 - \mu_M^1\right)(T-t)\right] \\ +\lambda(t) \exp\left[\left(\sigma_M^2 - \mu_M^2\right)(T-t)\right] \end{array} \right\}, \qquad (3.49) \\
\sigma_R(t) &= \frac{F^2(t) - F^1(t)}{\left(F^1(t) + \lambda(t) F^2(t)\right)^2} \lambda(t) \bar{\mu}_M. \qquad (3.50)
\end{aligned}$$

The drift appears to be ambiguously related to heterogeneity in beliefs (and other variables), but could potentially generate interesting behavior on the part of the nominal interest rate. By substituting (3.42), we can express μ_R as a function of R itself; for some parameter values, this function is decreasing, meaning that our model can generate mean-reversion of the nominal interest rate. The diffusion coefficient reveals that the nominal short rate is positively correlated with pure monetary uncertainty, and its volatility is made stochastic and increased by the heterogeneity in beliefs. When a positive shock occurs to the money supply, the price of money drops as the investor expecting higher money growth becomes more wealthy, and the nominal interest rate increases to make up for this, and ensure that the real interest received from nominal bonds (Rp) remains commensurate with the quantity of money in the economy. In our model, the real return from nominally riskless bonds is negatively correlated with money supply and inflation, as is consistent with both intuition and empirical studies (see, e.g., Bakshi and Chen (1996) and the references therein).

In this separable logarithmic case, nominal zero-coupon yields for all future maturities can be computed explicitly. Proposition 6 characterizes the term structure in our economy.

Proposition 6 *The nominal time t-price of a nominal discount bond with maturity τ, paying one unit of money at time $t+\tau$, is given by:*

$$B_m(t,\tau) = \frac{1}{F^1(t) + \lambda(t) F^2(t)} \left\{ \begin{array}{c} F^1(t+\tau) \exp\left[\left(\sigma_M^2 - \mu_M^1\right)\tau\right] \\ +\lambda(t) F^2(t+\tau) \exp\left[\left(\sigma_M^2 - \mu_M^2\right)\tau\right] \end{array} \right\}. \tag{3.51}$$

The zero-coupon yield with maturity is given by: $R(t,\tau) = -\frac{1}{\tau} \ln B_m(t,\tau)$.

It is difficult to provide general results on the term structure in this model, and how it is affected by heterogeneity in beliefs, in particular because when heterogeneity in beliefs increases, it could be either that the more optimistic agent becomes more optimistic, or the less optimistic agent becomes even less optimistic, or both, and the effect on the term structure is different in each case.

Accordingly, most comparative statics are ambiguous. Nonetheless, numerical analyses suggest that, when the more optimistic agent is more wealthy, nominal zero-coupon yields increase with heterogeneity in beliefs, and when he is less wealthy, yields decrease with heterogeneity in beliefs. This can be explained intuitively. We can rewrite the price of a discount bond as follows:

$$B_m(t,\tau) = \frac{M(t)}{F^1(t) + \lambda(t) F^2(t)} \left\{ \begin{array}{c} F^1(t+\tau) E_t^1\left[\frac{1}{M(t+\tau)}\right] \\ +\lambda(t) F^2(t+\tau) E_t^2\left[\frac{1}{M(t+\tau)}\right] \end{array} \right\}. \quad (3.52)$$

In this setup, real interest rates are constant, and so the only type of risk affecting these nominal bonds is that related to money supply growth. Assume that agent 1 is more optimistic. If he is more wealthy (low λ), the quantity that plays the most important role in pricing the bond is $E_t^1[1/M(t+\tau)]$, driven by his expectation of money supply growth μ_M^1. When heterogeneity in beliefs increases, μ_M^1 generally increases, $E_t^1[1/M(t+\tau)]$ decreases, and agent 1's valuation of the bond's payoff (one unit of money at time $t+\tau$) decreases: since there is expected to be more money available in the future, this payoff is less valuable, and the bond's price drops, leading to an increase in its yield. A symmetric argument applies when the more pessimistic agent is more wealthy, and thus plays a dominant role in setting bond prices. When heterogeneity in beliefs increases and his expectation of money supply drops, his valuation of future nominal payoffs increases and yields decrease. As is the case for other quantities in this economy, the price of a discount bond is essentially a weighted average of the two agents' valuations of the corresponding payoffs. Other observations that can be made based on numerical simulations are that the yield curve is generally increasing, and that the higher the heterogeneity in beliefs, the less steep its slope is (to the point that, in some cases, the yield curve can be decreasing for high heterogeneity in beliefs).

3.5 The Case of Separable Power Preferences

In this Section, we maintain Condition 2, but replace Condition 1 with Condition 3. While an explicit computation of all monetary quantities is not possible any more, interesting implications for real asset prices can be derived.

Condition 3 *Both investors have separable, constant relative risk aversion preferences ($i = 1, 2$):*

$$u^i\left(c^i; pm^i\right) = \phi \frac{\left(c^i\right)^{1-\alpha}}{1-\alpha} + (1-\phi)\frac{\left(pm^i\right)^{1-\beta}}{1-\beta}, \tag{3.53}$$

where α, $\beta > 1$ and $\phi \in (0, 1)$.

Condition 3 implies that the representative agent utility function is given by

$$U(c, pm; \lambda) = \phi\left(1 + \lambda^{\frac{1}{\alpha}}\right)^{\alpha} \frac{c^{1-\alpha}}{1-\alpha} + (1-\phi)\left(1 + \lambda^{\frac{1}{\beta}}\right)^{\beta} \frac{(pm)^{1-\beta}}{1-\beta}. \tag{3.54}$$

3.5.1 Characterization of Equilibrium

Proposition 7 characterizes the equilibrium.

Proposition 7 *In equilibrium, the consumption and money holdings, state-*

price densities, money price and nominal interest rate are as follows:

$$c^1(t) = \frac{\varepsilon(t)}{1+\lambda(t)^{\frac{1}{\alpha}}}, \quad c^2(t) = \frac{\lambda(t)^{\frac{1}{\alpha}}\varepsilon(t)}{1+\lambda(t)^{\frac{1}{\alpha}}}, \tag{3.55}$$

$$m^1(t) = \frac{M(t)}{1+\lambda(t)^{\frac{1}{\beta}}}, \quad m^2(t) = \frac{\lambda(t)^{\frac{1}{\beta}}M(t)}{1+\lambda(t)^{\frac{1}{\beta}}}, \tag{3.56}$$

$$\xi^1(t) = \frac{\phi\left(1+\lambda(t)^{\frac{1}{\alpha}}\right)^{\alpha}}{\varepsilon(t)^{\alpha}}, \quad \xi^2(t) = \frac{\lambda(0)}{\lambda(t)}\xi^1(t), \tag{3.57}$$

$$\begin{aligned} p(t) &= \frac{\phi}{1-\phi}\left(\frac{\varepsilon(t)}{1+\lambda(t)^{\frac{1}{\alpha}}}\right)^{\alpha} \\ &\quad * E_t^1\left[\int_t^T \left(\frac{1+\lambda(s)^{\frac{1}{\alpha}}}{M(s)}\right)^{\beta} p(s)^{1-\beta}\, ds\right], \end{aligned} \tag{3.58}$$

$$R(t) = \frac{1-\phi}{\phi}\frac{\left(1+\lambda(t)^{\frac{1}{\beta}}\right)^{\beta}}{\left(1+\lambda(t)^{\frac{1}{\alpha}}\right)^{\alpha}}\frac{\varepsilon(t)^{\alpha}}{(p(t)M(t))^{\beta}}, \tag{3.59}$$

where the weighting process is given by

$$\frac{d\lambda(t)}{\lambda(t)} = -\bar{\mu}_M(t)\, d\omega_M^1(t), \tag{3.60}$$

and $\lambda(0)$ solves agent 2's static budget constraint:

$$\begin{aligned} & E^2\left[\int_0^T \xi^2(t)\left(c^2(t) + R(t)p(t)m^2(t)\right)dt\right] \\ = \ & \xi^2(0)a^2S(0) + b^2E^2\left[\int_0^T \xi^2(t)R(t)p(t)M(t)\,dt\right]. \end{aligned} \tag{3.61}$$

Consumption and money balances are somewhat similar to the logarithmic case: agents share aggregate consumption and money supply according to their weights in the economy ($1/\left(1+\lambda(t)^{1/\alpha}\right)$ and $\lambda(t)^{1/\alpha}/\left(1+\lambda(t)^{1/\alpha}\right)$), that evolve stochastically, depending on the behavior of the money supply, and whose beliefs prove to be the most accurate. The sharing is not directly proportional to wealth, however, but depends on relative risk aversions. Unlike in the logarithmic case, the price of money does not "separate out" of the utility; the marginal utility of money holdings depends on the price of money, so does the real value of the interest received from nominal short term bonds, and future values of p appear in the equation for the price of money (3.58), which cannot be solved explicitly. Similarly, the price of money affects the nominal interest rate and as a result, none of the monetary asset prices can be solved in closed-form. Real asset prices, however, can be solved explicitly. Proposition 8 provides the market prices of risk and the real short rate.

Proposition 8 *In equilibrium, the market prices of risk are as follows*

$$\begin{aligned} \theta_\varepsilon(t) &= \alpha\sigma_\varepsilon \\ \theta_M^1(t) &= \frac{\lambda(t)^{\frac{1}{\alpha}}}{1+\lambda(t)^{\frac{1}{\alpha}}}\bar{\mu}_M, \qquad \theta_M^2(t) = -\frac{1}{1+\lambda(t)^{\frac{1}{\alpha}}}\bar{\mu}_M, \end{aligned} \tag{3.62}$$

and the real interest rate is given by

$$r(t) = \alpha\mu_\varepsilon - \frac{1}{2}\alpha(1+\alpha)\sigma_\varepsilon^2 + \frac{1}{2}\frac{\alpha-1}{\alpha}\frac{\lambda(t)^{\frac{1}{\alpha}}}{\left(1+\lambda(t)^{\frac{1}{\alpha}}\right)^2}\bar{\mu}_M^2. \tag{3.63}$$

As in the logarithmic preferences case, the market price of consumption risk is as in a standard economy, but the market prices of monetary risk are individual-specific; the investor who expects higher money supply growth also faces a higher market price of monetary risk. Unlike in the logarithmic case, however, the interest rate is also affected, and is increased by the heterogeneity in beliefs. This is due to the investors' subjective assessment of their consumption growth being increased by the possibility to "bet" against one another:

the higher the heterogeneity in beliefs, the more each agent believes the other is "wrong", and the more he expects to win from his bet, raising his expected consumption growth and, correspondingly, the real interest rate. Mathematically, one can see from equation (3.55) that expected consumption increases with heterogeneity in beliefs: for example, c^1 is convex in λ, and so expected consumption increases under higher volatility for the stochastic weighting.

3.5.2 Stock Price, Stock Volatility and Equity Premium

For the remainder of this section, we assume, with little loss of generality, that the investors' relative risk aversion, α, is an integer.[9] This makes an explicit computation of the stock price possible, as reported in Proposition 9.

Proposition 9 *If is an integer, the stock price is given by*

$$S(t) = \frac{\varepsilon(t)}{\left(1+\lambda(t)^{\frac{1}{\alpha}}\right)^{\alpha}} \sum_{i=0}^{a} \binom{i}{a} \lambda(t)^{\frac{i}{\alpha}} \frac{\exp\{f(i)(T-t)\}-1}{f(i)}, \tag{3.64}$$

where $\binom{i}{a}$ *denotes the binomial coefficient* $a!/(a-i)!$ *and* $f(i) \equiv (1-\alpha)\left(\mu_\varepsilon \frac{1}{2}\alpha\sigma_\varepsilon^2\right)$ $-\frac{\alpha i - i^2}{2\alpha^2}\bar{\mu}_M^{\,2}$. *The stock volatility coefficients are given by*

$$\begin{aligned}
\sigma_{S\varepsilon}(t) &= \sigma_\varepsilon, \\
\sigma_{SM}(t) &= \left(\frac{\lambda(t)^{\frac{1}{\alpha}}}{1+\lambda(t)^{\frac{1}{\alpha}}} - \frac{\sum_{i=0}^{a} \frac{i}{\alpha}\binom{i}{a}\lambda(t)^{\frac{i}{\alpha}} \frac{\exp\{f(i)(T-t)\}-1}{f(i)}}{\sum_{i=0}^{a} \binom{i}{a}\lambda(t)^{\frac{i}{\alpha}} \frac{\exp\{f(i)(T-t)\}-1}{f(i)}} \right) \bar{\mu}_M.
\end{aligned} \tag{3.65}$$

The stock price is reduced by the presence of heterogeneity in beliefs. The intuition is similar to that for the increased real interest rate: both agents

[9] In the case where is not an integer, one could use a Taylor series approximation around the results presented here.

expect to profit from the heterogeneity in beliefs and "win their bets", leading to higher expected consumption growth, and a lower value placed on future dividends; hence a decrease in the stock price. The total stock volatility $\left(\sigma_{S\varepsilon}(t)^2 + \sigma_{SM}(t)\right)^{1/2}$ is unambiguously raised by the heterogeneity in beliefs. Under homogeneous beliefs, the stock volatility would equal that of aggregate consumption, but the heterogeneity in beliefs generates stochastic volatility, as the stock price becomes affected by pure monetary risk. Even though this risk is uncorrelated with aggregate consumption and dividends, as agents bet against each other, "trading risk" affects consumptions, stochastic discount factors and eventually the stock price. The sign of σ_{SM} depends on which investor, the more optimistic or the more pessimistic, is the more wealthy. If the more optimistic investor is more wealthy, $\sigma_{SM} > 0$, and conversely when the more pessimistic agent is more wealthy. The less wealthy investor, due to his lower consumption and much steeper marginal utility is the one driving the stock price volatility, despite the fact that his weight in the economy is lower (for relative risk aversion more than one). When he is "right" about the money supply (e.g., he is relatively optimistic and there is a positive shock in the money supply), his expected future consumptions increase, leading to a decreased valuation for future dividends and a lower stock price. The stock return in our model could be either positively or negatively correlated with the money supply, but is more likely to be positively correlated with it (which is consistent with empirical evidence), as

$$cov\left(dS/S,\ dM/M\right) = \sigma_M \left(\rho\sigma_{S\varepsilon}\left(t\right) + \left(1-\rho^2\right)^{1/2}\sigma_{SM}\left(t\right)\right) dt.$$

The magnitude of this effect is significant (although, most likely, not sufficient to explain the stock volatility levels observed in real-life markets). For example, assuming the following parameter values: $\sigma_M = 3.5355\%$, $\mu_\varepsilon = 2.06\%$, $\sigma_\varepsilon = 1.49\%$ (all as in Basak and Gallmeyer (1999)), $\rho = 0.4$, $T = 50$ years, $\alpha = 2$, $\lambda = 1/3$ (i.e., the more optimistic investor is more wealthy, given that $\bar{\mu_M} > 0$), for $\bar{\mu_M} = 1$, the total stock volatility is 7.76% (vs. 1.49% in the absence of heterogeneity in beliefs). This amount of heterogeneity in beliefs seems plausible and not an extreme case: $\bar{\mu_M} = 1$ means that $\mu_M^1 - \mu_M^2 = 3.24\%$, a value that should be compared to a historical average of $\mu_M = 4.89\%$ (as reported in Basak and Gallmeyer (1999)), and that is sufficient to cause a five-fold increase in stock volatility.

To study the stock expected return, it is more intuitive to focus on the equity premium, which follows readily from our earlier results. Adopting the "full information" perspective of an observer who would know the true expected money supply growth,[10] the equity premium equals

$$\mu_S\left(t\right) - r\left(t\right) = \alpha\sigma_\varepsilon^2 + \frac{\sigma_{SM}\left(t\right)}{\sigma_{MM}}\left\{\mu_M - \left[\frac{1}{1+\lambda\left(t\right)^{\frac{1}{\alpha}}}\mu_M^1 + \frac{\lambda\left(t\right)^{\frac{1}{\alpha}}}{1+\lambda\left(t\right)^{\frac{1}{\alpha}}}\mu_M^1\right]\right\}. \tag{3.66}$$

The first term provides compensation for consumption risk, and equals the value of the equity premium in a standard economy without heterogeneity in beliefs. The term in the square bracket is essentially a wealth-weighted average

[10] In other words, we compute the stock expected return under the objective, "historical" probability measure that generates actual data, as opposed to the agents' subjective beliefs. This is the value of the equity premium that should be taken to the data to test our model.

of the agents' beliefs on money supply growth. Thus, how the equity premium in our model departs from this standard value depends on two things, the sign of $\sigma_{SM}(t)$ (studied above) on the one hand, and the difference between the average beliefs of the investors and the true value of money supply growth on the other hand. The equity premium is higher than in a standard economy if either the investors who are optimistic on money supply growth are more wealthy (so that $\sigma_{SM} > 0$) and the wealth-weighted average beliefs on μ_M are lower than the true value, or if the pessimistic investors are more wealthy and the average beliefs are higher. This is intuitive. Take the case where $\sigma_{SM} > 0$. If the agents' average beliefs are lower than the true value of μ_M, it is more likely that agents will see (given their overly pessimistic beliefs) positive shocks to the money supply, leading to increases in the stock price. And conversely when $\sigma_{SM} < 0$. Thus, our model could potentially play a role in explaining the equity premium puzzle, as the second term in the right hand-side of (3.66) is typically of a much higher magnitude than the first, standard term. For example, for the above parameters, assuming that the more optimistic agents have correct beliefs (i.e., $\mu_M^1 = \mu_M = 4.89\%$), the heterogeneity in beliefs makes the equity premium more than 70 times higher than in a standard model (2.83% vs. 0.04%).

3.6 Empirical Analysis

In this Section, we empirically examine one of the main implications of our model, the impact of heterogeneous beliefs on money growth on the stock market volatility.

3.6.1 Data

Since data on heterogeneity in beliefs is not readily available, we use the Survey of Professional Forecasters (SPF), published by the Philadelphia Fed, to construct a quarterly index of heterogeneity in beliefs. The SPF provides forecasts by 30 economists (on average) of various economic variables (including real and nominal GDP, 3-month T-Bill rate, etc.) Since these do not include money supply growth, we use changes in the nominal GDP as a proxy.

This approximation can be justified by invoking the quantity theory of money: according to this theory, we have $MV = PY$, where M denotes the money supply, V is the velocity of money, P the price of goods and services and Y the real output. If we assume the velocity of money to be constant, an assumption typical in the macroeconomic literature (Ball, Mankiw and Romer (1988)), growth in nominal output (PY) is equal to money supply growth. To further verify that we are testing the effect of disagreement on money supply (rather than disagreement on output), we will also investigate the effect of disagreement on real output; if the results are significantly different from the effect of disagreement on nominal output, it must be that there is disagreement related to the monetary side of the economy, and that the effect of this disagreement is significant.

Our index of heterogeneity in beliefs is constructed as follows. Given individual i's forecast for the nominal GDP, we can calculate its growth rate, $GNGDP_t^i$ and the median $GNGDP_t^m$ of all forecasts. If $GNGDP_t^i$ is greater than $GNGDP_t^m$, we call individual i "optimistic"; if not, he is "pessimistic". We calculate the median forecast for all optimistic (pessimistic) investors' forecasts, denoted by $GNGDP_t^O$ ($GNGDP_t^P$). The difference in beliefs index is then given by $DB_{M,t} = GNGDP_t^O - GNGDP_t^P$ (proxy for $\mu_M^1 - \mu_M^2$ in our theoretical model).

We conduct our analysis (with quarterly data) for the period lasting from the first quarter of 1988 (in order for our results not to be contaminated by the 1987 stock market crash) to the last quarter of 2005. Figure 1 provides the evolution of our index of heterogeneity in beliefs over this period. We see that there are three peaks during the period: 1988, 1990-1991 and 2002. These dates loosely correspond to recessions in the US economy. This is intuitive: the economy is more uncertain during recessions.

INSERT FIGURE 1 HERE

The quarterly stock market volatility is given by

$$s_t = \left(\sum_{d=1}^{90} \left(r_{d,t} - \bar{r}_t \right)^2 \right)^{1/2}, \tag{3.67}$$

where $\bar{r}_t = \frac{1}{90} \sum_{d=1}^{90} r_{d,t}$ and $r_{d,t}$ denotes the daily S&P 500 return.

3.6.2 Model Specification

According to our theoretical model, the stock volatility is given by $\left({\sigma_{S\varepsilon}}^2 + \sigma_{SM}\right)^{1/2}$, where $\sigma_{S\varepsilon}$ and σ_{SM} are as in equation (3.65). Since we expect the effect of consumption volatility ($\sigma_{S\varepsilon}$) to be relatively small relative to that of heterogeneous beliefs, and σ_{SM} is linear in heterogeneity in beliefs (equation (3.65)), a linear regression model should capture well the effect of heterogeneous beliefs on stock volatility. Thus, we propose the following model specification:

$$s_t = a + bDB_{M,t} + e_t \qquad (Model\ 1)\,, \tag{3.68}$$

To check that our results are not due mainly to the presence of heterogeneous beliefs on real output, we also perform the following regressions involving $DB_{R,t}$, an index of difference in beliefs on real output, constructed using the same procedure as $DB_{M,t}$ above (also using SPF data):

$$s_t = a + bDB_{R,t} + e_t \qquad (Model\ 2)\,, \tag{3.69}$$

$$s_t = a + bDB_{M,t} + cDB_{R,t} + e_t \qquad (Model\ 3)\,. \tag{3.70}$$

3.6.3 Results

Our regression results are provided in the Table I. We see that the results of the Model 1 regression are consistent with our theoretical model, in that heterogeneity in beliefs on money growth has a positive impact on stock market volatility, and that this effect is highly significant (at the 0% significance level). The R^2, however, is relatively low, suggesting that heterogeneity in beliefs only accounts for a limited share of stock volatility.

INSERT TABLE 1 HERE

The results of Model 2 and Model 3 further underscore the role of heterogeneous beliefs on money growth in explaining stock volatility: performing a regression of volatility on disagreement on real output (Model 2) does not yield a significant coefficient. This suggests that heterogeneous beliefs on money growth play a specific role, and that our results are not simply due to a general higher level of uncertainty and disagreement in the economy; if this were the case, Model 2 would also yield a significant coefficient. Furthermore, since the effect of disagreement on nominal output is significant while that of disagreement on real output is not, this suggests that what drives our results is not disagreement on output itself, but rather disagreement on the difference between nominal and real output, the price of money, which is directly related to money growth. This provides some justification for our choice of disagreement on nominal output as a proxy for disagreement on money growth. When volatility is regressed jointly on $DB_{M,t}$ and $DB_{R,t}$ (Model 3), the results of Model 1 are hardly affected. The effect of disagreement on money growth remains significant and its coefficient changes very little, while the effect of disagreement on real output remains insignificant.[11]

While these results are no doubt preliminary, they suggest that disagreement on money growth plays a significant, robust role in generating stock mar-

[11] Other robustness checks that were performed included: splitting the 1988-2005 into two subperiods; and using differences of opinion on earnings as an additional independent variable. These analyses had little impact on our results.

ket volatility. More interestingly perhaps, this role seems specific to money growth, in that disagreement on real output growth does not appear to play such a role.

3.7 Conclusion

This paper investigates the effects of heterogeneity in beliefs on future monetary policy on the pricing of money and financial assets. Under heterogeneous beliefs, investors use security markets to place bets on the future money supply against each other; thus, the weights of the two agents in the economy evolve stochastically, depending on which agent proves the better forecaster. This has profound implications on asset pricing, because it makes agents' consumptions and stochastic discount factors more volatile and correlated with the money supply (and in some cases, can break the neutrality of money). In particular, inflation is made more volatile: when a positive shock affects the money supply, not only does the increase generate inflation, but it also gives more weight to those agents who had higher expectations for money supply growth (as they win their bet) and so expected higher inflation in the first place; this generates yet extra inflation. Stock prices are also affected. In particular, the stock volatility is significantly increased by the heterogeneity in beliefs, as it creates a much greater dependence between monetary policy and the stock market. This conclusion is supported by our empirical analysis.

On top of its implications for asset pricing, our model sheds some new light on the issue of how much transparency is optimal on the part of central banks, because it is intuitive that greater transparency should decrease the amount of heterogeneity in beliefs on monetary policy, which, according to our model, would reduce stock market and inflation volatility (as well as decrease real interest rates). Natural extensions of our model would include assuming more sophisticated and realistic dynamics for the aggregate consumption and money supply (with mean-reversion and/or regime switches in monetary policy), taking into account agents updating their beliefs over time, and a more thorough investigation of the implications for the price of money (that obeys a backward stochastic differential equation that is, in general, quite intractable) and interest rates. With more realistic, less tractable assumptions, our model could complement the growing, recent literature on the interrelation between monetary policy and bond yields (e.g., Piazzesi (2005)). As far as empirical work is concerned, a particularly interesting investigation could consist in performing international comparisons taking into account differences in the transparency of monetary policy, which is likely to lead to reduced heterogeneity in beliefs; our model provides tools to understand the impact of these differences.

Proof of Proposition 1: The proof is an adaptation of Cox and Huang (1989), Karatzas, Lehoczky and Shreve (1987) to the MIUF case, and is similar to Basak and Gallmeyer (1999). Q.E.D.

Proof of Proposition 2: It can be easily checked that, for $\lambda = u_c^1/u_c^2$, the solution of the representative agent's problem coincides with the equilibrium allocation (as agents 1 and 2's optimally conditions and clearing in the good market hold). The stochastic weighting dynamics (3.28) then follow from applying Ito's lemma to the state-price density dynamics (3.12). The envelope theorem (applied to the optimization problem in the definition of the representative agent (3.25)) shows that $U_c = u_c^1 = y^1\xi$, and the expressions for the state-price densities follow. The investors' consumptions and money holdings obtain by applying Proposition 1, (3.29) and (3.30) follow, respectively, from agents' first-order conditions and from (3.13) and (3.14). Q.E.D.

Proof of Proposition 3: The expressions follow from applying Ito's lemma to the state-price density in (3.28). Q.E.D.

Proof of Proposition 4: The expressions obtain readily by substituting logarithmic utility into the expressions in Proposition 2. Q.E.D.

Proof of Proposition 5: The expressions follow from applying Ito's lemma to the state-price density in (3.40). Q.E.D.

Proof of Proposition 6: The bond pays one unit of money, worth $p(t+\tau)$ in real terms, at time $t+\tau$. Applying the standard present value formula, its nominal price is given by:

$$\begin{aligned} B_m(t,\tau) &= \frac{1}{p(t)} E_t^1 \left[\frac{\xi^1(t+\tau)}{\xi^1(t)} p(t+\tau) \right] \\ &= \frac{M(t)}{F^1(t) + \lambda(t) F^2(t)} \left\{ \begin{array}{c} F^1(t+\tau) E_t^1 \left[\frac{1}{M(t+\tau)} \right] \\ +\lambda(t) F^2(t+\tau) E_t^2 \left[\frac{1}{M(t+\tau)} \right] \end{array} \right\}, \end{aligned}$$

where the second equality follows by substituting the results in Proposition 4. The expectations can be computed explicitly due to both $1/M$ and λ/M being lognormally distributed. Q.E.D.

Proof of Proposition 7: The expressions obtain readily by substituting power utility into the expressions in Proposition 2. Q.E.D.

Proof of Proposition 8: The expressions follow from applying Ito's lemma to the state-price density in (3.57). Q.E.D.

Proof of Proposition 9: The stock price obeys a standard present value formula:

$$\begin{aligned} S(t) &= \frac{1}{\xi^1(t)} E_t^1 \left[\int_t^T \xi^1(s) \varepsilon(s) \, ds \right] \\ &= \frac{\varepsilon(t)^\alpha}{\left(1 + \lambda(t)^{\frac{1}{\alpha}}\right)^\alpha} E_t^1 \left[\int_t^T \left(1 + \lambda(s)^{\frac{1}{\alpha}}\right)^\alpha \varepsilon(s)^{1-\alpha} \, ds \right] \\ &= \frac{\varepsilon(t)^\alpha}{\left(1 + \lambda(t)^{\frac{1}{\alpha}}\right)^\alpha} \int_t^T E_t^1 \left[\left(1 + \lambda(s)^{\frac{1}{\alpha}}\right)^\alpha \right] E_t^1 \left[\varepsilon(s)^{1-\alpha} \right] ds, \end{aligned}$$

where the second equality follows from substituting the results in Proposition 7, and the third equality from Fubini's theorem, and the fact that λ and ε are independent. Using the fact that, when α is an integer,

$$\left(1 + \lambda(s)^{\frac{1}{\alpha}}\right)^\alpha = \sum_{i=0}^{a} \binom{i}{a} \lambda(s)^{\frac{i}{\alpha}},$$

algebraic manipulation yields (3.64). Applying Ito's lemma then yields the volatility coefficients. Q.E.D.

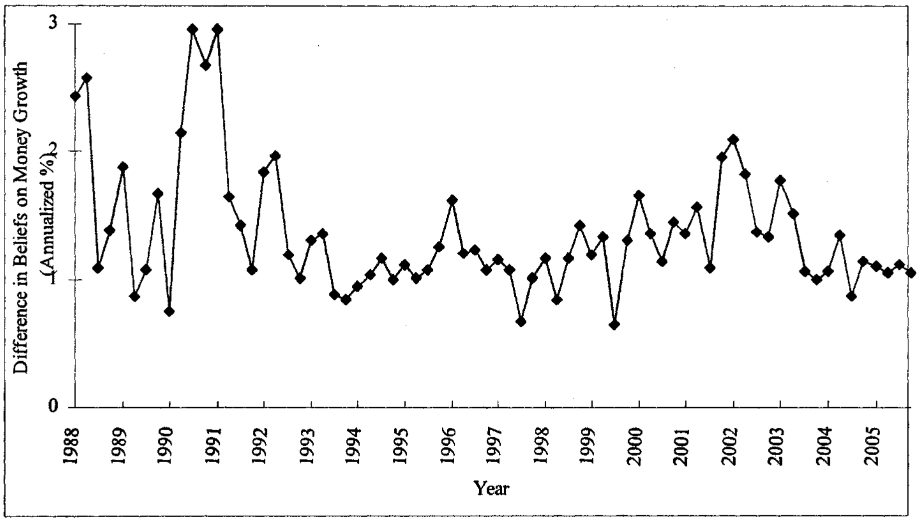


Figure 1: Difference in Beliefs about Monetary Policy.

Figure 1 plots the evolution of difference in beliefs on the future money growth from the first quarter of 1988 to the last quarter of 2005. During a recession, the economy is more uncertain; investors have higher difference in beliefs on the future monetary policy. During an expansion, the economy is more certain, they have a lower difference in beliefs. Therefore the difference in beliefs on the future monetary policy changes with the business cycle. Data resource: Survey of Professional Forecasters by the Federal Reserve Bank of Philadelphia.

Table I

The Effects of Difference in Beliefs on Stock Market Volatility,

1988 Q1-2005 Q4

The table summarizes the results of the estimation of following three models: $s_t = a + bDB_{M,t} + e_t$ (Model 1); $s_t = a + bDB_{R,t} + e_t$ (Model 2); $s_t = a + bDB_{M,t} + cDB_{R,t} + e_t$ (Model 3). Where s_t is the standard deviation of quarterly stock return, and $DB_{M,t}$ and $DB_{R,t}$ denote investors difference in beliefs on future money growth and real output. The associated p-values calculated using Huber-White sandwich estimators of variance are reported in parentheses. The data are from the Survey of Professional Forecasters provided by the Federal Bank of Philadelphia and the CRSP.

	Model 1	Model 2	Model 3
Intercept	0.475 (0.000)	0.619 (0.000)	0.508 (0.000)
DB about money growth	0.589 (0.000)		0.688 (0.014)
DB about real output		0.320 (0.270)	-0.169 (0.636)
R^2(%)	7.7	1.4	8.2

Chapter 4

ASSET PRICING AND WELFARE ANALYSIS WITH BOUNDED RATIONAL INVESTOR

Abstract

Motivated by the fact that investors have limited ability and insufficient knowledge to process the wealth of information, I model investors' bounded rational behavior in processing information and investigate its implications on asset pricing. Bounded rational investors perceive "correlated" information (which consists of news items that are correlated with fundamentals, but provide no information on them) as "fundamental" information (which consists of news items about expected changes in fundamentals). This process generates "bounded rational risk". Asset prices and volatilities of asset returns are derived. Specifically, the equity premium and the stock volatility are increased under some conditions. I also analyze the welfare impact of investors' bounded rational behavior.

4.1 Introduction

Investors employ available information to make their consumption and portfolio decisions, but whether they process information rationally is still an open question. Simon (1955, 1987) doubts the full rationality of human behavior in making decisions and formally defines bounded rationality as *"rational choice that takes into account the cognitive limitations of the decision maker – limitations of both knowledge and computational capacity."* This implies that human beings have limited ability to process information and therefore make sub-optimal decisions.

In financial markets, investors face a wealth of information, which require a large amount of knowledge and ability to process. For example, investors can find voluminous information or data about listed firms on the Internet, but it is hard for them to exploit this information. Among these pieces of information, some provide valuable insight about expected stock returns, while others are only correlated with stock prices and provide no information about expected stock returns. For simplicity, I call these two types of information "fundamental" information and "correlated" information, respectively. To clarify the difference between the two types of information, we could think that "fundamental" information provides investors with information about expected changes in fundamentals (e.g., dividends, earnings, and cash flows), while "correlated" information (e.g., unemployment rate) is only correlated with fundamentals and provides no information on them; it may, however, provide investors with information about shocks to fundamentals. Due to investors' insufficient knowledge and limited ability to process information, some investors cannot distinguish between these two types of information. I call the investor who distinguishes them the rational investor; the one who cannot is a bounded rational investor.

The objective of this paper is to analyze the effect of the bounded rational investor on asset pricing. I provide a general equilibrium model in a continuous-time economy with both types of investor. Given that only "correlated" information exists in the economy[1], investors estimate the unobserved dividend growth by observing the realizations of dividends and "correlated" information. Since investors have different perceptions of information, they have different estimates about dividend growth and therefore heterogeneity in beliefs is generated endogenously.

Investors are assumed to have CRRA preferences with a coefficient of relative risk aversion greater than one, and choose a nonnegative consumption process and a portfolio process to maximize their utility. I employ the martingale representation technique (Cox and Huang (1989), Karatzas, Lehoczky and Shreve (1987)) to solve the investors' optimization problem. To solve for the equilibrium, I use the aggregation technique (Cuoco and He (1994), Basak and Cuoco (1998)) to construct a representative agent with stochastic weights. Generally, in the context of incomplete information, it is usually impossible

[1] I can extend this setup by including fundamental information in the model; however, the conclusions still hold.

to characterize the dynamics of asset prices (e.g., volatility, risk premium) explicitly. To make this analysis possible, I use the Clark-Ocone formula and the Martingale Representation Theorem (Ocone and Karatzas (1991), Nualart (1995)) to decompose the volatilities of assets and to derive the excess return of assets, and employ Monte Carlo simulation to compare my model with the benchmark economy in which all investors are rational (e.g., they can distinguish between the two types of information).

The main results are as follows. First, the presence of the bounded rational investor increases stock price when the drift of signal process significantly exceeds dividend growth. Intuitively, this means that when the drift of signal process is higher (lower) than dividend growth, the bounded rational investor is more optimistic (pessimistic) than the rational investor since his estimate of dividend growth is equal to the drift of signal process, which in turn makes him perceive the stock more favorably (unfavorably), and therefore the stock price is increased (reduced). This is consistent with Hirshleifer, Subrahmanyam and Titman (2004), who provide a model integrating irrational investors and feedback effects from stock prices to cash flows. They find that stock prices tend to increase when irrational investors buy and irrational trading does not provide profit opportunities to rational investors.

Second, stock volatility is increased when the drift of signal process is small enough relative to dividend growth. When the drift of signal process is low, "learning risk" generated by the rational investor dominates the economy and tends to increase stock volatility, making volatility higher than that in the benchmark economy. As the drift of signal process increases, "bounded rational risk" generated by the bounded rational investor offsets the effect of "learning risk" and may lead to a low level of stock volatility. Moreover, stock volatility may change with the arrival of new information, which is consistent with the finding of Anderson and Bollerslev (1998).

Third, expected equity premium is increased when the drift of signal process is small enough relative to dividend growth. The intuition for this is very similar to that of decreased stock price, because the expected equity premium and the stock price reflect two sides of the same economic story.

Finally, the rational investor may experience welfare loss from the trading with the bounded rational investor when the drift of signal process significantly exceeds dividend growth. This is consistent with Hirshleifer, Subrahmanyam and Titman (2004), who suggest that irrational trading may not provide profit opportunities to rational investors.

This paper is related to three streams of literature. First, it is related to limited information-processing capacity or rational inattention (e.g., see Sims (2003)). Peng and Xiong (2005) and Peng (2005) examine two learning models in which investors have limited time and attention to process information, and fail to respond appropriately to information until they pay attention to it. When there are multiple sources of uncertainty, investors optimally allocate their attention across different sources of uncertainty to minimize the total uncertainty of their portfolio. Even though investors are not attention-constrained, they may still make mistakes in processing information because they cannot distinguish between the two types of information. At the market level, I analyze the effect of investors' bounded rational behavior on asset pricing rather than the cross-sectional properties of stock returns.

Second, this paper can be compared to other papers that study investor psychology. Several behavioral models[2] suggest that investors' behavior deviates from rationality in various ways. Barberis, Shleifer, and Vishny (1998) provide a learning model in which actual earnings follow a random walk, but individuals believe that earnings either follow a steady growth path in the long run or are mean-reverting. Daniel, Hirshleifer and Subrahmanyam (1998) stress that investors are overconfident about the precision of private information. In Hong and Stein (1999), the bounded rationality indicates that investors are only able to process a subset of available information. All of these models explain the phenomena of over- and under-reaction of asset prices to news and analyze the cross-sectional properties of stock returns. In contrast, I provide a model in which the rational investor can distinguish between two types of information, while the bounded rational investor cannot. In this context, I investigate the impact of investors' bounded rational behavior on asset pricing rather than over- and under-reaction of stock prices to news.

Scheinkman and Xiong (2003) present a model in which overconfidence generates disagreements among investors, where overconfidence means investors overestimate the correlation between innovations in the signal and innovations

[2]Barberis and Thaler (2003) and Hirshleifer (2001) survey the recent developments of behavioral finance and the impacts of investor psychology on asset pricing, respectively.

in the unobserved variables. With short-selling constraints, they analyze links between asset prices, trading volume, and price volatility. Dumas, Kurshev and Uppal (2005) extend this work to provide a model in which there are two types of investor: one has correct beliefs about the signal and the other is overconfident in it. They analyze the implications of the overconfident investor for asset pricing and hedging strategies. In Dumas, Kurshev and Uppal (2005), the overconfident investor adds "noise" for which all investors require a risk premium and therefore the price levels of assets are reduced, and the volatility and the risk premium of assets increase. In this paper, the bounded rational investor incorrectly assumes that "correlated" information provides information about fundamentals and hence believes that dividend growth is equal to the drift of signal process. I find that these conclusions hold only under some conditions.

Berrada (2006b) considers a model with unobserved dividend growth, in which some investors display learning bias and over- and under-react to the arrival of new information. In his setting, the over- (under-) reactive investor allocates a high (low) weight to his updated belief, and thus stock volatility is increased (decreased). In this paper, the mechanism affecting stock volatility is different from that of Berrada (2006b). Due to investors' bounded rational behavior, distribution of wealth across investors changes over time, and therefore stock volatility is affected. Moreover, although the bounded rational investor incorrectly assumes that "correlated" information provides information on fundamentals, he is still a Bayesian learner and optimally uses his observations to update his estimate of unobserved dividend growth.

Boswijk, Hommes and Manzan (2006) present an asset pricing model, in which investors have homogeneous beliefs about future cash flows, but disagree on the speed of reversion of stock prices towards the intrinsic valuation. Given that the overvaluation of stock is common knowledge, "fundamentalists" believe that stock price will move back towards its fundamental value, while "trend followers" expect that the price trend will continue in the short run. Boswijk, Hommes and Manzan (2006) estimate the model to annual U.S. stock price data and offer an explanation for the recent stock price run-up. In my paper, the dividend growth is unobservable and investors disagree on the expected dividend growth by observing the realizations of dividends and available information. Within this framework, I analyze the implications of the bounded rational investor for stock volatility, equity premium and rational investor's welfare changes.

Third, this paper is connected to heterogeneous beliefs. Detemple and Murthy (1994), Zapatero (1998) and Basak (2000, 2005) investigate the impact of differences in opinion regarding some fundamental and non-fundamental aspects of financial economy on asset pricing. This paper differs from the preceding studies in that heterogeneity in beliefs is generated endogenously by investors' limited ability to process information rather than being exogenously specified (e.g., prior belief difference). In previous models, investors are assumed to have logarithmic preferences, and therefore the stock price and its volatility are not affected by heterogeneity in belief. Buraschi and Jiltsov

(2006) provide a model in which all investors are rational and investigate the impacts of disagreement about fundamentals on option pricing and trading volume.

The rest of this chapter is organized as follows: Section 4.2 describes the economy and Section 4.3 characterizes the equilibrium. Section 4.4 analyzes the implications of investors' bounded rational behavior on asset pricing (e.g., stock price, stock volatility and expected equity premium), and rational investor's welfare. Section 4.5 provides an extension to a more realistic case. Conclusions are discussed in section 4.6 and the Appendix provides all proofs, simulation parameters and figures.

4.2 The Economy

I consider a continuous-time, pure-exchange economy with a finite horizon, in which two types of investor are endowed with an exogenous dividend flow. Uncertainty is represented by a complete probability space $(\Omega, \mathcal{F}, P)$ and a two-dimensional Brownian motion process $W = (W_1,\ W_2)^T$ defined on $(\Omega, \mathcal{F})$. Letting $\{\mathcal{F}_t^W\}$ denote the augmented filtration generated by W and H, a $\sigma-$field independent of $\mathcal{F}_T^W$. The complete information filtration $\{\mathcal{F}_t\}$ is the augmentation of filtration $H \times \{\mathcal{F}_t^W\}$.

4.2.1 Dividend Process and Information Structure

I assume that the economy is populated by two types of investor, indexed by $i = A, B$. They observe the same exogenous aggregate dividend process D, which follows the dynamics

$$d\ln D(t) = \mu_D dt + \sigma_D^T dW(t), \tag{4.1}$$

where dividend growth μ_D and its volatility $\sigma_D = (\sigma_{D1}, \sigma_{D2})^T$ are constant, and μ_D is unobservable. $W = (W_1,\ W_2)^T$ is a standard two-dimensional Brownian motion, and W_1 and W_2 are independent.

I assume that, besides the dividend process D, investors also observe a signal e. The process of e satisfies

$$de(t) = \mu_e dt + \sigma_e^T dW(t), \tag{4.2}$$

where the drift of signal process μ_e and its volatility $\sigma_e = (\sigma_{e1}, \sigma_{e2})^T$ are constant, μ_e is observable and $\mu_e \neq \mu_D$. In section 4.5, I will extend this to a more realistic case where μ_e is unobservable. Equation (4.2) means that the signal e provides information about Brownian motion W, which affects the dividend process, but does not provide information about the unobserved dividend growth (e.g., see Zapatero (1998)). For convenience, if the signal provides information about dividend growth (e.g., $\mu_e = \mu_D$, see Detemple (2002)), I call it "fundamental" information. Otherwise, if it is only correlated with the dividends and provides no information about dividend growth, I call it "correlated" information (as in (4.2)). In this case, the signal and the dividends are driven by same shocks and/or factors. At each period, investors observe the realizations of dividends and signal to deduce their estimates of dividend growth.

The Rational Investor's Evolution of Beliefs

Investor A is assumed to be rational, in the sense that he knows that the signal e provides information about W, but not about μ_D, and he updates his estimate of μ_D and its variance via $\mu_D^A(t) \equiv E^A[\mu_D(t)|\mathcal{F}(t)]$ and $v^A(t) \equiv E_t^A\left[\left(\mu_D^A(t) - \mu_D(t)\right)^2 |\mathcal{F}(t)\right]$ (Liptser and Shiryayev (2001)). Therefore, investor A perceives the processes of dividends and signal as follows:

$$\begin{aligned} d\ln D(t) &= \mu_D^A(t)\,dt + \sigma_D^T dW^A(t), \\ de(t) &= \mu_e dt + \sigma_e^T dW^A(t), \end{aligned} \tag{4.3}$$

where the innovation process induced by investor A's estimate and filtration is

$$\begin{aligned} dW_1^A(t) &= dW_1(t) + \frac{\sigma_{e2}\left(\mu_D - \mu_D^A(t)\right)}{\sigma_{D1}\sigma_{e2} - \sigma_{D2}\sigma_{e1}}dt, \\ dW_2^A(t) &= dW_2(t) - \frac{\sigma_{e1}\left(\mu_D - \mu_D^A(t)\right)}{\sigma_{D1}\sigma_{e2} - \sigma_{D2}\sigma_{e1}}dt. \end{aligned} \tag{4.4}$$

By Girsanov's theorem, $W^A \equiv \left(W_1^A,\ W_2^A\right)^T$ is a two-dimensional Brownian motion in the space $(\Omega, \mathcal{F}^A, P^A)$. Under regularity conditions, μ_D^A and v^A evolve over time with dynamics

$$\begin{aligned} d\mu_D^A(t) &= \frac{\sigma_{e2}v^A(t)}{\sigma_{D1}\sigma_{e2} - \sigma_{D2}\sigma_{e1}}\left[\frac{\sigma_{e2}\left(\mu_D - \mu_D^A(t)\right)}{\sigma_{D1}\sigma_{e2} - \sigma_{D2}\sigma_{e1}}dt + dW_1(t)\right], \\ dv^A(t) &= -\left(\frac{\sigma_{e2}}{\sigma_{D1}\sigma_{e2} - \sigma_{D2}\sigma_{e1}}\right)^2 v^A(t)^2\,dt. \end{aligned} \tag{4.5}$$

Equation (4.5) means that investor A's estimate of dividend growth will converge to actual growth in the long run since v^A will converge to zero. According to the incomplete information literature[3], if investors have incomplete information about economic variables and update their beliefs dynamically, "learning risk" is generated in the economy and affects the dynamics of asset prices. In Sections 4.3 and 4.4, I will show that "learning risk" has an important role in asset pricing.

The Bounded Rational Investor's Evolution of Beliefs

Bounded rational investor B has limited ability or insufficient knowledge to process information, and therefore he incorrectly believes that the signal e provides information about dividend growth and assumes that $\mu_D = \mu_e$. I call investor B a bounded rational investor. From the perspective of investor B, the processes of dividends and signal follow the dynamics

$$\begin{aligned} d\ln D(t) &= \mu_e dt + \sigma_D^T dW^B(t), \\ de(t) &= \mu_e dt + \sigma_e^T dW^B(t), \end{aligned} \tag{4.6}$$

[3]Theoretically, Brennan (1998) shows that the estimation risk or the risk of "learning" affects investors' investments in stock. Barberis (2000) suggests that the process of learning entices investors to hedge more estimation risk than systematic uncertainty. The literature of learning also includes Detemple (1986), Dothan and Feldman (1986), Gennotte (1986), Wang (1993), Veronesi (1999, 2000), Brennan and Xia (2001), Buraschi and Jiltsov (2006), and David (2006). Empirically, Pastor and Veronesi (2003) investigate the impact of learning uncertainty on stock profitability to explain the market-to-book premium. David and Veronesi (2006) find that when investors are highly uncertain about fundamentals, their expectations tend to react more swiftly to news affecting both variance and covariance of asset returns. Massa and Simonov (2005) assess how the uncertainty induced by investors' learning about fundamental variables affects stock returns. They construct a price-based measure of uncertainty and find that learning uncertainty affects the time-variation of economic risk premiums.

where $W^B \equiv \left(W_1^B,\ W_2^B\right)^T$ is the innovation process induced by investor B's estimate and filtration given by

$$\begin{aligned} dW_1^B(t) &= dW_1(t) + \frac{\sigma_{e2}(\mu_D - \mu_e)}{\sigma_{D1}\sigma_{e2} - \sigma_{D2}\sigma_{e1}}dt, \\ dW_2^B(t) &= dW_2(t) - \frac{\sigma_{e1}(\mu_D - \mu_e)}{\sigma_{D1}\sigma_{e2} - \sigma_{D2}\sigma_{e1}}dt. \end{aligned} \tag{4.7}$$

Given equations (4.4) and (4.7), I have the following relationship

$$dW^B(t) = dW^A(t) + \Phi^{(M)}(t)\,dt, \tag{4.8}$$

where $\Phi^{(M)} \equiv \left(\Phi_1^{(M)},\ \Phi_2^{(M)}\right)^T$ satisfies

$$\Phi_1^{(M)}(t) \equiv \frac{\sigma_{e2}\left(\mu_D^A(t) - \mu_e\right)}{\sigma_{D1}\sigma_{e2} - \sigma_{D2}\sigma_{e1}}, \quad \Phi_2^{(M)}(t) \equiv -\frac{\sigma_{e1}}{\sigma_{e2}}\Phi_1^{(M)}(t). \tag{4.9}$$

Equation (4.8) means that investors have different perceptions about the innovation process. Moreover, the bounded rational investor's incorrect updating of his beliefs generates "bounded rational risk". In Sections 4.3 and 4.4, I will demonstrate that "bounded rational risk" plays an important role in asset pricing.

Remark 1 (*Benchmark*) *If investor B is rational, I have the following relationship*

$$dW^B(t) = dW^A(t) + \Phi^{(B)}(t)\,dt, \tag{4.10}$$

where

$$\begin{aligned} \Phi_1^{(B)}(t) &\equiv \frac{\sigma_{e2}\left(\mu_D^A(t) - \mu_D^B(t)\right)}{\sigma_{D1}\sigma_{e2} - \sigma_{D2}\sigma_{e1}}, \\ \Phi_2^{(B)}(t) &\equiv -\frac{\sigma_{e1}}{\sigma_{e2}}\Phi_1^{(B)}(t), \end{aligned} \tag{4.11}$$

where μ_D^B *has same dynamics as* μ_D^A *in equation* (4.5). *In this case, the disagreement process* $\Phi^{(B)}$ *is deterministic (Basak (2000)), and therefore one investor is always optimistic and the other pessimistic. However, in this paper, the disagreement process* $\Phi^{(M)}$ *follows a stochastic process, so the current optimistic investor can become pessimistic investor in the future and vice versa.*

4.2.2 Security Market

To hedge risk, investors can trade continuously in three financial assets: a riskless bond B, a stock S, and a long-term bond B_T. The riskless bond is in zero net supply and pays the real interest rate r, the stock is in net supply of 1 and represents a claim on the aggregate dividends D, and the long-term bond has payoff 1 at maturity T with price dynamics

$$dB(t) = r(t) B(t) dt, \tag{4.12}$$

$$\begin{aligned} dS(t) + D(t) dt &= S(t) \left[\mu_S(t) dt + \sigma_S(t)^T dW(t)\right] \\ &= S(t) \left[\mu_S^i(t) dt + \sigma_S(t)^T dW^i(t)\right], \end{aligned} \tag{4.13}$$

$$\begin{aligned} dB_T(t) &= B_T(t) \left[\mu_B(t) dt + \sigma_B(t)^T dW(t)\right] \\ &= B_T(t) \left[\mu_B^i(t) dt + \sigma_B(t)^T dW^i(t)\right]. \end{aligned} \tag{4.14}$$

Equations (4.13) and (4.14) represent the asset dynamics perceived by investor i. The assets' expected return $\mu^i \equiv (\mu_S^i,\ \mu_B^i)$, volatility $\tilde{\sigma} \equiv \left(\sigma_S{}^T, \sigma_B{}^T\right)$ and the real interest rate r are determined endogenously in equilibrium. Investors can observe the prices of the stock and the long-term bond, but have different estimates of assets' expected return. The relationship between investors' expected returns on assets satisfies

$$\begin{pmatrix} \mu_S^B(t) - \mu_S^A(t) \\ \mu_B^B(t) - \mu_B^A(t) \end{pmatrix} = \begin{pmatrix} \sigma_S(t) \\ \sigma_B(t) \end{pmatrix} \Phi^{(M)}(t) . \tag{4.15}$$

From market completeness, there exists a unique state-price density ξ^i (or a Stochastic Discount Factor) for each investor. Under the no-arbitrage, it follows that

$$d\xi^i(t) = -\xi^i(t) \left[r(t)\, dt + \theta_1^i(t)\, dW_1^i(t) + \theta_2^i(t)\, dW_2^i(t) \right] , \tag{4.16}$$

where $\theta^i \equiv \left(\theta_1^i,\, \theta_2^i\right)^T = \tilde{\sigma}^{-1} \left[\mu_D^i - r\underline{1}\right]$ is the market price of risk (or Sharpe ratio) perceived by investor i. Using (4.15) I can derive the relationship between the market prices of risk perceived by investors

$$\theta^A(t) - \theta^B(t) = \Phi^{(M)}(t) , \tag{4.17}$$

which means investors disagree on the market price of risk.

4.2.3 Investors' Preferences and Optimization

Investor i is endowed, at time t, with $\alpha_i(t)$ shares of stock where $\alpha_i(t) > 0$ and $\alpha_A(t) + \alpha_B(t) = 1$. He chooses a nonnegative consumption process c_i and a portfolio process π_i satisfying $\int_t^T c_i(s)\, ds < \infty$ and $\int_t^T \|\pi_i(s)\|^2 ds < \infty$. The dynamic budget constraint satisfies

$$\begin{aligned} dX_i(t) &= [X_i(t) r(t) - c_i(t)] dt \\ &\quad + X_i(t) \pi_i^S(t) \left[\left(\mu_S^i(t) - r(t) \right) dt + \sigma_S(t)^T dW^i(t) \right] \\ &\quad + X_i(t) \pi_i^B(t) \left[\left(\mu_B^i(t) - r(t) \right) dt + \sigma_B(t)^T dW^i(t) \right], \end{aligned} \tag{4.18}$$

where $W^i \equiv (W_1^i, W_2^i)^T$ and X_i is bounded from below, satisfying $X_i(T) \geq 0$ $a.s.$.

Investors are assumed to have CRRA preference with a coefficient of relative risk aversion γ (> 1). Following the martingale techniques developed by Cox and Huang (1989) and Karatzas, Lehoczky and Shreve (1987), investor i's dynamic consumption-portfolio problem can be converted into the following static problem:

$$\max_{c_i} E_t^i \int_t^T \frac{c_i(s)^{1-\gamma}}{1-\gamma} ds \quad s.t. \; E_t^i \left[\int_t^T \xi^i(s) c_i(s) ds \right] \leq \xi^i(t) \alpha_i(t) \varepsilon_i(t), \tag{4.19}$$

where $E_t^i[.]$ denotes the expectation conditional on the information structure $(\Omega, \mathcal{F}^i, P^i)$.

4.3 Equilibrium Characterization

In the presence of the bounded rational investor, the equilibrium is defined as follows:

Definition 2 *An equilibrium is a price system* (r, S, B_T) *and an admissible consumption-portfolio process* (c_i, π_i) *such that* (i) *investors choose their opti-*

mal consumption-portfolio strategies given their perceived price processes; (ii) asset prices are consistent across investors, that is,

$$\begin{pmatrix} \mu_S^B(t) - \mu_S^A(t) \\ \mu_B^B(t) - \mu_B^A(t) \end{pmatrix} = \begin{pmatrix} \sigma_S(t) \\ \sigma_B(t) \end{pmatrix} \Phi^{(M)}(t), \tag{4.20}$$

and (iii) markets of good and assets clear, that is

$$c_A(t) + c_B(t) = D(t), \tag{4.21}$$

$$\begin{aligned} \pi_A^S(t) X_A(t) + \pi_B^S(t) X_B(t) &= S(t), \\ X_A(t) + X_B(t) &= S(t), \\ \pi_A^B(t) X_A(t) + \pi_B^B(t) X_B(t) &= 0. \end{aligned} \tag{4.22}$$

To solve for the equilibrium, I introduce a representative agent (Cuoco and He (1994), Basak and Cuoco (1998)) endowed with the aggregate dividends and with utility function

$$U(c(t); \lambda(t)) \equiv \max_{c_A(t)+c_B(t)=c(t)} \frac{c_A(t)^{1-\gamma}}{1-\gamma} + \lambda(t) \frac{c_B(t)^{1-\gamma}}{1-\gamma}, \tag{4.23}$$

where $\lambda > 0$ denotes the relative weight of bounded rational investor B. In equilibrium, I can obtain the optimal consumption and the state-price density for each investor. The results are summarized in the following proposition:

Proposition 10 *In Equilibrium, the investors' state-price densities are given by*

$$\xi^A(t) = \frac{1}{y_A} \left(\frac{1 + \lambda(t)^{\frac{1}{\gamma}}}{D(t)} \right)^{\gamma}, \quad \xi^B(t) = \frac{y_A}{y_B \lambda(t)} \xi^A(t), \tag{4.24}$$

where y_i $(i = A, B)$ satisfies investor i's static budget constraint with equality (expression (4.19)), and the relative weight λ satisfies

$$d\lambda(t) = -\lambda(t)\,\Phi^{(M)}(t)^T\,dW^A(t). \tag{4.25}$$

The optimal consumptions are

$$c_A(t) = (1-\omega(t))\,D(t), \qquad c_B(t) = \omega(t)\,D(t), \tag{4.26}$$

where $\omega(t) \equiv \frac{\lambda(t)^{\frac{1}{\gamma}}}{1+\lambda(t)^{\frac{1}{\gamma}}}$ is the consumption share of bounded rational investor B.

Remark 2 (*Benchmark*) *If investor B can distinguish between the two types of information, the relative weight λ satisfies*

$$d\lambda(t) = -\lambda(t)\,\Phi^{(B)}(t)^T\,dW^A(t). \tag{4.27}$$

Equations (4.24) and (4.26) characterize investors' state-price density and optimal consumption, which depend on the relative weight λ of the bounded rational investor. Equation (4.25) shows that λ plays an important role in determining optimal consumption: the higher λ is, the higher the wealth allocated to the bounded rational investor and so the higher his optimal consumption.

Long-run effect: equation (4.5) implies that the rational investor's estimate of dividend growth converges to actual growth, μ_D, while the bounded rational investor's estimate is equal to the drift of signal process μ_e. Therefore, the difference in market price of risk across investors $\Phi^{(M)}$ will be different from zero in the long run. If the drift of signal process is lower than dividend growth, the rational investor is more optimistic than the bounded rational investor in the long run, and his consumption will tend to increase over time. However, whether or not the bounded rational investor is driven out of markets by the rational investor is ambiguous[4]. In a benchmark model where all investors are rational, the difference in their market price of risk $\Phi^{(B)}$ will converge to zero, and so the investors' sharing of consumption is identical in the long run.

Short-run effect: in this paper, the current optimistic investor may become pessimistic in the future, so investors' consumption may increase now and decrease in the future. However, in the benchmark model, the difference in market price of risk across investors $\Phi^{(B)}$ is deterministic; therefore one investor is always optimistic and the other pessimistic.

In the following proposition, I will characterize the market prices of risk perceived by investors and the real interest rate.

Proposition 11 *The market prices of risk perceived by investors are given by*

$$\begin{aligned} \theta^A(t) &= \gamma\sigma_D + \omega(t)\,\Phi^{(M)}(t), \\ \theta^B(t) &= \gamma\sigma_D - (1-\omega(t))\,\Phi^{(M)}(t), \end{aligned} \tag{4.28}$$

and the real interest rate is

$$\begin{aligned} r(t) &= \gamma\left[(1-\omega(t))\,\mu_D^A(t) + \omega(t)\,\mu_e\right] - \frac{1}{2}\gamma^2\left(\sigma_{D1}^2 + \sigma_{D2}^2\right) \\ &\quad + \frac{\gamma-1}{2\gamma}\omega(t)\,(1-\omega(t))\left(\Phi_1^{(M)}(t)^2 + \Phi_2^{(M)}(t)^2\right). \end{aligned} \tag{4.29}$$

[4]Friedman (1953) suggests that investors with incorrect beliefs will eventually be driven out of markets by those with rational expectations. Yan (2006) demonstrates that this selection process is excessively slow, although the investor with incorrect beliefs can not survive in the long run. Berrada (2006), De Long, Shleifer, Summers and Waldmann (1990), and Kogan, Ross, Wang and Westerfield (2006) have additional interesting results regarding this issue.

Remark 3 *(Benchmark) If investor B can distinguish between the two types of information, $\Phi^{(M)}(t)$ is replaced by $\Phi^{(B)}(t)$ in equations (4.28) and (4.29).*

The real interest rate consists of three terms. The first term reflects investors' relative risk aversion: as the coefficient of relative risk aversion increases, investors demand more riskless assets, which increases the real interest rate. The second term reflects the wealth effect on consumption: as investors' expected dividend growth increases, they will save less and the real interest rate will increase. The third term captures the effect of the difference in market price of risk across investors: when the difference in market price of risk across investors is higher, the real interest rate tends to increase.

Typically, the real interest rate will increase in the presence of the bounded rational investor. The intuition for this is that investors have different estimates of unobserved dividend growth and bet against each other on available future dividends. Each investor expects to win the bet and has a higher expectation of consumption growth, which decreases his saving motive and hence increases the real interest rate.

Given the state-price density ξ^i $(i = A, B)$, the stock price and the long-term bond price are equal to

$$\begin{aligned} S(t) &= E_t^i\left[\int_t^T \frac{\xi^i(s)}{\xi^i(t)}\varepsilon(s)\,ds\right], \\ B_T(t) &= E_t^i\left[\frac{\xi^i(T)}{\xi^i(t)}\right]. \end{aligned} \tag{4.30}$$

(4.30) is the standard expression for asset prices, since the stock is a claim on the aggregate dividends D and the long-term bong has payoff 1 at maturity time T.

In order to analyze the implications of the bounded rational investor on asset pricing, I need to solve asset prices and to derive their volatility and expected return. In an incomplete information setting, it is usually impossible to characterize the dynamics of asset prices explicitly. However, I can decompose the asset volatility by using the Clark-Ocone formula and the Martingale Representation Theorem (Ocone and Karatzas (1991)). The following proposition illustrates the decomposition of asset volatility.

Proposition 12 *The volatility coefficients of the stock price and the long-term bond price are given by* $(j = 1, 2)$

$$\begin{aligned}\sigma_{Sj}(t) &= \sigma_{Dj} + \omega_S(t)\,\Phi_j^{(M)}(t)\\ &\quad + \frac{1}{S(t)}E_t^A\left[\int_t^T (1-\gamma)(s-t)\,D(s)\,\frac{\xi^A(s)}{\xi^A(t)}ds\right]\frac{\sigma_{ej}\nu^A(t)}{\sigma_{D1}\sigma_{e2}-\sigma_{D2}\sigma_{e1}}\\ &\quad + \frac{1}{S(t)}E_t^A\left[\int_t^T f_j(t,s)\,c_B(s)\,\frac{\xi^A(s)}{\xi^A(t)}ds\right],\end{aligned} \tag{4.31}$$

$$\begin{aligned}\sigma_{Bj}(t) &= \omega_B(t)\,\Phi_j^{(M)}(t)\\ &\quad - \frac{1}{B_T(t)}E_t^A\left[\gamma(T-t)\,\frac{\xi^A(T)}{\xi^A(t)}\right]\frac{\sigma_{ej}\nu^A(t)}{\sigma_{D1}\sigma_{e2}-\sigma_{D2}\sigma_{e1}}\\ &\quad + \frac{1}{B_T(t)}E_t^A\left[f_j(t,T)\,\omega(T)\,\frac{\xi^A(T)}{\xi^A(t)}\right],\end{aligned} \tag{4.32}$$

where

$$\begin{aligned}\omega_S(t) &\equiv \omega(t) - \frac{X_B(t)}{S(t)},\\ \omega_B(t) &\equiv \omega(t) - \frac{1}{B_T(t)}E_t^A\left[\omega(T)\,\frac{\xi^A(T)}{\xi^A(t)}\right],\end{aligned} \tag{4.33}$$

$$
\begin{aligned}
f_j(t,s) &\equiv -\frac{\sigma_{Dj}\nu^A(t)}{\left(\sigma_{D1}^2+\sigma_{D2}^2\right)\left(\sigma_{D1}\sigma_{e2}-\sigma_{D2}\sigma_{e1}\right)} \\
&\quad * \left[\begin{array}{c} \int_t^s \left(\sigma_{e2}+\frac{\sigma_{e1}^2}{\sigma_{e2}}\right)\Phi_j^{(M)}(u)\,du \\ +\int_t^s \sigma_{e2}dW_1^A(u) - \int_t^s \sigma_{e1}dW_2^A(u) \end{array}\right], \qquad (4.34)
\end{aligned}
$$

where $X_B(t) = E_t^A\left[\int_t^T \frac{\xi^A(s)}{\xi^A(t)} c_B(s)\,ds\right]$ *is investor* B*'s wealth at time* t.

Remark 4 (*Benchmark*) *If investor* B *can distinguish between the two types of information, the volatilities of the stock price and the long-term bond price are given by*

$$
\begin{aligned}
\sigma_{Sj}(t) &= \sigma_{Dj}+\omega_S(t)\Phi_j^{(B)}(t)+\frac{1}{S(t)} * \\
&\quad E_t^A\left[\int_t^T (1-\gamma)(s-t)D(s)\frac{\xi^A(s)}{\xi^A(t)}ds\right]\frac{\sigma_{ej}\nu^A(t)}{\sigma_{D1}\sigma_{e2}-\sigma_{D2}\sigma_{e1}} \qquad (4.35)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{Bj}(t) &= \omega_B(t)\Phi_j^{(B)}(t) \\
&\quad -\frac{1}{B_T(t)}E_t^A\left[\gamma(T-t)\frac{\xi^A(T)}{\xi^A(t)}\right]\frac{\sigma_{ej}\nu^A(t)}{\sigma_{D1}\sigma_{e2}-\sigma_{D2}\sigma_{e1}}. \qquad (4.36)
\end{aligned}
$$

The *stock volatility* σ_{Sj} caused by Brownian motion W_j consists of four terms. The first term is the dividend volatility, which represents the impact of the shock dW_j on the dividend process. The second term is the impact of the shock dW_j on current and future cash flows and is equal to the product of the difference between current consumption share and current wealth share of the bounded rational investor, ω_S, and the difference in market price of risk across investors, $\Phi^{(M)}$. These two terms are standard and consistent with existing asset pricing literature. The third term captures the effect of "learning risk" caused by the rational investor, and the fourth term reflects the effect of "bounded rational risk" generated by the bounded rational investor.

"Learning risk": Brennan (1998) claims that, if investors have incomplete information and update their beliefs dynamically, "learning risk" is generated in the economy. Veronesi (1999) theoretically demonstrates that investors' uncertainty about fundamentals tends to increase stock volatility. David and Veronesi (2006) empirically find that when investors are highly uncertain about fundamentals, their expectations tend to react more swiftly to news affecting both variances and covariance of asset returns. These evidences show that "learning risk" tends to increase stock volatility. *"Bounded rational risk"*: the bounded rational investor's incorrect updating of his beliefs generates "bounded rational risk". However, the effect of "bounded rational risk" on stock volatility is ambiguous.

Massa and Simonov (2005) construct two proxies that separately identify learning uncertainty (or learning risk) and dispersion of beliefs, and find that these two factors affect the time-variation of economic risk premiums. In my paper, the bounded rational investor inaccurately estimates dividend growth and so the dispersion of beliefs is generated endogenously. Therefore, equations (4.31) and (4.32) imply that I can disentangle the effect of "bounded rational risk" on asset volatility from that of "learning risk". Once I derive the volatilities of assets, the expected premium for the stock and the long-term bond perceived by the rational investor can be obtained by

$$\begin{pmatrix} \mu_S^A(t) - r(t) \\ \mu_B^A(t) - r(t) \end{pmatrix} = \begin{pmatrix} \sigma_{S1}(t) & \sigma_{S2}(t) \\ \sigma_{B1}(t) & \sigma_{B2}(t) \end{pmatrix} \begin{pmatrix} \theta_1^A(t) \\ \theta_2^A(t) \end{pmatrix}. \tag{4.37}$$

4.4 Numerical Analysis

Given investors' state-price density, I numerically calculate the stock price, the stock volatility and the expected equity premium through Monte Carlo simulation[5]. Using parameters calibrated in Buraschi and Jiltsov (2006), I establish that $\mu_D = 3\%$, $\sigma_{D1} = \sigma_{D2} = 3.5\%$ and investors' investment horizon is $T = 30$ years. The variance of rational investor A's estimate of dividend growth at time 0, $\nu^A(0)$, is set to be $(3\%)^2 = 0.09\%$. In order to implement the simulations, 100,000 paths of equilibrium variables are simulated using the Milestein discretization scheme. Moreover, when I analyze the effects of the consumption share of the bounded rational investor on asset pricing and rational investor's welfare, the drift of signal process is set at $\mu_e = 2\%$ and 4% respectively, and rational investor A's initial estimate of dividend growth is 3%. Table I illustrates all parameters for simulation analysis. To better understand the impact of investors' bounded rational behavior on asset pricing, I will compare my model with a benchmark economy in which all investors are rational.

[5]The same simulation methodology applies to analysis of the bond market.

4.4.1 Stock Price

Panel A of Figure 1 illustrates the changes in stock price against the difference between the drift of signal process and dividend growth, $\mu_e - \mu_D$. I find that the stock price is higher than that in the benchmark economy when the drift of signal process significantly exceeds dividend growth. The intuition for this is: when the drift of signal process is higher (lower) than dividend growth, the bounded rational investor is more optimistic (pessimistic) than the rational investor since his estimate of dividend growth equals the drift of signal process, which in turn makes him perceive the stock more favorably (unfavorably), and therefore the stock price is increased (reduced). This is roughly consistent with Panel A of Figure 1.

Hirshleifer, Subrahmanyam and Titman (2004) provide a model in which irrational investors do not anticipate the feedback effect from stock prices to cash flows, and find that stock prices tend to increase when irrational investors buy and irrational trading does not provide profit opportunities to rational investors. Panel A of Figure 1 shows that, if the difference between the drift of signal process and dividend growth is more than 0.28%, the stock price is increased relative to the benchmark economy, which is consistent with the finding of Hirshleifer, Subrahmanyam and Titman (2004).

Dumas, Kurshev and Uppal (2005) present a model in which the overconfident investor overestimates the correlation between the innovations in the signal and the innovations in the unobserved variables. They demonstrate that stock price is reduced since the overconfident investor adds "noise" for which all investors require a risk premium. In my paper, the bounded rational investor (they call him "overconfident investor") correctly estimates the correlation between innovations, but incorrectly assumes that "correlated" information provides information about fundamentals. I find that the stock price is reduced by the presence of the bounded rational investor only when the difference between the drift of signal process and dividend growth is less than 0.28%.

INSERT FIGURE 1 HERE

Panel B of Figure 1 illustrates the changes in stock price against the consumption share of the bounded rational investor. It shows that the stock price in this paper is lower than that in the benchmark model when the drift of signal process is 2% ($< \mu_D$). Moreover, as the consumption share of the bounded rational investor increases from 0 to 1, the stock price decreases for $\mu_e = 2\%$, while it tends to increase for $\mu_e = 4\%$ ($> \mu_D$).

4.4.2 Stock Volatility

Panel A of Figure 2 illustrates stock volatility against the difference between the drift of signal process and dividend growth, and demonstrates that stock volatility in my model is higher than that in the benchmark model when the drift of signal process is small enough relative to dividend growth.

In Dumas, Kurshev and Uppal (2005), stock volatility increases because the overconfident investor generates "noise" in the economy. My paper demonstrates that both "learning risk" and "bounded rational risk" affect stock volatility: when the difference between the drift of signal process and dividend growth is small, "learning risk" dominates the economy and tends to increase stock volatility, making volatility higher than that in the benchmark economy. As the drift of signal process increases, "bounded rational risk" offsets the effect of "learning risk" and may lead to a low level of stock volatility.

Berrada (2006b) considers a model in which dividend growth is unobservable and some investors display learning bias and over- and under-react to the arrival of new information. In his setting, the over-reactive (under-reactive) investor allocates a high (low) weight to his updated belief and consequently stock volatility is increased (reduced). In my model, the mechanism affecting stock volatility is different from that of Berrada (2006b). There are two types of investor, and distribution of wealth among them changes over time; therefore, stock volatility is affected. Moreover, although the bounded rational investor incorrectly assumes that "correlated" information provides information about dividend growth, he is still a Bayesian learner and optimally uses his prior observations to update his estimate of unobserved dividend growth.

Panel A of Figure 2 shows that the stock volatility changes with the drift of signal process, which may change with the arrival of new information. This is consistent with the finding of Andersen and Bollerslev (1998) that news or announcements affects stock volatility.

INSERT FIGURE 2 HERE

Panel B of Figure 2 plots stock volatility against the consumption share of the bounded rational investor. I find that the stock volatility in my model is higher than that in the benchmark model for $\mu_e = 2\%$ and that it is a concave function of the consumption share of the bounded rational investor, while I obtain opposite results for $\mu_e = 4\%$. This is consistent with the previous analysis: stock volatility is increased only when the drift of signal process is small enough relative to dividend growth.

4.4.3 Expected Equity Premium

Theoretically, stock price and expected equity premium should reflect the same economic story if the real interest rate is constant: when the current stock price decreases, expected equity premium should increase. Equation (4.29) shows that, as the drift of signal process approaches dividend growth from below, the real interest rate is roughly constant. The reason is that the effect of investor's relative risk aversion, $\gamma\left[(1-\omega(t))\,\mu_D^A(t)+\omega(t)\,\mu_e\right]$, and the effect of the difference in market price of risk across investors, $\frac{\gamma-1}{2\gamma}\omega(t)\,(1-\omega(t))\left(\Phi_1^{(M)}(t)^2+\Phi_2^{(M)}(t)^2\right)$, offset each other[6]. When the drift of signal process exceeds dividend growth, the real interest rate will unambiguously increase and therefore expected eq-

[6]If I set $\mu_D^A(t) = \mu_D = 3\%$, as μ_e approaches μ_D from below, $\gamma\left[(1-\omega(t))\,\mu_D^A(t)+\omega(t)\,\mu_e\right]$ increases and $\Phi_1^{(M)}(t)^2 + \Phi_2^{(M)}(t)^2 = \left[1+\left(\frac{\sigma_{e1}}{\sigma_{e2}}\right)^2\right]\left[\frac{\sigma_{e2}\left(\mu_D^A(t)-\mu_e\right)}{\sigma_{D1}\sigma_{e2}-\sigma_{D2}\sigma_{e1}}\right]^2$ decreases.

uity premium will decrease. This analysis is roughly consistent with Panel A of Figures 2 and 3, except that the difference between the drift of signal and dividend growth lies in the interval [−0.7%, 0.25%].

INSERT FIGURE 3 HERE

Panel B of Figure 3 plots expected equity premium against the consumption share of the bounded rational investor. I find that the expected equity premium is higher than that in the benchmark model for $\mu_e = 2\%$, while it is lower for $\mu_e = 4\%$. Therefore, the presence of the bounded rational investor tends to increase expected equity premium only when the drift of signal process is significantly less than dividend growth.

4.4.4 Welfare Analysis

In the presence of the bounded rational investor, the rational investor's optimal consumption will change, and so will his welfare. In this section, I will explore the rational investor's welfare change, which is defined as the percentage change in his utility given by

$$WG = \frac{E_t^A \left[\int_t^T c_A^{(M)} (s)^{1-\gamma} \, ds \right]}{E_t^A \left[\int_t^T c_A^{(B)} (s)^{1-\gamma} \, ds \right]} - 1, \tag{4.38}$$

where $c_A^{(M)}(.)$ and $c_A^{(B)}(.)$ represent the rational investor's optimal consumption in the presence and the absence of bounded rational investor respectively.

Panel A of Figure 4 plots the changes in the rational investor's welfare with the difference between the drift of signal process and dividend growth.

It demonstrates that the rational investor gains from the trading with the bounded rational investor when the difference is less than 0.4%, and his welfare change reaches the maximum when the difference is around -0.35%.

INSERT FIGURE 4 HERE

Panel B of Figure 4 illustrates the changes in the rational investor's welfare with the consumption share of the bounded rational investor. When the drift of signal process is 2%, the rational investor gains from trading and his welfare gain increases with the consumption share of the bounded rational investor. Inversely, he loses when the drift of signal process is 4% and his welfare loss decreases more as the bounded rational investor's consumption share increases. Therefore, I conclude that the rational investor may experience a decrease in welfare in the presence of the bounded rational investor, which is consistent with Hirshleifer, Subrahmanyam and Titman (2004), who suggest that irrational trading may not provide profit opportunities to rational investors.

4.5 The Unobserved Drift of Signal Process

In the previous sections, I analyze the impact of investors' bounded rational behavior on asset pricing when the drift of signal process is observable. In this section, I will analyze a more realistic case: the unobserved drift of signal process.

In the same economy of Section 4.2, the exogenous processes of aggregate dividends and signal satisfy

$$
\begin{aligned}
d \ln D(t) &= \mu_D dt + {\sigma_D}^T dW(t), \\
de(t) &= \mu_e dt + \sigma_e^T dW(t),
\end{aligned} \tag{4.39}
$$

where both μ_D and μ_e are unobservable and $\mu_e \neq \mu_D$. At time t, investors observe the realizations of dividends and signal, and deduce their estimates of μ_D and μ_e. In the following, I will derive their estimates of μ_D and μ_e respectively.

4.5.1 Investor's Evolution of Beliefs

Rational Investor

I assume that rational investor A knows that signal e provides information about W, but not about μ_D; that updates his estimate of μ_D, μ_e, and their covariance via standard filtering theory. Therefore, he perceives the dividend process as follows

$$
\begin{aligned}
d \ln D(t) &= \mu_D^A(t)\, dt + \sigma_D^T dW^A(t), \\
de(t) &= \mu_e^A(t)\, dt + \sigma_e^T dW^A(t),
\end{aligned} \tag{4.40}
$$

and his estimates of μ_D and μ_e satisfy

$$
\begin{aligned}
d\mu_D^A(t) &= d\mu_e^A(t) = v^A \begin{bmatrix} g_1^A \left(\mu_D - \mu_D^A(t)\right) dt \\ +g_2^A \left(\mu_e - \mu_e^A(t)\right) dt \\ +h_1^A dW_1(t) + h_2^A dW_2(t) \end{bmatrix}, \\
v^A(t) &= \frac{\delta v^A(0)}{\delta + v^A(0)\, t},
\end{aligned} \tag{4.41}
$$

where

$$
\begin{aligned}
g_1 &\equiv \frac{\sigma_{e1}\left(\sigma_{e1}-\sigma_{D1}\right)+\sigma_{e2}\left(\sigma_{e2}-\sigma_{D2}\right)}{\left(\sigma_{D1}\sigma_{e2}-\sigma_{e1}\sigma_{D2}\right)^2}, \\
g_2 &\equiv \frac{\sigma_{D1}\left(\sigma_{D1}-\sigma_{e1}\right)+\sigma_{D2}\left(\sigma_{D2}-\sigma_{e2}\right)}{\left(\sigma_{D1}\sigma_{e2}-\sigma_{e1}\sigma_{D2}\right)^2}, \\
h_1 &\equiv \frac{\sigma_{e2}-\sigma_{D2}}{\sigma_{D1}\sigma_{e2}-\sigma_{e1}\sigma_{D2}}, \\
h_2 &\equiv \frac{\sigma_{e1}-\sigma_{D1}}{\sigma_{e1}\sigma_{D2}-\sigma_{D1}\sigma_{e2}}, \\
\delta &\equiv \left(\sigma_{e1}-\sigma_{D1}\right)^2+\left(\sigma_{e2}-\sigma_{D2}\right)^2. \qquad (4.42)
\end{aligned}
$$

(4.41) means that rational investor A has correct estimates of μ_D and μ_e in the long run since v^A will converge to zero.

Bounded Rational Investor

The Bounded rational investor B believes that the signal e provides information about μ_D, and then assumes $\mu_D^B(t)=\mu_e^B(t)$. Thus, from the perspective of investor B, the processes of dividends and signal follow the dynamics

$$
\begin{aligned}
d\ln D(t) &= \mu_D^B(t)\,dt+\sigma_D^T dW^B(t), \\
de(t) &= \mu_D^B(t)\,dt+\sigma_e^T dW^B(t), \qquad (4.43)
\end{aligned}
$$

and his estimate of μ_D evolves over time given by

$$
\begin{aligned}
d\mu_D^B(t) &= v^B(t)\left[\begin{array}{c}\left(g_1+g_2\right)\left(\frac{g_1\mu_D+g_2\mu_e}{g_1+g_2}-\mu_D^B(t)\right)dt \\ +h_1 dW_1(t)+h_2 dW_2(t)\end{array}\right], \\
v^B(t) &= \frac{\delta v^B(0)}{\delta+v^B(0)\,t}. \qquad (4.44)
\end{aligned}
$$

(4.44) means that bounded rational investor B believes that dividend growth converges to $\frac{g_1\mu_D+g_2\mu_e}{g_1+g_2}$, which deviates from actual growth μ_D. This is not surprising, since he incorrectly believes that the signal provides information on dividend growth.

4.5.2 Numerical Analysis

Using the methodologies adopted in Section 4.3, I can derive investors' optimal consumptions. However, it is impossible to decompose asset volatility (as (4.31) and (4.32)), and therefore I cannot derive expected premium for assets. However, I can analyze the impacts of investors' bounded rational behavior on stock price and the rational investor's welfare changes using the same parameters and simulation technologies used in Section 4.4.

Stock Price

In the case of unobserved drift of signal process, the impact of investors' bounded rational behavior on stock price is similar to that in the case of observed drift of signal process, except that the stock price in the benchmark economy is not constant. The reason is that investors need to estimate unobserved drift of signal process by observing the realizations of dividends and information, this generates the second dimension of "learning risk", the stock price in the benchmark economy for the case of unobserved drift of signal process is lower than that for the observed drift of signal process. Panel A of Figure 5 plots the changes in stock price with the difference between the drift of signal process and dividend growth. The stock price is lower than that in the benchmark economy when the difference between the drift of signal process and dividend growth is below 0.5%. The intuition for this is very similar to that for the case of observed drift of signal process, since the bounded rational investor's estimate of dividend growth deviates from actual growth in the long run. Panel B of Figure 5 plots the changes in stock price against the consumption share of the bounded rational investor, and the results are quite similar to the case of observed drift of signal process.

INSERT FIGURE 5 HERE

Welfare Analysis

Intuitively, when the drift of signal process is unobservable, the rational investor's welfare gains should not change significantly since all investors update their estimates of the drift of signal process (e.g., see Figure 6). The only difference is that the rational investor gains from trading with the bounded rational investor when the difference between the drift of signal process and dividend growth is lower than 0.5% rather than 0.4%. This has a trivial impact on rational investor's welfare changes.

INSERT FIGURE 6 HERE

4.6 Conclusion

In an incomplete information setting, this paper investigates the impact of investors' bounded rational behavior on asset pricing. The bounded rational investor perceives "correlated" information (which consists of news items that are correlated with fundamentals, but provide no information on them) as "fundamental" information (which consists of news items about expected changes in fundamentals). This process generates "bounded rational risk". Asset prices and volatilities of asset returns are derived. Specifically, the equity premium and the stock volatility are increased only when the drift of signal (e.g., "correlated" information) is small enough relative to dividend growth.

A natural extension of this paper would be to analyze the impact of investors' bounded rational behavior on asset prices when information processing is costly. Another possible extension would be to consider a case in which both dividend growth and the drift of signal process follow a mean-reverting process and some investors overestimate their correlation, making it possible to analyze the impact of over-reactive investors on asset pricing.

Appendix: Proof

1. Volatilities of Asset Prices

Let $M(t)$ be the $\mathcal{L}^2(P_1)$ - martingale

$$M(t) = E^A\left[\int_0^T \varepsilon(s)^{1-\gamma}\left(1+\lambda(s)^{\frac{1}{\gamma}}\right)^\gamma ds|\mathcal{F}(t)\right]. \tag{4.45}$$

Following the martingale representation theorem,

$$M(t) = M(0) + \int_0^T \left[\phi_1(t)\, dW_1^A(t) + \phi_2(t)\, dW_2^A(t)\right] dt, \tag{4.46}$$

where ϕ_j $(j = 1, 2)$ is the optional projection of the derivative of $M(t)$. Therefore, stock price is given by

$$S(t) = D(t)^\gamma \left(1+\lambda(t)^{\frac{1}{\gamma}}\right)^{-\gamma}\left[M(t) - \int_0^t D(s)^{1-\gamma}\left(1+\lambda(s)^{\frac{1}{\gamma}}\right)^\gamma ds\right], \tag{4.47}$$

applying Ito's lemma to above equation and matching the diffusion coefficients with the dynamics of S, the volatilities of stock price are given by $(j = 1, 2)$

$$\sigma_{Sj} = \gamma\sigma_{Dj} + \frac{\lambda(t)^{\frac{1}{\gamma}}}{1+\lambda(t)^{\frac{1}{\gamma}}}\Phi_j^{(M)}(t) + \frac{\phi_j(t)}{\xi^A(t)\, S(t)}. \tag{4.48}$$

Following the Clark-Ocone formula and the Malliavin derivatives (e.g., Nualart, 1995), I have

$$\phi_j(t) = E_t^A\left[\int_t^T D(s)^{1-\gamma}\left(1+\lambda(s)^{\frac{1}{\gamma}}\right)^\gamma \left[\begin{array}{c} \frac{\lambda(s)^{\frac{1}{\gamma}}}{1+\lambda(s)^{\frac{1}{\gamma}}}\frac{D_t^j\lambda(s)}{\lambda(s)} \\ +(1-\gamma)\frac{D_t^j D(s)}{D(s)} \end{array}\right] ds\right], \tag{4.49}$$

Similarly, the bond volatility is given by

$$\sigma_{Bj} = \gamma\sigma_{Dj} + \frac{\lambda(t)^{\frac{1}{\gamma}}}{1+\lambda(t)^{\frac{1}{\gamma}}}\Phi_j^{(M)}(t) + \frac{\psi_j(t)}{\xi^A(t)B_T(t)}, \tag{4.50}$$

$$\psi_j(t) = E_t^A\left[\left(\frac{1+\lambda(T)^{\frac{1}{\gamma}}}{D(T)}\right)^{\gamma}\begin{pmatrix}\frac{\lambda(T)^{\frac{1}{\gamma}}}{1+\lambda(T)^{\frac{1}{\gamma}}}\frac{D_t^j\lambda(T)}{\lambda(T)} \\ -\gamma\frac{D_t^jD(T)}{D(T)}\end{pmatrix}\right]. \tag{4.51}$$

I solve for (4.1) and (4.25) to derive the following Malliavin derivatives

$$D_t^jD(s) = \sigma_{Dj}D(s)\left[1+\frac{(s-t)\nu(t)}{\sigma_{D1}^2+\sigma_{D2}^2}\right], \tag{4.52}$$

$$D_t^j\lambda(s) = \lambda(s)\left[\kappa_j(t,s) - \Phi_j^{(M)}(t)\right] \tag{4.53}$$

where

$$\kappa_j(t,s) \equiv -\frac{\sigma_{Dj}\nu(t)}{(\sigma_{D1}^2+\sigma_{D2}^2)(\sigma_{D1}\sigma_{e2}-\sigma_{D2}\sigma_{e1})}\begin{bmatrix}\frac{\sigma_{e1}^2+\sigma_{e2}^2}{\sigma_{e2}}\int_t^s\Phi_j^{(M)}(u)\,du \\ +\int_t^s\sigma_{e2}dW_1^A(u) \\ -\int_t^s\sigma_{e1}dW_2^A(u)\end{bmatrix}. \tag{4.54}$$

Rearranging (4.48)-(4.54) to derive asset volatility. Q.E.D.

2. Investor's Evolution of Beliefs (unobserved drift of signal process)

2.1 Rational Investor A

The processes of dividends and signal perceived by the rational investor satisfy

$$d\ln D = \mu_D^A dt + \sigma_{D1}dW_1^A + \sigma_{D2}dW_2^A.$$

$$de = \mu_e^A dt + \sigma_{e1}dW_1^A + \sigma_{e2}dW_2^A. \tag{4.55}$$

Therefore, I have

$$\begin{aligned} dW_1^A &= dW_1 + \frac{\sigma_{e2}\left(\mu_D - \mu_D^A\right) - \sigma_{D2}\left(\mu_e - \mu_e^A\right)}{\sigma_{D1}\sigma_{e2} - \sigma_{D2}\sigma_{e1}} dt, \\ dW_2^A &= dW_2 + \frac{\sigma_{D1}\left(\mu_e - \mu_e^A\right) - \sigma_{e1}\left(\mu_D - \mu_D^A\right)}{\sigma_{D1}\sigma_{e2} - \sigma_{D2}\sigma_{e1}} dt. \end{aligned} \tag{4.56}$$

If I define $\mu(t) \equiv \binom{\mu_D}{\mu_e}$, $\mu^A(t) \equiv \binom{\mu_D^A}{\mu_e^A} \equiv E^A\left[\mu(t)\,|\mathcal{F}(t)\right]$, $\sigma \equiv \begin{pmatrix} \sigma_{D1} & \sigma_{D2} \\ \sigma_{e1} & \sigma_{e2} \end{pmatrix}$,

and $v^A(t) \equiv \begin{pmatrix} v_{11}^A & v_{12}^A \\ v_{21}^A & v_{22}^A \end{pmatrix} \equiv E_t^A\left[\left(\mu_D^A(t) - \mu_D(t)\right)^2 |\mathcal{F}(t)\right]$, then

$$\begin{aligned} d\mu^A(t) &= v^A(t)\left(\sigma\sigma^T\right)^{-1}\begin{pmatrix} \sigma_{D1} dW_1^A + \sigma_{D2} dW_2^A \\ \sigma_{e1} dW_1^A + \sigma_{e2} dW_2^A \end{pmatrix}, \\ dv^A(t) &= -v^A(t)\left(\sigma\sigma^T\right)^{-1} v^A(t)^T dt, \end{aligned} \tag{4.57}$$

I can verify that $v_{ij}^A(t) \equiv v^A(t)$ solves above ODEs, then

$$\frac{dv^A(t)}{dt} = -\delta v^A(t)^2 \Rightarrow v^A(t) = \frac{\delta v^A(0)}{\delta + v^A(0)\,t}, \tag{4.58}$$

where $\delta \equiv \left(\sigma_{e1} - \sigma_{D1}\right)^2 + \left(\sigma_{e2} - \sigma_{D2}\right)^2$. Therefore,

$$d\mu_D^A = d\mu_e^A = v^A \begin{bmatrix} g_1\left(\mu_D - \mu_D^A\right) dt + g_2\left(\mu_e - \mu_e^A\right) dt \\ +h_1 dW_1 + h_2 dW_2 \end{bmatrix}, \tag{4.59}$$

where

$$\begin{aligned} g_1 &\equiv \frac{\sigma_{e1}\left(\sigma_{e1} - \sigma_{D1}\right) + \sigma_{e2}\left(\sigma_{e2} - \sigma_{D2}\right)}{\left(\sigma_{D1}\sigma_{e2} - \sigma_{e1}\sigma_{D2}\right)^2}, \\ g_2 &\equiv \frac{\sigma_{D1}\left(\sigma_{D1} - \sigma_{e1}\right) + \sigma_{D2}\left(\sigma_{D2} - \sigma_{e2}\right)}{\left(\sigma_{D1}\sigma_{e2} - \sigma_{e1}\sigma_{D2}\right)^2}, \\ h_1 &\equiv \frac{\sigma_{e2} - \sigma_{D2}}{\sigma_{D1}\sigma_{e2} - \sigma_{e1}\sigma_{D2}}, \\ h_2 &\equiv \frac{\sigma_{D1} - \sigma_{e1}}{\sigma_{D1}\sigma_{e2} - \sigma_{e1}\sigma_{D2}}, \\ \delta &\equiv \left(\sigma_{e1} - \sigma_{D1}\right)^2 + \left(\sigma_{e2} - \sigma_{D2}\right)^2. \end{aligned} \tag{4.60}$$

Q.E.D.

2.2 Bounded Rational Investor B

The processes of dividends and signal perceived by the bounded rational investor are

$$\begin{aligned} d\ln D &= \mu_D^B dt + \sigma_{D1} dW_1^B + \sigma_{D2} dW_2^B, \\ de &= \mu_D^B dt + \sigma_{e1} dW_1^B + \sigma_{e2} dW_2^B, \end{aligned} \tag{4.61}$$

therefore, I have

$$\begin{aligned} dW_1^B &= dW_1 - \frac{\sigma_{e2}\left(\mu_D - \mu_D^B\right) - \sigma_{D2}\left(\mu_e - \mu_D^B\right)}{\sigma_{e1}\sigma_{D2} - \sigma_{e2}\sigma_{D1}} dt, \\ dW_2^B &= dW_2 + \frac{\sigma_{e1}\left(\mu_D - \mu_D^B\right) - \sigma_{D1}\left(\mu_e - \mu_D^B\right)}{\sigma_{e1}\sigma_{D2} - \sigma_{e2}\sigma_{D1}} dt. \end{aligned} \tag{4.62}$$

His estimate of unobserved dividend growth is given by

$$\begin{aligned} d\mu_D^B &= v^B \begin{pmatrix} 1 & 1 \end{pmatrix} \left(\sigma\sigma^T\right)^{-1} \begin{pmatrix} \sigma_{D1} dW_1^B + \sigma_{D2} dW_2^B \\ \sigma_{e1} dW_1^B + \sigma_{e2} dW_2^B \end{pmatrix} \\ &= v^B(t) \begin{bmatrix} (g_1 + g_2)\left(\frac{g_1\mu_D + g_2\mu_e}{g_1 + g_2} - \mu_D^B\right) dt \\ + h_1 dW_1 + h_2 dW_2 \end{bmatrix}, \end{aligned} \tag{4.63}$$

and its variance is

$$dv^B(t) = -\delta v^B(t)^2 \Rightarrow v^B(t) = \frac{\delta v^B(0)}{\delta + v^B(0)\, t}. \tag{4.64}$$

Q.E.D.

Table I: Parameters of Simulation

Parameters	*Symbol*	*Value*
Growth rate of dividends	μ_D	3%
Dividend volatility caused by W_1	σ_{D1}	3.5%
Dividend volatility caused by W_2	σ_{D2}	3.5%
Drift of signal process	μ_e	2% and 4%
Signal volatility caused by W_1	σ_{e1}	0
Signal volatility caused by W_2	σ_{e2}	3.5%
Relative relative risk aversion of investors	γ	3
Investment horizon	T	30 years
Investor A's initial estimate of	$\mu_D^A(0)$	3%
dividend growth rate		
Investor A's initial estimate of	$\nu^A(0)$	$(3\%)^2$
variance of dividend growth		

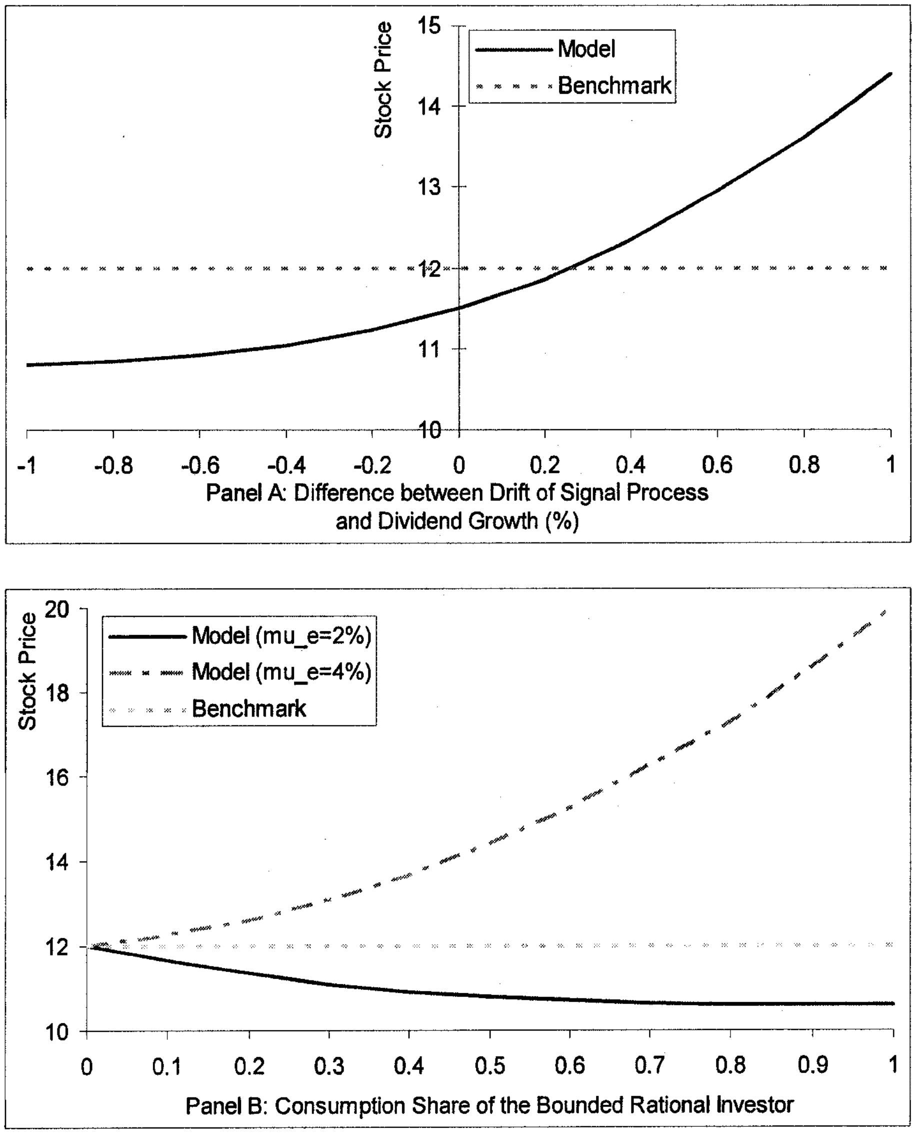


Figure 1: Panel A plots the changes in stock price with the difference between the drift of signal process and dividend growth. All investors have identical initial consumption share; i.e., w = 0.5. The rational investor has a correct initial estimate of dividend growth. Panel B plots the changes in stock price with the consumption share of the bounded rational investor. μ_e=2% or 4% represents that the drift of signal process is 2% or 4%, respectively.

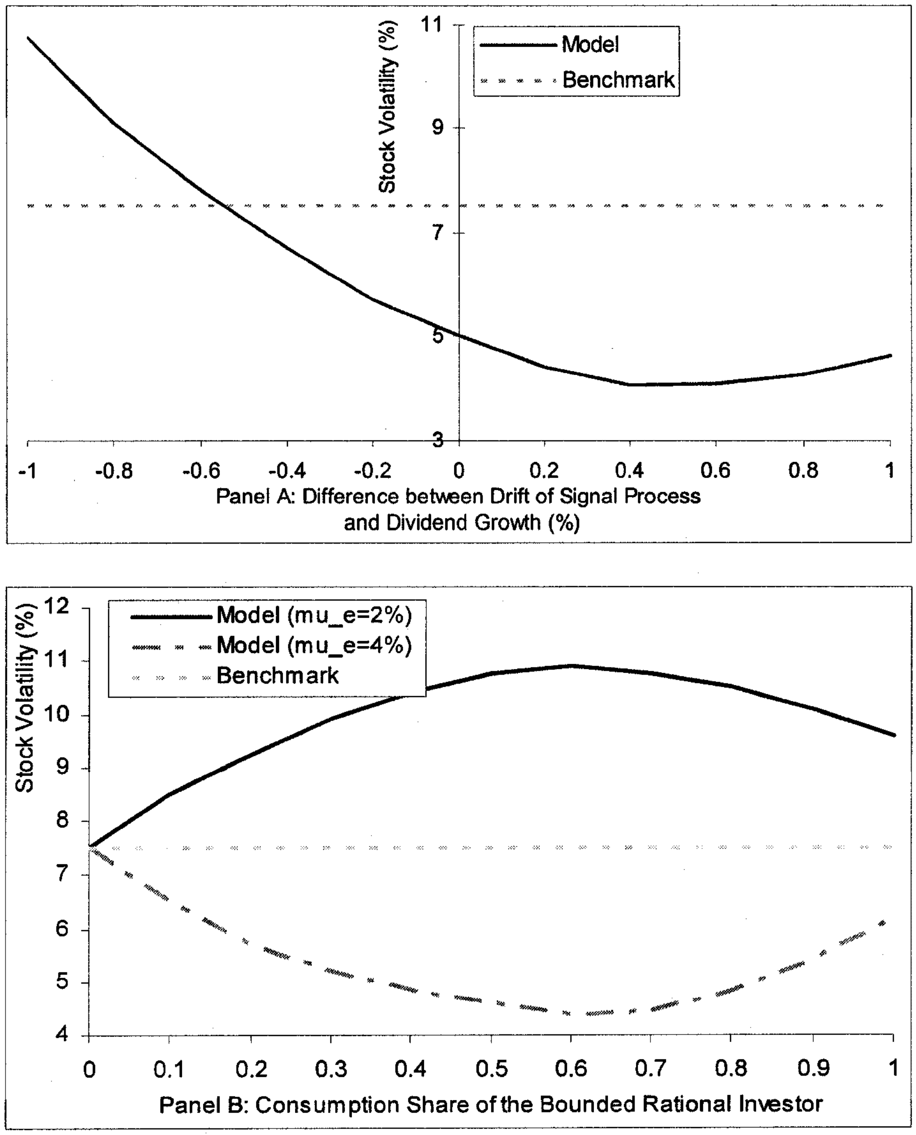


Figure 2: Panel A plots the changes in stock volatility with the difference between the drift of signal process and dividend growth. All investors have identical initial consumption share; i.e., $w = 0.5$. The rational investor has a correct initial estimate of dividend growth. Panel B plots the changes in stock volatility with the consumption share of the bounded rational investor. μ_e=2% or 4% represents that the drift of signal process is 2% or 4%, respectively.

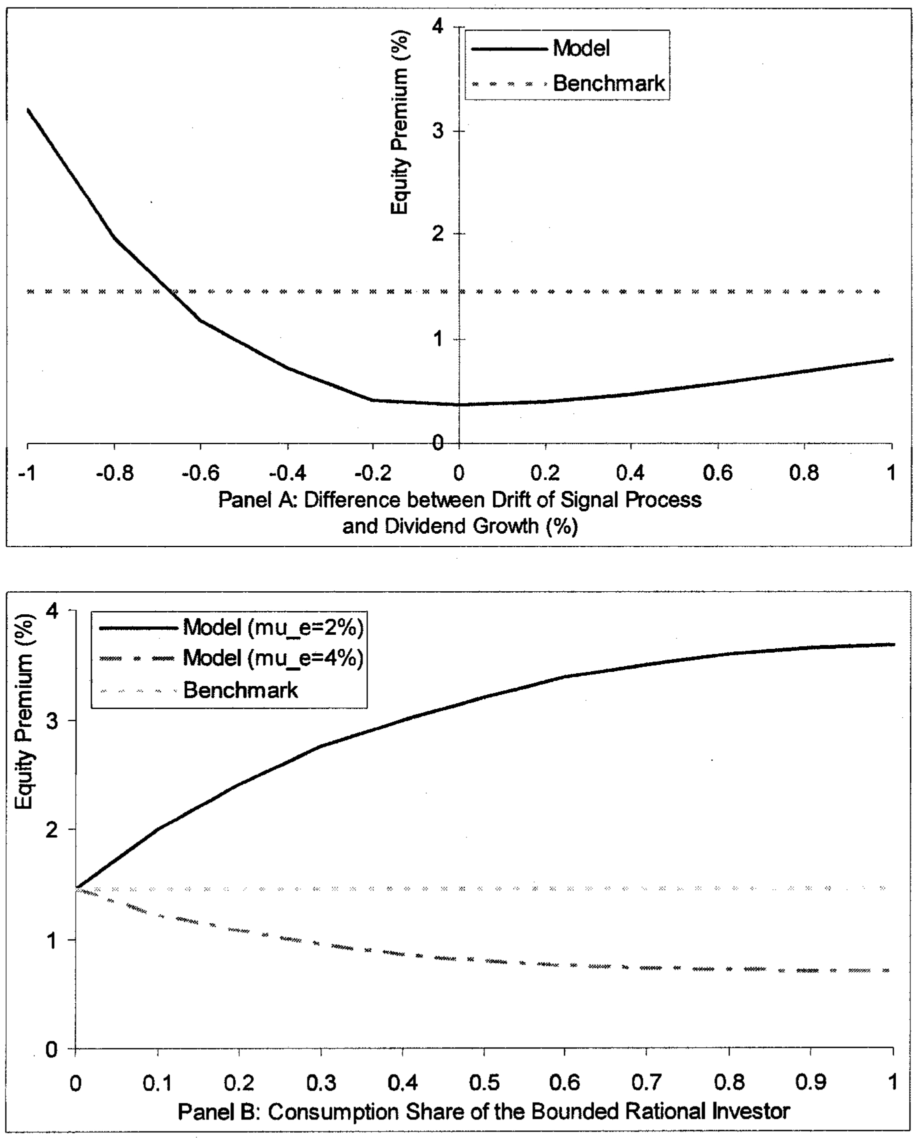
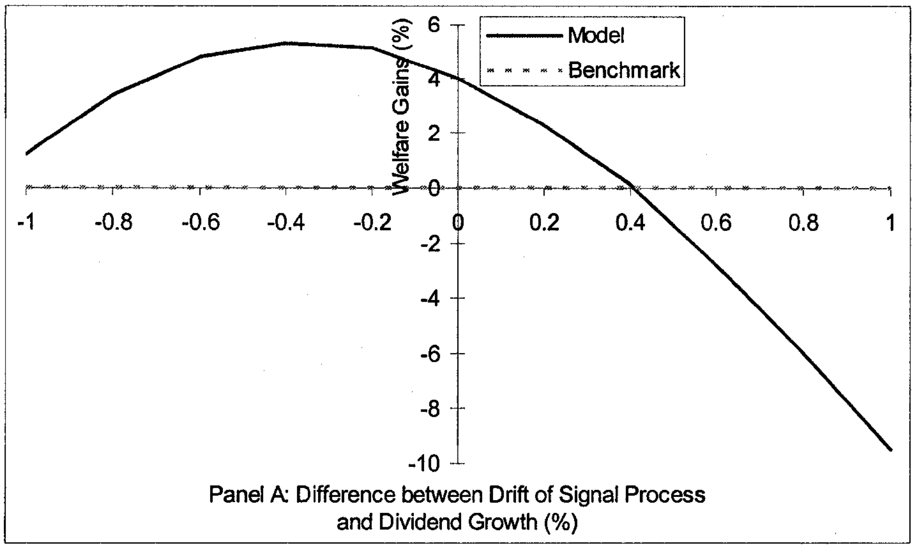


Figure 3: Panel A plots the changes in expected equity premium with the difference between the drift of signal process and dividend growth. All investors have identical initial consumption share; i.e., $w = 0.5$. The rational investor has a correct initial estimate of dividend growth. Panel B plots the changes in expected equity premium with the consumption share of the bounded rational investor. μ_e=2% or 4% represents that the drift of signal process is 2% or 4%, respectively.



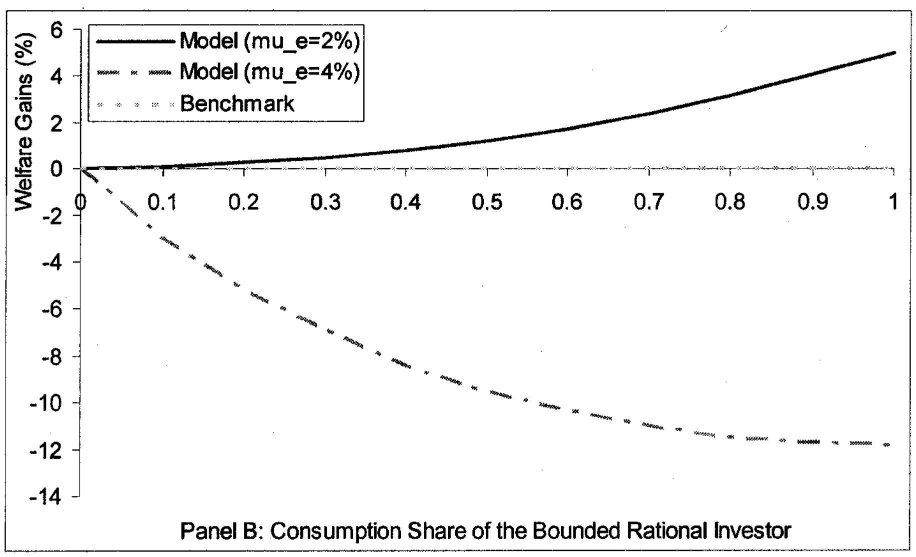
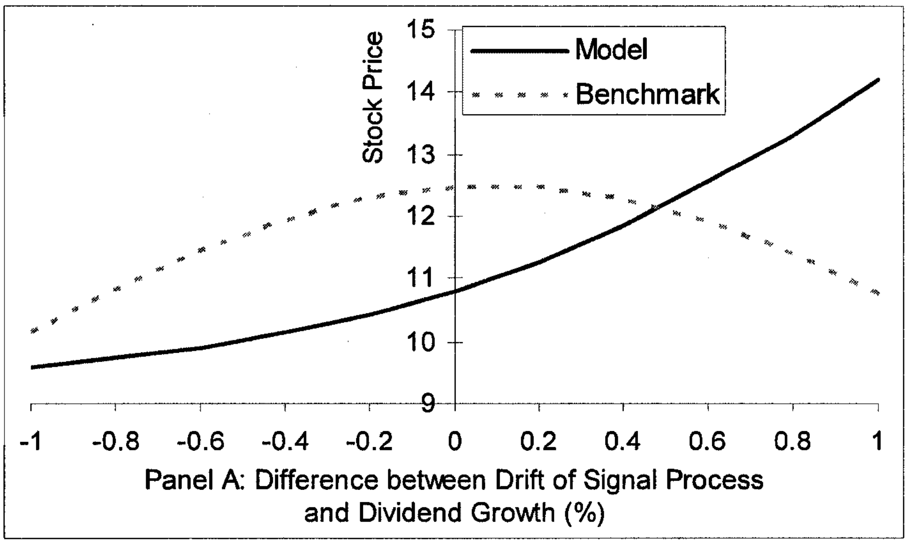


Figure 4: Panel A plots the changes in the rational investor's welfare with the difference between the drift of signal process and dividend growth. All investors have identical initial consumption share; i.e., $w = 0.5$. The rational investor has a correct initial estimate of dividend growth. Panel B plots the changes in the rational investor's welfare with the consumption share of the bounded rational investor. μ_e=2% or 4% represents that the drift of signal process is 2% or 4%, respectively.



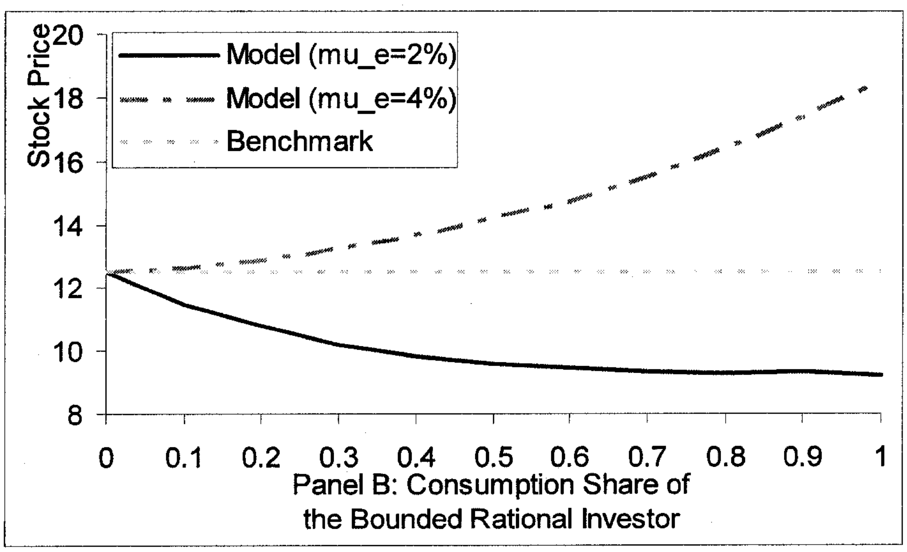


Figure 5: Panel A plots the changes in stock price with the difference between the drift of signal process and dividend growth. All investors have identical initial consumption share; i.e., $w = 0.5$. The rational investor has a correct initial estimate of dividend growth and the drift of signal process, and the bounded rational investor has a correct estimate of drift of signal process. Panel B plots the changes in stock price with the consumption share of the bounded rational investor. μ_e=2% or 4% represents that the drift of signal

process is 2% or 4%, respectively.

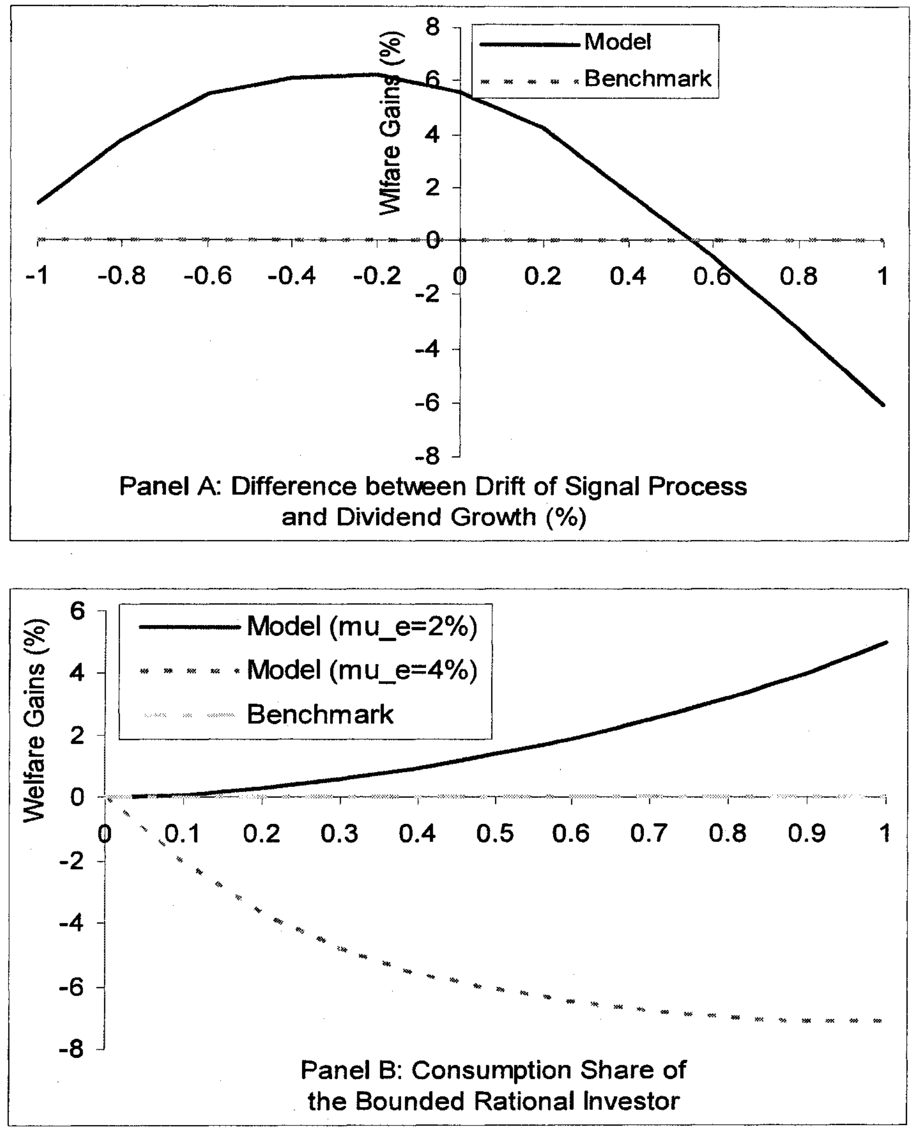


Figure 6: Panel A plots the changes in the rational investor's welfare with the difference between the drift of signal process and dividend growth. All investors have identical initial consumption share; i.e., $w = 0.5$. The rational investor has a correct initial estimate of dividend growth and the drift of signal process, and the bounded rational investor has a correct estimate of the drift of signal process. Panel B plots changes in the rational investor's welfare with the

consumption share of the bounded rational investor. μ_e=2% or 4% represents that the drift of signal process is 2% or 4%, respectively.

Chapter 5

SUMMARY AND CONCLUSION

In this chapter, I summarize the main findings of this thesis and its contributions to the literature on asset pricing. I also propose relevant research topics for future studies. The objective of this thesis is to investigate the effects of heterogeneous beliefs about future monetary policy and of bounded rational investors on asset pricing (e.g., stock price, expected equity premium and stock volatility). Moreover, I also investigate the impact of bounded rational investor on rational investor's welfare.

The main findings of Chapter III are as follows.

1. In a general framework, without making particular assumptions about the dynamics of aggregate consumption and money supply and investors' preferences, we find that the equilibrium economic quantities are all affected by the presence of heterogeneous beliefs. The most important feature of the model is that, under heterogeneous beliefs, agents place bets against each other on the money supply; therefore, shocks in the money supply affect the distribution of wealth in the economy. Thus, money supply shocks affect consumption: heterogeneity in beliefs makes money non-neutral. The additional "trading risk" leads to an extra factor in the pricing of assets that subsists even when preferences are separable.

One interesting result is that, when investors' utilities are additively separable, assets' risk premia are only affected by their exposure to monetary risk in the presence of heterogeneous beliefs. This shows that heterogeneity in beliefs has a profound effect on the equilibrium, because the structure of the expressions is affected. This result is independent of the specific shape of the investors' utilities and implies that it is possible to obtain novel, interesting implications in a tractable setup, with separable preferences.

2. If investors have logarithmic utility preferences, we can solve the expected inflation, nominal interest rate and its term structure explicitly. Nominal interest rates are driven by investors' expectations of future monetary policy. The most robust implication is that heterogeneity in beliefs increases the volatility of inflation: when a positive shock to the money supply occurs, not only does the extra amount of money cause inflation, but those investors who expect higher money supply growth and higher future inflation "win their bet" and so their weight in the economy increases, which in turn generates extra inflation.

3. If investors have separable, constant relative risk aversion (CRRA) utility functions, we can provide an explicit formula for the stock price and its volatility. It is rare to be able to explicitly compute a stock price that is not trivial (for example, equal to the amount of aggregate dividends) and exhibits interesting properties, as is the case here. In particular, heterogeneity in beliefs on monetary policy generates a much higher volatility for the stock, and real interest rates are also increased. The implications for the equity premium are ambiguous; nevertheless, for some plausible parameter values, the equity premium is much increased over a standard model.

4. Highlighting the significant impact of heterogeneous beliefs about monetary policy on volatility is the key empirical implication of this paper, and we perform an empirical analysis of this relationship. Using economists' forecasts

as provided by the Survey of Professional Forecasters and S&P 500 returns, the prediction of this theoretical model is strongly supported by the data, with a significant positive relationship between heterogeneity in beliefs and volatility.

This model sheds some new light on the issue of how much transparency is optimal on the part of central banks, because it is intuitive that greater transparency should decrease the amount of heterogeneity in beliefs about monetary policy, which, according to this model, would reduce stock market and inflation volatility (as well as decrease real interest rates).

Natural extensions of this model would include assuming more sophisticated and realistic dynamics for the aggregate consumption and money supply (with mean-reversion and/or regime switches in monetary policy), taking into account agents updating their beliefs over time, and a more thorough investigation of the implications for the price of money (which obeys a backward stochastic differential equation that is, in general, quite intractable) and interest rates. With more realistic, less tractable assumptions, this model could complement the growing, recent literature on the interrelation between monetary policy and bond yields (e.g., Piazzesi (2005)). As far as empirical work is concerned, a particularly interesting investigation could include international comparisons, taking into account differences in the transparency of monetary policy, which is likely to lead to reduced heterogeneity in beliefs; the present model provides tools to understand the impact of these differences.

The main results of Chapter IV are as follows. The rational investors update their beliefs accurately, which generates "learning risk", while the bounded-rational investors' incorrect updating of their beliefs generates "bounded rational risk". Compared to a benchmark economy in which all investors are rational,

1. The presence of the bounded rational investor tends to increase stock price when the drift of signal process significantly exceeds dividend growth. The intuition is: when the drift of signal process is higher (lower) than dividend growth, the bounded rational investor is more optimistic (pessimistic) than the rational investor since his estimate of dividend growth is equal to the drift of signal process, which in turn makes him perceive the stock more favorably (unfavorably) and therefore stock price is increased (reduced).

2. Stock volatility is increased when the drift of signal process is small enough relative to dividend growth, a condition under which the "learning risk" tends to dominate the economy and increase stock volatility. As the drift of signal process increases, "bounded rational risk" offsets the effect of "learning risk" and stock volatility is reduced. Moreover, stock volatility may change with the arrival of new information.

3. Expected equity premium is increased when the drift of signal process is small enough relative to dividend growth, which has a similar explanation to that of decreased stock price.

4. The rational investor experiences a decrease in welfare in the presence of the bounded rational investor when the drift of signal process significantly exceeds dividend growth.

A natural extension of this paper would be to analyze the impact of investors' bounded rational behavior on asset prices when information processing is costly. Another possible extension would be to consider a case in which both dividend growth and the drift of signal process follow a mean-reverting process and investors overestimate their correlation, making it possible to analyze the impact of over-reactive investors on asset pricing.

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