INFLUENCE OF TUBE ORIENTATION ON COMBINED FREE AND FORCED LAMINAR CONVECTION HEAT TRANSFER

by

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ABSTRACT

Combined free and forced-convection inside inclined circular tubes is studied theoretically. The case considered is that of fully developed laminar flow with constant pressure gradient, and constant heat flux. Fluid properties are considered constant except for the variation of density in the buoyancy terms. Upward flow only is considered. Velocity and temperature fields are calculated by perturbation analysis in terms of power series of Rayleigh numbers. A detailed analysis of the final equations is made to determine the range of values of non-dimensional parameter such as Rayleigh and Reynolds numbers over which the mathematical results are valid. Nusselt numbers are calculated based on bulk temperature difference and in final form are also expressed in terms of power series of Rayleigh numbers. Rayleigh number appears to be the most dominant parameter in equations of velocity and temperature fields and Nusselt number. However, Rayleigh and Reynolds number product and Prandtl number also influence the equations independently. As the tube inclination varies from horizontal, the Nusselt number increases up to a maximum which may occur before the vertical position is reached. The angle at which this maximum occurs appears to be a function of Rayleigh, Reynolds and Prandtl number, and in most instances lies between 20° and 60° of tube inclination.

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NOMENCLATURE

Roman Letter Symbols

A = axial temperature gradient (assumed constant), $\partial t/\partial x$ °F/ft.

a = tube radius in feet.

C = $(\partial p/\partial x + \rho_w g_x)$, axial pressure gradient in fluid, lb/ft^3.

c_p = specific heat of fluid at constant pressure, BTU/lb. °F.

g = acceleration due to gravity, ft/sec^2.

g_x, g_r, g_θ = components of acceleration due to gravity in three coordinate directions, ft/sec^2.

h = $q/(T_w - T_b)$ = heat transfer coefficient for fully developed flow based on bulk temperature difference, BTU/sec.ft^2 °F.

N_{Gr} = $\beta g A a^4/\nu^2$ Grashof number, dimensionless.

N_{Nu} = $2ah/k$ Nusselt number, dimensionless.

N_{Nu_b} = Nusselt number based on bulk temperature difference, dimensionless.

N_{Pr} = $c_p\mu/k$, Prandtl number, dimensionless.

N_{Ra} = $N_{Gr} \times N_{Pr}$ = $\beta g A a^4 \rho^2 c_p/\mu k$, Rayleigh number, dimensionless

N_{Re} = $-Ca^3/(4\rho\nu^2)$, Reynolds number based on pipe diameter, dimensionless.

p = $p'x + P(r,\theta)$, static fluid pressure, lb/ft^2.

q = wall heat flux density, average over the circumference BTU/sec.ft^2.

R = r/a, radial distance from the centre line of the tube, dimensionless.

r = radial distance in cylindrical coordinate system, measured from the centre line of the tube, ft.

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R = r/a, radial distance from the centre line of the tube, dimensionless.

r = radial distance in cylindrical coordinate system, measured from the centre line of the tube, ft.

T = temperature of the fluid at any point, ^{o}F.

T_b = bulk temperature of the fluid at any section, ^{o}F.

T_o = temperature of the tube wall at beginning of the fully developed flow, ^{o}F.

T^* = $(T_w - T)/(A\ a\ N_{Pr})$, difference between the wall temperature and any point of the fluid at same section, dimensionless.

T^*_b = $(T_w - T_b)/(A\ a\ N_{Pr})$, difference between the wall temperature and bulk temperature of the fluid at the same section, dimensionless.

v = velocity of a fluid particle, ft/sec.

v_r = fluid velocity measured along the radial coordinate of tube, ft/sec.

v_x = axial velocity of fluid measured along the x-axis of tube, ft/sec.

v_θ = angular velocity of fluid measured along the angular coordinate of tube, ft/sec.

$v_{x_{av}}$ = average velocity along the x-axis of tube, ft/sec.

V_x = $v_x/(\nu/a)$, axial velocity, dimensionless.

$V_{x_{av}}$ = $v_{x_{av}}/(\nu/a)$, average axial velocity, dimensionless.

x = axial coordinate of the tube measured in upward direction (direction of flow), ft.

<u>Greek Letter Symbols</u>

α = tube inclination measured from the horizontal position, degrees.

β = volumetric coefficient of thermal expansion of the fluid, $1/^{o}F$.

Δ = difference between two points.

∇^2 = Laplacian in cylindrical coordinates.

$$= \frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial \theta^2}$$

∇^4 = Laplacian of the Laplacian in cylindrical coordinates

$$= \partial^4/\partial R^4 + (2/R)\,\partial^3/\partial R^3 - (1/R^2)\,\partial^2/\partial R^2 + (1/R^3)\,\partial/\partial R + (2/R^2)\,\partial^4/\partial\theta^2\partial R^2$$
$$- (2/R^3)\,\partial^3/\partial\theta^2\partial R + (4/R^4)\,\partial^2/\partial\theta^2 + (1/R^4)\,\partial^4/\partial\theta^4$$

k = thermal conductivity of fluid, BTU/sec.sq.ft. ^{0}F/ft.

θ = angular position in cylindrical coordinate system, measured from the top point of the tube circumference, degrees.

μ = dynamic viscosity of fluid, lb/ft.sec.

ν = kinematic viscosity of fluid, ft^2/sec.

ρ = mass density of the fluid, lb.sec^2/ft^4.

ρ_w = mass density of the fluid at wall, lb.sec^2/ft^4.

ψ = Stokes stream function, dimensionless.

<u>Subscripts</u>

0, 1, 2 - refer to zero, lst and 2nd order perturbations.

0, 1, 2 - refer to points at various positions along the axis of the tube.

INTRODUCTION

In any convective heat transfer process, density differences arise due to differences in temperature, and under the influence of a gravitational force field natural-convection effects result. In a forced-convection case, associated with large Reynolds numbers connected with large flow velocities, where the forces and momentum transport rates are very large, the effects of natural-convection are negligible. If, on the other hand, buoyancy forces arising from density differences are relatively large, (as exemplified by large Grashof numbers) the forced convection effects may be ignored. However, in many cases of practical interest, both the effects of forced-convection and natural-convection may be of comparable order. An indication of the relative magnitude of the two effects can be obtained from the differential equations describing the flow. With the comparatively small velocities associated with laminar motion, the heat transfer is substantially affected by buoyancy forces and the resulting velocity fields. In this circumstance, in addition to Grashof, Reynolds, and Prandtl numbers, the parameters describing the geometry of heat transfer surface and flow orientation to the gravitational field are also important. The problem that has been most extensively studied is that of vertical round tubes, where gravitational force is parallel to the tube axis. Various aspects of combined free and forced-convection inside vertical circular tubes, ducts and channels were studied in references [1-20][1]. References [21-27] deal with the influence

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[1] Numbers in brackets refer to similarly numbered references in bibliography at end of paper.

of free-convection on forced flow in horizontal circular tubes and channels. Numerous studies of the influence of free-convection on forced-flow for external flows of boundary layer type are also available in the literature cited [28-31]. The case of pure free-convection inside vertical and inclined tubes, with both ends closed or open, has been investigated by references [32-34].

The present analytical study of combined free and forced-convection inside inclined tubes springs from an interest in its applications to flat plate solar collectors, which are normally placed at an inclined position. It is desired to know the influence of tube inclination on heat transfer. The case considered is that of uniform heat flux, which results in uniform temperature gradient along the wall. This happens to be approximately so in solar collectors. The analysis, however, is a generalized one.

FORMULATION OF THE PROBLEM

Consider a tube of radius "a" inclined at an angle α to the horizontal, as shown in Fig. 1. There is a uniform heat flux "q" around the circumference and per unit length of the tube. This heat flux could be due to solar energy absorbed by the flat plate collector in solid contact with the tube, or resistance heating of the tube, etc. On the slow laminar motion of the fluid, flowing under external pressure, buoyancy forces are superimposed due to differences in density arising out of differences in temperature. These buoyancy forces create a secondary flow, distorting the normal Poiseuille flow to a form of helical motion as shown in Fig. 2. Due to the buoyancy effects and the circulation of the fluid inside the tube, the circumferential distribution of tube temperature at any section will be no longer constant. The fact, however, that the thermal conductivity of the tube material is usually much higher than that of the fluid, will tend to minimize the circumferential temperature variation, so that a constant tube wall temperature at any section may be assumed. In addition, the specific heat, thermal conductivity and viscosity can be considered constant throughout the fluid. Density is also considered constant throughout except for its variation in the buoyancy term. Pressure gradient is assumed constant. For the case of uniform heat flux and under the above conditions, the temperature gradient within the fluid and at the tube surface becomes constant beyond the entrance length, as shown in Fig. 3. Also the temperature difference $(T_w - T)$, between tube wall and fluid at any section, is constant along the tube, barring the entrance length. Since this temperature difference creates buoyancy forces, resulting in secondary flow that gives rise to radial and angular velocities, therefore v_r, v_θ and v_x., (the radial, angular and axial velocities respectively) become independent of the axial distance x.

ANALYSIS

Using the cylindrical polar coordinate system where θ is measured from the top vertical position of the circumference, x is measured from the point of fully developed flow, and with the assumptions made in the previous section, the governing equations for laminar flow can be written as:

Continuity Equation

$$\frac{\partial (r\, v_r)}{\partial r} + \frac{\partial v_\theta}{\partial \theta} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Momentum Equations

Momentum equations in r, θ, and x directions are respectively:

$$\rho\left(v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r}\right) =$$

$$-\frac{\partial p}{\partial r} + \mu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right) - \rho g_r \quad . \quad . \quad . \quad (2)$$

$$\rho\left(v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r}\right) =$$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r^2}\right) + \rho g_\theta \quad . \quad . \quad (3)$$

$$\rho\left(v_r \frac{\partial v_x}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_x}{\partial \theta}\right) =$$

$$-\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial r^2} + \frac{1}{r}\frac{\partial v_x}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_x}{\partial \theta^2}\right) - \rho g_x \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

In the above equations, density in the buoyancy terms is to be expanded as $\rho = \rho_w \left\{1 + \beta (T_w - T)\right\}$, where β is the coefficient of thermal expansion of fluid. Buoyancy force is calculated using the difference between fluid temperature T and the temperature of the wall T_w. Gravitational components g_r, g_θ and g_x, are g Cosα Cosθ , g Cosα Sinθ and g Sinα respectively. Pressure gradient along the tube is considered constant, while across a section it is a function of r and θ. Temperature of the wall at any section is calculated as $T_w = T_o + (\partial t/\partial x)x$. Where T_o is the wall temperature at the point starting the fully developed flow, and $\partial T/\partial x = A$ being a constant.

Energy Equation

The energy equation is:

$$c_p \left(v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \cdot \frac{\partial T}{\partial \theta} + v_x \frac{\partial T}{\partial x} \right) = k \left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial T}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2 T}{\partial \theta^2} \right) \quad . \quad . \quad . \qquad (5)$$

In this equation, conduction along tube axis, dissipation and pressure terms are ignored.

The pressure terms from equation (2) and (3) can be eliminated by differentiating equation (2) with respect to θ and equation (3) with respect to r and then reducing them to one equation. The resulting equation, along with equations (1), (4) and (5) can now be reduced and non-dimensionalized with the help of the stream function Ψ expressed as:

$$r\, v_r / \nu = \partial \Psi / \partial \theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6a)$$

$$v_\theta / \nu = - \partial \Psi / \partial r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (6b)$$

and the parameters,

dimensionless radius $R = r/a$ . . . . (7a)

dimensionless axial velocity $V_x = v_x/(\nu/a)$ . . . . (7b)

dimensionless temperature $T^* = (T_W - T)/(A\ a\ N_{pr})$ . . . . (7c)

The momentum equations (2) and (3) become:

$$\nabla^4 \Psi + \frac{1}{R} \left(\frac{\partial \Psi}{\partial R} \cdot \frac{\partial}{\partial \theta} - \frac{\partial \Psi}{\partial \theta} \cdot \frac{\partial}{\partial R} \right) \nabla^2 \Psi =$$

$$N_{Ra} \left(\frac{\partial T^*}{\partial R} \sin\theta + \frac{1}{R} \frac{\partial T^*}{\partial \theta} \cos\theta \right) \cos\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (8)$$

Equation (4) is reduced to:

$$\nabla^2 V_x + \frac{1}{R}\left(\frac{\partial \Psi}{\partial R} \cdot \frac{\partial}{\partial \Theta} - \frac{\partial \Psi}{\partial \Theta} \cdot \frac{\partial}{\partial R} \right) V_x + 4 N_{Re} - N_{Ra} T^* \operatorname{Sin}\alpha = 0 \quad . \quad (9)$$

and the energy equation becomes:

$$\nabla^2 T^* + \frac{N_{Pr}}{R}\left(\frac{\partial \Psi}{\partial R} \cdot \frac{\partial}{\partial \Theta} - \frac{\partial \Psi}{\partial \Theta} \cdot \frac{\partial}{\partial R} \right) T^* + V_x = 0 \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The boundary conditions for the above equations are:

$$T^* = V_x = \partial \Psi / \partial R = \partial \Psi / \partial \Theta = 0 \text{ at } R = 1 \quad . \quad . \quad . \quad (11)$$

and

$$T^*, V_x, R^{-1} \partial \Psi / \partial \Theta, \partial \Psi / \partial R \text{ are finite, at } R = 0 \quad . \quad . \quad . \quad (12)$$

Equations (8), (9) and (10) are non-linear, simultaneous partial differential equations. Their solution is extremely difficult, however a perturbation approach similar to Morton's [26] is made.

In the absence of any exact solution for the dependent variables Ψ, V_x and T^* in equations (8), (9) and (10), these unknowns are expanded in a power series of N_{Ra}. In literature Rayleigh number is generally chosen for such an expansion, since this is the significant parameter indicative of velocity and temperature fields with free-convection effects.

$$\Psi = \Psi_o + N_{Ra} \Psi_1 + N_{Ra}^2 \Psi_2 + N_{Ra}^3 \Psi_3 + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

$$V_x = V_{x_o} + N_{Ra} V_{x_1} + N_{Ra}^2 V_{x_2} + N_{Ra}^3 V_{x_3} + \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$T^* = T_o^* + N_{Ra} T_1^* + N_{Ra}^2 T_2^* + N_{Ra}^3 T_3^* \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

To ensure convergence of the series, the numerical value of N_{Ra} must be small. For a desired accuracy of the result, number of terms chosen in the series depends upon the numerical value of N_{Ra}.

The equations (13) to (15) are now substituted in equations (8) to (10) and the terms of the like powers of N_{Ra} are grouped together. Taking the terms up to second order of Rayleigh number only, the three groups of equations are:

The terms with zero order of N_{Ra} are:

$$\nabla^2 V_{x_o} + 4 N_{Re} = 0 \quad \ldots \quad (16)$$

$$\nabla^2 T_o^* + V_{x_o} = 0 \quad \ldots \quad (17)$$

The terms with first power of N_{Ra} are:

$$\nabla^4 \Psi_1 = \left(\frac{\partial T_o^*}{\partial R} \operatorname{Sin}\theta + \frac{1}{R} \frac{\partial T_o^*}{\partial \theta} \operatorname{Cos}\theta \right) \operatorname{Cos}\alpha \quad \ldots \quad (18)$$

$$\nabla^2 V_{x_1} + \frac{1}{R} \left(\frac{\partial \Psi_1}{\partial R} \cdot \frac{\partial V_{x_o}}{\partial \theta} - \frac{\partial \Psi_1}{\partial \theta} \cdot \frac{\partial V_{x_o}}{\partial R} \right) - T_o^* \operatorname{Sin}\alpha = 0 \quad \ldots \quad (19)$$

$$\nabla^2 T_1^* + \frac{N_{Pr}}{R} \left(\frac{\partial \Psi_1}{\partial R} \cdot \frac{\partial T_o^*}{\partial \theta} - \frac{\partial \Psi_1}{\partial \theta} \cdot \frac{\partial T_o^*}{\partial R} \right) + V_{x_1} = 0 \quad \ldots \quad (20)$$

The terms with 2nd power of N_{Ra} are:

$$\nabla^4 \Psi_2 + \frac{1}{R} \left\{ \frac{\partial \Psi_1}{\partial R} \cdot \frac{\partial (\nabla^2 \Psi_1)}{\partial \theta} - \frac{\partial \Psi_1}{\partial \theta} \cdot \frac{\partial (\nabla^2 \Psi_1)}{\partial R} \right\} =$$

$$\operatorname{Cos}\alpha \left(\frac{\partial T_1^*}{\partial R} \operatorname{Sin}\theta + \frac{1}{R} \frac{\partial T_1^*}{\partial \theta} \operatorname{Cos}\theta \right) \quad \ldots \quad (21)$$

$$\nabla^2 V_{x_2} + \frac{1}{R}\left\{ \frac{\partial \Psi_2}{\partial R} \cdot \frac{\partial V_{x_0}}{\partial \theta} + \frac{\partial \Psi_1}{\partial R} \cdot \frac{\partial V_{x_1}}{\partial \theta} \right\} =$$

$$+ \frac{1}{R}\left\{ \frac{\partial \Psi_2}{\partial \theta} \cdot \frac{\partial W_0}{\partial R} + \frac{\partial \Psi_1}{\partial \theta} \cdot \frac{\partial V_{x_1}}{\partial R} \right\} + T_1^* \operatorname{Sin}\alpha \quad . \quad . \quad (22)$$

$$\nabla^2 T_2^* + \frac{N_{Pr}}{R}\left\{ \frac{\partial \Psi_2}{\partial R} \cdot \frac{\partial T_0^*}{\partial \theta} + \frac{\partial \Psi_1}{\partial R} \cdot \frac{\partial T_1^*}{\partial \theta} \right\} =$$

$$+ \frac{N_{Pr}}{R}\left\{ \frac{\partial \Psi_2}{\partial \theta} \cdot \frac{\partial T_0^*}{\partial R} + \frac{\partial \Psi_1}{\partial \theta} \cdot \frac{\partial T_1^*}{\partial R} \right\} - V_{x_2} \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The boundary conditions, equations (11) and (12) apply to all equations (16) through (23).

Equations (16) and (17) are for normal Poiseuille flow, without free-convection effects. Their solution is available in standard texts. The remaining equations are solved by substituting the forced flow solution in the equations of first order in Rayleigh number and proceeding step by step.

The final equations up to second order in N_{Ra} are rather lengthy and are given in the Appendix A as equations (A1), (A2) and (A3).

For the case of $\alpha = 0$, these equations reduce to those of Morton (26) except for some slight numerical differences and more importantly, differences in some minus signs in the temperature equations which will probably affect his final expression for the mean Nusselt number.

CALCULATION OF NUSSELT NUMBER

Nusselt number is the dimensionless parameter indicative of the rate of energy convection from a surface.

$$N_{Nu_b} = \frac{2a}{k} h \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (24)$$

Where h, the convective heat transfer coefficient is based on bulk temperature of the fluid.

Nusselt number can also be expressed as:

$$N_{Nu_b} = \frac{2a}{k} \frac{q}{(T_w - T_b)} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (25)$$

$$\text{Where } T_b = \frac{\int_0^{2\pi}\int_0^{a} T(r,\theta)\, v_x(r,\theta)\, r dr d\theta}{\int_0^{2\pi}\int_0^{a} v_x(r,\theta)\, r dr d\theta} \quad . \; . \; . \; . \; . \; . \; . \qquad (26)$$

$$\text{And in dimensionless form, } T^*_b = \frac{\int_0^{2\pi}\int_0^{1} T^*(R,\theta)\, V_x(R,\theta)\, R dR d\theta}{\int_0^{2\pi}\int_0^{1} V_x(R,\theta)\, R dR d\theta} \quad . \; . \qquad (27)$$

From the equation (A2) the dimensionless average axial velocity is evaluated as:

$$V_{x_{av}} = \frac{\int_0^{2\pi}\int_0^{1} V_x(R,\theta)\, R dR d\theta}{\int_0^{2\pi}\int_0^{1} R dR d\theta} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \qquad (28)$$

$$= 2N_{Re} \Big\{ 1/4 - (4.125/576)\, N_{Ra} \operatorname{Sin}\alpha - 0.052486 (N_{Ra} N_{Re} \operatorname{Cos}\alpha/4608)^2 + 4540.8 (N_{Ra} \operatorname{Sin}\alpha/4608)^2 + \text{-------} \Big\} . \qquad (29)$$

$$\text{Volume flow through tube} = \left\{\begin{matrix}\text{Cross-sectional}\\ \text{area of tube}\end{matrix}\right\} \times \left\{\begin{matrix}\text{Average}\\ \text{velocity}\end{matrix}\right\} = \pi a \nu V_{x_{av}} \quad . \; . \; . \qquad (30)$$

Making an energy balance over two cross-sections of the tube, Δx apart gives,

$$q = \frac{\pi \, a \, \nu \, c_p \, V_{x_{av}} \, (T_{b_2} - T_{b_1})}{2 \pi \, a \, \Delta x}$$

$$= (1/2) \nu \rho \, c_p \, A \, V_{x_{av}} \quad \ldots\ldots\ldots \quad (31)$$

From equation (25) and (31),

$$N_{Nu_b} = \frac{V_{x_{av}}}{T_b^{*}} \quad \ldots\ldots\ldots \quad (32)$$

Equation (32) is given in detail in Appendix B.

DISCUSSION

Equations A1, A2 and A3 indicate that the three velocity components as well as the temperature at any location on a cross-section of the tube are a function of N_{Ra}, N_{Re}, N_{Pr} and of the tube inclination. In order to obtain a clear physical picture of the limitations of these equations, numerical calculations were performed to study the convergence of these equations for various parameters. It should be stressed here that the criterion used was that of reasonably rapid convergence, considering that the solutions obtained are developed only up to the second order in N_{Ra}.

The analysis showed that convergence limits for the velocity, temperature and Nusselt number equations may be considerably different. In the present case, for instance, limits of convergence of the Nusselt number and temperature equations may, in some cases (higher N_{Pr}), be considerably lower than those of the velocity equation. This also appears to be the case in Morton's analysis (Ref. 26 , which considers the problem of horizontal tubes). A detailed check of his results indicates, moreover, that his final Nusselt number equation is convergent in a range of values of $N_{Ra} \times N_{Re}$ in which the temperature equation is divergent, and therefore his results have doubtful physical meaning in this range. It was to avoid this mathematical pitfall that a study of validity limits of equations A1, A2, A3 and B1 was made.

This study revealed that, for inclined tubes, both the Nusselt number and the temperature equations were convergent within practically the same limits of the dimensionless parameters, while for the velocity equation the limits were considerably higher.

Figure 4 shows quantitatively the results obtained for temperature and Nusselt number equations at two values of Prandtl number (0.75 and 5.0). To interpret these figures, the reader should select a radial line corresponding to the tube inclination. The intersection of this line with the limit lines will give the upper limit of N_{Ra} (read on the vertical axis) and $N_{Ra} \times N_{Re}$ (on the horizontal axis). At large inclinations (nearly vertical tubes), N_{Ra} is the only controlling parameter and all the equations are valid within the same limits. At lower inclinations (nearly horizontal), the significant parameters are the Prandtl number and the product $N_{Ra} \times N_{Re}$. The upper limit of $N_{Ra} \times N_{Re}$ for temperature and Nusselt number equations decreases rapidly as N_{Pr} increases, as clearly shown in Figure 4.

The dependence on Prandtl number is not so pronounced in the velocity expression. In Figure 5, the Rayleigh number limitations are about the same as in the case of N_{Nu} and T^* equation. The range of $N_{Ra} \times N_{Re}$ is, however, much higher.

Figures 6 - 9 show the variations of temperature and axial velocity at the vertical tube centre line for Prandtl numbers 5.0 and 0.75. Figure 10 is a contour map showing a typical temperature distribution across the tube. These plots show that Prandtl number has a strong influence on distortion of temperature and velocity profiles. Maximum temperature and velocity occur below the centre line for the horizontal position, but as tube inclination increases, the location of the maximum values shifts upward. Depending upon the value of Prandtl number, this maximum could be located above the centre line for some tube inclinations. As the inclination is increased, the distortion of the profiles is reduced until they become symmetrical about the centre line in the vertical position. Exact solutions [5, 13] available for this case (heating in vertical upward flow) indicate that the effect of natural convection is to decrease the centre line velocity and temperature. A similar result is also obtained by the present perturbation analysis.

Equation B1 in Appendix B shows the influence of the various parameters on the Nusselt number. For the horizontal case ($\alpha = 0^\circ$), Nusselt number depends on the Rayleigh number only.

Figures 11 and 12 show the plots of Nusselt number against Rayleigh number for various values of $N_{Ra} \times N_{Re}$. Figure 11 gives the results for $N_{Pr} = 5.0$ and Figure 12 for $N_{Pr} = 0.75$. At low Prandtl numbers, the heat transfer is affected only slightly by the product $N_{Ra} \times N_{Re}$, but at higher values this dependence is very significant for low tube inclinations. The influence of both N_{Pr} and $N_{Ra} \times N_{Re}$ diminishes with tube inclination, until it disappears completely at $\alpha = 90^\circ$ (vertical tubes).

Figures 13 and 14 show that for any combination of Rayleigh, Reynolds and Prandtl numbers, there is an optimum value of tube inclination that gives the maximum value of Nusselt number. For most instances this maximum appears to lie between 20° and 60° of tube inclination. This is somewhat similar to Larson and Hartnett's [33] findings for free convection inside inclined tube closed at both ends.

The optimum tube inclination mentioned in the preceding paragraph is plotted against the product $N_{Ra} \times N_{Re}$ in Figures 15 and 16. The curves are similar at both Prandtl numbers investigated, but an interesting effect of Rayleigh number can be observed. At high values of $N_{Ra} \times N_{Re}$ (for a given N_{Pr}), the optimum tube inclination increases slightly when N_{Ra} is increased, but the reverse is true in the low $N_{Ra} \times N_{Re}$ range.

CONCLUSIONS

The analysis shows that for inclined tubes, the velocity and temperature profiles as well as Nusselt numbers are functions of Rayleigh, Reynolds and Prandtl numbers. At low values of these parameters, for the horizontal and vertical case, the results seem to be in agreement with other theoretical work already published. Nusselt number increases with tube inclination, compared to the horizontal case. For most combinations of N_{Ra}, N_{Re} and N_{Pr}, the maximum value of Nusselt number seems to lie between 20° and 60° of tube inclination.

APPENDIX A

The three equations in dimensionless form, up to second order in Rayleigh number are:

Equation of Stokes Stream Function

$$\psi = \psi_o + N_{Ra}\psi_1 + N_{Ra}^2\psi_2 + --- \qquad \text{where } \psi_o = 0$$

$$= \frac{N_{Ra}\ N_{Re}\ \cos\alpha}{4608}(-10R + 21R^3 - 12R^5 + R^7)\ \sin\theta$$

$$+ \frac{N_{Ra}^2\ N_{Re}^2\ \cos^2\alpha}{(4608)^2}(1.204108R^2 - 2.061786R^4 + 0.1500R^6 + 1.10R^8 - 0.425R^{10} + 0.034285R^{12} - 0.001607R^{14})\ \sin 2\theta$$

$$+ \frac{N_{Ra}^2\ N_{Re}^2\ \cos^2\alpha}{(4608)^2}N_{Pr}(3.75643R^2 - 9.160715R^4 + 7.5R^6 - 2.6R^8 + 0.5625R^{10} - 0.06R^{12} + 0.001785R^{14})\ \sin 2\theta$$

$$+ \frac{N_{Ra}^2\ N_{Re}\ \sin 2\alpha}{36864}(1.244166R - 2.652083R^3 + 1.583333R^5 - 0.1875R^7 + 0.0125R^9 - 0.000416R^{11})\ \sin\theta + --- \qquad \text{..... (A1)}$$

Velocity Equation in Axial Direction

$$V_x = V_{x_o} + N_{Ra}\ V_{x_1} + N_{Ra}^2\ V_{x_2} + ---$$

$$= N_{Re}(1 - R^2) + \frac{N_{Ra}\ N_{Re}^2\ \cos\alpha}{184320}\ \cos\theta(-49R + 100R^3 - 70R^5 + 20R^7 - R^9)$$

$$+ \frac{N_{Ra}\ N_{Re}\ \sin\alpha}{576}(-19 + 27R^2 - 9R^4 + R^6)$$

$$+ \frac{N_{Ra}^2\ N_{Re}^3\ \cos^2\alpha}{(4608)^2}(\cos 2\theta)(0.115356R^2 - 0.340952R^4 + 0.432723R^6 - 0.31125R^8 + 0.13125R^{10} - 0.029464R^{12} + 0.00241R^{14} - 0.00073R^{16})$$

APPENDIX B

Detailed expression for Nusselt number.

$$N_{Nu_b} = V_{x_{av}}/T_b^{*} \quad \ldots \quad (32)$$

$$= \frac{48}{11} \frac{(1 + 1RN + 2RN + ---)^2}{\left\{1 + 34.909116\ (1RD + 2RD) + ---\right\}} \quad \ldots\ldots \quad (B1)$$

Where

1RN = First order effect in Rayleigh number in the numerator

$$= -(N_{Ra}\ Sin\alpha)\ 16.5/576 \quad \ldots\ldots \quad (B2)$$

2RN = Second order effect in Rayleigh number in the numerator

$$= \left\{-0.209944\ (N_{Ra}\ N_{Re}\ Cos\alpha)^2 + 18163.2\ (N_{Ra}\ Sin\alpha)^2\right\}/(4608)^2 \quad \ldots\ldots \quad (B3)$$

1RD = First order effect in Rayleigh number in the denominator

$$= 1RD1 + 1RD2 \quad \ldots\ldots \quad (B4)$$

2RD = Second order effect in Rayleigh number in the denominator

$$= 2RD1 + 2RD2 + 2RD3 + 2RD4 + 2RD5 \quad \ldots\ldots \quad (B5)$$

Where

$$1RD1 = 63.06666\ N_{Ra}\ Sin\alpha/36864 + 543.048214\ (N_{Ra}\ Sin\alpha/4608)^2 \quad \ldots\ldots \quad (B6)$$

$$1RD2 = (0.00105 + 0.003281\ N_{Pr})\ (N_{Ra}\ N_{Re}\ Cos\alpha/4608)^2 \quad \ldots\ldots \quad (B7)$$

$$2RD1 = \left\{1086.096428 - 32.47384\ N_{Ra}\ Sin\alpha + 10308464.2\ (N_{Ra}\ Sin\alpha/4608)^2\right\}(N_{Ra}\ N_{Re}\ Sin\alpha/4608)^2 \quad \ldots\ldots \quad (B8)$$

$$2RD2 = (0.00034 - 0.000618\ N_{Pr} + 0.000255\ N_{Pr}^2) \times N_{Ra}\ Sin\alpha\ (N_{Ra}\ N_{Re}\ Cos\alpha/4608)^2 \quad \ldots\ldots \quad (B9)$$

$$2RD3 = -(196.29141 - 276.640184\ N_{Pr} + 128.267224\ N_{Pr}^2)\ x$$

$$x\ (N_{Re}\ \mathrm{Sin}\alpha\ \mathrm{Cos}\alpha)^2\ (N_{Ra}/4608)^4\ \ldots.. \quad (B10)$$

$$2RD4 = -(0.016054 + 0.003028\ N_{Pr} + 0.009552\ N_{Pr}^2)\ x$$

$$x\ (N_{Ra}\ N_{Re}\ \mathrm{Cos}\alpha/4608)^2 \quad \ldots.. \quad (B11)$$

$$2RD5 = (0.005553 + 0.001188\ N_{Pr} + 0.003826\ N_{Pr}^2 + 0.000095\ N_{Pr}^3)\ x$$

$$x\ (N_{Ra}\ N_{Re}\ \mathrm{Cos}\alpha/4608)^4 \quad \ldots.. \quad (B12)$$

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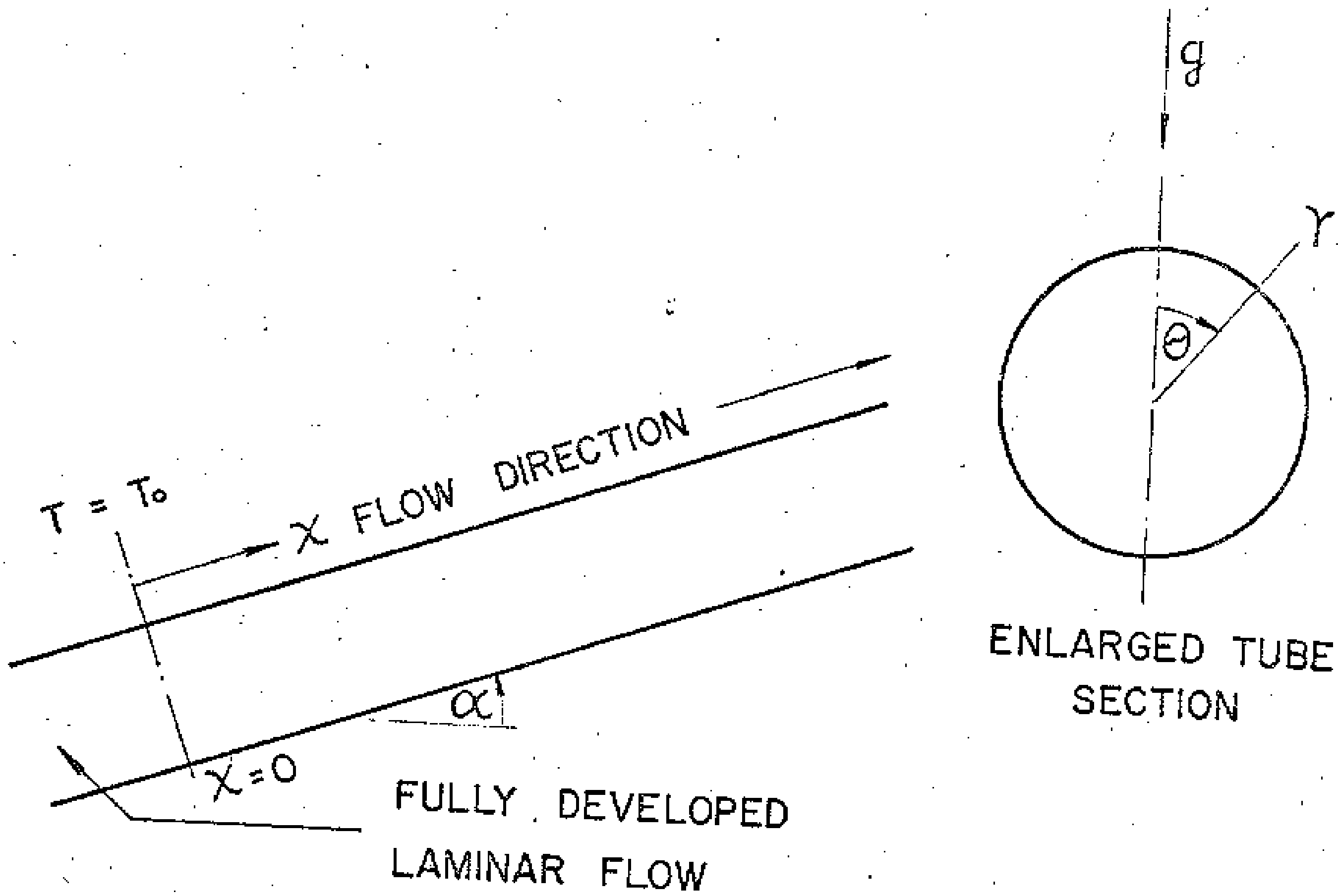

Fig. 1 · COORDINATE SYSTEM

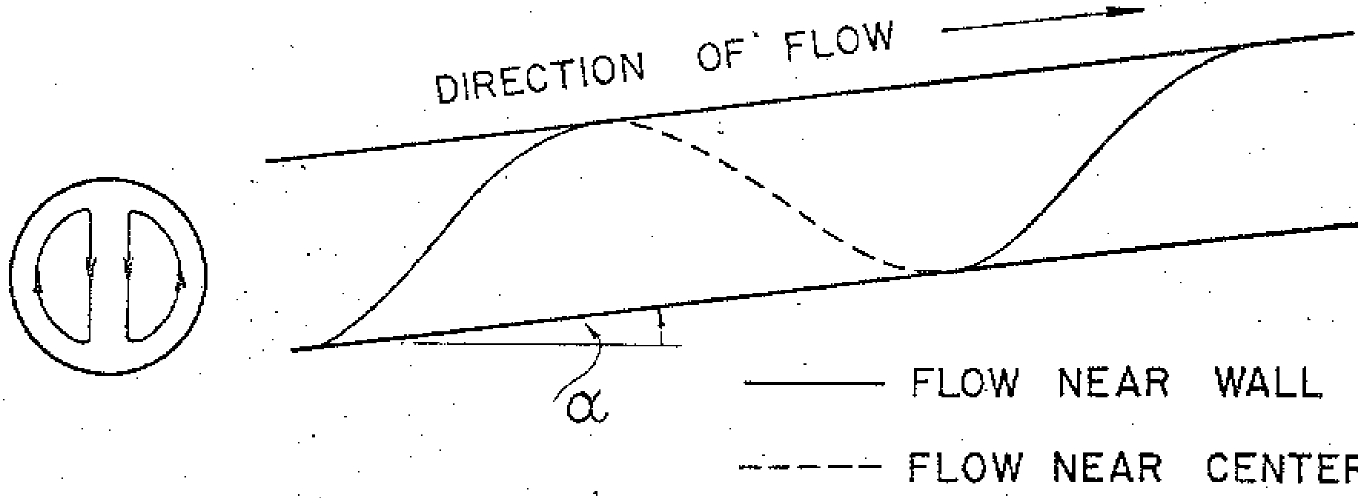

Fig. 2 FLOW VISUALIZATION

CONSTANT HEAT FLUX

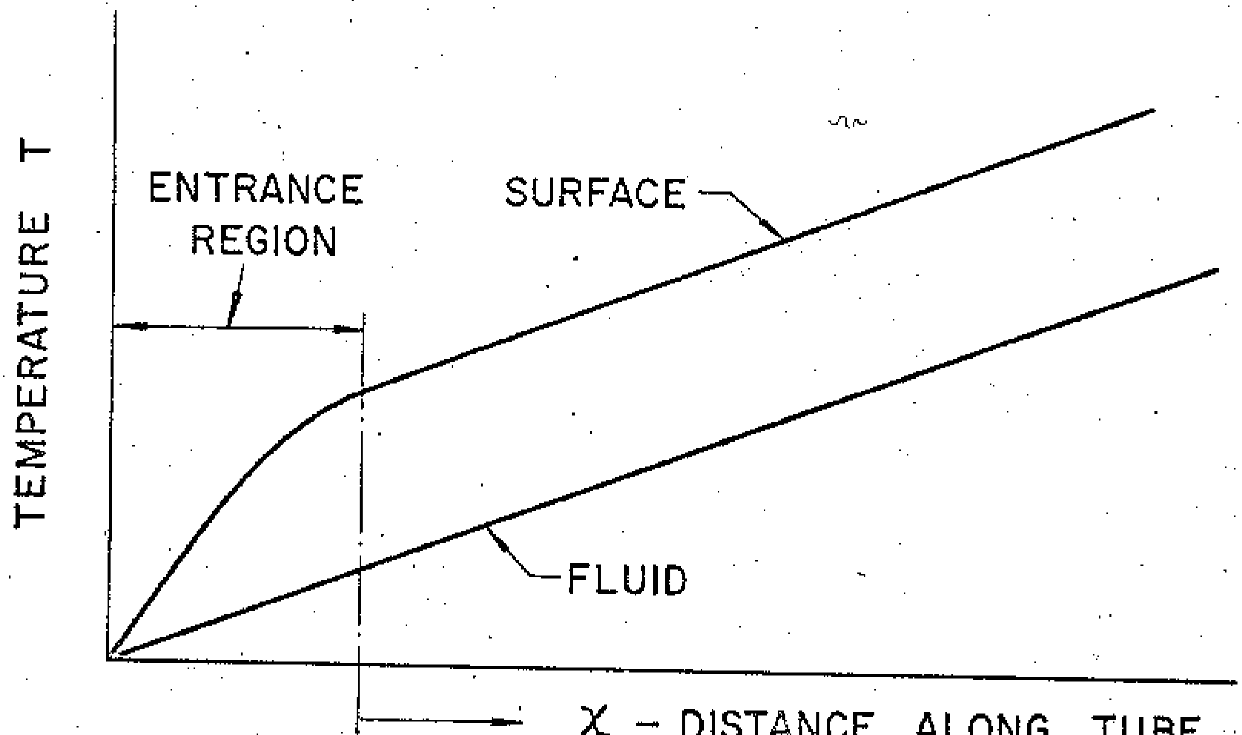

Fig. 3 SURFACE FLUID TEMPERATURE RISE

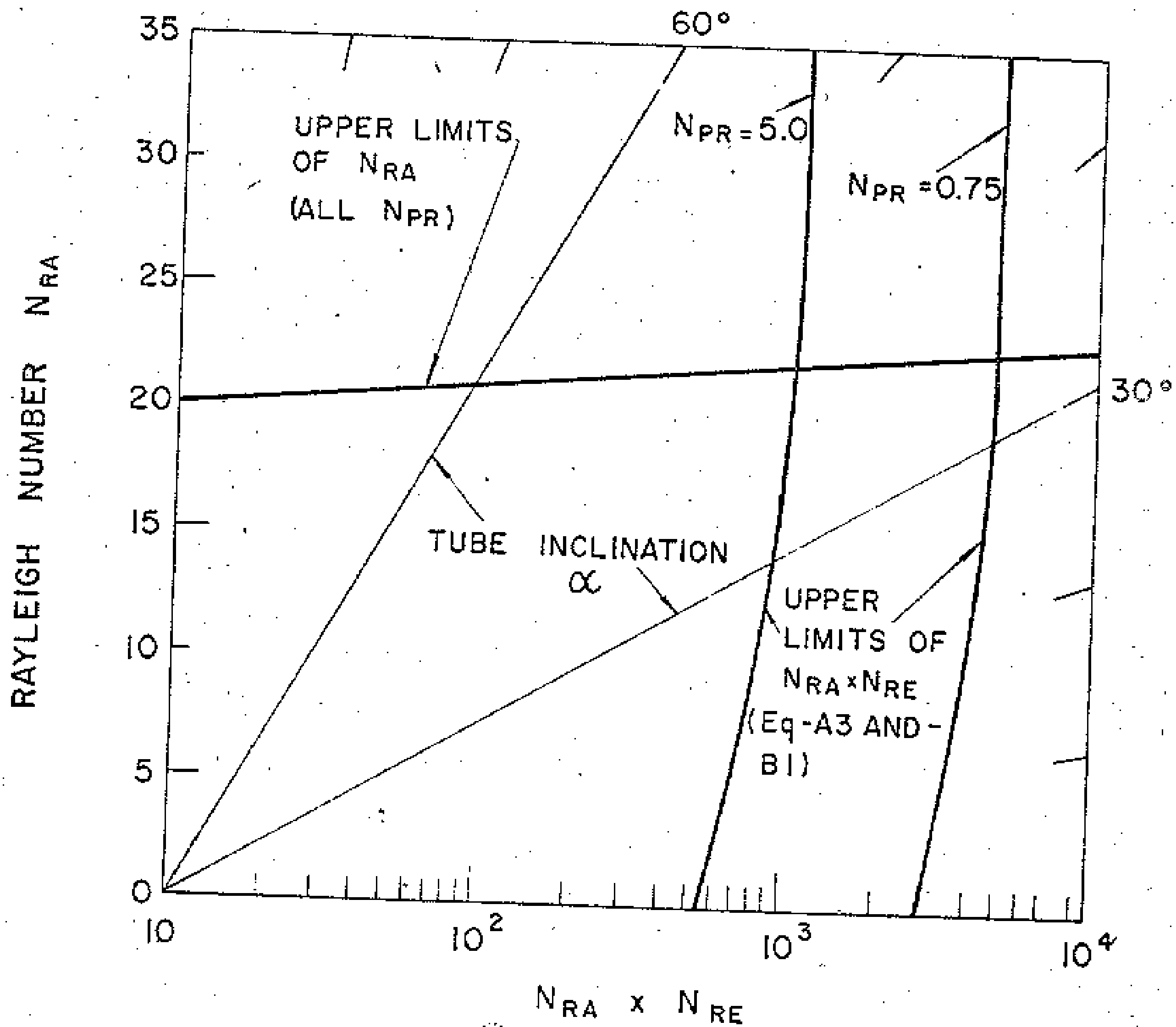

Fig.4 SUGGESTED LIMITS OF N_{RA} AND $N_{RA} \times N_{RE}$ FOR TEMPERATURE AND NUSSELT NUMBER EQUATIONS A3 AND B1 RESPECTIVELY

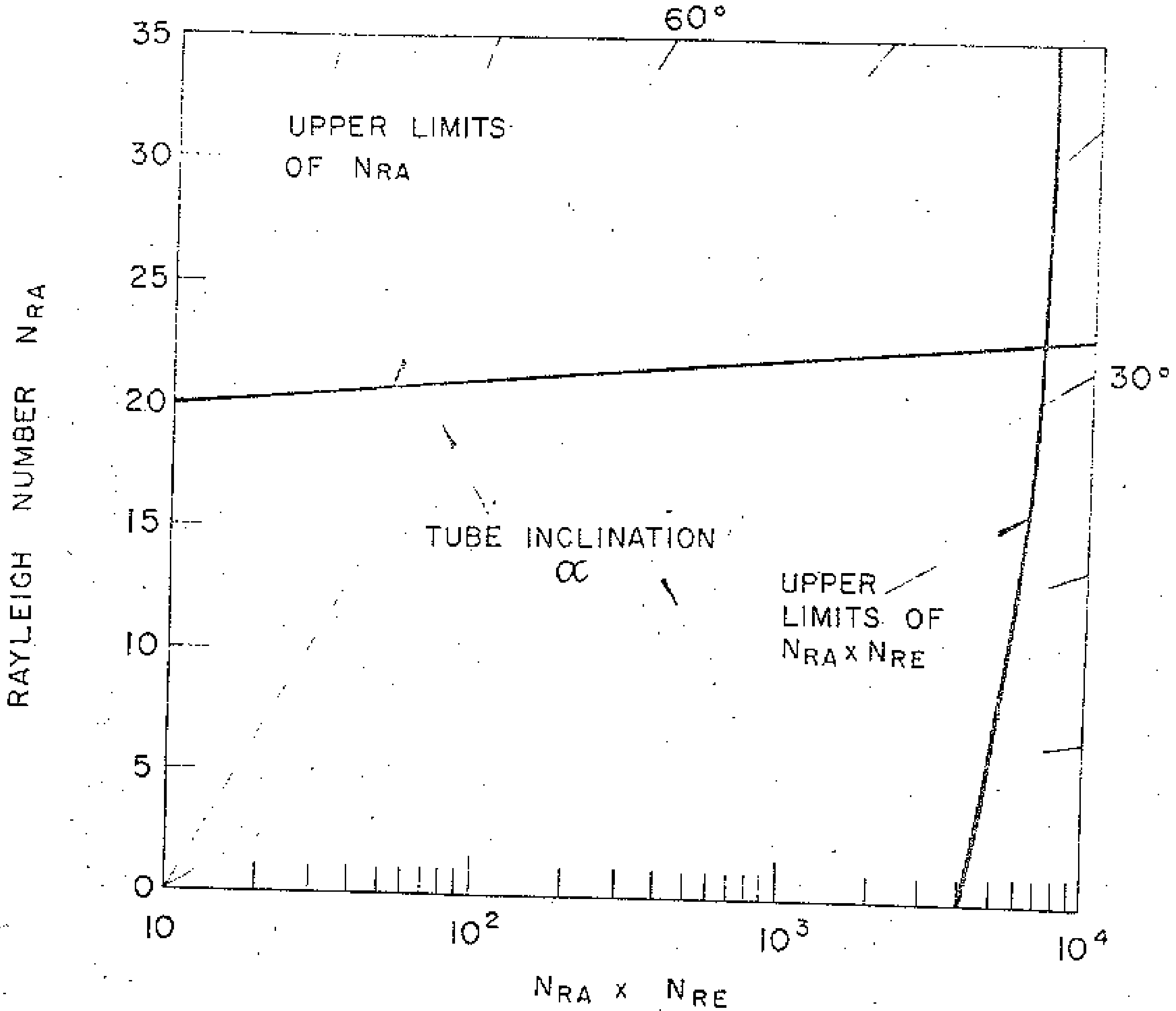

Fig. 5 SUGGESTED LIMITS OF N_{RA} AND $N_{RA} \times N_{RE}$ FOR AXIAL VELOCITY EQUATION A2. LIMITS VALID FOR $N_{PR} = 0.75$ TO 5.0

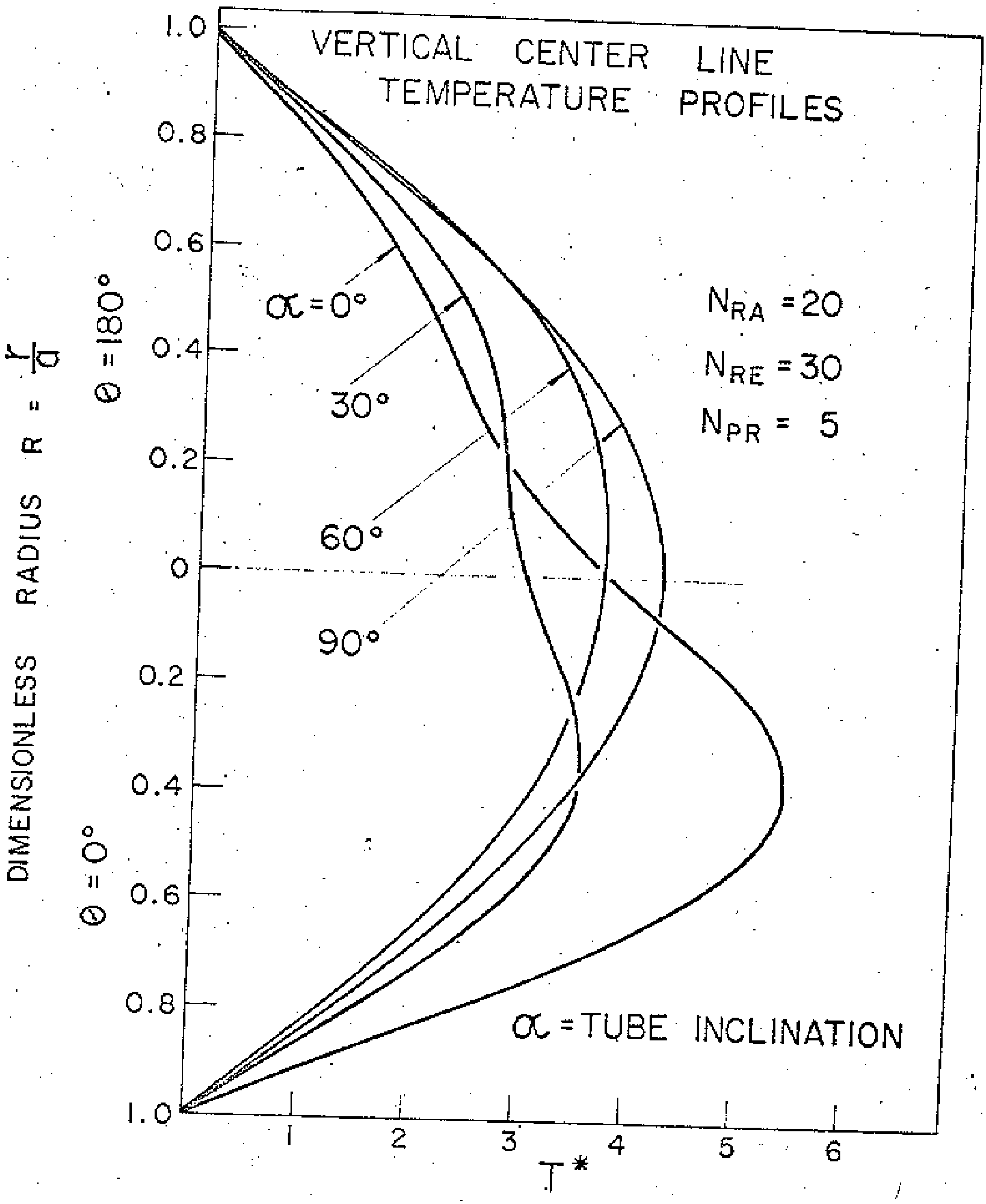

Fig. 6 DIMENSIONLESS TEMPERATURE DIFFERENCE VARIATION AGAINST DIMENSIONLESS RADIUS AT VERTICAL ℄ FOR VARIOUS TUBE INCLINATIONS.

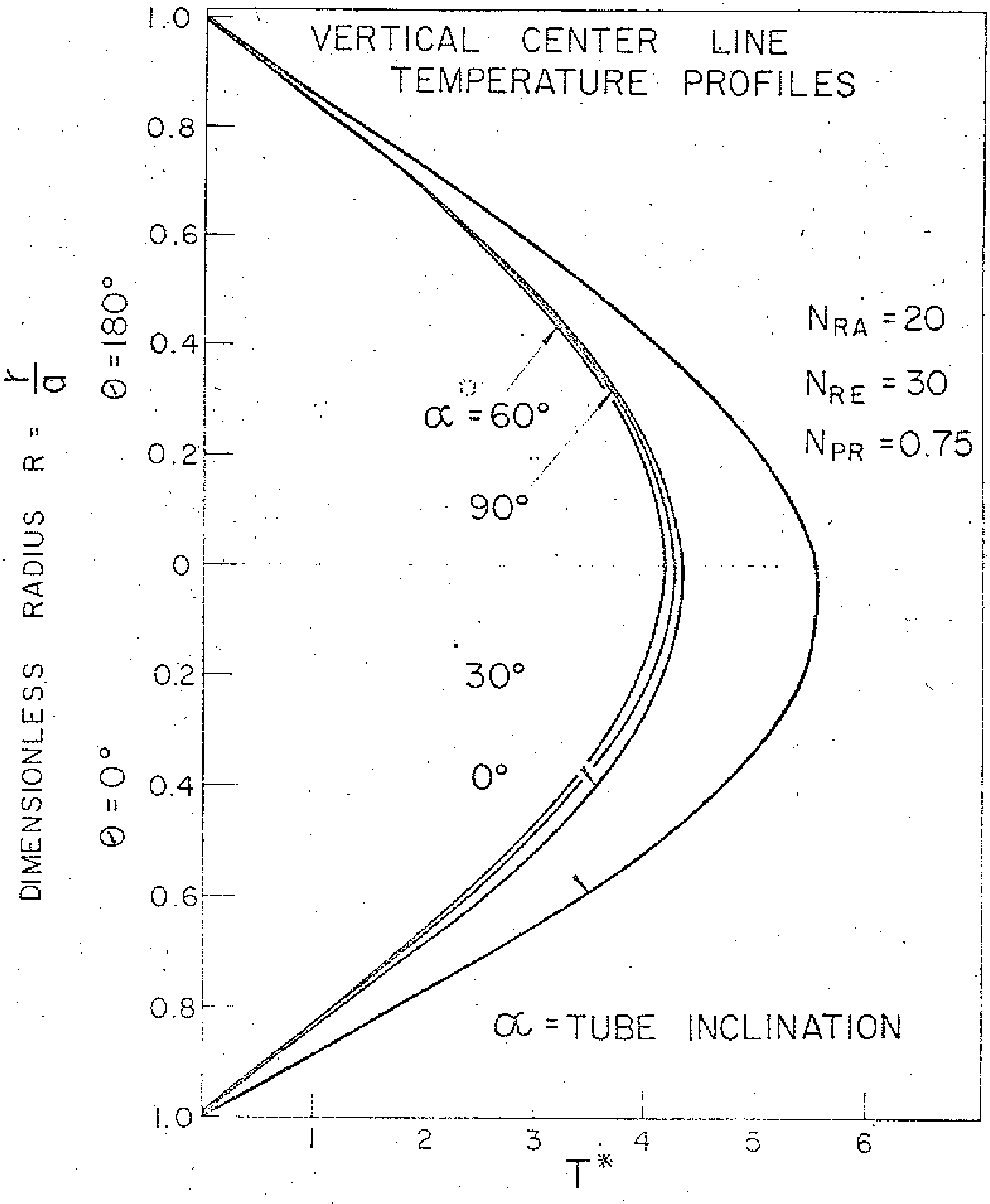

Fig. 7 DIMENSIONLESS TEMPERATURE DIFFERENCE VARIATION AGAINST DIMENSIONLESS RADIUS AT VERTICAL ℄ FOR VARIOUS TUBE INCLINATIONS.

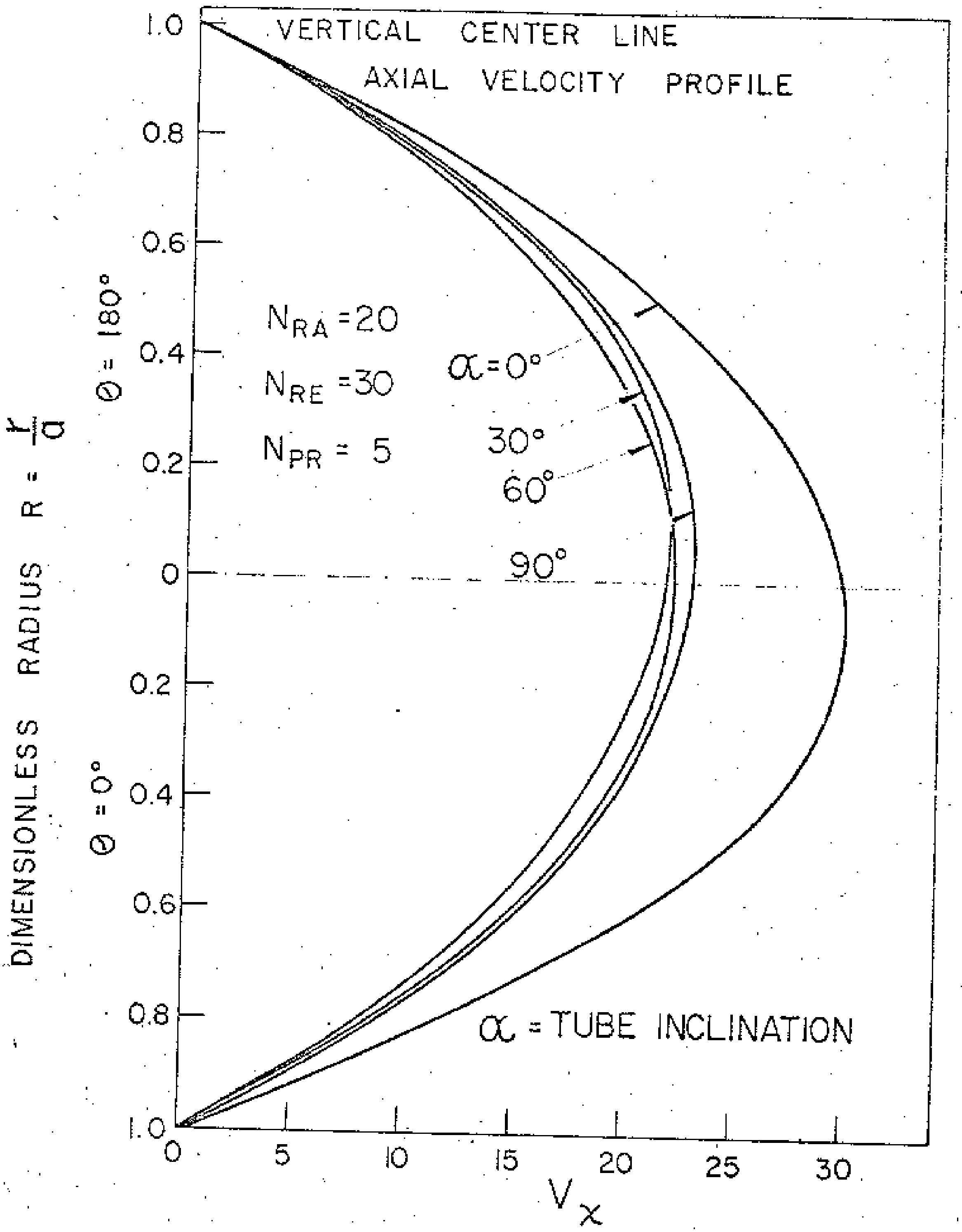

Fig. 8 DIMENSIONLESS AXIAL VELOCITY VARIATION AGAINST DIMENSIONLESS RADIUS AT VERTICAL ℄ FOR VARIOUS TUBE INCLINATIONS.

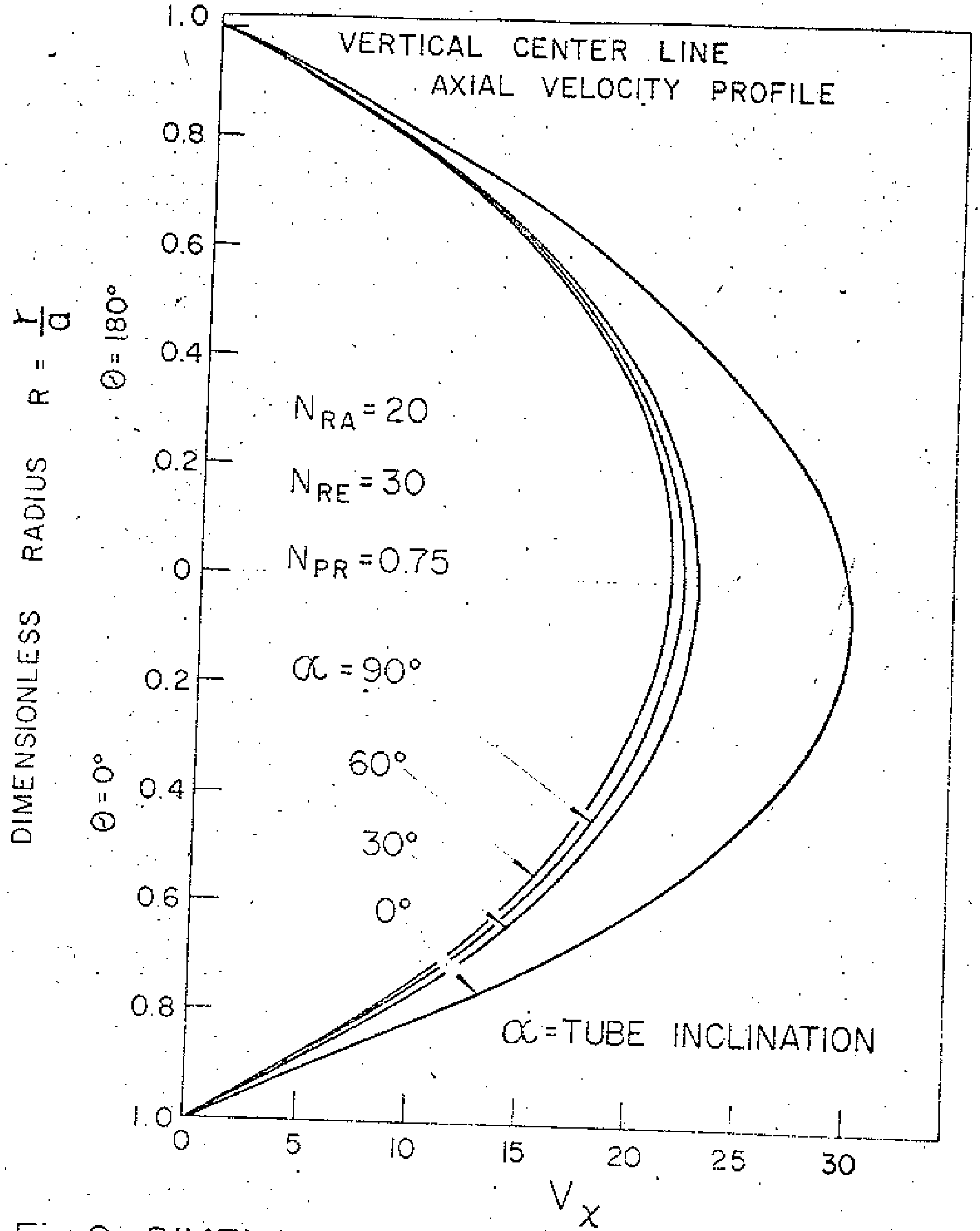

Fig. 9 DIMENSIONLESS AXIAL VELOCITY VARIATION AGAINST DIMENSIONLESS RADIUS AT VERTICAL ℄ FOR VARIOUS TUBE INCLINATIONS.

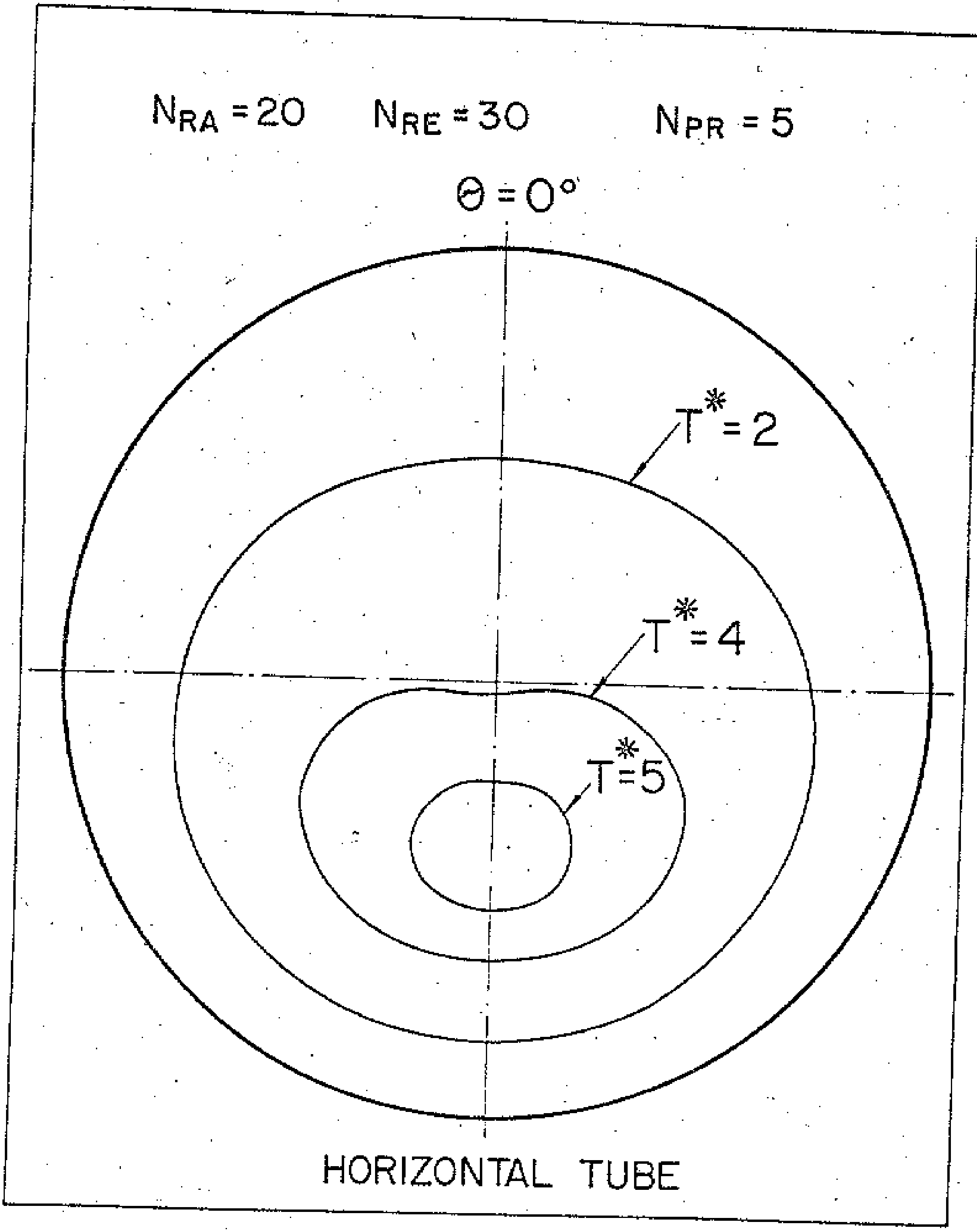

Fig. 10 CONTOUR MAP OF DIMENSIONLESS TEMPERATURE T^* IN A HORIZONTAL TUBE

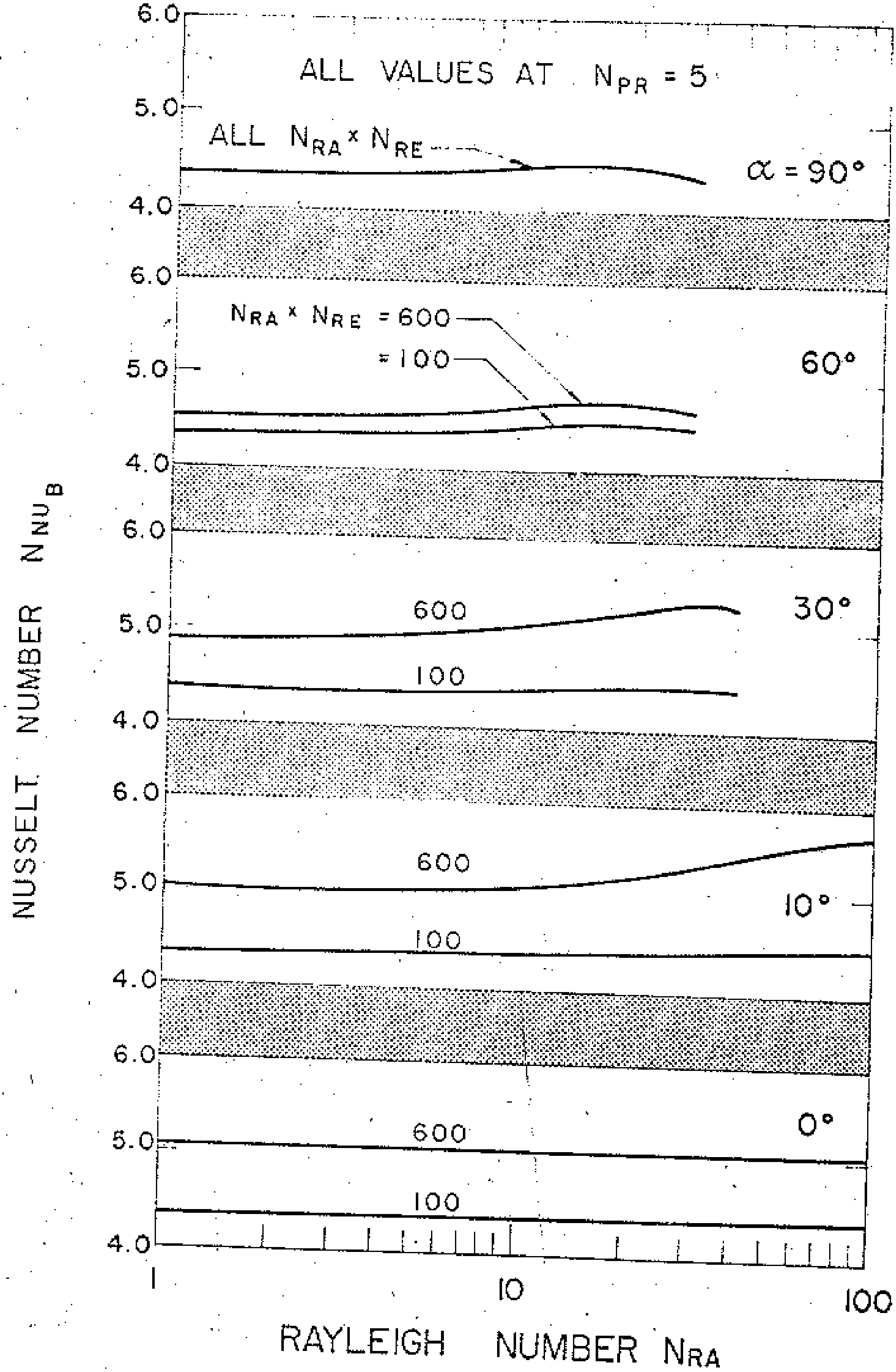

Fig. II NUSSELT NUMBER AGAINST RAYLEIGH NUMBER AT VARIOUS TUBE INCLINATIONS.

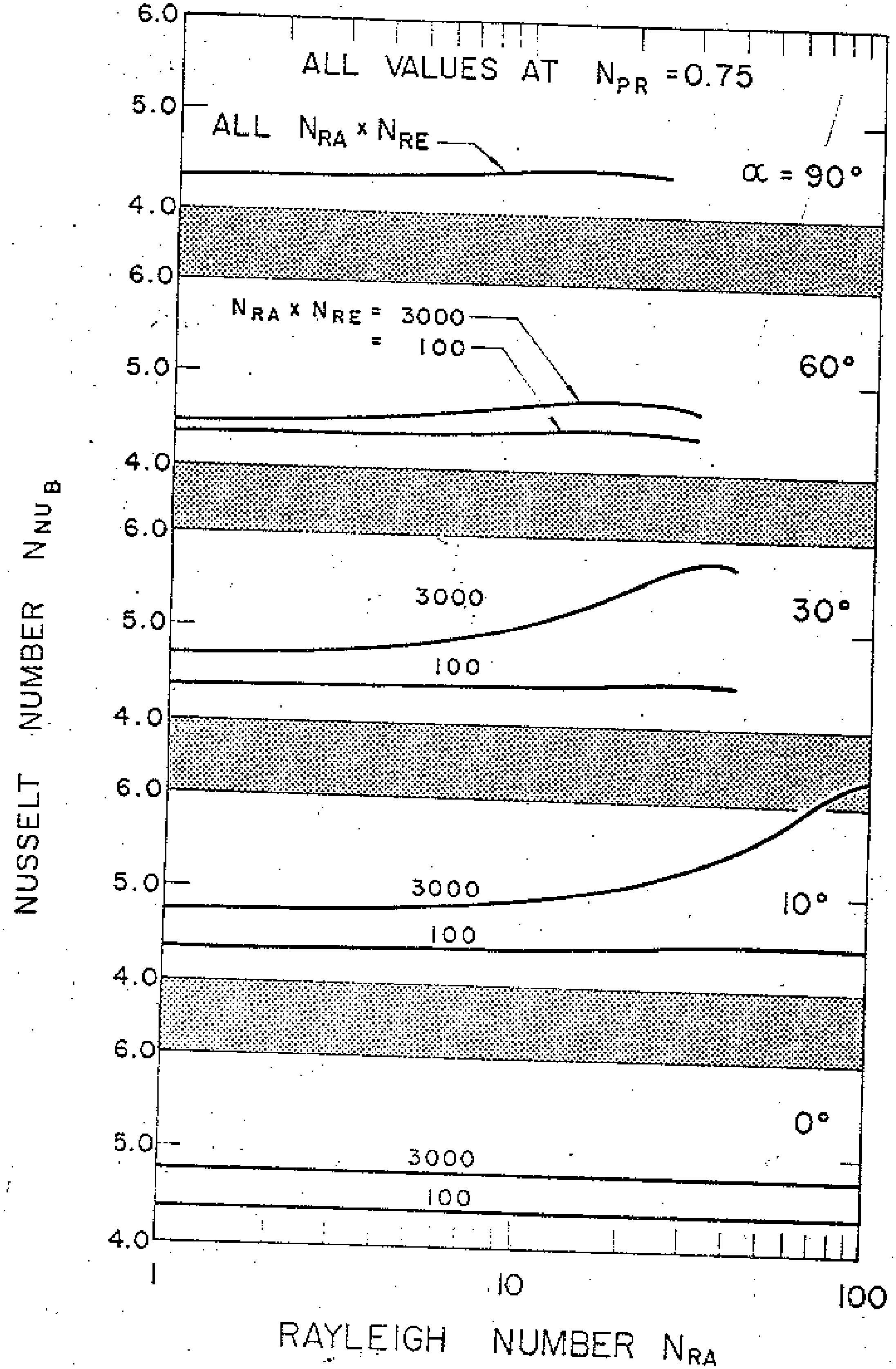

Fig. 12 NUSSELT NUMBER AGAINST RAYLEIGH NUMBER AT VARIOUS TUBE INCLINATIONS.

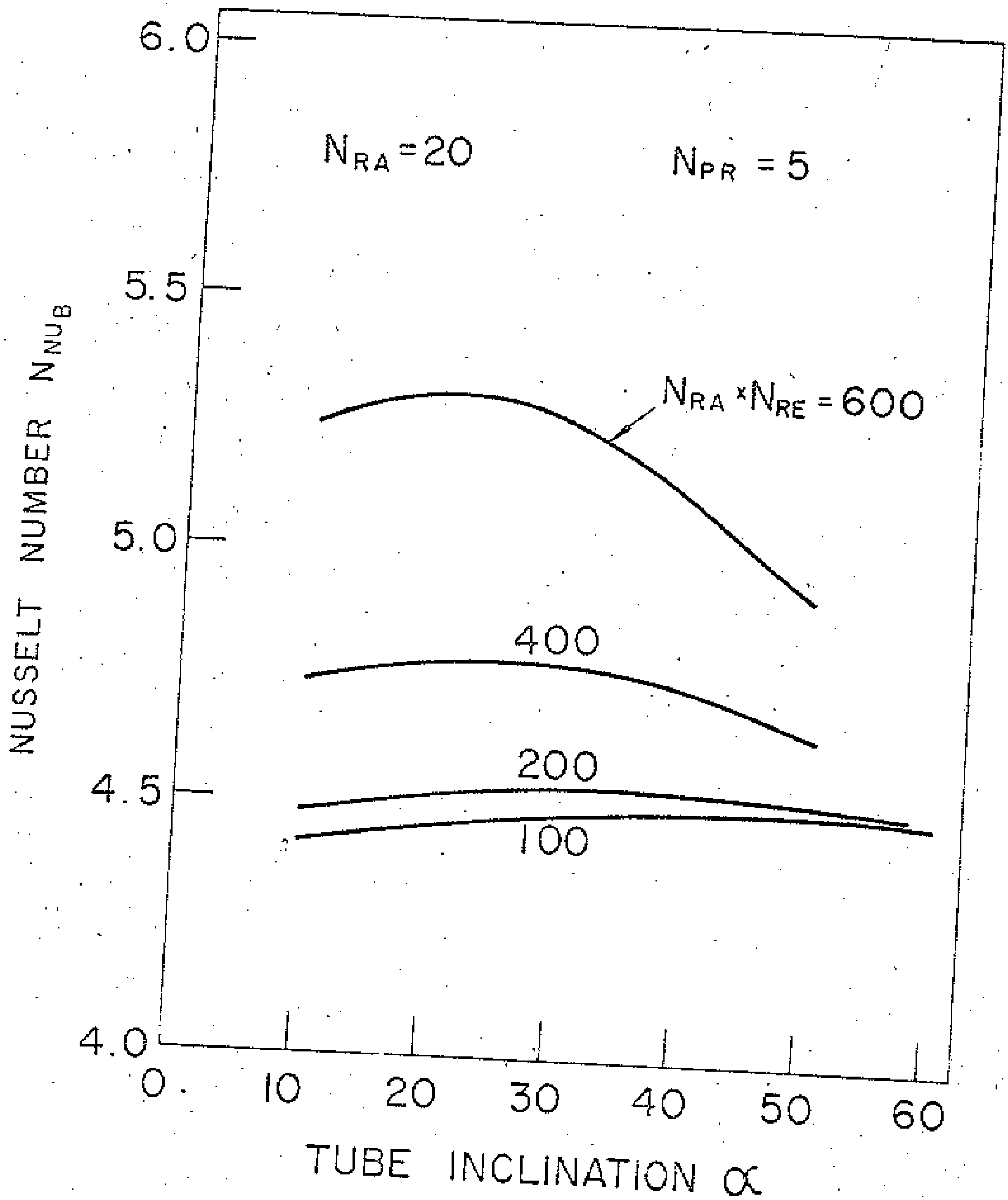

Fig. 13 VARIATION OF NUSSELT NUMBER WITH TUBE INCLINATION AT A PRANDTL NUMBER OF 5.0

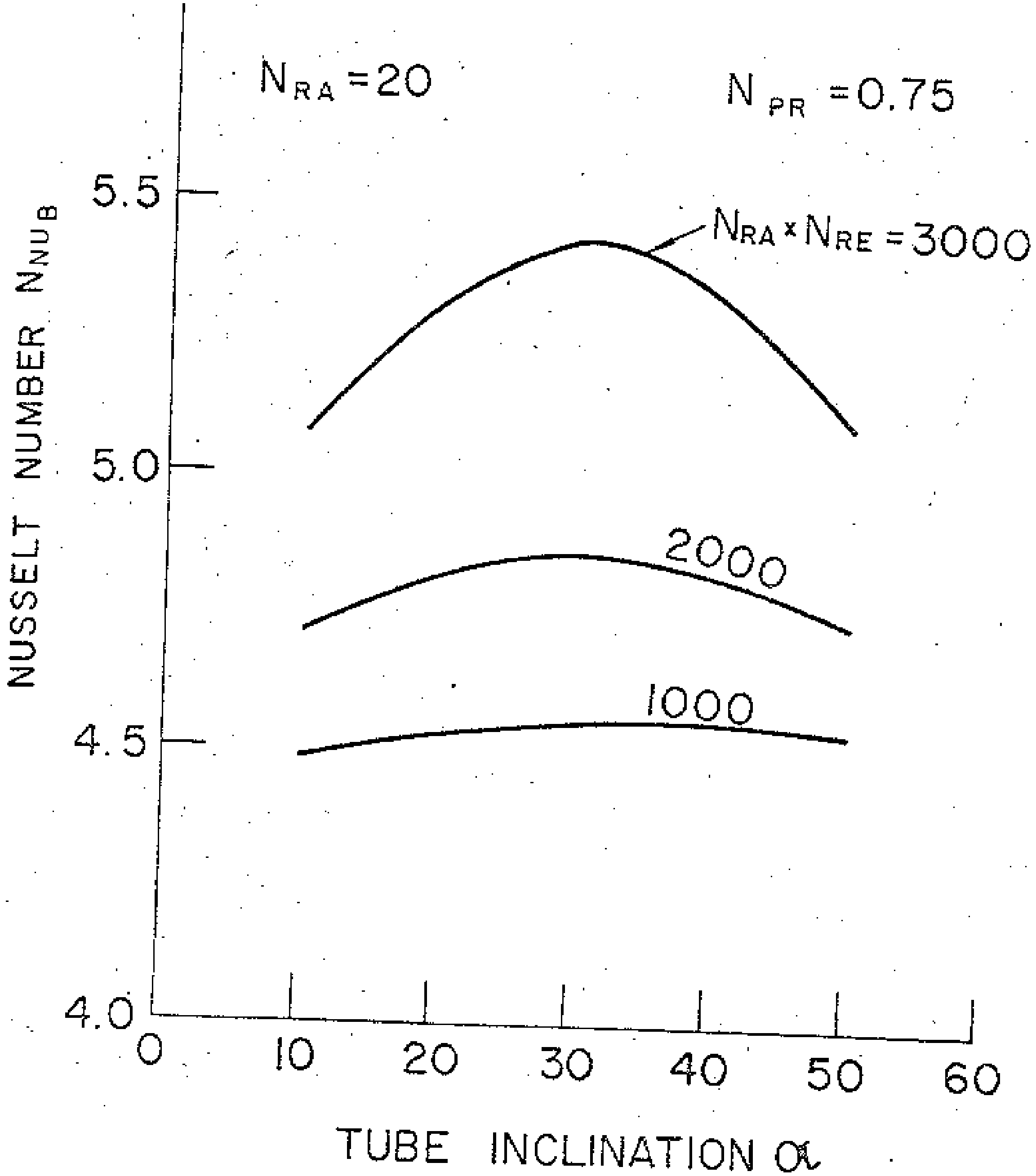

Fig. 14 VARIATION OF NUSSELT NUMBER WITH TUBE INCLINATION AT A PRANDTL NUMBER OF 0.75

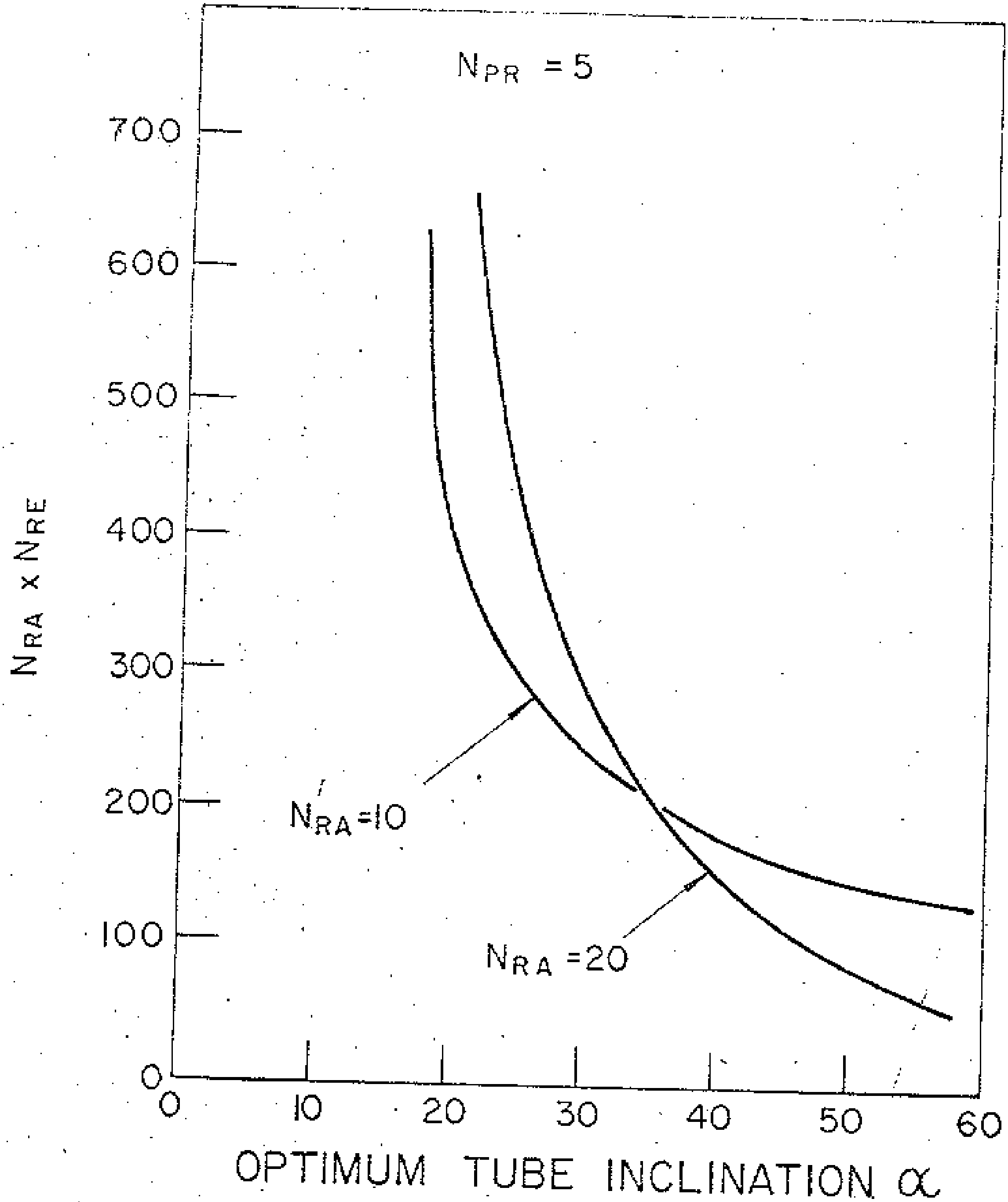

Fig. 15 VARIATION OF OPTIMUM TUBE INCLINATION FOR MAXIMUM NUSSELT NUMBER AGAINST VARIOUS VALUES OF N_{RA} AND $N_{RA} \times N_{RE}$

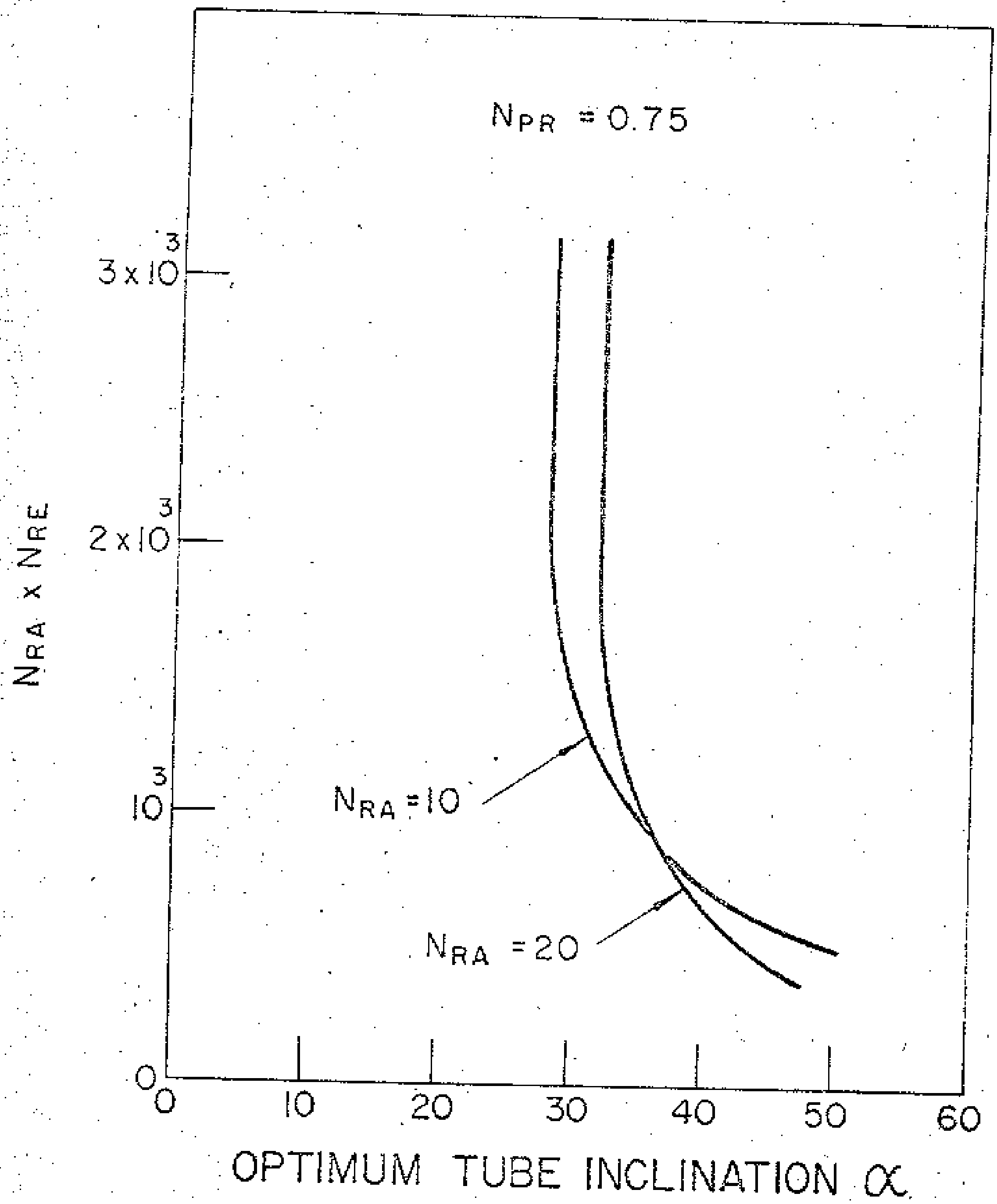

Fig.16 VARIATION OF OPTIMUM TUBE INCLINATION FOR MAXIMUM NUSSELT NUMBER AGAINST VARIOUS VALUES OF N_{RA} AND $N_{RA} \times N_{RE}$