Concentration-compactness Principle for Anisotropic Variable Exponent Sobolev Spaces and Some Applications

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Contents

Abstract. When attempting to solve nonlinear elliptic equations of critical growth, the concentration compactness principle of P.L. Lions has proven to be a very useful tool. In 2009-2010, Bonder & Silva and Fu independently generalized this result to variable exponent spaces. Lately, more attention has also been given to anisotropic variable exponent differential equations, as they seem to provide a good model for describing the filtration of incompressible fluids through a porous medium. The first aim of this research is to generalize the results of Bonder & Silva and Fu to the anisotropic variable exponent case. In order to do this, we first had to prove a critical Sobolev embedding theorem for the anisotropic variable exponent Sobolev spaces, which, to our knowledge, is non-existent in the literature. Finally, we apply these results to prove the existence of a weak solution to an anisotropic variable exponent Laplace type operator with critical growth.

Résumé. Afin de résoudre une équation elliptique non linéaire avec croissance critique, le principe de compacité-concentrée développé par P.L. Lions s'avère être très utile. En 2009-2010, Bonder & Silva et Fu ont indépendamment généralisé ce résultat aux espaces avec exposants variables. Récemment, une attention particulière est orientée vers les équations anisotropiques à exposants variables, puisqu'elles semblent bien décrire la filtration d'un liquide incompressible à travers un medium poreux. Le premier objectif de ce travail est de généralisé les résultats de Bonder & Silva et Fu aux espaces anisotropiques de Sobolev à exposants variables. Afin de réussir cela, nous avons dû démontrer un théorem sur le plongement de ces espaces avec exposant critique, ce qui était, à notre connaissance, inexistant dans la littérature. Finalement, nous appliquons ces résultats pour prouver l'existence d'une solution faible à une équation aux dérivées partielle avec croissance critique et un opérateur de Laplace anisotropique à exposant variable.

1. Introduction

In recent years, partial differential equations using variable exponents have been of increasing interest, notably due to its applications to fluid dynamics, more specifically electrorheological fluids ([45][1]), and to image processing ([13]). In most cases, it comes down to solving an equation of the form:

$$\begin{cases} -\operatorname{div}\left[a(x,u,\nabla u)\right] = f(x,u), & \text{in } \Omega \subset \mathbb{R}^n \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where f is a non-linear source such that $|f(u(x),x)| \leq C(1+|u(x)|^{q(x)}$ for all $x \in \Omega$, and $q : \Omega \to [1,\infty)$. Of particular importance, we have $a(x,u,\nabla u) = |\nabla u|^{p(x)-2}\nabla u$, where $p : \Omega \to [1,\infty)$ is often assumed to be at least Lipschitz continuous. Then we call its divergence the $p(x)$-Laplacian operator $\Delta_{p(x)} = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$. In the case where $p(x) = 2$, this becomes the classical Laplace operator. We refer to [23] for some results related to this operator.

For example, in image processing, the exponent $p(x)$ is a convex function between 1 and 2, related to the probability that the point x is an edge (see [13]). Similar models are also useful with regards to the thermistor problem that models an electrical current in a conductor influenced by a non-constant temperature field, where $u(x)$ is the electric potential and $p(x)$ is the temperature ([48]).

Now if we let $f(x,u) = u|u|^{q(x)-2}$, we obtain from (1)

$$\begin{cases} -\Delta_{p(x)} = u|u|^{q(x)-2}, & \text{in } \Omega \subset \mathbb{R}^n \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

Then we may define, as in the constant exponent case, a notion of critical exponents. One may refer to the survey by Rădulescu [41] in order to see the substantial amount of research on (2). We will define the critical exponent pointwise as $p^*(x) = \dfrac{np(x)}{n-p(x)}$, called the critical Sobolev exponent for variable exponent spaces.

Next we may come up, assuming some continuity on $p(x)$, with analogs of the Sobolev embeddings, the Rellich-Kondrachov theorem and the Poincaré inequality, which will be crucial in solving (2). From this, there are already several results to (2) in the subcritical case $q(x) < p^*(x) - \varepsilon$, such as [47]. There exists also some results for the critical case, i.e. $q(x) = p^*(x)$ on some subset of Ω, such as those obtained in [28][10].

The main objective of this thesis will be to extend the latter results to the anisotropic case. More specifically, we want to prove existence of a weak solution to the following equation

$$\begin{cases} -\text{div}\left(\sum_{i=1}^{n} a_i(x)|\partial_i u(x)|^{p_i(x)-2}\partial_i u(x)\right) + a_0(x)|u(x)|^{p_0(x)-2}u(x) = f(u(x),x), & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{$*$}$$

where the functions $a_i(x)$ are positive and $a_0(x)$ is nonnegative. As pointed out in [5], equations of the form $(*)$ arise from studying the process of filtration of an incompressible fluid through a porous medium. Although the study of the anisotropic equations is not as extensive, compared with equations of the type (1), we may still cite some papers on subcritical equations similar to $(*)$, such as [4], [32], [34] and [6].

To our knowledge, equations like these, where f has critical growth, have not been studied, nor has there been a critical exponent embedding theorem. When we have fixed exponents, we define the critical Sobolev exponent as the critical exponent of the harmonic mean, i.e. $\overline{p}^* = \frac{n\overline{p}}{n-\overline{p}}$, where $\overline{p}$ is the harmonic mean of $\{p_i\}$.

Thus when exponents are variable, we define the harmonic mean and critical exponent pointwise,

$$\overline{p}(x) = \frac{n}{\sum_{i=1}^{n} \frac{1}{p_i(x)}} \qquad \text{and} \qquad \overline{p}^*(x) = \frac{n\overline{p}(x)}{n - \overline{p}(x)}.$$

This exponent is defined, amongst others, in [19], in which the author presents many theorems related to anisotropic Sobolev spaces, notably a compact embedding theorem for subcritical exponnents, a Poincaré inequality as well as a density theorem. Yet the author states that the embedding into the critical exponent Lebesgue space remains an open problem. Therefore our first main result, theorem 3.12, is proving there exists a continuous embedding of the type

$$W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{\overline{p}^*(x)}(\Omega),$$

where $W^{1,\vec{p}(\cdot)}(\Omega)$ is an anisotropic Sobolev space, defined in Section 3, and $L^{\overline{p}^*(x)}(\Omega)$ the Lebesgue space, defined in Section 2, with critical exponent. We follow this later by an anisotropic variable exponent concentration-compactness principle, Theorem 4.1 and finally by showing existence of a weak solution, Theorem 6.4, under additional assumptions on f.

1.1. **Organization of the Thesis.** The first part of this thesis, Section 2, will be to establish the bases on which we will build our work, i.e., defining the variable exponent Lebesgue and Sobolev spaces as well as their important properties. Notably, we will give proofs of some theorems that are classical in the fixed exponent spaces, such as embedding theorems (Lebesgue and Sobolev), the Rellich-Kondrachov theorem and the Poincaré inequality. The results in this section have now been well established in the litterature. In fact one may refer to the book [17], where the authors do a very extensive survey of all the results related to variable exponent spaces. We have included proofs of these main theorems for three reasons. First, in order to make this thesis as much self-contained as possible. Second, because we have reworked some of the proofs in order to simplify them and/or to adapt them to our setting. Third, the techniques used in these proofs will be useful when proving our main results, hence they may help the reader familiarize himself to the particularities of variable exponents.

Section 3 will deal with the anisotropic variable exponent Sobolev spaces. We were first surprised that, to our knowledge, there is not yet a first order critical exponent Sobolev embedding for these spaces. Thus we present here such a theorem, which generalizes the one presented in the previous section, for order one ($k = 1$). We then proceed to present already existing results, such as the Rellich-Kondrachov and Poincaré analogues.

After having established the bases for these Sobolev spaces, we present, in Section 4, a new concentration-compactness principle, for bounded domains, which is a generalization of the one presented by Fu ([27]) and Bonder & Silva ([12]).

In Section 6, equipped with the new critical embedding theorem and concentration compactness theorem, we may proceed, after establishing a few restrictions on $\{a_i(x)\}$ and f, to using the Mountain Pass theorem (MP) in order to prove existence of a weak solution to $(*)$, with critical growth on f. Hence, before getting to this application, we will present, in Section 5, a proof of (MP). In that section, we have also included a proof of the deformation theorem, which is the main tool in proving (MP), as well as an analogue of Picard's theorem for solving an ordinary differential equation.

1.2. **Aknowledgements.** I would like to acklowledge of my supervisors, Dr. Jérôme Vétois and Dr. Pengfei Guan, for their meticulous revisions of my work, suggestions, advices and overall support for the writing of this thesis.

1.3. **Contribution to Original Knowledge.** To our knowledgs, Theorems 3.12, 4.1 and 6.4 are contributions to original knowledge.

1.4. **Contribution of Author.** All chapters in this thesis were done by the author alone.

2. Preliminaries

The following definitions and properties are taken from [17]. The reader may also refer to the latter and [15] for much more properties of these spaces, as well as the case when the exponent is unbounded, which will not be treated in this thesis.

2.1. **Variable Exponent Lebesgue Spaces.** Let Ω be a domain in $\mathbb{R}^n$ and μ be a nonnegative Borel measure on Ω. We define the following class of functions:

$$\mathcal{P}(\Omega) = \{p : \Omega \to [1, \infty) \text{ is measurable } : \operatorname{ess\,sup}_{x\in\Omega}\{p(x)\} =: p^+ < \infty\}.$$

We define the variable exponent Lebesgue space as follows:

$$L_\mu^{p(x)}(\Omega) = \{u \in L^1_{loc}(\Omega, \mu) : \int_\Omega |u(x)|^{p(x)} d\mu < \infty\},$$

and equip it with the norm

$$\|u\|_{p(x)} = \|u\|_{L_\mu^{p(x)}(\Omega)} = \inf_{\lambda>0} \left\{ \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} d\mu \leq 1 \right\}.$$

Note that the variable exponent Lebesgue spaces are a special case of Musielak-Orlicz spaces, with the Luxembourg norm. Hence some of the properties outlined further, such as reflexivity or completeness, are simply a consequence of this. The reader may refer to [17], Chapter 2, for further information.

Although it is also possible to define the Lebesgue spaces with unbounded exponents, they lose many of the porperties that the classical Lebesgue spaces have, such as the density of simple functions (when the measure is separable, for example any Radon measure on $\mathbb{R}^n$). Furthermore, the following spaces may not be equivalent to $L_\mu^{p(x)}(\Omega)$ when $p(x)$ is unbounded, but are equivalent for $p \in \mathcal{P}(\Omega)$:

$$\{u \text{ is measurable: } \exists\,\lambda > 0 \text{ s.t. } \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} d\mu < \infty\},$$

$$\{u \text{ is measurable: } \int_\Omega |u(x)|^{p(x)} d\mu < \infty\},$$

$$\{u \text{ is measurable: } \forall\,\lambda > 0 \text{ s.t. } \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} d\mu < \infty\},$$

$$\{u \text{ is measurable: } \lim_{\lambda\to\infty} \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} d\mu = 0\}.$$

Throughout this thesis, we will use the following notation:

$$p^+(E) = \sup_{x\in E}\{p(x)\},$$

$$p^-(E) = \inf_{x\in E}\{p(x)\}$$

and we will write simply p^+ and p^- whenever $E = \Omega$.

Here are some properties of $L_\mu^{p(x)}(\Omega)$:

- It is a Banach space.
- Simple functions are dense and the space is separable if μ is separable.
- Let $p^- > 1$ and $p'(x) = \frac{p(x)}{p(x)-1}$. Then $L_\mu^{p(x)}(\Omega)$ is reflexive with
$$(L_\mu^{p(x)}(\Omega))^* \cong L_\mu^{p'(x)}(\Omega).$$
- Analogues of Fatou's Lemma, the Monotone Convergence Theorem and the Dominated Convergence theorem hold.

Let $\rho(u) = \int_\Omega |u(x)|^p(x) d\mu$. This defines a modular on $L_\mu^{p(x)}(\Omega)$ which behaves similarly to the norm, as can be seen by the following:

- $\rho(u_j) \to 0 \iff \|u_j\|_{p(x)} \to 0$
- If $u \neq 0$, then $\left[\rho\left(\frac{u(x)}{\lambda}\right) = 1 \iff \lambda = \|u\|_{p(x)}\right]$
- $\rho(u) < 1 (= 1, > 1) \iff \|u\|_{p(x)} < 1 (= 1, > 1)$
- If $\|u\|_{p(x)} > 1$, then $\|u\|_{p(x)}^{p^-} \leq \rho(u) \leq \|u\|_{p(x)}^{p^+}$

- If $\|u\|_{p(x)} < 1$, then $\|u\|_{p(x)}^{p^+} \le \rho(u) \le \|u\|_{p(x)}^{p^-}$
- If $\mu(A) \ge 1$, then $\mu(A)^{1/p^+} \le \|\chi_A\|_{p(x)} \le \mu(A)^{1/p^-}$
- If $\mu(A) \le 1$, then $\mu(A)^{1/p^-} \le \|\chi_A\|_{p(x)} \le \mu(A)^{1/p^+}$

Notice that by the first point, the topology induced by the norm is the same as the topology of convergence of the integral, which is an essential tool in proving convergence of sequences. Note that this may not be the case when $p(x)$ is unbounded. The second point from the list above will often be used, such as in the next theorem. Finally we will prove Hölder's inequality, embeddings of Lebesgue spaces, and another very useful theorem.

Theorem 2.1. *Let $p, q \in \mathcal{P}(\Omega)$ be such that $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$, $f \in L_\mu^{p(x)}(\Omega)$ and $g \in L_\mu^{q(x)}(\Omega)$. Then*

$$\int\limits_\Omega f(x)g(x)\,d\mu \le 2\,\|f\|_{p(x),\mu}\,\|g\|_{q(x),\mu}\,.$$

Proof. First, note that since $p, q \in \mathcal{P}(\Omega)$, then $p^+ < \infty$ and $q^+ < \infty$, which implies $p^- > 1$ and $q^- > 1$. Let $\lambda = \|f\|_{p(x),\mu}$ and $\eta = \|g\|_{p(x),\mu}$. Then by Young's inequality, we have

$$\begin{aligned}\int\limits_\Omega \frac{f(x)g(x)}{\lambda\eta}\,d\mu &\le \int\limits_\Omega \frac{\left|\frac{f(x)}{\lambda}\right|^{p(x)}}{p(x)}\,d\mu + \int\limits_\Omega \frac{\left|\frac{g(x)}{\eta}\right|^{q(x)}}{q(x)}\,d\mu \\ &\le \int\limits_\Omega \left|\frac{f(x)}{\lambda}\right|^{p(x)} d\mu + \int\limits_\Omega \left|\frac{g(x)}{\eta}\right|^{q(x)} d\mu = 2.\end{aligned}$$

This completes the proof. $\square$

Note that the constant in Hölder's inequality can be made sharper, by setting it to be $\frac{1}{p^-} + \frac{1}{q^-}$, which gives back the constant 1 in the fixed exponent case. For the purposes of our work, we will use the constant 2, in order to avoid it depending on $p(x)$, since we do not need this constant to be sharp.

Theorem 2.2. *If $\mu(\Omega) < \infty$ and $p, q \in \mathcal{P}(\Omega)$ with $p(x) \le q(x)$, then we have*

$$L_\mu^{q(x)}(\Omega) \hookrightarrow L_\mu^{p(x)}(\Omega)$$

Proof. Let $r(x)$ be such that $\frac{1}{p(x)} = \frac{1}{q(x)} + \frac{1}{r(x)}$. Then, since we have by assumption $0 < \frac{p(x)}{q(x)} \le 1$, we obtain

$$r(x) = \frac{p(x)q(x)}{q(x) - p(x)} = \frac{p(x)}{1 - \frac{p(x)}{q(x)}} \ge p(x) \ge 1,$$

and $r(x) \in \mathbb{R}$ on $A = \{x \in \Omega : p(x) < q(x)\}$. Hence we also have $0 < \frac{p(x)}{r(x)} \leq 1$.

Now by applying Young's inequality, we obtain for $f \in L_\mu^{q(x)}(\Omega)$ with $\|f\|_{L_\mu^{q(x)}(\Omega)} = 1$

$$\begin{aligned}
\int_A |f(x)|^{p(x)} d\mu &\leq \int_A \frac{1}{\frac{r(x)}{p(x)}} d\mu + \int_A \frac{\left| |f(x)|^{p(x)} \right|^{\frac{q(x)}{p(x)}}}{\frac{q(x)}{p(x)}} d\mu \\
&\leq |A| + \int_A |f(x)|^{q(x)} d\mu \\
&\leq \mu(\Omega) + 1 := D < \infty.
\end{aligned}$$

Since $D > 1$

$$\int_A \left| \frac{f(x)}{D} \right|^{p(x)} d\mu \leq \int_A \frac{|f(x)|^{p(x)}}{D} d\mu \leq 1.$$

Hence by definition of the norm, we have $\|f\|_{L_\mu^{p(x)}(A)} \leq D$.

Hence for any $f \in L_\mu^{q(x)}(\Omega)$, we get $\|f\|_{L_\mu^{p(x)}(A)} \leq D\|f\|_{L_\mu^{q(x)}(A)} \leq D\|f\|_{L_\mu^{q(x)}(\Omega)}$ and so

$$\begin{aligned}
\|f\|_{L_\mu^{p(x)}(\Omega)} &\leq \|f\|_{L_\mu^{p(x)}(\Omega \backslash A)} + \|f\|_{L_\mu^{p(x)}(A)} \\
&= \|f\|_{L_\mu^{q(x)}(\Omega \backslash A)} + \|f\|_{L_\mu^{p(x)}(A)} \\
&\leq \|f\|_{L_\mu^{q(x)}(\Omega \backslash A)} + D\|f\|_{L_\mu^{q(x)}(\Omega)} \\
&\leq 2D\|f\|_{L_\mu^{q(x)}(\Omega)},
\end{aligned}$$

which completes the proof. □

Note that in the previous theorem, the embedding constant depends only on the measure of Ω. Also observe that we avoided using Hölder's inequality, since $r(x)$ may be unbounded, which is not a class of exponents that we consider here. In the case where $\inf\limits_{x \in \Omega} \{q(x) - p(x)\} > 0$, we may use Hölder's inequality and get a sharper constant, which would depend on $p(x)$ and $q(x)$ and $\mu(\Omega)$. Again, for our purposes, we will avoid it depending on the exponents, but as in the fixed exponent case, we cannot avoid it depending on the measure of Ω.

The following theorem will be important after having applied Hölder's inequality. Observe that in the fixed exponent case, we have $s^+ = s^-$, which gives an equality.

Theorem 2.3. *Let s be a measurable function and $r \in \mathcal{P}(\Omega)$, with $0 < s^- \leq s(x) \leq r(x)$. If $f \in L_\mu^{r(x)}(\Omega)$, then we have*

$$\min\left\{ \|f\|_{r(x),\mu}^{s^+}, \|f\|_{r(x),\mu}^{s^-} \right\} \leq \left\| |f|^{s(x)} \right\|_{\frac{r(x)}{s(x)},\mu} \leq \max\left\{ \|f\|_{r(x),\mu}^{s^+}, \|f\|_{r(x),\mu}^{s^-} \right\}.$$

Proof. Let $\lambda = \|f\|_{r(x),\mu} \leq 1$, then by properties of the norm, we have

$$\int_{\Omega} \left| \frac{|f(x)|^{s(x)}}{\lambda^{s^-}} \right|^{\frac{r(x)}{s(x)}} d\mu \leq \int_{\Omega} \left| \frac{f(x)}{\lambda} \right|^{r(x)} d\mu = 1 \implies \left\| |f|^{s(x)} \right\|_{\frac{r(x)}{s(x)},\mu} \leq \|f\|_{r(x),\mu}^{s^-}.$$

Likewise, we get $\left\| |f|^{s(x)} \right\|_{\frac{r(x)}{s(x)},\mu} \geq \|f\|_{r(x),\mu}^{s^+}$. Using the same technique for $\|f\|_{r(x),\mu} > 1$, we get

$$\|f\|_{r(x),\mu}^{s^-} \leq \left\| |f|^{s(x)} \right\|_{\frac{r(x)}{s(x)},\mu} \leq \|f\|_{r(x),\mu}^{s^+},$$

establishing the proof. □

This theorem will be particularly usefull in the following setting.

Corollary 2.4. *Let $p \in \mathcal{P}(\Omega)$, with $p^- > 1$, and $f \in L_\mu^{p(x)}(\Omega)$, then*

$$\min\left\{ \|f\|_{p(x),\mu}^{p^+-1}, \|f\|_{p(x),\mu}^{p^--1} \right\} \leq \left\| |f|^{p(x)-1} \right\|_{p'(x),\mu} \leq \max\left\{ \|f\|_{p(x),\mu}^{p^+-1}, \|f\|_{p(x),\mu}^{p^--1} \right\}.$$

For the corollary, use the previous theorem with $s(x) = p(x) - 1$ and $r(x) = p(x)$, which gives $\frac{r(x)}{s(x)} = p'(x)$.

2.2. **Variable Exponent Sobolev Spaces.** Up to know, we have dealt with an abstract Borel measure on $\Omega \subset \mathbb{R}^n$. We will need to use some of these properties in the proof of the concentration-compactness principle, where we will use measures that are not the Lebesgue measure. But for the remainder of this section, as well as the next one (on anisotropic spaces), we will only deal with the Lebesgue measure, so we omit the symbol μ and use instead dx for integration and $|\cdot|$ for the measure of a set.

We define the variable exponent Sobolev spaces as such:

$$W^{k,p(x)}(\Omega) = \{u \in W_{loc}^{k,1}(\Omega) : \partial^\alpha u \in L^{p(x)}(\Omega) \quad \forall |\alpha| \leq k\}$$

$$W_0^{k,p(x)}(\Omega) = \overline{C_c^\infty(\Omega)}^{W^{k,p(x)}(\Omega)},$$

where the norm is defined by:

$$\|u\|_{W^{k,p(x)}(\Omega)} = \sum_{|\alpha| \leq k} \|\partial^\alpha\|_{L^{p(x)}(\Omega)}.$$

Equivalently, for $W^{1,p(x)}(\Omega)$, $\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}$.
If $p^- > 1$, then $W^{k,p(x)}(\Omega)$ and $W_0^{k,p(x)}(\Omega)$ are reflexive, separable Banach spaces. We define pointwise the critical Sobolev exponent for $W^{k,p(x)}(\Omega)$ and $p \in \mathcal{P}(\Omega)$ such that $p^+ < \frac{n}{k}$:

$$p^*(x) = \frac{np(x)}{n - kp(x)}.$$

Before going on to prove our main theorems, we will present a lemma that will prove useful in using $L^{p(x)}$ embedding theorem and the classical fixed exponents results on $W^{k,p}$ to prove results for $W^{k,p(x)}$.

Definition 2.5. *We say that an open set $\Omega \subset \mathbb{R}^n$ has the cone property if there exists a cone C such that for every $x \in \Omega$, there exists a cone $C_x \subset \Omega$ with apex at x that is identical to C under rigid motion (translations and rotations).*

As mentioned in Adams [2], bounded domains with Lipschitz boundary have the cone condition.

Lemma 2.6. *Let $\Omega \subset \mathbb{R}^n$ be an open set with the cone property. Then for any $x \in \overline{\Omega}$ and $0 < r < R$, we can build a domain U_x with the cone property, such that $B(x,r) \cap \Omega \subset U_x \subset B(x,R) \cap \Omega$.*

Furthermore, if R is fixed for every $x \in \overline{\Omega}$ and r are bounded above by some $r_0 < R$, then every U_x can satisfy the cone condition using the same cone, under rigid motion.

Proof. Fix x and r as in the statement of the theorem. Let C be the cone that gives the cone property to Ω. Denote by C_x the cone congruent to C with apex at x such that $C_x \subset \Omega$. Now notice that C is a bounded domain with Lipschitz boundary, thus C has itself the cone condition.

Let $U_x = \left(\bigcup_{y \in B(x,r)} (R-r)C_y \right) \cup B(x,r)$, where $(R-r)C_y = C_y \cap B(y, R-r)$, which is itself a cone. Then we can take a cone $\widetilde{C}$ that gives the cone condition to $(R-r)C$, thus, by the triangle inequality, it follows that $U_x \subset B(x,R) \cap \Omega$ has the cone condition with $\widetilde{C}$. Furthermore, if R is fixed for all x and r are bounded above by some $r_0 < R$, then we may use the same cone (under rigid motion) for all sets U_x, by using $(R - r_0)C$.

$\square$

Remark 2.7. Observe that if Ω is bounded, then for any cover of $\overline{\Omega}$ by balls $\{B(x, r_x)\}_{x \in \overline{\Omega}}$, there exists a finite subcover $\{B(x_i, r_{x_i}/2)\}_{i=1}^k$ of Ω. Then by the previous lemma, we can take a finite cover U_{x_i} such that for each i, U_{x_i} has the cone property and $U_{x_i} \subset B(x_i, r_{x_i}) \cap \Omega$.

If on the other hand, if Ω is unbounded, then we can bound $\{r_x\}$ uniformly from above over Ω and use the same R for all x in order to create a countable cover U_{x_i} of Ω, where all sets of the cover have the cone condition with the same cone, under rigid motion, and $\sup_i \{|U_{x_i}|\} < \infty$.

2.3. **Sobolev Embeddings.** In this section we will prove the Sobolev embeddings for the variable exponent Sobolev spaces as defined previously. First, we will state the one for fixed exponents, as presented in Adams [2], Theorem 5.4 (p.97). We will omit the part of the theorem that deals with subspaces of lesser dimension, since we will not be dealing with these spaces here. Also, we will write $p^* = \frac{np}{n-kp}$ (the k^{th}-order critical exponent),

$$C_b^j(\Omega) = \{f \in C^j(\Omega) : \partial^\alpha f \text{ is bounded for all } |\alpha| \leq j\}$$

and

$$C^{j,\lambda}(\Omega) = \{f \in C^j(\Omega), 0 < \lambda \leq 1 : |\partial^\alpha f(x) - \partial^\alpha(y)| \leq |x-y|^\lambda \text{ for all } |\alpha| \leq j\}.$$

We equip the above spaces with the norms

$$\|f\|_{C_b^j(\Omega)} = \max_{0\leq|\alpha|\leq j} \sup_{x\in\Omega} |\partial^\alpha f(x)|$$

and

$$\|f\|_{C^{j,\lambda}(\Omega)} = \max_{0\leq|\alpha|\leq j} \sup_{x,y\in\Omega} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x-y|^\lambda}.$$

Note that we will be needing the strong local Lipschitz condition for part 2. For this definition, we refer to [2] Chapter 4. In case of a bounded domain, it simply refers to a Lipschitz boundary. In case of an unbounded domain, we essentially require the existence of a "good" cover $\{U_i\}$ of $\partial\Omega$ such that for each i, $\partial\Omega \cap U_i$ is represented by a Lipschitz function f_i and $\sup_i\{\text{Lip}(f_i)\} < \infty$.

Theorem 2.8 (Adams [2]). *Let $\Omega \subset \mathbb{R}^n$ be a domain, $1 \leq p < \infty$, $k \in \mathbb{N}$ and $j \in \mathbb{N} \cup \{0\}$.*

Part 1: *If Ω has the cone property, then the following embeddings are continuous, with constants depending only on Ω, n, k, j and the chosen cone;*

<u>*Case A:*</u> *If $kp < n$, then*

$$W^{j+k,p}(\Omega) \hookrightarrow W^{j,q}(\Omega), \qquad \textit{for } p \leq q \leq p^*$$

or

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega), \qquad \textit{for } p \leq q \leq p^*.$$

<u>*Case B:*</u> *If $kp = n$, then*

$$W^{j+k,p}(\Omega) \hookrightarrow W^{j,q}(\Omega), \qquad \textit{for } p \leq q < \infty$$

or

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega), \qquad \textit{for } p \leq q < \infty.$$

Moreover we have the special case $p = 1$ and $k = n$ for which

$$W^{k+j,1}(\Omega) \hookrightarrow C_b^j(\Omega).$$

<u>*Case C:*</u> *If $kp > n$, then*

$$W^{k+j,p}(\Omega) \hookrightarrow C_b^j(\Omega).$$

Part 2: *If Ω has the strong local Lipschitz Property, then we can refine case C as follows:*

<u>*Case C':*</u> *If $kp > n > (k-1)p$, then*

$$W^{k+j,p}(\Omega) \hookrightarrow C^{j,\lambda}(\overline{\Omega}), \qquad \text{for } 0 < \lambda \leq k - \frac{n}{p}.$$

<u>*Case C":*</u> *If $(k-1)p = n$, then*

$$W^{k+j,p}(\Omega) \hookrightarrow C^{j,\lambda}(\overline{\Omega}), \qquad \text{for } 0 < \lambda < 1.$$

If $p = 1$, then the above holds for $\lambda = 1$ as well.

Part 3: *If we replace W by W_0, then the statements in part 1 hold for arbitrary domains.*

When dealing with variable exponents, we will include Case B in case A and we will relabel Case C as Case B.

This is because we now have the case when the exponent the $p(x)$ crosses (or reaches) the threshold n/k, i.e. $p^- \leq \frac{n}{k} \leq p^+$. The problem here will be that $p^*(x)$ is unbounded. The Lebesgue spaces with unbounded exponents lose many of their good properties, so we will avoid going there.

In order to go around this problem, whenever $p^- \leq \frac{n}{k} \leq p^+$, we will have an embedding that holds for any $q \in \mathcal{P}(\Omega)$ such that $q(x) \leq p^*(x)$. Notice that $q \in \mathcal{P}(\Omega)$ implies $q \in L^\infty$, thus we will never have $q(x) = p^*(x)$ for all $x \in \Omega$ in this scenario, but we may take q to be as large as we want on the set where $p(x) \geq n$, or as p approaches n.

Now we present the modified version of Theorem 2.8 for variable exponent. For the remainder of the thesis, we will denote the Lipschitz continuous functions on $\overline{\Omega}$ by $\mathrm{Lip}(\overline{\Omega})$.

Theorem 2.9 (Sobolev Embedding Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be a domain, $p \in \mathcal{P}(\Omega)$, $k \in \mathbb{N}$ and $j \in \mathbb{N} \cup \{0\}$.*

Part 1: *If Ω has the cone property, then the following embeddings are continuous, with constants depending only on Ω, n, k, j and the chosen cone;*

<u>*Case A:*</u> *If* $1 < p^- \leq \frac{n}{k}$ *and* $p \in \mathrm{Lip}(\overline{\Omega})$*, then*

$$W^{j+k,p(x)}(\Omega) \hookrightarrow W^{j,q(x)}(\Omega), \qquad \text{for } q \in \mathcal{P}(\Omega) \text{ such that } p(x) \leq q(x) \leq p^*(x)$$

or

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega), \qquad \text{for } q \in \mathcal{P}(\Omega) \text{ such that } p(x) \leq q(x) \leq p^*(x).$$

with the embedding constant also depending on the Lipschitz constant of p *and on* p^-*.*

<u>*Case B:*</u> *If* $kp^- > n$*, then*

$$W^{k+j,p(x)}(\Omega) \hookrightarrow C_b^j(\Omega).$$

Part 2:*If* Ω *has the strong local Lipschitz Property, then we can refine case B as follows:*

<u>*Case B':*</u> *If* $kp^- > n > (k-1)p^-$*, then*

$$W^{k+j,p(x)}(\Omega) \hookrightarrow C^{j,\lambda}(\overline{\Omega}), \qquad \text{for } 0 < \lambda \leq k - \frac{n}{p^-}.$$

Part 3: *If we replace* W *by* W_0*, then the statements in part 1 hold for arbitrary domains.*

Part 4: *If* $\mu(\Omega) < \infty$*, the we can replace the lower bound on* $q(x)$ *by* 1 *in Case A.*

Before presenting the proof, we give a lemma that will greatly simplify the proof of Case A.

Lemma 2.10. *If the embedding* $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ *of Case A holds, then all the embeddings* $W^{k+j,p(x)} \hookrightarrow W^{j,q(x)}(\Omega)$ *also hold.*

As in the case of fixed exponents, if one can prove the embeddings above for the special cases $j = 0$ and $k = 1$, then we can use induction to prove it for larger values of j and k.

Proof. In fact if the embeddings hold for any $k \in \mathbb{N}$ and $j = 0$ and $q(x)$ as above, then we have, for any other $j \in \mathbb{N}$, that if $u \in W^{k+j,p(x)}(\Omega)$, then $\partial^\alpha u \in W^{k,p(x)}(\Omega)$ for all $|\alpha| \leq j$, thus

$$\|u\|_{W^{j,q(x)}(\Omega)} = \sum_{|\alpha|\leq j} \|\partial^\alpha u\|_{L^{q(x)}(\Omega)} \leq C_1 \sum_{|\alpha|\leq j} \|\partial^\alpha u\|_{W^{k,p(x)}(\Omega)} \leq C_2 \|u\|_{W^{k+j,p(x)}(\Omega)}.$$

So we showed that we only need to prove the case $j = 0$. Now we will use induction to show we only need to prove the case $k = 1$.

Assume the embeddings holds up to some $k \geq 1$ and $j = 0$, i.e $W^{k,p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$, for any $q \in \mathcal{P}(\Omega)$ such that $q(x) \leq r(x) = \frac{np(x)}{n-kp(x)}$. Note that $r(x)$ may be infinite. Then set $s(x) = \frac{np(x)}{n-(k+1)p(x)}$, then by simple calculations, we get $q^*(x) \leq r^*(x) = \frac{nr(x)}{n-r(x)} = s(x)$. If $u \in W^{k+1,p(x)}(\Omega)$, then for any $1 \leq i \leq n$, u and $\partial_i u$ are in $W^{k,p(x)}(\Omega)$. Thus we have:

$$\begin{aligned}\|u\|_{W^{1,q(x)}(\Omega)} &= \|u\|_{L^{q(x)}(\Omega)} + \sum_{1\leq i\leq n} \|\partial_i u\|_{L^{q(x)}(\Omega)} \\ &\leq C_1 \|u\|_{W^{k,p(x)}(\Omega)} + \sum_{1\leq i\leq n} \|\partial_i u\|_{W^{k,p(x)}(\Omega)} \leq C_2 \|u\|_{W^{k+1,p(x)}(\Omega)}.\end{aligned}$$

Then because $q^*(x) \leq r^*(x)$, we get

$$\|u\|_{L^{q^*(x)}(\Omega)} \leq C_0 \|u\|_{W^{1,r(x)}(\Omega)} \leq C \|u\|_{W^{k+1,p(x)}(\Omega)}.$$

Therefore by induction, we only need to prove the case $j = 0$ and $k = 1$. □

Observe that when $p^+ < n$, we only need to consider the case $q(x) = p^*(x)$, since for any $q \in \mathcal{P}(\Omega)$ such that $p(x) \leq q(x) \leq p^*(x)$, we have for $u \in W^{1,p(x)}(\Omega)$ with $\|u\|_{W^{1,p(x)}(\Omega)} = 1$,

$$\int_\Omega |u|^{q(x)}\, dx \leq \int_\Omega |u|^{p(x)}\, dx + \int_\Omega |u|^{p^*(x)}\, dx \leq 1 + C,$$

and so for any $u \in W^{1,p(x)}(\Omega)$, so by using techniques as in Theorem 2.2, the above leads to

$$\|u\|_{L^{q(x)}(\Omega)} \lesssim \|u\|_{W^{1,p(x)}(\Omega)}.$$

Furthermore, if $p^+ \geq n$, then for any $q \in \mathcal{P}(\Omega)$, there exists $\delta > 0$ such that $q^+ < (n-\delta)^*$. By Lemma 2.6 (1), we can cover the open set $\Omega_1 = \{x \in \Omega : p(x) > n - \frac{\delta}{2}\}$ by countably many sets U_x, such that the union of those sets, denoted E is contained in $\{x \in \Omega : p(x) > n - \delta\}$. Since p is Lipschitz, we may also uniformly control the diameter of the sets U_x and thus also the cone condition. Therefore E is a domain with the cone condition by construction, on which we have, by Theorem 2.8,

$$\|u\|_{L^{q(x)}(E)} \leq \|u\|_{L^{(n-\delta)^*}(E)} \leq \|u\|_{W^{p(x)}(E)} \leq \|u\|_{W^{p(x)}(\Omega)}.$$

Then, we may likewise cover $\Omega \backslash \Omega_1$ by a domain D with the cone condition such that $D \subset \{x \in \Omega : p(x) < n - \frac{\delta}{4}\}$. Therefore, with the previous lemma, we just showed that the problem of proving Case A is reduced to proving the embedding

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega),$$

when $p^+ < n$. Before proving this, we will demonstrate this useful lemma, which we will also need in the anisotropic case later on. The proof of this is actually part of the proof given by Fan & Zhao in [25] for the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega)$.

Lemma 2.11. *Let $p, r, q \in \mathcal{P}(\Omega)$ such that for all $x \in \Omega$ and some $\varepsilon > 0$, we have $p(x)+\varepsilon \leq r(x) \leq q(x) - \varepsilon$. Then for any $u \in L^{q(x)}(\Omega)$ with $\|u\|_{q(x)} = 1$ and any $C > 0$, there exists $\widetilde{C}$ such that*

$$\int_\Omega |u(x)|^{r(x)} |\log |u(x)|| \, dx \leq \widetilde{C} \int_\Omega |u(x)|^{p(x)} \, dx + \frac{1}{C}.$$

Proof. Fix $C > 0$. Since $\lim\limits_{t\to\infty} t^{r(x)-q(x)} \leq \lim\limits_{t\to\infty} t^{-\varepsilon} = 0$, then we can find $t_0 > 1$ such that for any $t > t_0$ and any $x \in \Omega$, the following inequality holds:

$$t^{r(x)} \leq \frac{1}{C} t^{q(x)}. \tag{3}$$

Then we can define

$$\widetilde{C} = \left(\sup_{t\in(0,t_0]} t^{\varepsilon} |\log t| \right) \left(\sup_{t\in(0,t_0], x\in\Omega} t^{r(x)-\varepsilon-p(x)} \right),$$

which is finite, since $r(x) - \varepsilon - p(x) \geq 0$ and $\lim\limits_{t\to 0^+} t^{\varepsilon} |\log t| = 0$.

Then we get

$$\sup_{t\in(0,t_0], x\in\Omega} t^{r(x)} |\log t| \leq \widetilde{C} t^{p(x)}. \tag{4}$$

Thus letting $\Omega_1 = \{x \in \Omega : |u(x)| \leq t_0\}$ and $\Omega_2 = \{x \in \Omega : |u(x)| > t_0\}$, we can calculate

$$\begin{aligned}
\int_\Omega |u(x)|^{r(x)} |\log |u(x)|| \, dx &\leq \int_{\Omega_1} |u(x)|^{r(x)} |\log |u(x)|| \, dx + \int_{\Omega_2} |u(x)|^{r(x)} |\log |u(x)|| \, dx \\
&\leq \widetilde{C} \int_\Omega |u(x)|^{p(x)} \, dx + \frac{1}{C} \int_\Omega |u(x)|^{q(x)} \, dx \\
&= \widetilde{C} \int_\Omega |u(x)|^{p(x)} \, dx + \frac{1}{C},
\end{aligned}$$

establishing the proof. □

Note that $\widetilde{C}$ in the above proof depends on the choice of ε. In fact, for a fixed C, as $\varepsilon \to 0$, we have $t_0 \to \infty$, hence $\widetilde{C} \to \infty$.

Proof of Part 1, Case A:

This proof mainly follows the steps presented by Fan & Zhao in [24]. First, assume $u \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ and has compact support in $\overline{\Omega}$. Thus we know $u \in L^{p^*(x)}(\Omega)$. Let $\lambda = \|u\|_{L^{p^*(x)}(\Omega)}$ and $f(x) = \left| \dfrac{u(x)}{\lambda} \right|^{\left(\frac{n-1}{n}\right)p^*(x)}$.

Then we get that $f \in L^{\frac{n}{n-1}}(\Omega)$ with $\|f\|_{L^{\frac{n}{n-1}}(\Omega)} = 1$, since

$$\int_\Omega f^{\frac{n}{n-1}} = \int_\Omega \left(\left| \frac{u(x)}{\lambda} \right|^{\left(\frac{n-1}{n}\right)p^*(x)} \right)^{\frac{n}{n-1}} = \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p^*(x)} = 1.$$

Since $p(x)$ is Lipschitz, then so is $p^*(x)$, thus $|\nabla p^*(x)|$ exists almost everywhere and is bounded by the Lipschitz constant. Hence for almost every $x \in \Omega$ and $1 \le i \le n$,

$$\begin{aligned}
\partial_i f(x) &= \left[\partial_i \left(p^*(x) \frac{n-1}{n} \log \left| \frac{u(x)}{\lambda} \right| \right) \right] f(x) \\
&= \left(\frac{n-1}{n} \right) \left(\partial_i p^*(x) \log \left| \frac{u(x)}{\lambda} \right| + p^*(x) \left| \frac{u(x)}{\lambda} \right|^{-1} \operatorname{sign}(u) \frac{\partial_i u(x)}{\lambda} \right) \left| \frac{u(x)}{\lambda} \right|^{\frac{n-1}{n} p^*(x)}
\end{aligned}$$

Therefore we get the inequality

$$\begin{aligned}
|\nabla f(x)| &\lesssim |\nabla p^*(x)| \left| \log \left| \frac{u(x)}{\lambda} \right| \right| \left| \frac{u(x)}{\lambda} \right|^{\frac{n-1}{n} p^*(x)} + |p^*(x)| \left| \frac{u(x)}{\lambda} \right|^{\frac{n-1}{n} p^*(x) - 1} \left| \frac{\nabla u}{\lambda} \right| \\
&\lesssim \left[|f(x)| \left| \log \left| \frac{u(x)}{\lambda} \right| \right| \right] + \left[\frac{1}{\lambda} \left| \frac{u(x)}{\lambda} \right|^{\frac{n-1}{n} p^*(x) - 1} |\nabla u(x)| \right] \\
&=: h(x) + g(x) \qquad (5)
\end{aligned}$$

Now, letting $p'(x) = \frac{p(x)}{p(x)-1}$, some elementary calculations yield

$$\frac{n-1}{n} p^*(x) - 1 = \frac{n(p(x)-1)}{n - p(x)}$$

and so

$$\left(\frac{n-1}{n} p^*(x) - 1 \right) p'(x) = p^*(x).$$

Therefore by applying Young's inequality, we obtain

$$\begin{aligned}
\int_\Omega g(x) &\le \frac{1}{\lambda} \left(\int_\Omega \left(\frac{1}{p'(x)} \right) \left| \frac{u(x)}{\lambda} \right|^{\left(\frac{n-1}{n} p^*(x) - 1\right) p'(x)} + \int_\Omega \frac{1}{p(x)} |\nabla u(x)|^{p(x)} \right) \\
&\le \frac{1}{\lambda} \left(\int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p^*(x)} + \int_\Omega |\nabla u(x)|^{p(x)} \right) \\
&= \frac{1}{\lambda} \left(1 + \int_\Omega |\nabla u(x)|^{p(x)} \right) < \infty, \qquad \text{since } u \in W^{1,p(x)}(\Omega) \qquad (6)
\end{aligned}$$

Now we want to work on $h(x)$. Since $p^- > 1$, then $p^*(x)\frac{n-1}{n} - p(x) \ge \frac{p^- - 1}{n - p^-} > 0$. So we can pick $\varepsilon > 0$ small enough such that $p^*(x)\frac{n-1}{n} > p(x) + \varepsilon$ and $p^*(x)\frac{n-1}{n} < p^*(x) - \varepsilon$. Fix a constant $C>0$ that we will define later and then use lemma 2.11 with $r(x)=p^*(x)\frac{n-1}{n}$ and $q(x)=p^*(x)$ to obtain

$$\int\limits_\Omega h(x) \leq \widetilde{C}\int\limits_\Omega |u|^{p(x)}+\frac{1}{3C}.$$

Therefore by (5), (6) and (67), we get

$$\int_\Omega |\nabla f(x)| \leq \int_\Omega g(x)+h(x) \leq \frac{1}{\lambda}\left(1+\int_\Omega |\nabla u(x)|^{p(x)}\right)+\widetilde{C}\int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)}+\frac{1}{3C}. \tag{7}$$

Now let $\overline{C}=\sup\limits_{t\in(0,t_0],x\in\Omega} t^{p^*(x)\frac{n-1}{n}-p(x)}$, we obtain

$$\begin{aligned}\int_\Omega |f(x)| &\leq \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p^*(x)\frac{n-1}{n}}\\ &\leq \int_{\Omega_1}\left|\frac{u(x)}{\lambda}\right|^{p^*(x)\frac{n-1}{n}}+\int_{\Omega_2}\left|\frac{u(x)}{\lambda}\right|^{p^*(x)\frac{n-1}{n}}\\ &\leq \overline{C}\int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)}+\frac{1}{3C} \end{aligned}\tag{8}$$

Combining (7) and (8) with the classical Sobolev inequality for $W^{1,1}(\Omega)\hookrightarrow L^{\frac{n}{n-1}}(\Omega)$, with constant C as in (22), gives us

$$\begin{aligned}1&=\|f\|_{L^{\frac{n}{n-1}}(\Omega)}\\ &\leq C\left(\frac{1}{\lambda}\left(1+\int_\Omega|\nabla u(x)|^{p(x)}\right)+\widetilde{C}\int_\Omega\left|\frac{u(x)}{\lambda}\right|^{p(x)}+\frac{1}{3C}+\overline{C}\int_\Omega\left|\frac{u(x)}{\lambda}\right|^{p(x)}+\frac{1}{3C}\right)\\ &\lesssim \frac{C}{\lambda}+\frac{C}{\lambda}\int_\Omega|\nabla u(x)|^{p(x)}+C\int_\Omega\left|\frac{u(x)}{\lambda}\right|^{p(x)}+\frac{1}{3}+C\int_\Omega\left|\frac{u(x)}{\lambda}\right|^{p(x)}+\frac{1}{3}\end{aligned}$$

Removing $\frac{2}{3}$ on each side and using a new constant C, we obtain

$$1\leq C\left(\int_\Omega\left|\frac{u(x)}{\lambda}\right|^{p(x)}+\frac{1}{\lambda}\int_\Omega|\nabla u(x)|^{p(x)}+\frac{1}{\lambda}\right).$$

If $\lambda>1$, then we get $\lambda^{-p(x)}\leq\lambda^{-1}$, thus

$$\|u\|_{L^{p^*(x)}(\Omega)}=\lambda\leq C\left(\int_\Omega|u(x)|^{p(x)}+\int_\Omega|\nabla u(x)|^{p(x)}+1\right). \tag{9}$$

We may assume that $C\geq1$, thus the above is also true for $\lambda\leq1$.

Now by (9), we can observe that

$$\left\| \frac{u}{\|u\|_{W^{1,p(x)}(\Omega)}} \right\|_{L^{p^*(x)}(\Omega)} \leq C \left(\int_\Omega \left| \frac{u(x)}{\|u\|_{W^{1,p(x)}(\Omega)}} \right|^{p(x)} + \int_\Omega \left| \frac{\nabla u(x)}{\|u\|_{W^{1,p(x)}(\Omega)}} \right|^{p(x)} + 1 \right) \leq 3C$$

Then, we get that

$$\|u\|_{L^{p^*(x)}(\Omega)} \leq 3C \|u\|_{W^{1,p(x)}(\Omega)} \tag{10}$$

Observe that C does not depend on the support of u, nor on it's $L^\infty(\Omega)$ norm. Now if Ω is bounded, then every $u \in W^{1,p(x)}(\Omega)$ has compact support in $\overline{\Omega}$. If it is unbounded, then take a sequence $\{\phi_j\} \subset \mathscr{D}(\mathbb{R}^n)$ such that $0 \leq \phi_j \leq$, $|\nabla \phi_j| \leq 1$ and $\phi_j = 1$ on $B(0, j)$. Let $u \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ and let $u_j = \phi_j u$, then u_j and has compact support in $\overline{\Omega}$. Furthermore, one can notice that $|u_j| \leq |u|$ and $|\nabla u_j| \leq |u \nabla \phi_j| + |\phi_j \nabla u| \leq |u| + |\nabla u|$.

So we apply (10) to u_j

$$\|u_j\|_{L^{p^*(x)}(\Omega)} \lesssim \|u_j\|_{L^{p(x)}(\Omega)} + \|\nabla u_j\|_{L^{p(x)}(\Omega)} \lesssim \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)} = \|u\|_{W^{1,p(x)}(\Omega)}.$$

By taking the limit $j \to \infty$ on the left hand side, we now have that (10), with a new C, holds for all $u \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$.

Now for any $u \in W^{1,p(x)}(\Omega)$, we define

$$u_k(x) = \begin{cases} u(x), & \text{if } |u(x)| \leq k \\ k \,\mathrm{sign}(u(x)), & \text{if } |u(x)| > k \end{cases} \tag{11}$$

Then $u_k \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, $\int_\Omega |u_k|^{p(x)} \leq \int_\Omega |u|^{p(x)}$ and $\int_\Omega |\nabla u_k|^{p(x)} \leq \int_\Omega |\nabla u|^{p(x)}$, thus

$$\|u_k\|_{L^{p^*(x)}(\Omega)} \lesssim \|u_k\|_{L^{p(x)}(\Omega)} + \|\nabla u_k\|_{L^{p(x)}(\Omega)} \leq \|u\|_{W^{1,p(x)}(\Omega)}.$$

So taking $k \to \infty$ on the left hand side finishes the proof.

Proof of Part 1, Case B:

By Lemma 2.6, fix a countable cover $\{U_m\}$ of Ω as in the second part of Remark 2.7. As mentioned in the remark, they all have the cone property with the same cone, under rigid motion, and furthermore, we can bound from above the measure of the sets U_m.

By the Lebesgue embedding and the Sobolev embedding for fixed exponents, for each m, we have

$$W^{k+j,p(x)}(U_m) \hookrightarrow W^{k+j,p^-}(U_m) \hookrightarrow C_b^j(U_m).$$

Since $\{U_m\}$ is an open cover of Ω, then it follows that for all $|\alpha| \leq j$, $\partial^\alpha u$ is continuous on Ω (up to a set of measure zero).

From the choice of this cover, since the constants in the above embeddings depend only on the cone, on the measure of U_m, on $p(x)$ and on the dimension, then it follows that there exists $C > 0$ such that for all m,

$$\|u\|_{C_b^j(U_m)} \leq C\|u\|_{W^{k+j,p(x)}(U_m)} \leq C\|u\|_{W^{k+j,p(x)}(\Omega)}.$$

From the definition of $\|\cdot\|_{C_b^j(\Omega)}$, it follows that

$$\|u\|_{C_b^j(\Omega)} \leq C\|u\|_{W^{k+j,p(x)}(\Omega)}.$$

Proof of Part 2:

Follows in the same manner as the proof of Case B. Since the strong local Lipschitz property implies the cone condition, by Lemma 2.6, we can build a cover of bounded domains with the cone condition. Then, by Theorem 4.8 in [2], we can cover each $\{U_m\}$ by finitely many bounded domains with uniformly bounded radius and with the strong local Lipschitz condition, whose constants will depend on the cone (and thus on the original strong local Lipschitz condition of Ω), on the dimension and on the measure (which is also uniformly bounded). Thus we can proceed to prove the embeddings as in the proof of Case B.

Proof of Part 3:

This simply follows from the fact that the cone property was only used in order to apply the classical embedding theorems, thus as in Theorem 2.8, for $W_0^{k,p(x)}(\Omega)$, the theorem holds for arbitrary domains.

Proof of Part 4:

This simply follows by Theorem 2.2, since we proved for $q(x) = p^*(x)$ and thus for any other $1 \leq q(x) \leq p^*(x)$, we have

$$W^{k,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

Remark 2.12. It is important to mention here that Theorem 2.9 may be further weakened. First, cases B and B' require only continuity of the exponent, i.e. the Lipschitz condition is not needed. Even for case A, the authors of [17] (chapter 8) proved the case $p^+ < n$ for a class of continuous functions called log-Hölder continuous, which is a weaker condition than α-Hölder continuous, for any $\alpha \in (0, 1)$.

The proof, however, relies on the Hardy–Littlewood–Sobolev fractional integration theorem, which bounds the $L^{p^*}(\Omega)$ norm of the Riesz potential of a function by its $L^p(\Omega)$ norm. They show that the log-Hölder condition allows us to generalize the latter inequality to the variable exponent case so that we can also bound the $L^{p^*(x)}(\Omega)$ norm of the Riesz potential of the gradient by the $L^{p(x)}(\Omega)$ norm of the gradient. In the anisotropic case, it is not quite clear how to solve the problem using this approach.

Furthermore, this proof allows us to use a more general class of domains than the ones with the cone condition, which are the α-John domains.

So although we could generalize case A, we have decided to present the proof for a Lipschitz continuous function, since it is similar to how we will prove that in anisotropic case, whereas the proof in [17] will not be useful for us.

Observe though, that our proof used the cone condition only in order to apply the classical Sobolev embedding for $W^{1,1}(\Omega)$. Since it was shown in [9] that part A of Theorem 2.8 holds for α-John domains, our proof is still valid for this class of domains.

Finally, we will remark that in the anisotropic case, even for fixed exponents, not much generality has been developped in terms of domain regularity. Up to our knowledge, the most general type of domain was described by Rakosnik in [40], but they are very similar to a finite union of rectangles, and in fact a rectangular domain is usually assumed in the litterature, as was mentionned by [19].

2.4. **Rellich-Kondrachov Theorem.** Here we present an analogue of the Rellich-Kondrachov theorem for variable exponents. In the case of the Sobolev embedding for $kp^+ < n$, we required p to be Lipschitz continuous on $\overline{\Omega}$ with $p^- > 1$. For the compactness of embeddings though, we will only need $p \in C(\overline{\Omega})$ and $p^- \geq 1$.

Theorem 2.13 (Rellich-Kondrachov, Fan & Zhao [25])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the cone property, $p \in C(\overline{\Omega}) \cap \mathcal{P}(\Omega)$ and $q \in \mathcal{P}(\Omega)$ such that $1 \leq p(x) < \frac{n}{k}$ and $\inf\limits_{x\in\overline{\Omega}} \{p^*(x) - q(x)\} > 0$. Then we have*

$$W^{k,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega).$$

Proof. Let $\delta = \inf\limits_{x\in\overline{\Omega}} \{p^*(x) - q(x)\}$ and fix $0 < \varepsilon < \delta$. Since $p \in C(\overline{\Omega})$, then so is p^*. Therefore, for every $x \in \overline{\Omega}$, pick $r_x > 0$ such that for every $y \in B(x, r_x)$, $|p^*(x) - p^*(y)| < \varepsilon/2$. By remark 2.7, it follows that we can cover Ω by finitely many domains $\{U_i\}$ with the cone property, such that $B(x_i, r_i/2) \cap \Omega \subset U_i \subset B(x_i, r_i) \cap \Omega$.

Therefore, for every i, for every $x, y \in U_i$,

$$p^*(x) > p^*(y) - \varepsilon > p^*(y) - \delta \geq q(y).$$

It follows that $(p^*)^-(U_i) := \inf\limits_{x\in U_i} p^*(x) > \sup\limits_{x\in U_i} q(x) =: q^+(U_i)$.

Next observe that

$$\begin{aligned}
(p^*)^-(U_i) &= \inf_{x\in U_i} p^*(x) \\
&= \inf_{x\in U_i} \frac{np(x)}{n-kp(x)} \\
&= \inf_{x\in U_i} \frac{n}{\frac{n}{p(x)}-k} \\
&= \frac{n}{\frac{n}{p^-(U_i)}-k} \\
&= (p^-(U_i))^*.
\end{aligned}$$

Now let $\{u_j\}$ be a bounded subset of $W^{k,p(x)}(\Omega)$. Then it is also bounded in $W^{k,p^-}(U_i)$ for each i, so by the Rellich-Kondrachev theorem for fixed exponents, there exists a subsequence, still denote $\{u_j\}$, that is a Cauchy sequence in $L^{q^+}(U_i)$ for all i. Therefore by the Lebesgue embedding theorem for variable exponents,

$$\|u_j-u_m\|_{L^{q(x)}(\Omega)} \le \sum_i \|u_j-u_m\|_{L^{q(x)}(U_i)} \le 2\left(|\Omega|+1\right)\sum_i \|u_j-u_m\|_{L^{q^+}(U_i)} \to 0.$$

It follows from completeness of $L^{q(x)}(\Omega)$, that $u_j \to u$ in $L^{q(x)}(\Omega)$ (or a subsequence), which proves the compactness of the embedding. $\square$

Remark 2.14. Observe that if one uses Theorem 2.9 and Lemma 3.4 presented in the next section, then from Remark 2.12, we can prove compactness for a log-Hölder continuous exponent and a John domain in a more direct way. Our choice of proof relied on the fact that we may have even more general exponents, since we showed that $p(x)$ only needs to be continuous. In fact, because of Lemma 3.4, it follows that the compactness for fixed exponents also holds on α-John domains, hence by using our proof, we could have also relaxed the domain condition in Theorem 2.13 without requiring log-Hölder continuity. In [17], Definition 7.4.1, we can see that we can cover the closure of a bounded α-John domain by finitely many α_i-John domains, thus the proof follows in the same manner.

Corollary 2.15. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the cone property, $p, q \in C(\overline{\Omega})\cap\mathcal{P}(\Omega)$ such that $1 \le p(x) < \frac{n}{k}$ and $1 \le q(x) < p^*(x)$. Then we have*

$$W^{k,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega).$$

Proof. The above implies by continuity on $\overline{\Omega}$ that $\inf\limits_{x\in\overline{\Omega}} \{p^*(x)-q(x)\} > 0$, hence we can simply apply the previous theorem. $\square$

Notice that the cone condition is needed in order to apply the classical embedding theorem for fixed exponent. Thus if we restrict ourselves to functions in $W_0^{k,p(x)}(\Omega)$, then we can drop the cone condition.

2.5. **Poincaré's Inequality.** Next, we present the Poincaré inequality for $W_0^{1,p(x)}(\Omega)$. As in the fixed exponent case, one can show by induction that this inequality implies that on $W_0^{k,p(x)}(\Omega)$, the following are equivalent norms:

$$\|u\|_{W^{k,p(x)}(\Omega)} \simeq \sum_{|\alpha|=k} \|\partial^\alpha u\|_{L^{p(x)}(\Omega)}.$$

Theorem 2.16 (Poincaré's Inequality, Fan & Zhao [25]). *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $p \in C(\overline{\Omega}) \cap \mathcal{P}(\Omega)$, then there exists $C > 0$ such that*

$$\|u\|_{p(x)} \leq C\|\nabla u\|_{p(x)}, \text{ for all } u \in W_0^{1,p(x)}(\Omega).$$

Proof. First, observe that for $1 \leq p < n$,

$$p^* = \frac{np}{n-p} \geq \frac{n}{n-1}p \tag{12}$$

Let $p^+ < n$ and define $p_0(x) = p(x)$ and $p_{i+1}(x) = \min\{p_i(x) - \frac{1}{n}, 1\}$ for $i \in \mathbb{N}$. It follows that for all $i \geq n^2$, $p_i(x) = 1$, since $p^+ < n$. Now by (12), we get that

$$p^*_{i+1}(x) - p_i(x) = p^*_{i+1}(x) - p_{i+1}(x) + p_{i+1}(x) - p_i(x) \geq \frac{1}{n-1}p^{i+1}(x) - \frac{1}{n} > \frac{1}{n^2} \tag{13}$$

And so we have $p^*_{i+1}(x) - \frac{1}{n^2} > p_i(x)$, so we can apply theorem 2.13 and thus

$$\|u\|_{L^{p_i(x)}(\Omega)} \leq \widetilde{C}_i \left(\|u\|_{L^{p_{i+1}(x)}(\Omega)} + \|\nabla u\|_{L^{p_{i+1}(x)}(\Omega)}\right) \leq C_i \left(\|u\|_{L^{p_{i+1}(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}\right) \tag{14}$$

So by repeating the process, after finitely many step, we can apply the Poincaré inequality for fixed exponents to obtain

$$\begin{aligned} \|u\|_{L^{p(x)}(\Omega)} &\leq \widetilde{C_M} \left(\|u\|_{L^1(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}\right) \\ &\leq C_M \left(\|\nabla u\|_{L^1(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}\right) \leq C\|\nabla u\|_{L^{p(x)}(\Omega)}, \end{aligned} \tag{15}$$

which is Poincaré's inequality for exponents in $C(\overline{\Omega}) \cap \mathcal{P}(\Omega)$ such that $p^+ < n$.

For general $p \in C(\overline{\Omega}) \cap \mathcal{P}(\Omega)$, we can pick $s \in [1, n)$, such that $s^* > p^+$, and define $r \in C(\overline{\Omega}) \cap \mathcal{P}(\Omega)$ by $r(x) = \min\{p(x), s\}$. Observe then that $r(x) \leq s < n$ and that

$r(x) \leq p(x) < r^*(x)$. Therefore we can apply theorem 2.13, the Lebesgue embedding theorem and (15) to $r(x)$, to obtain

$$\begin{aligned}\|u\|_{L^{p(x)}(\Omega)} &\leq C_1 \|u\|_{W^{1,r(x)}(\Omega)} \leq C_2 \left(\|u\|_{L^{r(x)}(\Omega)} + \|\nabla u\|_{L^{r(x)}(\Omega)} \right) \\ &\leq C_3 \|\nabla u\|_{L^{r(x)}(\Omega)} \leq C \|\nabla u\|_{L^{p(x)}(\Omega)}\end{aligned}$$

□

Corollary 2.17 (Poincaré-Sobolev Inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $p \in \mathrm{Lip}(\overline{\Omega}) \cap \mathcal{P}(\Omega)$ be such that $1 < p^- \leq p(x) \leq p^+ < \frac{n}{k}$. Then there exists $C > 0$ such that*

$$\|u\|_{p^*(x)} \leq C \|\nabla u\|_{p(x)}, \;\; \textit{for all} \;\; u \in W_0^{1,p(x)}(\Omega).$$

Proof. It follows from the Poincaré inequality and the Sobolev embedding theorem Case A:

$$\|u\|_{p^*(x)} \leq C_1 \|u\|_{W^{1,p(x)}(\Omega)} = C_1(\|u\|_{p(x)} + \|\nabla u\|_{p(x)}) \leq C \|\nabla u\|_{p(x)}$$

□

In the next section, we will discuss anisotropic Sobolev spaces with variable exponents. Note that these are generalizations of the variable exponent Sobolev spaces, except in the fact that we will need to restrict ourselves to rectangular-like domains. Therefore, the proofs presented in the next section will also work for the present section, if we restrict ourselves to more regular domains. That is why we have decided to present the proofs separately, since we can benefit from greater generality of domains for the isotropic spaces. Although, the reader may note that the proof of the anisotropic Poincaré inequality, different from the one for Theorem 3.16, will also hold for the isotropic case, regardless of the domain.

3. Anisotropic Variable Exponent Sobolev Spaces

The overall aim of this work is to prove and use the Concentration-Compactness principle for anisotropic variable exponent Sobolev spaces. In order to even consider this, we must first have a critical embedding theorem, like the one we have in the previous section. To our surprise, up to our knowledge, this critical embedding has not yet been proven. In fact it was also stated as an open question by Fan in [19].

We will now establish some notation.

Let $\vec{p}(\cdot) = (p_1(\cdot), \dots, p_n(\cdot))$, with each $p_i \in \mathcal{P}(\Omega)$. Define

$$p_M(x) = \max_{1 \leq i \leq n} \{p_i(x)\};$$

$$p_m(x) = \min_{1 \leq i \leq n} \{p_i(x)\};$$

$$p^+ = \max_{1 \leq i \leq n} p_i^+ = \sup_{x \in \overline{\Omega}} \{p_M(x)\};$$

$$p^- = \min_{1\le i\le n} p_i^- = \inf_{x\in\overline{\Omega}}\{p_m(x)\};$$

$$\overline{p}(x) = \frac{n}{\sum\limits_{i=1}^{n} \frac{1}{p_i(x)}}$$

and

$$\overline{p}^*(x) = \frac{n\overline{p}(x)}{n-\overline{p}(x)}.$$

Observe that when $p_i(x) = p(x)$ for all $x \in \overline{\Omega}$ and $1 \le i \le n$, then $\overline{p}(x) = p(x) = p_m(x) = p_M(x)$ and $\overline{p}^*(x) = p^*(x)$. Furthermore, we will have equivalence of the norm $\|\cdot\|_{W^{1,p(\cdot)}(\Omega)}$ with the norm for the anisotropic Sobolev spaces defined below. Hence the anisotropic variable exponent Sobolev space is a generalization of the isotropic one.

As in the isotropic case, if $p^- > 1$, then $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are reflexive and uniformly convex Banach spaces (see theorem 2.1 of [19]).

In the remainder of the thesis, it is to be understood that if $\overline{p}(x) \ge n$, then $\overline{p}^*(x) = \infty$.

We define the anisotropic variable exponent Sobolev space as

$$W^{1,\{p_0(\cdot)\vec{p}(\cdot)\}}(\Omega) = \left\{u \in L^{p_0(\cdot)}(\Omega) : \partial_i u \in L^{p_i(\cdot)}(\Omega), \quad \text{for all } 1 \le i \le n\right\}.$$

We define the norm

$$\|u\|_{W^{1,\{p_0(\cdot)\vec{p}(\cdot)\}}(\Omega)} = \|u\|_{L^{p_0(\cdot)}(\Omega)} + \sum_{i=1}^{n} \|\partial_i u\|_{L^{p_i(\cdot)}(\Omega)}.$$

We will also define $W_0^{1,\{p_0(\cdot),\vec{p}(\cdot)\}}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ with the $W^{1,\{p_0(\cdot),\vec{p}(\cdot)\}}(\Omega)$ norm, i.e.

$$W_0^{1,\{p_0(\cdot),\vec{p}(\cdot)\}}(\Omega) = \overline{C_c^\infty}(\Omega).$$

As mentionned before, we will require more restriction on the regularity of Ω, except when working with $W_0^{1,\{p_0(\cdot),\vec{p}(\cdot)\}}(\Omega)$, since we may then use the zero extension to assume regularity of the domain. Hence we have the following definition.

Definition 3.1. *Ω is a rectangular domain if it is bounded and can be written as a product of intervals, i.e.,*

$$\Omega = (a_1, b_1) \times \cdots \times (a_n, b_n).$$

$\Omega \subset \mathbb{R}^n$ is a rectangle-like domain if for any $m \in \mathbb{N}$, $\Omega \cap (-m, m)^n$ is a finite union of rectangular domains.

Observe that when Ω is bounded and rectangle-like, then it is simply a union of finitely many rectangular domains. Thus for bounded domains, we will only consider, without loss of generality, rectangular domains instead of rectangular-like domains.

Before presenting our proof of the critical exponent embedding for the anisotropic variable exponent Sobolev spaces, we will give the embedding theorem for their fixed exponent counterparts.

Theorem 3.2 (Fan [19]). *Let Ω be a rectangular domain and $\vec{p_i} \in [1,\infty)^n$.*

(1) If $\overline{p} < n$, then $W^{1,\vec{p}}(\Omega) \hookrightarrow L^{\overline{p}^*}(\Omega)$ and $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $1 \le q < \overline{p}^*$.
(2) If $\overline{p} = n$, then $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for all $1 \le q < \infty$.
(3) If $\overline{p} > n$, then $W^{1,\vec{p}}(\Omega) \hookrightarrow C^{0,\beta}(\Omega)$, with

$$0 < \beta = \frac{\alpha}{\frac{n}{p_m} + \alpha} \quad \text{and} \quad \alpha = 1 - \frac{n}{\overline{p}}.$$

Furthermore, if we restrict ourselves to $W_0^{1,\vec{p}}(\Omega)$, then the above holds on any domain.

3.1. **Rellich-Kondrachov Theorem.** As in the previous section with isotropy, we can fairly easily prove the compactness of embeddings for subcritical exponents by using the classical embedding theorems for fixed anisotropic exponents. The following proofs are adapted from [19].

Lemma 3.3. *Let Ω be a rectangular domain, $p_0 \in C(\overline{\Omega}) \cap \mathcal{P}(\Omega)$ and $\vec{p}(x) \in \left(C(\overline{\Omega}) \cap \mathcal{P}(\Omega)\right)^n$, $q(x) = \max\{p_0(x), \overline{p}^*(x)\}$ and $r \in \mathcal{P}(\Omega)$ such that*

$$\inf_{x\in\Omega} \{q(x) - r(x)\} \ge \alpha > 0,$$

for some $\alpha \in (0,1)$. Then there exists a finite collection of cubes $\{Q_k\} \subset \Omega$ that cover Ω and such that for all k, one of the following holds:

(1) $r^+(Q_k) < \left(\overline{p^-}(Q_k)\right)^*$, where $\overline{p^-}(Q_k) = \dfrac{n}{\sum\limits_{i=1}^{n} \frac{1}{p_i^-(Q_k)}}$
(2) $r^+(Q_k) < (p_0)^-(Q_k)$

Proof. For each $x \in \overline{\Omega}$ such that $\overline{p}(x) < n$, by continuity of $\vec{p}$ and p_0, we may pick a small cube Q_x such that $q^+(Q_x) \le q^-(Q_x) + \frac{\alpha}{2}$. Hence we have

$$r^+(Q_x) < q^+(Q_x) - \frac{\alpha}{2} \le q^-(Q_x).$$

Then if $q(x) = p_0(x)$, we have shown Q_x satisfies (2). Otherwise, we may take Q_x to be even smaller, such that $\overline{p^+}(Q_x) < \overline{p^-}(Q_x) + \varepsilon$. If ε is small enough, then we get

$$q^+(Q_x) = (\overline{p}^*)^+(Q_x) = (\overline{p}^+)^*(Q_x) \le (\overline{p^+})^*(Q_x) \le (\overline{p^-})^*(Q_x) + \frac{\alpha}{2}.$$

Observe that when $\overline{p}(x) \geq n$, we can choose a $\delta > 0$ such that $r^+ < (n-\delta)^* - \frac{\alpha}{2}$, and a cube small enough such that $\overline{p^-}(Q_x) > n - \delta$, which implies (1).

Since Ω is bounded, then we may take a finite cover of such cubes, which completes the proof. $\square$

Lemma 3.4. *Let Ω be a bounded domain with the cone condition. If $q \in \mathcal{P}(\Omega)$ such that $W^{1,\{p_0(\cdot),\vec{p}(\cdot)\}}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, then for any $r \in \mathcal{P}(\Omega)$ such that $\inf_{x\in\Omega}\{q(x) - r(x)\} = \varepsilon > 0$,*

$$W^{1,\{p_0(\cdot),\vec{p}(\cdot)\}}(\Omega) \hookrightarrow\hookrightarrow L^{r(x)}(\Omega).$$

Proof. First, observe that the conditions imply $q^- \geq r^- + \varepsilon \geq 1 + \varepsilon > 1$. Then we may assume that $r^- > 1$, otherwise we can take $s(x) = \max\left\{r(x), 1 + \frac{\varepsilon}{2}\right\}$ and $L^{s(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$. By Lebesgue embeddings and Theorem 2.3, $W^{1,\{p_0(\cdot),\vec{p}(\cdot)\}}(\Omega) \hookrightarrow W^{1,1}(\Omega) \hookrightarrow\hookrightarrow L^1(\Omega)$. If $\{u_j\}$ is bounded in $W^{1,\{p_0(\cdot),\vec{p}(\cdot)\}}(\Omega)$, then it converges to some $u \in L^1(\Omega)$. Then by Hölder's inequality, with $v_j = u_j - u$,

$$\begin{aligned}
\int_\Omega |v_j|^{r(x)}\,dx &= \int_\Omega |v_j|^{\frac{q(x)-r(x)}{q(x)-1}} |v_j|^{\frac{q(x)(r(x)-1)}{q(x)-1}}\,dx \\
&\leq 2\left\| |v_j|^{\frac{q(x)-r(x)}{q(x)-1}} \right\|_{\frac{q(x)-1}{q(x)-r(x)}} \left\| |v_j|^{\frac{q(x)(r(x)-1)}{q(x)-1}} \right\|_{\frac{q(x)-1}{r(x)-1}} \\
&\leq 2M^{\frac{q^+(r^+-1)}{q^--1}} \|v_j\|_1^{\frac{\varepsilon}{q^+-1}} \to 0,
\end{aligned}$$

where $M = \sup_j \left\{\|v_j\|_{q(x)} + 1\right\} < \infty$, since boundedness in $W^{1,\{p_0(\cdot),\vec{p}(\cdot)\}}(\Omega)$ implies weak convergence in $L^{q(x)}(\Omega)$, because of the continuous embedding, which implies boundedness in $L^{q(x)}(\Omega)$. Also note that we used the exponent $\frac{\varepsilon}{q^+-1}$ in the last term, since $\|v_j\|_1 \to 0$, we may assume $\|v_j\|_1 < 1$. This completes the proof. $\square$

Then from the critical embedding proved before and Lemma 3.4, we get the following compactness results.

Theorem 3.5 (Fan [19])**.** *Let Ω be a rectangular domain, $p_0 \in C(\overline{\Omega}) \cap \mathcal{P}(\Omega)$ and $\vec{p}(x) \in \left(C(\overline{\Omega}) \cap \mathcal{P}(\Omega)\right)^n$.*

(1) If $r \in \mathcal{P}(\Omega)$ and there exists $\varepsilon > 0$ such that $r(x) < \max\{p_0(x), \overline{p}^*(x)\} - \varepsilon$ for all $x \in \overline{\Omega}$, then we have the continuous compact embedding

$$W^{1,\{p_0(x),\vec{p}(\cdot)\}}(\Omega) \hookrightarrow\hookrightarrow L^{r(x)}(\Omega).$$

(2) If $\overline{p}(x) > n$, then there exists $\beta \in (0,1)$ such that

$$W^{1,\{p_0(x),\vec{p}(\cdot)\}}(\Omega) \hookrightarrow C^{0,\beta}(\Omega) \hookrightarrow\hookrightarrow C(\overline{\Omega}).$$

Furthermore, if we restrict ourselves to $W_0^{1,\{p_0(\cdot),\vec{p}(\cdot)\}}(\Omega)$, then the above holds on any domain.

Proof. Let $\{u_j\}$ be a bounded sequence in $W^{1,\{p_0(\cdot),\vec{p}(\cdot)\}}(\Omega)$. Let $\{Q_k\}$ be a collection of cubes as in lemma 3.3. Then $\{u_j\}$ is also bounded in $W^{1,\{p_0(\cdot),\vec{p}(\cdot)\}}(Q_k)$ for each k.

If Q_k satisfies (1), then by the fixed exponent theorems and the fact that , we have that

$$W^{1,\{p_0(\cdot),\vec{p}(\cdot)\}}(Q_k) \hookrightarrow W^{1,(\vec{p})^-}(Q_k) \hookrightarrow\hookrightarrow L^{r^+(Q_k)}(Q_k) \hookrightarrow L^{r(x)}(Q_k),$$

where $(\vec{p})^- = (p_1^-, \ldots, p_n^-)$. If Q_k satisfies (2), then by lemma 3.4

$$W^{1,\{p_0(\cdot),\vec{p}(\cdot)\}}(Q_k) \hookrightarrow\hookrightarrow L^{r(x)}(Q_k).$$

Thus it follows that in both cases, $\{u_j\}$ (or a subsequence) is Cauchy in $L^{r(x)}(Q_k)$, hence we have

$$\|u_j - u_i\|_{r(x)} \leq \sum_k \|u_j - u_i\|_{r(x),Q_k} \to 0$$

This proves part (1) of the theorem. For part 2, we may construct cubes similarly to lemma 3.3 to get embedding into $C^{0,\beta_k}(Q_k)$ for each k and then take $\beta = \min\{\beta_k\}$ to complete the proof. □

3.2. **Critical Exponent Sobolev Embedding.** We now finally arrive at a new result. As stated previously, in [19], Fan stated the question of whether there is a critical embedding of the anisotropic variable exponent Sobolev space, when $p_M(x) < \overline{p}^*(x)$, as an open question, which, up to our knowledge, has not yet been answered.

Before proving so, we will need some preliminaries. First off, we will assume $p_0(x) = 1$ and $p_0(x) = p_m(x)$ in the bounded and unbounded domain cases, respectively. We will then adopt the shorter notation

$$W^{1,\{p_0(\cdot),\vec{p}(\cdot)\}}(\Omega) = W^{1,\vec{p}(\cdot)}(\Omega).$$

Furthermore, we will now always assume that $\inf_{x\in\Omega}\{\overline{p}^*(x) - p_M(x)\} > 0$. There are two main reasons for this. The first is that even in the fixed exponent case, the authors in [35] gave an example of a $\vec{p}$ with $p_M > \overline{p}^*$ and Ω is a cube, for which there is no embedding $W^{1,\{1,\vec{p}\}}(\Omega) \hookrightarrow L^{\overline{p}^*}(\Omega)$. On the other hand, in [40], it was shown that when $p_M < \overline{p}^*$, then the embedding exists for rectangular domains. Hence, we may assume that it is also the case with variable exponents.

The second is that it is still unknown, for the variable exponents, if there is a Poincaré inequality when $p_M \geq \overline{p}^*$, as it was pointed out by Fan ([19]). Since we will be needing this inequality to prove the concentration-compactness principle, we will stay away from this scenario.

In our proof, we will use density of smooth functions. Hence we present the following theorem, from [19].

Theorem 3.6 (Fan [19]). *Let $\Omega \subset \mathbb{R}^n$ be a rectangular domain and $\vec{p}(\cdot) \in \left(\mathrm{Lip}(\overline{\Omega}) \cap \mathcal{P}(\overline{\Omega})\right)^n$, then*

$$C^\infty(\overline{\Omega}) \text{ is dense in } W^{1,\vec{p}(\cdot)}(\Omega)$$

Note that the above is a stronger version of the theorem, since it actually holds for a larger class of continuous exponents, namely log-Hölder continuous exponents, as discussed in Remark 2.12.

Now in order to prove the critical embedding we define the following exponents, which relate the critical exponent to each component of $\vec{p}(\cdot)$:

$$\sigma_i(x) = \frac{1}{n(n-1)}\left(n - 1 + \sum_{j=1}^{n} \frac{1}{p_j(x)} - \frac{n}{p_i(x)}\right).$$

Lemma 3.7. *If $\vec{p}(\cdot) \in \left(Lip(\overline{\Omega})\right)^n$, $p^- > 1$, $p^+ < n$ and $\inf_{x\in\Omega}\{\overline{p}^*(x) - p_M(x)\} = \alpha > 0$, then:*

(1) $\sigma_i(x) > 0$;

(2) $\sum_{i=1}^{n} \sigma_i(x) = 1$;

(3) $[q(x)\sigma_i(x)(n-1) - 1]\left(\frac{p_i(x)}{p_i(x)-1}\right) = q(x)$; and

(4) there exists $\varepsilon > 0$ such that for any $i \in \{1, \dots, n\}$ and $x \in \Omega$,

$$p_i(x) + \varepsilon < q(x)\sigma_i(x)(n-1) < q(x) - \varepsilon.$$

Proof. Notice that by definition, $p_M(\cdot), \overline{p}(\cdot), q(\cdot)$ and $\sigma_i(\cdot)$ are also Lipschiptz continuous on $\overline{\Omega}$, for any i.

For (1), observe that

$$(n-1)\sigma_i(x) = 1 - \frac{1}{n} + \frac{1}{\overline{p}(x)} - \frac{1}{p_i(x)} = 1 + \frac{1}{\overline{p}^*(x)} - \frac{1}{p_i(x)} \geq 1 - \frac{1}{p^-} > 0. \tag{16}$$

For (2) and (3), we calculate directly:

$$\sum_{i=1}^{n}\sigma_i(x)=\sum_{i=1}^{n}\left(\frac{1}{n}+\left(\frac{1}{n(n-1)}\right)\left(\sum_{j=1}^{n}\frac{1}{p_j(x)}-\frac{n}{p_i(x)}\right)\right)$$
$$=1+\frac{1}{n(n-1)}\left(n\sum_{j=1}^{n}\frac{1}{p_j(x)}-\sum_{i=1}^{n}\frac{n}{p_i(x)}\right)=1$$

Since $\frac{1}{q(x)}=\frac{1}{\overline{p}(x)}-\frac{1}{n}$, then simple calculations yield

$$[q(x)\sigma_i(x)(n-1)-1]=q(x)\left(1-\frac{1}{n}+\frac{1}{\overline{p}(x)}-\frac{1}{p_i}\right)-\frac{q(x)}{q(x)}$$
$$=q(x)\left(1+\frac{1}{q(x)}-\frac{1}{p_i(x)}-\frac{1}{q(x)}\right)$$
$$=q(x)\left(\frac{p_i(x)-1}{p_i(x)}\right)$$

Thus we obtain (3) from the above.

Finally, to prove (4), we use (3) and the fact that $q(x)>p_i(x)+\alpha$ to get:

$$q(x)\sigma_i(x)(n-1)=\left(\frac{p_i(x)-1}{p_i(x)}\right)q(x)+1=q(x)-\frac{q(x)}{p_i(x)}+1<q(x)-\frac{\alpha}{p^+}.$$

We again use (3) and the inequality above to get:

$$\frac{p_i(x)}{p_i(x)-1}=\frac{q(x)}{q(x)\sigma_i(x)(n-1)-1}>\frac{q(x)\sigma_i(x)(n-1)}{q(x)\sigma_i(x)(n-1)-1}\implies p_i(x)<q(x)\sigma_i(x)(n-1).$$

Then, by using (16), we can see that for any x and i,

$$(n-1)\sigma_i(x)\leq 1-\frac{\alpha}{p^+(\overline{p}^+)^*}.$$

By taking $\varepsilon<\min\left\{\frac{\alpha}{p^+},\frac{q^-\alpha}{p^+(\overline{p}^+)^*}\right\}$, the proof is complete. □

The following lemma is similar to Lemma 2.11 and it allows us to use $u\in L^{p_0(x)}$ in our definition of $W^{1,\vec{p}(\cdot)}(\Omega)$, which will in turn prove that we may use, instead of $p_0(\cdot)$, any $r\in\mathcal{P}(\Omega)$ that satisfies $1\leq r(x)\leq p_M(x)$ in the bounded case or $p_m(x)\leq r(x)\leq p_M(x)$ in the unbounded case.

Note that the next result should be somewhat standard, since it is very similar to Young's inequality, but we have not seen it used directly in the litterature, which is why we give a proof here.

Lemma 3.8. *Let $\Omega\subset\mathbb{R}^n$, $p,q,r\in\mathcal{P}(\Omega)$ satisfiying $0<\varepsilon=\operatorname{ess\,inf}_{x\in\Omega}\{q(x)-r(x)\}$, $p(x)\leq r(x)$ and $u\in L^{p(x)}(\Omega)\cap L^{q(x)}(\Omega)$. Then for any $C>0$, there exists $\widetilde{C}>0$ such that*

$$\|u\|_{L^{r(x)}(\Omega)} \le \widetilde{C}\|u\|_{L^{p(x)}(\Omega)} + \frac{1}{C}\|u\|_{L^{q(x)}(\Omega)}$$

Proof. If $C \le 1$, then we can use the pointwise inequality to get

$$\begin{aligned}\int_\Omega |u(x)|^{r(x)}\,dx &\le \int_{\{x\in\Omega:|u(x)|\le 1\}} |u(x)|^{p(x)}\,dx + \int_{\{x\in\Omega:|u(x)|>1\}} |u(x)|^{q(x)}\,dx\\ &\le \frac{\widetilde{C}}{2}\int_\Omega |u(x)|^{p(x)}\,dx + \frac{1}{2C^{q^+}}\int_\Omega |u(x)|^{q(x)}\,dx,\end{aligned}$$

with $\widetilde{C} = 2$.

If $C > 1$, then set $t_0 = (2C^{q^+})^{1/\varepsilon}$. Hence for any $t \ge t_0$,

$$t^{r(x)-q(x)} \le t^{-\varepsilon} \le t_0^{-\varepsilon} = \frac{1}{2C^{q^+}}.$$

Let $\widetilde{C} = 2t_0^{r^+}$, then for all $t < t_0$,

$$t^{r(x)-p(x)} \le t_0^{r^+} = \frac{\widetilde{C}}{2}.$$

We can assume that $\widetilde{C} \ge 2$, thus from the two inequalities above, we obtain as previously

$$\begin{aligned}\int_\Omega |u(x)|^{r(x)}\,dx &\le \frac{\widetilde{C}}{2}\int_\Omega |u(x)|^{p(x)}\,dx + \frac{1}{2C^{q^+}}\int_\Omega |u(x)|^{q(x)}\,dx\\ &\le \frac{1}{2}\left(\int_\Omega |\widetilde{C}u(x)|^{p(x)}\,dx + \int_\Omega \left|\frac{u(x)}{C}\right|^{q(x)}\,dx\right).\end{aligned} \tag{17}$$

Since $\widetilde{C}$ does not depend on u, lettting $\lambda = \|\widetilde{C}u\|_{L^{p(x)}(\Omega)} + \|\frac{u}{C}\|_{L^{q(x)}(\Omega)}$ and using $\frac{u}{\lambda}$ in place of u in (17), then by properties of the norm, we obtain

$$\int_\Omega \left|\frac{u(x)}{\lambda}\right|^{r(x)}\,dx \le 1, \tag{18}$$

which implies that

$$\|u\|_{L^{r(x)}(\Omega)} \le \widetilde{C}\|u\|_{L^{p(x)}(\Omega)} + \frac{1}{C}\|u\|_{L^{q(x)}(\Omega)}. \tag{19}$$

Observe that the inequality of the previous lemma can also be stated in terms of integrals, as was shown in the proof. □

Observe that the preceding lemma proves the continuous embedding

$$L^{q(x)}(\Omega) \cap L^{p(x)}(\Omega) \hookrightarrow L^{r(x)}(\Omega).$$

Now we are ready to start proving the embedding theorem. For lighter notation, we will use $q(x) = \overline{p}^*(x)$ in the following proof only.

Because of the density Theorem 3.6 and the rectangular domain, we are able to follows to steps of the proof as outlined by Adams in [2], Lemma 5.7. We had to define the exponents $\sigma_i(x)$ in order to use the Gagliardo lemma. These exponents were in turn inspired by those defined by Rákosník in [40], when he proved the critical embedding theorem for anisotropic fixed exponents. Finally, we inspired ourselves as well from the proof of the critical embedding for the isotropic variable exponent case given by Fan & Zhao in [25], in that we used Lemma 2.11.

Lemma 3.9. *Let $\Omega \subset \mathbb{R}^n$ be a rectangular domain and $\vec{p}(\cdot) \in \left(\mathcal{P}(\Omega) \cap Lip(\overline{\Omega})\right)^n$ such that for all $x \in \overline{\Omega}$, $1 < p_m(x) \le p_M(x) < q(x) = \overline{p}^*(x)$ and $\overline{p}(x) < n$. Then the following embedding is continuous:*

$$W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{\overline{p}^*(x)}(\Omega).$$

Proof. Fix $u \in C^\infty(\overline{\Omega})$ such that $\|u\|_{L^{q(x)}(\Omega)} = 1$. We define the following:

$$x^{(i)} = (x_1, \ldots, x_{i-1}, x_{1+1}, \ldots, x_n);$$

$$\Omega_i(x) = \{y \in \Omega : y^{(i)} = x^{(i)}\};$$

$$\Omega_i = \text{the projection of } \Omega \text{ onto } \{x_i = 0\}; \text{ and}$$

$$v_i(x^{(i)}) = \sup_{y \in \Omega_i(x)} |u(y)|^{q(y)\sigma_i(y)}.$$

We know that $v_i(\cdot)$ is well defined since $u \in C^\infty(\overline{\Omega})$ and, as mention previously, $\overline{p}^*(\cdot)$ and $\sigma_i(\cdot)$ are Lipschitz continous on $\overline{\Omega}$ (see Lemma 5.20). In fact we even know that $v_i \in L^\infty(\Omega_i)$

Therefore, we can use the famous lemma by Galiardo in conjunction with (2) of Lemma 3.7 to obtain:

$$
\begin{aligned}
1 &= \left(\int_\Omega |u(x)|^{q(x)}\, dx \right)^{(n-1)} = \left(\int_\Omega \prod_{i=1}^n |u(x)|^{q(x)\sigma_i(x)}\, dx \right)^{(n-1)} \\
&\leq \left(\int_\Omega \prod_{i=1}^n v_i(x^{(i)})\, dx \right)^{(n-1)} \\
&\leq \prod_{i=1}^n \int_{\Omega_i} \left[v_i(x^{(i)}) \right]^{(n-1)}\, dx^{(i)} \qquad (20)
\end{aligned}
$$

By a translation and scaling argument, since Ω is rectangular, we can assume without loss of generality that it is a cube of sidelength 2. [1]

For any $y \in \Omega$ and $1 \leq i \leq n$, fix a unit vector e_i parallel to the axis $x^{(i)} = 0$, such that $y + (1-t)e_i \in \Omega$ for all $t \in [0,1]$.

For a fixed $x \in \Omega$ and a corresponding e_i, let $f_i(t) = |u(x+(1-t)e_i)|^{q(x+(1-t)e_i)\sigma_i(x+(1-t)e_i)(n-1)}$. Then by integration by parts, we have:

$$
\int_0^1 f(t)\, dt = f(1) - \int_0^1 t f'(t)\, dt. \qquad (21)
$$

Note that $f(1) = |u(x)|^{q(x)\sigma_i(x)(n-1)}$ and also

$$
\int_0^1 f(t)\, dt \leq \int_{\Omega_i(x)} |u(y)|^{q(y)\sigma_i(y)(n-1)}\, dy
$$

Now we calculate

$$
\begin{aligned}
t f'(t) &= t(1-n) \left[\left(\partial_i [q\sigma_i] \log|u| + (q\sigma_i) \mathrm{sign}(u) |u|^{-1} \partial_i u \right) |u|^{q\sigma_i(n-1)} \right] (x + (1-t)e_i) \\
&\leq C_0 \left[\left(|\log|u|| + |u|^{-1} |\partial_i u| \right) |u|^{q\sigma_i(n-1)} \right] (x + (1-t)e_i),
\end{aligned}
$$

where C_0 depends on n and on the Lipschitz norm of $q\sigma_i$. Hence we may choose it to be uniform for all i. Combining (21) with the two inequalities above, and denoting $s(x) = q(x)\sigma_i(x)(n-1)$, we obtain

[1]Note that in the isotropic case, we can nicely transform a domain Ω with the cone condition into uniform cubes, but we cannot use this with the anisotropy, since the transformation involved in the integral will also affect the anisotropy. That is probably why there is not much more generality to be added in terms of the shape of the domain, although some authors did, such as [40], in the fixed exponent case, they are still quite resctritive, so we have decided to stick to rectangular-like domains.

$$
\begin{aligned}
f(1) &= |u(x)|^{s(x)} \\
&\le C_0 \int_{\Omega_i(x)} \left(|u(y)|^{s(y)} + |u(y)|^{s(y)}|\log|u(y)|| + |u(y)|^{s(y)-1}|\partial_i u(y)|\right)\, dy \\
&= C_0 \left(\int_{\Omega_i(x)} |u(y)|^{s(y)}|\log|u(y)||\, dy + \int_{\Omega_i(x)} |u(y)|^{s(y)-1}\left(|u(y)| + |\partial_i u(y)|\right)\, dy \right)
\end{aligned}
$$

Now we can take the supremum over all elements of $\Omega_i(x)$ on the left hand side and then integrate both sides of the previous inequality over Ω_i to obtain:

(22)

$$
\int_{\Omega_i} [v_i(x^{(i)})]^{(n-1)}\, dx^{(i)} \le C_0 \left(\int_{\Omega} |u(x)|^{s(x)}|\log|u(x)||\, dx + \int_{\Omega} |u(x)|^{s(x)-1}\left(|u(x)| + |\partial_i u(x)|\right)\, dx \right).
$$

Recall that by Lemma 3.7, $[q(x)\sigma_i(x)(n-1)-1]\frac{p_i(x)}{p_i(x)-1} = q(x)$, hence by Lemma 2.3,

$$
\| |u|^{q(x)\sigma_i(x)(n-1)-1} \|_{L^{\frac{p_i(x)}{p_i(x)-1}}} = \|u\|_{L^{q(x)}(\Omega)} = 1.
$$

Thus, we can use Hölder's inequality and the triangle inequality for the second term on the right hand side of (22) and use Lemma 2.11, with $r(x) = q(x)\sigma_i(x)(n-1)$ and $p(x) = 1$, for the first term, where $C = 2C_0$. This yields

$$
\int_{\Omega_i} [v_i(x^{(i)})]^{(n-1)}\, dx^{(i)} \le C \left(\widetilde{C} \int_{\Omega} |u(x)|\, dx + \frac{1}{3C} + \|u\|_{L^{p_i(x)}(\Omega)} + \|\partial_i u\|_{L^{p_i(x)}(\Omega)} \right)
$$

Using now Lemma 3.8 with $p(x) = 1$ and $r(x) = p_i(x)$, we get, by using a new $\widetilde{C}$ that is uniform over all i,

(23)

$$
\begin{aligned}
\int_{\Omega_i} [v_i(x^{(i)})]^{(n-1)}\, dx^{(i)} &\le C \left(\widetilde{C} \int_{\Omega} |u(x)|\, dx + \frac{1}{3C} + \widetilde{C}\|u\|_{L^1(\Omega)} + \frac{1}{3C} + \sum_{i=1}^{n} \|\partial_i u\|_{L^{p_i(x)}(\Omega)} \right) \\
&\le C \left(\widetilde{C}\, \|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} + \frac{2}{3C} \right)
\end{aligned}
$$

Using (20) gives us

$$
1 \le \prod_{i=1}^{n} \int_{\Omega_i} [v_i(x^{(i)})]^{(n-1)}\, dx^{(i)} \le C^n \left(\widetilde{C}\, \|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} + \frac{2}{3C} \right)^n.
$$

Reworking the inequality and using a new constant C, we now have

$$1 \leq C \, \|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} \tag{24}$$

Note that this constant depends only on p^+, p^-, n and the max Lipschitz constants of $\{p_i\}_{i=1}^n$.

Recall that this stands for any $u \in C^\infty(\overline{\Omega})$ with $\|u\|_{L^{q(x)}(\Omega)} = 1$.

Now pick an arbitrary $u \in C^\infty(\overline{\Omega})$ and let $\lambda = \|u\|_{L^{q(x)}(\Omega)}$. Putting $\frac{u}{\lambda}$ into (24) yields

$$1 \leq C \left\| \frac{u}{\lambda} \right\|_{W^{1,\vec{p}(\cdot)}(\Omega)}, \tag{25}$$

which leads to

$$\|u\|_{L^{q(x)}(\Omega)} \leq C \|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)}. \tag{26}$$

By density of $C^\infty(\overline{\Omega})$ (Theorem 3.6), the result holds for any $u \in W^{1,\vec{p}(\cdot)}(\Omega)$. $\square$

Remark 3.10. Since Ω is bounded, it follows that for any $q \in \mathcal{P}(\Omega)$ such that $q(x) \leq p^*(x)$, the above embedding is true. Hence we will now stop using $q(x) = \overline{p}^*(x)$, so that we may define our embeddings in more general terms. Notably, we may have embeddings into exponents that are critical only on a proper subset of Ω.

For the following, recall that when $\overline{p}(x) \geq n$, $p^*(x) = \infty$.

Lemma 3.11. *Let $\Omega \subset \mathbb{R}^n$ be a rectangular domain and $\vec{p}(\cdot) \in \left(\mathcal{P}(\Omega) \cap Lip(\overline{\Omega})\right)^n$ and $q \in C(\overline{\Omega})$ such that for all $x \in \overline{\Omega}$, $1 < p_m(x) \leq p_M(x) < p^*(x)$ and $1 \leq q(x) \leq p^*(x)$. Then the following embedding is continuous:*

$$W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{q(x)}(\Omega).$$

Proof. Denote $\Omega_\delta = \{x \in \Omega : \overline{p}(x) > n - \delta\}$. Since $q \in C(\overline{\Omega})$ and Ω is bounded, then $q^+ < \infty$, thus we can pick $\delta > 0$ such that $(n-\delta)^* > q^+$. Now fix some $\varepsilon \in (0, \delta)$. Since the closure of Ω_ε is compact, we can cover it by finitely many rectangles $\{Q_k\}$ such that $\overline{\Omega_\varepsilon} \cap \Omega \subset Q = \bigcup_k Q_k \subset \Omega_\delta$. Therefore for every k and every $x \in Q_k$, $q(x) < (n-\delta)^* < \overline{p}^*(x)$.

Hence by the compact embedding Theorem 3.5, for every k and every $u \in W^{1,\vec{p}(x)}(\Omega)$,

$$\|u\|_{L^{q(x)}(Q_k)} \leq C_k \|u\|_{W^{1,\vec{p}(x)}(Q_k)}.$$

Therefore we have

$$\|u\|_{L^{q(x)}(Q)} \leq \sum_k \|u\chi_{Q_k}\|_{L^{q(x)}(Q)} = \sum_k \|u\|_{L^{q(x)}(Q_k)} \leq \sum_k C_k\|u\|_{W^{1,\vec{p}(x)}(Q_k)} \leq C'\|u\|_{W^{1,\vec{p}(x)}(Q)}.$$

It then follows that $\Omega^* = \Omega\backslash\overline{Q} \subset \Omega\backslash\overline{\Omega_\varepsilon} = \{x \in \Omega : \overline{p}(x) < n - \varepsilon\}$ is a union of finitely many rectangles R_j. Hence by Lemma 3.9, for any j and $u \in W^{1,\vec{p}(x)}(\Omega)$,

$$\|u\|_{L^{q(x)}(R_j)} \leq C_j\|u\|_{W^{1,\vec{p}(x)}(R_j)}.$$

Once again we can combine all j to obtain

$$\|u\|_{L^{q(x)}(R)} \leq \widetilde{C}\|u\|_{W^{1,\vec{p}(x)}(R)}.$$

Now $Q \cup R$ covers Ω up to a set of measure zero, thus we combine to obtain

$$\|u\|_{L^{q(x)}(\Omega)} \leq C\|u\|_{W^{1,\vec{p}(x)}(\Omega)}.$$

□

Finally we can combine to obtain the most general result we have for critical exponent anisotropic Sobolev embeddings on rectangular domains.

Theorem 3.12. *Let $\Omega \subset \mathbb{R}^n$ be a rectangular domain and $\vec{p}(\cdot) \in \left(Lip(\overline{\Omega})\right)^n$ such that for all $x \in \overline{\Omega}$ and $1 < p_m(x) \leq p_M(x) < p^*(x)$. Then for any $q \in \mathcal{P}(\Omega)$ that satisfies $q(x) \leq p^*(x)$ for all $x \in \overline{\Omega}$, the following embedding is continuous:*

$$W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{q(x)}(\Omega).$$

Proof. Since $q \in L^\infty(\Omega)$, there exists $k \in \mathbb{N}$ such that $q(x) < k$ for all $x \in \Omega$. Thus we can take q in Lemma 3.11 to be the truncation of $p^*(x)$ at k, denoted $T_k[\overline{p}^*](x)$. Then by boundedness of the domain and the fact that $q(x) \leq T_k[\overline{p}^*](x)$, we get the desired result. □

By the zero extension and the fact that we can always extend a Lipschitz function on a compact set to a Lipschitz function on $\mathbb{R}^n$ with the same upper and lower bounds (see (40) in the next section), we get

Corollary 3.13. *Let $\vec{p}(\cdot)$ and $q(x)$ satisfy the assumptions of the previous theorem, then for any bounded domain, the following embedding is continuous:*

$$W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{q(x)}(\Omega).$$

Although it would be nice to obtain such embedding when $p^- = 1$, it was not necessary in our case, since in order to prove the concentration-compactness principle, we need to assume $p^- > 1$ in order to use the reflexivity of the space.

The next theorem shows that in our definition of $W^{1,\vec{p}(\cdot)}(\Omega)$, we could have taken any exponent in $[1, \overline{p}^*(x)]$ and we would have gotten the same space, with equivalent norms.

Theorem 3.14. *Under the assumptions of theorem 3.12, we get that for any $r \in \mathcal{P}(\Omega)$ such that $1 \le r(x) \le \overline{p}^*(x)$ for all $x \in \Omega$,*

$$W^{1,\{r(x),\vec{p}(x)\}}(\Omega) = W^{1,\vec{p}(x)}(\Omega),$$

where $W^{1,\{r(x),\vec{p}(x)\}}(\Omega) = \left\{u \in L^{r(\cdot)}(\Omega) : \partial_i u \in L^{p_i(\cdot)}(\Omega), \quad \text{for all } 1 \le i \le n\right\}$ and

$$\|u\|_{W^{1,\{r(x),\vec{p}(x)\}}(\Omega)} = \|u\|_{r(x)} + \sum_{i=1}^{n} \|\partial_i u\|_{p_i(x)}.$$

This follows directly form Theorem 3.12 and the Lebesgue embedding theorem. Now we will prove the embedding for unbounded domain, even though it will not be necessary in subsequent section. Recall that we define the space $W^{1,\vec{p}(\cdot)}(\Omega)$ with $p_0(x) = p_m(x)$ for unbounded domains.

Theorem 3.15. *Let $\Omega \subset \mathbb{R}^n$ be a rectangular-like domain and $\vec{p}(\cdot) \in \left(Lip(\overline{\Omega})\right)^n$ such that $1 < p^- \le p^+ < n$ and $\inf_{x\in\overline{\Omega}}\{\overline{p}^*(x) - p_M(x)\} > 0$. Then for any $q \in \mathcal{P}(\Omega)$ that satisfies $p_m(x) \le q(x) \le p^*(x)$ for all $x \in \overline{\Omega}$, the following embedding is continuous:*

$$W^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{q(x)}(\Omega).$$

Proof. We start the proof exactly as in lemma 3.9 for the case $p^+ < n$ and $q(x) = \overline{p}^*(x)$. Fix $u \in C_c^\infty(\Omega)$. Using lemma 2.11 with $p(x) = p_m(x)$ and $r(x) = q(x)\sigma_i(x)(n-1)$ and lemma 3.8 with $p(x) = p_m$ and $r(x) = p_i(x)$, equation (23) becomes

$$\int_{\Omega_i} [v_i(x^{(i)})]^{(n-1)}\, dx^{(i)} \le C\left(\widetilde{C}\int_\Omega |u(x)|^{p_m(x)}\, dx + \frac{1}{3C} + \widetilde{C}\|u\|_{L^{p_m(x)}(\Omega)} + \frac{1}{3C} + \sum_{i=1}^{n}\|\partial_i u\|_{L^{p_i(x)}(\Omega)}\right)$$

Using (20) gives us

$$1 \le \prod_{i=1}^{n}\int_{\Omega_i} [v_i(x^{(i)})]^{(n-1)}\, dx^{(i)} \le C^n\left(\widetilde{C}\int_\Omega |u|^{p_m(x)}\, dx + \widetilde{C}\,\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} + \frac{2}{3C}\right)^n.$$

Reworking the inequality and using a new constant C, we now have

$$1 \leq C \left(\int\limits_{\Omega} |u|^{p_m(x)}\, dx + \|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} \right). \tag{27}$$

Note that this constant depends only on p^+, p^-, n and the max Lipschitz constants of $\{p_i\}_{i=1}^n$.

Recall that this stands for any $u \in C_c^\infty(\Omega)$ with $\|u\|_{L^{q(x)}(\Omega)} = 1$.

Now pick an arbitrary $u \in C_c^\infty(\Omega)$ and let $\lambda = \|u\|_{L^{q(x)}(\Omega)}$. Putting this into (27), we get for $\lambda \geq 1$,

$$\begin{aligned}
1 &\leq \left(\int\limits_{\Omega} \left|\frac{u}{\lambda}\right|^{p_m(x)} dx + \left\|\frac{u}{\lambda}\right\|_{W^{1,\vec{p}(\cdot)}(\Omega)} \right) \\
&\leq \frac{C}{\lambda} \left(\int\limits_{\Omega} |u|^{p_m(x)}\, dx + \|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} \right) \\
&\leq \frac{C}{\lambda} \left(\int\limits_{\Omega} |u|^{p_m(x)}\, dx + \|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} + 1 \right)
\end{aligned}$$

Note that by multiplying both sides by λ and assuming $C > 1$, we now have an inequality that also trivially holds for $\lambda < 1$. Thus we write for any $u \in C_c^\infty(\Omega)$,

$$\|u\|_{L^{q(x)}(\Omega)} \leq C \left(\max_{1\leq i\leq n} \left\{ \int_{\Omega} |u(x)|^{p_i(x)}\, dx \right\} + \|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} + 1 \right) \tag{28}$$

Finally, we replace u by $\frac{u}{\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)}}$ to obtain

$$\left\| \frac{u}{\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)}} \right\|_{L^{q(x)}(\Omega)} \leq C \left(\int_{\Omega} \left| \frac{u(x)}{\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)}} \right|^{p_m(x)} dx + \left\| \frac{u}{\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)}} \right\|_{W^{1,\vec{p}(\cdot)}(\Omega)} + 1 \right) \leq 3C$$

Hence by the above and by Theorem 3.6, we have the following inequality for all $u \in W^{1,\vec{p}(\cdot)}(E)$, where E is the support of u. Because the constant in the inequality does not depend on u nor on E, then we may take any $u \in W^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ multiplied by a sequence of smooth cutoff functions $\phi_k \to 1$ in $L^\infty(\Omega)$. Thus we get that the theorem hold for all $u \in W^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$. Taking now an arbitrary $u \in W^{1,\vec{p}(\cdot)}(\Omega)$, we may take its truncations, which will converge to it in the norm, thus completing the first statement of the proof. Then, for any $q \in \mathcal{P}(\Omega)$ such that $p_m(x) \leq q(x) \neq \overline{p}^*(x)$ everywhere, we use the fact that

$$\int_\Omega |u|^{q(x)}\,dx \leq \int_\Omega |u|^{p_m(x)}\,dx + \int_\Omega |u|^{\overline{p}^*(x)}\,dx.$$

By techniques similar to the proof of Lemma 3.8, we get $\|u\|_{q(x)} \leq 2\left(\|u\|_{p_m(x)} + \|u\|_{\overline{p}^*(x)}\right)$, which completes the proof.

□

By using Lemma 3.8, we can show, as in Theorem 3.14, that when the domain is unbounded, we can choose any $p_0(x) \in [p_m(x), p_M(x)]$ and obtain the same space with equivalent norms.

3.3. **Poincaré-Sobolev Inequality.** The Poincaré-Sobolev inequality is another major component of the concentration compactness principle. In the case of the anisotropic Sobolev space, a new special case arise, which remains as mentionned previously, quite unanswered, even in the fixed exponent spaces, i.e. when there exists $p_M(x) > \overline{p}^*(x)$. Thus, recall that we will always assume here that $p_M(x) < \overline{p}^*(x)$.

Theorem 3.16 (Poincaré-Sobolev Inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\vec{p} \in \left(\mathcal{P}(\Omega) \cap \operatorname{Lip}(\overline{\Omega})\right)^n$ be such that $1 < p^- \leq p_M(x) < \overline{p}^*(x)$ and $q \in \mathcal{P}(\Omega)$ such that $1 \leq q(x) \leq \overline{p}^*(x)$, for all $x \in \overline{\Omega}$. Then there exists $S > 0$ such that for every $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$:*

$$S\|u\|_{q(x)} \leq \sum_{i=1}^n \|\partial_i u\|_{p_i(x)}$$

Proof. First, note that by Theorem 3.12, we only need to prove it for $q \equiv 1$, i.e. that there exists $S > 0$ such that for all $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$,

$$S\|u\|_1 \leq \sum_{i=1}^n \|\partial_i u\|_{p_i(x)}. \tag{29}$$

Assume the theorem doesn't hold, then there exists a sequence $\{u_k\}$ such that for all $k \in \mathbb{N}$,

$$\|u_k\|_1 > k\sum_{i=1}^n \|\partial_i u_k\|_{p_i(x)}. \tag{30}$$

We may also assume without loss of generality that $\|u_k\|_{W^{1,\vec{p}(\cdot)}(\Omega)} = 1$ for all $k \in \mathbb{N}$. Then by the Rellich-Kondrachov theorem and reflexivity, we get that there exists $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ with $\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} = 1$ such that, up to a subsequence, $u_k \to u$ strongly in $L^1(\Omega)$ and weakly in $W_0^{1,\vec{p}(\cdot)}(\Omega)$. Also we rework (30) to get

$$\sum_{i=1}^n \|\partial_i u_k\|_{p_i(x)} < \frac{1}{k}.$$

Let $\Lambda \in \left(W_0^{1,\vec{p}(\cdot)}(\Omega)\right)^*$ be defined by $\langle \Lambda, v\rangle = \sum_{i=1}^{n} \int_\Omega |\partial_i u|^{p_i(x)-2}\partial_i u \partial_i v \, dx$. Then, since $\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} = 1$,

$$\begin{aligned}\sum_{i=1}^{n} \int_\Omega |\partial_i u|^{p_i(x)} \, dx = \langle \Lambda, u\rangle &= \lim_{k\to\infty} \langle \Lambda, u_k\rangle \\ &\leq \lim_{k\to\infty} \sum_{i=1}^{n} \left\| |\partial_i u|^{p_i(x)-1}\right\|_{p_i'(x)} \|\partial_i u_k\|_{p_i(x)} \\ &\leq \lim_{k\to\infty} \sum_{i=1}^{n} \|\partial_i u_k\|_{p_i(x)} \leq \lim_{k\to\infty} \frac{1}{k} = 0\end{aligned}$$

Therefore $\nabla u \equiv 0$ almost everywhere. Since $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, this implies $u \equiv 0$ almost everywhere, which contradicts the assumption that $\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)} = 1$. Hence we have that (29) holds.

Now we use Theorem 3.12 to complete the proof. □

4. Concentration-Compactness Principle

We are now at the section where we prove our main theorem. Up to our knowledge, there is no concentration-compactness principle for anisotropic variable exponents Sobolev spaces in the litterature. This principle was first developped by P.L. Lions ([36]), for fixed exponents, and has proven quite useful in applications. We will start by stating our concentration compactness principle for anisotropic variable exponent Sobolev spaces, and then we will build up to the proof at the end of this section.

The concentration-compactness principle for isotropic variable exponents was first developped by Fu in [27] and by Bonder & Silva in [12] independently. Our proof is greatly inspired by a combination of the two proofs from the latter papers. Drawing from the strengths of each, we were able to give a more direct proof, which works also in the isotropic case. Because of the critical embedding theorem, we had to restrict ourselves to Lipschitz continuous exponents, rather than log-Hölder continuous ones, as was done by Bonder & Silva. In the proof, we only use the Lipschitz assumption in order to apply the critical embedding. Hence by using our proof for the isotropic case, we may as well assume the log-Hölder condition.

Theorem 4.1 (Concentration-Compactness Principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\vec{p} \in \left(\mathcal{P}(\Omega) \cap \mathrm{Lip}(\overline{\Omega})\right)^n$ and $q \in C(\overline{\Omega})$, such that $1 < p^- \leq p_M(x) < p^*(x)$ and $1 \leq q(x) \leq p^*(x)$ for all $x \in \overline{\Omega}$.*
If $\{u_j\}$ is a bounded sequence in $W_0^{1,\vec{p}(\cdot)}(\Omega)$, then there exists measures $\nu, \mu \in \mathcal{M}^b(\overline{\Omega})$, a subsequence, still denoted with indices j, and an element $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that

(1) $u_j \to u$ *weakly in* $W_0^{1,\vec{p}(\cdot)}(\Omega)$*;*

(2) $u_j \to u$ *strongly in* $L^{r(x)}(\Omega)$ *for all* $r \in \mathcal{P}(\Omega)$ *such that* $\inf_{x\in\Omega}\{p^*(x) - r(x)\} > 0$*;*

(3) if $|u_j|^{q(x)}dx = d\nu_j$*, then* $\nu_j \to \nu \quad \text{weak}^*$ *in* $\mathcal{M}^b(\overline{\Omega})$ *with* $d\nu = |u|^{q(x)}\,dx + \sum_{\ell\in L} \nu^\ell \delta_{x_\ell}$*;*

(4) if $\sum_{i=1}^n |\partial_i u_j|^{p_i(x)}dx = \sum_{i=1}^n d\mu_{i,j}$*, then for all* i*,* $\mu_{i,j} \to \mu_i$ weak* *in* $\mathcal{M}^b(\overline{\Omega})$ *with* $\sum_{i=1}^n \mu_i = \mu$

(5) $d\mu \geq \sum_{i=1}^n |\partial_i u|^{p_i(x)}dx + \sum_{\ell\in L} \mu^\ell \delta_{x_\ell}$*,*

with $\{x_\ell\}_{\ell\in L} \subset \mathcal{A} = \{x \in \overline{\Omega} : p^*(x) = q(x)\}$*,* L *is countable and* $S(\nu^\ell)^{1/q(x_\ell)} \leq (\mu^\ell)^{1/p(x_\ell)}$*, where* S *is the best Sobolev constant as in theorem 3.16 and* δ_{x_ℓ} *is the Dirac Delta function for* x_ℓ*.*

4.1. **Convergence of Borel Measures.** Before going any further, we will clarify the notion of weak* convergence in the sense of measures.

For a compact Hausdorff space X, we denote the collection of $\mathbb{R}$-valued regular Borel measures on X with bounded variation by $\mathcal{M}^b(X)$. Note that if μ is a nonnegative measure on X, then $\mu(X) < \infty \iff \mu \in \mathcal{M}^b(X)$. From the Riesz-Representation theorem (Theorem 2.14 in [44]), we get that

$$(C(X))^* \cong \mathcal{M}^b(X).$$

Thus, from the Banach-Aloaglu theorem, if $\{\mu_i\}$ is a bounded sequence in $\mathcal{M}^b(X)$, then there exists a subsequence (that we still denote with indice i) and a $\mu \in \mathcal{M}^b(X)$ such that

$$\int_X f\, d\mu_i \to \int_X f\, d\mu \qquad \forall f \in C(X). \tag{31}$$

Hence, if $\mu_i \geq 0$ for all i, then μ is also nonnegative.

Now let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\{u_j\}$ be a bounded sequence in $W_0^{k,p(x)}(\Omega)$ and $q \in \mathcal{P}(\Omega)$ with $q(x) \leq p^*(x)$ almost everywhere, then by the embedding theorems, the measures defined by $\nu_j(E) = \int_E |u_j|^{q(x)}dx$, for any Borel set $E \subset \overline{\Omega}$, form a bounded sequence in $\mathcal{M}^b(\overline{\Omega})$. Although $\{u_j\}$ and $q(x)$ are only defined on Ω, the measures can be extended to $\overline{\Omega}$ by setting $\nu_j(\partial\Omega) = 0$. The regularity follows from the fact that by the above, for every j, ν_j is absolutely continuous to the Lebesgue measure.

Therefore there exists, after possibly taking a subsequence, a nonnegative $\nu \in \mathcal{M}^b(\Omega)$ such that if $|u_j|^{q(x)}dx = d\nu_j$, then

$$\nu_j \to \nu \qquad \text{weak}^* \text{ in the sense of measure,}$$

i.e. (31) is satisfied.

Similarly, we can do the same for every $1 \leq i \leq n$, i.e. find a $\mu \in \mathcal{M}^b(\Omega)$ such that if $|\partial_i u_j|^{p(x)}dx = d\mu_{i,j}$, then, up to a subsequence,

$$\mu_{i,j} \to \mu_i \qquad \text{weak}^* \text{ in the sense of measure.}$$

4.2. **The Concentration-Compactness Lemma.** We will first show a weaker version of the concentration-compactness principle, in that we will assume $u_j \to 0$ weakly in $W_0^{1,\vec{p}(\cdot)}(\Omega)$. In order to do so, we will need to build specific sequences of functions converging to any relatively open set in $\overline{\Omega}$.

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $U \subset \overline{\Omega}$ be a relatively open set. Then there exists a sequence $\{\phi_k\}_{k\in\mathbb{N}} \subset C^\infty(\overline{\Omega})$ such that $0 \leq \phi_k \leq \chi_U$ for every $k \in \mathbb{N}$ and $\phi_k(x) \to \chi_U(x)$ for every $x \in \overline{\Omega}$.*

Proof. Define the following continuous functions on $\overline{\Omega}$:

$$f_\varepsilon(x) = \begin{cases} 1, & \text{if } \operatorname{dist}(x, \overline{\Omega}\backslash U_\varepsilon) \geq \varepsilon \\ \frac{\operatorname{dist}(x,\overline{\Omega}\backslash U_\varepsilon)}{\varepsilon}, & \text{if } \operatorname{dist}(x, \overline{\Omega}\backslash U_\varepsilon) < \varepsilon \end{cases} \tag{32}$$

where $U_\varepsilon = \{x \in U : \operatorname{dist}(x, \overline{\Omega}\backslash U) > \varepsilon\}$.

Then let φ be a bump function such that $\widetilde{\phi}_\varepsilon = f_\varepsilon * \varphi_\varepsilon$ is a standard mollification, then $\{\phi_k = \widetilde{\phi}_{1/k} \cdot \chi_{\overline{\Omega}}\}$ satisfies the conditions of the lemma, since for all k large enough, $\phi_k \equiv 0$ on $\overline{\Omega}\backslash U$, $\phi_k \equiv 1$ on $U_{2/k}$ and $\bigcup_{k\in\mathbb{N}} U_{2/k} = U$. ☐

The following is a well known result in the constant exponent case, known as the Brezis-Lieb lemma. It is also well known for variable exponents, but it is often stated without proof. We have included a proof here, since it uses the Vitali convergence theorem, which we will use in the section on applications of the concentration-compactness principle.

It uses some other important measure theory notions, such as the Lebesgue dominated convergence theorem and Fubini's theorem. The reader may refer to Fitzpatrick [42], chapters 4 and 5, and Bogachev [8], Chapters 2,3 and 4. In our case, i.e., in a bounded domain, with a sequence that converges pointwise almost everywhere to another element of the space, the Vitali convergence theorem essentially reduces to proving equi-integrability of the sequence, which we will explain in the proof.

Lemma 4.3. *Let μ be a finite Borel measure on a subset Ω of $\mathbb{R}^n$, $p \in \mathcal{P}(\Omega)$ and $\{u_m\}$ be a bounded sequence in $L_\mu^{p(x)}(\Omega)$ such that $u_m \to u$ pointwise almost everwhere. Then we have*

$$\lim_{m\to\infty} \left(\int_\Omega |u_j|^{p(x)} d\mu + \int_\Omega |u_j - u|^{p(x)} d\mu \right) = \int_\Omega |u|^{p(x)} d\mu. \tag{33}$$

Proof. We know, by Fatou's lemma and boundedness of $\{u_j\}$, that $u \in L^{p(x)}_\mu(\Omega)$, since

$$\int_\Omega |u|^{p(x)}\,d\mu \le \liminf_{j\to\infty} \int_\Omega |u_j|^{p(x)}\,d\mu \le \sup_{j\in\mathbb{N}} \int_\Omega |u|^{p(x)}\,d\mu < \infty.$$

We will split this problem into two cases: $p(x) > 1$ for all $x \in \Omega$ and $p(x) = 1$ for all $x \in \Omega$. Observe that if we prove (33) for those two cases, then we may simply split Ω into two measurable sets $A = \{x \in \Omega : p(x) > 1\}$ and $B = \{x \in \Omega : p(x) = 1\}$. Then $A \cup B = \Omega$ and $A \cap B = \emptyset$. Thus the result will follow by spliting the integral and applying (33) to A and B.

Let $f_j = u(u_j - \eta u)|u_j - \eta u|^{p(x)-2}$. Note that when $p(x) < 2$, we actually have

$$(u_j(x) - \eta u(x))|u_j(x) - \eta u(x)|^{p(x)-2} = \mathrm{sign}[u_j - \eta u](x)\,|u_j(x) - \eta u(x)|^{p(x)-1}.$$

So for the first case, let $p(x) > 1$ for all $x \in \Omega$. We will use the Vitali convergence theorem to show that for all $\eta \in (0,1)$, we have $f_j \to f = (1-\eta)|u|^{p(x)}$ in $L^1_\mu(\Omega)$.

Since μ is a finite Borel measure, this means that we need to show that $f_j \to f$ in measure and that $\{f_j\}$ is uniformly bounded in $L^1_\mu(\Omega)$ and equi-integrable, i.e.

$$\lim_{\mu(E)\to 0} \int_\Omega f_j\,d\mu \to 0 \text{ uniformly.}$$

First, we know that $f_j \to f$ in measure, since $u_j \to u$ pointwise implies that $f_j \to f$ pointwise, which implies convergence in measure. We will then show boundedness and equi-integrability at the same time. Before doing so, we will just state the facts that $p \in (0,1)$, we have the inequality $\int |u+v|^p \le \int |u|^p + \int |v|^p$. When $p \ge 1$, we use convexity to show that $|u+v|^p \le 2^{p-1}(|u|^p + |v|^p)$.

Let E be any μ-measurable subset of Ω and $E_s = \{x \in E : p(x) > s > 1\}$. Set $M = \sup_j \left\{\|u_j\|_{p(x)} + 1\right\}$. Then by Hölder's inequality and theorem 2.3, we obtain

$$\begin{aligned}
\int_{E_s} |f_j|\,d\mu &= \int_{E_s} \left|u(u_j - \eta u)|u_j - \eta u|^{p(x)-2}\right|\,d\mu = \int_{E_s} |u|\,|u_j - \eta u|^{p(x)-1}\,d\mu \\
&\le 2^{p^+-1}\left(\int_{E_s} \eta |u|^{p(x)}\,d\mu + \int_{E_s} |u|\,|u_j|^{p(x)-1}\,d\mu\right) \\
&\le 2^{p^+-1}\left(\int_E |u|^{p(x)}\,d\mu + 2\,\|u\|_{p(x),E_s}\left\||u_j|^{p(x)-1}\right\|_{p'(x),E_s}\right) \\
&\le 2^{p^+-1}\left(\int_E |u|^{p(x)}\,d\mu + 2M^{p^+-1}\,\|u\|_{p(x),E}\right)
\end{aligned}$$

Letting $s \to 1$ on the left hand side, we get that $\int_E f_j \, d\mu \to 0$ as $\mu(E)$ goes to zero, since $|u|^{p(x)} \in L^1_\mu(\Omega)$, hence the right hand side goes to zero. Note that by placing $E = \Omega$, this shows uniform boundedness of $\{f_j\}$ in $L^1_\mu(\Omega)$.

Therefore, we have that $f_j \to f$ in $L^1_\mu(\Omega)$. Using basic calculus, the Fubini theorem and the Lebesgue dominated convergence theorem, we can now prove the desired result.

$$
\begin{aligned}
\lim_{j\to\infty}\left(\int_\Omega |u_j|^{p(x)}d\mu + \int_\Omega |u_j - u|^{p(x)}d\mu\right) &= \lim_{j\to\infty}\int_\Omega \left[|u_j - \eta u|^{p(x)}\right]_1^0 \, d\mu \\
&= \lim_{j\to\infty} - \int_\Omega \int_0^1 \frac{d}{d\eta}\left(|u_j - \eta u|^{p(x)}\right) \, d\eta \, d\mu \\
&= \lim_{j\to\infty} \int_\Omega \int_0^1 p(x)u(u_j - \eta u)|u_j - \eta u|^{p(x)-2} \, d\eta \, d\mu \\
&= \int_0^1 \left(\lim_{j\to\infty}\int_\Omega p(x)u(u_j - \eta u)|u_j - \eta u|^{p(x)-2} \, d\mu\right) \, d\eta \\
&= \int_0^1 \int_\Omega p(x)(1-\eta)^{p(x)-1}|u|^{p(x)} \, d\mu \, d\eta \\
&= \int_\Omega |u|^{p(x)}\left(\int_0^1 p(x)(1-\eta)^{p(x)-1} \, d\eta\right) \, d\mu \\
&= \int_\Omega |u|^{p(x)}d\mu \qquad (34)
\end{aligned}
$$

Now for the case $p(x) = 1$, we will simply use the Lebesgue dominated convergence theorem. Observe that by the reverse triangle inequality, for any $x \in \Omega$ and any j, we have

$$
\left| |u_j(x)| - |u_j(x) - u(x)| \right| \le |u(x)|,
$$

so we may take the limit inside the integral, which will be $|u|$. This completes the proof. □

Observe that when $p^- > 1$, then by reflexivity, for a bounded sequence, there always exists a function $u \in L^{p(x)}(\Omega)$ such that $u_j \to u$ pointwise almost everywhere.

Lemma 4.4 (Concentration-Compactness Lemma). *With the assumptions of Theorem 4.1, if $u_j \to 0$ weakly, then there exists a subsequence, still denoted with indices j, such that*

(1) $u_j \to 0$ strongly in $L^{r(x)}(\Omega)$ for all $r \in \mathcal{P}(\Omega)$ with $\inf_{x\in\Omega}\{p^*(x) - r(x)\} > 0$;

(2) if $|u_j|^{q(x)}dx = d\nu_j$*, then* $\nu_j \to \nu = \sum_{\ell\in L} \nu^\ell \quad \text{weak}^*$ *in* $\mathcal{M}^b(\overline{\Omega})$*;*

(3) if $\sum_{i=1}^n |\partial_i u_j|^{p_i(x)}dx = \sum_{i=1}^n d\mu_{i,j}$*, then for all* i*,* $\mu_{i,j} \to \mu_i$ weak* *in* $\mathcal{M}^b(\overline{\Omega})$ *with* $\sum_{i=1}^n \mu_i = \mu$*,*

where $\{x_\ell\}_{\ell\in L} \subset \mathcal{A} = \{x \in \overline{\Omega} : p^*(x) = q(x)\}$ *and* L *is countable.*

We will prove this in 4 steps. Note that although the elements of $\{u_j\}$ and $\{\partial_i u_j\}$ are only defined a.e. on Ω, we define the measures $\{\nu_j\}$ and $\{\mu_j\}$ to be zero on $\partial\Omega$.

Claim 1: For all $\phi \in C^\infty(\overline{\Omega})$,

$$S\,\|\phi\|_{q(x),\nu} \leq \sum_{i=1}^n \|\phi\|_{p_i(x),\mu_i} \tag{35}$$

Proof. Fix non trivial $\phi \in C^\infty(\overline{\Omega})$, then $\phi u_j \in W_0^{1,p(x)}(\Omega)$ for any j. Thus by Theorem 3.16

$$S\,\|\phi\|_{q(x),\nu_j} = S\,\|\phi u_j\|_{q(x)} \leq \sum_{i=1}^n \|\partial_i(\phi u_j)\|_{p_i(x)} \leq \sum_{i=1}^n \left(\|\phi\partial_i u_j\|_{p_i(x)} + \|u_j\partial_i\phi\|_{p_i(x)}\right)$$

$$= \sum_{i=1}^n \left(\|\phi\|_{p_i(x),\mu_{i,j}} + \|u_j\partial_i\phi\|_{p_i(x)}\right) \tag{36}$$

By the compact embedding theorem, up to a subsequence, $u_j \to 0$ in $L^{p_i(x)}$ for all $i \in \{1,\ldots,n\}$, so as $j\to\infty$,

$$\|u_j\partial_i\phi\|_{p_i(x)} \leq \|\partial_i\phi\|_\infty \|u_j\|_{p_i(x)} \to 0 \tag{37}$$

Now fix $\varepsilon \in (0, \|\phi\|_{q(x),\nu})$ and let $\lambda = \|\phi\|_{q(x),\nu} + \varepsilon$, then by weak* convergence in measure, as $j\to\infty$, we get

$$\int_\Omega \left|\frac{\phi}{\lambda}\right|^{q(x)} d\nu_j \to \int_\Omega \left|\frac{\phi}{\lambda}\right|^{q(x)} d\nu < 1.$$

Observe that for the above to be true, we need $\left|\frac{\phi}{\lambda}\right|^{q(x)}$ to be continuous, i.e. we need continuity of the exponent. Then, for all j large enough, we have

$$\int_\Omega \left|\frac{\phi}{\lambda}\right|^{q(x)} d\nu_j < 1,$$

which implies that $\limsup_{j\to\infty} \|\phi\|_{q(x),\nu_j} \leq \lambda = \|\phi\|_{q(x),\nu} + \varepsilon$. This holds for all such ε.

Similarly, if we let $\lambda = \|\phi\|_{q(x),\nu} - \varepsilon$, then we obtain $\liminf_{j\to\infty} \|\phi\|_{q(x),\nu_j} \geq \lambda = \|\phi\|_{q(x),\nu} - \varepsilon$, for all such ε. Thus, we obtain $\lim_{j\to\infty} \|\phi\|_{q(x),\nu_j} = \|\phi\|_{q(x),\nu}$. We can apply the same proof to show that $\lim_{j\to\infty} \|\phi\|_{q(x),\mu_{i,j}} = \|\phi\|_{q(x),\mu_i}$ for all $i \in \{1,\ldots,n\}$. Then taking $j\to\infty$ on the left and right hand side in (47), then we have proven the claim.

□

Claim 2: $\nu = 0$ on $\overline{\Omega}\backslash\mathcal{A}$.

Proof. By continuity of p and q, we get that $\overline{\Omega}\backslash\mathcal{A}$ is relatively open. Then for any $x \in \Omega\backslash\mathcal{A}$, we can pick an open rectangular neighborhood $U \subset \Omega\backslash\mathcal{A}$ such that

$$\inf_{y\in U}\{\overline{p}^*(y) - q(y)\} > 0. \tag{38}$$

Therefore $W^{1,\vec{p}(x)}(U)$ is compactly embedded into $L^{q(x)}(U)$.

Thus we can pick a sequence ϕ_k as in Lemma 4.2 for U, and we compute for any $k \in \mathbb{N}$:

$$0 \le \int_{\overline{\Omega}} \phi_k\, d\nu = \lim_{j\to\infty}\int_{\overline{\Omega}} \phi_k\, d\nu_j = \lim_{j\to\infty}\int_{\overline{\Omega}} \phi_k |u_j|^{q(x)}dx \le \lim_{j\to\infty}\int_U |u_j|^{q(x)}dx = 0. \tag{39}$$

For any $x \in \partial\Omega\backslash\mathcal{A}$, we can pick a rectangular neighbourhood $U' \subset \mathbb{R}^n$ such that $U = U' \cap \overline{\Omega} \subset \left(\overline{\Omega}\backslash\mathcal{A}\right)$ and (38) is satisfied. We use the zero extension on $\{u_j\}$ to make them functions on U'. We can extend each p_i to $\widetilde{p_i} \in \mathrm{Lip}(\overline{U'}) \cap \mathcal{P}(U')$, with $\widetilde{p_i}^- = p_i^- > 1$. We do it by defining $\widetilde{p_i} : \overline{U'} \to \mathbb{R}$ as such:

$$\widetilde{p_i}(x) = \max\left\{\sup_{y\in\overline{\Omega}}\{p_i(y) - C|y-x|\}\,,\, p_i^-\right\}, \tag{40}$$

where C is the Lipschitz norm of p_i. If needed, we can also make U' smaller so that $\widetilde{p}_M(x) < \overline{\widetilde{p}}^*(x)$ for all $x \in U'$.

We can also extend q to a function q_1 that is continuous on $\overline{U'}$, as shown by Diemling (Proposition 1.1 in [16]). Now if we let

$$\widetilde{q}(x) = \max\left\{\min\left\{q_1(x)\,,\, \overline{\widetilde{p}}^*(x)\right\} - \mathrm{dist}(x,\overline{\Omega})\,,\, 1\right\},$$

then $\widetilde{q} \in C(\overline{\Omega})$ with $\widetilde{q} \equiv q$ on $\overline{\Omega}$ and $1 \le \widetilde{q}(x) < \overline{\widetilde{p}}^*$. Therefore $W^{1,\vec{p}(x)}(U')$ is compactly embedded into $L^{q(x)}(U')$, i.e. $u_j \to 0$ in $L^{q(x)}(U')$.

Thus, similarly to (39), we have

$$0 \le \int_{\overline{\Omega}} \phi_k\, d\nu \le \lim_{j\to\infty}\int_U |u_j|^{q(x)}dx = \lim_{j\to\infty}\int_{U'} |u_j|^{q(x)}dx = 0. \tag{41}$$

Hence in (39) and (41), for any $k \in \mathbb{N}$, $\int_{\overline{\Omega}} \phi_k\, d\nu = 0$, so taking $k \to \infty$ gives $\nu(U) = 0$. We do it for every $x \in \overline{\Omega}\backslash\mathcal{A}$ and take a countable cover of $\overline{\Omega}\backslash\mathcal{A}$ to prove the claim.

Note that since $q \in C(\overline{\Omega})$, $q^+ < \infty$, then there exists $\delta > 0$ such that $(n-\delta)^* > q^+$, which means that $\{x \in \overline{\Omega} : \overline{p}(x) \ge n - \delta\} \subset \overline{\Omega}\backslash\mathcal{A}$. Hence on $\mathcal{A}$, we have that $\overline{p}(x) < n$.

$\square$

Claim 3: $\nu \ll \mu$

Proof. First, recall that by properties of the of the norm, for any measurable set $A \subset \overline{\Omega}$, we have

$$\min\{\nu(A)^{1/q^-}, \nu(A)^{1/q^+}\} \leq \|\chi_A\|_{q(x),\nu} \leq \max\{\nu(A)^{1/q^-}, \nu(A)^{1/q^+}\} \tag{42}$$

The same holds for μ_i and the norms $\|\cdot\|_{p_i(x),\mu_i}$. Now fix some relatively open set $U \subset \overline{\Omega}$ and a corresponding sequence ϕ_k as in Lemma 4.2. By (35), (42) and the dominated convergence theorem, we obtain:

$$\begin{aligned} S\min\{\nu(U)^{1/q^-}, \nu(U)^{1/q^+}\} &\leq S\,\|\chi_U\|_{q(x),\nu} \\ &= S \lim_{k\to\infty} \|\phi_k\|_{q(x),\nu} \\ &\leq \lim_{k\to\infty} \sum_{i=1}^{n} \|\phi_k\|_{p_i(x)\mu_i} \\ &\leq \sum_{i=1}^{n} \|\chi_U\|_{p_i(x),\mu_i} \\ &\leq \sum_{i=1}^{n} \max\{\mu_i(U)^{1/p_i^-}, \mu_i(U)^{1/p_i^+}\} \end{aligned} \tag{43}$$

So if $\mu(U) = 0$, then for all i, $\mu_i(U) = 0$, hence $\nu(U) = 0$. Since ν and $\{\mu_i\}_{i=1}^n$ are regular Borel measures, then so is μ, hence this proves the claim. □

Claim 4: $\nu = \sum_{\ell\in L} \nu^\ell \delta_{x_\ell}$ with $\{x_\ell\} \subset \mathcal{A}$ and L countable.

Proof. By Claim 3 and the Radon-Nikodym theorem (see Rudin theorem 6.10 in [44]), there exists a unique nonnegative $f \in L^1_\mu(\overline{\Omega})$ such that $d\nu = f\,d\mu$.
By Claim 2, we already know that $f(x) = 0$ for all $x \in \overline{\Omega}\backslash\mathcal{A}$. Fix $x \in \mathcal{A}$, then we can pick $R > 0$ such that $\inf_{y\in B_R(x)}\{q(y)\} = q_x^- > p_x^+ = \sup_{y\in B_R(x)}\{p(y)\}$.

Assume x is not an atom of μ, then it's not an atom of ν. Then by (43), there exist $r \in (0, R)$ such that $\max\{\nu(B_r(x)), \mu(B_r(x))\} \leq 1$, from (43) again, we get

$$S\nu(B_r(x))^{\frac{1}{q_x^-}} \leq \sum_{i=1}^{n} \mu_i(B_r(x))^{\frac{1}{p_x^+}} \leq n\mu(B_r(x))^{\frac{1}{p_x^+}} \tag{44}$$

where $\frac{q_x^-}{p_x^+} > 1$.

If for some $r > 0$, $\mu(B_r(x)) = 0$, then trivially, $f(x) = 0$. Otherwise, by (43) and the Lebesgue differentiation theorem, for μ-almost every $x \in \overline{\Omega}$, hence also ν-almost everyhwere by absolute continuity,

$$0 \leq f(x) = \lim_{r\to 0} \frac{\nu(B_r(x))}{\mu(B_r(x))} \leq \lim_{r\to 0} \left(\frac{n}{S}\right)^{q_x^-} \mu(B_r(x))^{\frac{q_x^-}{p_x^+}-1} = 0.$$

If x is an atom of μ, but it is not an atom of ν, then, letting $\alpha = \mu(\{x\})$, we have

$$0 \leq f(x) \leq \lim_{r\to 0} \frac{\nu(B_r(x))}{\alpha} = 0.$$

Therefore $f(x) \neq 0 \iff x$ is an atom of ν and μ and is contained in $\mathcal{A}$. Therefore we have that

$$\lim_{j\to\infty} |u_j|^{q(x)}\, dx = \sum_{\ell\in L} \nu^\ell. \tag{45}$$

We know L is countable since it is contained in the collection of atoms of μ and $\mu(\overline{\Omega}) < \infty$.

□

4.3. **Proof of the Concentration-Compactness Principle.** We are now in a good position to complete the proof of Theorem 4.1. By Lemmas 4.3 and 4.4, since $u_j \to u$ weakly, which implies that $(u_j - u) \to 0$ weakly, we get for every $\phi \in C(\overline{\Omega})$,

$$\begin{aligned}
\int_{\overline{\Omega}} \phi\, d\nu &= \lim_{j\to\infty} \int_{\overline{\Omega}} \phi |u_j|^{q(x)} dx \\
&= \int_{\overline{\Omega}} \phi |u|^{q(x)} dx + \lim_{j\to\infty} \int_{\overline{\Omega}} \phi |u_j - u|^{q(x)} dx \\
&= \int_{\overline{\Omega}} \phi |u|^{q(x)} dx + \sum_{\ell\in L} \nu^\ell \phi(x_\ell).
\end{aligned}$$

Thus using sequences $\{\phi_k\}$ for all relatively open sets in $\overline{\Omega}$ and the regularity of ν, we obtain $d\nu = |u|^{q(x)} dx + \sum_{\ell\in L} \nu^\ell \delta_{x_\ell}$.

By continuity, for all $\ell \in L$, $\lim_{r\to 0} q^+(B_r(x_\ell)) = \lim_{r\to 0} q^-(B_r(x_\ell)) = q(x_\ell)$. And so taking $r \to 0$ in (43) with x_ℓ, we obtain

$$S(\nu^\ell)^{1/q(x_\ell)} \leq \sum_{i=1}^n (\mu_i^\ell)^{1/p_i(x_\ell)}.$$

Therefore it is only left to show the inequality in point (5) of the theorem. Using Lemma 4.2 again for U, by Fatou's lemma, we get

$$\sum_{i=1}^n \int_{\overline{\Omega}} \phi_k |\partial_i u|^{p_i(x)} dx \leq \sum_{i=1}^n \liminf_{k\to\infty} \int_{\overline{\Omega}} \phi_k |\partial_i u_j|^{p_i(x)} dx = \int_{\overline{\Omega}} \phi_k d\mu \leq \mu(U)$$

Taking $k \to \infty$ on the lefthand side, we get by regularity that $\mu \geq \sum_{i=1}^n |\partial_i u|^{p_i(x)} dx$.

Trivially, $\mu \geq \sum_{\ell \in L} \mu^\ell$ and since $\sum_{i=1}^{n} |\partial_i u|^{p_i(x)} dx$ is non-atomic, we get

$$\mu \geq \sum_{i=1}^{n} |\partial_i u|^{p_i(x)} dx + \sum_{\ell \in L} \mu^\ell.$$

5. Mountain Pass Theorem

The objective of the section is to provide a proof of the Mountain pass theorem as self-contained as possible. In fact, the Mountain pass theorem is a consequence of the deformation theorem, which is much more difficult to prove. Rabinowitz presents a more general proof of the latter in [39], Annex A, but we have included a slightly different proof here. The reason is that much of the proof relies on building a locally Lipschitz map from $X \to X$, where X is a Banach space, and although the author did not prove directly the local Lipschitz condition, it doesn't appear to be so trivial. Hence we have shown a direct proof of this property, and followed the rest of the proof along the same lines as in [39].

Furthermore, the aforementionned map is used to solve an ordinary differential equation for Banach space valued functions. A proof of this is difficult to come by in the literature, and so we have adapted from classical proofs given, when $X = \mathbb{R}$ and $X = \mathbb{R}^n$, to this more general setting.

Before going over the proof of the Mountain Pass theorem, we will first start by setting the structure which allows us to integrate Banach space valued functions in a similar fashion as we do over $\mathbb{R}$ with the Lebesgue measure.

5.1. **Banach Space Valued Integration.**

Definition 5.1. *Let X be a Banach space with norm $\|\cdot\|$ and G be a Lebesgue measurable subset of $\mathbb{R}$. A function $s : G \to X$ is simple if it can be written as*

$$s(t) = \sum_{i=1}^{N} \chi_{E_i}(t) u_i,$$

where $\{u_i\} \subset X$, $\{E_i\}$ are Lebesgue measurable subsets of G and χ_E is the characteristic function of the set E.

Then we can define the integral operator on simple functions as such:

$$\int_G s(t)\, dt = \sum_{i=1}^{N} m(E_i) u_i,$$

where $m(\cdot)$ denotes the Lebesgue measure.

Now we may extend this operator to all the functions that can be pointwise approximated by simple functions, i.e. $f : G \to X$ such that for all $t \in G$, $\|f(t) - s_k(t)\| \to 0$ as $k \to \infty$, where $\{s_k\}$ are simple functions. If f is such a function, we say that it is strongly measurable. The next theorem will give a nice characterization of such functions. It is taken from [46], chapter V, section 4.

Theorem 5.2 (Pettis). *Let X' be the topological dual of X and $\langle \cdot, \cdot \rangle$ denote the duality pairing, then*

$$f \text{ is strongly measurable } \iff \begin{cases} \bullet\ g(t) = \langle \phi, f(t) \rangle \text{ is Lebesgue measurable for all } \phi \in X'; \\ \bullet\ \exists \text{ a measurable set } E \subset G \text{ such that } m(G \backslash E) = 0 \\ \qquad \text{and } f(E) \subset X \text{ is separable.} \end{cases}$$

Here we use the classical definition of Lebesgue measurable functions, i.e. $g : G \to \mathbb{R}$ is Lebesgue measurable if $g^{-1}(B)$ is a Lebesgue measurable set for every Borel set B. By this definition, all continuous functions are measurable.

Remark 5.3. Let $I \subset \mathbb{R}$ be a possibly unbounded interval, closed, open or half-closed, and let $f : I \to X$ be continuous, then if $t_n \to t$ in I, $\|f(t_n) - f(t)\| \to 0$ in X. Therefore, since norm convergence implies weak convergence, it follows that for any $\phi \in X'$, $\langle \phi, f(t_n) - f(t) \rangle \to 0$, i.e. $\langle \phi, f(t) \rangle$ is continuous. Furthermore, the continuity of f implies that $\{f(t)\}_{t \in \mathbb{Q}}$ is dense in $f(I)$. Hence, f is strongly measurable.

Definition 5.4. *We say that a function $f : G \to X$ is integrable if it is strongly measurable and there exists a sequence of simple functions $\{s_k\}$ such that*

$$\int_G \|f(t) - s_k(t)\| \, dt \to 0 \text{ as } k \to \infty,$$

Then we define

$$\int_G f(t) \, dt = \lim_{k \to \infty} \int_G s_k(t) \, dt.$$

The following theorem, taken again in [46], gives us familiar properties of the integral.

Theorem 5.5 (Bochner).

$$f \text{ is integrable } \iff \int_G \|f(t)\| \, dt < \infty.$$

Furthermore, we have the following properties,

$$\left\| \int_G f(t) \, dt \right\| \le \int_G \|f(t)\| \, dt \text{ and}$$

$$\left\langle \phi, \int_G f(t) \, dt \right\rangle = \int_G \langle \phi, f(t) \rangle \, dt \text{ for all } \phi \in X'.$$

We can also define the derivative of a Banach space valued function, in such a way:

Definition 5.6. *If there exists $u \in X$ such that*

$$\lim_{\varepsilon \to 0} \left\| \frac{f(t+\varepsilon) - f(t)}{\varepsilon} - u \right\| = 0,$$

then we say that f is differentiable at t and write $f'(t) = u$.

Remark 5.7. Observe that if f is differentiable at t, then $\lim\limits_{\varepsilon \to 0^+} \frac{\|f(t+\varepsilon)-f(t)\|}{\varepsilon} = \|v\|$, so we must have that $\lim\limits_{n \to \infty} \|f(t_n) - f(t)\| = 0$ whenever $t_n \to t$, i.e. f is continuous.

From the last property of Theorem 5.5, we can write an analogue to the Fundamental Theorem of Calculus (FTC).

Corollary 5.8. *Let X be a Banach space and $f : [0,T] \to X$.*

(1) If f is differentiable in $[0,T]$, then for any $a, b \in [0,T]$,

$$f(b) - f(a) = \int_a^b f'(t)\, dt.$$

(2) If f if continuous and $F(t) = \int\limits_0^t f(s)\, ds$, then

$$F'(t) = f(t).$$

Proof. By remarks 5.3 and 5.7, we know that f and F are measurable. For proof of (1), observe that from definition 5.6, we have for any $\phi \in X'$,

$$\begin{aligned}
\left| \frac{d}{dt} \langle \phi, f(t) \rangle - \langle \phi, f'(t) \rangle \right| &= \lim_{\varepsilon \to 0} \left| \frac{\langle \phi, f(t+\varepsilon) \rangle - \langle \phi, f(t) \rangle}{\varepsilon} - \langle \phi, f'(t) \rangle \right| \\
&= \lim_{\varepsilon \to 0} \left| \left\langle \phi, \frac{f(t+\varepsilon) - f(t)}{\varepsilon} - f'(t) \right\rangle \right| \\
&\leq \|\phi\|_{X'} \lim_{\varepsilon \to 0} \left\| \frac{f(t+\varepsilon) - f(t)}{\varepsilon} - f'(t) \right\| = 0 \qquad (46)
\end{aligned}$$

Therefore, for any $\phi \in X'$,

$$\left\langle \phi, \int_a^b f'(t)\, dt \right\rangle = \int_a^b \langle \phi, f'(t) \rangle \ dt = \langle \phi, f(b) - f(a) \rangle \, . \qquad (47)$$

Note that we could have simply mentionned that (47) follows from the fact that strong convergence implies weak convergence.

For (2), first observe that if f is continuous with respect to the norm topology of X, then it is also continuous with respect to the weak topology, therefore for any $\phi \in X'$, the mapping $t \mapsto \langle \phi, f(t) \rangle$ is continuous.

Then for any $\phi \in X'$, by (46) and the Lebesgue differentiation theorem, we have

$$\langle \phi, F'(t) \rangle = \lim_{\varepsilon \to 0} \left\langle \phi, \frac{1}{\varepsilon} \int_t^{t+\varepsilon} f(s)\, ds \right\rangle = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} \langle \phi, f(s) \rangle \, ds = \langle \phi, f(t) \rangle . \tag{48}$$

By Hanh Banach's theorem, if $u \in X$ is not 0, then there exists $\phi \in X'$ such that $\langle \phi, u \rangle = \|u\| > 0$. Thus from (47) and (48), we have proven the corollary. $\square$

5.2. **Picard's Theorem.** Now we are in a good position to prove an analogue of Picard's theorem, also know as Picard-Lindelöf's theorem or Cauchy-Lipschitz theorem. It states the existence and uniqueness of a solution to a nonlinear ordinary differential equation (ODE). The proof of this theorem will use Picard's iteration process, as can be found in [29], where the ODE is solved for $X = \mathbb{R}$. We will adapt this method to our needs here. Note that Khalil has proven a similar theorem for $X = \mathbb{R}^n$ in [31] by using a fixed point theorem. But with the tools defined in the preceding theorems, we will have enough to prove this theorem for general Banach spaces.

Before doing so, we will present the Gronwall-Bellman inequality, for which a more general version can be found in [31], Appendix A.

Lemma 5.9 (Gronwall-Bellman). *Let $C, L > 0$ and $w : [0,T] \to \mathbb{R}$ be a continuous function such that for any $t \in [0,T]$,*

$$w(t) \leq C + L \int_0^t w(s)\, ds,$$

then for every $t \in [0,T]$, $w(t) \leq Ce^{Lt}$.

Proof. Since w is continuous, then the function $v(t) = C + L \int_0^t w(s)\, ds$ is differentiable, with $v'(t) = Lw(t)$ and $w(t) \leq v(t)$, for all $t \in [0,T]$. Then

$$\frac{d}{dt} e^{-Lt} v(t) = e^{-Lt} L w(t) - e^{-Lt} L v(t) = e^{-Lt} L (w(t) - v(t)) \leq 0.$$

So we get $e^{-Lt} v(t) \leq v(0) = C$, hence $w(t) \leq Ce^{Lt}$ for all $t \in [0,T]$. $\square$

Observe that if $0 \leq w(t)$ and $C = 0$, then $w(t) = 0$.

Theorem 5.10 (Picard). *Let X be a Banach space, $u_0 \in X$, $T > 0$ and $f : [0,T] \times X \to X$ be continuous with respect to $[0,T]$. Assume there exists $r >$ such if we let $B = \{u \in X : \|u - u_0\| \leq r\}$, we have*

$$M := \sup_{t \in [0,T], u \in B} \|f(t,u)\| < \infty \qquad \text{and} \qquad L := \sup_{t \in [0,T], u,v \in B} \frac{\|f(t,u) - f(t,v)\|}{\|u - v\|} < \infty.$$

Then there exists $c \in (0,T]$ and a solution $u : [0,c] \to X$, to the ODE

$$\begin{cases} u'(t) = f(t,u(t)), & \textit{on } [0,c], \\ u(0) = u_0. \end{cases} \tag{49}$$

Proof. First, we observe that if $u(t)$ is a solution to (49), then by Remark 5.7, u is continuous. Since f is continuous in t and u is continuous, then so is $f(\cdot, u(\cdot))$. Observe also that for any bounded function g, if v is given by $v(t) = v_0 + \int_0^t g(s)\,ds$, then $\|v(t+\varepsilon) - v(t)\| \leq \sup\limits_{s\in[0,t]} \{g(s)\}\varepsilon$, so v is continuous if g is bounded.

Note that by Corollary 5.8, $u : [0,c] \to X$ is a solution to (49) if and only if

$$u(t) = u_0 + \int_0^t f(s, u(s))\,ds$$

for all $t \in [0,c]$. By the previous comment, all the functions involved are measurable, so the integration is well-defined.

Now we use Picard's iteration method, i.e. we write $u_0(t) = u_0$ and for all $n \in \mathbb{N}$,

$$u_{n+1}(t) = \int_0^t f(s, u_n(s))\,ds.$$

Again, by previous comments, all these functions are measurable.

We first want to show that $\{u_n\}$ converges to a solution u. We define the constants $C = \max\{M, L\}$ and $c = \frac{\ln(r+1)}{C}$. Note that c depends on r and that the assumptions of the theorem are analoguous to a local Lipschitz condition.

By the Taylor series of $e^x - 1$, we know that for any $t \in [0,c]$ and $N \in \mathbb{N}$,

$$\sum_{j=1}^{N} \frac{(Ct)^n}{n!} < e^{Ct} - 1 \leq r.$$

Then for any $t \in [0,c]$,

$$\|u_1(t) - u_0\| \leq \int_0^t \|f(s,u_0)\|\,dt \leq Mt \leq Ct < r.$$

Hence we know that $u_1(t) \in B$ for all $t \in [0,c]$. Thus, we can show that $u_2(t) \in B$ for all $t \in [0,c]$ as well, since

$$\|u_2(t) - u_1(t)\| \leq \int_0^t \|f(s,u_1(s)) - f(s,u_0)\|\,ds \leq L\int_0^t \|u_1(s) - u_0\|\,ds \leq C^2 \int_0^t s\,ds = \frac{(Ct)^2}{2!}$$

and so

$$\|u_2(t) - u_0\| \le \|u_2(t) - u_1(t)\| + \|u_1(t) - u_0\| \le Ct + \frac{(Ct)^2}{2!} < r.$$

By repeating the process as above, we obtain

$$\begin{aligned}\|u_n(t) - u_{n-1}(t)\| &\le \int_0^t \|f(s, u_{n-1}(s)) - f(s, u_{n-2}(s))\| \, ds \le L \int_0^t \|u_{n-1}(s) - u_{n-2}(s)\| \, ds \\ &\le \cdots \le \frac{(Ct)^n}{n!}\end{aligned}$$

and so

$$\|u_n(t) - u_0\| \le \sum_{j=1}^{n} \|u_j(t) - u_{j-1}(t)\| \le \sum_{j=1}^{n} \frac{(Ct)^j}{j!} < r.$$

Therefore the sequences $\{u_n(t)\} \subset B$ for all $t \in [0, c]$. Now for any $m, n \in \mathbb{N}$, assuming without loss of generality that $m > n$, we get for any $t \in [0, c]$,

$$\|u_m(t) - u_n(t)\| \le \sum_{j=n+1}^{m} \|u_j(t) - u_{j-1}(t)\| \le \sum_{j=n+1}^{m} \frac{(Ct)^j}{j!} \longrightarrow 0 \text{ as } n, m \to \infty.$$

Therefore for every $t \in [0, c]$, $\{u_n(t)\}$ is a Cauchy sequence in X, so it converges to a unique $u(t) \in X$.

Since $\|u(t) - u_0\| < \|u(t) - u_n(t)\| + r$, then $\{u(t)\}_{t \in [0,c]} \subset B$ and for every $s \in [0, c]$ and $n \in \mathbb{N}$ large enough, we have $\|f(s, u(s) - f(s, u_n(s))\| \le L \|u(s) - u_n(s)\| \le L$, so by the Lebesgue dominated convergence theorem (LDCT),

$$\begin{aligned}\lim_{n\to\infty} \left\| \int_0^t f(s, u(s) \, dt - \int_0^t f(s, u_n(s)) \, dt \right\| &\le \lim_{n\to\infty} \int_0^t \|f(s, u(s) - f(s, u_n(s))\| \, dt \\ &\le L \int_0^t \lim_{n\to\infty} \|u(s) - u_n(s)\| \, dt = 0.\end{aligned}$$

Therefore, taking the limit in the norm topology of X, we get for every $t \in [0, c]$,

$$u(t) = \lim_{n\to\infty} u_{n+1}(t) = u_0 + \lim_{n\to\infty} \int_0^t f(s, u_n(s)) \, ds = u_0 + \int_0^t f(s, u(s)) \, ds.$$

Therefore $u(t)$ is a solution to (49). □

We now define a property that we will use in a corollary to the previous theorem.

Definition 5.11. *Let $f : \mathbb{R} \times X \to X$. We say that f is uniformly locally Lipschitz continuous if for every $u \in X$, there exists $L(u) \in (0, \infty)$ and a neighbordhood $U(u)$ such that*

$$\sup_{t\in\mathbb{R}} \sup_{v \in U(u)} \left\{ \frac{\|f(t, u) - f(t, v)\|}{\|u - v\|} \right\} = L(u).$$

Corollary 5.12. *Let X be a Banach space and $f : \mathbb{R}\times X \to X$ be uniformly locally Lipschitz continuous, with $\|f(t,u)\| \leq M < \infty$ for all $(t,u) \in \mathbb{R}\times X$.*
Then the solution to (49) is unique and defined for all $t \in \mathbb{R}$.

Proof. First, look at a finite interval $[0,T]$. Because f is uniformly locally Lipschitz and bounded, we can satisfy the conditions of Theorem 5.10 for any initial condition, for which we know there exists at least one solution. Assume that we have two distinct solutions $u(t)$ and $\widetilde{u}(t)$ to (49) with initial value u_0 and $c = T$. Hence we know that $u(t)$ and $\widetilde{u}(t)$ are in B for all $t \in [0,T]$. Set $w(t) = \|u(t) - \widetilde{u}(t)\|$, then for all $t \in [0,T]$,

$$w(t) \leq \int_0^t \|f(s,u(s)) - f(s,\widetilde{u}(s))\|\, ds \leq L\int_0^t \|u(s) - \widetilde{u}(s)\|\, ds = L\int_0^t w(s)\, ds.$$

Observe that $|w(t+\varepsilon) - w(t)| \leq \int_t^{t+\varepsilon} \|f(s,u(s)) - f(s,\widetilde{u}(s))\|\, ds \leq 2M\varepsilon$.

So w is continuous, thus by Lemma 5.9, we have $w(t) = 0$, which proves uniqueness of the solution. Likewise, if $u(t)$ is a solution on $[0,T_1]$ and $\widetilde{u}(t)$ is a solution on $[0,T_2]$, with $T_1 < T_2$, then as above, $u(t) = \widetilde{u}(t)$ on $[0,T_1]$. Therefore it follows that if a solution exists on $\mathbb{R}$, then it is unique (we can use $g(-t,u) = f(t,u)$ for $t \leq 0$).

Next, we want to show that the solution is actually defined on all of $\mathbb{R}$. As mentionned above, it will suffice to show this for $[0,\infty)$. If the solution is defined on $[0,T]$, then we may pose a new ODE in the form of

$$\begin{cases} v'(t) = f(t,v(t)), & \text{on } [0,T_1], \\ v(0) = u(T), \end{cases} \tag{50}$$

for some $T_1 > 0$. Since f is uniformly locally Lipschitz uniformly over all $t \in \mathbb{R}$, then we can satisfy the conditions of Theorem 5.10 with any initial value. Hence there exists $\{v(t)\} \subset X$ that satisfy (50) on $[0,\delta_1]$ for some $\delta_1 > 0$. Now set

$$\widetilde{u}(t) = \begin{cases} u(t), & \text{on } [0,T], \\ v(t-T), & \text{on } (T, T+\delta_1]. \end{cases} \tag{51}$$

Then for $t \in (T, T+\delta_1]$, we get

$$\begin{aligned}\widetilde{u}(t) &= u(T) + \int_T^t f(s, v(s-T))\, ds = u_0 + \int_0^T f(s,u(s))\, ds + \int_T^t f(s,v(s-T))\, ds \\ &= u_0 + \int_0^t f(s,\widetilde{u}(s))\, ds,\end{aligned}$$

so $\widetilde{u}$ is a solution to (49) on $[0, T+\delta_1]$.

Now we may repeat this infinitely to get a solution $u(t)$ to (49) on $\left[0, T + \sum_{n\in\mathbb{N}} \delta_n\right)$.

Assume that $c = T + \sum_{n\in\mathbb{N}} \delta_n < \infty$ and that it is maximal, i.e. $u(t)$ is not defined or is not a solution at $t = c$. Then we can take a strictly increasing sequence $\{t_n\}$ such that $t_n \to c$. Then

$$\|u(t_m) - u(t_n)\| = \left\| \int\limits_{t_n}^{t_m} f(s, u(s))\, ds \right\| \leq 2M|t_m - t_n| \to 0,$$

as $n, m \to \infty$. Thus $\{u(t_n)\}$ is Cauchy in X, so it converges to some unique u_c. Then using again LDCT, we get

$$u(c) := u_c = \lim_{n\to\infty} u(t_n) = u_0 + \lim_{n\to\infty} \int_0^{t_n} f(s, u(s))\, ds = u_0 + \int_0^c f(s, u(s))\, ds.$$

We have thus a solution on $[0, c]$, which is a contradiction, therefore u is defined on all $[0, \infty)$. □

5.3. **Semigroup Properties of Solutions.** From Theorem 5.10, we can write a function $\eta : \mathbb{R} \times X \to X$ such that for a fixed $u_0 \in X$, $\eta(t, u_0) = \eta_{u_0}(t) = u(t)$ is the solution to (49) with initial value u_0 and with the added conditions of Corollary 5.12. Thus we can define the operators $\eta_t : X \to X$ by $\eta_t(u) = \eta(t, u)$. Now we will show that these operators form a semigroup.

Definition 5.13 ([18]). *A family $\{\eta_t\}_{t\in I\subset\mathbb{R}}$ of operators $X \to X$ is called a semigroup if it satifies the following for all $u \in X$:*

- *$\eta_0(u) = u$;*
- *$\eta_{t+s}(u) = \eta_t(\eta_s(u)) = \eta_s(\eta_t(u))$; and*
- *the mapping $t \mapsto \eta_t(u)$ is continuous on I.*

Proposition 5.14. *If $\{\eta_t\}_{t\in\mathbb{R}}$ are the operators $X \to X$ defined by the solutions to (49), then they form a semigroup.*

Proof. The first property is immediate from (49). The second property can be seen from our construction of v and $\widetilde{u}$ in (50) and (51). There, we had, for $s = t - T$, $\eta_{T+s}(u_o) = \widetilde{u}(T+s) = \widetilde{u}(t) = v(s) = \eta_s(u(T)) = \eta_s(\eta_T(u_0))$. A similar argument shows $\eta_{T+s}(u_0) = \eta_T(\eta_s(u_0))$.

The third property follows from the FTC and the norm inequality of the integral, i.e.

$$\|\eta_t(u) - \eta_{t_0}(u)\| \leq \int_{t_0}^t \|f(s, \eta_s(u))\|\; ds \leq M|t - t_0|.$$

□

We will know show how every element of this family of operators is in fact a homeomorphism.

Theorem 5.15. *If $\{\eta_t\}_{t\in\mathbb{R}}$ is a semigroup of operators $X \to X$, then for any fixed $t \in \mathbb{R}$, $\eta_t : X \to X$ is a homeomorphism.*

Proof. Fix $t \in \mathbb{R}$. Using the first and second properties of semigroups, if $w = \eta_t(u) = \eta_t(v)$, then $u = \eta_{-t+t}(u) = \eta_{-t}(w) = \eta_{-t+t}(v) = v$, therefore η_t is injective.

Now fix $u \in X$. Let $v = \eta_{-t}(u)$, then by the first two properties of semi-groups, $\eta_t(v) = u$, so it is surjective.

By the third property of semigroups, η_t is continuous, and so is η_{-t}. By the first and second properties, we have that $(\eta_t)^{-1} = \eta_{-t}$, thus for every fixed $t \in \mathbb{R}$, $\eta_t : X \to X$ is a continuous bijective map with continuous inverse, i.e. it is a homeomorphism. $\square$

5.4. **Differentiation of Banach Space Functionals.** Previously, we defined a way of differentiating a function $\mathbb{R} \to X$ in order to solve a specific ODE. Now we will define ways to differentiate a functional $I : X \to \mathbb{R}$, where X is a Banach space, with norm $\|\cdot\|$, and the values of the derivative are in X', the topological dual of X, which has norm $\|\cdot\|_*$.

Before giving the definition, we will establish a convention. When we write that a $\lim\limits_{w\to 0} F(w) = 0$ for some functional $F : X \to \mathbb{R}$, we mean that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|w\| < \delta$, then $|F(w)| < \varepsilon$.

Also note that $\langle \cdot, \cdot \rangle$ refers to the duality pairing between $X' - X$.

Definition 5.16 ([3])**.** *If there exists $v \in X'$ such that*

$$\lim_{w\to 0} \frac{|I[u+w] - I[u] - \langle v, w\rangle\,|}{\|w\|} = 0, \tag{52}$$

then we say that I is Fréchet differentiable at u, with $I'[u] = v$.

If there exists $v \in X'$ such that for any $w \in X$,

$$\lim_{\varepsilon\to 0^+} \frac{|I[u+\varepsilon w] - I[u] - \langle v, \varepsilon w\rangle\,|}{\varepsilon} = 0, \tag{53}$$

then we say that I is Gateaux differentiable at u, with $I'[u] = v$.

Essentially, the Gateaux derivative restricts the limits to directional ones, whereas the Fréchet derivative includes all possible limits. Hence Fréchet differentiable implies Gateaux differentiable and the derivatives coincide, but the converse is not always true (see [3], section 1.1).

Proposition 5.17. *If I is Fréchet or Gateaux differentiable at u, then $I'[u]$ is unique.*

Proof. Let v and z both satisfy (78) for a fixed $u \in X$. Then we have

$$\begin{aligned}\lim_{w\to 0}\frac{|\langle v-z,w\rangle|}{\|w\|} &= \lim_{w\to 0}\frac{|(I[u+w]-I[u]-\langle z,w\rangle)-(I[u+w]-I[u]-\langle v,w\rangle)|}{\|w\|}\\ &\le \lim_{w\to 0}\frac{|I[u+w]-I[u]-\langle z,w\rangle|}{\|w\|}+\lim_{w\to 0}\frac{|I[u+w]-I[u]-\langle v,w\rangle|}{\|w\|}=0\end{aligned}$$

The above shows that for every $w \in X$ with $\|w\| = 1$,

$$|\langle v-z,w\rangle| = 0.$$

Therefore $\|v-z\|_* = \sup_{\|w\|=1} |\langle v-z,w\rangle| = 0$, i.e. $v = z$.

Observe that the same proof works for Gateaux diffrentiable, thus the proof is complete. □

We can now define some classes of functionals. We say that $I \in C^1(X;\mathbb{R})$ if $I : X \to \mathbb{R}$ is Fréchet differentiable at every $u \in X$ and $I' : X \to X'$ is continuous with respect to the norm topologies. Note that similarly to Remark 5.7, we can show that if I is Fréchet/Gateaux differentiable at $u \in X$, then it is continuous at u.

We also define the class $\mathcal{C}$ to be functionals in $C^1(X;\mathbb{R})$ such that I' is bounded on bounded sets.

Finally, before presenting the main theorems of this section, we will present an analogue of the mean value theorem.

Theorem 5.18 ([3]). *Let $I : X \to \mathbb{R}$ be Gateaux differentiable everywhere, then for any $u, v \in X$, there exists $\lambda \in (0,1)$ such that*

$$I[u] - I[v] = \langle I'[\lambda u + (1-\lambda)v], u-v\rangle\,. \tag{54}$$

This leads to the estimate

$$|I[u]-I[v]| \le \left(\sup_{\lambda\in[0,1]}\{\|I'[\lambda u+(1-\lambda)v]\|\}\right)\|u-v\|\,. \tag{55}$$

Proof. Let $\gamma(t) = tu + (1-t)v$ and $h(t) = I[\gamma(t)]$, for $t \in [0,1]$. Then, since $\gamma(t+\varepsilon) = tu + (1-t)v + \varepsilon(u-v) = \gamma(t) + \varepsilon(u-v)$, by the definition of the Gateaux derivative, for every $t \in [0,1]$,

$$\begin{aligned}h'(t) &= \lim_{\varepsilon\to 0}\frac{h(t+\varepsilon)-h(t)}{\varepsilon}\\ &= \lim_{\varepsilon\to 0}\frac{I[\gamma(t)+\varepsilon(u-v)]-I[\gamma(t)]}{\varepsilon} = \langle I'[\gamma(t)], u-v\rangle\,.\end{aligned}$$

By the classical mean value theorem, there exist $\lambda \in (0,1)$ such that

$$I[u] - I[v] = h(1) - h(0) = h'(\lambda) = \langle I'[\gamma(\lambda)], u - v\rangle = \langle I'[\lambda u + (1-\lambda)v], u - v\rangle \,. \tag{56}$$

Then we obtain the inequality

$$|I[u] - I[v]| \le |\langle I'[\lambda u + (1-\lambda)v], u - v\rangle| \le \left(\sup_{\lambda\in[0,1]} \{\|I'[\lambda u + (1-\lambda)v]\|\}\right) \|u - v\| \,. \tag{57}$$

□

Corollary 5.19. *If $I \in \mathcal{C}$, then I is Lipschitz on bounded sets, i.e. for every $R > 0$, there exists $M_R \in (0,\infty)$ sur that for any $u, v \in B_R = \{w \in X : \|w\| \le R\}$,*

$$|I[u] - I[v]| \le M_R \|u - v\| \,.$$

Proof. Since $I \in \mathcal{C}$, then for any $R > 0$ there exists M_R such that $\sup_{u\in B_R} \{\|I'[u]\|_*\} \le M_R$. Since $u, v \in B_R$ implies $\|\lambda u + (1-\lambda)v\| \le R$, we obtain

$$\sup_{\lambda\in[0,1]} \{\|I'[\lambda u + (1-\lambda)v]\|\} \le M_R.$$

Applying this to (57) completes the proof. □

5.5. **Lipschitz Functions.** Soon we will present the two main theorems of this section. Once we are done with the deformation theorem, then the Mountain Pass theorem becomes an easy consequence of the former. Before jumping in, we will be needing the following two lemmas concerning the Lipschitz property.

Lemma 5.20. *Let X be a normed vector space with norm $\|\cdot\|_X$. Let $\{f_i : X \to \mathbb{R}\}_{i=1}^n$ be a finite collection of Lipschitz function, with Lipschitz constants $\{L_i\}_{i=1}^n$.*
Then we have that $g = \sum_{i-1}^{n} f_i$ is Lipschitz with constant at most $\sum_{i=1}^{n} L_i$.
If $\{|f_i|\}$ are bounded above with bounds $\{M_i\}$, then $h = \prod_{i=1}^{n} f_i$ is Lipschitz with constant at most

$$\left(\prod_{i=1}^{n} M_i\right)\left(\sum_{i=1}^{n} L_i\right).$$

If $|f_i|$ and $|f_i + f_j|$ are bounded below by $r_i > 0$ and $r_{i,j} > 0$ respectively, then $\frac{1}{f_i}$ and $\frac{f_i}{f_i+f_j}$ are Lipschitz with constants at most $\frac{L_i}{r_i^2}$ and $\frac{\max\{L_i,L_j\}}{r_{i,j}}$ respectively.

Proof. For the first part, we simply use the triangle inequality:

$$|g(x) - g(y)| = \left|\sum_{i=1}^{n} f_i(x) - \sum_{i=1}^{n} f_i(y)\right| \le \sum_{i=1}^{n} |f_i(x) - f_i(y)| \le \left(\sum_{i=1}^{n} L_i\right) \|x - y\| \,.$$

For the second part, we may use induction. First, we start with 1 and 2.

$$\begin{aligned}|f_1(x)f_2(x) - f_1(y)f_2(y)| &\leq |f_1(x)|\,|f_2(x) - f_2(y)| + |f_2(y)|\,|f_1(x) - f_1(y)|\\ &\leq (M_1L_2 + M_2L_1)\,\|x-y\|\,.\end{aligned}$$

Now substitute f_1 by f_1f_2 and f_2 by f_3 in the above, to obtain

$$\begin{aligned}|f_1(x)f_2(x)f_3(x) - f_1(y)f_2(y)f_3(y)| &\leq (M_1M_2)L_3 + M_3(M_1L_2 + M_2L_1)\,\|x-y\|\\ &\leq \left(\prod_{i=1}^{3} M_i\right)\left(\sum_{i=1}^{3} L_i\right)\|x-y\|\,.\end{aligned}$$

Assume the last inequality to be true up to $j-1$, then by denoting $h_{j-1} = \prod_{i=1}^{j-1} f_i$, we get

$$\begin{aligned}|f_j(x)h_{j-1}(x) - f_j(y)h_{j-1}(y)| &\leq M_j\left(\prod_{i=1}^{j-1} M_i\right)\left(\sum_{i=1}^{j-1} L_i\right) + \left(\prod_{i=1}^{j-1} M_i\right)L_j\,\|x-y\|\\ &\leq \left(\prod_{i=1}^{j} M_i\right)\left(\sum_{i=1}^{j} L_i\right)\|x-y\|\,.\end{aligned}$$

Finally, we will prove the last part. First,

$$\left|\frac{1}{f_i(x)} - \frac{1}{f_i(y)}\right| \leq \frac{|f_i(x) - f_i(y)|}{|f_i(x)|\,|f_i(y)|} \leq \frac{L_i}{r_i^2}\,\|x-y\|\,.$$

For the final one, we will first remark the following, for any $a,b,c,d \geq 0$ with $a+b>0$ and $c+d>0$, we have

$$\begin{aligned}\left|\frac{a}{a+b} - \frac{c}{c+d}\right| &= \frac{|ad-bc|}{(a+b)(c+d)} = \frac{|ad-ab+ba-bc|}{(a+b)(c+d)} = \frac{a|b-d| + b|a-c|}{(a+b)(c+d)}\\ &\leq \frac{\max\{|b-d|, |a-c|\}}{c+d}\end{aligned} \tag{58}$$

Setting $a = f_i(x)$, $b = f_j(x)$, $c = f_i(y)$ and $d = f_j(y)$, we get from (58),

$$\left|\frac{f_i(x)}{f_i(x)+f_j(x)} - \frac{f_i(y)}{f_i(y)+f_j(y)}\right| \leq \frac{\max\{L_i, L_j\}}{r_{i,j}}\,\|x-y\|\,.$$

□

Observe that the previous lemma also applies to the local Lipschitz condition, since we are dealing with finitely many compositions of functions, and any finite intersection of open sets are open.

Lemma 5.21 ([39], Annex A)**.** *Let X be a Banach Space, $I \in C^1(X;\mathbb{R})$ and $E = \{u \in X : I'[u] \neq 0\}$. Then there exists a locally Lipschitz continuous map $W : E \to X$ that satisfies*

(1) $\|W(u)\| \leq 2\,\|I'[u]\|_*$ *and*

(2) $\langle I'[u], W(u)\rangle \geq \|I'[u]\|_*^2$,

for every $u \in E$.

The following proof is the one given in [39], but we have added the proof that W is locally Lipschitz, which was stated without proof by the author.

Proof. Note that by construction, E is open. Since X is a Banach space, then E is a metric space, so it is paracompact (see [43]), i.e. any open cover $\{\widetilde{U}_\alpha\}$ has a refinement $\{U_\alpha\}$ such that every $u \in E$, there exists a neighborhood of u which intersects only finitely many elements of $\{U_\alpha\}$.

By definition of the dual norm, we know that for any $u \in E$, there exists $w \in X$ with $\|w\| = 1$ such that $\langle I'[u], w\rangle > \frac{2}{3}\|I'[u]\|_*$. Set $z = \frac{3}{2}\|I'[u]\|_* w$, then we see that

$$\|z\| < 2\|I'[u]\|_* \tag{59}$$

and

$$\langle I'[u], z\rangle = \frac{3}{2}\|I'[u]\|_* \langle I'[u], w\rangle > \|I'[u]\|_*^2. \tag{60}$$

By continuity of I', we know there exists a neighborhood N_u of u such that (1) and (2) are satisfies for all $v \in N_u$. Doing so for all $u \in E$ provides and open cover of E, of which we can take a local refinement. Denote this refinement by $\{M_\alpha\}$, for some index set $\mathcal{A}$. Then for each $\alpha \in \mathcal{A}$, there exists $u \in E$ such that $M_\alpha \subset N_u$.

Define for each α the function $\rho_\alpha : E \to [0, \infty)$ by $\rho_\alpha(u) = \operatorname{dist}(u, E \backslash M_\alpha)$. Thus if $u \notin M_\alpha$, then $\rho_\alpha(u) = 0$.

Now define $\beta_\alpha = \frac{\rho_\alpha(u)}{\sum_{\gamma\in\mathcal{A}} \rho_\gamma(u)}$ Note that because $\{M_\alpha\}$ is a local refinement, then the sum in the denominator of β_α is actually a finite sum, so the mapping is well-defined. Furthermore, observe that $\sum_{\alpha\in\mathcal{A}} \beta_\alpha(u) = 1$ for all $u \in E$. Finally, we define z_α to be the element of E that satisfies (59) and (60) for N_{u_α} and

$$W(u) = \sum_{\alpha\in\mathcal{A}} z_\alpha \beta_\alpha.$$

Now fix $u \in E$ and denote by $\widetilde{U}$ its open neighborhood which intersects only finitely many elements of $\{M_\alpha\}$, which we will denote by $\{M_j\}_{j\in J}$. Then we have

$$\|W(u)\| \leq \sum_j \|z_j\| \beta_j \leq 2\|I'[u]\|_* \sum_j \beta_j = 2\|I'[u]\|_*$$

and

$$\langle I'[u], W(u)\rangle = \sum_j \langle I'[u], z_j\rangle \beta_j \geq \|I'[u]\|_*^2 \sum_j \beta_j = \|I'[u]\|_*^2.$$

Therefore, since the above holds for any $u \in E$, it is only left to prove the local Lipschitz property. Define

$$U = \bigcap_j \left(M_j \cap \widetilde{U} \right).$$

Because j's are finite, then U is open, hence we can pick $r > 0$ such that $B_{2r} = \{v \in E : \|u - v\| < 2r\} \subset U$. Thus for all $v \in B_r$, $\rho_\alpha(v) = 0$ whenever $\alpha \notin J$ and $r < \rho_j(v)$ for all $j \in J$. Since the distance function is Lipschitz with constant 1, letting $N \in \mathbb{N}$ be the cardinality of the index set J associated to u, we get by lemma 5.20 that for each $j \in J$,

$$\mathrm{Lip}(\beta_j) = \mathrm{Lip}\left(\frac{\rho_j}{\rho_j + \sum\limits_{i \neq j} \rho_i} \right) \leq \frac{\max\left\{ \mathrm{Lip}(\rho_j), \mathrm{Lip}\left(\sum\limits_{i \neq j} \rho_i \right) \right\}}{\inf\limits_{v \in B_r} \left\{ \sum\limits_j \rho_j(v) \right\}} \leq \frac{(N-1)}{Nr}.$$

Applying again lemma 5.20, we get our final inequality

$$\|W(u) - W(v)\| \leq \sum_j \|z_j\| \, |\beta_j(u) - \beta_j(v)| \leq 2 \, \|I'[u]\|_* \left(\frac{N-1}{r} \right) \|u - v\|, \tag{61}$$

for any $v \in B_r$. Since $u \in E$ was arbitrary, then W has the local Lipschitz condition. □

We also need to define a condition on $I \in \mathcal{C}$ in order to ensure some compactness.

Definition 5.22 (PS)**.** *We say that a functional $I \in C^1(X;\mathbb{R})$ satisfies the local Palais-Smale (PS) condition for energy level $c > 0$ if for every sequence $\{u_k\} \subset X$ such that*

- $\lim\limits_k \{|I[u_k]|\} = c$ *and*
- $\lim\limits_{k \to \infty} I'[u_k] = 0$,

$\{u_k\}$ is precompact in X.

5.6. **The Deformation Theorem.** As in the classical setting of real numbers, we define critical points and values.

Definition 5.23. *We say that $u \in X$ is a critical point of the functional I if $I'[u] = 0$.*

We say that $c \in \mathbb{R}$ is a critical value, if there exists a critical point u such that $I[u] = c$.

We will denote the following sets for some $c \in \mathbb{R}$:

- $A_c = \{u \in X : I[u] \leq c\}$ and
- $K_c = \{u \in X : I[u] = c \text{ and } I'[u] = 0\}$.

The following theorem is known as a deformation theorem. It says that if c is not a critical value, then we can "nicely" deform the set $A_{c+\delta}$ into $A_{c-\delta}$, for some $\delta > 0$. The general idea is to solve an ODE of the form (49) with a proper choise of f, that is somewhat related to I'.

Theorem 5.24 (Deformation, [18], [39]). *Let $I \in \mathcal{C}$ satisfy the local Palais-Smale condition for energy level $c \in \mathbb{R}$ and*

$$K_c = \emptyset.$$

Then for any $\varepsilon > 0$ sufficiently small, there exists a $\delta \in (0,1)$ and a function

$$\eta : C([0,1] \times X; X)$$

such that the mappings

$$\eta_t(u) = \eta(t,u), \qquad (t \in [0,1],\, u \in X)$$

satisfy

(i) $\eta_0(u) = u,$ *for all* $u \in X$*;*
(ii) $\eta_1(u) = u,$ *for all* $u \notin I^{-1}[c-\varepsilon, c+\varepsilon]$*;*
(iii) $I[\eta_t(u)] \leq I[u],$ *for all* $u \in X$ *and* $t \in [0,1]$ *and*
(iv) $\eta_1(A_{c+\delta}) \subset A_{c-\delta}$.

The following proof is a combination of the one given by Evans in [18] for Hilbert Spaces and the one given by Rabinowitz in [39] for a more general version. Once again, we have added the proof that V is locally Lipschitz.

Proof. First, we want to show that because c is not a critical value, then there exist $\varepsilon > 0$ and $\sigma > 0$ such that

$$\text{if } I[u] \in [c-\varepsilon, c+\varepsilon], \text{ then } \|I'[u]\| \geq \sigma \tag{62}$$

Assume not, then we get a sequence $\varepsilon_k \to 0$, $\sigma_k \to 0$ and $\{u_k\} \subset X$ such that $c - \varepsilon_k \leq I[u_k] \leq c + \varepsilon_k$ and $\|I'[u_k]\| < \sigma_k$. Therefore by the (PS) condition, there exists $u \in X$ such that $u_k \to u$ in the norm (by extracting a subsequence if necessary). Hence by continuity of I and I', $I[u] = c$ and $I'[u] = 0$, i.e. $K_c \neq \emptyset$, which is a contradiction.

Now we will fix $\varepsilon > 0$ and $\sigma \in (0,1)$ that satisfy the claim above and then fix some δ such that

$$0 < \delta < \min\left\{\varepsilon, \frac{\sigma^2}{2}\right\}. \tag{63}$$

We now define the sets

$$A = \{u \in X : I[u] \leq c - \varepsilon \text{ or } I[u] \geq c + \varepsilon\},$$

$$B = \{u \in X : c - \delta \leq I[u] \leq c + \delta\} \text{ and}$$

$$D = X \backslash (A \cup B).$$

We now claim that the function

$$g(u) := \frac{\operatorname{dist}(u,A)}{\operatorname{dist}(u,A)+\operatorname{dist}(u,B)} \tag{64}$$

is Lipschitz on bounded domains, which implies that it is locally Lipschitz.

First, we already know that the distance function is Lipschitz with constant 1. Therefore, in order to apply part 3 of Lemma 5.20, we want to show that for any $r > 0$, there exists $\alpha_r > 0$ such that for any $u \in X$ with $\|u\| \le r$,

$$d(u) := \operatorname{dist}(u,A) + \operatorname{dist}(u,B) \ge \alpha_r > 0. \tag{65}$$

By corollary 5.19, we have for any $v \in A$ and $w \in B$ with $\max\{\|v\|, \|w\|\} \le r+1$,

$$0 < \varepsilon - \delta \le |I[v] - I[w]| \le M_{r+1} \|v - w\|.$$

By taking infimums in the above, we get that for any $u \in A \cup B$, $d(u) \ge \frac{\varepsilon-\delta}{M_{r+1}}$.

Now fix $u \in D$ with $\|u\| \le r$ and $d(u) < 1$. By definition of the distance function, we can pick sequences $\{v_k\} \subset A$ and $\{w_k\} \subset B$ such that for k large enough,

$$\|u - v_k\| < \frac{1}{2k} + \operatorname{dist}(u,A) \le 1, \quad \text{and } \|u - w_k\| < \frac{1}{2k} + \operatorname{dist}(u,B) \le 1.$$

Thus, for k large enough, $\max\{\|v_k\|, \|w_k\|\} \le r+1$. Furthermore, we have that

$$d(u) \ge \|u - v_k\| + \|u - w_k\| - \frac{1}{k} \ge \|v_k - w_k\| - \frac{1}{k} \ge \frac{\varepsilon-\delta}{M_{r+1}} - \frac{1}{k}. \tag{66}$$

Taking $k \to \infty$ in (66), we obtain (65) with

$$\alpha_r = \min\left\{\frac{\varepsilon-\delta}{M_{r+1}}, 1\right\} > 0.$$

So by Lemma 5.20, we get that g is bounded on bounded sets, with Lipschitz constant at most α_r^{-1} on $\{u \in X : \|u\| \le r\}$.

Finally, we observe that $0 \le g \le 1$, $g \equiv 0$ on A and $g \equiv 1$ on B.

Now let

$$h(t) = \begin{cases} 1, & \text{if } t \in [0,1] \\ \dfrac{1}{t}, & \text{if } t > 1. \end{cases} \tag{67}$$

Note that $|h| \le 1$ for all $t \in [0,\infty)$.

Next, we fix a mapping $W : E = \{u \in X : I'[u] \ne 0\} \to X$ as in Lemma 5.21. Next, define $W(u) = 0$ if $I'[u] = 0$. This extension of W is also locally Lipschitz, since for each $u \in E$, there exists r such that $B_r(u) \subset E$. Hence if $v \notin E$, then

$$\|W(u) - W(v)\| = \|W(u)\| \le 2\|I'[u]\|_* \le 2\|I'[u]\|_* \frac{N-1}{r} r \le 2\|I'[u]\|_* \frac{N-1}{r} \|u - v\|.$$

Finally, we set

$$V(u) = -g(u)h\left(\left\|I'[u]\right\|\right)W(u). \tag{68}$$

We already know that g is locally Lipschitz. Thus if we can show that $h\left(\left\|I'[u]\right\|\right)W(u)$ is locally Lipschitz, then by Lemma 5.20, we will have that V is locally Lipschitz.

First observe that if $\left\|I'[u]\right\|_* < 1$, then by continuity of I', we can pick a small neighborhood U such that for all $v \in U$, $\left\|I'[v]\right\|_* \leq 1$, in which case we have $h\left(\left\|I'[v]\right\|\right)W(v) = W(v)$ for all $v \in U$, which is locally Lipschitz.

Now if $u > 1$, then we can also pick a neighborhood U such that for all $v \in U$, $\left\|I'[v]\right\|_* \geq 1$. We can intersect that neighbordhood with $B_r(u)$ as in the proof of Lemma 5.21. By (61), we have for all $v \in B_r(u) \cap U$,

$$\begin{aligned}
\left\| \frac{W(u)}{\left\|I'[u]\right\|_*} - \frac{W(v)}{\left\|I'[v]\right\|_*} \right\| &\leq \sum_j \|z_j\| \left| \frac{\beta_j(u)}{\left\|I'[u]\right\|_*} - \frac{\beta_j(v)}{\left\|I'[v]\right\|_*} \right| \\
&\leq 2\sum_j |\beta_j(u) - \beta_j(v)| \leq 2\frac{N-1}{r}\|u - v\|
\end{aligned} \tag{69}$$

If $\left\|I'[u]\right\|_* = 1$, then $h\left(\left\|I'[u]\right\|\right)W(u) = W(u)$ and we can pick a small neighborhood U for which (69) hold for all $v \in U$ with $\left\|I'[v]\right\|_* \geq 1$. This completes the proof that V is locally Lipschitz.

Set $f(t,u) = V(u)$ for all $t \in \mathbb{R}$, then it is trivially continuous on $\mathbb{R}$ and is is locally Lipschitz on X uniformly over all t's. Since $\|W(u)\| \leq 2\left\|I'[u]\right\|_*$ and $|g(u)| \leq 1$ for all $u \in X$, we get that $\sup_{t\in\mathbb{R}} \|f(t,u)\| = \|V(u)\| \leq 2$ for all $u \in X$. Therefore f satisfies all the conditions of Corollary 5.12.

Therefore by Proposition 5.14, there exists a unique semigroup of continuous operators $\{\eta_t\}_{t\in\mathbb{R}} : X \to X$ such that for every $u \in H$, $\eta_t(u) = \eta(t,u)$ is the solution to the ODE

$$\begin{cases} \frac{d}{dt}\eta(t,u) = V(u), & \text{on } \mathbb{R}, \\ \eta(0,u) = u. \end{cases} \tag{70}$$

By construction, η_t satisfy (i). If u is such that $I[u] \notin [c-\varepsilon, c+\varepsilon]$, then $u \in A$, so $g(u) = 0$, which implies $V(u) = 0$. Then, when applying the Picard iteration process, we get

$u_1(t) = u + \int\limits_0^t V(u)\,dt = u$ and more generally $u_n(t) = u$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$, thus the solution $\eta(t,u) = u$ for all $t \in \mathbb{R}$. Hence we have satified (ii).

For (iii), it will suffice to show that $\frac{d}{dt}I[\eta(t,u)] \leq 0$ for all $t \in [0,1]$ and $u \in X$, since this will imply $I[\eta(t,u)] \leq I[\eta(0,u)] = I[u]$. In order to calculate this derivative, we will use the definition of the Gateaux derivative given by (53), with $u = \eta(t,u)$ and $w = \eta(t+\varepsilon,u)-\eta(t,u)$. Then we have

$$\begin{aligned} \frac{d}{dt}I[\eta(t,u)] &= \lim_{\varepsilon\to 0}\frac{I[\eta(t+\varepsilon,u)] - I[\eta(t,u)]}{\varepsilon} \\ &= \lim_{\varepsilon\to 0}\left\langle I'[\eta(t,u)], \frac{\eta(t+\varepsilon,u)-\eta(t,u)}{\varepsilon}\right\rangle \\ &= \left\langle I'[\eta(t,u)], \frac{d}{dt}\eta(t,u)\right\rangle \\ &= \left\langle I'[\eta(t,u)], V(\eta(t,u))\right\rangle \\ &= -g(\eta(t,u))h\left(\|I'[\eta(t,u)]\|\right)\langle I'[\eta(t,u)], W(\eta(t,u))\rangle \\ &\leq -g(\eta(t,u))h\left(\|I'[\eta(t,u)]\|\right)\|I'[\eta(t,u)]\|^2 \qquad (71) \\ &\leq 0 \end{aligned}$$

So now it is only left to prove (iv) in order to conclude the proof. First, observe that by (71), if $u \in X$, is such that there exists some $t \in [0,1]$ with $\eta(t,u) \in A_{c-\delta}$, then $I[\eta(1,u)] \leq I[\eta(t,u)] \leq c - \delta$, so $\eta(1,u) \in A_{c-\delta}$.

Otherwise, if $u \in A_{c+\delta}$ is such that for all $t \in [0,1]$, $\eta(t,u) \in B$. Then $g(\eta(t,u)) = 1$ for all $t \in [0,1]$, hence by (71),

$$\frac{d}{dt}I[\eta(t,u)] \leq -h\left(\|I'[\eta(t,u)]\|\right)\|I'[\eta(t,u)]\|^2, \qquad \text{for all } t \in [0,1].$$

By (62), we have for $\|I'[\eta(t,u)]\| \leq 1$,

$$\frac{d}{dt}I[\eta(t,u)] \leq -\|I'[\eta(t,u)]\|^2 \leq -\sigma^2$$

and for $\|I'[\eta(t,u)]\| > 1$,

$$\frac{d}{dt}I[\eta(t,u)] \leq -\|I'[\eta(t,u)]\| \leq -1 < -\sigma^2.$$

Then by the FTC and (63), we get

$$I[\eta(1,u)] = I[u] + \int_0^1 \frac{d}{dt}I[\eta(s,u)]\,ds \leq (c+\delta) + (-\sigma^2) < c + \delta - 2\delta = c - \delta.$$

Therefore we have showed that $\eta_1(A_{c+\delta}) \subset A_{c-\delta}$, i.e. (iv), which concludes the proof. $\square$

Remark 5.25. By Theorem 5.15, we see that $\{\eta_t\}_{t\in[0,1]}$ is a family of homeomorphisms $X \to X$.

Now we are finally in a good position to prove the Mountain Pass theorem, which can be viewed as a generalization of the Morse lemma, as seen in [38], to infinite dimensional Hilbert spaces.

Theorem 5.26 (Mountain Pass, [18])**.** *Assume that*

(i) $I[0] = 0$,

(ii) *there exists constants* $r, a > 0$ *such that*

$$I[u] \geq a \ \ if \ \ \|u\| = r$$

and

(iii) *there exists* $v \in X$ *with*

$$\|v\| > r \ \ and \ \ I[v] \leq 0.$$

Define

$$\Gamma := \{g \in C([0,1]; X) : g(0) = 0 \ \ and \ \ g(1) = v\}.$$

and

$$c = \inf_{g\in\Gamma} \max_{t\in[0,1]} I[g(t)]$$

If $I \in \mathcal{C}$ *satisfies the (PS) condition for energy level c, then c is a non-zero critical point of* I*.*

Imagine that I is a mapping that sends $\mathbb{R}^2 \to \mathbb{R}$, then its graph would ressemble a landscape. Then Γ is the collection of paths that join the point 0 to the point v. The conditions (ii) and (iii) "impose" a mountain between 0 and v, so we are looking for the path with the least altitude. The highest point of that path will then either be a saddle point or a maximal point, if we have just one hill.

Proof. Fix $g \in \Gamma$. Since it is continuous, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|t - s| < \delta \implies \|g(t) - g(s)\| < \varepsilon.$$

Because $|\,\|g(t)\| - \|g(s)\|\,| \leq \|g(t) - g(s)\|$, then it follows that $f(t) := \|g(t)\|$ is continuous. Therefore by the intermediate value theorem (IVT), since $f(0) = 0$ and $f(1) > r$ (by (iii)), then there exists $t \in [0,1]$ such that $\|g(t)\| = r$. By (ii), $I[g(t)] \geq a > 0$. Since it is true for any $g \in \Gamma$, then

$$c \geq a > 0. \tag{72}$$

Assume that $K_c = \emptyset$ and pick $\varepsilon \in (0, \frac{a}{2})$. By the deformation theorem, we can pick ε small enough such that there exists $\delta > 0$ and a homeomorphism η such that $\eta(A_{c+\delta}) \subset A_{c-\delta}$ and $\eta(u) = u$ if $u \notin I^{-1}([c-\varepsilon, c+\varepsilon])$.

By definition of c as an infimum, we can pick $g \in \Gamma$ such that

$$\max_{t\in[0,1]} I[g(t)] \leq c + \delta.$$

Because $\eta \in C(X; X)$ and $g \in C([0,1]; X)$, it follows that $\widehat{g} := \eta \circ g$ is in $C([0,1]; X)$. Furthermore, by choice of ε, $I[0] < c - \frac{a}{2} < c - \varepsilon$, so $g(0) = 0 \notin I^{-1}([c-\varepsilon, c+\varepsilon])$, thus $\widehat{g}(0) = \eta(0) = 0$.

Similarly, by (iii), $I[v] \leq 0 < c - \varepsilon$, so $\widehat{g}(1) = \eta(v) = v$.

Therefore $\widehat{g} \in \Gamma$. But by choice of g,

$$\widehat{g}([0,1]) = \eta(g([0,1])) \subset \eta(A_{c+\delta}) \subset A_{c-\delta}.$$

This means that $\max\limits_{t\in[0,1]} I[\widehat{g}(t)] \leq c - \delta < c$, which in turn contradicts that c is an infimum.

This contradiction means that $K_c \neq \emptyset$, i.e. c is a critical value, so there exists $u \in X$ such that $I'[u] = 0$ and $I[u] = c > 0$, implying that $u \neq 0$ by (i), and $I'[u] = 0$. $\square$

6. Applications of the Concentration-Compactness Principle

We will now see how the mountain pass theorem can prove the existence of a weak solution to a anisotropic variable exponent PDE problem. We will always assume $n \geq 3$.

6.1. Preliminaries.

For this section, we will assume the following:

- $\Omega \subset \mathbb{R}^n$ is a bounded domain with $n \geq 3$;
- $\vec{p}(x) = (p_1(x), p_2(x), \ldots, p_n(x)) \in (\mathcal{P}(\Omega) \cap \mathrm{Lip}(\overline{\Omega}))^n$;
- $1 \leq p_0(x) \leq p_M(x)$ for all $x \in \overline{\Omega}$;
- $1 < p_m(x) \leq p_M(x) < \overline{p}^*(x)$ for all $x \in \overline{\Omega}$.

Note that in this section, $p_0(x)$ can take on any values between 1 and $p_M(x)$, contrary to Section 3.2.

We will use the notation $a(x) \ll b(x)$, which means that $\inf\limits_{x\in\overline{\Omega}}\{b(x) - a(x)\} > 0$ and S will always be the best constant for the Sobolev embeddings, for any valid exponent. Finally, we will use the convention $\|u\| = \|u\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)}$.

Let f be a function mapping $\mathbb{R} \times \overline{\Omega} \to \mathbb{R}$. We will define here our first important assumption:

(A1) $f \in C(\mathbb{R} \times \overline{\Omega}; \mathbb{R})$ such that for all $(z, x) \in \mathbb{R} \times \overline{\Omega}$, $|f(z,x)| \leq C + c(x)|z|^{q(x)-1}$, where $C \geq 0$, $c \in L^\infty(\Omega)$ is nonnegative and $q \in \mathcal{P}(\Omega)$ satisfies $p_M(x) \ll q(x) \leq \overline{p}^*(x)$ for all $x \in \overline{\Omega}$.

For this section, we will also define

$$F(z,x) = \int_0^z f(t,x)\,dt,$$

which implies that for all $x \in \overline{\Omega}$, $F(0,x) = 0$ and $\frac{dF}{dz}(z,x) = f(z,x)$.

Furthermore, one can observe that if f satisfies (A1), then we have for $z \geq 0$,

$$|F(z,x)| \leq \int_0^z |f(t,x)|\,dt \leq |z|(C + c(x)|z|^{q(x)-1}) = C|z| + c(x)|z|^{q(x)}.$$

and likewise for $z < 0$, by a change of variable $s = -t$,

$$|F(z,x)| = \left| \int_0^{|z|} -f(-s,x)\,ds \right| \leq \int_0^{|z|} |f(-s,x)|\,ds \leq |z|(C + c(x)|z|^{q(x)-1}) = C|z| + c(x)|z|^{q(x)}.$$

The following proof is adapted from Proposition B.1 in [39] to our setting with variable exponants.

Theorem 6.1. *Let f satisfy (A1). Define $J(u) = \int_\Omega F(u(x),x)\,dx$.*

Then the mapping $\widetilde{f} : L^{q(x)}(\Omega) \to L^{q'(x)}(\Omega)$ defined by $\widetilde{f}[u](x) = f(u(x),x)$ is continuous, $J \in C^1(L^{q(x)}(\Omega); \mathbb{R})$ and we have the duality pairing, for all $v \in L^{q(x)}(\Omega)$,

$$\langle J'(u), v \rangle = \int_\Omega f(u(x),x)v(x)\,dx.$$

Furthermore, J' is bounded on bounded sets.

Proof. First, we will show that for any $u \in L^{q(x)}(\Omega)$, $\widetilde{f}[u] \in L^{q'(x)}(\Omega)$. We may assume $C \geq 1$, thus

$$\int_\Omega |f(u(x),x)|^{q'(x)}\,dx \leq \int_\Omega |C + c(x)|u(x)|^{q(x)-1}|^{q'(x)}\,dx$$

$$\leq 2^{(q')^+ - 1} \left(C^{(q')^+}|\Omega| + \max\{(c^+)^{(q')^+}, (c^+)^{(q')^-}\} \int_\Omega |u(x)|^{q(x)}\,dx \right) < \infty \tag{73}$$

Fix some $w \in L^{q(x)}(\Omega)$ and some small $u \in L^{q(x)}(\Omega)$. Let $\Omega_1(u) = \{x \in \overline{\Omega} : |u(x)| \leq \widetilde{\delta}\}$ and $A(w) = \{x \in \Omega : |w(x)| \leq K\}$, for some $\widetilde{\delta} \in (0,1)$ and some large $K > 0$. By continuity, for any $K > 0$, f is uniformly continuous on $[-K-1, K+1] \times \overline{\Omega}$. This implies that for any $K > 0$ and $\widetilde{\varepsilon} > 0$, there exists $\widetilde{\delta}$ such that $|f(u(x) + w(x), x) - f(w(x), x)| < \widetilde{\varepsilon}$ on $\Omega_1(u) \cap A(w)$. For a fixed $\varepsilon > 0$, pick $\widetilde{\varepsilon}$ small enough such that $|\Omega|\widetilde{\varepsilon}^{(q')^-} < \frac{\varepsilon^{(q')^+}}{6}$.

Since $w \in L^1(\Omega)$, then as $K \to \infty$, $|\Omega_1(u)\backslash A(w)| \leq |\Omega\backslash A(w)| \to 0$. Then by (73), there exists K large enough such that

$$\int_{\Omega_1(u)\backslash A(w)} |f(u(x)+w(x),x) - f(w(x),x)|^{q'(x)}\,dx < \frac{\varepsilon^{(q')^+}}{6} \tag{74}$$

Fix this K and then choose $\widetilde{\delta}$ small enough such that

$$\int_{\Omega_1(u)\cap A(w)} |f(u(x)+w(x),x) - f(w(x),x)|^{q'(x)}\,dx < \frac{\varepsilon^{(q')^+}}{6} \tag{75}$$

Combining (74) and (75), we get

$$\int_{\Omega_1(u)} |f(u(x)+w(x),x) - f(w(x),x)|^{q'(x)}\,dx < \frac{\varepsilon^{(q')^+}}{3} \tag{76}$$

Note that now we have fixed an $\varepsilon > 0$ and a $\widetilde{\delta} \in (0,1)$. Let $\Omega_2(u) = \overline{\Omega}\backslash\Omega_1(u) = \{x \in \overline{\Omega} : |u(x)| > \widetilde{\delta}\}$ and assume $\|u\|_{q(x)} < \delta$ for some $\delta \in (0,1)$. Then we have

$$|\Omega_2(u)|\widetilde{\delta}^{q^+} < \int_{\Omega_2(u)} |u(x)|^{q(x)}\,dx \leq \|u\|_{q(x)}^{q^-} < \delta^{q^-}.$$

Since $\widetilde{\delta}$ is fixed without depending on δ, we get that

$$|\Omega_2(u)| < \frac{1}{\widetilde{\delta}^{q^+}}\delta^{q^-} \to 0, \text{ as } \delta \to 0.$$

Now by convexity of exponents and similar calculations as in (73), we get

$$\begin{aligned}
\int_{\Omega_2(u)} &|f(u(x)+w(x),x) - f(w(x),x)|^{q'(x)}\,dx \\
&\leq 2^{(q')^+-1} \int_{\Omega_2(u)} |f(u(x)+w(x),x)|^{q'(x)} + |f(w(x),x)|^{q'(x)}\,dx \\
&\leq \widetilde{C}(q,C,c)\left(|\Omega_2(u)| + \int_{\Omega_2(u)} |u(x)|^{q(x)}\,dx + \int_{\Omega_2(u)} |w(x)|^{q(x)}\,dx\right) \\
&\leq \widetilde{C}\left(\left(\frac{1}{\widetilde{\delta}^{q^+}}+1\right)\delta^{q^-} + \int_{\Omega_2(u)} |w(x)|^{q(x)}\,dx\right)
\end{aligned}$$

Since $|w(x)|^{q(x)} \in L^1(\Omega)$, then $\int\limits_{\Omega_2(u)} |w(x)|^{q(x)}\,dx \to 0$ as $\delta \to 0$. Thus we can pick δ small enough such that

$$\max\left\{\left(\frac{1}{\widetilde{\delta}^{q^+}}+1\right)\delta^{q^-}, \int\limits_{\Omega_2(u)} |w(x)|^{q(x)}\,dx\right\} < \frac{\varepsilon^{(q')^+}}{3\widetilde{C}}.$$

So we have

$$\|f(u(x)+w(x),x)-f(w(x),x)\|_{q'(x)} \le \left(\int\limits_{\Omega} |f(u(x)+w(x),x)-f(w(x),x)|^{q'(x)}\,dx\right)^{\frac{1}{(q')^+}} < \varepsilon.$$

Letting $u = v - w$ for some $v \in L^{q(x)}(\Omega)$, then for any $\varepsilon > 0$, we can choose some $\delta > 0$ such that if $\|v-w\|_{q(x)} < \delta$, then

$$\|f(v(x),x)-f(w(x),x)\|_{q'(x)} < \varepsilon,$$

which proves that $f \in C(L^{q(x)}(\Omega); L^{q'(x)}(\Omega))$.

Now observe that for any $u, w \in L^{q(x)}(\Omega)$, $g : [0,1] \to \mathbb{R}$ defined by

$$g(s) = \|f(u(x)+s(w(x)-u(x)),x)-f(u(x),x)\|_{q'(x)}$$

is continuous, since for any $s, t \in [0,1]$,

$$|g(s)-g(t)| = \|f(u(x)+s(w(x)-u(x)),x)-f(u(x)+t(w(x)-u(x)),x)\|_{q'(x)}$$

and $\|u(x)+s(w(x)-u(x))-u(x)-t(w(x)-u(x))\| = |s-t|\,\|w(x)-u(x)\|$. Therefore, if we fix u, then for any $w \in L^{q(x)}(\Omega)$, there exists $s_w \in [0,1]$ such that for any $s \in [0,1]$,

(77)

$$\|f(u(x)+s(w(x)-u(x)),x)-f(u(x),x)\|_{q'(x)} \le \|f(u(x)+s_w(w(x)-u(x)),x)-f(u(x),x)\|_{q'(x)}.$$

By the fundamental theorem of calculus, we have

$$F(w(x),x)-F(u(x),x) = \int_0^1 (w(x)-u(x))\,f(u(x)+s(w(x)-u(x)),x)\,ds.$$

Therefore by Fubini's theorem, Hólder's inequality and the above, we get

(78)

$$\left|J(w) - J(u) - \int_\Omega f(u(x),x)(w(x)-u(x))\,dx\right|$$

$$= \left|\int_0^1 \int_\Omega (w(x)-u(x))\left(f(u(x)+s(w(x)-u(x)),x) - f(u(x),x)\right)\,dx\,ds\right|$$

$$\leq \int_0^1 \int_\Omega |w(x)-u(x)|\,|f(u(x)+s(w(x)-u(x)),x) - f(u(x),x)|\,dx\,ds$$

$$\leq 2\int_0^1 \|w-u\|_{q(x)} \|f(u(x)+s(w(x)-u(x)),x) - f(u(x),x)\|_{q'(x)}\,ds$$

(79) $$\leq 2\|w-u\|_{q(x)} \|f(u(x)+s_w(w(x)-u(x)),x) - f(u(x),x)\|_{q'(x)}$$

Since $\|u+s_w(w-u)-u\|_{q(x)} = s_w\|w-u\|_{q(x)} \leq \|w-u\|_{q(x)}$ for all $w \in L^{q(x)}(\Omega)$, then as $\|w-u\|_{q(x)} \to 0$,

$$\|f(u(x)+s_w(w(x)-u(x)),x) - f(u(x),x)\|_{q'(x)} \to 0,$$

we therefore get that

$$\lim_{w\to u} \frac{|J(w)-J(u)-\int_\Omega f(u(x),x)(w(x)-u(x))\,dx|}{\|w-u\|_{q(x)}} = 0.$$

Since u and w are arbitrary, this proves that J is Fréchet diffrentiable and that

$$\langle J'(u),v\rangle = \int_\Omega f(u(x),x)v(x)\,dx.$$

Furthermore, since $\widetilde{f} \in C(L^{q(x)}(\Omega);L^{q'(x)}(\Omega))$, then $J \in C^1(L^{q(x)}(\Omega);\mathbb{R})$. Hence to complete the proof, it is left to show that $J' : L^{q(x)}(\Omega) \to L^{q'(x)}(\Omega)$ is bounded on bounded sets. Let $\|v\|_{q(x)} = 1$, then by Hölder's inequality, we have

$$\|J'(u)\|_{q(x)} \sim \|J'(u)\|_* \leq \langle J'(u),v\rangle \leq 2\|f(u(x),x)\|_{q'(x)}.$$

By (73) and the above, we know that $\|f(u(x),x)\|_{q'(x)}$ is bounded on bounded sets. $\square$

Now we will add some more assumptions on f.

(A2) There exists $\gamma \in \mathcal{P}(\Omega)$ such that $p^+ < \gamma^-$ and $\gamma(x)F(z,x) \leq zf(z,x)$ for all $(z,x) \in \mathbb{R}\times\overline{\Omega}$.

(A3) There exists an open set $\Omega_0 \subset \Omega$ such that for all $x \in \Omega_0$ and $z \neq 0$, $zf(z,x) > 0$.

Observe that for $z > 0$ and $x \in \Omega_0$, $f(z,x) > 0$, thus $F(z,x) > 0$. For $z < 0$ and $x \in \Omega_0$, by a change of variable, we get

$$F(z,x) = \int_0^z f(t,x)\,dt = -\int_0^{|z|} f(-s,x)\,ds = \int_0^{|z|} \frac{(-s)f(-s,x)}{s}\,ds > 0.$$

Therefore, if f satisfies (A2), then for all $x \in \Omega_0$ and $z \neq 0$, $F(z,x) > 0$.

Proposition 6.2. *If f satisfies (A1) and (A2), then for any $z \in \mathbb{R}$, $x \in \overline{\Omega}$ and $t \geq 1$,*

$$F(tz,x) \geq t^{\gamma(x)} F(z,x).$$

Proof. Fix any $x \in \overline{\Omega}$ and let $g(z) = \frac{F(z,x)}{|z|^{\gamma(x)}}$. Then

$$g'(z) = \frac{z}{|z|^{\gamma(x)+2}} \left(zf(z,x) - \gamma(x)F(z,x) \right) \begin{cases} \geq 0, & \text{when } z > 0 \\ \leq 0, & \text{when } z < 0 \end{cases}$$

Therefore, as $|z| \to 0$, $g(z)$ decreases, i.e.

$$\frac{F(tz,x)}{|tz|^{\gamma(x)}} \geq \frac{F(z,x)}{|z|^{\gamma(x)}} \implies F(tz,x) \geq t^{\gamma(x)} F(z,x).$$

□

Corollary 6.3. *If f satisfies (A1), (A2) and (A3), and $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ is such that it has compact support in Ω_0 and $u \geq 1$ on a ball $B \subset \Omega_0$, then*

$$I_2(u) = \int\limits_\Omega F(u,x)\,dx \geq \int\limits_B F(1,x)\,dx.$$

Proof. By (A3), we get that $F(u,x) \geq 0$ for all $x \in \Omega$, hence $\int\limits_\Omega F(u,x)\,dx \geq \int\limits_B F(u,x)\,dx$. Then by Proposition 6.2, $F(u,x) \geq |u|^{\gamma(x)} F(1,x)$ for all $x \in B$, which concludes the proof. which concludes the proof. □

6.2. **Existence problem.** In this section, we will use the Mountain Pass theorem to prove the following theorem.

Theorem 6.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\vec{p} \in \left(\mathrm{Lip}(\overline{\Omega}) \cap \mathcal{P}(\Omega)\right)$, $p^- > 1$ and $1 \leq p_0(x) \leq p_M(x) < \overline{p}^*(x)$ for all $x \in \Omega$.*

Consider the following equation:

$$(*) \quad \begin{cases} -\mathrm{div}\left(\sum\limits_{i=1}^n a_i(x)|\partial_i u(x)|^{p_i(x)-2}\partial_i u(x) \right) + a_0(x)|u(x)|^{p_0(x)-2}u(x) = f(u(x),x), & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

Assume $\vec{a}(x) = (a_1(x), \ldots, a_n(x))$ satisfies $0 < a^- \leq a_i(x) \leq a^+ < \infty$, for all $i \in \{1,2,\ldots,n\}$ and $x \in \Omega$, $a_0(x) \geq 0$ and f satisfies (A1), (A2) and (A3). Then there exists a weak solution to $(*)$ if we have one of the following:

- *$q(x) < \overline{p}^*(x)$ for all $x \in \overline{\Omega}$; or*
- *There exists $x^0 \in \Omega_0$ and $D > 0$, such that $\overline{p}(x^0) < n$ and $F(1, x_0) \geq D + c(x^0)$, and $c^+ < K$, where $K > 0$ is a constant that depends only on $\vec{p}$, $\vec{a}$, q, γ, n and D.*

Recall that in $(*)$, we used $\partial_0 u = u$. We must first mention that although the previous theorem provides new knowledge, this section is influenced by the articles of Fu [27] and Bonder & Silva [12], who obtained somewhat similar results in the isotropic case. Furthermore, we will give a concrete example of Theorem 6.4 in Corollary 6.6.

In order to show existence of a weak solution, we need to find $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that for all $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$,

$$\sum_{i=0}^{n} \int_{\Omega} a_i(x)\partial_i u|\partial_i u(x)|^{p_i(x)-2}\partial_i v(x)\,dx = \int_{\Omega} f(u(x), x)v(x)\,dx.$$

Note that for simplicity, we will write $f(u(x), x) = f(u, x)$ and $F(u(x), x) = F(u, x)$. Also, we used the convention that $\partial_0 u = u$.

Letting $I_2(u) = \int_{\Omega} F(u, x)\,dx$, by the continuity of the embedding $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ and (A1), we can apply Theorem 6.1 to get that for all $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, $\langle I_2'(u), v\rangle = \int_{\Omega} f(u, x)v\,dx$.

Let $g_i(z, x) = a_i(x)z|z|^{p_i(x)-2}$, $G_i(z, x) = \int_0^z g_i(t, x)\,dt = a_i(x)\frac{|z|^{p_i(x)}}{p_i(x)}\,dx$ and $J_i(u) = \int_{\Omega} G_i(u, x)\,dx$, then by theorem 78, we know that $J_i \in C^1(L^{p_i(x)}(\Omega); \mathbb{R})$ and $\langle J_i'(u), v\rangle = \int_{\Omega} a_i(x)u|u|^{p_i(x)-2}v\,dx$. Since the mapping $W_0^{1,\vec{p}(\cdot)}(\Omega) \to L^{p_i(x)}(\Omega)$ defined by $u \to \partial_i u$ is linear and continuous, then if we define $I_1(u) = \sum_{i=0}^{n} J_i(\partial_i u) = \sum_{i=0}^{n} \int_{\Omega} a_i(x)\frac{|\partial_i u|^{p_i(x)}}{p_i(x)}\,dx$, we have that $I_1 \in C^1(W_0^{1,\vec{p}(\cdot)}(\Omega); \mathbb{R})$ with

$$\langle I_1'(u), v\rangle = \sum_{i=0}^{n} \int_{\Omega} a_i(x)\partial_i u|\partial_i u|^{p_i(x)-2}\partial_i v\,dx, \qquad \text{for all } v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$$

Thus, if we let $I(u) = I_1(u) - I_2(u)$, then it follows that if $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ satisfies $I'(u) = 0$, then u is a weak solution to $(*)$. Furthermore, by applying Hölder's inequality, because of (A1) and Theorem 2.3, it follows that I' is bounded on bounded sets, i.e. $I \in \mathcal{C}$.

So now we want to show that I satisfies the conditions of the Mountain pass theorem, in order to find a weak solution. Since we already showed that $I \in \mathcal{C}$, it is left to show that

(i) there exists $r \in (0, 1)$ such that $\inf_{\|u\|=r} I(u) > 0$;

(ii) there exists $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that $\|v\| > r$ and $I(v) \leq 0$; and

(iii) I satisfies the local Palais-Smale condition for energy level $c_0 = \inf_{g\in\Gamma} \max_{t\in[0,1]} I[g(t)]$, with Γ defined as in Theorem 5.26 for some v chosen in (ii).

Concretely, to prove (iii), we want to show that any sequence $\{u_m\} \subset W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that $\lim_{m\to\infty} |I(u_m)| = c_0$ and $\lim_{m\to\infty} \|I'(u_m)\|_{(W_0^{1,\vec{p}(\cdot)}(\Omega))^*} = 0$ is precompact in $W_0^{1,\vec{p}(\cdot)}(\Omega)$.

Note that the type of sequence outlined here will be referred to as a Palais-Smale sequence.

Before proving each point, we want to remark that by convexity of the function $(\cdot)^p$, for $p \geq 1$, and induction, we know there exists $\widehat{C}$ such that for any finite subset $\{x_\ell\} \subset \mathbb{R}$, $\left(\sum_\ell x_\ell\right)^p \leq \widehat{C}^{-1} \sum_\ell x_\ell^p$.

By the Poincaré-Sobolev inequality, we may use $\|u\| = \sum_{i=1}^n \|\partial_i u\|_{p_i(x)}$. For any $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, since for any $1 \leq i \leq n$, we have $\|u\| \geq \|\partial_i u\|_{p_i(x)}$, then if $\lambda = \|u\|$, we have $\left\|\frac{\partial_i u}{\lambda}\right\|_{p_i(x)} \leq 1$. Therefore, we get

$$
\begin{aligned}
\max\left\{\frac{1}{\lambda^{p^-}}, \frac{1}{\lambda^{p^+}}\right\} \sum_{i=0}^n \int_\Omega |\partial_i u|^{p_i(x)}\, dx &\geq \sum_{i=1}^n \int_\Omega \left|\frac{\partial_i u}{\lambda}\right|^{p_i(x)} dx \geq \sum_{i=1}^n \left\|\frac{\partial_i u}{\lambda}\right\|_{p_i(x)}^{p^+} \\
&\geq \widehat{C}\left(\sum_{i=1}^n \left\|\frac{\partial_i u}{\lambda}\right\|_{p_i(x)}\right)^{p^+} = \widehat{C}\left\|\frac{u}{\lambda}\right\|^{p^+} = \widehat{C}
\end{aligned}
$$

Hence we have the inequality

$$
\sum_{i=0}^n \int_\Omega |\partial_i u|^{p_i(x)}\, dx \geq \widehat{C} \min\left\{\|u\|^{p^+}, \|u\|^{p^-}\right\}. \tag{80}
$$

<u>Proof of (i):</u>

Let $\|u\| = 1$ and $r < 1$, then by (A1), proposition 6.2, with $t = \frac{1}{r}$, and (80) we obtain

$$
\begin{aligned}
I(ru) &= \sum_{i=0}^n \int_\Omega a_i(x) \frac{|r\partial_i u|^{p_i(x)}}{p_i(x)}\, dx - \int_\Omega F(ru, x)\, dx \\
&\geq r^{p^+} \frac{a^-}{p^+} \sum_{i=1}^n \int_\Omega |\partial_i u|^{p_i(x)}\, dx - r^{\gamma^-} \int_\Omega F(u, x)\, dx \\
&\geq r^{p^+}\left(\widehat{C}\frac{a^-}{p^+}\|u\|^{p^+} - r^{\gamma^- - p^+}\left(C\|u\|_{L^1(\Omega)} + c^+ \max\{\|u\|_{q(x)}^{q^-}, \|u\|_{q(x)}^{q^+}\}\right)\right) \\
&\geq r^{p^+}\left(\widehat{C}\frac{a^-}{p^+}\|u\|^{p^+} - r^{\gamma^- - p^+}\left(CS\|u\| + c^+ \max\{(S\|u\|)^{q^-}, (S\|u\|)^{q^+}\}\right)\right) \\
&\geq r^{p^+}\left(\widehat{C}\frac{a^-}{p^+} - r^{\gamma^- - p^+}\left(CS + c^+ \max\{S^{q^-}, S^{q^+}\}\right)\right)
\end{aligned}
$$

Hence, in order to prove (i), we can pick

$$r < \min\left\{1, \left(\frac{\widehat{C}\,(\frac{a^-}{p^+})}{CS + c^+ \max\{S^{q^-}, S^{q^+}\}}\right)^{\frac{1}{\gamma^- - p^+}}\right\}. \tag{81}$$

<u>Proof of (ii)</u> To prove this part, we will make good use of (A3) by choosing $\widetilde{u} \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ with compact support in Ω_0, so that $F(\widetilde{u}, x) \geq 0$, and $\|\widetilde{u}\| = r$ as given by (i).

Let $t > 1$, then by proposition 6.2, we get

$$\begin{aligned}
I(t\widetilde{u}) &= \sum_{i=0}^{n} \int_{\Omega_0} a_i(x) t^{p_i(x)} |\partial_i \widetilde{u}|^{p_i(x)}\, dx - \int_{\Omega_0} F(t\widetilde{u}, x)\, dx \\
&\leq t^{p^+} \sum_{i=0}^{n} \int_{\Omega_0} a_i(x) |\partial_i \widetilde{u}|^{p_i(x)}\, dx - t^{\gamma^-} \int_{\Omega_0} F(\widetilde{u}, x)\, dx \\
&= t^{p^+} \left(\sum_{i=0}^{n} \int_{\Omega_0} a_i(x) |\partial_i \widetilde{u}|^{p_i(x)}\, dx - t^{\gamma^- - p^+} \int_{\Omega_0} F(\widetilde{u}, x)\, dx \right) \\
&\longrightarrow -\infty \qquad \text{as } t \to \infty
\end{aligned}$$

Since $g(t) = I(t\widetilde{u})$ is continuous on $[1, \infty)$ and $g(1) > 0$, then we can pick $t > 1$ such that $I(t\widetilde{u}) = 0$. We will write $v = t\widetilde{u}$ to prove (ii).

Remark 6.5. Since $I(t\widetilde{u}) = 0$, we have $0 = I(t\widetilde{u}) \leq t^{p^+} I_1(\widetilde{u}) - t^{\gamma^-} I_2(\widetilde{u})$, which leads to

$$t \leq \left(\frac{I_1(\widetilde{u})}{I_2(\widetilde{u})}\right)^{\frac{1}{\gamma^- - p^+}} \leq \left(\frac{a^+ r^{p^-}}{I_2(\widetilde{u})}\right)^{\frac{1}{\gamma^- - p^+}}.$$

For any $s \in (0, 1)$, using the fact that $\|\widetilde{u}\| = r$ and the inequality (81), we have

$$\begin{aligned}
I(st\widetilde{u}) \leq I_1(st\widetilde{u}) &\leq (n + \widetilde{S}) a^+ t^{p^+} r^{p^-} \\
&\leq (n + \widetilde{S}) a^+ \left(\frac{a^+ r^{p^-}}{I_2(\widetilde{u})}\right)^{\frac{1}{\gamma^- - p^+}} r^{p^-} \\
&= (n + \widetilde{S}) (a^+)^{1 + \frac{1}{\gamma^- - p^+}} \left(\frac{1}{I_2(\widetilde{u})}\right)^{\frac{1}{\gamma^- - p^+}},
\end{aligned}$$

where $\widetilde{S} = \max\left\{S^{p_0^+}, S^{p_0^-}\right\}$. Let $K_1 = (n + \widetilde{S})(a^+)^{1 + \frac{1}{\gamma^- - p^+}}$.

Setting $g(s) = sv$, where $v = t\widetilde{u}$, we get that $g \in \Gamma$, as presented in the mountain pass theorem. Hence we get

$$c_0 \leq \max_{s \in [0,1]} I(st\widetilde{u}) \leq K_1 \left(\frac{1}{I_2(\widetilde{u})}\right)^{\frac{1}{\gamma^- - p^+}}. \tag{82}$$

Proof of (iii): For this part, we will first need to show that a Palais-Smale sequence is bounded in $W_0^{1,\vec{p}(\cdot)}(\Omega)$. Let $\{u_m\}$ be a Palais-Smale sequence with energy level c_0, then

$$
\begin{aligned}
I(u_m) &\geq \frac{1}{p^+}\sum_{i=0}^{n}\int_\Omega a_i(x)|\partial_i u_m|^{p_i(x)}\,dx - \frac{1}{\gamma^-}\int_\Omega \gamma(x)F(u_m,x)\,dx \\
&\geq \frac{1}{p^+}\sum_{i=0}^{n}\int_\Omega a_i(x)|\partial_i u_m|^{p_i(x)}\,dx - \frac{1}{\gamma^-}\int_\Omega f(u_m,x)u_m\,dx \\
&= \left(\frac{1}{p^+}-\frac{1}{\gamma^-}\right)\sum_{i=0}^{n}\int_\Omega a_i(x)|\partial_i u_m|^{p_i(x)}\,dx + \frac{1}{\gamma^-}\left(\sum_{i=0}^{n}\int_\Omega a_i(x)|\partial_i u_m|^{p_i(x)}\,dx - \int_\Omega f(u_m,x)u_m\,dx\right) \\
&= \left(\frac{1}{p^+}-\frac{1}{\gamma^-}\right)\sum_{i=0}^{n}\int_\Omega a_i(x)|\partial_i u_m|^{p_i(x)}\,dx + \frac{1}{\gamma^-}\langle I'(u_m),u_m\rangle
\end{aligned}
\tag{83}
$$

Since $\|I'(u_m)\|_{(W_0^{1,\vec{p}(\cdot)}(\Omega))^*}\to 0$, then for m large enough, we get $|\langle I'(u_m),u_m\rangle|\leq \|u_m\|$. If $\|u_m\|\leq 1$, then it is bounded. Thus we will assume that $\|u_m\|>1$.

Thus we can combine (80) and (83) to obtain

$$
\begin{aligned}
a^-\left(\frac{1}{p^+}-\frac{1}{\gamma^-}\right)\widehat{C}\,\|u_m\|^{p^-} &\leq \left(\frac{1}{p^+}-\frac{1}{\gamma^-}\right)\sum_{i=0}^{n}\int_\Omega a_i(x)|\partial_i u_m|^{p_i(x)}\,dx \\
&\leq I(u_m)-\frac{1}{\gamma^-}\langle I'(u_m),u_m\rangle \\
&\leq M+\frac{1}{\gamma^-}|\langle I'(u_m),u_m\rangle| \\
&\leq M+\frac{1}{\gamma^-}\|u_m\| \leq \left(M+\frac{1}{\gamma^-}\right)\|u_m\|
\end{aligned}
$$

Therefore for all m large enough, we have

$$
\|u_m\|\leq \max\left\{1,\left(\frac{\left(M+\frac{1}{\gamma^-}\right)}{a^-\left(\frac{1}{p^+}-\frac{1}{\gamma^-}\right)\widehat{C}}\right)^{\frac{1}{p^--1}}\right\}
\tag{84}
$$

The next step is to prove that under additional assumptions on f, $u_m\to u$ in $L^{q(x)}(\Omega)$. By the boundedness of $\{u_m\}$, we have strong convergence of $u_m\to u$ in $L^{r(x)}(\Omega)$ for any $r(x)\ll \overline{p}^*(x)$ and we can apply the concentration compactness principle to get

$$
\int_\Omega \phi(x)|u_m|^{q(x)}\,dx \to \int_\Omega \phi(x)|u|^{q(x)}\,dx + \sum_j \phi(x^j)\nu_j(x^j) \text{ and}
$$

$$\sum_{i=1}^{n}\int_{\Omega}\phi(x)|\partial_i u_m|^{p_i(x)}\,dx \to \int_{\Omega}\phi(x)\,d\mu \geq \sum_{i=1}^{n}\int_{\Omega}\phi(x)|\partial_i u|^{p_i(x)}\,dx + \sum_j \phi(x^j)\mu(x^j),$$

for any $\phi \in C(\overline{\Omega})$, for some countable set $\{x^j\} \subset \mathcal{A}$ and with the relation

$$S\nu(x^j)^{\frac{1}{q(x^j)}} \leq \sum_{i=1}^{n}\mu_i(x^j)^{\frac{1}{p_i(x^j)}}. \tag{85}$$

If the set of atoms of ν is empty, including the case when $q(x) < \overline{p}^*(x)$ for all $x \in \overline{\Omega}$ (which implies a compact embedding $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$), then by the Brezis-Lieb Lemma 4.3, we get that $u_m \to u$ in $L^{q(x)}(\Omega)$. Otherwise, fix any x^j that is an atom of ν. Let $\phi \subset C_c^1(\mathbb{R}^n)$ be nonnegative with compact support in the cube $Q = \left(-\frac{1}{2}, \frac{1}{2}\right)^n$, such that $\|\phi\|_\infty = 1$, $\max\limits_{1\leq i\leq n}\{\|\partial_i\phi\|_\infty\} =: M_\phi < \infty$ and $\phi(0) = 1$.

Now let $\phi_\varepsilon(x) = \phi\left(\left(\frac{x_1 - x_1^j}{\varepsilon^{\eta_1(x^j)}}, \ldots, \frac{x_n - x_n^j}{\varepsilon^{\eta_1(x^j)}}\right)\right)$, where $\eta_i(x) = \frac{1}{p_i(x)} - \frac{1}{\overline{p}^*(x)} > 0$. We get for every small $\varepsilon > 0$ that $\phi_\varepsilon(x^j) = 1$ and the support of ϕ_ε is contained in the rectangle

$$R_\varepsilon = \left(x_1^j - \frac{\varepsilon^{\eta_1(x^j)}}{2}, x_1^j + \frac{\varepsilon^{\eta_1(x^j)}}{2}\right) \times \cdots \times \left(x_n^j - \frac{\varepsilon^{\eta_n(x^j)}}{2}, x_n^j + \frac{\varepsilon^{\eta_n(x^j)}}{2}\right),$$

which has measure going to zero.

Furthermore, the functions η_i are Lipschitz continuous on $\overline{\Omega}$ by Lemma 5.20 and satisfy for all $x \in \{y \in \Omega : \overline{p}(y) < n\}$

$$\sum_{i=1}^{n}\eta_i(x) = \sum_{i=1}^{n}\left(\frac{1}{p_i(x)} - \frac{1}{\overline{p}(x)} + \frac{1}{n}\right) = \left(\sum_{i=1}^{n}\frac{1}{p_i(x)}\right) - \frac{n}{\overline{p}(x)} + 1 = 1 \tag{86}$$

and

$$s_i(x) := p_i(x)\left(\frac{\overline{p}^*(x)}{p_i(x)}\right)' = \frac{p_i(x)\overline{p}^*(x)}{\overline{p}^*(x) - p_i(x)} = \frac{1}{\eta_i(x)}. \tag{87}$$

Because x^j is an atom, we know that it is in $\mathcal{A}$, so $\overline{p}(x^j) < n$. Therefore we may assume that ε is small enough such that $\overline{p}^+(R_\varepsilon) < n$, i.e. $R_\varepsilon \subset \{y \in \Omega : \overline{p}(y) < n\}$.

Since $I'(u_m) \to 0$, then we get

$$\begin{aligned}\sum_{i=0}^{n}\int_{\Omega} a_i(x)|\partial_i u_m|^{p_i(x)-2}\partial_i u_m \partial_i(\phi_\varepsilon u_m) &= \int_{\Omega} f(u_m, x)\phi_\varepsilon u_m\,dx + o(1)\\ &\leq C\int_{\Omega}|u_m|\phi_\varepsilon\,dx + c^+\int_{\Omega}|u_m|^{q(x)}\phi_\varepsilon\,dx + o(1).\end{aligned} \tag{88}$$

On the right hand side, taking $m \to \infty$ first and then taking $\varepsilon \to 0$, because we get convergence to u in $L^1(\Omega)$, we obtain

$$c^+\nu(x^j). \tag{89}$$

Observe that if c is continuous, then we may keep $c(x)$ in the integral when taking the limit, giving us the sharper estimate $c(x^j)\nu(x^j)$ in place of (89).

On the left hand side, we have

$$\sum_{i=0}^{n}\int_{\Omega} a_i(x)|\partial_i u_m|^{p_i(x)-2}\partial_i u_m \partial_i(\phi_\varepsilon u_m)$$
$$= \left(\sum_{i=0}^{n}\int_{\Omega} a_i(x)\phi_\varepsilon|\partial_i u_m|^{p_i(x)}\,dx + \sum_{i=1}^{n}\int_{\Omega} a_i(x)|\partial_i u_m|^{p_i(x)-2}\partial_i u_m(u_m\partial_i\phi_\varepsilon)\,dx\right).$$

For the first term on the right hand side of the above, we can put a lower bound

$$\sum_{i=0}^{n}\int_{\Omega} a_i(x)\phi_\varepsilon|\partial_i u_m|^{p_i(x)}\,dx \geq a^- \sum_{i=1}^{n}\int_{\Omega}\phi_\varepsilon|\partial_i u_m|^{p_i(x)}\,dx. \tag{90}$$

Taking $m \to \infty$ first and then taking $\varepsilon \to 0$ in (90), we obtain

$$a^-\sum_{i=1}^{n}\mu_i(x^j) = a^-\mu(x^j). \tag{91}$$

Now we will show that

$$\lim_{\varepsilon\to 0}\left(\lim_{m\to\infty}\sum_{i=1}^{n}\int_{\Omega} a_i(x)|\partial_i u_m|^{p_i(x)-2}\partial_i u_m(u_m\partial_i\phi_\varepsilon)\,dx\right) = 0. \tag{92}$$

First, since u is bounded in $W_0^{1,\vec{p}(\cdot)}(\Omega)$, set $M := \sup_{m\in\mathbb{N}}\{\|\partial_i u_m\| + 1\}^{p^+-1} < \infty$. Then by Hölder's inequality, the Rellich-Kondrachov theorem and the fact that $0 < \eta_i < 1$, by (86), we have for any ε small,

$$\lim_{m\to\infty}\left|\sum_{i=1}^{n}\int_{\Omega} a_i(x)|\partial_i u_m|^{p_i(x)-2}\partial_i u_m(u_m - u)\partial_i\phi_\varepsilon)\,dx\right| \leq \lim_{m\to\infty}\frac{2M_\phi}{\varepsilon}M\sum_{i=1}^{n}\|u_m - u\|_{p_i(x)} = 0.$$

Thus we can rewrite (92) as

$$\lim_{\varepsilon\to 0}\left(\lim_{m\to\infty}\sum_{i=1}^{n}\int_{\Omega} a_i(x)|\partial_i u_m|^{p_i(x)-2}\partial_i u_m(u\partial_i\phi_\varepsilon)\,dx\right). \tag{93}$$

Now we will evaluate the following integral, for any ε small enough:

$$\int_\Omega |\partial_i \phi_\varepsilon|^{s_i(x)}\,dx = \int_{R_\varepsilon} \left| \frac{\partial_i \phi\left(\left(\frac{x_1 - x_1^j}{\varepsilon^{\eta_1(x^j)}}, \dots, \frac{x_n - x_n^j}{\varepsilon^{\eta_1(x^j)}} \right) \right)}{\varepsilon^{\eta_i(x^j)}} \right|^{s_i(x)} dx$$

$$\leq \frac{1}{\varepsilon^{s_i^+(R_\varepsilon)\eta_i(x^j)}} \left(\int_{R_\varepsilon} \left| \partial_i \phi \left(\frac{x_1 - x_1^j}{\varepsilon^{\eta_1(x^j)}}, \dots, \frac{x_n - x_n^j}{\varepsilon^{\eta_1(x^j)}} \right) \right|^{s^+} dx \dots \right. \tag{94}$$

$$\left. \dots + \int_{R_\varepsilon} \left| \partial_i \phi \left(\frac{x_1 - x_1^j}{\varepsilon^{\eta_1(x^j)}}, \dots, \frac{x_n - x_n^j}{\varepsilon^{\eta_1(x^j)}} \right) \right|^{s^-} dx \right)$$

$$= \frac{\varepsilon^{\sum_i \eta_i(x^j)}}{\varepsilon^{s_i^+(R_\varepsilon)\eta_i(x^j)}} \left(\int_Q |\partial_i \phi(x)|^{s^+}\,dx + \int_Q |\partial_i \phi(x)|^{s^-}\,dx \right)$$

$$= \varepsilon^{1 - s_i^+(R_\varepsilon)\eta_i(x^j)} \left(\int_Q |\partial_i \phi(x)|^{s^+}\,dx + \int_Q |\partial_i \phi(x)|^{s^-}\,dx \right). \tag{95}$$

(96)

Let $r := \min_{1\leq i\leq n} \{\eta_i(x^j)\} > 0$, so that for all $\varepsilon \in (0,1)$ and $1 \leq i \leq n$, we have $\varepsilon^{\eta_i(x^j)} \leq \varepsilon^r$.

By Lemma 5.20, we know that s_i is Lipschitz, hence there exists $C > 0$ such that for all $x \in R_\varepsilon$, we have

$$|s_i(x) - s_i(x^j)|^2 \leq \frac{C^2}{n}|x - x^j|^2 = \frac{C^2}{n}\sum_{i=1}^n |x_1 - x_1^j|^2 \leq \frac{C^2}{n}\sum_{i=1}^n \varepsilon^{2\eta_i(x^j)} \leq C^2\varepsilon^{2r}, \tag{97}$$

which implies that $|s_i^+(R_\varepsilon) - s_i(x^j)| \leq C\varepsilon^r$.

Using (87) for x^j and (97), we can calculate the limit

$$0 \leq \lim_{\varepsilon\to 0^+} (s_i^+(R_\varepsilon)\eta_i(x^j) - 1)\ln\frac{1}{\varepsilon} = \lim_{\varepsilon\to 0^+} \eta_i(x^j)(s_i^+(R_\varepsilon) - s_i(x^j))\ln\frac{1}{\varepsilon}$$

$$\leq C\eta_i(x^j) \lim_{\varepsilon\to 0^+} \varepsilon^r \ln\frac{1}{\varepsilon} = 0. \tag{98}$$

This in turn implies that $\lim_{\varepsilon\to 0^+} \varepsilon^{1-s^+(R_\varepsilon)\eta_i(x^j)} = 1$, hence from (95) and Theorem 2.3, there exists some $C \in (0,\infty)$ such that

$$\limsup_{\varepsilon\to 0^+} \left\| |\partial_i \phi_\varepsilon|^{p_i(x)} \right\|_{\frac{\overline{p}^*(x)}{\overline{p}^*(x) - p_i(x)}} \leq C. \tag{99}$$

Now we can show by Hölder's inequality that for any $1 \leq i \leq n$ and ε small enough,

$$\int_\Omega |u|^{p_i(x)}|\partial_i \phi_\varepsilon|^{p_i(x)}\,dx \leq 2\left\| |u|^{pi(x)}\chi_{R_\varepsilon} \right\|_{\frac{\overline{p}^*(x)}{p_i(x)}} \left\| |\partial_i \phi_\varepsilon|^{p_i(x)} \right\|_{\frac{\overline{p}^*(x)}{\overline{p}^*(x) - p_i(x)}} \leq 2C\left\| u\chi_{R_\varepsilon} \right\|_{\overline{p}^*(x)}^{p^+}.$$

Since $u \in L^{\overline{p}^*(x)}(\Omega)$, then the above implies that

$$\lim_{\varepsilon\to 0^+} \||u|\partial_i\phi_\varepsilon\|_{p_i(x)} = 0. \tag{100}$$

So now using again Hölder's inequality combined with (100) on (93) , we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+}\left(\lim_{m\to\infty}\left|\sum_{i=1}^{n}\int_\Omega a_i(x)|\partial_i u_m|^{p_i(x)-2}\partial_i u_m(u\partial_i\phi_\varepsilon)\,dx\right|\right)\\
&\quad\le 2a^+\lim_{\varepsilon\to 0^+}\left(\lim_{m\to\infty}\sum_{i=1}^{n}\left\||\partial_i u_m|^{p_i(x)-1}\right\|_{p_i'(x)}\||u|\partial_i\phi_\varepsilon\|_{p_i(x)}\right)\\
&\quad\le 2a^+M\sum_{i=1}^{n}\left(\lim_{\varepsilon\to 0^+}\||u|\partial_i\phi_\varepsilon\|_{p_i(x)}\right) = 0.
\end{aligned}$$

Using the above, (89), (91) and (88), we finally arrive at the inequality

$$\frac{a^-}{c^+}\mu(x^j) \le \nu(x^j). \tag{101}$$

Combining with (85), we have

$$\begin{aligned}
\frac{a^-}{c^+}\mu(x^j) &\le \left(\frac{1}{S}\sum_{i=1}^{n}\mu_i(x^j)^{\frac{1}{p_i(x^j)}}\right)^{q(x^j)}\\
&\le \left(\frac{n}{S}\mu(x^j)^{\frac{1}{p_\ell(x^j)}}\right)^{q(x^j)},
\end{aligned}$$

where $p_\ell(x^j) \in \{p_i(x^j)\}_{i=1}^n$ is the exponent that maximizes $\mu(x^j)$. Rearranging the above, we obtain, for some K_2 that depends on $\vec{p}$, q, $\vec{a}$ and n,

$$K_2\left(c^+\right)^{\frac{-p_\ell(x^j)}{q(x^j)-p_\ell(x^j)}} \le \mu(x^j). \tag{102}$$

Going back to the inequality (83) and combining with (102), we can now take limits to obtain, for a Palais-Smale sequence of energy level c,

$$\begin{aligned}
c_0 = \lim_{m\to\infty} I(u_m) &\ge \lim_{m\to\infty}\left(\frac{1}{p^+}-\frac{1}{\gamma^-}\right)\sum_{i=0}^{n}\int_\Omega a_i(x)|\partial_i u_m|^{p_i(x)}\,dx + \lim_{m\to\infty}\frac{1}{\gamma^-}\langle I'(u_m), u_m\rangle\\
&\ge a^-\left(\frac{1}{p^+}-\frac{1}{\gamma^-}\right)\left(\sum_j \mu(x^j) + \sum_{i=1}^{n}\int_\Omega |\partial_i u|^{p_i(x)}\,dx\right)\\
&\ge a^-\left(\frac{1}{p^+}-\frac{1}{\gamma^-}\right)\mu(x^j)\\
&\ge a^-\left(\frac{1}{p^+}-\frac{1}{\gamma^-}\right)K_2\left(c^+\right)^{\frac{-p_\ell(x^j)}{q(x^j)-p_\ell(x^j)}}\\
&= K_3\left(c^+\right)^{\frac{-p_\ell(x^j)}{q(x^j)-p_\ell(x^j)}}, \qquad (103)
\end{aligned}$$

where K_3 depends on γ, $\vec{p}$, q, $\vec{a}$ and n.

So now we can use (103) and (82) to obtain

$$K_3 \left(c^+\right)^{\frac{-p_\ell(x^j)}{q(x^j)-p_\ell(x^j)}} \leq K_1 \left(\frac{1}{I_2(\widetilde{u})}\right)^{\frac{1}{\gamma^- - p^+}}.$$

Rearranging all, we can get a constant $K_4 > 0$ that depends only on γ, $\vec{p}$, q, $\vec{a}$ and n, such that

$$K_4 \leq \left(\frac{1}{I_2(\widetilde{u})}\right) \left(c^+\right)^{\frac{p_\ell(x^j)(\gamma^- - p^+)}{q(x^j)-p_\ell(x^j)}}.$$

By assumptions of theorem 6.4, we can fix $x^0 = (x_1^0, \ldots, x_n^0) \in \Omega_0$ such that $\overline{p}(x^0) < n$ and $F(1, x^0) \geq D + c(x^0)$, for some $D > 0$. Denote $Q_b = \prod_{i=1}^n (x_i^0 - b, x_i^0 + b)$ for $b > 0$. Fix $a > 0$ such that $Q_{2a} \subset \Omega_0$. Pick $\psi \in C_c^\infty(Q_{2a})$ such that $0 \leq \psi \leq 1$ and $\psi \equiv 1$ on Q_a. If $\|\psi\| \leq r$, then we may take $\widetilde{u} = R\psi$, where $R \geq 1$ is such that $\|\widetilde{u}\| = r$, then $\widetilde{u} \geq 1$ on Q_a.

Otherwise, similarly as was done for ϕ, we define $\psi_\varepsilon(x) = \psi\left(\frac{x_1}{\varepsilon^{\frac{1}{p_1(x_0)}}}, \ldots, \frac{x_n}{\varepsilon^{\frac{1}{p_n(x_0)}}}\right)$. Then, since $\sum\limits_{i=1}^n \frac{1}{p_i(x^0)} = \frac{n}{\overline{p}(x^0)}$, we obtain

$$\int\limits_\Omega |\partial_i \psi_\varepsilon|^{p_i(x)}\, dx \leq \varepsilon^{\frac{-p_i^+(Q_{2\varepsilon})}{p_i(x_0)}} \int\limits_\Omega \left|\partial_i \psi\left(\frac{x_1}{\varepsilon^{\frac{1}{p_1(x_0)}}}, \ldots, \frac{x_n}{\varepsilon^{\frac{1}{p_n(x_0)}}}\right)\right|^{p_i(x)}$$

$$\leq \varepsilon^{\frac{n}{\overline{p}(x^0)} - \frac{p_i^+(Q_{2\varepsilon})}{p_i(x_0)}} \left(\int\limits_\Omega |\partial_i \psi|^{p^+}\, dx + \int\limits_\Omega |\partial_i \psi|^{p^-}\, dx\right).$$

By continuity, $\lim\limits_{\varepsilon\to 0} \frac{p_i^+(Q_\varepsilon)}{p_i(x^0)} = 1$, therefore $\|\psi_\varepsilon\| \to 0$. So we can pick ε such that $\|\psi_\varepsilon\| = r$. Let $\widetilde{u} = \psi_\varepsilon$, then $\widetilde{u} = 1$ on some ball $B \subset \Omega_0$. Then by corollary 6.3, $I_2(\widetilde{u}) \geq \int\limits_B F(1, x)\, dx$. By contitnuity, we can make B smaller if necessary such that $\inf\limits_{x\in B}\{F(1,x)\}\, dx \geq \frac{D}{2}$, giving us $I_2(\widetilde{u}) \geq |B|\frac{D}{2}$. Finally, we get a constant K, that depends on γ, $\vec{p}$, q, $\vec{a}$, n, D and $|B|$, such that

$$K \leq c^+. \tag{104}$$

Notice that the choice of B depends on $\vec{p}$, r and Ω_0. By (81) and $F(1, x^0) \geq D$, which implies that $C \neq 0$, if we make c^+ smaller, then we may pick a larger r, hence also a bigger ball B. Since Ω_0 depends only on where $F > 0$ and not on how big or small F is, then making c^+ smaller will not affect $|B|$.

Therefore if c^+ is small enough, then the above inequality may not hold, which would imply that $\mu(x^j) = 0$ for all atoms x^j, i.e. $d\eta = |u|^{q(x)}\, dx$, which implies that $u_m \to u$ in

$L^{q(x)}(\Omega)$. Now in order to finish the proof of the theorem, it is left to show that $u_m \to u$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$.

Note that by weak convergence, we know that $\partial_i u_m \to \partial_i u$ pointwise. Hence, to apply the Vitali convergence theorem to

$$\sum_{i=0}^{n} \int_{\Omega} a_i(x)|\partial_i u_m|^{p_i(x)-2}\partial_i u_m \partial_i u \, dx,$$

we only need to prove equi-integrability. By Hölder's inequality, we have

$$\lim_{|E|\to 0} \left| \sum_{i=0}^{n} \int_{E} a_i(x)|\partial_i u_m|^{p_i(x)-2}\partial_i u_m \partial_i u \, dx \right| \leq 2a^+ M \sum_{i=0}^{n} \left(\lim_{|E|\to 0} \|\partial_i u \chi_E\|_{p_i(x)} \right) = 0.$$

This implies that

$$\lim_{m\to\infty} \sum_{i=0}^{n} \int_{\Omega} a_i(x)|\partial_i u_m|^{p_i(x)-2}\partial_i u_m \partial_i u \, dx = \lim_{m\to\infty} \sum_{i=0}^{n} \int_{\Omega} a_i(x)|\partial_i u|^{p_i(x)} \, dx. \tag{105}$$

Now, observe that since $\{u_m\}$ is bounded in $W_0^{1,\vec{p}(\cdot)}(\Omega)$,

$$\begin{aligned}
\lim_{m\to\infty} \left| \sum_{i=0}^{n} \int_{\Omega} a_i(x)|\partial_i u_m|^{p_i(x)-2}\partial_i u_m \partial_i (u-u_m) \, dx - \int_{\Omega} f(u_m,x)(u-u_m)\, dx \right| \\
&= \lim_{m\to\infty} |\langle I'(u_m), (u-u_m)\rangle| \\
&\leq \lim_{m\to\infty} \|I'(u_m)\|_* (\|u\| + M) \\
&= 0.
\end{aligned}$$

Since we proved that $u_m \to u$ in $L^{q(x)}(\Omega)$, then by continuity of f, we get that $\widetilde{M} = \sup_m \{\|f(u_m,x)\|_{q'(x)}\} < \infty$. Considering that $I'(u_m) \to 0$, we have

$$\begin{aligned}
\lim_{m\to\infty} \left| \sum_{i=0}^{n} \int_{\Omega} a_i(x)|\partial_i u_m|^{p_i(x)-2}\partial_i u_m \partial_i (u-u_m) \, dx \right| &= \lim_{m\to\infty} \left| \int_{\Omega} f(u_m,x)(u-u_m)\, dx \right| \\
&\leq \lim_{m\to\infty} 2\widetilde{M} \|u-u_m\|_{q(x)} = 0. \qquad (106)
\end{aligned}$$

Using (105) with (106), we get

$$\lim_{m\to\infty} \sum_{i=0}^{n} \int_{\Omega} a_i(x)|\partial_i u_m|^{p_i(x)} \, dx = \sum_{i=0}^{n} \int_{\Omega} a_i(x)|\partial_i u|^{p_i(x)} \, dx. \tag{107}$$

Fix i, then by lemma 4.3 (Brezis-Lieb), we get

$$\int_{\Omega} a_i(x)|\partial_i u - \partial_i u_m|^{p_i(x)} \, dx = \int_{\Omega} a_i(x)|\partial_i u|^{p_i(x)} \, dx - \int_{\Omega} a_i(x)|\partial_i u_m|^{p_i(x)} \, dx.$$

Summing over all i, by (107), we get

$$\lim_{m\to\infty}\sum_{i=0}^{n}\int_{\Omega} a_i(x)|\partial_i u-\partial_i u_m|^{p_i(x)}\,dx=0.$$

which implies that $u_m\to u$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$, since $a^->0$ and $a_0(x)\geq 0$ for all $x\in\Omega$.

Hence if c^+ is small enough, then I satisfies the local Palais-Smale condition for energy level c_0, and therefore also the conditions of the Mountain pass theorem, which completes the proof of existence of a weak solution.

Corollary 6.6. *Under the assumption on Ω, $\vec{p}$ and $\vec{a}$ for Theorem 6.4, consider the equation*

$$\begin{cases}-\operatorname{div}\left(\sum\limits_{i=1}^{n}a_i(x)|\partial_i u|^{p_i(x)-2}\partial_i u\right)=\lambda(x)|u|^{r(x)-2}u+c(x)|u|^{q(x)-2}u, & x\in\Omega\\ u(x)=0, & x\in\partial\Omega,\end{cases}\tag{108}$$

where $\lambda,c\in C(\overline{\Omega})$ are positive, $r,q\in C(\overline{\Omega})$, with $p^+<r^-\leq r(x)<q(x)\leq\overline{p}^*(x)$.

Then there exists a $K\in(0,\infty]$, that depends only on $\vec{p}$, $\vec{a}$, q, r, λ and n, such that if,

$$c^+<K,$$

there exists a weak solution to (108).

Proof. From the above, we have the source function

$$f(u(x),x)=\lambda(x)|u(x)|^{r(x)-2}u(x)+c(x)|u(x)|^{q(x)-2}u(x).$$

Then we can see that f satisfies (A3) on all of Ω, since $zf(z,x)=\lambda(x)|z|^{r(x)}+c(x)|z|^{q(x)}$.

Furthermore, we get

$$F(z,x)=\lambda(x)\frac{|z|^{r(x)}}{r(x)}+c(x)\frac{|z|^{q(x)}}{q(x)}.$$

Setting $\gamma(x)=r(x)$, we get

$$\gamma(x)F(z,x)=\lambda(x)|z|^{r(x)}+\frac{r(x)}{q(x)}|z|^{q(x)}<\lambda(x)|z|^{r(x)}+|z|^{q(x)}=zf(z,x),$$

since $r(x)<q(x)$. This proves that f satisfies (A2).

As in the proof of Lemma 3.8, for any $\varepsilon<0$, the can find $C(\varepsilon)$ such that $|u|^{r(x)}\leq C(\varepsilon)+\varepsilon|u|^{q(x)}$. Thus we get (A1), since

$$|f(u,x)|\leq C(\varepsilon)\left\|\lambda\right\|_{\infty}+(c(x)+\varepsilon)|u|^{q(x)-1}$$

and by the continuity of λ,c,q and r, f is continuous.

Now observe that we can take any $\varepsilon>0$, so we could define c^+ in (104) with this $c(x)$, no matter what $\lambda(x)$ is.

If $\overline{p}(x) \geq n$ for all $x \in \Omega$, then it follows that $q(x) < \overline{p}^*(x) = \infty$ for all $x \in \overline{\Omega}$. Therefore by Theorem 6.4, we have a weak solution, hence we can take $K = \infty$.

Otherwise, there exists x^0 such that $\overline{p}(x^0) < n$. Also, for all $x \in \Omega$, $F(1,x) = \lambda(x) + c(x) \geq \lambda^- + c(x)$. Hence we can make $D = \lambda^-$. So by Theorem 6.4, there exists $K \in (0,\infty)$ such that if $c^+ < K$, then there exists a weak solution to (108). $\square$

Discussion

The critical embedding Theorem 3.12 allows us to now solve a bigger class of partial differential equations. Equipped with this embedding, it may now be possible to generalize many differential equation results obtained either in the variable exponent isotropic case, or in the fixed exponent anisotropic case. Here we proved only existence of a weak solution, hence we may also look at uniqueness and regularity of such solutions.

Another relevant question to pursue is whether the embeddings holds true on the class of log-Hölder continuous functions, as it does in the isotropic variable exponent case. The class of domains for which the anisotropic embeddings hold is another important avenue, but it is a problem still precarious even for the fixed exponent case.

In [37], the authors showed that if an exponent is critical only on parts of the domain, then under some conditions related to how "fast" it approches the critical set, we may still have compact embedding of the isotropic variable exponent Sobolev space into the Lebesgue space. Therefore it may be interesting to pursue this question in the anisotropic spaces. Furthermore, one may wonder if the spread of $\vec{p}$, i.e. how far apart the p_i exponents functions are, affects how quickly the critical function must approach the critical set.

The concentration-compactness principle, Theorem 4.1, will also be useful in solving differential equations. As a follow-up, one may also, as did Bonder, Saintier & Silva in [11], following [12], study the existence of extremals of a critical Sobolev embedding, i.e. the existence of a $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$\frac{\|u\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)}}{\|u\|_{q(x)}} = \inf_{v \in W_0^{1,\vec{p}(\cdot)}(\Omega)} \left\{ \frac{\|v\|_{W_0^{1,\vec{p}(\cdot)}(\Omega)}}{\|v\|_{q(x)}} \right\}.$$

Conclusion

The main objective of this thesis was to prove weak existence of a partial differential equation involving an anisotropic variable exponent Laplacian type operator and a source with critical growth. The first challenge was to prove that there even is an embedding with critical exponents, which was achieved mainly by adapting existing proofs, for the classical isotropic fixed exponent case, to our needs. We then proceeded to prove the concentration-compactness principle, by combining the best of both existing proofs for the isotropic case

([27],[12]). Finally, we were able to use the Mountain Pass theorem in order to achieve our main goal.

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