Descent for Cocomplete Categories

by

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Abstract

This thesis investigates descent for the 2-fibration of cocomplete categories over toposes and geometric morphisms. Change of base within this 2-fibration is given by the left adjoint to the restriction functor. Pitts' pullback theorem [Pi] is important to descent in this context, and a new and more natural proof of it is obtained. As in [Pi], the proof herein depends on Paré's [P2] results on generated topologies. The present context is 2-categorical, and an abstract 2-descent theorem is obtained. Its first use is to show that a geometric morphism which is of effective descent for cocomplete categories remains so for toposes.

Studying toposes as cocomplete categories is analogous to studying locales as sup-lattices. Pure geometric morphisms are introduced in terms of the cocontinuous dual of a cocomplete category. They are shown to be of effective descent for cocomplete categories. Hence, a new proof of Moerdijk's [M5] version of a classification theorem for toposes originally due to Bunge [B4] is obtained.

Résumé

Dans cette thèse on étudie la descente pour la 2-catégorie fibrée des catégories cocomplètes par rapport aux topos et aux morphismes géométriques. Pour cette 2-categorie fibrée le changement de base est donné par l'adjoint à gauche du foncteur restriction. Le théorème du produit fibré de Pitts [Pi] est important pour la descente dans ce contexte ; on en donne une démonstration nouvelle et plus naturelle. Comme celle de [Pi], notre preuve utilise les résultats de Paré [P2] sur les topologies engendrées. Notre contexte étant 2-catégoriel, on obtient un théorème de 2-descente abstrait. On se servira en premier lieu de ce résultat pour montrer qu'un morphisme géométrique de descente effective pour les catégories cocomplètes reste de descente effective pour les topos.

L'étude des topos en tant que catégories cocomplètes est analogue à celle des locales en tant que treillis complets. On introduit la notion de morphisme géométrique pur et on montre que ces morphismes sont de descente effective pour les catégories cocomplètes. On obtient ainsi une nouvelle démonstration de la version de Moerdijk [M5] du théorème de classification des topos de Bunge [B4].

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Introduction

Principally, this thesis addresses the issue of descent along a geometric morphism between toposes in the context of cocomplete categories over toposes. These investigations find their application primarily within toposes and geometric morphisms, but they are also of interest in themselves.

Descent theory was introduced by J. Giraud [Gi1,2] in connection with his work on non-abelian cohomology. Working within the framework of fibrations over a Grothendieck topos, he formalized 'descent' as typified in situations arising in Grothendieck [Gr1]. Subsequently, M. Bunge and R. Paré [BP] developed descent over an elementary topos $\mathcal{S}$, for the regular epimorphism topology. They worked within the context of indexed categories, and this is the context in which the present work investigates descent theory.

Descent theory is formulated within the theory of fibered categories, or fibrations. A fibration consists of a *base* category together with a collection of *fiber* categories; a fiber category is assigned to each object of the base category. One is also given a rule for 'changing base' within the fibration. Change of base within the fibration occurs along morphisms which belong to the base category. To say that an object of a fiber category 'descends' along a morphism of the base category is to say that it essentially arises via change of base. A morphism of the base category may have the property that every object which comes equipped with *descent data*, does in fact descend along the morphism. Such a morphism is referred to as one of *effective descent.* The question of descent then asks which of the morphisms in the base category are of effective descent.

In some situations, descent data can be thought as a 'local' presentation of an object which is thereupon retrieved by a 'glueing' operation. This would constitute the descent of the local information. For example, let us take the base category to be that of spaces and continuous maps.[1] The fiber category over a typical space $\mathbf{X}$ is taken to be that of all spaces over $\mathbf{X}$; change of base is taken to be the operation of pullback. A continuous map $\mathbf{X}\rightarrow\mathbf{Y}$ is then of effective descent if it has the property that every space $\mathbf{Z}$ over $\mathbf{X}$ which comes equipped with descent data is necessarily of the form $\mathbf{X}\times_{\mathbf{Y}}\mathbf{W}$ for some space $\mathbf{W}$ over $\mathbf{Y}$. If the map $\mathbf{X}\rightarrow\mathbf{Y}$ is an open surjection, then it is of effective descent. This result was proved by Joyal and Tierney [JT] and it falls within the descent theory they had developed for locales and sup-lattices.

Descent theory for locales and sup-lattices is of interest in itself; however, it

[1] To follow [JT], this category is, by definition, the opposite of the category of locales and localic maps. The 'locale/frame' terminology of [J5] is not used in this thesis.

was put to use in [JT] in the study of Grothendieck toposes. They used their descent theory to establish a representation theorem for Grothendieck toposes. They proceeded first by using the descent theory for locales to establish a descent theorem for toposes. This theorem states that geometric morpisms which are *open surjections* are of effective descent. In this context of descent, the base category is taken to be that of Grothendieck toposes, and the fiber above a given topos is the topos itself. Change of base is accomplished under the inverse image functors. With this result in hand, the representation theorem is then established by using the fact that any Grothendieck topos can be 'covered' by an open surjection such that the covering topos is spatial.

As one may have guessed from the rather broad description of descent given above, 'descent theorems' and their applications can vary greatly from one context to the next. In a recent work of M. Bunge [B4], the descent theorem for toposes of [JT] was used to establish a classification theorem for toposes. She proved that the topos of etale $\mathbf{G}$-spaces classifies $\widehat{\mathbf{G}}$-torsors, where $\mathbf{G}$ is a groupoid in the category of spaces and $\widehat{\mathbf{G}}$ is the etale completion of $\mathbf{G}$. She proceeds by observing that two certain fibrations are each the stack completion of (the fibration determined by) $\widehat{\mathbf{G}}$. One concludes that these two fibrations are equivalent. In yet another application of [JT], Bunge [B3] has defined, under certain assumptions, the fundamental groupoid of a topos. In general, this is a (totally disconnected) spatial groupoid.

In the theory of locales and sup-lattices, the base category is in fact a 2-category (with non-trivial 2-cells since the 'hom' categories are posets). The same is true of the fiber categories; they are posetal 2-categories. However, in this case the 2-structure plays a neutral role, and the resulting descent theorem for toposes of Joyal and Tierney can be regarded as within 1-dimensional category theory. This changes when one considers Moerdijk's [M5] recent descent theorem. Here, the base category remains that of Grothendieck toposes, but the fiber category over a topos in the base is taken to be the 2-category of toposes over that topos. He shows that the descent theorem for toposes from [JT] can be combined with his stability theorem [M3] to yield a 2-descent theorem. In this 2-categorical context, he proves that open surjections are of effective descent. As a direct consequence, he obtains a classification theorem [Chap. 4, Eg. 3.3] for toposes which is the 2-dimensional version of that previously obtained by Bunge [B4]. Moerdijk's descent theorem (the spatial case) is a consequence of the descent theorem [Chap. 4, Th. 4.5] of this thesis.

A good part of Chapter 4 of this thesis is spent developing a formal 2-descent theorem [Chap. 4, Th. 2.13] as the context of descent in this thesis is a

2-categorical one. In this regard one could say that a morphism is of effective descent *at the level of objects*, if the comparison functor, now actually a 2-functor, is 2-fully faithful. The descent theorem for toposes from [JT] asserts, when stated now in the terms of 2-descent, that open surjections are of effective descent at the level of objects.

Another descent theorem has been recently established by M. Zawadowski [Z1,2]. Grothendieck toposes are related to categorical logic (as originally observed by Lawvere [L2], see below). In fact, a site can be thought of as a 'theory', and the topos of sheaves on that site can be thought of as the 'embodiment' of that theory. In connection with these ideas one has the notion of a pretopos, and Zawadowski's theorem is in the context of pretoposes. He considers *lax*-descent, but at the level of objects.

Topos theory represents the confluence of essentially two streams of thought which both have their origins in the early to mid 1960's.[2] The first stream originates with the work of A. Grothendieck. He generalized the notion of a system of open coverings, and extended the definition of sheaves on a topological space to that of sheaves on a site. Sheaves on a site define a category, called a Grothendieck topos. Thus, a topos is in this sense a generalized topological space.

The other stream of ideas to which topos theory owes its existence has its origins in the work of F. W. Lawvere [L2]. He observed that a Grothendieck topos has an internal logic, and that therefore one might be able to 'free' the theory of its dependence on 'external' notions. Subsequently, he and M. Tierney laid down the axioms of *elementary* topos theory. A topos is thus a theory of 'variable' sets in which one can 'do' mathematics.

Given that mathematics can be done in a topos, and if the dependence on classical set theory is to be 'entirely' removed, then a 'large' part of the theory is needed. One would like to be able to do mathematics *over* a topos. This programme, which also originates with Lawvere [L3], focuses on the notion of a 'family of objects' indexed by an elementary topos. From the point of view of fibered categories, such a theory has been extensively studied by Bénabou (see [Be], for example). Then in the mid 1970's, Paré and D. Schumacher [PS] published a monograph on indexed-category theory, taking as its goal the adjoint functor theorems.

Thus, one can do category theory over an elementary topos. A category over $\mathcal{S}$ is herein taken to mean an $\mathcal{S}$*-indexed* category. The principal source for the basics of indexed category theory is the aforementioned work of Paré and

[2]See P. T. Johnstone [J1] for a historical survery.

Schumacher. (Since this is to be indexed category theory, the word 'indexed' shall henceforth be entirely omitted. The emphasis is instead on which base topos one is working over.) In this thesis, the approach to descent for toposes is via cocomplete categories over an elementary topos. Assigned to each object of the base category, which is to say assigned to each topos, is the fiber category of cocomplete categories over the topos. It is within this context, that the question of descent is herein addressed. Which geometric morphisms are of effective descent in the context of cocomplete categories?

According to A. Pitts [Pi], 'one may be able to describe Grothendieck toposes in terms of cocomplete categories in a way analogous to that in which Joyal and Tierney have described the theory of locales as part of the "commutative algebra" of complete lattices and arbitrary sup preserving maps'. To support his contention, he proves that the pullback of a Grothendieck $\mathcal{S}$-topos $\mathcal{F}$ along a geometric morphism $\mathcal{E} \xrightarrow{\mathsf{p}} \mathcal{S}$ coincides with that cocomplete category over $\mathcal{E}$ obtained from $\mathcal{F}$ by changing base along p . This result, which is important in the treatment of descent in this thesis, is analogous to the fact that the pushout of a diagram

$$\begin{array}{ccc} \mathbf{A} & \longrightarrow & \mathbf{C} \\ \downarrow & & \\ \mathbf{B} & & \end{array}$$

of locales and localic maps coincides with the tensor product of $\mathbf{C}$ and $\mathbf{B}$ as calculated in $\mathbf{A}$-modules. Questions about cocomplete categories over $\mathcal{S}$, and their relevance to toposes, can thus be motivated in this manner; however, unlike the situation for locales and sup-lattices, no characterization of toposes within cocomplete categories is known, but it is still profitable to operate at the level of cocomplete categories and then restrict one's attention (using any pertinent considerations) to toposes. Thus, one can study toposes within cocomplete categories as analogous to the study of locales within sup-lattices, but genuine difficulties appear in the process. Proofs of theorems about toposes and cocomplete categories are quite different to those of their counterparts in locales and sup-lattices.

The descent theorem for locales and sup-lattices from [JT] states that a localic map $\mathbf{A} \rightarrow \mathbf{B}$ is of effective descent if and only if it is *pure*. By 'pure' is meant that for any $\mathbf{A}$-module $\mathbf{M}$, the universal morphism

$$\mathbf{M} \rightarrow \mathbf{B} \otimes_{\mathbf{A}} \mathbf{M}$$

is faithful. Directly adapting this property to geometric morphisms is a possible approach, but perhaps not the best. The approach that is taken in this thesis is to move to the *cocontinuous dual*, and then define in this context the appropriate notion of purity. The main result of this thesis [Chap. 4, Th. 4.5] asserts that pure geometric morphisms are of effective descent for cocomplete categories.

In keeping with Pitts [Pi], the notation $\text{COCTS}_{\mathcal{S}}$ is here used to denote the 2-category of cocomplete locally small categories over an elementary topos $\mathcal{S}$. In sections §0 through §4 of Chapter 1, the basic definitions concerning the 2-category $\text{COCTS}_{\mathcal{S}}$ are to be found. The concept of a *stack* comes up in Chapter 2, in connection with sheaves in an arbitrary category over $\mathcal{S}$. An important fact is that any $\mathcal{A} \in \text{COCTS}_{\mathcal{S}}$ is a stack [Th. 4.7]. The proof of this uses the (Beck-) Bénabou-Roubaud [BR] characterization of stacks in terms of tripleability [Th. 1.3]. Section §5 reviews the restriction 2-functor

$$(\ )_{\mathsf{p}}\ : \text{COCTS}_{\mathcal{E}} \longrightarrow \text{COCTS}_{\mathcal{S}}$$

induced by a geometric morphism $\mathcal{E} \xrightarrow{\mathsf{p}} \mathcal{S}$. A topos over $\mathcal{S}$ is an object of $\text{COCTS}_{\mathcal{S}}$, and one has in fact a 2-embedding

$$\wp : \text{TOP}_{\mathcal{S}}{}^{op} \longrightarrow \text{COCTS}_{\mathcal{S}}\ .$$

Some basic results [Ths. 6.7 and 6.8] are derived in §6 of Chapter 1 as to when a category over $\mathcal{S}$ is in the essential image of $\wp$. These results are later put to use in Chapter 4 [Th. 3.4]. Among those properties possessed by a category within the essential image of $\wp$ is that of having universal coproducts. This property is formally introduced, and referred to, as coproducts which satisfy *Frobenius reciprocity* [Def. 6.2]. Related to Frobenius reciprocity is a result [Eg. 4.3] which goes back to Bénabou. This result is extended in [Eg. 4.3]. It illustrates the interplay of internal category theory with the general theory.

Chapter 2 begins with a discussion of internal diagrams on a small category $\mathbf{C}$ taking their values in a category $\mathcal{A}$. Let us denote this category by $\mathcal{A}^{\mathbf{C}}$. The main point here is that $\mathcal{A}^{\mathbf{C}}$ is viewed as over $\mathcal{S}^{\mathbf{C}}$, which differs from [PS] where internal diagrams are viewed as over $\mathcal{S}$. As a category over $\mathcal{S}^{\mathbf{C}}$, $\mathcal{A}^{\mathbf{C}}$ is locally small and cocomplete if $\mathcal{A}$ has these properties over $\mathcal{S}$. Section §2 is essentially a review of Paré's paper [P2]. These results are important for Chapter 3, although the main theorem from that paper is in fact not used anywhere in this thesis (except for Eg. 1.18-2 of Chapter 3). A proof of this theorem [Th. 2.11] has been included. This proof is essentially that which is found in [P2]. The single theorem of section §4 [Th. 4.3] is of particular note because of its connection (see Chap. 4, §5) with the cocontinuous dual of a topos. This result does not, however, enter into the proof that pure geometric morphisms are of effective descent.

A principal result [Th. 1.16] coming from Chapter 3 is that for $\mathcal{E} \xrightarrow{\mathsf{p}} \mathcal{S}$ bounded, the restriction 2-functor $(\ )_{\mathsf{p}}$ has a right adjoint,

$$\textsc{Cocts}_{\mathcal{S}}(\mathcal{E},\_) : \textsc{Cocts}_{\mathcal{S}} \longrightarrow \textsc{Cocts}_{\mathcal{E}} \ . \quad \bullet$$

This is established in §1 of Chapter 3 by first [Prop. 1.1] showing that the adjointness holds for arbitrary (that is, not necessarily locally small) cocomplete categories. As an immediate consequence, one obtains at this point that which is introduced as the *change of base formula.* This is an invaluable tool. It is used, for example, to show that if $\mathcal{A}$ is locally small, then so is $\textsc{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{A})$ in the case that p is bounded [Th. 1.7], and hence one obtains • above. It is also used to show that the adjointness $(\ )_{\mathsf{p}} \dashv \textsc{Cocts}_{\mathcal{S}}(\mathcal{E},\_)$ satisfies the *Beck(-Chevalley)* condition. Consequently, the left adjoint (where defined) also satisfies the Beck condition. This is important in the treatment of descent in Chapter 4.

The principal result from [Pi] states that the tensor product (that is, the left adjoint of $(\ )_{\mathsf{p}}$) of a bounded topos with $\mathcal{E}$ exists, and coincides with the pullback as contructed in toposes. A new proof of this result is given in Chapter 3 [Th. 2.11]. The present proof differs from Pitts' in that the result is here seen to follow directly from the fact that $(\ )_{\mathsf{p}} \dashv \textsc{Cocts}_{\mathcal{S}}(\mathcal{E},\_)$ satisfies the Beck condition. In turn, the proof of the Beck condition takes advantage of the change of base formula and also of Paré's work on sheaves and generated topologies [P2]. Pitts' original proof relies on [P2] as well.

Paré [P2] has introduced the notion [Chap. 2, Def. 2.1] of a *j-sheaf* in an arbitrary category $\mathcal{A}$ over $\mathcal{S}$, where j is a arbitrary topology on $\mathcal{S}$. This coincides with the usual notion in the case that $\mathcal{A}$ is $\mathcal{S}$. Let $sh_j(\mathcal{A})$ denote the full sub-category of $\mathcal{A}$ whose objects consist of the j-sheaves. In §2 of Chapter 3, it is proved [Cor. 2.18] that under certain conditions, $sh_j(\mathcal{A})$ is the tensor product of $\mathcal{A}$ with $\mathcal{E}$ over $\mathcal{S}$, where $\mathcal{E}$ denotes the topos of j-sheaves in $\mathcal{S}$. One of these conditions is that $sh_j(\mathcal{A})$ be a reflective sub-category of $\mathcal{A}$, and this presents an avenue of further investigation perhaps leading to an improved theorem about $sh_j(\mathcal{A})$ as the tensor product. These questions are pursued no further in this thesis (in any case, Cor. 2.18 is not used in Chapter 4).

It is well known that descent can be rephrased in terms of cotripleability (dually, tripleability) if the Beck condition is satisfied. Since this is the case in the present context of locally small cocomplete categories, rather than going through a lengthy translation, the definition of an effective descent morphism is here given directly in terms of cotripleability. This is, however, 2-dimensional category theory; the base category and the fiber categories are 2-categories. Thus, a 2-dimensional cotripleability theorem is required. Given the corresponding result in the 1-dimensional case, which is well known, such a 2-dimensional theorem is readily obtainable. A 2-cotriple is herein taken to be a *strong* cotriple (see, for example, [B1]). In section §2 of Chapter 4 a cotripleability theorem [Th. 2.13] for strong cotriples is derived. This result hinges on the 'correct' definition being given of a *split equalizer* in a 2-category [Def. 1.3]. Logically prior to this is the notion of an *equalizer* in a 2-category [Def. 1.2]. An example of that notion of an equalizer which is adopted in Chapter 4 can be found in [M3]. The 2-category in that case is that of Grothendieck toposes. As in the 1-dimensional case, a split equalizer is an equalizer. Armed with this fact, the proof of strong cotripleability proceeds in a manner analogous to the 1-dimensional case. In fact, strong cotripleability subsumes the 1-dimensional theorem. Generalizing the notion of a split equalizer is that of a *semi-split* equalizer [Def. 1.5]. However, unlike a split equalizer, a semi-split equalizer is not in general an equalizer. A theorem [Th. 1.6] is then given specifying conditions on the 2-category under which a semi-split equalizer is an equalizer. This result is used in §4 to show that pure geometric morphisms are of effective descent.

The definition of a morphism of effective descent for cocomplete categories is introduced in section §3 [Def. 3.1]. There is a minor hitch here in that the tensor product is possibly not everywhere defined. However, with the help of the Beck condition, one can easily 'fix' this. This fix-up is more for the purposes of ease of exposition than anything else, and it makes available the cotripleability theorem of §2. It is then shown, by using the cotripleability theorem, that a morphism of effective descent for cocomplete categories remains so at the level of Grothendieck toposes [Th. 3.4]. Results obtained about when a category comes from a topos [Chap. 1, Ths. 6.7 and 6.8] are also used for this purpose.

Section §4 introduces the notion of a *pure* geometric morphism [Def. 4.4]. The class of pure morphisms is contained in, but distinct from, the class of surjections. A surjection which is also *locally connected* is pure, but more importantly, a spatial open surjection is pure. To conclude §4, it is shown, by using the semi-split equalizer theorem [Th. 1.6], that pure geometric morphisms are of effective descent for cocomplete categories [Th. 4.5]. Finally, as an application of the cosheaf theorem [Chap. 2, Th. 4.3], a result about the cocontinuous dual of a topos is given in §5.

CHAPTER 1

Locally Small Cocomplete Categories

1.0 Categories over $\mathcal{S}$

Let $\mathcal{S}$ denote an elementary topos. A *category over* $\mathcal{S}$ shall mean, in the terminology of [PS], an $\mathcal{S}$-indexed category.[1] Often just *a category* will be used if the base topos is clear. The 'indexed' terminology will not be used. Of course the same goes for functors and natural transformations. That is, they are functors and natural transformations *over* $\mathcal{S}$.

0.1 DEFINITION A *category* $\mathcal{A}$ *over* $\mathcal{S}$ is given by the following data:

1. for each object $I \in \mathcal{S}$, a category $\mathcal{A}^I$, sometimes referred to as the *fiber category* over I ,
2. for each morphism $I \xrightarrow{\alpha} K$ in $\mathcal{S}$, a functor $\mathcal{A}^K \xrightarrow{\alpha^*} \mathcal{A}^I$, called the *substitution functor* for α,
3. for each pair α, β of composable morphisms of $\mathcal{S}$, a natural isomorphism $\phi_{\alpha,\beta} : \beta^*\alpha^* \simeq (\alpha\beta)^*$,
4. for each $I \in \mathcal{S}$, a natural isomorphism $\tau_I : \mathbf{1}_I \simeq (1_I)^*$ of the identity functor on $\mathcal{A}^I$ with the substitution functor for $I \xrightarrow{1} I$.

This data is to be subject to the following coherence conditions:

$$\begin{array}{ccc} \gamma^*\beta^*\alpha^* & \xrightarrow{\gamma^*\phi_{\alpha,\beta}} & \gamma^*(\alpha\beta)^* \\ {\scriptstyle \phi_{\beta,\gamma}\alpha^*}\downarrow & & \downarrow{\scriptstyle \phi_{\alpha\beta,\gamma}} \\ (\beta\gamma)^*\alpha^* & \xrightarrow[\phi_{\alpha,\beta\gamma}]{} & (\alpha\beta\gamma)^* \end{array} \qquad \begin{array}{ccc} \alpha^*(1_I)^* & & \\ {\scriptstyle \phi_{1_I,\alpha}}\downarrow & \searrow{\scriptstyle \alpha^*\tau_I} & \\ (1_I\alpha)^* & \longrightarrow & \alpha^*\mathbf{1}_I \end{array}$$

where the bottom arrow in the triangle is equality. There is a third condition, one similiar to the triangle but with α^* on the right, which is a consequence of the two above (see [MP]).

A *functor over* $\mathcal{S}$, $\mathcal{A} \xrightarrow{\mathsf{F}} \mathcal{B}$, is given by the data:

1. for each object $I \in \mathcal{S}$, a functor $\mathsf{F}^I : \mathcal{A}^I \longrightarrow \mathcal{B}^I$,
2. for each morphism $K \xrightarrow{\alpha} I$, a natural isomorphism $\theta_\alpha : \alpha^*\mathsf{F}^I \simeq \mathsf{F}^K\alpha^*$,

[1] In terms of fibrations, an $\mathcal{S}$-indexed category consists of a fibration and a chosen cleavage, see [Gi2].

subject to the coherence condition

$$\begin{array}{ccc}
\beta^*\alpha^*\mathsf{F}^I & \xrightarrow{\phi_{\alpha,\beta}\mathsf{F}^I} & (\alpha\beta)^*\mathsf{F}^I \\
\beta^*\theta_\alpha \downarrow & & \\
\beta^*\mathsf{F}^L\alpha^* & & \downarrow \theta_{\alpha\beta} \\
\theta_\beta\alpha^* \downarrow & & \\
\mathsf{F}^K\beta^*\alpha^* & \xrightarrow[\mathsf{F}^K\phi_{\alpha,\beta}]{} & \mathsf{F}^K(\alpha\beta)^*,
\end{array}$$

where $K \xrightarrow{\beta} L \xrightarrow{\alpha} I$ in $\mathcal{S}$. One might also expect that the condition

$$\mathsf{F}^I\tau_I \cdot \theta_{1_I} = \tau_I\mathsf{F}^I$$

be required to hold for every $I \in \mathcal{S}$. This condition is a consequence of those coherence conditions already given (see [MP]).

A *natural transformation over* $\mathcal{S}$, $\mathsf{F} \xrightarrow{t} \mathsf{G}$, consists of for every $I \in \mathcal{S}$ a natural transformation $t^I : \mathsf{F}^I \to \mathsf{G}^I$ such that for every morphism $K \xrightarrow{\alpha} I$ in $\mathcal{S}$ one has $\alpha^*t^I = t^K\alpha^*$. This 'equality' expresses the commutivity of the following diagram.

$$\begin{array}{ccc}
\alpha^*\mathsf{F}^I & \xrightarrow{\alpha^*t^I} & \alpha^*\mathsf{G}^I \\
\theta_\alpha \downarrow & & \downarrow \theta_\alpha \\
\mathsf{F}^K\alpha^* & \xrightarrow[t^K\alpha^*]{} & \mathsf{G}^K\alpha^*
\end{array}$$

One verifies directly that categories, functors and natural transformations over $\mathcal{S}$ comprise a 2-category.

0.2 DEFINITION A functor $\mathcal{A} \xrightarrow{\mathsf{F}} \mathcal{B}$ over $\mathcal{S}$ is said to be:

1. *faithful* if for every $I \in \mathcal{S}$, F^I is faithful.

2. *fully faithful* if F^I if fully faithful for all $I \in \mathcal{S}$.

3. *essentially surjective* if for every I and every $B \in \mathcal{B}^I$ there exists an epimorphism $H \xrightarrow{\beta} I$ in $\mathcal{S}$ and an $A \in \mathcal{A}^H$ such that $\mathsf{F}^H(A) \simeq \beta^*(B)$.

4. a *weak equivalence* if F is fully faithful and essentially surjective.

An *equivalence* and an *adjointness* in the context of categories over an elementary topos shall mean those notions available in any 2-category. In the present context, it follows (see [BP]) that a functor F is an equivalence if and only if every F^I is an equivalence. See Appendix B for a review of the concept of an adjointness in the context of categories over a topos.

The topos $\mathcal{S}$ itself becomes a category over $\mathcal{S}$ by letting $\mathcal{S}^I = \mathcal{S}_{/I}$ and by defining the substitution functors to be pullback.

Following [PS], for a category $\mathcal{A}$ over $\mathcal{S}$ and for $I \in \mathcal{S}$, the *localization of* $\mathcal{A}$ *at* I shall be denoted by $\mathcal{A}_{/I}$. It is a category over $\mathcal{S}_{/I}$ with

$$(\mathcal{A}_{/I})^\alpha = \mathcal{A}^K, \text{ for } K \xrightarrow{\alpha} I .$$

For $\alpha \xrightarrow{f} \beta$ in $\mathcal{S}_{/I}$, the substitution functor for f over $\mathcal{S}_{/I}$ is by definition f^* as originally supplied by $\mathcal{A}$. Localization can be used to simplify a given demonstration if the properties and constructions involved are stable under localization.

1.1 Stacks

A stack is a '2-dimensional sheaf'. Let $\mathcal{A}$ be a category over $\mathcal{S}$, and let $H \xrightarrow{\alpha} I$ be an arbitrary morphism in $\mathcal{S}$. Let K denote the kernel pair of α . That is, let

$$\begin{array}{ccc} K & \xrightarrow{\pi_0} & H \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ H & \xrightarrow[\alpha]{} & I \end{array}$$

be a pullback. There is then the category $Des_{\mathcal{A}}(\alpha)$ defined as follows. Its objects are pairs (A, θ) where $A \in \mathcal{A}^H$ and $\pi_0^* A \xrightarrow{\theta} \pi_1^* A$ (π_0^* means of course $(\pi_0)^*$) is an isomorphism in $\mathcal{A}^K$ which satisfies the *cocycle* condition,

$$\pi_{01}^*(\theta) \cdot \pi_{12}^*(\theta) = \pi_{02}^*(\theta) \ ,$$

where π_{01}, π_{12} and π_{02} are the projections from $H \times_I H \times_I H$ to K . The appropriate canonical isomorphisms must be inserted for this 'equality' to make sense. It follows that the *unit* condition, $\delta^*(\theta) = 1_A$, is satisfied, where $H \xrightarrow{\delta} K$ is the diagonal. Conversely, given a morphism $\pi_0^* A \xrightarrow{\theta} \pi_1^* A$ satisfying the unit and cocycle conditions, it follows that θ must be an isomorphism.

The isomorphism θ is referred to as *descent data*, and so one says that the objects of $Des_{\mathcal{A}}(\alpha)$ are objects of $\mathcal{A}^H$ equipped with descent data. Morphisms in $Des_{\mathcal{A}}(\alpha)$ are by definition morphisms in $\mathcal{A}^H$ commuting with descent data.

If $A \in \mathcal{A}^I$, then $\alpha^* A$ comes equipped with canonical descent data given by the composite

$$\pi_0^* \alpha^* A \simeq (\alpha\pi_0)^* A = (\alpha\pi_1)^* A \simeq \pi_1^* \alpha^* A$$

of canonical isomorphisms. In other words, α^* factors as

$$\begin{array}{ccc} \mathcal{A}^I & \xrightarrow{\hat{\alpha}} & Des_{\mathcal{A}}(\alpha) \\ & {\scriptstyle \alpha^*} \searrow & \downarrow {\scriptstyle U} \\ & & \mathcal{A}^H \end{array}$$

where U is the forgetful functor, and $\hat{\alpha}$ sends A to $\alpha^* A$ equipped its canonical descent data.

1.1 DEFINITION $\mathcal{A}$ is said to be a *stack* if for all epimorphisms $H \overset{\alpha}{\twoheadrightarrow} I$ in $\mathcal{S}$,

$$\hat{\alpha} : \mathcal{A}^I \longrightarrow Des_{\mathcal{A}}(\alpha)$$

is an equivalence. One says in this case that objects (morphisms) in $\mathcal{A}^H$ equipped (commuting) with descent data 'descend' uniquely to $\mathcal{A}^I$.

One can speak of a category having the *stack property* with respect to a single given epimorphism.

Clearly the forgetful functor U is faithful and reflects isomorphisms, so if $\mathcal{A}$ is a stack then α^* is faithful and reflects isomorphisms for any epimorphism α .

1.2 DEFINITION $\mathcal{A}$ is said to have Σ *satisfying the Beck condition*, alternatively *small coproducts* or $\mathcal{S}$*-coproducts*, if for every morphism $I \xrightarrow{\alpha} J$, the substitution functor α^* has a left adjoint Σ_α such that if

$$\begin{array}{ccc} I \times_J K & \xrightarrow{\pi_0} & I \\ {\scriptstyle \pi_1} \downarrow & & \downarrow {\scriptstyle \alpha} \\ K & \xrightarrow[\beta]{} & J \end{array}$$

is a pullback in $\mathcal{S}$, then the canonical morphism $\Sigma_{\pi_1} \pi_0^* \rightarrow \beta^* \Sigma_\alpha$ is an isomorphism.

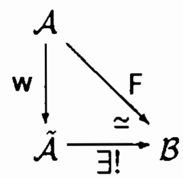
1.3 Theorem (Beck-Bénabou-Roubaud) *Assume that* $\mathcal{A}$ *has* Σ *satisfying the Beck condition. Then* $\mathcal{A}$ *is a stack if and only if for every epimorphism* $H \overset{\alpha}{\twoheadrightarrow} I$ *, the substitution functor*

$$\mathcal{A}^I \xrightarrow{\alpha^*} \mathcal{A}^H$$

is tripleable.

A proof of 1.3 can be found in [BP].

As with sheaves, one can speak of the *associated stack*, or the *stack completion* of a given category. By this is meant a stack $\tilde{\mathcal{A}}$ and a functor $\mathcal{A} \xrightarrow{\mathsf{w}} \tilde{\mathcal{A}}$ which is universal in the sense that any functor $\mathcal{A} \xrightarrow{\mathsf{F}} \mathcal{B}$, with $\mathcal{B}$ a stack, has an essentially unique factorization through w.



The following is due to M. Bunge, a proof of which can be found in [B2].

1.4 Theorem *Given $\mathcal{A} \xrightarrow{\mathsf{F}} \mathcal{B}$, then $\mathcal{B}$ is the stack completion of $\mathcal{A}$ if and only if $\mathcal{B}$ is a stack and* F *is a weak equivalence.*

1.2 Locally small categories

Let $\mathcal{A}$ be a category over $\mathcal{S}$.

2.1 Definition $\mathcal{A}$ is said to be *locally small* (or in the terminology of [PS], to have *small homs*) if for every $I \in \mathcal{S}$ and every $A, B \in \mathcal{A}^I$ there is an object $\mathcal{A}^I(A, B)$ in $\mathcal{S}_{/I}$ such that for every $K \xrightarrow{\alpha} I$ there is a bijection

$$\frac{\alpha \to \mathcal{A}^I(A,B) \text{ in } \mathcal{S}_{/I}}{\alpha^* A \to \alpha^* B \text{ in } (\mathcal{A}_{/I})^\alpha = \mathcal{A}^K}$$

which is natural in α . The object $\mathcal{A}^I(A, B)$ shall be referred to as *the object in $\mathcal{S}_{/I}$ which represents morphisms $A \to B$ in $\mathcal{A}_{/I}$.*

If $\mathcal{A}$ is locally small over $\mathcal{S}$ then $\mathcal{A}_{/I}$ is locally small over $\mathcal{S}_{/I}$, as follows directly from the definition.

The corresponding notion for a morphism between locally small categories is referred to as that of a *strong* functor. This is a functor whose action is suitably internalized. As it turns out, an arbitrary functor between locally small categories is automatically strong, as are natural transformations. For a proof of this, and for a precise formulation of these ideas, the reader is referred to [PS].

Let $\text{CAT}_\mathcal{S}$ denote the 2-category of locally small categories, functors and natural transformations over $\mathcal{S}$. For $\mathcal{A}$ and $\mathcal{B}$ in $\text{CAT}_\mathcal{S}$, let $\text{FUNCT}_\mathcal{S}(\mathcal{A}, \mathcal{B})$ denote the category whose objects are the functors from $\mathcal{A}$ to $\mathcal{B}$ over $\mathcal{S}$. The morphisms are the natural transformations over $\mathcal{S}$.

Bunge has shown (see [B2]) that the stack completion of a locally small category always exists. By using this fact, one deduces that locally small categories have the following important property.

2.2 Proposition *If $\mathcal{A}$ is locally small, then for any epimorphism $H \xrightarrow{\beta} I$, the substitution functor β^* reflects isomorphisms.*

PROOF Let $\mathcal{A} \xrightarrow{\mathbf{w}} \tilde{\mathcal{A}}$ denote the stack completion of $\mathcal{A}$, where $\mathbf{w}$ is a weak equivalence. Given $H \xrightarrow{\beta} I$, there is then the commutative square

$$\begin{array}{ccc} \mathcal{A}^I & \xrightarrow{\mathbf{w}^I} & \tilde{\mathcal{A}}^I \\ {\scriptstyle \beta^*}\downarrow & \simeq & \downarrow{\scriptstyle \beta^*} \\ \mathcal{A}^H & \xrightarrow[\mathbf{w}^H]{} & \tilde{\mathcal{A}}^H \end{array}$$

where $\mathbf{w}^I$ and $\mathbf{w}^H$ are fully faithful. With respect to $\tilde{\mathcal{A}}$, β^* reflects isomorphisms because $\tilde{\mathcal{A}}$ is a stack. It follows therefore that β^*, with respect to $\mathcal{A}$ now, reflects isomorphisms. □

1.3 Internal diagrams

Let $\mathbf{C} = (C_0, C_1)$ be an internal category in $\mathcal{S}$, an elementary topos. The notation

$$C_2 \;\overset{\pi_0}{\underset{\pi_1}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{m}}}}}\; C_1 \;\overset{\delta_0}{\underset{\delta_1}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xleftarrow{e}}}}}\; C_0$$

will be used for an internal category, where δ_0 is the domain map, δ_1 is the codomain, m is the composition of $\mathbf{C}$ and e is the 'identities' map. C_2 is the object of composable pairs of morphisms, as in the following pullback.

$$\begin{array}{ccc} C_2 & \xrightarrow{\pi_1} & C_1 \\ {\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \delta_0} \\ C_1 & \xrightarrow[\delta_1]{} & C_0 \end{array}$$

Such categories shall also be referred to as *small.* This is with respect to $\mathcal{S}$.

There is then a category over $\mathcal{S}$ associated with $\mathbf{C}$ called the externalization of $\mathbf{C}$. A typical arrow in its fiber above $I \in \mathcal{S}$ is by definition a diagram:

$$\begin{array}{ccccc} & & I & & \\ & {}^{b}\swarrow & \downarrow \beta & \searrow^{c} & \\ & & C_1 & & \\ & \swarrow_{\delta_0} & & \searrow_{\delta_1} & \\ C_0 & & & & C_0 \end{array} ,$$

such that $\delta_0\beta = b$ and $\delta_1\beta = c$. That is, β is an arrow with domain b and codomain c . For $K \xrightarrow{\alpha} I$ in $\mathcal{S}$, the substitution functor for α is defined to be composition with α . Of course, internal functors and natural transformations have their corresponding external descriptions.

No notational distinction shall be made between the internal 'world' and its externalization. One writes, for example, $\mathbf{C}^I$ to denote the fiber above I of (the externalization of) $\mathbf{C}$. This is a minor break with tradition.

The localization $\mathbf{C}_{/I}$ is isomorphic to $I^*\mathbf{C}$, so 'smallness' is stable under localization.

The topos of internal diagrams on $\mathbf{C}$ with values in $\mathcal{S}$ is denoted by

$$\mathcal{S}^{\mathbf{C}} \xrightarrow{\mathsf{C}} \mathcal{S} ,$$

where $\mathsf{C} = (\varprojlim, \mathbf{C}^*)$ is the canonical geometric morphism. Also referred to as a *discrete opfibration*, a typical object of $\mathcal{S}^{\mathbf{C}}$ shall often be written as $\begin{array}{c}\mathbf{X}\\ \downarrow \mathsf{x}\\ \mathbf{C}\end{array}$, or as $\mathsf{x} = (x_0, x_1)$, where $X_0 \xrightarrow{x_0} C_0$ is the 'rule' for objects and $X_1 \xrightarrow{x_1} C_1$ is the rule for morphisms.

3.1 Definition Let $\mathcal{A}$ be an arbitrary category over $\mathcal{S}$, and let $\mathbf{C}$ be a small category.

1. The following data comprises an ordinary category, denoted $\mathcal{A}^{\mathbf{C}}$, which is called the category of *internal diagrams on* $\mathbf{C}$ *with values in* $\mathcal{A}$. Its objects are pairs (A, θ) , where $A \in \mathcal{A}^{C_0}$ and $\delta_0^* A \xrightarrow{\theta} \delta_1^* A$ in $\mathcal{A}^{C_1}$ is the 'action' map which is required to satisfy:

   (a) $e^*(\theta) = 1_A$ (preservation of identities)

(b) $\pi_1^*(\theta) \cdot \pi_0^*(\theta) = m^*(\theta)$ (preservation of composition).

The appropriate canonical isomorphisms must be inserted for these equalities to make sense. Morphisms $(A, \theta) \xrightarrow{f} (B, \phi)$ are morphisms $A \xrightarrow{f} B$ in $\mathcal{A}^{C_0}$ commuting with the action maps.

2. $\mathcal{A}^{\mathbf{C}}$ is regarded as a category over $\mathcal{S}$ by defining the fiber above $I \in \mathcal{S}$ to be

$$(\mathcal{A}^{\mathbf{C}})^I = \mathcal{A}^{\mathbf{C} \times I} ,$$

where $\mathcal{A}^{\mathbf{C} \times I}$ pertains to the first part of the definition. For $I \xrightarrow{\alpha} K$ in $\mathcal{S}$, the substitution functor

$$(\mathcal{A}^{\mathbf{C}})^K \xrightarrow{\alpha^*} (\mathcal{A}^{\mathbf{C}})^I$$

is defined to be

$$\alpha^* : (A, \theta) \rightsquigarrow ((1_{C_0} \times \alpha)^* A, ((1_{C_1} \times \alpha)^* \theta) .$$

This definition extends in the obvious way to morphisms of $(\mathcal{A}^{\mathbf{C}})^K$.

3.2 Examples

1. With $\mathcal{A} = \mathcal{S}$ in the above definition, one obtains the topos of discrete opfibrations previously mentioned.

2. Let $H \xrightarrow{\beta} I$ be an epimorphism in $\mathcal{S}$, and let H_1 denote the kernel pair of β . Then $\mathbf{H}_\beta = (H, H_1)$ is a small category, and there is a functor $\mathbf{H}_\beta \xrightarrow{\beta} \mathbf{I}$, regarding $\mathbf{I}$ as a discrete category. Then $Des_{\mathcal{A}}(\beta)$ is equivalent to $\mathcal{A}^{\mathbf{H}_\beta}$, and $\mathcal{A}$ has the stack property with respect to β if and only if the induced functor

$$\beta^* : \mathcal{A}^{\mathbf{I}} \longrightarrow \mathcal{A}^{\mathbf{H}_\beta}$$

is an equivalence.

As is well known, along with internal categories and functors, an internal diagram has its external description. That is, there is an isomorphism

$$\mathcal{A}^{\mathbf{C}} \simeq \text{FUNCT}_{\mathcal{S}}(\mathbf{C}, \mathcal{A}) ,$$

of categories over $\mathcal{S}$ which identifies the substitution functor α^* with composition with (the externalization of) α .

It is not hard to check that the construction $\mathcal{A}^{\mathbf{C}}$ is stable under, or commutes with, localization over $\mathcal{S}$. By this is meant, in this case anyway, that for any $I \in \mathcal{S}$, there is a canonical equivalence

$$(\mathcal{A}_{/I})^{\mathbf{C}_{/I}} \cong \mathcal{A}^{\mathbf{C}}{}_{/I} ,$$

of categories over $\mathcal{S}_{/I}$.

This discussion on internal diagrams is continued in Chapter 2 where they are realized as a category not over $\mathcal{S}$, but over $\mathcal{S}^{\mathbf{C}}$.

1.4 Cocomplete categories

Let $\mathcal{A}$ be an arbitrary category over $\mathcal{S}$. For any small category $\mathbf{C}$ there is the constancy functor

$$\mathsf{C}^* : \mathcal{A} \longrightarrow \mathcal{A}^{\mathbf{C}} ,$$

which sends $A \in \mathcal{A}$ to $(C_0^*(A), C_1^*(1_A))$. A finite colimit in the fiber $\mathcal{A}^I$ is said to be *stable* if it is preserved by the substitution functors.

4.1 DEFINITION $\mathcal{A}$ is said to be *cocomplete* if for all $I \in \mathcal{S}$:

1. the fiber $\mathcal{A}^I$ has finite stable colimits, and

2. for all small categories $\mathbf{D}$ in $\mathcal{S}_{/I}$, the constancy functor

$$\mathsf{D}^* : \mathcal{A}_{/I} \longrightarrow (\mathcal{A}_{/I})^{\mathbf{D}}$$

   has a left adjoint (over $\mathcal{S}_{/I}$).

Cocompleteness is stable under localization as follows directly from the definition.

Definition 4.1 is rather troublesome to work with as it is often easier to handle colimits in terms of coproducts and coequalizers. The notion of coproducts has already been defined (definition 1.2), where it was also referred to as 'Σ satisfying the Beck condition'. A proof of the following theorem can be found in [PS].

4.2 Theorem *$\mathcal{A}$ is cocomplete if and only if $\mathcal{A}$ has Σ satisfying the Beck condition, and for every $I \in \mathcal{S}$ the fiber $\mathcal{A}^I$ has stable finite colimits.*

4.3 Example Let $\mathbf{M} = (M, M_1)$ be an internal poset. The map

$$\downarrow seg : M \longrightarrow \Omega^M$$

is by definition the exponential transpose of the classifying map of the sub-object $M_1 \overset{(\delta_0,\delta_1)}{\hookrightarrow} M\times M$. Then $\downarrow seg$ is a poset map and $\mathbf{M}$ is, by definition, an internal *sup-lattice* if $\downarrow seg$ has a left adjoint. If such is the case the left adjoint is denoted by

$$\bigvee : \Omega^M \to M .$$

This concept is preserved under pullback. That is, for $I \in \mathcal{S}$, $I^*(\downarrow seg)$ is equal to $\downarrow seg_{I^*\mathbf{M}}$, and therefore $I^*\mathbf{M}$ is a sup-lattice with $\bigvee_{I^*\mathbf{M}} = I^*(\bigvee_{\mathbf{M}})$.

To be shown in this example is that *a small poset* $\mathbf{M}$ *is an internal sup-lattice if and only if* $\mathbf{M}$ *is cocomplete regarded as a category over* $\mathcal{S}$. This result goes back to Bénabou, and a proof of it using the adjoint functor theorems can be found in [PS]. The methods used here allow the result to be extended. It will also be shown here that *an internal sup-lattice is a* locale *if and only if it satisfies* Frobenius reciprocity *as a category over* $\mathcal{S}$. See section §6 of this chapter for the definition of 'Frobenius reciprocity'.

To show the first claim, assume first that $\mathbf{M}$ is an internal sup-lattice. One can calculate finite supremums in $\mathbf{M}$ as,

$$M\times M \overset{\vee}{\to} M$$

$$(m,n) \rightsquigarrow \bigvee\{y \mid \forall x(((x\geq n)\wedge(x\geq m)) \Rightarrow x\geq y)\} ,$$

and then for $m,n \in \mathbf{M}^I$, $m\vee n$ is

$$I \overset{(m,n)}{\to} M\times M \overset{\vee}{\to} M$$

in $\mathbf{M}^I$.

For $I \in \mathcal{S}$, by definition the substitution functor

$$I^* : \mathbf{M}^1 \longrightarrow \mathbf{M}^I$$

sends $1 \overset{n}{\to} M$ to $I \to 1 \overset{n}{\to} M$. Define

$$\Sigma_I : \mathbf{M}^I \longrightarrow \mathbf{M}$$

by letting $\Sigma_I(I \overset{m}{\to} M)$ be the composite

$$1 \overset{\sigma_m}{\to} \Omega^M \overset{\bigvee}{\to} M ,$$

where σ_m is the exponential transpose of the classifying map of (the image of) m . To be shown is that $\Sigma_I \dashv I^*$. For $n \in \mathbf{M}^1$ and $m \in \mathbf{M}^I$ one has $\Sigma_I(m) = \bigvee\cdot\sigma_m \leq n$ if and only if $\sigma_m \leq \downarrow seg\cdot n$ if and only if m factors through the sub-object of $\mathbf{M}$ corresponding to $\downarrow seg \cdot n$. Let us denote that sub-object by $\downarrow seg(n) \hookrightarrow M$. Observe that in the commutative diagram

$$\begin{array}{ccccc}
\downarrow seg(n) & \longrightarrow & M_1 & \longrightarrow & 1 \\
\big\downarrow & & \big\downarrow (\delta_0,\delta_1) & & \big\downarrow t \\
M\times 1 & \xrightarrow[1\times n]{} & M\times M & \xrightarrow[\widetilde{\downarrow seg}]{} & \Omega
\end{array}$$

the left square is a pullback since the right and outer squares are. Now $M_1 \overset{(\delta_0,\delta_1)}{\hookrightarrow} M\times M \underset{\pi_1}{\overset{\vee}{\rightrightarrows}} M$ is an equalizer, and therefore m factors through $\downarrow seg(n)$ if and only if

$$I \overset{m}{\to} M\times 1 \overset{1\times n}{\to} M\times M \underset{\pi_1}{\overset{\vee}{\rightrightarrows}} M$$

commutes, which is true if and only if $m \leq I^*(n)$. This proves that $\Sigma_I \dashv I^*$. To get Σ_α in general, for $K \overset{\alpha}{\to} I$, one can proceed by localizing. It is not difficult to then verify that the Beck condition holds.

Assume now that $\mathbf{M}$ is cocomplete in the external sense. Our aim is to exibit a poset map $\bigvee : \Omega^M \to M$, which is left adjoint to $\downarrow seg$. The method of the generic element can be used to do this. Let Z denote Ω^M. Then $Z^*\mathbf{M} = \mathbf{M}_{/Z}$ is a poset in $\mathcal{S}_{/Z}$ which is also cocomplete in the external sense. There is the 'generic' global section

$$1 \overset{\delta}{\to} Z^*(Z) \simeq (Z^*\Omega)^{Z^*M}$$

in $\mathcal{S}_{/Z}$, which gives us a sub-object $s \overset{i}{\hookrightarrow} Z^*M$. Since $\mathbf{M}$ is assumed to be cocomplete, there is given

$$\Sigma_s : (Z^*\mathbf{M})^s \longrightarrow (Z^*\mathbf{M})^1 ,$$

the left adjoint of s^* over $\mathcal{S}_{/Z}$. This gives the global section

$$\Sigma_s(i) : 1 \to Z^*M .$$

Define $\bigvee : \Omega^M \to M$ to be the transpose of $\Sigma_s(i)$ with respect to $\Sigma_Z \dashv Z^*$. To be verified now is that $\bigvee \dashv \downarrow seg$. Given generalized elements $I \overset{\alpha}{\to} \Omega^M = Z$ and $I \overset{x}{\to} M$, one wants to show that

$$\frac{\bigvee \cdot \alpha \leq x}{\alpha \leq \downarrow seg \cdot x} ,$$

which, upon transposing to $\mathcal{S}_{/Z}$, is true if and only if

$$\frac{Z^*(\bigvee) \cdot \delta \cdot !_\alpha \leq \dot{x}}{\delta \cdot !_\alpha \leq \downarrow seg_{Z^*\mathbf{M}} \cdot \dot{x}} ,$$

where $!_\alpha$ is the unique map $\alpha \to 1$, and where $\alpha \overset{\dot{x}}{\to} Z^*M$ denotes the transpose of x in $\mathcal{S}_{/Z}$. By the definition of the morphism $\bigvee$, one has

$$Z^*(\bigvee) \cdot \delta = \Sigma_s(i) \ ,$$

and therefore

$$Z^*(\bigvee) \cdot \delta{\cdot}!_\alpha = \alpha^*\Sigma_s(i) = \Sigma_{\pi_0}\pi_1^*(i) \ ,$$

where

$$\begin{array}{ccc} \alpha \times s & \xrightarrow{\pi_1} & s \\ {\scriptstyle \pi_0}\downarrow & & \downarrow \\ \alpha & \longrightarrow & 1 \end{array}$$

is a pullback in $\mathcal{S}_{/Z}$. Hence

$$\frac{\dfrac{\dfrac{Z^*(\bigvee) \cdot \delta{\cdot}!_\alpha \leq \dot{x}}{\Sigma_{\pi_0}\pi_1^*(i) \leq \dot{x}}}{\pi_1^*(i) \leq \pi_0^*(\dot{x})}}{\delta{\cdot}!_\alpha \leq\downarrow seg_{Z^*\mathrm{M}} \cdot \dot{x}} \ .$$

The last equivalence in the above series could use some clarification. As interpreted in $Z^*\mathrm{M}$, $\pi_1^*(i) = i \cdot \pi_1$ and $\pi_0^*(\dot{x}) = \dot{x} \cdot \pi_0$. Transposing to $(\mathcal{S}_{/Z})_{/\alpha}$, with respect to $\Sigma_\alpha \dashv \alpha^*$, produces

$$\frac{\pi_1^*(i) \leq \pi_0^*(\dot{x}) \text{ in } \mathcal{S}_{/Z}}{\widehat{i \cdot \pi_1} \leq \pi_0^*(\hat{\dot{x}}) \text{ in } (\mathcal{S}_{/Z})_{/\alpha} \ ,}$$

where

$$\widehat{i \cdot \pi_1} : \pi_0 \rightarrow \alpha^* Z^* M$$

and

$$\pi_0^*(\hat{\dot{x}}) : \pi_0 \rightarrow 1 \xrightarrow{\hat{\dot{x}}} \alpha^* Z^* M$$

are generalized elements of $\alpha^* Z^*\mathrm{M}$ in $(\mathcal{S}_{/Z})_{/\alpha}$. By a previous argument (the one which showed that $\sigma_m \leq\downarrow seg \cdot n$ if and only if $m \leq I^*(n)$) one obtains

$$\frac{\widehat{i \cdot \pi_1} \leq \pi_0^*(\hat{\dot{x}})}{\sigma \leq\downarrow seg_{(Z\alpha)^*M} \cdot \hat{\dot{x}} \ ,}$$

σ being the exponential transpose of the classifying map of $\widehat{i \cdot \pi_1}$. That σ corresponds to $\delta{\cdot}!_\alpha$ under $\Sigma_\alpha \dashv \alpha^*$, thereby concluding the proof of the first claim, is by the following lemma. Its proof is left to the reader.

Lemma Given a pullback

$$\begin{array}{ccc} A\times B & \xrightarrow{\pi_1} & B \\ {\scriptstyle \pi_0}\downarrow & & \downarrow \\ A & \longrightarrow & 1 \end{array}$$

in a topos $\mathcal{T}$, and a sub-object $B \overset{b}{\hookrightarrow} X$, let $1 \xrightarrow{\sigma} A^*(\Omega^X)$ denote the exponential transpose of the classifying map of the sub-object $\pi_0 \overset{\widehat{b\pi_1}}{\hookrightarrow} A^*X$, where $\frown$ means transpose under $\Sigma_A \dashv A^*$. Then $\sigma = \widehat{\beta \cdot !_A}$, where $1 \xrightarrow{\beta} \Omega^X$ is the exponential transpose of the classifying map of b .

To apply the lemma, take $\mathcal{T}$ to be $\mathcal{S}_{/Z}$, b to be i , and A to be α .

The above methods can be used to prove that an internal sup-lattice is a *locale* if and only if it satisfies *Frobenius reciprocity* (see §6 of this chapter). An internal sup-lattice $\mathbf{M} = (M, M_1)$ is by definition a *locale* if

$$\begin{array}{ccc} \Omega^M \times M & \xrightarrow{\bigvee \times 1} & M \times M \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \wedge} \\ \Omega^M & \xrightarrow[\bigvee]{} & M \end{array}$$

commutes, where

$$\tau(Y, x) = \{\, x \wedge y \mid y \in Y \,\}, \ (Y, x) \in \Omega^M \times M .$$

It is not hard to see that if $\mathbf{M}$ is a locale, then as a category over $\mathcal{S}$, $\mathbf{M}$ satisfies Frobenius reciprocity.

Let us assume that $\mathbf{M}$ satisfies Frobenius reciprocity, and show that $\mathbf{M}$ is a locale. One always has $\bigvee \cdot \tau \leq \wedge \cdot (\bigvee \times 1)$, and so it suffices to show that

$$(4.4) \qquad \wedge \cdot (\bigvee \times 1) \leq \bigvee \cdot \tau .$$

As before, let Z denote Ω^M . Let $\widehat{\ }$ denote transpositon with respect to $\Sigma_Z \dashv Z^*$. Transposing to $\mathcal{S}_{/Z}$, 4.4 is true if and only if

$$(4.5) \qquad \wedge \cdot \widehat{(\bigvee \times 1)} \leq \widehat{\bigvee \cdot \tau} .$$

Writing Z^*M as $Z \times M \xrightarrow{p} Z$, note that 4.5 is a statement about elements of Z^*M at stage p . Form the product

$$\begin{array}{ccc} p\times s & \xrightarrow{\pi_1} & s \\ {\scriptstyle \pi_0}\downarrow & & \downarrow \\ p & \longrightarrow & 1 \end{array}$$

in $\mathcal{S}_{/Z}$. We have the two 'elements'

$$s \overset{i}{\hookrightarrow} Z^*M \ , \ p \overset{1}{\rightarrow} Z^*M$$

of Z^*M , and it follows that

$$\wedge \cdot (\widehat{\bigvee \times 1}) = 1 \wedge p^*\Sigma_s(i) \, .$$

By using the Beck condition and Frobenius reciprocity, $1 \wedge p^*\Sigma_s(i)$ is equal to

$$1 \wedge \Sigma_{\pi_0}\pi_1^*(i) = \Sigma_{\pi_0}(\pi_0^*1 \wedge \pi_1^*i) \, .$$

Thus, the question comes down to showing that

$$(4.6) \qquad \pi_0^*1 \wedge \pi_1^*i \leq \pi_0^*(\widehat{\bigvee \cdot \tau}) \ ,$$

as elements of Z^*M at stage $p\times s$ in $\mathcal{S}_{/Z}$. The idea is to transpose 4.6 back to $\mathcal{S}$ and verify the resulting inequality there. The element $\pi_0^*1 \wedge \pi_1^*i$ is

$$p\times s \overset{(\pi_0, i\pi_1)}{\longrightarrow} Z^*M \times Z^*M \overset{\wedge}{\rightarrow} Z^*M \ ,$$

and its transpose is

$$M\times \in \xrightarrow{1\times \hat{i}} M \times M \overset{\wedge}{\rightarrow} M \ .$$

Note that the transpose of i is

$$\hat{i} : \in \hookrightarrow \Omega^M \times M \rightarrow M \ ,$$

where $\in$ is that sub-object classified by the 'evaluation' map $\Omega^M \times M \rightarrow \Omega$. Also note that

$$\Sigma_Z(p\times s) = (\Omega^M \times M) \times_{\Omega^M} \in = M\times \in \ .$$

Hence,

$$\wedge \cdot (1\times \hat{i})\,(x, (Y, y)) = x \wedge y \ ,$$

where $y \in Y$. The element $\pi_0^*(\widehat{\bigvee \cdot \tau})$ is

$$p\times s \overset{\pi_0}{\rightarrow} p \xrightarrow{\widehat{\bigvee \cdot \tau}} Z^*M \ ,$$

and its transpose is

$$M\times \in \xrightarrow{\Sigma_Z(\pi_0)} \Omega^M \times M \xrightarrow{\bigvee\cdot\tau} M \ .$$

Note that $\Sigma_Z(\pi_0)(x,(Y,y)) = (Y,x)$, and hence,

$$\bigvee\cdot\tau\cdot\Sigma_Z(\pi_0)\,(x,(Y,y)) = \bigvee x\wedge v \ ,$$

as v runs over Y . Thus,

$$\wedge\cdot(1\times\hat{\imath}) \leq \bigvee\cdot\tau\cdot\Sigma_Z(\pi_0) \ .$$

This proves 4.6, which concludes the proof that **M** is a locale. This concludes this example.

A functor $\mathcal{A}\xrightarrow{\mathsf{F}}\mathcal{B}$ shall be said to be *cocontinuous* if it preserves all those small colimits which may exist in $\mathcal{A}$. Most often it will be the case that $\mathcal{A}$ and $\mathcal{B}$ are cocomplete, and if such is the case, then F is cocontinuous if and only if F commutes with the Σ's, and for every $I\in\mathcal{S}$, F^I preserves finite colimits. Denote by $\text{C}\textsc{octs}_\mathcal{S}(\mathcal{A},\mathcal{B})$ the full sub-category of $\text{F}\textsc{unct}_\mathcal{S}(\mathcal{A},\mathcal{B})$ whose objects are those functors from $\mathcal{A}$ to $\mathcal{B}$ that are cocontinuous.

Let $\text{C}\textsc{octs}_\mathcal{S}$ denote the 2-category whose objects (or 0-cells) are the locally small cocomplete categories over $\mathcal{S}$. For $\mathcal{A}$ and $\mathcal{B}$ in $\text{C}\textsc{octs}_\mathcal{S}$, the category of 1-cells and 2-cells from $\mathcal{A}$ to $\mathcal{B}$ is $\text{C}\textsc{octs}_\mathcal{S}(\mathcal{A},\mathcal{B})$, as above.

An important fact about $\text{C}\textsc{octs}_\mathcal{S}$ is that its objects are stacks.

4.7 Theorem *Any locally small cocomplete category is a stack.*

PROOF Let $\mathcal{A}\in\text{C}\textsc{octs}_\mathcal{S}$. Since $\mathcal{A}$ has Σ satisfying the Beck condition, $\mathcal{A}$ is a stack if and only if for every epimorphism $I\xrightarrow{\alpha}K$, the substitution functor

$$\mathcal{A}^K\xrightarrow{\alpha^*}\mathcal{A}^I$$

is tripleable (1.3). Now α^* has a left adjoint, and recall (2.2) that since $\mathcal{A}$ is locally small, α^* reflects isomorphisms. Moreover, $\mathcal{A}^K$ has all coequalizers and α^* preserves them. So the result follows by Beck's tripleability theorem. □

4.8 Corollary *Let $\mathcal{A}$ and $\mathcal{B}$ be objects of* $\text{C}\textsc{octs}_\mathcal{S}$. *Then any weak equivalence $\mathcal{A}\xrightarrow{\mathsf{W}}\mathcal{B}$ is an equivalence.*

PROOF This follows by 4.7 and 1.4. □

This section is concluded with a special mention of a certain type of colimit.

4.9 Definition $\mathcal{A}$ is said to have *small copowers* if:

1. For any $I \xrightarrow{\alpha} J \in \mathcal{S}$ and any $A \in \mathcal{A}^J$, there exists an object $\alpha.A$, the *α-copower* of A, such that for any $B \in \mathcal{A}^J$, there is a bijection:

$$\frac{\alpha.A \to B}{\alpha^* A \to \alpha^* B\,,}$$

natural in B. Let $\alpha^* A \xrightarrow{\delta_A} \alpha^*(\alpha.A)$ (called the diagonal) denote the morphism corresponding to $\alpha.A \xrightarrow{1} \alpha.A$.

2. The copowers $\alpha.A$ are *stable*. By this is meant that the morphism, call it μ, arising from the following series of bijections:

$$\begin{array}{c} (I\times K).A \xrightarrow{\pi_0.A} I.A \\ \hline (I\times K)^* A \to (I\times K)^* I.A \\ \hline \pi_1^* K^* A \to \pi_1^* K^* I.A \\ \hline \pi_1.(K^* A) \xrightarrow{\mu} K^*(I.A) \end{array}$$

is an isomorphism for any $K \in \mathcal{S}$, where $I\times K \xrightarrow{\pi_0} I$ and $I \times K \xrightarrow{\pi_1} K$ are the projections. (Actually, what it means for the copower $I.A \in \mathcal{A}^1$ to be stable under K^* has been defined here.)

Given $H \xrightarrow{\alpha} K$, the morphism $H.A \xrightarrow{\alpha.A} K.A$ is by definition that morphism corresponding to $\alpha^*\delta_A$, plus some canonical isomorphisms.

If $\mathcal{A}$ has coproducts, then $\mathcal{A}$ has copowers with $\alpha.A = \Sigma_\alpha \alpha^* A$. Furthermore, in this case $H.A \xrightarrow{\alpha.A} K.A$ can be calculated (to within canonical isomorphism) as $\Sigma_K(\varepsilon_{K^* A})$, where ε is the counit of $\Sigma_\alpha \dashv \alpha^*$. In particular, if α^* is fully faithful, then $\alpha.A$ is an isomorphism.

A category over $\mathcal{S}$ possessing small copowers is in some sense an '$\mathcal{S}$-module', the 'action' being given by copowers.

Any $A \in \mathcal{A}^1$ determines an ordinary functor from $\mathcal{S}^J$ to $\mathcal{A}^J$, $\alpha \rightsquigarrow \alpha.J^* A$, which shall be denoted as $(\Phi A)^J$. Then by stability, the functors $(\Phi A)^J$ define a functor *over* $\mathcal{S}$,

$$\Phi A : \mathcal{S} \longrightarrow \mathcal{A}\,.$$

Moreover, ΦA is easily seen to be cocontinuous. Similarly, any $A \in \mathcal{A}^K$ defines a cocontinuous functor

$$\Phi^K A : \mathcal{S}_{/K} \longrightarrow \mathcal{A}_{/K}$$

over $\mathcal{S}_{/K}$.

The following simple fact from [Pi] is of basic importance.

4.10 Proposition *Let $\mathcal{A}$ be an arbitrary category over $\mathcal{S}$, and assume that $\mathcal{A}$ has small copowers. Then the passage $A \rightsquigarrow \Phi A$ is an equivalence $\mathcal{A} \cong \mathrm{C}\text{OCTS}_{\mathcal{S}}(\mathcal{S}, \mathcal{A})$, which is natural in $\mathcal{A}$.*

PROOF The map $A \rightsquigarrow \Phi A$ defined above is obviously functorial. In fact, given $A \xrightarrow{f} B$ one defines a natural transformation Φf as follows. At 1, for example, let $(\Phi f)^1_X = X.f$, the morphism corresponding to $\delta_B \cdot X^* f$,

$$\frac{X^*A \xrightarrow{X^*f} X^*B \xrightarrow{\delta_B} X^*X.B}{X.A \xrightarrow{X.f} X.B} ,$$

for $X \in \mathcal{S}$. Going the other way, define a functor Ψ by letting $\Psi \mathsf{F} = \mathsf{F}(1)$. One then routinely verifies that $\Phi \cdot \Psi \simeq \mathbf{1}$, and that $\Psi \cdot \Phi \simeq \mathbf{1}$. For example, if $\mathsf{F} \in \mathrm{C}\text{OCTS}_{\mathcal{S}}(\mathcal{S}, \mathcal{A})$, then

$$\Phi \cdot \Psi(\mathsf{F})(X) = X.\mathsf{F}(1) \simeq \mathsf{F}(X)$$

for $X \in \mathcal{S}$. □

The equivalence of 4.10 is an equivalence over $\mathcal{S}$, where by definition

$$\mathrm{C}\text{OCTS}_{\mathcal{S}}(\mathcal{S}, \mathcal{A})^I = \mathrm{C}\text{OCTS}_{\mathcal{S}_{/I}}(\mathcal{S}_{/I}, \mathcal{A}_{/I}),\ I \in \mathcal{S} .$$

In the future, the notation ΦA shall not be used to denote the cocontinuous functor corresponding to A , instead simply A shall be used.

Let $\mathcal{A} \in \mathrm{C}\text{AT}_{\mathcal{S}}$, the 2-category of locally small categories over $\mathcal{S}$. Then for any $A \in \mathcal{A}$ there is the 'hom' functor:

$$(A, \_) : \mathcal{A} \longrightarrow \mathcal{S} ,$$

which has a left adjoint if and only if $\mathcal{A}$ has copowers of A . That is, one has $\mathsf{A} \dashv (A, \_)$.

1.5 Restriction of scalars

Let $\mathcal{E}$ be an arbitrary topos, and let $\mathcal{E} \xrightarrow{\mathsf{p}} \mathcal{S}$ be an arbitrary geometric morphism. One should think of $\mathcal{E}$ as an 'extension' of $\mathcal{S}$, rather that as an 'object' over $\mathcal{S}$. If $\mathcal{B}$ is a category over $\mathcal{E}$, then one can *restrict* $\mathcal{B}$ along p thereby obtaining a category over $\mathcal{S}$ which shall be denoted by $\mathcal{B}_{\mathsf{p}}$. Let us write $\mathcal{B}^I_{\mathsf{p}}$ for the fiber above I of $\mathcal{B}_{\mathsf{p}}$. By definition,

$$\begin{array}{ccc} \mathcal{B}_{\mathsf{p}}^{I} & \xrightarrow{\alpha_{\mathsf{p}}^{*}} & \mathcal{B}_{\mathsf{p}}^{K} \\ & \| & \\ \mathcal{B}^{\mathsf{p}^*I} & \xrightarrow{(\mathsf{p}^*\alpha)^*} & \mathcal{B}^{\mathsf{p}^*K} \end{array}$$

for $K \xrightarrow{\alpha} I$ in $\mathcal{S}$. Similarly, for a functor $\mathcal{B} \xrightarrow{\mathsf{F}} \mathcal{C}$ over $\mathcal{E}$, define

$$\mathsf{F}_{\mathsf{p}}^{I} = \mathsf{F}^{\mathsf{p}^*I} .$$

For a natural transformation $\mathsf{F} \xrightarrow{t} \mathsf{G}$, let

$$t_{\mathsf{p}}^{I} = t^{\mathsf{p}^*I} ,$$

where $I \in \mathcal{S}$. It is quite routine to verify that this is all well defined over $\mathcal{S}$.

Next, observe that if $\mathcal{B}$ is locally small over $\mathcal{E}$, then $\mathcal{B}_{\mathsf{p}}$ is locally small over $\mathcal{S}$. In fact, given $I \in \mathcal{S}$, let $\gamma = \mathcal{B}^{\mathsf{p}^*I}(B, C)$, the object in $\mathcal{E}_{/\mathsf{p}^*I}$ which represents morphisms $B \rightarrow C$ in $\mathcal{B}_{/\mathsf{p}^*I}$. Then the left side of the pullback

$$\begin{array}{ccc} P & \longrightarrow & \mathsf{p}_*\square \\ \downarrow & & \downarrow \mathsf{p}_*\gamma \\ I & \longrightarrow & \mathsf{p}_*\mathsf{p}^*I \end{array}$$

is the object in $\mathcal{S}_{/I}$ which represents morphisms $B \rightarrow C$ in $(\mathcal{B}_{\mathsf{p}})_{/I}$. The bottom morphism of this pullback is the unit of $\mathsf{p}^* \dashv \mathsf{p}_*$. There is thus a 2-functor

$$\begin{array}{rcl} (\ )_{\mathsf{p}} : \mathrm{CAT}_{\mathcal{E}} & \longrightarrow & \mathrm{CAT}_{\mathcal{S}} \\ \mathcal{B} & \rightsquigarrow & \mathcal{B}_{\mathsf{p}} \end{array}$$

which is referred to as the *restriction* functor, or as the *restriction of scalars along* p.

If $\mathcal{B}$ is a cocomplete category over $\mathcal{E}$, then $\mathcal{B}_{\mathsf{p}}$ is cocomplete over $\mathcal{S}$. This follows by 4.2 and since p^* is left exact. Also, if F is a cocontinuous functor between *cocomplete* categories over $\mathcal{E}$, then F_{p} is cocontinuous. This gives us a 2-functor

$$(\ )_{\mathsf{p}} : \mathrm{COCTS}_{\mathcal{E}} \longrightarrow \mathrm{COCTS}_{\mathcal{S}} .$$

5.1 Examples

1. Let $\mathcal{A}$ be a category over $\mathcal{S}$, and let $I \in \mathcal{S}$. The notation used to denote the fiber above I , $\mathcal{A}^I$, shall also be used to denote the restriction along $\mathcal{S}_{/I} \xrightarrow{\mathsf{I}} \mathcal{S}$ of the localization $\mathcal{A}_{/I}$. That is, by definition

$$\mathcal{A}^I = (\mathcal{A}_{/I})_{\mathsf{I}} .$$

The category $\mathcal{A}^I$ is over $\mathcal{S}$. This notation follows [PS], although $\mathcal{A}^I$ was not introduced there as $(\mathcal{A}_{/I})_{\mathsf{I}}$. Observe that if $\mathcal{A}$ is locally small (cocomplete) then $\mathcal{A}^I$ is locally small (cocomplete).

2. Given a geometric morphism $\mathcal{E} \xrightarrow{\mathsf{p}} \mathcal{S}$ and a topos over $\mathcal{E}$, say $\mathcal{F} \xrightarrow{\mathsf{f}} \mathcal{E}$ (see the next section for the details of how a topos over $\mathcal{S}$ is viewed as a category over $\mathcal{S}$), then the restriction $\mathcal{F}_{\mathsf{p}}$ is obtained by composing with p . For toposes, let us just write $\mathcal{F}$ again when it is clear that the restriction is along p .

3. If $\mathbf{C}$ is a small category in $\mathcal{E}$, then $\mathbf{C}_{\mathsf{p}}$ is small in $\mathcal{S}$. In fact, $\mathbf{C}_{\mathsf{p}}$ is isomorphic to $\mathsf{p}_*\mathbf{C}$.

1.6 Toposes as cocomplete categories

$\textsc{Top}_{\mathcal{S}}$ shall denote the 2-category of toposes over $\mathcal{S}$. A typical object in $\textsc{Top}_{\mathcal{S}}$ shall be written

$$\begin{array}{c}\mathcal{F}\\ \downarrow \mathsf{f}\\ \mathcal{S}\end{array}$$

where $\mathcal{F}$ is a topos and $\mathsf{f} = (\mathsf{f}_*, \mathsf{f}^*)$ is a geometric morphism. A morphism between toposes over $\mathcal{S}$, say from $\mathcal{F}$ to $\mathcal{H}$, is a pair (k, a) where $\mathcal{F} \xrightarrow{\mathsf{k}} \mathcal{H}$ is a geometric morphism and $\mathsf{h} \cdot \mathsf{k} \overset{a}{\simeq} \mathsf{f}$ is a natural isomorphism. A 2-cell of such morphisms $(\mathsf{k}, a) \xrightarrow{t} (\mathsf{l}, b)$, is a natural transformation $\mathsf{k}^* \xrightarrow{t} \mathsf{l}^*$ (equivalently, a natural transformation $\mathsf{k}_* \xrightarrow{\hat{t}} \mathsf{l}_*$) such that $t\mathsf{h}^* \cdot a = b$.

Given a topos over $\mathcal{S}$ as above, the topos $\mathcal{F}$ is in particular a category over itself, in fact a locally small cocomplete one. Hence, its restriction along f gives rise to a locally small cocomplete category over $\mathcal{S}$. If morphisms between toposes are sent to their inverse images, then this passage extends to a 2-functor

$$\wp : \textsc{Top}_{\mathcal{S}}{}^{op} \longrightarrow \textsc{Cocts}_{\mathcal{S}} .$$

Upon giving it a name, let us write instead of $\wp\mathcal{F}$ just $\mathcal{F}$ again to denote the category over $\mathcal{S}$ coming from the topos $\mathcal{F}$. Explicitly then, as a category over $\mathcal{S}$,

$$\mathcal{F}^I = \mathcal{F}_{/\mathsf{f}^*I} \; .$$

For a morphism α in $\mathcal{S}$, the substitution functor is given by pulling back along $\mathsf{f}^*\alpha$. Morphisms $(\mathsf{k}, a) : \mathcal{F} \longrightarrow \mathcal{H}$ are sent by $\wp$ to the cocontinuous functor $\wp(\mathsf{k}, a)$ such that

$$\wp(\mathsf{k}, a)^I(x) = a^I \cdot \mathsf{k}^*x \; ,$$

where $X \xrightarrow{x} \mathsf{h}^*I$ is a typical object of $\mathcal{H}^I$. This definition extends in an obvious way to morphisms in $\mathcal{H}^I$. The notation $\wp(\mathsf{k}, a)$ is unnecessary, so let us just write k^* to denote the cocontinuous functor coming from the geometric morphism (k, a) . It is easy to see that as a functor over $\mathcal{S}$, k^* has a right adjoint over $\mathcal{S}$.

Lastly, for natural transformations $(\mathsf{k}, a) \xrightarrow{t} (\mathsf{l}, b)$ over $\mathcal{S}$, let

$$(\wp t)^I_x = t_X \; ,$$

where $X \xrightarrow{x} \mathsf{h}^*I$ is an object of $\mathcal{H}^I$.

The 2-functor $\wp$ is evidently a *2-embedding*, in the sense that:

1. for any toposes $\mathcal{F}$ and $\mathcal{H}$ over $\mathcal{S}$,

$$\wp_{\mathcal{F}\mathcal{H}} : \mathrm{TOP}_{\mathcal{S}}(\mathcal{F}, \mathcal{H}) \longrightarrow \mathrm{COCTS}_{\mathcal{S}}(\mathcal{H}, \mathcal{F})$$

   is fully faithful, and

2. $\wp$ is full on equivalences, which means that for all equivalences e in $\mathrm{COCTS}_{\mathcal{S}}(\mathcal{H}, \mathcal{F})$, there is an equivalence d and a natural isomorphism $\wp(d) \overset{\alpha}{\cong} e$.

A category (functor) over $\mathcal{S}$ shall be said to *come from a topos (geometric morphism)* if it is in the essential image of $\wp$. The rest of this section is concerned with when a category comes from a topos.

The reader is advised to read Appendix A before proceeding.

6.1 DEFINITION A category $\mathcal{A}$ is said to have *small coproducts which satisfy Frobenius reciprocity at $1 \in \mathcal{S}$* , if $\mathcal{A}$ has small coproducts which in addition satisfy the property that for any $I \in \mathcal{S}$ and any pullback diagram

$$\begin{array}{ccc} P & \longrightarrow & \Sigma_I C \\ \big\downarrow & & \big\downarrow {\scriptstyle b} \\ A & \xrightarrow[a]{} & B \end{array}$$

in $\mathcal{A}^I$, the pullback

$$\begin{array}{ccc} Q & \xrightarrow{\pi_0} & C \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \hat{b}} \\ I^*A & \xrightarrow[I^*a]{} & I^*B \end{array}$$

exists in $\mathcal{A}^I$ and the induced map $(\Sigma_I \pi_0, \widehat{\pi_1}) : \Sigma_I Q \rightarrow P$ is an isomorphism.

Intuitively, Frobenius reciprocity says that if $A \times \sum C_i$ exists, then it is equal to $\sum(A \times C_i)$.

6.2 DEFINITION A category $\mathcal{A}$ is said to have *small coproducts which satisfy Frobenius reciprocity* if for every $I \in \mathcal{S}$, $\mathcal{A}_{/I}$ has small coproducts which satisfy Frobenius reciprocity at $1 \in \mathcal{S}_{/I}$.

There is some redundancy in definition 6.2 in that if $\mathcal{A}$ has coproducts over $\mathcal{S}$, then $\mathcal{A}_{/I}$ has coproducts over $\mathcal{S}_{/I}$.

6.3 DEFINITION $\mathcal{A}$ is said to have *small coproducts such that Σ reflects isomorphisms at 1* if $\mathcal{A}$ has small coproducts such that Σ_I reflects isomorphisms for every $I \in \mathcal{S}$.

6.4 Proposition *Assume that $\mathcal{A}^1$ has a terminal object 1. Then the following are equivalent:*

1. *the fibers of $\mathcal{A}$ have the form $\mathcal{A}^I \cong (\mathcal{A}^1)_{/I.1}$, identifying the substitution functors α^* with pulling back along $\alpha.1$. In which case, $I.A$ is the product $I.1 \times A$, for every $A \in \mathcal{A}^1$ and every $I \in \mathcal{S}$.*

2. *$\mathcal{A}$ has small coproducts which satisfy Frobenius reciprocity at 1 and are such that Σ reflects isomorphisms at 1.*

PROOF That *1.* implies *2.* is left to the reader.

Assuming the conditions of *2.*, let $I \xrightarrow{\alpha} K$ be a morphism in $\mathcal{S}$. Let

$$\Phi^I : \mathcal{A}^I \longrightarrow (\mathcal{A}^1)_{/I.1}$$

denote the (ordinary) functor which sends an object $X \in \mathcal{A}^I$ to $\Sigma_I(!)$, where $X \xrightarrow{!} I^*1$ is the unique arrow from X to the terminal object I^*1 in $\mathcal{A}^I$. There is a natural isomorphism

$$
(6.5)\qquad
\begin{array}{ccc}
\mathcal{A}^I & \xrightarrow{\Phi^I} & (\mathcal{A}^I)_{/I.1} \\
\Sigma_\alpha \big\downarrow & \simeq & \big\downarrow \Sigma_{\alpha.1} \\
\mathcal{A}^K & \xrightarrow[\Phi^K]{} & (\mathcal{A}^I)_{/K.1}
\end{array}
$$

where $\Sigma_{\alpha.1}$ is composition with the morphism $I.1 \xrightarrow{\alpha.1} K.1$. The natural isomorphism in 6.5 arises as follows. If ε denotes the counit of $\Sigma_\alpha \dashv \alpha^*$, then

$$\Sigma_\alpha I^* 1 \simeq \alpha.(K^* 1) \xrightarrow{\varepsilon} K^* 1$$

is the unique map from $\Sigma_\alpha I^* 1$ to the terminal object in $\mathcal{A}^K$. Applying Σ_K to this map yields the morphism $\alpha.1$ down in $\mathcal{A}^I$, and from this it follows immediately that there is a natural isomorphism as in 6.5. By the results in Appendix A, the functors Φ^I and Φ^K are equivalences, and by 6.5 above, α^* is therefore identified with pulling back along $\alpha.1$. □

Categories of the form described in 6.4 can be thought of as categories with a terminal object and with universal disjoint coproducts. The corresponding result for functors between such categories is as follows.

6.6 Proposition *Let $\mathcal{A} \xrightarrow{\mathsf{F}} \mathcal{B}$ be a functor between categories as in 6.4, and assume that F^I preserves the terminal object 1 in $\mathcal{A}^I$. Then the following are equivalent:*

1. *F^I preserves all small copowers of 1 , and F has the form $\mathsf{F}^I \simeq (\mathsf{F}^I)_{/I.1}$. By this is meant that $\mathsf{F}^I(A \xrightarrow{a} I.1)$ is isomorphic to the object*

   $$\mathsf{F}^I a : \mathsf{F}^I A \to \mathsf{F}^I (I.1) \simeq I.1$$

   in $\mathcal{B}^I$, naturally in a .

2. *F preserves coproducts at 1 .*

PROOF Clearly *1.* implies *2.*.

Conversely, assume that F preserves coproducts at *1*. Let $A \xrightarrow{a} I.1$ be an arbitrary object of $\mathcal{A}^I$, and write $\mathsf{F}^I(a \xrightarrow{!} 1)$ as

$$
\begin{array}{ccc}
B & \xrightarrow{b} & I.1 \\
b \big\downarrow & \swarrow 1 & \\
I.1 & &
\end{array}
$$

in $\mathcal{B}^I$. Then $\mathsf{F}^I a$ is equal to the object $B \xrightarrow{b} I.1$. On the other hand, $\mathsf{F}^I a$ is equal to

$$\mathsf{F}^I(A \xrightarrow{a} I.1) = \mathsf{F}^I \Sigma_I(a \xrightarrow{!} 1) \simeq \Sigma_I \mathsf{F}^I(a \xrightarrow{!} 1) .$$

This is the morphism $B \xrightarrow{b} I.1$. □

The following two theorems will be used in Chapter 4, §3.

6.7 Theorem *A category $\mathcal{A}$ over $\mathcal{S}$ comes from a topos if and only if*

1. *$\mathcal{A}^1$ is an elementary topos,*
2. *$\mathcal{A}$ has small coproducts which satisfy Frobenius reciprocity at 1 and are such that Σ reflects isomorphisms at 1, and*
3. *$\mathcal{A}$ is locally small (although small homs at 1 will do).*

PROOF Given a category $\mathcal{A}$ over $\mathcal{S}$ which satisfies the three conditions, let $\mathcal{F} = \mathcal{A}^1$. Also, let

$$\mathsf{f}^* I = I.1 \; ; \; I \in \mathcal{S}$$

$$\mathsf{f}_* X = \mathcal{A}^1(1, X) \; ; \; X \in \mathcal{F} \; ,$$

where 1 is the terminal object in $\mathcal{F}$. Since $\mathcal{A}$ is assumed to have small homs at *1* it follows that $\mathsf{f}^* \dashv \mathsf{f}_*$, and the left exactness of f^* follows from the Beck conditon. Thus, $\mathsf{f} = (\mathsf{f}_*, \mathsf{f}^*)$ is a geometric morphism, and by 6.4, $\mathcal{A}$ therefore comes from the topos $\mathcal{F} \xrightarrow{\mathsf{f}} \mathcal{S}$. □

6.8 Theorem *A cocontinuous functor F between categories which come from toposes, comes from a geometric morphism if and only if the ordinary functor F^1 is left exact and has an ordinary right adjoint.*

PROOF This follows by 6.6. □

Chapter 2

Internal Diagrams and Sheaves

2.1 Internal diagrams

In Chapter 1, the category of internal diagrams was defined as a category over the base topos $\mathcal{S}$. This definition shall now be revised slightly in that internal diagrams shall be regarded as a category over $\mathcal{S}^{\mathbf{C}}$.

Let $\mathbf{C}$ be an internal category in $\mathcal{S}$.

1.1 Definition Let $\mathcal{A}$ be an arbitrary category over $\mathcal{S}$. Regard $\mathcal{A}^{\mathbf{C}}$ as a category over $\mathcal{S}^{\mathbf{C}}$ by defining the fiber above $\begin{matrix}\mathbf{X}\\ \downarrow \mathsf{x}\\ \mathbf{C}\end{matrix} \in \mathcal{S}^{\mathbf{C}}$ to be

$$(\mathcal{A}^{\mathbf{C}})^{\mathsf{x}} = \mathcal{A}^{\mathbf{X}} ,$$

where $\mathcal{A}^{\mathbf{X}}$ is the (ordinary) category of internal diagrams on $\mathbf{X}$, as defined in Chapter 1. For $\mathsf{x} \xrightarrow{\alpha} \mathsf{y}$ in $\mathcal{S}^{\mathbf{C}}$, the substitution functor

$$(\mathcal{A}^{\mathbf{C}})^{\mathsf{y}} \xrightarrow{\alpha^*} (\mathcal{A}^{\mathbf{C}})^{\mathsf{x}}$$

is defined to be

$$\alpha^* : (A, \theta) \rightsquigarrow (\alpha_0^* A, \alpha_1^* \theta) .$$

This definition extends in an obvious manner to morphisms of $(\mathcal{A}^{\mathbf{C}})^{\mathsf{y}}$.

The notation $(\mathcal{A}^{\mathbf{C}})_{\mathsf{C}}$ is used for the restriction of $\mathcal{A}^{\mathbf{C}}$ along $\mathcal{S}^{\mathbf{C}} \xrightarrow{\mathsf{C}} \mathcal{S}$, and this is in fact simply $\mathcal{A}^{\mathbf{C}}$ as previously regarded over $\mathcal{S}$.

Recall that $\mathcal{A}_{/I}$ denotes the *localization* of $\mathcal{A}$ at I . $\mathcal{A}_{/I}$ is a category over $\mathcal{S}_{/I}$, and it is a special case of the more general construction given above. That is, $\mathcal{A}_{/I} = \mathcal{A}^{\mathbf{I}}$ for the discrete category $\mathbf{I}$. (This is not to be confused with $\mathcal{A}^{I}$, which denotes $(\mathcal{A}_{/I})_{\mathsf{I}}$, the restriction along $\mathcal{S}_{/I} \xrightarrow{\mathsf{I}} \mathcal{S}$ of $\mathcal{A}_{/I}$, see Chapter 1, §5.) It is not hard to check that the construction $\mathcal{A}^{\mathbf{C}}$ is stable under, or commutes with, localization over $\mathcal{S}$. By this is meant, in this case anyway, that for any $I \in \mathcal{S}$ there is an canonical equivalence

$$(\mathcal{A}_{/I})^{\mathbf{C}_{/I}} \cong \mathcal{A}^{\mathbf{C}}{}_{/\mathbf{C} \cdot I} ,$$

of categories over $(\mathcal{S}_{/I})^{\mathbf{C}_{/I}} \cong \mathcal{S}^{\mathbf{C}}{}_{/\mathbf{C} \cdot I}$.

Observe that for $\begin{matrix}\mathbf{X}\\ \downarrow \mathsf{x}\\ \mathbf{C}\end{matrix} \in \mathcal{S}^{\mathbf{C}}$, $\mathbf{X}$ is a small category in its own right. Therefore, according to 1.1 one has the category $\mathcal{A}^{\mathbf{X}}$ over $\mathcal{S}^{\mathbf{X}}$. This is precisely the localization $\mathcal{A}^{\mathbf{C}}{}_{/\mathsf{x}}$ over $\mathcal{S}^{\mathbf{C}}{}_{/\mathsf{x}}$.

If $\mathcal{A}$ is locally small, then, as is shown in [PS], the restriction $(\mathcal{A}^{\mathbf{C}})_C$ is locally small over $\mathcal{S}$. In fact, $\mathcal{A}^{\mathbf{C}}$ is locally small over $\mathcal{S}^{\mathbf{C}}$. This is shown in the paragraphs that follow.

Let us show first that $\mathcal{A}^{\mathbf{C}}$ has small homs at $1 \in \mathcal{S}^{\mathbf{C}}$. Given $\mathsf{a} = (A, \theta)$ and $\mathsf{b} = (B, \phi)$ in $\mathcal{A}^{\mathbf{C}}$, an object $\mathcal{A}^{\mathbf{C}}(\mathsf{a},\mathsf{b})$ in $\mathcal{S}^{\mathbf{C}}$ is to be exhibited together with a bijection

$$\frac{\mathsf{x}^*\mathsf{a} \to \mathsf{x}^*\mathsf{b} \text{ in } (\mathcal{A}^{\mathbf{C}})^{\mathsf{x}}}{\mathsf{x} \to \mathcal{A}^{\mathbf{C}}(\mathsf{a},\mathsf{b}) \text{ in } \mathcal{S}^{\mathbf{C}}}$$

which is natural in $\begin{array}{c}\mathbf{X}\\ \downarrow \mathsf{x}\\ \mathbf{C}\end{array} \in \mathcal{S}^{\mathbf{C}}$. Recall that $\mathcal{S}^{\mathbf{C}}$ is cotripleable over $\mathcal{S}_{/C_0}$ with the forgetful functor

$$U : \mathcal{S}^{\mathbf{C}} \longrightarrow \mathcal{S}_{/C_0}$$

sending $\mathsf{x} = (x_0, x_1)$ to x_0 . Let G denote the right adjoint of U . The composite UG is equal to $\Pi_{\delta_0}\delta_1^*$.

Let (A, B) denote the hom-object in $\mathcal{S}_{/C_0}$ which represents morphisms $A \to B$ in $\mathcal{A}_{/C_0}$. Let

$$(\delta_0^* A, \phi) : (\delta_0^* A, \delta_0^* B) \to (\delta_0^* A, \delta_1^* B)$$

$$(\theta, \delta_1^* B) : (\delta_1^* A, \delta_1^* B) \to (\delta_0^* A, \delta_1^* B)$$

denote the morphisms in $\mathcal{S}_{/C_1}$ corresponding to composition with ϕ and θ respectively. Define $\mathcal{A}^{\mathbf{C}}(\mathsf{a},\mathsf{b})$ to be the equalizer in $\mathcal{S}^{\mathbf{C}}$ of

$$(1.2) \qquad \begin{array}{ccc} G(A,B) & \xrightarrow{G\rho} & G(\Pi_{\delta_0}(\delta_0^* A, \delta_1^* B)) \\ & {\scriptstyle \eta_{G(A,B)}} \searrow \quad \nearrow {\scriptstyle G(\Pi_{\delta_0}(\theta, \delta_1^* B))} & \\ & GUG(A,B) & \end{array}$$

where ρ is the morphism corresponding to $(\delta_0^* A, \phi)$ under the adjointness $\delta_0^* \dashv \Pi_{\delta_0}$, and η is the unit of $U \dashv G$. Also, note that

$$UG(A,B) = \Pi_{\delta_0}\delta_1^*(A,B) \simeq \Pi_{\delta_0}(\delta_1^* A, \delta_1^* B) \; .$$

Given $\mathsf{x} \xrightarrow{\hat{f}} G(A,B)$ in $\mathcal{S}^{\mathbf{C}}$, let $U\mathsf{x} \xrightarrow{f} (A,B)$ denote its transpose under $U \dashv G$. Now transpose 1.2 together with $\hat{f}$, first with respect to $U \dashv G$, and then with respect to $\delta_0^* \dashv \Pi_{\delta_0}$. One obtains the square

$$
(1.3)\qquad
\begin{array}{ccc}
\delta_0^*(U\mathsf{x}) & \xrightarrow{\delta_0^* f} & \delta_0^*(A,B) \\
{\scriptstyle \overline{U\hat{f}}}\downarrow & & \downarrow{\scriptstyle (\delta_0^* A, \phi)} \\
(\delta_1^* A, \delta_1^* B) & \xrightarrow[(\theta, \delta_1^* B)]{} & (\delta_0^* A, \delta_1^* B)
\end{array}
$$

in $\mathcal{S}_{/C_1}$, where $\overline{U\hat{f}}$ corresponds to $U\hat{f}$ under $\delta_0^* \dashv \Pi_{\delta_0}$. Moreover, 1.3 commutes if and only if $\hat{f}$ equalizes 1.2. But $\hat{f} = G(f) \cdot \eta_{\mathsf{x}}$, and hence $U\hat{f}$ is the morphism

$$
U\hat{f} : x_0 \stackrel{U\eta_{\mathsf{x}}}{\rightarrow} \Pi_{\delta_0}\delta_1^*(x_0) \stackrel{\Pi_{\delta_0}\delta_1^* f}{\rightarrow} \Pi_{\delta_0}\delta_1^*(A,B) .
$$

Therefore, one has

$$
\overline{U\hat{f}} : x_1 = \delta_0^*(x_0) \stackrel{\overline{U\eta_{\mathsf{x}}}}{\rightarrow} \delta_1^*(x_0) \stackrel{\delta_1^* f}{\rightarrow} \delta_1^*(A,B) .
$$

Transposing this once again, with respect to $\Sigma_{\delta_1} \dashv \delta_1^*$, one arrives at

$$
\Sigma_{\delta_1}(x_1) \stackrel{\delta_1}{\rightarrow} x_0 \stackrel{f}{\rightarrow} (A,B) .
$$

Observe that the transpose of $\overline{U\eta_{\mathsf{x}}}$ *is* δ_1. The upshot is that $\overline{U\hat{f}}$ represents the bottom composite morphism in the square

$$
(1.4)\qquad
\begin{array}{ccccccc}
x_1^*\delta_0^* A & \simeq & \delta_0^* x_0^* A & \xrightarrow{\delta_0^* f} & \delta_0^* x_0^* B & \simeq & x_1^*\delta_0^* B \\
{\scriptstyle x_1^*\theta}\downarrow & & & & & & \downarrow{\scriptstyle x_1^*\phi} \\
x_1^*\delta_1^* A & \simeq & \delta_1^* x_0^* A & \xrightarrow[\delta_1^* f]{} & \delta_1^* x_0^* B & \simeq & x_1^*\delta_1^* B
\end{array}
$$

in $\mathcal{A}^{X_1}$, where the same symbol $x_0^* A \stackrel{f}{\rightarrow} x_0^* B$ is being used to denote the arrow in $\mathcal{A}^{X_0}$ represented by $x_0 \stackrel{f}{\rightarrow} (A,B)$. The top morphism in 1.4 is represented by

$$
x_1 = \delta_0^*(U\mathsf{x}) \stackrel{\delta_0^* f}{\rightarrow} \delta_0^*(A,B) ,
$$

and hence 1.3 commutes if and only if 1.4 commutes. This establishes the following series of bijections.

$$
\frac{\dfrac{\mathsf{x}^*\mathsf{a} \rightarrow \mathsf{x}^*\mathsf{b} \text{ in } (\mathcal{A}^{\mathbf{C}})^{\mathsf{x}}}{U\mathsf{x} \rightarrow (A,B) \text{ in } \mathcal{S}_{/C_0} \text{ such that 1.4 commutes}}}{\mathsf{x} \rightarrow \mathcal{A}^{\mathbf{C}}(\mathsf{a},\mathsf{b}) \text{ in } \mathcal{S}^{\mathbf{C}}}
$$

This bijection is natural in x, and this proves that $\mathcal{A}^{\mathbf{C}}$ has small homs over $\mathcal{S}^{\mathbf{C}}$ at $1 \in \mathcal{S}^{\mathbf{C}}$. To show that $\mathcal{A}^{\mathbf{C}}$ has small homs at $\begin{smallmatrix}\mathbf{X}\\ \downarrow \mathsf{x}\\ \mathbf{C}\end{smallmatrix}$ in $\mathcal{S}^{\mathbf{C}}$ just repeat the above argument replacing $\mathcal{S}^{\mathbf{C}}$ by $\mathcal{S}^{\mathbf{X}}$ and $\mathcal{A}^{\mathbf{C}}$ by $\mathcal{A}^{\mathbf{X}}$.

1.5 Example Taking $\mathcal{A} = \mathcal{S}$ in the above argument produces a proof that $\mathcal{S}^{\mathbf{C}}$ is cartesian closed.

Recall that 'small coproducts', '$\mathcal{S}$-coproducts' and 'Σ satisfying the Beck condition' all mean one and the same thing. This property, that of having small coproducts, 'lifts' to discrete opfibrations. That is, if $\mathcal{A}$ has $\mathcal{S}$-coproducts then $\mathcal{A}^{\mathbf{C}}$ has $\mathcal{S}^{\mathbf{C}}$-coproducts. In fact, let $\mathsf{x} \xrightarrow{\alpha} \mathsf{y}$ be an arbitrary morphism in $\mathcal{S}^{\mathbf{C}}$. Define Σ_α to be

$$\Sigma_\alpha : (\mathcal{A}^{\mathbf{C}})^{\mathsf{x}} \longrightarrow (\mathcal{A}^{\mathbf{C}})^{\mathsf{y}}$$
$$(A,\theta) \quad \rightsquigarrow \quad (\Sigma_{\alpha_0} A, \Sigma_{\alpha_0}\theta) \ ,$$

where $\Sigma_{\delta_1}\delta_0^* A \xrightarrow{\theta} A$, now regarding the fiber $\mathcal{A}^{\mathbf{X}}$ as the category of algebras for the triple (whose functor part is) $\Sigma_{\delta_1}\delta_0^*$. δ_0 is the domain map of X, and δ_1 the codomain. This definition obviously extends to morphisms of algebras. Also note that one has to insert canonical isomorphisms for the definition to make sense. For example, the action map of $\Sigma_\alpha(A,\theta)$ is really

$$\Sigma_{\delta_1}\delta_0^*\Sigma_{\alpha_0} A \simeq \Sigma_{\delta_1}\Sigma_{\alpha_1}\delta_0^* A \simeq \Sigma_{\alpha_0}\Sigma_{\delta_1}\delta_0^* A \xrightarrow{\Sigma_{\alpha_0}\theta} \Sigma_{\alpha_0} A \ .$$

Then $\Sigma_\alpha \dashv \alpha^*$, and the Beck condition is satisfied over $\mathcal{S}^{\mathbf{C}}$.

If the fibers of $\mathcal{A}$ have (stable) finite colimits, then this too is true for $\mathcal{A}^{\mathbf{C}}$. For example, pushouts in $(\mathcal{A}^{\mathbf{C}})^1$ are calculated as

$$\begin{array}{ccc} (A,\theta) & \longrightarrow & (B,\phi) \\ \downarrow & & \downarrow \\ (D,\rho) & \rightarrow & (D+_A B, \rho+_\theta \phi) \end{array}$$

which means that $D+_A B$ is calculated in $\mathcal{A}^{C_0}$, and $\rho+_\theta\phi$ in $\mathcal{A}^{C_1}$. Thus, one has the following proposition.

1.6 Proposition *Suppose that $\mathcal{A}$ is cocomplete, and let* C *be a small category in $\mathcal{S}$. Then $\mathcal{A}^{\mathbf{C}}$ is cocomplete over $\mathcal{S}^{\mathbf{C}}$.*

PROOF Cocompleteness is equivalent to having small coproducts and stable finite colimits in the fibers. □

1.7 Example Let $\mathcal{A}$ be a cocomplete category over $\mathcal{S}$. As a category over $\mathcal{S}^{\mathbf{C}}$, the left adjoints to the substitution functors of $\mathcal{A}^{\mathbf{C}}$ can be 'externally' described as left Kan extensions. Indeed, let $\mathsf{x} \xrightarrow{\alpha} \mathsf{y}$ be in $\mathcal{S}^{\mathbf{C}}$. Then given $\mathbf{g} \in (\mathcal{A}^{\mathbf{C}})^{\mathsf{x}}$ one would like to describe $\Sigma_\alpha(\mathbf{g}) \in (\mathcal{A}^{\mathbf{C}})^{\mathsf{y}}$ as the left Kan extension

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\alpha} & \mathbf{Y} \\ & {}_{\mathbf{g}}\searrow \;\nearrow & \downarrow \Sigma_\alpha(\mathbf{g}) \\ & & \mathcal{A}\,. \end{array}$$

Frequently an argument can be abbreviated by localizing. That is, if the definitions and constructions relevant to the argument are stable under localization, then it suffices to prove the given statement at the terminal object $1 \in \mathcal{S}$. One first defines $\Sigma_\alpha(\mathbf{g})$ at $1 \in \mathcal{S}$. Let $1 \xrightarrow{y} Y_0 \in \mathbf{Y}^1$, and form the small category α/y. For example, $(\alpha/y)_0$ is defined as:

$$\begin{array}{ccccc} (\alpha/y)_0 & \longrightarrow & Y_1/y & \longrightarrow & 1 \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow y \\ \gamma_0 & & Y_1 & \xrightarrow[\delta_1]{} & Y_0 \\ \Big\downarrow & & \Big\downarrow \delta_0 & & \\ X_0 & \xrightarrow[\alpha_0]{} & Y_0 & & \end{array}$$

where both squares are pullbacks. Then let $\Sigma_\alpha(\mathbf{g})(y) = \varinjlim(\mathbf{g}\gamma)$ where γ is the obvious functor. The construction α/y is stable under localization, as is the construction $\mathcal{A}^{\mathbf{C}}$and as is the cocompleteness of $\mathcal{A}$. Therefore, the definition of $\Sigma_\alpha(\mathbf{g})$ can be considered to be complete. To define $\Sigma_\alpha(\mathbf{g})^I$, $I \in \mathcal{S}$, one proceeds as above, only now working over $\mathcal{S}_{/I}$.

We have established that if $\mathcal{A} \in \textsc{Cocts}_{\mathcal{S}}$, then $\mathcal{A}^{\mathbf{C}} \in \textsc{Cocts}_{\mathcal{S}^{\mathbf{C}}}$. The 'internal diagrams' construction constitutes a 2-functor

$$(\ )^{\mathbf{C}} : \textsc{Cocts}_{\mathcal{S}} \longrightarrow \textsc{Cocts}_{\mathcal{S}^{\mathbf{C}}}$$

$$\mathcal{A} \rightsquigarrow \mathcal{A}^{\mathbf{C}}\,.$$

To conclude this section, a discussion of connectedness is included. This notion will come up in Chapter 4, §5.

The coequalizer of $C_1 \underset{\delta_0}{\overset{\delta_1}{\rightrightarrows}} C_0$ in $\mathcal{S}$ represents the 'number' of connected components of $\mathbf{C}$. Thus, one could say that $\mathbf{C}$ is *weakly* or *internally* connected if this coequalizer is isomorphic to $1 \in \mathcal{S}$. This is true if and only if $\mathcal{S} \xrightarrow{\mathbf{C}^*} \mathcal{S}^{\mathbf{C}}$ is fully faithful. There is a stronger external notion, to be simply called *connected*, which is as follows.

1.8 DEFINITION A small category $\mathbf{C}$ is said to be *connected* if:

1. the unique map $C_0 \rightarrow 1$ is an epimorphism, and

2. for any objects $c, d \in \mathbf{C}^I$, there is an epimorphism $K \xrightarrow{\varepsilon} I$ in $\mathcal{S}$ and morphisms $f_1, \ldots, f_n \in \mathbf{C}^K$ connecting $\varepsilon^* c$ with $\varepsilon^* d$,

where no particular direction of any of the arrows $f_1, \ldots, f_n$ is intended.

1.9 Proposition *Let $\mathcal{A}$ be a category over $\mathcal{S}$ (not necessarily locally small or cocomplete), and assume that $\mathcal{A}$ is a* stack. *Let $\mathbf{C}$ be a small connected category. Then*

$$\mathsf{C}^* : \mathcal{A} \longrightarrow (\mathcal{A}^{\mathbf{C}})_{\mathbf{C}}$$

is full and faithful.

PROOF Connectedness is stable under localization, as is the property of being a stack. Therefore, it suffices to prove the proposition at $1 \in \mathcal{S}$. Recall that the category $(\mathcal{A}^{\mathbf{C}})^{1}_{\mathbf{C}}$ is isomorphic to the category of functors from $\mathbf{C}$ to $\mathcal{A}$ over $\mathcal{S}$. To be shown is that C^{*1} is fully faithful. Suppress the '1' notation writing, for example, C^* for C^{*1}. Then, by definition, for $A \xrightarrow{f} B$ in $\mathcal{A}$ and $c \in \mathbf{C}^I$ one has

$$\begin{array}{ccc} (\mathsf{C}^* A)^I(c) & & I^* A \\ (\mathsf{C}^* f)^I_c \big\downarrow & = & \big\downarrow I^* f \\ (\mathsf{C}^* B)^I(c) & & I^* B \end{array} .$$

So suppose there is given a natural transformation $\mathsf{C}^* A \xrightarrow{t} \mathsf{C}^* B$. That is, for every $c \in \mathbf{C}^I$, there is given an arrow $I^* A \xrightarrow{t^I_c} I^* B$. The naturality of t says that if there is an arrow $c \xrightarrow{\alpha} d$ in $\mathbf{C}^I$, then $t^I_c = t^I_d$. The idea is to show that $t^I_c = t^I_d$ for all $c, d \in \mathbf{C}^I$. So fix $c, d \in \mathbf{C}^I$. By using condition 2. of 'connected' it follows that

$t^K_{\varepsilon^* c} = t^K_{\varepsilon^* d}$ for some epimorphism $K \xrightarrow{\varepsilon} I$ in $\mathcal{S}$. Therefore, the diagram

$$\begin{array}{ccccc} \varepsilon^*I^*A & \simeq & K^*A & \simeq & \varepsilon^*I^*A \\ \varepsilon^*(t^I_d)\downarrow & & \downarrow & & \downarrow\varepsilon^*(t^I_c) \\ \varepsilon^*I^*B & \simeq & K^*B & \simeq & \varepsilon^*I^*B \end{array}$$

commutes, where the center arrow is $t^K_{\varepsilon^*c} = t^K_{\varepsilon^*d}$. The isomorphisms cancel out to give $\varepsilon^*(t^I_d) = \varepsilon^*(t^I_c)$, and therefore $t^I_d = t^I_c$ since $\mathcal{A}$ is a stack. By using this, taking $I = C_0 \times C_0$ and c, d to be the two projections $C_0 \times C_0 \to C_0$, it follows that

$$\begin{array}{ccccc} \pi_0^* C_0^* A & \simeq & (C_0 \times C_0)^* A & \simeq & \pi_1^* C_0^* A \\ \pi_0^*(t_1^{C_0})\downarrow & & \downarrow & & \downarrow \pi_1^*(t_1^{C_0}) \\ \pi_0^* C_0^* B & \simeq & (C_0 \times C_0)^* B & \simeq & \pi_1^* C_0^* B \end{array}$$

commutes, where the center arrow is $t^{C_0 \times C_0}_{\pi_0} = t^{C_0 \times C_0}_{\pi_1}$. The 'object' 1 is the identity on C_0 . The horizontal composite isomorphisms are the canonical descent data of $C_0^* A$ and $C_0^* B$. Since $\mathcal{A}$ is assumed to be a stack, and since $C_0 \to 1$ is, by hypothesis, an epimorphism, one has $t_1^{C_0} = C_0^* f$ for a unique $A \xrightarrow{f} B$. It follows that $t = \mathsf{C}^* f$. This concludes the proof. □

By taking $\mathcal{A} = \mathcal{S}$ in the above proposition we see that a small connected category is weakly connected.

1.10 Corollary *Let $\mathcal{A} \in$ COCTS$_\mathcal{S}$, and let* **C** *be a small connected category. Then* $\varinjlim_{\mathsf{C}} \cdot \mathsf{C}^* \simeq \mathbf{1}_\mathcal{A}$.

PROOF One has $\varinjlim_{\mathsf{C}} \dashv \mathsf{C}^*$ and, since $\mathcal{A}$ is necessarily a stack, 1.9 applies. □

If **C** has a terminal object, in the internal sense (see [J1], p. 74), then **C** is connected.

2.2 Sheaves

Let $\mathcal{A}$ be a category over $\mathcal{S}$, and let j be a topology on $\mathcal{S}$. In [P2], the following definition is made.

2.1 DEFINITION An object $A \in \mathcal{A}^1$ is said to be a *sheaf* for the topology j, or simply a *j-sheaf*, if for any j-dense monomorphism $S \overset{s}{\hookrightarrow} T$ and any $B \in \mathcal{A}^T$ the function

$$s^* : [B, T^* A] \to [s^* B, s^* T^* A]$$

is a bijection.

The full sub-category of $\mathcal{A}^I$ whose objects are sheaves shall be denoted by $sh_j(\mathcal{A})^I$. One considers sheaves as a category over $\mathcal{S}$ by defining:

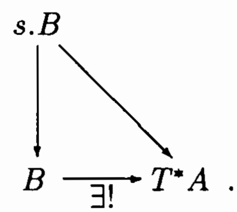
$$sh_j(\mathcal{A})^I \;= sh_{I^*j}(\mathcal{A}_{/I})^I,$$

for $I \in \mathcal{S}$. Explicitly, this means that an object of the fiber $sh_j(\mathcal{A})^I$ is an object $A \in \mathcal{A}^I$ such that for every $T \xrightarrow{\xi} I$ in $\mathcal{S}_{/I}$, every j-dense monomorphism $S \overset{s}{\hookrightarrow} T$, and every $B \in \mathcal{A}^T$, the function

$$s^* : [B, \xi^* A] \rightarrow [s^* B, s^* \xi^* A]$$

is a bijection. One sees directly that the substitution functors restrict to sheaves giving us the category $sh_j(\mathcal{A})$ over $\mathcal{S}$.

If $\mathcal{A}$ has small copowers, then definition 2.1 is equivalent to the following more 'sheaf-like' condition on A: for any dense monomorphism $S \overset{s}{\hookrightarrow} T$ and any $B \in \mathcal{A}^T$, every morphism $s.B \rightarrow T^* A$ lifts uniquely along the canonical morphism $s.B \rightarrow B$,



Furthermore, in this case the category $sh_j(\mathcal{A})^I$, which is $(sh_j(\mathcal{A})_{/I})_{|}$, is equivalent to $sh_j(\mathcal{A}^I\,)$, which is $sh_j((\mathcal{A}_{/I})_{|})$, over $\mathcal{S}$. The reader is referred to [P2] for a proof of this.

2.2 Example The basic example is that of sheaves in the base topos $\mathcal{S}$. Let $\mathcal{E}$ denote the sub-topos of j-sheaves, in the original sense, in $\mathcal{S}$. Then $\mathcal{E}$ is considered as a category over $\mathcal{S}$ (see Chapter 1, §6) by letting $\mathcal{E}^I = \mathcal{E}_{/\mathrm{i}^* I}$, where $\mathcal{E} \xrightarrow{\;\mathrm{i}\;} \mathcal{S}$ is the inclusion. Then, as is shown in [P2], $\mathcal{E} \cong sh_j(\mathcal{S})$ as categories over $\mathcal{S}$.

Defined to be a full-subcategory of $\mathcal{A}$, $sh_j(\mathcal{A})$ is automatically locally small if $\mathcal{A}$ is. However, more is true, for any sheaf A and any B , the representing object (B, A) is itself a sheaf. The following proposition is from [P2].

2.3 Proposition *Let $\mathcal{A}$ be locally small, and let $A \in \mathcal{A}$. Then A is a j-sheaf if and only if for every $I \in \mathcal{S}$ and every $B \in \mathcal{A}^I$, $(B, I^* A)^I$ is an $I^* j$-sheaf in $\mathcal{S}_{/I}$.*

The intention here is to consider $sh_j(\mathcal{A})$ as a category over $\mathcal{E}$, and in doing so proposition 2.3 would then tell us that $sh_j(\mathcal{A})$ is locally small over $\mathcal{E}$.

2.4 Definition $sh_j(\mathcal{A})$ is regarded as a category over $\mathcal{E}$ by defining

$$sh_j(\mathcal{A})^K = sh_j(\mathcal{A})^{\mathrm{i}_* K} \ ,$$

for every $K \in \mathcal{E}$.

Then under this definition, by 2.3 $sh_j(\mathcal{A})$ is in indeed locally small over $\mathcal{E}$ if $\mathcal{A}$ is locally small over $\mathcal{S}$.

Given definition 2.4, one is led to consider the category $sh_j(\mathcal{A})_{\mathrm{i}}$, its restriction back to $\mathcal{S}$. It would be desirable that this category be equivalent (by an equivalence natural in $\mathcal{A}$) to $sh_j(\mathcal{A})$ in its original definition as a category over $\mathcal{S}$. Why indeed should this be desirable? The answer lies in the fact that such an equivalence can be interpreted as an 'equivariance' of the j-sheaves construction. See Chapter 3 for a further discussion on this. For now, let us be content with the observation that such an equivalence entails that $sh_j(\mathcal{A})^K \xrightarrow{f^*} sh_j(\mathcal{A})^I$ be an equivalence for every j-bidense morphism $I \xrightarrow{f} K$. To analyze this condition let us begin with the following definition.

2.5 Definition $\mathcal{A}$ is said to be a *j-stack* if $\mathcal{A}$ has the stack property with respect to every j-bidense epimorphism.

Observe that the true stacks are then in fact the t-stacks. Also, if $j \leq j\prime$ then any $j\prime$-stack is j-stack.

In the following proposition, $sh_j(\mathcal{A})$ is to be regarded as a category over $\mathcal{S}$ in its original definition.

2.6 Proposition *Let $\mathcal{A}$ be an arbitrary category over $\mathcal{S}$. Assume that $sh_j(\mathcal{A})$ has small coproducts. Then the following are equivalent:*

1. $sh_j(\mathcal{A})^H \xrightarrow{g^*} sh_j(\mathcal{A})^I$ *reflects isomorphisms for every j-bidense epimorphism* $I \xrightarrow{g} H$,
2. $sh_j(\mathcal{A})^K \xrightarrow{f^*} sh_j(\mathcal{A})^I$ *is an equivalence for every j-bidense morphism* $I \xrightarrow{f} K$,
3. $sh_j(\mathcal{A})$ *is a j-stack.*

PROOF (*1.⇒2.*) Let $I \xrightarrow{f} K$ be an arbitrary bidense morphism. If f is factored as an epimorphism followed by a monomorphism $I \xrightarrow{g} H \xhookrightarrow{h} K$, then g and h are both bidense. Let us first consider h . Observe that

$$h^* : [B, A] \to [h^* B, h^* A]$$

is a bijection for any sheaf $A \in \mathcal{A}^K$ and any $B \in \mathcal{A}^K$. In particular, when restricted to sheaves in $\mathcal{A}^K$, h^* is fully faithful. Moreover, h^* is essentially surjective because for any $B \in sh_j(\mathcal{A})^H$, the unit $B \rightarrow h^*\Sigma_h B$ is an isomorphism. This follows because

$$\begin{array}{ccc} H & \xrightarrow{1} & H \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle h} \\ H & \xrightarrow[h]{} & K \end{array}$$

is a pullback and the Beck condition is satisfied. Thus, h^* is an equivalence.

Let P denote the kernel pair of g.

$$\begin{array}{ccc} P & \xrightarrow{\pi_0} & I \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ I & \xrightarrow[g]{} & H \end{array}$$

Let $I \xrightarrow{\delta} P$ denote the factorization of the diagonal through P. Then δ is a j-dense monomorphism and

$$\delta^* : [B, \pi_1^* A] \rightarrow [\delta^* B, \delta^* \pi_1^* A]$$

is a bijection for any sheaf $A \in \mathcal{A}^I$ and any $B \in \mathcal{A}^P$. In particular, for any sheaves $A, C \in \mathcal{A}^I$ one has following series of bijections:

$$\frac{\dfrac{\dfrac{\dfrac{g^*\Sigma_g C \rightarrow A}{\Sigma_{\pi_1}\pi_0^* C \rightarrow A}}{\pi_0^* C \rightarrow \pi_1^* A}}{\delta^*\pi_0^* C \rightarrow \delta^*\pi_1^* A}}{C \rightarrow A}$$

which is given by composition with the unit $C \xrightarrow{\eta_C} g^*\Sigma_g C$. Therefore, there is a morphism $g^*\Sigma_g C \xrightarrow{\rho} C$ such that $\rho \cdot \eta_C = 1_C$. Then $\eta_C \cdot \rho$ is the identity on $g^*\Sigma_g C$, since both $\eta_C \cdot \rho$ and the identity correspond to η_C under the above bijection. Therefore, η_C is an isomorphism. To see that the counit of $\Sigma_g \dashv g^*$ is an isomorphism, observe that for any sheaf $A \in \mathcal{A}^H$, by what has just been shown, η_{g^*A} is an isomorphism. But the inverse of η_{g^*A} is $g^*(\varepsilon_A)$, and by our hypothesis, the counit ε_A is therefore an isomorphism. Thus, g^* is an equivalence.

(*2.*$\Rightarrow$*3.*) Let $I \xrightarrow{g} H$ be an arbitrary j-bidense epimorphism. Then $sh_j(\mathcal{A})$ has the stack property with respect to g, trivially so in fact, because g^* is assumed to be an equivalence.

(3.⇒1.) If $sh_j(\mathcal{A})$ is a j-stack then in particular the substitution functors reflects isomorphisms along j-bidense epimorphisms. □

Let $\mathcal{E} \xrightarrow{\mathsf{i}} \mathcal{S}$ denote the inclusion of $\mathcal{E}$ into $\mathcal{S}$. Then under the conditions of 2.6 and under definition 2.4 one has, for any $I \in \mathcal{S}$,

$$sh_j(\mathcal{A})_{\mathsf{i}}^I = sh_j(\mathcal{A})^{\mathsf{i}^* I} = sh_j(\mathcal{A})^{\mathsf{i}_* \mathsf{i}^* I} \cong sh_j(\mathcal{A})^I ,$$

since $I \to \mathsf{i}_* \mathsf{i}^* I$ is bidense. Since this equivalence is over $\mathcal{S}$, it can be written as

$$(2.7) \qquad sh_j(\mathcal{A})_{\mathsf{i}} \cong sh_j(\mathcal{A}) .$$

Line 2.7 means that the restriction back to $\mathcal{S}$ of $sh_j(\mathcal{A})$ regarded as a category over $\mathcal{E}$, is equivalent to its original formulation as a category over $\mathcal{S}$. This shall be referred to as the *equivariance* of sheaves.

This section is concluded with a review of some facts from [P2]. Let $\mathcal{A}$ be an arbitrary category over $\mathcal{S}$, and assume for this discussion that $\mathcal{A}$ has small powers. For α in $\mathcal{S}_{/H}$ and $A \in \mathcal{A}^H$, let A^α denote the α-power of A .

Let $\mathbf{D}$ be a full sub-category of Ω . That is, for every $I \in \mathcal{S}$, a collection $\mathbf{D}^I$ of sub-objects of I is given such that for any $K \overset{k}{\hookrightarrow} I \in \mathbf{D}^I$ and any $H \overset{\alpha}{\to} I$ one has $\alpha^* k \in \mathbf{D}^H$. A sub-object of $S \overset{s}{\hookrightarrow} \Omega$ determines such a category $\mathbf{S}$ by letting

$$\mathbf{S}^I = \{ A \overset{a}{\hookrightarrow} I \mid \text{the characteristic map of } a \text{ factors through } s \} .$$

2.8 Definition $A \in \mathcal{A}$ is said to be a $\mathbf{D}$-*sheaf* if for every $Y \in \mathcal{S}$, A has the sheaf property with respect to every monomorphism $X \overset{x}{\hookrightarrow} Y$ in $\mathbf{D}^Y$.

Let $S \overset{s}{\hookrightarrow} T$ be a given fixed monomorphism. Let $(\mathbf{s})$ denote the full sub-category of Ω generated by s in the sense that

$$(\mathbf{s})^I = \{\alpha^* s \mid I \overset{\alpha}{\to} T\} .$$

2.9 Proposition *For a given $A \in \mathcal{A}$, the following are equivalent.*

1. *A is an $(\mathbf{s})$-sheaf.*

2. *A has the sheaf property with respect to the monomorphism s .*

3. *The canonical map $T^* A \to (T^* A)^s$ is an isomorphism.*

Denote by $1 \xrightarrow{d} J$ the factorization of $1 \xrightarrow{t} \Omega$ through J, where J is the sub-object of Ω classified by j. Then $\mathbf{J} = (\mathbf{d})$, and therefore $A \in \mathcal{A}$ is a j-sheaf if and only if $J^*A \to (J^*A)^d$ is an isomorphism.

Given $\mathbf{D}$, denote by $\tilde{\mathbf{D}}$ the full sub-category of Ω whose fiber at $T \in \mathcal{S}$ is defined to be

$$\tilde{\mathbf{D}}^T = \{S \xhookrightarrow{s} T \mid \text{ there exists } Y \xrightarrow{e} T \text{ with } e^*s \in \mathbf{D}^Y\} .$$

$\tilde{\mathbf{D}}$ is the *stack completion* of $\mathbf{D}$. Also, recall (Chap. 1, §1) that if $\mathcal{A}$ is a stack, then for any epimorphism $Y \xrightarrow{e} T$, e^* reflects isomorphisms.

2.10 Proposition *Assume that $\mathcal{A}$ has the property that e^* reflects isomorphisms for any epimorphism e. Then $A \in \mathcal{A}$ is a $\mathbf{D}$-sheaf if and only if A is a $\tilde{\mathbf{D}}$-sheaf.*

The reader if referred to [P2] for proofs of 2.9 and 2.10. The main result from that paper is the next theorem. The proof included here is essentially that which is found in [P2]. It uses the following two facts, which will also be needed in the next chapter. As usual, (j, J) denotes a topology on $\mathcal{S}$.

1. If a sub-object $S \hookrightarrow \Omega$ generates J, then any $A \in \mathcal{A}$ is a j-sheaf if and only if it is an $\mathbf{S}$-sheaf.
2. Let $\mathcal{F} \xrightarrow{\mathrm{f}} \mathcal{S}$ denote an arbitrary topos over $\mathcal{S}$. Let $K \hookrightarrow \Omega_{\mathcal{F}}$ denote the image of the characteristic map of f^*d. Then

   $$\mathbf{K} = (\widetilde{\mathrm{f}^*\mathbf{d}}) .$$

   One says that $\mathbf{K}$ consists of those monomorphisms which are locally pullbacks of f^*d.

With $\mathcal{F} \xrightarrow{\mathrm{f}} \mathcal{S}$ as above, let $\hat{J}$ denote the topology on $\mathcal{F}$ generated by K, with characteristic map $\hat{j}$. Then $sh_{\hat{j}}(\mathcal{F})$ is a category over $\mathcal{F}$.

2.11 Theorem (Paré) *As categories over $\mathcal{S}$, $(sh_{\hat{j}}(\mathcal{F}))_{\mathrm{f}} = sh_j(\mathcal{F})$.*

PROOF The topos $\mathcal{F}$ is a stack and it has small powers, so the two preceeding propositions apply. Let us demonstrate the expressed equality at $1 \in \mathcal{S}$. An object $X \in \mathcal{F}$ is a $\hat{j}$-sheaf if and only if X is a $\mathbf{K}$-sheaf if and only if X is a $(\mathrm{f}^*\mathbf{d})$-sheaf if and only if

$$(\mathrm{f}^*J)^*X \to ((\mathrm{f}^*J)^*X)^{\mathrm{f}^*d}$$

is an isomorphism if and only if

$$J^*X \to (J^*X)^d$$

is an isomorphism, where $\mathcal{F}$ is now being viewed as a category over $\mathcal{S}$. This last statement is true if and only if X is a j-sheaf. □

2.12 Corollary *In the notation of the above theorem, we have $sh_j(\mathcal{F}) \cong \mathcal{E} \times_{\mathcal{S}} \mathcal{F}$ as categories over $\mathcal{S}$.*

PROOF The topos of $\hat{j}$-sheaves in $\mathcal{F}$, in the ordinary sense, is the pullback $\mathcal{E} \times_{\mathcal{S}} \mathcal{F}$ in toposes. The result follows by example 2.2 and theorem 2.11. □

In the terminology of Chapter 1 §6, the above corollary says that if $\mathcal{A}$ comes from a topos, then so does $sh_j(\mathcal{A})$.

2.13 Example In this example it is shown that the category of sheaves in a small category is, with an additional assumption, again small. As an application of the General Adjoint Functor Theorem, in [PS] it is shown that a small category is complete if and only if it is cocomplete. Since the present context is within cocomplete categories, the following result is stated in these terms.

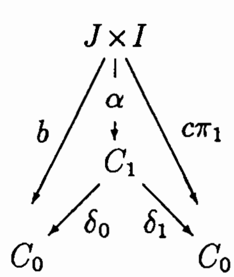
Let $\mathbf{C}$ be a small cocomplete category. Then $sh_j(\mathbf{C})$ is small and cocomplete. Furthermore, the object of objects $sh_j(\mathbf{C})_0$ is itself a sheaf, as is the object of morphisms $sh_j(\mathbf{C})_1$.

Observe that the second statement in italics says that $sh_j(\mathbf{C})$ is a small category in $\mathcal{E} = sh_j(\mathcal{S})$.

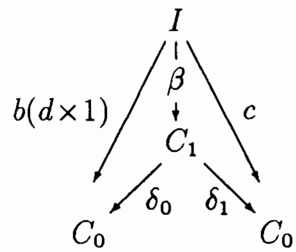
To prove our claims, one can use the fact that a category is small if and only if it is locally small and there exists an object of objects (for a proof of this see [PS]). Thus, our aim is to construct the object of sheaves, those elements of C_0 that are sheaves. Let c be an arbitrary object in $\mathbf{C}^I$, that is, a morphism $I \xrightarrow{c} C_0$. We work over $\mathcal{S}_{/I}$. By 2.9, the object c is a sheaf if and only if the canonical morphism $(I^*J)^*c \rightarrow ((I^*J)^*c)^{I^*d}$ is an isomorphism. By the universal property of powers, and since $J \cdot d = 1$, this is true if and only if application of $(I^*d)^*$ gives a bijection:

$$\frac{b \xrightarrow{\alpha} (I^*J)^*c}{(I^*d)^*b \xrightarrow{\beta} c}$$

for any $b \in \mathbf{C}^{I \times J}$. In concrete terms this means precisely that composition with $d \times 1$ gives a bijection between commuting diagrams:



and commuting diagrams:



Thus, the sheaves in $\mathbf{C}$ can be described using the internal language of the topos. Define $sh_j(\mathbf{C})_0$ to be the following sub-object of C_0:

$$[\![c \in C_0 \mid c \text{ is a sheaf}]\!]$$

$$= [\![c \in C_0 \mid \forall b \in C_0^J \; \forall \beta \in C_1(((\delta_0 \beta = bd) \wedge (\delta_1 \beta = c))$$
$$\Rightarrow \exists! \alpha \in C_1^J((\delta_0 \alpha = b) \wedge (\delta_1 \alpha = cJ) \wedge (\beta = \alpha d)))]\!]$$

One now verifies that a given $I \xrightarrow{c} C_0$ is a sheaf if and only if it factors through $sh_j(\mathbf{C})_0$. One concludes that there is therefore an object of morphisms $sh_j(\mathbf{C})_1$ in $\mathcal{S}$. As for the cocompleteness of $sh_j(\mathbf{C})$, this follows because $sh_j(\mathbf{C})$ is complete since $\mathbf{C}$ is, and being small, $sh_j(\mathbf{C})$ is therefore cocomplete.

It remains to show that the objects $sh_j(\mathbf{C})_0$ and $sh_j(\mathbf{C})_1$ are *themselves* sheaves. Let $S \overset{s}{\hookrightarrow} T$ be a j-dense monomorphism. As a general comment observe that if $\mathcal{A}$ is an arbitrary category, then the substitution functor s^* is fully faithful when restricted to sheaves. In our case $\mathbf{C}$ is complete, and therefore so is $sh_j(\mathbf{C})$. In particular, $sh_j(\mathbf{C})$ has Π satisfying the Beck condition, and Π is always fully faithful along monomorphisms. Thus, restricted to sheaves, s^* must be an equivalence,

$$sh_j(\mathbf{C})^T \cong sh_j(\mathbf{C})^S ,$$

for the j-dense monomorphism $S \overset{s}{\hookrightarrow} T$. Since s^* is composition with s , it follows that the objects $sh_j(\mathbf{C})_0$ and $sh_j(\mathbf{C})_1$ are both sheaves.

2.3 Sheafification

As in the previous section, $\mathcal{S}$ is the base topos, j is a topology on $\mathcal{S}$, and $\mathcal{E} \xrightarrow{i} \mathcal{S}$ is the sub-topos of sheaves. For a category $\mathcal{A}$ over $\mathcal{S}$, $sh_j(\mathcal{A})$ denotes the full sub-category of $\mathcal{A}$ whose objects are the j-sheaves.

3.1 Definition $\mathcal{A}$ is said to *admit j-sheafification* if $sh_j(\mathcal{A})$ is a reflective sub-category of $\mathcal{A}$.

When $\mathcal{A}$ does admit j-sheafification, the left adjoint of the inclusion functor shall be denoted by a,

$$sh_j(\mathcal{A}) \underset{\mathsf{i}}{\overset{\mathsf{a}}{\leftrightarrows}} \mathcal{A}$$

where $\mathsf{a} \dashv \mathsf{i}$. These are functors over $\mathcal{S}$.

Let us say that $\mathcal{A}$ *admits sheafification* if $\mathcal{A}$ admits j-sheafification for all topologies j on $\mathcal{S}$.

3.2 Examples

1. Let $\mathcal{B}$ denote an arbitrary category over $\mathcal{E}$. Then the restriction $\mathcal{B}_{\mathsf{i}}$ admits j-sheafification. Trivially in fact, because every object in $\mathcal{B}_{\mathsf{i}}$ is a sheaf. That is, $sh_j(\mathcal{B}_{\mathsf{i}}) = \mathcal{B}_{\mathsf{i}}$.

2. A topos over $\mathcal{S}$ admits sheafification. Recall that from the previous section (corollary 2.12) that for $\mathcal{F}$ a topos over $\mathcal{S}$, $sh_j(\mathcal{F})$ is equivalent to the pullback $\mathcal{E} \times_{\mathcal{S}} \mathcal{F}$ in toposes. This is a reflective sub-category of $\mathcal{F}$.

3. Any small cocomplete category admits sheafification. In fact, let $\mathbf{C}$ be such a category, and let j be a topology on $\mathcal{S}$. $\mathbf{C}$ is necessarily complete, and therefore so is $sh_j(\mathbf{C})$ as limits in $sh_j(\mathbf{C})$ are just computed in $\mathbf{C}$. (This second statement being true quite generally.) As was seen in the previous section, $sh_j(\mathbf{C})$ is small. Therefore it satisfies the solution set of objects condition (see Appendix B). Thus, since the inclusion functor is continuous, it must have a left adjoint by the General Adjoint Functor Theorem.

Recall that $sh_j(\mathcal{A})$ can be regarded as a category over $\mathcal{E}$ by defining the fiber at $K \in \mathcal{E}$ to be $sh_j(\mathcal{A})^{\mathsf{i}_* K}$. Under this definition one has the following.

3.3 Theorem *Let $\mathcal{A} \in$ COCTS$_{\mathcal{S}}$, admitting j-sheafification. Then $sh_j(\mathcal{A}) \in$ COCTS$_{\mathcal{E}}$, and is equivariant as explained in the previous section.*

PROOF If $sh_j(\mathcal{A})$ is a reflective sub-category of $\mathcal{A}$ over $\mathcal{S}$, then $sh_j(\mathcal{A})$ is cocomplete over $\mathcal{S}$ since $\mathcal{S}$-colimits in $sh_j(\mathcal{A})$ can be calculated in $\mathcal{A}$ and then reflected into $sh_j(\mathcal{A})$. Obviously, $sh_j(\mathcal{A})$ is then cocomplete as regarded over $\mathcal{E}$. By 2.3, $sh_j(\mathcal{A})$ is locally small over $\mathcal{E}$. The equivariance follows by 2.6 since, being cocomplete and locally small over $\mathcal{S}$, $sh_j(\mathcal{A})$ is a stack. Alternatively, since $\mathcal{A}$ is a stack it follows in any case that $sh_j(\mathcal{A})$ is also. Since $sh_j(\mathcal{A})$ is assumed to be a reflective sub-category of $\mathcal{A}$, $sh_j(\mathcal{A})$ has coproducts, and 2.6 again applies. □

The general problem of when $\mathcal{A}$ admits sheafification is an interesting one in itself, but to be pursued no further here. In the event that $sh_j(\mathcal{A})$ is a reflective sub-category of $\mathcal{A}$, in the next chapter it is seen that $sh_j(\mathcal{A})$ is in some sense a 'module of fractions'; it has a universal property analogous to that of a module of fractions in rings and modules. Theorem 3.3 is the first step towards proving such a result.

In the study of sheaves in the opposite category, or 'cosheaves', to follow in the next section, it is shown that $\mathcal{F}^{op}$ admits sheafification for $\mathcal{F}$ a topos over $\mathcal{S}$. More importantly, it will be seen that the category of cosheaves in a topos form a topos.

2.4 Cosheaves and open topologies

Let $\mathcal{A}$ be a category over $\mathcal{S}$, and let j be a topology on $\mathcal{S}$. Having defined previously the notion of a sheaf in an arbitrary category, consider now the category $(sh_j(\mathcal{A}^{op}))^{op}$, which is a full sub-category of $\mathcal{A}$. Let us simplify the notation, and write $sh_j(\mathcal{A}^{op})^{op}$ for this category.

4.1 DEFINITION The category $sh_j(\mathcal{A}^{op})^{op}$ shall be called the category of *j-cosheaves* in $\mathcal{A}$.

Let us begin this section by investigating the category of j-cosheaves in the base topos $\mathcal{S}$.

Let $U \hookrightarrow 1$ be a sub-object of 1. Such an object is sometimes referred to as an *open* object. The morphism

$$\Omega \overset{U\times I}{\to} \Omega\times\Omega \overset{\Rightarrow}{\to} \Omega$$

is a topology on $\mathcal{S}$ which is denote by j^o_U . It is called the *open topology* associated with U. Furthermore, there is an equivalence $sh_{j^o_U}(\mathcal{S}) \cong \mathcal{S}_{/U}$ which identifies the inclusion functor with Π_U. The associated j^o_U-sheaf of an object I is I^U. These facts are proved in [J1]. The map $U \rightsquigarrow j^0_U$ is order reversing from the lattice of sub-objects of 1 to the lattice of topologies on $\mathcal{S}$.

For an arbitrary topology j on $\mathcal{S}$ define the *interior* of j to be the equalizer of $1 \overset{\hat{1}}{\to} \Omega^\Omega$ and $1 \overset{\hat{j}}{\to} \Omega^\Omega$. Denote this open object by *int(j)*. There is a (contravariant) Galois correspondence (see [J1]) between the lattice of sub-objects of Ω and itself, which shall be denoted by $l \dashv r$. If U is an arbitrary open object then $U\times\Omega$ is a sub-object of Ω, and it is the sub-object classifier in the topos $sh_{j^o_U}(\mathcal{S})$. Therefore, $(U\times\Omega)^l$ is the sub-object of Ω classified by j^o_U . There is then the following series of equivalences:

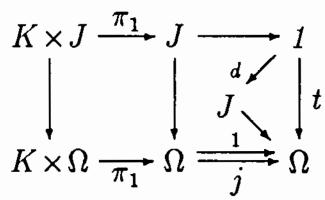
$$
\begin{array}{c}
U \leq int(j) \\
\hline
U \times \Omega \xrightarrow{\pi_1} \Omega \underset{j}{\overset{1}{\rightrightarrows}} \Omega \text{ commutes} \\
\hline
U \times \Omega \leq \Omega_j = J^r \\
\hline
J \leq (U \times \Omega)^l \\
\hline
j \leq j_U^o
\end{array}
$$

for any j and any open U, where J denotes the sub-object of Ω classified by j. A (contravariant) Galois correspondence has been established between the open objects and the lattice of topologies on $\mathcal{S}$ ([J1] page 102, ex. 10).

A characterization of cosheaves in $\mathcal{S}$ can now be obtained.

4.2 Proposition *An object $I \in \mathcal{S}$ is a j-cosheaf if and only if $I \in int(j)$.*

PROOF First observe that in the following diagram,



$dJ\pi_1 = \pi_1$ if and only if $j\pi_1 = \pi_1$, where K is any object.

Fix $I \in \mathcal{S}$. A cosheaf is by definition a sheaf in the opposite category, and so by proposition 2.9 one sees that I is a cosheaf if and only if the morphism $I \overset{(1,dI)}{\to} I \times J$ is an isomorphism, in which case its inverse is the projection $I \times J \xrightarrow{\pi_0} I$. Let $U \hookrightarrow 1$ denote the support of I. Then one has the following series of equivalences:

$$
\begin{array}{c}
I \overset{(1,dI)}{\to} I \times J \text{ is an isomorphism} \\
\hline
U \overset{(1,dU)}{\to} U \times J \text{ is an isomorphism} \\
\hline
(1, dU)\pi_0 = 1, \text{ where } U \times J \xrightarrow{\pi_0} U \\
\hline
\pi_1(1, dU)\pi_0 = \pi_1, \text{ where } U \times J \xrightarrow{\pi_1} J \\
\text{is the projection, a monomorphism} \\
\hline
dJ\pi_1 = \pi_1, \\
\text{since } \pi_1(1, dU)\pi_0 = dJ\pi_1 \\
\hline
j\pi_1 = \pi_1, \text{ as in the diagram above} \\
\text{taking } K = U \\
\hline
U \leq int(j) \\
\hline
I \in int(j)
\end{array}
$$

□

Thus, the category of j-cosheaves in $\mathcal{S}$ is isomorphic to $\mathcal{S}_{/int(j)}$, by an isomorphism which identifies the inclusion functor with $\Sigma_{int(j)}$.

Cosheaves are an example of a 'unity of opposites'. Let $V = int(j)$. The adjoint pair

$$\mathcal{S} \underset{(\ )^V}{\overset{V\times(\ )}{\rightleftarrows}} \mathcal{S}$$

induces an equivalence of categories $\mathbf{Fix}(\varepsilon) \cong \mathbf{Fix}(\eta)$, where ε is the counit, and η is the unit of the above adjointness. For an arbitrary object I , since V is a sub-object of 1, it follows that $V \times I^V \simeq V \times I$. By proposition 4.2, I is a cosheaf if and only if $I \simeq V \times I$, and therefore if and only if the counit $V \times I^V \to I$ is an isomorphism. Thus, I is a j-cosheaf if and only if $I \in \mathbf{Fix}(\varepsilon)$. Similarly, one sees that I is a j^o_V-sheaf if and only if $I \in \mathbf{Fix}(\eta)$, and so the 'unity of opposites' says that the category of j-cosheaves is equivalent to the category of j^o_V-sheaves.

Any object $I \in \mathcal{S}$ has an *associated* cosheaf, it being the object $int(j) \times I$. In particular, the associated cosheaf of 1 is $int(j)$. In this regard one can think of $int(j)$ as the union of those sub-objects of 1 that are cosheaves.

Let $\mathsf{c}I$ denote the associated cosheaf of I . Since $j \leq j^o_{int(j)}$, every j-dense monomorphism is $j^o_{int(j)}$-dense. Therefore, c carries all j-dense monomorphisms to isomorphisms since it does so with the $j^o_{int(j)}$-dense ones. Hence, as we shall see in the next chapter, c factors through i^* , where $\mathcal{E} \xrightarrow{\mathsf{i}} \mathcal{S}$ is the inclusion of j-sheaves into $\mathcal{S}$. It follows that

$$\mathcal{E} \xrightarrow{\mathsf{i}_*} \mathcal{S} \xrightarrow{\mathsf{c}} \mathcal{S}$$

is cocontinuous. Moreover, the functor $\mathsf{c} \cdot \mathsf{i}_*$ is the terminal object in $\mathrm{C{\scriptsize OCTS}}_{\mathcal{S}}(\mathcal{E}, \mathcal{S})$. This fact plays a role in the study of the cocontinuous dual of $\mathcal{E}$ (see Chap. 4, §4 and §5). The following theorem, in which 4.2 is generalized to an arbitrary topos over $\mathcal{S}$, will be used in those investigations.

4.3 Theorem *Let $\mathcal{F}$ be an arbitrary topos over $\mathcal{S}$, j a topology on $\mathcal{S}$. Let $\hat{j}$ denote the topology on $\mathcal{F}$ induced by j. Then there is an isomorphism of categories*

$$sh_{\hat{j}}(\mathcal{F}^{op})^{op} \cong \mathcal{F}_{/V}$$

identifying the inclusion functor with Σ_V, where V denotes the interior of $\hat{j}$.

A proof of 4.3 shall be given in the examples at the end of section §1 of the next chapter.

In particular, the category of cosheaves in a topos is a topos.

4.4 Example Since $\mathcal{A}^{op}$ admits j-sheafification if and only if $sh_j(\mathcal{A}^{op})^{op}$ is a coreflective sub-category of $\mathcal{A}$, by 4.3 $\mathcal{F}^{op}$ admits sheafification for any topos $\mathcal{F}$ over $\mathcal{S}$.

Cosheaves are related to the category $\textsc{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathcal{A})$, the study of which begins the next chapter.

CHAPTER 3

The Adjoints of Restriction

3.1 The right adjoint

Let $\mathcal{E} \xrightarrow{\mathsf{p}} \mathcal{S}$ be a topos over $\mathcal{S}$, and let $\mathcal{A}$ denote an arbitrary cocomplete category over $\mathcal{S}$. Then $\textsc{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathcal{A})$, the category of $\mathcal{S}$-cocontinuous functors from $\mathcal{E}$ to $\mathcal{A}$, can be viewed as a category over $\mathcal{E}$. In fact, the fiber above $K \in \mathcal{E}$ is defined to be

$$\textsc{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathcal{A})^K = \textsc{Cocts}_{\mathcal{S}}(\mathcal{E}^K, \mathcal{A}) .$$

For $K \xrightarrow{\alpha} L$ in $\mathcal{E}$, the substitution functor α^* is defined to be composition with Σ_α (pertaining to $\mathcal{E}$),

$$\alpha^*(\mathsf{F}) = \mathsf{F} \cdot \Sigma_\alpha ,$$

for $\mathsf{F} \in \textsc{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathcal{A})^K$.

It is not hard to see that $\textsc{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathcal{A})$ is cocomplete over $\mathcal{E}$. In fact, for any morphism α of $\mathcal{S}$, the substitution functor α^* has a left adjoint, which is given by composition with the pullback functor in $\mathcal{E}$,

$$\Sigma_\alpha(\mathsf{F}) = \mathsf{F} \cdot \alpha^* .$$

Furthermore, one easily verifies that the Beck condition holds. Finite colimits in the fiber $\textsc{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathcal{A})^K$ are computed 'pointwise', which the substitution functors are seen to preserve.

If $\mathsf{F} \in \textsc{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathcal{A})$ and $K \in \mathcal{E}$, then the K-copower of F, which is denoted by $K.\mathsf{F}$, is the cocontinuous functor

$$\mathcal{E} \xrightarrow{\mathsf{K}^*} \mathcal{E}_{/K} \xrightarrow{\Sigma_K} \mathcal{E} \xrightarrow{\mathsf{F}} \mathcal{A} .$$

Thus, $(K.\mathsf{F})(X) = \mathsf{F}(K.X)$ for any $X \in \mathcal{E}$.

The principal fact about the 2-functor $\mathcal{A} \rightsquigarrow \textsc{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathcal{A})$ is that it is the right adjoint of the restriction functor $(\ )_{\mathsf{p}}$.

1.1 Proposition *For any cocomplete categories $\mathcal{B}$ over $\mathcal{E}$ and $\mathcal{A}$ over $\mathcal{S}$, there is an equivalence of categories*

$$\textsc{Cocts}_{\mathcal{S}}(\mathcal{B}_{\mathsf{p}}, \mathcal{A}) \cong \textsc{Cocts}_{\mathcal{E}}(\mathcal{B}, \textsc{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathcal{A})) .$$

This equivalence is natural in $\mathcal{A}$ and $\mathcal{B}$.

PROOF The desired equivalence expresses a 2-adjointness for which the unit and counit are as follows. The unit shall be denoted by

$$\Lambda_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathrm{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathcal{B}_{\mathsf{p}}) .$$

Let us write Λ for $\Lambda_{\mathcal{B}}$, dropping the '$\mathcal{B}$'. For $K \in \mathcal{E}$, to be defined is

$$\Lambda^{K} : \mathcal{B}^{K} \longrightarrow \mathrm{Cocts}_{\mathcal{S}}(\mathcal{E}^{K}, \mathcal{B}_{\mathsf{p}}) .$$

So for $B \in \mathcal{B}^{K}$ and $I \in \mathcal{S}$ define

$$\Lambda^{K}(B)^{I} : \mathcal{E}_{/\mathsf{p}^{*}I \times K} \longrightarrow \mathcal{B}^{\mathsf{p}^{*}I}$$

by setting

$$\Lambda^{K}(B)^{I}(\alpha, \beta) = \Sigma_{\alpha}\beta^{*}(B) ,$$

where $M \overset{(\alpha,\beta)}{\to} \mathsf{p}^{*}I \times K$ is a typical object of $\mathcal{E}_{/\mathsf{p}^{*}I \times K}$. It follows that $\Lambda^{K}(B)$ is a cocontinuous functor over $\mathcal{S}$, and that Λ is cocontinuous over $\mathcal{E}$.

Turning now to the counit, it is a cocontinuous functor over $\mathcal{S}$ which shall be denoted by

$$\Xi_{\mathcal{A}} : \mathrm{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathcal{A})_{\mathsf{p}} \longrightarrow \mathcal{A} .$$

Writing Ξ for $\Xi_{\mathcal{A}}$, define Ξ^{I} for $I \in \mathcal{S}$ by setting

$$\Xi^{I}(\mathsf{F}) = \mathsf{F}^{I}(\delta_{I}) ,$$

where $\mathsf{F} \in \mathrm{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathcal{A})^{I}_{\mathsf{p}} = \mathrm{Cocts}_{\mathcal{S}}(\mathcal{E}^{I}, \mathcal{A})$, and where δ_{I} is the diagonal $\mathsf{p}^{*}I \to \mathsf{p}^{*}I \times \mathsf{p}^{*}I$ as an object of $(\mathcal{E}^{I})^{I} = \mathcal{E}_{/\mathsf{p}^{*}(I \times I)}$. Ξ is indeed cocontinuous. Let us show, for example, that Ξ preserves coproducts. That is, considering the unique map $I \to 1$ (the general case $I \to K$ is similiar), let us show that

$$\begin{array}{ccc} \mathrm{Cocts}_{\mathcal{S}}(\mathcal{E}^{I}, \mathcal{A}) & \xrightarrow{\ \Xi^{I}\ } & \mathcal{A}^{I} \\ {\scriptstyle \Sigma_{I}}\downarrow & \simeq & \downarrow{\scriptstyle \Sigma_{I}} \\ \mathrm{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathcal{A}) & \xrightarrow[\ \Xi\ ]{} & \mathcal{A} \end{array}$$

commutes. If $\mathsf{F} \in \mathrm{Cocts}_{\mathcal{S}}(\mathcal{E}^{I}, \mathcal{A})$, then one has

$$\Sigma_{I} \cdot \Xi^{I}(\mathsf{F}) = \Sigma_{I}(\mathsf{F}^{I}\delta_{I}) \simeq \mathsf{F}(\Sigma_{I}\delta_{I}) = \mathsf{F}(1) ,$$

where 1 is $\mathsf{p}^{*}I \xrightarrow{1} \mathsf{p}^{*}I$, the terminal object in $\mathcal{E}^{I}$. Going the other route,

$$\Xi \cdot \Sigma_{I}(\mathsf{F}) = \Xi(\Sigma_{I}\mathsf{F}) = (\Sigma_{I}\mathsf{F})(\delta_{1}) = (\mathsf{F} \cdot I^{*})(\delta_{1}) \simeq \mathsf{F}(1) .$$

Our task is to verify that $\textsc{Cocts}_{\mathcal{S}}(\mathcal{E},\Xi_{\mathcal{A}})\cdot\Lambda_{\textsc{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{A})}$ is isomorphic over $\mathcal{E}$ to the identity functor on $\textsc{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{A})$, and that $\Xi_{(\mathcal{B}_{\mathsf{p}})}\cdot(\Lambda_{\mathcal{B}})_{\mathsf{p}}$ is isomorphic over $\mathcal{S}$ to the identity functor on $\mathcal{B}_{\mathsf{p}}$. Let us turn first to the functor $\textsc{Cocts}_{\mathcal{S}}(\mathcal{E},\Xi_{\mathcal{A}})\cdot\Lambda_{\textsc{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{A})}$ over $\mathcal{E}$. For the remainder of this paragraph let $\hat{\mathcal{A}}$ denote $\textsc{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{A})$, Λ denote $\Lambda_{\hat{\mathcal{A}}}$, and Ξ denote $\Xi_{\mathcal{A}}$. For $K\in\mathcal{E}$ and $\mathsf{F}\in\hat{\mathcal{A}}^K = \textsc{Cocts}_{\mathcal{S}}(\mathcal{E}^K,\mathcal{A})$, to be shown is that $\textsc{Cocts}_{\mathcal{S}}(\mathcal{E},\Xi)^K\cdot\Lambda^K(\mathsf{F})\simeq\mathsf{F}$, where

$$\textsc{Cocts}_{\mathcal{S}}(\mathcal{E},\Xi)^K : \textsc{Cocts}_{\mathcal{S}}(\mathcal{E}^K,\hat{\mathcal{A}}_{\mathsf{p}})\longrightarrow\textsc{Cocts}_{\mathcal{S}}(\mathcal{E}^K,\mathcal{A})$$

$$\mathsf{G}\quad\rightsquigarrow\quad\Xi\cdot\mathsf{G}\,,$$

and

$$\Lambda^K : \hat{\mathcal{A}}^K\longrightarrow\textsc{Cocts}_{\mathcal{S}}(\mathcal{E}^K,\hat{\mathcal{A}}_{\mathsf{p}})\,.$$

So, to be shown is that

$$\Xi\cdot(\Lambda^K\mathsf{F}) : \mathcal{E}^K\longrightarrow\hat{\mathcal{A}}_{\mathsf{p}}\longrightarrow\mathcal{A}$$

is isomorphic to F over $\mathcal{S}$. Let $I\in\mathcal{S}$, and let $M\overset{(\alpha,\beta)}{\to}\mathsf{p}^*I\times K$ be an arbitrary object of $(\mathcal{E}^K)^I$. Then

$$(\Xi\cdot(\Lambda^K\mathsf{F}))^I(\alpha,\beta)=\Xi^I((\Lambda^K\mathsf{F})^I(\alpha,\beta))=((\Lambda^K\mathsf{F})^I(\alpha,\beta))^I(\delta_I)\,.$$

At this point observe that from the definitions $(\Lambda^K\mathsf{F})^I(\alpha,\beta)$ is the cocontinuous functor

$$\mathcal{E}^I\xrightarrow{\alpha^*}\mathcal{E}^M\xrightarrow{\Sigma_\beta}\mathcal{E}^K\xrightarrow{\mathsf{F}}\mathcal{A}\,.$$

Evaluating this functor at δ_I (at stage I) yields $\mathsf{F}^I(\alpha,\beta)$ because $\Sigma^I_\beta(\alpha^{*I}(\delta_I))\simeq(\alpha,\beta)$.

Regarding $\Xi_{(\mathcal{B}_{\mathsf{p}})}\cdot(\Lambda_{\mathcal{B}})_{\mathsf{p}}$ over $\mathcal{S}$, let us write Ξ for $\Xi_{\mathcal{B}_{\mathsf{p}}}$, and Λ for $\Lambda_{\mathcal{B}}$. Let $I\in\mathcal{S}$, and let $B\in(\mathcal{B}_{\mathsf{p}})^I=\mathcal{B}^{\mathsf{p}^*I}$. Then

$$\Xi^I(\Lambda^{\mathsf{p}^*I}(B))=(\Lambda^{\mathsf{p}^*I}(B))^I(\delta_I)=\Sigma_1 1^*(B)\simeq B\,,$$

since $\delta_I=(1,1)$, where 1 is the identity on p^*I. This concludes the proof. □

If $\mathcal{F}$ is a topos over $\mathcal{E}$, then by the above proposition, one obtains that which shall be referred to as the *change of base formula*:

$$(1.2)\qquad \textsc{Cocts}_{\mathcal{S}}(\mathcal{F},\mathcal{A})\cong\textsc{Cocts}_{\mathcal{E}}(\mathcal{F},\textsc{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{A}))\,,$$

for any cocomplete category $\mathcal{A}$ over $\mathcal{S}$. There is a small abuse of notation here in that on the left, $\mathcal{F}$ has been written instead of $\mathcal{F}_{\mathsf{p}}$.

Recall that $\mathcal{A}_{/I}$ denotes the localization of $\mathcal{A}$ at $I\in\mathcal{S}$.

1.3 Proposition *For any $I \in \mathcal{S}$, there is an equivalence $\mathcal{A}_{/I} \cong \text{Cocts}_{\mathcal{S}}(\mathcal{S}_{/I}, \mathcal{A})$ of categories over $\mathcal{S}_{/I}$, which is natural in $\mathcal{A}$.*

PROOF Define a functor

$$\Phi : \mathcal{A}_{/I} \longrightarrow \text{Cocts}_{\mathcal{S}}(\mathcal{S}_{/I}, \mathcal{A})$$

by defining for $A \in (\mathcal{A}_{/I})^1 = \mathcal{A}^I$

$$\Phi A : \mathcal{S}_{/I} \longrightarrow \mathcal{A}$$

to be

$$(\Phi A)^J(\alpha, \beta) = \Sigma_\alpha \beta^* A \ ,$$

where $J \in \mathcal{S}$ and $K \overset{(\alpha,\beta)}{\to} J \times I$ is a typical object of $(\mathcal{S}_{/I})^J = \mathcal{S}_{/J\times I}$. Then ΦA is a cocontinuous functor over $\mathcal{S}$. This defines Φ at $1 \in \mathcal{S}_{/I}$, and the general definition is obtained by localizing. One then routinely verifies that Φ is indeed a functor over $\mathcal{S}_{/I}$.

Next, define a functor

$$\Psi : \text{Cocts}_{\mathcal{S}}(\mathcal{S}_{/I}, \mathcal{A}) \longrightarrow \mathcal{A}_{/I}$$

by setting

$$\Psi \mathsf{F} = \mathsf{F}^I(\delta_I) \ ,$$

where $I \overset{\delta_I}{\to} I \times I$ is the diagonal regarded as an object of $(\mathcal{S}_{/I})^I$. Let us verify that Ψ is in fact a functor over $\mathcal{S}_{/I}$. To keep it simple let us consider the unique arrow $m \to 1$ in $\mathcal{S}_{/I}$, where $M \overset{m}{\to} I$. Then

$$(m^* \cdot \Psi)(\mathsf{F}) = m^*(\mathsf{F}^I(\delta_I)) = \mathsf{F}^M((1 \times m)^*(\delta_I)) \ .$$

On the other hand,

$$(\Psi^M \cdot m^*)(\mathsf{F}) = (\mathsf{F} \cdot \Sigma_m)^M(\delta_M) = \mathsf{F}^M(\Sigma_{m\times 1}(\delta_M)) \ .$$

Now observe that $\Sigma_{m\times 1}(\delta_M) \simeq (1 \times m)^*(\delta_I)$.

It remains to verify that Ψ and Φ are mutual inverses, which is a routine calculation. □

Thus, by 1.3 and 1.1 applied to $\mathcal{S}_{/I} \longrightarrow \mathcal{S}$, localization is right adjoint to restriction in the sense that

$$(1.4) \qquad \text{Cocts}_{\mathcal{S}}(\mathcal{B}_{|}, \mathcal{A}) \cong \text{Cocts}_{\mathcal{S}/I}(\mathcal{B}, \mathcal{A}_{/I}) \ .$$

natural in cocomplete categories $\mathcal{A}$ and $\mathcal{B}$. One can *a priori* view $\text{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{A})$ as a category over $\mathcal{S}$ by defining

$$\text{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{A})^I = \text{Cocts}_{\mathcal{S}/I}(\mathcal{E}_{/I},\mathcal{A}_{/I})\ , I\in\mathcal{S}\ .$$

By 1.4, this category is equivalent to $\text{Cocts}_{\mathcal{S}}((\mathcal{E}_{/I})_{\mathsf{l}},\mathcal{A})$. Recall (Chapter 1, §5) that $(\mathcal{E}_{/I})_{\mathsf{l}}$ is denoted by $\mathcal{E}^I$. Therefore, one has

$$\text{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{A})^I \cong \text{Cocts}_{\mathcal{S}}(\mathcal{E}^I,\mathcal{A})\ , I\in\mathcal{S}\ .$$

The right side of this equivalence is, by definition, $\text{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{A})^I_{\mathsf{p}}$, and so suppressing the variable I , one has

$$(1.5)\qquad \text{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{A}) \cong \text{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{A})_{\mathsf{p}}\ .$$

The equivalence 1.5 says that regarding $\text{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{A})$ as a category over $\mathcal{E}$ and then restricting back to $\mathcal{S}$ is the same as *a priori* viewing $\text{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{A})$ as a category over $\mathcal{S}$. This is the equivariance of $\text{Cocts}_{\mathcal{S}}(\mathcal{E},\_)$ regarded as an endofunctor of $\text{Cocts}_{\mathcal{S}}$. By analogy in locales and sup-lattices, if $\mathbf{A}\xrightarrow{f}\mathbf{B}$ is a homomorphism of locales and $\mathbf{M}$ an $\mathbf{A}$-module, then $Hom_{\mathbf{A}}(\mathbf{B},\mathbf{M})$ is an $\mathbf{A}$-module in two equivalent ways.

Also, the equivalence 1.5 identifies $\Xi_{\mathcal{A}}$ with p^* in the sense that

$$(1.6)\qquad \begin{array}{ccc} \text{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{A})_{\mathsf{p}} & \xrightarrow{\ \Xi_{\mathcal{A}}\ } & \mathcal{A} \\ \wr\| & & \wr\| \\ \text{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{A}) & \longrightarrow & \text{Cocts}_{\mathcal{S}}(\mathcal{S},\mathcal{A}) \end{array}$$

commutes up to natural isomorphism, where the bottom arrow is composition with p^* . This identification is of basic importance, and it shall in the future be made sometimes without notice.

It is not clear that $\text{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{A})$ is locally small if $\mathcal{A}$ is. Our next aim is to establish the following important fact.

1.7 Theorem *Assume that $\mathcal{E}\xrightarrow{\mathsf{p}}\mathcal{S}$ is bounded. Then for any $\mathcal{A}\in\text{Cocts}_{\mathcal{S}}$, the category $\text{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{A})$ is locally small over $\mathcal{E}$.*

1.8 Corollary *With $\mathcal{E}$ as in 1.7, if $\mathcal{F}$ denotes an arbitrary topos over $\mathcal{S}$, then $\text{Top}_{\mathcal{S}}(\mathcal{F},\mathcal{E})$ is locally small over $\mathcal{E}$.*

PROOF $\text{Top}_{\mathcal{S}}(\mathcal{F},\mathcal{E})$ is a full sub-category of $\text{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{F})$. □

The proof of 1.7 uses the following two propositions.

1.9 Proposition *Let $\mathcal{A}$ be a cocomplete category over $\mathcal{S}$, and let* $\mathbf{C}$ *denote a small category in $\mathcal{S}$. Then composition with the Yoneda embedding gives an equivalence* $\mathrm{Cocts}_{\mathcal{S}}(\mathcal{S}^{\mathbf{C}}, \mathcal{A}) \cong \mathcal{A}^{\mathbf{C}^{op}}$ *as categories over* $\mathcal{S}^{\mathbf{C}}$.

A proof of 1.9 can be found in [Pi] with the difference here that these categories are regarded as over $\mathcal{S}^{\mathbf{C}}$. The category $\mathcal{A}^{\mathbf{C}^{op}}$ is regarded as over $\mathcal{S}^{\mathbf{C}}$ by letting $(\mathcal{A}^{\mathbf{C}^{op}})^{\mathsf{x}} = \mathcal{A}^{\mathbf{X}^{op}}$, for $\begin{array}{c}\mathbf{X}\\ \downarrow \mathsf{x}\\ \mathbf{C}\end{array} \in \mathcal{S}^{\mathbf{C}}$. For a morphism $\mathsf{x} \xrightarrow{\alpha} \mathsf{y}$ in $\mathcal{S}^{\mathbf{C}}$, the substitution functor α^* for $\mathcal{A}^{\mathbf{C}^{op}}$ is given by composition with α^{op} (regarding α as a functor between the total categories of x and y). To indicate that the equivalence of 1.9 is indeed an equivalence over $\mathcal{S}^{\mathbf{C}}$, let $\begin{array}{c}\mathbf{X}\\ \downarrow \mathsf{x}\\ \mathbf{C}\end{array} \in \mathcal{S}^{\mathbf{C}}$. Let us consider the unique map $\mathsf{x} \to 1$ in $\mathcal{S}^{\mathbf{C}}$ (the general case $\mathsf{x} \to \mathsf{y}$ is similiar), and show that

$$(1.10) \qquad \begin{array}{ccc} \mathrm{Cocts}_{\mathcal{S}}(\mathcal{S}^{\mathbf{C}}, \mathcal{A}) & \cong & \mathcal{A}^{\mathbf{C}^{op}} \\ {\scriptstyle \mathsf{x}^*}\downarrow & & \downarrow{\scriptstyle \mathsf{x}^*} \\ \mathrm{Cocts}_{\mathcal{S}}(\mathcal{S}^{\mathbf{X}}, \mathcal{A}) & \cong & \mathcal{A}^{\mathbf{X}^{op}} \end{array}$$

commutes (up to natural isomorphism). First note that

$$\begin{array}{ccc} \mathbf{X}^{op} & \xrightarrow{\mathsf{x}^{op}} & \mathbf{C}^{op} \\ {\scriptstyle Y}\downarrow & \simeq & \downarrow{\scriptstyle Y} \\ \mathcal{S}^{\mathbf{X}} & \xrightarrow[\Sigma_{\mathsf{x}}]{} & \mathcal{S}^{\mathbf{C}} \end{array}$$

commutes, where Σ_{x} sends a discrete opfibration $\begin{array}{c}\mathbf{Y}\\ \downarrow \mathsf{y}\\ \mathbf{X}\end{array}$ to $\mathsf{x} \cdot \mathsf{y}$. Then if $\mathsf{F} \in \mathrm{Cocts}_{\mathcal{S}}(\mathcal{S}^{\mathbf{C}}, \mathcal{A})$, by going the top route in 1.10 one gets $\mathsf{F} \cdot Y \cdot \mathsf{x}^{op} \simeq \mathsf{F} \cdot \Sigma_{\mathsf{x}} \cdot Y$, which is the result going the other way in 1.10.

Proposition 1.3 is actually a special case of 1.9.

1.11 Proposition *Let j be a topology on $\mathcal{S}$, and let $\mathcal{E} \xrightarrow{\mathsf{i}} \mathcal{S}$ denote the sub-topos of j-sheaves. Then:*

1. *For any cocomplete category $\mathcal{A}$ over $\mathcal{S}$,*

$$\Xi_{\mathcal{A}} : \mathrm{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathcal{A})_{\mathsf{i}} \longrightarrow \mathcal{A}$$

is fully faithful. Its essential image consists of those $A \in \mathcal{A}$ with the property that if $S \overset{s}{\hookrightarrow} T$ is an arbitrary j-dense monomorphism then $S.A \xrightarrow{s.A} T.A$ is an isomorphism.

2. If in addition $\mathcal{A}$ is locally small, and if $S \overset{s}{\hookrightarrow} T$ is a given monomorphism such that $\widetilde{(\mathrm{s})}$ (see Chapter 2, §2) generates J, then the essential image of $\Xi_{\mathcal{A}}$ consists of those $A \in \mathcal{A}$ such that $S.A \overset{s.A}{\rightarrow} T.A$ is an isomorphism.

3. For any cocomplete $\mathcal{B}$ over $\mathcal{E}$, the unit

$$\Lambda_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathrm{C}\textsc{octs}_{\mathcal{S}}(\mathcal{E}, \mathcal{B}_{\mathsf{i}})$$

is an equivalence.

PROOF *1.* In view of 1.6 (occurring just before Theorem 1.7), $\Xi_{\mathcal{A}}$ is fully faithful since composition with i^* clearly is. Regarding the statement about the essential image of $\Xi_{\mathcal{A}}$, it is equivalent to the statement that $\mathsf{G} \in \mathrm{C}\textsc{octs}_{\mathcal{S}}(\mathcal{S}, \mathcal{A})$ factors through i^* if and only if G takes j-dense monomorphisms to isomorphisms. The necessity of this latter statement is clear. For the sufficiency, let $\mathcal{S} \overset{\mathsf{G}}{\longrightarrow} \mathcal{A}$ be any cocontinuous functor taking j-dense monomorphisms to isomorphisms. Then G takes j-bidense morphisms to isomorphisms. In fact, an epimorphism $I \overset{\alpha}{\twoheadrightarrow} K$ is bidense if and only if the inclusion of I into the kernel pair of α is j-dense. It follows that $\mathsf{G}\alpha$ must be an isomorphism because G preserves coequalizers. Thus, there is a natural isomorphism

$$\mathsf{G} \simeq \mathsf{G} \cdot \mathsf{i}_* \cdot \mathsf{i}^* ,$$

since $1_{\mathcal{S}} \rightarrow \mathsf{i}_* \cdot \mathsf{i}^*$ is bidense. So if $\mathsf{G} \cdot \mathsf{i}_*$ is shown to be cocontinuous, the proof of *1.* will be complete. But this follows easily. Indeed, let $\gamma \in \mathcal{E}^{\mathbf{D}}$ be a diagram in $\mathcal{E}$, where $\mathbf{D}$ is a small category in $\mathcal{S}$. Let σ correspond under $\mathsf{i}^* \dashv \mathsf{i}_*$ to the canonical isomorphism

$$\mathsf{i}^*(\varinjlim(\mathsf{i}_* \cdot \gamma)) \simeq \varinjlim(\mathsf{i}^* \cdot \mathsf{i}_* \cdot \gamma) \simeq \varinjlim \gamma \, .$$

Then σ is bidense, and therefore

$$\varinjlim(\mathsf{G} \cdot \mathsf{i}_* \cdot \gamma) \simeq \mathsf{G}(\varinjlim(\mathsf{i}_* \cdot \gamma)) \overset{\mathsf{G}\sigma}{\rightarrow} \mathsf{G} \cdot \mathsf{i}_*(\varinjlim \gamma)$$

is an isomorphism. That is, $\mathsf{G} \cdot \mathsf{i}_*$ is cocontinuous.

2. Let $\mathcal{S} \overset{\mathsf{G}}{\longrightarrow} \mathcal{A}$ be any cocontinuous functor taking the monomorphism s to an isomorphism. Since $\mathcal{A}$ is assumed to be locally small, G has a right adjoint, $\mathsf{G} \dashv \mathsf{R}$. Then for any $A \in \mathcal{A}$, $\mathsf{R}A$ is seen to have the sheaf property with respect to s because $\mathsf{G}s$ is an isomorphism. By the results from [P2], which were reviewed in Chapter 2, $\mathsf{R}A$ is therefore an $\widetilde{(\mathrm{s})}$-sheaf. Hence, $\mathsf{R}A$ is a sheaf since it is assumed that $\widetilde{(\mathrm{s})}$ generates J, and so

$$\mathsf{R} \simeq \mathsf{i}_* \cdot (\mathsf{i}^* \cdot \mathsf{R}) \, . \qquad (1.12)$$

It follows that $\mathsf{G} \cdot \mathsf{i}_* \dashv \mathsf{i}^* \cdot \mathsf{R}$, and upon taking left adjoints of 1.12 one has

$$\mathsf{G} \simeq (\mathsf{G} \cdot \mathsf{i}_*) \cdot \mathsf{i}^* \ .$$

This concludes the proof of *2.*

3. Let $\mathcal{B}$ be a cocomplete category over $\mathcal{E}$. Observe that the counit $\Xi_{(\mathcal{B}_\mathsf{i})}$ is an equivalence. In fact, by *1.* above, it is fully faithful. To see that $\Xi_{(\mathcal{B}_\mathsf{i})}$ is essentially surjective let $S \overset{s}{\hookrightarrow} T$ be j-dense. Then

$$s^* : \mathcal{B}_\mathsf{i}^T \longrightarrow \mathcal{B}_\mathsf{i}^S$$

is an equivalence since $\mathsf{i}^*(s)$ is an isomorphism. It follows (see Chapter 1, §4, the discussion on copowers) that $S.B \overset{s.B}{\to} T.B$ is an isomorphism for every $B \in \mathcal{B}_\mathsf{i}$, which shows that $\Xi_{(\mathcal{B}_\mathsf{i})}$ is essentially surjective. Now $\Xi_{(\mathcal{B}_\mathsf{i})} \cdot (\Lambda_\mathcal{B})_\mathsf{i}$ is isomorphic to the identity functor on $\mathcal{B}_\mathsf{i}$, and therefore $(\Lambda_\mathcal{B})_\mathsf{i}$ is an equivalence. It is easy to see that $(\ )_\mathsf{i}$ reflects equivalences, and therefore, $\Lambda_\mathcal{B}$ is an equivalence. □

Observe, by *3.* above, that $(\ )_\mathsf{i}$ is 2-fully faithful, which means that

$$\begin{array}{ccc} \text{Cocts}_\mathcal{E}(\mathcal{A},\mathcal{B}) & \longrightarrow & \text{Cocts}_\mathcal{S}(\mathcal{A}_\mathsf{i},\mathcal{B}_\mathsf{i}) \\ \mathsf{F} & \rightsquigarrow & \mathsf{F}_\mathsf{i} \end{array}$$

is an equivalence for any cocomplete categories $\mathcal{A}$ and $\mathcal{B}$ over $\mathcal{E}$.

Proposition 1.11 has the following useful corollary.

1.13 Corollary *Let $\mathcal{F}$ be a topos over $\mathcal{S}$, and j a topology on $\mathcal{F}$ with $\mathcal{E} = sh_j(\mathcal{F})$. Then for any cocomplete $\mathcal{A}$, $\text{Cocts}_\mathcal{S}(\mathcal{E},\mathcal{A})$ is equivalent to the full sub-category of $\text{Cocts}_\mathcal{S}(\mathcal{F},\mathcal{A})$ whose objects consist of those cocontinuous functors taking j-dense monomorphisms to isomorphisms.*

PROOF We have

$$\text{Cocts}_\mathcal{S}(\mathcal{E},\mathcal{A}) \cong \text{Cocts}_\mathcal{F}(\mathcal{E}, \text{Cocts}_\mathcal{S}(\mathcal{F},\mathcal{A}))\ .$$

Therefore, by 1.11 $\text{Cocts}_\mathcal{S}(\mathcal{E},\mathcal{A})$ is equivalent to the full sub-category of $\text{Cocts}_\mathcal{S}(\mathcal{F},\mathcal{A})$ determined by those cocontinuous $\mathcal{F} \overset{\mathsf{F}}{\longrightarrow} \mathcal{A}$ such that

$$S.\mathsf{F} \overset{s.\mathsf{F}}{\to} T.\mathsf{F}$$

is an isomorphism, for any j-dense monomorphism $S \overset{s}{\hookrightarrow} T$. By recalling how copowers are calculated in $\text{Cocts}_\mathcal{S}(\mathcal{F},\mathcal{A})$ over $\mathcal{F}$, the result follows. □

Corollary 1.13 will be put to use (see Chapter 4, §4) in the case that $\mathcal{F}$ is a topos of presheaves.

Let us now return to the proof of 1.7.

PROOF (of 1.7) By the change of base formula 1.2, that is since

$$\text{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathcal{A}) \cong \text{Cocts}_{\mathcal{S}^{\mathbf{C}}}(\mathcal{E}, \text{Cocts}_{\mathcal{S}}(\mathcal{S}^{\mathbf{C}}, \mathcal{A})) ,$$

one can separate the cases $\mathcal{E} = \mathcal{S}^{\mathbf{C}}$ for $\mathbf{C}$ a small category in $\mathcal{S}$, and $\mathcal{E}$ a sub-topos of $\mathcal{S}$. Let us first take the case $\mathcal{E} = \mathcal{S}^{\mathbf{C}}$. Of course, if $\mathcal{A}$ is locally small, then $\mathcal{A}^{op}$ is also locally small. Therefore, $(\mathcal{A}^{op})^{\mathbf{C}}$ is locally small over $\mathcal{S}^{\mathbf{C}}$. This was proved in Chapter 2, §2. The opposite category of $(\mathcal{A}^{op})^{\mathbf{C}}$ is

$$\mathcal{A}^{\mathbf{C}^{op}} \cong \text{Cocts}_{\mathcal{S}}(\mathcal{S}^{\mathbf{C}}, \mathcal{A}) ,$$

and therefore, $\text{Cocts}_{\mathcal{S}}(\mathcal{S}^{\mathbf{C}}, \mathcal{A})$ is locally small over $\mathcal{S}^{\mathbf{C}}$.

Let us now turn to the case that $\mathcal{E}$ is the sub-topos of sheaves for a topology j on $\mathcal{S}$. Observe that $\text{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathcal{A})$ is locally small over $\mathcal{S}$ since it is a full sub-category of $\mathcal{A}$. But for a cocontinuous functor $\mathcal{E} \xrightarrow{\mathsf{A}} \mathcal{A}$, the hom-object $(A, X) \in \mathcal{S}$ is a j-sheaf, where $A = \Xi_{\mathcal{A}}(\mathsf{A})$ and X is any object of $\mathcal{A}$. In fact, let $S \overset{s}{\hookrightarrow} T$ be an arbitrary j-dense monomorphism. Then $S.A \overset{s.A}{\to} T.A$ is an isomorphism, and one has the following series of bijections:

$$\frac{\frac{\frac{\frac{\frac{S \to (A, X)}{S^* A \to S^* X}}{S.A \to X}}{T.A \to X}}{T^* A \to T^* X}}{T \to (A, X)\ .}$$

This bijection is given by composition with s , and therefore the object (A, X) is a sheaf. Thus, $\text{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathcal{A})$ is locally small over $\mathcal{E}$ in the case of an inclusion. This concludes the proof of the proposition. □

Recall that $\text{Cocts}_{\mathcal{S}}$ denotes the 2-category of locally small cocomplete categories over $\mathcal{S}$. Thus, for $\mathcal{E} \xrightarrow{\mathsf{p}} \mathcal{S}$ bounded, $(\ )_{\mathsf{p}}$ has a right adjoint. It is the 2-functor

$$\text{Cocts}_{\mathcal{S}}(\mathcal{E}, \_) : \text{Cocts}_{\mathcal{S}} \longrightarrow \text{Cocts}_{\mathcal{E}}\ .$$

A key fact about the adjointness $(\ )_{\mathsf{p}} \dashv \text{Cocts}_{\mathcal{S}}(\mathcal{E}, \_)$ is that it satisfies the Beck condition.

1.14 Proposition *Let* $\mathcal{E} \xrightarrow{\mathsf{p}} \mathcal{S}$ *be a bounded geometric morphism, and suppose that*

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\mathsf{r}} & \mathcal{F} \\ {\scriptstyle \mathsf{q}}\downarrow & & \downarrow{\scriptstyle \mathsf{f}} \\ \mathcal{E} & \xrightarrow[\mathsf{p}]{} & \mathcal{S} \end{array}$$

is a pullback square of toposes. Then for any $\mathcal{B} \in \text{COCTS}_{\mathcal{F}}$ *, there is a canonical equivalence*

$$\text{COCTS}_{\mathcal{F}}(\mathcal{P}, \mathcal{B})_{\mathsf{q}} \cong \text{COCTS}_{\mathcal{S}}(\mathcal{E}, \mathcal{B}_{\mathsf{f}})$$

of categories in $\text{COCTS}_{\mathcal{E}}$ *.*

PROOF First assume that $\mathcal{E} = \mathcal{S}^{\mathbf{C}}$, for $\mathbf{C}$ a small category in $\mathcal{S}$. Then $\mathcal{P} \cong \mathcal{F}^{\mathsf{f}^*\mathbf{C}}$, and if $\begin{matrix}\mathbf{X}\\ \downarrow \mathsf{x}\\ \mathbf{C}\end{matrix} \in \mathcal{S}^{\mathbf{C}}$, then $\mathsf{q}^*\mathsf{x} = \begin{matrix}\mathsf{f}^*\mathbf{X}\\ \downarrow \mathsf{f}^*\mathsf{x}\\ \mathsf{f}^*\mathbf{C}\end{matrix}$. Let $\mathcal{B} \in \text{COCTS}_{\mathcal{F}}$. Then by definition,

$$\text{COCTS}_{\mathcal{F}}(\mathcal{P}, \mathcal{B})^{\mathsf{x}}_{\mathsf{q}} = \text{COCTS}_{\mathcal{F}}(\mathcal{P}_{/\mathsf{q}^*\mathsf{x}}, \mathcal{B})$$

which is equivalent to

$$\text{COCTS}_{\mathcal{F}}(\mathcal{F}^{\mathsf{f}^*\mathbf{X}}, \mathcal{B}) \cong \mathcal{B}^{\mathsf{f}^*\mathbf{X}^{op}} .$$

On the other hand,

$$\text{COCTS}_{\mathcal{S}}(\mathcal{S}^{\mathbf{C}}, \mathcal{B}_{\mathsf{f}})^{\mathsf{x}} \cong \text{COCTS}_{\mathcal{S}}(\mathcal{S}^{\mathbf{X}}, \mathcal{B}_{\mathsf{f}}) \cong (\mathcal{B}_{\mathsf{f}})^{\mathbf{X}^{op}} ,$$

and this last category is in fact *equal* to $\mathcal{B}^{\mathsf{f}^*\mathbf{X}^{op}}$. Thus, the fibers at $\mathsf{x} \in \mathcal{S}^{\mathbf{C}}$ are equivalent, and one only needs to verify that this constitutes an equivalence over $\mathcal{S}^{\mathbf{C}}$.

Now assume that $\mathsf{p} = \mathsf{i}$ is an inclusion with $\mathcal{E} = sh_j(\mathcal{S})$. Let J denote the sub-object of Ω characterized by j, and let $1 \xrightarrow{d} J$ denote the factorization of $1 \xrightarrow{t} \Omega$ through J . Let $K \hookrightarrow \Omega_{\mathcal{F}}$ denote the image of the characteristic map of f^*d , and let $\hat{J}$ denote the topology on $\mathcal{F}$ generated by K . Then, as is well known, $\mathcal{P}$ is equivalent to the topos of $\hat{J}$-sheaves. Furthermore, $\mathbf{K}$, the full sub-category of Ω determined by K , consists of those monomorphisms which are locally pullbacks of f^*d . That is, $\mathbf{K} = (\widetilde{\mathsf{f}^*\mathbf{d}})$ (see Chap. 2, §2). Being 2-fully faithful, $(\ )_{\mathsf{i}}$ is full on equivalences, and so it suffices to show that

$$(\text{COCTS}_{\mathcal{F}}(\mathcal{P}, \mathcal{B})_{\mathsf{q}})_{\mathsf{i}} \cong \text{COCTS}_{\mathcal{S}}(\mathcal{E}, \mathcal{B}_{\mathsf{f}})_{\mathsf{i}} ,$$

and hence that

$$(1.15) \qquad (\text{COCTS}_{\mathcal{F}}(\mathcal{P},\mathcal{B})_{\mathsf{r}})_{\mathsf{f}} \cong \text{COCTS}_{\mathcal{S}}(\mathcal{E},\mathcal{B}_{\mathsf{f}})_{\mathsf{i}} .$$

By 1.11, the essential image of

$$\Xi_{(\mathcal{B}_{\mathsf{f}})} : \text{COCTS}_{\mathcal{S}}(\mathcal{E},\mathcal{B}_{\mathsf{f}})_{\mathsf{i}} \longrightarrow \mathcal{B}_{\mathsf{f}}$$

consists of those $B \in \mathcal{B}_{\mathsf{f}}$ such that $B \overset{d.B}{\to} J.B$ is an isomorphism. As calculated in $\mathcal{B}_{\mathsf{f}}$, this means that $(\mathsf{f}^*d).B$ is an isomorphism. Consider now the essential image of

$$\Xi_{\mathcal{B}} : \text{COCTS}_{\mathcal{F}}(\mathcal{P},\mathcal{B})_{\mathsf{r}} \longrightarrow \mathcal{B}$$

over $\mathcal{F}$. It consists, again by 1.11, of those $B \in \mathcal{B}$ such that $(\mathsf{f}^*d).B$ is an isomorphism. Here was used the fact that $(\widetilde{\mathbf{f}^*\mathbf{d}})$ generates $\hat{J}$. Thus, the essential images of $\Xi_{(\mathcal{B}_{\mathsf{f}})}$ and $(\Xi_{\mathcal{B}})_{\mathsf{f}}$ in $\mathcal{B}_{\mathsf{f}}$ coincide, as in the following diagram.

$$\begin{array}{ccc} & & \text{COCTS}_{\mathcal{S}}(\mathcal{E},\mathcal{B}_{\mathsf{f}})_{\mathsf{i}} \\ & & \downarrow \Xi_{(\mathcal{B}_{\mathsf{f}})} \\ (\text{COCTS}_{\mathcal{F}}(\mathcal{P},\mathcal{B})_{\mathsf{r}})_{\mathsf{f}} & \xrightarrow[(\Xi_{\mathcal{B}})_{\mathsf{f}}]{} & \mathcal{B}_{\mathsf{f}} \end{array}$$

This proves 1.15 (strictly speaking, 1.15 has only been demonstrated at $1 \in \mathcal{S}$). The proof of the proposition is concluded by using the change of base formula. □

The following theorem summarizes the results so far.

1.16 Theorem *Let $\mathcal{E} \xrightarrow{\mathsf{p}} \mathcal{S}$ be a bounded geometric morphism. Then*

$$(\ )_{\mathsf{p}} : \text{COCTS}_{\mathcal{E}} \longrightarrow \text{COCTS}_{\mathcal{S}}$$

has a right adjoint which is

$$\text{COCTS}_{\mathcal{S}}(\mathcal{E},\_) : \text{COCTS}_{\mathcal{S}} \longrightarrow \text{COCTS}_{\mathcal{E}} .$$

Furthermore, the Beck condition is satisfied as explained in 1.14.

To conclude this section, the dual of $\text{COCTS}_{\mathcal{S}}(\mathcal{E},\mathcal{A})$ is calculated in the case that $\mathcal{E} \xrightarrow{\mathsf{p}} \mathcal{S}$ is bounded.

1.17 Proposition *Let $\mathbf{C}$ be a small category in $\mathcal{S}$, j a topology on $\mathcal{S}^{\mathbf{C}}$, and $\mathcal{E}$ the topos of j-sheaves. Then for any $\mathcal{A} \in \text{COCTS}_{\mathcal{S}}$, the dual of $\text{COCTS}_{\mathcal{S}}(\mathcal{E},\mathcal{A})$ is equivalent to $sh_j((\mathcal{A}^{op})^{\mathbf{C}})$ over $\mathcal{E}$.*

PROOF Consider first the case that $\mathsf{p} = \mathsf{i}$ is an inclusion, with $\mathcal{E} = sh_j(\mathcal{S})$. By 1.11 the essential image of $\Xi_{\mathcal{A}}$ consists of those $A \in \mathcal{A}$ such that $S.A \xrightarrow{s.A} T.A$ is an isomorphism, for every j-dense monomorphism $S \overset{s}{\hookrightarrow} T$. But, as was seen in the proof of 1.7 (second paragraph), this is true if and only if the hom-object $\mathcal{A}(A, X)$ is a sheaf in $\mathcal{S}$ for every $X \in \mathcal{A}$, which is true if and only if $\mathcal{A}^{op}(X, A)$ is a sheaf for every $X \in \mathcal{A}^{op}$, which is true if and only if A is a sheaf in $\mathcal{A}^{op}$. In other words,

$$\Xi_{\mathcal{A}} : \mathrm{C{\scriptsize OCTS}}_{\mathcal{S}}(\mathcal{E}, \mathcal{A})_{\mathsf{i}} \longrightarrow sh_j(\mathcal{A}^{op})^{op}$$

is an equivalence over $\mathcal{S}$. Both of these categories have a natural $\mathcal{E}$ structure, with respect to which $\Xi_{\mathcal{A}}$ is an equivalence

$$\mathrm{C{\scriptsize OCTS}}_{\mathcal{S}}(\mathcal{E}, \mathcal{A}) \cong sh_j(\mathcal{A}^{op})^{op}$$

over $\mathcal{E}$.

As for the general case, the change of base formula gives

$$\mathrm{C{\scriptsize OCTS}}_{\mathcal{S}}(\mathcal{E}, \mathcal{A}) \cong \mathrm{C{\scriptsize OCTS}}_{\mathcal{S}^{\mathbf{C}}}(\mathcal{E}, \mathrm{C{\scriptsize OCTS}}_{\mathcal{S}}(\mathcal{S}^{\mathbf{C}}, \mathcal{A}))\ ,$$

of which the right side is, by the first paragraph of this proof, equivalent to

$$sh_j(\mathrm{C{\scriptsize OCTS}}_{\mathcal{S}}(\mathcal{S}^{\mathbf{C}}, \mathcal{A})^{op})^{op} \cong sh_j((\mathcal{A}^{op})^{\mathbf{C}})^{op}\ .$$

□

1.18 Examples

1. We are in a position to prove, as promised, theorem 4.3 from Chapter 2. Let j be a topology on $\mathcal{S}$, with $\mathcal{E}$ the sub-topos of j-sheaves. Let $\mathcal{F} \xrightarrow{\mathsf{f}} \mathcal{S}$ be an arbitrary topos over $\mathcal{S}$. By our work on cosheaves and by using 1.17, one has

$$\mathcal{F}_{/V} \cong sh_{\hat{j}}(\mathcal{F}^{op})^{op} \cong \mathrm{C{\scriptsize OCTS}}_{\mathcal{F}}(\mathcal{P}, \mathcal{F})_{\mathsf{r}}\ ,$$

over $\mathcal{F}$. Here $V = int(\hat{j})$, the interior of the topology on $\mathcal{F}$ induced by j, and

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\ \mathsf{r}\ } & \mathcal{F} \\ {\scriptstyle \mathsf{q}}\downarrow & & \downarrow{\scriptstyle \mathsf{f}} \\ \mathcal{E} & \xrightarrow[\ \mathsf{i}\ ]{} & \mathcal{S} \end{array}$$

is a pullback. Now restrict this equivalence along f. By line 1.15, we have

$$\mathcal{F}_{/V} \cong \mathrm{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{F})_{\mathsf{i}} \cong sh_j(\mathcal{F}^{op})^{op} \ .$$

The second equivalence is again by 1.17, but now over $\mathcal{S}$. Furthermore, these equivalences identify the following three functors:

$$\Sigma_V : \mathcal{F}_{/V} \longrightarrow \mathcal{F} \ , \ \ \Xi_{\mathcal{F}} : \mathrm{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{F})_{\mathsf{i}} \longrightarrow \mathcal{F} \ , \ \ sh_j(\mathcal{F}^{op})^{op} \longrightarrow \mathcal{F} \ .$$

The last one is the inclusion of cosheaves into $\mathcal{F}$. In particular, $\mathrm{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{F})_{\mathsf{i}}$ is a topos.

2. With $\mathcal{E}$ as in the statement of 1.17, if $\mathcal{A} \in \mathrm{Cocts}_{\mathcal{S}}$ is *complete*, then

$$sh_j(\mathcal{A}^{\mathbf{C}})^{op} \cong \mathrm{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{A}^{op}) \ .$$

In particular, if $\mathcal{F}$ is a topos over $\mathcal{S}$, then since $sh_j(\mathcal{F}^{\mathbf{C}})$ is equivalent to the pullback $\mathcal{E} \times_{\mathcal{S}} \mathcal{F}$ (Chap. 2, Cor. 2.12), one has

$$(\mathcal{E} \times_{\mathcal{S}} \mathcal{F})^{op} \cong \mathrm{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{F}^{op}) \ .$$

That is, the dual of $\mathcal{E} \times_{\mathcal{S}} \mathcal{F}$ is equivalent to the category of cocontinuous functors from $\mathcal{E}$ into the dual of $\mathcal{F}$. Moreover, as shall be shown in the next section, if $\mathcal{F}$ is bounded, then $\mathcal{E} \times_{\mathcal{S}} \mathcal{F}$ is the tensor product $\mathcal{E} \otimes_{\mathcal{S}} \mathcal{F}$ (i.e., the left adjoint of $(\ )_{\mathsf{p}}$). Hence, the duality 'formula'

$$(\mathcal{E} \otimes_{\mathcal{S}} \mathcal{F})^{op} \cong \mathrm{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{F}^{op})$$

holds.

3. Let $\mathcal{E} \xrightarrow{\mathsf{p}} \mathcal{S}$ be a topos over $\mathcal{S}$. Let $\mathbf{D}$ denote a small cocomplete category in $\mathcal{S}$. To remind the reader, no notational distinction is made between a small category and its externalization.

   It is not clear that $\mathrm{Cocts}_{\mathcal{S}}(\mathcal{E},\mathbf{D})$ is small in $\mathcal{E}$. However, in the case that $\mathcal{E} = sh(X)$, X a space in $\mathcal{S}$, $\mathrm{Cocts}_{\mathcal{S}}(\mathcal{E},\mathbf{M})$ is small for $\mathbf{M}$ is a sup-lattice in $\mathcal{S}$. The intention is to show this. Fix $\mathbf{M}$, a sup-lattice in $\mathcal{S}$. A sup-lattice $\mathbf{N}$ in $\mathcal{E}$ shall be produced such that for any $K \in \mathcal{E}$, there is a bijection

$$(1.19) \qquad \frac{\mathrm{Cocts}_{\mathcal{S}}(\mathcal{E},\mathbf{M})^K}{K \to \mathbf{N}}$$

as posets, which is natural in K . The following fact is from [JT].

- Let $\vartheta(X)$ denote the locale associated with the space X. Then there is an equivalence

$$\Phi_X : Mod(\vartheta(X)) \longrightarrow sl(\mathcal{E})$$

of the category of $\vartheta(X)$-modules with that of the sup-lattices in $\mathcal{E}$. Furthermore, for any $\mathbf{L} \in sl(\mathcal{S})$ there is a bijection

$$\frac{\vartheta(X) \rightarrow \mathbf{L} \text{ in } sl(\mathcal{S})}{1 \rightarrow \Phi_X(Hom(\vartheta(X), \mathbf{L})) \text{ in } \mathcal{E}}$$

as posets, noting that $Hom(\vartheta(X), \mathbf{L})$ is an $\vartheta(X)$-module.

Let $\mathbf{N} = \Phi_X(Hom(\vartheta(X), \mathbf{M}))$. By 1.13 and the above fact one has

$$\begin{array}{c} \text{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathbf{M})^{1} \\ \hline \text{cocontinuous } \mathcal{S}^{\vartheta(X)^{op}} \longrightarrow \mathbf{M} \text{ taking} \\ \text{dense monos to isomorphisms} \\ \hline \text{sup-lattice maps} \\ \vartheta(X) \rightarrow \mathbf{M} \text{ in } sl(\mathcal{S}) \\ \hline 1 \rightarrow \Phi_X(Hom(\vartheta(X), \mathbf{M})) \text{ in } \mathcal{E} , \end{array}$$

and this proves 1.19 at $1 \in \mathcal{E}$. To get the general case, one can proceed by localizing as follows. Fix $K \in \mathcal{E}$, and let Y be a space in $\mathcal{S}$ such that $\mathcal{E}_{/K} \cong sh(Y)$. Then by our above work,

$$\frac{\text{Cocts}_{\mathcal{S}}(\mathcal{E}_{/K}, \mathbf{M})}{1 \rightarrow \Phi_Y(Hom(\vartheta(Y), \mathbf{M})) \text{ in } \mathcal{E}_{/K} .}$$

However, $\Phi_Y(Hom(\vartheta(Y), \mathbf{M})) = \Phi_K(Hom(\Omega^K, \mathbf{N}))$, where

$$\Phi_K : Mod(\Omega^K) \longrightarrow sl(\mathcal{E}_{/K})$$

is an equivalence, regarding K as a discrete space in $\mathcal{E}$. Then, since

$$\begin{array}{c} 1 \rightarrow \Phi_K(Hom(\Omega^K, \mathbf{N})) \text{ in } \mathcal{E}_{/K} \\ \hline \Omega^K \rightarrow \mathbf{N} \text{ in } sl(\mathcal{E}) \\ \hline K \rightarrow \mathbf{N} \text{ in } \mathcal{E} , \end{array}$$

one obtains 1.19.

3.2 The tensor product

Let $\mathcal{E} \xrightarrow{\mathsf{p}} \mathcal{S}$ be a topos over $\mathcal{S}$. Recall that $(\ )_{\mathsf{p}}$ denotes the 2-functor taking a category over $\mathcal{E}$ to its restiction over $\mathcal{S}$. Let $\mathcal{A} \in \text{Cocts}_{\mathcal{S}}$.

2.1 Definition The *tensor product of $\mathcal{A}$ with $\mathcal{E}$ over $\mathcal{S}$* is a pair $(\mathcal{E}\otimes_{\mathcal{S}}\mathcal{A}, \eta_{\mathcal{A}})$, where $\mathcal{E}\otimes_{\mathcal{S}}\mathcal{A}$ is an object of $\text{COCTS}_{\mathcal{E}}$ and $\eta_{\mathcal{A}} : \mathcal{A} \longrightarrow (\mathcal{E}\otimes_{\mathcal{S}}\mathcal{A})_{\mathsf{p}}$ is a cocontinuous functor, such that the functor

$$(2.2) \qquad \text{COCTS}_{\mathcal{E}}(\mathcal{E}\otimes_{\mathcal{S}}\mathcal{A}, \mathcal{B})_{\mathsf{p}} \longrightarrow \text{COCTS}_{\mathcal{S}}(\mathcal{A}, \mathcal{B}_{\mathsf{p}})$$

$$\mathsf{G} \quad \rightsquigarrow \quad \mathsf{G}_{\mathsf{p}}\cdot\eta_{\mathcal{A}}$$

is an equivalence for every category $\mathcal{B} \in \text{COCTS}_{\mathcal{E}}$.

Observe that the passage $\mathsf{G} \rightsquigarrow \mathsf{G}_{\mathsf{p}}\cdot\eta_{\mathcal{A}}$ is a functor between categories over $\mathcal{S}$. The definition then requires that this be an equivalence over $\mathcal{S}$. However, as shall soon be seen, that a given category be the tensor product it suffices that 2.2 be an equivalence at $1 \in \mathcal{S}$, for every $\mathcal{B} \in \text{COCTS}_{\mathcal{E}}$.

Thus, the left adjoint of $(\ )_{\mathsf{p}}$, if it exists, shall be called the tensor product of $\mathcal{E}$ with $\mathcal{S}$. Pitts (see [Pi]) denotes this category by $\mathsf{p}^{\sharp}\mathcal{A}$, a notation which shall not be used here. He also talks about a tensor product, but instead meaning a category over $\mathcal{S}$ which would represent cocontinuous bimorphisms. Here, however, the primary interest is in change of base, so 'tensor product' will always mean the left adjoint of $(\ )_{\mathsf{p}}$.

To begin, the following basic fact is from [PS].

2.3 Proposition *Let $I \in \mathcal{S}$, and let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary categories over $\mathcal{S}$ and $\mathcal{S}_{/I}$ respectively. Then there is an equivalence*

$$\text{FUNCT}_{\mathcal{S}/I}(\mathcal{A}_{/I}, \mathcal{B}) \cong \text{FUNCT}_{\mathcal{S}}(\mathcal{A}, \mathcal{B}_{\mathsf{I}}) ,$$

natural in $\mathcal{A}$ and $\mathcal{B}$.

Since this is actually a slight improvement on [PS], in that there $\mathcal{B}$ was taken to be of the form $\mathcal{C}_{/I}$ for $\mathcal{C}$ a category over $\mathcal{S}$, a proof is included.

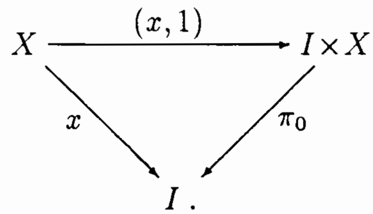
PROOF The proposition expresses an adjointness for which the unit and counit are as follows. For $\mathcal{B}$ a category over $\mathcal{S}_{/I}$, the counit

$$\epsilon : (\mathcal{B}_{\mathsf{I}})_{/I} \longrightarrow \mathcal{B} \ ,$$

is a functor over $\mathcal{S}_{/I}$ defined by setting

$$\epsilon^{x} = (x, 1)^{*} \ , x \in \mathcal{S}_{/I} \ ,$$

where



For $\mathcal{A}$ a category over $\mathcal{S}$, define the unit

$$\eta : \mathcal{A} \longrightarrow (\mathcal{A}_{/I})_{|}$$

at stage $H \in \mathcal{S}$ by setting

$$\eta^H = \pi_1^* ,$$

where $I \times H \xrightarrow{\pi_1} H$. It is left to the reader to verify that the expressed equivalence holds. □

2.4 Proposition *For $\mathcal{A}$ and $\mathcal{B}$ cocomplete, the equivalence of* 2.3 *restricts to cocontinuous functors,*

$$\text{COCTS}_{\mathcal{S}/I}(\mathcal{A}_{/I}, \mathcal{B}) \cong \text{COCTS}_{\mathcal{S}}(\mathcal{A}, \mathcal{B}_{|}) .$$

PROOF The restriction of a cocontinuous functor between cocomplete categories is cocontinuous, and likewise for localization. The proposition now follows since the unit and counit as defined in 2.3 are cocontinuous functors. □

Thus, $\mathcal{A}_{/I}$ is the tensor product $\mathcal{S}_{/I} \otimes_{\mathcal{S}} \mathcal{A}$ for any $\mathcal{A} \in \text{COCTS}_{\mathcal{S}}$. It now follows that, as previously mentioned, for a given $\mathcal{A} \in \text{COCTS}_{\mathcal{S}}$, if $\mathcal{X} \in \text{COCTS}_{\mathcal{E}}$ satisfies 2.2 at $1 \in \mathcal{S}$ for every $\mathcal{B} \in \text{COCTS}_{\mathcal{E}}$, then $\mathcal{X}$ is the tensor product $\mathcal{E} \otimes_{\mathcal{S}} \mathcal{A}$.

2.5 Example From the previous section, $\mathcal{A}_{/I}$ is also equivalent to $\text{COCTS}_{\mathcal{S}}(\mathcal{S}_{/I}, \mathcal{A})$. Thus, for any $I \in \mathcal{S}$, there is a canonical equivalence

$$(2.6) \qquad \text{COCTS}_{\mathcal{S}}(\mathcal{S}_{/I}, \mathcal{A}) \cong \mathcal{S}_{/I} \otimes_{\mathcal{S}} \mathcal{A} \ \ , \mathcal{A} \in \text{COCTS}_{\mathcal{S}} .$$

Line 2.6 is reminiscent of the fact that in locales and sup-lattices,

$$(2.7) \qquad Hom(\Omega^I, \mathrm{M}) \simeq \Omega^I \otimes \mathrm{M} ,$$

where $I \in \mathcal{S}$, and **M** is a sup-lattice in $\mathcal{S}$. Indeed, 2.7 is a special case of 2.6, for if **D** is a small category in $\mathcal{S}$, then $\mathbf{D}_{/I}$ is a small category in $\mathcal{S}_{/I}$. In particular, if $\mathbf{D} = \mathbf{M}$ is a sup-lattice in $\mathcal{S}$, then $\mathcal{S}_{/I} \otimes_{\mathcal{S}} \mathbf{M} \simeq \mathbf{M}_{/I}$ is a sup-lattice in $\mathcal{S}_{/I}$. Hence,

$$\mathcal{S}_{/I} \otimes_{\mathcal{S}} \mathrm{M} \simeq \Phi_I(\Omega^I \otimes \mathrm{M})$$

since amongst sup-lattices they share the same universal property. The functor Φ_I is the equivalence

$$\Phi_I : Mod(\Omega^I) \longrightarrow sl(\mathcal{S}_{/I}) .$$

By 2.4, the property of being the tensor product is stable under localization over $\mathcal{E}$. More precisely, if $\mathcal{E}\otimes_S\mathcal{A}$ exists, then for any $K \in \mathcal{E}$,

$$(2.8) \qquad (\mathcal{E}\otimes_S\mathcal{A})_{/K} \cong \mathcal{E}_{/K}\otimes_S\mathcal{A}$$

over $\mathcal{E}_{/K}$. Recall that $\mathcal{X}^K$, $K \in \mathcal{E}$ and $\mathcal{X} \in \mathrm{Cocts}_{\mathcal{E}}$, is shorthand for the category $(\mathcal{X}_{/K})_{\mathsf{K}}$. This is a category over $\mathcal{E}$, namely the restriction of the localization of $\mathcal{X}$ at K . For any $\mathcal{A} \in \mathrm{Cocts}_S$, let $\mathcal{E}^K\otimes_S\mathcal{A}$ denote $(\mathcal{E}_{/K}\otimes_S\mathcal{A})_{\mathsf{K}}$. $\mathcal{E}^K\otimes_S\mathcal{A}$ is a category over $\mathcal{E}$. Thus, by 2.8 we have

$$(2.9) \qquad (\mathcal{E}\otimes_S\mathcal{A})^K \cong \mathcal{E}^K\otimes_S\mathcal{A}$$

over $\mathcal{E}$. In the case that $K = \mathsf{p}^*I$, let us write $\mathcal{E}^I\otimes_S\mathcal{A}$ for $\mathcal{E}^{\mathsf{p}^*I}\otimes_S\mathcal{A}$. Thus, $\mathcal{E}\otimes_S\mathcal{A}$ can be regarded as a category over $\mathcal{S}$ by setting

$$(\mathcal{E}\otimes_S\mathcal{A})^I = (\mathcal{E}^I\otimes_S\mathcal{A})^I .$$

However, by 2.9 this is none other than the category $(\mathcal{E}\otimes_S\mathcal{A})_{\mathsf{p}}$. Therefore, it is meaningful and correct to write

$$(\mathcal{E}\otimes_S\mathcal{A})_{\mathsf{p}} \cong \mathcal{E}\otimes_S\mathcal{A} ,$$

as categories over $\mathcal{S}$.

Since localization is both left and right adjoint to $(\ )_{\mathsf{I}}$, it follows that

$$\mathrm{Cocts}_S(\mathcal{A}^I, \mathcal{C}) \cong \mathrm{Cocts}_S(\mathcal{A}, \mathcal{C}^I) ,$$

for any cocomplete categories $\mathcal{A}$ and $\mathcal{C}$ over $\mathcal{S}$, and any $I \in \mathcal{S}$. From this one sees that if $\mathcal{E}\otimes_S\mathcal{A}$ exists, then $(\mathcal{E}\otimes_S\mathcal{A})^{\mathsf{p}^*I}$ is the tensor product $\mathcal{E}\otimes_S(\mathcal{A}^I)$. This establishes the second equivalence of the following proposition. The first equivalence of this proposition has already been established above.

2.10 Proposition *Assume that $\mathcal{E}\otimes_S\mathcal{A}$ exists, for $\mathcal{A} \in \mathrm{Cocts}_S$. Then both $\mathcal{E}^I\otimes_S\mathcal{A}$ and $\mathcal{E}\otimes_S(\mathcal{A}^I)$ exist, and for every $I \in \mathcal{S}$ we have*

$$\mathcal{E}^I\otimes_S\mathcal{A} \cong (\mathcal{E}\otimes_S\mathcal{A})^{\mathsf{p}^*I} \cong \mathcal{E}\otimes_S(\mathcal{A}^I)$$

over $\mathcal{E}$.

Proposition 2.10 expresses the equivariance over $\mathcal{S}$ of the tensor product. This phenomenon has its obvious analogy in sup-lattices and locales.

In [Pi] Pitts has shown that the tensor product over $\mathcal{S}$ of a bounded topos with $\mathcal{E}$ exists and that it coincides with the pullback as constructed in toposes.

2.11 Theorem (Pitts) *Let $\mathcal{F} \xrightarrow{f} \mathcal{S}$ be a bounded topos. Then the tensor product of $\mathcal{F}$ with $\mathcal{E}$ is the pullback $\mathcal{E} \times_{\mathcal{S}} \mathcal{F}$ as constructed in toposes. The universal morphism $\eta_{\mathcal{F}}$ is the inverse image functor of the projection $\mathcal{E} \times_{\mathcal{S}} \mathcal{F} \longrightarrow \mathcal{F}$.*

PROOF Let $\mathcal{P}$ denote the pullback

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\mathsf{r}} & \mathcal{E} \\ {\scriptstyle \mathsf{q}}\downarrow & & \downarrow{\scriptstyle \mathsf{p}} \\ \mathcal{F} & \xrightarrow[\mathsf{f}]{} & \mathcal{S} \end{array}$$

with f bounded. By 1.14,

$$\text{(2.12)} \qquad \text{Cocts}_{\mathcal{E}}(\mathcal{P}, \mathcal{B})_{\mathsf{q}} \cong \text{Cocts}_{\mathcal{S}}(\mathcal{F}, \mathcal{B}_{\mathsf{p}})$$

for any $\mathcal{B} \in \text{Cocts}_{\mathcal{E}}$. This is an equivalence of categories in $\text{Cocts}_{\mathcal{F}}$ which when carried to $\text{Cocts}_{\mathcal{S}}$ under $(\ )_{\mathsf{f}}$ gives the desired result. In fact, applying $(\ )_{\mathsf{f}}$ to the left side of 2.12 yields

$$(\text{Cocts}_{\mathcal{E}}(\mathcal{P}, \mathcal{B})_{\mathsf{q}})_{\mathsf{f}} \cong (\text{Cocts}_{\mathcal{E}}(\mathcal{P}, \mathcal{B})_{\mathsf{r}})_{\mathsf{p}} \cong \text{Cocts}_{\mathcal{E}}(\mathcal{P}, \mathcal{B})_{\mathsf{p}} \ .$$

Observe that the equivariance of $\text{Cocts}_{\mathcal{E}}(\mathcal{P}, \_)$ has been used here. Applying $(\ )_{\mathsf{f}}$ to the right side of 2.12 gives $\text{Cocts}_{\mathcal{S}}(\mathcal{F}, \mathcal{B}_{\mathsf{p}})$ again, but now over $\mathcal{S}$. This is the equivariance of $\text{Cocts}_{\mathcal{S}}(\mathcal{F}, \_)$. □

The above constitutes a new proof of Pitts' result. As presented here, this theorem should be viewed as a corollary of 1.14.

2.13 Example As is shown in [Pi], theorem 2.11 can be used to show that locally connected geometric morphisms are stable under pullback, and that the Beck condition is satisfied. Indeed, let

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\mathsf{q}} & \mathcal{F} \\ {\scriptstyle \mathsf{r}}\downarrow & & \downarrow{\scriptstyle \mathsf{f}} \\ \mathcal{E} & \xrightarrow[\mathsf{p}]{} & \mathcal{S} \end{array}$$

be a pullback of toposes with f bounded and locally connected. By the naturality of η ,

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{f^*} & \mathcal{F} \\ {\scriptstyle \eta_{\mathcal{S}}}\downarrow & & \downarrow{\scriptstyle \eta_{\mathcal{F}}} \\ \mathcal{E}\otimes_{\mathcal{S}}\mathcal{S} & \xrightarrow[\mathcal{E}\otimes_{\mathcal{S}} f^*]{} & \mathcal{E}\otimes_{\mathcal{S}}\mathcal{F} \end{array} \qquad \begin{array}{ccc} \mathcal{S} & \xleftarrow{f_!} & \mathcal{F} \\ {\scriptstyle \eta_{\mathcal{S}}}\downarrow & & \downarrow{\scriptstyle \eta_{\mathcal{F}}} \\ \mathcal{E}\otimes_{\mathcal{S}}\mathcal{S} & \xleftarrow[\mathcal{E}\otimes_{\mathcal{S}} f_!]{} & \mathcal{E}\otimes_{\mathcal{S}}\mathcal{F} \end{array}$$

commute up to natural isomorphism, where $f_! \dashv f^*$ over $\mathcal{S}$. The '$(\ )_p$' notation has been omitted in these diagrams since everything is over $\mathcal{S}$. Let $r_! = \mathcal{E}\otimes_{\mathcal{S}} f_!$. It must be that $r^* \simeq \mathcal{E}\otimes_{\mathcal{S}} f^*$, and hence $r_! \dashv r^*$. Moreover, the right-hand square above says that

$$p^* \cdot f_! \simeq r_! \cdot q^* ,$$

since $\eta_{\mathcal{F}}$ is identified with q^* and $\eta_{\mathcal{S}}$ with p^* . That is, the Beck condition is satisfied.

The Beck condition (in the context of cocomplete categories) holds for the tensor product. This fact will be useful in Chapter 4.

2.14 Proposition *Suppose that*

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{r} & \mathcal{F} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle f} \\ \mathcal{E} & \xrightarrow[p]{} & \mathcal{S} \end{array}$$

is a pullback square of toposes, with $\mathcal{E} \xrightarrow{p} \mathcal{S}$ bounded. Let $\mathcal{A} \in \mathrm{COCTS}_{\mathcal{E}}$, and assume that $\mathcal{P}\otimes_{\mathcal{E}}\mathcal{A}$ exists. Then $(\mathcal{P}\otimes_{\mathcal{E}}\mathcal{A})_r$ is the tensor product of $\mathcal{A}_p$ with $\mathcal{F}$ over $\mathcal{S}$.

PROOF This is a purely formal consequence of the fact that the Beck condition holds for the adjointness $(\ )_p \dashv \mathrm{COCTS}_{\mathcal{S}}(\mathcal{E}, \_)$, see 1.14. □

Although the existence of the tensor product is in general unclear, by imposing further conditions on $\mathcal{A}$ some results in this regard can be achieved when $\mathcal{E}$ is the sub-topos of sheaves for a topology j on $\mathcal{S}$. This is the intention in the paragraphs that follow. Let $\mathcal{E} \xrightarrow{i} \mathcal{S}$ denote the inclusion of $\mathcal{E}$ into $\mathcal{S}$.

Up till now $sh_j(\mathcal{A})$ has been regarded as a category over $\mathcal{S}$. Recall from Chapter 2 that $sh_j(\mathcal{A})$ is made into a category over $\mathcal{E}$ by defining

$$sh_j(\mathcal{A})^K = sh_j(\mathcal{A})^{i_* K} .$$

Under this definition, it was seen that if $\mathcal{A}$ is locally small then $sh_j(\mathcal{A})$ is locally small over $\mathcal{E}$. Also recall from Chapter 2 that $\mathcal{A}$ is said to admit j-sheafification if $sh_j(\mathcal{A})$ is a reflective sub-category of $\mathcal{A}$, in which case the reflection functor is denoted by a (also called the sheafification functor). It was shown that if $\mathcal{A} \in \mathrm{COCTS}_{\mathcal{S}}$ admits sheafification, then $sh_j(\mathcal{A})$ is cocomplete over $\mathcal{S}$ and hence over $\mathcal{E}$, and so $sh_j(\mathcal{A}) \in \mathrm{COCTS}_{\mathcal{E}}$ in this case. Moreover, $sh_j(\mathcal{A})$ was seen to be equivariant in the sense that

$$(2.15) \qquad sh_j(\mathcal{A})_{\mathsf{i}} \cong sh_j(\mathcal{A})$$

over $\mathcal{S}$. This means that if one views $sh_j(\mathcal{A})$ as over $\mathcal{E}$, then restricting back to $\mathcal{S}$ gives the original category. This amounted to $sh_j(\mathcal{A})^I \xrightarrow{\alpha^*} sh_j(\mathcal{A})^K$ being an equivalence for every j-bidense morphism $K \xrightarrow{\alpha} I$ in $\mathcal{S}$.

Recall (1.11) that for any $\mathcal{B} \in \mathrm{COCTS}_{\mathcal{E}}$, the unit of the adjointness $(\ )_{\mathsf{i}} \dashv \mathrm{COCTS}_{\mathcal{S}}(\mathcal{E},\_)$,

$$\Lambda_{\mathcal{B}} : \mathcal{B} \longrightarrow \mathrm{COCTS}_{\mathcal{S}}(\mathcal{E}, \mathcal{B}_{\mathsf{i}}) \ ,$$

is an equivalence. Let $\mathcal{A} \in \mathrm{COCTS}_{\mathcal{S}}$, and assume that $\mathcal{A}$ admits j-sheafification. Then $sh_j(\mathcal{A}) \in \mathrm{COCTS}_{\mathcal{E}}$, and therefore, there is an equivalence

$$\Lambda : sh_j(\mathcal{A}) \longrightarrow \mathrm{COCTS}_{\mathcal{S}}(\mathcal{E}, sh_j(\mathcal{A})_{\mathsf{i}}) \ ,$$

over $\mathcal{E}$. Now regard this equivalence, by restriction, as over $\mathcal{S}$.

$$\Lambda_{\mathsf{i}} : sh_j(\mathcal{A})_{\mathsf{i}} \longrightarrow \mathrm{COCTS}_{\mathcal{S}}(\mathcal{E}, sh_j(\mathcal{A})_{\mathsf{i}})_{\mathsf{i}}$$

By 2.15, there is therefore an equivalence

$$sh_j(\mathcal{A}) \cong \mathrm{COCTS}_{\mathcal{S}}(\mathcal{E}, sh_j(\mathcal{A}))$$

over $\mathcal{S}$. This equivalence says that every cocontinuous functor $\mathcal{S} \xrightarrow{\mathsf{A}} sh_j(\mathcal{A})$ factors through i^* ,

$$\begin{array}{ccc} \mathcal{S} & & \\ {\scriptstyle \mathsf{i}^*}\downarrow & \searrow^{\mathsf{A}} & \\ \mathcal{E} & \xrightarrow{\cong} & sh_j(\mathcal{A}) \ . \end{array}$$

Now let $sh_j(\mathcal{A}) \xrightarrow{\mathsf{F}} \mathcal{C}$ be an arbitrary cocontinuous functor over $\mathcal{S}$. Then for all $A \in sh_j(\mathcal{A})$,

$$\mathcal{S} \xrightarrow{\mathsf{A}} sh_j(\mathcal{A}) \xrightarrow{\mathsf{F}} \mathcal{C}$$

factors through i^*. In other words, F itself factors through

$$\Xi_{\mathcal{C}} : \textsc{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{C})_{\mathsf{i}} \longrightarrow \mathcal{C} ,$$

and this factorization must be essentially unique because $\Xi_{\mathcal{C}}$ is fully faithful.

Thus, for any $\mathcal{C} \in \textsc{Cocts}_{\mathcal{S}}$ there is the functor

$$\mathsf{a}^* : \textsc{Cocts}_{\mathcal{S}}(sh_j(\mathcal{A}),\mathcal{C}) \longrightarrow \textsc{Cocts}_{\mathcal{S}}(\mathcal{A}, \textsc{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{C})_{\mathsf{i}})$$

sending F to $\mathsf{F} \cdot \mathsf{a}$, which evidently is fully faithful. If $\mathcal{A}$ is assumed to satisfy the hypothesis of the Special Adjoint Functor Theorem (see Appendix B), then a^* is an equivalence. All the hypothesis are summed up in the following theorem.

2.16 Theorem *Let $\mathcal{A} \in \textsc{Cocts}_{\mathcal{S}}$, admitting j-sheafification. Assume that $\mathcal{A}$ has a generating family and is cowell-powered. Then composition with the sheafification functor induces an equivalence of categories*

$$\textsc{Cocts}_{\mathcal{S}}(sh_j(\mathcal{A}),\mathcal{C}) \longrightarrow \textsc{Cocts}_{\mathcal{S}}(\mathcal{A}, \textsc{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{C})_{\mathsf{i}}),$$

for any $\mathcal{C} \in \textsc{Cocts}_{\mathcal{S}}$.

PROOF It remains to show that a^* is essentially surjective. Let $\mathcal{A} \xrightarrow{\mathsf{G}} \textsc{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{C})_{\mathsf{i}}$ be an arbitrary cocontinuous functor over $\mathcal{S}$. Then by the Special Adjoint Functor Theorem, G has a right adjoint which shall be denoted by R. It follows that for all $C \in \textsc{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{C})_{\mathsf{i}}$, $\mathsf{R}C$ is a sheaf. In fact, if $U \overset{u}{\hookrightarrow} V$ is j-dense and if $A \in \mathcal{A}^V$ is arbitrary, then first note that

$$\mathsf{G}^V(u.A) \simeq u.\mathsf{G}^V(A) \simeq \mathsf{G}^V(A)$$

regarding the j-dense monomorphism $u \hookrightarrow 1$ in $\mathcal{S}_{/V}$. This follows because it is true in $\mathcal{C}$, and because $\Xi_{\mathcal{C}}$ is fully faithful and cocontinuous. There is therefore the following series of bijections:

$$\frac{\dfrac{\dfrac{u.A \rightarrow V^*(\mathsf{R}C)}{\mathsf{G}^V(u.A) \rightarrow V^*C}}{\mathsf{G}^V(A) \rightarrow V^*C}}{A \rightarrow V^*(\mathsf{R}C)},$$

which is given by composition with $u.A \rightarrow A$, establishing that $\mathsf{R}C$ is a sheaf. In other words, R factors through

$$sh_j(\mathcal{A}) \xrightarrow{\mathsf{i}} \mathcal{A} .$$

Hence $\mathsf{R} \simeq \mathsf{i} \cdot \mathsf{a} \cdot \mathsf{R}$ and $\mathsf{G} \cdot \mathsf{i} \dashv \mathsf{a} \cdot \mathsf{R}$. In particular, $\mathsf{G} \cdot \mathsf{i}$ is cocontinuous. By taking left adjoints of $\mathsf{R} \simeq \mathsf{i} \cdot \mathsf{a} \cdot \mathsf{R}$, one has $\mathsf{G} \simeq \mathsf{G} \cdot \mathsf{i} \cdot \mathsf{a}$. □

2.17 Example As pointed out in [Pi], the exponential adjointness

$$\text{FUNCT}_{\mathcal{S}}(\mathcal{E} \times \mathcal{A}, \mathcal{C}) \cong \text{FUNCT}_{\mathcal{S}}(\mathcal{A}, \text{FUNCT}_{\mathcal{S}}(\mathcal{E}, \mathcal{C}))$$

restricts to

$$\text{BIM}_{\mathcal{S}}(\mathcal{E}, \mathcal{A}; \mathcal{C}) \cong \text{COCTS}_{\mathcal{S}}(\mathcal{A}, \text{COCTS}_{\mathcal{S}}(\mathcal{E}, \mathcal{C})) ,$$

where the category on the left has as its objects those functors cocontinuous in each variable separately, the so-called cocontinuous bimorphisms (see [Pi] for a precise definition). Therefore, under the hypothesis of 2.16, $sh_j(\mathcal{A})$ represents cocontinuous bimorphisms over $\mathcal{S}$. That is,

$$\text{BIM}_{\mathcal{S}}(\mathcal{E}, \mathcal{A}; \mathcal{C}) \cong \text{COCTS}_{\mathcal{S}}(sh_j(\mathcal{A}), \mathcal{C}) ,$$

by an equivalence which is natural in $\mathcal{C}$. This result can be extended (the details have been omitted) to $\mathcal{E}$ bounded. This can be done by using the the change of base formula, and the fact (from [Pi]) that if $\mathbf{C}$ is a small category in $\mathcal{S}$, then

$$\text{BIM}_{\mathcal{S}}(\mathcal{S}^{\mathbf{C}}, \mathcal{A}; \mathcal{C}) \cong \text{COCTS}_{\mathcal{S}}((\mathcal{A}^{\mathbf{C}})_{\mathbf{C}}, \mathcal{C}) ,$$

by an equivalence which is natural in $\mathcal{C}$.

2.18 Corollary *Under the conditions of* 2.16, $(sh_j(\mathcal{A}),\mathsf{a})$ *is the tensor product of* $\mathcal{A}$ *with* $\mathcal{E}$.

PROOF Given an arbitrary $\mathcal{B} \in \text{COCTS}_{\mathcal{E}}$ one has

$$\begin{aligned} &\text{COCTS}_{\mathcal{E}}(sh_j(\mathcal{A}), \mathcal{B}) \\ \cong\ &\text{COCTS}_{\mathcal{S}}(sh_j(\mathcal{A})_{\mathsf{i}}, \mathcal{B}_{\mathsf{i}}) \ , \ (\ )_{\mathsf{i}} \text{ is 2-fully faithful, see 1.11} \\ \cong\ &\text{COCTS}_{\mathcal{S}}(sh_j(\mathcal{A}), \mathcal{B}_{\mathsf{i}}) \ , \ \text{by } 2.15 \\ \cong\ &\text{COCTS}_{\mathcal{S}}(\mathcal{A}, \text{COCTS}_{\mathcal{S}}(\mathcal{E}, \mathcal{B}_{\mathsf{i}})_{\mathsf{i}}) \ , \ \text{by } 2.16 \\ \cong\ &\text{COCTS}_{\mathcal{S}}(\mathcal{A}, \mathcal{B}_{\mathsf{i}}) \ , \ (\ )_{\mathsf{i}} \text{ is 2-fully faithful.} \end{aligned}$$

This equivalence is natural in $\mathcal{B}$, which establishes that $sh_j(\mathcal{A})$ is the tensor product. □

2.19 Examples

1. Any small cocomplete category satisfies the hypothesis of 2.16; that such a category admits sheafification was shown in Chapter 2, §3.

2. Let j be an arbitrary topology on $\mathcal{S}$, with $\mathcal{E}$ the sub-topos of j-sheaves. Recall (Chap. 2, §2) that $\mathcal{F}^{op}$admits sheafification, for $\mathcal{F}$ an arbitrary topos over $\mathcal{S}$. If $\mathcal{F}$ is bounded, then $\mathcal{F}$ has a *co*generating family (see [PS], p. 102). $\mathcal{F}$ is well-powered as well, and so by 2.18, $sh_j(\mathcal{F}^{op})$ is the tensor product $\mathcal{E}\otimes_{\mathcal{S}}(\mathcal{F}^{op})$. Recall (1.18) that the dual of $\text{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{F})$ is equivalent to $sh_j(\mathcal{F}^{op})$. Therefore, one has the duality 'formula'

$$\text{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{F})^{op} \cong \mathcal{E}\otimes_{\mathcal{S}}(\mathcal{F}^{op}) ,$$

for $\mathcal{F}$ a bounded topos over $\mathcal{S}$.

3. Let X be an arbitrary space in $\mathcal{S}$, and let $\mathcal{E} = sh(X)$. Then, as was seen in §1 of this chapter, $\text{Cocts}_{\mathcal{S}}(\mathcal{E},\mathrm{M}^{op})$ is a small sup-lattice in $\mathcal{E}$, represented by $\Phi_X(Hom(\vartheta(X),\mathrm{M}^{op}))$. Φ_X is the equivalence

$$\Phi_X : Mod(\vartheta(X)) \longrightarrow sl(\mathcal{E}) .$$

A fact from [JT] is that

$$Hom(\vartheta(X),\mathrm{M}^{op})^{op} = \vartheta(X)\otimes\mathrm{M}$$

as $\vartheta(X)$-modules. Also, by 1.17 we have

$$\text{Cocts}_{\mathcal{S}}(\mathcal{E},\mathrm{M}^{op})^{op} \simeq sh_j(\mathrm{M}^{\vartheta(X)^{op}}) ,$$

where j denotes the canonical topology on $\mathcal{S}^{\vartheta(X)^{op}}$. Thus, $sh_j(\mathrm{M}^{\vartheta(X)^{op}})$ is a small sup-lattice in $\mathcal{E}$, represented by $\Phi_X(\vartheta(X)\otimes\mathrm{M})$.

CHAPTER 4

Descent

4.1 Equalizers and semi-split equalizers in a 2-category

Let $\mathcal{K}$ denote an arbitrary 2-category, and consider the following diagram

$$(1.1) \qquad B \underset{\pi_1}{\overset{\overset{\lambda}{\longleftarrow}}{\underset{\pi_0}{\longrightarrow}}} C \overset{\rho_0}{\underset{\rho_2}{\overset{\rho_1}{\longrightarrow}}} D$$

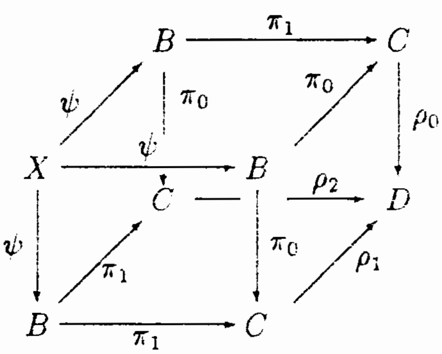
of 0-cells and 1-cells in $\mathcal{K}$. Furthermore, suppose that there are given 2-isomorphisms

$$\rho_0 \cdot \pi_1 \overset{l}{\simeq} \rho_2 \cdot \pi_0 \qquad \rho_0 \cdot \pi_0 \overset{m}{\simeq} \rho_1 \cdot \pi_0 \qquad \rho_2 \cdot \pi_1 \overset{n}{\simeq} \rho_1 \cdot \pi_1$$
$$\lambda \cdot \pi_0 \overset{u}{\simeq} 1_B \qquad \lambda \cdot \pi_1 \overset{v}{\simeq} 1_B \, .$$

1.2 DEFINITION For any 0-cell $X \in \mathcal{K}$, by a *cone* from X to the diagram 1.1 is meant a pair (ψ, x) where $X \xrightarrow{\psi} B$ is a 1-cell and $\pi_0 \cdot \psi \overset{x}{\simeq} \pi_1 \cdot \psi$ is a 2-isomorphism satisfying the *unit* and *cocycle* conditions:

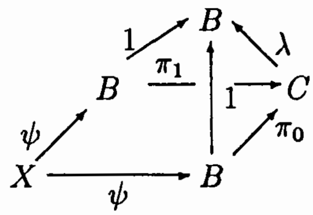
$$v\psi \cdot \lambda x = u\psi \qquad \rho_1 x = n\psi \cdot \rho_2 x \cdot l\psi \cdot \rho_0 x \cdot m\psi \, .$$

For a cone (ψ, x), the cocycle condition is sometimes easier to work with when visualized as a commutative 'cube':



where the 2-isomorphisms belonging to the faces have been omitted from this diagram.[1] These isomorphisms are l, x, x, m, n, x in the order: back face, left, top, right, bottom, and front face. In the future, when such a cube is given, the 2-cells belonging to the faces will always be given in this order. The unit condition expresses the commutativity of the following diagram.

[1]This is due to typographical difficulties. In any case, it is clear where they go and their directions are immaterial because they are isomorphisms.



The 2-isomorphisms that belong in this figure are x, u and v.

The cones from X to the diagram 1.1 are the objects of a category, where a morphism $(\psi, x) \xrightarrow{f} (\phi, y)$ of two such cones is defined to be a 2-cell $\psi \xrightarrow{f} \phi$ such that

$$\begin{array}{ccc} \pi_0 \cdot \psi & \overset{x}{\cong} & \pi_1 \cdot \psi \\ {\scriptstyle \pi_0 f}\downarrow & & \downarrow{\scriptstyle \pi_1 f} \\ \pi_0 \cdot \phi & \underset{y}{\cong} & \pi_1 \cdot \phi \end{array}$$

commutes. Let us denote this category by CONE_X.

1.3 DEFINITION An *equalizer* of diagram 1.1 is defined to be a universal cone. To be precise, it is a 0-cell A and a cone (θ, k) from A to diagram 1.1 such that for any 0-cell X in $\mathcal{K}$, composition with θ,

$$[X, \theta] : \mathcal{K}[X, A] \longrightarrow \text{CONE}_X ,$$

induces an equivalence of $\mathcal{K}[X, A]$ with the category of cones from X to 1.1.

1.4 DEFINITION A *split equalizer* in the 2-category $\mathcal{K}$ is a cone (θ, k) to 1.1,

$$A \xrightarrow{\theta} B \underset{\pi_1}{\overset{\overset{\lambda}{\longleftarrow}}{\underset{\longrightarrow}{\overset{\pi_0}{\longrightarrow}}}} C \underset{\rho_2}{\overset{\rho_0}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xrightarrow{\rho_1}}}}} D$$

with in addition, 1-cells

$$A \xleftarrow[r]{} B \xleftarrow[s]{} C \xleftarrow[t]{} D$$

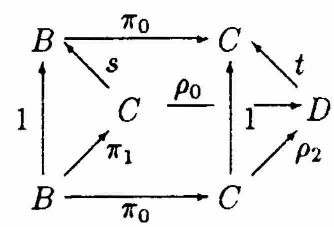
and 2-isomorphisms

$$\begin{array}{lll} r \cdot \theta \overset{i}{\cong} 1_A & s \cdot \pi_1 \overset{d}{\cong} 1_B & s \cdot \pi_0 \overset{e}{\cong} \theta \cdot r \\ t \cdot \rho_2 \overset{a}{\cong} 1_C & t \cdot \rho_1 \overset{b}{\cong} \pi_1 \cdot s & t \cdot \rho_0 \overset{c}{\cong} \pi_0 \cdot s \end{array}$$

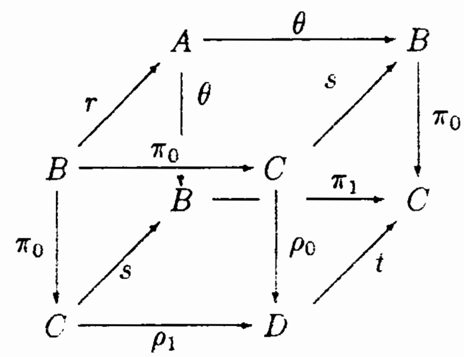
satisfying the following three coherence conditions:

1. $a\pi_0 \cdot tl = \pi_0 d \cdot c\pi_1$
2. $a\pi_1 \cdot tn = \pi_1 d \cdot b\pi_1$
3. $kr \cdot \pi_0 e \cdot c\pi_0 = \pi_1 e \cdot b\pi_0 \cdot tm$.

To visualize these coherence conditions, the first one requires that the following 'prism' commute.



As usual, the 2-isomorphisms have been omitted from the diagram, which in this case are c, l, d and a . The second condition, involving isomorphisms b, n, d and a , is similiar to this. The third condition expresses the commutativity of the following cube.



The 2-isomorphisms belonging to this cube are, in the usual order: k, e, e, c, b and m .

The intention is to show that a split equalizer is an equalizer. Given a split equalizer $(A, \theta, k, \ldots)$ as defined above, it must be shown that for any 0-cell X in $\mathcal{K}$, $[X, \theta]$ is an equivalence. Composition with r induces a functor

$$[X, r] : \mathrm{CONE}_X \longrightarrow \mathcal{K}[X, A] ,$$

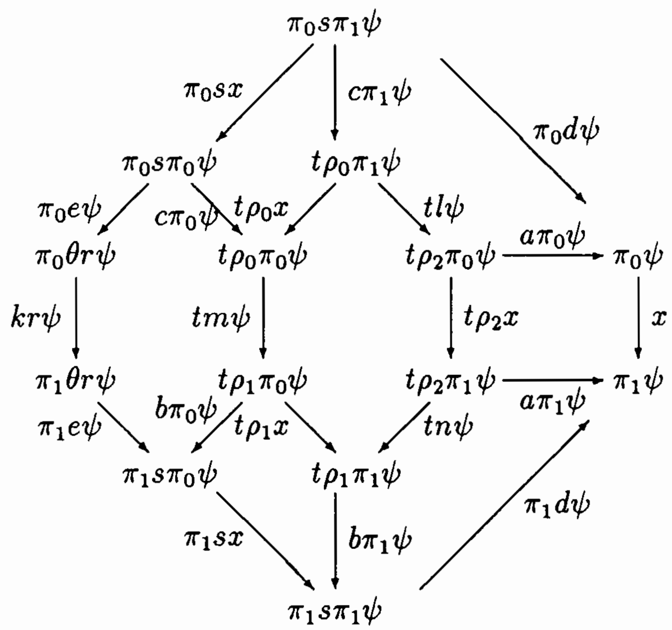
and since $r \cdot \theta \doteq 1$, $[X, r] \cdot [X, \theta]$ is therefore isomorphic to the identity functor on $\mathcal{K}[X, A]$. To show that $[X, \theta] \cdot [X, r]$ is isomorphic to the identity on CONE_X, let (ψ, x) denote an arbitrary cone from X . By definition,

$$[X, \theta] \cdot [X, r](\psi, x) = (\theta \cdot r \cdot \psi, k \cdot r \cdot \psi) ,$$

and there is an isomorphism

$$\psi \overset{d\psi}{\simeq} s \cdot \pi_1 \cdot \psi \overset{sx}{\simeq} s \cdot \pi_0 \cdot \psi \overset{e\psi}{\simeq} \theta \cdot r \cdot \psi \ ,$$

which is seen to be an isomorphism of cones as follows. First apply t to the cocycle condition. Then by putting together the appropriate prisms and cubes one deduces that $e\psi \cdot sx \cdot d\psi$ is indeed an isomorphism of cones. Alternatively, this derivation can be expressed with the single commutative diagram given below. Note that the arrows in this diagram are all 2-isomorphisms, so no particular direction of any of them is intended.



The center hexagon is t applied to the cocycle condition for (ψ, x) . The hexagon on the left is the third coherence condition given by the split equalizer, applied to ψ . The upper right and lower right squares are the other two coherence conditions applied to ψ . The remaining three squares commute by the axioms of a 2-category. Thus, the perimeter of the above diagram commutes, which says precisely that $e\psi \cdot sx \cdot d\psi$ is an isomorphism of the cones (ψ, x) and $(\theta r\psi, kr\psi)$. This proves that $[X, \theta] \cdot [X, r]$ is isomorphic to the identity functor on CONE_X .

Thus, a split equalizer is an equalizer, one which, by its equational nature, is preserved under any 2-functor.

The following generalization of a split equalizer will be used in section §4.

1.5 Definition A *semi-split equalizer* in a 2-category $\mathcal{K}$ consists of the same data as a split equalizer with the difference that the 2-cells i, a and d are not required to be isomorphisms. They are required to be 2-cells

$$r \cdot \theta \xrightarrow{i} 1_A \qquad\qquad s \cdot \pi_1 \xrightarrow{d} 1_B \qquad\qquad t \cdot \rho_2 \xrightarrow{a} 1_C$$

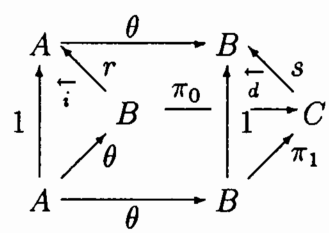
with the property that

$$r \cdot \theta \cdot r \cdot \theta \underset{r\theta i}{\overset{ir\theta}{\rightrightarrows}} r \cdot \theta \xrightarrow{\ i\ } 1_A \qquad s \cdot \pi_1 \cdot s \cdot \pi_1 \underset{s\pi_1 d}{\overset{ds\pi_1}{\rightrightarrows}} s \cdot \pi_1 \xrightarrow{\ d\ } 1_B$$

be coequalizers, in the categories $\mathcal{K}[A, A]$ and $\mathcal{K}[B, B]$ respectively. Furthermore, a fourth coherence condition,

$$4.\ \ d\theta \cdot sk = \theta i \cdot e\theta\ ,$$

is required to hold. This condition can be visualized as



where the 2-isomorphisms k and e have been omitted from the diagram.

With some assumptions on $\mathcal{K}$, semi-split equalizers are equalizers.

1.6 Theorem *Given a semi-split equalizer* $(A, B, \ldots, \theta, \pi_0, \ldots)$, *assume that* $\mathcal{K}$ *satisfies the following.*

1. *For every 0-cell* $X \in \mathcal{K}$, $\mathcal{K}[X, A]$ *has coequalizers.*

2. *For every 1-cell* $X \xrightarrow{\alpha} A$, *the functor*

$$[\alpha, A] : \mathcal{K}[A, A] \longrightarrow \mathcal{K}[X, A]$$

preserves coequalizers.

3. *For every cone* $(X \xrightarrow{\psi} B, x)$, *the functor*

$$[\psi, B] : \mathcal{K}[B, B] \longrightarrow \mathcal{K}[X, B]$$

preserves coequalizers.

4. For every 0-cell X , the functors

$$[X,\theta] : \mathcal{K}[X,A] \longrightarrow \mathcal{K}[X,B]$$

and

$$[X,\pi_0] : \mathcal{K}[X,B] \longrightarrow \mathcal{K}[X,C]$$

preserve coequalizers.

Then $(A, B, \ldots, \theta, \pi_0, \ldots)$ *is an equalizer.*

To prove the theorem, let X be an arbitrary 0-cell in $\mathcal{K}$. Define a functor

$$\Gamma : \text{CONE}_X \longrightarrow \mathcal{K}[X,A]$$

as follows. Given a cone (ψ, x) , denote the composite 2-cell

$$\theta \cdot r \cdot \psi \overset{e\psi}{\simeq} s \cdot \pi_0 \cdot \psi \overset{sx}{\simeq} s \cdot \pi_1 \cdot \psi \overset{d\psi}{\to} \psi$$

by $\kappa_{(\psi,x)}$. In the proof that a split equalizer is an equalizer, it was shown that $\kappa_{(\psi,x)}$ is a morphism of the cones (ψ, x) and $(\theta r\psi, kr\psi)$. That proof remains valid here, though now a and d are not necessarily isomorphisms. Let $\Gamma(\psi, x)$ be the coequalizer

$$r\theta r\psi \underset{r\kappa_{(\psi,x)}}{\overset{ir\psi}{\rightrightarrows}} r\psi \xrightarrow{\quad q \quad} \Gamma(\psi, x)$$

in $\mathcal{K}[X,A]$. To see that $\Gamma \cdot [X,\theta]$ is isomorphic to the identity on $\mathcal{K}[X,A]$, let $X \xrightarrow{\alpha} A$ be arbitrary. Then by assumption *2*,

$$r\theta r\theta\alpha \underset{r\theta i\alpha}{\overset{ir\theta\alpha}{\rightrightarrows}} r\theta\alpha \xrightarrow{\quad i\alpha \quad} \alpha$$

is a coequalizer in $\mathcal{K}[X,A]$. By definition, $\Gamma \cdot [X,\theta](\alpha)$ is the coequalizer of $r\kappa_{(\theta\alpha,k\alpha)}$ and $ir\theta\alpha$. However, the fourth coherence condition of a semi-split equalizer says that $\kappa_{(\theta,k)} = \theta i$, and therefore

$$r\kappa_{(\theta\alpha,k\alpha)} = r\kappa_{(\theta,k)}\alpha = r\theta i\alpha\,.$$

Hence, $\Gamma \cdot [X,\theta](\alpha)$ is isomorphic to α .

To show that $[X,\theta] \cdot \Gamma$ is isomorphic to the identity on CONE_X, let (ψ, x) denote an arbitrary cone from X . The 1-cell component of $[X,\theta] \cdot \Gamma(\psi, x)$ is by definition $\theta\Gamma(\psi, x)$ as in the diagram

$$(1.7)\quad \theta r\theta r\psi \;\underset{\theta r\kappa_{(\psi,x)}}{\overset{\theta i r\psi}{\rightrightarrows}}\; \theta r\psi \xrightarrow{\theta q} \theta\Gamma(\psi,x)\,.$$

The 2-cell component of $[X,\theta]\cdot\Gamma(\psi,x)$ is $k\Gamma(\psi,x)$. Diagram 1.7 is coequalizer by assumption *4*. By assumption *3*,

$$(1.8)\qquad s\pi_1 s\pi_1\psi \;\underset{s\pi_1 d\psi}{\overset{ds\pi_1\psi}{\rightrightarrows}}\; s\pi_1\psi \xrightarrow{d\psi} \psi$$

is a coequalizer in $\mathcal{K}[X,B]$. Our task will be first to show that the parallel pair in 1.7 is isomorphic to the parallel pair in 1.8, and then that the resulting isomorphism $\psi\simeq\theta\Gamma(\psi,x)$ is a morphism of cones. Let z denote the 2-isomorphism $sx\cdot e\psi$, and let w denote the 2-isomorphism $sk\cdot e\theta$. Then $d\psi\cdot z=\kappa_{(\psi,x)}$, and $d\theta\cdot w=\kappa_{(\theta,k)}=\theta i$. We shall show that the squares

$$(1.9)\qquad \begin{array}{ccccc}
\theta r\theta r\psi & \xrightarrow{wr\psi} & s\pi_1\theta r\psi & \xrightarrow{s\pi_1 z} & s\pi_1 s\pi_1\psi\\
{\scriptstyle \theta i r\psi}\downarrow & & & & \downarrow{\scriptstyle ds\pi_1\psi}\\
\theta r\psi & & \xrightarrow{\quad z\quad} & & s\pi_1\psi
\end{array}$$

and

$$(1.10)\qquad \begin{array}{ccccc}
\theta r\theta r\psi & \xrightarrow{wr\psi} & s\pi_1\theta r\psi & \xrightarrow{s\pi_1 z} & s\pi_1 s\pi_1\psi\\
{\scriptstyle \theta r\kappa_{(\psi,x)}}\downarrow & & & & \downarrow{\scriptstyle s\pi_1 d\psi}\\
\theta r\psi & & \xrightarrow{\quad z\quad} & & s\pi_1\psi
\end{array}$$

commute. To see that 1.9 commutes, draw in the arrow

$$d\theta r\psi : s\pi_1\theta r\psi\rightarrow\theta r\psi\,,$$

while noting that

$$d\theta r\psi\cdot wr\psi=(d\theta\cdot w)r\psi=\theta i r\psi\,.$$

As for 1.10, rewrite it as

$$\begin{array}{ccccccc}
\theta r\theta r\psi & \xrightarrow{e\theta r\psi} & s\pi_0\theta r\psi & \xrightarrow{skr\psi} & s\pi_1\theta r\psi & \xrightarrow{s\pi_1 z} & s\pi_1 s\pi_1\psi\\
{\scriptstyle \theta r\kappa_{(\psi,x)}}\downarrow & \bullet & \downarrow{\scriptstyle s\pi_0\kappa_{(\psi,x)}} & & \bullet\bullet & & \downarrow{\scriptstyle s\pi_1 d\psi}\\
\theta r\psi & \xrightarrow[e\psi]{} & s\pi_0\psi & & \xrightarrow[sx]{\qquad} & & s\pi_1\psi\,.
\end{array}$$

In this figure, ●● commutes because, after removing s , it is simply the diagram stating that $\kappa_{(\psi,x)}$ is a morphism of cones, noting that $d\psi \cdot z = \kappa_{(\psi,x)}$. The square ● commutes by the axioms of a 2-category. Thus, 1.9 and 1.10 commute. Therefore, there is an isomorphism h making

$$\begin{array}{ccc} \theta r\psi & \xrightarrow{\theta q} & \theta\Gamma(\psi,x) \\ {\scriptstyle z}\downarrow & & \downarrow{\scriptstyle h} \\ s\pi_1\psi & \xrightarrow[d\psi]{} & \psi \end{array}$$

commute. It remains to show that h is a morphism of cones. This follows by a simple diagram chase involving the following three facts:

1. $h \cdot \theta q = d\psi \cdot z = \kappa_{(\psi,x)}$ is a morphism of cones,
2. θq is a morphism of cones, and
3. since $[X, \pi_0]$ is assumed to preserve coequalizers, the 2-cell $\pi_0\theta q$ is an epimorphism in $\mathcal{K}[X, C]$.

This proves that $[X, \theta] \cdot \Gamma$ is isomorphic to the identity on CONE_X, and concludes the proof of the theorem.

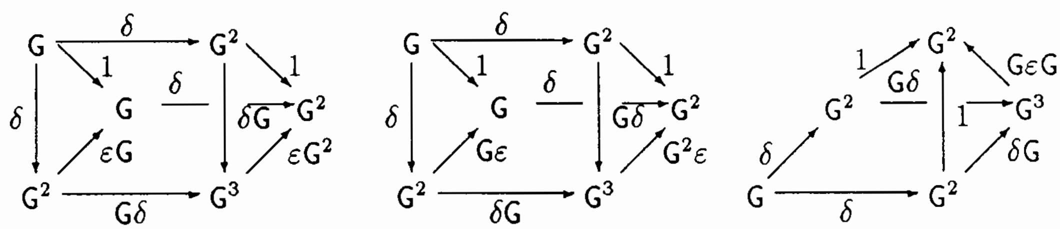
4.2 2-Cotriples

A 2-cotriple is herein taken to be a *strong* 2-cotriple (see [B1]). There is no need to keep using the prefix '2-', so a 2-cotriple shall henceforth be referred to as simply a cotriple. Let $\mathcal{K}$ denote an arbitrary 2-category.

2.1 DEFINITION A *cotriple* G on $\mathcal{K}$ is a 6-tuple $(\mathsf{G}, \varepsilon, \delta, p, q, w)$ where G is a 2-endofunctor on $\mathcal{K}$, and $\mathsf{G} \xrightarrow{\varepsilon} 1$ and $\mathsf{G} \xrightarrow{\delta} \mathsf{G}^2$ are 2-natural transformations together with modifying isomorphisms p, q and w as in the following diagrams.

$$\begin{array}{ccc} \mathsf{G} & \xrightarrow{\delta} & \mathsf{G}^2 \\ {\scriptstyle \delta}\downarrow & \overset{w}{\cong} & \downarrow{\scriptstyle \mathsf{G}\delta} \\ \mathsf{G}^2 & \xrightarrow[\delta\mathsf{G}]{} & \mathsf{G}^3 \end{array} \qquad \begin{array}{ccc} \mathsf{G} & \xrightarrow{\delta} & \mathsf{G}^2 \\ & {\scriptstyle 1}\searrow \; \overset{p}{\cong} & \downarrow{\scriptstyle \varepsilon\mathsf{G}} \\ & & \mathsf{G} \end{array} \qquad \begin{array}{ccc} \mathsf{G} & \xrightarrow{\delta} & \mathsf{G}^2 \\ & {\scriptstyle 1}\searrow \; \overset{q}{\cong} & \downarrow{\scriptstyle \mathsf{G}\varepsilon} \\ & & \mathsf{G} \end{array}$$

In these diagrams, G^n means G applied n times. Furthermore, the following coherence conditions are to be satisfied.



The modifying isomorphisms that belong in these diagrams have been omitted. For example, in the prism on the left go the isomorphisms p, $p\mathsf{G}$, w and ε_δ , which belong to the faces: left, right, front and bottom respectively. The isomorphism ε_δ is that supplied by ε as in the following square.

$$\begin{array}{ccc} \mathsf{G}^2 & \xrightarrow{\varepsilon\mathsf{G}} & \mathsf{G} \\ {\scriptstyle \mathsf{G}\delta}\downarrow & \overset{\varepsilon_\delta}{\cong} & \downarrow{\scriptstyle \delta} \\ \mathsf{G}^3 & \xrightarrow[\varepsilon\mathsf{G}^2]{} & \mathsf{G}^2 \end{array}$$

2.2 DEFINITION A *coalgebra* (again, to not use '2-coalgebra') for the cotriple **G** is a quadruple (B, θ, k, i) where B is a 0-cell, θ is a 1-cell, and k and i are 2-isomorphisms as in

$$\begin{array}{ccc} B & \xrightarrow{\theta} & \mathsf{G}B \\ {\scriptstyle \theta}\downarrow & \overset{k}{\cong} & \downarrow{\scriptstyle \mathsf{G}\theta} \\ \mathsf{G}B & \xrightarrow[\delta_B]{} & \mathsf{G}^2B \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{\theta} & \mathsf{G}B \\ & {\scriptstyle 1}\searrow \; \overset{i}{\cong} & \downarrow{\scriptstyle \varepsilon_B} \\ & & B \end{array}$$

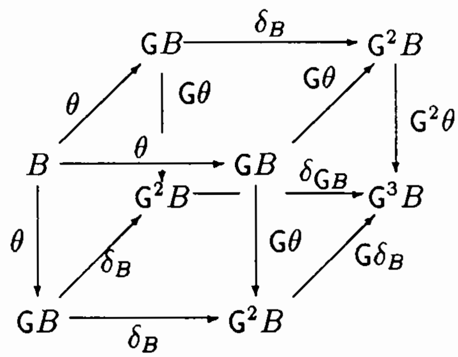
satisfying the unit and cocycle conditions,

$$q\theta \cdot \mathsf{G}(\varepsilon_B)k = \mathsf{G}(i)\theta \qquad\qquad \mathsf{G}(\delta_B)k = w_B\theta \cdot \delta_{\mathsf{G}B}k \cdot \delta_\theta\theta \cdot \mathsf{G}^2(\theta)k \cdot \mathsf{G}(k)\theta \; .$$

The 2-isomorphism δ_θ is that supplied by δ as in

$$\begin{array}{ccc} \mathsf{G}B & \xrightarrow{\delta_B} & \mathsf{G}^2B \\ {\scriptstyle \mathsf{G}\theta}\downarrow & \overset{\delta_\theta}{\cong} & \downarrow{\scriptstyle \mathsf{G}^2\theta} \\ \mathsf{G}^2B & \xrightarrow[\delta_{\mathsf{G}B}]{} & \mathsf{G}^3B \, . \end{array}$$

The cocycle condition expresses the commutivitity of the cube

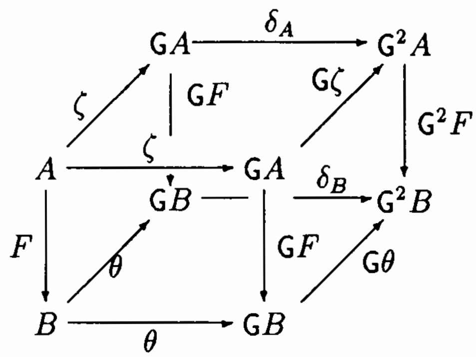


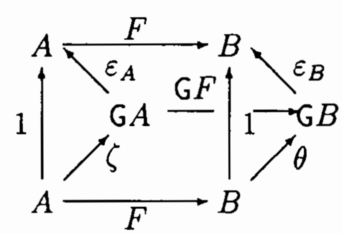
where the 2-isomorphisms belonging to the back, left, top, right, bottom and front faces are δ_θ, k, k, $\mathsf{G}(k)$, w_B and k respectively.

A morphism (1-cell) of coalgebras is a pair

$$(F, f) : (A, \zeta, j, h) \longrightarrow (B, \theta, k, i)$$

where $A \xrightarrow{F} B$ is a 1-cell and $\theta F \overset{f}{\simeq} \mathsf{G}(F)\zeta$ is a 2-isomorphism such that the diagrams





commute. The 2-isomorphisms belonging to the faces of the cube are, in their usual order: δ_F, f, j, $\mathsf{G}(f)$, k and f . In the prism, the isomorphisms which belong to the triangular faces are h and i , and to the three square faces are the identity, ε_F and f .

Finally, a 2-cell $(G, g) \xrightarrow{\alpha} (F, f)$ between morphisms of coalgebras is a 2-cell $G \xrightarrow{\alpha} F$ such that

$$\begin{array}{ccc} \theta G & \overset{g}{\simeq} & \mathsf{G}(G)\zeta \\ {\scriptstyle \theta\alpha}\downarrow & & \downarrow{\scriptstyle \mathsf{G}(\alpha)\zeta} \\ \theta F & \underset{f}{\simeq} & \mathsf{G}(F)\zeta \end{array}$$

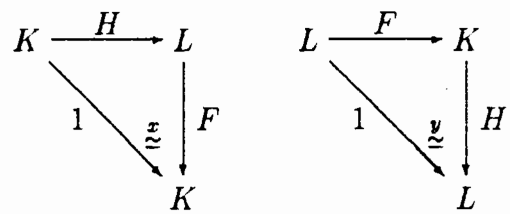
commutes. Coalgebras in $\mathcal{K}$ for the cotriple $\mathbf{G}$ form a 2-category which shall be denoted by $\mathcal{K}_{\mathbf{G}}$. This ends definition 2.2.

Let

$$\mathsf{U_G} : \mathcal{K}_{\mathsf{G}} \longrightarrow \mathcal{K}$$

denote the forgetful functor. $\mathsf{U_G}$ sends (B, θ, k, i) to B , (F, f) to F and α to α .

The intention is to establish a cotripleability theorem in the context of 2-categories. The first step is to observe that $\mathsf{U_G}$ reflects equivalences (an equivalence between two objects K, L, in $\mathcal{K}$ is a quadruple (F, H, x, y) with



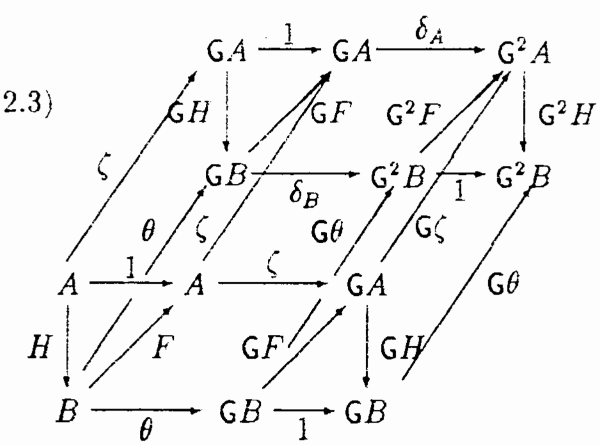
in $\mathcal{K}$). Indeed, assume that there is given a morphism of coalgebras

$$(F, f) : (B, \theta, k, i) \longrightarrow (A, \zeta, j, h) ,$$

and a 1-cell $A \xrightarrow{H} B$ along with 2-isomorphisms $F \cdot H \overset{x}{\simeq} 1_A$ and $H \cdot F \overset{y}{\simeq} 1_B$ in $\mathcal{K}$. A 2-isomorphism $\mathsf{G}H \cdot \theta \overset{g}{\simeq} \zeta \cdot H$ must be exibited making (H, g) a morphism of coalgebras. Furthermore, x and y must be shown to be 2-isomorphisms in $\mathcal{K}_{\mathsf{G}}$. Regarding x and y, a simple diagram chase shows that $\mathsf{U_G}$ reflects 2-isomorphisms, so if x and y are shown to be legitimate 2-cells, then so are their inverses.

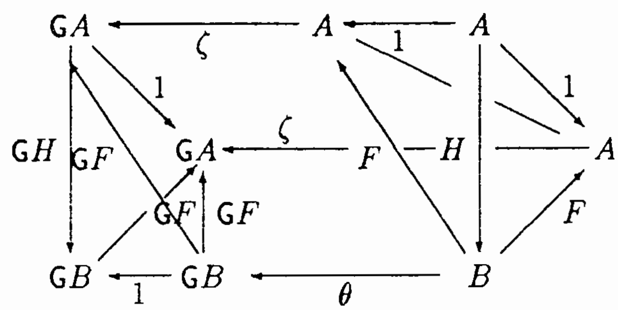
Let g denote the composite 2-isomorphism from $\mathsf{G}H \cdot \zeta$ to $\theta \cdot H$ as in the front face of the following cube. The commutativity of this cube is required for (H, g) to be a coalgebra morphism.

(2.3)



The inside cube in 2.3, with the appropriate 2-isomorphisms inserted, commutes because (F, f) is a coalgebra morphism. If the two prisms on either side of it were to commute, then this would show that (H, g) is a coalgebra morphism.

Furthermore, the commutativity of the left prism would say that the 2-isomorphism x is a 2-cell of coalgebra morphisms. A similiar prism would show the same thing for y, and in fact, G applied to that prism would be the right prism in the diagram above. Thus, the question comes down to showing that the left prism in 2.3 commutes. To see this, partition the prism's left face (the left face of 2.3, that is) in the same way that the front face of 2.3 is, to give the following figure of now just the prism.

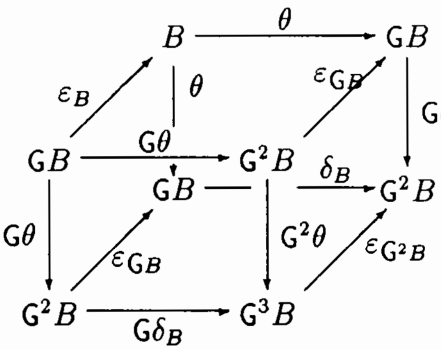


In this diagram, the two triangular pyramids on the ends commute, as does the inside prism. Therefore the whole thing commutes, and hence does 2.3. This concludes the proof that U_{G} reflects equivalences.

The next step is to observe that coalgebras are split equalizers as defined in the previous section. In fact, if (B, θ, k, i) is a typical coalgebra, then

$$(2.4) \qquad B \xrightarrow{\ \theta\ } \mathsf{G}B \underset{\delta_B}{\overset{\overset{\mathsf{G}\varepsilon_B}{\longleftarrow}}{\underset{\mathsf{G}\theta}{\rightrightarrows}}} \mathsf{G}^2B \underset{\delta_{\mathsf{G}B}}{\overset{\mathsf{G}^2\theta}{\underset{\mathsf{G}\delta_B}{\longrightarrow\!\!\!\!\longrightarrow\!\!\!\!\longrightarrow}}} \mathsf{G}^3B$$

is a split equalizer in $\mathcal{K}$, with, in the notation of the previous section, $\lambda = \mathsf{G}\varepsilon_B$, $\pi_0 = \mathsf{G}\theta$, $\pi_1 = \delta_B$, $\rho_0 = \mathsf{G}^2\theta$, $\rho_1 = \mathsf{G}\delta_B$, $\rho_2 = \delta_{\mathsf{G}B}$, and with r, s, t equal to $\varepsilon_B, \varepsilon_{\mathsf{G}B}, \varepsilon_{\mathsf{G}^2B}$ respectively. Furthermore, the 2-isomorphisms a, b, c, d, e and l, m, n are $p_{\mathsf{G}B}$, ε_{δ_B}, $\varepsilon_{\mathsf{G}\theta}$, p_B, ε_θ and δ_θ, $\mathsf{G}k$, w_B respectively. All the commuting conditions are routinely verified. For example, let us verify the three coherence conditions of a split equalizer. Of those three, the first one follows by the naturality of p for θ . The second condition is by the first coherence diagram (at B) in the definition of a cotriple. The third condition requires that



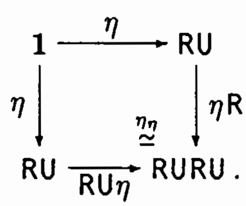
commute, which is true by the naturality of ε. The 2-isomorphisms in this cube are, in the usual order: k, ε_θ, ε_θ, $\varepsilon_{\mathsf{G}\theta}$, ε_{δ_B} and $\mathsf{G}k$. Thus, 2.4 is a split equalizer.

2.5 DEFINITION By an *adjointness* $\mathsf{U} \dashv \mathsf{R}$ between 2-categories $\mathcal{K}$ and $\mathcal{L}$ shall be meant a 6-tuple $(\mathsf{U}, \mathsf{R}, \varepsilon, \eta, \beta, \gamma)$, where $\mathcal{L} \xrightarrow{\mathsf{U}} \mathcal{K}$ and $\mathcal{K} \xrightarrow{\mathsf{R}} \mathcal{L}$ are 2-functors, $\mathbf{1} \xrightarrow{\eta} \mathsf{RU}$ and $\mathsf{UR} \xrightarrow{\varepsilon} \mathbf{1}$ are 2-natural transformations, and β and γ are modifying isomorphisms:

$$\varepsilon\mathsf{U} \cdot \mathsf{U}\eta \overset{\beta}{\simeq} 1_{\mathsf{F}} \quad \mathsf{R}\varepsilon \cdot \eta\mathsf{R} \overset{\gamma}{\simeq} 1_{\mathsf{R}} .$$

This notion is referred to as an *i-weak quasi-adjointness* in [Gra2] (p. 168).

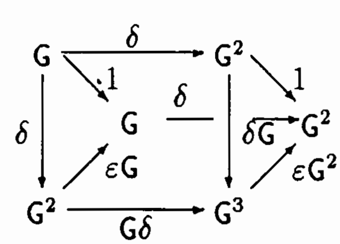
As expected, an adjointness between 2-categories gives rise to a cotriple. The notation η_η denotes the modifying isomorphism supplied by η *at* η as in



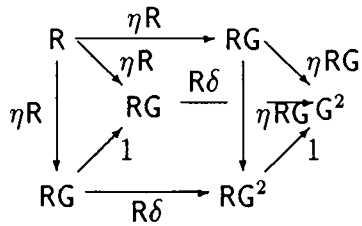
2.6 Proposition *Given an adjointness* $\mathsf{U} \dashv \mathsf{R}$ *with* $\mathcal{L} \xrightarrow{\mathsf{U}} \mathcal{K}$*, then* $(\mathsf{UR}, \varepsilon, \mathsf{U}\eta\mathsf{R}, \beta\mathsf{R}, \mathsf{U}\gamma, \mathsf{U}\eta_\eta\mathsf{R})$ *is a cotriple on* $\mathcal{K}$.

PROOF To verify the coherence conditions of a cotriple, let us denote the data $(\mathsf{UR}, \varepsilon, \mathsf{U}\eta\mathsf{R}, \beta\mathsf{R}, \mathsf{U}\gamma, \mathsf{U}\eta_\eta\mathsf{R})$ by $(\mathsf{G}, \varepsilon, \delta, p, q, w)$. The condition

(2.7)



with modifications p, $p\mathsf{G}$, w and ε_δ, transposes to



where $\eta_\eta \mathsf{R}$ is the modification in both the front and top faces. The other faces have the identity as their modifying isomorphisms, and therefore this figure commutes, which proves that 2.7 commutes. The other coherence conditions are verified in a similiar way. □

The cotriple of 2.6 shall be called the cotriple induced by, or associated with, $\mathsf{U} \dashv \mathsf{R}$.

Given an adjointness $\mathsf{U} \dashv \mathsf{R}$ with $\mathcal{L} \xrightarrow{\mathsf{U}} \mathcal{K}$, and with the induced cotriple $\mathbf{G}$ on $\mathcal{K}$, the *comparison* 2-functor,

$$\Phi : \mathcal{L} \longrightarrow \mathcal{K}_{\mathbf{G}} ,$$

is then defined to be, for 0-cells, 1-cells, and 2-cells respectively,

$$L, F, f \quad \rightsquigarrow \quad (\mathsf{U}L, \mathsf{U}\eta_L, \mathsf{U}\eta_{\eta_L}, \beta_L), \mathsf{U}F, \mathsf{U}f .$$

It is easily verified that Φ is indeed a 2-functor into the category of $\mathbf{G}$-coalgebras. U is said to be *cotripleable* if Φ is a 2-equivalence, in the following sense.

1. Φ is 2-fully faithful, which is to say that for all $L, M \in \mathcal{L}$, the functor

   $$\Phi_{LM} : \mathcal{L}[L, M] \longrightarrow \mathcal{K}_{\mathbf{G}}[\Phi L, \Phi M]$$

   is an equivalence.

2. Φ is 2-essentially surjective, which is to say that every coalgebra is equivalent in $\mathcal{K}_{\mathbf{G}}$ to ΦL for some $L \in \mathcal{L}$.

For any $L \in \mathcal{L}$, the diagram

$$\mathsf{RU}L \begin{array}{c} \xleftarrow{\mathsf{R}\varepsilon_{\mathsf{U}L}} \\ \xrightarrow{\mathsf{RU}\eta_L} \\ \xrightarrow[\eta_{\mathsf{RU}L}]{} \end{array} (\mathsf{RU})^2 L \begin{array}{c} \xrightarrow{(\mathsf{RU})^2\eta_L} \\ \xrightarrow{\mathsf{RU}\eta_{\mathsf{RU}L}} \\ \xrightarrow[\eta_{(\mathsf{RU})^2 L}]{} \end{array} (\mathsf{RU})^3 L \tag{2.8}$$

is one of the form 1.1, where $\pi_0 = \mathsf{RU}\eta_L$, $\rho_0 = (\mathsf{RU})^2\eta_L$ and so on. The 2-isomorphisms l, m, n and u, v are $\eta_{\mathsf{RU}\eta_L}$, $\mathsf{RU}\eta_{\eta_L}$, $\eta_{\eta_{\mathsf{RU}_L}}$ and $\mathsf{R}\gamma_L$, $\beta_{\mathsf{U}L}$ respectively. Furthermore, (η_L, η_{η_L}) is a cone (see 1.2) from L to the above diagram, and U applied to this cone along with 2.8 yields the split equalizer in $\mathcal{K}$ arising from the coalgebra ΦL.

In the context of 2-categories, the cotripleability theorems are then as follows. In these theorems, which are the next three, $\mathbf{G} = (\mathsf{G}, \varepsilon, \delta, p, q, w)$ denotes the cotriple induced by $\mathsf{U} \dashv \mathsf{R}$.

2.9 Theorem *The comparison 2-functor Φ is 2-fully faithful if and only if for every $L \in \mathcal{L}$ the cone (η_L, η_{η_L}) is an equalizer of diagram* 2.8.

PROOF For the sufficiency of the given condition, fix 0-cells M, N in $\mathcal{L}$. To be shown is that

$$\Phi_{MN} : \mathcal{L}[M,N] \longrightarrow \mathcal{K}_{\mathbf{G}}[\Phi M, \Phi N]$$

is an equivalence. Let CONE_M denote the category of cones from M to the diagram of the form 2.8 with N in the place of L. Define a functor

$$\Psi : \mathcal{K}_{\mathbf{G}}[\Phi M, \Phi N] \longrightarrow \text{CONE}_M \cong \mathcal{L}[M,N] ,$$

where the first arrow sends a coalgebra morphism $\Phi M \xrightarrow{F} \Phi N$ to the cone whose 1-cell component is $\mathsf{R}F \cdot \eta_M$. The equivalence in the definition of Ψ is by hypothesis. Now for a given $M \xrightarrow{H} N$ both H and $\Psi\Phi_{MN}H$ give rise to the cone $\mathsf{RU}H \cdot \eta_M$ in CONE_M as in the following diagram:

$$\begin{array}{ccc} M & \xrightarrow{\eta_M} & \mathsf{RU}M \\ {\scriptstyle \Psi\Phi_{MN}H}\Big\downarrow\Big\downarrow{\scriptstyle H} & \simeq & \Big\downarrow{\scriptstyle \mathsf{RU}H} \\ N & \xrightarrow[\eta_N]{} & \mathsf{RU}N . \end{array}$$

It follows that $\Psi\Phi_{MN}$ is isomorphic to the identity on $\mathcal{L}[M,N]$. On the other hand, given a coalgebra morphism $\Phi M \xrightarrow{F} \Phi N$, by the definition of ΨF there is an isomorphism of cones

$$\begin{array}{ccc} M & \xrightarrow{\eta_M} & \mathsf{RU}M \\ {\scriptstyle \Psi F}\Big\downarrow & \simeq & \Big\downarrow{\scriptstyle \mathsf{R}F} \\ N & \xrightarrow[\eta_N]{} & \mathsf{RU}N . \end{array}$$

Applying U to this square gives

$$
(2.10)\qquad
\begin{array}{ccc}
\mathsf{U}M & \xrightarrow{\mathsf{U}\eta_M} & \mathsf{GU}M \\
{\scriptstyle \mathsf{U}\Psi F}\big\downarrow & \simeq & \big\downarrow{\scriptstyle \mathsf{G}F} \\
\mathsf{U}N & \xrightarrow[\mathsf{U}\eta_N]{} & \mathsf{GU}N\,,
\end{array}
$$

where observe that the horizontal arrows in 2.10 are the universal cones for the split equalizers in $\mathcal{K}$ arising from the coalgebras ΦM and ΦN . Being a coalgebra map, F also makes 2.10 commute. Therefore, $\mathsf{U}\eta_N \cdot F \simeq \mathsf{U}\eta_N \cdot \mathsf{U}\Psi F$ as cones from $\mathsf{U}M$. Thus, F is isomorphic to $\mathsf{U}\Psi F = \Phi_{MN}\Psi F$.

For the converse, fix an $L \in \mathcal{L}$. To be shown is that composition with the cone (η_L, η_{η_L}) gives, for any $X \in \mathcal{L}$, an equivalence

$$\mathcal{L}[X, L] \cong \textsc{Cone}_X \,.$$

The categories $\mathcal{K}_{\mathbf{G}}[\Phi X, \Phi L]$ and $\textsc{Cone}_X$ are sub-categories (not full) of $\mathcal{K}[\mathsf{U}X, \mathsf{U}L]$ and $\mathcal{L}[X, \mathsf{RU}L]$ respectively, and it is not hard to see that the equivalence

$$\mathcal{K}[\mathsf{U}X, \mathsf{U}L] \cong \mathcal{L}[X, \mathsf{RU}L]$$

coming from the adjointness $\mathsf{U} \dashv \mathsf{R}$ restricts to

$$\mathcal{K}_{\mathbf{G}}[\Phi X, \Phi L] \cong \textsc{Cone}_X \,.$$

By hypothesis, Φ_{XL} is an equivalence, and this concludes the proof. □

2.11 Theorem *A coalgebra (B, θ, k, i) is in the 2-essential image of Φ if the equalizer of*

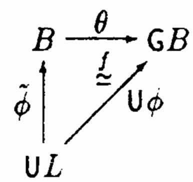
$$
(2.12)\qquad
\mathsf{R}B \;
\begin{array}{c}
\xleftarrow{\;\mathsf{R}\varepsilon_B\;} \\
\xrightarrow{\;\mathsf{R}\theta\;} \\
\xrightarrow[\;\eta_{\mathsf{R}B}\;]{}
\end{array}
\; \mathsf{RG}B \;
\begin{array}{c}
\xrightarrow{\;\mathsf{RG}\theta\;} \\
\xrightarrow{\;\mathsf{R}\delta_B\;} \\
\xrightarrow[\;\eta_{\mathsf{RG}B}\;]{}
\end{array}
\; \mathsf{RG}^2B
$$

exists in $\mathcal{L}$ and U preserves it.

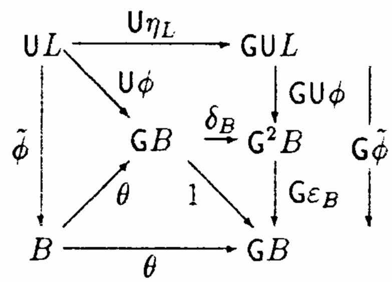
PROOF Let (B, θ, k, i) denote an arbitrary coalgebra. By hypothesis, there exists a universal cone (ϕ, j) from say L to diagram 2.12. Applying U to diagram 2.12 and the cone (ϕ, j) gives

$$
\mathsf{U}L \xrightarrow{\;\mathsf{U}\phi\;} \mathsf{G}B \;
\begin{array}{c}
\xleftarrow{\;\mathsf{G}\varepsilon_B\;} \\
\xrightarrow{\;\mathsf{G}\theta\;} \\
\xrightarrow[\;\delta_B\;]{}
\end{array}
\; \mathsf{G}^2B \;
\begin{array}{c}
\xrightarrow{\;\mathsf{G}^2\theta\;} \\
\xrightarrow{\;\mathsf{G}\delta_B\;} \\
\xrightarrow[\;\delta_{\mathsf{G}B}\;]{}
\end{array}
\; \mathsf{G}^3B\,.
$$

Since (θ, k) is an equalizer of this diagram, in fact making it a split equalizer, there exists



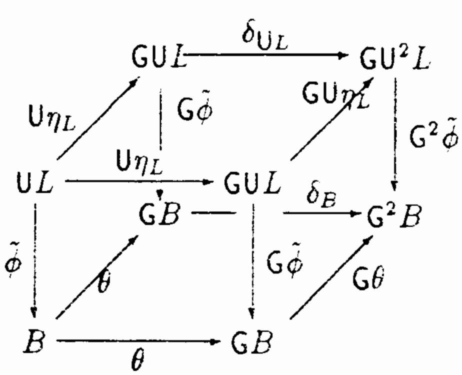
where f is an isomorphism of the cones $(\mathsf{U}\phi, \mathsf{U}j)$ and $(\theta \cdot \tilde{\phi}, k \cdot \tilde{\phi})$. Furthermore, $\tilde{\phi}$ can be taken to be *equal* to $\varepsilon_B \cdot \mathsf{U}\phi$, the transpose of ϕ, because ε_B is the 'r' of the split equalizer $(B, \theta, k, \ldots)$. However, $\tilde{\phi}$ must be an equivalence in $\mathcal{K}$ because U is assumed to preserve the equalizer (ϕ, j). Now consider the following figure.



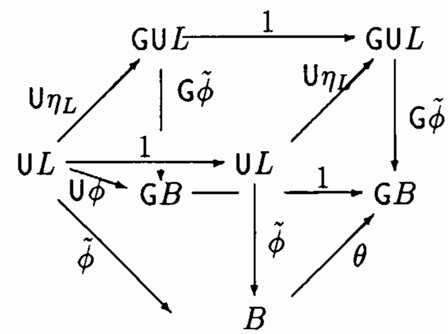
In this figure, f is the 2-isomorphism in the left triangle, $\mathsf{U}\eta_\phi$ goes in the square, and $\mathsf{U}\gamma_B$ goes in the lower right triangle. Denote the composite 2-isomorphism in the above diagram from $\theta \cdot \tilde{\phi}$ to $\mathsf{G}\tilde{\phi} \cdot \mathsf{U}\eta_L$ by x. The claim is that

$$(\tilde{\phi}, x) : \Phi L \longrightarrow (B, \theta, k, i)$$

is a morphism of coalgebras. So, to be shown is the commutativity of



Insert the morphism $\mathsf{U}L \xrightarrow{\mathsf{U}\phi} \mathsf{G}B$ in as a diagonal on the left and front faces of the above cube. The cube is thus divided into two figures. One, which consists of the lower left corner together with the bottom face, commutes because f is a morphism of cones. The other figure, which consist of the cube *less* the lower left corner, is entirely within the image of U, which means that every 0-cell, 1-cell and 2-isomorphism in it is preceeded by a 'U'. Upon removing U from this figure and transposing it back to $\mathcal{K}$, the following figure results.

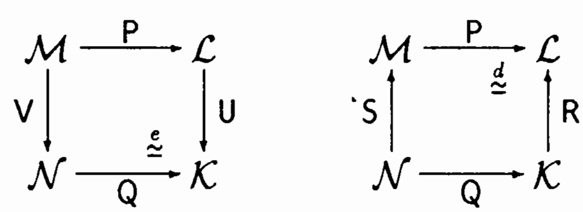


This figure commutes, which proves that $(\tilde{\phi}, x)$ is coalgebra morphism. Since $\mathsf{U}_{\mathbf{G}}$ reflects equivalences, (B, θ, k, i) is therefore equivalent to ΦL in $\mathcal{K}_{\mathbf{G}}$. □

2.13 Theorem $\mathsf{U} \dashv \mathsf{R}$ *is cotripleable if and only if for all* $L \in \mathcal{L}$ *the cone* (η_L, η_{η_L}) *is an equalizer of diagram* 2.8, *and for every coalgebra* (B, θ, k, i) , *the equalizer of* 2.12 *exists in* $\mathcal{L}$ *and* U *preserves it.*

PROOF All that remains to prove is that if $\mathsf{U} \dashv \mathsf{R}$ is cotripleable, then for every coalgebra (B, θ, k, i) , the equalizer of 2.12 exists in $\mathcal{L}$ and U preserves it. Since Φ is 2-essentially surjective it suffices to show this for every coalgebra of the form ΦL . But, since Φ is 2-fully faithful, L with the cone (η_L, η_{η_L}) is the equalizer of 2.12, where $B = \mathsf{U}L$, $\theta = \mathsf{U}\eta_L$ and so on. As previously remarked, U applied to this equalizer is a split equalizer in $\mathcal{K}$. □

Theorem 2.13 shall find its first application in the next section with the following observation in mind. Suppose that



are 2-categories and 2-functors, with natural equivalences e and d . Also, suppose that $\mathsf{U} \dashv \mathsf{R}$ and $\mathsf{V} \dashv \mathsf{S}$, and that d corresponds to $\mathsf{Q}\varepsilon \cdot e\mathsf{S}$ under $\mathsf{U} \dashv \mathsf{R}$, where ε

denotes the counit of $\mathsf{V} \dashv \mathsf{S}$. Then, assuming that U is cotripleable, if P preserves and reflects equalizers and if Q reflects equalizers, then V is cotripleable.

4.3 Descent for cocomplete categories

Throughout this section $\mathcal{E} \xrightarrow{\mathsf{p}} \mathcal{S}$ shall denote a fixed arbitrary bounded geometric morphism. Let $\mathcal{E}^n$ denote the n-fold product of $\mathcal{E}$ in $\text{TOP}_{\mathcal{S}}$. Then $\mathcal{E}^n$, $n = 1, 2, \ldots$, exists and is the tensor product $\overbrace{\mathcal{E}\otimes_{\mathcal{S}}\mathcal{E}\otimes_{\mathcal{S}}\ldots\mathcal{E}}^{n}$.

The immediate difficulty with descent in the context of locally small cocomplete categories is that the tensor product is possibly not everywhere defined. However, by using the Beck condition, this difficulty can be 'done away with' as follows. Let $\text{COCTS}_{\mathsf{p}^\circ}$ denote the full sub-2-category of $\text{COCTS}_{\mathcal{S}}$ whose objects are those $\mathcal{A} \in \text{COCTS}_{\mathcal{S}}$ such that $\mathcal{E}^n \otimes_{\mathcal{S}} \mathcal{A}$ exists for $n = 1, 2, 3, \ldots$.

$$|\, \text{COCTS}_{\mathsf{p}^\circ} \,| = \{\mathcal{A} \in \text{COCTS}_{\mathcal{S}} \;\;|\;\; \mathcal{E}^n \otimes_{\mathcal{S}} \mathcal{A} \text{ exists, } n = 1, 2, 3, \ldots\}$$

Similarly, let $\text{COCTS}_{\mathsf{p}}$ denote the full sub-2-category of $\text{COCTS}_{\mathcal{E}}$ whose objects are

$$|\, \text{COCTS}_{\mathsf{p}} \,| = \{\mathcal{B} \in \text{COCTS}_{\mathcal{E}} \;\;|\;\; \mathcal{E}^{n+1} \otimes_{\mathcal{E}} \mathcal{B} \text{ exists, } n = 1, 2, 3, \ldots\} .$$

Here, $\mathcal{E}^{n+1} \otimes_{\mathcal{E}} \mathcal{B}$ means the tensor product taken along any one of the $n+1$ projections $\mathcal{E}^{n+1} \longrightarrow \mathcal{E}$. Observe that it exists along one projection if and only if it exists along all $n+1$ of them.

Thus, there is a 2-functor

$$\mathcal{E}\otimes_{\mathcal{S}} : \text{COCTS}_{\mathsf{p}^\circ} \longrightarrow \text{COCTS}_{\mathsf{p}} .$$

Since the Beck condition (see Chap. 3, §2) is satisfied, it follows that the right adjoint of $\mathcal{E}\otimes_{\mathcal{S}}$ restricts to these 2-categories,

$$(\;)_{\mathsf{p}} : \text{COCTS}_{\mathsf{p}} \longrightarrow \text{COCTS}_{\mathsf{p}^\circ} .$$

For example, if $\mathcal{B} \in \text{COCTS}_{\mathsf{p}}$, then $\mathcal{E}\otimes_{\mathcal{S}}(\mathcal{B}_{\mathsf{p}})$ exists and is equivalent to $(\mathcal{E}^2 \otimes_{\mathcal{E}} \mathcal{B})_{\pi_1}$, where

$$\begin{array}{ccc} \mathcal{E}^2 & \xrightarrow{\pi_1} & \mathcal{E} \\ {\scriptstyle \pi_0}\downarrow & \simeq & \downarrow{\scriptstyle \mathsf{p}} \\ \mathcal{E} & \xrightarrow[\mathsf{p}]{} & \mathcal{S} \end{array} .$$

is a pullback of toposes.

3.1 Definition The geometric morphism p is said to be of *effective descent for cocomplete categories* if

$$\mathcal{E}\otimes_{\mathcal{S}} \; : \text{COCTS}_{\mathsf{p}^0} \longrightarrow \text{COCTS}_{\mathsf{p}}$$

is cotripleable (in the sense developed in §2).

Normally, one would say that p is of effective descent if objects in $\text{COCTS}_{\mathcal{E}}$ equipped with descent data descend uniquely to $\text{COCTS}_{\mathcal{S}}$. Descent data would be an equivalence

$$\theta : \mathcal{E}^2 \otimes_{\mathcal{E}} \mathcal{B} \; \cong \mathcal{B} \otimes_{\mathcal{E}} \mathcal{E}^2$$

which satisfies the cocycle condition (in this context, up to coherent isomorphism). The same would be required of 1-cells and 2-cells in $\text{COCTS}_{\mathcal{E}}$ which commute with descent data; they would be required to descent uniquely to $\text{COCTS}_{\mathcal{S}}$. However, since the Beck condition is satisfied, this formulation of descent would then be equivalent to 3.1. This 'translation' has been omitted.

Let $\text{GTOP}_{\mathcal{S}}$ denote those toposes over $\mathcal{S}$ which are bounded. The 2-embedding (see Chap. 1, §6)

$$\wp : \text{GTOP}_{\mathcal{S}}{}^{op} \longrightarrow \text{COCTS}_{\mathcal{S}}$$

factors through $\text{COCTS}_{\mathsf{p}^0}$, and in this way one regards $\text{GTOP}_{\mathcal{S}}{}^{op}$ as a sub-2-category of $\text{COCTS}_{\mathsf{p}^0}$. Similarly, $\text{GTOP}_{\mathcal{E}}{}^{op}$ is regarded as a sub-2-category of $\text{COCTS}_{\mathsf{p}}$. Moreover, by Pitts' theorem (Chap. 3, §2), the adjointness $\mathcal{E}\otimes_{\mathcal{S}} \dashv (\;)_{\mathsf{p}}$ restricts to bounded toposes. It would then be written as $\Sigma_{\mathsf{p}} \dashv \mathcal{E}\times_{\mathcal{S}}$, or as $\Sigma_{\mathsf{p}} \dashv \mathsf{p}\times$.

3.2 Definition The geometric morphism p is said to be of *effective descent for toposes* if

$$\mathcal{E}\times_{\mathcal{S}} \; : \text{GTOP}_{\mathcal{S}} \longrightarrow \text{GTOP}_{\mathcal{E}}$$

is tripleable.

Let $\mathcal{S}et$ denote the topos of sets, and for the rest of this section assume that $\mathcal{S} \in \text{GTOP}_{\mathcal{S}et}$; assume that $\mathcal{S}$ is a Grothendieck topos.

3.3 Example Pure geometric morphisms are introduced in §4, and they are shown to be of effective descent for cocomplete categories. If p is a spatial open surjection (the definitions of these notions are given in §4), then it is seen in §4 that p is pure. Hence, the following theorem is established.

Theorem *Any spatial open surjection is of effective descent for cocomplete categories.*

By the theorem which follows this example, the following result due to Moerdijk [M5] is thereby established.

Theorem (Moerdijk) *Any spatial*[2] *open surjection is of effective descent for toposes.*

Moerdijk's theorem has as a direct consequence the 2-dimensional version of a classification theorem for toposes originally due to Bunge [B4]. Let us state and prove this classification theorem.

Let $\mathbf{G} = (G_0, G_1)$ be a spatial groupoid in $\mathcal{S}$. Assume that the domain and codomain maps d_0 and d_1 are open. Define $\mathbf{BG}$ to be the coequalizer

$$sh(G_1 \times_{G_0} G_1) \underset{\pi_1}{\overset{\pi_0}{\underset{m}{\rightrightarrows}}} sh(G_1) \underset{d_1}{\overset{\lambda}{\underset{d_0}{\rightrightarrows}}} sh(G_0) \xrightarrow{\mathsf{p}} \mathbf{BG}$$

in $\text{GTop}_{\mathcal{S}}$. That this coequalizer exists is shown later in this section. The same symbols have been used to denote the geometric morphisms which correspond to the continuous maps. The geometric morphism m is (that which corresponds to) the composition of the groupoid. From the construction of $\mathbf{BG}$ (see below), it is clear that p is a surjection, and since d_0 and d_1 are assumed to be open, it follows that p is open (see [M3]). Since $sh(G_0)$ is spatial over $\mathcal{S}$, p is spatial over $\mathbf{BG}$. Therefore, p is of effective descent for toposes.

Let $\widehat{\mathbf{G}}$ denote the spatial groupoid obtained as the 2-kernel pair of p . In other words, $\widehat{G}_1$ is defined as the pullback

$$\begin{array}{ccc} sh(\widehat{G}_1) & \longrightarrow & sh(G_0) \\ \big\downarrow & \simeq & \big\downarrow \mathsf{p} \\ sh(G_0) & \xrightarrow[\mathsf{p}]{} & \mathbf{BG} \end{array}$$

in $\text{GTop}_{\mathcal{S}}$. In [M3], $\widehat{\mathbf{G}} = (G_0, \widehat{G}_1)$ is called the etale-completion of $\mathbf{G}$. Let

$$\text{GTop}_{\mathcal{S}}{}^{\widehat{\mathbf{G}}}$$

denote the 2-category of algebras for the triple on $\text{GTop}_{sh(G_0)}$ induced by the adjointness $\Sigma_{\mathsf{p}} \dashv \mathsf{p}\times$. An object of $\text{GTop}_{\mathcal{S}}{}^{\widehat{\mathbf{G}}}$ is a topos over $sh(G_0)$ equipped

[2] He proves the general case. His methods are entirely different from those used here.

with a continuous action by $\widehat{\mathbf{G}}$. The (2-dimensional version of the) classification theorem is as follows.

Theorem (Bunge) *Pullback along* p *induces an equivalence*

$$\mathrm{GTOP}_{\mathbf{BG}} \cong \mathrm{GTOP}_{\mathcal{S}}{}^{\widehat{\mathbf{G}}} .$$

PROOF p is of effective descent for toposes. □

One says that $\mathbf{BG}$ classifies $\widehat{\mathbf{G}}$-toposes. This concludes this example.

In the remainder of this section, the intention is to establish that a morphism which is of effective descent for cocomplete categories remains so at the level of toposes.

3.4 Theorem *If* p *is of effective descent for cocomplete categories, then* p *is of effective descent for toposes.*

To prove 3.4, let us begin by examining equalizers in $\mathrm{COCTS}_{\mathcal{S}}$. Given a diagram

$$(3.5)\qquad \mathcal{B}\ \begin{matrix}\xleftarrow{\ \lambda\ }\\[-4pt] \xrightarrow{\ \pi_0\ }\\[-4pt] \xrightarrow[\ \pi_1\ ]{}\end{matrix}\ \mathcal{C}\ \begin{matrix}\xrightarrow{\ \rho_0\ }\\[-4pt] \xrightarrow{\ \rho_1\ }\\[-4pt] \xrightarrow[\ \rho_2\ ]{}\end{matrix}\ \mathcal{D}$$

of locally small cocomplete categories and cocontinuous functors with 2-isomorphisms

$$\rho_0 \cdot \pi_1 \overset{l}{\simeq} \rho_2 \cdot \pi_0 \qquad \rho_0 \cdot \pi_0 \overset{m}{\simeq} \rho_1 \cdot \pi_0 \qquad \rho_2 \cdot \pi_1 \overset{n}{\simeq} \rho_1 \cdot \pi_1$$
$$\lambda \cdot \pi_0 \overset{u}{\simeq} 1_{\mathcal{B}} \qquad \lambda \cdot \pi_1 \overset{v}{\simeq} 1_{\mathcal{B}}$$

all over $\mathcal{S}$, consider first only π_0 and π_1 . For $I \in \mathcal{S}$, let $\mathcal{Z}^I$ denote the sub-category of $\mathcal{B}^I$ which has as its objects all pairs (B, k) where $B \in \mathcal{B}^I$ and $\pi_0^I B \overset{k}{\simeq} \pi_1^I B$ is an isomorphism in $\mathcal{C}$. A morphism $(A, j) \xrightarrow{f} (B, k)$ in $\mathcal{Z}^I$ is defined to be a morphism $A \xrightarrow{f} B$ in $\mathcal{B}^I$ such that

$$\begin{array}{ccc} \pi_0^I A & \overset{j}{\simeq} & \pi_1^I A \\ {\scriptstyle \pi_0^I f}\downarrow & & \downarrow{\scriptstyle \pi_1^I f} \\ \pi_0^I B & \underset{k}{\simeq} & \pi_1^I B \end{array}$$

commutes. One sees easily that the substitution functors of $\mathcal{B}$ restrict to the categories $\mathcal{Z}^I$ thereby giving us the category $\mathcal{Z}$ over $\mathcal{S}$. $\mathcal{Z}$ is locally small. In fact, the object in $\mathcal{S}_{/I}$ which represents morphisms $(A,j) \xrightarrow{f} (B,k)$ in $\mathcal{Z}_{/I}$ can be calculated as the equalizer of

$$\begin{array}{ccc} \mathcal{B}^I(A,B) & \xrightarrow{\pi_0^I} & \mathcal{C}^I(\pi_0^I A, \pi_0^I B) \\ {\scriptstyle \pi_1^I}\downarrow & & \downarrow{\scriptstyle k\circ} \\ \mathcal{C}^I(\pi_1^I A, \pi_1^I B) & \xrightarrow[\circ j]{} & \mathcal{C}^I(\pi_0^I A, \pi_1^I B), \end{array}$$

where $k\circ$ and $\circ j$ are composition with k and respectively j . Furthermore, since π_0 and π_1 are cocontinuous it follows directly that $\mathcal{Z}$ is cocomplete, with colimits being created by the inclusion of $\mathcal{Z}$ into $\mathcal{B}$.

Now consider the full sub-category $\mathcal{A}$ of $\mathcal{Z}$ determined by those objects (A,j) which satisfy the unit and cocycle conditions:

$$v_A \cdot \lambda(j) = u_A \qquad\qquad \rho_1(j) = n_A \cdot \rho_2(j) \cdot l_A \cdot \rho_0(j) \cdot m_A \ .$$

Then $\mathcal{A}$ is locally small since it is a full sub-category of a locally small category. Moreover, $\mathcal{A}$ is cocomplete since $\lambda, \rho_0, \rho_1, \rho_2$ are cocontinuous, with colimits being created by the inclusion of $\mathcal{A}$ into $\mathcal{B}$. It follows that $\mathcal{A}$ is the equalizer in $\text{COCTS}_{\mathcal{S}}$ of diagram 3.5. Thus, $\text{COCTS}_{\mathcal{S}}$ has equalizers.

The next step is to show that the 2-embedding

$$\wp : \text{GTOP}_{\mathcal{S}}{}^{op} \longrightarrow \text{COCTS}_{\mathcal{S}}$$

creates equalizers. Let

$$(3.6) \qquad \mathcal{A} \xrightarrow{\theta} \mathcal{B} \;\underset{\pi_1^*}{\overset{\lambda^*,\ \pi_0^*}{\rightleftarrows\!\!\to}}\; \mathcal{C} \;\underset{\rho_2^*}{\overset{\rho_0^*,\ \rho_1^*}{\rightrightarrows\!\!\to}}\; \mathcal{D}$$

be the equalizer in $\text{COCTS}_{\mathcal{S}}$ of toposes and (the inverse images of) geometric morphisms in $\text{GTOP}_{\mathcal{S}}$. It must be shown that $\mathcal{A}$ comes from a topos and θ from a geometric morphism over $\mathcal{S}$, and furthermore, that $\mathcal{A}$ is the coequalizer in $\text{GTOP}_{\mathcal{S}}$. The intention is to use the theorem (Th. 6.7) developed in Chapter 1 about when a category comes from a topos. First note that finite limits in $\mathcal{A}$ are created by θ since all the functors in 3.6, except ostensibly θ , are left exact. But then θ must be left exact.

The first requirement of theorem 6.7 from Chapter 1 is that the fiber category $\mathcal{A}^1$ be an elementary topos. Consider the data of 3.6 at $1 \in \mathcal{S}$.

$$(3.7) \qquad \mathcal{A}^1 \xrightarrow{\theta^1} \mathcal{B}^1 \; \underset{\pi_1^{*1}}{\overset{\overset{\lambda^{*1}}{\longleftarrow}}{\underset{}{\overset{\pi_0^{*1}}{\rightrightarrows}}}} \; \mathcal{C}^1 \; \underset{\rho_2^{*1}}{\overset{\rho_0^{*1}}{\underset{}{\overset{\rho_1^{*1}}{\rightrightarrows}}}} \; \mathcal{D}^1 \, .$$

In so forgetting the $\mathcal{S}$ structure, 3.7 is simply a diagram in $\text{GTOP}_{\mathcal{S}et}$, with ostensibly the exception of $\mathcal{A}^1$ and θ^1. Given a diagram in $\text{GTOP}_{\mathcal{S}et}$ such as 3.7, it follows that $\mathcal{A}^1$ is in $\text{GTOP}_{\mathcal{S}et}$. A direct demonstration of this can be found in [M3], and the reader is referred there for the details. Thus, the first requirement of 6.7 is satisfied. The second requirement, that $\mathcal{A}$ have coproducts which satisfy Frobenius reciprocity and which reflect isomorphisms at 1, is satisfied because $\mathcal{A}$ inherits coproducts and finite limits from $\mathcal{B}$. The third requirement, that $\mathcal{A}$ be locally small, has already been shown to be satisfied. Thus, $\mathcal{A}$ does indeed come from a topos over $\mathcal{S}$ (which is bounded over $\mathcal{S}$ since $\mathcal{A}^1 \in \text{GTOP}_{\mathcal{S}et}$). That is, $\mathcal{A} \in \text{GTOP}_{\mathcal{S}}$.

The ordinary functor θ^1 is cocontinuous in the ordinary sense over $\mathcal{S}et$ because θ is cocontinuous over $\mathcal{S}$. Therefore, θ^1 is the inverse image of a geometric morphism. However, the point is that one wants θ to come from a geometric morphism over $\mathcal{S}$. By 6.8 of Chapter 1, this is the case. Also, given a cone in $\text{GTOP}_{\mathcal{S}}$ from an $\mathcal{X} \in \text{GTOP}_{\mathcal{S}}$ to 3.6, the induced functor from $\mathcal{X}$ to $\mathcal{A}$ is left exact, and it is clear that it must come from a geometric morphism over $\mathcal{S}$. Hence, $\mathcal{A}$ is the coequalizer in $\text{GTOP}_{\mathcal{S}}$. This proves that $\wp$ creates equalizers.

Let us return now to the geometric morphism $\mathcal{E} \xrightarrow{\mathsf{p}} \mathcal{S}$, and the full sub-2-category $\text{COCTS}_{\mathsf{p}^0}$ of $\text{COCTS}_{\mathcal{S}}$. The 2-embedding $\wp$ factors through $\text{COCTS}_{\mathsf{p}^0}$,

$$\wp : {\text{GTOP}_{\mathcal{S}}}^{op} \longrightarrow \text{COCTS}_{\mathsf{p}^0} \, .$$

By regarding $\wp$ as a functor into $\text{COCTS}_{\mathsf{p}^0}$, we see that it preserves equalizers since it creates them in $\text{COCTS}_{\mathcal{S}}$. A 2-embedding which preserves equalizers must also reflect them (if they exist). Therefore, $\wp$ reflects equalizers from $\text{COCTS}_{\mathsf{p}^0}$. The same is true at the level of $\mathcal{E}$. That is,

$$\wp : {\text{GTOP}_{\mathcal{E}}}^{op} \longrightarrow \text{COCTS}_{\mathsf{p}}$$

preserves and reflects equalizers. Given the work done in the previous section (see the paragraph following 2.13), the proof of 3.4 is now complete.

4.4 Pure geometric morphisms

The intention in this section is to show that pure geometric morphisms, to be defined presently, are of effective descent for cocomplete categories.

In [JT], a morphism of locales $\mathbf{A} \xrightarrow{f} \mathbf{B}$ is said to be pure if for every $\mathbf{A}$-module $\mathbf{M}$, the universal morphism

$$\eta_{\mathbf{M}} : \mathbf{M} \rightarrow \mathbf{B} \otimes_{\mathbf{A}} \mathbf{M}$$

is faithful. The category of $\mathbf{A}$-modules is self-dual via the functor $\mathbf{M} \rightsquigarrow \mathbf{M}^{op}$. The $\mathbf{A}$-module $\mathbf{M}^{op}$ is the opposite sup-lattice of $\mathbf{M}$, where supremums in $\mathbf{M}^{op}$ are infimums in $\mathbf{M}$. This passage sends an $\mathbf{A}$-module map $\mathbf{N} \xrightarrow{\alpha} \mathbf{M}$ to its right adjoint which can be denoted by α^{o}. One shows (see [JT]) that a pure localic map f is of effective descent by obtaining a retract of every $\eta_{\mathbf{M}}$. This can be done essentially because the category of $\mathbf{A}$-modules is self-dual. This line of argument will not work in the context of cocomplete categories. However, the right adjoint of $\eta_{\mathbf{M}}^{o}$ is of course $\eta_{\mathbf{M}}$, and therefore, f is pure if and only if every $\eta_{\mathbf{M}}^{o}$ has a faithful right adjoint. With this in mind, let us proceed to define the notion of a pure geometric morphism.

To begin, the cocontinuous dual of a cocomplete category shall take the place of the opposite category of a sup-lattice.

4.1 Definition For any cocomplete category $\mathcal{A}$, let $\mathcal{A}^{\star}$ denote the category $\text{Cocts}_{\mathcal{S}}(\mathcal{A}, \mathcal{S})$. The fiber above $I \in \mathcal{S}$ is

$$\mathcal{A}^{\star\, I} = \text{Cocts}_{\mathcal{S}_{/I}}(\mathcal{A}_{/I}, \mathcal{S}_{/I}) \cong \text{Cocts}_{\mathcal{S}}(\mathcal{A}, \mathcal{S}_{/I}) .$$

For a morphism $H \xrightarrow{\alpha} I$ in $\mathcal{S}$, the substitution functor α^{*} is given by composition with the pullback functor $\mathcal{S}_{/I} \longrightarrow \mathcal{S}_{/H}$. $\mathcal{A}^{\star}$ shall be referred to as the *cocontinuous dual* of $\mathcal{A}$.

The category $\mathcal{A}^{\star}$ is cocomplete over $\mathcal{S}$, though it may not be locally small (even if $\mathcal{A}$ is).

For any cocontinuous $\mathcal{A} \xrightarrow{\mathsf{F}} \mathcal{B}$, composition with F gives us a cocontinuous functor

$$\mathsf{F}^{\star} : \mathcal{B}^{\star} \longrightarrow \mathcal{A}^{\star} .$$

Let

$$\Delta_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}^{\star\star}$$

denote the functor which sends an $A \in \mathcal{A}$ to the cocontinuous functor

$$\Delta_{\mathcal{A}}(A) : \mathcal{A}^{\star} \longrightarrow \mathcal{S}$$

$$\varphi \rightsquigarrow \varphi(A) .$$

Then $\Delta_{\mathcal{A}}$ is itself cocontinuous, and Δ is natural in $\mathcal{A}$.

Let $\mathcal{E} \xrightarrow{\mathsf{p}} \mathcal{S}$ denote an arbitrary geometric morphism between elementary toposes.

Recall that

$$\eta_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{E} \otimes_{\mathcal{S}} \mathcal{A}$$

is used to denote the universal morphism associated with the tensor product of $\mathcal{A}$ with $\mathcal{E}$ (wherever defined). Then there is a 2-natural transformation

$$\eta^{\star} : (\mathcal{E} \otimes_{\mathcal{S}} (\ ))^{\star} \longrightarrow (\ )^{\star}$$

whose component at $\mathcal{A}$ is $(\eta_{\mathcal{A}})^{\star}$. A 2-natural transformation can have an adjoint, and it is in the following sense that this is meant.

4.2 Definition Given 2-functors $\mathcal{K} \xrightarrow{F} \mathcal{L}$ and $\mathcal{L} \xrightarrow{G} \mathcal{K}$, and a 2-natural transformation $F \xrightarrow{t} G$, let us say that t *has a right adjoint* if for every 0-cell $K \in \mathcal{K}$ there is given an adjointness $(t_K, s_K, \varepsilon_K, \eta_K)$ with $t_K \dashv s_K$. Furthermore, it is required that for every 1-cell $J \xrightarrow{f} K$ in $\mathcal{K}$, the 2-cell corresponding to

$$t_K \cdot F(f) \cdot s_J \xrightarrow{t_f s_J} G(f) \cdot t_J \cdot s_J \xrightarrow{G(f)\varepsilon^J} G(f)$$

under $t_K \dashv s_K$ be an isomorphism.

If $\eta^{\star}$ has a right adjoint, then let us denote the right adjoint of the component $(\eta_{\mathcal{A}})^{\star}$ by $\mathsf{R}_{\mathcal{A}}$.

4.3 Definition Let us say that $\eta^{\star}$ has a *cocontinuous (resp. faithful)* right adjoint if $\eta^{\star}$ has a right adjoint such that every component $\mathsf{R}_{\mathcal{A}}$ is cocontinuous (resp. faithful).

4.4 Definition A geometric morphism $\mathcal{E} \xrightarrow{\mathsf{p}} \mathcal{S}$ shall be called *pure* if $\eta^{\star}$ has a cocontinuous faithful right adjoint.

One could introduce the notion of an $\mathcal{S}$-*pure* geometric morphism $\mathcal{E} \longrightarrow \mathcal{F}$ between toposes over $\mathcal{S}$. It would not be hard to see that $\mathcal{S}$-pure geometric morphisms would then compose and would be stable under pullback. As these developments are not needed here, they are omitted.

The central result of this thesis is the following.

4.5 Theorem *An arbitrary (bounded) pure geometric morphism is of effective descent for cocomplete categories. In other words, if $\mathcal{E} \xrightarrow{\mathsf{p}} \mathcal{S}$ is such a geometric morphism, then*

$$\mathcal{E} \otimes_{\mathcal{S}} : \mathrm{C{\scriptsize OCTS}}_{\mathsf{p}^{\circ}} \longrightarrow \mathrm{C{\scriptsize OCTS}}_{\mathsf{p}}$$

is cotripleable.

To remind the reader, 'cotripleable' means the 2-categorical sense as developed in section §2 of this chapter. The 2-categories $\mathrm{COCTS}_{\mathsf{p}^0}$ and $\mathrm{COCTS}_{\mathsf{p}}$ were introduced in §3.

4.6 Corollary *Pure geometric morphisms are of effective descent for Grothendieck toposes.*

PROOF This is a consequence of the above theorem, and of 3.4. □

A geometric morphism p is said to be a *surjection* if p^* is faithful. A geometric morphism p is said to be *locally connected* if p^* has a left adjoint over $\mathcal{S}$. If such is the case, then the left adjoint shall be denoted by $\mathsf{p}_!$.

4.7 Corollary *Any (bounded) locally connected geometric morphism which is a surjection is of effective descent for cocomplete categories.*

PROOF We will see that any locally connected surjection is pure. □

A geometric morphism p is said to be *open* if the unique localic map

$$\Omega \to (\Omega_{\mathcal{E}})_{\mathsf{p}}$$

in $\mathcal{S}$ has a left adjoint (see [J3]). The locale $(\Omega_{\mathcal{E}})_{\mathsf{p}}$ is the restriction along p of the sub-object classifier in $\mathcal{E}$. It is equal to the locale $\mathsf{p}_*(\Omega_{\mathcal{E}})$. A geometric morphism p is said to be *spatial*[1] if $\mathcal{E}$ is equivalent to $sh(X)$ for a space X in $\mathcal{S}$. If this is the case, then $(\Omega_{\mathcal{E}})_{\mathsf{p}}$ is equal to $\vartheta(X)$, the locale associated with the space X .

4.8 Corollary *Any spatial geometric morphism which is an open surjection is of effective descent for cocomplete categories.*

PROOF We will see that any spatial open surjection is pure. □

Let us begin the analysis of pure geometric morphisms with the following observation. As endofunctors of $\mathrm{COCTS}_{\mathcal{S}}$, $\mathcal{E}\otimes_{\mathcal{S}}$ is left adjoint to $\mathrm{COCTS}_{\mathcal{S}}(\mathcal{E},\_)$. Therefore, $\mathrm{COCTS}_{\mathcal{S}}(\mathcal{A},\mathcal{E}^*) \cong (\mathcal{E}\otimes_{\mathcal{S}}\mathcal{A})^*$, and there are natural isomorphisms between functors over $\mathcal{S}$ as in the following diagram. The center arrow in this diagram is the functor $\mathrm{COCTS}_{\mathcal{S}}(\mathcal{A},\Xi_{\mathcal{S}})$.

[1] This terminology follows [JT]. The term *localic*, which corresponds to the 'locale/frame' terminology, is also used for such a geometric morphism.

$$\begin{array}{ccccc} \textsc{Cocts}_{\mathcal{S}}(\mathcal{E},\mathcal{A}^\star) & \cong & \textsc{Cocts}_{\mathcal{S}}(\mathcal{A},\mathcal{E}^\star) & \cong & (\mathcal{E}\otimes_{\mathcal{S}}\mathcal{A})^\star \\ & \searrow^{\Xi_{\mathcal{A}^\star}}_{\simeq} & \downarrow{\simeq} & \swarrow^{(\eta_{\mathcal{A}})^\star} & \\ & & \mathcal{A}^\star & & \end{array} \tag{4.9}$$

Recall from Chapter 3 that $\Xi_{\mathcal{A}^\star}$ is the counit of the adjointness $(\ )_{\mathsf{p}} \dashv \textsc{Cocts}_{\mathcal{S}}(\mathcal{E},\_)$ at $\mathcal{A}^\star$. The equivalence on the left in 4.9 results because both categories are equivalent to the category of cocontinuous bimorphisms

$$\textsc{Bim}_{\mathcal{S}}(\mathcal{A},\mathcal{E};\mathcal{S})\ .$$

Thus, $(\eta_{\mathcal{A}})^\star$ is isomorphic to $\Xi_{\mathcal{A}^\star}$. In particular, since $\eta_{\mathcal{S}}$ is equal to p^*, we can identify

$$(\mathsf{p}^*)^\star : \mathcal{E}^\star \longrightarrow \mathcal{S}^\star$$

with $\Xi_{\mathcal{S}^\star}$. Furthermore, since $\mathcal{S}^\star \cong \mathcal{S}$, $(\mathsf{p}^*)^\star$ can therefore be identified with $\Xi_{\mathcal{S}}$. That is, if $(\mathsf{p}^*)^\star$ is regarded as a functor to $\mathcal{S}$, then

$$(\mathsf{p}^*)^\star(\mathsf{F}) \simeq \Xi_{\mathcal{S}}(\mathsf{F}) = \mathsf{F}(1_{\mathcal{E}})\ ,$$

where $\mathsf{F} \in \mathcal{E}^\star$.

Pure geometric morphisms are surjections. The following lemma can be used to see this.

Lemma Let $\mathcal{B} \xrightarrow{\mathsf{G}} \mathcal{A}$ be a cocontinuous functor with a cocontinuous right adjoint F. Assume that epimorphisms in $\mathcal{A}$ are coequalizers. Then if F is faithful, so is $\mathsf{G}^\star$.

PROOF $\mathsf{G}^\star$ is right adjoint to $\mathsf{F}^\star$. Let ε denote the counit of this adjointness. It suffices to show that $\mathsf{F}^\star(\mathsf{G}^\star(\varphi)) \xrightarrow{\varepsilon_\varphi} \varphi$ is an epimorphism for every $\varphi \in \mathcal{A}^\star$. For this to be true, it suffices that for every $A \in \mathcal{A}$

$$(\varepsilon_\varphi)_A : \mathsf{F}^\star(\mathsf{G}^\star(\varphi))(A) \to \varphi(A)$$

be an epimorphism in $\mathcal{S}$. By definition, $\mathsf{F}^\star(\mathsf{G}^\star(\varphi))(A) = \varphi(\mathsf{G}(\mathsf{F}(A)))$ and $(\varepsilon_\varphi)_A = \varphi(\omega_A)$, where ω denotes the counit of $\mathsf{G} \dashv \mathsf{F}$. By hypothesis, ω_A is a coequalizer, and therefore so is $(\varepsilon_\varphi)_A$ since φ is cocontinuous. Thus, $(\varepsilon_\varphi)_A$ is an epimorphism. □

Now let p be a pure geometric morphism. Then $(\mathsf{p}^*)^\star$ has a faithful cocontinuous right adjoint. By taking G to be $(\mathsf{p}^*)^\star$ in the lemma, one concludes that $(\mathsf{p}^*)^{\star\star}$ is faithful. But $\Delta_{\mathcal{E}} \cdot \mathsf{p}^* \simeq (\mathsf{p}^*)^{\star\star} \cdot \Delta_{\mathcal{S}}$, and $\Delta_{\mathcal{S}}$ is an equivalence. Hence $\Delta_{\mathcal{E}} \cdot \mathsf{p}^*$ is faithful, and therefore p^* is also.

Diagram 4.9 can be used to obtain the the following characterization of the existence of a cocontinuous right adjoint to $\eta^\star$.

4.10 Proposition *The following are equivalent:*

1. *$\eta^\star$ has a cocontinuous right adjoint.*

2. *$\eta^\star$ has a right adjoint.*

3. *$(\mathsf{p}^*)^\star$ has a right adjoint.*

4. *$\mathcal{E}^\star$ has a terminal object.*

PROOF That *1.* implies *2.*, and that *2.* implies *3.* are trivial.

If $(\mathsf{p}^*)^\star$ has a right adjoint R, then $\mathsf{R}(1)$ is the terminal object in $\mathcal{E}^\star$, and so *3.* implies *4.*.

Assume now that $\mathcal{E}^\star$ has a terminal object, which shall be denoted by $\top$. View $\mathcal{E}^\star$ as a full sub-category of $\mathrm{FUNCT}_{\mathcal{S}}(\mathcal{E},\mathcal{S})$ as in the following diagram.

$$\begin{array}{ccc} \mathcal{E}^\star & & \\ \downarrow & \searrow^{(\mathsf{p}^*)^\star} & \\ \mathrm{FUNCT}_{\mathcal{S}}(\mathcal{E},\mathcal{S}) & \xrightarrow[\mu]{} & \mathcal{S} \end{array}$$

The functor μ sends an $\mathsf{F} \in \mathrm{COCTS}_{\mathcal{S}}(\mathcal{E},\mathcal{S})$ to $\mathsf{F}(1_{\mathcal{E}})$, and it has a right adjoint κ such that

$$\kappa(X) : E \rightsquigarrow X$$

for all $X \in \mathcal{S}$ and $E \in \mathcal{E}$. Also, observe that for any $X \in \mathcal{S}$ and $\mathsf{F} \in \mathcal{E}^\star$, the copower $X.\mathsf{F}$ in $\mathcal{E}^\star$ is equal to $\kappa(X) \times \mathsf{F}$ as calculated in $\mathrm{FUNCT}_{\mathcal{S}}(\mathcal{E},\mathcal{S})$. Define a functor

$$\mathsf{R} : \mathcal{S} \longrightarrow \mathcal{E}^\star$$

by letting $\mathsf{R}(X) = X.\top$ for every $X \in \mathcal{S}$. R is a cocontinuous functor over $\mathcal{S}$. Moreover, the following series of bijections, natural in $\mathsf{F} \in \mathcal{E}^\star$ and $X \in \mathcal{S}$, show that R is right adjoint to $(\mathsf{p}^*)^\star$.

$$\frac{\frac{(\mathsf{p}^*)^\star(\mathsf{F}) = \mu(\mathsf{F}) \to X}{\mathsf{F} \to \kappa(X)}}{\mathsf{F} \to \kappa(X) \times \top = X.\top}$$

Then $\textsc{Cocts}_{\mathcal{S}}(\mathcal{A}, \mathsf{R})$ is right adjoint to $\textsc{Cocts}_{\mathcal{S}}(\mathcal{A}, (\mathsf{p}^*)^\star)$. Moreover, $\textsc{Cocts}_{\mathcal{S}}(\mathcal{A}, \mathsf{R})$ is cocontinuous. By 4.9, it follows that $\eta^\star$ has a cocontinuous right adjoint. □

As the proof of 4.10 illustrates, if $(\mathsf{p}^*)^\star$ has a (cocontinuous) right adjoint, then the terminal object and the right adjoint correspond under the equivalence $\mathcal{E}^\star \cong \textsc{Cocts}_{\mathcal{S}}(\mathcal{S}, \mathcal{E}^\star)$, provided that one identifies $\mathcal{S}$ with $\mathcal{S}^\star$.

4.11 Proposition *The following are equivalent for a geometric morphism* $\mathcal{E} \xrightarrow{\mathsf{p}} \mathcal{S}$.

1. p *is pure.*

2. $\eta^\star$ *has a faithful right adjoint.*

3. $(\mathsf{p}^*)^\star$ *has a faithful right adjoint.*

4. $\mathcal{E}^\star$ *has a terminal object* $\top$ *, and the unique map* $\top(1_{\mathcal{E}}) \to 1$ *is an epimorphism.*

PROOF That *1.* implies *2.*, and that *2.* implies *3.* are trivial.

Assuming *3.*, let R denote the faithful right adjoint of $(\mathsf{p}^*)^\star$. R is cocontinuous by 4.10. Let ε denote the counit of $(\mathsf{p}^*)^\star \dashv \mathsf{R}$. As usual, 1 denotes the terminal object in $\mathcal{S}$, and $1_{\mathcal{E}}$ denotes the terminal object in $\mathcal{E}$. Identifying $\mathcal{S}$ with $\mathcal{S}^\star$, let $\top$ denote $\mathsf{R}(1)$. $\mathsf{R}(1)$ is the terminal object in $\mathcal{E}^\star$. We have

$$(\mathsf{p}^*)^\star(\mathsf{R}(1)) = (\mathsf{p}^*)^\star(\top) \simeq \top(1_{\mathcal{E}}) .$$

The counit $(\mathsf{p}^*)^\star(\mathsf{R}(1)) \xrightarrow{\varepsilon_1} 1$ is an epimorphism since R is assumed to be faithful. Thus, *3.* implies *4.*.

To prove *1.* from *4.*, one proceeds as in 4.10. That is, first one shows that $(\mathsf{p}^*)^\star$ has a cocontinuous right adjoint R . Then $\mathsf{R}(1)$ is isomorphic to $\top$, and therefore, $(\mathsf{p}^*)^\star(\mathsf{R}(1)) \simeq \top(1_{\mathcal{E}})$. Thus, the counit

$$\varepsilon_1 : (\mathsf{p}^*)^\star(\mathsf{R}(1)) \to 1$$

is an epimorphism. Then for any $I \in \mathcal{S}$, ε_I is an epimorphism since

$$(\mathsf{p}^*)(\mathsf{R}(I)) \simeq I \times (\mathsf{p}^*)^\star(\mathsf{R}(1)) .$$

This isomorphism identifies ε_I with $I \times \varepsilon_1$. Therefore, R is faithful. It follows, by using the fact that epimorphisms in $\mathcal{S}$ are coequalizers, that the cocontinuous right adjoint of $(\eta_{\mathcal{A}})^*$, which is identified with $\mathrm{Cocts}_{\mathcal{S}}(\mathcal{A}, \mathsf{R})$ (see diagram 4.9), is also faithful. □

4.12 Proposition *An arbitrary locally connected surjection is pure.*

PROOF Let p be a locally connected geometric morphism with $\mathsf{p}_! \dashv \mathsf{p}^*$ over $\mathcal{S}$. Then

$$(\mathsf{p}^*)^* \dashv (\mathsf{p}_!)^*,$$

and consequently the terminal object in $\mathcal{E}^*$ is $(\mathsf{p}_!)^*(1)$. This is equal to $\mathsf{p}_!$. Also, the unique morphism

$$\mathsf{p}_!(1_{\mathcal{E}}) \simeq \mathsf{p}_!(\mathsf{p}^*(1)) \to 1$$

is an epimorphism since p^* is assumed to be faithful. Hence, by 4.11 p is pure. □

4.13 Proposition *An arbitrary spatial open surjection is pure.*

PROOF Let $\mathcal{E} \xrightarrow{\mathsf{p}} \mathcal{S}$ denote an arbitrary open surjection, with $\mathcal{E} = sh(X)$. Our aim is to show first that $\mathcal{E}^*$ has a terminal object. Recall (Chap. 3, §1) that composition with sheafification identifies $\mathcal{E}^*$ with the full sub-category of $\mathrm{Cocts}_{\mathcal{S}}(\mathcal{S}^{\vartheta(X)^{op}}, \mathcal{S})$ whose objects are those cocontinuous functors

$$\mathcal{S}^{\vartheta(X)^{op}} \longrightarrow \mathcal{S}$$

which take dense (for the canonical topology) monomorphisms to isomorphisms. Also recall that there is an equivalence

$$\mathrm{Cocts}_{\mathcal{S}}(\mathcal{S}^{\vartheta(X)^{op}}, \mathcal{S}) \cong \mathcal{S}^{\vartheta(X)},$$

which is given by composition with the Yoneda embedding. This equivalence restricts to an equivalence

$$\mathcal{E}^* \cong X\text{-}Cocts(\mathcal{S}^{\vartheta(X)}). \tag{4.14}$$

The category on the right in 4.14 is by definition the full sub-category of $\mathcal{S}^{\vartheta(X)}$ whose objects are those functors

$$\vartheta(X) \xrightarrow{D} \mathcal{S}$$

such that for every $U \in \vartheta(X)$ and every covering sieve

$$\begin{array}{ccc} R & \longrightarrow & \vartheta(X)_{/U} \\ & \searrow & \downarrow \\ & & \vartheta(X)\,, \end{array}$$

the colimit of the diagram

$$R \longrightarrow \vartheta(X) \xrightarrow{D} \mathcal{S}$$

is $D(U)$.

Next, the claim is that for any sup-lattice map (not necessarily localic)

$$\vartheta(X) \xrightarrow{f} \vartheta(Y)$$

between locales, the induced functor

$$[f, \mathcal{S}] : \mathcal{S}^{\vartheta(Y)} \longrightarrow \mathcal{S}^{\vartheta(X)}$$

restricts to

$$[f, \mathcal{S}] : Y\text{–}Cocts(\mathcal{S}^{\vartheta(Y)}) \longrightarrow X\text{–}Cocts(\mathcal{S}^{\vartheta(X)})\,.$$

To see this, let R be an arbitrary covering sieve of $U \in \vartheta(X)$. Let $D \in Y\text{–}Cocts(\mathcal{S}^{\vartheta(Y)})$. To be shown is that $D(f(U))$ is the colimit of the diagram

$$R \longrightarrow \vartheta(X) \xrightarrow{f} \vartheta(Y) \xrightarrow{D} \mathcal{S}\,.$$

Let $[fR]$ denote the sieve generated by the image of R under f. Since f preserves suprema, $[fR]$ is a covering sieve of $f(U)$. We have the following diagram.

$$\begin{array}{ccc} R & \xrightarrow{\;f\;} & [fR] \\ \downarrow & & \downarrow \\ \vartheta(X)_{/U} & & \vartheta(Y)_{/fU} \\ \downarrow & & \downarrow \\ \vartheta(X) & \xrightarrow[f]{} & \vartheta(Y) \end{array}$$

Thus, $D(f(U))$ is the colimit of the diagram

$$[fR] \longrightarrow \vartheta(Y) \xrightarrow{D} \mathcal{S}\,.$$

The claim now follows since the poset map $R \xrightarrow{f} [fR]$ is cofinal (see [J1], p. 74).

Since p is assumed to be open, there is a sup-lattice map

$$\vartheta(X) \xrightarrow{\exists} \Omega$$

which is left adjoint to the unique map

$$\Omega \xrightarrow{!} \vartheta(X) .$$

Therefore,

$$[\exists, \mathcal{S}] : \mathit{1}\text{--}\mathit{Cocts}(\mathcal{S}^{\vartheta(\mathit{1})}) \longrightarrow X\text{--}\mathit{Cocts}(\mathcal{S}^{\vartheta(X)})$$

is right adjoint to $[!, \mathcal{S}]$, where $\mathit{1}$ is the terminal in $\mathcal{S}$ and $\Omega = \vartheta(\mathit{1})$. Of course,

$$\mathcal{S} \cong \mathrm{C{\scriptstyle OCTS}}_{\mathcal{S}}(\mathcal{S}, \mathcal{S}) \cong \mathit{1}\text{--}\mathit{Cocts}(\mathcal{S}^{\vartheta(\mathit{1})}) ,$$

and therefore, $\mathit{1}\text{--}\mathit{Cocts}(\mathcal{S}^{\vartheta(\mathit{1})})$ has a terminal object. In fact, it is the functor

$$sub : \Omega \longrightarrow \mathcal{S}$$

which takes an 'element' $I \xrightarrow{a} \Omega$ of Ω at 'stage' I to the sub-object of I classified by a. Thus, $X\text{--}\mathit{Cocts}(\mathcal{S}^{\vartheta(X)})$ has a terminal object; it is the functor

$$\vartheta(X) \xrightarrow{\exists} \Omega \xrightarrow{sub} \mathcal{S} .$$

We have shown that $\mathcal{E}^{\star}$ has a terminal object, which we shall denote by $\top$. Our task now is to show that the unique map

$$\top(\mathit{1}_{\mathcal{E}}) \rightarrow \mathit{1}$$

is an epimorphism (assuming that p is a surjection). Let t_X denote the top element of $\vartheta(X)$. It follows that the support of $\top(\mathit{1}_{\mathcal{E}})$ is isomorphic to the sub-object

$$sub \cdot \exists(t_X) \hookrightarrow \mathit{1}$$

in $\mathcal{S}$. As is well known, p is a surjection if and only if $\exists(t_X) \simeq t$, where $\mathit{1} \xrightarrow{t} \Omega$ is the top element of Ω. So if p is a surjection, then

$$sub \cdot \exists(t_X) \simeq sub(t) \simeq \mathit{1} .$$

Thus, $\top(\mathit{1}_{\mathcal{E}}) \rightarrow \mathit{1}$ is an epimorphism. The converse is true as well. That is, if $\top(\mathit{1}_{\mathcal{E}}) \rightarrow \mathit{1}$ is an epimorphism, then the open geometric morpism p is a surjection. This concludes the proof. □

The next proposition is a characterization of pure morphisms which will be used in the proof of 4.5. Before getting to that proposition, let us set down some notation. For any $\mathsf{F} \in \mathrm{C{\scriptsize OCTS}}_{\mathcal{S}}(\mathcal{E}, \mathcal{S})$ and $\mathcal{A} \in \mathrm{C{\scriptsize OCTS}}_{\mathcal{S}}$ such that $\mathcal{E} \otimes_{\mathcal{S}} \mathcal{A}$ exists, let

$$\mathsf{F} \otimes \mathcal{A} \ : (\mathcal{E} \otimes_{\mathcal{S}} \mathcal{A})_{\mathsf{p}} \longrightarrow \mathcal{A}$$

denote the cocontinuous functor $\Xi_{\mathcal{A}} \cdot (\widehat{\mathcal{A}^{\mathsf{F}}})_{\mathsf{p}}$, where

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\ \eta_{\mathcal{A}}\ } & (\mathcal{E} \otimes_{\mathcal{S}} \mathcal{A})_{\mathsf{p}} \\ & \searrow^{\mathcal{A}^{\mathsf{F}}} \ \simeq & \Big\downarrow (\widehat{\mathcal{A}^{\mathsf{F}}})_{\mathsf{p}} \\ & & \mathrm{C{\scriptsize OCTS}}_{\mathcal{S}}(\mathcal{E}, \mathcal{A})_{\mathsf{p}} \end{array} \qquad \begin{array}{c} \mathcal{E} \otimes_{\mathcal{S}} \mathcal{A} \\ \Big\downarrow \widehat{\mathcal{A}^{\mathsf{F}}} \\ \mathrm{C{\scriptsize OCTS}}_{\mathcal{S}}(\mathcal{E}, \mathcal{A})\,. \end{array}$$

The cocontinuous functor $\mathcal{A}^{\mathsf{F}}$ is given by composition with F subject to the identification

$$\mathcal{A} \cong \mathrm{C{\scriptsize OCTS}}_{\mathcal{S}}(\mathcal{S}, \mathcal{A})\,.$$

In making this identification, one has

$$\mathcal{A}^{\mathsf{F}}(A)(E) \simeq (\mathsf{F}(E)).A\,,$$

for any $E \in \mathcal{E}$ and $A \in \mathcal{A}$. Observe that $\mathrm{C{\scriptsize OCTS}}_{\mathcal{S}}(\mathcal{E}, \mathcal{A})$ must be in $\mathrm{C{\scriptsize OCTS}}_{\mathcal{E}}$, in particular locally small, for the functor $\mathsf{F} \otimes \mathcal{A}$ to exist. As demonstrated in the previous chapter, this is the case if p is bounded. Then for any $A \in \mathcal{A}$,

$$\mathsf{F} \otimes \mathcal{A} \cdot \eta_{\mathcal{A}}(A) \simeq \Xi_{\mathcal{A}} \cdot \mathcal{A}^{\mathsf{F}}(A) \simeq (\mathsf{F}(1_{\mathcal{E}})).A$$

by an isomorphism which is natural in A . $1_{\mathcal{E}}$ is the terminal object in $\mathcal{E}$.

4.15 Proposition *A bounded geometric morphism* p *is pure if and only if* $\mathcal{E}^{*}$ *has a terminal object* $\top$ *, and for every* $\eta_{\mathcal{A}}$ *there is a natural transformation* $\top \otimes \mathcal{A} \cdot \eta_{\mathcal{A}} \xrightarrow{\ i\ } 1_{\mathcal{A}}$ *such that (writing* η *for* $\eta_{\mathcal{A}}$*)*

$$(4.16) \quad (\top \otimes \mathcal{A} \cdot \eta)^2(A) \underset{\top \otimes \mathcal{A} \cdot \eta(i_A)}{\overset{i(\top \otimes \mathcal{A} \cdot \eta(A))}{\rightrightarrows}} \top \otimes \mathcal{A} \cdot \eta(A) \xrightarrow{\ i_A\ } A$$

is a (stable) coequalizer for all $A \in \mathcal{A}$.

PROOF Assume that p is pure. Let R denote the faithful cocontinuous right adjoint of $(\mathsf{p}^*)^\star$. Let ε denote the counit of $(\mathsf{p}^*)^\star \dashv \mathsf{R}$. As usual, 1 denotes the terminal object in $\mathcal{S}$, and $1_{\mathcal{E}}$ denotes the terminal object in $\mathcal{E}$. As in 4.15, let $\top$ denote $\mathsf{R}(1)$, which is the terminal object in $\mathcal{E}^\star$. As before, for this to make sense one must identify $\mathcal{S}$ with $\mathcal{S}^\star$. One has

$$(\mathsf{p}^*)^\star(\mathsf{R}(1)) = (\mathsf{p}^*)^\star(\top) = \top(1_{\mathcal{E}}) .$$

The counit $(\mathsf{p}^*)^\star(\mathsf{R}(1)) \xrightarrow{\varepsilon_1} 1$ is an epimorphism since R is assumed to be faithful. Therefore,

$$(4.17) \quad \mathsf{R}(\top(1_{\mathcal{E}}))(1_{\mathcal{E}}) \underset{\mathsf{R}(\varepsilon_1)(1_{\mathcal{E}})}{\overset{\varepsilon_{\top(1_{\mathcal{E}})}}{\rightrightarrows}} \top(1_{\mathcal{E}}) \xrightarrow{\varepsilon_1} 1$$

is a coequalizer in $\mathcal{S}$. In fact,

$$\mathsf{R}(\top(1_{\mathcal{E}}))(1_{\mathcal{E}}) \simeq \top(1_{\mathcal{E}}).(\mathsf{R}(1)(1_{\mathcal{E}})) = \top(1_{\mathcal{E}}).\top(1_{\mathcal{E}}) = \top(1_{\mathcal{E}}) \times \top(1_{\mathcal{E}}) ,$$

by an isomorphism which identifies $\varepsilon_{\top(1_{\mathcal{E}})}$ and $\mathsf{R}(\varepsilon_1)(1_{\mathcal{E}})$ with the two projections

$$\top(1_{\mathcal{E}}) \times \top(1_{\mathcal{E}}) \underset{\pi_0}{\overset{\pi_1}{\rightrightarrows}} \top(1_{\mathcal{E}}) \to 1 .$$

Hence, 4.17 is a coequalizer since ε_1 is an epimorphism.

For any $A \in \mathcal{A}$, one can form the copower in $\mathcal{A}$ of 4.17 with A. The following coequalizer in $\mathcal{A}$ results.

$$(4.18) \qquad \mathsf{R}(\top(1_{\mathcal{E}}))(1_{\mathcal{E}}).A \underset{\mathsf{R}(\varepsilon_1)(1_{\mathcal{E}}).A}{\overset{\varepsilon_{\top(1_{\mathcal{E}})}.A}{\rightrightarrows}} \top(1_{\mathcal{E}}).A \xrightarrow{\varepsilon_1 .A} A$$

Recall that for any $A \in \mathcal{A}$,

$$\top \otimes \mathcal{A} \cdot \eta_{\mathcal{A}}(A) \simeq \Xi_{\mathcal{A}} \cdot \mathcal{A}^{\top}(A) \simeq \top(1_{\mathcal{E}}).A ,$$

and hence

$$\top \otimes \mathcal{A} \cdot \eta_{\mathcal{A}} \cdot \top \otimes \mathcal{A} \cdot \eta_{\mathcal{A}}(A) \simeq \top(1_{\mathcal{E}}).(\top \otimes \mathcal{A} \cdot \eta_{\mathcal{A}}(A))$$
$$\simeq \top(1_{\mathcal{E}}).(\top(1_{\mathcal{E}}).A) \simeq \mathsf{R}(\top(1_{\mathcal{E}}))(1_{\mathcal{E}}).A .$$

Define i_A to be the composite

$$\mathrm{T}\otimes\mathcal{A}\cdot\eta_{\mathcal{A}}(A) \simeq \mathrm{T}(1_{\mathcal{E}}).A \xrightarrow{\varepsilon_1 . A} A\,.$$

Then i is natural in A, and diagram 4.16 is a coequalizer since diagram 4.18 is.

Assuming the given condition, to see that p is pure take $\mathcal{A} = \mathcal{S}$ and $A = 1$. Then since

$$\mathrm{T}\otimes\mathcal{S}\cdot\eta_{\mathcal{S}}(1) \simeq \mathrm{T}(1_{\mathcal{E}})\,,$$

it follows that the unique map $\mathrm{T}(1_{\mathcal{E}}) \rightarrow 1$ is an epimorphism. Now use 4.11. □

If p is pure, then a simple diagram chase using 4.16 shows that every $\eta_{\mathcal{A}}$ is faithful. It also follows that every $\eta_{\mathcal{A}}$ reflects isomorphisms. This gives another proof, since $\eta_{\mathcal{S}} = \mathsf{p}^*$, that (bounded) pure geometric morphisms are surjections. It also proves that surjections are not in general pure. In fact, surjections are not in general pullback stable, and if

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\ \mathsf{q}\ } & \mathcal{F} \\ \downarrow & & \downarrow \\ \mathcal{E} & \xrightarrow[\ \mathsf{p}\ ]{} & \mathcal{S} \end{array}$$

is a bounded pullback of toposes with p pure, then $\mathsf{q}^* = \eta_{\mathcal{F}}$ is faithful.

We can now proceed with the proof of 4.5.

PROOF (of 4.5) Let $\mathcal{E} \xrightarrow{\mathsf{p}} \mathcal{S}$ be an arbitrary bounded pure geometric morphism. Then $\mathcal{E}^n$, $n = 1, 2, \ldots$, exists and is the tensor product $\overbrace{\mathcal{E}\otimes_{\mathcal{S}}\mathcal{E}\otimes_{\mathcal{S}}\ldots\mathcal{E}}^{n}$. Also, recall the categories $\mathrm{COCTS}_{\mathsf{p}^0}$ and $\mathrm{COCTS}_{\mathsf{p}}$. These categories were introduced in section §3, and our attention will be focused on them.

For this proof only, denote

$$\mathcal{E}\otimes_{\mathcal{S}} : \mathrm{COCTS}_{\mathsf{p}^0} \longrightarrow \mathrm{COCTS}_{\mathsf{p}}$$

and

$$(\ )_{\mathsf{p}} : \mathrm{COCTS}_{\mathsf{p}} \longrightarrow \mathrm{COCTS}_{\mathsf{p}^0}$$

by U and R respectively. We have $\mathrm{U} \dashv \mathrm{R}$, and the unit of this adjointness is η. Let $\mathbf{G} = (\mathrm{G}, \varepsilon, \delta, p, q, w)$ denote the cotriple on $\mathrm{COCTS}_{\mathsf{p}}$ induced by the adjointness $\mathrm{U} \dashv \mathrm{R}$. The theorem from §1 on semi-split equalizers (Th. 1.6) will be put to use here. Of course, our work on cotripleability in §2 will also be used.

Let $\mathcal{A}$ be in $\text{COCTS}_{\mathcal{P}^{\circ}}$. Since coequalizers are computed pointwise in $\text{COCTS}_{\mathcal{S}}(\mathcal{A},\mathcal{A})$, proposition 4.15 says that there is a natural transformation $\top\otimes\mathcal{A}\cdot\eta_{\mathcal{A}} \xrightarrow{i} \mathbf{1}_{\mathcal{A}}$ such that

$$(\top\otimes\mathcal{A}\cdot\eta_{\mathcal{A}})^2 \underset{(\top\otimes\mathcal{A}\cdot\eta_{\mathcal{A}})i}{\overset{i(\top\otimes\mathcal{A}\cdot\eta_{\mathcal{A}})}{\rightrightarrows}} \top\otimes\mathcal{A}\cdot\eta_{\mathcal{A}} \xrightarrow{\ i\ } \mathbf{1}_{\mathcal{A}}$$

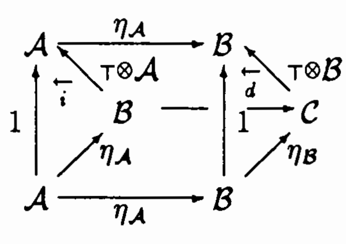
is a coequalizer in $\text{COCTS}_{\mathcal{S}}(\mathcal{A},\mathcal{A})$. Similiarly, writing $\mathcal{B}$ for $\text{RU}\mathcal{A}$, there is a natural transformation $\top\otimes\mathcal{B}\cdot\eta_{\mathcal{B}} \xrightarrow{d} \mathbf{1}_{\mathcal{B}}$ such that

$$(\top\otimes\mathcal{B}\cdot\eta_{\mathcal{B}})^2 \underset{(\top\otimes\mathcal{B}\cdot\eta_{\mathcal{B}})d}{\overset{d(\top\otimes\mathcal{B}\cdot\eta_{\mathcal{B}})}{\rightrightarrows}} \top\otimes\mathcal{B}\cdot\eta_{\mathcal{B}} \xrightarrow{\ d\ } \mathbf{1}_{\mathcal{B}}$$

is a coequalizer in $\text{COCTS}_{\mathcal{S}}(\mathcal{B},\mathcal{B})$. There is also a natural transformation $\top\otimes\mathcal{C}\cdot\eta_{\mathcal{C}} \xrightarrow{a} \mathbf{1}_{\mathcal{C}}$, where $\mathcal{C}$ denotes $(\text{RU})^2\mathcal{A}$. It follows that the data

$$\text{(4.19)} \qquad \mathcal{A} \xrightarrow{\eta_{\mathcal{A}}} \text{RU}\mathcal{A} \;\underset{\eta_{\text{RU}\mathcal{A}}}{\overset{\overset{\text{R}\varepsilon_{\text{U}\mathcal{A}}}{\longleftarrow}}{\underset{\text{RU}\eta_{\mathcal{A}}}{\rightrightarrows}}}\; (\text{RU})^2\mathcal{A} \;\underset{\eta_{(\text{RU})^2\mathcal{A}}}{\overset{(\text{RU})^2\eta_{\mathcal{A}}}{\underset{\text{RU}\eta_{\text{RU}\mathcal{A}}}{\rightrightarrows}}}\; (\text{RU})^3\mathcal{A}$$

along with $\top\otimes\mathcal{A}$, $\top\otimes\mathcal{B}$ and $\top\otimes\mathcal{C}$, comprises a semi-split equalizer in $\text{COCTS}_{\mathcal{P}^{\circ}}$. For example, the fourth coherence condition in the definition of a semi-split equalizer requires that



commute, where the back arrow $\mathcal{B}\longrightarrow\mathcal{C}$ is $\text{RU}\eta_{\mathcal{A}}$. This prism does commute because the construction of i is natural in $\mathcal{A}$. One readily verifies that the 2-category $\text{COCTS}_{\mathcal{S}}$ satisfies the conditions of 1.6 ensuring that 4.19 is an equalizer in $\text{COCTS}_{\mathcal{S}}$, and hence an equalizer in $\text{COCTS}_{\mathcal{P}^{\circ}}$. For example, the second condition of 1.6 is satisfied because coequalizers are computed pointwise in $\mathrm{C}\textsc{octs}_{\mathcal{S}}(\mathcal{X},\mathcal{A})$, for any $\mathcal{X}\in\mathrm{C}\textsc{octs}_{\mathcal{S}}$.

The proof of the other half of the cotripleability equation, that the comparison functor is 2-essentially surjective, now follows. Let $(\mathcal{B},\theta,k,i)$ be an arbitrary coalgebra for the cotriple $\mathbf{G}$ on $\mathrm{C}\textsc{octs}_{\mathsf{p}}$ induced by the adjointness $\mathrm{U}\dashv\mathrm{R}$. Define $\mathcal{A}$ as the equalizer

$$(4.20)\qquad \mathcal{A}\xrightarrow{\ \phi\ }\mathrm{R}\mathcal{B}\ \underset{\eta_{\mathrm{R}\mathcal{B}}}{\overset{\overset{\mathrm{R}\varepsilon_{\mathcal{B}}}{\longleftarrow}}{\underset{\longrightarrow}{\overset{\mathrm{R}\theta}{\longrightarrow}}}}\ \mathrm{RG}\mathcal{B}\ \underset{\eta_{\mathrm{RG}\mathcal{B}}}{\overset{\overset{\mathrm{RG}\theta}{\longrightarrow}}{\underset{\longrightarrow}{\overset{\mathrm{R}\delta_{\mathcal{B}}}{\longrightarrow}}}}\ \mathrm{RG}^2\mathcal{B}$$

in $\mathrm{C}\textsc{octs}_{\mathcal{S}}$. It will be shown that for any $\mathcal{X}\in\mathrm{C}\textsc{octs}_{\mathcal{E}}$, the (ordinary) functor

$$\Phi_{\mathcal{X}}:\mathrm{C}\textsc{octs}_{\mathcal{E}}(\mathcal{B},\mathcal{X})^{\prime}\longrightarrow\mathrm{C}\textsc{octs}_{\mathcal{S}}(\mathcal{A},\mathrm{R}\mathcal{X})^{\prime}$$

$$\mathsf{F}\quad\rightsquigarrow\quad\mathrm{R}(\mathsf{F})\cdot\phi$$

is an equivalence. This would show that the pair $(\mathcal{B},\phi)$ is the tensor product $\mathrm{U}\mathcal{A}$. (Recall that here $\mathcal{E}\otimes_{\mathcal{S}}\mathcal{A}$ is being denoted by $\mathrm{U}\mathcal{A}$.) Moreover, the canonical coalgebra structure of $\mathrm{U}\mathcal{A}$ would then be identified with that given for $\mathcal{B}$. Of course, one would also have that $\mathcal{A}\in\mathrm{C}\textsc{octs}_{\mathsf{p}^{\mathrm{o}}}$ because $\mathcal{B}$ is in $\mathrm{C}\textsc{octs}_{\mathsf{p}}$.

Fix an arbitrary $\mathcal{X}\in\mathrm{C}\textsc{octs}_{\mathcal{E}}$. The first step is to show that 4.20 is a semi-split equalizer. Consider the diagram

$$(4.21)\qquad\begin{array}{ccccccc}
\mathcal{A} & \xrightarrow{\ \phi\ } & \mathrm{R}\mathcal{B} & \overset{\mathrm{R}\theta}{\underset{\eta_{\mathrm{R}\mathcal{B}}}{\rightrightarrows}} & \mathrm{RG}\mathcal{B} & \overset{\mathrm{RG}\theta,\ \mathrm{R}\delta_{\mathcal{B}}}{\underset{\eta_{\mathrm{RG}\mathcal{B}}}{\Rrightarrow}} & \mathrm{RG}^2\mathcal{B}\\
{\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\eta_{\mathrm{R}\mathcal{B}}} & & \downarrow{\scriptstyle\eta_{\mathrm{RG}\mathcal{B}}} & & \downarrow{\scriptstyle\eta_{\mathrm{RG}^2\mathcal{B}}}\\
\mathrm{R}\mathcal{B} & \xrightarrow{\ \mathrm{R}\theta\ } & \mathrm{RG}\mathcal{B} & \overset{\mathrm{RG}\theta}{\underset{\mathrm{R}\delta_{\mathcal{B}}}{\rightrightarrows}} & \mathrm{RG}^2\mathcal{B} & \overset{\mathrm{RG}^2\theta,\ \mathrm{RG}\delta_{\mathcal{B}}}{\underset{\mathrm{R}\delta_{\mathrm{G}\mathcal{B}}}{\Rrightarrow}} & \mathrm{RG}^3\mathcal{B}\\
{\scriptstyle r}\downarrow & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle y}\\
\mathcal{A} & \xrightarrow{\ \phi\ } & \mathrm{R}\mathcal{B} & \overset{\mathrm{R}\theta}{\underset{\eta_{\mathrm{R}\mathcal{B}}}{\rightrightarrows}} & \mathrm{RG}\mathcal{B} & \overset{\mathrm{RG}\theta,\ \mathrm{R}\delta_{\mathcal{B}}}{\underset{\eta_{\mathrm{RG}\mathcal{B}}}{\Rrightarrow}} & \mathrm{RG}^2\mathcal{B}
\end{array}$$

where s , t and y denote respectively $\tau\otimes(\mathrm{R}\mathcal{B})$, $\tau\otimes(\mathrm{RG}\mathcal{B})$ and $\tau\otimes(\mathrm{RG}^2\mathcal{B})$. Then r is the induced morphism. Let us write η for $\eta_{\mathrm{R}\mathcal{B}}$. By 4.15, there is a natural transformation $s\eta\xrightarrow{d}\mathbf{1}_{\mathrm{R}\mathcal{B}}$ such that the diagram

$$(4.22) \qquad s\eta s\eta \overset{ds\eta}{\underset{s\eta d}{\rightrightarrows}} s\eta \xrightarrow{\quad d \quad} \mathbf{1}_{\mathrm{R}\mathcal{B}}$$

is a coequalizer in $\mathrm{Cocts}_{\mathcal{S}}(\mathrm{R}\mathcal{B}, \mathrm{R}\mathcal{B})$. The composite natural transformation

$$(4.23) \qquad \phi r \phi \simeq s\eta\phi \overset{d\phi}{\to} \phi ,$$

where the isomorphism arises as in 4.21, is a morphism of cones from $\mathcal{A}$ to $\mathrm{R}\mathcal{B}$. Since 4.20 is an equalizer, the morphism 4.23 must be of the form ϕi for some $r\phi \overset{i}{\to} \mathbf{1}_{\mathcal{A}}$. Moreover, it follows that

$$(4.24) \qquad r\phi r\phi \overset{ir\phi}{\underset{r\phi i}{\rightrightarrows}} r\phi \xrightarrow{\quad i \quad} \mathbf{1}_{\mathcal{A}}$$

is a coequalizer in $\mathrm{Cocts}_{\mathcal{S}}(\mathcal{A}, \mathcal{A})$. In fact, ϕ applied to 4.24 yields a diagram which is isomorphic to

$$(4.25) \qquad s\eta s\eta\phi \overset{ds\eta\phi}{\underset{s\eta d\phi}{\rightrightarrows}} s\eta\phi \xrightarrow{\quad d\phi \quad} \phi .$$

Diagram 4.25 is a coequalizer (in $\mathrm{Cocts}_{\mathcal{S}}(\mathcal{A}, \mathrm{R}\mathcal{B})$) because 4.22 is a coequalizer. Then, since ϕ reflects coequalizers, we conclude that 4.24 is a coequalizer. Hence, the data $(\mathcal{A}, \mathrm{R}\mathcal{B}, \ldots, \phi, \mathrm{R}\theta, \ldots)$ is a semi-split equalizer in $\mathrm{Cocts}_{\mathcal{S}}$.

Consider now, solely for the purposes of this argument, the (meta) 2-category whose 0-cells are all finitely cocomplete categories (not over $\mathcal{S}$). Let us denote this 2-category by $\mathcal{CO}$. The 1-cells of $\mathcal{CO}$ are taken to be all finite colimit preserving functors, and its 2-cells are all natural transformations.

For any 2-category $\mathcal{K}$, let $\mathcal{K}^{op}$ denote the 2-category obtained by reversing the 1-cells of $\mathcal{K}$. Then a semi-split *co*equalizer in $\mathcal{K}$ is, by definition, a semi-split equalizer in $\mathcal{K}^{op}$. Also, let us say that a 2-functor $\mathcal{K} \overset{F}{\longrightarrow} \mathcal{L}$ preserves colimits at the level of 1-cells and 2-cells if for all 0-cells $X, Y \in \mathcal{K}$, the functor

$$F_{XY} : \mathcal{K}[X,Y] \longrightarrow \mathcal{L}[FX, FY]$$

preserves colimits.

Lemma 1 Any semi-split coequalizer in $\mathcal{CO}$ is a coequalizer.

Lemma 2 For any $\mathcal{C} \in \text{Cocts}_{\mathcal{S}}$, the 2-functor

$$\text{Cocts}_{\mathcal{S}}(\_,\mathcal{C})^{1} : \text{Cocts}_{\mathcal{S}}{}^{op} \longrightarrow \mathcal{CO}$$

preserves coequalizers at the level of 1-cells and 2-cells.

The first lemma follows directly from 1.6. $\mathcal{CO}^{op}$ satisfies the conditions of 1.6 because finite colimits in the categories $\mathcal{CO}(\mathcal{V},\mathcal{U})$ are computed pointwise. The second lemma follows just as easily since in the ordinary category $\text{Cocts}_{\mathcal{S}}(\mathcal{D},\mathcal{C})^{1}$, finite colimits are also computed pointwise.

Let $\mathbf{H}$ denote the 2-functor $\text{Cocts}_{\mathcal{S}}(\_,\mathrm{R}\mathcal{X})^{1}$, as in Lemma 2. Let $\mathbf{K}$ denote the 2-functor

$$\text{Cocts}_{\mathcal{E}}(\_,\mathcal{X})^{1} : \text{Cocts}_{\mathsf{p}}{}^{op} \longrightarrow \mathcal{CO}\,.$$

By carrying the semi-split equalizer 4.20 to $\mathcal{CO}$ under $\mathbf{H}$, one gets the bottom half of the following commutative diagram.

$$(4.26)\qquad \begin{array}{ccccccc}
\mathbf{K}(\mathcal{B}) & \xleftarrow{\mathbf{K}(\theta)} & \mathbf{K}(\mathrm{G}\mathcal{B}) & \underset{\mathbf{K}(\delta_{\mathcal{B}})}{\overset{\mathbf{K}(\mathrm{G}\theta)}{\longleftleftarrows}} & \mathbf{K}(\mathrm{G}^2\mathcal{B}) & \longleftleftarrows & \mathbf{K}(\mathrm{G}^3\mathcal{B}) \\
\Phi_{\mathcal{X}}\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbf{H}(\mathcal{A}) & \xleftarrow{\mathbf{H}(\phi)} & \mathbf{H}(\mathrm{R}\mathcal{B}) & \underset{\mathbf{H}(\eta_{\mathrm{R}\mathcal{B}})}{\overset{\mathbf{H}(\mathrm{R}\theta)}{\longleftleftarrows}} & \mathbf{H}(\mathrm{RG}\mathcal{B}) & \longleftleftarrows & \mathbf{H}(\mathrm{RG}^2\mathcal{B})
\end{array}$$

By Lemma 2, the bottom half of 4.26 is a semi-split coequalizer since 4.20 is semi-split. By Lemma 1, the bottom half of 4.26 is a coequalizer in $\mathcal{CO}$. The top half of 4.26 is $\mathbf{K}$ applied to the split equalizer

$$\mathcal{B} \xrightarrow{\ \theta\ } \mathrm{G}\mathcal{B} \;\underset{\delta_{\mathcal{B}}}{\overset{\mathrm{G}\theta}{\rightrightarrows}}\; \mathrm{G}^2\mathcal{B} \;\underset{\delta_{\mathrm{G}\mathcal{B}}}{\overset{\mathrm{G}^2\theta,\ \mathrm{G}\delta_{\mathcal{B}}}{\rightrightarrows}}\; \mathrm{G}^3\mathcal{B} \qquad (\mathrm{G}\varepsilon_{\mathcal{B}} : \mathrm{G}^2\mathcal{B} \to \mathrm{G}\mathcal{B})$$

in $\text{Cocts}_{\mathsf{p}}$. Therefore, the top half is also a coequalizer in $\mathcal{CO}$ since split equalizers are preserved under any 2-functor, in this case $\mathbf{K}$. The three vertical arrows in 4.26 with no labels are equivalences which arise from the adjointness $\mathrm{U} \dashv \mathrm{R}$. Therefore, $\Phi_{\mathcal{X}}$ is an equivalence. The proof of the theorem is complete. □

4.5 The cocontinuous dual

As the previous section indicates, the *completeness* properties of the cocontinuous dual are of some interest. $\mathcal{E}^\star$ may not in general be complete; it may not, for example, have a terminal object. If $\mathcal{E} \xrightarrow{p} \mathcal{S}$ is spatial and open, or locally connected, then it was shown in §4 that $\mathcal{E}^\star$ does have a terminal object. Moreover, in both these cases, p is a surjection if and only if the unique map $\top(1_{\mathcal{E}}) \to 1$ is an epimorphism, where $\top$ denotes the terminal object in $\mathcal{E}^\star$.

Assume that p is bounded. In this case, $\mathcal{E}^\star$ is locally small (see Chap. 3, §1).

To begin, factor p as

$$\mathcal{E} \xrightarrow{i} \mathcal{S}^{\mathbf{C}} \xrightarrow{\mathsf{C}} \mathcal{S},$$

with $\mathcal{E} = sh_j(\mathcal{S}^{\mathbf{C}})$, where j is a topology on $\mathcal{S}^{\mathbf{C}}$. It will be shown that $(\mathcal{E}^\star)^{\mathbf{C}}$ is a locally connected (bounded) topos.

Let $\mathcal{F}$ denote the category $\textsc{Cocts}_{\mathcal{S}}(\mathcal{S}^{\mathbf{C}}, \mathcal{S}^{\mathbf{C}})$. $\mathcal{F}$ is equivalent to the topos $\mathcal{S}^{\mathbf{C} \times \mathbf{C}^{op}}$ over $\mathcal{S}^{\mathbf{C}}$. Then we have

$$(5.1) \qquad \mathcal{E}^\star = \textsc{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathcal{S}) \longrightarrow \textsc{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathcal{S}^{\mathbf{C}}) \cong \textsc{Cocts}_{\mathcal{S}^{\mathbf{C}}}(\mathcal{E}, \mathcal{F}).$$

The first arrow in 5.1 is $\textsc{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathsf{C}^*)$, where

$$\mathsf{C}^* : \mathcal{S} \longrightarrow \mathcal{S}^{\mathbf{C}}$$

is the constancy functor. The equivalence in 5.1 is by the adjointness $(\ )_{\mathsf{C}} \dashv \textsc{Cocts}_{\mathcal{S}}(\mathcal{S}^{\mathbf{C}}, \_)$. Recall (Eg. 1.18 1, Chap. 3) that the category $\textsc{Cocts}_{\mathcal{S}^{\mathbf{C}}}(\mathcal{E}, \mathcal{F})$ is a topos. In fact, it is equivalent to the topos of cosheaves $sh_j(\mathcal{F}^{op})^{op}$ (see Chap. 2, §4). Furthermore, there is an equivalence (see [Pi])

$$\textsc{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathcal{S}^{\mathbf{C}}) \cong \textsc{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathcal{S})^{\mathbf{C}} = (\mathcal{E}^\star)^{\mathbf{C}}$$

which identifies $\textsc{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathsf{C}^*)$ with the constancy functor

$$(5.2) \qquad \mathsf{C}^* : \mathcal{E}^\star \longrightarrow (\mathcal{E}^\star)^{\mathbf{C}}.$$

If $\mathbf{C}$ were chosen to be connected, and this can certainly be done, then 5.2 is fully faithful. (Actually, for 5.2 to be fully faithful, weakly connected would suffice since 5.2 is identified with $\textsc{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathsf{C}^*)$. See Chapter 2, §1 for a discussion of connectedness.) The following theorem is now established.

5.3 Theorem *$\mathcal{E}^\star$ is a full reflective cocontinuous sub-category of a locally connected (bounded) topos.*

PROOF Let $\mathcal{T} \xrightarrow{\mathsf{t}} \mathcal{S}$ denote the topos

$$\mathrm{Cocts}_{\mathcal{S}}(\mathcal{E}, \mathcal{S}^{\mathbf{C}}) \cong (\mathcal{E}^{\star})^{\mathbf{C}} .$$

Then $\mathbf{t}$ is locally connected since $\mathcal{T}$ is a slice topos of $\mathcal{F}$. Moreover, $\mathbf{t}_!$ is identified with $(\mathsf{p}^*)^{\star}$. That is, there is a natural isomorphism

$$\begin{array}{ccc} \mathcal{E}^{\star} & \xrightarrow{\mathsf{C}^*} & \mathcal{T} \\ {\scriptstyle (\mathsf{p}^*)^{\star}}\downarrow & \simeq & \downarrow{\scriptstyle t_!} \\ \mathcal{S}^{\star} & \cong & \mathcal{S} . \end{array}$$

□

Thus, although $\mathcal{E}^{\star}$ may not in general have a terminal object, one can 'pick up' the terminal object by moving to a category of internal diagrams.

Conclusion

The approach to descent for toposes in this thesis can be considered as 'algebraic'. This is in contrast to the 'geometric' approach as exemplified by Moerdijk.

One aspect of the 'algebraic' approach is the use of the fact that the Beck condition holds. Related to this, is the use of the important theorem [Chap. 3, Th. 2.11] of Pitts [Pi].

Another aspect of this approach is the use of the cocontinuous dual. Clearly, the cocontinuous dual is related to descent theory for cocomplete categories, and hence for toposes. Of course, the very definition of the cocontinuous dual [Chap. 4, Def. 4.1], and of pure geometric morphisms [Chap. 4, Def. 4.4], necessitates the introduction of cocomplete categories. The study of geometric morphisms through their cocontinuous duals remains largely untouched.

A third aspect of the cocomplete categories approach is its 2-categorical nature. Of course, this does not distinguish it from all other 'descent theorems'. For example, Zawadowski [Z1,2] has entertained the notion of lax-descent which requires a 2-categorical setting. The methods used in this thesis do not preclude, in fact they invite, further investigation along these lines (for cocomplete categories).

To close, two open questions shall be mentioned. In the spatial case, an open surjection is pure. The general case remains unanswered. The other question is to characterize those geometric morphisms which are of effective descent for cocomplete categories.

All results herein not due to the author were so indicated. All others are original.

APPENDIX A: **Frobenius reciprocity**

Let $\mathbf{B} \xrightarrow{U} \mathbf{C}$ be an ordinary functor, and assume that U has a right adjoint R.

A.1 DEFINITION $U \dashv R$ is said to satisfy *Frobenius reciprocity* if for all pullback diagrams

$$\begin{array}{ccc} P & \longrightarrow & UB \\ \big\downarrow & & \big\downarrow b \\ C & \xrightarrow[a]{} & D \end{array}$$

which exist in $\mathbf{C}$, the pullback

$$\begin{array}{ccc} Q & \xrightarrow{\pi_0} & B \\ \pi_1 \big\downarrow & & \big\downarrow \tilde{b} \\ RC & \xrightarrow[Ra]{} & RD \end{array}$$

exists in $\mathbf{B}$, and the induced map $(U\pi_0, \widetilde{\pi_1}) : UQ \rightarrow P$ is an isomorphism.

A.2 **Theorem** *Given* $\mathbf{B} \xrightarrow{U} \mathbf{C}$, *the following are equivalent:*

1. $\mathbf{B} \cong \mathbf{C}_{/D}$ *for some* $D \in \mathbf{C}$ *such that* U *is identified with the forgetful functor, and* $\mathbf{C}$ *has products* $C \times D$ *for every* $C \in \mathbf{C}$.

2. U *has a right adjoint* R *such that* $U \dashv R$ *satisfies Frobenius reciprocity,* U *reflects isomorphisms, and* $\mathbf{B}$ *has a terminal object.*

PROOF It is easy to check that *1.* implies *2.*.

For the converse, let 1 denote the terminal object in $\mathbf{B}$. Define functors

$$\hat{R} : \mathbf{C}_{/U1} \longrightarrow \mathbf{B} \ ; \ \ \hat{U} : \mathbf{B} \longrightarrow \mathbf{C}_{/U1}$$

as follows. For an object B of $\mathbf{B}$, let $\hat{U}B = UB \xrightarrow{U!} U1$ and for a morphism f, let $\hat{U}f = Uf$. For $C \xrightarrow{c} U1$, let $\hat{R}c$ be given as the pullback

$$\begin{array}{ccc} \hat{R}c & \xrightarrow{!} & 1 \\ \pi_1 \big\downarrow & & \big\downarrow \eta_1 \\ RC & \xrightarrow[Rc]{} & RU1 \end{array}$$

which exists in **B** because

$$\begin{array}{ccc} C & \xrightarrow{c} & U1 \\ {\scriptstyle 1}\downarrow & & \downarrow{\scriptstyle 1} \\ C & \xrightarrow[c]{} & U1 \end{array}$$

is a pullback in **C** and Frobenius reciprocity applies. η is the unit of $U \dashv R$.

It follows that $\hat{U} \dashv \hat{R}$ where the counit of this adjointness at an object $C \xrightarrow{c} U1$ is $\widetilde{\pi_1}$, where π_1 is as in the definition of $\hat{R}c$. Observe that $\widetilde{\pi_1}$ is the induced map $(U!, \widetilde{\pi_1})$ which, by Frobenius reciprocity, is an isomorphism.

The unit of $\hat{U} \dashv \hat{R}$ at an object $B \in \mathbf{B}$ is given as the induced map from B to the pullback

$$\begin{array}{ccc} \hat{R}\hat{U}B & \xrightarrow{*} & 1 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \eta_1} \\ RUB & \xrightarrow[RU!]{} & RU1 \end{array}$$

That is, $\hat{\eta}_B = (!, \eta_B)$, where $B \xrightarrow{!} 1$. By Frobenius reciprocity, the induced map

$$(U*, \widetilde{\pi_1}) : U\hat{R}\hat{U}B \to UB$$

is an isomorphism. But this map is $\widetilde{\pi_1}$, and its inverse is $U\hat{\eta}_B$. Since U is assumed to reflect isomorphisms, $\hat{\eta}_B$ is therefore an isomorphism. □

APPENDIX B: **The adjoint functor theorems**

Let $\mathcal{S}$ denote an elementary topos. Here, all categories and functors are over $\mathcal{S}$.

A right adjoint to a functor $\mathcal{A} \xrightarrow{\mathsf{F}} \mathcal{B}$ is a functor $\mathcal{B} \xrightarrow{\mathsf{R}} \mathcal{A}$ over $\mathcal{S}$ and natural transformations $\mathsf{FR} \xrightarrow{\varepsilon} \mathbf{1}_{\mathcal{B}}$ and $\mathbf{1}_{\mathcal{A}} \xrightarrow{\eta} \mathsf{RF}$ over $\mathcal{S}$ such that $\varepsilon\mathsf{F} \cdot \mathsf{F}\eta = 1_{\mathsf{F}}$ and $\mathsf{R}\varepsilon \cdot \eta\mathsf{R} = 1_{\mathsf{R}}$.

B.1 Theorem *A functor $\mathcal{A} \xrightarrow{\mathsf{F}} \mathcal{B}$ has a right adjoint if and only if there is given for every $I \in \mathcal{S}$ an adjointness $(\mathsf{F}^I, \mathsf{R}^I, \eta^I, \varepsilon^I)$ such that for every morphism $J \xrightarrow{\alpha} I$ in $\mathcal{S}$, the canonical morphism*

$$\alpha^*\mathsf{R}^I \xrightarrow{\eta^J\alpha^*\mathsf{R}^I} \mathsf{R}^J\mathsf{F}^J\alpha^*\mathsf{R}^I \xrightarrow{\mathsf{R}^J\theta_\alpha^{-1}\mathsf{R}^I} \mathsf{R}^J\alpha^*\mathsf{F}^I\mathsf{R}^I \xrightarrow{\mathsf{R}^J\alpha^*\varepsilon^I} \mathsf{R}^J\alpha^*$$

is an isomorphism. (The isomorphism $\alpha^*\mathsf{F}^I \xrightarrow{\theta_\alpha} \mathsf{F}^J\alpha^*$ is that supplied by F.*)*

PROOF Assume the given condition. Given $J \xrightarrow{\alpha} I$, let $\theta_\alpha^{\mathsf{R}}$ denote the canonical morphism in the statement of the theorem. At issue is the legitimacy of R, ε , and η . Regarding ε , it must be shown that

$$(2) \qquad \varepsilon^J\alpha^* \cdot \mathsf{F}^J\theta_\alpha^{\mathsf{R}} \cdot \theta_\alpha\mathsf{R}^I = \alpha^*\varepsilon^I .$$

Let z denote the natural transformation $\alpha^*\varepsilon^I \cdot \theta_\alpha^{-1}\mathsf{R}^I$. Then

$$\theta_\alpha^{\mathsf{R}} = \mathsf{R}^J z \cdot \eta^J\alpha^*\mathsf{R}^I ,$$

and hence

$$\mathsf{F}^J\theta_\alpha^{\mathsf{R}} = \mathsf{F}^J\mathsf{R}^J z \cdot \mathsf{F}^J\eta^J\alpha^*\mathsf{R}^I .$$

By the naturality of ε^J ,

$$z \cdot \varepsilon^J\mathsf{F}^J\alpha^*\mathsf{R}^I = \varepsilon^J\alpha^* \cdot \mathsf{F}^J\mathsf{R}^J z .$$

Therefore, the left side of 2 is equal to

$$z \cdot \varepsilon^J\mathsf{F}^J\alpha^*\mathsf{R}^I \cdot \mathsf{F}^J\eta^J\alpha^*\mathsf{R}^I \cdot \theta_\alpha\mathsf{R}^I$$

which is equal to

$$z \cdot \theta_\alpha\mathsf{R}^I = \alpha^*\varepsilon^I .$$

This proves that ε is a natural transformation over $\mathcal{S}$. That the same is true of η can be shown in a similiar fashion. Now that we know the counit and unit are legitimate, it follows that the $\theta_\alpha^{\mathsf{R}}$'s satisfy the required coherence condition (see Chap. 1, §1) because the θ_α's do.

Conversely, if $\mathsf{F} \dashv \mathsf{R}$ over $\mathcal{S}$, then it follows that the isomorphism $\theta_\alpha^{\mathsf{R}}$ is equal to the canonical morphism in the statement of the theorem. □

B.3 DEFINITION A functor $\mathcal{A}\xrightarrow{\mathsf{F}}\mathcal{B}$ satisfies the *solution set of objects* condition if for every $I\in\mathcal{S}$ and every $B\in\mathcal{B}^I$, there exists $J\in\mathcal{S}$ and $A\in\mathcal{A}^J$ such that for every $C\in\mathcal{A}^I$ and every $\mathsf{F}^I C\xrightarrow{x} B$, there exists $I\xrightarrow{i} J$, $\mathsf{F}^I i^*A\xrightarrow{b} B$, and $C\xrightarrow{a} i^*A$ such that $b\cdot(\mathsf{F}^I a)=x$.

The following is Freyd's General Adjoint Functor Theorem in the context of categories over $\mathcal{S}$.

B.4 Theorem (GAFT) *Let $\mathcal{A}\xrightarrow{\mathsf{F}}\mathcal{B}$ be a functor with $\mathcal{A}$ and $\mathcal{B}$ locally small and with $\mathcal{A}$ cocomplete. Then F has a right adjoint if and only if F is cocontinuous and satisfies the solution set of objects condition.*

If $\mathcal{A}$ has small coproducts, then the following can be taken as a definition of a generating family in $\mathcal{A}$.

B.5 DEFINITION $\mathcal{A}$ is said to have a *generating family* if there is an $I\in\mathcal{S}$ and an object $G\in\mathcal{A}^I$ such that for every $J\in\mathcal{S}$ and every $A\in\mathcal{A}^J$ there exists $K\xrightarrow{(\alpha,\beta)} I\times J$ and an epimorphism $\Sigma_\beta\alpha^*G\twoheadrightarrow A$ in $\mathcal{A}^J$.

It is shown in [PS] that a topos over $\mathcal{S}$ is bounded if and only if it has a generating family when regarded as a category over $\mathcal{S}$. Also, any small category has a generating family.

B.6 DEFINITION $\mathcal{A}$ is said to be *cowell-powered* if for every I and every $A\in\mathcal{A}^I$, there is an object $X\xrightarrow{Q^I A} I$ in $\mathcal{S}_{/I}$ such that for every $J\xrightarrow{\alpha} I$ there is a natural bijection of morphisms $\alpha\rightarrow Q^I A$ in $\mathcal{S}_{/I}$ with the stable quotient objects of α^*A in $\mathcal{A}^J$.

A topos over $\mathcal{S}$ is cowell-powered, as is any small category.

The Special Adjoint Functor Theorem is as follows.

B.7 Theorem (SAFT) *Let $\mathcal{A}\xrightarrow{\mathsf{F}}\mathcal{B}$ be a functor with $\mathcal{A}$ and $\mathcal{B}$ locally small and with $\mathcal{A}$ cocomplete. Assume also that $\mathcal{A}$ has a generating family and is cowell-powered. Then F has a right adjoint if and only if F is cocontinuous.*

See [PS] for proofs of theorems B.4 and B.7.

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