Conformally Covariant Operators and Conformal Invariants

Peter Yuen

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Department of Mathematics and Statistics

McGill University

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ABSTRACT

In this thesis, we give a survey of results in conformal geometry that give rise to conformal invariants, including zero and negative eigenvalues of conformally covariant operators; nodal sets of eigenfunctions in the kernel of those operators; conformally invariant maps into projective space; and finally, conformal invariants arising from the component functions of the Weyl tensor. We also discuss the case of products of Riemann surfaces, and explore the connections to spectral theory of the hyperbolic Laplacian on Riemann surfaces.

ABRÉGÉ

Dans cette thèse, nous donnons une revue des résultats en géométrie conforme qui définissent les invariants conformes, y compris les valeurs propres nulles et négatives des opérateurs conformément covariants; les ensembles nodaux des fonctions propres dans le noyau de ces opérateurs; applications invariantes conformément covariants à l'espace projectif; et, finalement, les invariants conformes donnés par des composantes du tenseur de Weyl. Nous discutons également le cas des produits des surfaces de Riemann et explorons les connexions à la théorie spectrale du Laplacien hyperbolique sur les surfaces de Riemann.

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CHAPTER 1
Introduction

The central idea of conformal geometry is to change metrics while preserving angles. Under these types of changes, some operators—known as conformally covariant operators—transform in specific ways. These operators, and the conformal invariants they give rise to, form the theme of this thesis.

Chapters 2 and 3 serve as a gradual progression to Chapter 4, whose primary focus aligns with the theme of this thesis. Chapter 2 starts with a subsection devoted to introducing the fundamental ideas of conformal geometry—like how, under a conformal change of a Riemannian metric, angles are preserved; important ideas like conformal classes and conformal transformations are also introduced. Then, after a few more subsections of background material, the Uniformization Theorem is discussed. While the theorem is not proved in this thesis, its equivalence with the statement that every Riemann surface admits, in its conformal class, a complete metric of constant curvature, is proven. This equivalent statement serves as a sort of precursor to the Yamabe problem, which is discussed in Chapter 3. The last two sections of the chapter serve as asides; one is devoted to how the Ricci flow on a 2-dimensional closed Riemmanian manifold can be used to take one from the given metric to a conformally equivalent metric that is of constant curvature; the other discusses how one can determine what the universal cover of a Riemann surface is.

In Chapter 3, we discuss the generalization of the Uniformization Theorem: the Yamabe problem, which asks if, for a compact Riemannian manifold of dimension $n \geq 3$, there is a metric, conformal to the starting metric, that has constant scalar curvature. The answer to the Yamabe problem is yes, though we only provide a rough outline of the proof, with some parts of the proof emphasized more than others. Apart from the useful statement that the Yamabe problem provides, it also serves to introduce the conformal Laplacian, which paves the way for the next chapter.

In Chapter 4, we clearly define conformally covariant operators and provide a few examples of such operators—one of which is the conformal Laplacian (Yamabe operator).

In Chapter 5, we discuss some of the conformal invariants that can be obtained from conformally covariant operators; the nodal sets and nodal domains of an eigenfunction in the kernel, maps into projective space (obtained by using eigenfunctions in the kernel as projective coordinates), and the number of negative eigenvalues are amongst the invariants discussed. The chapter closes with a short section on spaces of conformal structures and a theorem which states that for a generic smooth metric on a closed n-dimensional manifold, zero is not an eigenvalue of the conformal Laplacian.

In Chapter 6, we explore the smallest number of negative eigenvalues of the conformal Laplacian on a product of two or more Riemann surfaces. The new results in this chapter are the joint effort of the author and Professor Dmitry Jakobson, where the ideas are largely due to Professor Jakobson and the computations are mostly due to the author. The proof of Proposition 6.2.1 in this chapter was communicated by Professor C. LeBrun; I would also like to thank Professor V. Apostolov for useful and stimulating conversations regarding these results. The results in this chapter are a work in progress; see [JY].

Over the next three chapters, we discuss the Weyl tensor. This discussion is largely inspired by how the simple transformation law for the Weyl tensor is similar to the transformation law for eigenfunctions in the kernel of a conformally covariant operator.

In Chapter 7, we survey basic results of the Weyl tensor. In particular, we show how the Weyl tensor is a term in the decomposition of the $(0,4)$-Riemann curvature tensor, and how it transforms under a conformal change of the metric.

In Chapter 8, the survey on results of the Weyl tensor continues but now specializes to two types of metrics on Lie groups: left-invariant metrics and bi-invariant metrics. A left-invariant metric allows the components of the Weyl tensor to be written in terms of the Lie algebra's structure constants, while a bi-invariant metric allows the Weyl tensor to be written in a relatively simple form.

In Chapter 9, we consider a product of two surfaces, where the metric on one surface is multiplied by a conformal factor, and then compute the ratios of the components of the Weyl tensor. We then discuss the behaviour of these ratios as the conformal factor degenerates at a point. The computations in this chapter are due to the author, while the ideas that inspired these computations are due to Professor Dmitry Jakobson.

In Chapter 10, we conclude the thesis and provide directions for further research in the form of several conjectures.

CHAPTER 2
The Uniformization Problem

The primary goal of this chapter is to prove that the Uniformization Theorem is equivalent to the statement that every Riemann surfaces admits a complete metric of constant curvature in its conformal class. Attempting to generalize this latter statement brings one to the Yamabe problem, which is the subject of Chapter 3.

2.1 Background

2.1.1 Conformal transformations

In this subsection, we quickly introduce a few important concepts from conformal geometry. To do this, we use the introductions of [YS], [YO], and [JNSS].

Consider an n-dimensional, C^∞ manifold M equipped with a Riemannian metric g. Consider two arbitrary, nonzero, tangent vectors from the tangent space of some point $p \in M$; that is, $x, y \in T_p(M)$. It is well-known that the angle θ between x and y is uniquely given through

$$\cos\theta = \frac{\langle x, y\rangle_g}{|x|_g |y|_g},$$

where $|x|_g = \langle x, x\rangle_g^{1/2} = g_p(x,x)^{1/2}$ (see [Lee2]).

Now, suppose $\tilde{g}$ is another Riemannian metric on M. If, at each point of the manifold, the angle between any pair of tangent vectors with respect to g

and $\tilde{g}$ is equal, then the two metrics are said to be *conformally equivalent* (or *conformally related* or *conformal* to each other).

Consider the case where we start with (M, g) and we multiply g by a positive function u that is defined on M. Then the angle between two tangent vectors with respect to the metric ug is uniquely given by

$$\cos\theta = \frac{\langle x, y\rangle_{ug}}{|x|_{ug}|y|_{ug}} = \frac{u(p)\langle x, y\rangle_g}{(u(p)^{1/2}|x|_g)(u(p)^{1/2}|y|_g)} = \frac{\langle x, y\rangle_g}{|x|_g|y|_g}.$$

So, the multiplication of g by a positive function u defined on M has not changed the angle between the two tangent vectors, meaning that the metric defined as $\tilde{g} := ug$ is conformal to g.

Expanding upon this, we can take any function $f \in C^\infty(M)$ and define a new metric $\tilde{g} := e^f g$ so that $\tilde{g}$ is conformal to g. Changing g in this manner, to obtain $\tilde{g}$, is referred to as a *conformal change* of the metric, and the function $e^{f(x)}$ is called the *conformal factor*. In fact, the necessary and sufficient condition for two metrics g and $\tilde{g}$ of M to be conformally equivalent is for there to exist a function f such that $\tilde{g} = e^f g$.

Definition 2.1.1. For a Riemannian metric g on a manifold M, its *conformal class* $[g]$ is the set of metrics $\{e^f g : f \in C^\infty(M)\}$.

Definition 2.1.2. Let (M, g) and (M', g') be two Riemannian manifolds and $\Psi : M \longrightarrow M'$ be a diffeomorphism. Then the pullback $\tilde{g} := \Psi^* g'$ is a Riemannian metric on M. If $\tilde{g}$ is conformally equivalent to g, then the map Ψ is called a *conformal transformation* (or, a *conformal map*).

Remark 2.1.3. In [Lee2], an *isometry* is defined as a diffeomorphism Ψ between two Riemannian manifolds (M, g) and (M', g') such that $g = \Psi^* g'$. Clearly then, an isometry preserves angles and, by the above definition, it may be thought of as a conformal transformation.

2.1.2 Riemann surfaces

For this subsection we shall follow [For] (chapter 1, section 1) and [FK] (chapter 1, section 1).

Definition 2.1.4. Let M be a 2-dimensional manifold. A *complex chart* on M is a homeomorphism $\varphi : U \longrightarrow V$, where $U \subset M$ is open and $V \subset \mathbb{C}$ is open. Two complex charts φ_i, φ_j are said to be *holomorphically compatible* if the map

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$$

is biholomorphic (holomorphic, bijective, and its inverse is also holomorphic).

Remark 2.1.5. A complex chart is also known as a local uniformizing variable; however, when we use this latter term, it shall be understood in the following sense: for a Riemann surface M, a *local uniformizing variable* at a point $p \in M$ is a homeomorphism $z_p : D_p \longrightarrow U_p$, where $D_p \subset \mathbb{C}$ is open and U_p is a neighbourhood of p.

Definition 2.1.6. A *complex atlas* on M is a collection of charts $\mathcal{A} = \{\varphi_i : U_i \longrightarrow V_i \, : \, i \in I\}$ such that the charts are all holomorphically compatible and M is covered by them, that is, $\bigcup_{i\in I} U_i = M$. Two complex atlases $\mathcal{A}$ and $\mathcal{A}'$ are said to be *analytically equivalent* if every chart of $\mathcal{A}$ is holomorphically compatible with every chart of $\mathcal{A}'$.

Definition 2.1.7. A *complex structure* on a 2-dimensional manifold M is an equivalence class of analytically equivalent atlases on M.

Definition 2.1.8. A *Riemann surface* is a pair (M, Σ), where M is a 2-dimensional connected manifold and Σ is a complex structure on M.

Alternatively, in the introduction of [MT], we see that a Riemann surface may also be defined in the following way:

Definition 2.1.9. A *Riemann surface* is a pair $(M, [g])$, where M is a 2-dimensional, connected, oriented, C^∞ manifold, and $[g]$ is a conformal class.

It is further noted in [MT] that these definitions are made equivalent via a well-known bijection between the set of conformal classes on M and the set of complex structures on M—it should be stressed that this bijection only exists in the case where M is 2-dimensional. As for the question of orientability in Definition 2.1.8, we look to [Mi] where it is noted that the complex charts induce a well-defined local orientation at each point of the Riemann surface; this, in turn, induces an orientation for the entire Riemann surface.

As one might expect from the equivalence of the above two definitions, we can (and will, in the coming sections) pass between Riemann surfaces and connected, oriented, 2-dimensional Riemannian manifolds.

Definition 2.1.10. Let M and N be Riemann surfaces. A continuous mapping $f : M \longrightarrow N$ is said to be *holomorphic* if for every pair of charts $\varphi_i : U_i \longrightarrow V_i$ on M and $\varphi_j : U_j \longrightarrow V_j$ on N, with $f(U_i) \subset U_j$, the mapping

$$\varphi_j \circ f \circ \varphi_i^{-1} : V_i \longrightarrow V_j$$

is holomorphic in the usual sense.

In the context of the above definition, a mapping $f : M \longrightarrow N$ is said to be *biholomorphic* (or, *conformal*) if it is bijective and both f and f^{-1} are holomorphic. Two Riemann surfaces M and N are said to be *biholomorphically equivalent* (or, *conformally equivalent*) if such a mapping exists between them. Note that the automorphism group $\mathrm{Aut}(M)$ of a Riemann surface M is the group whose elements are conformal mappings from M to itself.

2.1.3 Covering spaces

For this subsection, we utilize [Mun] (chapter 9, section 53; and chapter 13, sections 80 and 81), and [Ahl] (chapter 9, section 5).

Definition 2.1.11. Let X and $\widetilde{X}$ be topological spaces. Let $\pi : \widetilde{X} \longrightarrow X$ be a continuous surjective map. An open set U of X is *evenly covered* by π if $\pi^{-1}(U)$ can be written as a union of disjoint open sets V_j from $\widetilde{X}$ such that for each j, the restriction of π to V_j is a homeomorphism of V_j onto U.

If every point of X has a neighbourhood that is evenly covered by π, then the map π is called a *covering map*, and $\widetilde{X}$ is called a *covering space* of X.

Definition 2.1.12. Let $\pi : \widetilde{X} \longrightarrow X$ be a covering map. If $\widetilde{X}$ is simply connected, then $\widetilde{X}$ is said to be a *universal covering space*.

Definition 2.1.13. Let $\pi : \widetilde{X} \longrightarrow X$ be a covering map. A *deck transformation* (or, *covering transformation*) is a homeomorphism $h : \widetilde{X} \longrightarrow \widetilde{X}$ such that $\pi = \pi \circ h$.

From [Ahl], we have the following theorem.

Theorem 2.1.14. *Apart from the identity, a deck transformation has no fixed points.*

The set of all deck transformations for a covering space $\widetilde{X}$ forms a group $G(\widetilde{X})$ under composition.

Definition 2.1.15. Let $\pi : \widetilde{X} \longrightarrow X$ be a covering map. The covering space $\widetilde{X}$ is said to be *regular* (or, *normal*) if for each $x \in X$ and each pair of points $\tilde{x}, \tilde{x}' \in \{\pi^{-1}(x)\}$, there is a deck transformation taking $\tilde{x}$ to $\tilde{x}'$.

We refer to [Mun] (Theorem 81.6) for the proof of the following theorem.

Theorem 2.1.16. *Let $\pi : \widetilde{X} \longrightarrow X$ be a covering map and let $\widetilde{X}$ be regular. Then the quotient $\widetilde{X}/G(\widetilde{X})$ is homeomorphic to X.*

Note that when we eventually use Theorem 2.1.16, we will take $\widetilde{X}$ to be the universal cover of X, since every universal cover is regular.

2.1.4 Constant curvature

For this subsection, we follow [Bo] (chapter 8) and [Lee2] (chapters 7 and 8), and focus primarily on sectional curvature. It should be noted that Ricci curvature and scalar curvature are not defined here but are instead defined in Section 2.3, where they are particularly relevant.

Definition 2.1.17. Let (M, g) be a Riemannian manifold and let $\mathfrak{X}(M)$ denote the set of C^∞ vector fields on M. The (1,3)-*Riemann curvature tensor* is defined by

$$\mathcal{R} : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M),$$

$$\mathcal{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

In local coordinates, its components are denoted as ${\mathcal{R}_{ijk}}^l$.

Remark 2.1.18. The map $\mathcal{R}$ is multilinear over $C^\infty(M)$, hence, it is indeed a (1,3)-tensor field (see Proposition 7.3 in [Lee2]).

Definition 2.1.19. The (0,4)-*Riemann curvature tensor*, denoted as Rm, is the (0,4)-tensor field obtained from lowering the last index of the $(1,3)$-Riemann curvature tensor $\mathcal{R}$. The action of Rm on vector fields is given by

$$Rm(X, Y, Z, W) = \langle \mathcal{R}(X, Y)Z, W\rangle_g.$$

In local coordinates, its components $(Rm)_{ijkl}$ are given by $(Rm)_{ijkl} = \mathcal{R}_{ijkl} = g_{lm}\mathcal{R}_{ijk}{}^m$.

At any point $p \in M$, a plane section σ (i.e. a two dimensional subspace of $T_p(M)$) is determined by any pair of mutually orthogonal unit vectors u, v at p.

Definition 2.1.20. For a plane section σ with orthonormal basis $u, v \in T_p(M)$, its *sectional curvature* $K_p(\sigma)$ is defined as

$$K_p(\sigma) = -Rm_p(u, v, u, v) = -\langle \mathcal{R}_p(u, v)u, v\rangle_g.$$

In the case of surfaces (2-dimensional Riemannian manifolds), the sectional curvature is the same as the Gaussian curvature of the surface.

Definition 2.1.21. If the sectional curvatures across all plane sections at all points are the same constant value, then the Riemannian manifold is said to be of *constant curvature* (or, *constant sectional curvature*).

In preparation for the next theorem, we recall the definition of hyperbolic space; more precisely, we state the hyperboloid model of hyperbolic space:

Fix $r > 0$ and suppose $n \geq 1$. Let $\mathbb{R}^{n,1}$ be Minkowski space, whose coordinates are $(x_1, \ldots, x_n, t)$ and whose metric is given by

$$q = dx_1{}^2 + \cdots + dx_n{}^2 - dt^2.$$

Then $\mathbb{H}^n_r$, the hyperboloid model of hyperbolic space of radius r, is the submanifold of $\mathbb{R}^{n,1}$ defined as all points which satisfy $x_1^2 + \cdots + x_n^2 - t^2 = -r^2$ with $t > 0$, and whose metric is given by i^*q, where $i : \mathbb{H}^n_r \to \mathbb{R}^{n,1}$ is the inclusion map.

We can now state the following theorem which provides us with some important, and soon to be relevant, examples of Riemannian manifolds with constant curvature.

Theorem 2.1.22. *The following n-dimensional Riemannian manifolds have the indicated constant curvatures:*

(i) $\mathbb{R}^n$ *with the Euclidean metric has constant curvature* 0*.*

(ii) The sphere S^n_r *of radius* $r > 0$*, with the standard round metric* $\bar{g}$*, has constant curvature* $1/r^2$*.*

(iii) Hyperbolic space $\mathbb{H}^n_r$ *of radius* $r > 0$ *has constant curvature* $-1/r^2$*.*

Note that for each fixed r, there are actually four mutually isometric models of hyperbolic space (see Theorem 3.7 in [Lee2]). When hyperbolic space next arises, we shall be making use of the model known as the Poincaré ball model $\mathbb{B}^n_r$, rather than the hyperboloid model that was previously defined.

For hyperbolic space of radius r, the Poincaré ball model $\mathbb{B}^n_r$ is the ball of radius r centred at the origin in $\mathbb{R}^n$, and its metric in the coordinates $(x_1, \ldots, x_n)$ is given by

$$g = 4r^4 \frac{dx_1{}^2 + \cdots + dx_n{}^2}{(r^2 - |x|^2)^2}.$$

We now end this subsection by stating the Killing-Hopf theorem (see Chapter 12 of [Lee2]).

Theorem 2.1.23. *Let (M, g) be a complete, simply connected, n-dimensional Riemannian manifold with constant curvature and $n \geq 2$. Then M is isometric to either $\mathbb{R}^n$, S^n_r, or $\mathbb{H}^n_r$.*

2.2 The Uniformization Theorem

To reach the Uniformization Theorem, we shall follow [Ab] (sections 2 and 3) and the introduction of [Ch]. We shall omit the proof of the theorem, but it is proved in both of these references.

We begin with the uniformization problem, which can be stated in the following way: *Let M be an arbitrary Riemann surface. Find all domains $D \subset \hat{\mathbb{C}}$ and holomorphic functions $t : D \longrightarrow M$ so that at each point $p \in M$, t is a local uniformizing variable at p.* In other words, for each $p \in M$, there is a neighbourhood U_p such that t restricted to $D_p := t^{-1}(U_p)$ is a homeomorphism.

A useful way of perceiving the uniformization problem is to view it from a covering space perspective. Let $D \subset \hat{\mathbb{C}}$, let M be a Riemann surface, and let $\widetilde{M}$ be the universal covering space of M with covering map $\pi : \widetilde{M} \longrightarrow M$. Using the well-known fact that the universal cover of a Riemann surface is also a Riemann surface, we see that $\widetilde{M}$ is a simply connected Riemann surface. Suppose $t : D \longrightarrow \widetilde{M}$ is a uniformizing map. Then with the covering map π, we can obtain a uniformization for M through the composition $\pi \circ t$.

So, if we can show that every simply connected Riemann surface is conformally equivalent to a subdomain of $\hat{\mathbb{C}}$ (so that $\widetilde{M}$ is conformally equivalent to D), then we are done. The following theorem shows that this can in fact be done, and since it essentially leads one to the solution of the uniformization problem, it has become known as the Uniformization Theorem.

Theorem 2.2.1. *Every simply connected Riemann surface is conformally equivalent to either the complex plane $\mathbb{C}$, the Riemann sphere $\hat{\mathbb{C}}$, or the unit disk D_1.*

Alternatively, the Uniformization Theorem can be stated in the following way.

Theorem 2.2.2. *Every Riemann surface admits, in its conformal class, a complete metric of constant curvature.*

Let us roughly prove the equivalence of the above two theorems.

Proof. First, we start with Theorem 2.2.2. Let M be our Riemann surface with conformal class $[g]$, and let $g' \in [g]$ be the complete metric of constant curvature. Now, we pass to the universal cover $\widetilde{M}$ by using the covering map $\pi : \widetilde{M} \longrightarrow M$ to pullback our complete metric; this gives us $\pi^* g'$ which is still of constant curvature. So, $\widetilde{M}$ is now a simply connected, complete surface with constant curvature. Applying the Killing-Hopf theorem, we see that $\widetilde{M}$ is isometric to either $\mathbb{R}^2$, S^2, or $\mathbb{H}^2$ (or rather, $\mathbb{B}^2$). Upon identifying $\widetilde{M}$ as a simply connected Riemann surface and identifying $\mathbb{R}^2$, S^2, and $\mathbb{B}^2$ as, respectively, $\mathbb{C}$, $\hat{\mathbb{C}}$, and D_1, we are able to realize the isometry as a biholomorphism. Thus, we have obtained Theorem 2.2.1.

Now, let us start with Theorem 2.2.1 and show that we can obtain Theorem 2.2.2 from it. Let M be an arbitrary Riemann surface. Its universal covering space $\widetilde{M}$ is a simply connected Riemann surface, meaning it is conformally equivalent to either $\mathbb{C}$, $\hat{\mathbb{C}}$, or D_1. In their defining conformal class, the spaces

$\mathbb{C}$, $\hat{\mathbb{C}}$, and D_1 admit the following metrics:

$$\begin{aligned}
\rho(z)|dz| &= |dz| \text{ for } \mathbb{C}, \\
\rho(z)|dz| &= \frac{2|dz|}{1+|z|^2} \text{ for } \hat{\mathbb{C}}, \\
\rho(z)|dz| &= \frac{2|dz|}{1-|z|^2} \text{ for } D_1.
\end{aligned}$$

From [FK] (chapter 4, section 8), we know that these metrics are complete and of constant curvature, where the curvature is 0 for $\mathbb{C}$, 1 for $\hat{\mathbb{C}}$, and -1 for D_1. So, $\widetilde{M}$ admits one of these metrics.

Now we would like to bring the complete constant curvature metric on $\widetilde{M}$ down to M. To do this, we use Theorem 2.1.16 to see that $M \simeq \widetilde{M}/G(\widetilde{M})$, where $G(\widetilde{M})$ is the group of deck transformations of $\widetilde{M}$. We next observe that $G(\widetilde{M})$ is a subgroup of the automorphism group $\mathrm{Aut}(\widetilde{M})$.

For the first case, suppose $\widetilde{M}$ is conformally equivalent to $\hat{\mathbb{C}}$. From [FK] (chapter 4, section 5), we know that

$$\mathrm{Aut}(\hat{\mathbb{C}}) \cong PSL(2,\mathbb{C}) \cong \left\{ z \longrightarrow \frac{az+b}{cz+d} : a,b,c,d \in \mathbb{C}, \text{ and } ad-bc=1 \right\}.$$

The only automorphisms of $\hat{\mathbb{C}}$ which act as isometries (i.e. preserve the metric) are those which belong to the following set (see [FK], chapter 4, section 8):

$$\left\{ z \longrightarrow \frac{az-\bar{c}}{cz+\bar{a}} : a,b \in \mathbb{C}, \text{ and } |a|^2+|c|^2=1 \right\} \cong SU(2)$$

However, for our purposes, we do not have to concern ourselves with the isometries of $\hat{\mathbb{C}}$. Indeed, suppose $h \in \mathrm{Aut}(\hat{\mathbb{C}})$. As noted in [Ahl] (chapter 10, section 6), such an h has at least one fixed point on $\hat{\mathbb{C}}$; then from Theorem 2.1.14, we see that if $h \in G(\hat{\mathbb{C}})$, then h must be the identity. Thus, when we push the metric for $\hat{\mathbb{C}}$ down via the covering map $\widetilde{M} \longrightarrow \widetilde{M}/G(\widetilde{M}) \simeq M$, it is still complete and of constant curvature.

For the second case, suppose $\widetilde{M}$ is conformally equivalent to D_1. From [Kr] (Section 0.2, Theorem 3), we know that every automorphism of D_1 is of the form $h(z) = \varphi_\theta(z) \circ \phi_a$, where

$$\phi_a(z) = \frac{z-a}{1-\bar{a}z}, \quad a \in D_1 \qquad \text{and} \qquad \varphi_\theta(z) = ze^{i\theta}, \quad \theta \in \mathbb{R}.$$

To see how the metric for D_1 behaves under such an h, we compute the pullback $h^*\rho(z)$; from [Kr] again, we know that $h^*\rho(z) = \rho(h(z)) \cdot |h'(z)|$.

(i) If $h(z) = \varphi_\theta(z)$, then $|h'(z)| = 1$ and

$$h^*\rho(z) = \rho(ze^{i\theta}) = \frac{2}{1-|ze^{i\theta}|^2} = \frac{2}{1-|z|^2} = \rho(z).$$

(ii) If $h(z) = \phi_a(z)$, then

$$|h'(z)| = \frac{1-|a|^2}{|1-\bar{a}z|^2},$$

and so

$$\begin{aligned} h^*\rho(z) &= \rho\left(\frac{z-a}{1-\bar{a}z}\right) \cdot |h'(z)| \\ &= \frac{2}{1-|\frac{z-a}{1-\bar{a}z}|^2} \cdot \frac{1-|a|^2}{|1-\bar{a}z|^2} \\ &= \frac{2(1-|a|^2)}{|1-\bar{a}z|^2 - |z-a|^2} \\ &= \frac{2(1-|a|^2)}{1-|z|^2-|a|^2+|a|^2|z|^2} \\ &= \frac{2}{1-|z|^2} \\ &= \rho(z). \end{aligned}$$

Since every $h \in \mathrm{Aut}(D_1)$ can be written as a composition of the above two cases, we see that our metric for D_1 is invariant under the action of $\mathrm{Aut}(D_1)$; thus, it is also invariant under the action of $G(D_1)$. So, when this metric gets pushed down via the covering map $\widetilde{M} \longrightarrow \widetilde{M}/G(\widetilde{M}) \simeq M$, it is still complete and of constant curvature.

For the final case, suppose $\widetilde{M}$ is conformally equivalent to $\mathbb{C}$. Then

$$\mathrm{Aut}(\mathbb{C}) \cong \{z \longmapsto az + b : a, b \in \mathbb{C}, a \neq 0\}.$$

Let $h \in \mathrm{Aut}(\mathbb{C})$. Then

$$h^*\rho(z) = \rho(h(z)) \cdot |h'(z)| = \rho(az + b) \cdot |a| = |a|.$$

Clearly then, the only automorphisms of $\mathbb{C}$ which act as isometries are those with a such that $|a| = 1$. Indeed, [FK] notes that the automorphisms which act as isometries are those which take the form

$$z \longmapsto e^{i\theta}z + b, \qquad \theta \in \mathbb{R}, b \in \mathbb{C}. \tag{2.1}$$

Now, in searching for the deck transformations of $\mathbb{C}$, we should disregard the $h \in \mathrm{Aut}(\mathbb{C})$ that have fixed points, since Theorem 2.1.14 says that deck transformations have no fixed ponts (except for the identity). So, suppose $h \in \mathrm{Aut}(\mathbb{C})$ has a fixed point: $h(w) = aw + b = w$. Then

$$w = \frac{b}{1-a},$$

which means that $a \neq 1$. Conversely, if $a \neq 1$, this expression gives us a fixed point for $h(z) = az + b$. Thus, $h \in \mathrm{Aut}(\mathbb{C})$ has a fixed point if and only if $a \neq 1$. So, disregarding these automorphisms, we are left with automorphisms of the form $h(z) = z + b$, which, according to 2.1, preserve the metric. Thus, we are able to conclude that our metric on $\mathbb{C}$ is invariant under the action of $G(\mathbb{C})$, and so when it gets pushed down via the covering map, it is still complete and of constant curvature.

So, as originally desired, we have obtained Theorem 2.2.2 from Theorem 2.2.1.

This finishes the proof that Theorem 2.2.1 and Theorem 2.2.2 are equivalent. $\square$

2.3 Ricci flow

For this section, we follow [CK] (primarily the beginning of chapter 5).

In the previous section, we showed how the Uniformization Theorem implies that every Riemann surface admits in its conformal class a complete metric of constant curvature. This guaranteed existence of a metric of constant curvature raises a new question: If we start with a 2-dimensional Riemannian manifold, is there an evolution equation that will conformally deform our starting metric to the point where we obtain a metric of constant curvature?

To answer this question, we need the concept of Ricci flow, which was first introduced by Hamilton in [Ha] with the intent of applying it to Thurston's Geometrization Conjecture. For our purposes, it will provide us with a way of evolving our metric to obtain the metric of constant curvature.

Before defining the Ricci flow, we use [Lee2] to recall the definition of the Ricci curvature.

Definition 2.3.1. Let (M, g) be an n-dimensional Riemannian manifold. The *Ricci curvature* (or, *Ricci tensor*) is the covariant 2-tensor field, denoted as Ric, whose action upon vector fields X and Y is defined as

$$\mathrm{Ric}(X, Y) = \mathrm{tr}(Z \longmapsto \mathcal{R}(Z, X)Y),$$

where $\mathcal{R}$ is the (1,3)-Riemann curvature tensor. In local coordinates, its components are given by $(\mathrm{Ric})_{ij} = \mathcal{R}_{kij}{}^{k} = g^{km}\mathcal{R}_{kijm}$.

Now that we have defined the Ricci curvature, it is convenient to define the scalar curvature.

Definition 2.3.2. The *scalar curvature* is the function R defined as the trace of the Ricci curvature:

$$R = \mathrm{tr}_g \mathrm{Ric} = g^{ij} (\mathrm{Ric})_{ij}.$$

Remark 2.3.3. From [Lee2] (Proposition 8.32) and [Bo] (chapter 8, section 3), we know that for an orthonormal basis $\{e_1, \ldots, e_n\}$ at a point p of a Riemannian manifold, the Ricci curvature is given by

$$\mathrm{Ric}_p(u,v) = \sum_{i=1}^{n} \langle \mathcal{R}_p(e_i,u)v, e_i\rangle_g = \sum_{i=1}^{n} Rm_p(e_i,u,v,e_i),$$

and the scalar curvature is given by

$$\begin{aligned} R(p) = \sum_{j=1}^{n} \mathrm{Ric}_p(e_j,e_j) &= \sum_{i,j=1}^{n} Rm_p(e_i,e_j,e_j,e_i) \\ &= \sum_{i,j=1}^{n} -Rm_p(e_j,e_i,e_j,e_i) = \sum_{i\neq j} K_p(e_i,e_j), \end{aligned}$$

where $K_p(e_i,e_j)$ is the sectional curvature of the plane section with orthonormal basis e_i, e_j.

Remark 2.3.4. In [Lee2] (Corollary 7.27), it is shown that in dimension 2, the Ricci curvature can be written as $\mathrm{Ric} = \frac{1}{2} Rg$.

Now we may continue with our discussion on the Ricci flow.

Definition 2.3.5. The *Ricci flow* is defined as

$$\frac{\partial}{\partial t} g = -2\mathrm{Ric}$$
$$g(0) = g_0,$$

where g_0 is the metric on the Riemannian manifold (M, g_0).

Definition 2.3.6. The *normalized Ricci flow* is defined as

$$\frac{\partial}{\partial t} g = -2\mathrm{Ric} + \frac{2}{n} \frac{\int_M R \, d\mu}{\int_M d\mu} g$$
$$g(0) = g_0,$$

where $d\mu$ is the volume form, g_0 is the metric on the n-dimensional Riemannian manifold (M, g_0), and R is the scalar curvature of g_0.

Remark 2.3.7. In general, the Ricci flow does not preserve volume. The normalized Ricci flow, however, will ensure that the volume is preserved; as shown in [Ha], it is obtained from the Ricci flow by reparametrizing in time and applying a change of scale in space.

Theorem 2.3.8. *If (M, g) is a 2-dimensional, closed (i.e. compact and without boundary) Riemannian manifold, then there exists a unique solution $g(t)$ of the normalized Ricci flow*

$$\frac{\partial}{\partial t} g = \left(\frac{\int_M R \, d\mu}{\int_M d\mu} - R \right) g$$
$$g(0) = g_0.$$

The solution exists for all time, and as $t \longrightarrow \infty$, the metrics $g(t)$ converge uniformly in any C^k-norm to a smooth metric g_∞ of constant curvature.

The proof of Theorem 2.3.8 is rather long and so we omit it from here; the full proof may be found in [CK] (chapter 5). We shall merely note that the proof is split into three parts, where each part depends upon the sign of

$$r := \frac{\int_M R\, d\mu}{\int_M d\mu}, \tag{2.2}$$

which is referred to as the *average scalar curvature*. In literature, however, the parts are typically distinguished by the sign of the Euler characteristic $\chi(M)$. To see why, observe that in dimension $n = 2$, the scalar curvature is twice the Gaussian curvature; then look at the Gauss-Bonnet theorem (Theorem 2.4.1 in the next section) to see that the sign of r coincides with the sign of $\chi(M)$.

So, given a 2-dimensional closed Riemannian manifold (M, g_0), the normalized Ricci flow will provide us with an evolution equation $g(t)$ that will take us from our starting metric g_0 to the metric of constant curvature $\tilde{g}$. The only thing left to do is to confirm that $\tilde{g}$ is indeed conformally equivalent to g_0.

Suppose $g(t)$ is the solution obtained from Theorem 2.3.8. Let f be the function $f(x,t) := r - R_{g(t)}(x)$, $x \in M$, $t \in [0,T)$, and r is defined in (2.2). Then the normalized Ricci flow becomes

$$\begin{aligned} &\frac{\partial}{\partial t} g(t) = f(x,t) g(t) \\ \Leftrightarrow\ &\frac{\partial}{\partial t} \ln g(t) = f(x,t) \\ \Longrightarrow\ &\ln g(t) = \ln g(0) + \int_0^T f(x,s)\, ds \\ \Longrightarrow\ &g(t) = \exp\left(\int_0^T f(x,s)\, ds \right) g_0. \end{aligned}$$

So, every metric that is coming from the solution $g(t)$ is conformally equivalent to our starting metric g_0. Hence, the constant curvature metric $\tilde{g}$ is also conformally equivalent to g_0.

2.4 Universal covers of Riemann surfaces

In section 3.2, the universal cover provides the link between the uniformization problem and the Uniformization Theorem. Given the importance of the universal cover, one is led to wonder if there is a simple way to determine what the universal cover of a Riemann surface is.

Specifically, let M be a Riemann surface and $\widetilde{M}$ be its universal covering space. Recall that $\widetilde{M}$ is a simply connected Riemann surface. Applying the Uniformization Theorem, we find that $\widetilde{M}$ is conformally equivalent to either the complex plane, the Riemann sphere, or the unit disk; but is there a simple way to know which one?

The answer is *yes*, but first we must impose some conditions. To start, we recall the Gauss-Bonnet theorem from [Lee2].

Theorem 2.4.1. *If (M, g) is a 2-dimensional closed Riemannian manifold, then*

$$\int_M K\, dA = 2\pi\chi(M), \tag{2.3}$$

where K is the Gaussian curvature of g and $\chi(M)$ is its Euler characteristic.

From the well-known classification theorem for surfaces, we know that any 2-dimensional, closed, orientable manifold M with genus k is homeomorphic to either the sphere ($k = 0$) or a connected sum of k-tori ($k \geq 1$). From this, we can easily compute the Euler characteristic of M by using $\chi(M) = 2 - 2k$.

So, to make use of these results, we suppose that our Riemann surface M is closed. For the sake of an example, suppose its genus is $k = 0$. Then $\chi(M) = 2$ which, from equation (2.3), implies that $K > 0$ for M. As a consequence, we also have $K > 0$ for the universal cover $\widetilde{M}$. Recall that $\widetilde{M}$ is conformally equivalent to either the complex plane, the Riemann sphere, or the unit disk.

Only the Riemann sphere has positive Gaussian curvature, thus $\widetilde{M}$ must be conformally equivalent to the Riemann sphere.

This argument can be easily repeated for the other cases. The results are summarized below.

Let M be a compact Riemann surface without boundary, k be its genus, and $\widetilde{M}$ be its universal covering space. Then:

(i) If $k = 0$, $\widetilde{M}$ is conformally equivalent to the Riemann sphere $\hat{\mathbb{C}}$ (or rather, S^2).

(ii) If $k = 1$, $\widetilde{M}$ is conformally equivalent to the complex plane $\mathbb{C}$ (or rather, $\mathbb{R}^2$)

(iii) If $k \geq 2$, $\widetilde{M}$ is conformally equivalent to the unit disk D_1 (or rather, $\mathbb{H}^2$).

CHAPTER 3
The Yamabe Problem

Naturally, one now seeks out a generalization of the Uniformization Theorem to higher dimensions. Doing so leads one to the Yamabe problem:

For a compact Riemannian manifold (M, g) *of dimension* $n \geq 3$*, is there a metric conformal to* g *that has constant scalar curvature?*

Ultimately, the answer to this question is *yes*. The first attempt to show that this is indeed the answer came from Yamabe in 1960, in the paper [Yam]. There was, however, an error in the paper which was later mended by Trudinger, though it came at the cost of having to introduce a condition on the manifold's Yamabe invariant $\lambda(M)$ (we define this later). Aubin then showed that Trudinger's condition can be stated in a simpler manner: if the n-dimensional manifold M satisfies $\lambda(M) < \lambda(S^n)$, then the problem can be solved. Aubin then showed that the condition is satisfied when $n \geq 6$ and M is not locally conformally flat. Then, in 1984, Schoen finished solving the problem by proving that the condition is satisfied for all other cases ($n = 3, 4$, or 5, or if M is locally conformally flat). In this chapter, we we follow [LeeP] and make use of [Aub] to explore some of these results.

Apart from being interesting on its own, the Yamabe problem introduces us to the conformal Laplacian (Yamabe operator) which, as we shall see in Chapter 4, is a conformally covariant differential operator.

3.1 Background

Definition 3.1.1. The *Hölder space* $C^{k,\alpha}(M)$, $0 < \alpha < 1$, is defined as

$$C^{k,\alpha}(M) = \{f \in C^k(M) : \|f\|_{C^{k,\alpha}} < \infty\},$$

where

$$\|f\|_{C^{k,\alpha}} = \|f\|_{C^k} + \sup_{x,y} \frac{|\nabla^k f(x) - \nabla^k f(y)|}{|x-y|^\alpha}.$$

The supremum is over all $x \neq y$ such that y is contained in a coordinate neighbourhood of x. The covariant derivative is denoted as ∇, and so when ∇^k acts on the function f, we get the k-tensor $\nabla^k f$; when we write $\nabla^k f(y)$, we mean the tensor at y obtained by parallel transport along the geodesic from x to y.

We now state the Sobolev embedding theorems for compact Riemannian manifolds.

Theorem 3.1.2. *Suppose M is a compact, n-dimensional Riemannian manifold (possibly with C^1 boundary).*

(i) If

$$\frac{1}{r} \geq \frac{1}{q} - \frac{k}{n}$$

then $W^{k,q}(M)$ is continuously embedded in $L^r(M)$.

(ii) (Rellich-Kondrakov Theorem) If strict inequality holds in (i), then the embedding $W^{k,q}(M) \hookrightarrow L^r(M)$ is compact.

(iii) Suppose $0 < \alpha < 1$, and

$$\frac{1}{q} \leq \frac{k-\alpha}{n}.$$

Then $W^{k,q}(M)$ is continuously embedded in $C^\alpha(M)$

From [BGV], we obtain a definition for what it means to be locally conformally flat.

Definition 3.1.3. A Riemannian manifold (M, g) is *locally conformally flat* if each point in M has a coordinate neighbourhood U which is conformal to Euclidean space $\mathbb{R}^n$. That is, there is a diffeomorphism $\Psi : V \subset \mathbb{R}^n \longrightarrow U$ such that the pullback $\Psi^* g$ is conformal to the standard Euclidean metric.

From [Can], we obtain a definition of the Laplace-Beltrami operator, which is a generalization of the usual Laplace operator for Riemannian manifolds.

Definition 3.1.4. Given a Riemannian manifold (M, g), the *Laplace-Beltrami operator* is defined as

$$\Delta_g : C^\infty(M) \longrightarrow C^\infty(M)$$
$$\Delta_g = -\mathrm{div}_g \circ \nabla_g.$$

In local coordinates, the operator takes the form,

$$\Delta_g = -\frac{1}{\sqrt{|\det g|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{|\det g|}\, g^{ij} \frac{\partial}{\partial x_j}\right). \tag{3.1}$$

Observe that this reduces to the usual Laplacian in Euclidean space, since the Euclidean metric has $|\det g| = 1$.

For this definition, the subscript of g has been used to emphasis a dependence on the metric. In what follows, this subscript is occasionally dropped, though the dependence on the metric should still be clear.

3.2 The Yamabe equation

Let (M, g) be a compact, connected Riemannian manifold of dimension $n \geq 3$. Let $\tilde{g}$ be conformal to g, which we can write as $\tilde{g} = e^{2f} g$ for some real-valued $f \in C^\infty(M)$. Let R and $\tilde{R}$ denote the scalar curvatures of g and $\tilde{g}$, respectively. The following transformation law is then satisfied:

$$\tilde{R} = e^{-2f}(R + 2(n-1)\Delta f - (n-1)(n-2)|\nabla f|^2),$$

where Δ is the Laplace-Beltrami operator and ∇f is the covariant derivative of f, both defined with respect to the metric g.

Before proceeding, make note of the following notation:

$$p = \frac{2n}{n-2}, \qquad a = 4\frac{n-1}{n-2}, \qquad \Box = a\Delta + R.$$

Now, we make the substitution $e^{2f} = \varphi^{p-2}$, where $\varphi \in C^\infty(M)$ and $\varphi > 0$. Then $\tilde{g} = \varphi^{p-2} g$ and

$$\tilde{R} = \varphi^{1-p}\left(4\frac{n-1}{n-2}\Delta\varphi + R\varphi\right). \tag{3.2}$$

So, we are led to the conclusion that $\tilde{g} = \varphi^{p-2} g$ has constant scalar curvature $\tilde{R} = \lambda$ if and only if φ satisfies

$$\Box\varphi = \lambda\varphi^{p-1}. \tag{3.3}$$

Equation (3.3) is referred to as the *Yamabe equation*.

So, solving the Yamabe problem is equivalent to solving (3.3), with the requirement that the solution φ be smooth and strictly positive.

3.3 The Yamabe invariant

Consider the functional

$$Q(g) = \frac{\int_M R\, dV_g}{\left(\int_M dV_g\right)^{(n-2)/n}}$$

where $dV_g = \sqrt{|\det g|}\, dx^1 \wedge \cdots \wedge dx^n$ is the volume form. When we evaluate this functional at $\tilde{g} = \varphi^{p-2} g$, we get what is known as the *Yamabe functional*:

$$Q(\tilde{g}) = \frac{\int_M \tilde{R}\, dV_{\tilde{g}}}{(\int_M dV_{\tilde{g}})^{2/p}} \quad \Leftrightarrow \quad Q_g(\varphi) = \frac{\int_M (a|\nabla\varphi|^2 + R\varphi^2) dV_g}{\|\varphi\|_p^2}, \tag{3.4}$$

where

$$\|\varphi\|_p = \left(\int_M |\varphi|^p dV_g \right)^{1/p}.$$

Yamabe noticed that (3.3) is the Euler-Lagrange equation for (3.4). To see this, start by setting

$$E(\varphi) := \int_M (a|\nabla\varphi|^2 + R\varphi^2) dV_g.$$

Then observe that

$$\begin{aligned}
\frac{d}{dt} E(\varphi + t\psi)\bigg|_{t=0} &= 2 \int_M (a\nabla\varphi \cdot \nabla\psi + R\varphi\psi)\, dV_g \\
&= 2 \int_M (-a \cdot \operatorname{div}(\nabla\varphi) + R\varphi)\psi\, dV_g \\
&= 2 \int_M (a\Delta\varphi + R\varphi)\psi\, dV_g,
\end{aligned}$$

where we used integration by parts to get the second line.

Further observe that

$$\begin{aligned}
\frac{d}{dt} \|\varphi + t\psi\|_p^{-2}\bigg|_{t=0} &= \left[-\frac{2}{p} \left(\int_M |\varphi + t\psi|^p \right)^{-\frac{2}{p}-1} \left(\int_M p|\varphi + t\psi|^{p-1}\psi\, dV_g \right) \right]\bigg|_{t=0} \\
&= -2 \left(\int_M |\varphi|^p \right)^{-\frac{2}{p}-1} \left(\int_M |\varphi|^{p-1}\psi\, dV_g \right) \\
&= -\frac{2\, \|\varphi\|_p^{-p} \int_M (|\varphi|^{p-1}\psi)\, dV_g}{\|\varphi\|_p^2}
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{d}{dt}Q_g(\varphi+t\psi)\bigg|_{t=0} &= \left[\left(\frac{d}{dt}E(\varphi+t\psi)\right)\|\varphi+t\psi\|_p^{-2} + E(\varphi+t\psi)\left(\frac{d}{dt}\|\varphi+t\psi\|_p^{-2}\right)\right]\bigg|_{t=0} \\
&= \frac{2}{\|\varphi\|_p^2}\int_M (a\Delta\varphi + R\varphi - \|\varphi\|_p^{-p} E(\varphi)\varphi^{p-1})\psi \, dV_g
\end{aligned}$$

From this, we see that φ satisfies the Yamabe equation (3.3) with $\lambda = E(\varphi)/\|\varphi\|_p^p$ if and only if it is a critical point of Q_g.

Let us now introduce the constant $\lambda(M)$, which is known as the *Yamabe invariant*:

$$\lambda(M) = \inf\{Q_g(\varphi) : \varphi \in C^\infty(M) \text{ and } \varphi > 0\}.$$

Proposition 3.3.1. *$\lambda(M)$ is an invariant of the conformal class $[g]$.*

Proof. We consider a conformal change of the metric g by $\tilde{g} = \varphi^{p-2}g$. We then have

$$dV_{\tilde{g}} = \varphi^{n(p-2)/2} dV_g = \varphi^p dV_g.$$

Now consider

$$Q_g(\varphi\psi) = \frac{\int_M (a|\nabla(\varphi\psi)|^2 + R\varphi^2\psi^2) dV_g}{\left(\int_M \varphi^p\psi^p \, dV_g\right)^{2/p}}.$$

Observe that

$$\int_M |\nabla(\varphi\psi)|^2 \, dV_g = \int_M (\varphi^2|\nabla\psi|^2 + \psi^2|\nabla\varphi|^2 + 2\varphi\psi\nabla\varphi\cdot\nabla\psi) \, dV_g. \tag{3.5}$$

Applying integration by parts to the second term on the right hand side of the above equation, we get

$$\begin{aligned}
\int_M \psi^2|\nabla\varphi|^2 \, dV_g &= \int_M (\psi^2\nabla\varphi)\cdot\nabla\varphi \, dV_g \\
&= -\int_M \varphi\cdot\operatorname{div}(\psi^2\nabla\varphi) \, dV_g \\
&= \int_M (\varphi\psi^2\Delta\varphi) \, dV_g - 2\int_M (\varphi\psi\nabla\psi\cdot\nabla\varphi) \, dV_g.
\end{aligned}$$

Inserting this back into (3.5) and then inserting that back into $Q_g(\varphi\psi)$, we get

$$Q_g(\varphi\psi) = \frac{\int_M (a\varphi^2|\nabla\psi|^2 dV_g) + \int_M (a\Delta\varphi + R\varphi)\varphi\psi^2 dV_g}{\left(\int_M \varphi^p\psi^p\, dV_g\right)^{2/p}}.$$

Then using (3.2) gives

$$Q_g(\varphi\psi) = \frac{\int_M (a\varphi^2|\nabla\psi|^2 dV_g) + \int_M \tilde{R}\varphi^p\psi^2 dV_g}{\left(\int_M \varphi^p\psi^p dV_g\right)^{2/p}}.$$

Now use the fact that

$$\varphi^2|\nabla\psi|^2\, dV_g = \varphi^{2-p}(\nabla\psi\cdot\nabla\psi)\, dV_{\tilde{g}} = \tilde{\nabla}\psi\cdot\tilde{\nabla}\psi\, dV_{\tilde{g}} = |\tilde{\nabla}\psi|^2\, dV_{\tilde{g}},$$

to arrive at

$$Q_g(\varphi\psi) = \frac{\int_M (a|\tilde{\nabla}\psi|^2 + \tilde{R}\psi^2) dV_{\tilde{g}}}{\left(\int_M \psi^p dV_{\tilde{g}}\right)^{2/p}} = Q_{\tilde{g}}(\psi).$$

This means that $\lambda(M)$, associated to the metric g, is equal to $\tilde{\lambda}(M)$, associated to the conformally equivalent metric $\tilde{g}$. Thus, $\lambda(M)$ is indeed an invariant of the conformal class $[g]$. □

The Yamabe invariant is crucial to solving the Yamabe problem. As we shall soon see, the way $\lambda(M)$ compares to $\lambda(S^n)$ is of particular importance, and so the next section is devoted to the Yamabe problem in the setting of the sphere.

3.4 The Yamabe problem on the sphere S^n

Let $\bar{g}$ denote the standard metric on the sphere S^n and recall that it is of constant scalar curvature.

Proposition 3.4.1. *There exists a positive, C^∞ function ψ on S^n satisfying $Q_{\bar{g}}(\psi) = \lambda(S^n)$.*

In other words, the above proposition is stating that $\lambda(S^n)$ is indeed attained by a metric g in the conformal class of $\bar{g}$. As a consequence, g has constant scalar curvature.

Now consider the following proposition, which essentially states that such a metric must be, up to a conformal diffeomorphism and a constant scale factor, the standard metric $\bar{g}$.

Proposition 3.4.2. *Suppose g is a metric on S^n that is conformal to the standard metric $\bar{g}$ and has constant scalar curvature. Then up to a constant scale factor, g is obtained from $\bar{g}$ by a conformal diffeomorphism of the sphere.*

The combination of the above two propositions provides us with the solution to the Yamabe problem on the sphere. This solution is stated in the following theorem:

Theorem 3.4.3. *The Yamabe functional* (3.4) *on $(S^n, \bar{g})$ is minimized by constant multiples of $\bar{g}$ and its images under conformal diffeomorphisms. These are the only metrics conformal to $\bar{g}$ that have constant scalar curvature.*

Lee and Parker go on in [LeeP] to show how the Yamabe problem on S^n is related to a problem concerning the Sobolev inequality; in doing so, they show that $\lambda(S^n) > 0$. They then proceed to prove the following lemma, which is due to Aubin.

Lemma 3.4.4. *Suppose M is a compact Riemannian manifold M of dimension $n \geq 3$. Then $\lambda(M) \leq \lambda(S^n)$.*

3.5 Minimizing the Yamabe functional

We now consider a general, compact Riemannian manifold M. We seek to minimize the Yamabe functional (3.4); one idea would be to construct a sequence of functions that causes the Yamabe functional to approach its infimum, and then seek out some subsequence which converges to an extremal function. But for reasons stated in [LeeP], this approach does not work.

Instead, Yamabe took a different approach and considered

$$Q^s(\varphi) = \frac{\int_M (a|\nabla\varphi|^2 + R\varphi^2) dV_g}{\|\varphi\|_s^2}$$

for $2 \le s \le p$. We then set $\lambda_s = \inf\{Q^s(\varphi) : \varphi \in C^\infty(M)\}$. Note that a function φ which minimizes the above functional and has $\|\varphi\|_s = 1$, satisfies

$$\Box\varphi = \lambda_s \varphi^{s-1}, \tag{3.6}$$

which is the Euler-Lagrange equation of Q^s. Equation (3.6) is known as the subcritical equation.

We now present a proposition which was proved in [Yam].

Proposition 3.5.1. *For $2 \le s < p$, there exists a smooth, positive solution φ_s to the subcritical equation* (3.6)*, for which $Q^s(\varphi_s) = \lambda_s$ and $\|\varphi_s\|_s = 1$.*

Proof. Let $\{u_i\} \subset C^\infty(M)$ be a sequence which minimizes the functional Q^s and satisfies $\|u_i\|_s = 1$. Observe that $Q^s(|u_i|) = Q^s(u_i)$, so we may replace u_i by $|u_i|$ and then assume that $u_i \ge 0$. Also observe that $\{u_i\}$ is bounded in the Sobolev space $H^1(M)$ (also denoted as $W^{1,2}(M)$):

$$\begin{aligned}
\|u_i\|_{H^1}^2 &= \int_M \left(|\nabla u_i|^2 + u_i^2\right) dV_g \\
&= \frac{1}{a} Q^s(u_i) + \int_M \left(1 - \frac{R}{a}\right) u_i^2 \, dV_g \\
&\le \frac{1}{a} Q^s(u_i) + C \|u_i\|_s^2 ,
\end{aligned}$$

where we have used Hölder's inequality to get the last line. Since $H^1(M)$ is a Hilbert space, we may use the well-known fact that a bounded sequence in a Hilbert space has a weakly convergent subsequence; thus, a subsequence of $\{u_i\}$ will weakly converge in $H^1(M)$.

Now we want to show that the embedding $H^1(M) \hookrightarrow L^s$ is compact. Observe that

$$\frac{1}{s} > \frac{1}{2} - \frac{1}{n} \quad \Leftrightarrow \quad s < \frac{2n}{n-2} = p$$

and since we assume that $s < p$, we may apply Theorem 3.1.2 to see that the embedding is indeed compact. Thus, the weakly convergent subsequence of $\{u_i\}$ will be mapped, via the compact embedding, to a sequence in L^s that strongly converges to a function $\varphi_s \in L^s$ (see [Con], Chapter 6, Proposition 3.3). Note that $\|\varphi_s\|_s = 1$.

Now, we can use Hölder's inequality to see that the L^2 norm is dominated by the L^s norm:

$$\|f\|_2^2 = \int_M (|f|^2 \cdot 1)\, dV_g \leq \left\| |f|^2 \right\|_{s/2} \|1\|_{s/(s-2)} = \|f\|_s^2 \operatorname{vol}_g(M)^{(s-2)/s}.$$

Thus, $\int_M Ru_i^2 \longrightarrow \int_M R\varphi_s^2$.

Weak convergence in H^1 implies that

$$\begin{aligned}\int_M |\nabla\varphi_s|^2\, dV_g &= \lim_{i\to\infty} \int_M \langle \nabla u_i, \nabla\varphi_s\rangle\, dV_g \\ &\leq \limsup_{i\to\infty} \left(\int_M |\nabla u_i|^2 dV_g\right)^{1/2} \left(\int_M |\nabla\varphi_s|^2 dV_g\right)^{1/2}.\end{aligned}$$

So, we arrive at $Q^s(\varphi_s) \leq \lim_{i\to\infty} Q^s(u_i) = \lambda_s$. But recall that λ_s was defined as the infimum of Q^s, and so we must have $Q^s(\varphi_s) = \lambda_s$. So, φ_s is a weak solution of the subcritical equation (3.6). By a regularity theorem in [LeeP] (labelled as Theorem 4.1), φ_s is C^∞ and positive. $\square$

We now turn our attention to the limit of φ_s as $s \longrightarrow p$, which leads us to the error in Yamabe's proof. Yamabe made the claim that the functions $\{\varphi_s\}$ are uniformly bounded as $s \longrightarrow p$; but in general, this is false, as in the case of the sphere. However, Proposition 3.5.3, which collects the work of Trudinger and Aubin, provides a uniform L^r bound that does allow the problem to solved, provided the condition $\lambda(M) < \lambda(S^n)$ holds. Before giving this proposition, we state the following lemma which describes the behaviour of λ_s.

Lemma 3.5.2. *If $\int_M dV_g = 1$, then $|\lambda_s|$ is nonincreasing as a function of $s \in [2, p]$ (i.e. if $s \leq s'$ then $|\lambda_{s'}| \leq |\lambda_s|$); furthermore, if $\lambda(M) \geq 0$, then λ_s is continuous from the left.*

As a result of this lemma, it is assumed, from this point on, that g is such that $\int_M dV_g = 1$; this can always be achieved by multiplying g by an appropriate constant.

Proposition 3.5.3. *Suppose $\lambda(M) < \lambda(S^n)$ and let $\{\varphi_s\}$ be the collection of functions given in Proposition (3.5.1). Then there are constants $s_0 < p$, $r > p$, and $C > 0$ such that $\|\varphi_s\|_r \leq C$ for all $s \geq s_0$.*

We can now state the following important theorem, which essentially states that the Yamabe problem can be solved if $\lambda(M) < \lambda(S^n)$.

Theorem 3.5.4. *Suppose $\lambda(M) < \lambda(S^n)$ and let $\{\varphi_s\}$ be the collection of functions given in Proposition (3.5.1). As $s \longrightarrow p$, a subsequence converges uniformly to a positive function $\varphi \in C^\infty(M)$ which satisfies*

$$Q_g(\varphi) = \lambda(M), \qquad \Box\varphi = \lambda(M)\varphi^{p-1}.$$

Thus, the metric $\tilde{g} = \varphi^{p-2}g$ has constant scalar curvature $\lambda(M)$.

Proof. From Proposition 3.5.3, we know that the functions $\{\varphi_s\}$ are uniformly bounded in L^r. The regularity theorem in [LeeP] may then be used to show that the functions are also uniformly bounded in $C^{2,\alpha}(M)$. Then an application of the Arzelà-Ascoli theorem gives us a subsequence which converges in $C^2(M)$ to a function $\varphi \in C^2(M)$. Thus, φ satisfies the Yamabe equation

$$\Box\varphi = \lambda\varphi^{p-1},$$

and $Q_g(\varphi) = \lambda$ where $\lambda = \lim_{s\to p}\lambda_s$. If $\lambda(M) \geq 0$, Lemma 3.5.2 says that $\lambda = \lambda(M)$. If $\lambda(M) < 0$, $|\lambda_s|$ nonincreasing implies that λ_s is increasing, which implies $\lambda \leq \lambda(M)$; but recall that $\lambda(M)$ is the infimum of Q_g and so we must have $\lambda = \lambda(M)$ once again. Another application of the regularity theorem in [LeeP] shows that $\varphi \in C^\infty(M)$ and that $\varphi > 0$. $\Box$

3.6 The condition on the Yamabe invariant $\lambda(M)$

Given Theorem 3.5.4, solving the Yamabe problem has been reduced to showing that $\lambda(M) < \lambda(S^n)$. Recall that $\lambda(S^n) > 0$; so, if $\lambda(M) < 0$, we are already done. This leaves us with the case of $\lambda(M) > 0$. For the sake of brevity, only an outline of how this final case is handled will be given here, but all details may be found in [LeeP].

The primary idea for the $\lambda(M) > 0$ case is to find a function ψ such that $Q_g(\psi) < \lambda(S^n)$. The first major step in doing this is to define generalized stereographic projections. Let ω denote the volume of the unit sphere. For $P \in M$, let Γ_P denote the Green function for $\Box$; that is, $\Box\Gamma_P = \delta_P$. Note that at each $P \in M$, Γ_P exists and is strictly positive (see Lemma 6.1 in [LeeP]).

Definition 3.6.1. Suppose (M, g) is a compact Riemannian manifold with $\lambda(M) > 0$. For $P \in M$ define the metric $\hat{g} = G^{p-2}g$ on $\hat{M} = M - \{P\}$, where

$$G = (n-2)\omega a \Gamma_P.$$

The manifold $(\hat{M}, \hat{g})$ together with the natural map $\sigma : M - \{P\} \longrightarrow \hat{M}$ is called the *stereographic projection* of M from P.

Now, on the manifold $\hat{M}$, one may define a function φ with a positive parameter α such that as $\alpha \to \infty$, $Q_{\hat{g}}(\varphi)$ becomes close to $\lambda(S^n)$. To know the behaviour of $Q_{\hat{g}}(\varphi)$ as $\alpha \to \infty$, one examines the average behaviour of $\hat{g}$ on $\hat{M}$ over large spheres. This average behaviour of $\hat{g}$ is measured by a constant μ which is called the *distortion coefficient* of $\hat{g}$. So, it is really μ that determines the behaviour of $Q_{\hat{g}}(\varphi)$ as $\alpha \to \infty$.

It turns out that if $\mu > 0$, one can obtain $Q_{\hat{g}}(\varphi) < \lambda(S^n)$ (see Proposition 7.1 in [LeeP]). Then, using the fact that

$$\lambda(M) = \inf_{\psi \in C_0^\infty(\hat{M})} Q_{\hat{g}}(\psi),$$

one approximates φ by a function $\psi \in C_0^\infty(\hat{M})$ to obtain $\lambda(M) < \lambda(S^n)$.

Theorem 3.6.2. *If (M, g) is a compact Riemannian manifold of dimenson $n \geq 3$ with $\lambda(M) > 0$, then $\lambda(M) < \lambda(S^n)$ if there is a generalized stereographic projection $\hat{M}$ of M with strictly positive distortion coefficient μ.*

Showing the positivity of μ is split into two cases. The first case concerns the scenario where $n \geq 6$ and M is not locally conformally flat. Once it is shown that $\mu > 0$ in this case, we get the following theorem, which was proved by Aubin.

Theorem 3.6.3. *If M has dimension $n \geq 6$ and is not locally conformally flat, then $\lambda(M) < \lambda(S^n)$.*

The second case concerns $n < 6$ or if M is locally conformally flat. Once it is shown that $\mu > 0$ here as well, we get the following theorem, which was proved by Schoen.

Theorem 3.6.4. *If M has dimension 3, 4, or 5, or if M is locally conformally flat, then $\lambda(M) < \lambda(S^n)$ unless M is conformal to the standard sphere.*

CHAPTER 4
Conformally Covariant Operators

In this chapter, we introduce conformally covariant differential operators and show that the conformal Laplacian (also known as the Yamabe operator) is such an operator.

From what is stated in [JNSS], [Ros], and [CGJP1], we can make the following definition for conformally covariant differential operators.

Definition 4.0.1. Let (M, g) be a Riemannian manifold, let P_g be a differential operator, and let $[g] = \{e^{2f}g : f \in C^\infty(M)\}$ be the conformal class for g. If there exists $\omega, \omega' \in \mathbb{R}$ such that for any $\tilde{g} \in [g]$, the differential operator P_g transforms according to

$$P_{\tilde{g}} = e^{-\omega' f} P_g e^{\omega f}, \tag{4.1}$$

then the operator is called a *conformally covariant differential operator* of biweight (ω, ω').

There are a few well-known conformally covariant operators: the conformal Laplacian, the Paneitz operator, and the Dirac operator. The class of conformally covariant operators known as the GJMS operators—which the conformal Laplacian and the Paneitz operator belong to—will be our primary concern. The Dirac operator is omitted here but a treatment of it may be found in [Hit].

4.1 GJMS operators

For this section, we follow some of [CGJP1].

Let (M, g) be a Riemannian manifold of dimension $n \geq 3$. The *GJMS operators* are defined through the following proposition.

Proposition 4.1.1. *For $k = 1, \ldots, \frac{n}{2}$ when n is even, and for all non-negative integers k when n is odd, there is a conformally invariant operator $P_k = P_{k,g}$ of biweight $(\frac{n}{2} - k, \frac{n}{2} + k)$ such that*

$$P_{k,g} = \Delta_g^{(k)} + \text{ lower order terms.} \tag{4.2}$$

4.1.1 The conformal Laplacian

In the case of $k = 1$, (4.2) gives us the *conformal Laplacian*,

$$P_{1,g} = \Delta_g + \frac{n-2}{4(n-1)} R_g, \tag{4.3}$$

where R_g is the scalar curvature.

Remark 4.1.2. Note that in the previous chapter, the Yamabe equation (3.3) was defined with the operator

$$\Box = 4\frac{n-1}{n-2}\Delta + R,$$

which is just $P_{1,g}$ multiplied by $4(n-1)/(n-2)$. In fact, Lee and Parker in [LeeP] refer to the operator $\Box$ as the conformal Laplacian.

Let's show that the operator (4.3) is indeed conformally covariant, with biweight $(\frac{n}{2} - 1, \frac{n}{2} + 1)$.

We start by considering a conformal change of $\tilde{g} = e^{2\varphi} g$. From (3.1), it is not too difficult to see that the conformal change results in the following transformation law:

$$\Delta_{\tilde{g}} f = e^{-2\varphi}(\Delta_g f - (n-2)\nabla^k\varphi\,\nabla_k f).$$

Now consider the action of Δ_g on $e^{a\varphi}f$, where a is some real number.

$$\begin{aligned}
\Delta_g(e^{a\varphi}f) &= -\nabla^k\nabla_k(e^{a\varphi}f) \\
&= -\nabla^k[e^{a\varphi}(\nabla_k f + af\nabla_k\varphi)] \\
&= -ae^{a\varphi}\nabla^k\varphi(\nabla_k f + af\nabla_k\varphi) + e^{a\varphi}(\Delta_g f + a(f\Delta_g\varphi - \nabla^k f\nabla_k\varphi)) \\
&= e^{a\varphi}[-2a\nabla^k\varphi\nabla_k f - a^2 f\nabla^k\varphi\nabla_k\varphi + \Delta_g f + af\Delta_g\varphi].
\end{aligned}$$

So,

$$\begin{aligned}
\Delta_{\tilde{g}}(e^{a\varphi}f) &= e^{-2\varphi}[\Delta_g(e^{a\varphi}f) - (n-2)\nabla^k\varphi\,\nabla_k(e^{a\varphi}f)] \\
&= e^{-2\varphi}[\Delta_g(e^{a\varphi}f) - e^{a\varphi}(n-2)\nabla^k\varphi\,(\nabla_k f + af\nabla_k\varphi)] \\
&= e^{(a-2)\varphi}[-2a\nabla^k\varphi\nabla_k f - a^2 f\nabla^k\varphi\nabla_k\varphi + \Delta_g f + af\Delta_g\varphi \\
&\quad - (n-2)\nabla^k\varphi\,(\nabla_k f + af\nabla_k\varphi)] \\
&= e^{(a-2)\varphi}[\Delta_g f + af\Delta_g\varphi - (n+2a-2)\nabla^k\varphi\nabla_k f \\
&\quad - (n+a-2)af\nabla^k\varphi\nabla_k\varphi].
\end{aligned}$$

If we let $a = -(\frac{n}{2}-1)$, then the above becomes

$$\Delta_{\tilde{g}}(e^{-(\frac{n}{2}-1)\varphi}f) = e^{-(\frac{n}{2}+1)\varphi}\left(\Delta_g - \frac{n-2}{4}\big(2\Delta_g\varphi - (n-2)\nabla^k\varphi\nabla_k\varphi\big)\right)f.$$

Now recall that under such a conformal change, the scalar curvature transforms according to:

$$R_{\tilde{g}} = e^{-2\varphi}(R_g + 2(n-1)\Delta_g\varphi - (n-1)(n-2)\nabla^k\varphi\nabla_k\varphi).$$

So,

$$R_{\tilde{g}}(e^{-(\frac{n}{2}-1)\varphi}f) = e^{-(\frac{n}{2}+1)\varphi}\bigg(R_g + (n-1)(2\Delta_g\varphi - (n-2)\nabla^k\varphi\nabla_k\varphi\bigg)f.$$

Clearly then, we have

$$e^{(\frac{n}{2}+1)\varphi}P_{1,\tilde{g}}\, e^{-(\frac{n}{2}-1)\varphi}f = e^{(\frac{n}{2}+1)\varphi}\bigg(\Delta_{\tilde{g}} + \frac{n-2}{4(n-1)}R_{\tilde{g}}\bigg)e^{-(\frac{n}{2}-1)\varphi}f = P_{1,g}f,$$

which is equivalent to

$$P_{1,\tilde{g}} = e^{-(\frac{n}{2}+1)\varphi}P_{1,g}\, e^{(\frac{n}{2}-1)\varphi}.$$

4.1.2 The Paneitz operator

In the case of $k = 2$, (4.2) gives us the *Paneitz operator*,

$$P_{2,g} = \Delta_g^2 + \delta V d + \frac{n-4}{2}\bigg(\frac{1}{2(n-1)}\Delta_g R_g + \frac{n}{8(n-1)^2}R_g^2 - 2|S|^2\bigg), \quad (4.4)$$

where

$$S = \frac{1}{n-2}\bigg(\mathrm{Ric}_g - \frac{R_g}{2(n-1)}g\bigg)$$

is the Schouten-Weyl tensor (see section 7.2 for more on this tensor), and V is the tensor

$$V = \frac{n-2}{2(n-1)}R_g g - 4S$$

which acts on 1-forms. The Paneitz operator is of biweight $(\frac{n}{2}-2, \frac{n}{2}+2)$.

CHAPTER 5
Conformal Invariants

Conformal invariants are things which are unaffected by conformal changes of the metric. In this chapter, we closely follow [CGJP1] to discuss several conformal invariants that arise from conformally covariant operators.

5.1 Nodal sets and nodal domains

The chief concern of this section is nodal sets—that is, zero loci. Closely related is the concept of nodal domains, which are the connected components of complements of nodal sets.

Let (M, g) be a Riemannian manifold of dimenson $n \geq 3$. Let P_g be a conformally covariant operator with biweight (ω, ω'). Then from the transformation law (4.1), we see that under a conformal change $\tilde{g} = e^{2f} g$, the kernel transforms according to

$$\ker P_{\tilde{g}} = e^{-\omega f} \ker P_g \tag{5.1}$$

In the case of a GJMS operator $P_{k,g}$, the dimension of its kernel is an invariant of the conformal class $[g]$ (see section 3 of [CGJP1] for the technical details). For the Dirac operator, see [Hit] (Proposition 1.3) for the proof that the dimension of its kernel is conformally invariant. In general, $\dim \ker P_g$ is conformally invariant if the conformally covariant operator P_g admits an endomorphism on some function space—from this point onwards, we assume this to be true.

Now suppose we have a conformally covariant operator P_g of biweight (ω, ω'), and a function $u_g \in C^\infty(M)$ with $x_0 \in M$ such that $u_g(x_0) = 0$. Further suppose that u_g belongs to a collection of functions $\{u_{\tilde{g}}\}_{\tilde{g} \in [g]} \subset C^\infty(M)$ that is parametrized by the conformal class $[g]$ in the following way:

$$u_{e^{2f}g}(x) = e^{-\omega f(x)} u_g(x), \qquad \forall f \in C^\infty(M, \mathbb{R}). \tag{5.2}$$

This means that for any $\tilde{g} \in [g]$, we have

$$u_{\tilde{g}}(x_0) = e^{-\omega f(x_0)} u_g(x_0) = 0.$$

Similarly, if $x_0 \in M$ is such that $u_{\tilde{g}}(x_0) = 0$, then we must have $u_g(x_0) = 0$ as well. Thus, we see that $u_{\tilde{g}}^{-1}(0) = u_g^{-1}(0)$, which means that the nodal set $u_g^{-1}(0)$ is an invariant of the conformal class $[g]$.

Observe that if we take $u_g \in \ker P_g$, then (5.1) will give us (5.2). Combining this nodal set result with the conformal invariance of $\dim \ker P_g$, we obtain the following proposition, which is stated in the context of GJMS operators.

Proposition 5.1.1. *Let $k \in \mathbb{N}$ and assume that $k \leq \frac{n}{2}$ if n is even.*

(i) If $\dim \ker P_{k,g} \geq 1$, then the nodal sets and nodal domains of any nonzero null-eigenfunction of $P_{k,g}$ give rise to invariants of the conformal class $[g]$.

(ii) If $\dim \ker P_{k,g} \geq 2$, then non-empty intersections of nodal sets of null-eigenfunctions of $P_{k,g}$, and their complements, are invariants of the conformal class $[g]$.

To make another observation, let us suppose that P_g is a conformally covariant operator with biweight $(0, \omega')$. Then from (5.1), we have

$$\ker P_{\tilde{g}} = \ker P_g$$

for any $\tilde{g} \in [g]$. So, if we take $u_g \in \ker P_g$, we see that all of its level sets

$$\{x \in M : u_g(x) = \lambda\}, \quad \text{where } \lambda \in \mathbb{C},$$

are independent of the metric that represents the conformal class. Thus, the level sets of a non-constant $u_g \in \ker P_g$ are invariants of $[g]$. The following proposition states this result in the context of GJMS operators.

Proposition 5.1.2. *Suppose n is even. If $\dim\ker P_{\frac{n}{2}} \geq 2$, then the level sets of any non-constant null-eigenfunction of $P_{\frac{n}{2}}$ are invariants of the conformal class $[g]$.*

For the final result of this section, we make use of [CGJP2]. Suppose $\dim\ker P_{k,g} = s \geq 1$ and let $\{u_{1,g}, \ldots, u_{s,g}\}$ serve as a basis for $\ker P_{k,g}$. We set $\mathcal{N} := \cap_{1\leq j\leq s} u_{j,g}^{-1}(0)$ and define the map $\Phi : M \setminus \mathcal{N} \longrightarrow \mathbb{RP}^{s-1}$ by

$$\Phi(x) := [u_{1,g}(x), \ldots, u_{s,g}(x)], \qquad \forall x \in M \setminus \mathcal{N}.$$

Recalling equation (5.1), we see that each $u_{j,g}$ satisfies equation (5.2). From this, we observe that for $x \in M \setminus \mathcal{N}$, the s-tuple $\{u_{1,g}, \ldots, u_{s,g}\}$ depends on g only up to positive scaling. This means that the projective vector $[u_{1,g}(x), \ldots, u_{s,g}(x)] \in \mathbb{RP}^{s-1}$ is independent of the metric chosen to represent the conformal class $[g]$. Furthermore, it means that for $\tilde{g} \in [g]$, we have

$$\frac{u_{i,\tilde{g}}(x)}{u_{j,\tilde{g}}(x)} = \frac{e^{-\omega f(x)} u_{i,g}(x)}{e^{-\omega f(x)} u_{j,g}(x)} = \frac{u_{i,g}(x)}{u_{j,g}(x)};$$

hence, the u_j-s may be used as conformally invariant projective coordinates. So, we have obtained the following proposition.

Proposition 5.1.3. *The map Φ, as defined above, is an invariant of the conformal class $[g]$.*

5.2 Negative eigenvalues

Let (M, g) be a compact Riemannian manifold of dimension $n \geq 3$ and let $k \in \mathbb{N}$ (and assume that $k \leq \frac{n}{2}$ when n is even). Let $\mathcal{G}$ be the set of Riemannian metrics on M.

As explained in [CGJP1], the spectrum of the GJMS operator $P_{k,g}$ consists of a sequence of real eigenvalues converging to ∞. Thus, the eigenvalues can be ordered as a non-decreasing sequence

$$\lambda_1(P_{k,g}) \leq \lambda_2(P_{k,g}) \leq \ldots,$$

where each eigenvalue is repeated according to multiplicity.

We now state a technical lemma which will be needed for the theorem that follows.

Lemma 5.2.1. *For every $j \in \mathbb{N}$, the function $g \longrightarrow \lambda_j(P_{k,g})$ is continuous on $\mathcal{G}$.*

Now, for a metric $g \in \mathcal{G}$, we would like to keep track of the number of negative eigenvalues that belong to the operator $P_{k,g}$. To do this, we define

$$\nu_k(g) := \#\{j \in \mathbb{N} : \lambda_j(P_{k,g}) < 0\}.$$

Additionally, we would like to keep track of the metrics in $\mathcal{G}$ that make $P_{k,g}$ have at least m negative eigenvalues, where $m \in \mathbb{N}$. To do this, we define

$$\mathcal{G}_{k,m} := \{g \in \mathcal{G} : P_{k,g} \text{ has at least } m \text{ negative eigenvalues}\}.$$

Observe that $\mathcal{G}_{k,m} = \{g \in \mathcal{G} : \nu_k(g) \geq m\} = \{g \in \mathcal{G} : \lambda_m(P_{k,g}) < 0\}$. This fact, combined with Lemma 5.2.1, means that $\mathcal{G}_{k,m}$ is an open subset of $\mathcal{G}$.

Theorem 5.2.2. *Let $g \in \mathcal{G}$. Then $\nu_k(g)$ is an invariant of the conformal class $[g]$. Furthermore, for the operator $P_{k,g}$, the sign of its first eigenvalue $\lambda_1(P_{k,g})$ is also an invariant of the conformal class $[g]$.*

Proof. Let $g \in \mathcal{G}$ and, for convenience, we set $m = \nu_k(g)$ and $s = \dim \ker P_{k,g}$. Then observe that $\lambda_j(P_{k,g}) < 0$ for $j \leq m$, $\lambda_j(P_{k,g}) = 0$ for $j = m+1, \ldots, m+s$, and $\lambda_j(P_{k,g}) > 0$ for $j \geq m+s+1$. Now let δ be a positive, real number which satisfies

$$\delta < \min\{\, |\lambda_m(P_{k,g})|\, ,\, \lambda_{m+s+1}(P_{k,g})\}.$$

Consider an arbitrary $\tilde{g} \in [g]$. From Lemma 5.2.1, we know that if $\tilde{g}$ is close enough to g, then we can have $\lambda_m(P_{k,\tilde{g}}) < -\delta$ and $\lambda_{m+s+1}(P_{k,\tilde{g}}) > \delta$. So, the only eigenvalues of $P_{k,\tilde{g}}$ that are contained in the interval $[-\delta, \delta]$ are $\lambda_{m+1}(P_{k,\tilde{g}}), \ldots, \lambda_{m+s}(P_{k,\tilde{g}})$.

From the previous section, we know that the dimension of $\ker P_{k,g}$ is an invariant of the conformal class $[g]$, meaning we have $\dim \ker P_{k,\tilde{g}} = s$ which in turn means that of all the $\lambda_j(P_{k,\tilde{g}})$, there are precisely s of them which are equal to 0. But recall that the s eigenvalues $\lambda_{m+1}(P_{k,\tilde{g}}), \ldots, \lambda_{m+s}(P_{k,\tilde{g}})$ are the only eigenvalues contained in the interval $[-\delta, \delta]$. For both of these facts to be simultaneously true, we must have $\lambda_{m+1}(P_{k,\tilde{g}}) = \cdots = \lambda_{m+s}(P_{k,\tilde{g}}) = 0$. From this, we observe that $\lambda_m(P_{k,\tilde{g}}) < 0$ and hence arrive at $\nu_k(\tilde{g}) = m$.

So, what we have shown is that when the map $g \longrightarrow \nu_k(g)$ is restricted to the conformal class $[g]$, it is locally constant. Now since $C^\infty(M, \mathbb{R})$ is path connected and $[g]$ is the range of this space under the continuous map $f \longrightarrow e^{2f} g$, we know that $[g]$ is path connected, and hence a connected subset of $\mathcal{G}$. Then, using the well-known fact that a locally constant function on a connected set is constant on the set, we can now conclude that $\nu_k(g)$ is constant for the entire conformal class $[g]$.

Now we examine the conformal invariance of the sign of the first eigenvalue of $P_{k,g}$. Observe the following:

(i) $\lambda_1(P_{k,g}) < 0$ if and only if $\nu_k(g) \geq 1$.

(ii) $\lambda_1(P_{k,g}) = 0$ if and only if $\nu_k(g) = 0$ and $\dim \ker P_{k,g} \geq 1$.

(iii) $\lambda_1(P_{k,g}) > 0$ if and only if $\dim \ker P_{k,g} = \nu_k(g) = 0$.

Combining these observations with the conformal invariance of both $\dim \ker P_{k,g}$ and $\nu_k(g)$, we see that the sign of $\lambda_1(P_{k,g})$ is a conformal invariant. $\square$

5.3 Manifolds with boundary

In this section we briefly remark upon the case of manifolds with boundary. Let M be an n-dimensional ($n \geq 3$) manifold with smooth boundary ∂M. Let P_g be a conformally covariant operator on $(M, \partial M)$ whose kernel transforms according to (5.1), and let g be a Riemannian metric such that $\dim \ker P_g \geq 1$. Given all of this, some of the previously discussed results will continue to hold. Specifically, the following statements are true (see [CJKS]):

(i) $k = \dim \ker P_g$ is an invariant of $[g]$.

(ii) The nodal sets and nodal domains of nonzero null-eigenfunctions $u \in \ker P_g$ are invariants of $[g]$.

(iii) If $k \geq 2$, then non-empty intersections of nodal sets of null-eigenfunctions and their complements are invariants of $[g]$

(iv) The number of negative eigenvalues of P_g is an invariant of $[g]$.

(v) Let $k \geq 2$, let $\{u_1, \ldots, u_k\}$ be a basis of $\ker P_g$, and let $\widetilde{M} = M \setminus (\cap_{i=1}^k u_i^{-1}(0))$. Define the map $\Phi_g : \widetilde{M} \longrightarrow \mathbb{RP}^{k-1}$ by $\Phi_g(x) = [u_1(x), \ldots, u_k(x)]$. Then the orbit of $\Phi_g(\widetilde{M})$ under the action of $GL_k(\mathbb{R})$ is conformally invariant.

5.4 Spaces of conformal structures

In [SS], we find the usual definitions for Teichmüller spaces and moduli spaces. Let M be a compact, oriented, C^∞, 2-dimensional manifold, and let $\mathcal{M}$ be the set of complex structures of M which agree with the orientation and the C^∞-structure. Let $\mathcal{D}$ be the group of diffeomorphic self-mappings of M, let $\mathcal{D}_0$ be the group consisting of all $f \in \mathcal{D}$ such that f is homotopic to the identity, and let $\mathcal{D}_+$ be the group of orientation preserving diffeomorphic self-mappings of M. With all of this in mind, we can now state the definitions.

Definition 5.4.1. The *Teichmüller space* of the surface M is defined by the quotient

$$T(M) = \mathcal{M}/\mathcal{D}_0.$$

Definition 5.4.2. The *moduli space* of the surface M is defined by the quotient

$$R(M) = \mathcal{M}/\mathcal{D}_+.$$

Observe that since we are dealing with surfaces, we could have defined $T(M)$ and $R(M)$ in terms of conformal classes—simply replace $\mathcal{M}$ by $\mathcal{G}/\mathcal{P}$, where $\mathcal{G}$ is the space of all Riemannian metrics on M and $\mathcal{P}$ is the group of conformal transformations.

If we pursue this approach with conformal classes and now let M be a compact, orientable, n-dimensional Riemannian manifold, we can make the following definitions.

Definition 5.4.3. For the manifold M, the *Teichmüller space of conformal structures* is defined by the quotient

$$\mathcal{T}(M) = \frac{\mathcal{G}/\mathcal{P}}{\mathcal{D}_0}.$$

Definition 5.4.4. For the manifold M, the *Riemannian moduli space of conformal structures* is defined by the quotient

$$\mathcal{R}(M) = \frac{\mathcal{G}/\mathcal{P}}{\mathcal{D}_+}.$$

Obviously, in dimension $n = 2$, we have $\mathcal{T}(M) = T(M)$ and $\mathcal{R}(M) = R(M)$.

A brief observation made in [CGJP1] provides us with one reason why the spaces $\mathcal{T}(M)$ and $\mathcal{R}(M)$ are of interest; namely, that the conformal invariants discussed earlier (nodal sets, negative eigenvalues, etc) can be thought of as functions on either $\mathcal{T}(M)$ or $\mathcal{R}(M)$.

The following theorem from [GHJL] provides us with another reason to be interested in $\mathcal{T}(M)$.

Theorem 5.4.5. *Let M be a closed, n-dimensional manifold. Then for a generic smooth metric g on M, zero is not an eigenvalue of the conformal Laplacian $P_{1,g}$.*

Let us clarify the meaning of this theorem and hence show the connection with $\mathcal{T}(M)$. First, observe that if g is a metric such that $P_{1,g}$ has zero as an eigenvalue, then for any $\tilde{g} \in [g]$, $P_{1,\tilde{g}}$ also has zero as an eigenvalue—by the transformation equation 4.1. So, we must change conformal classes if we hope to find metrics for which $P_{1,g}$ does not have zero as an eigenvalue; this gives us the first indication that working with $\mathcal{T}(M)$ could be useful.

Continuing in this direction, let $\mathcal{G}_0$ be the set of all metrics on M such that for $g \in \mathcal{G}_0$, zero is an eigenvalue of $P_{1,g}$ with a multiplicity of at least one. Then let

$$\mathcal{T}_0(M) = \frac{\mathcal{G}_0/\mathcal{P}}{\mathcal{D}_0}.$$

So, the true meaning of Theorem 5.4.5 is the following theorem, which is proved in [GHJL].

Theorem 5.4.6. *The complement $\mathcal{T}_0^c$ of the set $\mathcal{T}_0(M)$ in $\mathcal{T}(M)$ is open and dense in $\mathcal{T}(M)$.*

CHAPTER 6
Products of Surfaces and Few Negative Eigenvalues

In this chapter, we consider the conformal Laplacian P_g on a Riemannian manifold (M, g), and seek to understand its smallest number of negative eigenvalues. We restrict ourselves to cases where M is a product of two or more Riemann surfaces.

6.1 Background

Using [Lee2], we start by describing the product of two arbitrary Riemannian manfiolds (M_1, g_1) and (M_2, g_2). The natural Riemannian metric for the product manifold $M_1 \times M_2$ is given by the *product metric* $g = g_1 \oplus g_2$. The product metric is defined by

$$g_{(p_1,p_2)}\big((v_1, v_2), (w_1, w_2)\big) = g_1\big|_{p_1}(v_1, w_1) + g_2\big|_{p_2}(v_2, w_2)$$

where $(v_1, v_2), (w_1, w_2) \in T_{p_1}(M_1) \oplus T_{p_2}(M_2)$, which can be identified with $T_{(p_1,p_2)}(M_1 \times M_2)$. With local coordinates $(x_1, \ldots, x_n)$ for M_1 and $(x_{n+1}, \ldots, x_{n+m})$, we have coordinates $(x_1, \ldots, x_{n+m})$ for $M_1 \times M_2$. Locally, the product metric is then given by $g = g_{ij}\, dx^i dx^j$, where (g_{ij}) is the matrix

$$(g_{ij}) = \begin{pmatrix} (g_1)_{ab} & 0 \\ 0 & (g_2)_{cd} \end{pmatrix}$$

. The indices a, b run from 1 to n, and the indices c, d run from $n+1$ to $n+m$.

Let us now calculate the scalar curvature of this product metric. We start with the well-known fact that the $(1, 3)$-Riemann curvature tensor in this case is given by

$$\mathcal{R}(X,Y)Z = (\mathcal{R}_1(X_1,Y_1)Z_1, \mathcal{R}_2(X_2,Y_2)Z_2),$$

where $X = (X_1, X_2)$, $X_i \in \mathfrak{X}(M_i)$, and likewise for Y and Z. The $(0,4)$-Riemann curvature tensor is then

$$\begin{aligned} Rm(X,Y,Z,W) &= \langle \mathcal{R}(X,Y)Z, W\rangle_g \\ &= \langle \mathcal{R}_1(X_1,Y_1)Z_1, W_1\rangle_{g_1} + \langle \mathcal{R}_2(X_2,Y_2)Z_2, W_2\rangle_{g_2} \\ &= Rm_1(X_1,Y_1,Z_1,W_1) + Rm_2(X_2,Y_2,Z_2,W_2). \end{aligned}$$

The components of the Ricci curvature $(\mathrm{Ric})_{ij} = g^{kl}(Rm)_{kijl}$ are then given by

$$(\mathrm{Ric})_{ij} = \begin{pmatrix} (\mathrm{Ric}_1)_{ab} & 0 \\ 0 & (\mathrm{Ric}_2)_{cd} \end{pmatrix}.$$

So, the scalar curvature is

$$\begin{aligned} R &= g^{ij}(\mathrm{Ric})_{ij} \\ &= g_1^{ab}(\mathrm{Ric}_1)_{ab} + g_2^{cd}(\mathrm{Ric}_2)_{cd} \\ &= R_1 + R_2. \end{aligned} \tag{6.1}$$

6.2 Product of k Riemann surfaces

We denote by $M(\gamma_1, \gamma_2, \ldots, \gamma_k)$ the $2k$-dimensional Riemannian manifold of the form $M = S_1 \times S_2 \times \ldots \times S_k$, where S_j are orientable Riemann surfaces of genus $\gamma_j \geq 2$. We put a hyperbolic metric g_j on S_j, and equip M with the metric $g = g_1 \oplus g_2 \oplus \ldots \oplus g_k$.

Each hyperbolic metric g_j is normalized so that its sectional curvature is -1. The scalar curvature is *twice* the sectional curvature and so $R_j = -2$. By applying equation (6.1) $k-1$ times, we find that the scalar curvature of the metric g is $R = -2k$. So, the conformal Laplacian of g takes the form

$$\begin{aligned} P_g &= \Delta_g + \frac{n-2}{4(n-1)}R, \\ &= \Delta_g - \frac{k(k-1)}{2k-1}. \end{aligned}$$

Let $\lambda_{j,i}$, $i = 0, 1, 2, \ldots$ denote the eigenvalues of Δ_{g_j} on S_j; note that the eigenvalue $\lambda_{j,0} = 0$ corresponds to the constant eigenfunction. It is well-known that the eigenvalues of Δ_g are of the form

$$\mu_{i_1,\ldots,i_k} := \sum_{j=1}^{k} \lambda_{j,i_j}. \tag{6.2}$$

Accordingly, the eigenvalues of P_g are then of the form

$$\omega_{i_1,\ldots,i_k} := \sum_{j=1}^{k} \lambda_{j,i_j} - \frac{k(k-1)}{2k-1},$$

and the number of negative eigenvalues of P_g is equal to

$$\#\left\{(i_1,\ldots,i_k) : \mu_{i_1,\ldots,i_k} < \frac{k(k-1)}{2k-1}\right\}. \tag{6.3}$$

We remark that if we let $i_j = 0$ for all $j = 1, \ldots, k$, then $\mu_{0,\ldots,0} = 0 < \frac{k(k-1)}{2k-1}$, so P_g has *at least one* negative eigenvalue.

We now prove the following general proposition.

Proposition 6.2.1. *The manifold $M(\gamma_1, \ldots, \gamma_k)$ does not admit any metrics of positive or zero constant scalar curvature.*

The author and Professor D. Jakobson would like to thank Professor C. LeBrun for communicating the following proof, and Professor V. Apostolov for useful conversations.

Proof. We first remark that there exist no metrics of positive curvature on $M(\gamma_1, \ldots, \gamma_k)$. We observe that $M = M(\gamma_1, \ldots, \gamma_k)$ is a spin manifold that is enlargeable, in the sense of [GL], i.e. for every $\epsilon > 0$ there exists a finite covering of M which is ϵ-hyperspherical and spin. Indeed, it is proven in [GL] that every compact hyperbolic manifold is enlargeable (see pg. 210 of [GL] or Corollary 3.9 of [GL] and the following discussion); and that the product of enlargeable manifolds is also enlargeable (see [GL], pg. 210), proving the statement for M. Hence, by [GL] (Theorem A), M does not admit a metric of positive scalar curvature, and any metric of nonnegative scalar curvature on M must be flat—in particular, it must be Ricci-flat. Finally, the method in [Boch] shows that for any Ricci-flat metric g, any harmonic 1-form must be parallel with respect to g, implying $b_1(M) \leq \dim(M)$. But we observe, since $\gamma_j \geq 2$ for all j, that $b_1(M) = 2(\sum_{j=1}^k \gamma_j) > 2k = \dim(M)$, which is a contradiction, proving that M does not admit any metrics of zero scalar curvature. □

6.3 Few negative eigenvalues of the conformal Laplacian

It is well-known (see [CGJP1], [CGJP2], [El]) that for a compact manifold of dimension $n \geq 3$, the number of negative eigenvalues of P_g cannot be uniformly bounded from above, if we are allowed to vary the conformal class. Accordingly, an interesting topological invariant seems to be the *smallest* number of negative eigenvalues of P_g on M. It will, of course, be zero if M admits metrics of non-negative scalar curvature. To study this, we make the following definition.

Definition 6.3.1. Assume that a manifold M of dimension $n \geq 3$ does not admit metrics of positive or zero scalar curvature. We denote by $\mathrm{MinNeg}(M)$ the smallest number of negative eigenvalues of the conformal Laplacian.

By Proposition 6.2.1, any product $M = M(\gamma_1, \dots, \gamma_k)$ does not admit metrics of positive or zero scalar curvature, so $\mathrm{MinNeg}(M) \geq 1$ is well-defined.

Now, denote by $\Lambda(\gamma)$ the supremum of $\lambda_1(\Delta_h)$ over all hyperbolic metrics h on a surface S of genus $\gamma \geq 2$. Let S_j have genus γ_j.

Theorem 6.3.2. *Assume that $\Lambda(\gamma) > \frac{k(k-1)}{2k-1}$. Let $M = (\gamma, k)$, denote the k-fold product of Riemann surfaces of genus γ. Then*

$$\mathit{MinNeg}(M(\gamma, k)) = 1.$$

Proof. Equip S_γ with the metric g_j where λ_1 attains the supremum $\Lambda(\gamma)$. According to (6.3), negative eigenvalues of P_g are in bijection with $(i_1, \dots, i_k)$ such that $\mu_{i_1,\dots,i_k} < k(k-1)/(2k-1)$. By assumption, this can only happen when $0 = i_1 = \dots = i_k$, and so P_g has only *one* negative eigenvalue. On the other hand, $\mathrm{MinNeg}(M) \geq 1$, finishing the proof. □

Proposition 6.3.3. *For $2 \leq k \leq 8$, $\mathit{MinNeg}(M(2, k)) = 1$.*

Proof. It is known ([SU]) that the Bolza surface provides $\Lambda(2) \approx 3.8$. We next remark that $f(x) := x(x-1)/(2x-1)$ is an increasing function for $x \geq 1$, since $f'(x) = (2x^2 - 2x + 1)/(2x-1)^2 > 0$. Accordingly, the sequence $f(k), k \geq 2$ is monotone increasing (i.e. $f(2) < f(3) < \dots$). So, to apply Theorem 6.3.2, we want to find the largest k such that $f(k) < 3.8$. An easy calculation shows that $f(8) \approx 3.73 < 3.8$, while $f(9) \approx 4.24 > 3.8$. Therefore, Theorem 6.3.2 applies for $\gamma = 2$ and $2 \leq k \leq 8$. □

It is clear that we can obtain upper bounds on $\mathrm{MinNeg}(M(2, k))$ for $k > 8$ from precise values of eigenvalues of the hyperbolic Laplacian on the Bolza surface.

The results in this chapter establish for the first time the value of $\mathrm{MinNeg}(M)$ for certian product manifolds M. A reasonable question is whether MinNeg is always attained on such manifolds by a product of hyperbolic metrics on the corresponding Riemann surfaces. If true, then very interesting recent results about the spectrum of the hyperbolic Laplacian on Riemann surfaces in the large genus limit (see [LS, Monk]) would provide a lot of information on MinNeg for product manifolds; we discuss a possible consequence in the Conclusion.

CHAPTER 7
The Weyl Tensor

In Chapter 5, we discussed conformally invariant maps from manifolds (possibly with a subset removed) into projective spaces, obtained by using eigenfunctions in the kernel of a GJMS operator as projective coordinates. The key observation was that conformal covariance of the operator implies an easy transformation law for eigenfunctions in the kernel under a conformal change of metric.

One can then notice that similar transformation laws exist for the components of the Weyl tensor. Given this, it seems natural to extend the methods seen in Chapter 5 to the setting of the Weyl tensor, and thus to obtain conformal invariants from ratios of the components of the Weyl tensor.

Our discussion on the Weyl tensor begins in this chapter with a survey of basic results, including how it transforms under a conformal change of the metric. Since these results apply in both the Riemannian and pseudo-Riemannian setting, we start with a section on pseudo-Riemannian metrics. Throughout this chapter, we follow [Lee2].

7.1 Pseudo-Riemannian metrics

Let V be a finite dimensional vector space, and let q be a symmetric covariant 2-tensor on V (i.e. a symmetric bilinear form). We can then define a linear map $\hat{q} : V \to V^*$ by

$$\hat{q}(v)(w) = q(v, w) \quad \text{for all } v, w \in V.$$

If $\hat{q}$ is an isomorphism, then we say that q is *nondegenerate*. This is equivalent to saying that for every nonzero $v \in V$, there is a $w \in V$ such that $q(v, w) \neq 0$. Yet another equivalent statement is that if $q = q_{ij}\eta^i\eta^j$ where $\{\eta^i\}$ is some basis of V^*, then the matrix (q_{ij}) is invertible.

A nondegenerate symmetric bilinear form on a finite dimensional vector space V is referred to as a *scalar product*. In the case where the scalar product is positive definite, it is referred to as an *inner product*.

Now, suppose (V, q) is an n-dimensional scalar product space. As shown in [Lee2] (Corollary 2.64), there is a basis $\{\beta^i\}$ of V^* such that q can be written as

$$q = (\beta^1)^2 + \cdots + (\beta^r)^2 - (\beta^{r+1})^2 - \cdots - (\beta^{r+s})^2,$$

where r and s are nonnegative integers which satisfy $r + s = n$. The nonnegative integers r and s are actually independent of the choice of basis. Together, they form an ordered pair (r, s) which is referred to as the *signature* of q.

Definition 7.1.1. Let M be a smooth manifold. A *pseudo-Riemannian metric* on M is a smooth, symmetric 2-tensor field g that is nondegenerate at each point of M and has the same signature everywhere.

The pseudo-Euclidean space of signature (r, s), denoted by $\mathbb{R}^{(r,s)}$, provides us with a simple example of pseudo-Riemannian manifolds. Specifically, this space is the manifold $\mathbb{R}^{r+s}$ with coordinates $(x_1, \ldots, x_r, t_1, \ldots, t_s)$ and pseudo-Riemannian metric $q^{(r,s)}$ defined by

$$q^{(r,s)} = dx_1^2 + \cdots + dx_r^2 - dt_1^2 - \cdots - dt_s^2.$$

The pseudo-Euclidean space $\mathbb{R}^{(n,1)}$ is especially well-known; it is called the $(n+1)$-dimensional *Minkowski space* and its metric is called the *Minkowski metric*.

The Minkowski metric belongs to an important class of pseudo-Riemannian metrics known as the *Lorentz metrics*; these pseudo-Riemannian metrics are characterized by having a signature of $(r, 1)$ (or $(1, r)$ in some literature). Note that Riemannian metrics constitute another important class of pseudo-Riemannian metrics.

7.2 Decomposition of the Riemann curvature tensor

In this section, we define the Weyl tensor and show that it is one of the terms in the decomposition of the $(0, 4)$-Riemann curvature tensor Rm. Before this is done, we must introduce the Kulkarni-Nomizu product.

Let $\mathcal{R}(V^*)$ denote the vector space of all covariant 4-tensors on the vector space V which have the same symmetries as the $(0, 4)$-Riemann curvature tensor; that is, such a tensor T should satisfy:

(i) $T(w, x, y, z) = -T(x, w, y, z)$

(ii) $T(w, x, y, z) = -T(w, x, z, y)$

(iii) $T(w, x, y, z) = T(y, z, w, x)$

(iv) $T(w, x, y, z) + T(x, y, w, z) + T(y, w, x, z) = 0.$

Let (V, g) be a scalar product space, and let $\Sigma^2(V^*)$ denote the space of symmetric 2-tensors on V. The trace of a covariant 2-tensor h with respect to g is given by $\mathrm{tr}_g(h) = g^{ij}h_{ij}$. In the case of a tensor from $\mathcal{R}(V^*)$, we let $\mathrm{tr}_g : \mathcal{R}(V^*) \longrightarrow \Sigma^2(V^*)$ be the trace, with respect to g, on the first and last indices. Given this, we see that $\mathrm{Ric} = \mathrm{tr}_g Rm$.

Definition 7.2.1. Given $h, k \in \Sigma^2(V^*)$, their *Kulkarni-Nomizu product* is a covariant 4-tensor denoted as $h \owedge k$ and defined by

$$h \owedge k(w, x, y, z) = h(w, z)k(x, y) + h(x, y)k(w, z) - h(w, y)k(x, z) - h(x, z)k(w, y).$$

The following lemma provides us with some useful properties of the Kulkarni-Nomizu product.

Lemma 7.2.2. *Let (V, g) be an n-dimensional scalar product space. Let $h, k \in \Sigma^2(V^*)$ and let $T \in \mathcal{R}(V^*)$.*

(i) $h \owedge k \in \mathcal{R}(V^*)$

(ii) $h \owedge k = k \owedge h$

(iii) $tr_g(h \owedge g) = (n-2)h + (tr_g h)g$.

(iv) $tr_g(g \owedge g) = 2(n-1)g$.

(v) $\langle T, h \owedge g\rangle_g = 4\langle tr_g T, h\rangle_g$.

Proposition 7.2.3. *Let (V, g) be an n-dimensional scalar product space with $n \geq 3$. Define a linear map $G : \Sigma^2(V^*) \to \mathcal{R}(V^*)$ by*

$$G(h) = \frac{1}{n-2}\left(h - \frac{tr_g h}{2(n-1)}g\right) \owedge g.$$

Then G is a right inverse for tr_g, and its image is the orthogonal complement of the kernel of tr_g in $\mathcal{R}(V^*)$.

Proof. Let $h \in \Sigma^2(V^*)$. Then, using (iii) and (iv) of Lemma 7.2.2, we have

$$\begin{aligned} \mathrm{tr}_g(G(h)) &= \frac{1}{(n-2)}\left(\mathrm{tr}_g(h \owedge g) - \frac{\mathrm{tr}_g h}{2(n-1)}\mathrm{tr}_g(g \owedge g)\right) \\ &= \frac{1}{(n-2)}\left((n-2)h + (\mathrm{tr}_g h)g - \frac{\mathrm{tr}_g h}{2(n-1)}2(n-1)g\right) \\ &= h, \end{aligned}$$

which shows that G is indeed a right inverse for tr_g. Given this, we see that G must be injective and tr_g must be surjective; this, in turn, reveals that $\dim \mathrm{Im}\,(G) = \dim \ker(\mathrm{tr}_g)^{\perp}$. Furthermore, if $T \in \mathcal{R}(V^*)$ is such that $\mathrm{tr}_g T = 0$, then an application of (v) of Lemma 7.2.2 results in $\langle T, G(h)\rangle_g = 0$. This, plus the dimensionality argument, leads us to conclude that $\mathrm{Im}(G) = \ker(\mathrm{tr}_g)^{\perp}$. $\square$

Definition 7.2.4. Let g be either a Riemannian or pseudo-Riemannian metric. The *Schouten tensor* (or, *Schouten-Weyl tensor*) of g is a symmetric 2-tensor field defined by:

$$S = \frac{1}{n-2}\left(\mathrm{Ric} - \frac{R}{2(n-1)}g\right),$$

where Ric is the Ricci curvature and R is the scalar curvature, both defined with respect to g.

Definition 7.2.5. Let g be either a Riemannian or pseudo-Riemannian metric. The *Weyl tensor* of g is the tensor from $\mathcal{R}(V^*)$ defined as

$$\begin{aligned} W &= Rm - S \owedge g \\ &= Rm - \frac{1}{n-2}\mathrm{Ric} \owedge g + \frac{R}{2(n-1)(n-2)} g \owedge g. \end{aligned}$$

Proposition 7.2.6. *For every Riemannian or pseudo-Riemannian manifold (M, g) of dimension $n \geq 3$, the trace of the Weyl tensor is zero, and*

$$Rm = W + S \owedge g$$

is the orthogonal decomposition of Rm corresponding to $\mathcal{R}(V^*) = \ker(tr_g) \oplus \ker(tr_g)^\perp$.

Proof. This follows easily by taking $h = \mathrm{Ric}$ in Proposition 7.2.3. Indeed, doing this shows that $G(\mathrm{Ric}) = S \owedge g$ and thus $S \owedge g \in \ker(\mathrm{tr}_g)^\perp$. Then, using the fact $\mathrm{Ric} = \mathrm{tr}_g Rm$ and that the map G is a right inverse for tr_g, we have

$$\begin{aligned}
\mathrm{tr}_g W &= \mathrm{tr}_g Rm - \mathrm{tr}_g(S \owedge g) \\
&= \mathrm{tr}_g Rm - \mathrm{tr}_g(G(\mathrm{tr}_g Rm)) \\
&= \mathrm{tr}_g Rm - \mathrm{tr}_g Rm = 0,
\end{aligned}$$

which shows that the Weyl tensor is traceless and hence $W \in \ker(\mathrm{tr}_g)$. $\square$

7.3 Conformal transformation of the Weyl tensor

Proposition 7.3.1. *Let (M, g) be an n-dimensional ($n \geq 3$) Riemannian or pseudo-Riemannian manifold (with or without boundary). Under a conformal change of the metric $\tilde{g} = e^{2f} g$, where $f \in C^\infty(M)$, the Weyl tensor transforms according to*

$$\widetilde{W} = e^{2f} W. \tag{7.1}$$

Proof. Under a conformal change $\tilde{g} = e^{2f} g$, the $(0,4)$-Riemann curvature tensor Rm, the Ricci curvature Ric, and the scalar curvature R transform in the following way (see [Lee2], Theorem 7.30):

$$\begin{aligned}
\widetilde{Rm} &= e^{2f}\left(Rm - (\nabla^2 f) \owedge g + (\nabla f \otimes \nabla f) \owedge g - \frac{1}{2}|\nabla f|^2 (g \owedge g)\right), \\
\widetilde{\mathrm{Ric}} &= \mathrm{Ric} - (n-2)(\nabla^2 f) + (n-2)(\nabla f \otimes \nabla f) + (\Delta f - (n-2)|\nabla f|^2) g, \\
\widetilde{R} &= e^{-2f}(R + 2(n-1)\Delta f - (n-1)(n-2)|\nabla f|^2).
\end{aligned}$$

With these transformations, the proof of the proposition is just a couple of simple calculations. First, we observe how the Schouten tensor S transforms under this conformal change:

$$\begin{aligned}\widetilde{S} &= \frac{1}{n-2}\left(\widetilde{\mathrm{Ric}} - \frac{\widetilde{R}}{2(n-1)}\tilde{g}\right) \\ &= S - \nabla^2 f + (\nabla f \otimes \nabla f) + \frac{\Delta f}{n-2}g - |\nabla f|^2 g - \frac{\Delta f}{n-2}g + \frac{1}{2}|\nabla f|^2 g \\ &= S - \nabla^2 f + (\nabla f \otimes \nabla f) - \frac{1}{2}|\nabla f|^2 g\end{aligned}$$

From this, we see that the last three terms of $\widetilde{S} \owedge \tilde{g} = e^{2f}(\widetilde{S} \owedge g)$ are identical to the last three terms of $\widetilde{Rm}$. Thus, we have

$$\widetilde{W} = \widetilde{Rm} - \widetilde{S} \owedge \tilde{g} = e^{2f} W.$$

□

Remark 7.3.2. Given the above proposition, we define the $(1,3)$-Weyl tensor $\mathcal{W}$ by defining its components as $\mathcal{W}_{ijk}{}^{l} = g^{lm} W_{ijkm}$. Now consider what happens to $\mathcal{W}$ under a conformal change of the metric $\tilde{g} = e^{2f} g$:

$$\widetilde{\mathcal{W}}_{ijk}{}^{l} = \tilde{g}^{lm} \widetilde{W}_{ijkm} = e^{-2f} g^{lm} e^{2f} W_{ijkm} = \mathcal{W}_{ijk}{}^{l}.$$

Thus, the $(1,3)$-Weyl tensor $\mathcal{W}$ is conformally invariant.

There is a useful corollary to be obtained from Propositon 7.3.1, but before we state and prove it, we shall need the following definition and theorem.

Definition 7.3.3. A Riemannian manifold is *flat* if it is locally isometric to Euclidean space. That is, every point on the manifold has a neighbourhood that is isometric to an open set in $\mathbb{R}^n$ with the Euclidean metric.

Theorem 7.3.4. *A Riemannian or pseudo-Riemannian manifold is flat if and only if its* $(0,4)$*-Riemann curvature tensor vanishes identically.*

Now we present the corollary from Propositon 7.3.1.

Corollary 7.3.5. *Let (M, g) be an n-dimensional $(n \geq 3)$ Riemannian or pseudo-Riemannian manifold. If g is locally conformally flat, then its Weyl tensor vanishes identically.*

Proof. If (M, g) is a Riemannian (pseudo-Riemannian) manifold, let g_0 denote the flat Riemannian (pseudo-Riemannian) metric on $\mathbb{R}^n$. Now suppose (M, g) is locally conformally flat. This means that for an arbitrary point $p \in M$, there is a neighbourhood U and a diffeomorphism $\Psi : U \longrightarrow \mathbb{R}^n$ such that the pullback $\tilde{g} = \Psi^* g_0$ satisfies $\tilde{g} = e^{2f} g$ for some $f \in C^\infty(M)$. Then, by Theorem 7.3.4, the $(0, 4)$-Riemann curvature tensor of $\tilde{g}$ is zero, meaning the Weyl tensor of $\tilde{g}$ is zero. Then, from (7.1), we conclude that the Weyl tensor of g must also be zero. $\square$

CHAPTER 8
The Weyl Tensor and Lie Groups

In this chapter, we continue the survey given in Chapter 7 but now specialize to the Weyl tensor for left-invariant metrics and bi-invariant metrics on Lie groups. In the former case, one finds that the components of the Weyl tensor may be written in terms of the structure constants. In the latter case, curvature formulas become pleasant and, as a consequence, so does the formula for the Weyl tensor.

8.1 Background

For this section, we follow [Lee1] and [Lee2].

8.1.1 Lie groups

Definition 8.1.1. A *Lie group* is a smooth manifold G, without boundary, that is also a group with the property that the multiplication map $m : G \times G \to G$ and the inversion map $i : G \to G$ are both smooth. Note that the multiplication map m and the inversion map i are given, respectively, by

$$m(a, h) = ah, \qquad i(a) = a^{-1}.$$

Definition 8.1.2. Let G be a Lie group. Given any $a \in G$, we can define two maps: the *left translation map* $L_a : G \to G$ and the *right translation map*

$R_a : G \to G$, respectively defined by

$$L_a(h) = ah, \qquad R_a(h) = ha.$$

Remark 8.1.3. In fact, $L_a : G \to G$ is a diffeomorphism because it is smooth and has the smooth inverse $L_{a^{-1}}$. Similarly, $R_a : G \to G$ is a diffeomorphism.

Definition 8.1.4. Let G be a Lie group. A Riemannian metric g on G is *left-invariant* if $L_a^* g = g$ for all $a \in G$ (i.e. it is invariant under all left translations). Similarly, g is *right-invariant* if $R_a^* g = g$ for all $a \in G$. If g is both left-invariant and right-invariant, it is said to be *bi-invariant*.

8.1.2 Lie algebras

Definition 8.1.5. A *Lie algebra*, over $\mathbb{R}$, is a real vector space $\mathfrak{g}$ equipped with a map (typically called the *bracket*) that is of the form

$$\begin{aligned} \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathfrak{g} \\ (X, Y) &\longmapsto [X, Y] \end{aligned}$$

and, for any $X, Y, Z \in \mathfrak{g}$, satisfies the following:

(i) Bilinearity: For any $a, b \in \mathbb{R}$,

$$\begin{aligned} [aX + bY, Z] &= a[X, Z] + b[Y, Z] \\ [Z, aX + bY] &= a[Z, X] + b[Z, Y]. \end{aligned}$$

(ii) Antisymmetry:

$$[X, Y] = -[Y, X].$$

(iii) Jacobi Identity:

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0.$$

Let us describe a well-known Lie algebra. Start by considering $\mathfrak{X}(M)$, the space of C^∞ vector fields on a C^∞ manifold M. Given any $X, Y \in \mathfrak{X}(M)$, their *Lie bracket* $[X, Y]$ is defined by

$$[X,Y]f = XYf - YXf,$$

for $f \in C^\infty(M)$. Lemma 8.25 and Proposition 8.28 of [Lee1] shows, respectively, that $[X, Y] \in \mathfrak{X}(M)$ and that the Lie bracket satisfies the three properties described in Definition 8.1.5. Thus, $\mathfrak{X}(M)$ with the Lie bracket is a Lie algebra.

For our purposes, we need to understand the Lie algebra that naturally comes with a given Lie group. To start, consider the following definition.

Definition 8.1.6. Let G be a Lie group. A vector field X on G is *left-invariant* if $(L_a)_*X = X$ for all $a \in G$. Similarly, X is *right-invariant* if $(R_a)_*X = X$ for all $a \in G$.

For a Lie group G, it turns out that the space of smooth, left-invariant vector fields is closed under the Lie bracket ([Lee1], Proposition 8.33). Thus, the space of these vector fields together with the Lie bracket determines a Lie algebra. This Lie algebra is known as the *Lie algebra of G* and is denoted by $\mathrm{Lie}(G)$.

8.1.3 The adjoint representation

Definition 8.1.7. If G and H are Lie groups, a *Lie group homomorphism* between them is a smooth map $\varphi : G \to H$ which is also a group homomorphism. If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, a *Lie algebra homomorphism* between them is a linear map $\psi : \mathfrak{g} \to \mathfrak{h}$ which preserves brackets (i.e. $\psi[X,Y] = [\psi X, \psi Y]$).

For a finite dimensional real or complex vector space V, let $\mathrm{GL}(V)$ denote the group of invertible linear maps from V to itself. Note that this group is isomorphic to either $\mathrm{GL}(n,\mathbb{R})$ or $\mathrm{GL}(n,\mathbb{C})$, where $n = \dim V$, and thus it is a Lie group.

Similarly, for a finite dimensional real or complex vector space V, let $\mathfrak{gl}(V)$ denote the Lie algebra of linear maps from V to itself (note that the bracket for this Lie algebra is defined by $[A,B] = A \circ B - B \circ A$).

Definition 8.1.8. For a Lie group G, a finite dimensional *representation of* G is a Lie group homomorphism $\varphi : G \to \mathrm{GL}(V)$, for some V. For a finite dimensional Lie algebra $\mathfrak{g}$, a finite dimensional *representation of* $\mathfrak{g}$ is a Lie algebra homomorphism $\psi : \mathfrak{g} \to \mathfrak{gl}(V)$, for some V.

With these definitions, we can now define what is known as the adjoint representation. To start, let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. For each $g \in G$, we may obtain a Lie group homomorphism $C_g : G \to G$ defined by $C_g(x) = gxg^{-1}$. From Theorem 8.44 of [Lee1], we know that each Lie group homomorphism induces a Lie algebra homomorphism; the Lie algebra homomorphism induced by C_g is denoted by $\mathrm{Ad}(g) = (C_g)_* : \mathfrak{g} \to \mathfrak{g}$.

Observe that because C_g is a Lie group homomorphism for each $g \in G$, we have $C_{g_1 g_2} = C_{g_1} \circ C_{g_2}$ and, as a consequence, $\mathrm{Ad}(g_1 g_2) = \mathrm{Ad}(g_1) \circ \mathrm{Ad}(g_2)$, for any $g_1, g_2 \in G$. From this, we see that $\mathrm{Ad}(g) \in \mathrm{GL}(\mathfrak{g})$, where its inverse is given by $\mathrm{Ad}(g^{-1})$. Additionally, we now see that $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is a group homomorphism. Upon showing that Ad is smooth, the following proposition will have been proven.

Proposition 8.1.9. *For a Lie group G with Lie algebra $\mathfrak{g}$, the map*

$$Ad : G \longrightarrow GL(\mathfrak{g})$$

is a Lie group representation, known as the adjoint representation of G.

We can also obtain an adjoint representation for Lie algebras. Consider a finite dimensional Lie algebra $\mathfrak{g}$. For each $X \in \mathfrak{g}$, we can define a map $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ by $\mathrm{ad}(X)Y = [X, Y]$.

Proposition 8.1.10. *For any Lie algebra $\mathfrak{g}$, the map $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra representation, known as the adjoint representation of $\mathfrak{g}$.*

Theorem 8.1.11. *Let G be a Lie group, $\mathfrak{g}$ be its Lie algebra, and $Ad : G \to GL(\mathfrak{g})$ be the adjoint representation of G. The induced Lie algebra representation $Ad_* : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is given by $Ad_* = ad$.*

8.2 Weyl tensor of a left-invariant metric

For this section, we primarily follow [Mil].

For a Lie group G, there is a bijection between left-invariant Riemannian metrics on G and inner products on $\mathrm{Lie}(G)$ (see [Lee2], Lemma 3.10). When there is no risk of confusion, the inner product corresponding to a left-invariant Riemmanian metric g will be simply denoted by $\langle \cdot, \cdot \rangle$.

Lemma 3.10 from [Lee2] also tells us that a Riemannian metric g on a Lie group G is left-invariant if and only if for all $X, Y \in \mathrm{Lie}(G)$, the function $g(X, Y)$ is constant on G. As a consequence, for any vector field Z, we have

$Z\langle X, Y\rangle = 0$. Knowing this, consider Koszul's formula:

$$\begin{aligned}\langle \nabla_X Y, Z\rangle = \frac{1}{2}(X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle \\ - \langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle + \langle X, [Z, Y]\rangle)\end{aligned}$$

Applying this formula to a Lie group with a left-invariant metric, and using left-invariant vector fields X, Y, and Z, the formula reduces to

$$\begin{aligned}\langle \nabla_X Y, Z\rangle &= \frac{1}{2}(-\langle Y, [X, Z]\rangle - \langle Z, [Y, X]\rangle + \langle X, [Z, Y]\rangle) \\ &= \frac{1}{2}(\langle [X, Y], Z\rangle - \langle [Y, Z], X\rangle + \langle [Z, X], Y\rangle). \qquad (8.1)\end{aligned}$$

Before continuing with this formula, we must now define the structure constants.

Definition 8.2.1. Let G be an n-dimensional Lie group with a left-invariant metric g. Let $e_1, \ldots, e_n$ be a basis for the vector space Lie(G) that is orthonormal with respect to the inner product associated to g. Then the *structure constants* α_{ijk} are defined by

$$\begin{aligned}[e_i, e_j] &= \sum_k \alpha_{ijk} e_k \\ \iff \quad \alpha_{ijk} &= \langle [e_i, e_j], e_k\rangle.\end{aligned}$$

Together, the structure constants form an $n \times n \times n$ array which describes the Lie algebra's structure.

We now return to equation (8.1) but with the orthonormal basis described in the above definition. This results in the following expression:

$$\begin{aligned}
\langle \nabla_{e_i} e_j, e_k \rangle &= \frac{1}{2}(\langle [e_i, e_j], e_k \rangle - \langle [e_j, e_k], e_i \rangle + \langle [e_k, e_i], e_j \rangle) \\
&= \frac{1}{2}(\alpha_{ijk} - \alpha_{jki} + \alpha_{kij}) \\
\implies \nabla_{e_i} e_j &= \sum_k \frac{1}{2}(\alpha_{ijk} - \alpha_{jki} + \alpha_{kij}) e_k.
\end{aligned}$$

Ultimately, this allows us to express the sectional curvature entirely in terms of the structure constants. To see this, we show how the sectional curvature of a plane section with orthonormal basis e_1, e_2 is calculated. Recalling Definition 2.1.20, and skipping over the tedious calculations, we obtain the following formula:

$$\begin{aligned}
K(e_1, e_2) &= -\langle \mathcal{R}(e_1, e_2) e_1, e_2 \rangle \\
&= \langle \nabla_{[e_1, e_2]} e_1 - \nabla_{e_1} \nabla_{e_2} e_1 + \nabla_{e_2} \nabla_{e_1} e_1, e_2 \rangle \\
&= \sum_k \Bigg(\frac{1}{2} \alpha_{12k}(-\alpha_{12k} + \alpha_{2k1} + \alpha_{k12}) \\
&\quad - \frac{1}{4}(\alpha_{12k} - \alpha_{2k1} + \alpha_{k12})(\alpha_{12k} + \alpha_{2k1} - \alpha_{k12}) - \alpha_{k11}\alpha_{k22} \Bigg)
\end{aligned}$$

Now, further recall Remark 2.3.3, which says that the scalar curvature R can be written as

$$R = \sum_{i \neq j} K(e_i, e_j).$$

From this formula, we see that the scalar curvature may also be written entirely in terms of the structure constants.

We can also express the components of the $(0, 4)$-Riemann curvature tensor Rm in terms of the structure constants and, as a consequence, we can do the same for the components of the Ricci curvature—by using the other formula in Remark 2.3.3:

$$\mathrm{Ric}(u, v) = \sum_{i=1}^{n} Rm(e_i, u, v, e_i)$$

While these expressions are somewhat impractical, they provide us with the valuable fact that, for a Lie group with a left-invariant metric, the components of the Weyl tensor may be written in terms of the structure constants.

8.3 Weyl tensor of a bi-invariant metric

For this section, we continue to follow [Mil].

First, we must recall what the adjoint of a linear transformation is. If L is a linear transformation between metric spaces, then its *adjoint* L^* is defined by

$$\langle Lx, y\rangle = \langle x, L^* y\rangle.$$

Additionally, L is said to be *skew-adjoint* if $L^* = -L$, meaning

$$\langle Lx, y\rangle = -\langle x, Ly\rangle.$$

We now state a lemma from [Mil] which provides us with a condition for determining if a left-invariant metric on a connected Lie group is bi-invariant.

Lemma 8.3.1. *Let G be a connected Lie group. A left-invariant metric on G is bi-invariant if and only if the linear transformation $ad(X)$ is skew-adjoint for every $X \in Lie(G)$.*

So, for the rest of this section, we assume that G is a connected Lie group and that g is a bi-invariant metric on G with corresponding inner product $\langle\cdot,\cdot\rangle$ on $\mathrm{Lie}(G)$. Furthermore, all vector fields are assumed to belong to $\mathrm{Lie}(G)$.

As a result of these assumptions and the above lemma, $\mathrm{ad}(X)$ is skew-adjoint for all $X \in \mathrm{Lie}(G)$, which means we have

$$\langle \mathrm{ad}(X)Y, Z\rangle = -\langle Y, \mathrm{ad}(X)Z\rangle.$$

Recalling that $\mathrm{ad}(X)Y = [X,Y]$, we see that this is equivalent to

$$\begin{aligned} &\langle [X,Y], Z\rangle = -\langle Y, [X,Z]\rangle \\ \Leftrightarrow\ &\langle [Y,X], Z\rangle = \langle Y, [X,Z]\rangle \end{aligned} \tag{8.2}$$

Applying this to equation (8.1), we get

$$\begin{aligned} \langle \nabla_X Y, Z\rangle &= \frac{1}{2}(\langle [X,Y], Z\rangle - \langle [Y,Z], X\rangle + \langle [Z,X], Y\rangle) \\ &= \frac{1}{2}(\langle [X,Y], Z\rangle - \langle Y, [Z,X]\rangle + \langle [Z,X], Y\rangle) \\ &= \frac{1}{2}\langle [X,Y], Z\rangle, \end{aligned}$$

and so,

$$\nabla_X = \frac{1}{2}\mathrm{ad}(X) \quad \Leftrightarrow \quad \nabla_X Y = \frac{1}{2}[X,Y] \tag{8.3}$$

With (8.3), we can obtain simple formulas for the curvatures. Starting with the $(1,3)$-Riemann curvature tensor $\mathcal{R}$, we have

$$\begin{aligned} \mathcal{R}(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &= \frac{1}{4}[X,[Y,Z]] - \frac{1}{4}[Y,[X,Z]] - \frac{1}{2}[[X,Y],Z] \\ &= -\frac{1}{4}[[X,Y],Z] \\ \Longrightarrow \mathcal{R}(X,Y) &= -\frac{1}{4}\mathrm{ad}([X,Y]). \end{aligned}$$

Note that we used the Jacobi identity to go from the second line to the third line in the calculation of $\mathcal{R}(X,Y)Z$.

The $(0,4)$-Riemann curvature tensor is then

$$\begin{aligned} Rm(X,Y,Z,W) &= \langle \mathcal{R}(X,Y)Z, W\rangle \\ &= -\frac{1}{4}\langle [[X,Y],Z],W\rangle \\ &= -\frac{1}{4}\langle [X,Y],[Z,W]\rangle, \end{aligned}$$

where we used (8.2), a consequence of skew-adjointness, to go from the second line to the third line.

It is now a simple matter to obtain the sectional curvature of a plane section with orthonormal basis u, v:

$$\begin{aligned} K(u,v) &= -Rm(u,v,u,v) \\ &= \frac{1}{4}\langle [u,v],[u,v]\rangle. \end{aligned}$$

Before obtaining an expression for the Ricci curvature, we consider the following definition.

Definition 8.3.2. Let $\mathfrak{g}$ be a Lie algebra, either over $\mathbb{R}$ or $\mathbb{C}$. The *Killing form* B of $\mathfrak{g}$ is the symmetric, bilinear form defined by

$$B(x,y) = \operatorname{tr}(\operatorname{ad}(x) \circ \operatorname{ad}(y)),$$

where $x, y \in \mathfrak{g}$.

Now, recall that the Ricci curvature is defined as the trace of

$$\begin{aligned} Z \longmapsto \mathcal{R}(Z,X)Y &= -\frac{1}{4}[[Z,X],Y] \\ &= -\frac{1}{4}[Y,[X,Z]] \\ &= -\frac{1}{4}\operatorname{ad}(Y)(\operatorname{ad}(X)Z) \\ &= -\frac{1}{4}(\operatorname{ad}(Y) \circ \operatorname{ad}(X))Z, \end{aligned}$$

and so

$$\begin{aligned}
\mathrm{Ric}(X,Y) &= \mathrm{tr}(Z \longmapsto \mathcal{R}(Z,X)Y) \\
&= -\frac{1}{4}\mathrm{tr}(\mathrm{ad}(Y) \circ \mathrm{ad}(X)) \\
&= -\frac{1}{4}\mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)) \\
&= -\frac{1}{4}B(X,Y)
\end{aligned}$$

We are now in a position to obtain an expression for the Weyl tensor W of a bi-invariant metric g on a connected Lie group G:

$$\begin{aligned}
W &= Rm - \frac{1}{n-2}\mathrm{Ric} \owedge g + \frac{R}{2(n-1)(n-2)} g \owedge g \\
\Longrightarrow W(X,Y,Z,K) &= -\frac{1}{4}\langle [X,Y],[Z,K] \rangle + \frac{1}{4(n-2)} B \owedge g(X,Y,Z,K) \\
&\quad - \frac{tr_g B}{8(n-1)(n-2))} g \owedge g(X,Y,Z,K)
\end{aligned}$$

CHAPTER 9
The Weyl Tensor and Degeneration of a Conformal Factor

In this chapter, we consider a 4-dimensional Riemannian manifold that is formed from taking the product of two surfaces, where the metric on one surface is multiplied by a conformal factor. We compute the ratios of the components of the Weyl tensor for this product manifold, and then discuss the behaviour of these ratios as the conformal factor degenerates at a point. Note that the components of the $(1,3)$-Riemann curvature tensor were computed in Maple.

9.1 The first manifold

The first factor in our product will be the 2-dimensional Riemannian manifold (M,h) where the metric h is hyperbolic and, in local coordinates (x,y), is given by

$$h = \frac{4}{(1-x^2-y^2)^2}(dx^2+dy^2) = \frac{4}{a}(dx^2+dy^2),$$

where $a = a(x,y) := (1-x^2-y^2)^2$.

The nonzero components of its $(1,3)$-Riemann curvature tensor are

$$\mathcal{R}_{212}{}^{1} = \mathcal{R}_{121}{}^{2} = \frac{4}{a}$$
$$\mathcal{R}_{122}{}^{1} = \mathcal{R}_{211}{}^{2} = -\frac{4}{a}.$$

For the $(0,4)$-Riemann curvature tensor Rm, recall that $(Rm)_{ijkl} = \mathcal{R}_{ijkl} = h_{lm}\mathcal{R}_{ijk}{}^{m}$, and so its nonzero components are

$$\mathcal{R}_{2121} = \mathcal{R}_{1212} = \frac{16}{a^2}$$
$$\mathcal{R}_{1221} = \mathcal{R}_{2112} = -\frac{16}{a^2}.$$

The nonzero components of its Ricci curvature $(\mathrm{Ric})_{ij} = R_{kij}{}^{k}$ are

$$(\mathrm{Ric})_{11} = (\mathrm{Ric})_{22} = -\frac{4}{a}.$$

The scalar curvature is then

$$R = h^{ij}(\mathrm{Ric})_{ij} = \frac{a}{4}\bigg(-\frac{4}{a}\bigg) + \frac{a}{4}\bigg(-\frac{4}{a}\bigg) = -2.$$

9.2 The second manifold

The second factor in our product will be the 2-dimensional Riemannian manifold (N, k) where the metric k, in local coordinates (z, w), is given by

$$k := \frac{4e^{2f}}{(1 - z^2 - w^2)^2}(dz^2 + dw^2) = \frac{4e^{2f}}{\rho}(dz^2 + dw^2) = e^{2f}\tilde{k},$$

where $\rho = \rho(z, w) := (1-z^2-w^2)^2$, $\tilde{k}$ is the usual hyperbolic metric of the same form as the metric h for the first manifold, and $f := f(z, w)$ is an arbitrary smooth function defined on N, that is, $f \in C^\infty(N)$.

Now, given that (N, k) will be the second factor in our product, we will use 3, 4 for our indices as opposed to 1, 2. With this in mind, the nonzero components of its $(1,3)$-Riemann curvature tensor are

$$\mathcal{R}_{434}{}^{3} = \mathcal{R}_{343}{}^{4} = \frac{\rho f_{ww} + 4 + \rho f_{zz}}{\rho}$$
$$\mathcal{R}_{344}{}^{3} = \mathcal{R}_{433}{}^{4} = \frac{-\rho f_{ww} - 4 - \rho f_{zz}}{\rho},$$

where the notation f_{zz} means $f_{zz} = \frac{\partial^2}{\partial z^2} f(z, w)$.

The nonzero components of its $(0,4)$-Riemann curvature tensor are

$$\begin{aligned}
\mathcal{R}_{4343} = \mathcal{R}_{3434} &= \frac{4e^{2f}(\rho f_{ww} + 4 + \rho f_{zz})}{\rho^2} \\
\mathcal{R}_{3443} = \mathcal{R}_{4334} &= \frac{4e^{2f}(-\rho f_{ww} - 4 - \rho f_{zz})}{\rho^2}.
\end{aligned}$$

The nonzero components of its Ricci curvature are

$$(\mathrm{Ric})_{33} = (\mathrm{Ric})_{44} = \frac{-\rho f_{ww} - 4 - \rho f_{zz}}{\rho}.$$

The scalar curvature is

$$\begin{aligned}
R &= \frac{e^{-2f}\rho}{4}\left(\frac{-\rho f_{ww} - 4 - \rho f_{zz}}{\rho}\right) + \frac{e^{-2f}\rho}{4}\left(\frac{-\rho f_{ww} - 4 - \rho f_{zz}}{\rho}\right) \\
&= -\frac{e^{-2f}(\rho f_{ww} + 4 + \rho f_{zz})}{2} \\
&= -\frac{\rho}{2e^{2f}}\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial w^2}\right) f - 2e^{-2f} \\
&= 2(\Delta_k f - e^{-2f}) \\
&= 2e^{-2f}(\Delta_{\tilde{k}} f - 1),
\end{aligned}$$

where $\Delta_k = e^{-2f}\Delta_{\tilde{k}}$ is the Laplace-Beltrami operator for (N, k).

9.3 The product manifold

We now consider the product of the two previously described Riemannian manifolds; that is, we consider the manifold $M \times N$ with the metric $g = h \oplus k$ which, in local coordinates (x, y, z, w), is given by

$$g = \frac{4}{a}(dx^2 + dy^2) + \frac{4e^{2f}}{\rho}(dz^2 + dw^2).$$

Recall that by equation 6.1, the scalar curvature is the sum of each factor's scalar curvature, and so

$$R = 2e^{-2f}(\Delta_{\tilde{k}} f - e^{2f} - 1).$$

Regarding the $(1,3)$-Riemann curvature tensor $\mathcal{R}_{ijl}{}^{m}$, its components are equal to those for (M,h) when the indices i,j,l,m take on values 1 or 2, equal to those for (N,k) when they take on 3 or 4, and equal to zero in all other cases; obviously, the same is true for the components of the $(0,4)$-Riemann curvature tensor $\mathcal{R}_{ijlm}$. Similarly, the components of the Ricci tensor $(\mathrm{Ric})_{ij}$ are equal to those for (M,h) when i,j take on values 1 or 2 and equal to those for (N,k) when they take on 3 or 4.

Now, to compute the Weyl tensor we will need to also compute $\mathrm{Ric} \owedge g$ and $g \owedge g$. The components of $\mathrm{Ric} \owedge g$ are given by

$$(\mathrm{Ric} \owedge g)_{ijlm} = (\mathrm{Ric})_{im} g_{jl} + (\mathrm{Ric})_{jl} g_{im} - (\mathrm{Ric})_{il} g_{jm} - (\mathrm{Ric})_{jm} g_{il}.$$

So, we have

$$
\begin{aligned}
(\mathrm{Ric} \owedge g)_{2121} &= (\mathrm{Ric} \owedge g)_{1212} = \frac{32}{a^2} \\
(\mathrm{Ric} \owedge g)_{1221} &= (\mathrm{Ric} \owedge g)_{2112} = -\frac{32}{a^2} \\
(\mathrm{Ric} \owedge g)_{4343} &= (\mathrm{Ric} \owedge g)_{3434} = \frac{8e^{2f}(\rho f_{ww} + 4 + \rho f_{zz})}{\rho^2} \\
(\mathrm{Ric} \owedge g)_{4334} &= (\mathrm{Ric} \owedge g)_{3443} = -\frac{8e^{2f}(\rho f_{ww} + 4 + \rho f_{zz})}{\rho^2} \\
(\mathrm{Ric} \owedge g)_{1313} &= (\mathrm{Ric} \owedge g)_{3131} = (\mathrm{Ric} \owedge g)_{1414} = (\mathrm{Ric} \owedge g)_{4141} \\
&= (\mathrm{Ric} \owedge g)_{2323} = (\mathrm{Ric} \owedge g)_{3232} = (\mathrm{Ric} \owedge g)_{2424} \\
&= (\mathrm{Ric} \owedge g)_{4242} = \frac{16e^{2f} + 4\rho f_{ww} + 4\rho f_{zz} + 16}{a\rho} \\
(\mathrm{Ric} \owedge g)_{1331} &= (\mathrm{Ric} \owedge g)_{3113} = (\mathrm{Ric} \owedge g)_{1441} = (\mathrm{Ric} \owedge g)_{4114} \\
&= (\mathrm{Ric} \owedge g)_{3223} = (\mathrm{Ric} \owedge g)_{2332} = (\mathrm{Ric} \owedge g)_{2442} \\
&= (\mathrm{Ric} \owedge g)_{4224} = -\frac{16e^{2f} + 4\rho f_{ww} + 4\rho f_{zz} + 16}{a\rho},
\end{aligned}
$$

Calculating the components of $g \owedge g$ follows the same procedure and yields the following:

$$
\begin{aligned}
(g \owedge g)_{2121} &= (g \owedge g)_{1212} = -\frac{32}{a^2} \\
(g \owedge g)_{1221} &= (g \owedge g)_{2112} = \frac{32}{a^2} \\
(g \owedge g)_{4343} &= (g \owedge g)_{3434} = -\frac{32e^{4f}}{\rho^2} \\
(g \owedge g)_{4334} &= (g \owedge g)_{3443} = \frac{32e^{4f}}{\rho^2} \\
(g \owedge g)_{1313} &= (g \owedge g)_{3131} = (g \owedge g)_{1414} = (g \owedge g)_{4141} \\
&= (g \owedge g)_{2323} = (g \owedge g)_{3232} = (g \owedge g)_{2424} \\
&= (g \owedge g)_{4242} = -\frac{32e^{2f}}{a\rho} \\
(g \owedge g)_{1331} &= (g \owedge g)_{3113} = (g \owedge g)_{1441} = (g \owedge g)_{4114} \\
&= (g \owedge g)_{3223} = (g \owedge g)_{2332} = (g \owedge g)_{2442} \\
&= (g \owedge g)_{4224} = \frac{32e^{2f}}{a\rho}.
\end{aligned}
$$

Finally, we can now calculate the Weyl tensor W which takes the following form for dimension $n = 4$:

$$
\begin{aligned}
W &= Rm - \frac{1}{n-2}\mathrm{Ric} \owedge g + \frac{R}{2(n-1)(n-2)} g \owedge g \\
&= Rm - \frac{1}{2}\mathrm{Ric} \owedge g + \frac{R}{12}\, g \owedge g
\end{aligned}
$$

So, the components of W for our product manifold $(M \times N, g)$ are

$$\begin{aligned}
W_{2121} &= W_{1212} = -\frac{8}{3a^2}R \\
W_{1221} &= W_{2112} = \frac{8}{3a^2}R \\
W_{4343} &= W_{3434} = -\frac{8e^{4f}}{3\rho^2}R \\
W_{3443} &= W_{4334} = \frac{8e^{4f}}{3\rho^2}R \\
W_{1313} &= W_{3131} = W_{1414} = W_{4141} = W_{2323} = W_{3232} \\
&= W_{2424} = W_{4242} = \frac{4e^{2f}}{3a\rho}R \\
W_{1331} &= W_{3113} = W_{1441} = W_{4114} = W_{2332} = W_{3223} \\
&= W_{2442} = W_{4224} = -\frac{4e^{2f}}{3a\rho}R
\end{aligned}$$

9.4 Ratios of the Weyl tensor and degeneration at a point

Let us now determine the ratios of the components of the Weyl tensor. The trivial ratios which give ± 1 are omitted; then, up to multiplication by -1, there are three ratios:

$$\begin{aligned}
T_1 &= \frac{W_{1221}}{W_{3443}} = \left(\frac{\rho}{ae^{2f}}\right)^2 \\
T_2 &= \frac{W_{1221}}{W_{4242}} = \frac{2\rho}{ae^{2f}} \\
T_3 &= \frac{W_{3443}}{W_{4242}} = \frac{2ae^{2f}}{\rho}
\end{aligned}$$

Observe that, up to reciprocals and multiplication by a constant, each ratio is just the ratio of $4/a$ and $4e^{2f}/\rho$ (though T_1 differs a bit more in that it additionally squares this ratio).

Now suppose that the conformal factor e^{2f} degenerates at a point; that is, $f(p_0) = -\infty$ at some point $p_0 = (z_0, w_0) \in N$. As a result of this, the metric

g on $M \times \{p_0\}$ will degenerate. Observe that such a degeneration is detected by the Weyl tensor in the sense that at p_0, the ratios T_1, T_2, and T_3 will either go to 0 or ∞.

CHAPTER 10
Conclusion

While being reasonably self-contained, this thesis provides an introduction to the theory of conformally covariant operators. Additionally, numerous conformal invariants obtained from such operators have been discussed in detail. In conclusion, we state several conjectures about conformal invariants, which provide directions for further research.

In Chapter 6 we considered the problem of understanding the smallest number of negative eigenvalues MinNeg(M) of the conformal Laplacian on a product M of k Riemann surfaces. An important question is whether MinNeg(M) for the product M is always attained by the product of hyperbolic metrics on each factor. If true, then recent results about the spectrum of the hyperbolic Laplacian on Riemann surfaces of large genus provide a lot of information on the behaviour of MinNeg. We discuss this below in more detail.

It can be shown that for any $\epsilon > 0$, there exists N_ϵ such that for $\gamma > N_\epsilon$, we have $\Lambda(\gamma) > 975/4096 - \epsilon$ (following the work of Kim and Sarnak [Kim], this result is presented in [Mond] as Theorem 1.1).

On the other hand, it was shown (see Part 9 of [LS], and [Monk]) that the number of eigenvalues of the hyperbolic Laplacian on a surface of genus γ lying in the interval $[1/4, 1/4 + \delta]$ grows proportionally to the volume of the surfaces (and hence, by the Gauss-Bonnet theorem, linearly in γ) as $\gamma \to \infty$.

Based on these results, the following conjecture seems reasonable.

Conjecture 1. *Fix $k > 0$. There exists $C = C(k) > 0$, such that in the limit $\gamma_1 + \gamma_2 + \ldots + \gamma_k \to \infty$, we have*

$$\frac{1}{C} \leq \frac{MinNeg(M(\gamma_1, \gamma_2, \ldots, \gamma_k))}{\gamma_1\gamma_2\cdots\gamma_k} \leq C.$$

These results will be discussed in more detail in the forthcoming paper [JY].

In Chapter 9, we explored how a conformal factor degenerating at a point can influence the ratios of the components of the Weyl tensor. These computations were done in dimension 4, but it seems interesting to make similar explorations in higher dimensions.

Apart from Conjecture 2, which is due to Colin Guillarmou but stated in [CGJP1], all remaining conjectures were originally posed in [CGJP1].

Let M be an n-dimensional manifold. Recall that $P_k = P_{k,g}$ denotes a GJMS operator, $\mathcal{T}(M)$ denotes the Teichmüller space of conformal structures, and that $\mathcal{R}(M)$ denotes the Riemannian moduli space of conformal structures.

Conjecture 2. *Assume the dimension n is odd. Then for any conformal class in $\mathcal{T}(M)$, there exists a constant $C > 0$ such that*

$$\dim\ker P_k \leq Ck^n \qquad \forall k \in \mathbb{N}.$$

Consider the *discriminant hypersurface* $\mathcal{H}_k$ (in either $\mathcal{T}(M)$ or $\mathcal{R}(M)$), which consists of conformal classes with nontrivial nullspace $\ker P_k \neq 0$.

Conjecture 3. *For a generic conformal class in $\mathcal{H}_k$,* $\dim\ker P_k = 1$.

For the last two conjectures, suppose the dimension n is even and consider the critical GJMS operator $P_{\frac{n}{2}}$. Also, note that in the $k = \frac{n}{2}$ case, the discriminant hypersurface $\mathcal{H}_{\frac{n}{2}}$ is defined as the set of conformal classes for which $\dim \ker P_{\frac{n}{2}} \geq 2$.

Conjecture 4. *For a generic conformal class in* $\mathcal{T}(M)$*, the nullspace* $\ker P_{\frac{n}{2}}$ *consists of constant functions.*

Conjecture 5. *For a generic conformal class in* $\mathcal{H}_{\frac{n}{2}}$*, the nullspace* $\ker P_{\frac{n}{2}}$ *has dimension 2.*

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