# THEORY OF RINGOIDS

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# ABSTRACT

The aim of this thesis is to present coherently the theory of ringoids from a purely ring theoretical point of view.

Chapter 1, 2 and 3 are devoted to extend the results in theory of non-commutative rings to ringoids.

In Chapter 4 we characterize torsion theories in abelian categories by using techniques in topos theory.

In Chapter 5 and 6 we apply the results in previous chapters to characterize torsion theories in ringoids and prove the additive Giraud's theorem.

# RESUME

Le but de cette thèse est de présenter de façon cohérente la théorie des annoides vue à travers la théorie des anneaux.

Dans les trois premiers chapitres nous prolongeons des résultats de la théorie des anneaux non-commutatifs à la théorie des annoides.

Dans le quatrième chapitre, nous donnons une caractérisation des théories de torsion en utilisant des méthodes de la théorie des topos.

Dans les chapitres 5 et 6, nous appliquons les résultats des chapitres antérieurs au problème de la caractérisation des théories de torsion et démontrons une version additive du théorème de Giraud.

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# PREFACE

The notion of ringoids was introduced by B. Mitchell in 1972 as a generalization of rings with unity. The aim of this thesis is to present coherently the theory of ringoids from a purely ring theoretical point of view. The language of category theory is adopted in the presentation. That is instead of considering a ring as a set with algebraic structures, we regard elements of a ring as maps between objects in a small preadditive category.

Chapters 1,2 and 3 are devoted to constructions such as tensor products, calculus of fractions, and characterizations of special objects such as the prime radical, the Jacobson radical, injectives, flat objects and such classical results as Watt's theorem and Morita equivalence in ringoids.

Chapter 4 is concentrated on the characterization of torsion theories in abelian categories. But I have adopted a topos theoretical approach, by observing the notion of a topology in Topos can be defined for any category with pullbacks!

In Chapters 5 and 6 I use the technique and results of Chapter 4 to characterize torsion theories in ringoids and prove the additive Giraud's theorem.

All results are original unless otherwise stated. I would like to point out that this work is inspired by B. Mitchell's paper "Rings with several objects" in 1972.

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CHAPTER 1

PRELIMINARIES

## §1. Basic definitions.

**Definition.** A ringoid $\mathcal{A}$ consists of a small category together with an abelian group structure on each of its hom sets such that composition is bilinear.

**Notation.** (1) The sets of objects $\mathcal{A}$ will be denoted by $|\mathcal{A}|$.

(2) Suppose $A, B \in |\mathcal{A}|, 0_{AB}$ denotes the zero element of the abelian group $\mathcal{A}(A,B)$ and $1_A$ denotes the identity map of $A \in |\mathcal{A}|$.

(3) The additive functor category $\mathbf{Ab}^{\mathcal{A}}$ will be denoted by mod-$\mathcal{A}$ and similarly $\mathbf{Ab}^{\mathcal{A}^{op}}$ will be denoted by $\mathcal{A}$-mod. Suppose $\mathcal{B}$ is another ringoid the additive functor category $\mathbf{Ab}^{\mathcal{A}^{op}\times\mathcal{B}}$ will be denoted by $\mathcal{A}$-mod-$\mathcal{B}$. We next obtain a few easy properties of $\mathcal{A}$.

**Proposition.** For every $a \in \mathcal{A}(A,B)$ and $b,c \in \mathcal{A}(B,C)$

(1) $a(b+c) = ab+ac$;

(2) $a0_{BC} = 0_{AC} = 0_{AB}b$;

(3) $a(-b) = (-a)b = -ab$;

(4) $(-a)(-b) = ab$.

**Proof.** (1) is trivial;

(2) $a0_{BC} = a(0_{BC}+0_{BC}) = a0_{BC}+a0_{BC}$ so $0_{AC} = a0_{BC}$. Similarly, $0_{AB}b = 0_{AC}$;

(3) $0_{AC} = a0_{BC} = a(b-b) = ab+a(-b)$, so $-ab = a(-b)$. Similarly $(-a)b = -ab$.

(4) Replacing a by $-a$ we have $(-a)(-b) = -(a(-b))$ $= -((-ab))$. ∎

**Remark.** The composite

$$A \xrightarrow{a} B \xrightarrow{b} C$$

is denoted by ab. (1) implies that for any $\mathcal{a}$ (which need not be small), there is a distinguished element $0_{\mathcal{a}}$.

If $A \in |\mathcal{a}|$, $\mathcal{a}(A,-)$ is denoted by $h_A$ and $\mathcal{a}(-,A)$ by $h^A$. Clearly $h_A \in \text{mod-}\mathcal{a}$ and $h^A \in \mathcal{a}\text{-mod}$.

**Definition. A left ideal** $L$ of $\mathcal{a}$ is simply a subobject of $h^A$ in $\mathcal{a}$-mod for some $A \in |\mathcal{a}|$. This means for every $B \in |\mathcal{a}|$, $L(B)$ is a subgroup of $\mathcal{a}(B,A)$ and if $a \in L(B)$, $b \in \mathcal{a}(B',B)$ then $ba \in L(B')$. Similarly, a right ideal

$R$ of $\mathcal{A}$ is a subobject of $h_A$ for some $A \in |\mathcal{A}|$ and an ideal $I$ is simply a subobject of $\mathcal{A}(-,-)$ in $\mathcal{A}$-mod-$\mathcal{A}$.

**Examples.**

1. $0_{\mathcal{A}}$ is an ideal.

2. If $a \in \mathcal{A}(A,B)$, the right ideal of $\mathcal{A}$ generated by a is given by

$$a\mathcal{A}(X) = \{ax \in \mathcal{A}(A,X) | x \in \mathcal{A}(B,X) \text{ for every } X \in |\mathcal{A}|.$$

Similarly the left ideal $\mathcal{A}a$ generated by a is given by

$$\mathcal{A}a(X) = \{xa \in \mathcal{A}(X,B) | x \in \mathcal{A}(X,A) \text{ for every } X \in |\mathcal{A}|.$$

The ideal [a] generated by a is given by

$$[a](A',B') = \{\Sigma_{\text{fin}(i)} a'ab' \in \mathcal{A}(A',B') | a' \in \mathcal{A}(A',A), b' \in \mathcal{A}(B,B')\}$$

where fin$(i)$ means over some finite index $i$. It is easy to check $a\mathcal{A} \subseteq h_{\mathcal{A}}$, $\mathcal{A}a \subseteq h^A$ and $[a] \subseteq \mathcal{A}(-,-)$.

3. Let $F:\mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor, then

$$(\ker F)(A,B) = \{a \in \mathcal{A}(A,B) | F(a) = 0_{\mathcal{B}}\}$$

is an ideal of $\mathcal{a}$.

Operations on ideals.

1. If $L \subseteq h^A$ is a left ideal and $R \subseteq h_B$ a right ideal, then the product $RL$ is given by

$$RL = \{\Sigma_{\text{fin}(i)} b_i a_i \mid b_i \in R(X_i), a_i \in L(X_i), X_i \in |\mathcal{a}|\}.$$

$RL$ is a subgroup of $R(A)$ and $L(B)$.

2. If $I$, $I'$ are ideals of $\mathcal{a}$, the product $II'$ is given by

$$(II')(A,B) = \{\Sigma_{\text{fin}(i)} a_i b_i \mid a_i \in I(A,X_i),\ b_i \in I(X_i,B),\ X_i \in |\mathcal{a}|\}.$$

$II'$ is also an ideal and $II' \subseteq I$, $II' \subseteq I'$.

**Remark.** In particular, if $a \in \mathcal{a}(A,B)$, $a' \in \mathcal{a}(A',B')$ then

$$(a'\mathcal{a})(\mathcal{a}a) = \{\Sigma_{\text{fin}(i)} x_i y_i \mid x_i \in \mathcal{a}(a'\mathcal{a})(X_i), y_i \in (\mathcal{a}a)(X_i), X_i \in |\mathcal{a}|\} = \Sigma_{\text{fin}(i)} a'uva \mid u \in \mathcal{a}(B',X_i), v \in \mathcal{a}(X_i,A), X_i \in |\mathcal{a}|\} \subseteq \mathcal{a}(A',B).$$

Hence if we regard $\mathcal{a}$ as a diagram, $(a'\mathcal{a})(\mathcal{a}a)$ is the abelian subgroup of $\mathcal{a}(A',B)$ generated by all finite paths

$$A' \xrightarrow{a'} B' \xrightarrow{u} X_i \xrightarrow{v} A \xrightarrow{a} B.$$

## §2. The category $\mathcal{A}$-mod.

**Lemma (Yoneda).** **For every $A \in |\mathcal{A}|$ and $T \in \mathcal{A}$-mod, there is a natural isomorphism**

$$\mathcal{A}\text{-mod}(h^A, T) \xrightarrow{\cong} T(A).$$

**Proof.** See [B. Stenström, 1972]. ■

**Notation.** (1) Given $T \in \mathcal{A}$-mod and $x \in T(A)$, the corresponding map $h^A \longrightarrow T$ will be denoted by $\langle x \rangle$. Then $\langle x \rangle(B)(a) = T(a)(x)$ for every $a \in h^A(B) = \mathcal{A}(B,A)$.

(2) If $f \in \mathcal{A}$-mod$(M,N)$ and $g \in \mathcal{A}$-mod$(N,K)$, the composition is denoted by $g \circ f$.

Suppose $M \in |\mathcal{A}\text{-mod}|$. A set $\{x_i \in M(A_i) \mid i \in I\}$ generates $M$ if for every $x \in M(A)$, there exists $a_i \in \mathcal{A}(A,A_i)$ such that $x = \Sigma_{\mathrm{fin}(i)} M(a_i)(x_i)$ i.e. the map $p: \Sigma_{i \in I} h^{A_i} \longrightarrow M$ is an epimorphism in $\mathcal{A}$-mod where $p$ is induced by $\langle x_i \rangle : h^{A_i} \longrightarrow M$. $M$ is finitely generated if $I$ can be finite and $M$ is cyclic if $I$ can be taken a singleton set. Notice that for every $M \in \mathcal{A}$-mod there always exists a set of generators $U(M) = \cup_{A \in |\mathcal{A}|} M(A)$ -- the underlying set of $M$.

**Proposition.** **For every $M \in \mathcal{A}$-mod, $M \cong \mathrm{colim}_{i \in I} h^{A_i}$ for some set of $I$.**

**Proof.** See [Mac Lane, 1971]. ■

In particular $M$ is free if $M \cong \Sigma_{i \in I} h^{A_i}$ for some set $I$.

Operations on $\mathcal{A}$-*mod*.

1. Suppose $M \in \mathcal{A}$-*mod* and $x \in M(A)$, the subobject $\mathcal{A}x$ generated by $x$ is given by

$$(\mathcal{A}x)(X) = \{M(a)(x) \in M(X) | a \in \mathcal{A}(X,A)\}.$$

2. If $M \in \mathcal{A}$-*mod*, $x \in M(A)$ and $L \subseteq h^A$ is a left ideal, the product $Lx$ is given by

$$(Lx)(X) = \{M(a)(x) \in M(X) | a \in L(X)\}.$$

Then $Lx \in \mathcal{A}$-mod and $Lx \subseteq M$.

3. If $M \in \mathcal{A}$-mod and $R \subseteq h_A$ a right ideal, the product $MR$ is given by

$$MR = \{\Sigma_{\mathrm{fin}(i)} M(a_i)(x_i) \in M(A) \mid a_i \in R(X_i), X_i \in |\mathcal{A}|\}.$$

Then $MR$ is a subgroup of $M(A)$.

4. If $M \in \mathcal{A}$-mod and $I$ an ideal, the product $IM$ is given by

$$(IM)(X) = \{\Sigma_{\mathrm{fin}(i)} M(a_i)(x_i) \mid a_i \in I(X, X'), x_i \in M(X_i), X_i \in |\mathcal{A}|\}.$$

Then $IM \in \mathcal{A}$-mod and $IM \subseteq M$.

5. If $M'$ is a subobject of $M$ in $\mathcal{A}$-mod and $x \in M(A)$, then the left ideal $[M':x] \subseteq h^A$ obtained by pulling back $M'$ along $\langle x \rangle$ is defined by

$$[M':x](X) = \{a \in \mathcal{A}(X, A) \mid M(a)(x) \in M'(X)\}.$$

## §3. Jacobson radical, Prime radical and the center.

$M \in \mathcal{A}$-mod is non-zero if there is an $A \in |\mathcal{A}|$ such that $M(A) \neq 0$.

**Lemma. Every non-zero finitely generated $M$ has a maximal proper subobject.**

**Proof.** Let $\varphi$ be the set of all proper subobject of $M$ partially ordered by inclusion. If $\psi$ is a totally ordered subset of $\varphi$, let $M'$ be the sum of all $N \in \psi$. If we can show that $M' \neq M$, then $M'$ will be an upper bound for $\psi$ in $\varphi$, and Zorn's lemma can be applied to give a maximal proper subobject of $M$. If $M' = M$, there exists a finite set of generators $\{x_1,\cdots,x_n\}$ for $M'$. Each $x_i$ lies in some object belonging to $\psi$, and then for some $N \in \psi$ we have $\{x_1,\cdots,x_n\} \subseteq U(N)$ since $\psi$ is totally ordered. This implies $N = M$ a contradiction. ∎

**Corollary.** **For every $A \in |\mathcal{A}|$, $h^A$ has a proper maximal left ideal.**

Now we shall extend some classical results of non-commutative ring theory to $\mathcal{A}$.

(1) The Jacobson radical (Mitchell).

Let $A \in |\mathcal{A}|$ and $J^A$ be the intersection of all proper maximal left ideals of $h^A$. Then we have

**Proposition.** **$J^A(B) = \{a \in \mathcal{A}(B,A) \mid 1_A - ba$ is left invertible for all $b \in \mathcal{A}(A,B)\}$**
**$= \{a \in \mathcal{A}(B,A) \mid 1_A - ba$ is invertible for all $b \in \mathcal{A}(A,B)\}$.**

Proof. If $1_A - ba$ has no left inverse for some $b \in \mathcal{A}(A,B)$, then $\mathcal{A}(1_A - ba)$ is a proper left ideal and it is contained in some proper maximal left ideal $L$. If $a \in L(B)$ then $ba \in L(A)$ which implies $1_A \in L(A)$. Hence a

$\notin J_A(B)$.

Conversely, if $a \notin J^A(B)$, there exists some proper maximal left ideal $L$ such that $a \notin L(B)$ which implies $\mathcal{Q}a+L = h^A$. In particular $1_A \in (\mathcal{Q}a+L)(A)$ there exists $b \in \mathcal{Q}(A,B)$ and $c \in L(A)$ such that $1_A = ba+c$. Since $L$ is proper, $c = 1_A-ba$ is not left invertible.

To complete the proof, it suffices to show that if $1_A-ba$ has a left inverse for all $b \in \mathcal{Q}(A,B)$, then the left inverse is also the right inverse.

If $x \in \mathcal{Q}(A,A)$ such that $x(1_A-ba) = 1_A$ we have $x = 1_A+(xb)a = 1_a-(-(xb))$ so $x$ also has a left inverse $y \in \mathcal{Q}(A,A)$. Hence $1_A(1_A-ba) = yx(1_A-ba) = y1_A = y$. This implies $x(1_A-ba) = xy = 1_A$. ■

**Lemma.** **Given $a \in \mathcal{Q}(B,A)$ and $b \in \mathcal{Q}(A,B)$, then $1_A-ba$ is right invertible if and only if $1_B-ab$ is right invertible.**

**Proof.** Suppose $x \in \mathcal{Q}(A,A)$ such that $(1_A-ba)x = 1_A$. Then $1_A = x-bax$ and

$$(1_B-ab)(1_B+axb) = 1_B-ab+axb-abaxb = 1_B-a(1_A-x+bax)b = 1_B. \blacksquare$$

If $J_B$ denotes the intersection of all proper maximal right ideals of $h_B$, then the last two results show $J^A(B) = J_B(A)$. Thus if we define $J(B,A) = J^A(B)$, $J$ is an ideal.

**Lemma (Nakayama). If $N \in \mathcal{A}$-mod is finitely generated and $JN = N$, then $N = 0$.**

**Proof.** Suppose $N(A) \neq 0$ for some $A \in |\mathcal{A}|$, and let $\{x_i \in N(A_i)\}_{i=1}^n$ be a minimal set of generators of $N$. Then $x_n \in (JN)(A_n)$, hence we have $x_n = \Sigma_{i=1}^n N(a_i)(x_i)$ where $a_i \in J(A_n,A_i)$. This implies

$$N(1_{A_n}-a_n)(x_n) = \Sigma_{i=1}^{n-1} N(a_i)(x_i).$$

Since $a_n \in J(A_n,A_n)$ so $1_{A_n}-a_n$ is invertible. Hence $N$ is generated by $\{x_i \in N(A_i)\}_{i=1}^{n-1}$, a contradiction. ∎

**Definition.** $a \in \mathcal{A}(A,A)$ is **nilpotent** if $a^n = 0$ for some $n \in \mathbb{N}$. A left ideal $L \subseteq h^A$ is nilpotent if every element of $L(A)$ is nilpotent.

**Proposition. Suppose a left ideal $L \subseteq h^A$ is nilpotent, then $L \subseteq J(-,A)$.**

**Proof.** Suppose $B \in |\mathcal{A}|$ and $a \in L(B)$. Let $b \in \mathcal{A}(A,B)$, then $ba \in L(A)$. Since $L$ is nilpotent, so $(ba)^n = 0$ for some $n \in \mathbb{N}$, and hence $(1_A - ba)^{-1} = \Sigma_{i=1}^{n-1}(ba)^i + 1_A$. This implies $a \in J(B,A)$. ■

(2) The Prime Radical.

**Definition.** An ideal $P$ is **prime** if given ideals $I$ and $I'$ such that $II' \subseteq P$ then $I \subseteq P$ or $I' \subseteq P$.

**Theorem.** **$P$ is a prime ideal if and only if whenever a $\in \mathcal{a}(A,B')$ and $b \in \mathcal{a}(A',B)$ such that $(a\mathcal{a})(\mathcal{a}b) \subseteq P(A,B)$, then a $\in P(A,B')$ or $b \in P(A',B)$.**

**Proof.** Suppose $P$ is a prime, a $\in \mathcal{a}(A,B')$ and $b \in \mathcal{a}(A',B)$ such that $(a\mathcal{a})(\mathcal{a}b) \subseteq P$. It suffices to show $[a][b] \subseteq P$, then simply observe a $\in [a](A,B')$ and $b \in [b](A',B)$. Recall

$$[a](X,Y) = \{\Sigma_{\text{fin}(i)} x_i a y_i \mid x_i \in \mathcal{a}(X,A), y_i \in \mathcal{a}(B',Y)\}$$

$$[b](X,Y) = \{\Sigma_{\text{fin}(j)} u_j b v_j \mid u_j \in \mathcal{a}(X,A'), v_j \in \mathcal{a}(B,Y)\}$$

$$[a][b](X,Y) = \{\Sigma_{\text{fin}(k)} e_k f_k \mid e_k \in [a](X,X_k), f_k \in [b](X_k,Y)\}$$

Since composition is distributive over addition enough to consider each summand of $[a][b](X,Y)$, we have the following diagram

$$X \longrightarrow A \xrightarrow{a} B' \longrightarrow X_k \longrightarrow A' \xrightarrow{b} B \longrightarrow Y$$

But

$$A \xrightarrow{a} B' \longrightarrow X_k \longrightarrow A' \xrightarrow{b} B$$

belongs to $(a\mathcal{A})(\mathcal{A}b) \subseteq P(A,B)$, and since $P$ is an ideal we have $[a][b] \subseteq P$.

Conversely, suppose $I$, $I'$ are ideals such that $II' \subseteq P$. If $I \nsubseteq P$, there exists $A,B \in |\mathcal{A}|$ such that $I(A,B) \nsubseteq P(A,B)$. Choose $a \in I(A,B)$ such that $a \notin P(A,B)$ and suppose $b \in I'(A',B')$, it suffices to show $(a\mathcal{A})(\mathcal{A}b) \subseteq P(A,B')$ since in that case we have $b \in P(A',B')$. We can use the same argument as before. Clearly we have $a\mathcal{A} \subseteq I(A,-)$, $[a] \subseteq I$ and $[b] \subseteq I'$ so $[a][b] \subseteq II' \subseteq P$. But $a \notin P(A,B)$ so $b \in [b](A',B') \subseteq P[A',B']$.

**Definition.** (1) $\mathrm{rad}(\mathcal{A}) = \cap\{P \mid P \text{ is a prime ideal of } \mathcal{A}\}$

(2) $a \in \mathcal{A}(A,B)$ is **strongly nilpotent** if every sequence $\{a_0, a_1, \cdots\}$ such that it is constructed by $a_0 = a$ and $a_{n+1} \in (a_n\mathcal{A})(\mathcal{A}a_n)$ is eventually zero. Obviously $a_n \in \mathcal{A}(A,B)$ for all $n \in \mathbf{N}$.

**Theorem. rad($\mathcal{A}$)($A,B$) consists of all the strongly nilpotent elements in $\mathcal{A}(A,B)$.**

**Proof.** Suppose $a \in \mathcal{A}(A,B)$ and $a \notin \mathrm{rad}(A,B)$, then there exists some prime ideal $P$ such that $a \notin P(A,B)$ so $(a\mathcal{A})(\mathcal{A}a) \not\subseteq P(A,B)$, select $a_1 \in (a\mathcal{A})(\mathcal{A}a)$ and $a_1 \notin P(A,B)$. Continue this construction so we have a sequence $\{a,a_1,a_2,a_3,\cdots\}$ such that $a_n \neq 0$ for all $n$, since $a_n \notin P(A,B)$, so a is not strongly nilpotent.

Conversely, if $a \in \mathcal{A}(A,B)$ is not strongly nilpotent, which means that there exists a sequence $\{a_0,a_1,a_2,a_3,\cdots\}$ such that $a_n \neq 0$ for all $n$ and $a_{n+1} \in (a_n\mathcal{A})(\mathcal{A}a_n)$. Let $\epsilon$ denote the sequence then consider the family $\mathcal{S} = \{I \text{ an ideal of } \mathcal{A} \mid I(A,B)\cap\epsilon = \emptyset\}$. Clearly $\mathcal{S} \neq 0$ since the zero ideal of $\mathcal{A}$ has nothing in common with $\epsilon$. Order $\mathcal{S}$ by inclusion, then the usual Zorn's lemma argument says $\mathcal{S}$ has a maximal element $P$. If we can show $P$ is prime we are done, let $I$ and $I'$ be ideals of $\mathcal{A}$ such that $I \not\subseteq P$ and $I' \not\subseteq P$. Since $P$ is maximal, $(I+P)\cap\epsilon \neq \emptyset$ and $(I'+P)\cap\epsilon \neq \emptyset$ hence $a_i \in (I+P)(A,B)$ and $a_j \in (I'+P)(A,B)$ for some $i,j \in \mathbb{N}$. Put $m = \max(i,j)$ then we have

$$a_{m+1} \in (a_m\mathcal{A})(\mathcal{A}a_m) \subseteq [(I+P)(I'+P)](A+B) \subseteq (II'+P)(A,B).$$

But $a_{m+1} \notin P(A,B)$ implies $a_{m+1} \in (II')(A,B)$, so $II' \not\subseteq P$. ■

Observe if $a \in \mathcal{A}(A,A)$ is strongly nilpotent, then it is nilpotent. Then the result in last example on the Jacobson radical says:

**Corollary.** $\mathrm{rad}(\mathcal{A}) \subseteq J$.

(3) The center of $\mathcal{A}$.

Before we consider the center of $\mathcal{A}$, recall the following result of ring theory: Let $R$ be a ring with unity and $1_{R\text{-mod}}$ denote the identity functor. Then there is an isomorphism

$$\varphi : \mathrm{Cen}(R) \xrightarrow{\cong} \mathrm{Nat}(1_{R\text{-mod}}, 1_{R\text{-mod}})$$

So we have this result to guide us.

Given $e \in \mathrm{Nat}(1_{\mathcal{A}\text{-mod}}, 1_{\mathcal{A}\text{-mod}})$ we shall to describe $e$ in terms of morphisms of $\mathcal{A}$. By definition $e$ should satisfy the condition: for every $M \in \mathcal{A}$-mod, there is a map $e_M : M \longrightarrow M$ such that for all $f \in \mathcal{A}\text{-mod}(M,N)$ the diagram

$$\begin{tikzcd} M \arrow[r, "e_M"] \arrow[d, "f"'] & M \arrow[d, "f"] \\ N \arrow[r, "e_N"] & N \end{tikzcd}$$

commutes. In particular, for every $A \in |\mathcal{A}|$, let $e_A$ denote $e_{h^A}$. Then $e_A \in \mathcal{A}\text{-mod}(h^A, h^A) \cong \mathcal{A}(A,A)$. Hence for every $e$ we get a set of endomorphisms $\{e_A \in \mathcal{A}(A,A) \mid A \in |\mathcal{A}|\}$. But $e$ also implies the following relationship among $e_A$: for every $b \in \mathcal{A}(A,B)$ $\cong \mathcal{A}\text{-mod}(h^A,h^B)$, we have the following commutative diagram:

$$\begin{tikzcd} h^A \arrow[r, "e_A"] \arrow[d, "\mathcal{A}(-{,}b)"'] & h^A \arrow[d, "\mathcal{A}(-{,}b)"] \\ h^B \arrow[r, "e_B"] & h^B \end{tikzcd}$$

Evaluating this diagram at $1_A \in \mathcal{A}(A,A)$, we have $e_A b = b e_B$. We define a set of endomorphisms $E = \{e_A | A \in |\mathcal{A}|\}$ is a central element if and only if for all $b \in \mathcal{A}(A,B)$, $e_A b = b e_B$. The collection $Z(\mathcal{A})$ of all central elements of $\mathcal{A}$ is called the center of $\mathcal{A}$.

Suppose $E = \{e_A | A \in |\mathcal{A}|\} \in Z(\mathcal{A})$ and $M \in \mathcal{A}$-mod, define $e_M : M \longrightarrow M$ if $x \in M(A)$ then $e_M(A)(x) = M(e_A)(x)$. Suppose $b \in \mathcal{A}(A,B)$ and $x \in M(B)$. Then

$$M(b) \circ e_M(B)(x) = M(b) \circ M(e_B)(x) = M(be_B)(x) = M(e_A b)(x)$$
$$= M(e_A) \circ M(b)(x) = e_M(A) \circ M(b)(x).$$

Hence $e_M \in \mathcal{A}\text{-mod}(M,M)$. Now suppose $f \in \mathcal{A}\text{-mod}(M,N)$, $A \in |\mathcal{A}|$ and $x \in M(A)$.

$$f(A)e_M(A)(X) = f(A)M(e_A)(X) = N(e_A)f(A)(X) = e_N(A)f(A)(X).$$

This implies $Z(\mathcal{A}) \cong \mathrm{Nat}(1_{\mathcal{A}\text{-mod}}, 1_{\mathcal{A}\text{-mod}})$.

Suppose $E = \{e_A \in \mathcal{A}(A,A) \mid A \in |\mathcal{A}|\} \in Z(\mathcal{A})$. Define an ideal $[E]$ by $[E](A,B) = \{e_A b \in \mathcal{A}(A,B) \mid b \in \mathcal{A}(A,B)\}$ $= \{be_B \mid b \in \mathcal{A}(A,B)\}$. If $a \in \mathcal{A}(A',A)$ and $C \in (B,C)$, it is easy to check the following diagram commutes.

$$\begin{array}{ccc} [E](A,B) & \longrightarrow & [E](A,C) \\ \downarrow & & \downarrow \\ [E](A',B) & \longrightarrow & [E](A',C) \end{array}$$

$E = \{e_A \mid A \in |\mathcal{A}|\} \in Z(\mathcal{A})$ is idempotent if $e_A^2 = e_A$ for every $A \in |\mathcal{A}|$. Let $B(\mathcal{A})$ denote the set of idempotent central elements of $\mathcal{A}$, then $B(\mathcal{A})$ has a Boolean algebra structure $(E \vee E' = E+E'-EE', E \wedge E' = EE)$ moreover $[E]$ is an idempotent ideal that means $[E][E]$ $= [E]$. Clearly $[E][E] \subseteq [E]$. On the other hand, if $be_B \in [E](A,B)$, then $be_B = be_B^2 = be_B^3 = (be_B)(e_B^2)$ $\in [E][E](A,B)$.

## §4. Calculus of fractions

Given $\mathcal{A}$, we shall construct the ringoid of fractions of $\mathcal{A}$ with respect to a certain set $\Gamma$ of morphisms $\mathcal{A}$, moreover given $M \in \mathcal{A}$-mod we shall contruct its functor of fractions.

**Definition.** A set $\Gamma$ of morphisms of $\mathcal{A}$ is called an RMC set (RMC stands for **right multiplicatively closed**) if the following conditions are satisfied:

(1) $\Gamma$ is closed under composition and $1_A \in \Gamma$ for all $A \in |\mathcal{A}|$.

(2) A diagram

$$\begin{tikzcd} & B' \arrow[d] \\ A \arrow[r, "s"] & B \end{tikzcd}$$

in $\mathcal{A}$ with $s \in \Gamma$ can be embedded in the commutative square

$$\begin{tikzcd} A' \arrow[r, "s'"] \arrow[d] & B' \arrow[d] \\ A \arrow[r, "s"] & B \end{tikzcd}$$

with $s' \in \Gamma$.

(3) For every $a \in \mathcal{A}(A,B)$ with $as = 0$ for some $s \in \Gamma \cap \mathcal{A}(B,B')$ there exists $s' \in \Gamma \cap \mathcal{A}(A',A)$ such that $s'a = 0$.

**Definition.** A non-empty category $\mathcal{D}$ is called **quasi-directed** if the following conditions are satisifed:

(a) Any diagram in the form of a co-angle

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \\ A'' & \end{tikzcd}$$

can be embedded in a commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & A' \arrow[d] \\ A'' \arrow[r] & A''' \end{tikzcd}$$

(b) For any $a, a' \in \mathcal{D}(A,B)$, there exists $b \in \mathcal{D}(B,B')$ such that $ab = a'b$.

Given ringoid $\mathcal{a}$, $\Gamma$ an RMC set of morphisms of $\mathcal{a}$ and $A \in |\mathcal{a}|$, let $(\Gamma,A)$ be the full subcategory of the comma category $(\mathcal{a},A)$, which means an object of $(\Gamma,A)$ is of the form $s{:}A' \longrightarrow A \in \Gamma$, and if $t{:}A'' \longrightarrow A \in \Gamma$ is another object then

$$(\Gamma,A)(s,t) = \{a \in \mathcal{a}(A'',A') \,|\, as = t\}.$$

**Lemma.** **If $\Gamma$ is an RMC set of morphisms in $\mathcal{a}$, then for any $A \in |\mathcal{a}|$, $(\Gamma,A)$ is quasi-directed.**

**Proof.** Suppose that every diagram in $\mathcal{a}$

$$
\begin{tikzcd}
 & & B'' \arrow[dl, "b''"'] \arrow[dd, "s''"] \\
 & B' \arrow[dr, "s'"] & \\
B \arrow[ur, "b"] \arrow[rr, "s"] & & A
\end{tikzcd}
$$

with both triangles commute and $s, s', s'' \in \Gamma$. Then it can be embedded in

$$
\begin{tikzcd}
B''' \arrow[rr] \arrow[dd, "t"'] & & B'' \arrow[dl, "b''"'] \arrow[dd, "s''"] \\
 & B' \arrow[dr, "s'"] & \\
B \arrow[ur, "b"] \arrow[rr, "s"] & & A
\end{tikzcd}
$$

in which the outer square commutes and $t \in \Gamma$. Then $ts \in \Gamma$, hence the co-angle has been embedded in a commutative square in $(\Gamma, A)$.

To show second condition of quasi-directedness, if in $(\Gamma, A)$ we have

$$
\begin{tikzcd}
B \arrow[dr, "s"'] & & B' \arrow[ll, shift right, "a"'] \arrow[ll, shift left, "a'"] \arrow[dl, "s'"] \\
 & A &
\end{tikzcd}
$$

such that $as = a's = s'$ with $s, s' \in \Gamma$. Since $\Gamma$ is RMC, there exists $t: B'' \longrightarrow B'$ such that $ta = ta'$. Clearly

$$B'' \xrightarrow{t} B' \xrightarrow{s'} A$$

belongs to $\Gamma$ and it equalizes a and a′ in $(\Gamma,A)$.

**Definition.** A category $\mathcal{E}$ is **connected** if it is non-empty and if for any two objects $i$ and $j$ there is is a finite set of objects $i_1,i_2,\cdots,i_n$ and a diagram

$$\begin{tikzcd}
i \arrow[dr] & & i_2 \arrow[dl] \arrow[dr] & & & & j \arrow[dl] \\
& i_1 & & i_3 & \cdots & i_n &
\end{tikzcd}$$

in $\mathcal{E}$.

This means that any two objects may be connected by a path containing a finite set of oriented morphisms. Obviously $(\Gamma,A)$ is connected for any $A \in |\mathcal{A}|$.

**Definition.** A category $\mathcal{D}$ is **directed** if the following conditions are satisfied:

(a) for any pair of objects $D$ and $D'$ of $\mathcal{D}$, there is a diagram

$$D' \longrightarrow D'' \longleftarrow D$$

(b) for every parallel pair of arrows $d, d' : D \longrightarrow D'$, there is a map $d'' : D' \longrightarrow D''$ such that $dd'' = d'd''$.

It is easy to show that if $D$ is quasi-directed and connected, it is directed. Hence $(\Gamma, A)$ is directed.

**Definition.** Let $\Gamma$ be a RMC set of morphism of $\mathcal{A}$, a **ringoid of fractions** of $\mathcal{A}$ with respect to $\Gamma$ is a ringoid $\Gamma^{-1}(\mathcal{A})$ together with an additive functor $\varphi : \mathcal{A} \longrightarrow \Gamma^{-1}(\mathcal{A})$ satsifying:

F1. $\varphi(s)$ is invertible for every $s \in \Gamma$.

F2. For every $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ such that $\varphi(s)$ is invertible in $\mathcal{B}$ for every $s \in \Gamma$ then there exists a unique $\sigma : \Gamma^{-1}(\mathcal{A}) \longrightarrow \mathcal{B}$ such that $\sigma \circ \varphi = \psi$.

Clearly if $\Gamma^{-1}(\mathcal{A})$ exists then it is unique up to isomorphism. Now we shall confirm its existence.

Let $A, B \in |\mathcal{A}|$ define $\epsilon(A,B) : (\Gamma, A) \longrightarrow \mathbf{Ab}$ by $\epsilon(A,B)(X \longrightarrow A) = \mathcal{A}(X,B)$ and if $x : X' \longrightarrow X$ is a map sending $(s : X \longrightarrow A)$ to $(s' : X' \longrightarrow A)$ in $(\Gamma, A)$ then $\epsilon(A,B)(x) = \mathcal{A}(x,B)$. Now we can construct $\Gamma^{-1}(\mathcal{A})$, let $|\Gamma^{-1}(\mathcal{A})| = |\mathcal{A}|$ and given $A, B \in |\Gamma^{-1}(\mathcal{A})|$, put

$$\Gamma^{-1}(\mathcal{A})(A,B) = \operatorname{colim}_{s \in (\Gamma, A)} \epsilon(A,B)(s : X \longrightarrow A)$$
$$= \operatorname{colim}_{s \in (\Gamma, A)} \mathcal{A}(X,B).$$

The explicit construction of $\Gamma^{-1}(\mathcal{A})(A,B)$ is given by putting

$$P = \Sigma_{s\in(\Gamma,A)}\mathcal{A}(X,B)$$

then impose an equivalence relation on $P$: if $b \in \epsilon(A,B)(s:X\longrightarrow A) = \mathcal{A}(X,B)$ and $b' \in \epsilon(A,B)(s':X'\longrightarrow A) = \mathcal{A}(X',B)$ then $b\sim b'$ if and only if there exist $(t:Y\longrightarrow A) \in (\Gamma,A)$ and maps $x:s\longrightarrow t$, $s'\longrightarrow t$ where $x \in \mathcal{A}(Y,X)$, $x' \in \mathcal{A}(Y,X')$ such that $\mathcal{A}(x,B)(b) = \mathcal{A}(x',B)(b')$ that is $x'b' = xb$. All this amounts to is that we have the following diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "x"'] \arrow[dr, "x'"] & \\
X \arrow[dd, "s"'] \arrow[ddrr, "b"' near end] & & X' \arrow[dd, "b'"] \arrow[ddll, "s'" near start] \\
 & & \\
A & & B
\end{tikzcd}$$

such that $x'b' = xb$ and $x's' = xs = t \in \Gamma$.

It is routine to check $\sim$ is an equivalence relation. Now we have a less confusing picture of $\Gamma^{-1}(\mathcal{A})(A,B)$, a map between $A$ and $B$ consists of the equivalence classes of a pair of maps in $\mathcal{A}$, namely $(b,s)$ with $s:x\longrightarrow A$, $b:X\longrightarrow B$ and $s \in \Gamma$.

Suppose $(b,s)$ and $(b',s') \in \Gamma^{-1}(\mathcal{A})(A,B)$ that is we have

$$\begin{tikzcd} X \arrow[d, "s"'] \arrow[dr, "b" near end] & X' \arrow[dl, "s'"' near start] \arrow[d, "b'"] \\ A & B \end{tikzcd}$$

in $\mathcal{A}$, since $\Gamma$ is RMC we have the following commutative diagram

$$\begin{tikzcd} X'' \arrow[r, "a"] \arrow[d, "t"'] & X' \arrow[d, "s'"] \\ X \arrow[r, "s"] & A \end{tikzcd}$$

in $\mathcal{A}$ with $t \in \Gamma$, the define $(b,s)+(b',s') = (ab'+tb, ts)$ $\in \Gamma^{-1}(\mathcal{A})(A,B)$.

Suppose $(b,s) \in \Gamma^{-1}(\mathcal{A})(A,B)$, $(c,t) \in \Gamma^{-1}(\mathcal{A})(B,C)$ that is we have

$$\begin{tikzcd} & X \arrow[dl, "s"'] \arrow[dr, "b"] & & Y \arrow[dl, "t"'] \arrow[dr, "c"] & \\ A & & B & & C \end{tikzcd}$$

in $\mathcal{A}$, since $\Gamma$ is RMC we have the following diagram

$$\begin{tikzcd} Z \arrow[r, "a"] \arrow[d, "u"'] & Y \arrow[d, "t"] \\ X \arrow[r, "b"] & B \end{tikzcd}$$

in $\mathcal{A}$ such that $ub = at$ and $u \in \Gamma$ then define $(b,s)(c,t) = (ac,us)$ with $(ac,us) \in \Gamma^{-1}(\mathcal{A})(A,C)$. It is routine to check these operations respect the equivalence relation $\sim$. Moreover $\Gamma^{-1}(\mathcal{A})(A,B)$ is an abelian group with $(0,1_B)$ as the zero element for addition and $(1_A,1_A)$ is the identity for composition.

Now we shall construct $\varphi:\mathcal{A} \longrightarrow \Gamma^{-1}(\mathcal{A})$. Obviously we put $\varphi(A) = (A)$ for $A \in |\mathcal{A}|$ and if $a \in \mathcal{A}(A,B)$ we put $\varphi(a) = (a,1_A)$. It is easy to verify that $\varphi$ is additive. Moreover if $s \in \mathcal{A}(A,B)$ and $s \in \Gamma$, then $\varphi(s) = (s,1_A)$ so it is represented by

$$\begin{tikzcd} & A \\ B & A \arrow[l, "s"'] \arrow[u, "1_a"'] \end{tikzcd}$$

in $\mathcal{A}$. Since $\Gamma$ is RMC we can embed it in

$$\begin{tikzcd} & X \arrow[r, "t"] \arrow[d, "a"'] & B \arrow[d, "1_B"] \\ A & A \arrow[l, "1_A"'] \arrow[r, "s"] & B \end{tikzcd}$$

with $t = as$, then $(a,t) \in \Gamma^{-1}(\mathcal{A})(B,A)$. Consider the composition $(s,1_A)(a,t) \in \Gamma^{-1}(\mathcal{A})(A,B)$. We have the following diagram

$$\begin{tikzcd}
 & & Y \arrow[dl, "t'"'] \arrow[dr, "b"] & & \\
 & A \arrow[dl, "1"'] \arrow[dr, "s"] & & X \arrow[dl, "t"'] \arrow[dr, "a"] & \\
A & & B \arrow[dr, "1_B"'] & & A \arrow[dl, "s"] \\
 & & & B &
\end{tikzcd}$$

with $t' \in \Gamma$ and both squares commute. That implies that $(s,1_A)(a,t) = (ba,t')$. Moreover we have $t's = bas$. Hence there exists $u:Z \longrightarrow Y \in \Gamma$ such that $ut' = uba$. This implies that $(ba,t') \sim (1_A,1_A)$ so $\varphi(s) = (s,1_A)$ is invertible in $\Gamma^{-1}(\mathcal{A})$. Now suppose $\psi:\mathcal{A} \longrightarrow \mathcal{B}$ is an additive functor such that $\psi(s)$ is invertible in $\mathcal{B}$ for all $s \in \Gamma$. We define an additive functor $\sigma:\Gamma^{-1}(\mathcal{A}) \longrightarrow \mathcal{B}$ as follows: $\sigma(A) = \psi(A)$ for all $A \in |\Gamma^{-1}(\mathcal{A})|$, if $(b,s) \in \Gamma^{-1}(\mathcal{A})(A,B)$ for some $b \in \mathcal{A}(X,B)$ and $s \in \mathcal{A}(X,A) \cap \Gamma$ then put $\sigma(b,s) = \psi(s)^{-1}\psi(b)$. This shows that $\Gamma^{-1}(\mathcal{A})$ together with $\varphi:\mathcal{A} \longrightarrow \Gamma^{-1}(\mathcal{A})$ has the desired properties.

If $\Gamma$ is an RMC set of morphisms in $\mathcal{A}$ and $M \in \mathcal{A}$-mod, we would like to construct the functor of fractions $\Gamma^{-1}(M) \in \Gamma^{-1}(\mathcal{A})$-mod.

**Definition.** Given a RMC set $\Gamma$ of morphisms in $\mathcal{A}$ and $M \in \mathcal{A}$-mod, then $\Gamma^{-1}(M) \in \Gamma^{-1}(\mathcal{A})$-mod is a **functor of fractions** of $M$ if there exists $\mu_M \in \mathcal{A}\text{-mod}(M, \Gamma^{-1}(M)\circ\varphi)$ with the following universal property: for every $N \in \Gamma^{-1}(\mathcal{A})$-mod and $f \in \mathcal{A}\text{-mod}(M, N\circ\varphi)$ there exists a unique $\sigma \in \Gamma^{-1}(\mathcal{A})\text{-mod}(\Gamma^{-1}(M), N)$ such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\mu_M"] \arrow[dr, "f"'] & \Gamma^{-1}(M)\circ\varphi \arrow[d, "\sigma"] \\ & N\circ\varphi \end{tikzcd}$$

Notice we regard $\sigma \in \mathcal{A}\text{-mod}(\Gamma^{-1}(M), N)$ by restriction and the definition of $\Gamma^{-1}(M)$ simply states that $\Gamma^{-1}(-) \dashv (-)\circ\varphi$.

Now we take on the construction of $\Gamma^{-1}(M)$. For $A \in |\mathcal{A}|$ consider the functor $\delta_A : (\Gamma, A) \longrightarrow \mathbf{Ab}$ given by $\delta_A(s : X \longrightarrow A) = M(X)$ and suppose $x \in \mathcal{A}(X', X)$ has the property that $\alpha : (s : X \longrightarrow A) \longrightarrow (s' : X' \longrightarrow A)$ in $(\Gamma, A)$. Then $\delta_A(x) = M(\alpha) : M(x) \longrightarrow M(X')$. We put

$$\Gamma^{-1}(M)(A) = \operatorname{colim}_{s\in(\Gamma,A)} \delta_A(s : X \longrightarrow A) \text{ for } A \in |\Gamma^{-1}(\mathcal{A})|$$
$$= \operatorname{colim}_{s\in(\Gamma,A)} M(X).$$

The explicit construction of $\Gamma^{-1}(M)(A)$ is given by letting

$$P = \Sigma_{s\in(\Gamma,A)} M(X)$$

and then imposing an equivalence relation in $P$: if $x \in \delta_A(s:X \longrightarrow A) = M(X)$ and $x' \in \delta_A(s':X' \longrightarrow A) = M(X')$ then $x \sim x'$ if an only if there exist $(s'':X'' \longrightarrow A) \in (\Gamma,A)$ and maps $\alpha:s \longrightarrow s''$, $\alpha':s' \longrightarrow s''$, where $\alpha \in \mathcal{A}(X'',X)$, $\alpha' \in \mathcal{A}(X',X)$ such that $M(\alpha)(x) = M(\alpha')(x')$.

Again it is routine to verify $\sim$ is an equivalence relation and elements of $\Gamma^{-1}(M)(A)$ consist of equivalence classes of pairs $(x,s)$ where $x \in M(X)$ and $s:X \longrightarrow A \in \Gamma$.

If $(x,s)$, $(x',s') \in \Gamma^{-1}(M)(A)$ then we have

$$X \xrightarrow{\ s\ } A \xleftarrow{\ s'\ } X'$$

in $\mathcal{A}$ which we can embed it in

$$\begin{array}{ccc} X'' & \xrightarrow{\ a\ } & X' \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle s'} \\ X & \xrightarrow[\ s\ ]{} & A \end{array}$$

with $ts = as'$ and $t \in \Gamma$, then we define $(x,s)+(x',s') = (M(t)(x)+M(a)(x'),ts)$.

If $(x,s) \in \Gamma^{-1}(M)(A)$ and $(b,t) \in \Gamma^{-1}(\mathcal{A})(B,A)$, we have

$$\begin{tikzcd} & Y \arrow[dl, "t"'] \arrow[dr, "b"] & & X \arrow[dl, "s"'] \\ B & & A & \end{tikzcd}$$

in $\mathcal{A}$, since $\Gamma$ is RMC we can embed it in

$$\begin{tikzcd} Z \arrow[r, "b'"] \arrow[d, "s'"'] & X \arrow[d, "s"] \\ Y \arrow[r, "b"] \arrow[d, "t"'] & A \\ B & \end{tikzcd}$$

with $s'b = b's$ and $s' \in \Gamma$, then define $\Gamma^{-1}(M)(b,t)(x,s) = (M(b')(x),s't) \in \Gamma^{-1}(M)(B)$.

Now we shall construct $\mu_M \in \mathcal{A}\text{-mod}(M,\Gamma^{-1}(M)\circ\varphi)$. Let $x \in M(A)$ and put $\mu_M(A)(x) = (x,1_A)$. If $a \in \mathcal{A}(B,A)$ we must show the following diagram commutes.

$$\begin{tikzcd} M(A) \arrow[r, "\mu_M(A)"] \arrow[d, "M(a)"'] & \Gamma^{-1}(M)\circ\varphi(A) \arrow[d, "\Gamma^{-1}(M)\circ(a)"] \\ M(B) \arrow[r, "\mu_M(B)"] & \Gamma^{-1}(M)\circ\varphi(B) \end{tikzcd}$$

If $a \in \mathcal{A}(B,A)$, then $\varphi(a) = (a,1_B)$, we have

$\Gamma^{-1}(M)\circ\varphi(a)\circ\mu_M(A)(x) = [\Gamma^{-1}(M)(a,1_B)](x,1_A)$

$= (M(a')(x),s)$

for every $x \in M(A)$ and a′ is obtained from the following diagram:

$$\begin{tikzcd} X \arrow[r, "a'"] \arrow[d, "s"'] & A \arrow[d, "1_A"] \\ B \arrow[r, "a"] & A \end{tikzcd}$$

with $s \in \Gamma$. Since $s1_B = 1_X s = s$,

$(M(a)(x),1_B)\sim(M(a')(x),s) = (M(sa)(x),s)$

$= (M(s)\circ M(a)(x),s)$.

Now if $N \in \Gamma^{-1}(\mathcal{A})$–mod and $f \in \mathcal{A}$–mod$(M,N\circ\varphi)$, we must construct $\sigma \in \Gamma^{-1}(\mathcal{A})$–mod$(\Gamma^{-1}(M),N)$. Suppose $A \in |\Gamma^{-1}(\mathcal{A})|$ and $(x,s) \in \Gamma^{-1}(M)(A)$ where $x \in M(X)$ and $s\colon X\longrightarrow A \in (\Gamma,A)$. Since $f \in \mathcal{A}$–mod$(M,N\circ\varphi)$ we have that the diagram

$$\begin{tikzcd}
M(A) \arrow[r, "f(A)"] \arrow[d, "M(s)"'] & N\circ\varphi(A) \arrow[d, "N\circ\varphi(s)"] \\
M(X) \arrow[r, "f(X)"] & N\circ\varphi(X)
\end{tikzcd}$$

commutes and $\varphi(s) = (s,1_X)$ is invertible, so define $\sigma(A)(x,s) = N(s,1_X)^{-1}\circ f(X)(x) \in N\circ\varphi(A) = N(A)$. Given $(a,t) \in \Gamma^{-1}(\alpha)(B,A)$ where $(a,t)$ is represented by

$$B \xleftarrow{\ t\ } Y \xrightarrow{\ a\ } A$$

with $t \in \Gamma$, we have to verify that the square

$$\begin{tikzcd}
\Gamma^{-1}(M)(A) \arrow[r, "\delta(A)"] \arrow[d, "\Gamma^{-1}(M)(a,t)"'] & N(A) \arrow[d, "N(a,t)"] \\
\Gamma^{-1}(M)(A) \arrow[r, "\delta(B)"] & N(B)
\end{tikzcd}$$

is commutative. Let $(x,s) \in \Gamma^{-1}(M)(A)$ be represented by $s: X \longrightarrow A$ and $x \in M(X)$. Then

$$N(a,t)\circ\delta(A)(x,s) = N(a,t)\circ N(s,1_X)^{-1}\circ f(X)(x).$$

On the other hand we have a diagram

$$\begin{tikzcd}
Z \arrow[r, "s'"] \arrow[d, "a'"'] & Y \arrow[r, "t"] \arrow[d, "a"] & B \\
Z \arrow[r, "s"] & A &
\end{tikzcd}$$

in $\mathcal{A}$ with $s'a = a's$ and $s' \in \Gamma$, hence

$$\delta(B)\circ\Gamma^{-1}(M)(a,t)(x,s) = \delta(B)(M(a')(x),s't)$$

$$= N(s't,1_Z)^{-1}\circ f(Z)\circ M(a')(x)$$

$$= N(s't,1_Z)^{-1}\circ [N\circ\varphi](a')\circ f(X)(x)$$

$$= N(s't,1)^{-1}\circ N(a',1_Z)\circ f(X)(x)$$

for $f \in \mathcal{A}\text{-mod}(M,N\circ\varphi)$. So we must show that $(a,t)(s,1_Z)^{-1}\sim(s't,t_Z)^{-1}(a',1_Z)$ in $\Gamma^{-1}(\mathcal{A})(B,X)$ or equivalently $(s't,1_Z)(a,t)\sim(a',1_Z)(s,1_Z)$ in $\Gamma^{-1}(\mathcal{A})(Z,A)$.

Observe the following diagram in $\mathcal{A}$

$$\begin{tikzcd}
 & & Z \arrow[dl, "1_Z"'] \arrow[dr, "a'"] & & \\
 & Z \arrow[dl, "1_Z"'] \arrow[dr, "a'"] & & X \arrow[dl, "1_X"'] \arrow[dr, "s"] & \\
Z & & X & & A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & Z \arrow[dl, "1_Z"'] \arrow[ddrr, "s'"] & & & \\
 & Z \arrow[dl, "1_Z"'] \arrow[dr, "s'"] & & & & \\
Z & & Y \arrow[dr, "t"] & & Y \arrow[dl, "t"'] \arrow[dr, "a"] & \\
 & & & B & & X
\end{tikzcd}$$

Using the fact that $s'a = a's$, we have $(s't,1_Z)(a,t) = (s'a,1_Z) = (a's,1_Z)$.

Hence $\sigma \in \Gamma^{-1}(\mathcal{Q})\text{-mod}(\Gamma^{-1}(M), N)$. Moreover if $X \in M(A)$ then

$$\sigma(A)\circ\mu_M(A)(x) = \sigma(A)(x, 1_A) = N(1_A, 1_A)^{-1}\circ f(A)(x)$$

$$= f(A)(x).$$

## CHAPTER 2

### §1. Projectivity and Injectivity

$P \in \mathcal{A}$-mod is a projective if $\mathcal{A}\text{-mod}(P,-):\mathcal{A}\text{-mod} \longrightarrow \mathbf{Ab}$ is exact, which means that for every epimorphism $\beta:M \longrightarrow N$ and every $\varphi:P \longrightarrow N$, there exists $\varphi':P \longrightarrow M$ such that $\beta \circ \varphi' = \varphi$.

Clearly, for all $A \in |\mathcal{A}|$, $h^A$ is projective.

**Lemma.** $\Sigma_{i \in I} P_i$ **is projective if and only if each** $P_i$ **is projective.**

**Proof.** A simple application of the isomorphism $\mathcal{A}\text{-mod}(\Sigma_{i \in I} P_i, X) \cong \Pi_{i \in I} \mathcal{A}\text{-mod}(P_i, X)$ ■

**Corollary. (a) Every free functor is projective.**
**(b) Every direct summand of a free functor is projective.**

**Proposition. The following are equivalent for** $P \in \mathcal{A}$**-mod**
**(a)** $P$ **is projective;**
**(b)** $P$ **is direct summand of a free functor;**
**(c) Every exact sequence** $0 \longrightarrow N \longrightarrow M \longrightarrow P \longrightarrow 0$ **splits.**

**Corollary.** $P$ **is projective if and only if there exists a set** $\{x_i \in P(A_i) \mid i \in I\}$ **and maps** $\{\varphi_i \in \mathcal{A}\text{-mod}(P, h^{A_i}) \mid i$

$\in I\}$ such that for all $A \in |\mathcal{a}|$ and $x \in P(A)$ one has $x = \Sigma_{i\in I}P(\varphi_i(x))(x_i)$, $\varphi(x) = 0_{\mathcal{a}}$ for all but a finite number of $i \in I$.

**Proof.** Let $\rho:\Sigma_{i\in I}h^{A_i} \longrightarrow P$ be an epimorphism of a free functor onto $P$. Since $\mathcal{a}\text{-mod}(\Sigma_{i\in I}(h^{A_i},P) \cong \Pi_{i\in I}\mathcal{a}\text{-mod}(h^{A_i},P) \cong \Pi_{i\in I}P(A_i)$, $\rho$ induces a set $\{x_i \in P(A_i)|i \in I\}$. If $P$ is projective, there exists $\varphi:P \longrightarrow \Sigma_{i\in I}h^{A_i}$ such that $\rho\varphi = 1_P$. But $\mathcal{a}\text{-mod}(P,\Sigma_{i\in I}h^{A_i}) = \Sigma_{i\in I}\mathcal{a}\text{-mod}(P,h^{A_i})$ says that $\varphi$ induces a set $\{\varphi_i \in \mathcal{a}\text{-mod}(P,h^{A_i})|i \in I\}$ with stated property. ∎

Note that the set $\{x_i \in P(A_i)|i \in I\}$ generates $P$, and that it can be chosen finite if $P$ is finitely generated.

$E \in \mathcal{a}$-mod is injective if $\mathcal{a}\text{-mod}(-,E):(\mathcal{a}\text{-mod})^{op} \longrightarrow \mathbf{Ab}$ is exact, which means that for every monomorphism $\alpha:M \longrightarrow N$ and every $\varphi:M \longrightarrow E$, there exists $\varphi':N \longrightarrow E$ such that $\varphi'\alpha = \varphi$. Dually we have the lemma:

**Lemma.** **A direct product $\Pi_{i\in I}E_i$ is injective if and only if each $E_i$ is injective.**

When determining whether a module is injective, it suffices to consider a very restricted clas of monomorphisms: (A generalization of Baer's testing lemma).

**Theorem.** $E \in \mathcal{A}$-mod is injective if and only if for all $A \in |\mathcal{A}|$ and $L \subseteq h^A$ a left ideal of $\mathcal{A}$, if $\varphi \in \mathcal{A}\text{-mod}(L,E)$ there exists an $x \in E(A)$ such that the diagram

$$\begin{tikzcd} L \arrow[r, tail] \arrow[d, "\varphi"'] & h^A \arrow[dl, "\langle x \rangle"] \\ E & \end{tikzcd}$$

commutes, which means that for all $x \in |\mathcal{A}|$, $a \in L(X)$, $\varphi(X)(a) = E(a)(x)$.

Proof. The stated condition means simply that every $L \subseteq h^A$ and $\varphi \in \mathcal{A}\text{-mod}(L,E)$ can be extended to $\varphi: h^A \longrightarrow E$, so it is of course a necessary condition. Assume that $E$ satisfies the condition. Let $\alpha: M \longrightarrow N$ be a monomorphism and $\varphi \in \mathcal{A}\text{-mod}(M,E)$. Consider the set $\epsilon = \{\varphi' : N' \longrightarrow E \mid M \subseteq N' \subseteq N$ and $\varphi'$ extends $\varphi\}$. $\epsilon$ can be partially ordered by declaring $\varphi' \leq \varphi''$ if $\varphi''$ further extends $\varphi'$. If $\mathcal{S}$ is a totally ordered subset of $\epsilon$, we define $\tilde{N}$ as the sum of all $N' \in \mathcal{S}$, and define $\tilde{\varphi}: \tilde{N} \longrightarrow E$ so that it extends all $\varphi' \in \mathcal{S}$. $\tilde{\varphi}$ then is an upper bound for $\mathcal{S}$. The set $\epsilon$ is thus inductive, and we can apply Zorn's Lemma to obtain a maximal $\varphi_0: N_0 \longrightarrow E$ in $\epsilon$. We must show that $N_0 = N$. Clearly $N_0 \subseteq N$. Let $x \in N(A)$ and $A \in |\mathcal{A}|$ put $L(X) = \{a$

$\in \mathcal{a}(X,A) | N(a)(x) \in N_0(X)\}$. It is easy $L \subseteq h^A$ and $L$ is a left ideal. There exists $\beta \in \mathcal{a}$-mod$(L,E)$ given by $\beta(X)(a) = \varphi_0(X)[N(a)(x)]$, and by hypothesis there exists $y \in E(A)$ such that $\beta(X)(a) = E(a)(y)$ for all $X \in |\mathcal{a}|$ and $a \in L(X)$. Recall that if $x \in N(A)$, $\mathcal{A}x$ the subfunctor functor generated by $x$ is given by $(\mathcal{a}x)(X) = \{N(a)(x) \in N(X) | a \in \mathcal{a}(X,A)\}$. Now define $\psi: N_0 + \mathcal{a}x \longrightarrow E$ as $\psi(X)(z + N(a)(x)) = \varphi_0(X)(z) + E(a)(y)$ where $z \in N_0(X)$. We claim that $\psi$ is well-defined. Suppose $z + N(a)(x) = 0$ in $(N_0 + \mathcal{a}x)(X)$, then $-z = N(a)(x) \in N_0(X)$ implies $a \in L(X)$, and then $E(a)(y) = \beta(X)(a) = \varphi_0(X)[N(a)(x)]$, so we have $\psi(X)(z + N(a)(x)) = \varphi_0(X)(z) + E(a)(y) = \varphi_0(X)(z) + \varphi_0(X)[N(a)(x)] = \varphi_0(X)[x + N(a)(x)] = 0$. Furthermore $\psi \in \mathcal{a}$-mod$(N_0 + \mathcal{a}x, E)$ and extends $\varphi_0$. This contradicts the choice of $N_0$. ■

**Definition.** $M \in \mathcal{a}$-mod is divisible if for every $0_{\mathcal{a}} \neq a \in \mathcal{a}(A,B)$ and $x \in M(A)$ there exists $y \in M(B)$ such that $x = M(a)(y)$.

**Corollary.** If $E \in \mathcal{a}$-mod is injective then $E$ is divisible.

**Proof.** If $a \in M(A,B)$ and $x \in M(A)$, we can define $\varphi \in \mathcal{a}$-mod$(\mathcal{a}a, E)$ by $\varphi(X)(ba) = M(b)(x)$ then apply the lemma. ■

§2. Tensor products.

Let $M \in$ mod-$\mathcal{A}$ and $N \in \mathcal{A}$-mod. We shall define the tensor product of $M$ and $N$ as an abelian group with universal mapping property.

Put $M \times N = \Sigma_{A \in |\mathcal{A}|}[M(A) \times N(A)]$ and let $G$ be an abelian group. A map $\varphi : M \times N \longrightarrow G$ will be called $\mathcal{A}$-bilinear if

(1) The composite

$$\varphi(A) : M(A) \otimes N(A) \xrightarrow{i_A} M \times N \xrightarrow{\varphi} G$$

is $\mathbb{Z}$-bilinear, which means $\varphi(A)$ factors through $M(A) \otimes_{\mathbb{Z}} N(A)$ for all $A \in |\mathcal{A}|$.

(2) For all $x \in M(A)$, $y \in N(B)$, $a \in \mathcal{A}(A,B)$ we have

$$\varphi(A)[x, N(a)(y)] = \varphi(B)[M(a)(x), y].$$

**Definition.** **A tensor product of $M$ and $N$ is an abelain group $T$ together with a $\mathcal{A}$-bilinear map $\tau : M \times N \longrightarrow G$, such that for every abelian group $G$ and $\mathcal{A}$-linear map $\varphi : M \times N \longrightarrow G$ there exists a unique homomorphism $\alpha : T \longrightarrow G$ such that $\alpha\tau = \varphi$.**

It is easy to check that the tensor product is unique up to isomorhpism. As for existence, let $T$

$= \Sigma_{A\in|\mathcal{A}|}[M(A)\otimes N(A)]/\sim$, where $\sim$ is the abelian subgroup generated by elements of the form $M(a)(x)\otimes y - x\otimes N(a)(y)$, $x \in M(A)$, $y \in N(B)$ and $a \in \mathcal{A}(A,B)$. It is routine to check that $\tau: M\times N \longrightarrow T$ has the desired universal property.

Because is it unique (up to ismorphisms), we will speak of the tensor product of $M$ and $N$ and denote it by $M\otimes_{\mathcal{A}}N$.

**Theorem.** For all $A \in |\mathcal{A}|$, $M \in$ mod-$\mathcal{A}$, $M\otimes_{\mathcal{A}}h^A \cong M(A)$.

**Proof.** First we notice $M\times h^A = \Sigma_{B\in|\mathcal{A}|}[M(B)\times h^A(B)] = \Sigma_{B\in|\mathcal{A}|}[M(B)\times\mathcal{A}(B,A)]$. Define $\tau: M\times h^A \longrightarrow M(A)$ by $\tau(B)[x,a] = M(a)(x)$, $B \in |\mathcal{A}|$. Since $M$ is additive $\tau(B)$ is bilinear. To check that $\tau$ satisfies (2), let $x \in M(X)$, $y \in h^A(Y) = \mathcal{A}(Y,A)$ and $b \in \mathcal{A}(X,Y)$ then we have

$$\tau(Y)[M(b)(x),y] = M(y)\circ M(b)(x) = M(by)(x) = \tau(X)[x,by]$$
$$= \tau(X)[x,h^a(b)(y)]$$

Hence $\tau$ is $\mathcal{A}$-bilinear. Suppose $\varphi: M\times h^A \longrightarrow G$ is also $\mathcal{A}$-bilinear. Define $\alpha: M(A) \longrightarrow G$ by $\alpha(z) = \varphi(A)[z,1_A]$, we have to verify $\alpha\circ\tau = \varphi$. It suffices to show for all $X \in |\mathcal{A}|$, $\alpha\circ\tau(X) = \varphi(X)$. Let $[x,y] \in M(X)\times h^A(X) = M(X)\times\mathcal{A}(X,A)$, then

$$\alpha\circ\tau(X)[x,y] = \alpha M(y)(x) = \varphi(A)[M(y)(x),1_A]$$

$= \varphi(X)[x, 1_A \circ y] = \varphi(X)[x,y]$. ■

Similarly, we can show $h_A \otimes_{\mathcal{Q}} N \cong N(A)$ for all $A \in |\mathcal{Q}|$, $N \in \mathcal{Q}$-mod.

**Corollary.** For all $A,B \in |\mathcal{Q}|$, $h_A \otimes_{\mathcal{Q}} h^B \cong \mathcal{Q}(A,B)$.

We will now look at tensor product from a functorial point of view and show that the tensor product may be considered as a functor mod-$\mathcal{Q}\times\mathcal{Q}$-mod$\longrightarrow$**Ab**. Let $\lambda \in \mathcal{Q}$-mod$(M,M')$ and $\mu \in \mathcal{Q}$-mod$(N,N')$. We claim that there is an induced homomorphism $\alpha: M\otimes_{\mathcal{Q}}N \longrightarrow M'\otimes_{\mathcal{Q}}N'$. Define $\varphi: M\times N \longrightarrow M'\otimes_{\mathcal{Q}}N'$ by $\varphi(X)[x,y] = \lambda(X)(x)\otimes_{\mathcal{Q}}\mu(X)(y)$. $\varphi(X)$ is clearly bilinear on $M(X)\times N(X)$. Suppose $x \in M(A)$, $y \in N(B)$, $a \in \mathcal{Q}(A,B)$ then

$$\varphi(A)[x,N(a)(y)] = \lambda(A)(x)\otimes\mu(A)\circ N(a)(y)$$
$$= \lambda(A)(x)\otimes N'(a)\circ\mu(B)(y) = M'(a)\circ\lambda(A)(x)\otimes\mu(B)(y)$$
$$= \lambda(B)\circ M(a)(X)\otimes\mu(B)(y) = \varphi(B)[M(a)(x),y].$$

Hence $\varphi$ is $\mathcal{Q}$-bilinear, so there exists a unique $\alpha: M\otimes_{\mathcal{Q}}N \longrightarrow M'\otimes_{\mathcal{Q}}N'$. The homomorphism $\alpha$ is denoted by $\lambda\otimes\mu$. Also observe in the construction of $M\otimes_{\mathcal{Q}}N$ it is generated by elements of the form $x\otimes y$ where $x \in M(X)$ and $y \in N(X)$ then $\lambda(x\otimes y) = (\lambda\otimes\mu)(x\otimes y)$ $= \lambda(X)(x)\otimes\mu(X)(y)$.

**Proposition.** If $M_i \in$ mod-$\mathcal{Q}$ for $i \in I$, and $N \in \mathcal{Q}$-mod, then there is a natural isomorphism

$$(\Sigma_{i\in I}M_i)\otimes_{\mathcal{A}}N \cong \Sigma_{i\in I}(M_i\otimes_{\mathcal{A}}N).$$

Proof. See [Mitchell, 1972]. ■

Moreover, Mitchell has shown:

Proposition. The tensor product is a right exact functor in each of the two variables.

Corollary. Suppose $A_i \in |\mathcal{A}|$, $i \in I$ and $N \in \mathcal{A}$–mod then

$$\Sigma_{i\in I}h_{A_i}\otimes_{\mathcal{A}}N \cong \Sigma_{i\in I}(h_{A_i}\otimes_{\mathcal{A}}N) \cong \Sigma_{i\in I}N(A_i).$$

Similarly if $M \in$ mod–$\mathcal{A}$

$$M\otimes_{\mathcal{A}}(\Sigma_{i\in I}h^{A_i}) \cong \Sigma_{i\in I}(M\otimes_{\mathcal{A}}h^{A_i} \cong \Sigma_{i\in I}M(A_i).$$

Corollary. If $M_i \in$ mod–$\mathcal{A}$ is a direct system of objects and $N \in \mathcal{A}$–mod, there is a natural isomorphism

$$(\mathrm{colim}M_i)\otimes_{\mathcal{A}}N \cong \mathrm{colim}(M_i\otimes_{\mathcal{A}}N).$$

Examples

1. Tensoring with cyclic functors.

Let $L \subseteq h^A$ be a left ideal and $M \in \mathcal{A}$-mod. From the exact sequence $0 \longrightarrow L \longrightarrow h^A \longrightarrow h^A/L \longrightarrow 0$ we get an exact sequence

$$M \otimes_{\mathcal{A}} L \xrightarrow{\alpha} M(A) \longrightarrow M \otimes_{\mathcal{A}} (h^A/L) \longrightarrow 0$$

where $\mathrm{Im}(\alpha)$

$= \{\Sigma_{\mathrm{fin}(i)} M(a_i)(x_i) \in M(A) \mid a_i \in L(X_i), x_i \in M(X_i)$ and $X_i \in |\mathcal{A}|\}$

$= ML.$

It follows that $M \otimes_{\mathcal{A}} (h^A/L) \cong M(A)/ML$.

2. Direct Unions.

If $M \in$ mod-$\mathcal{A}$ is a direct union of subfunctors $M_i$ for $i \in I$, then $M \otimes_{\mathcal{A}} N = \mathrm{colim}(M_i \otimes_{\mathcal{A}} N)$, but it should be noted that $M_i \otimes_{\mathcal{A}} N$ is not in general a subgroup of $M \otimes N$.

3. Split exact sequences.

If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is a split exact sequence of functors in $\mathcal{A}$-mod, then it follows that the sequence of abelian groups

$0 \longrightarrow M' \otimes_{\mathcal{Q}} N \longrightarrow M \otimes_{\mathcal{Q}} N \longrightarrow M'' \otimes_{\mathcal{Q}} N \longrightarrow 0$ is also split exact.

## 4. Dual functors

Let $M \in$ mod-$\mathcal{Q}$ and $G \in$ **Ab** the dual functor $M[G]$ $\in \mathcal{Q}$-mod is defined by $M[G](A) = \mathbf{Ab}(M(A),G)$ for $A \in |\mathcal{Q}|$. If $N \in \mathcal{Q}$-mod we also have $N[G] \in \mathcal{Q}$-mod defined by $N[G](A) = \mathbf{Ab}(N(A),G)$ for $A \in |\mathcal{Q}|$.

**Proposition.** **If $M \in$ mod-$\mathcal{Q}$, there is a natural isomorphism**

$$\mathbf{Ab}(M \otimes_{\mathcal{Q}} N, G) \cong \mathcal{Q}\text{-mod}(N, M[G])$$

**which means $M \otimes_{\mathcal{Q}} \dashv M[-]$. Similarly, if $N \in \mathcal{Q}$-mod we have the adjoint functors $- \otimes_{\mathcal{Q}} N \dashv N[-]$.**

**Proof.** Since $M \otimes_{\mathcal{Q}} -$ preserves direct limits, it suffices to show the isomorphism for the representables. For $A \in |\mathcal{Q}|$, we have $h^A \in \mathcal{Q}$-mod and then

$\mathbf{Ab}(M \otimes_{\mathcal{Q}} h^A, G) \cong \mathbf{Ab}(M(A), G) = M[G](A) \cong \mathcal{Q}\text{-mod}(h^A, M[G])$. ■

## §3. Characterization of flatness.

**Definition.** $F \in \mathcal{Q}$-mod is flat if the functor $- \otimes_{\mathcal{Q}} F$ is exact.

Since the tensor product is always right exact, $F$ is flat if and only if $-\otimes_{\mathcal{Q}}F$ preserves monomorphisms.

**Proposition.** If $F_i \in \mathcal{Q}$-mod, then $\Sigma_{i\in I}F_i$ is flat if and only if each $F_i$ is flat.

**Proof.** If $M' \longrightarrow M$ be a monomorphism of functors in mod-$\mathcal{Q}$ there is a commutative diagram

$$\begin{array}{ccc} M'\otimes_{\mathcal{Q}}(\Sigma_{i\in I}F_i) & \longrightarrow & M\otimes_{\mathcal{Q}}(\Sigma_{i\in I}F_i) \\ \downarrow & & \downarrow \\ \Sigma_{i\in I}(M'\otimes_{\mathcal{Q}}F_i) & \longrightarrow & \Sigma_{i\in I}(M\otimes_{\mathcal{Q}}F_i) \end{array}$$

The upper row is a monomorphism if and only if the lower one is, and this happens if and only if each $M'\otimes_{\mathcal{Q}}F_i \longrightarrow M\otimes_{\mathcal{Q}}F_i$ is a monomorphism. ■

**Corollary.** Every projective $P \in \mathcal{Q}$-mod is flat.

**Proof.** For all $A \in |\mathcal{Q}|$, $h^A$ is flat since $M\otimes h^A \cong M(A)$ for $M$ in mod-$\mathcal{Q}$. It follows that every free functor in $\mathcal{Q}$-mod is flat. Then use the fact that every projective is a direct summand of a free functor. ■

**Corollary.** Every direct limit of flat functors is flat.

**Proof.** Direct limits are exact in $\mathcal{A}$-mod. ■

**Definition.** *$N \in \mathcal{A}$-mod the character functor of $N$ is given by $N[\mathbb{Q}/\mathbb{Z}] \in \mathcal{A}$-mod.*

Since $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator of Ab, there is an intimate connection between flat functors in $\mathcal{A}$-mod and injective functors in mod-$\mathcal{A}$.

**Theorem.** *$F \in \mathcal{A}$-mod is flat if and only if $F[\mathbb{Q}/\mathbb{Z}]$ is an injective functor in mod-$\mathcal{A}$.*

**Proof.** If $F \in \mathcal{A}$-mod is flat, which means $-\otimes_{\mathcal{A}}F$ is exact. But we know $-\otimes_{\mathcal{A}}F \dashv F[-]$ and $\mathbb{Q}/\mathbb{Z}$ is injective in Ab, hence $F[-]$ sends injective to injectives, in particular $F[\mathbb{Q}/\mathbb{Z}]$ is injective in mod-$\mathcal{A}$.

Assume instead that $F[\mathbb{Q}/\mathbb{Z}] \in$ mod-$\mathcal{A}$ is injective. Let $\alpha: M' \longrightarrow M$ be an arbitrary monomorphism in mod-$\mathcal{A}$. It induces a commutative diagram

$$\begin{array}{ccc}
\text{mod-}\mathcal{A}(M,F[\mathbb{Q}/\mathbb{Z}]) & \xrightarrow{\text{mod-}\mathcal{A}(\alpha,1_{F[\mathbb{Q}/\mathbb{Z}]})} & \text{mod-}\mathcal{A}(M',F[\mathbb{Q}/\mathbb{Z}]) \\
\cong\big\downarrow & & \big\downarrow\cong \\
\text{Ab}(M\otimes_{\mathcal{A}}F,\mathbb{Q}/\mathbb{Z}) & \xrightarrow{\text{Ab}(\alpha\otimes_{\mathcal{A}}1_F,1_{\mathbb{Q}/\mathbb{Z}})} & \text{Ab}(M'\otimes_{\mathcal{A}}F,\mathbb{Q}/\mathbb{Z})
\end{array}$$

Since $F[\mathbb{Q}/\mathbb{Z}]$ is injective, mod-$\mathcal{A}(\alpha,1_{F[\mathbb{Q}/\mathbb{Z}]})$ is an epimorphism, and so is $\text{Ab}(\alpha\otimes_{\mathcal{A}}1_F,1_{\mathbb{Q}/\mathbb{Z}})$. But $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator which implies that $\alpha\otimes_{\mathcal{A}}1_F$ is a monomorphism in Ab, which is what we want. ■

**Corollary.** $F \in \mathcal{A}$-mod is flat if and only if for all $A \in |\mathcal{A}|$ and $R \leq h^A$ the cannonical map $R \otimes_A F \longrightarrow h_A \otimes_{\mathcal{A}} F \cong F(A)$ is a monomorphism in Ab.

**Proof.** A simple application of the dual version of Baer's testing lemma. ■

**Corollary.** $(\Sigma_{A \in |\mathcal{A}|} h^A)[\mathbb{Q}/\mathbb{Z}]$ is an injective cogenerator for $\mathcal{A}$-mod.

**Proof.** Clearly $(\Sigma_{A \in |\mathcal{A}|} h^A)[\mathbb{Q}/\mathbb{Z}]$ is injective in $\mathcal{A}$-mod. We have to show that if $M \in \mathcal{A}$-mod and $M \neq 0$ then $\mathcal{A}\text{-mod}(M, (\Sigma_{A \in |\mathcal{A}|} h^A)[\mathbb{Q}/\mathbb{Z}]) \neq 0$. But

$$\text{mod-}\mathcal{A}(M, (\Sigma_{A \in |\mathcal{A}|} h^A)[\mathbb{Q}/\mathbb{Z}]) = \mathrm{Ab}(M \otimes_{\mathcal{A}} (\Sigma_{A \in |\mathcal{A}|} h^A), \mathbb{Q}/\mathbb{Z})$$

$$= \mathrm{Ab}(\Sigma_{A \in |\mathcal{A}|} (M \otimes_{\mathcal{A}} h^A), \mathbb{Q}/\mathbb{Z}) = \mathrm{Ab}(\Sigma_{A \in |\mathcal{A}|} M(A), \mathbb{Q}/\mathbb{Z})$$

then $\mathrm{Ab}(\Sigma_{A \in |\mathcal{A}|} M(A), \mathbb{Q}/\mathbb{Z}) = 0$ implies $\Sigma_{A \in |\mathcal{A}|} M(A) = 0$ which means that for all $A \in |\mathcal{A}|$, $M(A) = 0$, i.e. $M = 0$, a contradiction. ■

**Corollary.** Every $M \in \mathcal{A}$-mod is a subfunctor of an injective.

**Proposition.** $F \in \mathcal{A}$-mod is flat if and only if for all $A \in |A|$ and $R \leq h^A$ such that $R$ is finitely generated right ideal the cannonical map $R\otimes_{\mathcal{Q}}F \longrightarrow h_A\otimes_{\mathcal{Q}}F \cong F(A)$ is a monomorphism in Ab.

Proof. It suffices to show that if $R\otimes_{\mathcal{Q}}F \longrightarrow F(A)$ is monic for finitely generated right ideals $R \subseteq h_A$, then it is monic for arbitrary right ideals. Let $R \subseteq h_A$ be an arbitrary right ideal and $\Sigma_{i=1}^{n} a_i \otimes_{\mathcal{Q}} x_i \in R\otimes_{\mathcal{Q}}F$ where $a_i \in R(A_i)$ and $x_i \in F(A_i)$. Then $\Sigma_{i=1}^{n} a_i \mathcal{Q} \subseteq h_A$ is finitely generated with $\Sigma_{i=1}^{n} a_i \mathcal{Q} \longrightarrow R \longrightarrow h_A$ and the composed map $\Sigma_{i=1}^{n} a_i \mathcal{Q}\otimes_{\mathcal{Q}}F \longrightarrow R\otimes_{\mathcal{Q}}F \longrightarrow F(A)$ is monic, then so is $\Sigma_{i=1}^{n} a_i \mathcal{Q}\otimes_{\mathcal{Q}}F \longrightarrow R\otimes_{\mathcal{Q}}F$. Hence if $\Sigma_{i=1}^{n} F(a_i)(x_i) = 0$ in $F(A)$, then $\Sigma_{i=1}^{n} a_i \otimes_{\mathcal{Q}} x_i = 0$ in $R\otimes_{\mathcal{Q}}F$. ■

**Lemma.** Let $\{y_i \in N(A_i) \mid i \in I\}$ be a set of generators of $N \in \mathcal{Q}$-mod and $\{x_i \in M(A_i) \mid i \in I\}$ be a set of elements of $M \in$ mod-$\mathcal{Q}$ such that all but finitely many $x_i$ are 0. Then $\Sigma_{i\in I} x_i \otimes_{\mathcal{Q}} y_i = 0$ in $M\otimes_{\mathcal{Q}}N$ if and only if there exists $\{u_j \in M(A_j) \mid j=1,\cdots,n\}$ and a family $\{a_{ji} \in \mathcal{Q}(A_j, A_i) \mid i \in I,\ j=1,\cdots,n\}$ such that:

(i) All but finitely many $A_{ji} = 0$,

(ii) $\Sigma_{i\in I} N(a_{ji})(y_i) = 0 \in N(A_j)$ for $j = 1,\cdots,n$,

(iii) $\Sigma_{j=1}^{n} M(a_{ji})(u_j) = x_i \in M(A_i)$ for $i \in I$.

Proof. It is clear that the given conditions are sufficient to make a $\Sigma_{i\in I}x_i\otimes_{\mathcal{Q}}y_i = 0$, because they give

$$\Sigma_{i\in I}x_i\otimes_{\mathcal{Q}}y_i = \Sigma_{i\in I}[(\Sigma_{j=1}^{n}M(a_{ji}))(u_j)\otimes_{\mathcal{Q}}y_i]$$
$$= \Sigma_{j=1}^{n}\Sigma_{i\in I}M(a_{ji})(u_j)\otimes_{\mathcal{Q}}y_i = \Sigma_{j=1}^{n}u_j\otimes_{\mathcal{Q}}(\Sigma_{i\in I}N(a_{ji})(y_i))$$
$$= \Sigma_{j=1}^{n}u_j\otimes_{\mathcal{Q}}0 = 0.$$

Suppose $\Sigma_{i\in I}x_i\otimes y_i = 0 \in M\otimes_{\mathcal{Q}}N$. Since $\{y_i \in N(A_i) | i \in I\}$ generates $N$, there is an epimorphism $\beta:\Sigma_{i\in I}h^{A_i}\longrightarrow N$ induced by $\langle y_i\rangle:h^{A_i}\longrightarrow N$ where $\langle y_i\rangle(A)(a_i) = N(a_i)(y_i)$, in particular $\langle y_i\rangle(A_i)(1_{A_i}) = N(1_{A_i})(y_i) = y_i$. Tensor the exact sequence

$$0\longrightarrow \ker(\beta)\xrightarrow{\alpha}\Sigma_{i\in I}h^{A_i}\xrightarrow{\beta}N\longrightarrow 0$$

with $M$. We have the exact sequence

$$M\otimes_{\mathcal{Q}}\ker(\beta)\xrightarrow{1_M\otimes_{\mathcal{Q}}\alpha}M\otimes_{\mathcal{Q}}\Sigma_{i=1}h^{A_i}\xrightarrow{1_M\otimes_{\mathcal{Q}}\beta}M\otimes_{\mathcal{Q}}N\longrightarrow 0$$

in Ab. The hypothesis $\Sigma_{i\in I}x_i\otimes_{\mathcal{Q}}y_i = 0$ implies $\Sigma_{i\in I}x_i\otimes_{\mathcal{Q}}1_{A_i} \in \ker(1_M\otimes_{\mathcal{Q}}\beta) = \mathrm{Im}(1_M\otimes_{\mathcal{Q}}\alpha)$, hence some finite $j = 1,\cdots,n$ such that $u_j \in M(A_j)$, $z_j \in \ker(\beta)(A_j)$ and $\Sigma_{j=1}^{n}u_j\otimes_{\mathcal{Q}}\alpha(A_j)(z_j) = \Sigma_{i\in I}x_i\otimes_{\mathcal{Q}}1_{A_i}$. But for each $j$, $\alpha(A_j)(z_j) \in \Sigma_{i=1}h^{A_i}(A_j) = \Sigma_{i\in I}\mathcal{Q}(A_j,A_i)$, so $\Sigma_{i\in I}a_{ji} = \alpha(A_j)(z_j)$ for some $a_{ji} \in \mathcal{Q}(A_j,A_i)$ and the fact that

$\beta\alpha = 0$ implies that

$$(\beta\circ\alpha)(A_j)(z_j) = \beta(A_j)(\Sigma_{i\in I}a_{ji}) = \Sigma_{i\in I}N(a_{ji})(y_i) = 0,$$

which is condition (ii). We also have

$$\Sigma_{i\in I}x_i\otimes_{\mathcal{A}}1_{A_i} = \Sigma_{j=1}^{n}u_j\otimes_{\mathcal{A}}\alpha(A_j)(z_j) = \Sigma_{j=1}^{n}u_j\otimes_{\mathcal{A}}\Sigma_{i\in I}a_{ji}$$

$$= \Sigma_{j=1}^{n}u_j\otimes_{\mathcal{A}}\Sigma_{i\in I}a_{ji}1_{A_i} = \Sigma_{i\in I}\Sigma_{j=1}^{n}u_j\otimes_{\mathcal{A}}a_{ji}1_{A_i}$$

$$= \Sigma_{i\in I}\Sigma_{j=1}^{n}M(a_{ji})(u_j)\otimes_{\mathcal{A}}1_{A_i}.$$

Under the isomorphism $M\otimes_{\mathcal{A}}(\Sigma_{i\in I}h^{A_i}) \cong \Sigma_{i\in I}M(A_i)$, this gives $x_i = \Sigma_{j=1}^{n}M(a_{ji})(u_j)$ for each $i \in I$, i.e. condition (iii). ∎

Now we can characterize flat functors internally.

**Theorem.** $F \in \mathcal{A}$**-mod is flat if and only if it satisfies: if $\Sigma_{i=1}^{n}F(a_i)(x_i) = 0$ for $a_i \in \mathcal{A}(A,A_i)$ and $x_i \in M(A_i)$, then there exist $u_j \in F(A_j)$, $j=1,\cdots,m$ and $b_{ij} \in \{\mathcal{A}(A_i,A_j)\,|\,1 \le i \le n, 1 \le j \le m\}$ such that**

**(a) $\Sigma_{i=1}^{n}a_ib_{ij} = 0 \in \mathcal{A}(A,A_j)$ for each $j$.**

**(b) $\Sigma_{j=1}^{m}F(b_{ij})(u_j) = x_i$ for each $i$.**

Proof. Suppose $F$ is flat and $\Sigma_{i=1}^{n}F(a_i)(x_i) = 0$ for some $a_i \in \mathcal{A}(A,A_i)$ and $x_i \in F(A_i)$. Put $R = \sum_{i=1}^{n} a_i \mathcal{A}$ and then $R \subseteq h^A$ is a finitely generated right ideal. The flatness of $F$ implies $R \times_{\mathcal{A}} F \longrightarrow F(A)$ is monic, so we must have $\sum_{i=1}^{n} a_i \otimes_{\mathcal{A}} x_i = 0$ in $R \otimes_{\mathcal{A}} F$ and we can apply the previous lemma.

Conversely, if $F \in \mathcal{A}$-mod satisfies the condition, let $R \subseteq h_A$ be an arbitrary right ideal and consider the map $R \otimes_{\mathcal{A}} F \longrightarrow F(A)$ in Ab. If $\sum_{i=1}^{n} a_i \otimes_{\mathcal{A}} x_i \in R \otimes_{\mathcal{A}} F$ where $a_i \in R(A_i)$ and $x_i \in F(A_i)$ goes to zero in $F(A)$, this means $\sum_{i=1}^{n} F(a_i)(x_i) = 0$. Then there exist $u_j \in F(A_j)\}$ for $j=1,\cdots,m$ and $b_{ij} \in (A_i,A)$ for $1 \leq i \leq n, 1 \leq j \leq m$ such that (a) and (b) are satisfied. We have

$$\sum_{i=1}^{n} a_i \otimes_{\mathcal{A}} x_i = \sum_{i=1}^{n} a_i \otimes_{\mathcal{A}} (\sum_{j=1}^{m} F(b_{ij})(u_j))$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i \otimes_{\mathcal{A}} F(b_{ij})(u_j)) = \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i b_{ij} \otimes_{\mathcal{A}} u_j)$$
$$= \sum_{j=1}^{m} (\sum_{i=1}^{n} a_i b_{ij} \otimes_{\mathcal{A}} u_j) = \sum_{j=1}^{m} (0 \otimes_{\mathcal{A}} u_j) = 0.$$

Hence the map is monic. ∎

## §4. Regular ringoids.

**Definition. (1) A ringoid $\mathcal{A}$ is v.N.-regular (for von Neumann) if for each $a \in \mathcal{A}(A,B)$ there exists some $b \in \mathcal{A}(B,A)$ such that $a = aba$.**

(2) $e \in \mathcal{A}(A,A)$ is idempotent if $e^2 = e$.

**Proposition.** The following are equivalent:

(i) $\mathcal{A}$ is v.N.-regular

(ii) Every principal right ideal of $\mathcal{A}$ is generated by an idempotent.

(iii) Every finitely generated right ideal of $\mathcal{A}$ is generated by an idempotent.

Proof. (iii)$\Longrightarrow$(ii) is obvious.

(i)$\Longrightarrow$(ii): Suppose $\mathcal{A}$ is v.N.-regular and $a \in \mathcal{A}(A,B)$, choose $b \in \mathcal{A}(B,A)$ such that $a = aba$ then $ab$ is idempotent since $(ab)^2 = (ab)(ab) = (aba)b = ab$. Now recall that $a\mathcal{A}(X) = \{ax \in \mathcal{A}(A,X) \mid x \in \mathcal{A}(B,X)\}$ and observe that $a\mathcal{A} = (aba)\mathcal{A} \subseteq (ab)\mathcal{A} \subseteq a\mathcal{A}$. Hence $(ab)\mathcal{A} = \mathcal{A}a$.

(ii)$\Longrightarrow$(i): Given $a \in \mathcal{A}(A,B)$ choose an idempotent $e \in \mathcal{A}(A,A)$ such that $e\mathcal{A} = a\mathcal{A}$. But $a \in e\mathcal{A}(B) = a\mathcal{A}(B)$ implies $a = ea'$ for some $a' \in (A,B)$ then $ea = e(ea') = e^2a' = ea' = a$. Similarly $e \in e\mathcal{A}(A) = a\mathcal{A}(A)$ implies $e = ab$ for some $b \in \mathcal{A}(B,A)$. Hence $a = ea = aba$.

(ii)$\Longrightarrow$(iii): It suffices to show that if $e,e' \in \mathcal{A}(A,A')$ are idempotent then $e\mathcal{A} + e'\mathcal{A}$ is principal. First we observe $e\mathcal{A} + (e' - ee')\mathcal{A} \subseteq e\mathcal{A} + e'\mathcal{A}$. On the other hannd if $ex + e'x' \in (e\mathcal{A} + e'\mathcal{A})(B)$ then

$ex + e'x' = ex + (1_A - e + e)e'x' = ex + (1_A - e)e'x'$
$+ ee'x' = e(x + e'x') + (e' - ee')x \subseteq (e\mathcal{A} + (e'$
$- ee')\mathcal{A})(B)$

Hence equality holds. Consider the principal right ideal $(e' - ee')\mathcal{A}$, there exists $f \in \mathcal{A}(A,A)$ such that $e' - ee' = (e' - ee)f(e' - ee')$. Put $f' = (e' - ee')f$, then $f'$ is idempotent and $(e' - ee')\mathcal{A} = f'\mathcal{A}$, moreover $ef' = e(e' - ee')f = ee'f - e^2e'f = 0$. Hence $(e + f' + f'e)f' = f'^2 = f'$. But we also have $(e + f' - f'e)e = e + f'e - f'e^2 = e$, so $(e\mathcal{A} + f'\mathcal{A}) \subseteq (e + f - f'e)\mathcal{A}$. This implies $e\mathcal{A} + e'\mathcal{A} = e\mathcal{A} + (e' - ee')\mathcal{A} = e\mathcal{A} + f'\mathcal{A} = (e + f' - f'e)\mathcal{A}$.

**Corollary.** **If $\mathcal{A}$ is v.N.-regular, then every $M \in \mathcal{A}$-mod is flat.**

**Proof.** If $\mathcal{A}$ is v.N.-regular, then condition (iii) in the previous proposition says that every finitely generated right ideal is generated by an idempotent. Let $R \subseteq h_A$ a finitely generated right ideal, then $R = e\mathcal{A}$ for some idempotent $e \in \mathcal{A}(A,A)$. Hence $(1_A - e)\mathcal{A}\Sigma_{e\mathcal{A}} \cong \;= h_A$. Now let $M \in \mathcal{A}$-mod and tensor it with $h_A$ we have $h_A \otimes_{\mathcal{A}} M \cong ((1_A - e)\mathcal{A}\otimes_{\mathcal{A}} M)\Sigma(e\mathcal{A}\otimes_{\mathcal{A}} M) \cong M(A)$ so $e\mathcal{A}\otimes_{\mathcal{A}} M \rightarrowtail M(A)$. ■

**Definition.** (1) A $*$-ringoid is a pair $(\mathcal{A},*)$ where $\mathcal{A}$ is a ringoid and $*: \mathcal{A} \longrightarrow \mathcal{A}^{op}$ is an additive functor such that $** = 1_{\mathcal{A}}$. Given $a \in \mathcal{A}(A,B)$ we shall denote $*(a) \in \mathcal{A}(B,A)$ by $a^*$.

(2) Let $a \in \mathcal{A}(A,A)$, then $a$ is a self-adjoint if $a^* = a$ and $a$ is a projection if it is both self-adjoint and idempotent.

Here are some useful properties of $*$:

Proposition. (a) $0^*_{AB} = 0_{BA}$ and $1^*_A = 1_A$ for all $A,B \in \mathcal{A}$.

(b) $(a+b)^* = a^* + b^*$ and $a^{**} = a$ for all $a,b \in \mathcal{A}(A,B)$.

(c) If $a \in \mathcal{A}(A,B)$ and $b \in \mathcal{A}(B,C)$, then $(ab)^* = b^*a^*$.

(d) If $a \in \mathcal{A}(A,B)$ and $a^* = 0 \in \mathcal{A}(B,A)$, then $a = a^{**} = 0_{AB}$.

(e) If $R \subseteq h_A$ is a right ideal, then $R^* \subseteq h^A$ is a left ideal where $R^*(B) = \{a^* \in \mathcal{A}(B,A) \mid a \in R(B) \subseteq \mathcal{A}(A,B)\}$.

Proof. The only statement that requires proof is (e): let $a \in R(B)$ and $b \in \mathcal{A}(B',B)$, then $a^* \in R^*(B) \subseteq \mathcal{A}(B,A)$ and we must show $ba^* \in R^*(B')$. But $ba^* = b^{**}a^* = (ab^*)^*$, hence $b^* \in \mathcal{A}(B,B')$ and $R$ being a right ideal implies that $a^*b \in R(B')$ and $ba^* \in R^*(B')$. ■

**Definition.** Let $\mathcal{a}$ be a ringoid, $a \in \mathcal{a}(A,B)$, $R \subseteq h_A$ a right ideal and $L \subseteq h^A$ a left ideal then

(i) The left annihilator of a is given by

$$\ell(a)(X) = \{x \in \mathcal{a}(X,A) \mid xa = 0\};$$

(ii) The right annihilator of a is given by

$$r(a)(X) = \{a \in \mathcal{a}(B,X) \mid ax = 0\}.$$

Moreover we can define

(iii) The left annihilator of $R$ is given by

$\ell(R)(X) = \{x \in \mathcal{a}(X,A) \mid$ for all $A' \in |\mathcal{a}|$ and $a \in R(A')$, $xa = 0\}$;

(iv) The right annihilator of $L$ is given by

$r(L)(X) = \{x \in \mathcal{a}(A,X) \mid$ for all $A' \in |\mathcal{a}|$ and $a \in L(A')$, $ax = 0\}$.

Remark. It is easy to verify that

(a) A left annihilator is a left ideal and a right annihilator is a right ideal.

(b) If $\mathcal{a}$ is a $*$-ringoid, then $r(a) = (\ell(a^*))^*$ and $\ell(a) = (r(a^*))^*$ for $a \in \mathcal{a}(A,B)$.

**Definition.** Given a $*$-ringoid $\mathcal{a}$, then

(i) $*$ is proper if for all $a \in \mathcal{a}(A,B)$, $aa^* = 0_A$ implies $a = 0_{AB}$. It follows that if $a^*a = 0_B$, then $a^* = 0$ and $a = 0$ as well.

(ii) $\mathcal{a}$ is Rickart if for all $a \in \mathcal{a}(A,B) \ell(a) = \mathcal{a}e$ for some projection $e \in \mathcal{a}(A,A)$. It follows that $r(a) = (\ell(a^*))^* = (\mathcal{a}e')^* = e'\mathcal{a}$.

**Lemma.** If $\mathcal{a}$ is a proper $*$-ringoid then $ab = 0$ if and only if $a^*ab = 0$.

**Proof.** $a^*ab =$ implies $b^*a^*ab = 0$ which implies that $(ab)^*ab = 0$ and hence that $ab = 0$. Similarly, $ab = 0$ if and only if $abb^* = 0$ if $*$ is proper. ∎

**Lemma.** If $\mathcal{a}$ is $*$-Rickart then $*$ is proper.

**Proof.** Let $a \in \mathcal{a}(A,B)$ and $a^*a = 0_B$, then choose $e \in \mathcal{a}(A,A)$ a projection such that $\ell(a) = \mathcal{a}e$. Hence $a^* \in \ell(a)(B) = (e\mathcal{a})(B)$. So $a^* = xe$ for some $x \in \mathcal{a}(B,A)$ which implies $a^*e = xe^2 = xe = a^*$, and hence $a = a^{**} = (a^*e)^* = e^*a = ea$. But $e \in (\mathcal{a}e)(A) = \ell(a)(A)$ so a

$= ea = 0$. ■

**Definition.** Given a $*$-ringoid $\mathcal{A}$ we say that $\mathcal{A}$ is $*$-regular if $*$ is proper and $\mathcal{A}$ is v.N.-regular. Clearly, if $\mathcal{A}$ is v.N.-regular and $*$-Rickart, then $\mathcal{A}$ is $*$-regular.

**Theorem.** The following are equivalent for a $*$-category $\mathcal{A}$:

(1) $\mathcal{A}$ is $*$-regular.

(2) For all $a \in \mathcal{A}(A,B)$ there exists a projection $e \in \mathcal{A}(A,A)$ such that $a\mathcal{A} = e\mathcal{A}$.

(3) $\mathcal{A}$ is v.N.-regular and $*$-Rickart.

Proof. (3)$\Longrightarrow$(1) is done.

To show (1)$\Longrightarrow$(2) we need the following lemma:

**Lemma.** If $\mathcal{A}$ is a proper $*$-category and $a \in \mathcal{A}(A,B)$, then $r(a) = r(\mathcal{A}a)$ and $\ell(a) = \ell(a\mathcal{A})$.

Proof. Clearly $r(a) \subseteq r(\mathcal{A}a)$. Let $x \in r(\mathcal{A}a)(X)$, then for all $A' \in |\mathcal{A}|$ and $a' = \mathcal{A}(A',A)$, $a'ax = 0$. In particular pick $A' = B$ and $a' = a^* \in \mathcal{A}(B,A)$ we have $a^*ax = 0$ so $ax = 0$. Hence $x \in r(a)$. Similarly $\ell(a) = \ell(a\mathcal{A})$. ■

Now let $a \in \mathcal{A}(A,B)$, then we can show $\mathcal{A}a = \mathcal{A}e$ for some idempotent $e \in \mathcal{A}(B,B)$ and then it is easy to verify $r(e) = (1_B - e)\mathcal{A}$ and $\mathcal{A}e = \ell(1_B - e)$. So we have $r(a) = r(\mathcal{A}a) = r\mathcal{A}e) = (1_B - e)\mathcal{A}$ and $\ell(r(a)) = \ell((1_B - e)\mathcal{A}) = \ell(1_B - e) = \mathcal{A}e = \mathcal{A}a$. Hence for all $a \in \mathcal{A}(A,B)\ell(r(a)) = \mathcal{A}a$, in particular for all $a \in \mathcal{A}(A,B)$, $\ell(r(a^*a)) = \mathcal{A}(a^*a)$. But since $ab = 0$ if and only if $a^*ab = 0$, $r(a) = r(a^*a)$ and $\mathcal{A}a = \mathcal{A}(a^*a)$. Since $a \in \mathcal{A}a(A) = (a^*a)\mathcal{A}(A)$ implies there exists $b \in \mathcal{A}(A,B)$ such that $a = ba^*a$. Now put $f = ba^*$ then $f^* = ab^*$ and $f = ba^* = b(ba^*a)^* = ba^*ab^* = ff^*$. This implies $f^* = (ff^*)^* = ff^* = f$ and $f^2 = ff = ff^* = f$. Moreover we have $a\mathcal{A} = f\mathcal{A}$ since $a = ba^*a = fa$ and $f = f^* = ab^*$.

(2)$\Longrightarrow$(3) Let $a \in \mathcal{A}(A,B)$ then $e\mathcal{A} = a\mathcal{A}$ for some projection $e \in \mathcal{A}(A,A)$. This implies $\ell(a) = \ell(a\mathcal{A}) = \ell(e\mathcal{A}) = \ell(e) = \mathcal{A}(1_A - e)$. So $\mathcal{A}$ is a Rickart $*$-ringoid. On the other hand, $e\mathcal{A} = a\mathcal{A}$ implies there exist $b \in \mathcal{A}(B,A)$ and $f \in \mathcal{A}(A,B)$ such that $e = ab$ and $a = ef$. So we have $ea = aba$ and $ea = e(ef) = ef = a$. Hence $a = aba$. ■

CHAPTER 3

## §1. The category $\mathcal{A}$-mod-$\mathcal{B}$.

Recall that if $\mathcal{A}$ and $\mathcal{B}$ are ringoids, then the additive functor category $\mathrm{Ab}^{\mathcal{A}^{op}\times\mathcal{B}}$ is denoted by $\mathcal{A}$-mod-$\mathcal{B}$. If $K \in \mathcal{A}$-mod-$\mathcal{B}$ and $N \in \mathcal{A}$-mod, the tensor product $K\otimes N \in \mathcal{A}$-mod is given by

$$(K\otimes N)(A) = K(A,-)\otimes_{\mathcal{B}}N \text{ for every } \mathcal{A} \in |\mathcal{A}|.$$

Similarly if $M \in \mathcal{A}$-mod, then $M\otimes_{\mathcal{A}}K \in$ mod-$\mathcal{B}$ is given by

$$(M\otimes K)(B) = M\otimes_{\mathcal{A}}K(-,B) \text{ for every } \mathcal{B} \in |\mathcal{B}|.$$

It is easy to verify the desired functorial property, moreover we have $K\otimes h^B \cong K(-,B)$ and $h_A\otimes K \cong K(A,-)$ for every $A \in |\mathcal{A}|$ and $B \in |\mathcal{B}|$.

If $K \in \mathcal{A}$-mod-$\mathcal{B}$ and $T \in \mathcal{A}$-mod, the hom functor $\mathrm{Hom}(K,T) \in \mathcal{B}$-mod is given by

$$\mathrm{Hom}(K,T)(B) = \mathcal{A}\text{-mod}(K(-,B),T) \text{ for every } B \in |\mathcal{B}|.$$

Similarly if $S \in$ mod-$\mathcal{B}$, then $\mathrm{Hom}(K,S) \in$ mod-$\mathcal{A}$

$$\mathrm{Hom}(K,S) = \mathcal{B}\text{-mod}(K(A,-),S) \text{ for every } A \in |\mathcal{A}|.$$

**Proposition.** If $M \in \mathcal{A}$-mod, $N \in \mathcal{B}$-mod and $K \in \mathcal{A}$-mod-$\mathcal{B}$, there is a natural isomorphism

$$\mathcal{A}\text{-mod}(K \otimes N, M) \xrightarrow{\varphi} \mathcal{B}\text{-mod}(N, \mathrm{Hom}(K,M)).$$

**Proof.** First we observe that it is routine to show $K\otimes-$ is right exact and preserves exact sums. Since $K\otimes-$ preserves direct limits, it suffices to show the isomorphism for the representable. If $N = h^B$ for some $B \in |\mathcal{B}|$, then $\mathcal{A}\text{-mod}(K\otimes h^B), M) \cong \mathcal{A}\text{-mod}(K, (-,B), M)$ $= \mathrm{Hom}(K,M)(B) \cong \mathcal{B}\text{-mod}(h^B, \mathrm{Hom}(K,M)$. ∎

The above proposition states that every $K \in \mathcal{A}$-mod-$\mathcal{B}$ induces a pair of adjoint functors:

$$\begin{tikzcd} \mathcal{B}\text{-mod} \arrow[r, shift left, "K\otimes-"] & \mathcal{A}\text{-mod} \arrow[l, shift left, "\mathrm{Hom}(K,-)"] \end{tikzcd}$$

The next proposition says every pair of adjoint functors as above is induced by some $K \in \mathcal{A}$-mod-$\mathcal{B}$.

**Proposition.** The following assertions are equivalent for a functor $\mathcal{B}\text{-mod} \longrightarrow \mathcal{A}\text{-mod}$:

(1) $F$ has a right adjoint.

(2) $F$ is right exact and preserves direct sums.

(3) $F \cong K\otimes-$ for some $K \in \mathcal{A}\text{-mod-}\mathcal{B}$ which is unique up to isomorphism.

Proof. (1)$\Rightarrow$(2) and (3)$\Rightarrow$(1) are clear. Suppose $F$ is right exact and preserves direct sums. Put $K(A,B) = F(h^B)(A)$, then clearly $K \in \mathcal{A}\text{-mod-}\mathcal{B}$. If $N \in \mathcal{B}\text{-mod}$, there is an exact sequence

$$\Sigma_{j\in J}h^{B_j} \longrightarrow \Sigma_{i\in I}h^{B_i} \longrightarrow N \longrightarrow 0.$$

Applying both functors $K\otimes-$ and $F$ to this sequence, we obtain the following commutative diagram with exact rows.

$$\begin{array}{ccccccc}
\Sigma_{j\in J}K(-,B_j) & \longrightarrow & \Sigma_{i\in I}K(-,B_i) & \longrightarrow & K\otimes N & \longrightarrow & 0 \\
\cong\downarrow & & \cong\downarrow & & & & \\
\Sigma_{j\in J}F(h^{b_j}) & \longrightarrow & \Sigma_{i\in I}F(h^{B_i}) & \longrightarrow & F(N) & \longrightarrow & 0
\end{array}$$

Hence there is an induced isomorphism $K\otimes N \longrightarrow F(N)$.

Corollary. If $M \in \mathcal{A}\text{-mod}$, $K \in \mathcal{A}\text{-mod-}\mathcal{B}$ and $N \in \mathcal{B}\text{-mod}$, there is a natural isomorphism

$$(M\otimes K)\otimes N \longrightarrow M\otimes(K\otimes N)$$

Proof. Let $G \in \mathbf{Ab}$ then for every $B \in |\mathcal{B}|$ we have

$\mathrm{Hom}(K,M[G](B)) = \mathcal{A}\text{-mod}(K(-,B),M[G]) \cong \mathrm{Ab}(M\otimes K(-,B),G)$
$= (M\otimes K)[G](B)$.

This implies that

$\mathrm{Ab}(M\otimes(K\otimes N),G) \cong \mathcal{A}\text{-mod}(K\otimes N,M[G]) \cong \mathcal{B}\text{-mod}(N,\mathrm{Hom}(K,M[G])$
$\cong \mathcal{B}\text{-mod}(N,(M\otimes K)[G]) \cong \mathrm{Ab}((M\otimes K)\otimes N,G)$. ■

If $K_1 \in \mathcal{A}\text{-mod-}\mathcal{B}$ and $K_2 \in \mathcal{B}\text{-mod-}\mathcal{C}$, the tensor product $K_1\otimes K_2 \in \mathcal{A}\text{-mod-}\mathcal{C}$ is given by

$(K_1\otimes K_2)(A,C) = K_1(A_1,-)\otimes K_2(-,C)$ for every $A \in |\mathcal{A}|$ and $C \in |\mathcal{C}|$

**Corollary. If $N \in \mathcal{C}$-mod, there is a natural isomorphism**

$$(K_1\otimes K_2)\otimes N \longrightarrow K_1\otimes(K_2\otimes N)$$

**in $\mathcal{A}$-mod.**

**Proof.** If $N' \in \mathcal{A}$-mod, we observe that for every $C \in |\mathcal{C}|$,

$\mathrm{Hom}(K_1\otimes K_2,N')(C) = \mathcal{A}\text{-mod}((K_1\otimes K_2)(-,C),N')$
$\cong \mathcal{A}\text{-mod}(K_1\otimes K_2(-,C),N') \cong \mathcal{B}\text{-mod}(K_2(-,C),\mathrm{Hom}(K_1,N'))$.

This implies that $\mathrm{Hom}(K_1\otimes K_2,N')$ $\cong \mathrm{Hom}(K_2,\mathrm{Hom}(K_1,N'))$ in $\mathcal{C}$-mod. Now if $N' \in \mathcal{A}$-mod we have

$$\mathcal{A}\text{-mod}((K_1 \otimes K_2) \otimes N, N') \cong \mathcal{C}\text{-mod}(N, \mathrm{Hom}(K_1 \otimes K_2, N'))$$
$$\cong \mathcal{C}\text{-mod}(N, \mathrm{Hom}(K_2, \mathrm{Hom}(K_1, N'))) \cong \mathcal{B}\text{-mod}(K_2 \otimes N, \mathrm{Hom}(K_1, N'))$$
$$\cong \mathcal{A}\text{-mod}(K_1 \otimes (K_2 \otimes N), N'). \blacksquare$$

If $K \in \mathcal{A}\text{-mod-}\mathcal{A}$, we shall construct a ringoid $\mathcal{A}(K)$ as follows: $|\mathcal{A}(K)| = |\mathcal{A}|$ and given $A, B \in |\mathcal{A}(K)|$ then

$$\mathcal{A}(K)(A,B) = \mathcal{A}(A,B) \times K(A,B) = \{(a,x) \mid a \in \mathcal{A}(A,B), x \in K(A,B)\}.$$

If $(a,x) \in \mathcal{A}(K)(A,B)$ and $(b,y) \in \mathcal{A}(K)(B,C)$, then

$$(a,x)(b,y) = (ab, K(A,b)(x) + K(a,C)(y)) \in \mathcal{A}(K)(A,C).$$

It is routine to verfiy the biadditivity of composition. If $(a,x) \in \mathcal{A}(K)(A,B)$, $(b,y) \in \mathcal{A}(K)(B,C)$ and $(c,z) \in \mathcal{A}(K)(C,D)$,

$$[(a,x)(b,y)](c,z) = (ab, K(A,b)(x) + K(a,C)(y)(c,z))$$
$$= ((ab)c, K(A,c)[K(A,b)(x) + K(a,C)(y)] + K(ab,D)(z))$$
$$= (a(bc), K(A,c)K(A,b)(x) + K(A,c)K(a,C)(y)$$
$$+ K(ab,D)(z)) = (a(bc), K(A,bc)(x) + K(a,D)K(B,c)(y)$$
$$+ K(a,D)K(b,D)(z)) = (a(bc), K(A,bc)(x)$$
$$+ K(a,D)[K(B,c)(y) + K(b,D)(z)]) = (a,x)(bc, K(B,c)(y)$$
$$+ K(b,D)(z)) = (a,x)[(b,y)(c,z)].$$

Hence composition is associative. The ringoid $\mathcal{A}(K)$ is called the trivial extension of $\mathcal{A}$ by $K$. There are two obvious additive functors $u: \mathcal{A} \longrightarrow \mathcal{A}(K)$ and

$\nu : \mathcal{A}(K) \longrightarrow \mathcal{A}$ such that $\nu \circ u = 1_{\mathcal{A}}$ where $u(a) = (a,0)$ and $\nu(a,x) = a$ for $a \in \mathcal{A}(A,B)$ and $(a,x) = \mathcal{A}(K)(A,B)$. Moreover we can embed $K$ as an ideal $\bar{K}$ in $\mathcal{A}(K)$ by sending $x \in K(A,B)$ to $(0,x) \in \mathcal{A}(K)(A,B)$ and $\overline{KK} = 0$. ■

## §2. Additive Kan extensions.

Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be an additive functor, then $F$ induces three additive functors.

$$\begin{tikzcd}[column sep=huge]
\mathcal{A}\text{-mod} \arrow[r, shift left=3, "F^*"] \arrow[r, shift right=3, "F^!"'] & \mathcal{B}\text{-mod} \arrow[l, "F^*"']
\end{tikzcd}$$

such that $F^* \dashv F_* \dashv F^!$.

(1) The construction of $F_*$ is quite easy. Let $N \in \mathcal{B}$-mod then put $F_*(N)(A) = N(F(A))$ and if $a \in \mathcal{A}(A,A')$ we have $F(a) \in \mathcal{B}(F(A),F(A'))$ so we simply put $F_*(N)(a) = N(F(a))$.

(2) Here is how we construct $F^!$. If $M \in \mathcal{A}$-mod and $B \in |\mathcal{B}|$ then put $F^!(M)(B) = \mathcal{A}\text{-mod}(F_*(h^B),M)$ and if $b \in \mathcal{B}(B,B') = \mathcal{B}\text{-mod}(h^B,h^{B'})$ so we simply put $F^!(M)(b) = \mathcal{A}\text{-mod}(F_*(b),M)$.

We show that $\mathcal{A}\text{-mod}(F_*(N),M) \cong \mathcal{B}\text{-mod}(N,F^!(M))$ for $M \in \mathcal{A}\text{-mod}, N \in \mathcal{B}\text{-mod}$. First if $N = h^B$ for some $B \in |\mathcal{B}|$, then

$$\mathcal{B}\text{-mod}(h^B,F^!(M)) \cong F^!(M)(B) = \mathcal{A}\text{-mod}(F_*(h^B),M).$$

The general case follows easily from the fact that $N$ can be written as a colimit of the representables.

(3) The construction of $F^*$: Let $M \in \mathcal{A}\text{-mod}$ and $B \in |\mathcal{B}|$ we put $F^*(M)(B) = (h_B \circ F) \otimes M$ and if $b \in \mathcal{B}(B,B') \cong \mathcal{B}\text{-mod}(h_{B'},h_B)$ we simply put $F^*(M)(b) = (b \circ F) \otimes 1_M$.

We show that $\mathcal{B}\text{-mod}(F^*(M),N) \cong \mathcal{A}\text{-mod}(M,F_*(N))$ for all $M \in \mathcal{A}\text{-mod}$, $N \in \mathcal{B}\text{-mod}$. Let $M = h^A$ for some $A \in |\mathcal{A}|$ then

$$\mathcal{A}\text{-mod}(H^A,F_*(N)) \cong F_*(N)(A) = N(F(A)) \cong \mathcal{B}\text{-mod}(h^{F(A)},N)$$
$$\cong \mathcal{B}\text{-mod}(F^*(h^A),N).$$

The last equality is obtained by calculating $F^*(h^A)$, since

$$F^*(h^A)(B) = (h_B \circ F) \otimes_{\mathcal{A}} h^A \cong (h_B \circ F)(A) = \mathcal{B}(B,F(A)).$$

Again the general case follows easily from the fact that $M$ can be written as a colimit of the representables

Since $F^* \dashv F_*$ there exists, by the previous theorem, $K \in \mathcal{B}\text{–mod–}\mathcal{A}$ such that $F^* \cong K\otimes-$, moreover $K(B,A) = F^*(h^A)(B) \cong \mathcal{B}(B,F(A))$ as we have calculated before. We also have $F_* \cong \mathrm{Hom}(K,-)$.

Example. Recall in Chapter 1, given $\mathcal{A}$ and $\Gamma$ a RMC set of morphisms in $\mathcal{A}$, we constructed $\Gamma^{-1}(\mathcal{A})$ the category of fractions of $\mathcal{A}$ together with an additive functor $\varphi:\mathcal{A}\longrightarrow\Gamma^{-1}(\mathcal{A})$. Moreover $\varphi$ induced a pair of adjoint functors $\Gamma^{-1}(-) \dashv \varphi_*$ between $\mathcal{A}$–mod and $\Gamma^{-1}(\mathcal{A})$–mod. This implies $\Gamma^{-1}(\mathcal{A}) \cong \varphi^*$. We also have $\Phi \in \Gamma^{-1}(-)\text{–mod–}\mathcal{A}$ with $\Phi(A,A') = \Gamma^{-1}(\mathcal{A})(A,\varphi(A'))$ $= \Gamma^{-1}(\mathcal{A})(A,A')$ such that $\Gamma^{-1}(-) \cong \Phi\otimes-$. Notice the maps on the second variable are induced from $\mathcal{A}$.

**Definition. An additive functor $F:\mathcal{A}\longrightarrow\mathcal{B}$ is geometric if $F^*$ is left exact.**

Since $F^* \cong K\otimes-$ where $K \in \mathcal{B}\text{–mod–}\mathcal{A}$ and $K(B,A)$ $= \mathcal{B}(B,F(A))$, it follows that if $F$ is geometric, then for all monomorphism $\alpha:M\longrightarrow N$ in $\mathcal{A}$–mod, $(K\otimes M)(B)\longrightarrow(K\otimes N)(B)$ is monic for all $B \in |\mathcal{B}|$. Equivalently $K(B,-)\otimes M\longrightarrow K(B,-)\otimes N$ is monic for all $B \in |\mathcal{B}|$. Hence we have

**Proposition.** $F:\mathcal{A}\longrightarrow\mathcal{B}$ **is geometric if and only if when**

$K \in \mathcal{B}\text{-mod-}\mathcal{A}$ defined by $K(B,A) = \mathcal{B}(B,F(A))$, then $K(B,-)$ is flat for all $B \in |\mathcal{B}|$.

**Proposition.** $\varphi:\mathcal{A} \longrightarrow \Gamma^{-1}(\mathcal{A})$ is geometric.

**Proof.** It suffices to show that if $\alpha:M \longrightarrow N$ is monic in $\mathcal{A}$-mod then $\Gamma^{-1}(\alpha):\Gamma^{-1}(M) \longrightarrow \Gamma^{-1}(N)$ is monic in $\Gamma^{-1}(\mathcal{A})$-mod. Let $(x,s) \in \Gamma^{-1}(M)(A)$ be represented by $s:X \longrightarrow A$ and $x \in M(X)$ then $\Gamma^{-1}(\alpha)(x,s) = (\alpha(X)(x),s)$ $\in \Gamma^{-1}(N)(A)$. Suppose $(\alpha(X)(x),s) \sim (0,t)$ in $\Gamma^{-1}(N)(A)$ for some $t:Y \longrightarrow A \in \Gamma$. This means there exist $u:Z \longrightarrow A$ $\in \Gamma$ and maps $b \in \mathcal{A}(X,Y)$, $b' \in \mathcal{A}(Z,X)$ such that $u = bt$ $= b's$ and $N(b') \circ \alpha(X)(x) = 0$ in $N(Z)$. But $\alpha$ $\in \mathcal{A}\text{-mod}(M,N)$ so $N(b') \circ \alpha(X)(x) = \alpha(Z) \circ M(b')(x) = 0$. Now use fact $\alpha$ is monic $M(b')(x) = 0$ in $M(Z)$ hence $(x,a) \sim (M(b')(x),u) = (0,u)$ in $\Gamma^{-1}(M)$. ∎

## §3. Morita Equivalences for ringoids.

Let $\mathcal{A}$ and $\mathcal{B}$ be two ringoids. In this section we will examine the significance and implications of an equivalence between the functor categories $\mathcal{A}$-mod and $\mathcal{B}$-mod. First of all we note the following lemma.

**Lemma.** Let $P$ be projective in $\mathcal{A}$-mod so that $\mathcal{A}\text{-mod}(P,-)$ preserves direct sums then $P$ is finitely generated.

**Proof.** Let $\alpha:\sum_{i \in I} h^{A_i} \longrightarrow P$ be an epimorphism in $\mathcal{A}$-mod.

Then there exists $\beta \in \mathcal{A}\text{-mod}(P, \Sigma_{i \in I} h^{A_i})$ such that $\alpha \circ \beta = 1_P$ since $P$ is projective. But $\mathcal{A}\text{-mod}(P, \Sigma_{i \in I} h^{A_i}) = \Sigma_{i \in I} \mathcal{A}\text{-mod}(P, h^{A_i})$ so $\beta = \beta_1 + \beta_2 + \cdots + \beta_n$ where $\beta_j \in \mathcal{A}\text{-mod}(P, h^{A_j})$. Thus we also have $\alpha \circ \beta_1 + \alpha \circ \beta_2 + \cdots + \alpha \circ \beta_n = 1_P$. This implies that if we restrict $\alpha$ to $\Sigma_{j=1}^{n} h^{A_j}$ it is still an epimorphism so $P$ is finitely generated. ■

**Definition.** An object $P$ in an Abelian category $\mathcal{C}$ is called small if $\mathcal{C}(P, -)$ preserve direct sums.

**Definition.** Two categories $\mathcal{C}$ and $\mathcal{D}$ are called equivalent if these are functors $\epsilon : \mathcal{C} \longrightarrow \mathcal{D}$ and $\delta : \mathcal{D} \longrightarrow \mathcal{C}$ such that $\epsilon \circ \delta \cong 1_{\mathcal{D}}$ and $\delta \circ \epsilon \cong 1_{\mathcal{C}}$.

**Proposition.** A functor $\epsilon : \mathcal{C} \longrightarrow \mathcal{D}$ is an equivalence if and only if $\epsilon$ full and faithful and for each $D \in |\mathcal{D}|$ there is a $C \in |\mathcal{C}|$ such that $\epsilon(C) \cong D$.

Proof. See [Pareigis, 1970]. ■

**Corollary.** If $\mathcal{C}$ and $\mathcal{D}$ are equivalent as in the definition then $\epsilon \dashv \delta$ and $\delta \dashv \epsilon$.

**Theorem.** (Freyd) $\mathcal{D}$ is equivalent to $\text{Ab}^{\mathcal{A}}$ for some ringoid $\mathcal{A}$ if and only if $\mathcal{D}$ is an Abelian category with coproducts and a faithful set of small projectives.

Proof (Sketch). If $\mathcal{D}$ is equivalent to $\text{Ab}^{\mathcal{A}}$ then the image of the set $\{h_A | A \in |\mathcal{A}|\}$ is a faithful set of small projectives.

Conversely, suppose $\tau = \{P_i \in |\mathcal{D}| \,|\, i \in I\}$ is a set of small projectives. Considering $\mathcal{B}$ as a full subcategory generated by $\tau$ of $\mathcal{D}$, let $T:\mathcal{D} \longrightarrow \mathrm{Ab}^{\mathcal{B}^{op}}$ be the functor defined by

$$T(D)(P_i) = \mathcal{D}(P_i, D) \text{ for } D \in |\mathcal{D}| \text{ and } P_i \in \tau.$$

Then it is routine to verify $T$ is an equivalence. ■

Now suppose that

$$\begin{tikzcd} \mathcal{A}\text{-mod} \arrow[r, shift left, "\epsilon"] & \mathcal{B}\text{-mod} \arrow[l, shift left, "\delta"] \end{tikzcd}$$

is an equivalence of categories. Then $\epsilon \dashv \delta$ implies there exist $\mathcal{E} \in \mathcal{B}\text{-mod-}\mathcal{A}$ such that $\mathcal{E}\otimes- \cong \epsilon$ and $\delta \cong \mathrm{Hom}(\mathcal{E},-)$. Since we also have $\delta \dashv \epsilon$, there exists $\nabla \in \mathcal{A}\text{-mod-}\mathcal{B}$ such that $\delta \cong \nabla\otimes-$ and $\epsilon \cong \mathrm{Hom}(\nabla,-)$. Moreover we have $\epsilon\circ\delta = 1_{\mathcal{B}\text{-mod}}$ so that given $N \in \mathcal{B}\text{-mod}$

$$\epsilon\circ\delta(N) \cong \epsilon(\nabla\otimes N) \cong \mathcal{E}\otimes(\nabla\otimes N) \cong (\mathcal{E}\otimes\nabla)\otimes N \cong N.$$

In particular, if $N = h^B$ for some $B \in |\mathcal{B}|$,

$$(\mathcal{E}\otimes\nabla)\otimes h^B \cong \mathcal{E}\otimes\nabla(-,B) \cong h^B.$$

By symmetry we obtain

$(\nabla\otimes\mathcal{E})\otimes M \cong M$ for $M \in \mathcal{A}$-mod and $(\nabla\otimes\mathcal{E})\otimes h^A = \nabla\otimes(\mathcal{E}\otimes h^A)$ $\cong \nabla\otimes\mathcal{E}(-,A) \cong h^A$.

**Theorem.** Let $\mathcal{A}$, $\mathcal{B}$ be ringoids and $\mathcal{E} \in \mathcal{A}$-mod-$\mathcal{B}$. Then the following assertions are equivalent:

(a) $\mathcal{E}\otimes-:\mathcal{A}\text{-mod} \longrightarrow \mathcal{B}\text{-mod}$ is an equivalence;

(b) $-\otimes\mathcal{E}:\text{mod-}\mathcal{B} \longrightarrow \text{mod-}\mathcal{A}$ is an equivalence;

(c) $\mathrm{Hom}(\mathcal{E},-):\mathcal{B}\text{-mod} \longrightarrow \mathcal{A}\text{-mod}$ is an equivalence;

(d) $\mathrm{Hom}(\mathcal{E},-):\text{mod-}\mathcal{A} \longrightarrow \text{mod-}\mathcal{B}$ is an equivalence;

(e) $\{\mathcal{E}(-,A) \mid A \in |\mathcal{A}|\}$ is a set of faithful small projectives and for each pair $A,A' \in |\mathcal{A}|$ there is an isomorphism

$$\mathcal{A}(A,A') \cong \mathcal{B}\text{-mod}(\mathcal{E}(-,A),\mathcal{E}(-,A'));$$

(f) $\{\mathcal{E}(B,-) \mid B \in |\mathcal{B}|\}$ is a set of faithful small projectives and for each pair $B,B' \in |\mathcal{B}|$, there is an isomorphism

$$\mathcal{B}(B,B') \cong \mathcal{A}\text{-mod}(\mathcal{E}(B',-),\mathcal{E}(B,-)).$$

Proof. (c)$\Longrightarrow$(a) and (d)$\Longrightarrow$(b) follow from the discussion preceding the theorem.

To show (a)$\Longrightarrow$(e) we let $h^A \in \mathcal{A}$-mod be a representable since $\mathcal{E}\otimes h^A = \mathcal{E}(-,A)$ and the fact that $\mathcal{E}\otimes-$ is an equivalence so $\{\mathcal{E}(-,A) | A \in |\mathcal{A}|\}$ form a set of faithful small projectives. Moreover if we let $A' \in |\mathcal{A}|$, since $\mathcal{E}\otimes_{\mathcal{A}}-$ is an equivalence and $\mathcal{E}\otimes- \dashv \mathrm{Hom}(\mathcal{E},-)$ so $\mathrm{Hom}(\mathcal{E},-)\circ(\mathcal{E}\otimes-)(h^{A'}) \cong h^{A'}$. But $\mathrm{Hom}(\mathcal{E},-)\circ(\mathcal{E}\otimes-)(h^{A'}) = \mathrm{Hom}(\mathcal{E},(-,A'))$. In particular, if $A \in |\mathcal{A}|$, $\mathcal{A}(A,A') = h^{A'}(A) = \mathrm{Hom}(\mathcal{E},(-,A'))(A) \cong \mathcal{B}\text{-mod}(\mathcal{E}(-,A),\mathcal{E}(-,A'))$.

To show (e)$\Longrightarrow$(c) let $\mathcal{B}'$ be the full subcategory of $\mathcal{B}$-mod with $\{\mathcal{E}(-,A) | A \in |\mathcal{A}|\}$ as the set of objects, then $T\colon \mathcal{B}\text{-mod} \longrightarrow \mathcal{B}'\text{-mod}$ defined by

$$T(N)(\mathcal{E}(-,A)) = \mathcal{B}\text{-mod}(\mathcal{E}(-,A),N) \text{ for } N \in \mathcal{B}\text{-mod}.$$

is an equivalence. But by the second condition of (e), $\mathcal{B}' \cong \mathcal{A}$ so $T \cong \mathrm{Hom}(\mathcal{E},-)$ is an equivalence.

Dually we have (b)$\Longrightarrow$(f) and (f)$\Longrightarrow$(d).

To show (a)$\Longrightarrow$(f) requires a little work. First we have $\mathrm{Hom}(\mathcal{E},-) \dashv \mathcal{E}\otimes-$ and $\mathcal{E}\otimes- \dashv \mathrm{Hom}(\mathcal{E},-)$ so there exists $\nabla \in \mathcal{A}\text{-mod-}\mathcal{B}$ such that $\mathrm{Hom}(\mathcal{E},-) \cong \nabla\otimes-$. Then the associativity of tensor products gives $\mathcal{E}\otimes\nabla \cong \mathcal{B}(-,-)$ and $\nabla\otimes\mathcal{E} \cong \mathcal{A}(-,-)$. In particular we obtain $\mathcal{E}(B,-)\otimes\nabla(-,B) \cong \mathcal{B}(B,B)$ and $\nabla(A,-)\otimes\mathcal{E} \cong h_A = \mathcal{A}(A,-)$ for $A \in |\mathcal{A}|$ and $B \in |\mathcal{B}|$. This implies that if $B \in |\mathcal{B}|$ we have $\sum^n_{i=1} e_i \otimes d_i$

$= 1_B$ for some $e_i \in \mathcal{E}(B,A_i)$ and $d_i \in \nabla(A_i,B)$. Let $\varphi:\mathcal{E}(B,-)\longrightarrow\Sigma_{i=1}^n h_{A_i}$ be given by $\varphi(A)(e) = \Sigma_{i=1}^n d_i\otimes e$ for $e \in \mathcal{E}(B,A)$ and let $\psi:\Sigma_{i=1}^n h_{A_i}\longrightarrow\mathcal{E}(B,-)$ be induced by $\psi_i:\nabla(A_i,-)\otimes\mathcal{E}(-,-)\longrightarrow\mathcal{E}(B,-)$, where $\psi_i$ is defined by: If $A \in |\mathcal{A}|$ and $\Sigma_{\mathrm{fin}(i)}d_j\otimes e_j \in \nabla(A_i,-)\otimes_{\mathcal{B}}\mathcal{E}(-,A)$ for some $d_j \in \nabla(A_i,B_j)$, $e_j\otimes\mathcal{E}(B_j,A)$, then $\psi_i(A)(\Sigma_{\mathrm{fin}(j)}d_j\otimes e_j)$ $= e_i\otimes(\Sigma_{\mathrm{fin}(j)}d_j\otimes e_j) = \Sigma_{\mathrm{fin}(j)}e_i\otimes(d_j\otimes e_j)$ $= \Sigma_{\mathrm{fin}(j)}(e_i\otimes d_j)\otimes e_j$. Notice that $\nabla \in \mathcal{E}(-,A) \cong h^A$ and $\mathcal{E}(B,-)\otimes_{\mathcal{A}}h^A = \mathcal{E}(B,A)$, so $\Sigma_{\mathrm{fin}(j)}(e_i\otimes d_j)\otimes e_j$ is indeed in $\mathcal{E}(B,A)$. Since $\Sigma_{i=1}^n e_i\otimes d_i = 1_B$ it is easy to check that $\mathcal{E}(B,-)\longrightarrow\Sigma_{i=1}^n h_{A_i}\longrightarrow\mathcal{E}(B,-) = 1_{\mathcal{E}(B,-)}$. This shows that $\mathcal{E}(B,-)$ is finitely generated and projective in mod-$\mathcal{A}$.

But in Grothendieck categories every finitely generated and projective object is small, so $\mathcal{E}(B,-)$ is small projective for each $B \in |\mathcal{B}|$.

Next we show that $\mathcal{A}$-mod$(\mathcal{E}(B',-),\mathcal{E}(B,-)) \cong \mathcal{B}(B,B')$. Clearly if $b \in \mathcal{B}(B,B')$, $b$ induces $\mathcal{E}(b,-)$ in $\mathcal{A}$-mod$(\mathcal{E}(B',-),\mathcal{E}(B,-))$. If we fix $B \in |\mathcal{B}|$ as before, $\Sigma_{i=1}^n e_i\otimes d_i = 1_B$ for some $e_i \in \mathcal{E}(B,A_i)$, $d \in \nabla(A_i,B)$ and then we can express $b = \Sigma_{i=1}^n\mathcal{E}(b,A_i)(e_i)\otimes d_i$. Suppose $0 = \mathcal{E}(b,-) \in \mathcal{A}$-mod$(\mathcal{E}(B',-),\mathcal{E}(B,-))$. Then in particular $\mathcal{E}(b,A_i) = 0$ for all $i$ which implies $b = 0$; i.e. the map $\mathcal{B}(B,B')\longrightarrow\mathcal{A}$-mod$(\mathcal{E}(B',-),\mathcal{E}(B,-))$ is faithful. On the other hand if $\beta \in \mathcal{A}$-mod$(\mathcal{E}(B',-),\mathcal{E}(B,-))$ and $e$

$\in \mathcal{E}(B',A)$, we note that $\mathcal{E}(B',-) \cong \mathcal{E}(B',-)\otimes(\nabla\otimes\mathcal{E})$ $\cong (\mathcal{E}(B',-)\otimes\nabla)\otimes\mathcal{E}$ and $\mathcal{A}\text{-mod}(\mathcal{E}(B',-),\mathcal{E}(B,-)) \in \mathcal{A}\text{-mod-}\mathcal{A}$ so $\beta(A)(e) = \Sigma_{j=1}^{m} e_j\otimes d_j\otimes\beta(A)(e) \in \mathcal{E}(B,A)$. But $\Sigma_{j=1}^{m} e_j\otimes d_j\otimes\beta(A)(e) = \Sigma_{j=1}^{m}\beta(A)(e_j)\otimes d_j\otimes e$, where $e_j$ $\in \mathcal{E}(B',A_j) d_j \in \nabla(A_j,B')$ such that $\Sigma_{j=1}^{m} e_j\otimes d_j = 1_B$, is obtained from the isomorphism $\mathcal{B}(B',B)$ $\cong \mathcal{E}(B',-)\otimes_{\mathcal{A}}\nabla(-,B')$. Since $\beta(A)(e_j) \in \mathcal{E}(B,A_j)$, we have $\Sigma_{j=1}^{m}\beta(A)(e_j)\otimes d_j \in \mathcal{E}(B,-)\otimes\nabla(-,B') \cong \mathcal{B}(B,B')$, so the map $\mathcal{B}(B,B') \longrightarrow \text{mod-}\mathcal{A}(\mathcal{E}(B',-),\mathcal{E}(B,-))$ is onto.

Trivially $\{\mathcal{E}(B,-) \mid B \in |\mathcal{B}|\}$ is a faithful set in mod-$\mathcal{A}$ since $\nabla(A,-)\otimes\mathcal{E} \cong \mathcal{A}(A,-) = h_A$ for each $A \in |\mathcal{A}|$ and $\{h_A \mid A \in |\mathcal{A}|\}$ form a faithful set of small projectives in mod-$\mathcal{A}$.

Dually we can show that (b)$\Longrightarrow$(e). ■

Remark. We note that the key to the argument lies in the existence of $\mathcal{E} \in \mathcal{B}\text{-mod-}\mathcal{A}$ and $\nabla \in \mathcal{B}\text{-mod-}\mathcal{A}$ such that

$$\nabla\otimes\mathcal{E} \cong \mathcal{B}(-,-) \quad \text{and} \quad \mathcal{E}\otimes\nabla \cong \mathcal{A}(-,-).$$

We called the 4-tuple $(\mathcal{A},\mathcal{E},\nabla,\mathcal{B})$ with the above property a Morita context.

**Corollary.** Given a Morita context $(\mathcal{A},\mathcal{E},\nabla,\mathcal{B})$, we have a ringoid $\mathcal{A}\vee\mathcal{B}$ with

$|\mathcal{A}\vee\mathcal{B}| = |\mathcal{A}|\vee|\mathcal{B}|$,

$(\mathcal{A}\vee\mathcal{B})(A,A') = \mathcal{A}(A,A')$,

$(\mathcal{A}\vee\mathcal{B})(B,B') = \mathcal{B}(B,B')$,

$(\mathcal{A}\vee\mathcal{B})(A,B) = \nabla(A,B)$, and

$(\mathcal{A}\vee\mathcal{B})(B,A) = \Xi(B,A)$,

for $A,A' \in |\mathcal{A}|$ and $B,B' \in |\mathcal{B}|$.

**Proof.** Apply the associativity of tensor products. ∎

## §4. Epimorphism of ringoids.

In this action we shall study the general properties of epimorphisms in the category of ringoids.

Recall that if $\mathcal{A}$ and $\mathcal{B}$ are two ringoids and $F:\mathcal{A}\longrightarrow\mathcal{B}$ an additive functor, a functor $F$ is an epimorphism if for any ringoid $\mathcal{C}$ and additive functors $G,H:\mathcal{B}\longrightarrow\mathcal{C}$, $G\circ F = H\circ F$ implies $G = H$.

If $F:\mathcal{A}\longrightarrow\mathcal{B}$ is epic and $A,A'\in|\mathcal{A}|$, it is well-known that the map $\mathcal{A}(A,A')\longrightarrow\mathcal{B}(FA,FA')$ induced by $F$ may not be onto.

**Lemma.** Let $F:\mathcal{A}\longrightarrow\mathcal{B}$ be epic, then for each $B \in |\mathcal{B}|$ there is some $A \in |\mathcal{A}|$ such that $F(A) = B$; i.e. $F$ is onto on objects.

**Proof.** Suppose there exists $B_0 \in |\mathcal{B}|$ such that $B_0 \notin \{F(A)\,|\,A \in |\mathcal{A}|\}$. Construct a ringoid $\tilde{\mathcal{B}}$ by letting $|\tilde{\mathcal{B}}|$

$= |\mathcal{B}| \cup \{*\}$ and for $B, B' \in |\tilde{\mathcal{B}}|$,

$$\tilde{\mathcal{B}}(B,B') = \begin{cases} \mathcal{B}(B,B'), & \text{if } B \neq * \text{ and } B' \neq *; \\ \mathcal{B}(B_0,B'), & \text{if } B = * \text{ and } B' \neq *; \\ \mathcal{B}(B,B_0), & \text{if } B \neq * \text{ and } B' = *; \\ \mathcal{B}(B_0,B_0), & \text{if } B = * \text{ and } B' = *. \end{cases}$$

Obviously $\tilde{B}$ is made by adding an identical copy of $B_0$ together with its additive structure in $\mathcal{B}$.

Now define $H: \mathcal{B} \longrightarrow \tilde{\mathcal{B}}$ by $H(B) = B$ if $B \in |\mathcal{B}|$ and $H(b) = b$ if $B \in \mathcal{B}(B,B')$.

Also define $G: \mathcal{B} \longrightarrow \tilde{\mathcal{B}}$ by

$$G(B) = \begin{cases} B, & \text{if } B \neq B_0 \\ *, & \text{if } B = B_0 \end{cases}$$

and if $b \in \mathcal{B}(B,B')$, $H(b) = b$.

Note that $G \neq H$, since $G(B_0) = *$ and $H(B_0) = B_0$. But $GF = HF$ since $B_0$ is not in the image of $F$ by assumption, which is a contradiction. ■

Suppose $F: \mathcal{A} \longrightarrow \mathcal{B}$ is epic. Define $Q \in \mathcal{B}\text{-mod-}\mathcal{A}$ and $K \in \mathcal{A}\text{-mod-}\mathcal{B}$ by $Q(B,A) = \mathcal{B}(B,FA)$, $K(A,B) = \mathcal{B}(FA,B)$ for $A \in |\mathcal{A}|$ and $B \in |\mathcal{B}|$. Note that the morphisms of $\mathcal{A}$ acts on $Q$ and $K$ via $F$.

Now consider $Q \otimes K \in \mathcal{B}$-mod-$\mathcal{B}$ and construct the trivial extension $\mathcal{B}(Q \otimes K)$ of $\mathcal{B}$ by $Q \otimes K$ (see Chapter 3, §1). Let $G, H: \mathcal{B} \longrightarrow \mathcal{B}(Q \otimes K)$ be two functors defined by $G(B) = B = H(B)$ if $B \in |\mathcal{B}|$ and suppose $b \in \mathcal{B}(B,B')$, $G(b) = (b,0)$ and $H(b) = (b, b \otimes 1_{B'}, -1_B \otimes b)$. It is easy to check that $G$ and $H$ are additive and note that $H$ makes sense since $F$ is onto on objects.

Observe that $H \circ F = G \circ F$ and $F$ is an epic implies $H = G$. Hence for all $b \in \mathcal{B}(B,B')$, $b \otimes 1_{B'} = 1_B \otimes b$.

Now suppose $F: \mathcal{A} \longrightarrow \mathcal{B}$ is onto on objects and $Q$, $K$ defined as above such that for all $b \in \mathcal{B}(B',B)$, $b \otimes 1_B = 1_B{}' \otimes b$ in $[Q \otimes K](B',B)$. But $F$ induces a pair of adjoint functors:

$$\mathcal{A}\text{-mod} \underset{F_*}{\overset{F^*}{\rightleftarrows}} \mathcal{B}\text{-mod}$$

with $F^* \dashv F_*$, where $F_*(N)(A) = N(FA)$ and morphisms of $\mathcal{A}$ act on $F_*B$ via $F$, for $N \in \mathcal{B}$-mod and $A \in |\mathcal{A}|$. (See § 2.)

Since $F$ is onto on objects, $\mathcal{B}\text{-mod}(N,N') \longrightarrow \mathcal{A}\text{-mod}(F_*N, F_*N')$ is faithful. Suppose $\varphi \in \mathcal{A}\text{-mod}(F_*N, F_*N')$ and $N, N' \in \mathcal{B}$-mod. Let $B \in |\mathcal{B}|$, $x \in N(B)$ and define $\varphi_x \in \mathcal{B}\text{-mod}([Q \otimes K](-,B), N')$ as follows: Let $B' \in |\mathcal{B}|$ and $b' \in Q(B',A) = \mathcal{B}(B',FA)$, $b \in K(A,B)$

$= \mathcal{B}(FA,B)$. Then $b'\otimes b \in [Q\otimes K](B',B)$ and we put $\varphi_x(B')(b'\otimes b) = N'(b'), \varphi(FA)\circ N(b)(x)$ which is simply the composite

$$N(B) \xrightarrow{N(b)} N(FA) \xrightarrow{\varphi(FA)} N'(FA) \xrightarrow{N'(b')} N'(B')$$

applied to $x$. $\varphi_x$ is well-defined since $\varphi$ is $\mathcal{A}$-linear. It is easy to see that $\varphi_x$ is additive and $\mathcal{B}$-linear.

Now let $b \in \mathcal{B}(B',B)$ and $A',A \in |\mathcal{A}|$ such that $FA' = B'$ and $FA = B$. Since $b\otimes 1_B = 1_{B'}\otimes b$, we obtain $\varphi_x(B')(b\otimes 1_B) = N'(b)\circ\varphi(B)(x) = \varphi(B')\circ N(b)(x) = \varphi_x(B')(1_{B'}\otimes b)$. Hence $\varphi \in \mathcal{B}\text{-mod}(N,N')$ as well.

Now suppose $F:\mathcal{A}\longrightarrow\mathcal{B}$ is onto on objects of $\mathcal{B}$ such that $F_*:\mathcal{B}\text{-mod}\longrightarrow\mathcal{A}\text{-mod}$ is full and faithful. Let $\mathcal{C}$ be any ringoid and $G,H:\mathcal{B}\longrightarrow\mathcal{C}$ be two additive functors such that $G\circ F = H\circ F$. If $B \in |\mathcal{B}|$ and $HB \in |\mathcal{C}|$, then $H_*(h^{HB}) \in \mathcal{B}\text{-mod}$ where $H_*:\mathcal{C}\text{-mod}\longrightarrow\mathcal{B}\text{-mod}$ is induced by $H$. We also have $F_*(h^B), F_*\circ H_*(h^{HB}) \in \mathcal{A}\text{-mod}$.

**Lemma.** $G$ induces a map in $\mathcal{A}\text{-mod}(F_*(h^B), F_*\circ H_*(h^{HB}))$.

**Proof.** If $A \in |\mathcal{A}|$,

$$F_*(h^B)(A) = h^A(FA) = \mathcal{B}(FA,B).$$

and

$$F_*H_*(h^{HB})(A) = h^{HB}(H(FA)) = \mathcal{C}(H(FA),HB).$$

Since $F$ is onto on objects of $\mathcal{B}$ and $G \circ F = H \circ F$, $H(B) = G(B)$ for any $B \in |\mathcal{B}|$. Then $GA$ simply sends $\mathcal{B}(FA,B)$ to $\mathcal{C}(G(FA),GB) = \mathcal{C}(H(FA),HB)$. It is easy to check that $G$ is $\mathcal{A}$-linear.

Since $F_*$ is full and faithful, $G \in \mathcal{B}\text{-mod}(h^B,H_*(h^{HB}))$. If $b \in \mathcal{B}(B',B)$ we have the commutative diagram.

$$\begin{tikzcd}
h^B(B) \arrow[r, "GB"] \arrow[d, "h^B(b)"'] & H_*(h^{HB})(B) \arrow[d, "H_*(h^{HB})(b)"] \\
h^B(B) \arrow[r, "GB'"] & H_*(h^{HB})(B')
\end{tikzcd}$$

which is the same as

$$\begin{tikzcd}
\mathcal{B}(B,B) \arrow[r, "GB"] \arrow[d, "\mathcal{B}(b,B)"'] & \mathcal{C}(HB,HB) \arrow[d, "\mathcal{C}(Hb,HB)"] \\
\mathcal{B}(B',B) \arrow[r, "GB'"] & \mathcal{C}(HB,HB)
\end{tikzcd}$$

Then for $1_B \in \mathcal{B}(B,B)$, we have

$$\mathcal{C}(Hb,HB) \circ GB(1_B) = \mathcal{C}(Hb,HB)(1_{HB}) = Hb$$

$$GB' \circ \mathcal{B}(b,B)(1_b) = GB'(b) = Gb.$$

Hence $H = G$ and so $F$ is am epimorphism. Combining these results:

Theorem. Let $F:\mathcal{A}\longrightarrow\mathcal{B}$ be an additive functor between two small preadditive categories. Then the following are equivalent:

(1) $F$ is an epimorphism.

(2) The following two conditions are satisfied:

(i) $F$ is onto on objects of $\mathcal{B}$.

(ii) If $Q \in \mathcal{B}$-mod-$\mathcal{A}$ and $K \in \mathcal{A}$-mod-$\mathcal{B}$ are defined by $Q(B,A) = \mathcal{B}(B,FA)$, $K(A,B) = \mathcal{B}(FA,B)$, then for all $b \in \mathcal{B}(B',B)$, $b\otimes 1_B = 1_{B'}\otimes b$ in $[Q\otimes K](B',B)$.

(3) $F$ is onto on objects of $\mathcal{B}$ and $F_*$ is full.

Corollary. Let $F:\mathcal{A}\longrightarrow\mathcal{B}$ be onto on objects of $\mathcal{B}$ and $Q \in \mathcal{B}$-mod-$\mathcal{A}$, $K \in \mathcal{A}$-mod-$\mathcal{B}$ be defined as before. Then the following are equivalent:

(1) $F_*$ is full.

(2) $\epsilon_N:F^*\circ F_*N\longrightarrow N$ is an isomorphism for each $N \in \mathcal{B}$-mod, where $\epsilon:F^*\circ F_*\longrightarrow 1_{\mathcal{B}}$ is the counit of the adjunction $F^* \dashv F_*$.

(3) The map $Q\otimes K \longrightarrow \mathcal{B}(-,-)$ defined for $b' \in \mathcal{B}(B',FA)$ and $b \in \mathcal{B}(FA,B)$ as $b'\otimes b \longrightarrow b'b$, is a natural isomorphism.

**Proof.** (1)$\Longrightarrow$(2). Since $F$ is onto on objects, $F_*$ is faithful as well.

(2)$\Longrightarrow$(3) Let $H^B \in \mathcal{B}$-mod and $A \in |\mathcal{A}|$, then $F_*(h^B)(A) = h^B(FA) = \mathcal{B}(FA,B)$. Hence $F_*(h^B) = K(-,B)$ and $F^*\circ F_*(h^B) = Q\otimes K(-,B)$.

(3)$\Longrightarrow$(1) Since for all $b \in \mathcal{B}(B',B)$, $b\otimes 1_B = 1_B\otimes b$ in $[Q\otimes K](B',B)$. Then apply the implication (2)$\Longrightarrow$(3) in the previous theorem. ■

**Remark.** In general, $F_*$ is full and faithful if and only if the counit $\epsilon: F^*\circ F_* \longrightarrow 1_{\mathcal{B}}$ is an isomorphism and they imply (3) in the corollary. These arguments do not depend on the fact $F$ is onto on objects of $\mathcal{B}$. But the following example will show the fact that $F_*$ is full and faithful does not guarantee that $F:\mathcal{A} \longrightarrow \mathcal{B}$ is an epimorphism in the category of ringoids.

Given a ringoid $\mathcal{A}$, let $\hat{\mathcal{A}}$ be the idempotent cover of $\mathcal{A}$ (introduced by P. Freyd.) where $|\hat{\mathcal{A}}| = \{(A,e) | A \in |\mathcal{A}|$ and $e$ an idempotent in $\mathcal{A}(A,A)\}$ and $\hat{\mathcal{A}}[(A',e'),(A,e)] = \{a \in \mathcal{A}(A',A) | e'ae = a\}$. If $a:(A',e') \longrightarrow (A,e)$ and $a':(A'',e'') \longrightarrow (A',e')$, composition in $\hat{\mathcal{A}}$ is same as composition in $\mathcal{A}$ and $e$ acts as an identity for $(A,e)$. There is an obvious functor $F:\mathcal{A} \longrightarrow \hat{\mathcal{A}}$ such that $F(A) = (A,1_A)$ for $A \in |\mathcal{A}|$ and which is the identity on morphisms of $\mathcal{A}$. $\hat{\mathcal{A}}$ has the following properties (see [Freyd, 1964]):

(1) $\hat{\mathcal{A}}$ is idempotent complete in the sense that every idempotent in $\hat{\mathcal{A}}$ splits.

(2) If $\mathcal{A}$ is a ringoid, so is $\hat{\mathcal{A}}$ and $F$ is an additive functor.

(3) If $\mathcal{E}$ is idempotent complete, the inclusion $F$ induces an equivalence of categories: Func$(\hat{\mathcal{A}}^{op},\mathcal{E})$ $\cong$ Func$(\mathcal{A}^{op},\mathcal{E})$.

Given $\mathcal{E}$ such that every idempotent may be factored into an epimorphism followed by a monomorphism, then it is easy to verify that $\mathcal{E}$ is idempotent complete. In particular **Ab** is idempotent complete and so $\hat{\mathcal{A}}$–mod $\cong$ $\mathcal{A}$–mod such that $(F)_*$ induces the equivalence.

Hence $(F)_*$ is full and faithful, but clearly $F:\mathcal{A} \longrightarrow \hat{\mathcal{A}}$ is not necessarily onto on objects of $\hat{\mathcal{A}}$.

# CHAPTER 4

This chapter is devoted to a comprehensive study of the general aspects of torsion theory in Abelian categories with injective effacements.

## §1. Topologies and radicals.

Let $\mathcal{A}$ be an Abelian category. We define $\mathrm{sub}:\mathcal{A}^{op} \longrightarrow \mathbf{Set}$ as follows:

For $A \in |\mathcal{A}|$, let $\mathrm{sub}(A)$ be the set of subobjects of $A$. If $a \in \mathcal{A}(B,A)$, $A' \in \mathrm{sub}(A)$ then $B' = \mathrm{sub}(a)(A')$ is given by the pullback

$$\begin{tikzcd} B' \arrow[r] \arrow[d] & B \arrow[d, "a"] \\ A' \arrow[r] & A \end{tikzcd}$$

in $\mathcal{A}$.

**Definition.** **A topology is a natural transformation $j:\mathrm{sub} \longrightarrow \mathrm{sub}$ satisfying**

**(1) $j$ is increasing i.e. if $A' \longrightarrow A$ in $\mathcal{A}$, it can be factored as $A' \longrightarrow j_A(A') \longrightarrow A$.**

(2) $j$ is idempotent, i.e. $j_A^2(A') = j_A(j_A(A')) = j_A(A')$ for $A' \in \mathrm{sub}(A)$.

(3) $j$ is monotone, i.e. if $A'' \longrightarrow A \longrightarrow A$ then $j_A(A'') \longrightarrow j_A(A') \longrightarrow A$.

**Lemma.** Suppose

$$\begin{tikzcd}
A' \arrow[r, tail] \arrow[d] & A \arrow[d, "f"] \\
B' \arrow[r, tail] & B
\end{tikzcd}$$

is a pullback in $\mathcal{A}$. Then

$$\begin{tikzcd}
A' \arrow[r, tail] \arrow[d] & j_A(A') \arrow[r, tail] \arrow[d] & A \arrow[d, "f"] \\
B' \arrow[r, tail] & j_A(B') \arrow[r, tail] & B
\end{tikzcd}$$

are pullback squares.

**Proof.** Since $j$ is a natural transformation we have that

$$\begin{tikzcd}
\mathrm{sub}(B) \arrow[r, "j_B"] \arrow[d, "\mathrm{sub}(f)"'] & \mathrm{sub}(B) \arrow[d, "\mathrm{sub}(f)"] \\
\mathrm{sub}(A) \arrow[r, "j_A"] & \mathrm{sub}(A)
\end{tikzcd}$$

commutes for all $A,B \in |\mathcal{Q}|$ and $f \in \mathcal{Q}(A,B)$.

Now follow $B' \in \mathrm{sub}(B)$ around two directions. Clockwise gives $\mathrm{sub}(f)(j_B(B'))$ such that the square in

$$\begin{tikzcd}
 & \mathrm{sub}(f)(j_B(B')) \arrow[r] \arrow[d] & A \arrow[d, "f"] \\
B' \arrow[r, tail] & j_B(B') \arrow[r, tail] & B
\end{tikzcd}$$

is a pullback. Counter-clockwise we have $j_A(\mathrm{sub}(f)(B')) = j_A(A')$ so we have the following commutative diagram:

$$\begin{tikzcd}
A' \arrow[r, tail] \arrow[d] & j_A(A') \arrow[r, tail] \arrow[d] & A \arrow[d, "f"] \\
B' \arrow[r, tail] & j_B(B') \arrow[r, tail] & B
\end{tikzcd}$$

in which the right hand square and the outer square are pullbacks.

It follows that the left hand square is a pullback. ∎

**Lemma.** If $A'' \leq A' \leq A$, then $j_A(A'') \leq j_{A'}(A'') \leq A$.

**Proof.** $A'' \leq A' \leq A$ implies

$$\begin{tikzcd}
A'' \arrow[r, tail] \arrow[d, "="'] & A' \arrow[d] \\
A'' \arrow[r, tail] & A
\end{tikzcd}$$

is a pullback in $\mathcal{A}$. Now apply the previous lemma. ■

**Definition.** **Let** $A' \subseteq A$ **then** $A'$ **is j-closed in** $A$ **if** $j_A(A') = A'$. $A'$ **is j-dense in** $A$ **if** $j_A(A') = A$.

Clearly, if $A'$ is both j-dense and j-closed in $A$ then $A = A'$.

**Lemma.** **Let** $A' \subseteq A$, **then** $A'$ **is j-dense in** $j_A(A')$ **and** $j_A(A')$ **is j-closed in** $A$.

**Proof.** Clearly $j_A(A')$ is j-closed in $A$. Since $A' \subseteq j_A(A') \subseteq A$

$$\begin{tikzcd}
A' \arrow[r, tail] \arrow[d, "1_{A'}"'] & j_A(A') \arrow[d] \\
A' \arrow[r, tail] & A
\end{tikzcd}$$

is a pullback this implies

$$\begin{tikzcd}
A' \arrow[r, tail] \arrow[d] & j_{j_A(A')}(A') \arrow[r, tail] \arrow[d] & j_A(A') \arrow[d] \\
A' \arrow[r, tail] & j_A(A') \arrow[r, tail] & A
\end{tikzcd}$$

are pullback squares. This gives $j_{j_A(A')}(A') = j_A(A')$.

■

**Lemma.** Given a pullback in $\mathcal{A}$,

$$\begin{tikzcd} B' \arrow[r, tail] \arrow[d] & B \arrow[d] \\ A' \arrow[r, tail] & A \end{tikzcd} .$$

Then (1) $A'$ j-closed in $A$ implies $B'$ is j-closed in $B$.

(2) $A'$ j-dense in $A$ implies $B'$ is j-dense in $B$.

**Proof.** The hypothesis implies we have the pullback squares:

$$\begin{tikzcd} B' \arrow[r, tail] \arrow[d] & j_B(B') \arrow[r, tail] \arrow[d] & B \arrow[d] \\ A' \arrow[r, tail] & j_A(A') \arrow[r, tail] & A \end{tikzcd}$$

from which both results follow easily. ■

**Lemma.** Given $A'' \subseteq A' \subseteq A$ such that $A''$ is j-dense in $A'$ and $A'$ is j-dense in $A$. Then $A''$ is j-dense in $A$.

**Proof.** By previous lemma $j_{A'}(A'') \subseteq j_A(A'')$. Since j is monotone we have $A = j_A(A') = A \cap j_A(j_{A'}(A''))$
$\subseteq j_A(j_A(A'')) = j_A(A'') \subseteq A$. ■

**Proposition.** Suppose $A''$, $A' \in \mathrm{sub}(A)$, then

$$j_A(A''\cap A') = j_A(A'')\cap j_A(A').$$

**Proof.** Since j is monotone, it follows easily that $j_A(A''\cap A') \subseteq j_A(A'')\cap j_A(A')$. Hence it is sufficient to show $A''\cap A'$ is j-dense in $j_A(A'')\cap j_A(A')$. First we have the pullback squares:

$$\begin{tikzcd} A''\cap A' \arrow[r, tail] \arrow[d] & j_{A''}(A''\cap A') \arrow[r, tail] \arrow[d] & A'' \arrow[d] \\ A' \arrow[r, tail] & j_A(A') \arrow[r, tail] & A \end{tikzcd}$$

which implies $A'\cap j_A(A') = j_A(A''\cap A')$ and $A''\cap A'$ is j-dense in it.

Similarly we have the pullback squares

$$\begin{tikzcd} A''\cap j_A(A') \arrow[r, tail] \arrow[d] & j_{j_A(A')}(A''\cap j_A(A')) \arrow[r, tail] \arrow[d] & j_A(A') \arrow[d] \\ A'' \arrow[r, tail] & j_A(A'') \arrow[r, tail] & A \end{tikzcd}$$

which implies $j_{j_A(A')}(A''\cap j_A(A')) = j_A(A'')\cap j_A(A')$ and $A''\cap j_A(A')$ is j-dense in it.

Hence $A''\cap A'$ is j-dense in $j_A(A'')\cap j_A(A')$. So we obtain

$j_A(A'') \cap j_A(A') = j_{j_A(A'') \cap j_A(A')}(A'' \cap A') \subseteq j_A(A'' \cap A')$
$\subseteq j_A(A'') \cap j_A(A')$. ■

Trivially, if $A'' \subseteq A' \subseteq A$, then $j_A(A'' \cap A') = j_A(A'')$ $= j_A(A'') \cap j_A(A') \subseteq j_A(A')$. Therefore the monotonicity axiom for j is equivalent to the above proposition.

**Definition.** **A radical is a pair $(\epsilon, \delta)$ such that $\epsilon : \mathcal{a} \longrightarrow \mathcal{a}$ is a left exact functor, $\delta : \epsilon \longrightarrow 1_{\mathcal{a}}$ is a natural transformation satisfying**

**(1) $\epsilon(A/\epsilon(A)) = 0$;**

**(2) $\epsilon^2(A) = \epsilon(\epsilon(A)) = \epsilon(A)$;**

**(3) $\delta(A) : \epsilon(A) \longrightarrow A$ is monic for all $A \in |\mathcal{a}|$.**

Given an Abelian category $\mathcal{a}$, we are going to show there is one to one correspondence between topologies and radicals.

Suppose j is a topology. Let $A \in |\mathcal{a}|$ and put $\epsilon_j(A) = j_A(0)$ and let $\delta_j(A)$ be the inclusion map.

Suppose $f \in \mathcal{a}(A,B)$ we form the pullback

$$
\begin{tikzcd}
\ker(f) \arrow[r, tail] \arrow[d] & A \arrow[d, "f"] \\
0 \arrow[r, tail] & B
\end{tikzcd}
$$

which implies we have the pullback squares

$$
\begin{tikzcd}
\ker(f) \arrow[r, tail] \arrow[d] & j_A(\ker(f)) \arrow[r, tail] \arrow[d] & A \arrow[d] \\
0 \arrow[r, tail] & j_B(0) \arrow[r, tail] & B
\end{tikzcd}
$$

But $0 \subseteq \ker(f) \subseteq A$ so $j_A(0) \subseteq j_A(\ker(f))$. Hence define $\epsilon_j(f)$ to be composition $j_A(0) \longrightarrow j_A(\ker(f)) \longrightarrow j_B(0)$. Under this definition $\delta_j$ is trivially a natural transformation.

To show $\epsilon_j$ is left exact we need the following lemma:

**Lemma.** If the commutative diagram

$$
\begin{tikzcd}
 & B_{11} \arrow[r] \arrow[d] & B_{12} \arrow[d] \arrow[dr] & \\
0 \arrow[r] & B_{21} \arrow[r] & B_{21} \arrow[r] & B_{23}
\end{tikzcd}
$$

is such that that the bottom row is exact, then the square is a pullback if and only if $0 \longrightarrow B_{11} \longrightarrow B_{12} \longrightarrow B_{23}$ is exact.

**Proof.** See [Freyd, 1964], p. 54. ■

Now suppose $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact. Then we have a commutative diagram

$$\begin{tikzcd}
 & j_A(0) \arrow[r] \arrow[d, "\delta_j(A)"'] & j_B(0) \arrow[r] \arrow[d, "\delta_j(B)"'] & j_C(0) \arrow[d, "\delta_j(C)"'] & \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

in which the bottom row is exact and the left hand square is a pullback so that $0 \longrightarrow j_A(0) \longrightarrow j_B(0) \longrightarrow 0$ exact. But $\delta_j(C)$ is monic, hence $\ker(j_B(0) \longrightarrow j_C(0) \longrightarrow C) = \ker(j_B(0) \longrightarrow j_C(0))$. This implies that $0 \longrightarrow j_A(0) \longrightarrow j_B(0) \longrightarrow j_C(0)$ is exact. ■

To show $\epsilon_j(A/\epsilon_j(A)) = 0$ we need the following lemma:

**Lemma.** If $A \subseteq B$, then $A$ is j-closed in $B$ if and only if $j_{A/B}(0) = 0$.

**Proof.** If $A$ is j-closed in $B$, consider the pullback

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d] & B \arrow[d] \\
0 \arrow[r, tail] & B/A
\end{tikzcd}$$

This gives the pullback squares

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & j_B(A) \arrow[r] \arrow[d] & B \arrow[d] \\
0 \arrow[r, tail] & j_{B/A}(0) \arrow[r, tail] & B/A
\end{tikzcd} .$$

But $j_B(A) = A$, so

$$0 \longrightarrow A \xrightarrow{1_A} A \longrightarrow j_{B/A}(0)$$

is exact. Since $B \longrightarrow B/A$ is epimorphic, so is $A \longrightarrow j_{B/A}(0)$, hence

$$0 \longrightarrow A \xrightarrow{1_A} A \longrightarrow j_{B/A}(A) \longrightarrow 0$$

is exact which implies $j_{B/A}(0) = 0$. The other implication is immediate.

Now given $A \in |\mathcal{A}|$, $j_A(0) = \epsilon_j(A)$ is clearly $j$-closed in $A$ and so $\epsilon_j(A/\epsilon_j(A)) = j_{A/j_A(0)}(0) = 0$.

Evidently $\epsilon_j^2(A) = \epsilon_j(A)$ and $\delta_j(A):\epsilon_j(A) \longrightarrow A$ is monic for all $A \in |\mathcal{A}|$.

Suppose $(\epsilon,\delta)$ is a radical and $A \subseteq B$ define

$$j_B^{\epsilon}(A) = \ker(B \longrightarrow B/A \longrightarrow (B/A)/\epsilon(B/A)).$$

Then we have an exact sequence

$$0 \longrightarrow j_B^\epsilon(A) \longrightarrow B \longrightarrow (B/A)/\epsilon(B/A) \longrightarrow 0.$$

Trivially there is a monic map $A \longrightarrow j_B^\epsilon(A)$, so $j_B^\epsilon$ is increasing and $(B/A)/\epsilon(B/A) = B/j_B^\epsilon(A)$. Notice $\epsilon(B/j_B^\epsilon(A)) = 0$ since $(\epsilon,\delta)$ is a radical.

By definition $j_B^\epsilon(j_B(A)) =$

$$\ker(B \longrightarrow B/j_B^\epsilon(A) \longrightarrow (B/j_B^\epsilon(A))/\epsilon(B/j_B^\epsilon(A)).$$

Since $\epsilon(B/j_B^\epsilon(A)) = 0$,

$$0 \longrightarrow (j_B^\epsilon)^2(A) \longrightarrow B \longrightarrow B/j_B^\epsilon(A) \longrightarrow 0$$

is exact, so $j_B^\epsilon$ is idempotent.

To show $j_A^\epsilon$ is monotone; if $A'' \subseteq A' \subseteq A$, then

$$0 \longrightarrow A'/A'' \longrightarrow A/A'' \longrightarrow A/A' \longrightarrow 0$$

is exact. Since $\epsilon$ is left exact,

$$0 \longrightarrow \epsilon(A'/A'') \longrightarrow \epsilon(A/A'') \longrightarrow \epsilon(A/A')$$

is exact, which induces the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \epsilon(A/A'') \arrow[r] \arrow[d] & A/A'' \arrow[r] \arrow[d] & (A/A'')/\epsilon(A/A'') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \epsilon(A/A') \arrow[r] & A/A' \arrow[r] & (A/A')/\epsilon(A/A') \arrow[r] & 0
\end{tikzcd}$$

This implies that the following diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "="'] & A/A'' \arrow[r] \arrow[d] & (A/A'')/\epsilon(A/A'') \arrow[r] \arrow[d] & 0 \\
A \arrow[r] & A/A' \arrow[r] & (A/A')/\epsilon(A/A') \arrow[r] & 0
\end{tikzcd}$$

is commutative. So we have a map $j_A^\epsilon(A'') \longrightarrow j_A^\epsilon(A')$ such that $j_A^\epsilon(A'') \longrightarrow j_A^\epsilon(A') \longrightarrow A = j_A^\epsilon(A'') \longrightarrow A$, hence $j_A^\epsilon(A'') \longrightarrow j_A^\epsilon(A')$ must be monic.

To show $j^\epsilon$ is a natural transformation, given $A' \subseteq A$ and $f \in \mathcal{Q}(B,A)$, we must show the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{sub}(A) \arrow[r, "j_A^\epsilon"] \arrow[d, "\mathrm{sub}(f)"'] & \mathrm{sub}(A) \arrow[d, "\mathrm{sub}(f)"] \\
\mathrm{sub}(B) \arrow[r, "j_B^\epsilon"] & \mathrm{sub}(B)
\end{tikzcd}$$

First we observe if $C' \subseteq C$ then $0 \longrightarrow \epsilon(C') \longrightarrow \epsilon(C) \longrightarrow \epsilon(C/C')$ is exact. Since $\epsilon(C/C') \longrightarrow C/C'$ is monic, $0 \longrightarrow \epsilon(C') \longrightarrow \epsilon(C) \longrightarrow C/C'$ is exact. From the previous lemma we have the pullback diagram

$$\begin{tikzcd} \epsilon(C') \arrow[r] \arrow[d] & \epsilon(C) \arrow[d] \\ C' \arrow[r] & C \end{tikzcd}$$

Now follow $A' \in \mathrm{sub}(A)$ around two directions: let $B' = \mathrm{sub}(f)(A')$; i.e. the following diagram is a pullback

$$\begin{tikzcd} B' \arrow[r] \arrow[d] \arrow[dr, phantom, "(*)"] & B \arrow[d, "f"] \\ A' \arrow[r] & A \end{tikzcd}$$

Then $j_B^{\epsilon}(\mathrm{sub}(f)(A')) = j_B^{\epsilon}(B')$. On the other hand we have a pullback

$$\begin{tikzcd} \mathrm{sub}(f)(j_B^{\epsilon}(A')) \arrow[r] \arrow[d] & B \arrow[d, "f"] \\ j_A^{\epsilon}(A') \arrow[r] & A \end{tikzcd}$$

Hence it suffices to show the following squares commute:

$$\begin{tikzcd} j_B(B') \arrow[r] \arrow[d] \arrow[dr, phantom, "(1)"] & B \arrow[d, "f"] \\ j_A(A') \arrow[r] & A \end{tikzcd} \qquad \begin{tikzcd} \mathrm{sub}(f)(j_A^{\epsilon}(A')) \arrow[r] \arrow[d] \arrow[dr, phantom, "(2)"] & B \arrow[d] \\ 0 \arrow[r] & (B/B'')/\epsilon(B/B') \end{tikzcd}$$

For (1), the fact that (*) is a pullback implies there is a monic map $B/B' \longrightarrow A/A'$ such that

$$\begin{tikzcd}
0 \arrow[r] & B' \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & B/B' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & A \arrow[r] & A/A' \arrow[r] & 0
\end{tikzcd}$$

commutes. This induces a monic map $(B/B')/\epsilon(B/B') \longrightarrow (A/A')/\epsilon(A/A')$ such that

$$\begin{tikzcd}
0 \arrow[r] & \epsilon(B/B') \arrow[r] \arrow[d] \arrow[dr, phantom, "(3)"] & B/B' \arrow[r] \arrow[d] & (B/B')/\epsilon(B/B') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \epsilon(A/A') \arrow[r] & A/A' \arrow[r] & (A/A')/\epsilon(A/A') \arrow[r] & 0
\end{tikzcd}$$

commutes, since (3) is a pullback. Then so does

$$\begin{tikzcd}
B \arrow[r] \arrow[d, "f"'] & B/B' \arrow[r] \arrow[d] & (B/B')/\epsilon(B/B') \arrow[d] \\
A \arrow[r] & A/A' \arrow[r] & (A/A')/\epsilon(A/A')
\end{tikzcd}$$

Hence (1) commutes. On the other hand, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{sub}(f)(j_A^\epsilon(A')) \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, "f"] & B/B' \arrow[r] \arrow[d] & (B/B')/\epsilon(B/B') \arrow[d] \\
j_A^\epsilon(A') \arrow[r] & A \arrow[r] & A/A' \arrow[r] & (A/A')/\epsilon(A/A')
\end{tikzcd}$$

But

$$j^{\epsilon}_{A}(A) \longrightarrow A \longrightarrow A/A' \longrightarrow (A/A')/\epsilon(A'A') = 0$$

then so is

$$\mathrm{sub}(f)(j^{\epsilon}_{A}(A')) \longrightarrow B \longrightarrow B/B' \longrightarrow (B/B'/\epsilon(B/B') = 0$$

since $(B/B')/\epsilon(B/B') \longrightarrow (A/A')/\epsilon(A/A')$ is monic. Hence (2) commutes and this shows that $j^{\epsilon}$ is indeed a topology.

Let $\mathbf{Top}(\mathcal{A})$ denote the collection of topologies on $\mathcal{A}$ and $\mathbf{R}(\mathcal{A})$ denotes the collection of radicals on $\mathcal{A}$. We are going to show the constructions we have done give a one to one correspondence between $\mathbf{Top}(\mathcal{A})$ and $\mathbf{R}(\mathcal{A})$.

Let $(\epsilon,\delta) \in \mathbf{R}(\mathcal{A})$ and $A \in |\mathcal{A}|$ then

$$j^{\epsilon}_{A}(0) = \ker(A \longrightarrow A/0 \longrightarrow (A/0)/\epsilon(A/0))$$
$$= \ker(A \longrightarrow A/\epsilon(A)) = \epsilon_{j_\epsilon}(A).$$

This implies $\epsilon(A) = \epsilon_{j_\epsilon}(A)$, so we have
$\mathbf{R}(\mathcal{A}) \longrightarrow \mathbf{Top}(\mathcal{A}) \longrightarrow \mathbf{R}(\mathcal{A}) = 1_{\mathbf{R}(\mathcal{A})}$.

On the other hand, let $j \in \mathbf{Top}(\mathcal{A})$ and $A \in |\mathcal{A}|$ then

$$j^{\epsilon_j}_{A}(0) = \ker(A \longrightarrow A/0 \longrightarrow (A/0)/\epsilon_j(A/0))$$
$$= \ker(A \longrightarrow A/\epsilon_j(A)) = \ker(A \longrightarrow A/j_A(0)).$$

This implies $j_A(0) = j_A^{\in J}(0)$ for all $A \in |\mathcal{A}|$. Now let $A' \subseteq A$ then

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & A \arrow[d] \\ 0 \arrow[r] & A/A' \end{tikzcd}$$

is a pullback, which implies

$$\begin{tikzcd} j_A(A') \arrow[r] \arrow[d] & A \arrow[d] \\ j_{A/A'}(0) \arrow[r] & A/A' \end{tikzcd} \qquad \begin{tikzcd} j_A^{\in J}(A') \arrow[r] \arrow[d] & A \arrow[d] \\ j_{A/A'}^{\in J}(0) \arrow[r] & A/A' \end{tikzcd}$$

are both pullbacks, since $j$ and $j^{\in J}$ are topologies. But $j_{A/A'}(0) = j_{A/A'}^{\in J}(0)$ so $j_A(A') \cong j_A^{\in J}(A')$ for any $A \in |\mathcal{A}|$ and $A' \in \mathrm{sub}(A)$. Hence $\mathrm{Top}(\mathcal{A}) \longrightarrow (\mathcal{A}) \longrightarrow \mathrm{Top}(\mathcal{A}) = 1_{\mathrm{Top}(\mathcal{A})}$. This establishes the one to one correspondence between $\mathrm{Top}(\mathcal{A})$ and $\mathbf{R}(\mathcal{A})$.

## §2. Construction of sheaves

**Definition.** A torsion theory on an Abelian category $\mathcal{C}$ is a pair $(\mathcal{T},\mathcal{F})$ of classes of objects of $\mathcal{C}$ satisfying

(1) $\mathcal{C}(\mathcal{T},\mathcal{F}) = 0$ for all $T \in \mathcal{T}$, $F \in \mathcal{F}$;

(2) If $C \in |\mathcal{C}|$ such that $\mathcal{C}(C,F) = 0$ for all $F \in \mathcal{F}$ then $C \in \mathcal{T}$;

(3) If $C \in |\mathcal{C}|$ such that $\mathcal{C}(T,C) = 0$ for all $T \in \mathcal{T}$ then $C \in \mathcal{F}$;

(4) $\mathcal{T}$ is closed under the subobjects.

$\mathcal{T}$ is called a torsion class and its objects are torsion objects while $\mathcal{F}$ is a torsion-free class consisting of torsion-free objects.

If $\mathcal{C}$ is a complete and cocomplete Abelian category, then we have the following well-known result:

**Theorem.** There is a one to one correspondence between

(1) Torsion theories on $\mathcal{C}$;

(2) Radicals on $\mathcal{C}$.

**Proof.** For a formal proof see [Stenström, 1975]. But we would like to point out the main idea of the proof. Given a radical $(\epsilon,\delta)$ we associate a torsion theory $(\mathcal{T}_\epsilon,\mathcal{F}_\epsilon)$ by putting $\mathcal{T}_\epsilon = \{C \in |\mathcal{C}| \mid \epsilon(C) = C\}$ and $\mathcal{F}_\epsilon = \{C \in |\mathcal{C}| \mid \epsilon(C) = 0\}$. On the other hand if $(\mathcal{T},\mathcal{F})$ is a torsion theory then it is easy to verify that $\mathcal{T}$ is closed under quotient objects and coproducts as well as subobjects. Now suppose $C \in |\mathcal{C}|$ and $\epsilon_{\mathcal{T}}(C)$ equals the sum of all subobjects of $C$ belonging to $\mathcal{T}$. Then clearly $\epsilon_{\mathcal{T}}(C) \in \mathcal{T}$. Finally, one verifies $\epsilon_{\mathcal{T}}$ is a radical with the usual inclusion $\delta_{\mathcal{T}}$. ■

Corollary. There is a one to one correspondence between torsion theories and Top($A$).

Similarly, given $j \in$ Top($A$) the corresponding torsion theory $(\mathcal{T}_j, \mathcal{F}_j)$ can be described by

$$\mathcal{F}_j = \{A \in |\mathcal{A}| \mid j_A(0) = 0\}$$

$$\mathcal{T}_j = \{A \in |\mathcal{A}| \mid j_A(0) = A\}.$$

Definition. Given an Abelian category $\mathcal{A}$, then $\mathcal{A}$ has injective effacements if for every $A \in |\mathcal{A}|$ there is an $I \in |\mathcal{A}|$ and a monic map $A \longrightarrow I$ such that the diagram

$$\begin{tikzcd} B' \arrow[r, tail] \arrow[d] & B \\ A \arrow[r, tail] & I \end{tikzcd}$$

can be completed to a commutative square

$$\begin{tikzcd} B' \arrow[r, tail] \arrow[d] & B \arrow[d] \\ A \arrow[r, tail] & I \end{tikzcd}$$

by some map $B \longrightarrow I$.

Example. If $\mathcal{A}$ has enough injectives, clearly $\mathcal{A}$ has injective effacements.

Given $\mathcal{A}$ with injective effacements and $j \in \mathrm{Top}(\mathcal{A})$ we shall construct a full reflexive subcategory $\mathrm{Sh}_j(\mathcal{A})$ with exact left adjoint in this section.

**Definition.** $A \in |\mathcal{A}|$ and $j \in \mathrm{Top}(\mathcal{A})$, then $A$ is a j-sheaf if

(1) $j_A(0) = 0$;

(2) If $A \in \mathrm{sub}(B)$ and $j_B(0) = 0$ then $j_B(A) = A$.

We let $\mathrm{Sh}_j(\mathcal{A})$ be the full subcategory of $\mathcal{A}$ generated by the collection of j-sheaves.

We shall construct the left adjoint $P:\mathcal{A} \longrightarrow \mathrm{Sh}_j(\mathcal{A})$ in two stages. First let $S:\mathcal{A} \longrightarrow \mathcal{A}$ defined by $S(A) = A/j_A(0)$. Evidently $S$ is a functor, since $\epsilon_j$ is a functor.

**Lemma.** $S$ preserves monomorphisms and $j_{S(A)}(0) = 0$ for all $A \in |\mathcal{A}|$.

**Proof.** If $A' \in \mathrm{sub}(A)$, we have a commutative diagram

$$\begin{tikzcd}
 & j_{A'}(0) \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d] & A'/j_{A'}(0) \arrow[d] \\
0 \arrow[r] & j_A(0) \arrow[r] & A \arrow[r] & A/j_A(0)
\end{tikzcd}$$

in which the left square is a pullback and the bottom row is exact. Hence $0 \longrightarrow j_{A'}(0) \longrightarrow A' \longrightarrow A/j_A(0)$ is exact so the induced map $A'/j_A(0) \longrightarrow A/j_A(0)$ is monic. Since $j_A(0)$ is j-closed in $A$, $j_{S(A)}(0) = j_{A/j_A(0)}(0) = 0$. ■

Now we restrict ourselves to $\mathcal{F}_j = \{A \in |\mathcal{Q}| \mid j_A(0) = 0\}$.

**Lemma. If $X \in \mathcal{F}_j$, then we can choose an injective effacement $I$ for $X$ such that $I \in \mathcal{F}_j$.**

**Proof.** Suppose $X \in \mathcal{F}_j$ and $X \rightarrowtail I$ be an injective effacement for $X$, then we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "="] \arrow[d] & SX \arrow[d] \\
I \arrow[r] & SI
\end{tikzcd}$$

in which $X \rightarrowtail I$ implies that $X = SX \rightarrowtail SI$, since $S$ preserves monic maps and if $j_X(0) = 0$ then $X = SX$. Now given a diagram

$$\begin{tikzcd}
Y' \arrow[r, tail] \arrow[d] & Y & \\
X \arrow[r, tail] & I \arrow[r, two heads] & SI
\end{tikzcd}$$

it can evidently be completed to a commutative square

$$\begin{tikzcd}
Y' \arrow[r, tail] \arrow[d] & Y \arrow[d] \\
X \arrow[r, tail] & SI
\end{tikzcd}$$

■

**Lemma.** If $A' \in \mathrm{sub}(A)$ and $A \in \mathcal{F}_j$, then $A' \in \mathcal{F}_j$.

**Proof.** $A' \in \mathrm{sub}(A)$ implies there is a pullback diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & A' \arrow[d] \\
0 \arrow[r] & A
\end{tikzcd}$$

Then we obtain a pullback

$$\begin{tikzcd}
j_{A'}(0) \arrow[r, tail] \arrow[d] & A' \arrow[d] \\
j_A(0) \arrow[r, tail] & A
\end{tikzcd}$$

But $j_A(0) = 0$ and $A' \leq A$ imply that $j_A'(0) = 0$. ■

**Lemma.** If $A \in \mathcal{F}_j$ and $B \in \mathcal{T}_j$, then $\mathcal{A}(B,A) = 0$.

**Proof.** Since j induces a radical $(\epsilon_j, \delta_j)$ we have a commutative diagram for any $f \in \mathcal{A}(B,A)$,

$$\begin{tikzcd}
\epsilon_j(B) \arrow[r, "\epsilon_j(f)"] \arrow[d] & \epsilon_j(A) \arrow[d] \\
B \arrow[r, "f"] & A
\end{tikzcd}$$

But $\epsilon_j(B) = B$ and $\epsilon_j(A) = 0$ so the above diagram reduces to

$$\begin{tikzcd}
B \arrow[r] \arrow[d, "="'] & 0 \arrow[d] \\
B \arrow[r, "f"] & A
\end{tikzcd}$$

from which the conclusion is evident. ■

**Lemma.** If $A' \in \mathrm{sub}(A)$, then $A'$ is j-dense in $A$ if and only if $j_{A/A'}(0) = A/A'$.

**Proof.** Since $j_A(A') = A$ and

$$j_A(A') = \ker(A \longrightarrow A/A' \longrightarrow (A/A')/j^{\epsilon}_{A/A'}(0)) \; (*)$$

we have an exact sequence

$$0 \longrightarrow A \xrightarrow{1} A \longrightarrow (A/A')/j_{A/A'}(0) \longrightarrow 0.$$

Hence $A/A' = j_{A/A'}(0)$. On the other hand (*) implies $0 \longrightarrow j_A(A') \longrightarrow A \longrightarrow 0$ is exact if $A/A' = j_{A/A'}(0)$. ■

**Corollary.** **If $A'' \in \mathrm{sub}(A)$ and $f: A \longrightarrow B$ are such that $A''$ is j-dense in $A$, $B \in \mathcal{F}_j$ and $A'' \rightarrowtail A \xrightarrow{f} B = 0$, then $f = 0$.**

**Proof.** $A'' \rightarrowtail A \xrightarrow{f} B = 0$ implies $A'' \subseteq A' = \ker(f)$. $A''$ j-dense in $A$ implies $A'$ is j-dense in $A$. Furthermore $f$ induces a commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & A/A' \arrow[r] & (A'/A')/j_{A/A'}(0) \arrow[d, dashed] \\
B \arrow[rr, "\cong"] & & B/j_B(0)
\end{tikzcd}$$

Hence $f = 0$. ■

Now we are ready for the second stage of the construction. Given $A \in \mathcal{F}_j$, let $A \rightarrowtail I$ be an injective effacement of $A$ such that $I \in \mathcal{F}_j$ (such choice of $I$ is always possible) put $T(A) = j_I(A)$. Notice $A$ is j-dense in $T(A)$ and $j_I(A) = T(A) \in \mathcal{F}_j$ since $T(A) \in \mathrm{sub}(I)$. Now we shall show $T(A)$ is in $\mathrm{Sh}_j(\mathcal{A})$.

**Lemma.** $T(A) \rightarrowtail I$ is an injective effacement for $T(A)$.

**Proof.** Let $T(A) \rightarrowtail I'$ be another injective effacement for $T(A)$ and $I' \in \mathcal{F}_j$. Then we have a commutative diagram

$$\begin{array}{ccccc} A & \longrightarrow & T(A) & \longrightarrow & I' \\ \| & & & & \downarrow \\ A & \longrightarrow & T(A) & \longrightarrow & I \end{array}$$

and the two maps $T(A) \longrightarrow I$ agree on the dense subobject $A$, hence are equal since $A \rightarrowtail I$ is an injective effacement of $A$. This implies $T(A) \rightarrowtail I$ $= T(A) \rightarrowtail T' \rightarrowtail I$ since $A$ is j-dense in $T(A)$. ■

**Lemma.** The following are equivalent for $A \in |\mathcal{A}|$

(1) $A \in \mathrm{Sh}_j(\mathcal{A})$;

(2) Any diagram of the form

$$\begin{array}{ccc} B'' & \rightarrowtail & B \\ \downarrow & & \\ A & & \end{array}$$

such that $B'$ is j-dense in $B$ can be completed to

$$\begin{tikzcd} B' \arrow[r, tail] \arrow[d] & B \arrow[dl] \\ A & \end{tikzcd}$$

with a unique map $B \longrightarrow A$.

**Proof.** (1)$\Rightarrow$(2) Choose an injective effacement $A \rightarrowtail I$ such that $I \in \mathcal{F}_j$. Then we have a commutative square

$$\begin{tikzcd} B'' \arrow[r, tail] \arrow[d] & B \arrow[d] \\ A \arrow[r] & I \end{tikzcd}$$

which gives a commutative diagram

$$\begin{tikzcd} B'' \arrow[dr] & & \\ & B' \arrow[r, tail] \arrow[d] & B \arrow[d] \\ & A \arrow[r, tail] & I \end{tikzcd}$$

in which the square is a pullback. This also implies $B'$ is $j$-dense in $B$. Then we have two pullback squares

$$\begin{tikzcd} B' \arrow[r, tail] \arrow[d] & j_B(B') \arrow[r, tail] \arrow[d] & B \arrow[d] \\ A \arrow[r, tail] & j_I(A) \arrow[r, tail] & I \end{tikzcd}$$

But $j_A(A) = A$ and $j_B(B') = B$ since $A \in \mathrm{Sh}_j(\mathcal{Q})$. So $B' \rightarrowtail B \longrightarrow A$ is our choice. The uniqueness follows from the fact $B''$ is j-dense in $B$.

$(2) \Longrightarrow (1)$ Since 0 is j-dense in $j_A(0)$ there is a unique map $j_A(0) \longrightarrow A$ such that $0 \longrightarrow j_A(0) \longrightarrow A$ $= 0 \longrightarrow A$. But both the cannonical injection and zero map have this property, so $j_A(0) \longrightarrow A = j_A(0) \longrightarrow 0 \longrightarrow A$ which implies $j_A(0) \longrightarrow 0$ and hence $j_A(0) = 0$. This shows that $A \in \mathcal{F}_j$.

Suppose $B \in \mathcal{F}_j$ and $A \in \mathrm{sub}(B)$, since $A$ is j-dense in $j_B(A)$ there is a unique map $j_B(A) \longrightarrow A$ such that

$$A \xrightarrow{1} A = A \overset{i}{\rightarrowtail} j_B(A) \longrightarrow A$$

This implies that $j_B(A) = A$. ∎

**Lemma.** Suppose $A \rightarrowtail I$ is an injective effacement for $A$ such that

(1) $A, I \in \mathcal{F}_j$;

(2) $A$ is j-closed in $I$.

Then $A \in \mathrm{Sh}_j(\mathcal{A})$.

Proof. Suppose we have a diagram

$$\begin{array}{ccc} B'' & \rightarrowtail & B \\ \downarrow & & \\ A & & \end{array}$$

such that $B''$ is j-dense in $B$. We can embed it in the commutative square

$$\begin{array}{ccc} B'' & \rightarrowtail & B \\ \downarrow & & \downarrow \\ A & \longrightarrow & I \end{array}$$

and follow the same argument as in the previous lemma to obtain a unique $B \longrightarrow A$ such that $B'' \longrightarrow A$ $= B'' \rightarrowtail B \longrightarrow A$. ■

Corollary. $T(A)$ is a sheaf.

Proposition. Let $A \in \mathcal{F}_j$ then

(1) there exists an object $T(A) \in \mathrm{Sh}_j(\mathcal{A})$ such that $A \in \mathrm{sub}(T(A))$ and $A$ is j-dense in $T(A)$;

(2) for any map $F: A \longrightarrow B$ such that $B \in \mathrm{Sh}_j(\mathcal{A})$, there exists a unique map $T(A) \longrightarrow B$ and $A \rightarrowtail B$ $= A \rightarrowtail T(A) \longrightarrow B$.

**Proof.** The proof is trivial. ■

This shows $\mathrm{Sh}_j(\mathcal{A})$ is a coreflexive subcategory of $\mathcal{F}_j$. It is shown in [Barr, 19??] that $\mathcal{F}_j$ is coreflective in $\mathcal{A}$, hence so is $\mathrm{Sh}_j(\mathcal{A})$. Let $\mathbb{P} = (P, \sigma, \tau)$ be the triple corresponding to the coreflector. Evidently $PV = TSX$ where $PX$ is simply $j_I(SX)$ in any injective effacement $SX \longrightarrow I$ such that $I \in \mathcal{F}_j$. Now we are going to show $P$ is exact.

**Lemma.** Let $A' \rightarrowtail A \rightarrowtail I$, where the second map is an injective effacement. Then so is the composite $A' \rightarrowtail I$.

**Proof.** The proof is trivial. ■

**Lemma.** If $A' \rightarrowtail A$, then $PA' \rightarrowtail PA$.

**Proof.** We have shown that $SA' \rightarrowtail SA$. Suppose $SA \rightarrowtail I$ is an injective effacement such that $I \in \mathcal{F}_j$ then so is $SA' \rightarrowtail SA \rightarrowtail I$. Hence we have

$$\begin{tikzcd} SA' \arrow[r, tail] \arrow[d, tail] & SA \arrow[d, tail] \\ PA' \arrow[d, tail] & PA \arrow[d, tail] \\ I \arrow[r, "="] & I \end{tikzcd}$$

where $PA' = j_I(SA')$ and $PA = j_I(SA)$ so we have $PA' \rightarrowtail PA$. ■

**Lemma.** The following are equal for a monic map $A' \rightarrowtail A$:

(1) $A/A' \in \mathcal{F}_j$;

(2) $j_A(A') = A$;

(3) $A' \rightarrowtail A$ is a regular monomorphism in $\mathcal{F}_j$ (or $\mathrm{Sh}_j(\mathcal{Q})$).

**Proof.** The only non-trivial implication is (3)⟹(1). If $0 \longrightarrow A' \longrightarrow A \longrightarrow B$ is exact and $B \in \mathcal{F}_j$, then $A/A' \in \mathrm{sub}(B)$. Hence $A/A' \in \mathcal{F}_j$ as well. ■

**Proposition.** $P$ is left exact.

**Proof.** Suppose $0 \longrightarrow A'' \longrightarrow A' \longrightarrow A$ is exact in $\mathcal{Q}$. We know that $PA'' \rightarrowtail PA'$. Note that in the above lemma we can replace $\mathcal{F}_j$ by $\mathrm{Sh}_j(\mathcal{Q})$ in (3) (since $A/A' \rightarrowtail T(A/A')$ is monic) so $PA'' \rightarrowtail PA'$ is a regular monomorphism in

$\mathrm{Sh}_j(\mathcal{A})$ as well. Hence it is the kernel in $\mathrm{Sh}_j(\mathcal{A})$ of its cokernel in $\mathrm{Sh}_j(\mathcal{A})$. The coreflector preserves colimits, so that the cokernel of $PA'' \rightarrowtail PA'$ is $PA' \longrightarrow PA$, and the sequence is exact in $\mathrm{Sh}_j(\mathcal{A})$. The inclusion preserves limits, so that the sequence is exact in $\mathcal{A}$. ■

Combining these results, we have

**Theorem.** Let $j \in \mathrm{Top}(\mathcal{A})$ and $\mathcal{A}$ have injective effacements. Then the full subcategory $\mathrm{Sh}_j(\mathcal{A})$ is coreflective such that the coreflector is exact.

**Corollary.** Let $\mathcal{A}$ be an Abelian category with injuctive effacements.

Then

(1) $\mathrm{Sh}_j(\mathcal{A})$ is Abelian;

(2) $\mathrm{Sh}_j(\mathcal{A})$ has injective effacements.

**Proof.** (1) See [Mitchell, 1965]. Mitchell has also shown that if $\mathcal{A}$ is cocomplete then so is $\mathrm{Sh}_j(\mathcal{A})$.

(2) Let $A \in \mathrm{Sh}_j(\mathcal{A})$ and $A \rightarrowtail I$ be an injective effacement with $I \in \mathcal{F}_j$. Then it is clear that $A \rightarrowtail I \rightarrowtail T(I)$ is an injective effacement in $\mathrm{Sh}_j(\mathcal{A})$.

■

Remark. If $\mathcal{A}$ is a complete Grothendieck category, then a full coreflective subcategory $\mathcal{B}$ with exact coreflector is called a Giraud subcategory of $\mathcal{A}$. It is also a complete Grothendieck category.

We have shown that a radical $(\epsilon,\delta)$ induces through $j^{\epsilon}$ a left exact idempotent triple $\mathbb{P} = (P^{\epsilon},\sigma^{\epsilon},\tau^{\epsilon})$ on $\mathcal{A}$. Now if we are given a left exact idempotent triple $\mathbb{P} = (P,\sigma,\tau)$, we would like to find our way back.

Let $\mathbb{P} = (P,\sigma,\tau)$ be a left exact idempotent triple on $\mathcal{A}$ and $A \in |\mathcal{A}|$. Put $\epsilon_p(A) = \ker(\sigma(A):A \longrightarrow P(A))$. If $f \in \mathcal{A}(A,B)$, we have a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle \sigma A}\downarrow & & \downarrow{\scriptstyle \sigma B} \\ PA & \xrightarrow[P(f)]{} & PB \end{array}$$

in $\mathcal{A}$. This induces a map $\epsilon_p(f):\epsilon_p(A) \longrightarrow \epsilon_p(B)$ such that $\epsilon_p(A) \rightarrowtail A \longrightarrow B = \epsilon_p(A) \longrightarrow \epsilon_p(B) \rightarrowtail B$. Obviously we let $\delta_p$ be the cannonical injection.

To show $\epsilon_p$ is left exact we suppose $0 \longrightarrow A \longrightarrow B \longrightarrow C$ is exact in $\mathcal{A}$. Then we have the following commutative diagram with two lower rows and all columns exact:

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
 & \epsilon_p(A) \arrow[r] \arrow[d] & \epsilon_p(B) \arrow[r] \arrow[d] & \epsilon_p(C) \arrow[d] \\
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\
0 \arrow[r] & PA \arrow[r] & PB \arrow[r] & PC
\end{tikzcd}$$

Then $PA \rightarrowtail PB$ and $0 \longrightarrow \epsilon_p(A) \longrightarrow A \longrightarrow PA$ exact imply $0 \longrightarrow \epsilon_p(A) \longrightarrow A \longrightarrow PB$ is exact; $0 \longrightarrow \epsilon_p(B) \longrightarrow B \longrightarrow PB$ exact implies

$$\begin{tikzcd}
\epsilon_p(A) \arrow[r] \arrow[d] & \epsilon_p(B) \arrow[d] \\
A \arrow[r] & B
\end{tikzcd}$$

is a pullback. This implies $0 \longrightarrow \epsilon_p(A) \longrightarrow \epsilon_p(B) \longrightarrow C$ is exact. But $\epsilon_p(C) \rightarrowtail C$, so we have $0 \longrightarrow \epsilon_p(A) \longrightarrow \epsilon_p(B) \longrightarrow \epsilon_p(C)$ is exact.

**Corollary.** If $A \in \text{sub}(B)$ then

$$\begin{tikzcd}
\epsilon_p(A) \arrow[r, tail] \arrow[d] & \epsilon_p(B) \arrow[d] \\
A \arrow[r, tail] & B
\end{tikzcd}$$

is a pullback.

**Corollary.** For any $A \in |\mathcal{A}|$, $\epsilon_p(\epsilon_p(A)) = \epsilon_p(A)$.

**Proof.** Apply the previous corollary to $\epsilon_p(A) \rightarrowtail A$. ∎

**Lemma.** For any $A \in |\mathcal{A}|$, $\epsilon_p(PA) = 0$.

**Proof.** The triple identity $\tau A \cdot \sigma(PA) = 1_{PA}$ implies that $\sigma(PA)$ is monic. But

$$0 \longrightarrow \epsilon_p(PA) \longrightarrow PA \xrightarrow{\sigma(PA)} P^2A$$

is exact, hence $\epsilon_p(PA) = 0$. ∎

Suppose $A \in |\mathcal{A}|$. Then $0 \longrightarrow \epsilon_p(A) \longrightarrow A \longrightarrow PA$ is exact, so we have a monic map $A/\epsilon_p(A) \longrightarrow PA$. Then the corollary implies that $\epsilon_p(A/\epsilon_p(A)) \in \mathrm{sub}(\epsilon_p(PA))$ so $\epsilon_p(A/\epsilon_p(A)) = 0$. This shows that $(\epsilon_p, \delta_p)$ is a radical. Note that in the construction we did not use the fact that $\mathbb{P} = (P,\sigma,\tau)$ is idempotent.

**Theorem.** A left exact triple $\mathbb{P} = (P,\sigma,\tau)$ induces a radical $(\epsilon_p, \delta_p)$.

**Corollary.** A left exact triple $\mathbb{P} = (P,\sigma,\tau)$ induces a topology $jP$ on $\mathcal{A}$.

Remark. The only assumption required for these results is that $\mathcal{A}$ be Abelian.

**Definition.** If $\mathcal{C}$ is an Abelian category, then a full replete coreflective subcategory $\mathcal{D}$ is a Giraud subcategory of $\mathcal{C}$ if the coreflector is exact.

Remark. If j is a topology on $\mathcal{A}$ where $\mathcal{A}$ is Abelian with injective effacements, then $\mathrm{Sh}_j(\mathcal{A})$ is a Giraud subcategory.

**Proposition.** Let $(\epsilon,\delta)$ be a radical on $\mathcal{A}$ where $\mathcal{A}$ is Abelian with injective effacements, $\mathrm{Sh}(\mathcal{A})$ be the corresponding Giraud subcategory with coreflector $P$ and $(\epsilon',\delta')$ be the radical induced by the left exact idempotent triple associated to $\mathrm{Sh}(\mathcal{A})$. Then $(\epsilon,\delta) = (\epsilon',\delta')$.

Proof. If $A \in |\mathcal{A}|$, then $\epsilon(A) = \ker(\sigma A\colon A \longrightarrow PA) = \epsilon'(A)$. ■

**Proposition.** Let $\mathcal{D}$ be a Giraud subcategory of $\mathcal{A}$ with coreflector $P'$ where $\mathcal{A}$ is Abelian with injective effacements, $(\epsilon,\delta)$ be the corresponding radical and $\mathrm{Sh}(\mathcal{A})$ be the Giraud subcategory of $\mathcal{A}$ with coreflector $P$. Thus $\mathcal{D}$ is equivalent to $\mathrm{Sh}(\mathcal{A})$.

Proof. We have a diagram

$$
\begin{tikzcd}
 & \mathcal{D} \arrow[dl, shift left, "P\circ i'"] \arrow[dr, shift left, "i'"] & \\
\mathrm{Sh}(\mathcal{A}) \arrow[ur, shift left, "P'\circ i"] \arrow[rr, shift right, "i"'] & & \mathcal{A} \arrow[ll, shift right, "P"'] \arrow[ul, shift left, "P'"]
\end{tikzcd}
$$

where $i$ and $i'$ are inclusion functors. We first note that $P'\circ i\circ P \cong P'$, for if $A \in |\mathcal{A}|$ then the exact sequence

$$0 \longrightarrow \ker(\sigma A) \longrightarrow A \xrightarrow{\sigma A} (i\circ P)(A) \longrightarrow \operatorname{coker}(\sigma A) \longrightarrow 0$$

gives $P'A \cong P'((i\circ P)(A))$ since $\ker(\sigma A)$ and $\operatorname{coker}(\sigma A)$ are torsion. Similarly we have $P\circ i'\circ P' \cong P$, because the adjunction $\sigma' : 1_{\mathcal{A}} \longrightarrow i'\circ P'$ gives the exact sequence

$$0 \longrightarrow \ker(\sigma' A) \longrightarrow A \xrightarrow{\sigma' A} (i'\circ P')(A) \longrightarrow \operatorname{coker}(\sigma' A) \longrightarrow 0$$

and $P'(\sigma' A)$ is an isomorphism, so $\ker(\sigma' A)$ and $\operatorname{coker}(\sigma' A)$ are torsion, and it follows that $P(\sigma' A)$ is an isomorphism. From these two natural equivalences we obtain $(P'\circ i)\circ(P\circ i') \cong P'\circ i' \cong 1_{\mathcal{D}}$ and $(P\circ i')\circ(P'\circ i) \cong P\circ i \cong 1_{\mathrm{Sh}(\mathcal{A})}$. ■

M. Barr has also shown

**Theorem. Let $\mathcal{A}$ be an exact category with injective effacements. There is a one to one correspondence**

between natural equivalence classes of left exact idempotent triples and Giraud subcategories of $\mathcal{E}$ which associates to each triple the category of algebras and to a Giraud subcategory the corresponding idempotent triple.

Proof. See [Barr, 1984]. But observe that given $\mathbb{T} = (T,\eta,\mu)$, $\mathcal{E}^{\mathbb{T}}$ is the full subcategory of $\mathcal{E}$ consisting of those objects $C$ for which $\eta C$ is an isomorphism. ■

Combining these results we have,

Theorem. Given an Abelian category $\mathcal{A}$ with injective effacements. There is a one to one correspondence between

(1) left exact idempotent triples on $\mathcal{A}$,

(2) radicals on $\mathcal{A}$,

(3) full coreflective subcategories of $\mathcal{A}$ with exact coreflector,

(4) topologies on $\mathcal{A}$.

Corollary. Let $\mathbb{P} = (P,\sigma,\tau)$ be a left exact idempotent triple on $\mathcal{A}$. The category $\mathcal{A}^{\mathbb{P}}$ has injective effacements.

## §3. Geometric morphisms.

In this section, we introduce the notion of a geometric morphism between Abelian categories and study its factorization.

**Definition. A geometric morphism $F: \mathcal{A} \longrightarrow \mathcal{B}$ between two Abelian categories is an additive functor $F_*: \mathcal{B} \longrightarrow \mathcal{A}$ which has an exact left adjoint $F^*: \mathcal{A} \longrightarrow \mathcal{B}$.**

Examples.

1. If j is a topology on $\mathcal{A}$, then $\mathrm{Sh}_j(\mathcal{A}) \xrightarrow{i} \mathcal{A}$ is a geometric morphism with left adjoint $P_j$.

2. If $G: \mathcal{C} \longrightarrow \mathcal{D}$ is a geometric functor between two ringoids, then $G_*: \mathcal{D}\text{-mod} \longrightarrow \mathcal{C}\text{-mod}$ induced by $G$ is a geometric morphism.

Let $\mathbb{P} = (P, \sigma, \tau)$ be a left exact triple on $\mathcal{A}$ and $A \in \mathcal{A}$. Then for any $A' \in \mathrm{sub}(A)$, we define $j_A A'$ so that the following diagram is a pullback.

$$\begin{tikzcd} j_A A' \arrow[r, tail] \arrow[d] & A \arrow[d, "\sigma A"] \\ PA' \arrow[r, tail] & PA \end{tikzcd}$$

Clearly j is increasing. Since $P$ is left exact,

$$\begin{tikzcd}
P\mathrm{j}_A A' \arrow[r, tail] \arrow[d] & PA \arrow[d, "P\sigma A"] \\
P^2A' \arrow[r, tail] & P^2A
\end{tikzcd}$$

is also a pullback and hence there is a unique map $PA \rightarrowtail P\mathrm{j}_A A'$ such that

$$\begin{tikzcd}
PA' \arrow[dr] \arrow[drr, bend left] \arrow[ddr, "P\sigma A'"'] & & \\
& P\mathrm{j}_A A' \arrow[r, tail] \arrow[d] & PA \arrow[d, "P\sigma A"] \\
& P^2A' \arrow[r, tail] & P^2A
\end{tikzcd}$$

commutes. Now for any $A' \subseteq A$ in $|\mathcal{A}|$, the outer square of

$$\begin{tikzcd}
PA' \arrow[r] \arrow[d] & PA \arrow[d] \\
P^2A' \arrow[r] \arrow[d] & P^2A \arrow[d] \\
PA' \arrow[r] & PA
\end{tikzcd}$$

is a pullback and the lower one is a monosquare, so that the upper one is a pullback. Comparing this with

$P$ applied to the square defining $j_A A'$, we see that $PA'$ $= PjA'$, from which it is immediate that $j$ is idempotent.

If $A'' \rightarrowtail A' \rightarrowtail A$, the map to the pullback in the square

$$\begin{tikzcd}
j_A A'' \arrow[drr] \arrow[dd] & & \\
& j_A A' \arrow[r, tail] \arrow[d] & A \arrow[d, "\sigma A"] \\
PA'' \arrow[r, tail] & PA' \arrow[r, tail] & PA
\end{tikzcd}$$

gives the required inclusion $j_A A'' \leq j_A A'$. Hence $j$ is monotone.

Finally we are going to show $j{:}\mathrm{sub}() \longrightarrow \mathrm{sub}()$ is natural; i.e. if $f{:}B \longrightarrow A$ then the diagram

$$\begin{tikzcd}
\mathrm{sub}A \arrow[r, tail, "j_A"] \arrow[d, "\mathrm{sub}(f)"'] & \mathrm{sub}A \arrow[d, "\mathrm{sub}(f)"] \\
\mathrm{sub}B \arrow[r, tail, "j_B"] & \mathrm{sub}B
\end{tikzcd}$$

commutes. Now follow $A' \in \mathrm{sub}A$ around two directions: let $B' = \mathrm{sub}(f)(A')$, i.e. the following diagram is a pullback

$$\begin{tikzcd}
B' \arrow[r, tail] \arrow[d] & B \arrow[d, "f"] \\
A' \arrow[r, tail] & A
\end{tikzcd}$$

Then $j_B \mathrm{sub}(f)(A') = j_B B'$. On the other hand we have a pullback

$$\begin{tikzcd}
\mathrm{sub}(f)(j_A A') \arrow[r, tail] \arrow[d] & B \arrow[d, "f"] \\
j_A A' \arrow[r, tail] & A
\end{tikzcd}$$

Hence it suffices to show the following squares commute:

$$\begin{tikzcd}
j_B B' \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "(1)"] & B \arrow[d, "f"] \\
j_A A' \arrow[r, tail] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{sub}(f)(j_A A') \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "(2)"] & B \arrow[d, "\sigma B"] \\
PB' \arrow[r, tail] & PB
\end{tikzcd}$$

To show (1) commutes; observe the following commutative diagram:

$$\begin{tikzcd}
f_B B' \arrow[r, tail] \arrow[d] & B \arrow[d, "\sigma B"'] \arrow[dr, "f"] & \\
PB' \arrow[r, tail] \arrow[d] & PB \arrow[d, "Pf"'] & A \arrow[dl, "\sigma A"] \\
PA' \arrow[r, tail] & PA &
\end{tikzcd}$$

so $j_B B' \longrightarrow PB' \longrightarrow PA' \rightarrowtail PA = j_B B' \rightarrowtail B \xrightarrow{f} A \longrightarrow PA$.

By definition of $j_A A'$ there is a unique map $j_B B' \longrightarrow j_A A'$ such that (1) commutes.

On the other hand since

$$\begin{tikzcd}
\mathrm{sub}(f)(j_A A') \arrow[r] \arrow[d] & B \arrow[d, "f"'] \arrow[dr, "\sigma B"] & \\
j_A A' \arrow[r, dash] \arrow[d] & A \arrow[d, "A"'] & PB \arrow[dl, "Pf"] \\
PA' \arrow[r] & PA &
\end{tikzcd}$$

commutes and

$$\begin{tikzcd}
PB' \arrow[r] \arrow[d] & PB \arrow[d, "Pf"] \\
PA' \arrow[r] & PA
\end{tikzcd}$$

is a pullback, there is a unique map $\mathrm{sub}(f)(j_A A') \longrightarrow PB$ such that (2) commutes and this shows that $j$ is a topology.

Next observe the radical induced by $j$ is precisely the radical $(\epsilon_P, \delta_P)$ induced by $\mathbb{P}$ and, since there is a one to one correspondence between radicals and topologies, we have:

**Theorem.** The topology $j^P$ induced by $(\epsilon_P, \delta_P)$ is $j$.

**Corollary.** A monic map $a: A' \rightarrowtail A$ in $\mathcal{A}$ is $j^P$-dense of and only if

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d] & A \arrow[d, "\sigma A"] \\ PA' \arrow[r, "P_a"] & PA \end{tikzcd}$$

is a pullback.

**Lemma.** Let $j$ be a topology on $\mathcal{A}$ with cannonical geometric morphism $i: \mathrm{Sh}_j(\mathcal{A}) \longrightarrow \mathcal{A}$. Then a geometric morphism $f: \mathcal{B} \longrightarrow \mathcal{A}$ factors through $i$ if and only if $F^*$ takes j-dense monics to isomorphisms i.e. $F^*(T) = 0$ for all $T \in \mathcal{T}_j$ in $\mathcal{A}$.

**Proof.** Let $a: A' \longrightarrow A$ be j-dense in $\mathcal{A}$, and $B$ be an object of $\mathcal{B}$. We have a commutative diagram

$$\begin{tikzcd} \mathcal{A}(A, F_*B) \arrow[r, "\cong"] \arrow[d] & \mathcal{B}(F^*A, B) \arrow[d] \\ \mathcal{A}(A', F_*B) \arrow[r, "\cong"] & \mathcal{B}(F^*A', B) \end{tikzcd}$$

where the left map is induced by a and the right by $F^*(a)$. The left map is an isomorphism of and only if $F_*B$ is a j-sheaf. The right map is an isomorphism for all $B \in |\mathcal{B}|$ if and only if $F^*(a)$ is an isomorphism. It follows that if $F$ factors through $i$, $F^*$ takes j-dense monics in $\mathcal{A}$ to isomorphisms.

Conversely, if $F^*$ has this property, then $F_*$ factors through $i$, let $F_* = i \circ u_*$ for some $u_*: \mathcal{B} \longrightarrow \mathrm{Sh}_j(\mathcal{A})$. Put $u^* = F^* \circ i$. Then $u^*$ is left exact and it is left adjoint to $u_*$ since for any $A \in |\mathrm{Sh}_j(\mathcal{A})|$, $B \in |\mathcal{B}|$ we have

$$\mathrm{Sh}_j(\mathcal{A})(A, u_* B) \cong \mathcal{A}(iA, F_* B) \cong \mathcal{B}(F^* \circ iA, B) = \mathcal{B}(u^* A, B). \blacksquare$$

**Lemma.** **Let j be the topology on $\mathcal{A}$ induced by the geometric morphism $F: \mathcal{B} \longrightarrow \mathcal{A}$. Then a monic $a: A' \rightarrowtail A$ in $\mathcal{A}$ is j-dense if and only if $F^*(a)$ is an isomorphism.**

**Proof.** As a corollary of previous lemma, since $i: \mathrm{Sh}_j(\mathcal{A}) \longrightarrow \mathcal{A}$ factors through itself, the sheafification of a dense monic is an isomorphism, so a j-dense implies $F^*(a)$ is an isomorphism. Conversely, if $F^*(a)$ is an isomorphism, let $\mathbf{P} = (P, \sigma, \tau)$ where $P = F_* \circ F^*$ is the left exact triple induced by $F$. Then $F_* \circ F^* A \cong F_* \circ F^* A'$, so $A'$ is j-dense in $A$. $\blacksquare$

**Theorem.** **Every geometric morphism $F: \mathcal{B} \longrightarrow \mathcal{A}$ can be factorized into $\mathcal{B} \xrightarrow{G} \mathcal{C} \xrightarrow{H} \mathcal{A}$ where $G^*$ reflects**

**isomorphism and $H_*$ is full and faithful.**

Proof. Let $H:\mathcal{E}\longrightarrow\mathcal{A}$ be $i:\mathrm{Sh}_j(\mathcal{A})\longrightarrow\mathcal{A}$ where j is the topology on $\mathcal{A}$ induced by $F$. By two previous lemmas $F$ factors as $H\circ G$ with $H$ full and faithful.

Suppose $F:A\longrightarrow B$ in $\mathrm{Sh}_j(\mathcal{A})$ is such that $G^*(f)$ is an isomorphism. In the following diagram, $\Delta$ is the diagonal map, $T = G_*\circ G^*$ and $d^0$ and $d^1$ are the projections from the pullback. All vertical maps are components of the unit $\eta$ corresponding to the adjunction of $G^*$ and $G_*$.

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d] & A\times_B B \arrow[r, shift left, "d^0"] \arrow[r, shift right, "d^1"'] \arrow[d] & A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
TA \arrow[r, "T\Delta"] & TA\times_{TB}TA \arrow[r, shift left, "Td^0"] \arrow[r, shift right, "Td^1"'] & TA \arrow[r] & TB
\end{tikzcd}$$

The composite across the top is $f$ and $Tf$ is an isomorphism by assumption, so $Td^0=Td^1$. Thus $T\Delta$ is an isomorphism. But $A \in \mathrm{Sh}_j(\mathcal{A})$, so $A \in \mathcal{F}_j$, and hence the left square is a pullback. That means $\Delta$ is an isomorphism, so $f$ is monic. But that implies $f$ is j-dense, and so $f = H^*\circ H_*f$ is an isomorphism. ■

Moreover, we have,

**Proposition.** let $F:\mathcal{B} \longrightarrow \mathcal{A}$ be a geometric morphism for which $F^*$ reflects isomorphisms and $F_*$ is full and faithful. Then $F_*$ and $F^*$ are adjoint equivalences.

**Proof.** Let $\mu$ and $\epsilon$ be the front and end adjunctions. Since $F_*$ is full and faithful, $\mu$ is an isomorphism, so $F^*\mu$ is an isomorphism. But $F^*$ reflects ismorphisms, so $\mu$ is also an isomorphism. ■

# CHAPTER 5

## §1. Basic definitions.

In this section we shall introduce the notion of Grothendieck topology on a ringoid $\mathcal{A}$ and show there is a one to one correspondence between Grothendieck topologies on $\mathcal{A}$ and topologies on $\mathcal{A}$-mod.

Suppose $\mathcal{C}$ is a Grothendieck category then it is well-known for each $C \in |\mathcal{C}|$ there is an essential monomorphism from $C$ to an injective object $E(C)$ where $E(C)$ is an injective envelope of $C$ and it is unique up to isomorphism. And we have an alternative definition of a torsion theory $(\mathcal{T},\mathcal{F})$ on $\mathcal{C}$ (see Chapter 4, §2).

**Lemma. Given $(\mathcal{T},\mathcal{F})$ a torsion theory on $\mathcal{C}$. Then the property that $\mathcal{T}$ is closed under subobjects is equivalent to $\mathcal{F}$ is closed under injective envelopes.**

Proof. Since $\mathcal{C}$ is a Grothendieck category and $(\mathcal{T},\mathcal{F})$ is a torsion theory on $\mathcal{C}$ we have a radical $(\epsilon,\delta)$ on $\mathcal{C}$ associated to $(\mathcal{T},\mathcal{F})$. Then $\epsilon(E(F)) \cap F = \epsilon(F) = 0$ for any $F \in \mathcal{F}$, which implies $\epsilon(E(F)) = 0$ that is $E(F) \in \mathcal{F}$.

Conversely, let $T \in \mathcal{T}$ and $C \in \text{sub}(T)$. Since $\mathcal{T}$ is closed under coproducts we can construct $t(C) \in \text{sub}(C)$ such that $t(C)$ is the sum of all torsion subobjects of

$C$ then $C/t(C) \in \mathcal{F}$. Hence there is a map $\pi : T \longrightarrow E(C/t(C))$ such that

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\pi"'] & T \arrow[d, "\pi'"] \\ C/t(C) \arrow[r] & E(C/t(C)) \end{tikzcd}$$

commutes. But $E(C/t(C)) \in \mathcal{F}$, so $\pi' = 0$. This implies $\pi = 0$ and hence $C = t(C) \in \mathcal{T}$. ■

**Lemma.** Let $\mathcal{S}$ be a class of objects in $\mathcal{C}$ closed under subobjects and quotients, $\mathcal{F} = \{C \in |\mathcal{C}| \mid \mathcal{C}(T,C) = 0$ for all $T \in \mathcal{S}\}$ and $\mathcal{T} = \{C \in |\mathcal{C}| \mid \mathcal{C}(C,F) = 0$ for all $F \in \mathcal{F}\}$. Then $(\mathcal{T},\mathcal{F})$ is a torsion theory on $\mathcal{C}$.

**Proof.** It suffices to show $\mathcal{F}$ is closed under injective envelopes. If $F \in \mathcal{F}$ and $f \in \mathcal{C}(T,E(F))$ for some $T \in \mathcal{S}$. Then $\mathrm{Im}(f) \in \mathcal{S}$ and $F \cap \mathrm{Im}(f) \in \mathrm{sub}(F)$. But $\mathcal{F}$ is closed under subobjects so $F \cap \mathrm{Im}(f) \in \mathcal{S} \cap \mathcal{F} = \{0\}$ so $E(F) \in \mathcal{F}$. ■

**Proposition.** The following properties of a class $\tau$ of objects of $\mathcal{C}$ are equivalent:

(a) $\tau$ is a torsion class for some torsion theory;

(b) $\tau$ is closed under subobjects, quotient objects, coproducts and extensions.

**Proof.** A class $\mathcal{S}$ is said to be closed under extension of for every exact sequence $0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow )$

with $C', C'' \in \mathcal{S}$, then $C \in \mathcal{S}$.

Suppose $(\tau, \varphi)$ is a torsion theory, $\epsilon$ is obviously closed under quotient objects, subobjects, and it is closed under coproducts because $\mathcal{C}(\Sigma_{i \in I} T_i, F) \cong \Pi_{i \in I} \mathcal{C}(T_i, F)$. Let $0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$ be exact in $\mathcal{C}$ with $C'', C' \in \tau$. If $F \in \varphi$ and $f \in \mathcal{C}(C, F)$, then $f$ is zero on $C$, so $f$ factors over $C''$. But also $\mathcal{C}(C'', F) = 0$, so $f = 0$. Hence $C \in \varphi$.

Conversely, assume (b) holds. let $\mathcal{F} = \{C \in |\mathcal{C}| \mid \mathcal{C}(t, C) = 0$ for all $T \in \tau\}$ and $\mathcal{T} = \{c \in |\mathcal{C}| \mid \mathcal{C}(C, F) = 0$ for all $F \in \mathcal{F}\}$, then $(\mathcal{T}, \mathcal{F})$ is a torsion theory. We must show that $\mathcal{T} = \tau$. Suppose $\mathcal{C}(C, F) = 0$ for all $F \in \mathcal{F}$. Since $\tau$ is closed under coporoducts, there is a largest subobject $T$ of $C$ belonging to $\tau$. To show $C = T$, it suffices to show $C/T \in \mathcal{F}$. Suppose $f \in \mathcal{C}(T', C/T)$ for some $T' \in \tau$. Then $\mathrm{Im}(f) \in \tau$, and if $f \neq 0$ then we would get a subibject of $C$ which strictly contains $T$ and belongs to $\tau$, since $\tau$ is closed under extensions. This contradicts the maximality of $T$, and so we must have $f = 0$, and $C/T \in \mathcal{F}$. ■

**Theorem. Suppose $\mathcal{C}$ is a Grothendieck category. Then there is a one to one correspondence between**

**(1) topologies on $\mathcal{C}$,**

**(2) radicals on $\mathcal{C}$,**

(3) torsion classes of objects of $\mathcal{C}$,

(4) torsion theories on $\mathcal{C}$.

Notice the previous proposition allows us to define a torsion class to be a class of objects closed under subobjects, quotient objects coproducts and extensions.

If $\mathcal{A}$ is a ringoid, then $\mathcal{A}$-mod is a Grothendieck category. From now on we shall restrict our attention to this category.

**Definition.** Suppose a $\in |\mathcal{A}|$ and $I \subseteq \mathrm{sub}(h^A)$ is a left ideal of $\mathcal{A}$. Then if a $\in \mathcal{A}(A',A)$, the left ideal $[I\!:\!a]$ $\in \mathrm{sub}(h^{A'})$ is defined by $[I\!:\!a](B) = \{b \in \mathcal{A}(B,A') \mid ba \in I(B)\}$. Equivalently, $[I\!:\!a]$ is the left ideal in the pullback diagram

$$\begin{tikzcd}
{[I\!:\!a]} \arrow[r] \arrow[d] & h^{A'} \arrow[d, "a"] \\
I \arrow[r] & h^A
\end{tikzcd}$$

in $\mathcal{A}$-mod.

**Definition.** A Grothendieck topology on $\mathcal{A}$ is a set $\{G(A) \mid A \in |\mathcal{A}|\}$ such that for each $A$, $G(A)$ is a set of subfunctors of $h^A$, that is the left ideals of $\mathcal{A}$, satisfying

(1) $h^A \in G(A)$ for each $A \in |\mathcal{Q}|$.

(2) If $I \in G(A)$ and $a \in \mathcal{Q}(A',A)$, then $[I:a] \in G(A')$

(3) If $I \in \text{sub}(h^A)$ and there exists $J \in G(A)$ such that for all $A' \in |\mathcal{Q}|$ and $a \in J(A')$, which implies $[I:a] \in G(A')$, then $I \in G(A)$ as well.

Note that $a \in J(A') \subseteq \mathcal{Q}(A',A)$.

**Lemma.** Suppose $\{G(A) \mid A \in |\mathcal{Q}|\}$ is a Grothendieck topology. Then

(1) If $I, J \in \text{sub}(h^A)$ such that $I \in G(A)$ and $I \rightarrowtail J$, then $J \in G(A)$.

(2) If $I, J \in G(A)$, then $I \cap J \in G(A)$.

**Proof.** (1) Suppose $I, J \in \text{sub}(h^A)$ such that $I \in G(A)$ and $I \rightarrowtail J$. Let $A' \in |\mathcal{Q}|$ and $a \in I(A') \subseteq \mathcal{Q}(A'A)$ then $[J:a](B) = \{b \in \mathcal{Q}(B,A') \mid ba \in J(B)\} = \mathcal{Q}(B,A)$, since $I$ is a left ideal. So $[J:a] = h^{A'} \in G(A')$, hence $J \in G(A)$.

(2) Suppose $I, J \in G(A)$. If $a \in I(A')$ then $[(I \cap J):a][I:a] \cap [J:a] = h^{A'} \cap [J:a] = [J:a] \in G(A')$ so $I \cap J \in G(A)$. ■

**Definition.** Suppose $M \in \mathcal{A}$-mod and $x \in M(A)$. The annihilator of $x$ is defined by

$$\mathrm{Ann}(x)(A') = \{a \in \mathcal{A}(A'A) \mid M(a)(x) = 0 \in M(A')\}$$

Equivalently, $\mathrm{Ann}(x) = \ker(\langle x\rangle)$. Clearly $\mathrm{Ann}(x) \in \mathrm{sub}(h^A)$ is a left ideal.

Suppose $\{G(A) \mid A \in |\mathcal{A}|\}$ is a Grothendieck topology on $\mathcal{A}$. Letting $\mathcal{T}_G = \{M \in \mathcal{A}\text{-mod} \mid \forall A \in |\mathcal{A}|,\ \forall x \in M(A),\ \mathrm{Ann}(x) \in G(A)\}$, we are going to show $\mathcal{T}_G$ forms a torsion class of objects.

Clearly, $\mathcal{T}_G$ is closed under subobjects. To show it is closed under quotient objects we need:

**Lemma.** Suppose $M,N \in \mathcal{A}$-mod, $f \in \mathcal{A}\text{-mod}(M,N)$ and $x \in M(A)$. Then the square

$$\begin{tikzcd} h^A \arrow[r, "\langle x\rangle"] \arrow[d, "1_{h^A}"'] & M \arrow[d, "f"] \\ h^A \arrow[r, "\langle f(A)(x)\rangle"] & N \end{tikzcd}$$

is commutative in $\mathcal{A}$-mod.

**Proof.** Let $a \in h^A(A') = \mathcal{A}(A',A)$. Then

$$F(A')\circ\langle x\rangle(A')(a) = f(A')\circ M(a)(x) = N(a)\circ f(A)(x)$$

$= \langle f(A)(X)\rangle(A')(a)$. ■

**Corollary.** **Suppose $M,N \in \mathcal{A}$-mod, $f \in \mathcal{A}$-mod$(M,N)$ and $x \in M(A)$. Then there is a monic map $\mathrm{Ann}(x) \rightarrowtail \mathrm{Ann}(f(A)(x))$.**

This corollary implies $\mathcal{J}_G$ is closed under quotient objects. Now if $M,N \in \mathcal{A}$-mod such that $M,N \in \mathcal{J}_G$ and suppose $(x,y) \in (M\oplus N)(A) = M(A)\oplus N(A)$, then $\mathrm{Ann}(x)$, $\mathrm{Ann}(y) \in G(A)$. But clearly $\mathrm{Ann}(x)\cap\mathrm{Ann}(y)$ is a subobject of $\mathrm{Ann}[(x,y)]$ this implies $\mathcal{J}_G$ is closed under coproducts. ■

**Lemma.** **Suppose $M \in \mathcal{A}$-mod, $x \in M(A)$ and $a \in \mathcal{A}(A',A)$. Then $\mathrm{Ann}(M(a)(x)) = [\mathrm{Ann}(x):a]$.**

**Proof.** $\mathrm{Ann}(M(a)(x))(B) = \{b \in \mathcal{A}(B,A') \mid M(b)\circ M(a)(X) = 0\} =$
$\{b \in \mathcal{A}(B,A') \mid M(ba)(x) = 0\} = \{b \in \mathcal{A}(B,A'') \mid b \in \mathrm{Ann}(x)(B)\} = [\mathrm{Ann}(x):a](B)$.

Now suppose $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ is exact in $\mathcal{A}$-mod such that $M'',M' \in \mathcal{J}_G$. Let $x \in M(A)$. Then $g(A)(x) \in M''(A)$ and $M'' \in \mathcal{J}_G$ implies $\mathrm{Ann}(g(A)(x)) \in G(A)$ where

$\mathrm{Ann}(g(A)(x))(B) = \{b \in \mathcal{A}(B,A) \mid M''(b)\circ g(A)(x) = 0\}$ for $B \in |\mathcal{A}|$.

Let $b \in \text{Ann}(g(A)(x))(B)$ then $M''(b)\circ g(A)(x) = g(B)\circ M(b)(x) = 0$. This implies $M(b)(x) \in \ker(g(B)) = \text{Im}(f(B))$ so there exists some $y \in M'(B)$ such that $f(B)(y) = M(b)(x)$. Since $M' \in \mathcal{J}_G$, $\text{Ann}(y) \in G(B)$. Since $\mathcal{J}_G$ is closed under homomorphic images, $\text{Ann}(f(B)(y)) \in G(B)$. Then $\text{Ann}(f(B)(y)) = \text{Ann}(M(b)(x)) = [\text{Ann}(x):b] \in G(B)$. Since this is true for any $b \in \text{Ann}(g(A)(x))(B)$, $B \in |\mathcal{a}|$, and $\text{Ann}(g(A)(x)) \in G(B)$ so $\text{Ann}(x) \in G(A)$. This shows $\mathcal{J}_G$ is closed under extensions.

Suppose $\mathcal{J}$ is a torsion class of objects of $\mathcal{a}$-mod. Then for each $A \in |\mathcal{a}|$ we let $G_{\mathcal{J}}(A) = \{I \in \text{sub}(h^A) \mid h^A/I \in \mathcal{J}\}$ and we shall show that $\{G_{\mathcal{J}}(A) \mid A \in |\mathcal{a}|\}$ forms a Grothedieck topology on $\mathcal{a}$.

Evidently, $h^A \in G_{\mathcal{J}}(A)$ for each $A \in |\mathcal{a}|$. Suppose $I \in G_{\mathcal{J}}(A)$ and $a \in \mathcal{a}(A',A)$. Since

$$\begin{tikzcd}
{[I:a]} \arrow[r] \arrow[d] & h^{A'} \arrow[d, "a"] \\
I \arrow[r] & h^A
\end{tikzcd}$$

is a pullback in $\mathcal{a}$-mod so $0 \longrightarrow [I:a] \longrightarrow h^{A'} \longrightarrow H^A/I$ is exact. Hence $h^{A'}/[I:a] \in \text{sub}(h^A/I)$ which implies $[I:a] \in G_{\mathcal{J}}(A')$.

**Lemma.** Suppose $i \in \mathrm{sub}(h^A)$ and $a \in \mathcal{Q}(A',A)$. Then

$$H^{A'}/[I:a] \cong \mathcal{Q}a/(I \cap \mathcal{Q}a).$$

**Proof.** Since $\mathcal{Q}a(B) = \{ba \in \mathcal{Q}(B,A) \mid b \in \mathcal{Q}(B,A')\}$, we always have an epimorphism $h^{A'} \longrightarrow \mathcal{Q}a$ given by composing with a, hence an epimorphism $h^{A'} \longrightarrow \mathcal{Q}a \longrightarrow \mathcal{Q}a/(I \cap \mathcal{Q}a)$.

Now suppose we let $K$ $= \ker(h^{a'} \longrightarrow \mathcal{Q}a \longrightarrow \mathcal{Q}a/(I \cap \mathcal{Q}a))$. Then

$K(B) = \{n \in \mathcal{Q}(B,A') \mid ba \in (I \cap \mathcal{Q}a)(B)\} = \{b \in \mathcal{Q}(B,A') \mid ba \in I(B)\} = [I:a]$. ■

**Corollary.** Suppose $i \in \mathrm{sub}(h^A)$ and $a \in \mathcal{Q}(A',A)$. Then

$$h^{A'}/[I:a] \cong \mathcal{Q}a/(I \cap \mathcal{Q}a) \cong (I+\mathcal{Q}a)/I.$$

Now suppose $I \in \mathrm{sub}(h^A)$ such that there exists some $J \in G_{\mathcal{J}}(A)$ and for all $a \in J(A')$, $[I:a] \in G_{\mathcal{J}}(A')$. We consider the exact sequence

$$0 \longrightarrow \frac{I+J}{I} \longrightarrow \frac{h^A}{I} \longrightarrow \frac{h^A}{I+J} \longrightarrow 0 .$$

Clearly $h^A/(I+J) \in \mathcal{J}$ since it is a quotient object of

$h^A/J \in \mathcal{T}$. Since $a \in J(A')$ implies

$$\frac{I+\mathcal{Q}a}{I} \cong \frac{h^{A'}}{[I:a]} \in \mathcal{T}$$

and $\mathcal{T}$ is closed under coproducts, $(I+J)/I \in \mathcal{T}$. Hence the fact that $\mathcal{T}$ is closed under extensions implies $h^A/I \in \mathcal{T}$, so $I \in G_{\mathcal{T}}(A)$ as well. ■

**Proposition.** Let $\mathcal{T}$ be a torsion class and $\{G_{\mathcal{T}}(A) | A \in |\mathcal{Q}|\}$ be its induced Grothendieck topology. Then $\mathcal{T} = \mathcal{T}_{G_\sigma}$.

**Proof.** If $M \in \mathcal{T}_{G_\sigma}$ then each cyclic subobject of $M$ is in $\mathcal{T}$ so $M \in \mathcal{T}$. Conversely, if $M \in \mathcal{T}$ and $x \in M(A)$ then $h^A/\mathrm{Ann}(x) \in \mathrm{sub}(M)$ so $h^A/\mathrm{Ann}(x) \in \mathcal{T}$, hence $\mathrm{Ann}(x) \in G_{\mathcal{T}}(A)$ and it follows $M \in \mathcal{T}_{G_\sigma}$. ■

**Lemma.** Suppose $I \in \mathrm{sub}(h^A)$. Then every cyclic subfunctor of $H^A/I$ has the form $h^{A'}/[I:a]$ for some $a \in \mathcal{Q}(A',A)$.

**Proof.** Let $x \in (h^A/I)(A')$ and $a \in \mathcal{Q}(A',A) = h^A(A')$ be a preimage of $x$ under the map $h^A(A') \twoheadrightarrow (h^A/I)(A')$. Then the diagram

$$\begin{tikzcd} h^{A'} \arrow[r, "a"] \arrow[dr, "\hat{x}"'] & h^{A} \arrow[d] \\ & h^{A}/I \end{tikzcd}$$

commutes in $\mathcal{A}$-mod. Hence

$$\mathrm{Ann}(x)(B) = \{b \in \mathcal{A}(B,A') \mid ba \in I(B)\} = [I:a](B). \blacksquare$$

**Proposition.** **Let $\{G(A) \mid A \in |\mathcal{A}|\}$ be the Grothendieck topology on $\mathcal{A}$ and $\mathcal{T}_G$ be the associated torsion class. Then $G(A) = G_{\mathcal{T}_G}(A)$ for all $A \in |\mathcal{A}|$.**

**Proof.** $I \in G(A)$ then by previous lemma $h^A/I \in \mathcal{T}_G$ so $h^A/I \in G_{\mathcal{T}_G}$. on the other hand, if $I \in G_{\mathcal{T}_G}(A)$ then $h^A/I \in \mathcal{T}_G$, so for each $x \in (h^A/I)(A')$, $\mathrm{Ann}(x) = [I:a] \in G(A)$ for some $a \in \mathcal{A}(A',A)$ and thus $I \in \mathcal{T}_G$. $\blacksquare$

Combining these results we have

**Theorem.** **There is a one to one correspondence between**
**(1) topologies on $\mathcal{A}$-mod,**
**(2) radicals on $\mathcal{A}$-mod,**
**(3) torsion classes of objects of $\mathcal{A}$-mod, and**

**(4) Grothendieck topologies on $\mathcal{A}$.**

Remark. Given $\{G(A) \mid A \in |\mathcal{A}|\}$ a Grothendieck topology on $\mathcal{A}$ then $\epsilon_G$ the associated radical is defined for $M \in \mathcal{A}$-mod by $(\epsilon_G M)(A) = \{x \in M(A) \mid \mathrm{Ann}(x) \in G(A)\}$. And $j_G$ the associated topology is defined for $M' \in \mathrm{sub}(M)$ by $j_M(M') = \ker(M \longrightarrow M/M' \longrightarrow (M/M')/\epsilon_G(M/M'))$.

**Remark.** Let $E \in \mathcal{A}$-mod be injective. Suppose $\mathcal{T}_E = \{M \in \mathcal{A}\text{-mod} \mid \mathcal{A}\text{-mod}(M,E) = 0\}$ and $\mathcal{F}_E = \{M \in \mathcal{A}\text{-mod} \mid \mathcal{A}\text{-mod}(T,M) = 0$ for all $T \in \mathcal{T}_E\}$. Then $(\mathcal{T}_E, \mathcal{F}_E)$ forms a torsion theory on $\mathcal{A}$-mod, since if $M' \in \mathrm{sub}(M)$ and $M \in \mathcal{T}_E$, then consider $f \in \mathcal{A}$-mod$(M'E)$. $f$ must factor through $M$ which implies $f = 0$.

On the other hand, if $(\mathcal{T}, \mathcal{F})$ is a torsion theory on $\mathcal{A}$-mod. Put $F(A) = \{I \in \mathrm{sub}(h^A) \mid h^A/I \in \mathcal{F}\}$ and $E = \Pi_{A \in |\mathcal{A}|} \Pi_{I \in F(A)} E(h^A/I)$. Then $E \in \mathcal{F}$, so $\mathcal{A}$-mod$(M,E) = 0$ for all $M \in \mathcal{T}$. Now if $M \notin \mathcal{T}$ there exists some cyclic subfunctor $C$ of $M$ with a non-zero $f \in \mathcal{A}$-mod$(C,F)$ for some $F \in \mathcal{F}$. The image of $f$ is cyclic and torsion free, so $f$ induces a map $C \longrightarrow E$ which can be extended to a non-zero map $M \longrightarrow E$. Hence $M \in \mathcal{T}$ if and only if $\mathcal{A}$-mod$(M,E) = 0$. Combining these results we have

**Proposition.** **$\mathcal{T}$ is a torsion class of objects in $\mathcal{A}$-mod if and only if there exists an injective $E \in \mathcal{A}$-mod such that**

$$\mathcal{T} = \{M \in \mathcal{Q}\text{-mod} \mid \mathcal{Q}\text{-mod}(M,E) = 0\}.$$

**Remark**

1. Principal Grothendieck topologies.

**Definition.** A Grothendieck topology $\{G(A) A \in |\mathcal{Q}|\}$ is principal if for each $I \in G(A)$ there exists $a \in I(A')$ such that $\mathcal{Q}a \in G(A)$.

Suppose $\{G(A) A \in |\mathcal{Q}|\}$ is a principal Grothendieck topology. Put $\Gamma(G) = \{S \in \mathcal{Q}(A',A) \mid \mathcal{Q}s \in G(A),\ A',A \in |\mathcal{Q}|\}$. Clearly for each $A \in |\mathcal{Q}|$, $1_A \in \Gamma(G)$. Suppose $s \in \mathcal{Q}(A',A)$ and $t \in \mathcal{Q}(A'',A')$ are such that $s,t \in \Gamma(G)$. Let $b \in h^{A'}(B) = \mathcal{Q}(B,A')$. Then for any $D \in |\mathcal{Q}|$, we have

$$[\mathcal{Q}(ts):bs](D) = \{d \in \mathcal{Q}(D,B) \mid dbs \in \mathcal{Q}(ts)(D)\}$$

$$\supseteq \{d \in \mathcal{Q}(D,B) \mid db \in (\mathcal{Q}t)(D)\} = [\mathcal{Q}t:b](D).$$

Since $\mathcal{Q}t \in G(A')$ so $[\mathcal{Q}t:b] \in G(B)$ and $[\mathcal{Q}(ts):bs] \in G(B)$. But this is true for all $bs \in \mathcal{Q}s(B)$, hence $\mathcal{Q}(ts) \in G(A)$ so $ts \in \Gamma(G)$.

On the other hand suppose $a \in \mathcal{Q}(A',A)$, $a' \in \mathcal{Q}(A'',A')$ such that $a'a \in \Gamma(G)$. Then since $\mathcal{Q}(a'a) \in \text{sub}(\mathcal{Q}a)$, we have $\mathcal{Q}a \in G(A)$ hence $a \in \Gamma(G)$.

If $s \in \Gamma(G) \cap \mathcal{a}(A',A)$ and $a \in \mathcal{a}(A'',A)$, then $[\mathcal{a}s:a] \in G(A'')$. Since $G$ is principal there exists $t \in [\mathcal{a}s:a](B)$ such that $\mathcal{a}t \in G(A'')$. Notice $\mathcal{a}t \in \mathrm{sub}([\mathcal{a}s:a])$. Consider the following diagram in $\mathcal{a}$-mod:

$$\begin{tikzcd}
\mathcal{a}t \arrow[r, tail] & {[\mathcal{a}s:a]} \arrow[r, tail] \arrow[d] & h^{A''} \arrow[d] \\
 & \mathcal{a}s \arrow[r, tail] & h^{A}
\end{tikzcd}$$

such that the square is a pullback. Clearly $t \in (\mathcal{a}t)(B)$ and $ta \in (\mathcal{a}s)(B)$ so there exists some $b \in \mathcal{a}(B,A')$ such that $ta = bs$.

Conversely, suppose $\Gamma$ is a set of morphism satisfying

(1) $1_A \in \Gamma$, for all $A \in |\mathcal{a}|$;

(2) $\Gamma$ is closed under composition;

(3) if $a \in \mathcal{a}(A',A)$, $a' \in \mathcal{a}(A'',A')$ such that $a'a \in \Gamma$, then $a \in \Gamma$;

(4) if $s \in \Gamma \cap \mathcal{a}(A',A)$ and $a \in \mathcal{a}(A'',A)$, then there exists $t \in \Gamma \cap \mathcal{a}(B,A'')$ and $b \in \mathcal{a}(B,A')$ such that $ta = bs$.

Put $G_\Gamma(A) = \{I \in \mathrm{sub}(h^A) \mid \text{for some } A' \in |\mathcal{a}|, I(A') \cap \Gamma \neq \emptyset\}$. Clearly $h^A \in G_\Gamma(A)$. Suppose $I \in G_\Gamma(A)$

then there exists $s \in I(A')\cap\Gamma$. Let $a \in \mathcal{Q}(A'',A)$. Since $s \in I(A') \subseteq \mathcal{Q}(A',A)$, there exists $t \in \Gamma\cap\mathcal{Q}(B,A'')$ and $b \in \mathcal{Q}(B,A')$ such that $ta = bs$. Now consider $[I{:}a](B) = \{d \in \mathcal{Q}(B,A'') \mid da \in I(B)\}$. This implies $t \in [I{:}a](B)$ so $[I{:}a] \in G_\Gamma(A'')$.

Now suppose $I,J \in \mathrm{sub}(h^A)$ and $J \in G_\Gamma(A)$ such that for all $a \in J(A')[I{:}a] \in G_\Gamma(A')$. Since $J \in G_\Gamma(A)$ there exists $B \in |\mathcal{Q}|$ such that $s \in \Gamma\cap J(B)$. Then in particular $[I{:}a] \in G_\Gamma(B)$, so there exists $D \in |\mathcal{Q}|$ such that $t \in \Gamma\cap[I{:}s](D)$. But $t \in [I{:}s](D)$ implies $ts \in I(D)$, and $ts \in \Gamma$ we have $I \in G(A)$. Hence we have shown $G_\Gamma$ is a Grothendieck topology; moreover it is principal.

**Proposition.** Let $G$ be a principal Grothendieck topology and $\Gamma_G$ be the set of morphisms associated to $G$. Then $G = G_{\Gamma_G}$.

**Proof.** Let $I \in G(A)$. Then there exists some $s \in I(A')$ such that $\mathcal{Q}s \in G(A)$, so $s \in \Gamma_G$. But clearly $\mathcal{Q}s \in G_{\Gamma_G}(A)$ and $\mathcal{Q}s \in \mathrm{sub}(I)$, so $I \in G_{\Gamma_G}(A)$. On the other hand, suppose $I \in G_{\Gamma_G}(A)$. There exists $s \in \Gamma_G\cap I(A')$. Clearly $\mathcal{Q}s \in G(A)$ and $\mathcal{Q}s \in \mathrm{sub}(I)$ so $I \in G(A)$. ■

**Proposition.** Let $\Gamma$ be a set of morphisms satisfying the four conditions and $G_\Gamma$ be the associated principal Grothendieck topology. Then $\Gamma = \Gamma_{G_\Gamma}$.

**Proof.** Let $s \in \Gamma \cap \mathcal{Q}(A',A)$. Then clearly $\mathcal{Q}s \in G_\Gamma(A)$ so $s \in \Gamma_{G_\Gamma}$. On the other hand if $s \in \Gamma_{G_\Gamma} \cap \mathcal{Q}(A',A)$ then $\mathcal{Q}s \in G_\Gamma(A)$ so there exists $t \in \mathcal{Q}s(B) \cap \Gamma$. Hence there exists $b \in \mathcal{Q}(B,A')$ such that $t = bs \in \Gamma$ so $s \in \Gamma$. ■

Combining these two results we have:

**Theorem.** There is a one to one correspondence between

(1) Principal Grothendieck topologies on $\mathcal{Q}$

(2) Sets $\Gamma$ of morphism of $\mathcal{Q}$ satisfying

(a) $1_A \in \Gamma$ for all $A \in |\mathcal{Q}|$;

(b) $\Gamma$ is closed under composition;

(c) if $a \in \mathcal{Q}(A',A)$, $a' \in \mathcal{Q}(A'',A')$ such that $a'a \in \Gamma$, then $a \in \Gamma$;

(d) if $s \in \Gamma \cap \mathcal{Q}(A',A)$ and $a \in \mathcal{Q}(A'',A)$ then there exist $t \in \Gamma \cap \mathcal{Q}(B,A'')$ and $b \in \mathcal{Q}(B,A')$ such that $ta = bs$.

Remark. (1) Notice in the construction of $G_\Gamma$ we did not need the property (c) of $\Gamma$, so in particular if $\Gamma$ is a RMC set of morphisms of $\mathcal{Q}$ then $G_\Gamma$ is a principal Grothendieck topology.

(2) let $G$ be a principal Grothendieck topology on $\mathcal{A}$ and $(\mathcal{T}_G,\mathcal{F}_G)$ be the associated torsion theory. Then $M \in \mathcal{T}_G$ and $x \in M(A)$ implies $\mathrm{Ann}(x) \in G(A)$, so there exists $s \in \mathcal{A}(A',A)\cap\Gamma_G$ such that $\mathcal{A}s \in \mathrm{sub}(\mathrm{Ann}(x))$. Hence $M \in \mathcal{T}_G$ if and only if for all $x \in M(A)$ there exists some $s \in \Gamma_G\cap\mathcal{A}(A',A)$ with $M(s)(x) = 0$.

## 2. Bounded topologies.

**Lemma.** **Suppose $\{G(A)\,|\,A \in |\mathcal{A}|\}$ is a Grothendieck topology on $\mathcal{A}$ and put $\eta_G(B,A) = \bigcap_{I\in G(A)} I(B)$. Then $\eta_G$ is an ideal of $\mathcal{A}$.**

Proof. Clearly $\eta_G(-,A) = \bigcap_{I\in G(A)} I \in \mathrm{sub}(h^A)$ for all $A \in |\mathcal{A}|$. On the other hand suppose $a \in \mathcal{A}(A,A')$. Then $[I':a] \in G(A)$ for any $I' \in G(A')$, so $\eta_g(B,A)$ $= \bigcap_{I\in G(A)} I(B) \subseteq \bigcap_{I'\in G(A')}[I':a](B)$. This implies that for any $b \in \eta(B,A)$ $ba \in \eta_G(B,A')$. And it is easy to check if $a \in \mathcal{A}(A,A')$ and $b \in \mathcal{A}(B',B)$ then the diagram

$$\begin{array}{ccc}
\eta_G(B,A) & \longrightarrow & \eta_G(B,A') \\
\downarrow & & \downarrow \\
\eta_G(B',A) & \longrightarrow & \eta_G(B',A')
\end{array}$$

commutes, so $\eta_G \subseteq \mathcal{A}(-,-)$ and hence $\eta_G$ is an ideal. ■

**Definition.** A Grothendieck topology $G$ is bounded if for all $A \in |\mathcal{A}|$, $\eta_G(-,A) \in G(A)$ as well.

Suppose $G$ is a bounded Grothendieck topology and $\eta_G$ be its associated ideal. Evidently $\eta_G^2 \subseteq \eta_G$. On the other hand if $a \in \eta_G(A',A)$ then $\eta_G(-,A') \subseteq [\eta_G^2(-,A):a]$ so $\eta_G^2(-,A) \subseteq G(A)$. This implies $\eta_G(-,A) \subseteq \eta_G^2(-,A)$, hence $\eta_G = \eta_G^2$.

Conversely, if $\eta$ is an idempotent ideal put $G_\eta(A) = \{I \in \mathrm{sub}(h^A) | \eta(-,A) \subseteq I\}$. Clearly $h^A \in G_\eta(A)$. Now suppose $I \in G_\eta(A)$ and $a \in \mathcal{A}(A',A)$ then $[I:a](B) = \{b \in \mathcal{A}(B,A') | ba \in I(B)\}$. Let $d \in \eta(B,A')$ we have $da \in \eta(B,A) \subseteq I(B)$ so $[I:a]$ contains $\eta(-,A')$, hence $[I:a] \in G_\eta(A')$.

Let $I$, $J \in \mathrm{sub}(h^A)$ and $J \in G_\eta(A)$ such that for all $a \in J(A')$ $[I:a] \in G_\eta(A')$ then we must show $I \in G_\eta(A)$. But $J \in G_\eta(A)$ implies $\eta(-,A) \subseteq J$ so without loss of generality we can take $J = \eta(-,A)$, aince $\eta(-,A) \in G_\eta(A)$. Hence for all $a \in \eta(A',A)$ $\eta(-,A') \subseteq [I:a]$. this implies $\eta^2(-,A) \subseteq I$. But $\eta$ is idempotent so $I \in G_\eta(A)$. This shows $G_\eta$ is a Grothendieck topology and evidently it is bounded by $\eta$ since $\cap_{I \in G_\eta(A)} I = \eta(-,A)$. This construction of $G_\eta$ also implies

**Proposition.** Let $\eta$ be an idempotent ideal and $G_\eta$ be the associated bounded Grothendieck topology. Then $\eta = \eta_G$.

**Proposition.** Let $G$ be a bounded Grothendieck topology on $\mathcal{A}$, $\eta_G$ be its associated idempotent ideal and $G'$ be the Grothendieck topology associated to $\eta_G$. Then $G = G'$.

**Proof.** Let $I \in G(A)$ so $\eta_G(-,A) \subseteq I$ which implies $I \in G'(A)$. On the other hand $I \in G'(A)$, by definition. $\eta(-,A) \subseteq I$. So $I \in G(A)$. ■

Combining these results we have

**Theorem.** There is a one to one correspondence between

(1) Bounded Grothendieck topologies on $\mathcal{A}$.

(2) Idempotent ideals of $\mathcal{A}$.

**Definition.** Given $M \in \mathcal{A}$-mod then the annihilator of $M$ is given by $\mathrm{Ann}(M)(B,A) = \{a \in \mathcal{A}(B,A) \mid \text{for all } x \in M(A). M(a)(x) = 0\}$.

It is easy to check $\mathrm{Ann}(M)$ is a two sided ideal of $\mathcal{A}$ and $\mathrm{Ann}(M)(B,A) = \cap_{x \in M(A)} \mathrm{Ann}(x)$. hence suppose $G$ is a bounded Grothendieck topology on $\mathcal{A}$, then $M \in \mathcal{T}_G$ if and only if $\eta_G \subseteq \mathrm{Ann}(M)$.

**Theorem.** The following are equivalent for $G$ a Grothendieck topology in $\mathcal{A}$.

(a) $G$ is bounded;

(b) $\mathcal{T}_G$ is closed under products;

(c) there is an ideal $\sigma_G$ such that $M \in \mathcal{T}_G$ if and only if $\sigma_G \subseteq \mathrm{Ann}(M)$;

(d) there is an idempotent ideal $\eta_G$ such that $M \in \mathcal{T}_G$ if and only if $\eta_G \subseteq \mathrm{Ann}(M)$.

**Proof.** (a)$\Longrightarrow$(d) and (d)$\Longrightarrow$(c) are clear. To show (b)$\Longrightarrow$(c)$\Longrightarrow$(d), consider the cannonical map $h^A \longrightarrow \Pi_{I \in G(A)} h^A/I$ then $\mathrm{Im}(\alpha)$ is also in $\mathcal{T}_G$ so $\cap_{I \in G(A)} I = \ker(\alpha) \in G(A)$. Hence put $\sigma_G(B,A) = \cap_{I \in G(A)} I(B)$ and we have shown $\sigma_G$ is an ideal; furthermore it is idempotent. This also shows (b)$\Longrightarrow$(a).

To show (d)$\Longrightarrow$(b), suppose $\{M_i \mid i \in I\} \subseteq \mathcal{T}_G$ so for all $i \in I$, $\eta_G \subseteq \mathrm{Ann}(M_i)$. This implies that $\eta_G \subseteq \mathrm{Ann}(\Pi_{i \in I} M_i)$ so $\Pi_{i \in I} M_i \in \mathcal{T}_G$. ■

**Remark.** Suppose $\{e_A \in \mathcal{A}(A,A) \mid A \in |\mathcal{A}|\} \in Z(\mathcal{A})$ (Recall $Z(\mathcal{A})$ is the center of $\mathcal{A}$) such that $e_A^2 = e_A$ for all $A \in |\mathcal{A}|$ i.e. it is a central idempotent of $\mathcal{A}$. Then the ideal generated by it is idempotent so we can associate each central idempotent element of $\mathcal{A}$ to a bounded Grothendieck topology on $\mathcal{A}$.

## §2. $GT(\mathcal{A})$ forms a Heyting algebra.

In this section we shall show the collection of Grothendieck topologies on $\mathcal{A}$ forms a complete Heyting algebra. But first we observe from §1:

**Proposition. Suppose $\mathcal{A}$ is a ringoid. Then the collection $GT(\mathcal{A})$ of Grothendieck topologies on $\mathcal{A}$ is a set.**

**Definition. Suppose $G$, $G' \in GT(\mathcal{A})$ then we say $G' \leq G$ if for all $A \in |\mathcal{A}|$, $G'(A) \subseteq G(A)$.**

**Lemma. The following are equivalent for $G'$, $G \in GT(\mathcal{A})$.**

**(1) $G' \leq G$;**

**(2) $\mathcal{T}_{G'} \subseteq \mathcal{T}_G$;**

**(3) $\mathcal{F}_G \subseteq \mathcal{F}_{G'}$.**

**Proof.** (1)$\Rightarrow$(2): if $M \in \mathcal{T}_{G'}$ then for all $x \in M(A)$ $\mathrm{Ann}(x) \in G'(A)$ so $M \in \mathcal{T}_G$. (2)$\Rightarrow$(1) is evident. (2)$\Rightarrow$(3) follows easily from the fact that $\mathcal{A}\text{-mod}(T,F) = 0$ for all $T \in \mathcal{T}_G$ and $F \in \mathcal{F}_G$. ∎

Suppose $\mathfrak{Y}$ is a collection of Grothendieck topologies we define $G_{\cap\mathfrak{Y}}(A) = \cap_{G\in\mathfrak{Y}}G(A)$. Then clearly $G_{\cap\mathfrak{Y}} \in GT(\mathfrak{a})$ and we have $\mathfrak{J}_{\cap\mathfrak{Y}} = \cap_{G\in\mathfrak{Y}}\mathfrak{J}_G$. It is easy to verify:

**Lemma.** If $G' \le G$ for all $G \in \mathfrak{Y}$, then $G' \le G_{\cap\mathfrak{Y}}$.

To define $\vee\mathfrak{Y}$ on $GT(\mathfrak{a})$, suppose $G \in \mathfrak{Y}$. We denote the generalization of $G$ by the set $\text{gen}(G) = \{G' \in GT(\mathfrak{a}) \mid G \le G'\}$ then put $\omega = \cap\{\text{gen}(G) \mid G \in \mathfrak{Y}\}$ and $G_{\vee\mathfrak{Y}} = G_{\cap\omega}$.

**Lemma.** If $G \le G'$ for all $G \in \mathfrak{Y}$, then $G_{\vee\mathfrak{Y}} \le G'$.

**Proof.** $G \le G'$ for all $G \in \mathfrak{Y}$ implies $G' \in \text{gen}(G)$ for all $G \in \mathfrak{Y}$, so $G \in \omega$ and hence $G_{\vee\mathfrak{Y}} \le G'$. ∎

Let $G' = G_{\vee\mathfrak{Y}}$. Then it is easy to verify that for all $G \in \mathfrak{Y}$, $G \le G'$ and $\mathfrak{F}_{G'} = \cap_{G\in\mathfrak{Y}}\mathfrak{F}_G$.

**Lemma.** Let $G \in GT(\mathfrak{a})$ and $\mathfrak{Y}$ be a non-empty collection of Grothendieck topologies on $\mathfrak{a}$. Then

$$G\wedge\vee\mathfrak{Y} = \vee\{G\wedge G' \mid G' \in \mathfrak{Y}\}.$$

**Proof.** Let $G'' = \vee\{G\wedge G' \mid G' \in \mathfrak{Y}\}$. Then $G\wedge G' \le G$ for all $G' \in \mathfrak{Y}$ and $G\wedge G' \le G'$ for all $G' \in \mathfrak{Y}$ so $G'' \le G$ and $G'' \le \vee\mathfrak{Y}$; hence $G'' \le G\wedge\vee\mathfrak{Y}$.

To show the reverse inequality we are going to show $\mathcal{J}_{G \wedge \vee \mathfrak{F}} \subseteq \mathcal{J}_{G''}$. Suppose $M \in \mathcal{J}_{G \wedge \vee \mathfrak{F}}$ and $M \notin \mathcal{J}_{G''}$. Without loss of generality we can assume $M \in \mathcal{F}_{G''}$, sonce $\mathcal{J}_{G \wedge \vee \mathfrak{F}}$ is closed under quotient objects. So $M \in \mathcal{J}_{\vee \mathfrak{F}}$ which implies there exists $G' \in \mathfrak{F}$ such that $M \notin \mathcal{F}_{G}$, since $\mathcal{F}_{G \wedge \mathfrak{F}} = \bigcap_{G' \in \mathfrak{F}} \mathcal{F}_{G'}$. Hence $0 \neq \epsilon_{G'} M \in \mathcal{J}_{G'}$, and since $\mathcal{J}_G$ is closed under subobjects $\epsilon_{G'}(M) \in \mathcal{J}_G$ so we have $\epsilon_{G'}(M) \in \mathcal{J}_{G' \wedge G}$. But $\epsilon_{G'}(M)$ is clearly in $\mathcal{F}_{G''} = \bigcap_{\varphi \in \Phi} \mathcal{F}_{\varphi}$ where $\Phi = \{G \wedge G' \mid G \in \mathfrak{F}\}$ so it is in $\mathcal{F}_{G' \wedge G}$ and $\mathcal{J}_{G' \wedge G'}$, which is a contradiction. ■

Evidently, $GT(\mathcal{a})$ has a unit with respect to $\wedge$ namely, $U(A) = \{$all left ideals of $h^A\}$and a unit with respect to $\vee$ namely $O(A) = \{$the zero ideal of $h^A\}$. Combining all these we have:

**Theorem.** $GT(\mathcal{a})$ **together with $\leq$ forms a complete Heyting algebra.**

§3. The category $\mathrm{Sh}_G(\mathcal{a}$-mod).

In this section we shall characterize the full reflective subcategory $\mathrm{Sh}_G(\mathcal{a}$-mod) with respect to a Grothendieck topology $G$ on $\mathcal{a}$.

**Lemma.** **Suppose** $G \in GT(\mathcal{a})$ **and** $M' \rightarrowtail M$ **in** $\mathcal{a}$**-mod. Then the topology** $j$ **corresponding to** $G$ **is given by**

$j_M(M')(A) = \{x \in M(A) \mid [M':x] \in G(A)\}$ for all $A \in |\mathcal{A}|$.

Recall that $[M:x](B) = \{b \in \mathcal{A}(B,A) \mid M(b)(x) \in M'(A)\}$; equivalently

$$\begin{tikzcd}
{[M':x]} \arrow[r] \arrow[d] & h^A \arrow[d] \\
M' \arrow[r] & M
\end{tikzcd}$$

is a pullback in $\mathcal{A}$-mod.

**Proof.** Recall that given $G \in GT(\mathcal{A})$, then the corresponding redical $(\epsilon,\sigma)$ is defined by $\epsilon(M)(A) = \{x \in M(A) \mid \mathrm{Ann}(x) \in G(A)\}$. And the associated topology $j$ is defined by: If $M'\mathrm{sub}(M)$ then

$$j_M(M') = \ker(M \longrightarrow M/M' \longrightarrow (M/M')/\epsilon(M/M')).$$

Now suppose $x \in j_M(M')(A)$. Then it induces $\langle x\rangle \in \mathcal{A}\text{-mod})h^A, j_M)M'))$ and pulling back along $M' \rightarrowtail j_M(M')$ the cannonical dense monomorphism. So the monic map $[M':x] \rightarrowtail h^A$ must be dense as well, hence $[M':x] \in G(A)$.

Conversely, suppose $x \in M(A)$ such that $[M':x] \in G(A)$. Consider the following commutative diagram in $\mathcal{A}$-mod

$$\begin{tikzcd}
{[M':x]} \arrow[r, tail] \arrow[d] & h^A \arrow[r] \arrow[d] & h^A/[M':x] \arrow[d] & \\
M' \arrow[r, tail] & M \arrow[r] & M/M' \arrow[r] & (M/M')/\epsilon(M/M')
\end{tikzcd}$$

such that the left hand square is a pullback. Then the composite $h^A/[M':x] \longrightarrow M/M' \longrightarrow (M/M')/\epsilon(M/M')$ is 0 since $h^A/[M':x] \in \mathcal{T}$ and $(M/M')/\epsilon(M/M') \in \mathcal{F}$. Hence $x \in \ker(M \longrightarrow M/M' \longrightarrow (M/M')/\epsilon(M/M')) = j_M(M')(A)$. ■

**Definition.** Suppose $G \in GT(\mathcal{A})$ and $M \in \mathcal{A}$-mod. Then $M$ is $G$-injective if and only if for every $I \in G(A)$ and $f \in \mathcal{A}\text{-mod}(I,M)$ there exists some $f' \in \mathcal{A}\text{-mod}(h^A,M)$ such that the diagram

$$\begin{tikzcd}
I \arrow[r, tail] \arrow[d, "f"'] & H^A \arrow[dl, "f'"] \\
M &
\end{tikzcd}$$

commutes in $\mathcal{A}$-mod.

**Remark.** (1) The existence of $f'$ is not necessarily unique, and by the Yoneda lemma, there exists some $x_f \in M(A)$ such that if $b \in I(B)$ then $f(B)(b) = M(b)(x_f) \in M(B)$.

(2) Equivalently, $M$ is $G$-injective if and only if for all $I \in G(A)$ the map $M(A)$

$\cong \mathcal{a}\text{-mod}(h^A,M) \longrightarrow \mathcal{a}\text{-mod}(I,M)$ is epic in Ab.

**Proposition.** The following are equivalent for $M \in \mathcal{a}$-mod:

(1) $M$ is $G$-injective;

(2) for all $N' \in \text{sub}(N)$ such that $j_N(N') = N$ and $f \in \mathcal{a}\text{-mod}(N',M)$, there exists some $f \in \mathcal{a}\text{-mod}(N,M)$ such that

$$\begin{tikzcd} N' \arrow[r, rightarrowtail] \arrow[d, "f"'] & N \arrow[dl, "f'"] \\ M & \end{tikzcd}$$

is commutative in $\mathcal{a}$-mod;

(3) $M$ is j-closed in its injective enevlope $E(M)$;

(4) If $M \in \text{sub}(M')$ then there exists $N' \in \text{sub}(M')$ such that $M \oplus N'$ is j-closed in $M'$.

**Proof.** (1)$\Longrightarrow$(2): Let $N' \in \text{sub}(N)$ such that $j_N(N') = N$ and $f \in \mathcal{a}\text{-mod}(N',M)$. Consider the set of all pairs $(K,\varphi)$ where $N' \leq K \leq N$ and $\varphi \in \mathcal{a}\text{-mod}(K,M)$ extends $f$. Order this set by putting $(K',\varphi') \leq (K,\varphi)$ if and only if $K' \leq K$ and $\varphi|K' = \varphi'$. Then this set is inductive under $\leq$ and so by Zorn's Lemma it has a maximal element $(\bar{K},\bar{\varphi})$. We must show $\bar{K} = N$. If not, there exists $A \in |\mathcal{Q}|$ such that $\bar{K}(A) \notin N(A)$. Let $x \in N(A)$ and $x \notin \bar{K}(A)$. Since $N'$ is j-dense in $N$ so $\bar{K}$ is j-dense in $N$. Hence $[\bar{K}:x] \in G(A)$. Define $\psi:[\bar{K}:x] \longrightarrow M$ as follows: Given $B$

$\in |\mathcal{Q}|$,

$$\psi(B)(b) = \bar{\varphi}(B)N(b)(x) \text{ for all } b \in [\bar{K}:x](B).$$

It is easy to check $\psi \in \mathcal{Q}\text{-mod}([\bar{K}:x],M)$, so there exists $y \in M(A)$ such that $\psi(B)(b) = \bar{\varphi}(B)N(b)(x)$ $= M(b)(y)$ for all $b \in [\bar{K}:x](B)$.

Now define $\varphi_1:\bar{K}+\mathcal{Q}x \longrightarrow M$ (Recall: $(\mathcal{Q}x)(B)$ $= \{N(b)(x) \mid b \in \mathcal{Q}(B,A)\}$) as follows: $\varphi_1(B)(z+N(b)(x))$ $= \bar{\varphi}(B)(z)+M(b)(y)$ for all $B \in |\mathcal{Q}|$. I claim $\varphi_1$ is well-defined: suppose $z \in \bar{K}(B)$ and $N(b)(x) \in (\mathcal{Q}x)(B)$ such that $z+N(b)(x) = 0$. Then $N(b)(x) = -z \in \bar{K}(B) \cap (\mathcal{Q}x)(B)$ $\subseteq \bar{K}(B)$ so $b \in [\bar{K}:x]$ which implies $M(b)(y)$ $= \bar{\varphi}(B)N(b)(x)$. Hence

$$\varphi_1(B)(z+N(b)(x)) = \bar{\varphi}(B)(z)+M(b)(y)$$
$$= \bar{\varphi}(B)(z)+\bar{\varphi}(B)N(b)(x) = \bar{\varphi}(B)(z+N(b)(x)) = 0.$$

It is easy to check $\varphi_1 \in \mathcal{Q}\text{-mod}(\bar{K}+\mathcal{Q}x,M)$ and evidently $\varphi_1$ extends $\bar{K}$ so $\bar{K} = N$.

(2)$\Longrightarrow$(3) If $M \rightarrowtail E(M)$ then we have $M \overset{\alpha}{\rightarrowtail} j_{E(M)}(M) \overset{\beta}{\rightarrowtail} E(M)$ and $\alpha$ is a j-dense monomorphism so there exists $\alpha' \in \mathcal{A}\text{-mod}(j_{E(M)}(M),M)$ such that $\alpha' \circ \alpha = 1_M$. Since $M \rightarrowtail E(M)$ is essential, so is $\alpha$, and hence $\alpha'$ is monic. Evidently $\alpha'$ is epic so $\alpha'$ is an isomorphism.

(3)$\Longrightarrow$(1) Suppose $f \in \mathcal{A}\text{-mod}(I,M)$ and $I \in G(A)$. Then since $E(M)$ is injective, there exists a unique $g \in \mathcal{A}\text{-mod}(h^A,E(M))$ such that

$$\begin{array}{ccc} I & \longrightarrow & h^A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ M & \longrightarrow & E(M) \end{array}$$

commutes in $\mathcal{A}$-mod. Since the top map is j-dense and the bottom map is j-closed, there exists a map $h^A \longrightarrow M$ with the required property.

(3)$\Longrightarrow$(4) If $M \in \text{sub}(M')$ then there exists $N \in \mathcal{A}\text{-mod}$ such that $E(M') = E(M) \oplus N$. If $N' = M' \cap N$ then $M'/(M \oplus N')$ is a subfunctor of $E(M')/(M \oplus N)$. But $E(M')/(M \oplus N) \cong (E(M)+N)/(M \oplus N) \cong E(M)/M$. Thus $E(M)/M \in \mathcal{F}$ inplies $M'/(M + N') \in \mathcal{F}$.

(4)$\Longrightarrow$(3) Since $M \longrightarrow E(M)$, there exists $N'\text{sub}(E(M))$ such that $M \oplus N'$ is j-closed in $E(M)$. But $M \rightarrowtail E(M)$ is essential hence $N' = 0$, so $M$ is j-closed in $E(M)$. ■

**Theorem.** The following are equivalent for $M \in \mathcal{A}_{\text{mod}}$.

(a) $M \in \text{Sh}_G(\mathcal{A}\text{-mod})$;

(b) $M \in \mathcal{F}$ and $M$ is $G$-injective;

(c) if $I \in G(A)$ and $f \in \mathcal{A}\text{-mod}(I,M)$, there exists a unique $f' \in \mathcal{A}\text{-mod}(h^A,M)$ such that

$$\begin{tikzcd} I \arrow[r, rightarrowtail] \arrow[d, "f"'] & h^A \arrow[dl, "f'"] \\ M & \end{tikzcd}$$

commutes in $\mathcal{A}$-mod.

**Proof.** Evidently (a)$\Longrightarrow$(b) and (a)$\Longrightarrow$(c).

(b)$\Longrightarrow$(a): Suppose $M \in \text{sub}(M')$ and $M' \in \mathcal{F}$. Then there exists $N' \in \text{sub}(M')$ such that $M \oplus N'$ is j-closed in $M'$. Consider the exact sequence

$$0 \longrightarrow \frac{M \oplus N'}{M} \longrightarrow \frac{M'}{M} \longrightarrow \frac{M'}{M \oplus N'} \longrightarrow 0$$

in $\mathcal{A}$-mod. But $(M \oplus N')/M \cong N' \in \mathcal{F}$ and $M'/(M \oplus N') \in \mathcal{F}$ so $M'/M \in \mathcal{F}$. Hence $M \in \text{Sh}_G(\mathcal{A}\text{-mod})$.

(c)$\Longrightarrow$(b) Clearly $M$ is $G$-injective. Suppose $x \in M(A)$ is such that $\mathrm{Ann}(x) \in G(A)$. Consider the composition $\mathrm{Ann}(x) \longrightarrow h^A \longrightarrow M$. There exists a unique map $h^A \longrightarrow M$ such that the diagram

$$\begin{tikzcd}
\mathrm{Ann}(x) \arrow[r] \arrow[d, tail] & h^A \arrow[ddl, dashed] \\
h^A \arrow[d, tail] & \\
M &
\end{tikzcd}$$

commutes in $\mathcal{A}$-mod. Thus $x = 0 \in M(A)$ and hence $\epsilon(M) = 0$. ∎

**Corollary.** **Suppose $M \in \mathrm{Sh}_G(\mathcal{A}\text{-mod})$. Then the following are equivalent for $N \in \mathrm{sub}(M)$:**

**(1) $N \in \mathrm{Sh}_G(\mathcal{A}\text{-mod})$;**

**(2) $N$ is j-closed in $M$;**

**(3) $N$ is $G$-injective.**

**Proof.** (3)$\Longrightarrow$(1) and (1)$\Longrightarrow$(2) are evident since $N \in \mathcal{F}$.

(2)$\Longrightarrow$(3) Suppose $k'$ is j-dense in $K$. Since $M$ is $G$-injective, if $f \in \mathcal{A}\text{-mod}(K', N)$ we have a map $f' : K \longrightarrow M$ such that the diagram

$$\begin{tikzcd} K' \arrow[r, tail] \arrow[d, "f"'] & K \arrow[d, "f'"] \\ N \arrow[r, tail] & M \end{tikzcd}$$

commutes in $\mathcal{A}$–mod. But the top map is j-dense and the bottom map is j-closed. Hence there exists a map $K \rightarrowtail N$ with the required property.

§4. The category $\mathcal{A}_G$.

In this section we shall construct the category of quotients $\mathcal{A}_G$ with respect to $G \in GT(\mathcal{A})$.

Suppose $M \in \mathcal{A}$–mod and $G \in GT(\mathcal{A})$. Let $F$ be the radical corresponding to $G$ and put $S_M = M/\epsilon(M)$, so $SM = \mathcal{F}$. Let $P\colon \mathcal{A}\text{–mod} \longrightarrow \mathrm{Sh}_G(\mathcal{A}\text{–mod})$ be the left exact reflector. Then $PM = j_{E(SM)}(SM)$ since $\mathcal{F}$ is closed under injective envelopes. So we have

$$PM(A) = \{x \in E(SM)(A) | [SM:x] \in G(A)\} \text{ for all } A \in |\mathcal{A}|.$$

In particular we shall denote $SA = h^A/\epsilon(h^A)$ and $PA = P(h^A)$ for $A \in |\mathcal{A}|$.

Now we can construct $\mathcal{A}_G$ the category of quotients; the objects of $\mathcal{A}_G$ are the same as those of $\mathcal{A}$. If $A'$, $A \in |\mathcal{A}|$ we put $\mathcal{A}_G(A',A) = PA(A')$. Evidently it is an abelian group and we note that $PA(A') \cong \mathcal{A}\text{–mod}(h^A,PA)$

$= \mathrm{Sh}_G(\mathcal{A}\text{-mod})(PA',PA)$ so if $a' \in \mathcal{A}_G(A',A)$ and $a''$ $\in \mathcal{A}_G(A'',A')$ then the composite $a'a''$ is simply defined by composing their corresponding maps in $\mathrm{Sh}_G(\mathcal{A}\text{-mod})$. And it is easy to check the composition is distributive over addition so $\mathcal{A}_G$ is small and preadditive.

We also have an additive functor $\psi:\mathcal{A} \longrightarrow \mathcal{A}_G$: If $A$ $\in |\mathcal{A}|$, $\psi(A) = A$. Suppose $a \in \mathcal{A}(A',A) = h^A(A')$. Let $\bar{a}$ be its image in $PA(A')$ then we put $\psi(a) = \bar{a}$ $\in \mathrm{Sh}_G(\mathcal{A}\text{-mod})(PA',PA)$. Note that $\bar{a}$ is an element of $E(SA)(A')$ such that $[SA:\bar{a}] \in G(A')$.

So $\psi$ induces a pair of adjoint functors

$$\begin{tikzcd} \mathcal{A}\text{-mod} \arrow[r, shift left, "\psi^*"] & \mathcal{A}_G\text{-mod} \arrow[l, shift left, "\psi_*"] \end{tikzcd}$$

and $\Psi \in \mathcal{A}_G\text{-mod-}\mathcal{A}$ such that $\psi^* \cong \Psi \otimes -$, $\psi_* = \mathrm{Hom})\Psi,-)$ and $\Psi(A',A) = \mathcal{A}_G(A',\psi A) = \mathcal{A}_G(A',A)$ for $A' \in |\mathcal{A}_G|$, $A \in |\mathcal{A}|$.

If $M \in \mathrm{Sh}_G(\mathcal{A}\text{-mod})$ we define $\tau M \in \mathcal{A}_G\text{-mod}$ by $\tau M(A)$ $= M(A)$. Note that $\tau M(A) \cong (h^A,M) \cong \mathrm{Sh}_G(\mathcal{A}\text{-mod})(PA,M)$. Then suppose $x \in \tau M(A)$ and $a \in \mathcal{A}_G(A',A)$ $\cong \mathrm{Sh}_G(\mathcal{A}\text{-mod})(PA',PA)$. We let $\tau M(a)(x)$ be the composition of maps in $\mathrm{Sh}_G(\mathcal{A}\text{-mod})$. So we have the following diagram

$$\begin{tikzcd}
\mathcal{A}\text{-mod} \arrow[rr, shift left, "\Psi^*"] \arrow[dr, shift left, "p"] & & \mathcal{A}_G\text{-mod} \arrow[ll, shift left, "\Psi_*"] \arrow[dl, "\tau"] \\
& \mathrm{Sh}_G(\mathcal{A}\text{-mod}) \arrow[ul, shift left, "i"] &
\end{tikzcd}$$

## CHAPTER 6

### §1. The Grothendieck topology by $\mathfrak{M}$.

Let $\mathcal{Q}$ be the ringoid $\mathfrak{M} \subseteq \mathcal{Q}$-mod (not necessarily a set) and $N \in \mathcal{Q}$-mod.

**Definition.** $N$ is $\mathfrak{M}$-torsion if for all $x \in N(A)$ and all $M \in \mathfrak{M}$ the cannnonical morphism

$$M(A) \cong \mathcal{Q}\text{-mod}(h^A,M) \longrightarrow \mathcal{Q}\text{-mod}(\ker(x),M)$$

is an isomorphism. The collection of $\mathfrak{M}$-torsion objects in $\mathcal{Q}$-mod is denoted by $\mathcal{T}_{\mathfrak{M}}$.

**Lemma.** $N \in \mathcal{T}_{\mathfrak{M}}$ if and only if for $f:L \longrightarrow N$ in $\mathcal{Q}$-mod and any $M \in \mathfrak{M}$ the cannonical map

$$\mathcal{Q}\text{-mod}(L,M) \longrightarrow \mathcal{Q}\text{-mod}(\ker f,M)$$

is an isomorphism.

**Proof.** The "if" direction is obvious. Now suppose $f:L \longrightarrow N$ and $N \in \mathcal{T}_{\mathfrak{M}}$. Let $I$ be the disjoint union of the underlying set of abelian groups $\{M(A) \mid A \in |\mathcal{Q}|\}$, and $A^I = \Sigma_{a\in I}\mathcal{Q}(-,A_a)$. Then there is an epimorphism $p:A^I \longrightarrow M$. Now consider the commutative diagram

$$\begin{tikzcd}
 & \ker(f\circ p\circ i_A) \arrow[r, "k_a"] \arrow[d, "f_a"'] & \mathcal{a}(-,A_a) \arrow[r, "i_a"] \arrow[dr, "a"] & A^I \arrow[d, "p"] & \\
0 \arrow[r] & \ker f \arrow[rr, "k"] & & L \arrow[r, "f"] & N
\end{tikzcd}$$

with exact bottom row, $k_a$, $i_a$, $k$ cannonical injections and $f_a$ induced by the bottom row.

Let $M \in \Phi$ amd $u:L \longrightarrow M$ such that $u\circ k = 0$. Then for any $a \in I$, $u\circ k\circ f_a = u\circ p\circ i_a\circ k_a = 0$. Since $N \in \mathcal{J}_\Phi$ we have $u\circ p\circ i_a = 0$ which implies $u\circ p = 0$. But $p$ is epic so $u = 0$. Hence $\mathcal{a}\text{-mod}(L,M) \longrightarrow \mathcal{a}\text{-mod}(\ker f,M)$ is monic.

Suppose $u:\ker\varphi \longrightarrow M$. We have $f_a\circ u:\ker(f\circ p\circ i_a) \longrightarrow M$ which implies there exists a unique $a_f \in M(A_a) \cong \mathcal{a}\text{-mod}(\mathcal{a}(-,A_a),M)$, such that $a_f\circ k_a = f_a\circ u$ for each $a \in I$. That is if $b \in \ker(f\circ p\circ i_a)(B)$ we have $\langle a_f\rangle(B)\circ K_a(B)(b) = M(b)(a_f) = u(B)\circ f_a(B)(b) = u(B)\circ L(b)(a)$. The set of morphisms $\{a_f | a \in I\}$ induces a unique map $\hat{f}:A^I \longrightarrow M$. Hence we must show $\hat{f}$ vanishes on $\ker p$. For a finite subset $J \subseteq I$, let $A^J = \Sigma_{j\in J}\mathcal{a}(-,A_a)$. Define $P_J:A^J \longrightarrow L$ to be the cannonical map induced by $p$ and $K_J = \ker P_J$ with $h_J:K_J \longrightarrow A^J$ the canonical inclusion. Also let $\pi_a:A^J \longrightarrow \mathcal{a}(-,A_a)$ and $i_a:\mathcal{a}(-,A_a) \longrightarrow A^I$ the canonical projections and inclusions such that $\Sigma_{a\in J} i_a\circ\pi_a = 1_{A^J}$. Hence if $b$

$\in K_J(B) \cong \mathcal{A}\text{-mod}(h^B, K_J)$ we have

$$P_J \circ h_J \langle b\rangle = P_J \circ (\Sigma_{a\in J} i_a \circ \pi_a) \circ h_J \circ \langle b\rangle$$

$$= \Sigma_{a\in J}(P_J \circ i_a) \circ (\pi_a \circ h_J \circ \langle b\rangle) = \Sigma_{a\in J} a b_a = \Sigma_{a\in J} L(b_a)(a) = 0$$

where $b_a = \pi_a \circ h_J \circ \langle b\rangle : h^B \longrightarrow \mathcal{A}(-, A_a)$ and so $b_a \in (B, A_a)$.

This implies $f(B) \circ (\Sigma_{a\in J} L(b_a)(a))$ $= f(B) \circ L(\Sigma_{a\in J} b_a)(a) = 0$. But $\ker(f \circ p)$ $= \mathrm{colim}_{\mathrm{fin}(J)} \ker(A^J \xrightarrow{p} L \xrightarrow{f} N)$ so $\Sigma_{a\in J} i_a \circ b_a$ $\leq \ker(f \circ P_J : A^J \longrightarrow L \longrightarrow N)$. Hence we have

$$\hat{f}_J \circ h_J \circ \langle b\rangle = \hat{f}_J \circ ((\Sigma_{a\in J} i_a \circ \pi_a) \circ h_J \circ \langle b\rangle) = \hat{f}_J \circ (\Sigma_{a\in J} i_a \circ b_a)$$

$$= \Sigma_{a\in J} u(B) \circ L(b_a)(a) = u(B) \circ \Sigma_{a\in J} L(ba)(a) = 0.$$

This completes the proof. ■

Obviously, if $N \in \mathcal{J}_{\Phi}$ and $N' \in \mathrm{sub}(N)$ then $N' \in \mathcal{J}_{\Phi}$ as well. Suppose $p : N \longrightarrow N'$ is an epimorphism and $f : h^A \longrightarrow N'$. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker f' \arrow[r, "i'"] \arrow[d, "p''"] & h^A \times_{N'} N \arrow[r, "f'"] \arrow[d, "p'"] & N \arrow[d, "p"] \\
0 \arrow[r] & \ker f \arrow[r, "k"] & h^A \arrow[r, "f"] & N'
\end{tikzcd}$$

The rows are exact and $p''$ is an isomorphism.

If $M \in \Phi$ and $u:h^A \longrightarrow M$ such that $u \circ i = 0$, then $u \circ i \circ p'' = u \circ p' \circ i' = 0$. But $N \in \mathcal{J}_\Phi$ so $u \circ p' = 0$ and $p'$ is epic implies $u = 0$.

Since $p'$ is epic and $h^A$ is projective in $\mathcal{A}$-mod there exists $q:h^A \longrightarrow h^A \times_{N'} N$ such that $p' \circ q = 1_{h^A}$. Then a simple diagram chase shows that $q \circ i \circ p'' = i'$. Now if $u:\ker f \longrightarrow M$, there exists $\overline{u}:h^A \times_{N'} N \longrightarrow M$ such that $\overline{u} \circ i' = u \circ p''$. This implies $\overline{u} \circ q:h^A \longrightarrow M$ and we have $up'' = \overline{u} \circ i' = \overline{u} \circ q \circ i \circ p''$ so $u = \overline{u} \circ q \circ i$. Hence $N' \in \mathcal{J}_\Phi$.

If $0 \longrightarrow N' \xrightarrow{i} N \xrightarrow{f} N'' \longrightarrow 0$ is exact in $\mathcal{A}$-mod with $N', N'' \in \mathcal{J}_\Phi$, $f:h^A \longrightarrow N$, then consider the commutative diagram

$$\begin{tikzcd}
 & & \ker f \arrow[dl, "j"'] \arrow[d, "k"] & & \\
 & \ker(p \circ f) \arrow[r, "i'"] \arrow[d, "f'"'] & h^A \arrow[d, "f"] & & \\
0 \arrow[r] & N' \arrow[r, "i"] & N \arrow[r, "p"] & N'' \arrow[r] & 0
\end{tikzcd}$$

where $(\ker f, j)$ is the kernel of $f'$, $(\ker f, k)$ is the kernel of $f$ and $(\ker(p \circ f), i')$ is the kernel of $p \circ f$.

If $M \in \Phi$ and $u:h^A \longrightarrow M$, then $u \circ k = 0$ implies

$u \circ i' \circ j = 0$, but $N' \in \mathcal{T}_\Phi$ implies $u \circ i' = 0$ and $N'' \in \mathcal{T}_\Phi$ implies $u = 0$.

If $u: \ker f \longrightarrow M$, then $N' \in \mathcal{T}_\Phi$ implies there exists $\bar{u}: \ker(p \circ f) \longrightarrow M$ such that $\bar{u} \circ j = u$ and $N'' \in \mathcal{T}_\Phi$ implies there exists $u': h^A \longrightarrow M$ such that $u' \circ i' = \bar{u}$. Hence $u = \bar{u} \circ j = u' \circ i' \circ j = u' \circ k$ so $N \in \mathcal{T}_\Phi$. It is routine to show by induction that $\mathcal{T}_\Phi$ is closed under coproducts. Hence $\mathcal{T}_\Phi$ forms a torsion class. There exists a Grothendieck topology $G$ corresponding to $\mathcal{T}_\Phi$ (see Chapter 5) such that $I \in G(A)$ if and only if for all $M \in |\Phi|$, the cannonical morphism $M(A) = \mathcal{A}\text{-mod}(h^A, M) \longrightarrow \mathcal{A}\text{-mod}(I, M)$ is an isomorphism. This implies $\Phi \subseteq \mathrm{Sh}_G(\mathcal{A}\text{-mod})$. In particular, when $\Phi = |\mathcal{A}| = \{h^A | A \in |\mathcal{A}|\}$. $G$ is known as the cannonical topology on $\mathcal{A}$.

When $\Phi$ is a full preadditive subcategory of $\mathcal{A}$-mod there is an obvious full and faithful additive functor $F_\Phi: \Phi \longrightarrow \mathrm{Sh}_G(\mathcal{A}\text{-mod})$ and we have a diagram of functors

$$\begin{tikzcd}
 & \Phi \arrow[dl, "F_\Phi"'] \arrow[dr, "F'"] & \\
\mathrm{Sh}_G(\mathcal{A}\text{-mod}) \arrow[rr, shift left, "i"] & & \mathcal{A}\text{-mod} \arrow[ll, shift left, "p"]
\end{tikzcd}$$

with $F' = i \circ F_{\Phi}$ and $p \circ F' = F_{\Phi}$.

## §2. Embedding Grothendieck categories.

Let $\mathcal{C}$ be any category, $C$ an object of $\mathcal{C}$ and $\alpha = \{f_i : C_i \longrightarrow C \mid i \in I\}$ a set of morphisms in $\mathcal{C}$ with codemain $C$. $\alpha$ is epimorphic in $\mathcal{C}$ if whenever $g, h : C \longrightarrow C'$ such that $h \circ f_i = g \circ f_i$ for each $I \in I$, then $g = h$. A set $\Gamma$ of objects of $\mathcal{C}$ is said to be a set of generators for $\mathcal{C}$ if for every $C \in |\mathcal{C}|$ the family of all morphisms with domains in $\Gamma$ and codomain,

$$\{f : A \longrightarrow C \mid A \in \Gamma\}$$

is an epimorphic family. Obviously if $\mathcal{C}$ is cocomplete then $\{f_i : C_i \longrightarrow C \mid i \in I\}$ is epimorphic if and only if $\Sigma_{i \in I} C_i \longrightarrow C$ induced by $f_i$'s is an epimorphism in $\mathcal{C}$.

Recall that a cocomplete abelain category $\mathcal{C}$ is a Grothendieck category if direct limits are exact in $\mathcal{C}$ and $\mathcal{C}$ has a set of generators.

**Proposition. The following statements are equivalent in a cocomplete abelian category $\mathcal{C}$:**

**(a) Direct limits are exact in $\mathcal{C}$;**

**(b) $\mathcal{C}$ satisfies AB5, that is directed unions preserve finite intersections;**

(c) for every morphism $f: C_1 \longrightarrow C_2$ and direct family $\{C_i \mid i \in I\}$ of subobjects of $C_2$ one has

$$f^{-1}(\Sigma_{i \in I} C_i) = \Sigma_{i \in I} f^{-1}(C_i)$$

**Proof.** See [B. Stenström, 1975] ∎

Suppose $\mathcal{C}$ is a Grothendieck category with a set of generators $\Gamma$ and $\mathcal{A}$ a small full preadditive subcategory (i.e. a ringoid) such that $\Gamma \subseteq |\mathcal{A}|$. We construct the functor $S: \mathcal{C} \longrightarrow \mathcal{A}$-mod by: if $C \in \mathcal{C}$, $A \in |\mathcal{A}|$, then $S(C)(A) = \mathcal{C}(A,C)$ and if $f: C \longrightarrow C'$ then $S(C)(f) = \mathcal{C}(A,f)$.

**Proposition.** $S$ **is full and faithful.**

**Proof.** Since the family of all morphisms $\{f: A \longrightarrow C \mid A \in \Gamma\}$ is epimorphic, then so is the family $\{f: A \longrightarrow C \mid A \in |\mathcal{A}|\}$. Hence $S$ is faithful.

Now suppose $\delta \in \mathcal{A}\text{-mod}(S(C), S(C'))$. We must show there exists a unique $f: C \longrightarrow C'$ such that for any a $\in S(C)(A) = \mathcal{C}(A,C)$, $\delta(A)(\mathrm{a}) = f \circ \mathrm{a}$. If $I$ is the set of all morhisms with domain in $|\mathcal{A}|$ and codemain $C$, then $I$ is epimorphic. Hence there is an epimorphism $p: \Sigma_{\mathrm{a} \in I} A_{\mathrm{a}} \longrightarrow C$ such that the diagram

$$\begin{tikzcd} A_a \arrow[d, "i_a"'] \arrow[dr, "a"] & \\ \Sigma_{A\in I} A_a \arrow[r, "p"] & C \end{tikzcd}$$

commutes for any $a \in I$. Then $\delta$ induces $q:\Sigma_{a\in I}A_a \longrightarrow C'$ such that the diagram

$$\begin{tikzcd} A_a \arrow[d, "i_a"'] \arrow[dr, "\delta(A_a)(a)"] & \\ \Sigma_{A\in I} A_a \arrow[r, "q"] & C' \end{tikzcd}$$

is commutative for each $a \in I$. If $q$ vanishes on $\ker(p)$ then there exists a unique $f:C \longrightarrow C'$ such that $f \circ p = q$ and hence $\delta(A)(a) = f \circ a$ for any $a \in I$. Let $J$ be any finite subset of $I$ and $K_J$ be the kernel of the cannonical morphism $P_J:\Sigma_{a\in J}A_a \longrightarrow C$ induced by $p$ with a cannonical monomorphism $k_J:K_J \longrightarrow \Sigma_{a\in J}A_a$. We also have the usual injections $i_a:A_a \longrightarrow \Sigma_{a\in J}A_a$, projections $\pi_a:\Sigma_{a\in J}A_a \longrightarrow A_a$ such that $\Sigma_{a\in J} i_a \circ \pi_a$ is the identity of $\Sigma_{a\in J}A_a$.

Suppose $B \in |\mathcal{A}|$ and $b:B \longrightarrow K_J$. We have

$$P_J\circ h_J\circ b = P_J\circ(\Sigma_{a\in J} i_a\circ \pi_a)h_J\circ b = \Sigma_{a\in J} P_J\circ i_a\circ \pi_a\circ h_J\circ b$$
$$= \Sigma_{a\in J} a\circ b_a = 0,$$

where $b_a = \pi_a\circ h_J\circ b \in \mathcal{C}(B,A_a)$. If $q_J:\Sigma_{a\in J}A_a \longrightarrow C'$ is the cannonical morphism indiced by $q$, we have

$$q_J\circ H_J\circ b = q_J\circ(\Sigma_{a\in J} i_a\circ \pi_a)\circ h_J\circ b = \Sigma_{a\in J}(q_J\circ i_a)\circ(\pi_a\circ h_J\circ b)$$
$$= \Sigma_{a\in J}\delta(A_a)(a)\circ b_a = \delta(B)(a)\circ(\Sigma_{a\in J} a\circ b_a) = 0.$$

We conclude that $a_J\circ h_J = 0$, and hence $K_J$ is contained in $\ker q$.

Now let $\Phi = \{S(C) \mid C \in |\mathcal{C}|\}$. There is a Grothendieck topology $G$ induced by $\Phi$. Since every $C \in |\mathcal{C}|$, $S(C)$ is a $G$-sheaf, there results a diagram of functors

$$\begin{tikzcd}
 & \mathcal{C} \arrow[dl, "S'"'] \arrow[dr, "S"] & \\
\mathrm{Sh}_G(\mathcal{A}\text{-mod}) \arrow[rr, shift left, "i"] & & \mathcal{A}\text{-mod} \arrow[ll, shift left, "P"]
\end{tikzcd}$$

with $S' = P\circ S$ and $i\circ S' = S$. Since $S$ is full and faithful, $S'$ is also a full and faithful. ■

**Proposition. $S'$ is exact.**

**Proof.** Since $S' = PS$ is left exact so it suffices to show $s'$ preserves epimorphisms. Hence if $p: C_1 \longrightarrow C_2$ is epic in $\mathcal{C}$ we must show coker$S(p) \in J_{\Phi}$. If $x$ $\in$ coker$(S(p))(A) \cong \mathcal{A}$-mod$(h^A,$coker$S(p))$ we have the following commutative diagram.

$$\begin{tikzcd}
 & {[\mathrm{Im}S(p):\hat{x}]} \arrow[r] \arrow[d] & h^A \arrow[d, "\hat{x}"'] \arrow[dr, "\langle x\rangle"] & & \\
0 \arrow[r] & \mathrm{Im}(p) \arrow[r] & S(C_2) \arrow[r] & \mathrm{coker}S(p) \arrow[r] & 0
\end{tikzcd}$$

With $\hat{x}$ induced by $x$ since $h^A$ is projective in $\mathcal{A}$-mod, the square is a pullback and the bottom row is exact. This implies $\ker(x) = [\mathrm{Im}S(p):x]$. Hence we have to show for all $C \in |\mathcal{C}|$ the cannonical morphism $S(C)(A) \cong \mathcal{A}\text{-mod}(h^A, S(C) \longrightarrow \mathcal{A}\text{-mod}([\mathrm{Im}S(p):x], S(C))$ is an isomorphism. Let $y \in S(C_2)(A) = \mathcal{C}(A, C_2)$ $\cong \mathcal{A}\text{-mod}(h^A, S(C_2))$ correspond to $\hat{x}$. Then we have the following commutative diagram in $\mathcal{C}$

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, "i'"] \arrow[d, "="'] & C_1 \times_{C_2} A \arrow[r, "p'"] \arrow[d] & A \arrow[r] \arrow[d, "y"] & 0 \\
0 \arrow[r] & K \arrow[r, "i"] & C_1 \arrow[r, "p"] & C_2 \arrow[r] & 0
\end{tikzcd}$$

with exact rows and the right hand square a pullback. Now apply the left exact functor $S$ we have the following commutative diagram in $\mathcal{A}$-mod

$$\begin{tikzcd}
0 \arrow[r] & S(K) \arrow[r, "S(i')"] \arrow[d, "="'] & S(C_1)\times_{S(C_2)} h^A \arrow[r, "S(p')"] \arrow[d] & h^A \arrow[d, "\hat{x}"] \\
0 \arrow[r] & S(K) \arrow[r] & S(C_1) \arrow[r, "S(p)"] & S(C_2)
\end{tikzcd}$$

with exact rows and the right hand square a pullback. Note that $S(A) = h^A$. Now consider the following commutative diagram in $\mathcal{A}$-mod.

$$\begin{tikzcd}
 & & & \mathrm{Im}S(p') \arrow[dr, "j"] & & & \\
0 \arrow[r] & S(K) \arrow[r, "S(i')"] \arrow[d, "="'] & S(C_1)\times_{S(C_2)} h^A \arrow[ur, "q"] \arrow[rr, "S(p')"] \arrow[d] & & h^A \arrow[d, "\hat{x}"'] \arrow[dr, "\langle x\rangle"] & & \\
0 \arrow[r] & S(K) \arrow[r] & S(C_1) \arrow[rr, "S(p)"] & & S(C_2) \arrow[r] & \mathrm{coker}S(p) \arrow[r] & 0
\end{tikzcd}$$

where

$$S(C_1\times_{C_2}A)=S(C_1)\times_{S(C_2)}h^A \xrightarrow{q} \mathrm{Im}S(p') \xrightarrow{j} h^A$$

is the epi-mono factorization of $S(p')$ in $\mathcal{A}$-mod. Hence $\mathrm{Im}S(p') = [\mathrm{Im}S(p):x]$.

Now suppose $u \in \mathcal{A}\text{-mod}(h^A,S(C)) = \mathcal{A}\text{-mod}(S(A),S(C))$ such that $u\circ j = 0$. Since $S$ is full and faithful, there exists a unique $u' \in \mathcal{E}(A,C)$ such that $S(u') = u$. Then $u\circ j = 0$ implies $u\circ j\circ q = u\circ S(p') = S(u')\circ S(p')$ $= S(u'\circ p') = 0$ hence $u'\circ p' = 0$ in $\mathcal{E}$.

But $p'$ is epic in $\mathcal{C}$ so $u' = 0$ and then $u = S(u') = 0$. Now suppose $u \in \mathcal{A}\text{-mod}(\mathrm{Im}S(p'), S(C))$. Then $u \circ q \in \mathcal{A}\text{-mod}(S(C_1 \times_{C_2} A), S(C))$. So there is a $u' \in \mathcal{C}(C_1 \times_{C_2} A, C)$ such that $S(u') = u \circ q$. But $q \circ S(i') = 0$ implies $u \circ q \circ S(i) = S(u') \circ S(i') = 0$, hence $u' \circ i' = 0$ in $\mathcal{C}$. So there exists $\overline{u}: A \longrightarrow C$ such that $u' = \overline{u} \circ p'$. Then $S(\overline{u}) \circ j \circ q = S(\overline{u}) \circ S(p') = S(\overline{u} \circ p') = S(u') = u \circ q$ since $q$ is epic in $\mathcal{A}$-mod, $S(\overline{u}) \circ j = u$. ■

**Proposition.** $S'$ preserves direct unions.

**Proof.** Let $\{C_i \mid i \in I\}$ be a directed family of subobjects of $C_0 \in |\mathcal{C}|$. We must show the cokernel of the monomorphism $\delta: \Sigma_{i \in I} S(C_i) \longrightarrow S(\Sigma_{i \in I} C_i)$ is in $\mathcal{T}_{\Phi}$. As in the proof of the previous proposition it suffices to show for all $x \in S(\Sigma_{i \in I} C_i)(A) \cong \mathcal{A}\text{-mod}(h^A, S(\Sigma_{i \in I} C_i))$ the cannonical morphism $S(C)(A) = \mathcal{A}\text{-mod}(h^A, S(C)) \longrightarrow \mathcal{A}\text{-mod}([\mathrm{Im}\delta : x], S(C))$ is an isomorphism for every $C \in |\mathcal{C}|$. Let $y \in \mathcal{C}(A, \Sigma_{i \in I} C_i)$ such that $S(y) = x$. Then if $C' = \Sigma_{i \in I} C_i$, we have, for all $i \in I$, the following pullback diagram.

$$\begin{tikzcd}
0 \arrow[r] & A \times_{C'} C_i \arrow[r, "\alpha'_i"] \arrow[d] & A \arrow[d, "y"] \\
0 \arrow[r] & C_i \arrow[r, "\alpha_i"] & C_i
\end{tikzcd}$$

with exact rows. Since $S$ is a left exact we have the pullback

$$\begin{tikzcd}
0 \arrow[r] & S(A\times_C C_i) \arrow[r, "S(\alpha'_i)"] \arrow[d] & h^A \arrow[d, "\langle x\rangle"] \\
0 \arrow[r] & S(C_i) \arrow[r, "S(\alpha_i)"] & SC'
\end{tikzcd}$$

for each $i \in I$ with exact rows. Since $\mathcal{A}$-mod satisfies AB5 the square

$$\begin{tikzcd}
0 \arrow[r] & \Sigma_{i\in I}S(A\times_C C_i) \arrow[r, "\alpha'"] \arrow[d] & h^A \arrow[d, "\langle x\rangle"] \\
0 \arrow[r] & \Sigma_{i\in I}S(C_i) \arrow[r, "\alpha"] & S(C')
\end{tikzcd}$$

is also a pullback with exact rows. Hence we have

$[\mathrm{Im}\delta : x] = \mathrm{Im}\alpha' = \Sigma_{i\in I}S(A\times_C C_i)$ and if $C \in |\mathcal{C}|$

$$\mathcal{A}\text{-mod}(\Sigma_{i\in I}S(A\times_C C_i), S(C)) \cong \lim_I \mathcal{A}\text{-mod}(S(A\times_C C_i), S(C))$$
$$\cong \lim \mathcal{C}(A\times_C C_i, C) \cong \mathcal{C}(\Sigma_{i\in I}A\times_C C_i, C) \cong \mathcal{C}(A,C)$$
$$\cong \mathcal{A}\text{-mod}(S(A), S(C)) \cong \mathcal{A}\text{-mod}(h^A, S(C)). \blacksquare$$

**Corollary.** $S'$ **preserves direct sums.**

Let $M \in \mathrm{Sh}_{\Phi}(\mathcal{A}\text{-mod}) \subseteq \mathcal{A}\text{-mod}$ we can choose an exact sequence

$$\Sigma_{i\in I}\mathcal{A}(-,A_i)\xrightarrow{p'}\Sigma_{j\in J}\mathcal{A}(-,A_j)\xrightarrow{p}M\longrightarrow 0$$

in $\mathcal{A}$-mod. But $p' \in \mathcal{A}\text{-mod}(\Sigma_{i\in I}\mathcal{A}(-,A_i),\Sigma_{j\in J}\mathcal{A}(-,A_j))$ $= \mathcal{A}\text{-mod}(\Sigma_{i\in I}S(A_i),\Sigma_{j\in J}S(A_j))$ $\cong \mathcal{A}\text{-mod}(S(\Sigma_{i\in I}A_i),S(\Sigma_{j\in J}A_j)) = \mathcal{E}(\Sigma_{i\in I}A_i,\Sigma_{j\in J}A_j)$ so there exists $u \in S(\Sigma_{i\in I}A_i,\Sigma_{j\in J}A_j)$ such that $S(u) = p'$. Now let $M$ = coker$(u)$ that is

$$\Sigma_{i\in I}A_i\xrightarrow{u}\Sigma_{j\in J}A_j\longrightarrow\tilde{M}\longrightarrow 0$$

is exact in $\mathcal{E}$ then so is

$$S'(\Sigma_{i\in I}A_i)\xrightarrow{S'(u)}S'(\Sigma_{j\in J}A_j)\longrightarrow S'(\tilde{M})\longrightarrow 0.$$

But $S'(A_i) = \mathcal{A}(-,A_i) = S(A_i)$ and $S'(A_j) = \mathcal{A}(-,A_j)$ $= S(A_j)$ for each $i \in I$, $j \in J$, $S'(u) = p'$ and since $P:\mathcal{A}\text{-mod}\longrightarrow \mathrm{Sh}_{\Xi}(\mathcal{A}\text{-mod})$ is exact so $S'(M) \cong P(M)$. But $M$ $\cong P(M)$ so $S''(M) \cong M$.

Combining these results we have

**Theorem.** Let $\mathcal{E}$ be a Grothendieck category with a set of generators $\Gamma$ and $\mathcal{A}$ be any small full preadditive subcategory such that $\Gamma \subseteq |\mathcal{A}|$, then there is an equivalence between $\mathcal{E}$ and $\mathrm{Sh}_G(\mathcal{A}\text{-mod})$ where $G$ is the Grothendieck topology induced by $\mathcal{E}$.

Corollary. The following assertions are equivalent for a category $\mathcal{C}$:

(1) $\mathcal{C}$ is a Grothendieck abelian category;

(2) There exists a ringoid $\mathcal{A}$ such that $\mathcal{C}$ is a left exact retract of the functor category $\mathcal{A}$-mod i.e. $\mathcal{C}$ is the category of sheaves for some topology $\mathcal{G}$.

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