Elliptic units in ray class fields of real quadratic number fields

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Abstract

Let K be a real quadratic number field. Let p be a prime which is inert in K. We denote the completion of K at the place p by K_p. Let $f > 1$ be a positive integer coprime to p. In this thesis we give a p-adic construction of special elements $u(r,\tau) \in K_p^\times$ for special pairs $(r,\tau) \in (\mathbb{Z}/f\mathbb{Z})^\times \times \mathcal{H}_p$ where $\mathcal{H}_p = \mathbb{P}^1(\mathbb{C}_p)\backslash\mathbb{P}^1(\mathbb{Q}_p)$ is the so called p-adic upper half plane. These pairs (r,τ) can be thought of as an analogue of classical Heegner points on modular curves. The special elements $u(r,\tau)$ are conjectured to be global p-units in the narrow ray class field of K of conductor f. The construction of these elements that we propose is a generalization of a previous construction obtained in [DD06]. The method consists in doing p-adic integration of certain $\mathbb{Z}$-valued measures on $\mathbb{X} = (\mathbb{Z}_p \times \mathbb{Z}_p)\backslash(p\mathbb{Z}_p \times p\mathbb{Z}_p)$. The construction of those measures relies on the existence of a family of Eisenstein series (twisted by additive characters) of varying weight. Their moments are used to define those measures. We also construct p-adic zeta functions for which we prove an analogue of the so called Kronecker's limit formula. More precisely we relate the first derivative at $s = 0$ of a certain p-adic zeta function with $-\log_p \mathbf{N}_{K_p/\mathbb{Q}_p} u(r,\tau)$. Finally we also provide some evidence both theoretical and numerical for the algebraicity of $u(r,\tau)$. Namely we relate a certain norm of our p-adic invariant with Gauss sums of the cyclotomic field $\mathbb{Q}(\zeta_f, \zeta_p)$. The norm here is taken via a conjectural Shimura reciprocity law. We also have included some numerical examples at the end of section 18.

Résumé

Soit K un corps de nombre quadratique réel. Soit p un nombre premier inerte dans K. Nous noterons par K_p la complétion de K en p. Soit $f > 1$ un entier positif copremier à p. Dans cette thèse nous donnons une construction p-adique de certains éléments $u(r,\tau) \in K_p^\times$ pour certaines paires $(r,\tau) \in (\mathbb{Z}/f\mathbb{Z})^\times \times \mathcal{H}_p$ où $\mathcal{H}_p = \mathbb{P}^1(\mathbb{C}_p)\backslash\mathbb{P}^1(\mathbb{Q}_p)$. Ces paires (r,τ) sont en quelque sorte des analogues des points de Heegner classiques sur les courbes modulaires. Nous avons conjecturé que les éléments $u(r,\tau)$ sont des p-unités dans le corps de classe de K au sens restreint de conducteur f. La construction de ces éléments que nous proposons est une généralisation d'une construction obtenue dans [DD06]. La méthode consiste essentiellement a faire de l'integration p-adique de certaines mesures sur $\mathbb{X} = (\mathbb{Z}_p \times \mathbb{Z}_p)\backslash(p\mathbb{Z}_p \times p\mathbb{Z}_p)$ à valeurs dans $\mathbb{Z}$. La construction de ces mesures repose essentiellement sur l'existence d'une famille de séries d'Eisenstein (tordues par des caractères additifs) avec le poids $k \geq 2$ qui varie. Les moments de ces séries d'Eisenstien sont utilisés pour définir ces mesures. Nous construisons aussi une fonction zeta p-adique pour laquelle nous prouvons un analogue de la formule limite de Kronecker. Plus précisément, nous relions la première dérivée en $s = 0$ d'une certaine fonction zeta p-adique avec $-\log_p \mathbf{N}_{K_p/\mathbb{Q}_p} u(r,\tau)$. Finalement nous donnons une bonne raison théorique de croire en l'algébricité de $u(r,\tau)$. À savoir, nous relions une certaine norme de notre invariant p-adique avec des sommes de Gauss contenues dans le corps cyclotomique $\mathbb{Q}(\zeta_f, \zeta_p)$. La norme ici est définie à l'aide d'une loi de réciprocité de Shimura conjecturale. Nous avons aussi inclu quelques résultats numériques à la fin de la section 18.

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Contents

Introduction

Let L be a number field. Define

$$(L^\times)^- = \{x \in L^\times : |x|_\nu = 1 \text{ for all infinite places } \nu \text{ of } L\}.$$

One can think of $(L^\times)^-$ as the intersection of the minus spaces of all complex conjugations on $L^\times$. A complex conjugation of L acts as -1 on $(L^\times)^-$. One can show that if L contains no CM field then $(L^\times)^- = \{\pm 1\}$. Moreover when L contains a CM field, if we denote by L_{CM} the largest CM field contained in L, then $(L^\times)^- \subseteq L_{CM}^\times$. From now on we assume that L is a CM field.

Let p be an odd rational prime number. The group of p-units of L is defined as $\mathcal{O}_L[\frac{1}{p}]^\times$. Dirichlet's units theorem tells us that

$$\mathcal{O}_L[\frac{1}{p}]^\times \simeq (L^\times)_{tor} \times \mathbb{Z}^{n-1+g} \tag{0.1}$$

where g is the number of prime ideals in L above p and $2n = [L : \mathbb{Q}]$. We define the group of *strong p-units of* L to be

$$\begin{aligned} U_p(L) &:= (L^\times)^- \cap \mathcal{O}_L[\frac{1}{p}]^\times \\ &= \{x \in L^\times : \text{for all places } \nu \text{ of } L \text{ (finite and infinite) such that } \nu \nmid p,\ |x|_\nu = 1\}. \end{aligned}$$

Clearly $U_p(L)$ is a subgroup of the group of p-units of L. If we let L^+ be the maximal real subfield of L and g^+ the number of prime ideals of L^+ above p then an easy

calculation shows that $rank_{\mathbb{Z}}(U_p(L)) = g - g^+$(see proposition 1.1). For example when the prime p splits completely in L we have $rank_{\mathbb{Z}}(U_p(L)) = [L : \mathbb{Q}]/2$.

Let $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic number field where $disc(K) = D$ and p be an odd prime number which is inert in K. We denote the completion of K at p by K_p. In [DD06] a p-adic construction of elements in $K_p^\times$ is proposed. Those elements are conjectured to be strong p-units in certain abelian extensions of K, namely the so called *narrow ring class fields of* K. We recall the main ideas of their construction. Let $N > 1$ be an integer coprime to D such that $\sigma_0(N) = \sum_{d|N} 1 > 2$. Let $\alpha(z)$ be a non constant modular unit on the modular curve $X_0(N)$ having no zero or pole at the cusp $i\infty$ (such modular units always exist). Let $\mathcal{H}_p = \mathbb{P}^1(\mathbb{C}_p)\backslash\mathbb{P}^1(\mathbb{Q}_p)$ be the p-adic upper half plane. For certain points $\tau \in \mathcal{H}_p \cap K$ they define a p-adic invariant $u(\alpha, \tau) \in K_p^\times$. When this p-adic invariant is non trivial, i.e. when $u(\alpha, \tau) \neq \pm 1$, the two authors have conjectured that $u(\alpha, \tau)$ lies in a certain *narrow ring class field* L_τ of K which depends on τ and α. More precisely they conjecture that $u(\alpha, \tau)$ is a strong p-unit in L_τ. If we assume that $u(\alpha, \tau)$ is a non trivial strong p-unit contained in L_τ then L_τ contains a CM field and therefore has at least one complex embedding. Note that any ring class field L^{ring} of K is Galois over $\mathbb{Q}$. Therefore by normality L_τ is totally complex. Because of the dihedral nature of $Gal(L^{ring}/\mathbb{Q})$ we see that having at least one complex embedding is equivalent for L^{ring} to be a CM field. We thus conclude that if $u(\alpha, \tau)$ is a non trivial p-unit inside L_τ then L_τ has to be a CM field.

The key idea in their construction is to use the periods of the modular unit $\alpha(z)$ in order to construct a family of $\mathbb{Z}$-valued measures on $\mathbb{P}^1(\mathbb{Q}_p)$. In their construction they use modular units of the form

$$\alpha(z) = \prod_{d|N} \Delta(dz)^{n_d}$$

where $\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24}$ (where $q = e^{2\pi iz}$) and the $n_d \in \mathbb{Z}$'s are integers subject to the conditions $\sum_{d|N} n_d = 0$ and $\sum_{d|N} n_d d = 0$. The "periods" considered come from the modular unit $\frac{\alpha(z)}{\alpha(pz)}$ and they are given by the formula

$$\frac{1}{2\pi i}\int_{c_1}^{c_2} \mathrm{dlog}\left(\frac{\alpha(z)}{\alpha(pz)}\right) \in \mathbb{Z} \tag{0.2}$$

where $c_1, c_2 \in \Gamma_0(N)(i\infty)$. These periods can be expressed in terms of Dedekind sums. A key feature of their method is the possibility of testing their conjecture since the p-adic integral defining the invariant $u(\alpha, \tau)$ can be computed in polynomial time. For a description of this algorithm see [Das05].

In this thesis we propose a generalization of their construction. Let N_0 and f be coprime positive integers such that $(pD, fN_0) = 1$. We call f the conductor and N_0 the level. We replace the function $\frac{\Delta(z)}{\Delta(d_0 z)}$ for $d_0 | N_0$ by a certain power of the Siegel function $g_{(\frac{r}{f},0)}(f d_0 \tau)$, see (3.1) for the definition.

The first novelty is that instead of working with one modular unit $\alpha(z)$ we work with a family of modular units $\{g_t(z)\}_{t \in T}$ indexed by the finite set $T = (\mathbb{Z}/f\mathbb{Z})^\times / \langle \overline{p} \rangle$ where $\langle \overline{p} \rangle$ corresponds to the group generated by the image of p inside $(\mathbb{Z}/f\mathbb{Z})^\times$. By construction those modular units are $\Gamma_0(fN_0)$-invariant in the sense that for all $\gamma \in \Gamma_0(fN_0)$ one has

$$g_{\gamma \star t}(\gamma z) = g_t(z)$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \star t = dt (mod\ f)$. So the matrix γ acts not only on the variable z but also on the indexing set T. Moreover every modular unit in this family has no zero or pole on the set $\Gamma_0(fN_0)(i\infty)$. The periods considered are of the form

$$\frac{1}{2\pi i} \int_{c_1}^{c_2} \mathrm{dlog} \left(\frac{g_t(z)}{g_t(pz)} \right) \in \mathbb{Z} \tag{0.3}$$

where $c_1, c_2 \in \Gamma_0(fN_0)(i\infty)$ and $t \in T$ which is the analogue (0.2).

Using equation (0.3) we define, for every triple

$$(c_1, c_2, t) \in \Gamma_0(fN_0)(i\infty) \times \Gamma_0(fN_0)(i\infty) \times T,$$

$\mathbb{Z}$-valued measures on $\mathbb{P}^1(\mathbb{Q}_p)$ denoted by

$$\mu_{g_t}\{c_1 \to c_2\}.$$

Using those measures, we propose a construction of elements in $K_p^\times$. For certain pairs $(r, \tau) \in (\mathbb{Z}/f\mathbb{Z})^\times \times \mathcal{H}_p$ we associate an invariant $u(r, \tau) \in K_p^\times$ which depends also

on the family of modular units $\{g_t(z)\}_{t\in T}$. Those elements are constructed as certain p-adic integrals of our measures. The element $u(r,\tau)$ is conjectured to be a *strong p-unit* in $K(f\infty)$, the narrow ray class field of K of conductor f. In particular, if we want to construct non trivial strong p-units inside $K(f\infty)$, it will be essential to assume beforehand that the latter field is totally complex. This shows the importance of working in the narrow sense and not just in the wide sense. We propose a conjectural Shimura reciprocity law (see conjecture 5.1) which says how the Galois group $Gal(K(f\infty)/K)$ should permute the elements $u(r,\tau)$. We also prove an analogue of the Kronecker limit formula relating our invariant $u(r,\tau)$ to the first derivative at $s=0$ of a certain p-adic zeta function. More precisely we prove that

(1) $3\zeta_p'(0) = -\log_p N_{K_p/\mathbb{Q}_p} u(r,\tau)$

(2) $3\zeta(0) = v_p(u(r,\tau))$

where $\zeta_p(s)$ is a p-adic zeta function interpolating special values at negative integers of a classical zeta function $\zeta(s)$, attached to K, deprived from its Euler factor at p, namely $(1-p^{-2s})\zeta(s)$.

Let us explain more precisely the main ideas involved in the construction of the invariant $u(r,\tau)$. In a very similar way to [DD06], our family of $\mathbb{Z}$-valued measures $\mu_{g_t}\{c_1 \to c_2\}$ on $\mathbb{P}^1(\mathbb{Q}_p)$ can be used to construct a 2-cocycle $\kappa \in Z^2(\Gamma_1, K_p^\times)$ (see definition 5.10) where

$$\Gamma_1 := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}[\frac{1}{p}]) : a \equiv 1(mod\, f), c \equiv 0(mod\, fN_0)\}$$

and $K_p^\times$ has trivial Γ_1-action. It turns out that the 2-cocycle κ is a 2-coboundary i.e. there exists a 1-cochain $\rho \in C^1(\Gamma_1, K_p^\times)$ such that $d(\rho) = \kappa$. Note that ρ is uniquely determined modulo $Z^1(\Gamma_1, K_p^\times) = Hom(\Gamma_1, K_p^\times)$ which turns out to be a finite group. In order to show the splitting of the 2-cocycle κ one is lead to lift the system of measures $\mu_{g_t}\{c_1 \to c_2\}$ to a system of measures on $\mathbb{X} := (\mathbb{Z}_p \times \mathbb{Z}_p)\backslash(p\mathbb{Z}_p \times p\mathbb{Z}_p)$. There is a natural $\mathbb{Z}_p^\times$-bundle map $\pi : \mathbb{X} \to \mathbb{P}^1(\mathbb{Q}_p)$ given by $\pi(x,y) = \frac{x}{y}$. In order to lift our measures from $\mathbb{P}^1(\mathbb{Q}_p)$ to $\mathbb{X}$ we use the periods of a family of Eisenstein

series twisted by additive characters with varying weight $k \geq 2$. When the weight k equals 2 then the corresponding Eisenstein series is the logarithmic derivative of our modular unit. We denote the unique lift of $\mu_{g_t}\{c_1 \to c_2\}$ to $\mathbb{X}$ (under certain conditions see theorem 6.1) by $\widetilde{\mu}_{g_t}\{c_1 \to c_2\}$. Note that by construction we have $\pi_*\widetilde{\mu}_{g_t}\{c_1 \to c_2\} = \mu_{g_t}\{c_1 \to c_2\}$. Using this lift one can give a "simple expression" for $u(r,\tau)$ namely

$$u(r,\tau) := \rho(\gamma_\tau) = \int\!\!\!\!\!\!\diagup_{\mathbb{X}} (x - \tau y) d\widetilde{\mu}_{g_r}\{i\infty \to \gamma_\tau i\infty\}(x,y) \tag{0.4}$$

where γ_τ is an oriented generator of the stabilizer of τ under the action of Γ_1. Therefore the invariant $u(r,\tau)$ is obtained from the evaluation of the 1-cochain ρ at γ_τ. The presence of the stabilizer γ_τ is accounted for the presence of endomorphisms of infinite order of the lattice $\mathbb{Z}+\tau\mathbb{Z}$ which is equivalent to the presence of units of infinite order in $\mathcal{O}_K$. When the element $\tau \in \mathcal{H}_p \cap K$ and τ is reduced, there is a natural bijection $\phi : \mathbb{X} \to \mathcal{O}_{K_p}^\times$ given by $(x,y) \mapsto x - \tau y$. If we let $\nu_r = \phi_*\widetilde{\mu}_{g_r}\{i\infty \to \gamma_\tau i\infty\}$ then (0.4) can be rewritten in a more functorial way as

$$u(r,\tau) = \int_{x \in \mathcal{O}_{K_p}^\times} x d\nu_r(x) \in K_p^\times.$$

This new point of view, which applies to any totally real number field, is the subject of a recent paper by Dasgupta, see [Das06].

We compute the various moments of $\widetilde{\mu}_{g_t}\{c_1 \to c_2\}$ i.e. integrals of the form

$$\int\!\!\!\!\!\!\diagup_{\mathbb{X}} x^n y^m d\widetilde{\mu}_{g_r}\{i\infty \to \gamma_\tau(i\infty)\}(x,y),$$

where m and n are positive integers, see proposition 11.5 for explicit formulas. Following [Das05], we also give explicit formulas for the measures $\widetilde{\mu}_{g_t}\{c_1 \to c_2\}$ evaluated on balls of $\mathbb{X}$, i.e. compact open sets of the form

$$(u + p^n\mathbb{Z}_p) \times (v + p^n\mathbb{Z}_p).$$

See proposition 14.1 for the formulas. Both of these formulas involve periods of Eisenstein series which can be expressed in terms of Dedekind sums. Having such formulas turns out to be essential for numerical verifications. We have included at the

end of section 18 a few numerical examples which support the conjectural algebraicity of $u(r,\tau)$.

Finally we give some theoretical evidence for the algebraicity of $u(r,\tau)$ by computing "their norm" and relating them to normalized Gauss sums, see theorem 17.1. Let f be an integer such that for all $q|f$, q is inert in K. Assume furthermore that $\overline{-1} \notin \langle \overline{p} \rangle \leq (\mathbb{Z}/f\mathbb{Z})^\times$, then we prove that

$$\mathbf{N}_{K(f\infty)^{Fr_\wp}/\mathbb{Q}(\zeta_f)^{Fr_p}}(u(r,\tau)) = S \ (mod\ \mu_F) \tag{0.5}$$

where $\wp = p\mathcal{O}_K$, S is a product of normalized Gauss sums in $F = \mathbb{Q}(\zeta_f)^{\langle Fr_p \rangle} \cdot \mathbb{Q}(\zeta_p) \subseteq \overline{\mathbb{Q}}_p$. The norm in (0.5) is taken via a Shimura reciprocity law which is still meaningful even if we don't assume the algebraicity of the element $u(r,\tau)$. Note that the left hand side of (0.5) lies necessarily in $K_p \cap F = \mathbb{Q}(\zeta_f)^{\langle Fr_p \rangle} \subseteq \overline{\mathbb{Q}}_p$. Because of the assumption $\overline{-1} \notin \langle \overline{p} \rangle \leq (\mathbb{Z}/f\mathbb{Z})^\times$ we see that $\mathbb{Q}(\zeta_f)^{Fr_p}$ is a CM field.

Notation

Let K be a number field and $\mathcal{O}$ an order of K. For a finite subset of places S of K we define $\mathcal{O}_S$ to be the ring of S-integers of $\mathcal{O}$

$$\mathcal{O}_S := \left\{ \frac{a}{b} \in K : a, b \in \mathcal{O}, \text{ and } \left|\frac{a}{b}\right|_\nu \leq 1 \text{ for } \nu \text{ finite and } \nu \notin S \right\}.$$

Let $M_\infty = \{\sigma_1, \ldots, \sigma_r\}$ be the set of real embeddings of K. Given an integral $\mathcal{O}_S$-ideal $\mathfrak{f}$ and a subset $M \subseteq M_\infty$ we define the following sets

(1) $I_{\mathcal{O}_S}(\mathfrak{f}) := \{\mathfrak{b} \subseteq K : \mathfrak{b} \text{ is an integral } \mathcal{O}_S\text{-ideal coprime to } \mathfrak{f}\}$,

(2) $Q_{\mathcal{O}_S,1}(\mathfrak{f}\infty_M) := \{\frac{\alpha}{\beta} \in K : \alpha, \beta \in \mathcal{O}_S, (\alpha\beta, \mathfrak{f}) = 1, \alpha \equiv \beta (mod\, \mathfrak{f}), \text{ and } \sigma_i(\frac{\alpha}{\beta}) > 0 \ \forall \sigma_i \in M\}$,

where we think of $\infty_M = \prod_{\sigma_i \in M} \infty_i$ where ∞_i is the infinite place corresponding to σ_i. Two ideals $\mathfrak{a}, \mathfrak{b} \in I_{\mathcal{O}_S}(\mathfrak{f})$ are said to be equivalent modulo $Q_{\mathcal{O}_S,1}(\mathfrak{f}\infty_M)$ if there exists an element $\lambda \in Q_{\mathcal{O}_S,1}(\mathfrak{f}\infty_M)$ such that $\lambda\mathfrak{a} = \mathfrak{b}$. This gives us a relation of

equivalence on $I_{\mathcal{O}_S}(\mathfrak{f})$. The quotient $I_{\mathcal{O}_S}(\mathfrak{f})/Q_{\mathcal{O}_S,1}(\mathfrak{f}\infty_M)$ is a generalized $\mathcal{O}_S$-ideal class group corresponding by class field theory to a certain abelian extension of K.

Let K be a quadratic number field. For $\tau \in K\backslash\mathbb{Q}$ then we define

$$\Lambda_\tau := \mathbb{Z} + \tau\mathbb{Z},$$

and

$$\mathcal{O}_\tau := End_K(\Lambda_\tau) = \{\lambda \in K : \lambda\Lambda_\tau \subseteq \Lambda_\tau\}.$$

Let N be a positive integer then we define

$$(1)\ \Gamma_0(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0(mod\ N)\},$$

$$(2)\ \Gamma_1(N) := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0(mod\ N), d \equiv 1(mod\ N)\}.$$

1 The $\mathbb{Z}$-rank of strong p-units

Let L be a number field. Remember that

$$(L^\times)^- = \{x \in L^\times : |x|_\nu = 1 \text{ for all infinite places } \nu \text{ of } L\}.$$

One can think of $(L^\times)^-$ as the intersection of the minus spaces of all complex conjugations of $(L^\times)^-$. A complex conjugation acts like -1 on $L^\times$. If L contains no CM field then a simple computation reveals that $(L^\times)^- = \{\pm 1\}$. In the case where L contains a CM field let us denote by L_{CM} the largest CM field contained in L. In this case one has that $(L^\times)^- \subseteq L_{CM}^\times$. For this reason we now assume that L is a CM field of degree $2n$ over $\mathbb{Q}$. Let K be a totally real subfield of L and let $\mathfrak{p}$ be a prime ideal of K. We define a relative group of strong $\mathfrak{p}$-units

$$U_\mathfrak{p}(L/K) :=$$
$$\{x \in L^\times : \text{for all places } \nu \text{ of } L \text{ (finite and infinite) such that } \nu \nmid \mathfrak{p},\ |x|_\nu = 1\}.$$

We would like to compute the $\mathbb{Z}$-rank of $U_{\mathfrak{p}}(L/K)$. Let S be the set of places of L containing exactly all the infinite places and all the finite ones above $\mathfrak{p}$. Let Y_S be the free abelian group generated by the elements of S. Let also X_S be the subgroup of Y_S of elements having degree 0. Let $\pi \in K$ be such that $\pi\mathcal{O}_K = \mathfrak{p}^m$ for some integer m. We have a natural map

$$\begin{aligned} \lambda : \mathbb{R} \otimes_{\mathbb{Z}} \mathcal{O}_L[\frac{1}{\pi}]^{\times} &\to \mathbb{R} \otimes_{\mathbb{Z}} Y_S \\ 1 \otimes \epsilon &\mapsto \sum_{\nu \in S} \log |\epsilon|_{\nu} \cdot [\nu] \end{aligned}$$

where $||_{\nu}$ denotes the normalized local absolute value for which we have the formula

$$|\alpha|_{\nu} = \begin{cases} \mathbf{N}_{F_{\nu}/\mathbb{R}}(\alpha) & \text{if } \nu \text{ is complex} \\ \mathbf{N}(\nu)^{-v_{\nu}(\alpha)} & \text{if } \nu \text{ is finite} \end{cases}$$

for any $\alpha \in L^{\times}$. Using Dirichlet's unit theorem and the product formula we see that λ induces an $\mathbb{R}$-linear isomorphism between $\mathbb{R} \otimes_{\mathbb{Z}} \mathcal{O}_L[\frac{1}{\pi}]^{\times}$ and $\mathbb{R} \otimes_{\mathbb{Z}} X_S$. Let τ_{∞} be the complex conjugation on L then τ_{∞} acts naturally on the left and right hand side of λ. Note that τ_{∞} acts always trivially on infinite places of S. One can verify that λ is τ_{∞}-equivariant. Let us denote by $S^+ = \{\nu \in S : \tau_{\infty}\nu = \nu\}$ and by $S^- = \{\nu \in S : \tau_{\infty}\nu \neq \nu\}$. It is easy to see that the -1 eigenspace of $\mathbb{R} \otimes_{\mathbb{Z}} X_S$ has dimension $\frac{\#S^-}{2}$ where a $\mathbb{R}$-linear basis is provided for examples by the elements $[\nu] - [\tau_{\infty}\nu]$ for $\nu \in S^-$. Therefore it follows that the $+1$ eigenspace of $\mathbb{R} \otimes_{\mathbb{Z}} X_S$ has dimension equal to $\#S^+ + \frac{\#S^-}{2} - 1$. Since the map λ is τ_{∞}-equivariant the same is true for $\mathbb{R} \otimes_{\mathbb{Z}} \mathcal{O}_L[\frac{1}{\pi}]^{\times}$. Note also that $\#S^+ + \#S^- = n + g$ where g is the number of prime ideals of L above $\mathfrak{p}$.

Proposition 1.1 *Let L^+ be the maximal real subfield of L and g^+ be the number of prime ideals of $\mathcal{O}_{L^+}$ above $\mathfrak{p}$ then one has*

$$rank_{\mathbb{Z}}(U_{\mathfrak{p}}(L/K)) = s = g - g^+. \tag{1.1}$$

Proof A small computation shows that $\#S^+ = n + 2g^+ - g$ and $\#S^- = 2(g - g^+)$. From this we get

$$dim_{\mathbb{R}}\left(\mathbb{R}\otimes_{\mathbb{Z}}\left(\mathcal{O}_L[\frac{1}{\pi}]^{\times}\right)^{-}\right) = dim_{\mathbb{R}}\left(\mathbb{R}\otimes_{\mathbb{Z}} X_S^{-}\right) = dim_{\mathbb{R}}\left((\mathbb{R}\otimes_{\mathbb{Z}} X_S)^{-}\right) = \frac{\#S^-}{2}$$
$$= g - g^+.$$

The second equality follows from the fact that the eigenvalues of τ_∞, which are ± 1, lie in $\mathbb{Z}$. Finally note that $\left(\mathcal{O}_L[\frac{1}{\pi}]^{\times}\right)^{-} = U_{\mathfrak{p}}(L/K)$ and in general for any finitely generated abelian group A we have $dim_{\mathbb{R}}(\mathbb{R}\otimes_{\mathbb{Z}} A) = rank_{\mathbb{Z}}(A)$. $\square$

Question Let K be a real quadratic field and $L = K(f\infty)$ be the narrow ray class field of conductor f of K. Let p be a prime number inert in K which is congruent to 1 modulo f and assume that $K(f\infty)$ is totally complex. Let $\mathfrak{p} = p\mathcal{O}_K$. If conjecture 5.1 is true, can we prove that the $\mathbb{Z}$-rank of the subgroup generated by our strong $\mathfrak{p}$-units is equal to $g - g^+ = [K(f\infty)/K]/2$?

2 Distributions on $\mathbb{P}^1(\mathbb{Q}_p)$ and holomorphic functions on the upper half plane

Let p be a prime number and $(A, +)$ be an abelian group. Let $\mathbb{P}^1(\mathbb{Q}_p)$ be the projective line over the field of p-adic numbers endowed with its natural topology induced from the one on $\mathbb{Q}_p$. The field $\mathbb{Q}_p$ has a natural normalized non Archimedean metric $|\ |_p$ where $|p|_p = \frac{1}{p}$. The group of matrices

$$GL_2^+(\mathbb{Z}[\frac{1}{p}]) = \{\gamma \in GL_2(\mathbb{Z}[\frac{1}{p}]) : det(\gamma) > 0\}$$

acts naturally on $\mathbb{P}^1(\mathbb{Q}_p)$ by the rule $x \mapsto \gamma x = \frac{ax+b}{cx+d}$ where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Z}[\frac{1}{p}])$ and $x \in \mathbb{P}^1(\mathbb{Q}_p)$. We define a ball in $\mathbb{P}^1(\mathbb{Q}_p)$ to be a translate of $\mathbb{Z}_p$ under some element of $GL_2^+(\mathbb{Z}[\frac{1}{p}])$. Therefore by definition all balls of $\mathbb{P}^1(\mathbb{Q}_p)$ can be

written as

$$\gamma\mathbb{Z}_p := \{\gamma x \in \mathbb{P}^1(\mathbb{Q}_p) : x \in \mathbb{Z}_p\}.$$

for some $\gamma \in GL_2^+(\mathbb{Z}[\frac{1}{p}])$. Given a ball $B \subseteq \mathbb{P}^1(\mathbb{Q}_p)$ one can show that there exists an element $a \in \mathbb{Z}[\frac{1}{p}]$ and $n \in \mathbb{Z}$ such that

$$B = \{x \in \mathbb{Q}_p : |x-a|_p \leq \frac{1}{p^n}\} \text{ or } B = \{x \in \mathbb{Q}_p : |x-a|_p \geq \frac{1}{p^n}\} \cup \{\infty\}.$$

This explains somehow the terminology for the word "ball". We denote the set of all balls of $\mathbb{P}^1(\mathbb{Q}_p)$ by $\mathcal{B}$.

An A-valued distribution on $\mathbb{P}^1(\mathbb{Q}_p)$ is a map

$$\mu : \{\text{Compact open sets of } \mathbb{P}^1(\mathbb{Q}_p)\} \to A$$

which is finitely additive i.e. for all finite disjoint union $\bigcup_{i=1}^n U_i$ of compact open sets of $\mathbb{P}^1(\mathbb{Q}_p)$ we have

$$\mu\left(\bigcup_{i=1}^n U_i\right) = \sum_{i=1}^n \mu(U_i),$$

where the summation on the right hand side takes place in the abelian group A. It thus follows that a distribution on $\mathbb{P}^1(\mathbb{Q}_p)$ is completely determined by its values on a topological basis of $\mathbb{P}^1(\mathbb{Q}_p)$. A topological basis of $\mathbb{P}^1(\mathbb{Q}_p)$ is given for example by its set of balls.

We say that a distribution on $\mathbb{P}^1(\mathbb{Q}_p)$ has total value 0 if $\mu(\mathbb{P}^1(\mathbb{Q}_p) = 0$. We would like to give a simple criterion to construct A-valued distributions on $\mathbb{P}^1(\mathbb{Q}_p)$ of total value 0. Before stating this criterion we need to introduce some notation.

Every ball $B = \gamma\mathbb{Z}_p$ can be expressed uniquely as a disjoint union of p balls

$$B = \bigcup_{i=0}^{p-1} B_i,$$

where $B_i := \gamma\left(\alpha_i\mathbb{Z}_p\right)$ and $\alpha_i = \begin{pmatrix} p & i \\ 0 & 1 \end{pmatrix}$. We can now state the criterion:

Lemma 2.1 *If μ is an A-valued function on $\mathcal{B}$ satisfying*

$$\mu(\mathbb{P}^1(\mathbb{Q}_p) - B) = -\mu(B), \;\; \mu(B) = \sum_{i=1}^{p-1} \mu(B_i)$$

for all $B \in \mathcal{B}$ then μ extends uniquely to an A-valued distribution on $\mathbb{P}^1(\mathbb{Q}_p)$ with total value 0.

The proof of this lemma can be made transparent by using the dictionary between measures on $\mathbb{P}^1(\mathbb{Q}_p)$ and harmonic cocycles on the Bruhat-Tits tree of $PGL_2(\mathbb{Q}_p)$. For a small introduction to the subject see chapter 5 of [Dar04]. From now on we will use the previous lemma freely.

For the sequel we would like to give a general procedure to construct A-valued distributions on $\mathbb{P}^1(\mathbb{Q}_p)$ from certain analytic functions on the complex upper half plane. In practice we are mainly interested in the case where $A = \mathbb{Z}$.

Let $\mathcal{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$ be the upper half-plane endowed with its usual metric $ds^2 = \frac{dx^2+dy^2}{y^2}$. For any *analytic function* $f : \mathcal{H} \to \mathbb{C}$ we define the multiplicative $U_{p,m}$ operator as

$$(U_{p,m}f)(\tau) := \prod_{j=0}^{p-1} f(\frac{\tau + j}{p}).$$

We say that f satisfies the multiplicative distribution relation at p or simply that f is a $U_{p,m}$-eigenvector (even if $U_{p,m}$ is not a linear operator) if there exists $\lambda \in \mathbb{C}^\times$ such that

$$(U_{p,m}f)(\tau) = \lambda f(\tau), \forall \tau \in \mathcal{H}. \tag{2.1}$$

We also call λ the eigenvalue of f with respect to the operator $U_{p,m}$. Similarly for any *meromorphic function* $g : \mathcal{H} \to \mathbb{C}$ we define an $\mathbb{C}$-linear operator $U_{p,a}$ (where the "a" stands for additive) as

$$(U_{p,a}g)(\tau) := \frac{1}{p} \sum_{j=0}^{p-1} g(\frac{\tau + j}{p}).$$

If we take the logarithmic derivative of (2.1) we get

$$U_{p,a}(\mathrm{dlog}f)(\tau) = \frac{1}{p}\sum_{j=0}^{p-1}(\mathrm{dlog}f)(\frac{\tau+j}{p}) = \mathrm{dlog}f(\tau).$$

Note that the constant λ has dropped out. In general if $g(\tau)$ is a *meromorphic function* on $\mathcal{H}$ we say that g satisfies the distribution relation at p if

$$\frac{1}{p}\sum_{i=0}^{p-1} g(\frac{\tau+j}{p}) \overset{(*)}{=} g(\tau) \tag{2.2}$$

for all $\tau \in \mathcal{H}$ where $(*)$ is defined, in other words $g(\tau)$ is an $U_{p,a}$-eigenvector with eigenvalue 1. We call (2.2) the additive distribution relation at p. When in addition the function $f(\tau)$ is invariant under translation by $\mathbb{Z}$, i.e. $f(\tau+1) = f(\tau), \forall \tau \in \mathcal{H}$, we find that for $(N,p) = 1$

$$(U_{p,m}f_N)(\tau) = \lambda f_N(\tau),$$

where $f_N(\tau) := f(N\tau)$. In this way get even more functions on the upper half plane satisfying the multiplicative distribution relation at p. Note that the multiplicative distribution relation is stable under standard multiplication of functions. Using the previous observation we get

$$\prod_{d|N} f(d\tau)^{n_d} \tag{2.3}$$

is also a $U_{p,m}$-eigenvector for arbitrary integers n_d's. Equation (2.3) is the basic tool for constructing $U_{p,m}$-eigenvectors from a given one. We remind also the reader that since $f(\tau+1) = f(\tau)$ for all $\tau \in \mathcal{H}$ then f admits a q-expansion at $i\infty$ of the form

$$\sum_{n\in\mathbb{Z}} a_n q_\tau^n,\ \tau \in \mathcal{H}$$

where $q_\tau = e^{2\pi i\tau}$ and $a_n \in \mathbb{C}$.

Assume that $f(\tau)$ satisfies additional symmetries and some boundary conditions namely that there exists an integer $N > 1$ coprime to p such that

(1) $f(\tau)$ is $\Gamma_0(pN)$-invariant and that it descends to a meromorphic function on $X_0(pN)$.

(2) $f(\tau)$ has no zeros or poles on the set of cusps $\Gamma_0(N)(i\infty)$, i.e. for all $\gamma \in \Gamma_0(N)$ we have $a_0^{(\gamma)} \neq 0$ and $a_n^{(\gamma)} = 0$ if $n < 0$ where $a_n^{(\gamma)}$ is defined by (2.4).

Using (1), one can define the q-expansion of $f(\tau)$ at any point $c \in \mathbb{P}^1(\mathbb{Q})$ by the following rule: First choose a matrix $\gamma \in SL_2(\mathbb{Z})$ such that $\gamma c = i\infty$. It is an exercise to verify that there always exists a matrix $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z})$ with $h > 0$ such that $\gamma \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \gamma^{-1} \in \Gamma_0(pN)$. Without lost of generality we can assume that $h > 0$ is minimal, we call it the width at the point c. It is easy to see that the width is constant on the orbit $\Gamma_0(pN)c$. It follows that $f(\gamma\tau)$ is holomorphic on $\mathcal{H}$ and invariant under the translation $z \mapsto z + h$. Therefore the function $f(\gamma\tau)$ admits a q-expansion at $i\infty$ of the form

$$f(\gamma\tau) = \sum_{n \in \mathbb{Z}} a_n^{(\gamma)} q_{\tau/h}^n, \ \tau \in \mathcal{H}. \tag{2.4}$$

Note that $a_n^{(\gamma)} = 0$ if n is small enough since f is meromorphic. The latter q-expansion is defined to be the q-expansion of $f(\tau)$ at the point c. If one chooses a γ' such that $\gamma' i\infty = \gamma i\infty = c$ then one can verify that for $n > 0$ $a_n^{(\gamma')} = \zeta a_n^{(\gamma)}$ for some h-th root of unity ζ depending on n. So up to a root of unity, a_n depends only on c. However, a_0 is uniquely determined by c. So formally speaking the q-expansion at c depends not just on c but also on the choice of the matrix $\gamma \in SL_2(\mathbb{Z})$ such that $\gamma i\infty = c$. However, this slight ambiguity will not create any problems for the applications we have in mind. Often we are only interested by the qualitative behaviour of the q-expansion at $c \in \mathbb{P}^1(\mathbb{Q})$ which is the same for all points in $c' \in \Gamma_0(pN)c$, as one can verify.

We can summarize so far the assumptions made on the holomorphic function $f : \mathcal{H} \to \mathbb{C}$:

(1) f descends to a meromorphic function on $X_0(pN)$,

(2) f is a $U_{p,m}$ eigenvector,

(3) f has no zeros and poles on the set of cusps $\Gamma_0(N)(i\infty)$.

where $(p, N) = 1$.

Having such a f one can associate $\mathbb{C}/\Omega$-valued distributions on $\mathbb{P}^1(\mathbb{Q}_p)$ of total value 0 where Ω is a finitely generated $\mathbb{Z}$-module of $\mathbb{C}$ defined by

$$\Omega := \langle \{ \frac{1}{2\pi i} \int_C \mathrm{dlog} f(\tau) : C \text{ is a small loop around a zero of } f \ \} \rangle,$$

where $\mathrm{dlog} f(\tau) = \frac{f'(\tau)}{f(\tau)} d\tau$. By small loop we mean a circle C such that C doesn't cross any zero of f and inside that circle f has only one zero. It is finitely generated because f descends to a meromorphic function on $X_0(pN)$. In particular if f has no zeros in $\mathcal{H}$ then $\Omega = \{0\}$.

In order to construct such distributions we need to introduce some notation first. Let us denote

$$\Gamma_0 := \left\{ \gamma \in \langle \Gamma_0(N), \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \rangle : det(\gamma) = 1 \right\}.$$

Note that the matrix $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \notin \Gamma_0$. A calculation shows that the natural image of Γ_0 in $PGL_2^+(\mathbb{Z}[\frac{1}{p}])$ has index two. It thus follows that the group Γ_0 splits the set of balls of $\mathbb{P}^1(\mathbb{Q}_p)$ into two orbits, the one equivalent to $\mathbb{Z}_p$ and the one equivalent to $\mathbb{P}^1(\mathbb{Q}_p) \backslash \mathbb{Z}_p$. Let $(c_1, c_2) \in \Gamma_0(N)(i\infty) \times \Gamma_0(N)(i\infty)$. One is lead naturally to the following definition

$$\mu_f\{c_1 \to c_2\}(\gamma \mathbb{Z}_p) := \int_{\gamma^{-1}c_1}^{\gamma^{-1}c_2} \mathrm{dlog} f(\tau), \tag{2.5}$$

$$\mu_f\{c_1 \to c_2\}(\gamma \left(\mathbb{P}^1(\mathbb{Q}_p) \backslash \mathbb{Z}_p\right)) := - \int_{\gamma^{-1}c_1}^{\gamma^{-1}c_2} \mathrm{dlog} f(\tau) \tag{2.6}$$

where $\gamma \in \Gamma_0$. The integral between the two cusps appearing in the bounds of the integral is taken to be along a curve C (containing its end points) which is assumed to be smooth, of finite length and doesn't cross any zeros of $f(\tau)$. Moreover, we also require that C agrees with the unique geodesic of $\mathcal{H}$ joining $\gamma^{-1}c_1$ to $\gamma^{-1}c_2$ on small enough neighbourhoods of $\gamma^{-1}c_1$ and $\gamma^{-1}c_2$. In particular if we let U_1 and U_2 be small enough open discs centred around $\gamma^{-1}c_1$ and $\gamma^{-1}c_2$ respectively we find that $U_1 \cap \mathcal{H} \cap C$ and $U_2 \cap \mathcal{H} \cap C$ are small arcs containing no zeros of f. Such neighbourhoods

always exist since f descends to a meromorphic function on $X_0(pN)$. Under these assumptions those integrals make sense since the functions $\frac{f'(\tau)}{f(\tau)}$ has no zero, pole on the path C and this latter path is well behaved in small neighbourhoods of its two endpoints. Note that the image of the integrals (2.5) and (2.6) in $\mathbb{C}/\Omega$ doesn't depend on the choice of such special paths since we mod out by the only obstruction coming from the poles of $\mathrm{dlog}f(\tau)$ which correspond to the zeros of f. Those poles are the only obstruction since $\mathrm{dlog}f(\tau)$ is a meromorphic 1-form on $\mathcal{H}$, therefore closed.

By definition the total value of $\mu_f\{c_1 \to c_2\}$ is zero. Moreover, since $Stab_{\Gamma_0}(\mathbb{Z}_p) = \Gamma_0(pN)$ (uses $(N,p)=1$) and $f(\tau)$ is $\Gamma_0(pN)$-invariant, one sees that (2.5) and (2.6) are well defined. Finally, the fact that $\mu_f\{c_1 \to c_2\}$ is a distribution follows from lemma 2.1. The condition of lemma 2.1 is verified since f is by assumption a $U_{p,m}$-eigenvector, see equation (2.2).

Remark 2.1 The reason why one needs to be careful about the endpoints of the path of integration comes from the observation that $f(\tau)$ could have infinitely many zeros or poles in a small real interval around the point $c \in \Gamma_0(N)(i\infty)$.

Remark 2.2 Note that the set $\{f(c) \in \mathbb{C} : c \in \Gamma_0(N)(i\infty)\}$ is finite since $f(\tau)$ is a $\Gamma_0(pN)$-invariant.

Remark 2.3 An important observation is that the the group generated by the matrix $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ gives rise to a nontrivial action on the set $\Gamma_0(N)(i\infty) = \Gamma_0(i\infty)$ by the rule

$$\begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix} : \Gamma_0(N)(i\infty) \to \Gamma_0(N)(i\infty)$$

$$c \mapsto \begin{pmatrix} p^n & 0 \\ 0 & 1 \end{pmatrix} c = p^n c.$$

where $n \in \mathbb{Z}$. In other words, the multiplication by p map reshuffles the set $\Gamma_0(N)(i\infty)$. Here it is crucial for N and p to be coprime. Philosophically the non triviality of this action combined with the special properties of $f(\tau)$ give rise to non trivial $\mathbb{C}/\Omega$-valued distributions on $\mathbb{P}^1(\mathbb{Q}_p)$.

An important property satisfied by the family of distributions $\mu_f\{c_1 \to c_2\}$ is a Γ_0-invariance. For all compact open set U and $\gamma \in \Gamma_0$ one has

$$\mu_f\{\gamma c_1 \to \gamma c_2\}(\gamma U) = \mu_f\{c_1 \to c_2\}(U).$$

This is a direct consequence of the definition of $\mu_f\{c_1 \to c_2\}$.

In general one imposes even stronger conditions on $f(\tau)$ in order to control the range of $\mu_f\{c_1 \to c_2\}$. We introduce the following useful definition:

Definition 2.1 *We say that $f(\tau)$ satisfies the real algebraicity condition on the set $\Gamma_0(N)(i\infty)$ if there exists a real number field $L \subseteq \mathbb{C}$ such that $f(x) \in L, \forall x \in \Gamma_0(N)(i\infty)$.*

In general, modular functions tend to have algebraic coefficients therefore the previous definition is not too hard to fulfil. For example suppose that all the "q-expansions" of $f(\tau)$ at the cusps $\Gamma_0(N)(i\infty)$ lie in $M[[q]]$ where M is a CM-field. Let M^+ be the maximal real subfield of M. Then taking the norm of $f(\tau)$ down to $M^+[[q]]$ gives rise to a modular unit satisfying the definition 2.1.

Assumptions: Form now on we assume that $f(\tau)$ satisfy the conditions (1), (2), (3), has no zeros in $\mathcal{H}$ and also that it satisfies the real algebraicity condition on the set $\Gamma_0(N)(i\infty)$.

We thus get that $\Omega = \{0\}$. Doing a small change of variables reveals that

$$\frac{1}{2\pi i}\int_{\gamma^{-1}c_1}^{\gamma^{-1}c_2} \mathrm{dlog} f = \frac{1}{2\pi i}\int_{f(\gamma^{-1}c_1)}^{f(\gamma^{-1}c_2)} \frac{dt}{t} \in \frac{\Lambda}{2\pi i} + \frac{1}{2}\mathbb{Z} \tag{2.7}$$

where $t = f(\tau)$ and Λ is the finitely generated additive subgroup of $\mathbb{R}$ (by remark 2.2) generated by $\log|f(c)|$ for $c \in \Gamma_0(N)(i\infty)$ (here we assume that there exists a $c \in \Gamma_0(N)(i\infty)$ such that $f(c) = 1$). The real part of the left hand side of (2.7) is half integral since the bounds of the integral appearing on the right hand side are real valued. Therefore if we define

$$\mu_f\{c_1 \to c_2\}(\mathbb{Z}) := Re\left(\frac{1}{2\pi i}\int_{\gamma^{-1}c_1}^{\gamma^{-1}c_2} \mathrm{dlog} f\right) \in \frac{1}{2}\mathbb{Z}$$

we obtain a $\frac{1}{2}\mathbb{Z}$-valued distribution on $\mathbb{P}^1(\mathbb{Q}_p)$.

Remark 2.4 Note that there are only finitely many possibilities for the bounds appearing in the right hand side of (2.7). On the other hand the image by f of the geodesic joining c_1 to c_2 and the one joining c_1' to c_2' will be in general different even if $f(c_1) = f(c_1')$ and $f(c_2) = f(c_2')$.

Definition 2.2 *We say that a $\mathbb{C}_p$-valued distribution μ on $\mathbb{P}^1(\mathbb{Q}_p)$ is a measure if there exists a constant $c \in \mathbb{R}_{>0}$ such that*

$$|\mu(U)|_p \leq c$$

for all compact open sets U of $\mathbb{P}^1(\mathbb{Q}_p)$.

Clearly a $\frac{1}{2}\mathbb{Z}$-valued distribution is a measure since we can take $c = 1$ if $p \neq 2$ and $c = 2$ if $p = 2$.

Remark 2.5 A $\mathbb{C}_p$-valued measure on $\mathbb{P}^1(\mathbb{Q}_p)$ allows oneself to integrate $\mathbb{C}_p$-valued continuous functions. In general $\mathbb{C}_p$-valued distributions only allow the integration of locally constant functions.

Suppose that f satisfies the real algebraicity condition on the set $\Gamma_0(N)(i\infty)$ for the number field L, i.e. for all $c \in \Gamma_0(N)(i\infty)$ we have $f(c) \in L \subseteq \mathbb{C}$. Note that L comes naturally equipped with an embedding in $\mathbb{R}$ by definition. One can also use the imaginary part of of (2.7) to construct $L^\times$-valued distributions on $\mathbb{P}^1(\mathbb{Q}_p)$. For every pair of cusps $(c_1, c_2) \in \Gamma_0(N)(i\infty)$, define a $L^\times$-valued distribution $\widetilde{\nu}_f\{c_1 \to c_2\}$ on $\mathbb{P}^1(\mathbb{Q}_p)$ by the rule

i) $\widetilde{\nu}_f\{c_1 \to c_2\}(\gamma\mathbb{Z}_p) := \frac{|f(\gamma^{-1}c_2)|}{|f(\gamma^{-1}c_1)|}$,

ii) $\widetilde{\nu}_f\{c_1 \to c_2\}(\gamma(\mathbb{P}^1(\mathbb{Q}_p)\backslash\mathbb{Z}_p)) := \left(\frac{|f(\gamma^{-1}c_2)|}{|f(\gamma^{-1}c_1)|}\right)^{-1}$.

for all $\gamma \in \Gamma_0$.

Using lemma 2.1 one can verify that $\widetilde{\nu}_f$ gives rise to an $L^\times$-valued distribution on $\mathbb{P}^1(\mathbb{Q}_p)$ of total value 1 (the abelian group A considered is multiplicative). Note that

the set

$$\{\mu_f\{c_1 \to c_2\}(B) : B \in \mathcal{B}\}$$

is finite.

Let us fix an embedding $\iota : L \hookrightarrow \overline{\mathbb{Q}}_p$ and let $\widehat{L}$ be the topological closure of $\iota(L)$ in $\overline{\mathbb{Q}}_p$. If we fix a p-adic branch of $\log_p$ by declaring $\log_p \pi = 0$ for some uniformizer in $\widehat{L}$ then we can define a $\widehat{L}$-valued measure on $\mathbb{P}^1(\mathbb{Q}_p)$ by

$$\nu_f\{c_1 \to c_2\} := \log_p \circ \widetilde{\nu}_f\{c_1 \to c_2\}.$$

We thus get that $\nu_f\{c_1 \to c_2\}$ is a $\widehat{L}$-valued measure. Unfortunately the measure $\nu_f\{c_1 \to c_2\}$ depends on ι and the choice of the p-adic branch of $\log_p$.

In the present paper we only explore the case where $f(c) = 1$ for all $c \in \Gamma_0(N)(i\infty)$. When f satisfies the latter hypothesis we see directly from (2.7) that $\mu_f\{c_1 \to c_2\}$ is $\mathbb{Z}$-valued and also that all possible $\widehat{L}$-valued measures $\nu_f\{c_1 \to c_2\}$ are trivial, i.e. equal to 0 on all compact open sets of $\mathbb{P}^1(\mathbb{Q}_p)$.

3 A review of the classical setting

Let $\mathcal{H}$ be the Poincaré upper half-plane and $X(N) = \mathcal{H}^*/\Gamma(N)$ be the modular curve of level N where $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$. A modular unit of level N is a function $u(\tau) \in \mathbb{Q}(\zeta_N)(X(N))$ with $\tau \in \mathcal{H}$, for which $div(u(\tau))$ is supported on $\mathbb{P}^1(\mathbb{Q})$. In particular modular units are non vanishing analytic functions on $\mathcal{H}$. Because of this latter property they can be written as an infinite product in the variable $q_\tau = e^{2\pi i \tau}$. The simplest example of a modular unit is provided by quotients of the form $\frac{\Delta(\tau)}{\Delta(N\tau)}$, where $\Delta(\tau) = \eta(\tau)^{24}$ and $\eta(\tau)$ is the famous Dedekind eta function defined by the infinite product

$$\eta(\tau) = q_{\frac{\tau}{24}} \prod_{n \geq 1} (1 - q_\tau^n).$$

The modular unit $\frac{\Delta(\tau)}{\Delta(N\tau)}$ is invariant under the larger group $\Gamma_0(N) \supseteq \Gamma(N)$. Let K be an imaginary quadratic number field. By evaluating those modular units on quadratic

irrationalities $\tau \in \mathcal{H} \cap K$ one gets points in certain ring class fields of K. In order to get points generating ray class fields of K, one needs to consider a more general type of modular units. These modular units can be obtained by taking suitable powers of Siegel functions.

For a pair of rational numbers $(a_1, a_2) \in (\frac{1}{N}\mathbb{Z})^2$ such that $(a_1, a_2) \neq (0,0)(mod\,\mathbb{Z})$, we define a Siegel function of level N as

$$g_{(a_1,a_2)}(\tau) := \mathfrak{k}_{(a_1,a_2)}(\tau)\eta(\tau)^2, \tag{3.1}$$

where $\mathfrak{k}_{(a_1,a_2)}(\tau)$ is the Klein form, see chapter 1 of [DK81] for the definition. The infinite product corresponding to the Siegel function is given by

$$g_{(a_1,a_2)}(\tau) = -e^{2\pi i a_2(a_1-1)/2} q_\tau^{\frac{1}{2}\widetilde{B}_2(a_1)}(1-q_z)\prod_{n\geq 1}(1-q_\tau^n q_z)(1-q_\tau^n q_{-z}), \tag{3.2}$$

where $z = a_1\tau + a_2$, $B_2(x) = x^2 - x + 1/6$ is the second Bernoulli polynomial, $\widetilde{B}_2(x) := B_2(\{x\})$ where $\{x\}$ stands for the fractional part of x, $q_\tau = e^{2\pi i\tau}, \tau \in \mathcal{H}$ and $q_z = e^{2\pi i z}, z \in \mathbb{C}$. Note that the infinite product in (3.2) converges since $Im(\tau) > 0$. Using the identity **K 2.** in chapter 2 of [DK81] we deduce for $(a_1, a_2) \equiv (b_1, b_2)(mod\,\mathbb{Z}^2)$ that

$$g_{(b_1,b_2)}(\tau) = (-1)^{n_1 n_2 + n_1 + n_2} e^{-2\pi i \frac{(n_1 a_2 - n_2 a_1)}{2}} g_{(a_1,a_2)}(\tau), \tag{3.3}$$

where $(b_1, b_2) - (a_1, a_2) = (n_1, n_2)$. We thus see that $g_{(a_1,a_2)}(\tau)$ and $g_{(b_1,b_2)}(\tau)$ differ only by a $2N$ root of unity. The function $g_{(a_1,a_2)}(\tau)$ is not too far from being a modular unit. Let $\gamma \in SL_2(\mathbb{Z})$ then

$$g_{(a_1,a_2)}(\gamma\tau) = \epsilon(\gamma) g_{(a_1,a_2)\gamma}(\tau), \tag{3.4}$$

where $\epsilon(\gamma)$ is defined by

$$\eta(\gamma\tau)^2 = \epsilon(\gamma)(c\tau + d)\eta(\tau)^2$$

for some $\epsilon(\gamma) \in \mu_{12}$. The subscript of the Siegel function $(a_1, a_2)\gamma$ on the right hand side of (3.4) is the usual multiplication of a row vector by a matrix.

From the identity (3.4) we deduce that for any $\gamma \in SL_2(\mathbb{Z})$ and any $r, s \in \mathbb{Z}$

$$g_{(\frac{r}{N},\frac{s}{N})}(\gamma\tau)^{12} = g_{(\frac{r}{N},\frac{s}{N})\gamma}(\tau)^{12}, \tag{3.5}$$

In particular when $\gamma \in \Gamma(N)$, using the identity (3.5) combined with (3.3), we see that the function

$$\tau \mapsto g_{(\frac{r}{N},\frac{s}{N})}(\tau)^{12N} \tag{3.6}$$

is invariant under the substitution $\tau \mapsto \gamma\tau$. We thus see that a suitable power of a Siegel function gives rise to a modular unit. A natural question that arises is: How large is the set of modular units of $X(N)$? This is answered by the following proposition:

Proposition 3.1 *The $\mathbb{Z}$-rank of the group of modular units of $X(N)$ modulo $\mathbb{Q}(\zeta_N)^\times$ is equal to*

$$\textit{number of cusps of } X(N) - 1. \tag{3.7}$$

Moreover, the subgroup generated by modular units as in (3.6) for all integers r, s has maximal rank.

Proof The proof consists essentially in showing that the divisors of Siegel functions of level N give rise to the universal even distribution on $\mathbb{Q}^2/\mathbb{Z}^2[N]$. See theorem 3.1. of chapter 2 in [DK81]. □

The -1 in (3.7) is explained by the trivial relation ($deg(div(u(\tau)))=0$) imposed on the divisor of any function on $X(N)$. In thus follows that the $\mathbb{Z}$-rank of modular units is as large as it could be. Beside their modular properties, the main interest of modular units reside in the fact that can be used to construct units in ray class fields of imaginary quadratic number fields.

Using equation (3.1) defining the Siegel functions, on can think of g^{12} as a function on $\mathbb{C} \times \mathcal{L}$ where $\mathcal{L}$ is the set of lattices of rank 2 in $\mathbb{C}$. It thus makes sense to write $g^{12}(t, \Lambda)$ for any $t \in \mathbb{C}$ and $\Lambda \in \mathcal{L}$. For $w \in \Lambda$, $g^{12}(t + w, \Lambda) = \epsilon(w)g^{12}(t, \Lambda)$ for some $\epsilon(w) \in S^1$. Therefore g^{12} modulo S^1 is well defined on pairs $(t + \Lambda, \mathbb{C}/\Lambda)$. This notation agrees with the previous definition of $g^{12}_{(a_1,a_2)}(\tau)$ given by (3.1) in the sense that $g^{12}(a_1\tau + a_2, \Lambda_\tau) = g_{(a_1,a_2)}(\tau)^{12}$ for any pair of real numbers (a_1, a_2) and $\tau \in \mathcal{H}$. Finally one should also point out that the function g^{12} is homogeneous of degree 0 meaning that $g^{12}(\lambda t, \lambda\Lambda) = g^{12}(t, \Lambda)$ for any $\lambda \in \mathbb{C}^\times$.

Now we would like to formulate one version of the theorem of complex multiplication for imaginary quadratic number fields. Let K be an imaginary quadratic number field and $\mathfrak{f}$ an integral ideal of K. Let $C(\mathfrak{f}) := I_{O_K}(\mathfrak{f})/Q_{\mathcal{O}_K,1}(\mathfrak{f})$ (resp. $K(\mathfrak{f})$) denote the ray class group of conductor $\mathfrak{f}$ (resp. the ray class field of conductor $\mathfrak{f}$)

Theorem 3.1 *Let $\mathfrak{f} = (f)$ for some $f \in \mathbb{Z}_{>0}$ and assume that f is divisible by at least two distinct primes of $\mathbb{Z}$. Let $a \in C(\mathfrak{f})$ and choose $\mathfrak{a}, \mathfrak{b} \in a$. Then*

$$u(a) := g^{12f}(1, f\mathfrak{a}^{-1}) = g^{12f}(1, f\mathfrak{b}^{-1}) \in \mathcal{O}_{K(\mathfrak{f})}^{\times} \tag{3.8}$$

If we let $rec^{-1} : C(\mathfrak{f}) \to Gal(K(\mathfrak{f})/K)$ then

$$(g^{12f}(1, f\mathfrak{a}^{-1}))^{rec^{-1}(\mathfrak{c})} = g^{12f}(1, f\mathfrak{a}^{-1}\mathfrak{c}^{-1})$$

for any $\mathfrak{c} \in I_{O_K}(\mathfrak{f})$.

Proof See theorem 3 of chapter 19 section 3 of [Lan94b] . □

Remark 3.1 Note that since $\mathfrak{a}$ is an integral ideal then $1 \in \mathfrak{a}^{-1}$. It is easy to see that $\mathfrak{a}^{-1}$ can always be written as $\mathfrak{a}^{-1} = \frac{1}{s}\Lambda_\tau$ for some $s \in \mathbb{Z}_{>0}$ and $\tau \in (\mathcal{H} \cap K)$. It thus follows by homogeneity of g^{12} that

$$g^{12f}(1, f\mathfrak{a}^{-1}) = g^{12f}(\frac{s}{f}, \Lambda_\tau) = g_{(0,\frac{s}{f})}(\tau)^{12f}. \tag{3.9}$$

Here we emphasize the fact that for any integral ideal $\mathfrak{a}$ coprime to f we can associate a pair $(s, \tau) \in \mathbb{Z} \times (\mathcal{H} \cap K)$ such that $\mathfrak{a}^{-1} = \frac{1}{s}\Lambda_\tau$. One can then evaluate the Siegel function on the pair (s, τ) using (3.9). Note that this procedure depends implicitly on the conductor $\mathfrak{f} = (f)$. If we take $\mathfrak{a}, \mathfrak{b} \in a \in C(\mathfrak{f})$ and let $\mathfrak{a}^{-1} = \frac{1}{s}\Lambda_\tau$, $\mathfrak{b}^{-1} = \frac{1}{s'}\Lambda_{\tau'}$, then equation (3.8) implies that

$$g_{(0,\frac{s}{f})}(\tau)^{12f} = g_{(0,\frac{s'}{f})}(\tau')^{12f}.$$

This is an easy consequence of the homogeneity of degree 0 of g^{12} plus the the fact that the function $\tau \mapsto g_{(0,\frac{s}{f})}(\tau)^{12f}$ is $\Gamma_1(f)$-modular.

One can relate the logarithm of the absolute value of (3.9) with the first derivative of a certain zeta function (depending only on the ideal class of $\mathfrak{a}$ modulo $Q_{\mathcal{O}_K,1}(\mathfrak{f})$)

evaluated at $s = 0$. This is the so called *second Kronecker's limit formula.* For the remaining of the section we introduce some notation in order to define a certain class of zeta functions associated to a positive definite quadratic form $Q(\underline{x})$ and a spherical function $P(\underline{x})$ with respect to $Q(\underline{x})$. We only need the case where $n = 2$, i.e. when $\underline{x} = (x_1, x_2)$. For the general case, see chapter 1 section 5 of [Sie80].

Let $z = a + ib \in \mathcal{H}$. We attach to z a 2 by 2 positive definite matrix $M_z = \frac{1}{b}\begin{pmatrix} 1 & a \\ a & |z|^2 \end{pmatrix}$. Note that M_z is normalized in the sense that $det(M_z) = 1$. We define

$$\widetilde{Q}_z(x_1, x_2) := \underline{x}^t M_z \underline{x} = b^{-1}|x_1 + zx_2|^2$$

where $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Note that $\widetilde{Q}_z(x_1, x_2)$ is normalized in the sense that $disc(\widetilde{Q}_z) = -4$. The vector $w := \begin{pmatrix} -\overline{z} \\ 1 \end{pmatrix}$ is an isotropic vector with respect to $\widetilde{Q}_z$ i.e. $\widetilde{Q}_z(-\bar{z}, 1) = 0$. The following homogeneous polynomial of degree g, $P(x_1, x_2) := (-i\underline{x}^t M_z w)^g = (x_1 + x_2\overline{z})^g$ is a spherical function with respect to $\widetilde{Q}_z(x_1, x_2)$. Following [Sie80] we can associate to such data a zeta function

$$\zeta(s, \underline{u}^*, \underline{v}^*, z, g) := b^s \sum_{\underline{m}+\underline{v}^* \neq \underline{0}} \frac{e^{2\pi i(m_1 u_2 - m_2 u_1)}(m_1 + v_1 + (m_2 + v_2)\overline{z})^g}{|m_1 + v_1 + (m_2 + v_2)z|^{2s+g}} \tag{3.10}$$

where $\underline{u}^*, \underline{v}^* \in \mathbb{Q}^2$. For any integer $g \geq 0$ and $s \in \mathbb{C}$ with $Re(s) > 1$ this function (with respect to the variable s) converges absolutely. It is a fact that this function admits a meromorphic continuation on all of $\mathbb{C}$ with at most a pole of order 1 at $s = 1$ (this occurs precisely when $g = 0$ and $\underline{u}^* \in \mathbb{Z}^2$). Moreover it satisfies a nice functional equation.

Define

$$Z(s, \underline{u}^*, \underline{v}^*, z, g) := \pi^{-s}\Gamma(s + g/2)\zeta(s, \underline{u}^*, \underline{v}^*, z, g).$$

Siegel shows the following functional equation (special case of equation (61) in [Sie80])

$$Z(s, \underline{u}^*, \underline{v}^*, z, g) = (-1)^g e^{2\pi i(u_1 v_2 - u_2 v_1)} Z(1 - s, \underline{v}^*, \underline{u}^*, z, g). \tag{3.11}$$

Let $g = 0$, $\underline{u}^* = \begin{pmatrix} r/f \\ t/f \end{pmatrix}$ and $\underline{v}^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ then after some rearrangements the equation (3.11) looks like

(3.12)

$$\sum_{\underline{m}\neq(0,0)} \frac{e^{2\pi i(m_1\frac{t}{f}-m_2\frac{r}{f})}}{|\widetilde{Q}_z(m_1,m_2)|^{2s}} = \frac{\pi^{-(1-s)}\Gamma(1-s)}{\pi^{-s}\Gamma(s)} f^{2(1-s)} \sum_{\underline{m}\equiv(r,t) \pmod f} \frac{1}{|\widetilde{Q}_z(m_1,m_2)|^{2(1-s)}}.$$

where $\underline{m}$ goes over $\mathbb{Z}^2$. For any $(\frac{r}{f},\frac{t}{f}) \in (\frac{1}{f}\mathbb{Z})^2$ and $z \in \mathcal{H}$, equation (3.12) motivates the following definitions

(1) $\zeta(s,(\frac{r}{f},\frac{t}{f}),z) := \sum_{(0,0)\neq\underline{m}} \frac{e^{2\pi i(m_1\frac{r}{f}-m_2\frac{t}{f})}}{|\widetilde{Q}_z(m_1,m_2)|^{2s}}$,

(2) $\widehat{\zeta}(s,(\frac{r}{f},\frac{t}{f}),z) := f^{2s} \sum_{\underline{m}\equiv(r,t) \pmod f} \frac{1}{|\widetilde{Q}_z(m_1,m_2)|^{2s}}$.

Using this notation equation (3.12) can be rewritten more compactly as

$$\zeta(s,(\frac{r}{f},\frac{t}{f}),z) = \frac{\pi^{-(1-s)}\Gamma(1-s)}{\pi^{-s}\Gamma(s)}\widehat{\zeta}(1-s,(\frac{r}{f},\frac{t}{f}),z).$$

We can now formulate the second Kronecker limit formula:

Theorem 3.2 *Let $(a_1,a_2) \in \mathbb{Q}^2$ be such that $(a_1,a_2) \notin \mathbb{Z}^2$ and $\tau \in \mathcal{H}$ then we have*

$$\begin{aligned}\widehat{\zeta}'(0,(a_1,a_2),\tau) &= -\log \mathbf{N}_{\mathbb{C}/\mathbb{R}}(g_{-a_2,a_1}(\tau)) \\ &= -\log \mathbf{N}_{\mathbb{C}/\mathbb{R}}(g_{a_1,a_2})\left(\frac{-1}{\tau}\right)).\end{aligned}$$

Proof For the first equality see chapter 20 section 5 of [Lan94b]. For the second equality we use the homogeneity property of g^{12}. □

4 Modular units and Eisenstein series

4.1 The Siegel function

In this section we define certain modular units that will be used in section 5 to construct $\mathbb{Z}$-valued measures on $\mathbb{P}^1(\mathbb{Q}_p)$. For a pair $(\frac{r}{f}, \frac{s}{f}) \in (\frac{1}{f}\mathbb{Z})^2$ we associate the Siegel function

$$g_{(\frac{r}{f},\frac{s}{f})}(\tau) = -e^{2\pi i \frac{s}{f}(\frac{r}{f}-1)/2} q_\tau^{\frac{1}{2}\widetilde{B}_2(\frac{r}{f})}(1-q_z)\prod_{n\geq 1}(1-q_\tau^n q_z)(1-q_\tau^n q_{-z})$$

where $z = \frac{r}{f}\tau + \frac{s}{f}$. As explained in the first section the function $g_{(\frac{r}{f},\frac{s}{f})}(\tau)^{12f}$ is a modular unit on $X(f)$. Let $N_0 > 0$ be a positive integer coprime to pf. From now on we will be mainly concerned by Siegel functions of the form

$$g_{(\frac{r}{f},0)}(d_0 f\tau) = q_{fd_0\tau}^{\frac{1}{2}\widetilde{B}_2(\frac{r}{f})}(1-q_{rd_0\tau})\prod_{n\geq 1}(1-q_{d_0 f\tau}^n q_{rd_0\tau})(1-q_{d_0 f\tau}^n q_{-rd_0\tau})$$

for some $d_0|N_0$, $d_0 > 0$. An easy computation shows that $g_{(\frac{r}{f},0)}(d_0 f\tau)^{12f}$ is a modular unit with respect to the group $\Gamma_1(f)\cap\Gamma_0(d_0) \supseteq \Gamma_1(f)\cap\Gamma_0(N_0)$. The following lemma gives an explicit formula for the divisor of $g_{(\frac{r}{f},0)}(d_0 f\tau)^{12f}$ when regarded as a function on $X(fN_0)$. It is more natural to work with $X(fN_0)$ since this curve is a Galois covering of $\mathbb{P}^1(\mathbb{C})$, therefore all its cusps have the same width namely fN_0.

Proposition 4.1 *Let f be a positive integer and $r \in \mathbb{Z}/f\mathbb{Z}$. Choose an integer N_0 coprime to pf. Then for every $d_0|N_0$ the function $g_{(\frac{r}{f},0)}(d_0 f\tau)^{12f}$ is $\Gamma_1(f)\cap\Gamma_0(d_0)$-invariant. In particular we can think of it as a function on the modular curve $X(fN_0)$ with its divisor supported on the set of cusps of $X(fN_0)$, denoted by $cusp(X(fN_0))$. A uniformizer at $i\infty$ for the group $\Gamma(fN_0)$ is given by $\tau \mapsto e^{\frac{2\pi i\tau}{fN_0}}$. One has*

$$div(g_{(\frac{r}{f},0)}(d_0 f\tau)^{12f}) = \sum_{[\frac{a}{c}]\in cusp(X(fN_0))} 6f\frac{N_0}{d_0}(fd_0,c)^2\widetilde{B}_2\left(\frac{rad_0}{(fd_0,c)}\right)\left[\frac{a}{c}\right]$$

where (fd_0, c) stands for the greatest common divisor between fd_0 and c. We say that the modular unit $g_{(\frac{r}{f},0)}(d_0 f\tau)^{12f}$ has primitive index if $(r, f) = 1$.

Proof This is a standard computation. □

Remark 4.1 Observe that the divisor of $g_{(\frac{r}{f},0)}(d_0 f\tau)^{12f}$ is always an integral multiple of $6f$. So it is natural to ask if such a unit is a $6f$ power of some modular unit in $\mathbb{C}(X(fN_0))$. In general the answer is no. However later on we will show that by taking suitable products of the $g_{(\frac{r}{f},0)}(d_0 f\tau)^{12f}$'s, one can extract an f-th root, see proposition 4.2.

4.2 Modular units associated to a good divisor

Definition 4.1 *For positive integers f, N_0 coprime we define $D(N_0, f)$ to be the free abelian group generated by the symbols*

$$\{[d_0, r] : 0 < d_0 | N_0, r \in \mathbb{Z}/f\mathbb{Z}\}.$$

If $\delta \in D(N_0, f)$ we call f the conductor of δ and N_0 the level of δ.

A typical element $\delta \in D(N_0, f)$ will be denoted by $\delta = \sum_{d_0,r} n(d_0, r)[d_0, r]$ where the sum goes over $d_0 | N_0$ $(d_0 > 0)$ and $r \in \mathbb{Z}/f\mathbb{Z}$ with $n(d_0, r) \in \mathbb{Z}$. We have a natural action of $(\mathbb{Z}/f\mathbb{Z})^\times$ on $D(N_0, f)$ given by $j \star [d_0, r] := [d_0, jr]$ and we extend this action $\mathbb{Z}$-linearly to all of $D(N_0, f)$.

Since $(p, f) = 1$, by reducing p modulo f, we get an action of p on $D(N_0, f)$. We denote by $D(N_0, f)^{\langle p \rangle}$ the subgroup of $D(N_0, f)$ which is fixed by multiplication by $p(mod\ f)$. Sometimes we will use the short hand notation

$$j(mod\ f) \star \delta =: \delta_j.$$

We want to define the notion of a good divisor with respect to the data N_0, f, p.

Definition 4.2 *We say that a divisor*

$$\delta = \sum_{d_0 | N_0, r \in \mathbb{Z}/f\mathbb{Z}} n(d_0, r)[d_0, r] \in D(N_0, f)$$

is a **good divisor** *if it is non zero, $p \star \delta = \delta$ and that for all $r \in \mathbb{Z}/f\mathbb{Z}$,*

(1) $\sum_{d_0 | N_0} n(d_0, r) \equiv 0(mod\ f)$,

(2) $\sum_{d_0|N_0} n(d_0,r)d_0 = 0$.

More concisely one sees that a good divisor is an non zero element of $D(N_0,f)^{\langle p\rangle}$ *which satisfies (1) and (2).*

Remark 4.2 Note that when $p \equiv 1(mod\ f)$ the condition $p \star \delta = \delta$ is automatically satisfied.

Proposition 4.2 *To a good divisor* $\delta = \sum_{d_0|N_0, r\in\mathbb{Z}/f\mathbb{Z}} n(d_0,r)[d_0,r] \in D(N_0,f)^{\langle p\rangle}$ *we associate the function*

$$\beta_\delta(\tau) := \prod_{d_0|N_0, r\in\mathbb{Z}/f\mathbb{Z}} g_{(\frac{r}{f},0)}(d_0 f\tau)^{12n(d_0,r)}. \tag{4.1}$$

This function is a modular unit which is $\Gamma_1(f)\cap\Gamma_0(N_0)$*-invariant. Moreover for all* $c \in \Gamma_0(fN_0)\{i\infty\}$ *we have* $\beta_\delta(c) = 1$.

Proof Using equation equation (3.3) with the fact that for all $r \in \mathbb{Z}/f\mathbb{Z}$

$$\sum_{d_0|N_0} n(d_0,r) \equiv 0(mod\ f),$$

we see that the ambiguity created by the f-th root of unity is cancelled. The latter observation combined with equation (3.5) shows that the right hand side of (4.1) is $\Gamma_1(f)\cap\Gamma_0(N_0)$-invariant. Using the explicit formula in proposition 4.1 combined with the fact that for all $r \in \mathbb{Z}/f\mathbb{Z}$, $\sum_{d_0|N_0} d_0 n(d_0,r) = 0$, we get that $ord_c(\beta_\delta(\tau)) = 0$ for any $c \in \Gamma_0(fN_0)\{i\infty\}$. Finally using the infinite product of the Siegel function plus its transformation formula, a calculation shows that for all $c \in \Gamma_0(fN_0)\{i\infty\}$, $\beta_\delta(c) = 1$. For this latter computation it is enough to work at $i\infty$ after a suitable shift. □

Remark 4.3 The first remark is that $div(g_{(\frac{r}{f},0)}(d_0 f\tau)^{12}) = div(g_{(\frac{-r}{f},0)}(d_0\tau)^{12})$, so they only differ by a root of unity. So we could well assume

$$\delta \in D(N_0,f)_+ := \{\delta \in D(N_0,f) : -1 \star \delta = \delta\}.$$

However we have chosen not to do this since later on we will associate Eisenstein series of odd weight to δ and forcing δ to be inside $D(N_0, f)_+$ would impose unnecessary restrictions.

Note also that a good divisor $\delta \in D(N_0, f)^{\langle p \rangle}$ gives rise to a family of modular units indexed by $(\mathbb{Z}/f\mathbb{Z})^\times/\langle p \rangle$ since for any $r \in (\mathbb{Z}/f\mathbb{Z})^\times$ we still have that δ_r is a good. If we denote the family by $\{\beta_{\delta_r}(\tau)\}_{r \in (\mathbb{Z}/f\mathbb{Z})^\times/\langle p \rangle}$, we see that the group $\Gamma_0(fN_0)$ acts transitively on it via $\tau \mapsto \gamma\tau$.

4.3 The dual modular unit

Definition 4.3 *Let $\delta = \sum_{d_0|N_0, r \in \mathbb{Z}/f\mathbb{Z}} n(d_0, r)[d_0, r] \in D(N_0, f)^{\langle p \rangle}$ be a good divisor then we define*

$$\beta_\delta^*(\tau) := \prod_{d_0|N_0, r \in \mathbb{Z}/f\mathbb{Z}} g_{(0, \frac{-r}{f})}(d_0\tau)^{12fn\left(\frac{N_0}{d_0}, r\right)}. \tag{4.2}$$

We call $\beta_\delta^*(\tau)$ the dual unit of $\beta_\delta(\tau)$.

Remark 4.4 Note that the modular unit now have an f in its exponent. Also for very divisor $d_0|N_0$ and $r \in \mathbb{Z}/f\mathbb{Z}$ the exponent is a multiple of $n\left(\frac{N_0}{d_0}, r\right)$. One can verify the formula

$$g_{(\frac{r}{f}, 0)}(fd_0 W_{fN_0}\tau)^{12f} = g_{(0, \frac{-r}{f})}\left(\frac{N_0}{d_0}\tau\right)^{12f}.$$

where $W_{fN_0} = \begin{pmatrix} 0 & -1 \\ fN_0 & 0 \end{pmatrix}$.

We have the analogue of proposition 4.1 with the same assumptions.

Proposition 4.3 *We have*

$$div(g_{(0, \frac{-r}{f})}(d_0\tau)^{12f}) = \sum_{[\frac{a}{c}] \in cusp(X(fN_0))} 6f^2 \frac{N_0}{d_0}(d_0, c)^2 \widetilde{B}_2\left(\frac{rc}{f(c, d_0)}\right)\left[\frac{a}{c}\right]. \tag{4.3}$$

Remark 4.5 Note that contrary to $g_{(\frac{r}{f},0)}(fd_0\tau)^{12f}$, the divisor in (4.3) is not necessarily a multiple of f so the f in the exponent of $g_{(0,\frac{-r}{f})}(d_0\tau)^{12f}$ is essential.

We also have an analogue of proposition 4.2.

Proposition 4.4 *The function $\beta^*_\delta(\tau)$ is $\Gamma_1(f)\cap\Gamma_0(N_0)$-invariant. Moreover for all $c\in\Gamma_0(fN_0)\{0\}$ we have $\beta^*_\delta(c)=1$.*

Proof The proof is identical to proposition 4.2 except that we use proposition 4.3 instead of proposition 4.1 and replace $\Gamma_0(fN_0)(i\infty)$ by $\Gamma_0(fN_0)(0)$. Note however that one doesn't need the assumption $\sum_{d_0|N_0} n(d_0,r)\equiv 0(mod\ f)$ for all $r\in\mathbb{Z}/f\mathbb{Z}$.

4.4 From $g_{(\frac{k}{N},0)}(N\tau)^{12}$ to $\frac{\Delta(\tau)}{\Delta(N\tau)}$

One can relate the modular unit $\beta_\delta(\tau)$ with the modular units used in [DD06]. We have for any positive integer N the identity

$$\prod_{j=1}^{N-1} g_{(\frac{j}{N},0)}(N\tau)^{12}=\zeta_N\frac{\Delta(\tau)}{\Delta(N\tau)}, \tag{4.4}$$

for some $\zeta_N\in\mu_N$. As in [DD06] choose a divisor $\delta=\sum_{d_0|N_0} n_{d_0}[d_0]$ such that

$$\sum_{d_0|N_0} n_{d_0}d_0=0 \text{ and } \sum_{d_0|N_0} n_{d_0}=0.$$

To such a divisor they associate the modular unit

$$\alpha_\delta(\tau)=\prod_{d_0|N_0}\left(\frac{\Delta(\tau)}{\Delta(d_0\tau)}\right)^{n_{d_0}}.$$

which is $\Gamma_0(N_0)$-invariant.

If we set $\delta'=\sum_{d_0|N_0,r\in\mathbb{Z}/f\mathbb{Z}} n(d_0,r)[d_0,r]\in D(N_0,f)$ with $n(d_0,r)=n_{d_0}$ for all $r\in\mathbb{Z}/f\mathbb{Z}$ then we readily see that δ' is a good divisor with respect to any prime p.

Using equation (4.4) with $N = f$ we find

$$\begin{aligned}\beta_{\delta'}(\tau) &= \zeta \prod_{d_0} \left(\frac{\Delta(d_0\tau)}{\Delta(d_0 f\tau)}\right)^{n_{d_0}} \\ &= \zeta \frac{\prod_{d_0} \left(\frac{\Delta(d_0\tau)}{\Delta(\tau)}\right)^{n_{d_0}}}{\prod_{d_0} \left(\frac{\Delta(fd_0\tau)}{\Delta(f\tau)}\right)^{n_{d_0}}} \\ &= \zeta \frac{\alpha_\delta(f\tau)}{\alpha_\delta(\tau)}.\end{aligned}$$

for some $\zeta \in \mu_f$.

Remark 4.6 Having in mind the construction of points in ring class fields of a real quadratic number field K as in [DD06], we see that once the modular unit is fixed one can vary the prime number p freely (as long as p is inert in K) since it doesn't depend on the choice of the modular unit. However in the ray class field case, for a general good divisor δ, the prime number p is related to the conductor of δ, i.e. f. Therefore one doesn't have the same freedom as in the ring class field case. The additional constraint is a congruence modulo f. For example we can always let p vary among the set of primes p congruent to 1 modulo f.

4.5 The p-stabilization of modular units

In order to construct measures one needs modular units that satisfy the distribution relation at p, i.e. modular units which are $U_{p,m}$-eigenvectors where

$$(U_{p,m}f)(\tau) := \prod_{j=0}^{p-1} f(\frac{\tau + j}{p}). \tag{4.5}$$

Note that by taking the logarithmic derivative of (4.5) one obtains the usual additive distribution relation for a measure on $\mathbb{Z}_p$.

For any $d_0 | N_0$ the function

$$\tau \mapsto \frac{g^{12f}_{(\frac{r}{f},0)}(d_0 f\tau)}{g^{12f}_{(\frac{p^{-1}r}{f},0)}(pd_0 f\tau)}$$

is $\Gamma_1(f) \cap \Gamma_0(pd_0)$-invariant (the notation $p^{-1}r$ should be interpreted as the class of $p^{-1}r$ modulo f). Moreover it is an eigenvector with eigenvalue 1 with respect to the multiplicative $U_{p,m}$-operator.

Proposition 4.5

$$U_{p,m}\left(\frac{g^{12f}_{(\frac{r}{f},0)}(d_0 f\tau)}{g^{12f}_{(\frac{p^{-1}r}{f},0)}(pd_0\tau)}\right) := \prod_{i=0}^{p-1} \frac{g^{12f}_{(\frac{r}{f},0)}(d_0 f(\frac{\tau+i}{p}))}{g^{12f}_{(\frac{p^{-1}r}{f},0)}(pd_0 f(\frac{\tau+i}{p}))} = \frac{g^{12f}_{(\frac{r}{f},0)}(d_0 f\tau)}{g^{12f}_{(\frac{p^{-1}r}{f},0)}(pd_0 f\tau)}. \tag{4.6}$$

Proof(sketch of the $U_{p,m}$-invariance) One has the identity

$$\frac{g^{12f}_{(\frac{r}{f},0)}(d_0 f\tau)}{g^{12f}_{(\frac{p^{-1}r}{f},0)}(pfd_0\tau)} = \prod_{i=0}^{p-1} g_{(\frac{r_i}{pf},0)}(pfd_0\tau)^{12f} \tag{4.7}$$

where $r_i \equiv i(mod\, p)$ and $r_i \equiv r(mod\, f)$. A direct calculation shows that every term on the right hand side of (4.7) is $U_{p,m}$-invariant i.e.

$$U_{p,m}\left(g_{(\frac{r_i}{pf},0)}(pfd_0\tau)^{12f}\right) = g_{(\frac{r_i}{pf},0)}(pfd_0\tau)^{12f} \tag{4.8}$$

for all i. The identity (4.8) relies heavily on the infinite product of the Siegel function. □

Remark 4.7 In section 4.8 we will give a more conceptual proof of the latter proposition using Eisenstein series, see equations (4.16) and (4.19).

Definition 4.4 *For a good divisor $\delta \in D(N_0, f)^{\langle p\rangle}$ we define*

$$\begin{aligned}\beta_{\delta,p}(\tau) &:= \frac{\beta_\delta(\tau)}{\beta_{p^{-1}*\delta}(p\tau)} \\ &= \frac{\beta_\delta(\tau)}{\beta_\delta(p\tau)}.\end{aligned}$$

Proposition 4.6 *Let $\delta \in D(N_0, f)^{\langle p\rangle}$ be a good divisor then $\beta_{\delta,p}(\tau)$ is $\Gamma_1(f) \cap \Gamma_0(pN_0)$-invariant, $U_{p,m}$-invariant and $\forall c \in \Gamma_0(fN_0)(i\infty)$ we have $\beta_{\delta,p}(c) = 1$.*

Proof We already know that $\beta_{\delta,p}(\tau)$ is $\Gamma_1(f) \cap \Gamma_0(pN_0)$-invariant. Since $\beta_\delta(x) = 1$ for all $x \in \Gamma_0(fN_0)(i\infty)$ and multiplication by p induces a permutation on the set $\Gamma_0(fN_0)(i\infty)$ we get that $\beta_{\delta,p}(x) = \frac{\beta_\delta(x)}{\beta_\delta(px)} = \frac{1}{1}$. Finally the $U_{p,m}$-invariance comes from the identity (4.6). □

Remark 4.8 A more careful study of the modular units on the curve $X(N_0pf)$ reveals that any modular unit

$$u(\tau) \in \langle \{ g_{(\frac{r}{pf},0)}(d_0 f\tau)^{12} : d_0 | N_0, r \in \mathbb{Z}/pf\mathbb{Z} \} \rangle \tag{4.9}$$

which is $\Gamma_1(f) \cap \Gamma_0(N_0p)$-invariant, $U_{p,m}$-invariant and has no zeros and poles at the set of cusps $\Gamma_0(fN_0)(i\infty)$ comes necessarily from a *good divisor* in the sense that there exists an integer m and a good divisor δ such that

$$div(u(\tau)^m) = div(\beta_{\delta,p}(\tau)).$$

So we don't lose much by assuming that the modular unit comes from a *good divisor.* The proof relies on the fact that $\widetilde{B}_2(x)$ is the the universal even distribution of degree 1 on $\mathbb{Q}/\mathbb{Z}$.

Definition 4.5 *Let $\delta \in D(N_0, f)^{\langle p \rangle}$ be a good divisor. We define*

$$\beta^*_{\delta,p}(\tau) := \frac{\beta^*_\delta(\tau)}{\beta^*_\delta(p\tau)}.$$

We have an analogue proposition 4.6 for the dual modular unit $\beta^*_{\delta,p}(\tau)$.

Proposition 4.7 *Let $\delta \in D(N_0, f)^{\langle p \rangle}$ be a good divisor then $\beta^*_{\delta,p}(\tau)$ is $\Gamma_1(f) \cap \Gamma_0(pN_0)$-invariant, $U_{p,m}$-invariant and $\forall c \in \Gamma_0(fN_0)(0)$ we have $\beta^*_{\delta,p}(c) = 1$.*

Proof It is similar to proposition 4.6 except for the $U_{p,m}$-invariance which will be a consequence of proposition 4.9. □

Remark 4.9 Note that there is no direct analogue of equation (4.6) for the dual modular unit $g_{(0,\frac{-r}{f})}(d_0\tau)^{12f}$ since

$$\frac{g_{(0,\frac{-r}{f})}(d_0\tau)^{12f}}{g_{(0,\frac{-r}{f})}(d_0p\tau)^{12f}} \text{ and } \frac{g_{(0,\frac{-r}{f})}(d_0\tau)^{12f}}{g_{(0,\frac{-p^{-1}r}{f})}(d_0p\tau)^{12f}}$$

are not $U_{p,m}$-invariant. Nevertheless it is still true that $\beta^*_{\delta,p}(\tau)$ is $U_{p,m}$-invariant.

4.6 The involution ι_N on $X_1(N)(\mathbb{C})$

As it is well known $Y_1(N)(\mathbb{C}) := \mathcal{H}/\Gamma_1(N)$ classifies pairs (P, E), up to equivalence, where P is a point of exact order N on an elliptic curve E defined over $\mathbb{C}$. We denote the equivalence class of a pair $(P, \mathbb{C}/\Lambda)$ by $[(P, \Lambda)]$. Any class can be represented by a pair of the form $(\frac{\omega_2}{N}(mod\ \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2), \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$ for $\omega_1, \omega_2 \in \mathbb{C}$. We define a map ι_N on such pairs by

$$\iota_N\left(\frac{\omega_2}{N}(mod\ \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2), \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2\right) := \left(\frac{-\omega_1}{N}\ (mod\ \mathbb{Z}\omega_1 + \mathbb{Z}\frac{\omega_2}{N}), \mathbb{Z}\omega_1 + \mathbb{Z}\frac{\omega_2}{N}\right).$$

It is easy to check that ι_N is well defined on such pairs and also on equivalence classes of $Y_1(N)(\mathbb{C})$. A small calculation reveals that ι^2 restricts to the identity on equivalence classes. Therefore when $N > 1$, ι gives a non trivial involution on $Y_1(N)(\mathbb{C})$. If we think of $g_{(\frac{r}{N},0)}(N\tau)^{12N}$ as a function on such pairs then a direct calculation shows that

$$\iota(g_{(\frac{r}{N},0)}(N\tau)^{12N}) = g_{(0,\frac{-r}{N})}(\tau)^{12N}.$$

One can investigate what properties of modular functions are preserved under this involution. For example let us look at the curve $X_1(pf)$ where $N = pf$. The property of being a $U_{p,m}$-eigenvector is in general not preserved by ι_{pf}. For example consider the modular unit

$$g_{(\frac{1}{fp},0)}(fp\tau)^{12pf}$$

which is a $U_{p,m}$-eigenvector (with eigenvalue 1). A calculation shows that

$$\iota_{pf}(g_{(\frac{1}{fp},0)}(fp\tau)^{12pf}) = g_{(0,\frac{-1}{pf})}(\tau)^{12pf}$$

is not a $U_{p,m}$-eigenvector. However, let us take a good divisor $\delta \in D(N_0, f)^{\langle p \rangle}$ and consider the $U_{p,m}$-eigenvector $\beta_\delta(\tau)$. Using proposition 4.9 we see that it is still true that

$$\iota_{fN_0}(\beta_{\delta,p}(\tau))^{\frac{1}{f}} = \beta^*_{\delta,p}(\tau)$$

is a $U_{p,m}$-eigenvector. In general, the properties of modular functions which are preserved under the involution ι_N can probably be made more transparent if one uses the adelic point of view by viewing them as functions on double cosets.

4.7 Bernoulli polynomials and Eisenstein series

We recall first some definitions for Bernoulli numbers and polynomials. The Bernoulli numbers are defined by the generating function

$$\frac{t}{e^t-1}=\sum_{n\geq 0}B_n\frac{t^n}{n!}.$$

Note that $B_{2n+1}=0$ if $n\geq 1$.

We also define Bernoulli polynomials as

$$\frac{te^{xt}}{e^t-1}=\sum_{n\geq 0}B_n(x)\frac{t^n}{n!}. \tag{4.10}$$

One can verify that

$$B_n(x)=\sum_{i=0}^{n}\binom{n}{i}B_i x^{n-i}.$$

From (4.10) one can deduce the useful formula $B_n(1-x)=(-1)^n B_n(x)$.

Definition 4.6 *For $n\geq 2$. We define the n-th periodic Bernoulli polynomial as*

$$\widetilde{B}_n(x):=B_n(\{x\})$$

where $\{x\}=x-[x]$. For $n=1$ we define

$$\widetilde{B}_1(x):=\{x\}-\frac{1}{2}+\frac{\mathbb{1}_{\mathbb{Z}}(x)}{2}.$$

Note that $\widetilde{B}_1(x)$ corresponds to the famous sawtooth function.

Computing the Fourier series of $\widetilde{B}_k(x)$ we find for $k\geq 1$ that

$$\widetilde{B}_k(x)=\frac{-k!}{(2\pi i)^k}\sum_{n\in\mathbb{Z}}{}'\frac{e^{2\pi inx}}{n^k}$$

where the prime of the summation means that we omit $n=0$. One easily verifies that

$$\widetilde{B}_n(-x)=(-1)^n\widetilde{B}_n(x).$$

Either by using the generating function of Bernoulli polynomials or the Fourier series one finds for any positive integer N

$$N^{k-1}\sum_{i=0}^{N-1}\widetilde{B}_k(\frac{x+i}{N}) = \widetilde{B}_k(x). \tag{4.11}$$

We are now ready to define a certain class of Eisenstein series for which the constant term of the q-expansion at $i\infty$ is a certain periodic Bernoulli polynomial evaluated at some rational number.

Definition 4.7 *For $r \in \mathbb{Z}/f\mathbb{Z}$ and an integer $k \geq 2$ we define*

(4.12)

$$\begin{aligned} E_k(r,\tau) &:= \left(\frac{(-1)^k(2\pi i)^k}{(k-1)!}\right)^{-1} {\sum_{m,n}}' \frac{e^{-2\pi i m\frac{r}{f}}}{(m+nf\tau)^k} \\ &= \frac{-\widetilde{B}_k(-r/f)}{k} + \sum_{b=0}^{f-1} e^{-2\pi i br/f}\frac{1}{f^k}\sum_{m\in\mathbb{Z}}\sum_{n\neq 0}\frac{1}{(m+b/f+n\tau)^k} \\ &= \frac{-\widetilde{B}_k(-r/f)}{k} + \frac{1}{f^k}\sum_{b=0}^{f-1} e^{-2\pi i br/f}\sum_{m\geq 1}\sum_{n\geq 1} m^{k-1}(q_{n\tau+b/f}^m + (-1)^k q_{n\tau-b/f}^m) \end{aligned}$$

where $q_z = e^{2\pi iz}$. The prime on the summation means that we omit the pair $(0,0)$.

Remark 4.10 When $k \geq 3$ the convergence of the right hand side of (4.12) is absolute. When $k = 2$ the convergence is not absolute, nevertheless the q-expansion is still meaningful.

Generally when the level f is fixed we simply write $E_k(r,\tau)$.

For any $\gamma \in \Gamma_0(f)$ we have the useful transformation formula

$$E_k(\gamma \star r, \gamma\tau)(d(\gamma\tau))^k = E_k(r,\tau)(d\tau)^k$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \star r := a^{-1}r \equiv dr (mod\ f)$. Observe also that the q-expansion at $i\infty$ of $E_{k,r}(\tau)$ is defined over $\mathbb{Q}(\zeta_f)$. In fact the q-expansion at any other cusps of $X_0(f)$ is also defined over $\mathbb{Q}(\zeta_f)$. One can think of the expression

$$E_k(r,\tau)(d\tau)^k$$

as a system of twisted k-fold differential on $X_0(f)$. We have used the word twisted since γ also acts on the index $r \in \mathbb{Z}/f\mathbb{Z}$. However $E_k(r,\tau)$ is a true k-fold differential on the curve $X_1(f)$.

For future reference we define $E_k^*(r,\tau)$ and we call it the dual of $E_k(r,\tau)$.

Definition 4.8 *For $r \in \mathbb{Z}/f\mathbb{Z}$ and an integer $k \geq 2$ we define*

$$E_k^*(r,\tau) := \left(\frac{(-1)^k(2\pi i)^k}{(k-1)!}\right)^{-1} \sum_{m,n}{}' \frac{e^{2\pi i n\frac{r}{f}}}{(m+n\tau)^k}.$$

The first thing to notice is that

$$E_k^*(r,\tau) = E_k\left(r, \frac{-1}{f\tau}\right)\left(\frac{-1}{\tau}\right)^k, \tag{4.13}$$

If we denote $W_f = \begin{pmatrix} 0 & -1 \\ f & 0 \end{pmatrix}$ then we can rewrite the previous identity as

$$E_k(r,\tau) = det(W_f) E_k\left(r, W_f\tau\right)\left(W_f\tau\right)^k. \tag{4.14}$$

Remark 4.11 Note that in the case where $f = 1$ we have that $E_k(\tau)$ is invariant under W_1 therefore $E_k(\tau)$ is self dual. When $f > 1$ it is not the case since $W_f = \begin{pmatrix} 0 & -1 \\ f & 0 \end{pmatrix} \notin \Gamma_1(f)$.

The q-expansion of $E_k^*(r,\tau)$ is given by

$$\begin{aligned} E_k^*(r,\tau) &:= \left(\frac{(-1)^k(2\pi i)^k}{(k-1)!}\right)^{-1} \sum_{m,n}{}' \frac{e^{2\pi i n\frac{r}{f}}}{(m+n\tau)^k} \\ &= \frac{-\widetilde{B}_k(r/f)}{k} + \sum_{b=0}^{f-1} e^{2\pi i br/f} \sum_{m\in\mathbb{Z}} \sum_{n\neq 0} \frac{1}{(m+(b+fn)\tau)^k} \\ &= \frac{-\widetilde{B}_k(r/f)}{k} + \sum_{b=0}^{f-1} e^{2\pi i br/f} \sum_{m\geq 1} \sum_{n\geq 1} m^{k-1}(q^m_{(fn+b)\tau} + (-1)^k q^m_{(fn-b)\tau}). \end{aligned}$$

As in the previous case, for any $\gamma \in \Gamma_0(f)$ we have the useful transformation formula

$$E_k^*(\gamma * r, \gamma\tau)(d(\gamma\tau))^k = E_k^*(r,\tau)(d\tau)^k$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} * r := ar \equiv d^{-1}r(mod\ f)$. This is exactly as in the previous case except that the action on the index $r \in (\mathbb{Z}/f\mathbb{Z})^\times$ is inverted. It is for this reason that we have denoted it by $*$ instead of $\star$.

4.8 Hecke operators on modular forms twisted by an additive character

In this section we discuss the theory of Hecke operators on such twisted modular forms.

Let $\psi : \mathbb{Z}\times\mathbb{Z} \to \mu_N$ be any additive character. For every lattice $\Lambda \subseteq \mathbb{C}$ fix a group homomorphism $\varphi(\Lambda) : \Lambda \to \mathbb{Z}\times\mathbb{Z}$. Denote this family of group homomorphisms by φ_*. Assume furthermore that φ_* is homogeneous of degree -1, i.e. $\varphi(\alpha\Lambda) = \varphi(\Lambda) \circ \alpha^{-1}$ for any lattice Λ and $\alpha \in \mathbb{C}^\times$.

In such a setting it makes sense to define a function E_{k,ψ,φ_*} on the set of all lattices by the rule

$$E_{k,\psi,\varphi_*}(\Lambda) := \sum_{w\in\Lambda-\{0\}} \frac{\psi(\varphi(\Lambda)(w))}{w^k}.$$

It is also convenient to define for any lattice $\Lambda' \subseteq \Lambda$

$$E_{k,\psi,\varphi(\Lambda)}(\Lambda') := \sum_{w\in\Lambda'-\{0\}} \frac{\psi(\varphi(\Lambda)(w))}{w^k}.$$

For any scalar $\alpha \in \mathbb{C}^\times$ we define

$$(\alpha \star E_{k,\psi,\varphi_*})(\Lambda) := E_{k,\psi,\varphi(\alpha\Lambda)}(\alpha\Lambda).$$

Since φ_* is homogeneous of degree -1 we have

$$E_{k,\psi,\varphi(\alpha\Lambda)}(\alpha\Lambda) = \alpha^{-k}E_{k,\psi,\varphi(\Lambda)}(\Lambda).$$

therefore

$$\alpha \star E_{k,\psi,\varphi_*} = \alpha^{-k}E_{k,\psi,\varphi_*}.$$

It thus gives an action of $\mathbb{C}^\times$ on such Eisenstein series.

We still have a notion of Hecke operators $T_k(n)$ (k stands for the weight of the Eisenstein series) where we define

$$(T_k(n)E_{k,\psi,\varphi_*})(\Lambda) = n^{k-1}\sum_{[\Lambda:\Lambda']=n} E_{k,\psi,\varphi(\Lambda)}(\Lambda').$$

We want to compute the action of the Hecke operators $T_k(p)$ on $E_{k,\psi,\varphi}$. We have

$$\begin{aligned}(T_k(p)E_{k,\psi,\varphi_*})(\Lambda) &= p^{k-1}\sum_{[\Lambda:\Lambda']=p} E_{k,\psi,\varphi(\Lambda)}(\Lambda') \\ &= p^{k-1}E_{k,\psi,\varphi(\Lambda)}(\Lambda) + p^{k-1}pE_{k,\psi,\varphi(\Lambda)}(p\Lambda) \qquad (4.15)\\ &= p^{k-1}E_{k,\psi,\varphi_*}(\Lambda) + E_{k,\psi^p,\varphi_*}(\Lambda).\end{aligned}$$

The equality (4.15) comes from that the fact the $\cup_{[[\Lambda:\Lambda']=p}\Lambda' = \Lambda$ and for two distinct lattices Λ', Λ'' of index p in Λ we have $\Lambda' \cap \Lambda'' = p\Lambda$. Note that if $p \equiv 1(mod\ N)$ then $T_k(p)E_{k,\psi,\varphi_*} = (1+p^{k-1})E_{k,\psi,\varphi_*}$, i.e. E_{k,ψ,φ_*} is an eigenvector with eigenvalue $1+p^{k-1}$.

In order to simplify the dependence of φ_* on the set of lattices , it is convenient to work with normalized oriented lattices $\Lambda_\tau := \mathbb{Z}+\tau\mathbb{Z}$ for $\tau \in \mathcal{H}$. For every τ in a fixed fundamental domain $\mathcal{D}$ of $\mathcal{H}$ modulo $SL_2(\mathbb{Z})$, we define $\varphi(\Lambda_\tau) : \Lambda_\tau \to \mathbb{Z}\times\mathbb{Z}$ to be the group homomorphism (in fact isomorphism) given by $m+n\tau \mapsto (m,n)$. Having fixed the fundamental domain $\mathcal{D}$ we can write any lattice Λ uniquely as $\Lambda = \lambda\Lambda_\tau$ for a certain $\tau \in \mathcal{D}$ and $\lambda \in \mathbb{C}^\times$. We define $\varphi(\Lambda) := \varphi(\Lambda_\tau)\circ\lambda^{-1}$. In this way φ_* is homogeneous of degree -1.

For any τ in the fundamental domain we thus have $\psi(\varphi(\Lambda_\tau)(m+n\tau)) = e^{2\pi i(m\frac{r}{N}+n\frac{s}{N})}$ for some integers r, s depending only on ψ(and on the choice of the fundamental domain $\mathcal{D}$). We define

$$G_k((r\ mod\ N, s\ mod\ N),\tau) = G_k((r,s),\tau) := {\sum_{m,n}}' \frac{e^{2\pi i(\frac{r}{N}m+\frac{s}{N}n)}}{(m+n\tau)^k}.$$

When the level is fixed we drop the $(mod\ N)$ notation.

Now we fix a level $N = f$ and a prime p coprime to f.

Definition 4.9 *We define*

$$\begin{aligned} E_{k,p}(r,\tau) &:= G_k((-r,0),f\tau) - p^{k-1}G_k((-p^{-1}r,0),pf\tau) \\ &= E_k(r,\tau) - p^{k-1}E_k(p^{-1}r,p\tau). \end{aligned}$$

Note that $E_k(r,\tau)$ is a $\Gamma_1(f)$-modular and $E_{k,p}(r,\tau)$ is $\Gamma_1(f)\cap\Gamma_0(p)$-modular both of weight k.

We define the additive Hecke operator $U_{p,a}$ on the set of meromorphic functions $g:\mathcal{H}\to\mathbb{C}$ to be:

$$U_{p,a}g(\tau) := \frac{1}{p}\sum_{j=0}^{p-1} g(\frac{\tau+j}{p}).$$

Proposition 4.8 *$E_{k,p}(r,z)$ is a $U_{p,a}$-eigenvector with eigenvalue 1 i.e.*

$$U_{p,a}E_{k,p}(r,\tau) = \frac{1}{p}\sum_{j=0}^{p-1} E_{k,p}(r,\frac{\tau+j}{p}) = E_{k,p}(r,\tau). \tag{4.16}$$

Proof We have

$$\begin{aligned} T_k(p)E_k(r,\Lambda_\tau) &= p^{k-1}\sum_{j=0}^{p-1} E_k(r,p\mathbb{Z}+(\tau+j)\mathbb{Z}) + p^{k-1}E_{k,r}(\mathbb{Z}+p\tau\mathbb{Z}) \\ &= \frac{1}{p}\sum_{j=0}^{p-1} E_k(pr,\Lambda_{\frac{\tau+j}{p}}) + p^{k-1}E_k(r,\Lambda_{p\tau}) \\ &= U_{p,a}E_k(pr,\tau) + p^{k-1}E_k(r,p\tau). \end{aligned}$$

Replacing r by $p^{-1}r$ in the last equality we find

$$T_k(p)E_k(p^{-1}r,\tau) = U_{p,a}E_k(r,\tau) + p^{k-1}E_k(p^{-1}r,p\tau) \tag{4.17}$$

We are now ready to compute the action of $U_{p,a}$ on

$$E_{k,p}(r,\tau) = E_k(r,\tau) - p^{k-1}E_k(p^{-1}r,p\tau).$$

We have:

$$U_{p,a}E_{k,p}(r,\tau) = U_{p,a}E_{k,p}(r,\tau) - p^{k-1}U_{p,a}E_k(p^{-1}r,p\tau). \tag{4.18}$$

Using (4.17) for the first term of the right hand side of (4.18) and the definition of $U_{p,a}$ for the second term we find

$$\begin{aligned}
&= T_k(p)E_k(p^{-1}r,\tau) - p^{k-1}E_k(p^{-1}r,p\tau) - p^{k-1}\frac{1}{p}\sum_{j=0}^{p-1}E_k(p^{-1}r,\tau+j)\\
&= T_k(p)E_k(p^{-1}r,\tau) - p^{k-1}E_k(p^{-1}r,p\tau) - p^{k-1}E_k(p^{-1}r,\tau)\\
&= p^{k-1}E_k(p^{-1}r,\tau) + E_k(r,\tau) - p^{k-1}E_k(p^{-1}r,p\tau) - p^{k-1}E_k(p^{-1}r,\tau)\\
&= E_{k,p}(r,\tau)
\end{aligned}$$

where in the third equality we have used (4.15). □

Definition 4.10 *We define*

$$E^*_{k,p}(r,\tau) := E^*_k(r,\tau) - p^{k-1}E^*_k(r,\tau)$$

Note that there is no twist by p on the second index.

Proposition 4.9 *We have*

$$U_{p,a}E^*_{k,p}(r,\tau) = E^*_k(pr,\tau) - p^{k-1}E^*_k(r,p\tau) \tag{4.19}$$

Proof This is a similar computation to what we did previously. □

Remark 4.12 Even if $E^*_{k,p}(r,\tau)$ is not an $U_{p,a}$-eigenvector (unless $p \equiv 1(mod\ f)$) it is still very close.

4.9 The q-expansion of $E_{k,p}(r,\tau)$

For any rational number $\frac{a}{b} \in \mathbb{Q}$ with $(b,f) = 1$ define $(\frac{a}{b})_f$ to be the unique representative modulo f between 0 and $f-1$ congruent to $\frac{a}{b}$.

The q-expansion of $E_{k,p}(\tau)$ is given by

$$
\begin{aligned}
&E_{k,p}(r,\tau)\\
&= \frac{-\widetilde{B}_k(-r/f)}{k} + \frac{1}{f^k}\sum_{b=0}^{f-1} e^{-2\pi ibr/f} \sum_{m\geq 1}\sum_{n\geq 1} m^{k-1}(q^m_{n\tau+b/f} + (-1)^k q^m_{n\tau-b/f}) -\\
&\quad p^{k-1}\left(\frac{-\widetilde{B}_k(-(p^{-1}r)_f/f)}{k}\frac{1}{f^k}\sum_{b=0}^{f-1} e^{-2\pi ibr/f} \sum_{m\geq 1}\sum_{n\geq 1} m^{k-1}(q^m_{np\tau+pb/f} + (-1)^k q^m_{np\tau-pb/f})\right)\\
&= \left(\frac{-\widetilde{B}_k(-r/f)}{k} + p^{k-1}\frac{\widetilde{B}_k(-(p^{-1}r)_f/f)}{k}\right) +\\
&\quad \frac{1}{f^k}\sum_{b=0}^{f-1} e^{-2\pi ibr/f} \sum_{\substack{m\geq 1\\(m,p)=1}}\sum_{n\geq 1} m^{k-1}(q^m_{n\tau+b/f} + (-1)^k q^m_{n\tau-b/f}).
\end{aligned}
$$

We thus readily see that for $k \equiv k'(mod\, p^n(p-1))$ all coefficients of $E_{k,p}(\tau)$ vary p-adically continuously when n goes to infinity. In particular for a fix congruence class a modulo $p-1$, if we look at all the integers $k \equiv a(mod\, p-1)$ and $k \equiv 0(mod\, p^n)$ with n going to infinity, we see that all the coefficients are analytic functions in the weight k. We thus have a one dimensional p-adic family of Eisenstein series.

4.10 Relation between Eisenstein series and modular units

A calculation shows that

(1) $\mathrm{dlog}(g_{(\frac{r}{f},0)}(f\tau)^{12}) = 12\cdot \mathrm{dlog}(g_{(\frac{r}{f},0)}(f\tau)) = -24\pi i f E_2(r,\tau)d\tau,$

(2) $\mathrm{dlog}(g_{(0,\frac{r}{f})}(\tau)^{12}) = 12\cdot \mathrm{dlog}(g_{(0,\frac{r}{f})}(\tau)) = -24\pi i E_2^*(r,\tau)d\tau$

where dlog stands for the logarithmic derivative with respect to the variable τ.

This motivates the following definition

Definition 4.11 *Let $\delta = \sum_{d_0|N_0, r\in\mathbb{Z}/f\mathbb{Z}} n(d_0,r)[d_0,r] \in D(N_0,f)^{\langle p\rangle}$ be a good divisor then we associate to this divisor two families of Eisenstein series. We set*

(1) $F_{k,\delta}(\tau) := \sum_{d_0,r} d_0 n(d_0,r) E_k(r, d_0\tau)$

(2) $F^*_{k,\delta}(\tau) := \sum_{d_0,r} d_0^{k-1} n(\frac{N_0}{d_0}, r) E^*_k(r, d_0\tau)$

and also

$$\begin{aligned} F_{k,\delta,p}(\tau) &:= \sum_{d_0,r} n(d_0,r) d_0 E_{k,p}(r, d_0\tau) \\ &= \sum_{d_0,r} d_0 n(d_0,r) E_k(r, d_0\tau) - p^{k-1} \sum_{d_0,r} d_0 n(d_0,r) E_k(p^{-1}r, d_0 p\tau) \\ &= F_{k,\delta}(\tau) - p^{k-1} F_{k,\delta}(p\tau). \end{aligned}$$

where the last equality uses the fact that $p \star \delta = \delta$. *Similarly we define*

$$\begin{aligned} F^*_{k,\delta,p}(\tau) &:= \sum_{d_0,r} n(N_0/d_0,r) d_0^{k-1} E^*_{k,p}(r, d_0\tau) \\ &= \sum_{d_0,r} d_0^{k-1} n(N_0/d_0,r) E^*_k(r, d_0\tau) - p^{k-1} \sum_{d_0,r} d_0^{k-1} n(N_0/d_0,r) E^*_k(r, d_0 p\tau) \\ &= F^*_{k,\delta}(\tau) - p^{k-1} F^*_{k,\delta}(p\tau). \end{aligned}$$

The motivation for the definition of $F_{k,\delta}(\tau)$ and $F^*_{k,\delta}(\tau)$ is justified by the next proposition

Proposition 4.10 *Let* $\delta = \sum_{d_0 | N_0, r \in \mathbb{Z}/f\mathbb{Z}} n(d_0,r)[d_0,r] \in D(N_0,f)^{\langle p \rangle}$ *be a good divisor then when the weight is equal to* $k = 2$ *we have*

(1) $dlog\, \beta_\delta(\tau) = -24\pi i f F_{2,\delta}(\tau) d\tau$

(2) $dlog\, \beta_{\delta,p}(\tau) = -24\pi i f F_{2,\delta,p}(\tau) d\tau$

and similarly

(1) $dlog\, \beta^*_\delta(\tau) = -24\pi i F^*_{2,\delta}(\tau) d\tau$

(2) $dlog\, \beta^*_{\delta,p}(\tau) = -24\pi i F^*_{2,\delta,p}(\tau) d\tau$

Finally $F_{k,\delta}(\tau)$ *and* $F^*_{k,\delta}(\tau)$ *are related by the formula*

$$F_{k,\delta}(W_{fN_0}\tau) = (-1)^k \tau^k N_0 F^*_{k,\delta}(\tau) \tag{4.20}$$

where $W_{fN_0} = \begin{pmatrix} 0 & -1 \\ fN_0 & 0 \end{pmatrix}$.

Proof Straight forward computations. □

Remark 4.13 For l a prime number coprime to fN_0 we have

$$T_k(l)F_{k,\delta}(z) = (1 + l^{k-1})F_{k,\delta}(z).$$

Similarly for l a prime number coprime to pfN_0 we have

$$T_k(l)F_{k,\delta,p}(z) = (1 + l^{k-1})F_{k,\delta,p}(z).$$

Moreover equation (4.16) shows that for any $k \geq 2$ we have

$$U_{p,a}F_{k,\delta,p}(z) = F_{k,\delta,p}(z).$$

The group $\Gamma_0(fN_0)$ (resp. $\Gamma_0(pfN_0)$) acts transitively on the family

$$\{F_{k,\delta_r}(\tau)\}_{r\in(\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle}$$

(resp. the family $\{F_{k,\delta_r,p}(\tau)\}_{r\in(\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle}$). The same thing also holds when we take the dual Eisenstein series. Everything is straight forward except the $U_{p,a}$-invariance. For the latter , we use the fact that $p \star \delta = \delta$ combined with equation (4.19).

Remark 4.14 Because $\delta = \sum_{d_0,r} n(d_0, r)[d_0, r]$ is good we have for every $r \in \mathbb{Z}/f\mathbb{Z}$ that $\sum_{d_0|N_0} n(d_0, r)d_0 = 0$. This latter condition implies that $F_{k,\delta}(z)$ is holomorphic at $i\infty$. Similarly we have that $F^*_{k,\delta}(z)$ is holomorphic at 0.

5 The $\mathbb{Z}$-valued measures $\mu_r\{c_1 \to c_2\}$ and the invariant $u(\delta_r, \tau)$

5.1 $\mathbb{Z}$-valued measures on $\mathbb{P}^1(\mathbb{Q}_p)$

Let $0 \neq \delta \in D(N_0, f)^{\langle p \rangle}$ be a good divisor. Consider the family of modular units

$$\{\beta_{\delta_r, p}(\tau)\}_{r \in (\mathbb{Z}/f\mathbb{Z})^\times / \langle p \rangle}.$$

To such a family we want to associate a family of measures. Before defining the measures we need to define some suitable subgroups of matrices of $GL_2(\mathbb{Z}[\frac{1}{p}])$.

Definition 5.1 *For quantities p, f, N_0 fixed we define*

(1) $\widetilde{\Gamma}_0 := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Z}[1/p]) : c \equiv 0 (mod\ fN_0)\}$,

(2) $\Gamma_0 = \{\gamma \in \widetilde{\Gamma}_0 : det(\gamma) = 1\} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}[\frac{1}{p}]) : c \equiv 0 (mod\ fN_0)\}$,

(3) $\Gamma_1 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}[\frac{1}{p}]) : a \equiv 1 (mod\ f), c \equiv 0 (mod\ fN_0)\}$.

(4) $\Gamma = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}[\frac{1}{p}]) : a, d \equiv 1 (mod\ f), b, c \equiv 0 (mod\ fN_0)\}$.

Obviously one has the inclusions $\widetilde{\Gamma}_0 \supseteq \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma$. Note that the group $\widetilde{\Gamma}_0 = \langle \Gamma_0(fN_0), P \rangle$ where $P = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$.

Remark 5.1 We have an almost transitive action of Γ_0 on $\mathcal{B}$ where $\mathcal{B}$ is defined as the set of balls of $\mathbb{P}^1(\mathbb{Q}_p)$. One has $\mathcal{B} = \Gamma_0(\mathbb{Z}_p) \coprod \Gamma_0(\mathbb{P}^1(\mathbb{Q}_p) \backslash \mathbb{Z}_p)$.

We can now define a family of measures. For the rest of the subsection we assume that $\delta \in D(N_0, f)^{\langle p \rangle}$ is fixed good divisor.

Definition 5.2 *Let* $(c_1, c_2, k) \in \Gamma_0(fN_0)(i\infty) \times \Gamma_0(fN_0)(i\infty) \times (\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle$. *Let* $B \in \mathcal{B}$ *be any ball of* $\mathbb{P}^1(\mathbb{Q}_p)$. *If* B *is inside the coset* $SL_2(\mathbb{Z}[\frac{1}{p}])(\mathbb{Z}_p)$ *set* $\epsilon = 1$ *otherwise set* $\epsilon = -1$. *If* $\epsilon = 1$ *choose* $\gamma \in \Gamma$ *s.t.* $\gamma\mathbb{Z}_p = B$. *If* $\epsilon = -1$ *choose* $\gamma \in \Gamma$ *such that* $\gamma\mathbb{Z}_p = \mathbb{P}^1(\mathbb{Q}_p)\backslash B$. *We define*

$$\mu_k\{c_1 \to c_2\}(B) = \epsilon\frac{1}{2\pi i}\int_{\gamma^{-1}c_1}^{\gamma^{-1}c_2} \mathrm{dlog}\beta_{\delta_{\gamma^{-1}\star k},p}(\tau). \tag{5.1}$$

where $\gamma \star k \equiv dk (\mathrm{mod}\ f)$ *for* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0 \subseteq SL_2(\mathbb{Z}[\frac{1}{p}])$. *It makes sense to reduce* d *modulo* f *because its denominator is at worst a power a* p *which is coprime to* f.

Note that $Stab_{\Gamma_0}(\mathbb{Z}_p) = \Gamma_0(pN_0) \cap \Gamma_0(f) = \Gamma_0(pfN_0)$. Therefore since the modular units in $\{\beta_{\delta_k,p}(\tau)\}_{k\in(\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle}$ are $\Gamma_0(pfN_0)$-invariant in the sense that

$$\beta_{\delta_{\gamma\star k},p}(\gamma\tau) = \beta_{\delta_k,p}(\tau),$$

we get that (5.1) is well defined.

We can now state the main theorem of the section

Theorem 5.1 *There exists a unique system of measures indexed by*

$$\Gamma_0(i\infty) \times \Gamma_0(i\infty) \times (\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle$$

satisfying the following properties: For all $(c_1, c_2, k) \in \Gamma_0(i\infty)\times\Gamma_0(i\infty)\times(\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle$

(1) $\mu_k\{c_1 \to c_2\}(\mathbb{P}^1(\mathbb{Q}_p)) = 0$,

(2) $\mu_k\{c_1 \to c_2\}(\mathbb{Z}_p) = \frac{1}{2\pi i}\int_{c_1}^{c_2} \mathrm{dlog}\beta_{\delta_k,p}(\tau)$,

(3) *(Γ_0-invariance property) For all* $\gamma \in \Gamma_0$ *and all compact open* $U \subseteq \mathbb{P}^1(\mathbb{Q}_p)$ *we have*

$$\mu_{\gamma\star k}\{\gamma c_1 \to \gamma c_2\}(\gamma U) = \mu_k\{c_1 \to c_2\}(U).$$

Proof The $U_{p,m}$-invariance of the $\beta_{\delta_k,p}(\tau)$'s implies that $\mu_k\{c_1 \to c_2\}$ are distributions on $\mathbb{P}^1(\mathbb{Q}_p)$. Also since $\forall c \in \Gamma_0(fN_0)(i\infty)$ we have $\beta_{\delta_k,p}(c) = 1$, the line integrals can be interpreted as the winding number with respect to the origin of a closed path $\beta_{\delta_k,p}(\mathcal{C})$ where $\mathcal{C}$ is an arbitrary path joining $\gamma^{-1}c_1$ to $\gamma^{-1}c_2$. So we really get $\mathbb{Z}$-valued measures. The Γ_0-invariance comes from the definition of the measures. Finally the uniqueness follows from the properties (1)-(3) combined with the fact that Γ_0 splits $\mathcal{B}$ into two orbits. $\square$

Remark 5.2 Theorem 5.1 gives us a partial modular symbol of $\mathbb{Z}$-valued measures on $\mathbb{P}^1(\mathbb{Q}_p)$ i.e.

$$\mu_{\_}\{\_ \to \_\} : \Gamma_0(i\infty) \times \Gamma_0(i\infty) \times (\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle \to \{\mathbb{Z}\text{-valued measures on } \mathbb{P}^1(\mathbb{Q}_p)\}$$
$$(c_1, c_2, k) \mapsto \mu_k\{c_1 \to c_2\}.$$

Note that the image of $\mu_{\_}\{\_ \to \_\}$ lies in the set of Γ_0-invariant measures.

In the next subsection using explicit formulas for the moments of those periods we will see that this modular symbol is odd in the sense that

$$\frac{1}{2\pi i}\int_{-c_1}^{-c_2} \mathrm{dlog}\beta_{\delta_{-k},p}(\tau) = -\frac{1}{2\pi i}\int_{c_1}^{c_2} \mathrm{dlog}\beta_{\delta_k,p}(\tau),$$

in other words $\mu_{-k}\{-c_1 \to -c_2\}(\mathbb{Z}_p) = -\mu_k\{c_1 \to c_2\}(\mathbb{Z}_p)$.

Remark 5.3 Finally it should be pointed out that the $U_{p,m}$-invariance of $\beta_{\delta_k,p}(z)$ combined with the fact that $p\star\delta = \delta$ implies that the measures constructed in theorem 5.1 are in fact $\widetilde{\Gamma}_0 := \langle\Gamma_0, \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}\rangle$-invariant, see Proposition 5.13 of [Dar01].

We have a notion of a dual family of measures.

Definition 5.3 *Let $c_1, c_2 \in \Gamma_0(0)$. We define $\mu_k^*\{c_1 \to c_2\}$ as:*

$$\mu_k^*\{c_1 \to c_2\}(B) = \epsilon\frac{1}{2\pi i}\int_{\gamma^{-1}c_1}^{\gamma^{-1}c_2} \mathrm{dlog}\beta^*_{\delta_{\gamma^{-1}\star k},p}(\tau). \tag{5.2}$$

where $\epsilon = 1$ if $B \in \Gamma_0\mathbb{Z}_p$ with $\gamma\mathbb{Z}_p = B$ and $\epsilon = -1$ if $B \in \Gamma_0(\mathbb{P}^1(\mathbb{Q}_p)\backslash\mathbb{Z}_p)$ and $\gamma\mathbb{Z}_p = \mathbb{P}^1(\mathbb{Q}_p)\backslash B$.

We have an analogue of theorem 5.1 except that $\gamma \star r$ is replaced by $\gamma * r$ and the set of cusps of $\Gamma_0(fN_0)(i\infty)$ by the set of cusps $\Gamma_0(fN_0)(0)$. Note that $\gamma * r = \gamma^{-1} \star r$.

The reader also will have no problem to formulate the analogue of theorem 5.1.

5.2 Periods of modular units and Dedekind sums

This section might be skipped at the first reading. We included it only for the sake of completeness. We use Dedekind sums to give explicit formulas for the periods of the modular units considered in theorem 5.1.

Let us start with a very general principal which comes from calculus

Proposition 5.1 *(general principle) Let $G \subseteq SL_2(\mathbb{Z})$ be a discrete subgroup. Let X be the two dimensional compact surface (real dimensions) defined by $X := \mathcal{H}^*/G$ where $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$. Let $f(\tau)$ be a C^∞-closed 1-form on $\mathcal{H} \cup \{\infty\}$ which is G-invariant. Then for any fixed $g \in G$ the quantity*

$$\int_x^{gx} f(\tau)d\tau$$

doesn't depend on the base point x when x varies inside $\mathcal{H} \cup G(i\infty)$.

Remark 5.4 In the previous proposition we assume the path of integration to "be nice", i.e. contained in $\mathcal{H} \cup G(i\infty)$ and contractible inside $\mathcal{H} \cup G(i\infty)$.

Proof First of all the integral does not depend on the path of integration since $f(\tau)$ is a closed C^∞ 1-form. Let $x, x' \in \mathcal{H} \cup G(i\infty)$ be arbitrary points. Let C, C' be arbitrary curves joining x, gx and x', gx' respectively. Let $\pi : \mathcal{H} \cup G(i\infty) \to X$ be the natural projection. Note that $\pi(C)$ and $\pi(C')$ in X are closed homotopic curves. Since the 1-form $f(\tau)d\tau$ is G-invariant it descends to a C^∞ 1-form on X. Finally the latter is a closed 1-form on the surface X. Apply Stoke's theorem to obtain the result. □

Let $\delta = \sum_{d_0,r} n(d_0, r)[d_0, r] \in D(N_0, f)^{\langle p \rangle}$ be a good divisor that we fix until the

end of the subsection. We want to give explicit formulas for

$$\frac{1}{2\pi i}\int_{c_1}^{c_2} d\log(u(z))dz$$

in the case where $u(z)$ is the modular unit $\beta_\delta(z)$ or $\beta_{\delta,p}(z)$.

Let $a=(a_1,a_2)$ be rational numbers contained in the interval $[0,1[$. Since $g_a(\tau)$ has no zeros in $\mathcal{H}$ we can define the logarithm of such modular units on $\mathcal{H}$. We fix a branch of $\log g_a(\tau)$ by setting

$$\log(g_a(\tau)) = \pi i B_2(a_1)\tau + \log(1-q_z) + \sum_{n\geq 1}(\log(1-q_\tau^n q_z) + \log(1-q_\tau^n q_{-z})).$$

For $|x|<1$ we define $\log(1-x) := -\sum_{n\geq 1} x^n/n$. Because of the assumption on a_1, a_2 we have that $0 \leq |q_\tau^n q_z| < 1$ and $0 \leq |q_\tau^n q_{-z}| < 1$ for $1 \leq n$ where $z = a_1\tau + a_2 \in \mathbb{C}, \tau \in \mathcal{H}$. We define $\sim: \mathbb{Q}^2 \longrightarrow [0,1[^2$ be the function for which $(u_1,u_2)\in\mathbb{Q}^2$ goes to $\widetilde{(u_1,u_2)} = (a_1,a_2)$ with $u_1 \equiv a_1 (mod\ \mathbb{Z})$ and $u_2 \equiv a_2 (mod\ \mathbb{Z})$.

Definition 5.4 *Let $(a_1,a_2)\in(\frac{1}{N}\mathbb{Z})^2$ and $\gamma\in SL_2(\mathbb{Z})$. We define the γ-period of the Siegel function $g_a(\tau)$ to be*

$$\pi_a(\gamma) := (\log g_{\tilde{a}}(\gamma\tau) - \log g_{\widetilde{a\gamma}}(\tau))\,|_{\tau=i\infty} \in \frac{\pi i}{12N}\mathbb{Z}. \tag{5.3}$$

Remark 5.5 Up to a multiple of $i\pi$ those periods are rational since

$$g_a(\gamma\tau)^{12N} = g_{a\gamma}(\tau)^{12N} \forall \gamma \in SL_2(\mathbb{Z}).$$

In fact, using the $\Gamma(N)$-invariance we see that any element $\tau\in\mathcal{H}\cup\Gamma(N)(i\infty)$ can be used to compute the period $\pi_a(\gamma)$. This is an application of proposition 5.1 to the 1-form $\frac{d}{d\tau}(\log g_{\tilde{a}}(\gamma\tau) - \log g_{\widetilde{a\gamma}}(\tau))$ on the curve $X(N)$. In practice we will take $\tau = i\infty$. Note also that (5.3) depends only on the image of a in $(\mathbb{Q}/\mathbb{Z})^2[N]$.

A property satisfied by those periods $\pi_a(\gamma)$ is the so called cocycle condition.

Proposition 5.2 *Let $\gamma_1,\gamma_2\in SL_2(\mathbb{Z})$ then $\pi_a(\gamma_1\gamma_2) = \pi_a(\gamma_1) + \pi_{a\gamma_1}(\gamma_2)$.*

Proof We have $\pi_a(\gamma_1\gamma_2) = \log g_{\tilde{a}}(\gamma_1\gamma_2\tau) - \log g_{\widetilde{a\gamma_1\gamma_2}}(\tau)$ where τ is any point in the upper half plane. We also have $\log g_{\tilde{a}}(\gamma_1\gamma_2\tau) - \log g_{\widetilde{a\gamma_1}}(\gamma_2\tau) = \pi_a(\gamma_1)$. It thus follows that $\pi_a(\gamma_1\gamma_2) = \pi_a(\gamma_1) + \log g_{\widetilde{a\gamma_1}}(\gamma_2\tau) - \log g_{\widetilde{a\gamma_1\gamma_2}}(\tau) = \pi_a(\gamma_1) + \pi_{a\gamma_1}(\gamma_2)$. □

Proposition 5.3 *(Schöneberg) Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ *and* $r, s \in \mathbb{Z}$, *not both congruent to* 0 *modulo* N *then*

$$\pi_{(\frac{r}{N},\frac{s}{N})}(\gamma) = \begin{cases} \pi i \left(\frac{a}{c}\widetilde{B}_2(\frac{r}{N}) + \frac{d}{c}\widetilde{B}_2(\frac{r'}{N}) - 2sgn(c)s^N_{(\frac{r}{N},\frac{s}{N})}(a,c) \right) & \text{if } c \neq 0; \\ \pi i \frac{b}{d}\widetilde{B}_2(\frac{r}{N}) & \text{if } c = 0 \end{cases}$$

where $\begin{pmatrix} \frac{r}{N} & \frac{s}{N} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \frac{r'}{N} & \frac{s'}{N} \end{pmatrix}$ *and* $s^N_{(\frac{r}{N},\frac{s}{N})}(a,c)$ *is a twisted Dedekind sum:*

$$s^N_{(\frac{r}{N},\frac{s}{N})}(a,c) = \sum_{i(mod\ c)} \widetilde{B}_1\left(\frac{r+iN}{cN}\right)\widetilde{B}_1\left(\frac{r'+aiN}{cN}\right).$$

Proof See p. 199 of [B. 74].

For $N > 1$ we have the double coset

$$SL_2(\mathbb{Z}) \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} SL_2(\mathbb{Z}) = \coprod_i SL_2(\mathbb{Z})x_i$$

where the x_i's can be chosen as upper triangular matrices of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $a, d > 0, ad = N$ and $0 \leq b \leq d-1$.

Definition 5.5 *For a matrix* $\gamma \in SL_2(\mathbb{Z})$ *we define* $T_N(\gamma)$ *and* $R_N(\gamma)$ *to be matrices such that* $\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \gamma = T_N(\gamma)R_N(\gamma)$ *where* $T_N(\gamma) \in SL_2(\mathbb{Z})$ *and* $R_N(\gamma)$ *is equal to a unique representative* x_i.

For any matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ it is also convenient to define

$$\gamma(N) := \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \gamma \begin{pmatrix} \frac{1}{N} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & bN \\ c/N & d \end{pmatrix}.$$

Remark 5.6 Note that the map $\gamma \mapsto \gamma(N)$ induces a group isomorphism from $\Gamma_0(N)$ to $\Gamma^0(N)$.

Proposition 5.4 *Let $\delta \in D(N_0, f)^{\langle p \rangle}$ and $r \in \mathbb{Z}/f\mathbb{Z}$. For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \in \Gamma_0(fN_0)$ we have*

$$\frac{1}{12}(\log \beta_{\delta_r}(\gamma\tau) - \log \beta_{\delta_r}(\tau)) = \sum_{k \in \mathbb{Z}/f\mathbb{Z}} \sum_{d_0 | N_0} n(d_0, k) \pi_{(\frac{rk}{f}, 0)}(T_{fd_0}(\gamma)). \tag{5.4}$$

and

$$\frac{1}{12}(\log \beta_{\delta_r}(p\gamma\tau) - \log \beta_{\delta_r}(p\tau)) = \sum_{k \in \mathbb{Z}/f\mathbb{Z}} \sum_{d_0 | N_0} n(d_0, k) \pi_{(\frac{rk}{f}, 0)}(T_{pfd_0}(\gamma)) \tag{5.5}$$

Proof We only prove the equality (5.5) since (5.4) can be proved in a similar but simpler way.

Let $\gamma \in \Gamma_0(fN_0)$. We compute:

$$\begin{aligned} &\log \beta_{\delta_r}(p\gamma\tau) - \log \beta_{\delta_r}(p\tau) \\ &= \sum_{k \in \mathbb{Z}/f\mathbb{Z}} \sum_{d_0 | N_0} n(d_0, k) \left(\log g_{\widetilde{(\frac{rk}{f}, 0)}}(fd_0 p\gamma\tau)^{12} - \log g_{\widetilde{(\frac{rk}{f}, 0)}}(fd_0 p\tau)^{12} \right) \end{aligned}$$

We can write $\begin{pmatrix} pfd_0 & 0 \\ 0 & 1 \end{pmatrix} \gamma = T_{pfd_0}(\gamma) R_{pfd_0}(\gamma)$ for $T_{pfd_0}(\gamma) \in SL_2(\mathbb{Z})$ and $R_{pfd_0}(\gamma)$ is some primitive upper triangular matrix of determinant pfd_0 that will be chosen later. We thus get

$$\begin{aligned} &\frac{1}{12} \sum_{k \in \mathbb{Z}/f\mathbb{Z}} \sum_{d_0 | N_0} n(d_0, k) \left(\log g_{\widetilde{(\frac{rk}{f}, 0)}}(fd_0 p\gamma\tau)^{12} - \log g_{\widetilde{(\frac{rk}{f}, 0)}}(fd_0 p\tau)^{12} \right) = \\ &\sum_{k \in \mathbb{Z}/f\mathbb{Z}} \sum_{d_0 | N_0} n(d_0, k) \left(\log g_{\widetilde{(\frac{rk}{f}, 0)}}(T_{pfd_0}(\gamma) R_{pfd_0}(\gamma)\tau) - \log g_{\widetilde{(\frac{rk}{f}, 0)}}(pfd_0\tau) \right). \end{aligned} \tag{5.6}$$

But

$$\log g_{\widetilde{(\frac{rk}{f}, 0)}}(T_{pfd_0}(\gamma) R_{pfd_0}(\gamma)\tau) = \log g_{\widetilde{(\frac{rk}{f}, 0) T_{pfd_0}(\gamma)}}(R_{pfd_0}(\gamma)\tau) + \pi_{(\frac{rk}{f}, 0)}(T_{pfd_0}(\gamma)). \tag{5.7}$$

Substituting (5.7) in (5.6) we get that the right side of (5.6)

$$(5.8) \quad = \sum_{k\in(\mathbb{Z}/f\mathbb{Z})^\times} \sum_{d_0|N_0} n(d_0,k)\pi_{(\frac{rk}{f},0)}(T_{pfd_0}(\gamma))+$$
$$\sum_{k\in\mathbb{Z}/f\mathbb{Z}} \sum_{d_0|N_0} n(d_0,k)\left(\log g_{\widetilde{(\frac{rk}{f},0)T_{pfd_0}(\gamma)}}(R_{pfd_0}(\gamma)\tau) - \log g_{(\frac{rk}{f},0)}(pfd_0\tau)\right).$$

It remains to evaluate the second term of (5.8).

If $p|c$ we can take $R_{pfd_0}(\gamma) = \begin{pmatrix} pfd_0 & 0 \\ 0 & 1 \end{pmatrix}$. However when $p \nmid c$ we take $R_{pfd_0}(\gamma) = \begin{pmatrix} fd_0 & fd_0j \\ 0 & p \end{pmatrix}$ where $1 \leq j \leq p-1$ is chosen in such a way that $i \equiv \frac{d}{c}(mod\, p)$. Note that j does not depend on d_0. In order to evaluate the second term of (5.8) we let $\tau \to i\infty$ and we use the explicit formula for the matrices $T_{pfd_0}(\gamma)$ and $R_{pfd_0}(\gamma)$.

Let $T_{pfd_0}(\gamma) = \begin{pmatrix} A_{d_0} & B_{d_0} \\ C_{d_0} & D_{d_0} \end{pmatrix}$ and $R_{pfd_0}(\gamma) = \begin{pmatrix} A'_{d_0} & B'_{d_0} \\ 0 & C'_{d_0} \end{pmatrix}$. Note that by assumption $B'_{d_0} = 0$ if $p|c$ and $B'_{d_0} = jd_0f$ if $p \nmid c$. One finds

$$\lim_{\tau\to i\infty} \sum_{k\in\mathbb{Z}/f\mathbb{Z}} \sum_{d_0|N_0} n(d_0,k)\left(\log g_{\widetilde{(\frac{rk}{f},0)T_{pfd_0}(\gamma)}}(R_{pfd_0}(\gamma)\tau) - \log g_{\widetilde{(\frac{rk}{f},0)}}(pfd_0\tau)\right) =$$
$$\lim_{\tau\to i\infty} \sum_{k\in(\mathbb{Z}/f\mathbb{Z})^\times} \sum_{d_0|N_0} n(d_0,k)\left(\log g_{\widetilde{(\frac{rkA_{d_0}}{f},0)}}(\frac{A'_{d_0}\tau + B'_{d_0}}{C'_{d_0}}) - \log g_{\widetilde{(\frac{rk}{f},0)}}(pfd_0\tau)\right) = 0.$$

For the last equality we have used the the fact that $\sum_{d_0} n(d_0,k)d_0 = 0$ for all $k \in \mathbb{Z}/f\mathbb{Z}$. $\square$

We end this subsection by rewriting the formulas obtained in proposition 5.4 in a more compact way.

Proposition 5.5 *Let* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(fN_0)$ *where* $c \neq 0$. *Then we have the*

following formulas:

$$(5.9)\qquad \begin{aligned}&\mu_j\{i\infty \to \gamma(i\infty)\}(\mathbb{Z}_p)\\ &= \frac{1}{2\pi i}\left(\log \beta_{\delta_j,p}(\gamma\tau) - \log \beta_{\delta_j,p}(\tau)\right)\\ &= -12\cdot sign(c)\sum_{d_0,r} n(d_0,r)\left(D_{1,1}^{rj(mod\ f)}(a,c/d_0) - D_{1,1}^{rj(mod\ f)}(pa,c/d_0)\right)\end{aligned}$$

For the definition of $D_{1,1}^{r(mod\ f)}(a,c)$ *see definition 11.1.*

5.3 The modular symbols are odd

Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \in \Gamma_0(fN_0)$ then we define the involution $\gamma^\iota = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$. Note that for $\gamma_1, \gamma_2 \in \Gamma_0(fN_0)$ we have $(\gamma_1\gamma_2)^\iota = \gamma_1^\iota\gamma_2^\iota$. Let also $z^\iota = -\bar{z}$ be the natural involution on $\mathcal{H}$ then we have $\gamma^\iota z^\iota = (\gamma z)^\iota$.

One can also verify that $T_N(\gamma)^\iota = T_N(\gamma^\iota)$. Remember also that the function $\widetilde{B}_1$ is odd. Using the previous observations we deduce the important equality

$$s^N_{(\frac{r}{N},0)}(a,-c) = -s^N_{(\frac{r}{N},0)}(a,c).$$

It thus follows that

$$(5.10)\qquad \pi_{(\frac{r}{N},0)}(\gamma^\iota) = -\pi_{(\frac{r}{N},0)}(\gamma).$$

This last equality is important because it tells us that the family of measures constructed in theorem 5.1 give rise to odd modular symbols. So using the explicit formulas in proposition 5.4 we get the following proposition:

Proposition 5.6 *Let* $u(z) = \beta_\delta(\tau)$ *or* $\beta_{\delta,p}(\tau)$. *Then we have*

$$\int_{-c_1}^{-c_2} d\log u(z) = -\int_{c_1}^{c_2} d\log u(z)$$

for any $c_1, c_2 \in \Gamma_0(fN_0)(i\infty)$.

Proof Let $\gamma_1, \gamma_2 \in \Gamma_0(fN_0)$ be such that $\gamma_1(i\infty) = c_1$ and $\gamma_2(i\infty) = c_2$. Since $\gamma_j^*(i\infty) = -\gamma_j(i\infty) = -c_j$ $(j = 1, 2)$ we find that

$$\begin{aligned}\int_{c_1}^{c_2} d\log u(z) &= -\int_{i\infty}^{c_1} d\log u(z) + \int_{i\infty}^{c_2} d\log u(z) \\ &= \int_{i\infty}^{-c_1} d\log u(z) - \int_{i\infty}^{-c_2} d\log u(z) \\ &= -\int_{-c_1}^{-c_2} d\log u(z).\end{aligned}$$

where the second equality follows from proposition 5.4 combined with (5.10).

5.4 From $\mathcal{H}$ to $\mathcal{H}_p^{\mathcal{O}}(N_0, f)$

We would like to generalize theorem 3.1 to real quadratic number fields. Unfortunately one cannot evaluate modular units on a real quadratic argument $\tau \in K$ since $K \cap \mathcal{H} = \emptyset$. What one does is to replace $\mathcal{H}$ by the p-adic upper half-plane $\mathcal{H}_p := \mathbb{P}^1(\mathbb{C}_p) - \mathbb{P}^1(\mathbb{Q}_p)$, equipped with its structure of a rigid analytic space. We take the opportunity here to introduce some useful notation that will be used for the sequel. For any $\mathbb{Z}$-module $M \subseteq \mathbb{C}$ and a prime number p, we define $M^{(p)} := M[\frac{1}{p}] \simeq M \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}]$.

Let p be a prime number inert in K. Choose a $\mathbb{Z}$-order $\mathcal{O} \subseteq K$ and fix a positive integer N coprime to p. In [DD06] they associate to such data the set

$$\mathcal{H}_p^{\mathcal{O}}(N) = \mathcal{H}_p^{\mathcal{O}} := \{\tau \in \mathcal{H}_p : \mathcal{O}_\tau^{(p)} = \mathcal{O}_{N\tau}^{(p)} = \mathcal{O}^{(p)}\} \tag{5.11}$$

where $\mathcal{O}_\tau = End_K(\Lambda_\tau)$ and $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$.

Remark 5.7 Note that (5.11) differs slightly from [DD06] since in there setting $\mathcal{O}$ is assumed to be $\mathbb{Z}[\frac{1}{p}]$-orders instead of $\mathbb{Z}$-orders. Therefore there is no need to tensor over $\mathbb{Z}[\frac{1}{p}]$. This has the obvious advantage of simplifying the notation in (5.11). However having in mind the definition of discriminants (covolume) of lattices, we have decided to work with $\mathbb{Z}$-modules.

Implicitly in the definition of $\mathcal{H}_p^{\mathcal{O}}$, there is a level N structure which is assumed to be fixed. One can verify that the set $\mathcal{H}_p^{\mathcal{O}}$ is nonempty iff there exists an $\mathcal{O}$-ideal $\mathfrak{a}$

such that $\mathcal{O}/\mathfrak{a} \simeq \mathbb{Z}/N$, this is the so called Heegner hypothesis. In the spirit of the remark 3.1 we propose the following generalization of $\mathcal{H}_p^{\mathcal{O}}(N) = \mathcal{H}_p^{\mathcal{O}}$.

Definition 5.6 *Let K be a real quadratic number field. Let p be a prime number inert in K. Fix a $\mathbb{Z}$-order $\mathcal{O}$ of K. Let f(called the conductor) be a positive integer coprime to $p \cdot disc(\mathcal{O})$. Let N_0(called the level) be a positive integer coprime to pf. To such data we associate the set*

$$\mathcal{H}_p^{\mathcal{O}}(N_0, f) := \{(r, \tau) \in (\mathbb{Z}/f\mathbb{Z})^\times \times \mathcal{H}_p : \mathcal{O}_\tau^{(p)} = \mathcal{O}_{N_0\tau}^{(p)} = \mathcal{O}^{(p)}, (\Lambda_\tau^{(p)}, f) = 1, \tau > \tau^\sigma\}$$

where $G_{K/\mathbb{Q}} = \{1, \sigma\}$.

The notation $(\Lambda_\tau^{(p)}, f) = 1$ means that $\Lambda_\tau^{(p)}$, as an $\mathcal{O}^{(p)}$-ideal, is coprime to $f\mathcal{O}^{(p)}$. We have a natural action of $\widetilde{\Gamma}_0 := \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Z}[\frac{1}{p}]) : c \equiv 0 (mod\, fN_0)\}$ on the set $\mathcal{H}_p^{\mathcal{O}}(N_0, f)$ given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \star (k, \tau) = (dk, \frac{a\tau + b}{c\tau + d}).$$

Note that the quotient $\mathcal{H}_p^{\mathcal{O}}(N_0, f)/\widetilde{\Gamma}_0$ is finite (This will be proved, see (5.13)). We now define a map that allows us to go from $\mathcal{H}_p^{\mathcal{O}}(N_0, f)$ to $\mathcal{O}^{(p)}$-ideals.

Definition 5.7 *We define a map Ω(which depends on $\mathcal{O}, p, N_0, f$)*

$$\Omega : \mathcal{H}_p^{\mathcal{O}}(N_0, f) \to I_{\mathcal{O}^{(p)}}$$

where $I_{\mathcal{O}^{(p)}}$ stands for the monoid of integral $\mathcal{O}^{(p)}$-ideals of K by the following rule:

$$(r, \tau) \mapsto A_r \Lambda_\tau^{(p)}$$

where $0 \neq A_r \in \mathbb{Z}$ is the smallest positive integer such that the following two properties hold.

(1) $A_r \equiv r (mod\, f)$

(2) $A_r \Lambda_\tau^{(p)}$ is $\mathcal{O}^{(p)}$-integral

Definition 5.8 *Let* $(r,\tau),(r',\tau') \in \mathcal{H}_p^{\mathcal{O}}(N_0,f)$. *Let* $\Omega(r,\tau) = A\Lambda_\tau^{(p)}$ *and* $\Omega(r',\tau') = A'\Lambda_{\tau'}^{(p)}$. *We say that* $(r,\tau) \sim (r',\tau')$ *iff there exists a* $\lambda \in Q_{\mathcal{O},1}(f\infty)$ *s.t.*

$$\left(A\Lambda_\tau^{(p)}, A\Lambda_{N_0\tau}^{(p)})\right) = \left(\lambda A'\Lambda_{\tau'}^{(p)}, \lambda A'\Lambda_{N_0\tau'}^{(p)}\right).$$

We have a natural identification of $\mathcal{H}_p^{\mathcal{O}}(N_0,f)/\sim$ with

$$\{(L,M) : \text{ pairs of } \mathbb{Z}[\tfrac{1}{p}]\text{-modules of rank 2 in } K, End_K(L) = End_K(M) = \mathcal{O}^{(p)} \text{ and } L/M \simeq \mathbb{Z}/N_0\}/Q_{\mathcal{O}^{(p)},1}(f\infty)\}$$

which again can be identified to

$$\{(L,M) : \text{ pairs of } \mathbb{Z}\text{-modules of rank 2 in } K, End_K(L) = End_K(M) = \mathcal{O} \text{ and } L/M \simeq \mathbb{Z}/N_0\}/\langle Q_{\mathcal{O},1}(f\infty),(p)\rangle.$$

This identification will allow us to view $\mathcal{H}_p^{\mathcal{O}}(N_0,f)/\sim$ as a disjoint union of finitely many copies of a certain generalized ideal class group attached to $\mathcal{O}^{(p)} = \mathcal{O}[\frac{1}{p}]$.

Let us assume the existence of an $\mathcal{O}^{(p)}$-ideal $\mathfrak{a}$ such that $\mathcal{O}^{(p)}/\mathfrak{a} \simeq \mathbb{Z}/N_0$. Then there exists an inclusion

$$I_{\mathcal{O}^{(p)}}(f)/Q_{\mathcal{O}^{(p)},1}(f\infty) \hookrightarrow \mathcal{H}_p^{\mathcal{O}}(N_0,f)/\sim \tag{5.12}$$

given by the following rule:

Choose an ideal $\mathfrak{a} \trianglelefteq \mathcal{O}^{(p)}$ coprime to f such that $\mathcal{O}^{(p)}/\mathfrak{a} \simeq \mathbb{Z}/N_0$. Then for an ideal $I \in I_{\mathcal{O}^{(p)}}(f)$ we associate the pair $(I,\mathfrak{a}I)$. A calculation shows that there always exists a $\lambda \in Q_{\mathcal{O}^{(p)},1}(f\infty)$ s.t. $(I,\mathfrak{a}I) = \lambda(A\Lambda_\tau^{(p)}, A\Lambda_{N_0\tau}^{(p)})$ for some integer A and $\tau \in K$. Obviously this map is an inclusion. However it is not canonical since it depends on the choice of the ideal $\mathfrak{a}$. The number of distinct inclusions as in (5.12) is in bijection with

$$\{\mathfrak{a} \trianglelefteq \mathcal{O} : \mathcal{O}/\mathfrak{a} \simeq \mathbb{Z}/N_0\}/Q_{\mathcal{O}^{(p)},1}(f\infty). \tag{5.13}$$

Class field theory gives us an isomorphism

$$I_{\mathcal{O}^{(p)}}(f)/Q_{\mathcal{O}^{(p)},1}(f\infty) \simeq I_{\mathcal{O}}(f)/\langle Q_{\mathcal{O},1}(f\infty),p\rangle \overset{rec^{-1}}{\to} G_{H_{\mathcal{O}}(f\infty)^{\langle Fr_{\mathfrak{p}}\rangle}/K}$$

where $H_{\mathcal{O}}(f\infty)$ is the abelian extension of K corresponding to $I_{\mathcal{O}}(f)/Q_{\mathcal{O},1}(f\infty)$ by class field theory. We let $L := H_{\mathcal{O}}(f\infty)^{\langle Fr_{\wp}\rangle}$ be the subfield of $H_{\mathcal{O}}(f\infty)$ fixed by the Frobenius at $p\mathcal{O} = \wp$. Therefore in this case we get an natural action of $G_{L/K}$ on $\mathcal{H}_p^{\mathcal{O}}(N_0, f)$ given by the following rule

$$rec^{-1}(\mathfrak{b}) \star (L, M) = (\mathfrak{b}L, \mathfrak{b}M).$$

Obviously this action is simple but in general not transitive since (5.13) could be of size larger than 1.

The next two lemmas show that $\sim$ is induced by the action of $\widetilde{\Gamma}_0$.

Lemma 5.1 *Let $(r, \tau) \in \mathcal{H}_p^{\mathcal{O}}(N_0, f)$ and $\gamma \in \widetilde{\Gamma}_0$. Let $\Omega(r, \tau) = A\Lambda_\tau^{(p)}$ and $\Omega(\gamma \star r, \gamma\tau) = B\Lambda_{\gamma\tau}^{(p)}$. Then the pairs $(A\Lambda_\tau^{(p)}, A\Lambda_{N_0\tau}^{(p)})$ and $(B\Lambda_{\gamma\tau}^{(p)}, B\Lambda_{N_0\gamma\tau}^{(p)})$ are congruent modulo $Q_{\mathcal{O},1}(f\infty)$. In other words the relation of equivalence $\sim$ on $\mathcal{H}_p^{\mathcal{O}}(N_0, f)$ is $\widetilde{\Gamma}_0$-invariant.*

Proof Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \widetilde{\Gamma}_0$, $\Omega(r, \tau) = A\Lambda_\tau^{(p)}$ and $\Omega(\gamma \star r, \gamma\tau) = B\Lambda_{\gamma\tau}^{(p)}$. Note that $B \equiv Ad(mod\, f)$. Because $(r, \tau) \in \mathcal{H}_p^{\mathcal{O}}(N_0, f)$ we have that $(\Lambda_\tau^{(p)}, f) = 1$ and $\mathcal{O}_\tau^{(p)} = \mathcal{O}^{(p)}$. If we let $Q_\tau(x, y) = Ux^2 + Vxy + Wy^2$ then the first equality says that $(U, f) = 1$. The second equality says that $U^2 - 4VW = p^n D (n \geq 0)$ where $D = disc(\mathcal{O})$. By assumption $(f, D) = 1$, $f|c$ and $det(\gamma) = ad - bc > 0$. We also have the identity

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau & \tau^\sigma \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \tau' & \tau'^\sigma \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c\tau + d & 0 \\ 0 & c\tau^\sigma + d \end{pmatrix}. \tag{5.14}$$

where $\tau - \tau^\sigma > 0$ and $\tau' - \tau'^\sigma > 0$. Combining all the previous observations we see that

$$\epsilon \frac{A}{B}(c\tau + d) \in Q_{\mathcal{O},1}(f\infty),$$

for $\epsilon = \pm 1$ chosen appropriately. Finally note that

(1) $\frac{A}{B}(c\tau + d)\Lambda_{\gamma\tau}^{(p)} = \Lambda_\tau^{(p)}$,

(2) $\frac{A}{B}(\frac{c}{N_0}N_0\tau + d)\Lambda_{\gamma(N_0)N_0\tau}^{(p)} = \Lambda_{N_0\tau}^{(p)}$

where $\gamma(N_0) = \begin{pmatrix} a & N_0 b \\ c/N_0 & d \end{pmatrix} \in GL_2(\mathbb{Z}[\frac{1}{p}])$. $\square$

We prove the converse of the previous lemma

Lemma 5.2 *Let $(r,\tau),(r',\tau') \in \mathcal{H}_p^{\mathcal{O}}(N_0,f)$ be equivalent then there exists a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \widetilde{\Gamma}_0$ such that*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \lambda \frac{k'}{k} \begin{pmatrix} \tau' \\ 1 \end{pmatrix},$$

for some $\lambda \in Q_{\mathcal{O},1}(f\infty)$ and integers such that $k \equiv r (mod\ f)$ and $k' \equiv r' (mod\ f)$.

Note that we obtain automatically that $d \equiv \frac{k'}{k} (mod\ f)$.

Proof Let $\Omega(r,\tau) = k\Lambda_\tau^{(p)}$ and $\Omega(r',\tau') = k'\Lambda_{\tau'}^{(p)}$. By definition of Ω we have that k, k' are integers congruent to r and r' respectively modulo f. Since $(r,\tau) \sim (r',\tau')$ there exists a $\lambda \in Q_{\mathcal{O},1}(f\infty)$ such that

$$(k\Lambda_\tau^{(p)}, k\Lambda_{N_0\tau}^{(p)}) = (\lambda k'\Lambda_{\tau'}^{(p)}, \lambda k'\Lambda_{N_0\tau'}^{(p)}). \tag{5.15}$$

By looking at the first component of (5.15) we get $k\Lambda_\tau^{(p)} = \lambda k'\Lambda_{\tau'}^{(p)}$. Therefore there exists a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}[\frac{1}{p}])$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \lambda \frac{k'}{k} \begin{pmatrix} \tau' \\ 1 \end{pmatrix}. \tag{5.16}$$

Let $\varphi : \Lambda_\tau^{(p)} \to \Lambda_{\tau'}^{(p)}$ be the natural $\mathbb{Z}[\frac{1}{p}]$-module isomorphism defined by $\varphi(1) = 1$ and $\varphi(\tau) = \tau'$. Note that the restriction $\varphi|_{\Lambda_{N_0\tau}^{(p)}}$ induces an isomorphism $\varphi|_{\Lambda_{N_0\tau}^{(p)}} : \Lambda_{N_0\tau}^{(p)} \to \Lambda_{N_0\tau'}^{(p)}$ which sends $1 \mapsto 1$ and $N_0\tau \mapsto N_0\tau'$. Let $\psi : \Lambda_{\tau'}^{(p)} \to \Lambda_\tau^{(p)}$ be the $\mathbb{Z}[\frac{1}{p}]$-module isomorphism induced by multiplication by $\lambda\frac{k'}{k}$, so $\psi(1) = \lambda\frac{k'}{k} \cdot 1$ and $\psi(\tau') = \lambda\frac{k'}{k}\tau'$. We thus have $\phi \circ \varphi \in Aut_{\mathbb{Z}[\frac{1}{p}]}(\Lambda_\tau)$. By equation (5.16) we readily see that the matrix corresponding to $\phi \circ \varphi$ with respect to the basis $\{1,\tau\}$ is $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Using the second equality of (5.15) we get $\lambda\frac{k}{k'}\Lambda^{(p)}_{N_0\tau'} = \Lambda^{(p)}_{N_0\tau}$. From this we deduce that $\phi\circ\varphi(\Lambda^{(p)}_{N_0\tau}) = \Lambda^{(p)}_{N_0\tau}$. In other words the automorphism $\psi\circ\varphi$ preserves the lattice $\Lambda^{(p)}_{N_0\tau}$. Therefore the matrix corresponding to $\psi\circ\varphi$ with respect to the basis $\{1, N_0\tau\}$ has coefficients in $\mathbb{Z}[\frac{1}{p}]$. But this matrix is nothing else than $\begin{pmatrix} a & bN_0 \\ c/N_0 & d \end{pmatrix}$. We thus conclude that $N_0|c$.

Because $(r,\tau) \in \mathcal{H}^{\mathcal{O}}_p(N_0, f)$ we have $(\Lambda^{(p)}_\tau, f) = 1$ and $\mathcal{O}^{(p)}_\tau = \mathcal{O}^{(p)}$. Let $Q_\tau(x,y) = Ax^2 + Bxy + Cy^2$. Since $(\Lambda^{(p)}_\tau, f) = 1$ we get that$(A, f) = 1$. Also because $\mathcal{O}^{(p)}_\tau = \mathcal{O}^{(p)}$ we have $B^2 - 4AC = Dp^n$ $(n \geq 0)$ where $D = disc(\mathcal{O})$. But by assumption $(f, disc(\mathcal{O})) = 1$. Note that $\tau = \frac{-B\pm\sqrt{D}}{A}$. Looking at (5.16) combined with the previous observations gives us the congruence,

$$c\tau + d = \lambda\frac{k'}{k} \equiv integer\ (mod\ f)$$

since $\lambda \equiv 1(mod\ f)$. Because $(A, f) = 1$ and $(D, f) = 1$ we deduce that $c \equiv 0(mod\ f)$. Finally using equation (5.14) combined with $\lambda \gg 0$ we see that $det(\gamma) > 0$. □

Corollary 5.1 *The relation of equivalence $\sim$ on $\mathcal{H}^{\mathcal{O}}_p(N_0, f)$ is induced by the action of $\widetilde{\Gamma}_0$.*

Corollary 5.2 *Let $(r,\tau) \in \mathcal{H}^{\mathcal{O}}_p(N_0, p)$. Since $\begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}\star(r,\tau) = (pr,\tau)$ we deduce that the first component is well defined modulo the action of p in the sense that $(pr,\tau) \sim (r,\tau)$.*

5.5 Construction of the $K_p^\times$-points $u(\delta_r, \tau)$

The family of measures constructed in theorem 5.1 will enable us to construct $K_p^\times$ points. Note that K_p is the unique quadratic unramified extension of $\mathbb{Q}_p$ so

$$\mathcal{O}^\times_{K_p} \simeq \mu_{p^2-1} \times (1 + p\mathcal{O}_{K_p}).$$

In this section we assume that $\delta \in D(N_0, f)^{(p)}$ is a fixed good divisor. We remind also the reader that

(1) $\widetilde{\Gamma}_0 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Z}[\frac{1}{p}]) : c \equiv 0(mod\ fN_0)\}$,

(2) $\Gamma_0 = \{\gamma \in \widetilde{\Gamma}_0 : det(\gamma) = 1\}$,

(3) $\Gamma_1 = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}[\frac{1}{p}]) : a \equiv 1(mod\ f), c \equiv 0(mod\ fN_0)\}$ and

(4) $\Gamma = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}[\frac{1}{p}]) : a, d \equiv 1(mod\ f), b, c \equiv 0(mod\ fN_0)\}$.

Definition 5.9 *Let $k \in (\mathbb{Z}/f\mathbb{Z})^\times$. Let also $c_1, c_2 \in \Gamma_0(i\infty)$ and $\tau_1, \tau_2 \in \mathcal{H}_p \cap K_p$. We define*

$$\int_{\tau_1}^{\tau_2}\int_{c_1}^{c_2} d\log\beta_{\delta_k,p}(\tau) := \int_{\mathbb{P}^1(\mathbb{Q}_p)} log_p\left(\frac{t-\tau_2}{t-\tau_1}\right) d\mu_k\{c_1 \to c_2\}(t).$$

where $\mu_k\{c_1 \to c_2\}$ is the measure of theorem 5.1 for the for the modular unit $\beta_{\delta,p}(\tau)$.

Since the measures $\mu_k\{c_1 \to c_2\}$ are $\mathbb{Z}$-valued it makes sense also to define a multiplicative integral

$$\prod_{\tau_1}^{\tau_2}\int_{c_1}^{c_2}\left(\frac{t-\tau_2}{t-\tau_1}\right) d\mu_k\{c_1 \to c_2\}(t) := \lim_{c=\{U_i\}}\prod_i\left(\frac{t_i-\tau_2}{t_i-\tau_1}\right)^{\mu_k\{c_1\to c_2\}(U_i)}$$

Where the limit goes over a set of covers that become finer and finer.

Definition 5.10 *Let $\tau \in \mathcal{H}_p \cap K_p$, fix an $x \in \Gamma_0(i\infty)$ and $k \in (\mathbb{Z}/f\mathbb{Z})^\times$, then for all $\gamma_1, \gamma_2 \in \Gamma_0$ we define*

$$\kappa_{x,(k,\tau)}(\gamma_1,\gamma_2) := \prod_{\tau}^{\gamma_1\tau}\int_{\gamma_1 x}^{\gamma_1\gamma_2 x} dlog(\beta_{\delta_k,p}(z)) \in K_p^\times.$$

We let the group Γ_0 acts trivially on $K_p^\times$. We have the following proposition.

Proposition 5.7 *The 2-cochain $\kappa_{x,(k,\tau)} \in C^2(\Gamma_0, K_p^\times)$ is a "twisted" 2-cocycle satisfying the following relation:*

$$(d\kappa_{x,(k,\tau)})(\gamma_1,\gamma_2,\gamma_3) = \kappa_{x,(k,\tau)}(\gamma_2,\gamma_2) - \kappa_{x,(\gamma_1^{-1}\star k,\tau)}(\gamma_2,\gamma_3)$$

for all $\gamma_1, \gamma_2, \gamma_3 \in \Gamma_0$. In particular $(d\kappa_{x,(k,\tau)})|_{\Gamma_1} = 0$, i.e. $\kappa_{x,(k,\tau)}|_{\Gamma_1} \in Z^2(\Gamma_1, K_p^\times)$.

Proof We compute:

$$
\begin{aligned}
&(d\kappa_{x,(k,\tau)})(\gamma_1,\gamma_2,\gamma_3)\\
&= \gamma_1\kappa_{x,(k,\tau)}(\gamma_2,\gamma_3) - \kappa_{x,(k,\tau)}(\gamma_1\gamma_2,\gamma_3) + \kappa_{x,(k,\tau)}(\gamma_1,\gamma_2\gamma_3) - \kappa_{x,(k,\tau)}(\gamma_1,\gamma_2)\\
&= \kappa_{x,(k,\tau)}(\gamma_2,\gamma_3) - \kappa_{x,(k,\tau)}(\gamma_1\gamma_2,\gamma_3) + \kappa_{x,(k,\tau)}(\gamma_1,\gamma_2\gamma_3) - \kappa_{x,(k,\tau)}(\gamma_1,\gamma_2)\\
&= \int_{\tau}^{\gamma_2\tau}\int_{\gamma_2 x}^{\gamma_2\gamma_3 x} \mathrm{dlog}\beta_{\delta_k,p}(z) - \int_{\tau}^{\gamma_1\gamma_2\tau}\int_{\gamma_1\gamma_2 x}^{\gamma_1\gamma_2\gamma_3 x} \mathrm{dlog}\beta_{\delta_k,p}(z)\\
&+ \int_{\tau}^{\gamma_1\tau}\int_{\gamma_1 x}^{\gamma_1\gamma_2\gamma_3 x} \mathrm{dlog}\beta_{\delta_k,p}(z) - \int_{\tau}^{\gamma_1\tau}\int_{\gamma_1 x}^{\gamma_1\gamma_2 x} \mathrm{dlog}\beta_{\delta_k,p}(z)
\end{aligned}
$$

where the second equality follows from the trivial action of Γ_0 on $K_p^\times$. Let $\pi_{11}(\gamma_1) = a \in \mathbb{Z}[\frac{1}{p}]$. Using the invariance property of the measures under Γ_0 we can multiply the bounds of the integral of the second term by γ_1^{-1} at the cost of replacing $k(mod\ f)$ by $ak(mod\ f)$. So we find

$$
\begin{aligned}
(5.17) \qquad &= \int_{\tau}^{\gamma_2\tau}\int_{\gamma_2 x}^{\gamma_2\gamma_3 x} \mathrm{dlog}\beta_{\delta_k,p}(z) - \int_{\gamma_1^{-1}\tau}^{\gamma_2\tau}\int_{\gamma_2 x}^{\gamma_2\gamma_3 x} \mathrm{dlog}\beta_{\delta_{ak},p}(z)\\
&+ \int_{\tau}^{\gamma_1\tau}\int_{\gamma_1 x}^{\gamma_1\gamma_2\gamma_3 x} \mathrm{dlog}\beta_{\delta_k,p}(z) - \int_{\tau}^{\gamma_1\tau}\int_{\gamma_1 x}^{\gamma_1\gamma_2 x} \mathrm{dlog}\beta_{\delta_k,p}(z).
\end{aligned}
$$

Rearranging the first two terms together and and the last two in (5.17) we get

$$
\begin{aligned}
&= \int_{\tau}^{\gamma_2\tau}\int_{\gamma_2 x}^{\gamma_2\gamma_3 x} (\mathrm{dlog}\beta_{\delta_k,p}(z) - d\log\beta^*_{\delta_{ak}}(z)) - \int_{\gamma_1^{-1}\tau}^{\tau}\int_{\gamma_2 x}^{\gamma_2\gamma_3 x} \mathrm{dlog}\beta^*_{\delta_{ak}}(\tau)\\
&\quad + \int_{\tau}^{\gamma_1\tau}\int_{\gamma_1\gamma_2 x}^{\gamma_1\gamma_2\gamma_3 x} \mathrm{dlog}\beta_{\delta_k,p}(z)\\
&= \int_{\tau}^{\gamma_2\tau}\int_{\gamma_2 x}^{\gamma_2\gamma_3 x} (\mathrm{dlog}\beta_{\delta_k,p}(z) - \mathrm{dlog}\beta^*_{\delta_{ak}}(z))\\
&= \kappa_{x,(k,\tau)}(\gamma_2,\gamma_3) - \kappa_{x,(ak,\tau)}(\gamma_2,\gamma_3).
\end{aligned}
$$

In particular if a is congruent to 1 modulo f the right hand side of the last equality vanishes. We thus have that $\kappa_{x,(k,\tau)}|_{\Gamma_1} \in Z^2(\Gamma_1, K_p^\times)$ and $\kappa_{x,(k,\tau)}|_{\Gamma} \in Z^2(\Gamma, K_p^\times)$. □

We can now state one the main theorem of the paper.

Theorem 5.2 *The 2-cocycle $\kappa_{x,(k,\tau)}|_{\Gamma_1}$ is a 2-coboundary i.e. there exists a 1-cochain $\rho_{x,(k,\tau)} \in C^1(\Gamma_1, K_p^\times)$ s.t. $d(\rho_{x,(k,\tau)}) = \kappa_{x,(k,\tau)}|_{\Gamma_1}$.*

Proof See theorem 6.2 where we give an explicit splitting of $\kappa_{x,(k,\tau)}|_{\Gamma_1}$. $\square$

The theorem 5.2 will allow us to define points in $K_p^\times$.

Let $\rho_{x,(k,\tau)} \in C^1(\Gamma_1, K_p^\times)$ be such that $d\rho_{x,(k,\tau)} = \kappa_{x,(k,\tau)}$. Let $\Gamma_{1,\tau} = \{\gamma \in \Gamma_1 : \gamma\tau = \tau\}$. By Dirichlet's theorem we can identify $\Gamma_{1,\tau}$ with $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$. Let γ_τ be the unique matrix in $\Gamma_{1,\tau}$ s.t.

$$\gamma_\tau \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \epsilon \begin{pmatrix} \tau \\ 1 \end{pmatrix}$$

where $1 < \epsilon$ is a positive generator of $\Gamma_{1,\tau}/\{\pm 1\}$. When $red(\tau) = v_0$ (see chapter 5 of [Dar04] for the definition of red) we have $\langle \pm\gamma_\tau \rangle = Stab_{\Gamma_1(fN_0)}(Q_\tau(x,y))$, see lemma 9.1. In particular, when $red(\tau) = v_0$, the matrix γ_τ has integral coefficients. We have a similar thing if we replace Γ_1 by Γ.

Proposition 5.8 *The 1-cochain $\rho_{x,(k,\tau)}|_{\Gamma_{1,\tau}}$ modulo $Hom(\Gamma_1, K_p^\times)|_{\Gamma_{1,\tau}}$ does not depend on x.*

Proof Let $x, y \in \Gamma_1(i\infty)$. So we want to show that

$$\rho_{x,(k,\tau)}|_{\Gamma_{1,\tau}} - \rho_{y,(k,\tau)}|_{\Gamma_{1,\tau}} \in Hom(\Gamma_1, K_p^\times)|_{\Gamma_{1,\tau}} = Z^1(\Gamma_1, K_p^\times)|_{\Gamma_{1,\tau}}.$$

This is equivalent to show that

$$(d\rho_{x,(k,\tau)})|_{\Gamma_{1,\tau}} - (d\rho_{y,(k,\tau)})|_{\Gamma_{1,\tau}} = 0.$$

The last equality means exactly that $(\kappa_{x,(k,\tau)} - \kappa_{y,(k,\tau)})|_{\Gamma_{1,\tau}} = 0$.

Let $\gamma_1, \gamma_2 \in \Gamma_1$. We have

$$\begin{aligned}
\kappa_{x,(k,\tau)}(\gamma_1,\gamma_2) - \kappa_{y,(k,\tau)}(\gamma_1,\gamma_2) &= \int_\tau^{\gamma_1\tau}\int_{\gamma_1 x}^{\gamma_1\gamma_2 x} d\log\beta_{\delta_k,p}(z) - \int_\tau^{\gamma_1\tau}\int_{\gamma_1 y}^{\gamma_1\gamma_2 y} d\log\beta_{\delta_k,p}(z) \\
&= \int_\tau^{\gamma_1\tau}\int_{\gamma_1 x}^{\gamma_1 y} d\log\beta_{\delta_k,p}(z) - \int_\tau^{\gamma_1\tau}\int_{\gamma_1\gamma_2 x}^{\gamma_1\gamma_2 y} d\log\beta_{\delta_k,p}(z) \\
&= \int_\tau^{\gamma_1\tau}\int_{\gamma_1 x}^{\gamma_1 y} d\log\beta_{\delta_k,p}(z) - \int_\tau^{\gamma_1\gamma_2\tau}\int_{\gamma_1\gamma_2 x}^{\gamma_1\gamma_2 y} d\log\beta_{\delta_k,p}(z) \\
&\quad + \int_{\gamma_1\tau}^{\gamma_1\gamma_2\tau}\int_{\gamma_1\gamma_2 x}^{\gamma_1\gamma_2 y} d\log\beta_{\delta_k,p}(z)
\end{aligned}$$

Now applying γ_1^{-1} to the bounds of the third term of the last equality (note that $\gamma_1^{-1} \star k = k$) and setting

$$c_{x,y}(\gamma) := \int_{\tau}^{\gamma\tau} \int_{\gamma x}^{\gamma y} d\log \beta_{\delta_k,p}(z) \in C^1(\Gamma_1, K_p^\times),$$

we get

$$\begin{aligned} &= c_{x,y}(\gamma_2) - c_{x,y}(\gamma_1\gamma_2) + c_{x,y}(\gamma_1) \\ &= (dc_{x,y})(\gamma_1, \gamma_2) \end{aligned}$$

We thus have proved that $d(\rho_{x,(k,\tau)} - \rho_{y,(k,\tau)} - c_{x,y}) = 0$ on Γ_1. So $\rho_{x,(k,\tau)} - \rho_{y,(k,\tau)} - c_{x,y} \in Hom(\Gamma_1, K_p^\times)$. Finally evaluating at γ_τ and using the observation that $c_{x,y}(\gamma_\tau) = 0$ proves the claim. $\square$

Remark 5.8 The group $Hom(\Gamma_1, K_p^\times)$ is finite group. This comes from the fact that $(\Gamma_1)^{ab} = \Gamma_1/[\Gamma_1, \Gamma_1]$ is finite, see [Men67] and [Ser70]. We thus have an injection $Hom(\Gamma_1, K_p^\times) \hookrightarrow \mu_{p^2-1}$.

It now makes sense to define the following $K_p^\times$ points:

Definition 5.11 *We define the $K_p^\times$ invariant*

$$u(k,\tau) = u(\delta_k, \tau) := \rho_{x,(k,\tau)}(\gamma_\tau) = \rho_{(k,\tau)}(\gamma_\tau) \in K_p^\times/\mu_{p^2-1},$$

where $\langle \pm\gamma_\tau \rangle = Stab_{\Gamma_1}(\tau)$ such that $c\tau + d > 1$ for $\gamma_\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Remark 5.9 Implicitly in the notation for $\rho_{(k,\tau)}$ and $u(k,\tau)$, a good divisor

$$\delta \in D(N_0, f)^{\langle p \rangle}$$

is fixed. So it is important to keep this in mind!

Proposition 5.9 *Let $(r,\tau), (r',\tau') \in \mathcal{H}_p^{\mathcal{O}}(N_0, f)$ be equivalent then*

$$\rho_{(r,\tau)}(\gamma_\tau) = \rho_{(r',\tau')}(\gamma_{\tau'}).$$

Proof By lemma 5.2 there exists a matrix $\eta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \widetilde{\Gamma}_0$ and integers k, k' such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \lambda \frac{k'}{k} \begin{pmatrix} \tau' \\ 1 \end{pmatrix}$$

where $k \equiv r(mod\ f)$ and $k' \equiv r'(mod\ f)$.

Now we want to exploit the $\widetilde{\Gamma}_0$-invariance of the measures in theorem 5.1(see remark 5.3 for the $\widetilde{\Gamma}_0$-invariance) to show that $\rho_{(k,\tau)}(\gamma_\tau) = \rho_{(k',\tau')}(\gamma_{\tau'})\ (mod\ \mu_{p^2-1})$. Remember that $\gamma_\tau, \gamma_{\tau'} \in \Gamma_1$. We compute. Let $\gamma_1, \gamma_2 \in \Gamma_1$ then

$$\begin{aligned} \kappa_{x,(k',\tau')}(\gamma_1, \gamma_2) &= \int_{\tau'}^{\gamma_1\tau'} \int_{\gamma_1 x}^{\gamma_1\gamma_2 x} \mathrm{dlog}\beta_{\delta_{k'},p}(z) \\ &= \int_{\eta\tau}^{\gamma_1\eta\tau} \int_{\gamma_1 x}^{\gamma_1\gamma_2 x} \mathrm{dlog}\beta_{\delta_{k'},p}(z) \end{aligned}$$

now multiplying the last equality by η^{-1} and using the fact that $\eta^{-1} \star k' \equiv k(mod\ f)$ we find

$$\begin{aligned} &\int_{\tau}^{\eta^{-1}\gamma_1\eta\tau} \int_{\eta^{-1}\gamma_1\eta\eta^{-1}x}^{\eta^{-1}\gamma_1\eta\eta^{-1}\gamma_2\eta\eta^{-1}x} \mathrm{dlog}\beta_{\delta_k,p}(z) = \\ &\int_{\tau}^{\eta^{-1}\gamma_1\eta\tau} \int_{\eta^{-1}\gamma_1\eta\eta^{-1}x}^{\eta^{-1}\gamma_1\eta\eta^{-1}\gamma_2\eta\eta^{-1}x} \mathrm{dlog}\beta_{\delta_k,p}(z) = \\ &\kappa_{\eta^{-1}x,(k,\tau)}(\eta^{-1}\gamma_1\eta, \eta^{-1}\gamma_2\eta). \end{aligned}$$

We thus deduce

$$\kappa_{x,(k',\tau')}(\gamma_1, \gamma_2) = \kappa_{\eta^{-1}x,(k,\tau)}(\eta^{-1}\gamma_1\eta, \eta^{-1}\gamma_2\eta). \tag{5.18}$$

Let $\rho_{\eta^{-1}x,(k,\tau)} \in C^1(\Gamma_1, K_p^\times)$ be a 1-cochain splitting $\kappa_{\eta^{-1}x,(k,\tau)}$ i.e. $d\rho_{\eta^{-1}x,(k,\tau)} = \kappa_{\eta^{-1}x,(k,\tau)}$. Then by proposition 5.8 we have

$$\rho_{\eta^{-1}x,(k,\tau)}|_{\Gamma_{1,\tau}} = \rho_{x,(k,\tau)}|_{\Gamma_{1,\tau}}\ (mod\ \mu_{p^2-1}).$$

If we define

$$\rho_{x,(k',\tau')}(\gamma) := \rho_{\eta^{-1}x,(k,\tau)}(\eta^{-1}\gamma\eta)$$

then using (5.18) one finds that $d(\rho_{\eta^{-1}x,(k',\tau')}) = \kappa_{x,(k',\tau')}$, so this definition makes sense.

Since $\gamma_{\tau'} = \eta\gamma_\tau\eta^{-1}$ we find that

$$\begin{aligned}\rho_{x,(k',\tau')}(\gamma_{\tau'}) &= \rho_{\eta^{-1}x,(k,\tau)}(\eta^{-1}\gamma_{\tau'}\eta) \\ &\overset{(3.6)}{=} \rho_{x,(k,\tau)}(\eta^{-1}\eta\gamma_\tau\eta^{-1}\eta) \ (mod\ \mu_{p^2-1}) \\ &= \rho_{x,(k,\tau)}(\gamma_\tau)\end{aligned}$$

□

Corollary 5.3 *Let $(r,\tau) \in \mathcal{H}_p^{\mathcal{O}}(N_0, f)$. Then invariant $u(r,\tau)$ depends only on the class of (r,τ) modulo $\sim$. Therefore by corollary 5.1*

$$u(r,\tau) = u(\gamma \star r, \gamma\tau)$$

for any $\gamma \in \widetilde{\Gamma}_0$.

We are now ready to formulate the main conjecture.

Conjecture 5.1 *Let $(r,\tau) \in \mathcal{H}_p^{\mathcal{O}}(N_0, f)$. Then*

$$u(r,\tau) \in \mathcal{O}_L[\frac{1}{p}]^\times,$$

where $L = H_{\mathcal{O}}(f\infty)^{\langle Fr_\wp\rangle}$ where $\wp = p\mathcal{O}_K$ and $H_{\mathcal{O}}(f\infty)$ is the abelian extension corresponding to the generalized ideal class group $I_{\mathcal{O}}(f)/Q_{\mathcal{O},1}(f\infty)$. Moreover we have a Shimura reciprocity law. Let

$$rec : G_{L/K} \to I_{\mathcal{O}}(f)/\langle Q_{\mathcal{O},1}(f\infty), p\rangle,$$

then for $\sigma \in G_{L/K}$ we have

$$u(k,\tau)^{\sigma^{-1}} = u(k',\tau')(mod\ \mu_{p^2-1})$$

where $rec(\sigma) \star (k,\tau) = (k',\tau')$. Furthermore, if we let c_∞ denotes the complex conjugation in $G_{L/K}$ then

$$u(r,\tau)^{c_\infty} = u(r,\tau)^{-1}.$$

Remark 5.10 The last equality is in accordance with the fact that the modular symbols defined in remark 5.2 are odd.

Remark 5.11 In [DD06], since the conductor $f = 1$, one is lead to consider various orders of K. However in our case, since f can vary, it is sufficient to consider only the case where $\mathcal{O} = \mathcal{O}_K$.

6 The measures $\widetilde{\mu}\{c_1 \to c_2\}$

6.1 From $\mathbb{P}^1(\mathbb{Q}_p)$ to $(\mathbb{Q}_p \times \mathbb{Q}_p)\backslash\{(0,0)\}$

The main ingredient in showing the splitting of the 2-cocycle $\kappa_{(k,\tau)}$ (see theorem 5.2)consists in the construction of a family of measures on $\mathbb{Q}_p^2\backslash(0,0)$ taking values in $\mathbb{Z}_p$. This family of measures encode the moments of some family of Eisenstein series of varying weight that are $U_{p,a}$-eigenvectors.

Following [DD06] we define $\mathbb{X} := \{(x,y) \in \mathbb{Z}_p^2 : (x,y) = 1\}$. The group $\widetilde{\Gamma}_0$ acts by left translation on $\mathbb{Q}_p^2\backslash(0,0)$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$. There is a $\mathbb{Z}_p^\times$-bundle map

$$\pi : \mathbb{X} \to \mathbb{P}^1(\mathbb{Q}_p) \text{ given by } (x,y) \mapsto x/y.$$

From now on we assume a fixed choice of a good divisor $\delta \in D(N_0, f)^{\langle p \rangle}$. Remember that for $r \in \mathbb{Z}/f\mathbb{Z}$ we have defined

$$E_k(r,\tau) = \left(\frac{(-1)^k(2\pi i)^k}{(k-1)!}\right)^{-1} \sum_{m,n} \frac{e^{2\pi i m r/f}}{(m+nf\tau)^k}$$

for any integer $k \geq 2$. In order to simplify the notation and have better looking formulas we renormalize our Eisenstein series.

Definition 6.1 *For every $j \in (\mathbb{Z}/f\mathbb{Z})^\times/\langle p \rangle$ we set*

$$\widetilde{F}_k(j,z) := -12fF_{k,\delta_j}(z) \qquad \widetilde{F}_{k,p}(j,z) := -12fF_{k,\delta_j,p}(z)$$
$$\widetilde{F}^*_k(j,z) := -12F^*_{k,\delta_j}(z) \qquad \widetilde{F}^*_{k,p}(j,z) := -12F^*_{k,\delta_j,p}(z)$$

Since δ is not appearing in this notation it is important to keep in mind that such a divisor δ is fixed from the beginning. The group $\Gamma_0(fN_0)$ acts transitively on the set

$$\{\widetilde{F}_k(j,z)\}_{j\in(\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle} \text{ and } \{\widetilde{F}_k^*(j,z)\}_{j\in(\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle}$$

Similarly the group $\Gamma_0(pfN_0)$ acts transitively on

$$\{\widetilde{F}_{k,p}(j,z)\}_{j\in(\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle} \text{ and } \{\widetilde{F}_{k,p}^*(j,z)\}_{j\in(\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle}.$$

where the action is induced by the change of variables $\tau \mapsto \gamma\tau$.

We can now state the key theorem which is used to show the splitting of the 2-cocycle.

Theorem 6.1 *There exists a unique collection of p-adic measures on $\mathbb{Q}_p \times \mathbb{Q}_p - (0,0)$ taking values in $\mathbb{Z}_p$ (in fact in $\mathbb{Z}$ see theorem 13.1) indexed by triples $(r,s,j) \in \widetilde{\Gamma}_0(i\infty) \times \widetilde{\Gamma}_0(i\infty) \times (\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle$, denoted by $\widetilde{\mu}_j\{r \to s\}$ such that:*

1. *For every homogeneous polynomial $h(x,y) \in \mathbb{Z}_p[x,y]$ of degree $k-2$,*

$$\int_{\mathbb{X}} h(x,y)d\widetilde{\mu}_j\{r\to s\}(x,y) = (1-p^{k-2})\int_r^s h(z,1)\widetilde{F}_k(j,z)dz$$

2. *For all $\gamma \in \widetilde{\Gamma}_0$ and all open compact $U \subseteq \mathbb{Q}_p^2\backslash(0,0)$,*

$$\widetilde{\mu}_j\{r\to s\}(U) = \widetilde{\mu}_{\gamma\star j}\{\gamma r \to \gamma s\}(\gamma U)$$

3. *(invariance under multiplication by p),*

$$\widetilde{\mu}_j\{r\to s\}(pU) = \widetilde{\mu}_j\{r\to s\}(U)$$

Furthermore the measure satisfies:

4. *For every homogeneous polynomial $h(x,y) \in \mathbb{Z}_p[x,y]$ of degree $k-2$,*

$$\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} h(x,y)d\widetilde{\mu}_j\{r\to s\}(x,y) = \int_r^s h(z,1)\widetilde{F}_{k,p}(j,z)dz$$

Remark 6.1 Note that (3) follows from (2) by taking the matrix $\gamma = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$.

Proof We prove it in section 12.

The family of measures constructed on $\mathbb{P}^1(\mathbb{Q}_p)$ in theorem 5.1 can be thought of as the pushforward of the measures in theorem 6.1. This is the content of the following lemma:

Lemma 6.1 *For all compact open $U \subseteq \mathbb{P}^1(\mathbb{Q}_p)$ we have*

$$\widetilde{\mu}_j\{r \to s\}(\pi^{-1}(U)) = \mu_j\{r \to s\}(U).$$

where $\pi : \mathbb{X} \to \mathbb{P}^1(\mathbb{Q}_p)$ is the $\mathbb{Z}_p^\times$-bundle given by $(x, y) \mapsto \frac{x}{y}$.

Proof Define a collection of measures $\nu_j\{r \to s\}$ on $\mathbb{P}^1(\mathbb{Q}_p)$ by the rule

$$\nu_j\{r \to s\}(U) = \widetilde{\mu}_j\{r \to s\}(\pi^{-1}(U))$$

for any compact open $U \subseteq \mathbb{P}^1(\mathbb{Q}_p)$. We claim that ν_j satisfy the three properties of theorem 5.1. Therefore by uniqueness we deduce that $\nu_j\{r \to s\} = \mu_j\{r \to s\}$.

Let us show the first property. Let $\mathbb{Z}_p \subseteq \mathbb{P}^1(\mathbb{Q}_p)$. Then $\pi^{-1}(\mathbb{Z}_p) = \mathbb{Z}_p \times \mathbb{Z}_p^\times$. We have

$$\begin{aligned}
\nu_j\{r \to s\}(\mathbb{Z}_p) = \widetilde{\mu}_j\{r \to s\}(\pi^{-1}(\mathbb{Z}_p)) &= \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} d\widetilde{\mu}_j\{r \to s\}(x, y) \\
&= \int_r^s \widetilde{F}_{2,p}(j, z)dz \\
&= \frac{1}{2\pi i}\int_r^s \mathrm{dlog}\beta_{\delta_j,p}(z) \\
&= \mu_j\{r \to s\}(\mathbb{Z}_p)
\end{aligned}$$

where the second equality follows from the 4th property of theorem 6.1 and third equality follows $2\pi i\widetilde{F}_{k,p}(j, z) = \mathrm{dlog}(\beta_{\delta_j,p}(z))$.

Let us show the second property.

$$\begin{aligned}
\nu_j\{r \to s\}(\mathbb{P}^1(\mathbb{Q}_p)) = \widetilde{\mu}_j\{r \to s\}(\pi^{-1}(\mathbb{P}^1(\mathbb{Q}_p))) &= \int_{\mathbb{X}} d\widetilde{\mu}_j\{r \to s\}(x, y) \\
&= 0 \\
&= \mu_j\{r \to s\}(\mathbb{P}^1(\mathbb{Q}_p))
\end{aligned}$$

where the third equality follows from the 1st property of theorem 6.1.

It remains to show the third property. We need to show that for all $\gamma \in \widetilde{\Gamma}_0$ one has

$$\nu_{\gamma\star j}\{\gamma r \to \gamma s\}(\gamma U) = \nu_j\{r \to s\}(U) \tag{6.1}$$

for any compact open set $U \subseteq \mathbb{P}^1(\mathbb{Q}_p)$. In order to prove the equality (6.1) we will brake the open set U on smaller open sets on which we have a better control on the p-adic valuation. Before starting note the $\pi^{-1}\gamma(U) \subseteq \mathbb{X}$ but in general $\gamma\pi^{-1}(U) \nsubseteq \mathbb{X}$. In order to show that both sets have the same measure we want to use the third property of theorem 6.1.

Since $\widetilde{\Gamma}_0$ is generated (without taking inverses) by the elements $\{P = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, P^{-1} = \begin{pmatrix} \frac{1}{p} & 0 \\ 0 & 1 \end{pmatrix}, \Gamma_0(fN_0)\}$ it is enough to prove (6.1) when $\gamma \in \Gamma_0(fN_0)$ or $\gamma = P$ or P^{-1}.

We define for $n \in \mathbb{Z}$

$$U_n = U \cap p^n\mathbb{Z}_p^\times = \{u \in U : \frac{1}{p^{n+1}} < |u|_p < \frac{1}{p^{n-1}}\}$$

Clearly the U_n's are disjoint and open. So in order to show equation (6.1) it is enough to show that

$$\nu_{\gamma\star j}\{\gamma r \to \gamma s\}(\gamma U_n) = \nu_j\{r \to s\}(U_n)$$

for any U_n.

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(fN_0)$ and $n < 0$ then

$$\begin{aligned}
\pi^{-1}(\gamma U_n) &= \pi^{-1}\left\{\frac{au+b}{cu+d} \in \mathbb{P}^1(\mathbb{Q}_p) : u \in U_n\right\} \\
&= \mathbb{Z}_p^\times\{((p^{-n}(au+b), p^{-n}(cu+d)) \in \mathbb{X} : u \in U_n)\} \\
&= p^{-n}\mathbb{Z}_p^\times\{(au+b, cu+d) \in (\mathbb{Q}_p \times \mathbb{Q}_p)\backslash(0,0) : u \in U_n\}
\end{aligned}$$

where $\mathbb{Z}_p^\times A := \{(ka_1, ka_2) \in \mathbb{X} : (a_1, a_2) \in A\}$ (the $\mathbb{Z}_p^\times$-saturation) for any subset $A \subseteq \mathbb{Q}_p \times \mathbb{Q}_p$.

On the other hand

$$\begin{aligned}\gamma\pi^{-1}(U_n) &= \gamma\mathbb{Z}_p^\times\{(p^{-n}u, p^{-n}) \in \mathbb{X} : u \in U_n\} \\ &= \mathbb{Z}_p^\times\{(ap^{-n}u + bp^{-n}, cp^{-n}u + dp^{-n}) \in \mathbb{X} : u \in U_n\} \\ &= p^{-n}\mathbb{Z}_p^\times\{(au + b, cu + d) \in (\mathbb{Q}_p \times \mathbb{Q}_p)\backslash(0,0) : u \in U_n\}\end{aligned}$$

In that special case we really get the same sets. The case $n \geq 0$ can be treated in a similar way.

Let us verify it for $\gamma = P^{-1}$ and U_0. We have

$$\begin{aligned}\pi^{-1}(\gamma U_0) &= \pi^{-1}\left\{\frac{u}{p} \in \mathbb{P}^1(\mathbb{Q}_p) : u \in U_0\right\} \\ &= \mathbb{Z}_p^\times\{(u, p) \in \mathbb{X} : u \in U_0\} \\ &= p\mathbb{Z}_p^\times\{(\frac{u}{p}, 1) \in (\mathbb{Q}_p \times \mathbb{Q}_p)\backslash(0,0) : u \in U_0)\}\end{aligned}$$

On the other hand

$$\begin{aligned}\gamma\pi^{-1}(U_0) &= \gamma\mathbb{Z}_p^\times\{(u, 1) \in \mathbb{X} : u \in U_0\} \\ &= \mathbb{Z}_p^\times\{(\frac{u}{p}, 1) \in (\mathbb{Q}_p \times \mathbb{Q}_p)\backslash(0,0) : u \in U_0\}\end{aligned}$$

By the third property we conclude that $\widetilde{\mu}_j\{r \to s\}(\gamma\pi^{-1}(U_0)) = \widetilde{\mu}_j\{r \to s\}(\pi^{-1}(\gamma U_0))$.

The remaining cases can be treated in a similar way. □

6.2 Splitting of the 2-cocycle

We are now ready to prove the splitting of the 2-cocycle. We show the splitting by giving an explicit formula for the 1-cochain $\rho_{x,(k,\tau)} \in C^1(\Gamma_1, K_p^\times)$ where $(k, \tau) \in \mathcal{H}_p^{\mathcal{O}}(N_0, f)$ and $x \in \Gamma_0(fN_0)(i\infty)$. We have $K_p^\times = p^{\mathbb{Z}} \times \mathcal{O}_{K_p}^\times$. The map

$$K_p^\times \to \mathbb{Z} \times \mathcal{O}_{K_p}^\times, \quad x \mapsto (ord_p(x), u_p(x) := \frac{x}{p^{ord_p(x)}})$$

is an isomorphism. Therefore in order to determine $\rho_{(k,\tau)}$ it is enough to give explicit formulas for $ord_p(\rho_{(k,\tau)})$ and $u_p(\rho_{(k,\tau)})$.

To each $v \in \mathcal{V}(\mathcal{T})$ we associate a well defined partial modular symbol $m_v\{r \to s\}$ on the set of cusps $\Gamma_0(fN_0)(i\infty)$ taking values in the set of $\widetilde{\Gamma}_0$-invariant measures on $\mathbb{P}^1(\mathbb{Q}_p)$. We define

$$m_{v_0,k}\{r \to s\} := \frac{1}{2\pi i}\int_r^s \mathrm{dlog}\beta_{\delta_k}(z), \qquad m_{\gamma v,\gamma\star k}\{\gamma r \to \gamma s\} = m_{v,k}\{r \to s\}.$$

for all $v \in \mathcal{V}(\mathcal{T})$, $\gamma \in \widetilde{\Gamma}_0$, $k \in (\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle$ and $r, s \in \Gamma_0(fN_0)(i\infty)$. Note that the assignment $v \mapsto m_{v,k}\{r \to s\}$ satisfies the following harmonicity property:

$$\sum_{d(v',v)=1} m_{v',k}\{r \to s\} = (p+1)m_{v,k}\{r \to s\}.$$

The latter equality comes from the fact that $\widetilde{F}_2(k,z)$ is an eigenvector with eigenvalue $(1+p)$ for the Hecke operator $T_2(p)$.

Lemma 6.2 *Let $(k,\tau) \in \mathcal{H}_p^{\mathcal{O}}(N_0,f)$ with $red(\tau) = v$. Let $\gamma \in \Gamma_1$ then*

$$ord_p(\rho_{x,(k,\tau)}(\gamma)) = m_{v,k}\{x \to \gamma x\}. \tag{6.2}$$

Proof Same proof as lemma 3.5 of [DD06]. □

Remark 6.2 Using proposition 5.4 we get an explicit formula for $m_{v,k}\{x \to \gamma x\}$ it terms of Dedekind sums.

Theorem 6.2 *Let $\gamma \in \Gamma_1$ then*

$$\rho_{(k,\tau)}(\gamma) = \prod_{\mathbb{X}} (x - \tau y)d\widetilde{\mu}_k\{i\infty \to \gamma(i\infty)\}(x,y). \tag{6.3}$$

Note that the multiplicative integral makes sense since $\widetilde{\mu}_k\{i\infty \to \gamma(i\infty)\}$ by theorem 13.1 takes values in $\mathbb{Z}$.

Proof We have decided to reproduce the proof of of proposition 4.6 of [DD06] because we find the calculation interesting.

Let $\gamma_1, \gamma_2 \in \Gamma_1$. Note that $\mathbb{X}$ and $\gamma_i\mathbb{X}$ $(i = 1,2)$ are both fundamental domains for

multiplication by p on $\mathbb{Q}_p^2\backslash\{(0,0)\}$. We have

$$\begin{aligned}(d\rho_{(k,\tau)})(\gamma_1,\gamma_2) &= \frac{\rho_{(k,\tau)}(\gamma_1)\rho_{(k,\tau)}(\gamma_2)}{\rho_{(k,\tau)}(\gamma_1\gamma_2)} \\ &= \frac{\oint_{\mathbb{X}}(x-y\tau)d\widetilde{\mu}_k\{i\infty\to\gamma_1 i\infty\}(x,y)\oint_{\mathbb{X}}(x-y\tau)d\widetilde{\mu}_k\{i\infty\to\gamma_2 i\infty\}(x,y)}{\oint_{\mathbb{X}}(x-y\tau)d\widetilde{\mu}_k\{i\infty\to\gamma_1\gamma_2 i\infty\}(x,y)} \\ &= \frac{\oint_{\mathbb{X}}(x-y\tau)d\widetilde{\mu}_k\{i\infty\to\gamma_2 i\infty\}(x,y)}{\oint_{\mathbb{X}}(x-y\tau)d\widetilde{\mu}_k\{\gamma_1 i\infty\to\gamma_1\gamma_2 i\infty\}(x,y)} \\ &= \frac{\oint_{\mathbb{X}}(x-y\tau)d\widetilde{\mu}_k\{\gamma_1 i\infty\to\gamma_1\gamma_2 i\infty\}(a,b)}{\oint_{\mathbb{X}}(x-y\tau)d\widetilde{\mu}_k\{\gamma_1 i\infty\to\gamma_1\gamma_2 i\infty\}(x,y)}\end{aligned}$$

where $\gamma_1\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}a\\b\end{pmatrix}$. For the latter equality we have used the fact that

$$\widetilde{\mu}_k\{c_1\to c_2\}(x,y)=\widetilde{\mu}_k\{\gamma c_1\to\gamma c_2\}(\gamma(x,y)).$$

Let $\gamma_1=\begin{pmatrix}A&B\\C&D\end{pmatrix}$, then we have $Da-Bb=x$ and $-Ca+Ab=y$. Therefore $x-\tau y=a(D+C\tau)+b(A\tau+B)=(C\tau+D)(a-b\gamma_1\tau)$. From this we deduce

$$\begin{aligned}&= \oint_{\mathbb{X}}\frac{(x-y\gamma_1\tau)}{(x-y\tau)}\widetilde{\mu}_k\{\gamma_1 i\infty\to\gamma_1\gamma_2 i\infty\}(x,y) \qquad (6.4)\\ &= \oint_{\mathbb{P}^1(\mathbb{Q}_p)}\left(\frac{t-\gamma_1\tau}{t-\tau}\right)\mu_k\{\gamma_1 i\infty\to\gamma_1\gamma_2 i\infty\}(t) \qquad (6.5)\\ &= \kappa_{(k,\tau)}(\gamma_1,\gamma_2)\end{aligned}$$

where for (6.4) we use the fact that the total measure on $\mathbb{X}$ is zero and for (6.5) the fact that $\pi_*\widetilde{\mu}=\mu$. $\square$

Remark 6.3 Note that using only the fact that $\widetilde{\mu}$ is $\mathbb{Z}_p$-valued one can still prove the splitting of $\kappa_{(k,\tau)}$. One can do the same calculation as theorem 6.2 by replacing the multiplicative integral by its additive integral (after having taken $\log_p$). Combining the additive version of (6.3) with (6.2) provides a explicit splitting of $\kappa_{(k,\tau)}$. Using this approach doesn't use the integrality of $\widetilde{\mu}$ but only the fact that it is a bounded distribution, i.e. a measure.

7 Archimedean zeta functions attached to totally real number fields

7.1 Zeta functions twisted by additive characters

For this section we let K be an arbitrary totally real number field.

Let K be a totally real number field of degree r. Let $\{\sigma_1, \ldots, \sigma_r\}$ be a complete set of real embeddings of K. Let $\mathfrak{d}$ be the different of K and $\Delta = N_{K/\mathbb{Q}}(\mathfrak{d})$ the discriminant of K. Let $\mathfrak{f}$ be an integral ideal of K. Let $\mathcal{O}_K(\mathfrak{f}\infty)^\times$ be the group of totally positive units of $\mathcal{O}_K$ that are congruent to 1 modulo $\mathfrak{f}$. Let w be a sign character of K i.e. a product of a subset of the characters

$$sign \circ \sigma_i : K^\times \to \mathbb{R}^\times \to \{\pm 1\}.$$

Let $\mathfrak{c}$ be an integral ideal of K coprime to $\mathfrak{f}$. Following [Sie68] we define

$$\Psi(\frac{\mathfrak{c}}{\mathfrak{d}\mathfrak{f}}, w, s) = N(\frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}})^s \sum_{\mathcal{O}_K(\mathfrak{f}\infty)^\times \backslash \{0 \neq \mu \in \frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}}\}} w(\mu) \frac{e^{2\pi i Tr(\mu)}}{|N(\mu)|^s}, \quad Re(s) > 1$$

where Tr and N are the usual trace and norm functions on K down to $\mathbb{Q}$. Note that for any $\epsilon \in \mathcal{O}_K(\mathfrak{f}\infty)^\times$ and $\mu \in \frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}}$ we have $\mu - \epsilon\mu \in \mathfrak{c}\mathfrak{d}^{-1} \subseteq \mathfrak{d}^{-1}$ thus $Tr(\mu - \epsilon\mu) \in \mathbb{Z}$. So the summation doesn't depend on the choice of representatives of $\{0 \neq \mu \in \frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}}\}$ modulo $\mathcal{O}_K(\mathfrak{f}\infty)^\times$.

Let $\rho \in K$ be such that $\rho\mathfrak{c} \subseteq \mathcal{O}_K$ and $(\rho\mathfrak{c}, \mathfrak{f}) = 1$, then a straight forward calculation shows that

$$\Psi(\rho\frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}}, w, s) = w(\rho)N(\frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}})^s \sum_{\mathcal{O}_K(\mathfrak{f}\infty)^\times \backslash \{0 \neq \mu \in \frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}}\}} w(\mu) \frac{e^{2\pi i Tr(\rho\mu)}}{|N(\mu)|^s}. \tag{7.1}$$

From this it follows that the first entry of Ψ depends only on the narrow ray class modulo $\mathfrak{f}$ in the sense that if $\mathfrak{a}, \mathfrak{b} \in I_{\mathcal{O}_K}(\mathfrak{f})$, $\rho \in K$, $\rho \equiv 1 (mod\, \mathfrak{f})$ and $\rho \gg 0$ is such that $\rho\mathfrak{a} = \mathfrak{b}$ then

$$\Psi(\frac{\mathfrak{a}}{\mathfrak{f}\mathfrak{d}}, w, s) = \Psi(\frac{\mathfrak{b}}{\mathfrak{f}\mathfrak{d}}, w, s). \tag{7.2}$$

Note that if there exists a $\rho \in \mathcal{O}_K^\times$ congruent to 1 modulo $\mathfrak{f}$ such that $w(\rho) = -1$ we find using (7.1) that $\Psi(\frac{\mathfrak{c}}{\mathfrak{f}\partial}, w, s) = 0$. The existence of such units should be avoided.

Remark 7.1 One can relate the zeta functions $\Psi(\frac{\mathfrak{a}}{\partial\mathfrak{f}}, w, s)$ to classical zeta functions $L(\chi, s)$ where χ is a character of the narrow ideal class group of conductor $\mathfrak{f}$. In order to do so we need to recall some properties of finite Hecke characters.

Definition 7.1 *We define*

(1) $I_{\mathcal{O}_K}(\mathfrak{f}) = \{$*Integral ideals of* $\mathcal{O}_K$ *which are coprime to* $\mathfrak{f}\}$

(2) $I_K(\mathfrak{f}) = \{$*fractional ideals of* $\mathcal{O}_K$ *which are coprime to* $\mathfrak{f}\}$

(3) $P_{K,1}(\mathfrak{f}\infty) = \{\alpha\mathcal{O}_K \subseteq K : \alpha \in K, \alpha \equiv 1 (mod\ f), \alpha \gg 0\}$

We identify the quotients $I_{\mathcal{O}_K}(\mathfrak{f})/P_{K,1}(\mathfrak{f}\infty)$ and $I_K(\mathfrak{f})/P_{K,1}(\mathfrak{f}\infty)$ with the narrow ideal class group of conductor $\mathfrak{f}$.

We have the following short exact sequence

$$1 \longrightarrow (\mathcal{O}_K/\mathfrak{f})^\times/(\mathcal{O}_K^\times(\infty)\ (mod\ \mathfrak{f})) \xrightarrow{\iota} I_K(\mathfrak{f})/P_{K,1}(\mathfrak{f}\infty) \longrightarrow I_K(1)/P_K(\infty) \longrightarrow 1,$$

where $\iota(a(mod\ \mathfrak{f})) = a\mathcal{O}_K$ where a is chosen to be totally positive. From this short exact sequence we see that every character $\chi : I_K(\mathfrak{f})/P_{K,1}(\mathfrak{f}\infty) \to S^1$ can be pulled back to a character

$$\chi_f := \chi \circ \iota : (\mathcal{O}_K/\mathfrak{f})^\times/(\mathcal{O}_K^\times(\infty)\ (mod\ \mathfrak{f})) \to S^1$$

where the subscript f stands for finite.

Let $\alpha \in K^\times$ be coprime to $\mathfrak{f}$ then we define $\chi_\infty(\alpha) := \chi((\alpha))/\chi_f(\alpha)$. When α is totally positive we have $\chi((\alpha)) = \chi_f(\alpha)$ therefore $\chi_\infty(\alpha) = 1$. However if α is not totally positive and β is a totally positive element such that $\alpha \equiv \beta\ (mod\ \mathfrak{f})$ then $\chi_f(\alpha) = \chi((\beta))$ therefore $\chi_\infty(\alpha) = \chi((\frac{\alpha}{\beta}))$. In thus follows that χ_∞ is a sign character since $\left(\frac{\alpha}{\beta}\right)^2$ is a totally positive element congruent to 1 modulo $\mathfrak{f}$. Thus every character

$$\chi : I_K(\mathfrak{f})/P_{K,1}(\mathfrak{f}\infty) \to S^1$$

when restricted to principal ideals (α) coprime to $\mathfrak{f}$ can be written uniquely as $\chi = \chi_\infty \chi_f$ where $\chi_\infty : (\mathbb{R} \otimes K)^\times \to S^1$ and $\chi_f : (\mathcal{O}_K/\mathfrak{f})^\times/(\mathcal{O}_K^\times(\infty)(mod\, \mathfrak{f})) \to S^1$. If we think of χ_f as a character on $(\mathcal{O}_K/\mathfrak{f})^\times$ then the pair of characters (χ_∞, χ_f) satisfies the identity

$$(*) \qquad \chi_f(\epsilon)\chi_\infty(\epsilon) = 1 \; \forall \epsilon \in \mathcal{O}_K^\times.$$

Conversely for every pair of characters $(w, \eta) \in (\widehat{(\mathbb{R} \otimes K)^\times}, \widehat{(\mathcal{O}_K/\mathfrak{f})^\times})$ satisfying $(*)$ there exists a lift $\psi : I_K(\mathfrak{f})/P_{K,1}(\mathfrak{f}\infty) \to S^1$ (the number of lifts is exactly h_K^+, the narrow class group of K) such that $\psi_f = \eta$ and $\psi_\infty = w$.

Let us assume that $\mathcal{O}_K^\times(\mathfrak{f}) = \mathcal{O}_K^\times(\mathfrak{f}\infty)$. In this case we have that $P_{K,1}(\mathfrak{f})/P_{K,1}(\mathfrak{f}\infty) \simeq (\mathbb{Z}/2)^r$. So the index of the wide ray class field of conductor $\mathfrak{f}$ in the narrow ray class field of conductor $\mathfrak{f}$ is 2^r. In order to simplify the notation we let $G_{\mathfrak{f}\infty} = I_K(\mathfrak{f})/P_{K,1}(\mathfrak{f}\infty)$ and $G_\mathfrak{f} = I_K(\mathfrak{f})/P_{K,1}(\mathfrak{f})$. We identify $\widehat{G}_\mathfrak{f}$ as a subgroup of $\widehat{G}_{\mathfrak{f}\infty}$ via π^* where

$$\pi : G_{\mathfrak{f}\infty} \to G_\mathfrak{f}$$

is the natural projection. Let $\eta_1, \ldots, \eta_r$ be generators of the group of characters of

$$P_{K,1}(\mathfrak{f})/P_{K,1}(\mathfrak{f}\infty)$$

defined in such a way that for $\mathfrak{a} \in P_{K,1}(\mathfrak{f})$ we let $\eta_i(\mathfrak{a}) = w_i(\alpha) = sign \circ \sigma_i(\alpha)$ where α is a generator of $\mathfrak{a}$ congruent to 1 modulo f. The η_i's are well defined because of the assumption on the units. For every i take an arbitrary lift of η_i to $I_K(\mathfrak{f})/P_{K,1}(\mathfrak{f}\infty)$ and denote it again by η_i. By construction $(\eta_i)_\infty = w_i = sign \circ \sigma_i$. It is easy to see that the group generated by the η_i's is a complete set of representatives of $\widehat{G}_{\mathfrak{f}\infty}$ modulo $\widehat{G}_\mathfrak{f}$. We thus have the disjoint union

$$\widehat{G}_{\mathfrak{f}\infty} = \bigcup_i \eta_i \widehat{G}_\mathfrak{f}.$$

Note that $\widehat{G}_\mathfrak{f}$ corresponds precisely to the set of characters $\chi \in \widehat{G}_{\mathfrak{f}\infty}$ such that $\chi_\infty = 1$.

7.2 Gauss sums for Hecke characters and Dirichlet characters

Let $\chi \in \widehat{G}_{\mathfrak{f}\infty}$ be a Hecke character and $\gamma \in K$ be such that $(\gamma) = \frac{\mathfrak{a}}{\partial \mathfrak{f}}$ where $(\mathfrak{a}, \mathfrak{f}) = 1$. For a $\xi \in \mathcal{O}_K$ we define

$$g_\gamma(\chi, \xi) := \overline{\chi_f}(\gamma) \sum_{\rho \ (mod\ \mathfrak{f})} \overline{\chi_f}(\rho) e^{2\pi i Tr(\gamma\rho\xi)}.$$

We define $\chi_f(\rho) = 0$ if $(\rho, \mathfrak{f}) \neq 1$. It is easy to see that $g_\gamma(\chi, \xi)$ doesn't depend on γ, so from now on we omit the subscript γ. When ξ is coprime to $\mathfrak{f}$ we have

$$g(\chi, \xi) = \chi_f(\xi) g(\chi, 1). \tag{7.3}$$

Furthermore when χ is primitive (7.3) remains valid for ξ not coprime to $\mathfrak{f}$ since $g(\chi, \xi) = 0$.

We also define Gauss sums for Dirichlet characters $\chi : (\mathcal{O}_K/\mathfrak{m})^\times \to S^1$ where $\mathfrak{m}$ is some integral and $y \in \frac{\mathcal{O}_K}{\mathfrak{m}\partial}$. We define the Gauss sum

$$\tau(\chi, y) := \sum_{\substack{x (mod\ \mathfrak{m}) \\ (x, \mathfrak{m}) = 1}} \bar{\chi}(x) e^{2\pi i Tr(xy)}.$$

Let $\chi \in \widehat{G}_{\mathfrak{f}\infty}$ be a Hecke character and χ_f be the Dirichlet character corresponding to the finite part of χ, then it is easy to see that

$$g_\gamma(\chi, 1) = \overline{\chi_f}(\gamma) \tau(\chi_f, \gamma)$$

where $(\gamma) = \frac{\mathfrak{a}}{\partial \mathfrak{f}}$.

7.3 Relation between $\Psi(\frac{\mathfrak{a}}{\partial \mathfrak{f}}, \chi_\infty, s)$ and $L(\chi, s)$

In this subsection we would like to relate the functions $\Psi(\frac{\mathfrak{a}}{\partial \mathfrak{f}}, \chi_\infty, s)$ to classical Artin L-functions $L(\chi, s)$ where χ is a primitive character. We essentially reproduce a proof that can be found in [Sie68].

Proposition 7.1 *Let $\chi : I_K(\mathfrak{f})/P_{K,1}(\mathfrak{f}\infty) \to S^1$ be a primitive character then*

$$\sum_{c \in I_K(\mathfrak{f})/P_{K,1}(\mathfrak{f})} \bar{\chi}(\mathfrak{a}_c) \Psi\left(\frac{\mathfrak{a}_c}{\partial \mathfrak{f}}, \chi_\infty, s\right) = g(\chi, 1) L(\chi, s)$$

where $\mathfrak{a}_c \in c$ is any integral ideal.

Proof We first extend χ to $I_K(1)$ by setting $\chi(\mathfrak{a}) = 0$ when $(\mathfrak{a}, \mathfrak{f}) \neq 1$. We have

$$L(s,\chi) = \sum_{\mathfrak{a}\subseteq\mathcal{O}_K} \frac{\chi(\mathfrak{a})}{\mathbf{N}(\mathfrak{a})^s}$$
$$= \sum_{a^{-1}\in I_K(1)/P_{K,1}(1)} \sum_{\substack{\mathfrak{b}\in a^{-1} \\ \mathfrak{b}\ integral}} \frac{\chi(\mathfrak{b})}{\mathbf{N}(\mathfrak{b})^s}.$$

For every class a we fix an integral ideal $\mathfrak{a}_a \in a$. We have a natural bijection between the elements $\mu \in \mathfrak{a}_a$ modulo $\mathcal{O}_K^\times$ and integral ideals $\mathfrak{b} \in a^{-1}$ given by $\mu \mapsto \mu\mathfrak{a}_a^{-1} \in a^{-1}$. Therefore

$$\sum_{a^{-1}\in I_K(1)/P_{K,1}(\infty)} \sum_{\{0\neq\mu\in\mathfrak{a}_a\}/\mathcal{O}_K^\times} \frac{\chi((\mu)\mathfrak{a}_a^{-1})}{\mathbf{N}((\mu)\mathfrak{a}_a^{-1})^s} =$$

$$\sum_{a^{-1}\in I_K(1)/P_{K,1}(1)} \mathbf{N}(\mathfrak{a}_a)^s\bar{\chi}(\mathfrak{a}_a) \sum_{\{0\neq\mu\in\mathfrak{a}_a\}/\mathcal{O}_K^\times} \frac{\chi_\infty(\mu)\chi_f(\mu)}{|\mathbf{N}(\mu)|^s} = \tag{7.4}$$

Note that if $(\mu, \mathfrak{f}) \neq 1$ then $\chi_f(\mu) = 0$. Remember that

$$g(\chi, 1) = g_\gamma(\chi, 1) = \overline{\chi_f}(\mathfrak{a})\tau(\chi_f, \gamma)$$

where $(\gamma) = \frac{\mathfrak{g}}{\partial\mathfrak{f}}$ with $(\mathfrak{g}, \mathfrak{f}) = 1$. For $\mu \in \mathcal{O}_K$ coprime to $\mathfrak{f}$ we have

$$\chi_f(\mu)\tau(\chi_f, \gamma) = \tau(\chi_f, \mu\gamma) \tag{7.5}$$

substituting in (7.4) we get

$$= \sum_{a^{-1}\in I_K(1)/P_{K,1}(\infty)} \mathbf{N}(\mathfrak{a}_a)^s\bar{\chi}(\mathfrak{a}_a) \sum_{\{0\neq\mu\in\mathfrak{a}_a,(\mu,\mathfrak{f})=1\}/\mathcal{O}_K^\times} \frac{\tau(\chi_f, \mu\gamma)\chi_\infty(\mu)}{\tau(\chi_f, \gamma)|\mathbf{N}(\mu)|^s}. \tag{7.6}$$

Now using the assumption that χ_f is primitive we can remove the restriction $(\mu, \mathfrak{f}) = 1$ under the last summation of (7.6) since (7.5) also holds for μ not coprime to $\mathfrak{f}$.

Rearranging a bit (7.6) we get

$$= \sum_{a^{-1}\in I_K(1)/P_{K,1}(1)} \mathbf{N}(\mathfrak{a}_a)^s \bar{\chi}(\mathfrak{a}_a) \sum_{\alpha \pmod{\mathfrak{f}}} \sum_{\substack{\{0\neq\mu\in\mathfrak{a}_a,(\mu,\mathfrak{f})=1 \\ ,\mu\equiv\alpha(mod\ \mathfrak{f})\}/\mathcal{O}_K^\times}} \frac{\tau(\chi_f,\mu\gamma)\chi_\infty(\mu)}{\tau(\chi_f,\gamma)|\mathbf{N}(\mu)|^s}$$

$$= \frac{1}{A_\mathfrak{f}} \sum_{a^{-1}\in I_K(1)/P_{K,1}(1)} \mathbf{N}(\mathfrak{a}_a)^s \bar{\chi}(\mathfrak{a}_a) \sum_{\alpha \pmod{\mathfrak{f}}} \sum_{\substack{\{0\neq\mu\in\mathfrak{a}_a,(\mu,\mathfrak{f})=1 \\ ,\mu\equiv\alpha(mod\ \mathfrak{f})\}/\mathcal{O}_K(\mathfrak{f})^\times}} \frac{\tau(\chi_f,\mu\gamma)\chi_\infty(\mu)}{\tau(\chi_f,\gamma)|\mathbf{N}(\mu)|^s}$$

$$= \frac{1}{A_\mathfrak{f}\tau(\chi_f,\gamma)} \sum_{a^{-1}\in I_K(1)/P_{K,1}(1)} \mathbf{N}(\mathfrak{a}_a)^s \bar{\chi}(\mathfrak{a}_a)$$

$$\cdot \sum_{\alpha \pmod{\mathfrak{f}}} \sum_{\substack{\{0\neq\mu\in\mathfrak{a}_a,(\mu,\mathfrak{f})=1 \\ ,\mu\equiv\alpha(mod\ \mathfrak{f})\}/\mathcal{O}_K^\times(\mathfrak{f}\infty)}} \sum_{\rho \pmod{\mathfrak{f}}} \bar{\chi}_f(\rho) e^{2\pi i Tr(\mu\rho\gamma)} \frac{\chi_\infty(\mu)}{|\mathbf{N}(\mu)|^s}$$

where in the second equality every μ is counted $A_\mathfrak{f} := |\mathcal{O}_K^\times \ (mod\ \mathfrak{f})|$ times and in the last summation we have used the fact that $\mathcal{O}_K(\mathfrak{f})^\times = \mathcal{O}_K(\mathfrak{f}\infty)^\times$. Rearranging a bit the latter expression we get

$$= \frac{1}{A_\mathfrak{f}\tau(\chi_f,\gamma)} \sum_{a^{-1}\in I_K(1)/P_{K,1}(1)} \sum_{\rho \pmod{\mathfrak{f}}} \mathbf{N}(\mathfrak{a}_a)^s \bar{\chi}(\mathfrak{a}_a)\mathbf{N}(\rho)^s \bar{\chi}((\rho))$$

$$\cdot \sum_{\alpha(mod\ \mathfrak{f})} \sum_{\substack{\{0\neq\mu\in\mathfrak{a}_a,(\mu,\mathfrak{f})=1 \\ ,\mu\equiv\alpha(mod\ \mathfrak{f})\}/\mathcal{O}_K(\mathfrak{f}\infty)^\times}} e^{2\pi i Tr(\mu\rho\gamma)} \frac{\chi_\infty(\mu\rho)}{|\mathbf{N}(\mu\rho)|^s}$$

$$= \frac{1}{A_\mathfrak{f}\tau(\chi_f,\gamma)} \sum_{a^{-1}\in I_K(1)/P_{K,1}(1)} \sum_{\rho \pmod{\mathfrak{f}}} \mathbf{N}(\mathfrak{a}_a)^s \bar{\chi}(\mathfrak{a}_a)\mathbf{N}(\rho)^s \bar{\chi}((\rho))$$

$$\cdot \sum_{\alpha(mod\ \mathfrak{f})} \sum_{\substack{\{0\neq\mu\in\rho\mathfrak{a}_a,(\mu,\mathfrak{f})=1 \\ ,\mu\equiv\alpha\rho(mod\ \mathfrak{f})\}/\mathcal{O}_K(\mathfrak{f}\infty)^\times}} e^{2\pi i Tr(\mu\gamma)} \frac{\chi_\infty(\mu)}{|\mathbf{N}(\mu)|^s}$$

$$= \frac{\chi_f(\mathfrak{g})}{A_\mathfrak{f}\tau(\chi_f,\gamma)} \sum_{a^{-1}\in I_K(1)/P_{K,1}(1)} \sum_{\rho \pmod{\mathfrak{f}}} \bar{\chi}(\rho\mathfrak{a}_a\mathfrak{g})\Psi\left(\rho\mathfrak{a}_a\gamma,\chi_\infty,s\right)$$

$$= \frac{1}{g(\chi,1)} \sum_{a\in I_K(\mathfrak{f})/P_{K,1}(\mathfrak{f})} \bar{\chi}(\mathfrak{a}_a\mathfrak{g})\Psi\left(\mathfrak{a}_a\gamma,\chi_\infty,s\right)$$

The last equality comes from the observation that the set of ideals $\{\rho\mathfrak{a}_a\}$ covers every element of $I_K(\mathfrak{f})/P_{K,1}(\mathfrak{f})$ exactly $A_\mathfrak{f}$ times. $\square$

7.4 Partial zeta functions $\zeta(\mathfrak{a}^{-1}, \mathfrak{f}, w, s)$ as the dual of $\Psi(\frac{\mathfrak{a}}{\mathfrak{d}\mathfrak{f}}, w, s)$

Let K be any totally real number field. Let $w : (K \otimes_{\mathbb{R}} \mathbb{Q})^{\times} \to \{\pm 1\}$ be a sign character. Let $\mathfrak{f}$ be an integral ideal of K and $\mathfrak{d}$ be the different. For a fractional ideal $\mathfrak{a}$ coprime to $\mathfrak{f}$ we define

$$\zeta(\mathfrak{a}, \mathfrak{f}, w, s) := \mathbf{N}(\mathfrak{a})^s \sum_{\mathcal{O}_K(\mathfrak{f}\infty)^{\times} \backslash \{\mu \in \mathfrak{a}, \mu \equiv 1 (mod\ \mathfrak{f})\}} \frac{w(\mu)}{|\mathbf{N}(\mu)|^s}.$$

Note that both functions depend only on the narrow class of $\mathfrak{a}$ modulo $\mathfrak{f}$. Observe also that if $\mathfrak{b} = \lambda\mathfrak{a}$ are integral ideals coprime to $\mathfrak{f}$ then

$$\zeta(\mathfrak{b}, \mathfrak{f}, w, s) = w(\lambda)\mathbf{N}(\mathfrak{a})^s \sum_{\mathcal{O}_K(f\infty)^{\times} \backslash \{0 \neq \mu \in \mathfrak{a}, \mu \equiv \lambda (mod\ \mathfrak{f})\}} \frac{w(\mu)}{|\mathbf{N}(\mu)|^s}. \tag{7.7}$$

Let $\{a_i\}_{i=1}^n$ be the parity of w then we define

$$F_w(s) := |d_K|^{s/2} \pi^{-ns/2} \prod_{i=1}^{n} \Gamma(\frac{s + a_i}{2})$$

where $n = [K : \mathbb{Q}]$ and d_K is the discriminant of K.

Theorem 7.1 *We have the following functional equation*

$$F_w(s)\Psi(\frac{\mathfrak{a}}{\mathfrak{f}\mathfrak{d}}, w, s) = i^{Tr(w)} F_w(1-s)\mathbf{N}(\mathfrak{f})^{1-s}\zeta(\mathfrak{a}^{-1}, \mathfrak{f}, w, 1-s) \tag{7.8}$$

where $Tr(w) = \sum_i a_i$.

Proof The proof follows Hecke's classical method and relies on the functional equation of the generalized theta function which is a direct consequence of the Poisson summation formula. In order to prove (7.8) one needs to introduce heavy notation and also the notion of ideal numbers that repair the failure of not having a principal ring. For these reasons we have decided to only prove (7.8) in the case where K is real quadratic using a nice trick of Hecke which simplifies the argument. Moreover this second proof is better suited for the applications we have in mind since it involves an integral of a classical Eisenstein series against a suitable power of a quadratic form. □

Remark 7.2 One can use theorem 7.1 in conjunction with the well known functional equation for $L(s,\chi)$ to give another proof of proposition 7.1. However some difficulties arise since in order to express the partial zeta functions $\zeta(\mathfrak{a},w,\mathfrak{f},s)$ as linear combinations of $L(s,\chi)$ one needs to deal with non primitive characters χ of $I_K(\mathfrak{f})/P_{K,1}(\mathfrak{f}\infty)$.

Applying the last theorem in the case where K is real quadratic and letting $\mathfrak{a}=a\Lambda_\tau$ and $\mathfrak{f}=(f)$ one sees that

$$F_{w_1}(s)\Psi(\frac{a\Lambda_\tau}{f\sqrt{D}},w_1,s)=-F_{w_1}(1-s)\mathbf{N}(f)^{1-s}\zeta((a\Lambda_\tau)^{-1},f,w_1,1-s) \tag{7.9}$$

and

$$F_{w_0}(s)\Psi(\frac{a\Lambda_\tau}{f\sqrt{D}},w_0,s)=F_{w_0}(1-s)\mathbf{N}(f)^{1-s}\zeta((a\Lambda_\tau)^{-1},f,w_0,1-s).$$

Note that if $\Lambda_\tau\Lambda_{\tau^\sigma}=(\frac{1}{A})$ then $(a\Lambda_\tau)^{-1}=\frac{A}{a}\Lambda_{\tau^\sigma}$.

We conclude the end of this section by discussing some parity conditions on special values of partial zeta functions at negative integers.

For integers $m\geq 2$ which are even the quantity

$$\frac{F_w(m)}{F_w(1-m)} \tag{7.10}$$

is equal to 0 unless $a_i=0$ for all i. Similarly for integers $m\geq 1$ odd the quantity

$$\frac{F_w(m)}{F_w(1-m)} \tag{7.11}$$

is 0 unless $a_i=1$ for all i. We define $w_0=1$ and $w_1=sign(N_{K/\mathbb{Q}})$. We thus see that for integers $m\geq 1$ the quantity $\zeta(\mathfrak{a},\mathfrak{f},w,1-m)$ can be different than 0 only when $w=w_0$ and m is even or $w=w_1$ and m is odd.

Let

$$\zeta(\mathfrak{a},\mathfrak{f}\infty,s):=N(\mathfrak{a})^s\sum_{\mathcal{O}_K^\times(\mathfrak{f}\infty)\backslash\{\lambda\in\mathfrak{a},\lambda\equiv 1(mod\ \mathfrak{f}),\lambda\gg 0\}}\frac{1}{|\mathbf{N}(\lambda)|^s}. \tag{7.12}$$

We simply call those partial zeta functions. Note here that the sum is restricted to totally positive elements.

Let $\sigma_1, \ldots, \sigma_n$ be the different embeddings of K and let $\{a_i\}_{i=1}^n$ be such that $a_i \in \{0,1\}$. We define

$$\zeta(\mathfrak{a}, \mathfrak{f}\infty, \{a_i\}_{i=1}^n, s) = \mathbf{N}(\mathfrak{a})^s \sum_{\substack{\{\lambda \in \mathfrak{a}: \lambda \equiv 1 (mod\ f) \\ \lambda^{\sigma_i} > 0\ if\ a_i = 0 \\ \lambda^{\sigma_i} < 0\ if\ a_i = 1\}/\mathcal{O}_K(f\infty)^\times}} \frac{1}{|\mathbf{N}(\lambda)|^s}.$$

Let $\lambda \in K^\times$ have parity $\{a_i\}_{i=1}^n$ then using orthogonality relations we get

$$\sum_{w \text{ is a sign character}} w(\lambda)\zeta(\mathfrak{a}, \mathfrak{f}, w, s) = 2^n \zeta(\mathfrak{a}, \mathfrak{f}\infty, \{a_i\}_{i=1}^n, s) \tag{7.13}$$

$$= 2^n \zeta(\mathfrak{a}\lambda^{-1}, \mathfrak{f}\infty, s)$$

Choosing $a_i = 0$ for all i with $\lambda = 1$ in (7.13) and combining it with (7.10) and (7.11)) we see that for even integers $m \geq 2$

$$\zeta(\mathfrak{a}, \mathfrak{f}, w_0, 1-m) = 2^n \zeta(\mathfrak{a}, \mathfrak{f}\infty, 1-m) \tag{7.14}$$

and that for odd integers $m \geq 1$

$$\zeta(\mathfrak{a}, \mathfrak{f}, w_1, 1-m) = 2^n \zeta(\mathfrak{a}, \mathfrak{f}\infty, 1-m). \tag{7.15}$$

Using again (7.13) we get that for any sign character η

$$\sum_{(\lambda) \in P_{K,1}(\mathfrak{f})/P_{K,1}(\mathfrak{f}\infty)} \eta(\lambda)\zeta(\mathfrak{a}\lambda^{-1}, \mathfrak{f}\infty, s) = \zeta(\mathfrak{a}, \mathfrak{f}, \eta, s).$$

We have the following well known theorem:

Theorem 7.2 *(Siegel,Klingen) For integers $k \geq 1$ the quantities*

$$\zeta(\mathfrak{a}, \mathfrak{f}\infty, 1-k)$$

are rational integers.

Proof See [Sie69].

Corollary 7.1 *For integers $k \geq 1$ we have*

$$\frac{F_w(k)}{F_w(1-k)} \Psi\left(\frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}}, w, k\right) \in \mathbb{Q},$$

where $F_w(s) = |d_K|^{s/2} \pi^{-ns/2} \prod_{i=1}^n \Gamma(\frac{s+a_i}{2})$.

Proof Use theorem 7.2 in combination with theorem 7.1. □

We finish the section by recording one more result:

Proposition 7.2 *Let* $(\lambda) \in P_{K,1}(\mathfrak{f})$ *where* λ *have parity* $\{a_i\}_{i=1}^n$ *then for odd integers* $k \geq 1$ *we have*

$$\zeta(\mathfrak{a}\lambda, \mathfrak{f}\infty, 1-k) = (-1)^{\sum a_i}\zeta(\mathfrak{a}, \mathfrak{f}\infty, 1-k),$$

and for even integers $k \geq 2$ *we have*

$$\zeta(\mathfrak{a}\lambda, \mathfrak{f}\infty, 1-k) = \zeta(\mathfrak{a}, \mathfrak{f}\infty, 1-k).$$

Proof We have

$$\zeta(\mathfrak{c}^{-1}, \mathfrak{f}, w_1, s) = \frac{F_{w_1}(s)}{F_{w_1}(1-s)}\Psi\left(\frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}}, w_1, 1-s\right) \tag{7.16}$$

Now let $(\lambda) \in P_{K,1}(\mathfrak{f})$ where have parity $\{a_i\}_{i=1}^n$. Without lost of generality assumes that $\lambda \in \mathcal{O}_K$. Then we have

$$\begin{aligned}\Psi\left(\frac{\mathfrak{c}\lambda}{\mathfrak{f}\mathfrak{d}}, w_1, s\right) &= w_1(\lambda)\Psi\left(\frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}}, w_1, s\right) \\ &= (-1)^{\sum a_i}\Psi\left(\frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}}, w_1, s\right)\end{aligned}$$

Substituting in (7.16) we find

$$\zeta(\mathfrak{c}^{-1}\lambda^{-1}, \mathfrak{f}, w_1, 1-s) = (-1)^{\sum a_i}\zeta(\mathfrak{c}^{-1}, \mathfrak{f}, w_1, 1-s).$$

Now using (7.15) with $s = k$ for $k \geq 1$ odd we deduce

$$\zeta(\mathfrak{c}^{-1}\lambda^{-1}, \mathfrak{f}\infty, 1-k) = (-1)^{\sum a_i}\zeta(\mathfrak{c}^{-1}, \mathfrak{f}\infty, 1-k).$$

The proof for an even integer $k \geq 2$ is similar. We do the same calculation with w_1 replaced by w_0. □

8 Archimedean zeta functions attached to real quadratic number fields

We now specialize to the case where K is a quadratic number field of discriminant D. Note that the different $\mathfrak{d}$ of K is $(\sqrt{D})$. Let f be some positive integer that we call the conductor and N_0 another positive integer that we call the level. In order to motivate the definitions of the various zeta functions attached to K we need to revisit the involution $*_{fN_0}$ on $X_1(fN_0)$.

8.1 Involution $*_{fN_0}$ on $X_1(fN_0)(\mathbb{C})$ revisited

For this subsection we assume that $K = \mathbb{Q}(\sqrt{D})$ is an imaginary quadratic number field. Let $[(\frac{r}{fN_0}, r\Lambda_\tau)]$ be a point of $Y_1(fN_0)(\mathbb{C})$. Using the definition on the involution ι_{fN_0} defined in section 4.6 and denoting it simply by $*$ we find that

$$[(\frac{r}{fN_0}, r\Lambda_\tau)]^* = [(-\frac{r\tau}{fN_0}, r\tau\mathbb{Z} + \frac{r}{fN_0}\mathbb{Z})] = [(\frac{-r}{fN_0}, r\Lambda_{\frac{1}{fN_0\tau}})].$$

In particular we can think of $*$ as sending $\frac{r}{fN_0} \mapsto \frac{-r}{fN_0}$ and $\tau \mapsto \frac{1}{fN_0\tau}$. Define $\mathcal{H}^{\mathcal{O}_K}(N_0, f)$ as in definition 5.6 but where $\mathcal{H}$ replaces $\mathcal{H}_p$. Let $(r, \tau) \in \mathcal{H}^{\mathcal{O}_K}(N_0, f)$ then $Q_\tau(x, y) = Ax^2 + Bxy + Cy^2$ where $(A, f) = 1$, $N_0|A$ and $B^2 - 4AC = D$. We readily see that $Q_{\frac{1}{fN_0\tau}}(x, y) = Cf^2N_0x^2 + Bfxy + \frac{A}{N_0}y^2$. Therefore we deduce that $disc(Q_{\frac{1}{fN_0\tau}}) = f^2 disc(Q_\tau)$. Note that the leading coefficient of $Q_{\frac{1}{fN_0\tau}}(x, y)$ is not coprime to f but its last coefficient $\frac{A}{N_0}$ is. Remember that the group $\Gamma_0(fN_0)$ acts naturally on $\mathcal{H}^{\mathcal{O}_K}(f, N_0)$ by the rule

$$\gamma \star (r, \tau) = (dr, \frac{a\tau + b}{c\tau + d})$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. There is a natural inclusion of $\mathcal{H}^{\mathcal{O}_K}(N_0, f)/\Gamma_0(fN_0) \subseteq Y_1(fN_0)(\mathbb{C})$ given by $(r, \tau) \mapsto [(\frac{-r}{fN_0}, r\Lambda_\tau)]$. If $(r, \tau) \sim (r', \tau')$ inside $\mathcal{H}^{\mathcal{O}_K}(N_0, f)/\Gamma_0(fN_0)$

then there exists a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(fN_0)$ such that $dr \equiv r'(mod\, f)$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} = (c\tau + d) \begin{pmatrix} \tau' \\ 1 \end{pmatrix}. \tag{8.1}$$

Rewriting (8.1) in term of $\tau^* = \frac{1}{fN_0\tau}$ and $\tau'^* = \frac{1}{fN_0\tau'}$ we find that

$$\begin{pmatrix} d & c/fN_0 \\ bfN_0 & a \end{pmatrix} \begin{pmatrix} \tau^* \\ 1 \end{pmatrix} = (bfN_0\tau^* + a) \begin{pmatrix} \tau'^* \\ 1 \end{pmatrix}.$$

If we let $* : \widetilde{\Gamma}_0 \to \widetilde{\Gamma}_0$ be the involution defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & c/fN_0 \\ bfN_0 & a \end{pmatrix}$$

then we derive the following rule

$$(\gamma \star [(r,\tau)])^* = (\gamma)^* \star^* [(r,\tau)]^*.$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \star^* (r,\tau) = (d^{-1}r, \frac{a\tau + b}{c\tau + d}).$$

and

$$[(r,\tau)]^* = [(-r,\tau^*)].$$

Note however that stricly speaking $[(-r,\tau^*)]$ doesn't belong to $\mathcal{H}^{\mathcal{O}_K}(N_0,f)$ since $End_K(\Lambda_{\tau^*}) = \mathbb{Z} + f\omega\mathbb{Z} \neq \mathbb{Z} + \omega\mathbb{Z} = \mathcal{O}_K$.

Letting $Stab_{\Gamma_1(fN_0)}Q_\tau(x,y) = \langle \pm\gamma_\tau \rangle$ where $\gamma_\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we also see that

$$Stab_{\Gamma_1(fN_0)}(\tau^*) := \gamma_{\tau^*} = \begin{pmatrix} d & c/fN_0 \\ bfN_0 & a \end{pmatrix} = (\gamma_\tau)^*. \tag{8.2}$$

This involution $*$ of level fN_0 will play an important role later on.

8.2 Zeta functions Ψ and Ψ^*

Let K be a real quadratic field with discriminant D. Let us suppose that $\mathfrak{f} = (f)$ where f is a positive integer. Let $[\mathfrak{a}] \in I_{\mathcal{O}_K}(f)/Q_{K,1}(f)$ (the *wide* ideal class group of conductor f) where $\mathfrak{a}$ is an integral ideal coprime to f. Let also

$$w : (K \otimes_{\mathbb{Q}} \mathbb{R})^{\times} \to \{\pm 1\}$$

be any sign character. We are mainly interested by the sign character

$$w_1 = sign \circ N_{K/\mathbb{Q}}.$$

Let $\Psi(\frac{\mathfrak{a}}{\sqrt{D}f}, w_1, s)$ be the zeta function defined in section 7.1. In this section we would like first to define a zeta function $\Psi^*(\frac{\mathfrak{a}}{\sqrt{D}f}, w_1, s)$ where the $*$ refers to the involution discussed in section 8.1 Second of all we would like to write down a functional equation for the zeta functions Ψ and Ψ^*. In order to achieve those two goals it is more convenient to take a $\mathbb{Z}$-basis for the integral ideal $\mathfrak{a}$.

We take a $\mathbb{Z}$-basis in the following way. There always exists an integer $a \in \mathbb{Z}_{>0}$, $(a, f) = 1$ and a $\tau \in K$ such that

$$a(\mathbb{Z} + \tau\mathbb{Z}) = \mathfrak{a}.$$

We let $Q_\tau(x, y) = Ax^2 + Bxy + Cy^2$ with $A > 0$ be the primitive quadratic form associated to τ. Since $End_K(\Lambda_\tau) = \mathcal{O}_K$ we have $B^2 - 4AC = D$. Without lost of generality we assume that $\tau = \frac{-B+\sqrt{D}}{2A}$. Note that the ideal $A\Lambda_\tau$ is an integral ideal (in fact A is the smallest positive integer n for which $n\Lambda_\tau$ is integral).

A small computation shows that

$$\Psi(\frac{a\Lambda_\tau}{f\sqrt{D}}, w_1, s) = \mathbf{N}\left(\frac{a\Lambda_\tau}{f\sqrt{D}}\right)^s \sum_{\mathcal{O}_K(f\infty)^\times \backslash \{0 \neq \mu \in \frac{a\Lambda_\tau}{f\sqrt{D}}\}} \frac{sign(\mathbf{N}_{K/\mathbb{Q}}(\mu)) e^{2\pi i Tr_{K/\mathbb{Q}}(\mu)}}{|\mathbf{N}(\mu)|^s}$$

$$= w_1(\sqrt{D}) \sum_{\langle \eta_\tau \rangle \backslash \{(m,n) \in \mathbb{Z}^2 \backslash (0,0)\}} \frac{sign(Q_\tau(m,n))}{|Q_\tau(m,n)|^s} e^{\frac{-2\pi i \frac{a}{A} n}{f}}, \quad Re(s) > 1. \tag{8.3}$$

where $w_1(\sqrt{D}) = -1$, $Q_\tau(x, y) = Ax^2 + Bxy + Cy^2 = A(x - \tau y)(x - \tau^\sigma y)$, $\tau = \frac{-B-\sqrt{D}}{2A}$, $\langle \pm\eta_\tau \rangle = Stab_{\Gamma_1(f)}(\tau)$. The action of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acting on the vector (x, y)

is given by $(ax+by, cx+dy)$. We choose η_τ in such a way that $c\tau+d>1$ where $\eta_\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Remark 8.1 Note that the variable appearing in the exponent of the exponential of (8.3) is the negative of the second variable of the quadratic form.

Lemma 8.1 *Let $\langle\epsilon\rangle = \mathcal{O}_K(f\infty)^\times$ where $\epsilon > 1$. Then the matrix $\eta_\tau \in \Gamma_1(f)$ corresponds to the matrix representation of the multiplication map by ϵ^n, for some $n \geq 1$, on the lattice Λ_τ with ordered basis $\{\tau, 1\}$. In particular one has that $\eta_\tau \in \Gamma(f)$.*

Proof Let $\mathcal{O}_K = \mathbb{Z}+\omega\mathbb{Z}$ and assume that $\omega = \sqrt{D}$. The case where $\omega = \frac{1+\sqrt{D}}{2}$ can be treated in a similar way. The element τ can be written as $\tau = \frac{r\sqrt{D}+s}{t}$ where t and r are coprime to f (this uses the assumption that $(\Lambda_\tau, f) = 1$). Let $\epsilon = u + fv\sqrt{D} > 1$ be a generator of $\mathcal{O}_K(f\infty)^\times$ and let $\eta_\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \tau \\ 1 \end{pmatrix} = (c\tau+d)\begin{pmatrix} \tau \\ 1 \end{pmatrix}$ we have by definition that $c\tau+d$ is a norm one algebraic integer with $f|c$ and $d \equiv 1 (mod\, f)$. Therefore $c\tau+d \in \mathcal{O}_K(f\infty)^\times$ (this uses the fact that $(t,f)=1$) and so $c\tau+d = \epsilon^n$ for some positive n. The matrix corresponding to ϵ with respect to the basis $\{\sqrt{D}, 1\}$ is $\begin{pmatrix} u & fvD \\ fv & u \end{pmatrix}$. It thus follows that the matrix corresponding to multiplication by ϵ for the basis $\{\tau, 1\}$ is given by

$$\begin{pmatrix} r & s \\ 0 & t \end{pmatrix}\begin{pmatrix} u & fvD \\ fv & u \end{pmatrix}\begin{pmatrix} 1/r & -s/rt \\ 0 & 1/t \end{pmatrix}.$$

Computing the upper right entry we find $-\frac{s^2fv}{rt} + \frac{rfvD}{t}$ which is divisible by f. It thus follows that the upper right entry of the matrix corresponding to multiplication by ϵ^n is divisible by f. $\square$

If we consider the ideal $\tau^\sigma a\Lambda_\tau = a\tau^\sigma\mathbb{Z} + \frac{a}{A}C\mathbb{Z} = \frac{aC}{A}(\mathbb{Z} + \frac{A}{C}\tau^\sigma\mathbb{Z})$ then using (7.1) we find that

$$\Psi(\frac{a\tau^\sigma\Lambda_\tau}{f\sqrt{D}}, w_1, s) = w_1(\tau^\sigma\sqrt{D}) \sum_{\langle\eta_\tau\rangle\backslash\{(m,n)\in\mathbb{Z}^2\backslash(0,0)\}} \frac{sign(Q_\tau(m,n))}{|Q_\tau(m,n)|^s} e^{\frac{2\pi i \frac{a}{A}m}{f}} \tag{8.4}$$

Remark 8.2 Note that this time the variable appearing in the exponent of the exponential of (8.4) is the first variable of the quadratic form. Observe also that the ideal $a\tau^{\sigma}\Lambda_{\tau}$ is coprime to f iff $f \nmid C$. The reader should keep in mind that one can pass from (8.3) to (8.4) by multiplying the ideal in the first entry of Ψ by τ^{σ}.

We want to define a Ψ^*-zeta function attached to the lattice $a\Lambda_{\tau}$ where the $*$ corresponds to a certain involution. The lattice $a\Lambda_{\tau}$ is equivalent to $\Omega(a,\Lambda_{\tau})\cap\mathcal{O}_K$ modulo $P_{K,1}(f\infty)$. Remember that we have an involution

$$*_{fN_0} : Y_1(fN_0) \to Y_1(fN_0)$$
$$(\frac{r}{fN_0}, r\Lambda_{\tau}) \mapsto (\frac{-r}{fN_0}, r\Lambda_{\frac{1}{fN_0}})$$

Let $(1,\tau) \in \mathcal{H}_p^{\mathcal{O}_K}(f,N_0)$ where τ is reduced and consider the integral primitive binary quadratic form of discriminant D attached to τ

$$Q_{\tau}(x,y) = Ax^2 + Bxy + Cy^2, \quad A > 0.$$

The lattice $A\Lambda_{\tau}$ is the integral $\mathcal{O}_K$-ideal corresponding to $Q_{\tau}(x,y)$. Consider also the primitive binary quadratic form of discriminant f^2D attached to $\tau^* = \frac{1}{fN_0\tau}$.

$$Q_{\tau^*}(x,y) = sign(C)\left(f^2CN_0x^2 + Bfxy + \frac{A}{N_0}\right).$$

The lattice $f^2CN_0\Lambda_{\tau^*}$ is the integral $\mathcal{O}_{K,f}$-ideal corresponding to $Q_{\tau^*}(x,y)$ where $\mathcal{O}_{K,f} = \mathbb{Z} + f\omega\mathbb{Z}$ is the order of conductor f. Note that

(1) $Tr(\mu) \in \frac{1}{f}\mathbb{Z}$ if $\mu \in \frac{A\Lambda_{\tau}}{f\sqrt{D}}$,

(2) $Tr(\mu) \in \frac{1}{f}\mathbb{Z}$ if $\mu \in \frac{Cf^2N_0\Lambda_{\tau^*}}{f^2\sqrt{D}} = \frac{CN_0\Lambda_{\tau^*}}{\sqrt{D}}$.

Note the appearance of f^2 in the denominator of the left hand side of the equality in (2). This is accounted for the fact that that the ring of endomorphisms of $Cf^2N_0\Lambda_{\tau^*}$ is $\mathcal{O}_{K,f}$. This I hope motivates the following definition

Definition 8.1 *Let $a\Lambda_{\tau}$ be an integral $\mathcal{O}_K$-ideal and let $Q_{\tau}(x,y) = Ax^2 + Bxy + Cy^2$ be the integral primitive binary quadratic form attched to τ. Note that $A|a$. We*

define

$$\Psi^*\left(\frac{a\Lambda_\tau}{f\sqrt{D}}, w, s\right) := \mathbf{N}\left(\frac{CN_0\Lambda_{\tau^*}}{\sqrt{D}}\right)^s \sum_{\mathcal{O}_K(f\infty)^\times\backslash\{0\neq\mu\in\frac{CN_0\Lambda_{\tau^*}}{\sqrt{D}}\}} \frac{sign(\mathbf{N}_{K/\mathbb{Q}}(\mu))e^{2\pi i Tr_{K/\mathbb{Q}}(\mu)}}{|\mathbf{N}(\mu)|^s}$$

(8.5)

$$= w_1(\sqrt{D}) \sum_{\langle\eta_{\tau^*}\rangle\backslash(\mathbb{Z}^2\backslash(0,0))} \frac{sign(Q_{\tau^*}(m,n))}{|Q_{\tau^*}(m,n)|^s} e^{\frac{2\pi i\frac{a}{A}n}{f}}$$

(8.6)

$$= w_1(\sqrt{D}) \sum_{\langle\eta_\tau(fN_0)\rangle\backslash(\mathbb{Z}^2\backslash(0,0))} \frac{sign(Q_{fN_0\tau}(m,n))}{|Q_{fN_0\tau}(m,n)|^s} e^{\frac{2\pi i\frac{a}{A}m}{f}}$$

where $\tau^* = \frac{1}{fN_0\tau} = \frac{-Bf+f\sqrt{D}}{f^2N_0C}$, $Q_{\tau^*}(x,y) = sign(C)\left(f^2CN_0x^2 + Bfxy + \frac{A}{N_0}y^2\right)$ *and*

$$\eta_{\tau^*} = \begin{pmatrix} d & c/fN_0 \\ bfN_0 & a \end{pmatrix}, \qquad \eta_\tau(fN_0) = \begin{pmatrix} a & bfN_0 \\ c/fN_0 & d \end{pmatrix}.$$

The third equality follows from the change of variable $(x,y) \mapsto (y,x)$ *and uses the identity* $|Q_{\tau^*}(m,n)| = |Q_{fN_0\tau}(n,m)|$. *Again* Ψ^* *depends only on the narrow ideal class modulo* f *of the integral ideal* $a\Lambda_\tau$.

Remark 8.3 It is interesting to point out that the third equality reflects the functional equation of a certain Eisenstein series. This is clear if one looks at the proof of lemma 9.2.

We want to define now dual zeta functions to Ψ and Ψ^* (dual in the sense of the functional equation).

Definition 8.2 *For* $s \in \mathbb{C}$ *such that* $Re(s) > 1$ *we define*

$$\widehat{\Psi}(\frac{a\Lambda_\tau}{\sqrt{D}f}, w_1, s) := f^{2s} \sum_{\langle\eta_\tau\rangle\backslash(0\neq(m,n)\equiv(\frac{a}{A},0)(mod\ f))} \frac{sign(Q_\tau(m,n))}{|Q_\tau(m,n)|^s}, \tag{8.7}$$

and

$$\widehat{\Psi}^*(\frac{a\Lambda_\tau}{\sqrt{D}f}, w_1, s) := f^{2s} \sum_{\langle\eta_{\tau^*}\rangle\backslash(0\neq(m,n)\equiv(\frac{a}{A},0)(mod\ f))} \frac{sign(Q_{\tau^*}(m,n))}{|Q_{\tau^*}(m,n)|^s}$$

where A *is the leading coefficient of the quadratic form* $Q_\tau(x,y) = Ax^2 + Bxy + Cy^2$.

Remark 8.4 Note that the matrices η_τ and $\eta_{\tau*}$ preserve the congruence

$$(\frac{a}{A}, 0)(mod\ f).$$

We can now write down the functional equation for Ψ and Ψ^*.

Theorem 8.1 *For $s \in \mathbb{C}$ such that $Re(s) < -1$ we have*

$$-F_{w_1}(s)\Psi(\frac{a\Lambda_\tau}{\sqrt{D}f}, w_1, s) = F_{w_1}(1-s)\widehat{\Psi}(\frac{a\Lambda_\tau}{f\sqrt{D}}, w_1, 1-s), \tag{8.8}$$

and

$$-F^*_{w_1}(s)\Psi^*(\frac{a\Lambda_\tau}{\sqrt{D}f}, w_1, s) = F^*_{w_1}(1-s)\widehat{\Psi}^*(\frac{a\Lambda_\tau}{f\sqrt{D}}, w_1, 1-s). \tag{8.9}$$

where $F_{w_1}(s) = disc(Q_\tau)^{s/2}\pi^{-s}\Gamma\left(\frac{s+1}{2}\right)^2$ and $F^*_{w_1}(s) = disc(Q_{\tau^*})^{s/2}\pi^{-s}\Gamma\left(\frac{s+1}{2}\right)^2$. Note that the left hand side of (8.8) and (8.9) make sense when $Re(s) < -1$ since Ψ and Ψ^* admit a meromorphic continuation to $\mathbb{C}$ (see corollary 8.1).

Remark 8.5 Later on we will relate special values of $\widehat{\Psi}$ and $\widehat{\Psi}^*$ at negative even integers (see proposition 9.4). At this stage it is not clear that $\widehat{\Psi}$ can be related to $\widehat{\Psi}^*$ in any obvious way.

We end this subsection with this useful lemma

Lemma 8.2 *We have*

$$\widehat{\Psi}(\frac{a\Lambda_\tau}{\sqrt{D}f}, w_1, s) = \mathbf{N}(f)^s\zeta\left((a\Lambda_\tau)^{-1}, f, w_1, s\right).$$

where the function on the right hand side is a partial zeta function of K weighted by the infinite character $w_1 = sign \circ \mathbf{N}_{K/\mathbb{Q}}$.

Proof We have

$$\begin{aligned}\widehat{\Psi}(\frac{a\Lambda_\tau}{\sqrt{D}f}, w, s) &= f^{2s} \sum_{\langle\eta_\tau\rangle\backslash(0\neq(m,n)\equiv(\frac{a}{A},0)(mod\ f))} \frac{sign(Q_\tau(m,n))}{|Q_\tau(m,n)|^s} \\ &= \frac{f^{2s}A^s}{a^{2s}} \sum_{\langle\eta_{\tau^\sigma}\rangle\backslash(0\neq(m,n)\equiv(\frac{a}{A},0)(mod\ f))} \frac{sign(Q_{\tau^\sigma}(m,n))}{|\frac{A}{a^2}Q_{\tau^\sigma}(m,n)|^s}\end{aligned}$$

$$= \mathbf{N}(f)^s \mathbf{N}\left(\frac{A}{a}\Lambda_{\tau^\sigma}\right)^s \sum_{\mathcal{O}_K(f\infty)^\times \setminus \{0 \neq \mu \in \frac{A}{a}\Lambda_{\tau^\sigma}, \mu \equiv 1 (mod\ f)\}} \frac{w_1(\mu)}{|\mathbf{N}(\mu)|^s}$$

$$= \mathbf{N}(f)^s \zeta\left(\frac{A}{a}\Lambda_{\tau^\sigma}, f, w_1, s\right)$$

$$= \mathbf{N}(f)^s \zeta\left((a\Lambda_\tau)^{-1}, f, w_1, s\right)$$

The last equality follows from the fact that $\Lambda_\tau \Lambda_{\tau^\sigma} = \left(\frac{1}{A}\right)$. The term on the right hand side of the last equality is nothing else than a partial zeta function twisted by the character w_1. Note that this coincides with equation (7.9). $\square$

8.3 Proof of the functional equation of Ψ for K quadratic real

The key idea in the proof of theorem 8.1 is a trick due to Hecke relating the zeta function of definite quadratic forms to the zeta function of indefinite quadratic form. After it is a matter of relating the functional equation appearing in (3.11) to the one appearing in theorem 8.1. We have decided to include the proof for the reader but essentially all the ingredients are already contained in [Sie68].

Proof of theorem 8.1 We only prove (8.8) since (8.9) can be proved in a similar manner. Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \in SL_2(\mathbb{Z})$ be an hyperbolic matrix fixing two real points τ and τ^σ such that $\tau > \tau^\sigma$. Assume furthermore that $a \equiv 1 (mod\ f)$ and $c \equiv 0\ (mod\ f)$.

Consider the normalized quadratic form $\widetilde{Q}_\tau(z,1) = (z-\tau)(z-\tau^\sigma) = \frac{1}{A}Q_\tau(z,1)$ where $Q_\tau(x,y) = Ax^2 + Bxy + Cy^2$. We find the transformation formula

$$sign(c\tau + d)\widetilde{Q}_\tau(z,1) = (cz+d)^2 \widetilde{Q}_\tau(\gamma z, 1).$$

We also define

$$\psi_r(s,z) = \sum_{\underline{m}}{}' \frac{e^{2\pi i m_2 r/f}}{|(m_2 z - m_1)|^{2(s-1)}(m_2 z - m_1)^2}$$

which makes sense for all $s \in \mathbb{C}$ such that $Re(s) > 1$. We will use the functional equation in s of $\psi_r(s,z)$ to deduce it for $\Psi(\frac{\mathfrak{a}}{\mathfrak{f}\mathfrak{d}}, w_1, s)$. It also satisfies the following

transformation formula in z

$$\psi_r(s,\alpha z) = |Pz+Q|^{2(s-1)}(Pz+Q)^2\psi_r(s,z) \tag{8.10}$$

for any $\begin{pmatrix} R & S \\ P & Q \end{pmatrix} = \alpha \in \Gamma_1(f)$. It thus follows that the C^∞ 1-form $|\widetilde{Q}_\tau(z)|^{s-1}\psi_r(s,z)dz$ is invariant under the transformation $z \mapsto \gamma z$. We take the opportunity here to mention the useful formula

$$(Pz+Q)(m_2(\alpha z) - m_1) = (m_2' z - m_1') \tag{8.11}$$

where $\alpha = \begin{pmatrix} R & S \\ P & Q \end{pmatrix}$ and $\begin{pmatrix} Q & -S \\ -P & R \end{pmatrix}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} m_1' \\ m_2' \end{pmatrix}$.

Lemma 8.3 *Let C be the half circle of the upper half plane joining τ and τ^σ. Let $\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \in \mathbb{Z}^2$ and $\gamma^{-1}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} m_1' \\ m_2' \end{pmatrix}$ (for some integer l) then*

$$\int_C \frac{1}{|m_2 z - m_1|^{2(s-1)}(m_2 z - m_1)^2}|\widetilde{Q}_\tau(z)|^{s-1}dz$$
$$= \int_C \frac{1}{|m_2' z - m_1'|^{2(s-1)}(m_2' z - m_1')^2}|\widetilde{Q}_\tau(z)|^{s-1}dz.$$

Note that C, as a subset of $\mathcal{H}$, is fixed (the orientation also is preserved) by the Moebius transformation $z \mapsto \gamma z$.

Proof of the lemma 8.3 We prove it for $l = 1$. We compute:

$$\int_C \frac{1}{|m_2 z - m_1|^{2(s-1)}(m_2 z - m_1)^2}|\widetilde{Q}_\tau(z)|^{s-1}dz \tag{8.12}$$
$$= \int_{\gamma^{-1}C} \frac{1}{|m_2\gamma z - m_1|^{2(s-1)}(m_2\gamma z - m_1)^2}|\widetilde{Q}_\tau(\gamma z)|^{s-1}d\gamma z \tag{8.13}$$
$$= \int_C \frac{1}{|m_2\gamma z - m_1|^{2(s-1)}(m_2\gamma z - m_1)^2}|\widetilde{Q}_\tau(\gamma z)|^{s-1}d\gamma z$$

where the second equality follows from the invariance of C by γ.

$$= \int_C \frac{1}{|(am_2 - cm_1)z - (-bm_2 + dm_1)|^{2(s-1)}((am_2 - cm_1)z - (-bm_2 + dm_1))^2}|\widetilde{Q}_\tau(z)|^{s-1}dz$$
$$= \int_C \frac{1}{|m_2' z - m_1'|^{2(s-1)}(m_2' z - m_1')^2}|\widetilde{Q}_\tau(z)|^{s-1}dz$$

where $\begin{pmatrix} m_1' \\ m_2' \end{pmatrix} = \gamma^{-1} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$. The second equality follows from (8.10) and (8.11). □

Let x be an arbitrary point on C. From the previous computation we see that for any $(m_1, m_2) \in \mathbb{Z}^2 \backslash \{(0,0)\}$ we have

$$\int_C \frac{1}{|m_2 z - m_1|^{2(s-1)}(m_2 z - m_1)^2} |\widetilde{Q}_\tau(z)|^{s-1} dz = \sum_{(n_1,n_2) \in \langle \gamma^{-1} \rangle (m_1,m_2)} \int_x^{\gamma x} \frac{1}{|n_2 z - n_1|^{2(s-1)}(n_2 z - n_1)^2} |\widetilde{Q}_\tau(z)|^{s-1} dz \tag{8.14}$$

where the last summation goes over all the γ^{-1}-translate of the fixed pair (m_1, m_2). Geometrically the small arc with end points x and $\gamma^{-1}x$ gives a tessellation of C under the action of the $\langle \gamma^{-1} \rangle$. Note that we were allowed to change the order of summation with integration because of absolute convergence since $Re(s) > 1$. Note that we can also replace γ^{-1} by γ without affecting (8.14).

Fix a complete set of representatives $\{(m_1, m_2)\}$ for the action of $\langle \gamma \rangle$ on $(\mathbb{Z}^2 \backslash \{(0,0)\})$. Then for every representative (m_1, m_2) multiply the left hand side of (8.12) by $e^{2\pi i m_2 r/f}$. Taking the summation and using (8.14) gives us

$$\sum_{(m_1,m_2) \in \langle \gamma \rangle \backslash (\mathbb{Z}^2 \backslash (0,0))} e^{2\pi i m_2 r/f} \int_C \frac{1}{|m_2 z - m_1|^{2(s-1)}(m_2 z - m_1)^2} |\widetilde{Q}_\tau(z)|^{s-1} dz = j \int_x^{\gamma x} |\widetilde{Q}_\tau(z)|^{s-1} \psi_r(s, z) dz. \tag{8.15}$$

The orientation of C is taken to be the orientation of the arc segment joining τ^σ to τ. It is therefore in the clockwise orientation. The quantity $j := sign(\frac{a+d}{c})$ takes care of this choice of orientation. By lemma 8.3 the summation on the left hand side doesn't depend on the set of representatives which is clear from the right hand side. Also since the left hand side doesn't depend on x we get that the right hand side is independent of the base point x.

We are thus lead to evaluate the following expression

$$\int_C \frac{1}{|m_2 z - m_1|^{2(s-1)}(m_2 z - m_1)^2} |\widetilde{Q}_\tau(z)|^{s-1} dz.$$

This is the content of the next lemma

Lemma 8.4 *For $Re(s) > 1$ we have*

(8.16)
$$\int_C \frac{1}{|m_2 z - m_1|^{2(s-1)}(m_2 z - m_1)^2} |\widetilde{Q}_\tau(z)|^{s-1} dz = \frac{\Gamma(\frac{s+1}{2})^2}{\Gamma(s+1)} disc(\widetilde{Q}_\tau)^{s-1/2} \frac{sign(\widetilde{Q}_\tau(m_1, m_2))}{|\widetilde{Q}_\tau(m_1, m_2)|^s}.$$

Proof We omit the proof since this calculation is done in [Sie68] and a bit later we do a similar calculation. □

Corollary 8.1 *Let $Re(s) > 1$ then*

$$\int_x^{\gamma x} |\widetilde{Q}_\tau(z)|^{s-1} \psi_r(s, z) dz = j \frac{\Gamma(\frac{s+1}{2})^2}{\Gamma(s+1)} disc(\widetilde{Q}_\tau)^{s-1/2} A^s \varphi(\frac{r}{f}, \tau, \gamma, s) \tag{8.17}$$

where

$$\varphi(\frac{r}{f}, \tau, \gamma, s) := \sum_{\langle\gamma\rangle\backslash(\mathbb{Z}^2\backslash\{(0,0)\})} \frac{sign(Q_\tau(m_1, m_2)) e^{2\pi i m_2 r/f}}{|Q_\tau(m_1, m_2)|^s},$$

$j = sign(\frac{a+d}{c})$, $Q_\tau(x, y) = A\widetilde{Q}_\tau(x, y) = A(x - \tau y)(x - \tau^\sigma y)$.

Proof Combine (8.15) with lemma 8.4. □

Remark 8.6 Note that $disc(Q_\tau) = disc(A\widetilde{Q}_\tau) = A^2 disc(\widetilde{Q}_\tau)$.

Lemma 8.5 *We have*

$$\int_x^{\gamma x} |\widetilde{Q}_\tau(z)|^{s-1} \psi_r(s, z) dz = j \frac{\Gamma(\frac{2-s}{2})^2 \pi^{2s-1}}{f^{2(s-1)} \Gamma(s+1)} \sum_{\langle\gamma\rangle\backslash(m_1,m_2)\equiv(r,0) \ (mod\ f)} \frac{sign(\widetilde{Q}_\tau(m_1, m_2))}{|\widetilde{Q}_\tau(m_1, m_2))|^{1-s}}$$

for any $s \in \mathbb{C}$ such that $Re(s) < 0$.

Remark 8.7 Note that the matrix $\gamma \in \Gamma_1(f)$ preserves the congruence $(r, 0)(mod\ f)$.

Proof of lemma 8.5 The proof uses the same ideas as lemma 8.1.

Let $\zeta(s,u^*,v^*,z,g)$ (see (3.10) for the definition) with $g=2$, $u^*=\begin{pmatrix}\frac{r}{f}\\0\end{pmatrix}$ and $v^*=\begin{pmatrix}0\\0\end{pmatrix}$ we get

$$\begin{aligned}\zeta(s,\begin{pmatrix}\frac{r}{f}\\0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix},z,2) &= y^s\sum_{\underline{m}\neq 0}\frac{e^{-2\pi i m_2 r/f}}{|(m_1+m_2z)|^{2(s-1)}(m_1+m_2z)^2}\\ &= y^s\sum_{\underline{m}\neq 0}\frac{e^{2\pi i m_2 r/f}}{|(m_2z-m_1)|^{2(s-1)}(m_2z-m_1)^2}\\ &= y^s\psi_r(s,z).\end{aligned}$$

Using the functional equation in (3.11) applied to $\psi_r(s,z)$ we get

$$\begin{aligned}&=\int_x^{\gamma x}|\widetilde{Q}_\tau(z)|^{s-1}\psi_r(s,z)dz\\ &=\frac{\pi^{2s-1}\Gamma(2-s)}{f^{2(s-1)}\Gamma(s+1)}\int_x^{\gamma x}\frac{|\widetilde{Q}_\tau(z)|^{s-1}}{im(z)^{2s-1}}\frac{\zeta(1-s,\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}\frac{r}{f}\\0\end{pmatrix},z,2)}{im(z)^{1-s}f^{2(1-s)}}dz\\ &=\frac{\pi^{2s-1}\Gamma(2-s)}{f^{2(s-1)}\Gamma(s+1)}\int_x^{\gamma x}\frac{|\widetilde{Q}_\tau(z)|^{s-1}}{im(z)^{2s-1}}\sum_{(m_1,m_2)\equiv(r,0)\ (mod\ f)}\frac{1}{|m_1-m_2z|^{-2s}(m_1-m_2z)^2}dz\\ &=j\frac{\pi^{2s-1}\Gamma(2-s)}{f^{2(s-1)}\Gamma(s+1)}\int_C\frac{|\widetilde{Q}_\tau(z)|^{s-1}}{im(z)^{2s-1}}\sum_{\langle\gamma\rangle\backslash(m_1,m_2)\equiv(r,0)\ (mod\ f)}\frac{1}{|m_1-m_2z|^{-2s}(m_1-m_2z)^2}dz\\ &=j\frac{\pi^{2s-1}\Gamma(2-s)}{f^{2(s-1)}\Gamma(s+1)}\sum_{\langle\gamma\rangle\backslash(m_1,m_2)\equiv(r,0)\ (mod\ f)}\int_C\frac{|\widetilde{Q}_\tau(z)|^{s-1}}{im(z)^{2s-1}}\frac{1}{|m_1-m_2z|^{-2s}(m_1-m_2z)^2}dz\end{aligned}$$

We are thus lead to evaluate the integral

$$\int_{\tau^\sigma}^{\tau}\left|\frac{\widetilde{Q}_\tau(z)}{(m-nz)^2}\right|^{s-1}\left|\frac{(m-nz)^2}{im(z)}\right|^{2s-1}\frac{dz}{(m-nz)^2}.$$

The two end points of C correspond to τ and τ^σ. We integrate from τ^σ to τ, therefore since $\tau>\tau^\sigma$ this is integrating along C in the clockwise orientation.

In order to compute this integral we do a change of variables. The variables are z and w and t and we fix a pair $(m,n)\in\mathbb{Z}^2\backslash\{(0,0)\}$. We let

$z = \frac{\tau w+\tau^\sigma}{w+1}$, $w = \frac{z-\tau^\sigma}{\tau-z}$ $\qquad$ $v = n\tau - m$, $v^\sigma = n\tau^\sigma - m$

$nz - m = \frac{vw+v^\sigma}{w+1}$ $\qquad$ $j = sign(\frac{a+d}{c})$

$\widetilde{Q}_\tau(\frac{\tau w+\tau^\sigma}{w+1}) = -(\tau-\tau^\sigma)^2\frac{w}{(w+1)^2}$ $\qquad$ $g = sign(vv^\sigma) = sign(\widetilde{Q}_\tau(m,n))$

$w = |\frac{v^\sigma}{v}|it$ $\qquad$ $(nz-m)^{-2}dz = i(\tau-\tau^\sigma)|vv^\sigma|^{-1}$
$\cdot(1+igt)^{-2}dt$

The of variables $z \mapsto \frac{\tau w+\tau^\sigma}{w+1}$ sends the hyperbolic triangle C on the positive y-axis. Applying this change of variable we find

$$
\begin{aligned}
&= \int_0^{i\infty} \left|\frac{-(\tau-\tau^\sigma)^2\frac{w}{(w+1)^2}}{(\frac{vw+v^\sigma}{w+1})^2}\right|^{s-1}\left|\frac{(\frac{vw+v^\sigma}{w+1})^2}{(\tau-\tau^\sigma)\frac{im(w)}{|w+1|^2}}\right|^{2s-1}(\tau-\tau^\sigma)(vw+v^\sigma)^{-2}dw \\
&= \int_0^{i\infty} \left|\frac{w}{(vw+v^\sigma)^2}\right|^{s-1}\left|\frac{(vw+v^\sigma)^2}{im(w)}\right|^{2s-1}(vw+v^\sigma)^{-2}dw \\
&= \int_0^{\infty} \left|\frac{(vv^\sigma)^{-1}it}{(1+igt)^2}\right|^{s-1}\left|\frac{(1+igt)^2}{(vv^\sigma)^{-1}it}\right|^{2s-1} i|vv^\sigma|^{-1}(1+igt)^{-2}dt \\
&= i|vv^\sigma|^{-(1-s)}\int_0^{\infty}\left|\frac{t}{1+t^2}\right|^{-s}(1+igt)^{-2}dt \\
&= i|vv^\sigma|^{-(1-s)}\int_0^{\infty}(\frac{t}{1+t^2})^{-s}(1+igt)^{-2}dt.
\end{aligned}
$$

So we need to evaluate the quantity $I := \int_0^\infty(\frac{t}{1+t^2})^{-s}(1+igt)^{-2}dt$. Doing the change of variables $t \mapsto \frac{1}{t}$ we find that $I = -\int_0^\infty(\frac{t}{1+t^2})^{-s}(1-igt)^{-2}dt = -\bar{I}$, so I is purely imaginary. Therefore $I = \frac{1}{2}\int_0^{i\infty}\left((1+igt)^{-2}-(1-igt)^{-2}\right)(\frac{t}{1+t^2})^{s-1}dt = -2ig\int_0^\infty \frac{t}{(1+t^2)^2}(\frac{t}{1+t^2})^{-s}dt = -2ig\int_0^\infty t^{-s+1}(1+t^2)^{-s-2}dt$.

Doing the change of variable $1+t^2 = u$ followed by $u \mapsto \frac{1}{u}$ we find that

$$-2ig\int_0^\infty t^{-s+1}(1+t^2)^{s-2}dt = -ig\int_0^1(1-u)^{-s/2}u^{-s/2}du.$$

Using the well know formula $\int_0^1 u^{x-1}(1-u)^{y-1}du = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ we get

$$-2ig\int_0^\infty t^{-s+1}(1+t^2)^{s-2}dt = -ig\frac{\Gamma(\frac{2-s}{2})^2}{\Gamma(2-s)}.$$

Combining everything together we find

$$\int_x^{\gamma x}|\widetilde{Q}_\tau(z)|^{s-1}\psi_r(s,z)dz \;=\; j\frac{\Gamma(\frac{2-s}{2})^2\pi^{2s-1}}{f^{2(s-1)}\Gamma(s+1)}\sum_{\langle\gamma\rangle\backslash(m,n)\equiv(r,0)\ (mod\ f)}\frac{sign(\widetilde{Q}_\tau(m,n))}{|\widetilde{Q}_\tau(m,n))|^{1-s}}$$

This completes the proof of lemma 8.5. □

Lemma 8.5 suggests the following definition:

Definition 8.3

$$\hat{\varphi}(\frac{r}{f},\tau,\gamma,s) := f^{2s} \sum_{\langle\gamma\rangle\backslash 0\neq(m,n)\equiv(r,0)\ (mod\ f)} \frac{sign(Q_\tau(m,n))}{|Q_\tau(m,n))|^s}.$$

Combining Lemma 8.1 with lemma 8.5 and rearranging a bit we obtain the functional equation

(8.18)
$$\Gamma(\frac{s+1}{2})^2 disc(\widetilde{Q}_\tau)^{\frac{s}{2}} A^s \varphi(\frac{r}{f},\tau,\gamma,0) = disc(\widetilde{Q}_\tau)^{\frac{1-s}{2}} \Gamma(\frac{2-s}{2})^2 \pi^{2s-1} A^{1-s} \hat{\varphi}(\frac{r}{f},\tau,\gamma,1-s).$$

But this is nothing else than the functional equation (8.8) of theorem 8.1. □

9 Relation between special values of zeta functions and Eisenstein series

9.1 Archimedean zeta function associated to a class in $\mathcal{H}_p^{\mathcal{O}}(N_0,f)/\widetilde{\Gamma}_0$

In this section we want to associate to any class in $\mathcal{H}_p^{\mathcal{O}}(N_0,f)/\widetilde{\Gamma}_0$ a well defined Archimedean zeta function. We first start by proving an elementary lemma.

Lemma 9.1 *Let $\tau \in \mathcal{H}_p \cap K$ s.t. $(disc(Q_\tau),p) = 1$ and let*

$$\langle \pm\gamma_\tau \rangle = Stab_{SL_2(\mathbb{Z}[\frac{1}{p}])}(\tau),$$

then $\gamma_\tau \in SL_2(\mathbb{Z})$. Note that γ_τ is well defined up to ± 1.

Proof Let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma_\tau$. Since $det(\gamma_\tau) = 1$ this implies that $N_{K/\mathbb{Q}}(c\tau + d) = 1$. Since the denominators of c and d are at most powers of p and $\mathbf{N}(c\tau + d) = 1$ we

have that $c\tau + d$ is norm 1 unit of $\mathcal{O}_K^{(p)^\times}$. But since p is inert in K we have $\mathcal{O}_K^{(p)^\times} \simeq \pm 1 \times p^{\mathbb{Z}} \times \mathcal{O}_K^\times$. Therefore this forces $c\tau + d \in \mathcal{O}_K^\times$. We also have $c\tau^2 + (d-a)\tau + b = 0$. Therefore there exists a rational number of the form $\frac{p^s}{m}$ $((m,p)=1)$ such that

$$\frac{p^s}{m}(cx^2 + (d-a)x + b) = Q_\tau(x) = Ax^2 + Bx + C. \tag{9.1}$$

let $E = B^2 - 4AC$ then wlog we can assume $\tau = \frac{-B+\sqrt{E}}{2A}$. We have $(c\tau + d) - (c\tau^\sigma + d) = c(\tau - \tau^\sigma) \in \mathcal{O}_K$. Therefore $\frac{c\sqrt{E}}{A} \in \mathcal{O}_K$. But $c/A = m/p^s$ we thus have $\frac{m\sqrt{E}}{p^s} \in \mathcal{O}_K$. Since $(E,p)=1$ and $(m,p)=1$ this forces $s=0$. Because A, B, C are integers, $(m,p)=1$ and $a,b,c,d \in \mathbb{Z}[\frac{1}{p}]$, we deduce from (9.1) that $c, b, (d-a) \in \mathbb{Z}$. Finally note that $A|c$ so we find that $c\tau \in \mathcal{O}_K$ and therefore $d \in \mathbb{Z}$. $\square$

We would like to attach now a zeta function to certain pairs $(r,\tau) \in \mathcal{H}_p^{\mathcal{O}}(N_0, f)$.

Remark 9.1 It is easy to show that if τ is reduced, i.e. if $red(\tau) = v_0$ where v_0 is the standard vertex on the Bruhat-Tits tree and red is the reduction map, then $(disc(Q_\tau), p) = 1$. However the converse is false. We can therefore think of the reduced requirement as a finer notion compare to the more naive condition $(disc(Q_\tau), p) = 1$.

Definition 9.1 *Let $(r,\tau) \in \mathcal{H}_p^{\mathcal{O}}(N_0, f)$. Assume that τ is reduced i.e. $red(\tau) = v_0$. We set*

$$\langle \pm\eta_\tau \rangle := Stab_{\Gamma_1(f\mathbb{Z}[\frac{1}{p}])}(\tau).$$

Then by lemma 9.1 we know that $\eta_\tau \in \Gamma_1(f)$, in other words $Stab_{\Gamma_1(f\mathbb{Z}[\frac{1}{p}])}(\tau) = Stab_{\Gamma_1(f)}(\tau)$. We choose $\eta_\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in such a way that $c\tau + d > 1$. We define several zeta functions associated to the pair (r,τ) by

(1) $\zeta((r,\tau), s) := \Psi\left(\frac{\Omega(r,\tau)\cap\mathcal{O}}{f\sqrt{D}}, w_1, s\right)$

(2) $\widehat{\zeta}((r,\tau), s) := \widehat{\Psi}\left(\frac{\Omega(r,\tau)\cap\mathcal{O}}{f\sqrt{D}}, w_1, s\right)$

and similarly

3) $\zeta^*((r,\tau), s) := \Psi^*\left(\frac{\Omega(r,\tau)\cap\mathcal{O}}{f\sqrt{D}}, w_1, s\right)$

4) $\widehat{\zeta}^*((r,\tau),s) := \widehat{\Psi}^*\left(\frac{\Omega(r,\tau)\cap\mathcal{O}}{f\sqrt{D}}, w_1, s\right)$

where $w_1 = sign \circ N_{K/\mathbb{Q}}$ *and* $\mathfrak{d} = (\sqrt{D})$. *Note that the map* Ω *(see definition 5.7) depends on the quantities* $\mathcal{O}, p, f$ *and* N_0.

Proposition 9.1 *If* $(r,\tau),(r',\tau') \in \mathcal{H}_p^{\mathcal{O}}(N_0,f)$ *satisfy the assumption of the definition 9.1, namely* $red(\tau) = red(\tau') = v_0$, *then if* $(r,\tau) \sim (r',\tau')$ *we have*

(1) $\zeta((r,\tau),s) = \zeta((r',\tau'),s)$,

(2) $\zeta^*((r,\tau),s) = \zeta^*((r',\tau'),s)$

Proof Let $\Omega(r,\tau) = A\Lambda_\tau^{(p)}$ and $\Omega(r',\tau') = A'\Lambda_{\tau'}^{(p)}$ (where the exponent $^{(p)}$ means that we have tensored the $\mathbb{Z}$-lattices $A\Lambda_\tau$ and $A'\Lambda_{\tau'}$ over $\mathbb{Z}[\frac{1}{p}]$). Since $(r,\tau) \sim (r',\tau')$ there exists a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Z}[\frac{1}{p}])$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \tau \\ 1 \end{pmatrix} = \begin{pmatrix} \tau' \\ 1 \end{pmatrix}. \tag{9.2}$$

with $f|c$ and $d \equiv \frac{A'}{A}(mod\ f)$. Because τ and τ' are reduced this forces $\gamma \in GL_2(\mathbb{Z}_p)$. It thus follows that $\gamma \in SL_2(\mathbb{Z})$. Using relation (9.2) we deduce that $A\Lambda_\tau \equiv A'\Lambda_{\tau'}(mod\ Q_{K,1}(f\infty))$. Finally since τ and τ' are reduced we have $A\Lambda_\tau^{(p)} \cap \mathcal{O} = A\Lambda_\tau$ and $A'\Lambda_{\tau'}^{(p)} \cap \mathcal{O} = A'\Lambda_{\tau'}$. It thus follows that

$$\Psi\left(\frac{\Omega(r,\tau)\cap\mathcal{O}}{f\sqrt{D}}, w, s\right) = \Psi\left(\frac{\Omega(r',\tau')\cap\mathcal{O}}{f\sqrt{D}}, w, s\right).$$

This proves (1). The proof of (2) is similar. □

We have thus succeeded to attach well defined Archimedean zeta functions to any class of $\mathcal{H}_p^{\mathcal{O}}(N_0,f)/\sim$.

So far we haven't used the level N_0-structure build in inside $\mathcal{H}_p^{\mathcal{O}}(N_0,f)$. The next object we define is a zeta function attached to a good divisor $\delta \in D(N_0,f)^{(p)}$ and a pair $(r,\tau) \in \mathcal{H}_p^{\mathcal{O}}(N_0,f)$. From now on, in order to simplify the calculation we make the following assumption

Assumption 9.1 We assume that the good divisor

$$\delta = \sum_{d_0,r} n(d_0,r)d_0 \in D(N_0,f)^{\langle p \rangle}$$

is primitive i.e. $n(d_0,r) = 0$ if $(r,f) \neq 1$.

Definition 9.2 *Let $\delta = \sum_{d_0,r} n(d_0,r)[d_0,r] \in D(N_0,f)^{\langle p \rangle}$ be a good divisor and $(j,\tau) \in \mathcal{H}_p^{\mathcal{O}}(N_0,f)$ with $red(\tau) = v_0$ then we define*

(1) $\zeta(\delta_j,(1,\tau),s) := \sum_{d_0|N_0, r\in(\mathbb{Z}/f\mathbb{Z})^\times} n(d_0,r)d_0^s\widehat{\zeta}((\frac{rj}{d_0}, d_0\tau),s)$

(2) $\zeta^*(\delta_j,(1,\tau),s) := \sum_{d_0|N_0, r\in(\mathbb{Z}/f\mathbb{Z})^\times} n(\frac{N_0}{d_0},r)d_0^s\widehat{\zeta^*}((\frac{rj}{N_0/d_0}, \frac{N_0}{d_0}\tau),s).$

It is an easy exercise to show that $\zeta(\delta_j,(1,\tau),s)$ and $\zeta^*(\delta_j,(1,\tau),s)$ depend only on the class of $(1,\tau)$ modulo $\sim$. We also have the useful formula $\zeta(\delta_{aj},(1,\tau),s) = \zeta(\delta_j,(a,\tau),s)$.

Remark 9.2 First of all note that there is a hat on zeta functions appearing on the right hand side of (1) and (2). Note also that the lattices $\Lambda_{d_0\tau}$ has endomorphism by $\mathcal{O}_K = \mathbb{Z} + \omega\mathbb{Z}$ and $\Lambda_{d_0\tau^*}$ has endomorphism by $\mathbb{Z} + f\omega\mathbb{Z}$ which is the order of conductor f of $\mathcal{O}_K$.

Remark 9.3 In the case where $f = 1$ as in [DD06] one has that $\zeta(\delta_j,\tau,s) = \zeta^*(\delta_j,\tau,s)$. In general if $f > 1$ then $\zeta(\delta_j,(1,\tau),s) \neq \zeta^*(\delta_j,(1,\tau),s)$. In proposition 9.4 we relate both of them under the assumption that the primes dividing f are inert in K.

9.2 Special values of $\zeta(\delta_r,(A,\tau),1-k)$ as integrals of Eisenstein series of even weight F_{2k}

We are now ready to relate periods of Eisenstein series with special values of the Archimedean zeta functions $\zeta(\delta_r,(1,\tau),s)$ and $\zeta^*(\delta_r,(1,\tau),s)$.

Lemma 9.2 *Let* $(j,\tau) \in \mathcal{H}_p^{\mathcal{O}}(N_0,f)$ *where* $Q_\tau(x,y) = Ax^2+Bxy+Cy^2$ *and* $red(\tau) = v_0$. *Then for all odd integers* $k \geq 1$ *we have*

(1) $3\zeta^*(\delta_j,(A,\tau),1-k) = \int_{\xi_2}^{\gamma_{\tau^*}\xi_2} Q_{\tau^*}(z,1)^{k-1}\widetilde{F}_{2k}^*(j,z)dz$
$= f^{2k-2}\int_{\xi_1}^{\gamma_\tau\xi_1} Q_\tau(z,1)^{k-1}\widetilde{F}_{2k}(-j,z)dz,$

(2) $3\zeta(\delta_j,(A,\tau),1-k) = \int_{\xi_2}^{\gamma_\tau\xi_2} Q_\tau(z,1)^{k-1}\widetilde{F}_{2k}^*(j,z)dz$

where $\xi_1 = i\infty$, $\xi_2 = 0$, $\langle\pm\gamma_\tau\rangle = Stab_{\Gamma_1}(\tau)$ *with* $c\tau+d>1$ *for* $\gamma_\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Remark 9.4 If we use proposition 5.1 we see that we can replace ξ_1 and ξ_2 by any point in $\mathcal{H}$ without changing the value of the integrals of (1) and (2). Note also that if $(disc(Q_\tau),p)=1$ then

$$Stab_{\Gamma_1}(\tau) = Stab_{\Gamma_1(f)}(\tau). \tag{9.3}$$

In particular (9.3) is true when $red(\tau) = v_0$.

Proof We only prove (1) since (2) can be proved in a similar way simply by replacing τ^* by τ in the argument. Using the remark 9.4 we can assume that $\xi_2 \in \mathcal{H}$. We compute

$$\int_{\xi_2}^{\gamma_{\tau^*}\xi_2} Q_{\tau^*}(z,1)^{k-1}\widetilde{F}_{2k}^*(j,z)dz = -12\int_{\xi_2}^{\gamma_{\tau^*}\xi_2} Q_{\tau^*}(z,1)^{k-1}\sum_{d_0,r} n(\frac{N_0}{d_0},r)d_0^{2k-1}E_{2k}^*(rj,d_0z)dz$$

$$= -12\left(\frac{(-1)^{2k}(2\pi i)^{2k}}{(2k-1)!}\right)^{-1}\sum_{d_0,r} n(\frac{N_0}{d_0},r)d_0^{2k-1}\int_{\xi_2}^{\gamma_{\tau^*}\xi_2} Q_{\tau^*}(z,1)^{k-1}\sum_{(0,0)\neq(m,n)}{}' \frac{e^{-2\pi inrj/f}}{(m+nd_0z)^{2k}}dz$$

(9.4)

$$= -12\left(\frac{(-1)^{k}(2\pi)^{2k}}{(2k-1)!}\right)^{-1}\sum_{d_0,r} n(\frac{N_0}{d_0},r)d_0^{2k-1}\int_{\xi_2}^{\gamma_{\tau^*}\xi_2} Q_{\tau^*}(z,1)^{k-1}\Theta_{rj}(k,d_0z)dz$$

where $\Theta_r(k,z) = \sum_{m,n}{}' \frac{e^{2\pi inr/f}}{(nz-m)^{2k}}$ which is defined for integers $k \geq 2$.

For every term in (9.4) indexed by a $d_0|N_0$, we replace the base point ξ_2 by $\frac{\xi_2}{d_0} = \xi_2$ and followed by a change of variables $z \mapsto \frac{z}{d_0}$. We thus get

(9.5)

$$= -12\left(\frac{(-1)^k(2\pi)^{2k}}{(2k-1)!}\right)^{-1}\sum_{d_0,r} d_0^{k-1} n(\frac{N_0}{d_0}, r)\int_{\xi_2}^{d_0\gamma_{\tau^*}\frac{1}{d_0}\xi_2} (d_0 Q_{\tau^*}\left(\frac{1}{d_0}z, 1\right))^{k-1}\Theta_{rj}(k,z)dz$$

Let $Q_\tau(x,y) = Ax^2 + Bxy + Cy^2$ then because $(r,\tau) \in \mathcal{H}_p^{\mathcal{O}}(N_0, f)$ we have $(\frac{r}{f}, \tau)^* = (-\frac{r}{f}, \tau^*) = (-\frac{r}{f}, \frac{1}{fN_0\tau})$ and therefore $Q_{\tau^*}(x,y) = f^2CN_0x^2 + Bfxy + \frac{A}{N_0}y^2$ and $Q_{d_0\tau^*}(x,y) = f^2C\frac{N_0}{d_0}x^2 + Bfxy + \frac{Ad_0}{N_0}y^2$. Therefore $Q_{d_0\tau^*}(d_0x, y) = d_0Q_{\tau^*}(x,y)$. Substituting in (9.5) the latter equality we get

$$= -12\left(\frac{(-1)^k(2\pi)^{2k}}{(2k-1)!}\right)^{-1}\sum_{d_0,r} n(\frac{N_0}{d_0}, r)d_0^{k-1}\int_{\xi_2}^{\gamma_{d_0\tau^*}\xi_2} Q_{d_0\tau^*}(z,1)^{k-1}\,\Theta_{rj}(k,z)dz.$$

Now applying Hilfsaz 1 in [Sie68] we find

(9.6)
$$\int_{\xi}^{\gamma_{d_0\tau^*}\xi} Q_{d_0\tau^*}(z,1)^{k-1}\Theta_{rj}(k,z)dz$$
$$= (-1)^{k-1}sign\left(\frac{a+d}{c}\right)\frac{\Gamma(k)^2}{\Gamma(2k)}disc(Q_{d_0\tau^*})^{k-\frac{1}{2}}\varphi((-\frac{rj}{f}, 0), d_0\tau^*, \gamma_{d_0\tau^*}, k)$$

where for integers $k \geq 2$

$$\varphi((u,v),\tau,\gamma,k) := \sum_{\langle\gamma\rangle\backslash(\mathbb{Z}^2\backslash(0,0))}\frac{e^{2\pi i(mu+nv)}}{Q_\tau(m,n)^k}\ (converges\ absolutely),$$

$\xi \in \mathcal{H}$ is an arbitrary point and $\gamma \in SL_2(\mathbb{Z})$ is such that $\begin{pmatrix} u & v \end{pmatrix}\gamma \equiv \begin{pmatrix} u & v \end{pmatrix}(mod\ \mathbb{Z}^2)$ for all $(u,v) \in \mathbb{Z}^2$. Set $\xi := \xi_2 \in \mathcal{H}$.

When $k \geq 3$ is odd we readily see that

(9.7)
$$\varphi((\frac{r}{f}, 0), d_0\tau^*, \gamma_{d_0\tau^*}, k) = -\zeta^*((\frac{rA}{N_0/d_0}, \frac{N_0}{d_0}\tau), k).$$

The -1 on the right hand side comes from $w_1(\sqrt{D}) = -1$ in the definition of Ψ^*. Using the functional equation

$$-F^*(s)\zeta^*((\frac{rA}{N_0/d_0}, \frac{N_0}{d_0}\tau), s) = F^*(1-s)\widehat{\zeta}^*((\frac{rA}{N_0/d_0}, \frac{N_0}{d_0}\tau), 1-s),$$

where $F^*(s) = \pi^{-s} disc(Q_{\tau^*})^{\frac{s}{2}} \Gamma(\frac{s+1}{2})^2$ in combination with equation (9.7) plus the observation that $(-1)^{k-1} = sign(a+d) = 1$ and $sign(c) = -1$ in (9.6) we obtain

$$-12\frac{F^*(1-k)}{F^*(k)}\left(\frac{(-1)^k(2\pi)^{2k}}{(2k-1)!}\right)^{-1} disc(Q_{\tau^*})^{k-\frac{1}{2}}\frac{\Gamma(k)^2}{\Gamma(2k)}\sum_{d_0,r} n(\frac{N_0}{d_0},r)d_0^{1-k}\widehat{\zeta}^*((\frac{jA}{N_0/d_0},\frac{N_0}{d_0}\tau),1-k).$$

Simplifying we get

$$\begin{aligned} &= 3\zeta^*(\delta_{jA},(1,\tau),1-k) \\ &= 3\zeta^*(\delta_j,(A,\tau),1-k). \end{aligned}$$

We want to treat now the case $k = 1$. Using the equation (8.17) we see that we need to evaluate $\lim_{s\to 1}\psi_r(s,z)$ (see equation (8.10) for the definition of $\psi_r(s,z)$) and compare it with

$$-4\pi^2 E_2^*(-r,z) = \sum_{\underline{m}\neq(0,0)} \frac{e^{2\pi i m_2 r/f}}{(m_2 z - m_1)^2}.$$

Note that originally $\psi_r(s,z)$ was only defined when $Re(s) > 1$ in order to have absolute convergence. Note that the limit when $s \to 1$ makes sense since $\psi_r(s,z)$ admits analytic continuation.

Lemma 9.3 *We have*

(1) $\lim_{s\to 1}\psi_r(s,z) = \left(\frac{-\pi}{Im(z)} - 4\pi^2 E_2^*(-r,z)\right)$ *if* $r \equiv 0 (mod\ f)$

(2) $\lim_{s\to 1}\psi_r(s,z) = -4\pi^2 E_2^*(-r,z)$ *if* $r \neq 0 (mod\ f)$

Proof See theorem 7 of chapter 3 in [B. 74]. □

With this lemma we thus get for any $r \in (\mathbb{Z}/f\mathbb{Z})^\times$ that

$$\lim_{s\to 1}\psi_r(s,z) = -4\pi^2 E_2^*(-r,z) = \Theta_r(1,z).$$

From this we deduce

$$\begin{aligned} 3\zeta^*(\delta_j,(A,\tau),0) &= -12\int_{\xi_2}^{\gamma_{\tau^*}\xi_2} F_2^*(j,z)dz \\ &= \int_{\xi_2}^{\gamma_{\tau^*}\xi_2} \widetilde{F}_2^*(j,z)dz. \end{aligned} \tag{9.8}$$

This concludes the proof for the case $k = 1$. It remains to prove the second equality of (1). For this we do the change of variables $z \mapsto \frac{1}{fN_0 z}$ in equation (1) of lemma 9.2 and we use the two identities

$$\widetilde{F}_m^*(j, \frac{1}{fN_0 z}) = -z^m (fN_0)^{m-1} \widetilde{F}_m(-j, z) \tag{9.9}$$

with $m = 2k$ and

$$Q_{f\tau}(fx, y) = fQ(x, y).$$

□

Remark 9.5 The relation (9.8) continues to hold even if the divisor δ is not primitive because the non holomorphic terms cancel since δ is good.

At this point in makes sense to draw the following corollary from (9.9)

Proposition 9.2 *Let $\xi_1 = i\infty$ and $\xi_2 = 0$. Then for $\gamma \in \Gamma_0(fN_0)$ and $k \geq n-2$ we have*

$$\int_{\xi_1}^{\gamma\xi_1} z^n \widetilde{F}_k(j, z) dz = -(fN_0)^{-n} \int_{\xi_2}^{\gamma^*\xi_2} z^{k-n-2} \widetilde{F}_k^*(-j, z) dz.$$

Note that if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *then* $\gamma^* = \begin{pmatrix} d & c/fN_0 \\ bfN_0 & a \end{pmatrix}$.

Proof Simply apply the change of variable $z \mapsto \frac{1}{fN_0}$ to the left hand side and use (9.9).

Proposition 9.3 *Let $\delta = \sum_{d_0, r} n(d_0, r)[d_0, r] \in D(N_0, f)$ be such that*

$$\sum_{d_0 | N_0} n(d_0, r) d_0 = \sum_{d_0 | N_0} n(d_0, r) \frac{N_0}{d_0} = 0 \text{ for all } r \in \mathbb{Z}/f\mathbb{Z},$$

then the Eisenstein series $\widetilde{F}_{k,\delta}(r, \tau)$ is holomorphic at the set of cusps $\Gamma_0(fN_0)\{0, i\infty\}$. Using proposition 5.1 we deduce

$$\begin{aligned} \int_{i\infty}^{\gamma(i\infty)} z^n F_{k,\delta}(r, z) dz &= \int_0^{\gamma(0)} z^n F_{k,\delta}(r, z) dz \\ &= \int_{i\infty}^{\gamma(0)} z^n F_{k,\delta}(r, z) dz + \int_0^{i\infty} z^n F_{k,\delta}(r, z) dz. \end{aligned}$$

If we let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *we get*

$$\int_{i\infty}^{\frac{a}{c}} z^n F_{k,\delta}(r,z)dz = \int_{i\infty}^{\frac{b}{d}} z^n F_{k,\delta}(r,z)dz + \int_0^{i\infty} z^n F_{k,\delta}(r,z)dz.$$

Remark 9.6 This is a reciprocity formula.

9.3 Some explicit formulas for $\zeta^*(\delta_r,(A,\tau),0)$

We record in this subsection a special value of particular importance namely $\zeta(\delta_j,(A,\tau),0)$. From lemma 9.2 we have

$$\zeta^*(\delta_r,(A,\tau),0) = \frac{1}{3}\cdot\int_{i\infty}^{\gamma_\tau i\infty} \widetilde{F}_{2,\delta}(-r,z)dz \tag{9.10}$$
$$= \frac{1}{3}\cdot\frac{1}{2\pi i}(\log\beta_{\delta_r}(\gamma_\tau z) - \log\beta_{\delta_r}(z))|_{z=i\infty}$$

since

$$\widetilde{F}_{2,\delta}(-r,\tau)dz = \widetilde{F}_{2,\delta}(r,\tau)dz = \frac{1}{2\pi i}\mathrm{dlog}\beta_{\delta_r}(\tau).$$

Let $\gamma_\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Using equation (5.4) of proposition 5.4 and using the fact that δ is a good divisor we deduce

$$\zeta^*(\delta_j,(A,\tau),0) = -4\cdot sign(c)\sum_{r\in\mathbb{Z}/f\mathbb{Z},d_0|N_0} n(d_0,r)s^f_{(\frac{jr}{f},0)}(a,\frac{c}{fd_0})$$
$$= -4\cdot sign(c)\sum_{r\in\mathbb{Z}/f\mathbb{Z},d_0|N_0} n(d_0,r)\sum_{i(mod\ c/fd_0)}\widetilde{B}_1(\frac{jr+if}{c/d_0})\widetilde{B}_1(\frac{jra+aif}{c/d_0})$$
$$= -4\cdot sign(c)\sum_{r\in\mathbb{Z}/f\mathbb{Z},d_0|N_0} n(d_0,r)\sum_{i(mod\ c/fd_0)}\widetilde{B}_1(\frac{jr/f+i}{c/fd_0})\widetilde{B}_1(a(\frac{jr/f+i}{c/fd_0}))$$
$$= -4\cdot sign(c)\sum_{r\in\mathbb{Z}/f\mathbb{Z},d_0|N_0} n(d_0,r)D_{1,1}^{rj(mod\ f)}(a,c/d_0). \tag{9.11}$$

9.4 Relation between $\zeta(\delta,(1,\tau),s)$ and $\zeta^*(\delta,(1,\tau),s)$

For later purposes it will be important to relate those two zeta functions. We have the following

Proposition 9.4 *Assume that f is divisible only by primes that are inert in K then we have*

$$\widehat{\Psi}^*(\frac{a\Lambda_\tau}{f\sqrt{D}}, w_1, s) = \sum_{u=0}^{f-1} \widehat{\Psi}(\lambda_{u,a}\frac{a\Lambda_\tau}{f\sqrt{D}}, w_1, s)$$

where $\lambda_{u,a}$ is an algebraic integer chosen so that $\lambda_{u,a} \equiv (u\frac{A/N_0}{a} + \tau^\sigma)(mod\ f)$, $\lambda_{u,a}$ is coprime to p and totally positive.

Corollary 9.1 *Using lemma 8.2 we deduce*

$$\sum_{u=0}^{f-1} \zeta(\lambda_{u,a} \star \delta, (a,\tau), s) = \zeta^*(\delta, (a,\tau), s).$$

Proof of proposition 9.4 We have

$$\begin{aligned} f^{-2s}\widehat{\Psi}^*(\frac{a\Lambda_\tau}{f\sqrt{D}}, w_1, s) &= \sum_{\langle\gamma_\tau(fN_0)\rangle\backslash\{0\neq(m,n)\equiv(0,\frac{a}{A})(mod\ f)\}} \frac{sign(Q_{fN_0\tau}(m,n))}{|Q_{fN_0\tau}(m,n)|^s} \\ &= \sum_{u=0}^{f-1} \sum_{\langle\gamma_{N_0\tau}\rangle\backslash\{0\neq(m,n)\equiv(u,\frac{a}{A})(mod\ f)\}} \frac{sign(Q_{N_0\tau}(m,n))}{|Q_{N_0\tau}(m,n)|^s}. \end{aligned}$$

Note that the discriminant of $Q_{N_0\tau}(x,y)$ is equal to D. Since $\gamma_{N_0\tau} \in \Gamma(f)$ the second summation makes sense since the congruences $(u, \frac{a}{A})(mod\ f)$ are preserved under the action of $\gamma_{N_0\tau}$. Now because the primes dividing f are inert in K we get automatically that the algebraic numbers $\{\lambda_{u,a}\}_{u=0}^{f-1}$ are coprime to fp. We have that the right hand side of the last equality equals to

$$\begin{aligned} &= \sum_{u=0}^{f-1} \mathbf{N}(\Lambda_{N_0\tau})^s \sum_{\mathcal{O}_K(f\infty)^\times\backslash\{0\neq\lambda\in\Lambda_{N_0\tau}, \lambda\equiv u+\frac{a}{A}N_0\tau(mod\ f)\}} \frac{w_1(\lambda)}{|\mathbf{N}(\lambda)|^s} \\ &= \sum_{u=0}^{f-1} \mathbf{N}\left(\frac{A/N_0}{a}\Lambda_{N_0\tau^\sigma}\right)^s \sum_{\mathcal{O}_K(f\infty)^\times\backslash\{0\neq\lambda\in\frac{A/N_0}{a}\Lambda_{N_0\tau^\sigma}, \lambda\equiv\frac{A/N_0}{a}u+\tau^\sigma(mod\ f)\}} \frac{w_1(\lambda)}{|\mathbf{N}(\lambda)|^s} \\ &= \sum_{u=0}^{f-1} \zeta\left(\lambda_{u,a}^{-1}\frac{A/N_0}{a}\Lambda_{N_0\tau^\sigma}, f, w_1, s\right) \\ &= \sum_{u=0}^{f-1} \zeta\left((\lambda_{u,a}a\Lambda_{N_0\tau})^{-1}, f, w_1, s\right). \end{aligned}$$

where for the last equality we have used the fact that $\Lambda_{N_0\tau}\Lambda_{N_0\tau^\sigma} = \left(\frac{1}{A/N_0}\right)$. □

10 P-adic zeta functions and p-adic Kronecker limit formula

Definition 10.1 *We define the p-adic zeta function attached to a good divisor*

$$\delta \in D(N_0, f)^{\langle p \rangle}$$

and a pair $(j, \tau) \in \mathcal{H}_p^{\mathcal{O}}(N_0, f)$ *with* $Q_\tau(x,y) = Ax^2 + Bxy + Cy^2$ *and* $(disc(Q_\tau), p) = 1$ *to be*

$$\zeta_p^*(\delta_j, (A,\tau), s) := \frac{1}{3}\int_{\mathbb{X}} \langle Q_{f\tau}(fx,y)\rangle^{-s} d\widetilde{\mu}_j\{i\infty \to \gamma_\tau(i\infty)\}(x,y) \tag{10.1}$$
$$= \frac{1}{3}\langle f\rangle^{-2s}\int_{\mathbb{X}} \langle Q_\tau(x,y)\rangle^{-s} d\widetilde{\mu}_j\{i\infty \to \gamma_\tau(i\infty)\}(x,y)$$

where $\langle x\rangle$ *denotes the unique element in* $1 + p\mathbb{Z}_p$ *that differs from* x *by a* $p-1$ *root of unity. This zeta function makes sense for any* $s \in \mathbb{Z}_p$. *As usual* $\langle \pm\gamma_\tau\rangle = Stab_{\Gamma_1}(\tau)$.

Corollary 10.1 *For* $n \geq 0$ *an even negative integer congruent to* 0 *modulo* $p-1$ *we have*

$$(1 - p^{-2n})\zeta^*(\delta_j, (1,\tau), n) = \zeta_p^*(\delta_j, (1,\tau), n).$$

Proof Combine (1) of lemma 9.2 with (1) of theorem 6.1. □

Remark 10.1 We thus see that our p-adic zeta function interpolates rational values of the Archimedean zeta function $\zeta^*(\delta_j, (A,\tau), s)$ at negative integers.

Lemma 10.1 *The derivative* $(\zeta_p^*)'(\delta_j, (A,\tau), 0)$ *at* $s = 0$ *is given by*

$$(\zeta_p^*)'(\delta_j, (A,\tau), 0) = -\frac{1}{3}\int_{\mathbb{X}} log_p(Q_\tau(x,y)) d\widetilde{\mu}_j\{\xi \to \gamma_\tau\xi\}(x,y)$$

where $\xi = i\infty$.

Proof This is a direct calculation using equation (10.1). Note that the integral over $\mathbb{X}$ of $\log_p(Q_{f\tau}(fx,y)) = \log_p f^2 + \log_p Q_\tau(x,y)$ is the same as $\log_p Q_\tau(x,y)$ since the total measure is zero so that the constant term $\log_p f^2$ vanishes. □

We can now deduce a p-adic Kronecker limit formula

Theorem 10.1 *Let $(r,\tau) \in \mathcal{H}_p^{\mathcal{O}}(N_0, f)$ with τ reduced, i.e. $red(\tau) = v_0$. Then we have*

$$3(\zeta_p^*)'(\delta_r, (A,\tau), 0) = -\log_p \mathbf{N}_{K_p/\mathbb{Q}_p}(u(\delta_r,\tau)). \tag{10.2}$$

Proof From theorem 6.2 we have

$$u(\delta_r,\tau) = \int\!\!\!\!\!\!\diagup_{\mathbb{X}} (x - \tau y) d\widetilde{\mu}_r\{i\infty \to \gamma_\tau(i\infty)\}(x,y) \tag{10.3}$$

Replacing τ by τ^σ in the previous identity gives us

$$u(\delta_r,\tau^\sigma) = \int\!\!\!\!\!\!\diagup_{\mathbb{X}} (x - \tau^\sigma y) d\widetilde{\mu}_r\{i\infty \to \gamma_{\tau^\sigma}(i\infty)\}(x,y) \tag{10.4}$$

But $\gamma_\tau = \gamma_{\tau^\sigma}$. Therefore multiplying (10.3) with (10.4) together and taking the p-adic logarithm of this product combined with lemma 10.1 gives us (10.2). □

We end this subsection with the following useful proposition

Proposition 10.1

$$3\zeta^*(\delta_r, (A,\tau), 0) = ord_p(u(\delta_r,\tau)).$$

Proof Use lemma 6.2 with equation (9.10). □

11 Dedekind sums and periods of Eisenstein series

11.1 Dedekind sums

In order to give explicit formulas we need to introduce certain Dedekind sums. Let $\widetilde{B}_n$ be the n-th periodic Bernoulli polynomial, see definition 4.6. It is easy to show that they satisfy the following distribution relations

$$N^{k-1} \sum_{i=0}^{N-1} \widetilde{B}_k(a\frac{x+Mi}{MN}) = \widetilde{B}_k(\frac{ax}{M}) \tag{11.1}$$

where M and N are nonzero integers and a is coprime to N.

Definition 11.1 *Let a and $c \neq 0$ be two integers, not necessarily coprime with $f|c$. Let $s, t \geq 1$ be integers and choose a residue class $r \in \mathbb{Z}/f\mathbb{Z}$. We define*

$$D_{s,t}^{r\ (mod\ f)}(a,c) := c^{s-1} \sum_{\substack{1 \leq h \leq c \\ h \equiv r (mod\ f)}} \frac{\widetilde{B}_s(h/c)}{s} \frac{\widetilde{B}_t(ha/c)}{t}. \tag{11.2}$$

where $\widetilde{B}_n$ is the n-th periodic Bernoulli polynomial, see definition 4.6.

When the level f is fixed we omit the *mod* f notation.

Lemma 11.1 *Those Dedekind sums satisfy the following useful identities:*

(1) $D_{s,t}^{r(mod\ f)}(ad, dc) = D_{s,t}^{r(mod\ f)}(a, c)$

(2) $d_0^{-(t-1)} D_{s,t}^{r(mod\ f)}(a, c/d_0) = \left(\frac{c}{d_0}\right)^{s-1} \sum_{\substack{h=1 \\ h \equiv r (mod\ f)}}^{c} \frac{\widetilde{B}_s(\frac{h}{c/d_0})}{s} \frac{\widetilde{B}_t(\frac{ha}{c})}{t}$

for any $a, c, d \in \mathbb{Z}$ s.t. $d_0 f | c$.

Proof Let us prove the first identity first.

$$\begin{aligned}
D_{s,t}^{r(mod\ f)}(ad, dc) &= (dc)^{s-1} \sum_{\substack{h (mod\ dc) \\ h \equiv r (mod\ f)}} \frac{\widetilde{B}_s(h/dc)}{s} \frac{\widetilde{B}_t(ha/c)}{t} \\
&= (dc)^{s-1} \sum_{\substack{i (mod\ c) \\ i \equiv r (mod\ f)}} \sum_{j=0}^{d-1} \frac{\widetilde{B}_s(\frac{i+cj}{dc})}{s} \frac{\widetilde{B}_t(\frac{a(i+cj)}{c})}{t} \\
&= (dc)^{s-1} \sum_{\substack{i (mod\ c) \\ i \equiv r (mod\ f)}} \sum_{j=0}^{d-1} \frac{\widetilde{B}_s(\frac{i+cj}{dc})}{s} \frac{\widetilde{B}_t(\frac{ai}{c})}{t} \\
&= (dc)^{s-1} \sum_{\substack{i (mod\ c) \\ i \equiv r (mod\ f)}} \frac{\widetilde{B}_t(\frac{ai}{c})}{t} \sum_{j=0}^{d-1} \frac{\widetilde{B}_s(\frac{i/c+j}{d})}{s} \\
&= \frac{(dc)^{s-1}}{d^{s-1}} \sum_{\substack{i (mod\ c) \\ i \equiv r (mod\ f)}} \frac{\widetilde{B}_t(\frac{ai}{c})}{t} \frac{\widetilde{B}_s(i/c)}{s}.
\end{aligned}$$

This completes the first part of the lemma. Let us prove the second part.

$$(\frac{c}{d_0})^{s-1} \sum_{\substack{h=1 \\ h\equiv r(mod\ f)}}^{c} \frac{\widetilde{B}_s(h/(c/d_0))}{s} \frac{\widetilde{B}_t(ha/c)}{t} =$$

$$(\frac{c}{d_0})^{s-1} \sum_{\substack{h=1 \\ h\equiv r(mod\ f)}}^{c/d_0} \sum_{j=0}^{d_0-1} \frac{\widetilde{B}_s(\frac{h+jc/d_0}{c/d_0})}{s} \frac{\widetilde{B}_t(a\frac{h+jc/d_0}{c})}{t} =$$

$$(\frac{c}{d_0})^{s-1} \sum_{\substack{h=1 \\ h\equiv r(mod\ f)}}^{c/d_0} \sum_{j=0}^{d_0-1} \frac{\widetilde{B}_s(\frac{h}{c/d_0})}{s} \frac{\widetilde{B}_t(a\frac{h+jc/d_0}{d_0(c/d_0)})}{t} =$$

$$(\frac{c}{d_0})^{s-1} \sum_{\substack{h=1 \\ h\equiv r(mod\ f)}}^{c/d_0} \frac{\widetilde{B}_s(\frac{h}{c/d_0})}{s} \sum_{j=0}^{d_0-1} \frac{\widetilde{B}_t(a\frac{h+jc/d_0}{d_0(c/d_0)})}{t} =$$

$$d_0^{-(t-1)}(\frac{c}{d_0})^{s-1} \sum_{\substack{h=1 \\ h\equiv r(mod\ f)}}^{c/d_0} \frac{\widetilde{B}_s(\frac{h}{c/d_0})}{s} \frac{\widetilde{B}_t(\frac{ah}{c/d_0})}{t} =$$

$$d_0^{-(t-1)} D_{s,t}^{r(mod\ f)}(a, c/d_0).$$

□

11.2 A technical lemma

Here is some technical lemma that will turn out to be essential later on.

Lemma 11.2 *Let $s, t \geq 1$. For any rational number $\frac{a}{c}$ (p could divide c), we have inside $\mathbb{Q}_p$ the following identity:*

$$\lim_{j\to\infty} D_{s+(p-1)p^j,t}^{r(mod\ f)}(a, c) = D_{s,t}^{r(mod\ f)}(a, c) - p^{s-1} D_{s,t}^{p^{-1}r(mod\ f)}(pa, c)$$

The proof is similar to [DD06] but it avoids the use of Dedekind reciprocity formulas.

Proof Let $x = \frac{a}{c} \in \mathbb{Q}$ with $(a, c) = 1$ and assume first that $p \nmid c$. Let b be an integer such that $abp \equiv 1(mod\ c)$.

Note that

$$D_{s,t}^{r(mod\ f)}(a,c) = c^{s-1} \sum_{\substack{1\le l\le c \\ l\equiv ar(mod\ f)}} \frac{\widetilde{B}_s(lbp/c)}{s}\frac{\widetilde{B}_t(l/c)}{t} \tag{11.3}$$

therefore

$$D_{s+(p-1)p^j,t}^{r(mod\ f)}(a,c) = c^{s-1+(p-1)p^j} \sum_{\substack{1\le l\le c \\ l\equiv ar(mod\ f)}} \frac{\widetilde{B}_{s+(p-1)p^j}(lbp/c)}{s}\frac{\widetilde{B}_t(l/c)}{t} \tag{11.4}$$

and similarly

$$D_{s+(p-1)p^j,t}^{r(mod\ f)}(pa,c) = c^{s-1} \sum_{\substack{1\le l\le c \\ l\equiv ar(mod\ f)}} \frac{\widetilde{B}_{s+(p-1)p^j}(lb/c)}{s}\frac{\widetilde{B}_t(l/c)}{t} \tag{11.5}$$

Write $y = \{lbp/c\}$ and $y' = \{lb/c\}$. Since $c^{(p-1)p^j} \to 1$, then subtracting p^{s-1} times (11.5) to (11.4) we see that it suffices to prove that

$$\lim_{j\to\infty} B_{s+(p-1)p^j}(y) = B_s(y) - p^{s-1}B_s(y'). \tag{11.6}$$

For $s > 0$, this follows from the proof of Theorem 3.2 of [You01]. In the course of the proof of Theorem 3.2 of [You01] he gets that for any positive integer b coprime to p the following equality

$$\begin{aligned}(b^{s+(p-1)p^j} - 1)\frac{B_{s+(p-1)p^j}(x) - p^{s-1+(p-1)p^j}B_{s+(p-1)p^j}(x')}{s+(p-1)p^j} \\ -(b^s-1)\frac{B_s(x)-p^{s-1}B_s(x')}{s} \equiv 0(modp^{j+1}\mathbb{Z}_p)\end{aligned} \tag{11.7}$$

where x' is such that $px' - x \in \{0, 1, \ldots, p-1\}$ and $s \geq 1$. The denominator of $\frac{B_n}{n}$ at p is well behaved. If $(p-1) \nmid n$ then $\frac{B_n}{n}$ is p-integral. If $(p-1)|n$ then $v_p(\frac{B_n}{n}) = -1 - v_p(n)$. Using the previous observation it follows that $\lim_{j\to\infty} p^{(p-1)p^j}B_{s+(p-1)p^j}(x') = 0$. Letting $j \to \infty$ in (11.7) we get that

$$(b^s-1)\lim_{j\to\infty}\frac{B_{s+(p-1)p^j}(x)}{s} = (b^s-1)\frac{B_s(x)-p^{s-1}B_s(x')}{s}. \tag{11.8}$$

When $s \geq 1$ we can always choose b such that $b^s - 1 \neq 0$. Therefore we can cancel the two factors $b^s - 1$ in (11.8) to get (11.6). It remains to treat the case where $s = 0$.

We have $v_p(y) \geq 1$. Let $g = (p-1)p^j$. Note that

$$B_g(y) = \sum_{k=0}^{g} \binom{g}{k} B_k y^{g-k} \tag{11.9}$$
$$= y^g + g\left(\sum_{k=1}^{g-1} \binom{g-1}{k-1} \frac{B_k}{k} y^{g-k}\right) + B_g.$$

If $(p-1) \nmid k$ then $\frac{B_k}{k} \in \mathbb{Z}_p$. If $(p-1)|k$ then we can write $k = (p-1)p^u m$ with $(m,p) = 1$. So $v_p(\frac{B_k}{k} y^{g-k}) \geq -1 - u + (p-1)p^u \geq 0$ since $p^{j-u} - 1 \geq m$. We thus deduce from (11.9) that $\lim_{j\to\infty} B_{(p-1)p^j}(y) = B_{(p-1)p^j}$.

Let ω be the Teichmüller character at p. If we look at $L_p(s)$ the p-adic L-function twisted by the trivial character. We have the formula

$$L_p(1-n) = -(1 - \omega^{-n}(p)p^{n-1})\frac{B_{n,\omega^{-n}}}{n}$$

Here ω^{-n} means the primitive character associated to ω^{-n} (so $\omega^{-n}(a)$ is not necessarily equal to $\omega(a)^{-n}$). So letting $n = (p-1)p^j$ therefore $\omega^{-n}(p) = 1$ we get

$$L_p(1-(p-1)p^j) = -(1 - p^{(p-1)p^j - 1})\frac{B_{(p-1)p^j}}{(p-1)p^j}$$

Now we know that $\lim_{s\to 1}(s-1)L_p(s) = 1 - \frac{1}{p}$. So letting $j \to \infty$ we get

$$\lim_{j\to\infty} B_{(p-1)p^j} = 1 - \frac{1}{p}.$$

this proves the claim for $s = 0$.

We need to treat now the case where $p|c$. This case turns out to be simple. Let us prove the following elementary lemma

Lemma 11.3 *Let h be any integer and $0 \neq c \in \mathbb{Z}$ such that $p|c$. Then we have the following:*

(1) $\lim_{j\to\infty} c^{s+g}\widetilde{B}_{s+g}(\frac{h}{c}) = c^s \widetilde{B}_s(\frac{h}{c})$, *if* $(h,p) = 1$.

(2) $\lim_{j\to\infty} c^{s+g}\widetilde{B}_{s+g}(\frac{h}{c}) = 0$, *if* $p|h$.

where $g = (p-1)p^j$.

Proof of lemma 11.3 Let us prove the first case. We have

$$c^{s+g}\widetilde{B}_{s+g}(\frac{h}{c}) = \sum_{k=0}^{s+g}\binom{s+g}{k}B_k h^{s+g-k}c^k \tag{11.10}$$

$$= \sum_{k=0}^{s}\binom{s+g}{k}B_k h^{s+g-k}c^k + \sum_{k=s+1}^{s+g}\binom{s+g}{k}B_k h^{s+g-k}c^k \tag{11.11}$$

Now since $|c|_p < 1, |h|_p = 1, |\binom{m}{k}|_p \leq 1$ and $|B_k|_p \leq p$, the limit in (11.11) exists when $j \to \infty$. Since $(h,p) = 1$ the limit of the first term is $c^s\widetilde{B}_s(\frac{h}{c})$ and the limit of the second term is 0. This proves the first part of the lemma.

Assume now that $p|h$. If $v_p(h) \geq v_p(c)$ then $\frac{h}{c} \in \mathbb{Z}_p$. In this case we know that $\lim_{j\to\infty}\widetilde{B}_{s+(p-1)p^j}(\frac{h}{c})$ exists by (11.6). Finally since $p|c$ it follows that $\lim_{j\to\infty} c^{s+g}\widetilde{B}_{s+g}(\frac{h}{c}) = 0$. Assume now that $v_p(c) > v_p(h) = m \geq 1$. Then by the first part of the lemma 11.3 we know that $\lim_{j\to\infty}(\frac{c}{p^m})^{s+g}\widetilde{B}_{s+g}(\frac{h/p^m}{c/p^m})$ exists. It follows $\lim_{j\to\infty} c^{s+g}\widetilde{B}_{s+g}(\frac{h}{c}) = 0$ since $m \geq 1$. □

With Lemma 11.3 it is now easy to prove Lemma 11.2 for the case where $p|c$. We have

$$\lim_{j\to\infty} D_{s+g,t}^{r(mod\ f)}(a,c) = \lim_{j\to\infty}\sum_{\substack{1\leq h\leq c\\ h\equiv r(mod\ f)}} c^{s+g-1}\widetilde{B}_{s+g-1}(\frac{h}{c})\widetilde{B}_t(\frac{ah}{c})$$

$$= \sum_{\substack{1\leq h\leq c\\ h\equiv r(mod\ f)\\ (p,h)=1}} c^{s-1}\widetilde{B}_{s-1}(\frac{h}{c})\widetilde{B}_t(\frac{ah}{c}) \tag{11.12}$$

On the other hand we have

$$
\begin{aligned}
&D_{s,t}^{r(mod\ f)}(a,c) - p^{s-1}D_{s,t}^{p^{-1}r(mod\ f)}(pa,c) \\
&= D_{s,t}^{r(mod\ f)}(a,c) - p^{s-1}D_{s,t}^{p^{-1}r(mod\ f)}(a,c/p) \\
&= \sum_{\substack{1\le h\le c \\ h\equiv r(mod\ f)}} c^{s-1}\widetilde{B}_{s-1}(\frac{h}{c})\widetilde{B}_t(\frac{ah}{c}) - \sum_{\substack{1\le h\le c/p \\ h\equiv p^{-1}r(mod\ f)}} p^{s-1}(\frac{c}{p})^{s-1}\widetilde{B}_{s-1}(\frac{h}{c/p})\widetilde{B}_t(\frac{ah}{c/p}) \\
&= \sum_{\substack{1\le h\le c \\ h\equiv r(mod\ f)}} c^{s-1}\widetilde{B}_{s-1}(\frac{h}{c})\widetilde{B}_t(\frac{ah}{c}) - \sum_{\substack{1\le h\le c \\ h\equiv r(mod\ f) \\ h\equiv 0(mod\ p)}} c^{s-1}\widetilde{B}_{s-1}(\frac{h}{c})\widetilde{B}_t(\frac{ah}{c}) \\
&= \sum_{\substack{1\le h\le c \\ h\equiv r(mod\ f) \\ (p,h)=1}} c^{s-1}\widetilde{B}_{s-1}(\frac{h}{c})\widetilde{B}_t(\frac{ah}{c}).
\end{aligned}
$$

Compare with (11.12). This concludes the proof of lemma 11.2. □

11.3 Moments of Eisenstein series

In this section we compute the moments of certain Eisenstein series. This will turn out to be essential for the proof of theorem 6.1.

We remind the reader that for $r \in \mathbb{Z}/f\mathbb{Z}$ we have defined

$$E_k(r,z) = \left(\frac{(-1)^k(2\pi i)^k}{(k-1)!}\right)^{-1} \sum_{m,n}{}' \frac{e^{-2\pi imr/f}}{(m+nf\tau)^k}.$$

where $r \in \mathbb{Z}/f\mathbb{Z}$.

Those Eisenstein series are modular with respect to the group $\Gamma_0(f)$, in the sense that for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(f)$ we have the transformation formula

$$E_k(\gamma \star r, \gamma\tau)(c\tau+d)^{-k} = E_k(r,\tau). \tag{11.13}$$

where $\gamma \star r = dr(mod\ f)$. In order to simplify the notation we define only for this section

$$E_{k,r}(\tau) := E_k(r,\tau).$$

Let

$$E_{k,r}(\tau) = \sum_{n\geq 0} a_{E_{k,r}}(n) q_\tau^n.$$

be the q-expansion at $i\infty$. We want to compute the behaviour $E_{k,r}(\tau)$ in a neighbourhood of a cusp $\frac{a}{c} \in \Gamma_0(f)(i\infty)$. For that we will use the transformation formula (11.13). Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(f)$. Then we have the identity

$$\gamma(it') = \frac{ait'+b}{cit'+d} = \frac{a}{c} - \frac{1}{c(cit'+d)}, \quad t' > 0.$$

From (11.13) we deduce

$$\begin{aligned} E_{k,r}(it') &= E_{k,dr}(\gamma(it'))(cit'+d)^{-k} \\ &= E_{k,dr}\left(\frac{a}{c} - \frac{1}{c(cit'+d)}\right)(cit'+d)^{-k} \\ &= \frac{E_{k,dr}\left(\frac{a}{c} + it'\right)}{(-ict')^{-k}} \end{aligned}$$

where $it = -\frac{1}{c(cit'+d)}$. When $t' \to \infty$, $t \to 0$. We thus deduce the formula

$$\lim_{t\to 0}\left(E_{k,r}(\frac{a}{c} + it) - \frac{i^k a_{E_{k,ar}}(0)}{(ct)^k}\right) = 0 \tag{11.14}$$

Observe that the convergence to 0 in (11.14) is exponential.

Let $\delta = \sum_{d_0|N_0, r\in\mathbb{Z}/f\mathbb{Z}} n(d_0,r)[d_0,r] \in D(N_0,f)^{\langle p\rangle}$ be a good divisor. Remember that

$$F_{k,\delta}(j,z) = \sum_{d_0,r} n(d_0,r) d_0 E_{k,jr}(d_0 z).$$

Because of the choice of the $n(d_0,r)$'s we have

$$\begin{aligned} F_{k,\delta}(j,z) &= \sum_{d_0,r} n(d_0,r) d_0 E_{k,jr}(d_0 z) \\ &= \sum_{d_0,r} n(d_0,r) d_0 \left(E_{k,jr}(d_0 z) - \frac{i^k a_{E_{k,ajr}}(0)}{(tc)^k}\right) \\ &= \sum_{d_0,r} n(d_0,r) d_0 \left(E_{k,jr}(d_0 z) - a_{E_{k,ajr}}(0)\right) \end{aligned}$$

For the last equality we have used the assumption $\sum_{d_0|N_0} n(d_0,r)d_0 = 0$ for every $r \in \mathbb{Z}/f\mathbb{Z}$. Let $\frac{a}{c} \in \Gamma_0(f)(i\infty)$. Since

$$\lim_{t\to 0}\left(E_{k,jr}(a/c+it) - \frac{i^k a_{E_{k,jr}}(0)}{(tc)^k}\right) = 0$$

we find

$$lim_{t\to 0}F_{k,\delta}(j,a/c+it) = 0. \tag{11.15}$$

The limit (11.15) is valid for any $j \in \mathbb{Z}/f\mathbb{Z}$, $\frac{a}{c} \in \Gamma_0(f)(i\infty)$ and it converges exponentially to 0 when $t \to 0^+$.

It thus makes sense to consider line integrals of the form

$$\int_{\frac{a}{c}}^{i\infty} F_{k,\delta}(j,z)z^{s-1}dz \tag{11.16}$$

since $F_{k,\delta}(j,z)$ tends to zero exponentially for both endpoints of the line integral. In the sequel we will compute (11.16) for the integers $1 \le s \le k-1$. In order to compute (11.16) it is enough to compute integrals as in the next proposition.

Proposition 11.1 *Let $\frac{a}{c} \in \Gamma_0(f)$ with $c \ge 1$ then we have*

$$\int_{t=0}^{\infty}\left(E_{k,r}(\frac{a}{c}+it) - a_{E_{k,r}}(0)\right)t^{s-1}dt = \frac{i^s}{f^{k-1}}D_{k-s,s}^{r(mod\ f)}(a,c).$$

Before the proof of proposition 6.1 we remind the reader the Fourier expansion of periodic Bernoulli polynomials:

$$\widetilde{B}_r(x) = B_r(\{x\}) = -\frac{r!}{(2\pi i)^r}\sum_{n\neq 0}\frac{e^{2\pi inx}}{n^r}.$$

For integers $s \ge 1$ we also define

$$Li_s(z) = \sum_{n\ge 1}\frac{z^n}{n^s}$$

A direct calculation shows that

$$Li_s(e^{2\pi ix}) + (-1)^s Li_s(e^{-2\pi ix}) = -\frac{(2\pi i)^s}{s!}\widetilde{B}_s(x).$$

Proof of proposition 11.1 We have

$$(11.17)\quad \int_{t=0}^{\infty}(E_{k,r}(\frac{a}{c}+it)-a_{E_{k,r}}(0))t^{s-1}dt=$$
$$\frac{1}{f^k}\sum_{b=0}^{f-1}e^{-2\pi ibr/f}\int_0^{\infty}\left[\sum_{m\geq 1}\sum_{n\geq 1}m^{k-1}(q^m_{n(\frac{a}{c}+it)+b/f}+(-1)^k q^m_{n(\frac{a}{c}+it)-b/f})\right]t^{s-1}dt$$

Let us evaluate first the interior of the integral on the right hand side.

$$\int_0^{\infty}\left[\sum_{m\geq 1}\sum_{n\geq 1}m^{k-1}(e^{2\pi im(n(\frac{a}{c}+it)+b/f)}+(-1)^k e^{2\pi im(n(\frac{a}{c}+it)-b/f)})\right]t^{s-1}dt$$
$$=\sum_{m\geq 1}\sum_{n\geq 1}m^{k-1}(e^{2\pi im(n\frac{a}{c}+b/f)}+(-1)^k e^{2\pi im(n\frac{a}{c}-b/f)})\int_0^{\infty}e^{-2\pi mnt}t^{s-1}dt$$
$$=\frac{\Gamma(s)}{(2\pi)^s}\sum_{m\geq 1}\sum_{n\geq 1}m^{k-1}(e^{2\pi im(n\frac{a}{c}+b/f)}+(-1)^k e^{2\pi im(n\frac{a}{c}-b/f)})\frac{1}{(mn)^s}$$
$$=\frac{\Gamma(s)}{(2\pi)^s}\sum_{m\geq 1}\sum_{n\geq 1}(e^{2\pi im(n\frac{a}{c}+b/f)}+(-1)^k e^{2\pi im(n\frac{a}{c}-b/f)})\frac{1}{m^{1-(k-s)}n^s}$$
$$=\frac{\Gamma(s)}{c^{1-(k-s)}(2\pi)^s}\sum_{n\geq 1}\sum_{l=1}^{c}\sum_{j\geq 0}\frac{1}{n^s(j+l/c)^{1-(k-s)}}(e^{2\pi i(cj+l)(n\frac{a}{c}+b/f)}+(-1)^k e^{2\pi i(cj+l)(n\frac{a}{c}-b/f)}).$$

Since $f|c$ we deduce

$$=\frac{\Gamma(s)}{c^{1-(k-s)}(2\pi)^s}\sum_{n\geq 1}\sum_{l=1}^{c}\sum_{j\geq 0}\frac{1}{n^s(j+l/c)^{1-(k-s)}}(e^{2\pi il(n\frac{a}{c}+b/f)}+(-1)^k e^{2\pi il(n\frac{a}{c}-b/f)})$$
$$=\frac{\Gamma(s)}{c^{1-(k-s)}(2\pi)^s}\sum_{n\geq 1}\sum_{l=1}^{c}\frac{(e^{2\pi il(n\frac{a}{c}+b/f)}+(-1)^k e^{2\pi il(n\frac{a}{c}-b/f)})}{n^s}\sum_{j\geq 0}\frac{1}{(j+l/c)^{1-(k-s)}}$$
$$=\frac{\Gamma(s)}{c^{1-(k-s)}(2\pi)^s}\sum_{n\geq 1}\sum_{l=1}^{c}\frac{(e^{2\pi il(n\frac{a}{c}+b/f)}+(-1)^k e^{2\pi il(n\frac{a}{c}-b/f)})}{n^s}\zeta(1-(k-s),\frac{l}{p})$$

where for the second equality we have used $\zeta(x,1-k)=-\frac{B_k(x)}{k}$ where $\zeta(x,s)$ is the

Hurwitz zeta function.

$$= \frac{-\Gamma(s)}{c^{1-(k-s)}(2\pi)^s} \sum_{n\geq 1} \sum_{l=1}^{c} \frac{(e^{2\pi i l(n\frac{a}{c}+b/f)} + (-1)^k e^{2\pi i l(n\frac{a}{c}-b/f)})}{n^s} \frac{\widetilde{B}_{k-s}(\frac{l}{p})}{k-s}$$

$$= \frac{-\Gamma(s)}{c^{1-(k-s)}(2\pi)^s} \sum_{l=1}^{c} [e^{2\pi i l b/f} + (-1)^k e^{-2\pi i l b/f}] \frac{\widetilde{B}_{k-s}(\frac{l}{p})}{k-s} \sum_{n\geq 1} \frac{e^{2\pi i n(\frac{la}{c})}}{n^s}$$

$$= \frac{-\Gamma(s)}{c^{1-(k-s)}(2\pi)^s} \sum_{l=1}^{c} [e^{2\pi i l b/f} + (-1)^k e^{-2\pi i l b/f}] \frac{\widetilde{B}_{k-s}(\frac{l}{p})}{k-s} Li_s(e^{2\pi i \frac{la}{c}}).$$

Substituting the latter expression in equation (11.17) we get that (11.17) is equal to

$$= \frac{-\Gamma(s)}{f^k c^{1-(k-s)}(2\pi)^s} \sum_{b=0}^{f-1} e^{-2\pi i b r/f} \sum_{l=1}^{c} [e^{2\pi i l b/f} + (-1)^k e^{-2\pi i l b/f}] \frac{\widetilde{B}_{k-s}(\frac{l}{p})}{k-s} Li_s(e^{2\pi i \frac{la}{c}})$$

$$= \frac{-\Gamma(s) f}{f^k c^{1-(k-s)}(2\pi)^s} \left(\sum_{\substack{l=1 \\ l\equiv r(mod\ f)}}^{c} Li_s(e^{2\pi i l a/c}) \frac{\widetilde{B}_{k-s}(l/c)}{k-s} + (-1)^k \sum_{\substack{l=1 \\ l\equiv -r(mod\ f)}}^{c} Li_s(e^{2\pi i l a/c}) \frac{\widetilde{B}_{k-s}(l/c)}{k-s} \right)$$

$$= \frac{-\Gamma(s) f}{f^k c^{1-(k-s)}(2\pi)^s} \sum_{\substack{l=1 \\ l\equiv r(mod\ f)}}^{c} \frac{\widetilde{B}_{k-s}(l/c)}{k-s} [Li_s(e^{2\pi i l a/c}) + (-1)^s Li_s(e^{-2\pi i l a/c})]$$

$$= \frac{i^s c^{(k-s)-1}}{f^{k-1}} \sum_{\substack{l=1 \\ l\equiv r(mod\ f)}}^{c} \frac{\widetilde{B}_{k-s}(l/c)}{k-s} \frac{\widetilde{B}_s(la/c)}{s}.$$

□

We take the opportunity here to prove a functional equation between the L-function of $\widetilde{F}_{k,\delta}$ and $\widetilde{F}^*_{k,\delta}$. Before we need to introduce some definitions and prove a analytic continuation result.

Proposition 11.2 *Let $f \in M_k(G, \mathbb{C})$ where G is a discrete subgroup of $SL_2(\mathbb{Z})$ and $\frac{a}{c} \in G(i\infty)$. Define*

$$A_f(s; a, c) := e^{\pi i s/2} c^{s-1} \int_0^\infty (f(it + a/c) - a_f(0)) t^{s-1} dt$$

then $A_f(s; a, c)$ admits a meromorphic continuation on $\mathbb{C}$.

Proof Since $lim_{t\to 0}\left(f(it+a/c)-\frac{a_f(0)}{(-cit)^k}\right)$ converges to 0 exponentially we find that

$$A_f(s;a,c)$$

is holomorphic for $Re(s)>k$. We want to extend it to $\mathbb{C}\backslash\{k,0\}$ where k is the weight of f. We have

$$\begin{aligned}
&A_f(s;a,c)\\
&=e^{\pi is/2}c^{s-1}\int_{t_0}^{\infty}(f(a/c+it)-a_f(0))t^{s-1}dt\\
&\quad+e^{\pi is/2}c^{s-1}\int_0^{t_0}\left(f(a/c+it)-\frac{a_f(0)}{(-cit)^k}+\frac{a_f(0)}{(-cit)^k}-a_f(0)\right)t^{s-1}dt\\
&=e^{\pi is/2}c^{s-1}\int_{t_0}^{\infty}(f(a/c+it)-a_f(0))t^{s-1}dt+e^{\pi is/2}c^{s-1}\int_0^{t_0}\left(f(a/c+it)-\frac{a_f(0)}{(-cit)^k}\right)t^{s-1}dt\\
&\quad+e^{\pi is/2}c^{s-1}a_f(0)\int_0^{t_0}\left(\frac{1}{(-cit)^k}-1\right)t^{s-1}dt\\
&=e^{\pi is/2}c^{s-1}\int_{t_0}^{\infty}(f(a/c+it)-a_f(0))t^{s-1}dt+e^{\pi is/2}c^{s-1}\int_0^{t_0}\left(f(a/c+it)-\frac{a_f(0)}{(-cit)^k}\right)t^{s-1}dt\\
&\quad+e^{\pi is/2}c^{s-1}a_f(0)\left(\frac{t_0^{s-k}}{(-ci)^k(s-k)}-\frac{t_0^s}{s}\right).
\end{aligned}$$

□

From this computation we can deduce a very nice functional equation between $\widetilde{F}_{\delta,k}$ and $\widetilde{F}^*_{\delta,k}$.

Corollary 11.1 *Define*

$$L(\widetilde{F}_{k,\delta},s):=\int_0^{\infty}\widetilde{F}_{k,\delta}(it)t^{s-1}dt$$

and

$$L(\widetilde{F}^*_{k,\delta},s):=\int_0^{\infty}\widetilde{F}^*_{k,\delta}(it)t^{s-1}dt$$

then $L(\widetilde{F}_{k,\delta},s)$ and $L(\widetilde{F}^*_{k,\delta},s)$ are entire functions in s related by the following functional equation

$$i^k(fN_0)^{s-1}L(\widetilde{F}_{k,\delta},s)=L(\widetilde{F}^*_{k,\delta},k-s). \tag{11.18}$$

Proof Since $\widetilde{F}_{k,\delta}(z)$ and $\widetilde{F}^*_{k,\delta}(z)$ decay exponentially to 0 when t tends to 0 and $i\infty$ we get that $s \mapsto L(\widetilde{F}_{k,\delta}, s)$ and $s \mapsto L(\widetilde{F}^*_{k,\delta}, s)$ are analytic on all of $\mathbb{C}$. Let us prove the functional equation.

Using the calculations in proposition 11.2 and setting $t_0 = \frac{1}{\sqrt{fN_0}}$ we deduce that

$$L(\widetilde{F}_{k,\delta}, s) = \int_{\frac{1}{\sqrt{fN_0}}}^{\infty} \widetilde{F}_{k,\delta}(it)t^{s-1}dt + \int_0^{\frac{1}{\sqrt{fN_0}}} \widetilde{F}_{k,\delta}(it)t^{s-1}dt$$

Now applying the change of variable $t \mapsto \frac{1}{fN_0 t}$ in the second term and using equation (9.9) we find

$$L(\widetilde{F}_{k,\delta}, s) = \int_{\frac{1}{\sqrt{fN_0}}}^{\infty} \widetilde{F}_{k,\delta}(it)t^{s-1}dt + (fN_0)^{-s+1}(-1)^k i^k \int_{\frac{1}{\sqrt{fN_0}}}^{\infty} \widetilde{F}^*_{k,\delta}(it)t^{k-s-1}dt \tag{11.19}$$

Doing a similar computation we find that

$$L(\widetilde{F}^*_{k,\delta}, s) = \int_{\frac{1}{\sqrt{fN_0}}}^{\infty} \widetilde{F}^*_{k,\delta}(it)t^{s-1}dt + (fN_0)^{k-s-1} i^k \int_{\frac{1}{\sqrt{fN_0}}}^{\infty} \widetilde{F}_{k,\delta}(it)t^{k-s-1}dt \tag{11.20}$$

Comparing (11.19) with (11.20) we obtain (11.18). □

Proposition 11.3 *Let $\frac{a}{c} \in \Gamma_1(f)(i\infty)$. For the integers $1 \leq s \leq k-1$ $A_{E_{k,r}}(s; a, c)$ admits rational values. More precisely we have*

$$A_{E_{k,r}}(s; a, c) = \frac{(-1)^s c^{s-1}}{f^{k-1}} D^{r(mod\ f)}_{k-s,s}(a, c).$$

Proof It is a direct consequence of proposition 11.1. □

We can now write down an explicit formula for the moments

$$\int_{\frac{a}{c}}^{i\infty} z^n F_{k,\delta}(j, z)dz.$$

Proposition 11.4 *Let $\frac{a}{c} \in \Gamma_0(f)$ and $1 \leq s \leq k-1$. Then we have*

$$\int_{\frac{a}{c}}^{i\infty} z^n F_k(j, z)dz = \frac{1}{f^{k-1}} \sum_{l=0}^{n} \binom{n}{l} (\frac{a}{c})^{n-l}(-1)^l \sum_{d_0, r} n(d_0, r) d_0^{-l} D^{jr(mod\ f)}_{k-l-1, l+1}(a, c/d_0)$$

Proof We have

$$\begin{aligned}
A_{F_{k,\delta}(j,z)}(s;a,c) &= e^{\pi is/2}c^{s-1}\int_0^{i\infty} F_{k,\delta}(j,\frac{a}{c}+it)t^{s-1}dt \\
&= e^{\pi is/2}c^{s-1}\int_0^{\infty} F_{k,\delta}(j,\frac{a}{c}+it)t^{s-1}dt \\
&= e^{\pi is/2}c^{s-1}\int_0^{\infty} \sum_{d_0,r} n(d_0,r)d_0 E_{k,rj}(d_0(\frac{a}{c}+it))t^{s-1}dt \\
&= e^{\pi is/2}c^{s-1}\int_0^{\infty} \sum_{d_0,r} n(d_0,r)E_{k,rj}((\frac{a}{c/d_0}+id_0t))t^{s-1}d(d_0t) \\
&= e^{\pi is/2}\sum_{d_0,r} n(d_0,r)(\frac{c}{d_0})^{s-1}\int_0^{\infty} E_{k,rj}((\frac{a}{c/d_0}+id_0t))(d_0t)^{s-1}d(d_0t) \\
&= \sum_{d_0,r} n(d_0,r)A_{E_{k,jr}}(s;a,c/d_0)
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_{\frac{a}{c}}^{\infty} z^n F_{k,\delta}(j,z)dz &= \int_0^{\infty}(\frac{a}{c}+it)^n F_{k,\delta}(j,\frac{a}{c}+it)idt \\
&= \sum_{l=0}^{n}\binom{n}{l}(\frac{a}{c})^{n-l}\int_0^{\infty}(it)^l F_{k,\delta}(j,\frac{a}{c}+it)idt \\
&= \sum_{l=0}^{n}\binom{n}{l}(\frac{a}{c})^{n-l}c^{-l}e^{\frac{\pi i}{2}(l+1)}c^{(l+1)-1}\int_0^{\infty} F_{k,\delta}(j,\frac{a}{c}+it)t^{(l+1)-1}dt \\
&= \sum_{l=0}^{n}\binom{n}{l}(\frac{a}{c})^{n-l}c^{-l}A_{F_{k,\delta}(j,z)}(l+1;a,c) \\
&= \sum_{l=0}^{n}\binom{n}{l}(\frac{a}{c})^{n-l}c^{-l}\sum_{d_0,r} n(d_0,r)A_{E_{k,jr}}(l+1;a,c/d_0) \\
&= \sum_{l=0}^{n}\binom{n}{l}(\frac{a}{c})^{n-l}\sum_{d_0,r} n(d_0,r)d_0^{-l}(\frac{c}{d_0})^{-(l+1)+1}A_{E_{k,jr}}(l+1;a,c/d_0)
\end{aligned}$$

Using proposition 11.3 we find

$$= \frac{1}{f^{k-1}}\sum_{l=0}^{n}\binom{n}{l}(\frac{a}{c})^{n-l}(-1)^l\sum_{d_0,r} n(d_0,r)d_0^{-l}D_{k-l-1,l+1}^{jr(mod\ f)}(a,c/d_0).$$

□

11.4 Moments of $\widetilde{\mu}_r\{c_1 \to c_2\}$

We can now give explicit formulas for the moments of $\widetilde{\mu}_j\{i\infty \to \frac{a}{c}\}$.

Proposition 11.5 *Let $\xi = \frac{a}{c} \in \Gamma_0(fN_0)(i\infty)$ with $c \geq 1$ and let $\widetilde{\mu}_j\{i\infty \to \frac{a}{c}\}$ be as in theorem 6.1. Then we have*

$$\int_{\mathbb{X}} x^n y^m d\widetilde{\mu}_j\{i\infty \to \frac{a}{c}\}(x,y) = (1-p^{n+m}) \int_{\xi}^{i\infty} z^n \widetilde{F}_{n+m+2}(j,z)dz =$$

(11.21)

$$\frac{-12}{f^{n+m}}(1-p^{n+m}) \sum_{l=0}^{n} \binom{n}{l} (\frac{a}{c})^{n-l}(-1)^l \sum_{d_0|N_0, r\in(\mathbb{Z}/f\mathbb{Z})^\times} n(d_0,r) d_0^{-l} D^{jr(mod\ f)}_{n+m-l+1,l+1}(a, c/d_0).$$

Proof Using the first property of theorem 6.1 we have

$$\int_{\mathbb{X}} x^n y^m \widetilde{\mu}_j\{i\infty \to \frac{a}{c}\}(x,y) = (1-p^{m+n}) \int_{\frac{a}{c}}^{i\infty} z^n \widetilde{F}_{m+n+2}(j,z)dz$$

$$= -\frac{12}{f^{n+m}}(1-p^{n+m}) \sum_{l=0}^{n} \binom{n}{l} (\frac{a}{c})^{n-l}(-1)^l \sum_{d_0,r} n(d_0,r) d_0^{-l} D^{jr(mod\ f)}_{n+m-l+1,l+1}(a, c/d_0)$$

where the second equality follows from proposition 11.4. □

Remark 11.1 In the case where $\delta = \sum_{d_0,r}[d_0,r] \in D(N_0,f)^+$ and $n+m \equiv 1(mod\ 2)$ we have

$$\sum_{d_0,r} n(d_0,r) d_0^{-l} D^{jr(mod\ f)}_{n+m-l+1,l+1}(a, c/d_0) = 0.$$

Similarly when $\delta = \sum_{d_0,r}[d_0,r] \in D(N_0,f)^-$ and $n+m \equiv 0(mod\ 2)$ we have

$$\sum_{d_0,r} n(d_0,r) d_0^{-l} D^{jr(mod\ f)}_{n+m-l+1,l+1}(a, c/d_0) = 0.$$

Proposition 11.6 *Let $\widetilde{\mu}_j\{i\infty \to \frac{a}{c}\}$ be as in theorem 6.1 with $c \geq 1$ then we have*

$$\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} x^n y^m d\widetilde{\mu}_j\{i\infty \to \frac{a}{c}\}(x,y) = \int_{\xi}^{i\infty} z^n \widetilde{F}_{n+m+2,p}(j,z)dz$$

$$= \frac{-12}{f^{n+m}} \sum_{l=0}^{n} \binom{n}{l} (\frac{a}{c})^{n-l}(-1)^l$$

$$\cdot \sum_{d_0|N_0, r\in(\mathbb{Z}/f\mathbb{Z})^\times} n(d_0,r) d_0^{-l} \left(D^{jr(mod\ f)}_{n+m-l+1,l+1}(a, c/d_0) - p^{n+m-l} D^{jr(mod\ f)}_{n+m-l+1,l+1}(pa, c/d_0). \right)$$

Proof Using the fourth property of theorem 6.1 we have

$$\begin{aligned}\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} x^n y^m \widetilde{\mu}_j\{i\infty \to \frac{a}{c}\}(x,y) &= \int_{\frac{a}{c}}^{i\infty} z^n \widetilde{F}_{m+n+2,p}(j,z)dz \\ &= \int_{\frac{a}{c}}^{i\infty} z^n (\widetilde{F}_{m+n+2}(j,z) - p^{m+n+1}\widetilde{F}_{m+n+2}(j,pz))dz \\ &= \int_{\frac{a}{c}}^{i\infty} z^n \widetilde{F}_{m+n+2}(j,z)dz - p^{m+n+1}\int_{\frac{a}{c}}^{i\infty} z^n \widetilde{F}_{m+n+2}(j,pz))dz \\ &= \int_{\frac{a}{c}}^{i\infty} z^n \widetilde{F}_{m+n+2}(j,z)dz - p^m \int_{\frac{pa}{c}}^{i\infty} z^n \widetilde{F}_{m+n+2}(j,z)dz\end{aligned}$$

and using proposition 11.4 and the assumption that $p \star \delta = \delta$ we deduce

$$\begin{aligned}&\frac{-12}{f^{n+m}} \sum_{l=0}^{n} \binom{n}{l} (\frac{a}{c})^{n-l}(-1)^l \\ &\cdot \sum_{d_0|N_0, r\in(\mathbb{Z}/f\mathbb{Z})^\times} n(d_0,r)d_0^{-l}\left(D^{jr(mod\ f)}_{n+m-l+1,l+1}(a,c/d_0) - p^{n+m-l}D^{jr(mod\ f)}_{n+m-l+1,l+1}(pa,c/d_0)\right).\end{aligned}$$

□

12 Proof of theorem 6.1

In this section we prove the theorem 6.1 following essentially the same steps as [DD06]. We brake the proof in four steps.

12.1 Measures on $\mathbb{Z}_p \times \mathbb{Z}_p$

Let $\xi = \frac{a}{c} \in \Gamma_0(fN_0)(i\infty)$ with $p \nmid c$. In this subsection we prove the following crucial lemma

Lemma 12.1 *There exists a unique family of $\mathbb{Z}_p$-valued measures on $\mathbb{Z}_p \times \mathbb{Z}_p$ indexed by $(\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle$ denoted by $\nu_{\xi,j}$ for some $j \in (\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle$ such that*

$$\int_{\mathbb{Z}_p\times\mathbb{Z}_p} h(x,y)d\nu_{\xi,j}(x,y) = (1-p^{k-2})\int_{\xi}^{i\infty} h(z,1)\widetilde{F}_k(j,z)dz, \tag{12.1}$$

for every homogeneous polynomial $h(x,y) \in \mathbb{Z}[x,y]$ *of degree* $k-2$. *Moreover we have*

$$\nu_{\xi,\gamma\star j}(ax+by, cx+dy) = \nu_{\xi,j}(x,y)$$

for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(fN_0)$.

If we use the equation (12.1) to the monomials $x^n y^m$ we get

$$\int_{\mathbb{Z}_p\times\mathbb{Z}_p} x^n y^m d\nu_{\xi,j}(x,y) = -12f(1-p^{n+m}) \int_{\xi}^{i\infty} z^n F_{n+m+2}(j,z)dz$$

(12.2)

$$= -\frac{12}{f^{n+m}}(1-p^{n+m}) \sum_{l=0}^{n} \binom{n}{l} (\frac{a}{c})^{n-l}(-1)^l \sum_{d_0,r} n(d_0,r) d_0^{-l} D^{jr(mod\ f)}_{n+m-l+1,l+1}(a, c/d_0)$$

for all integers $n, m \geq 0$. We set

$$I_{n,m}(j) := (1-p^{n+m}) \sum_{l=0}^{n} \binom{n}{l} (\frac{a}{c})^{n-l}(-1)^l \sum_{d_0,r} n(d_0,r) d_0^{-l} D^{jr(mod\ f)}_{n+m-l+1,l+1}(a, c/d_0).$$

Our key tool in showing the existence and uniqueness of $\{\nu_{\xi,j}\}_{j\in(\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle}$ is the following result, which is a two variables version of a classical theorem of Mahler.

Lemma 12.2 *Let* $b_{n,m} \in \mathbb{Z}_p$ *be constants indexed by integers* $n, m \geq 0$. *There exists a unique measure* ν *on* $\mathbb{Z}_p \times \mathbb{Z}_p$ *such that*

$$\int_{\mathbb{Z}_p\times\mathbb{Z}_p} \binom{x}{n}\binom{y}{m} d\nu(x,y) = b_{n,m}.$$

We define rational numbers $c_{n,i}$'s to be

$$\binom{x}{n} = \sum_{i=0}^{n} c_{n,i} x^i.$$

for any $0 \leq n$ and $0 \leq i \leq n$.

For $j \in (\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle$ we define

$$J_{n,m}(j) = \sum_{i=0}^{n}\sum_{i=0}^{m} c_{n,i} c_{m,i'} I_{i,i'}(j).$$

So in order to prove lemma 12.1 it is enough to show that $J_{n,m}(j) \in \mathbb{Z}_p$. Note that we can ignore the denominator f in the expression in (12.2) since f is coprime to p.

Proof of the p-integrality of $J_{n,m}(j)$ In order to show the p-integrality of the terms $J_{n,m}(j)$ we need to analyze more closely the terms $I_{n,m}(j)$. By definition we have

$$I_{n,m}(r) = (1-p^{n+m}) \sum_{l=0}^{n} \binom{n}{l} (\frac{a}{c})^{n-l} (-1)^l \sum_{r \in (\mathbb{Z}/f\mathbb{Z})^\times} \sum_{d_0 | N_0} n(d_0, r) d_0^{-l} D^{jr(mod\ f)}_{n+m-l+1,l+1}(a, c/d_0).$$

Let us concentrate on the terms $\sum_{d_0|N_0} n(d_0,r) d_0^{-l} D^{jr(mod\ f)}_{n+m-l+1,l+1}(a,c/d_0)$. Using (2) of lemma 11.1 we find

$$(12.3) \quad d_0^{-l} D^{r(mod\ f)}_{k-l-1,l+1}(a, c/d_0) = (\frac{c}{d_0})^{k-l-2} \sum_{\substack{h=1 \\ h \equiv jr (mod\ f)}}^{c} \frac{\widetilde{B}_{k-l-1}(h/(c/d_0))}{k-l-1} \frac{\widetilde{B}_{l+1}(ha/c)}{l+1}.$$

We would like to think of (12.3) as the coefficients of a generating function. For the sequel we construct such a generating function.

We have by definition

$$\frac{xe^{\theta x}}{e^x - 1} = \sum_{n \geq 0} B_n(\theta) \frac{x^n}{n!}$$

We set $\theta(h) = \{\frac{ha}{c}\}$ for $1 \leq h \leq c$. For the value of h in this range we let $F_h = 1/2$ when $h = c$ and 0 otherwise. We thus have

$$\frac{xe^{\theta(h)x}}{e^x - 1} + xF_h = \sum_{t \geq 0} \widetilde{B}_{t+1}(\frac{ha}{c}) \frac{x^{t+1}}{(t+1)!}.$$

Rearranging a bit we find that

$$\frac{e^{\theta(h)x}}{e^x - 1} - \frac{1}{x} + F_h = \sum_{s \geq 0} \frac{\widetilde{B}_{s+1}(\frac{ha}{c})}{t+1} \frac{x^t}{t!} =: B(h,x)$$

We also define for $1 \leq h \leq c$

$$A(r,h,y) := \sum_{d_0 | N_0} n(d_0, r) \left(\frac{e^{\{hd_0/c\}(y/d_0)}}{e^{y/d_0} - 1} - \frac{1}{y/d_0} + F_h \right)$$

$$= \sum_{s \geq 0} \sum_{d_0 | N_0} n(d_0, r) \frac{\widetilde{B}_{s+1}(h/(c/d_0))}{s+1} (\frac{y}{d_0})^s.$$

We have

$$A(r,h,y)B(h,x) = \sum_{s,t\geq 0}\sum_{d_0|N_0} n(d_0,r)\frac{\widetilde{B}_{s+1}(h/(c/d_0))}{s+1}\frac{\widetilde{B}_{t+1}(ha/c)}{t+1}\frac{(\frac{y}{d_0})^s}{s!}\frac{(x)^t}{t!}$$

So

$$\begin{aligned}
&\sum_{\substack{1\leq h\leq c\\ h\equiv jr(\mathrm{mod}\ f)}} A(r,h,y)B(h,x)\\
&= \sum_{s,t\geq 0}\sum_{d_0|N_0} n(d_0,r)c^{-s}(c/d_0)^s \sum_{\substack{1\leq h\leq c\\ h\equiv jr(\mathrm{mod}\ f)}} \frac{\widetilde{B}_{s+1}(h/(c/d_0))}{s+1}\frac{\widetilde{B}_{t+1}(ha/c)}{t+1}\frac{y^s}{s!}\frac{x^t}{t!}\\
&= \sum_{s,t\geq 0}\sum_{d_0|N_0} n(d_0,r)c^{-s}d_0^{-t}D_{s+1,t+1}^{rj(\mathrm{mod}\ f)}(a,c/d_0)\frac{y^s}{s!}\frac{x^t}{t!}. \qquad (12.4)
\end{aligned}$$

Now taking the summation over the r's in $(\mathbb{Z}/f\mathbb{Z})^\times$ of equation (12.4) we find

$$\begin{aligned}
&\sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times}\sum_{\substack{1\leq h\leq c\\ h\equiv jr(\mathrm{mod}\ f)}} A(r,h,y)B(h,x) \qquad (12.5)\\
&= c^{-s}\sum_{s,t\geq 0}\sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times}\sum_{d_0|N_0} n(d_0,r)d_0^{-t}D_{s+1,t+1}^{rj(\mathrm{mod}\ f)}(a,c/d_0)\frac{y^s}{s!}\frac{x^t}{t!}.
\end{aligned}$$

In the summation (12.5) some cancellations occur and it is important to take them into account. We have

$$\begin{aligned}
&\sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times}\sum_{\substack{1\leq h\leq c\\ h\equiv jr(\mathrm{mod}\ f)}} A(r,h,y)B(h,x)\\
&= \sum_{\substack{1\leq h\leq c\\ (h,f)=1}} B(h,x)A(hj^{-1},h,y)\\
&= \sum_{\substack{1\leq h\leq c\\ (h,f)=1}} \left(\frac{e^{\theta(h)x}}{e^x-1}-\frac{1}{x}+F_h\right)\sum_{d_0|N_0} n(d_0,hj^{-1})\left(\frac{e^{\{hd_0/c\}(y/d_0)}}{e^{y/d_0}-1}-\frac{1}{y/d_0}+F_h\right)\\
&= \sum_{\substack{1\leq h\leq c\\ (h,f)=1}} \left(\frac{e^{\theta(h)x}-1}{e^x-1}+g(x)\right)\sum_{d_0|N_0} n(d_0,hj^{-1})\left(\frac{e^{\{hd_0/c\}(y/d_0)}}{e^{y/d_0}-1}-\frac{1}{y/d_0}\right) \qquad (12.6)
\end{aligned}$$

where $g(x) := \frac{1}{e^x-1}-\frac{1}{x}$. Note that the term F_h have vanished since $f|c$ and $(h,f)=1$.

Now we want to use the fact that $\sum_{d_0|N_0} n(d_0,r)d_0 = 0$ for all $r \in (\mathbb{Z}/f\mathbb{Z})^\times$.

Expanding (12.6) we find

$$(12.7) \qquad = \sum_{\substack{1\le h\le c\\(h,f)=1}} \left(\frac{e^{\theta(h)x}-1}{e^x-1}\right) \sum_{d_0|N_0} n(d_0,hj^{-1})\left(\frac{e^{\{hd_0/c\}(y/d_0)}}{e^{y/d_0}-1} - \frac{1}{y/d_0}\right) +$$

$$g(x) \sum_{\substack{1\le h\le c\\(h,f)=1}} \sum_{d_0|N_0} n(d_0,hj^{-1})\left(\frac{e^{\{hd_0/c\}(y/d_0)}}{e^{y/d_0}-1} - \frac{1}{y/d_0}\right).$$

Because $\sum_{d_0|N_0} n(d_0,hj^{-1})d_0 = 0$ the first term in (12.7) is equal to

$$\sum_{\substack{1\le h\le c\\(h,f)=1}} \left(\frac{e^{\theta(h)x}-1}{e^x-1}\right) \sum_{d_0|N_0} n(d_0,hj^{-1})\left(\frac{e^{\{hd_0/c\}(y/d_0)}}{e^{y/d_0}-1}\right).$$

For the second term of (12.7) we find

$$g(x) \sum_{\substack{1\le h\le c\\(h,f)=1}} \sum_{d_0|N_0} n(d_0,hj^{-1}) d_0^{-t} \frac{\widetilde{B}_{t+1}(h/(c/d_0))}{t+1}\frac{y^t}{t!} =$$

$$g(x) \sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times} \sum_{\substack{1\le h\le c\\ h\equiv jr (\mathrm{mod}\ f)}} \sum_{d_0|N_0} n(d_0,hj^{-1}) d_0^{-t} \frac{\widetilde{B}_{t+1}(h/(c/d_0))}{t+1}\frac{y^t}{t!} =$$

$$g(x) \sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times} \sum_{d_0|N_0} d_0^{-t} \sum_{\substack{1\le h\le c\\ h\equiv jr (\mathrm{mod}\ f)}} n(d_0,hj^{-1}) \frac{\widetilde{B}_{t+1}(h/(c/d_0))}{t+1}\frac{y^t}{t!} =$$

$$(12.8) \qquad g(x) \sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times} \sum_{d_0|N_0} \sum_{\substack{1\le h\le \frac{c}{d_0}\\ h\equiv jr (\mathrm{mod}\ f)}} d_0 n(d_0,hj^{-1}) d_0^{-t} \frac{\widetilde{B}_{t+1}(h/(c/d_0))}{t+1}\cdot\frac{y^t}{t!}$$

But using equation (11.1) we have

$$\sum_{\substack{1\le h\le \frac{c}{d_0}\\ h\equiv jr (\mathrm{mod}\ f)}} \frac{\widetilde{B}_{t+1}(h/(c/d_0))}{t+1} = \sum_{\substack{h (\mathrm{mod}\ f\frac{c}{fd_0})\\ h\equiv jr (\mathrm{mod}\ f)}} \frac{\widetilde{B}_{t+1}(h/(c/d_0))}{t+1}$$

$$= \left(\frac{c}{fd_0}\right)^{-t} \frac{\widetilde{B}_{t+1}(\frac{jr}{f})}{t+1}$$

It thus follows that (12.8) vanishes completely since $\sum_{d_0|N_0} n(d_0,hj^{-1})d_0 = 0$.

So with all those cancellations we find the important identity

$$\sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times}\sum_{\substack{1\le h\le c\\ h\equiv jr(mod\ f)}} A(r,h,y)B(h,x) = \sum_{\substack{1\le h\le c\\ (h,f)=1}}(\frac{e^{\theta(h)x}-1}{e^x-1})\sum_{d_0|N_0} n(d_0,hj^{-1})(\frac{e^{\beta(d_0h)(y/d_0)}}{e^{y/d_0}-1})$$

$$= c^{-s}\sum_{s,t\ge 0}\sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times}\sum_{d_0|N_0} n(d_0,r)d_0^{-t}D_{s+1,t+1}^{rj(mod\ f)}(a,c/d_0)\frac{y^s}{s!}\frac{x^t}{t!}$$

$$=: H_j(x,y).$$

where $\beta(d_0h) := \{\frac{h}{c/d_0}\}$. We have

$$\frac{\partial^s}{\partial y^s}\frac{\partial^t}{\partial x^t}H_j(x,y) = c^{-s}\sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times}\sum_{d_0|N_0} n(d_0,r)d_0^{-t}D_{s+1,t+1}^{rj(mod\ f)}(a,c/d_0)$$

and also

$$\frac{\partial^s}{\partial y^s}\frac{\partial^t}{\partial x^t}H_j(x^p,y^p) = p^{s+t}c^{-s}\sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times}\sum_{d_0|N_0} n(d_0,r)d_0^{-t}D_{s+1,t+1}^{rj(mod\ f)}(a,c/d_0).$$

Combining all this we get that

$$I_{n,m}(j) = (1-p^{n+m})\sum_{l=0}^{n}\binom{n}{l}(\frac{a}{c})^{n-l}(-1)^l\sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times}\sum_{d_0|N_0} n(d_0,r)d_0^{-l}D_{n+m-l+1,l+1}^{jr(mod\ f)}(a,c/d_0)$$

$$= (1-p^{n+m})\sum_{l=0}^{n}\binom{n}{l}(\frac{a}{c})^{n-l}(-1)^l c^{n+m-l}\frac{\partial^{n+m-l}}{\partial y^{n+m-l}}\frac{\partial^{l+1}}{\partial x^{l+1}}(H_j(x,y)+H_j(x^p,y^p))|_{(0,0)}.$$

In order to ease the notation we let $H_j^*(x,y) := H_j(x,y)+H_j(x^p,y^p)$ and $D_x = \frac{\partial}{\partial x}$ and $D_y = \frac{\partial}{\partial y}$. With this notation we get

$$(1-p^{n+m})\sum_{l=0}^{n}\binom{n}{l}(\frac{a}{c})^{n-l}(-1)^l c^{n+m-l}D_y^{n+m-l}D_x^l H_j^*(x,y)|_{(0,0)} =$$

$$(1-p^{n+m})\sum_{l=0}^{n}\binom{n}{l}a^{n-l}(-1)^l c^m D_y^{n+m-l}D_x^l H_j^*(x,y)|_{(0,0)} =$$

$$(1-p^{n+m})(cD_y)^m\sum_{l=0}^{n}\binom{n}{l}a^{n-l}(-1)^l c^m D_y^{n-l}D_x^l H_j^*(x,y)|_{(0,0)} =$$

$$(1-p^{n+m})(cD_y)^m(aD_y-D_x)^n H_j^*(x,y)|_{(0,0)}.$$

Now we do a change of variable, we set $u = e^x$ and $v = e^y$ so $D_x = \frac{\partial}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\frac{\partial v}{\partial x} = u\frac{\partial}{\partial u} := D_u$ and similarly and $D_y = \frac{\partial}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\frac{\partial v}{\partial y} = v\frac{\partial}{\partial v} := D_v$. Note that

$$H_j(u,v) = \sum_{\substack{1\leq h\leq c\\(h,f)=1}} \left(\frac{u^{\theta(h)}-1}{u-1}\right) \sum_{d_0|N_0} n(d_0, hj^{-1}) \left(\frac{v^{\beta(d_0h)/d_0}}{v^{1/d_0}-1}\right)$$

So $H_j(u,v)$ is a rational function in $u^{1/c}$ and $v^{1/c}$.

We do another change of variable we set $(u,v) = (\frac{1}{z}, w^c z^a)$ so $(z,w) = (\frac{1}{u}, u^{a/c}v^{1/c})$. Also $D_u = u\frac{\partial}{\partial u} = u(\frac{\partial}{\partial z}\frac{\partial z}{\partial u} + \frac{\partial}{\partial w}\frac{\partial w}{\partial u}) = -z\frac{\partial}{\partial z} + \frac{a}{c}w\frac{\partial}{\partial w}$ and $D_v = v\frac{\partial}{\partial v} = v(\frac{\partial}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial}{\partial w}\frac{\partial w}{\partial v}) = \frac{1}{c}w\frac{\partial}{\partial w}$. If we set $D_w = w\frac{\partial}{\partial w}$ and $D_z = z\frac{\partial}{\partial z}$ we get that

$$I_{n,m}(j) = (1-p^{n+m})D_w^m D_z^n H_j^*(u,v)|_{(1,1)}.$$

Consequently we have

$$J_{n,m}(j) = \binom{D_w}{m}\binom{D_z}{z} H_j^*(u,v)|_{(1,1)}.$$

Now the p-integrality of $J_{n,m}(j)$ is a direct consequence of the following lemma

Lemma 12.3 *Consider the subset R of $\mathbb{Z}_p(u^{1/c}, v^{1/c})$ defined by*

$$R := \left\{ \frac{P}{Q} \text{ where } P, Q \in \mathbb{Z}_p[u^{1/c}, v^{1/c}] \text{ and } Q(1,1) \in \mathbb{Z}_p^\times \right\}$$

then R is a ring stable under the operators $\binom{D_w}{m}$ and $\binom{D_z}{n}$. Furthermore $H(u,v) \in R$.

Proof The same as in [DD06]. □

It thus follows that the expression in (12.2) lies in $\mathbb{Z}_p$. This concludes the proof of lemma 12.1. □

12.2 A partial modular symbol of measures on $\mathbb{Z}_p \times \mathbb{Z}_p$

In this subsection, we use the family of measures $\{\nu_{\xi,j}\}_{j\in(\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle}$ of lemma 12.1 to construct a family of partial modular symbols (supported on the set of cusps $\Gamma_0(fN_0)(i\infty)$) of measures on $\mathbb{Z}_p \times \mathbb{Z}_p$ encoding the periods of $\{\widetilde{F}_k(j,z)\}_{j\in(\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle}$. Note that $\mathbb{Z}_p \times \mathbb{Z}_p$ is stable under the action of $\Gamma_0(fN_0)$.

Lemma 12.4 *There exists a unique family of partial modular symbols* $\{\nu_j\}_{j\in(\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle}$ *supported on the set of cusps* $\Gamma_0(fN_0)(i\infty)$ *of* $\mathbb{Z}_p$*-valued measures on* $\mathbb{Z}_p\times\mathbb{Z}_p$ *such that*

$$\int_{\mathbb{Z}_p\times\mathbb{Z}_p} h(x,y)d\nu_j\{r\to s\}(x,y) = (1-p^{k-2})\int_r^s h(z,1)\widetilde{F}_k(j,z)dz$$

for $r,s\in\Gamma_0(fN_0)(i\infty)$ *and every homogeneous polynomial* $h(x,y)\in\mathbb{Z}[x,y]$ *of degree* $k-2$. *Furthermore if* $\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0(fN_0)$ *then* $\nu_j\{r\to s\}(U)=\nu_{\gamma\star j}\{\gamma r\to\gamma s\}(\gamma U)$. *So in this sense the measures are* $\Gamma_0(fN_0)$*-invariant.*

Proof Uniqueness is easy. We must show the existence. Let M denote the $\Gamma_0(fN_0)$-module of degree zero divisors on the set $\Gamma_0(fN_0)(i\infty)=\Gamma(i\infty)$. Let $M'\subseteq M$ be the set of divisors $m\in Div_0(\Gamma_0(fN_0)(i\infty))$ for which there exists a family of $\mathbb{Z}_p$-valued measures indexed by $(\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle$, $\{\nu_j\{m\}\}_{j\in(\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle}$ on $\mathbb{Z}_p\times\mathbb{Z}_p$, such that

$$\int_{\mathbb{Z}_p\times\mathbb{Z}_p} h(x,y)d\nu_j\{m\}(x,y) = (1-p^{k-2})\int_m h(z,1)\widetilde{F}_k(j,z)dz.$$

Here $\int_m$ is defined by $\int_{[c_1]-[c_2]}:=\int_{c_1}^{c_2}$, and extend by linearity. We must show that $M'=M$.

It is clear that M' is a subgroup of M. We will show that M' is $\Gamma_0(fN_0)$-stable. Let $m\in M'$ and $\gamma=\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in\Gamma_0(fN_0)$; for compact open $U\subseteq\mathbb{Z}_p\times\mathbb{Z}_p$ we define

$$\nu_j\{\gamma m\}(U):=\nu_{\gamma^{-1}\star j}\{m\}(\gamma^{-1}U)$$

Define also a right action of $\Gamma_0(fN_0)$ on the space of polynomials in two variables by

$$h|_\gamma(x,y)=h(Ax+By,Cx+Dy).$$

We calculate

$$\begin{aligned}
&\int_{\mathbb{Z}_p\times\mathbb{Z}_p} h(u,v)d\nu_j\{\gamma m\}(u,v) \\
&= \int_{\mathbb{Z}_p\times\mathbb{Z}_p} h(u,v)d\nu_{\gamma^{-1}\star j}\{m\}(\gamma^{-1}(u,v)) \\
&= \int_{\mathbb{Z}_p\times\mathbb{Z}_p} h|_\gamma(x,y)d\nu_{\gamma^{-1}\star j}\{m\}(x,y)
\end{aligned}$$

where $\gamma^{-1}(u,v) = (x,y)$.

$$\begin{aligned}
&= \int_{\mathbb{Z}_p\times\mathbb{Z}_p} h|_\gamma(x,y)d\nu_{\gamma^{-1}\star j}\{m\}(x,y) \\
&= (1-p^{k-2})\int_m h|_\gamma(z,1)\widetilde{F}_k(\gamma^{-1}\star j,z)dz \\
&= (1-p^{k-2})\int_{\gamma m} h(z,1)\widetilde{F}_k(j,z)dz
\end{aligned}$$

where in the last line we use the change of variables $u = \gamma z$ and the fact that

$$\widetilde{F}_k(\gamma^{-1}\star j,\gamma^{-1}z)d(\gamma^{-1}z) = \widetilde{F}_k(j,z)dz.$$

Therefore M' is a $\Gamma_0(fN_0)$-submodule of M. Lemma 12.1 shows that

$$[a/c] - [i\infty] \in M'$$

when p does not divide c. Finally we claim that

$$\mathbb{Z}[\Gamma_0(fN_0)]\{[a/c] - [i\infty]\}_{p\nmid c} = M.$$

. Let us prove this last assertion. Let us take $1, fN_0 \in \mathbb{Z}$ which a are obviously coprime. Note that $p \nmid fN_0$ so $[i\infty] - [\frac{1}{fN_0}] \in M'$. Let $\frac{a}{c} \in \Gamma_0(fN_0)(i\infty)$ with $p|c$. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(fN_0)$ so $\gamma(i\infty) = \frac{a}{c}$. Then

$$\gamma([i\infty] - \left[\frac{1}{fN_0}\right]) = \left[\frac{a}{c}\right] - \left[\frac{a+bfN_0}{c+dfN_0}\right]$$

Note that $p \nmid (c+dfN_0)$. We thus have

$$\left[\frac{a}{c}\right] - [i\infty] = \gamma([i\infty] - \left[\frac{1}{fN_0}\right]) + (\left[\frac{a+bfN_0}{c+dfN_0}\right] - [i\infty]) \in M'$$

Finally the $\Gamma_0(fN_0)$-invariance of $\nu_j\{m\}$ follows from its definition. □

12.3 From $\mathbb{Z}_p \times \mathbb{Z}_p$ to $\mathbb{X}$

In this section we show that the family of measures $\{\nu_j\{x \to y\}\}_{j\in(\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle}$ (lemma 12.4) are supported on the set $\mathbb{X} \subseteq \mathbb{Z}_p \times \mathbb{Z}_p$ of primitive vectors. We start with some lemma.

Lemma 12.5 *Let $c_1, c_2 \in \Gamma(i\infty)$. We have*

$$\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} h(x,y)d\nu_j\{c_1 \to c_2\}(x,y) = \int_{c_1}^{c_2} h(z,1)\widetilde{F}_{k,p}(j,z)dz$$

for every homogeneous polynomial $h(x,y) \in \mathbb{Z}[x,y]$ of degree $k-2$.

Proof The characteristic function of the open set $\mathbb{Z}_p \times \mathbb{Z}_p^\times$ is $(x,y) \mapsto lim_{j\to\infty} y^{(p-1)p^j}$. Let $\xi = \frac{a}{c} \in \Gamma_0(fN_0)(i\infty)$ then for $n, m \geq 0$, we have

$$\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} x^n y^m d\nu\{i\infty \to \xi\}(x,y) = lim_{j\to\infty} \int_{\mathbb{Z}_p\times\mathbb{Z}_p} x^n y^{m+(p-1)p^j} d\nu_j\{\xi \to \infty\}(x,y)$$

$$= \lim_{j\to\infty} \frac{-12}{f^{n+m+(p-1)p^j}}(1 - p^{n+m+(p-1)p^j})$$

$$\cdot \sum_{l=0}^{n} \binom{n}{l} (\frac{a}{c})^{n-l}(-1)^l \sum_{d_0,r} n(d_0,r) d_0^{-l} D^{jr(mod\ f)}_{n+m-l+1+(p-1)p^j, l+1}(a, c/d_0)$$

(12.9)

$$= \frac{-12}{f^{n+m}} \sum_{l=0}^{n} \binom{n}{l} (\frac{a}{c})^{n-l}(-1)^l \sum_{d_0,r} n(d_0,r) d_0^{-l} \lim_{j\to\infty} D^{jr(mod\ f)}_{n+m-l+1+(p-1)p^j, l+1}(a, c/d_0)$$

Meanwhile we calculate

$$\int_{\frac{a}{c}}^{i\infty} z^n \widetilde{F}_{k,p}(j,z)dz$$

$$= \int_{\frac{a}{c}}^{i\infty} z^n \widetilde{F}_k(j,z)dz - p^{k-1}\int_{\frac{a}{c}}^{i\infty} z^n \widetilde{F}_k(j,pz)dz$$

$$= \int_{\frac{a}{c}}^{i\infty} z^n \widetilde{F}_k(j,z)dz - p^{k-n-2}\int_{\frac{pa}{c}}^{i\infty} z^n \widetilde{F}_k(j,z)dz$$

$$= \frac{12}{f^{n+m}}\sum_{l=0}^{n}\binom{n}{l}(\frac{a}{c})^{n-l}(-1)^l \sum_{d_0,r} n(d_0,r)d_0^{-l}D^{jr(mod\ f)}_{n+m-l+1,l+1}(a,c/d_0)$$

$$\frac{-12}{f^{n+m}}p^{(n+m+2)-n-2}\sum_{l=0}^{n}\binom{n}{l}(\frac{pa}{c})^{n-l}(-1)^l \sum_{d_0,r} n(d_0,r)d_0^{-l}D^{jr(mod\ f)}_{n+m-l+1,l+1}(pa,c/d_0)$$

(12.10)

$$= \frac{12}{f^{n+m}}\sum_{l=0}^{n}\binom{n}{l}(\frac{a}{c})^{n-l}(-1)^l$$

$$\cdot \sum_{d_0,r} n(d_0,r)d_0^{-l}(D^{jr(mod\ f)}_{n+m-l+1,l+1}(a,c/d_0) - p^{n+m-l}D^{jr(mod\ f)}_{n+m-l+1,l+1}(pa,c/d_0))$$

Combining lemma 11.2 with the assumption that $p \star \delta = \delta$ gives us that (12.9) is equal to (12.10). $\square$

Let $r, s \in \Gamma(i\infty)$. We want to show that the measures $\nu_j\{r \to s\}$ are supported on

the set $\mathbb{X} \subseteq \mathbb{Z}_p \times \mathbb{Z}_p$ of primitive vectors.

Lemma 12.6 *Let $r, s \in \Gamma(i\infty)$. Then the measures $\nu_j\{r \to s\}$ are supported on $\mathbb{X}$.*

Proof Let $\gamma \in \Gamma_0(fN_0)$. Let $h(x,y) \in \mathbb{Z}[x,y]$ be a homogeneous polynomial of degree $k-2 = m+n-2$.

$$\begin{aligned}\int_{\gamma(\mathbb{Z}_p \times \mathbb{Z}_p^\times)} h(x,y) d\nu_j\{r \to s\}(x,y) &= \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} h|_\gamma(x,y) d\nu_j\{r \to s\}(\gamma(x,y)) \\ &= \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} h|_\gamma(x,y) d\nu_{\gamma^{-1}\star j}\{\gamma^{-1}r \to \gamma^{-1}s\}(x,y) \\ &= \int_{\gamma^{-1}r}^{\gamma^{-1}s} h|_\gamma(z,1)\widetilde{F}_{k,p}(\gamma^{-1}\star j, z)dz \\ &= \int_r^s h(z,1)\widetilde{F}_{k,p}(\gamma^{-1}\star j, \gamma^{-1}z)d(\gamma^{-1}z)\end{aligned}$$

In terms of matrices, if we let $SL_2(\mathbb{Z})\backslash M(p) = \{\gamma_i\}_{i=1}^{p+1}$ be a complete set of representatives with $\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ then

$$T_k(p)E_k(j,z) = p^{k-1}\sum_{i=1}^{p+1} E_k(d_i j, \gamma_i z)\frac{1}{(c_i z + d_i)^k} \tag{12.11}$$

On the other hand using equation (4.15) gives us

$$T_k(p)E_k(j,z) = p^{k-1}E_k(j,z) + E_k(pj,z). \tag{12.12}$$

Let $P = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and $\{\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}\}$ be a complete set of representatives of $\Gamma_0(fN_0)/\Gamma_0(pfN_0)$. Then

$$\begin{aligned}
&\frac{-1}{12f}\sum_{i=1}^{p+1}\widetilde{F}_{k,p}(\gamma_i^{-1}\star j,\gamma_i^{-1}z)d(\gamma_i^{-1}z)\\
&=\sum_{i=1}^{p+1}\sum_{d_0,r}n(d_0,r)d_0E_k(\gamma_i^{-1}\star rj,d_0\gamma_i^{-1}z)d(\gamma_i^{-1}z)-\\
&\quad p^{k-1}\sum_{i=1}^{p+1}\sum_{d_0,r}n(d_0,r)d_0E_k(\gamma_i^{-1}\star p^{-1}rj,pd_0\gamma_i^{-1}z))d(\gamma_i^{-1}z)\\
&=\sum_{i=1}^{p+1}\sum_{d_0,r}n(d_0,r)d_0E_k(\gamma_i^{-1}\star rj,d_0\gamma_i^{-1}z)d(\gamma_i^{-1}z)-\\
&\quad p^{k-1}\sum_{i=1}^{p+1}\sum_{d_0,r}n(d_0,r)d_0E_k(\gamma_i^{-1}\star p^{-1}rj,pd_0\gamma_i^{-1}z))d(\gamma_i^{-1}z)\\
&=\sum_{d_0,r}n(d_0,r)d_0\sum_{i=1}^{p+1}E_k(a_irj,d_0\gamma_i^{-1}z)d(\gamma_i^{-1}z)-\qquad (12.13)\\
&\quad p^{k-1}\sum_{d_0,r}n(d_0,r)d_0\sum_{i=1}^{p+1}E_k(p^{-1}a_irj,pd_0\gamma_i^{-1}z)d(\gamma_i^{-1}z).
\end{aligned}$$

Note that in the last equation we could get rid of p^{-1} since $p\star\delta=\delta$.

Since $E_k(r,\gamma z)d(\gamma z)=E_k(\gamma^{-1}\star r,z)dz$ for any $\gamma\in\Gamma_0(f)$ we find that

$$\begin{aligned}
E_k(a_irj,d_0\gamma_i^{-1}z)d(\gamma_i^{-1}z)&=E_k(\gamma_i\star(a_irj),d_0z)dz\\
&=E_k(jr,d_0z)dz.
\end{aligned}$$

Note also that $pd_0\gamma_i^{-1}z=d_0P\gamma_i^{-1}z$. The set $\{P\gamma_i^{-1}\}_{i=1}^{p+1}$ is a complete set of representatives of $SL_2(\mathbb{Z})\backslash M(p)$. Using that (12.11) is equal to (12.12) we obtain that

$$\begin{aligned}
p^{k-1}\sum_{i=1}^{p+1}E_k(a_irj,d_0P\gamma_i^{-1}z)d(\gamma_i^{-1}z)&=p^{k-1}\sum_{i=1}^{p+1}E_k(a_irj,d_0P\gamma_i^{-1}z)d(P\gamma_i^{-1}z)\\
&=T_k(p)E_k(rj,z)dz\\
&=(p^{k-1}E_k(rj,z)+E_k(prj,z))dz.
\end{aligned}$$

Using the fact that $p \star \delta = \delta$ we find that

$$\sum_{d_0,r} n(d_0,r)d_0(p^{k-1}E_k(rj,d_0z) + E_k(prj,d_0z))dz = (p^{k-1}+1)\sum_{d_0,r} n(d_0,r)d_0E_k(rj,d_0z).$$

Substituting the latter expression in (12.13) we get

$$-12f((p+1)-(p^{k-1}+1))\sum_{d_0,r} n(d_0,r)d_0E_k(rj,d_0) = (p-p^{k-1})\widetilde{F}_k(j,z).$$

Finally note that $\cup_{i=1}^{p+1}\gamma_i(\mathbb{Z}_p \times \mathbb{Z}_p^\times)$ is a degree p cover of $\mathbb{X}$. Hence we get

$$\begin{aligned}
p\int_{\mathbb{X}} h(x,y)d\nu_j\{r\to\}(x,y) &= \sum_{i=1}^{p+1}\int_{\gamma_i(\mathbb{Z}_p\times\mathbb{Z}_p^\times)} h(x,y)d\nu_j\{r\to\}(x,y)\\
&= \sum_{i=1}^{p+1}\int_r^s h(z,1)\widetilde{F}_{k,p}(\gamma_i^{-1}\star j,\gamma_i^{-1}z)d(\gamma^{-1}z)\\
&= (p-p^{k-1})\int_r^s h(z,1)\widetilde{F}_{k,p}(j,z)dz\\
&= (p-p^{k-1})\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} h(x,y)d\nu_j\{r\to s\}(x,y)\\
&= p\int_{\mathbb{Z}_p\times\mathbb{Z}_p} h(x,y)d\nu_j\{r\to s\}(x,y)
\end{aligned}$$

Since this holds for any h homogeneous of degree k we get that the support of $\nu_j\{r\to s\}$ is included in $\mathbb{X}$. $\square$

12.4 The measure $\widetilde{\mu}_r\{c_1\to c_2\}$ is $\widetilde{\Gamma}_0$ invariant

The compact open set $\mathbb{X}$ is a fundamental domain for the action of multiplication by p on $\mathbb{Q}_p^2\backslash\{(0,0)\}$, $\begin{pmatrix} p & 0\\ 0 & 1\end{pmatrix}(x,y) = (px,py)$. Hence if we define for compact open $U\subseteq\mathbb{X}$:

$$\widetilde{\mu}_j\{r\to s\}(U) := \nu_j\{r\to s\}(U)$$

then $\widetilde{\mu}_j$ extends uniquely to a $\Gamma_0(fN_0)$-invariant partial modular symbol of $\mathbb{Z}_p$-valued measures on $\mathbb{Q}_p^2\backslash\{0\}$ which is invariant under the action of multiplication by p:

$$\widetilde{\mu}_j\{r\to s\}(pU) = \widetilde{\mu}_j\{r\to s\}(U)$$

for all compact open $U \subseteq \mathbb{Q}_p^2 \backslash \{(0,0)\}$. This almost proves Theorem 8. It remains to show that $\widetilde{\mu}_j$ is $\widetilde{\Gamma}_0$-invariant i.e. for all compact open set $U \subseteq \mathbb{Q}_p^2 \backslash \{(0,0)\}$

$$\widetilde{\mu}_{\gamma * j}\{\gamma r \to \gamma s\}(\gamma U) = \widetilde{\mu}_j\{r \to s\}(U).$$

Note that $\widetilde{\Gamma}_0 = \langle \Gamma_0(fN_0), P \rangle$ where $P = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$.

Lemma 12.7 *The partial modular symbol $\widetilde{\mu}_j$ is invariant under $\widetilde{\Gamma}_0$.*

Proof Since $\widetilde{\Gamma}_0$ is generated by $\Gamma_0(fN_0)$ and $P = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$, and that $\widetilde{\mu}_j$ is $\Gamma_0(fN_0)$-invariant, it suffices to show that $\widetilde{\mu}_j$ is invariant for the action of P. For a homogeneous polynomial $h(x,y)$ of degree $k-2$, we have

$$\begin{aligned}\int_{\mathbb{X}} h(x,y) d\widetilde{\mu}_{P^{-1}*j}\{P^{-1}r \to P^{-1}s\}(P^{-1}(x,y)) &= \int_{\mathbb{X}} h(x,y) d\widetilde{\mu}_j\{\frac{r}{p} \to \frac{s}{p}\}(x/p, y) \\ &= \int_{P^{-1}\mathbb{X}} h(px,y) d\widetilde{\mu}_j\{\frac{r}{p} \to \frac{s}{p}\}(x,y)\end{aligned}$$

Writing $P^{-1}\mathbb{X}$ as a disjoint union

$$\begin{aligned} P^{-1}\mathbb{X} &= (\mathbb{Z}_p \times \mathbb{Z}_p^\times) \bigsqcup (\frac{1}{p}\mathbb{Z}_p^\times \times \mathbb{Z}_p) \\ &= (\mathbb{Z}_p \times \mathbb{Z}_p^\times) \bigsqcup \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}^{-1} (\mathbb{Z}_p^\times \times p\mathbb{Z}_p) \end{aligned} \tag{12.14}$$

Using the invariance of $\widetilde{\mu}_j$ under multiplication by p, (12.14) becomes

$$\begin{aligned}&\int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} h(px,y) d\widetilde{\mu}_j\{r/p \to s/p\}(x,y) \quad + \int_{\mathbb{Z}_p^\times \times p\mathbb{Z}_p} h(x,y/p) d\widetilde{\mu}_j\{r/p \to s/p\}(x,y) \\ &= (p^{2-k} + (1-p^{2-k})) \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} h(px,y) d\widetilde{\mu}_j\{r/p \to s/p\}(x,y) \\ &\quad + p^{2-k} \int_{\mathbb{Z}_p^\times \times p\mathbb{Z}_p} h(px,y) d\widetilde{\mu}_j\{r/p \to s/p\}(x,y) \\ &= p^{2-k} \int_{\mathbb{X}} h(px,y) d\widetilde{\mu}_j\{r/p \to s/p\}(x,y) \\ &\quad + (1-p^{2-k}) \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} h(px,y) d\widetilde{\mu}_j\{r/p \to s/p\}(x,y)\end{aligned}$$

$$= p^{2-k}(1-p^{2-k})\int_{r/p}^{s/p} h(pz,1)\widetilde{F}_k(j,z)dz \quad + (1-p^{2-k})\int_{r/p}^{s/p} h(pz,1)\widetilde{F}_k(j,z)^* dz$$

$$= (p^{2-k}-1)\int_{r/p}^{s/p} h(pz,1)\widetilde{F}_k(j,z)dz$$

$$+ (1-p^{2-k})\int_{r/p}^{s/p} h(pz,1)(\widetilde{F}_k(j,z) - p^{k-1}\widetilde{F}_k(j,pz))dz$$

$$= p^{k-1}(1-p^{2-k})\int_{r/p}^{s/p} h(pz,1)\widetilde{F}_k(j,pz)dz$$

$$= (1-p^{k-2})\int_{r}^{s} h(w,1)\widetilde{F}_k(j,w)dw$$

$$= \int_{\mathbb{X}} h(x,y)d\widetilde{\mu}_j\{r\to s\}(x,y)$$

This concludes the proof of theorem 6.1. □

13 The measure $\widetilde{\mu}_r\{c_1 \to c_2\}$ is $\mathbb{Z}$-valued

In this section we want to prove the integrality of the measures $\widetilde{\mu}_j\{i\infty \to \frac{a}{c}\}$ of theorem 6.1 for $\frac{a}{c} \in \Gamma_0(fN_0)(i\infty)$. We use the same approach as in [Das05].

Let $e \geq 1$ be a positive integer divisible by fN_0 but not by p and let

$$Z = \varprojlim_n \mathbb{Z}/ep^n\mathbb{Z} \simeq \mathbb{Z}/e\mathbb{Z} \times \mathbb{Z}_p \tag{13.1}$$

Definition 13.1 *Let $\delta = \sum_{d_0|N_0, r\in(\mathbb{Z}/f\mathbb{Z})^\times} n(d_0,r)[d_0,r] \in D(N_0,f)^{\langle p\rangle}$ be a good divisor. For each integer $k \geq 1$ and $r \in (\mathbb{Z}/f\mathbb{Z})^\times$ we define a distribution on Z by the rule*

$$\mathcal{F}_{k,r}(a+ep^nZ) := \sum_{d_0|N_0} n(d_0,r)\left(\frac{ep^n}{d_0}\right)^{k-1}\frac{\widetilde{B}_k(\frac{a}{ep^n/d_0})}{k}.$$

where a is any integer.

We have a natural action of $(\mathbb{Z}/f\mathbb{Z})^\times$ on the measures $\mathcal{F}_{k,r}$ given simply by $j\star\mathcal{F}_{k,r} = \mathcal{F}_{k,rj}$. Note that for any compact open set $U \subseteq Z$ we have

$$\mathcal{F}_{k,r}(pU) = p^{k-1}\mathcal{F}_{k,r}(U). \tag{13.2}$$

For $x \in Z$ we let x_p denote the projection of x on $\mathbb{Z}_p$.

Proposition 13.1 *The distributions $\mathcal{F}_{k,r}$ are $\mathbb{Z}_p$-valued measures, and for every compact open set $U \subseteq Z$ and every $k \geq 1, r \in (\mathbb{Z}/f\mathbb{Z})^\times$ we have*

$$\mathcal{F}_{k,r}(U) = \int_U x_p^{k-1} d\mathcal{F}_{1,r}(x).$$

Proof See [Das05]. □

Theorem 13.1 *The measures $\widetilde{\mu}_j\{\infty \to \frac{a}{c}\}$ take values in $\mathbb{Z}$.*

Proof We know already that the measure $\nu_j := \widetilde{\mu}_j\{\infty \to \frac{a}{c}\}$ takes values in $\mathbb{Z}_p$ so it is enough to show that it takes also values in $\mathbb{Z}[\frac{1}{p}]$. First we want to find a closed formula for the compact open sets of the form $\mathbb{Z}_p \times (v + p^s\mathbb{Z}_p)$ when $(v,p) = 1$. Let V be any compact open subset of $\mathbb{Z}_p^\times$. We claim that for any ball of the form $v + p^s\mathbb{Z}_p$ we can always find a sequence of polynomials $\{h_i(y)\}$ in $\mathbb{Q}_p[y]$ such that $\lim_i h_i(y) = \mathbb{1}_{(v+p^s\mathbb{Z}_p)}(y)$. We can write down explicitly such a sequence by setting

$$h_i(y) = \left(\frac{1}{p^\nu} \prod_{j \neq v (mod\ p^s)} (y - j) \right)^{p^i(p-1)}$$

where $\nu = v_p((p^s - 1)!)$. Using the latter observation and the fact that any compact open set $V \subseteq \mathbb{Z}_p^\times$ can be decomposed as a finite disjoint union of balls we find that there exists a sequence of polynomials $\{f_i(y)\}$ in $\mathbb{Q}_p[y]$ such that $\lim_i f_i(y) = \mathbb{1}_V(y)$. Let $f_i(y) = \sum_{n=0}^{d_i} c_n(i) y^n$. We have

$$\begin{aligned}
\int_{\mathbb{Z}_p \times V} d\nu_j(x,y) &= \int_{\mathbb{X}} \lim_i f_i(y) d\nu_j(x,y) \\
&= \lim_i \int_{\mathbb{X}} f_i(y) d\nu_j(x,y) \\
&= \lim_i \sum_{n=0}^{d_i} c_n(i) \int_{\mathbb{X}} y^n d\nu_j(x,u) \\
&= \lim_i -12 \sum_{n=0}^{d_i} c_n(i) \frac{1-p^n}{f^n} \sum_{\substack{d_0 | N_0 \\ r \in (\mathbb{Z}/f\mathbb{Z})^\times}} n(d_0, r) D_{n+1,1}^{jr mod\ f}(a, c/d_0) \qquad (13.3)
\end{aligned}$$

where the last equality uses proposition 11.5. Now using (2) of lemma 11.3 we find

(13.4)

$$\lim_i -12\sum_{n=0}^{d_i} c_n(i)\frac{1-p^n}{f^n}\sum_{\substack{d_0|N_0\\ r\in(\mathbb{Z}/f\mathbb{Z})^\times}} n(d_0,r)\left(\frac{c}{d_0}\right)^{s-1}\sum_{\substack{h=1\\ h\equiv jr(\mathrm{mod}\ f)}}^{c}\frac{\widetilde{B}_{n+1}(\frac{h}{c/d_0})}{n+1}\widetilde{B}_1(\frac{ha}{c}) =$$

$$-12\lim_i\sum_{n=0}^{d_i}\frac{c_n(i)}{f^n}(1-p^n)\sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times}\sum_{\substack{h=1\\ h\equiv jr(\mathrm{mod}\ f)}}^{c}\widetilde{B}_1(\frac{ha}{c})\sum_{d_0|N_0}\left(\frac{c}{d_0}\right)^n n(d_0,r)\frac{\widetilde{B}_{n+1}(\frac{h}{c/d_0})}{n+1} =$$

$$-12\lim_i\sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times}\sum_{\substack{h=1\\ h\equiv jr(\mathrm{mod}\ f)}}^{c}\widetilde{B}_1(\frac{ha}{c})\sum_{n=0}^{d_i}\frac{c_n(i)}{f^n}(1-p^n)\sum_{d_0|N_0}\left(\frac{c}{d_0}\right)^n n(d_0,r)\frac{\widetilde{B}_{n+1}(\frac{h}{c/d_0})}{n+1} =$$

$$-12\sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times}\sum_{\substack{h=1\\ h\equiv jr(\mathrm{mod}\ f)}}^{c}\widetilde{B}_1(\frac{ha}{c})\int_{h+cZ}\lim_i(f_i(\frac{x_p}{f})-f_i(\frac{px_p}{f}))d\mathcal{F}_{1,r}(x).$$

where $c=ep^t$ with $(e,p)=1$. Note that $fN_0|e$. For the last equality we have used the definition of $\mathcal{F}_{k,r}$ combined with proposition 13.1 and (13.2).

We specialize $V=v+p^s\mathbb{Z}_p$ for $v\in\mathbb{Z}$ coprime to p and $s\geq t$. With this special choice of V the limit as $i\to\infty$ for $f_i(\frac{x_p}{f})$ approaches 1 or 0 according to whether $\frac{x_p}{f}\in V$ or not, and $f_i(\frac{px_p}{f})$ approaches 0. Therefore

$$\nu_j(\mathbb{Z}_p\times V)=-12\sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times}\sum_{\substack{h=1\\ h\equiv jr(\mathrm{mod}\ f)\\ h\equiv fv(\mathrm{mod}\ p^t)}}^{ep^t}\widetilde{B}_1\left(\frac{ha}{ep^t}\right)\mathcal{F}_{1,r}(\{x\in h+ep^sZ : x_p\in fV\}) =$$

$$\nu_j(\mathbb{Z}_p\times V)=-12\sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times}\sum_{\substack{h=1\\ h\equiv jr(\mathrm{mod}\ f)\\ h\equiv fv(\mathrm{mod}\ p^t)}}^{ep^t}\widetilde{B}_1\left(\frac{ha}{ep^t}\right)\sum_{d_0|N_0}n(d_0,r)\widetilde{B}_1\left(\frac{y_h}{ep^s/d_0}\right) \quad (13.5)$$

where $y_h \equiv h(mod\, e)$ and $y_h \equiv fv(mod\, p^s)$.

$$= -12 \sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times} \sum_{\substack{h=1 \\ h\equiv jr(mod\ f) \\ h\equiv fv(mod\ p^t)}}^{ep^t} \widetilde{B}_1\left(\frac{ha}{ep^t}\right) \sum_{d_0|N_0} n(d_0,r)\widetilde{B}_1\left(\frac{y_h}{ep^s/d_0}\right)$$

$$= -12 \sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times} \sum_{\substack{h=1 \\ h\equiv jr(mod\ f) \\ h\equiv fv(mod\ p^t)}}^{ep^t} \left(\frac{ha}{ep^t} - \left[\frac{ha}{ep^t}\right] - \frac{1}{2}\right) \sum_{d_0|N_0} n(d_0,r)\left(\frac{y_h}{ep^s/d_0} - \left[\frac{y_h}{ep^s/d_0}\right] - \frac{1}{2}\right)$$

So we get

$$- \sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times} \sum_{\substack{h=1 \\ h\equiv jr(mod\ f) \\ h\equiv fv(mod\ p^t)}}^{ep^t} \left(2\frac{ha}{ep^t} - 2\left[\frac{ha}{ep^t}\right] - 1\right) \sum_{d_0|N_0} n(d_0,r)\left(2\frac{y_h}{ep^s/d_0} - 2\left[\frac{y_h}{ep^s/d_0}\right] - 1\right)$$

$$\equiv \frac{2a}{ep^t} \sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times} \sum_{\substack{h=1 \\ h\equiv jr(mod\ f) \\ h\equiv fv(mod\ p^t)}}^{ep^t} h \sum_{d_0|N_0} n(d_0,r)(2\left[\frac{y_h}{ep^s/d_0}\right] - 1)\ (mod\, \mathbb{Z}) \tag{13.6}$$

So if the right hand side of (13.6) is in $\mathbb{Z}$ for $a = 1$, it will be in $\mathbb{Z}$ for any integer a. So in order to prove the integrality of the measure ν_j it is enough to show that the right hand side of (13.6) is in $\mathbb{Z}$ for $a = 1$. If we go back to (13.5) and set $a = 1$ we can rewrite it as

(13.7)

$$\nu_j(\mathbb{Z}_p \times V) = -12 \sum_{d_0|N_0} \sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times} n(d_0,r) \sum_{\substack{h=1 \\ h\equiv jr(mod\ f) \\ h\equiv fv(mod\ p^t)}}^{ep^t} \widetilde{B}_1\left(\frac{h}{ep^t}\right) \widetilde{B}_1\left(\frac{y_h}{ep^s/d_0}\right)$$

$$= -12 \sum_{d_0|N_0} \sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times} n(d_0,r) \sum_{\substack{h=1 \\ h\equiv jr(mod\ f) \\ h\equiv fv(mod\ p^t)}}^{ep^t/d_0} \widetilde{B}_1\left(\frac{h}{ep^t/d_0}\right) \widetilde{B}_1\left(\frac{y_h}{ep^s/d_0}\right).$$

where the second equality comes from the fact that if $h \equiv h'(mod\, ep^t/d_0)$ then $y_h \equiv y_{h'}(mod\, ep^s/d_0)$ and that $\sum_{\mu(mod\ d_0)} \widetilde{B}_1(x + \frac{\mu}{d_0}) = \widetilde{B}_1(d_0 x)$.

Now fix $r \in (\mathbb{Z}/f\mathbb{Z})^\times$ and look at the term

$$\sum_{d_0|N_0} n(d_0,r) \sum_{\substack{h=1 \\ h\equiv jr(mod\ f) \\ h\equiv fv(mod\ p^t)}}^{ep^t/d_0} \widetilde{B}_1\left(\frac{h}{ep^t/d_0}\right)\widetilde{B}_1\left(\frac{y_h}{ep^s/d_0}\right) =$$

$$\sum_{d_0|N_0} n(d_0,r) \sum_{\mu=1}^{e/fd_0} \widetilde{B}_1\left(p^{s-t}\frac{p^s f\mu + eA}{ep^s/d_0}\right)\widetilde{B}_1\left(\frac{p^s f\mu + eA}{ep^s/d_0}\right), \tag{13.8}$$

where $A \in \mathbb{Z}[\frac{1}{f}]$ is chosen in such a way that $eA \equiv jr(mod\ f)$ and $eA \equiv fv(mod\, p^s)$. Note in particular that $eA \in \mathbb{Z}$ and $(eA, f) = 1$. We obtain (13.8) by doing the change of variable $h = p^s f\mu + eA$ and $y_h = h$. This is justified for the following reasons:

$$\left\{h(mod\ ep^t/d_0) : h \equiv jr(mod\ f), h \equiv fv(mod\, p^t)\right\} = \left\{p^s f\mu + eA\ (mod\ \frac{ep^t}{d_0})\right\}_{1\le\mu\le\frac{e}{fd_0}}.$$

This equality comes from the fact that $p^s f\mu \equiv p^s f\mu'(mod\ \frac{ep^t}{d_0})$ iff $\frac{e}{fd_0}|(\mu-\mu')$. Finally since $y_h = h$ we readily see that $y_h \equiv h(mod\, e)$ and also that $y_h \equiv fv(mod\, p^s)$. Therefore this change of variables makes sense and this proves (13.8).

Using the notation in [Hal85] we find that

$$\sum_{\mu=1}^{\frac{e}{fd_0}} \widetilde{B}_1\left(p^{s-t}\frac{p^s f\mu + eA}{ep^s/d_0}\right)\widetilde{B}_1\left(\frac{p^s f\mu + eA}{ep^s/d_0}\right) = C(1,1,p^{s-t}, e/fd_0, \frac{eA}{fp^s}, 0). \tag{13.9}$$

Using the Dedekind reciprocity formulas for such sums for the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (see [Hal85] for the exact formula) we find

$$\begin{aligned} C(1,1,p^{s-t},\frac{e}{fd_0},\frac{eA}{fp^s},0) = &\sum_{\mu(mod\ p^{s-t})} \widetilde{B}_1(\frac{\mu}{p^{s-t}})\widetilde{B}_1(\frac{eA}{fp^s} - \frac{\frac{e}{fd_0}\mu}{p^{s-t}}) \\ &+ \frac{1}{2}\sum_{\mu(mod\ p^{s-t})} \frac{1}{\frac{e}{fd_0}}\widetilde{B}_2\left(\frac{eA}{fp^s} - \frac{\frac{e}{fd_0}\mu}{p^{s-t}}\right) \\ &+ \frac{1}{2}\frac{p^{s-t}}{\frac{e}{fd_0}}\widetilde{B}_2\left(\frac{eA}{fp^s}\right) + \frac{1}{2}\frac{\frac{e}{fd_0}}{p^{s-t}}\widetilde{B}_2(0) \\ &- \widetilde{B}_1\left(\frac{eA}{fp^s}\right) \end{aligned}$$

Since $(d_0, p^{s-t}) = 1$ and $eA \equiv y(mod\, fp^s)$ for an integer y (not depending on d_0) such that $y \equiv jr(mod\, f)$ and $y \equiv fv(mod\, p^s)$ we can rewrite the right hand side as

$$\sum_{\mu(mod\ p^{s-t})} \widetilde{B}_1(\frac{\mu}{p^{s-t}})\widetilde{B}_1(\frac{y}{fp^s} - \frac{\frac{e}{f}\mu}{p^{s-t}}) + \frac{1}{2}\sum_{\mu(mod\ p^{s-t})} \frac{1}{\frac{e}{fd_0}}\widetilde{B}_2\left(\frac{y}{fp^s} - \frac{\frac{e}{f}\mu}{p^{s-t}}\right) + \frac{1}{2}\frac{p^{s-t}}{\frac{e}{fd_0}}\widetilde{B}_2\left(\frac{y}{fp^s}\right) + \frac{1}{2}\frac{\frac{e}{fd_0}}{p^{s-t}}\widetilde{B}_2(0) - \widetilde{B}_1\left(\frac{y}{fp^s}\right). \tag{13.10}$$

Taking the summation of (13.10) over $d_0|N_0$ weighted by $n(d_0, r)$ we see that the second summation and the third term of (13.10) vanish since $\sum_{d_0} n(d_0, r)d_0 = 0$. The fourth term doesn't contribute to any denominator dviding f. It remains to deal with the first summation and the last term. Taking the summation over all $r \in (\mathbb{Z}/f\mathbb{Z})^\times$ then we get cancellations for the prime dividing f by pairing the elements $rj(mod\, f)$ with their additive inverse $-jr(mod\, f)$. This uses the fact that $\widetilde{B}_1(-x) = -\widetilde{B}_1(x)$.

Therefore (13.7) lies in $\mathbb{Z}[\frac{1}{p}]$. This implies that $\nu_j(\mathbb{Z}_p \times V) \in \left(\mathbb{Z}[\frac{1}{p}]\right) \cap \mathbb{Z}_p = \mathbb{Z}$. Since the $\widetilde{\Gamma}_0$ translates of the sets $\mathbb{Z}_p \times V$ form a basis of compact opens for $(\mathbb{Q}_p^2\backslash\{(0,0)\}/p^{\mathbb{Z}}) \simeq \mathbb{X}$, the $\widetilde{\Gamma}_0$-invariance of the $\{\nu_j\}_{j\in(\mathbb{Z}/f\mathbb{Z})^\times}$ therefore implies that $\{\nu_j\}_{j\in(\mathbb{Z}/f\mathbb{Z})^\times}$ are $\mathbb{Z}$-valued. $\square$

Remark 13.1 Note that we have never used the fact that $\sum_{d_0|N_0} n(d_0, r) \equiv 0(mod\, f)$ for all $r \in (\mathbb{Z}/f\mathbb{Z})^\times$. In fact this condition can be dropped.

14 Explicit formulas of $\widetilde{\mu_r}\{c_1 \to c_2\}$ on balls of $\mathbb{X}$

We want to give an explicit formula of those measures on the compact open sets of the form $(u + p^s\mathbb{Z}_p) \times (v + p^s\mathbb{Z}_p)$ for $u, v \in \mathbb{Z}$ and $(u, v, p) = 1$.

Proposition 14.1 *Let $u, v \in \mathbb{Z}$ such that $(u, v) \in \mathbb{X}$. For a positive integer s, let $U_{u,v,s}$ denote the ball of radius $\frac{1}{p^s}$ around $(u, v) \in \mathbb{X}$ i.e.*

$$U_{u,v,s} = (u + p^s\mathbb{Z}_p) \times (v + p^s\mathbb{Z}_p) \subseteq \mathbb{X}.$$

Let $\frac{a}{c} \in \Gamma_0(fN_0)(i\infty)$. Let A_r be an integer such that $A_r \equiv fv(mod\, p^s)$ and $A_r \equiv$

$r(mod\ f)$ *then*

(14.1)

$$\widetilde{\mu}_j\{i\infty \to \frac{a}{c}\}(U_{u,v,s})$$

(14.2)

$$= -12 \sum_{d_0|N_0} \sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times} n(d_0,r) \sum_{h=1}^{c/fd_0} \widetilde{B}_1\left(\frac{a}{c/fd_0}\left(h+\frac{A_{rj}}{fp^s}\right)-\frac{d_0fu}{p^s}\right)\widetilde{B}_1\left(\frac{1}{c/fd_0}\left(h+\frac{A_{rj}}{fp^s}\right)\right)$$

$$= -12\sum_{d_0,r} n(d_0,r) \sum_{\substack{1\le h\le p^sc/d_0\\ h\equiv fv(mod\ p^s)\\ h\equiv rj(mod\ f)}} \widetilde{B}_1\left(\frac{ah}{p^sc/d_0}-\frac{d_0fu}{p^s}\right)\widetilde{B}_1(\frac{h}{p^sc/d_0}).$$

where as usual $\{x\}$ *denotes the fractional part of a real number. Note that if we replace* $v \mapsto v+p^s$ *or* $u \mapsto u+p^s$ *the quantity is unchanged as expected.*

Proof The proof follows essentially from the explicit formula obtained in equation (13.7) for balls of the form $\mathbb{Z}_p \times V$. We just sketch the proof. Let $c = ep^t$ where $(e,p)=1$. Assume that $p \nmid v$. Let l be such that $u \equiv lv(mod\, p^s)$. In the case where $s \ge t$ using (13.8) we have

(14.3)

$$\widetilde{\mu}_j\{i\infty \to \frac{a}{c}\}(\mathbb{Z}_p \times (v+p^s\mathbb{Z}_p)) = -12\sum_{d_0|N_0} n(d_0,r) \sum_{\mu=1}^{e/fd_0} \widetilde{B}_1\left(a\frac{p^sf\mu+eA}{ep^t/d_0}\right)\widetilde{B}_1\left(\frac{p^sf\mu+eA}{ep^s/d_0}\right)$$

Similarly when $s \le t$ we have

(14.4) $$\widetilde{\mu}_j\{i\infty \to \frac{a}{c}\}(\mathbb{Z}_p \times (v+p^s\mathbb{Z}_p))$$

(14.5) $$= -12\sum_{d_0|N_0} n(d_0,r) \sum_{\mu=1}^{ep^{t-s}/fd_0} \widetilde{B}_1\left(a\frac{p^sf\mu+eA}{ep^t/d_0}\right)\widetilde{B}_1\left(\frac{p^sf\mu+eA}{ep^t/d_0}\right).$$

Consider the matrix $\gamma = \begin{pmatrix} p^s & l \\ 0 & 1 \end{pmatrix}$ and observe that

$$\gamma(\mathbb{Z}_p \times (v+p^s\mathbb{Z}_p)) = U_{u,v,s}.$$

The $\widetilde{\Gamma}_0$-invariance of $\widetilde{\mu}_j\{i\infty \to \frac{a}{c}\}$ implies that

$$\widetilde{\mu}_j\{i\infty \to \frac{a}{c}\}(U_{u,v,s}) = \widetilde{\mu}_j\{i\infty \to \frac{a-lc}{cp^s}\}(\mathbb{Z}_p \times (v + p^s\mathbb{Z}_p))$$

Let $(a-lc, cp^s) = p^m$ and assume that $t - m \geq 0$ then using (14.4) we deduce

$$\begin{aligned}
&\widetilde{\mu}\{i\infty \to \frac{(a-lc)/p^m}{ep^{s+t-m}}\}(U_{u,v,s}) \\
&= -12 \sum_{d_0|N_0,r} n(d_0,r) \sum_{\mu=1}^{ep^{t-m}/fd_0} \widetilde{B}_1\left(\frac{(a-lc)}{p^m} \cdot \frac{p^s f\mu + eA}{ep^{s+t-m}/d_0}\right) \widetilde{B}_1\left(\frac{p^s f\mu + eA}{ep^{s+t-m}/d_0}\right) \\
&= -12 \sum_{d_0|N_0,r} n(d_0,r) \sum_{\mu=1}^{c/fd_0} \widetilde{B}_1\left(\frac{(a-lc)}{c/d_0} \cdot \frac{p^s f\mu + eA}{p^s}\right) \widetilde{B}_1\left(\frac{p^s f\mu + eA}{cp^s/d_0}\right) \\
&= -12 \sum_{d_0|N_0,r} n(d_0,r) \sum_{\mu=1}^{c/fd_0} \widetilde{B}_1\left(\frac{a}{c/fd_0} \cdot (\mu + \frac{eA}{fp^s}) - ld_0\frac{eA}{p^s}\right) \widetilde{B}_1\left(\frac{\mu + \frac{eA}{fp^s}}{c/fd_0}\right)
\end{aligned}$$

which is nothing else than (14.1). The second equality follows from the distribution relation of $\widetilde{B}_1(x)$. A similar computation holds when $t - m \leq 0$. To handle the case when $p|v$ one can use the exact same idea as in [Das05]. This concludes the proof. □

Let us verify if (14.1) is in accordance with 4) of theorem 6.1 on a simple compact open set. First note that

$$(\mathbb{Z}_p^\times \times p\mathbb{Z}_p) \coprod (\mathbb{Z}_p \times \mathbb{Z}_p^\times) = \mathbb{X}.$$

It thus follows that

$$-\tilde{\mu}_j\{i\infty \to \frac{a}{c}\}(\mathbb{Z}_p^\times \times p\mathbb{Z}_p) = \tilde{\mu}_j\{i\infty \to \frac{a}{c}\}(\mathbb{Z}_p \times \mathbb{Z}_p^\times). \tag{14.6}$$

Since $\coprod_{u=1}^{p-1} U_{u,0,1} = (\mathbb{Z}_p^\times \times p\mathbb{Z}_p)$ we deduce from (14.1) and (14.6) that

$$\tilde{\mu}_j\{i\infty \to \frac{a}{c}\}(\mathbb{Z}_p \times \mathbb{Z}_p^\times) = 12\sum_{d_0,r} n(d_0,r) \sum_{\substack{1\le h\le pc/d_0 \\ h\equiv jr(mod\ f) \\ h\equiv 0(mod\ p)}} \sum_{u=1}^{p-1} \widetilde{B}_1(\frac{ha}{pc/d_0} + \frac{d_0 fu}{p})\widetilde{B}_1(\frac{h}{pc/d_0})$$

$$= 12\sum_{d_0,r} n(d_0,r) \sum_{\substack{1\le h\le c/d_0 \\ ph\equiv jr(mod\ f)}} \sum_{u=1}^{p-1} \widetilde{B}_1(\frac{pha}{pc/d_0} + \frac{u}{p})\widetilde{B}_1(\frac{ph}{pc/d_0})$$

$$= 12\sum_{d_0,r} n(d_0,r) \sum_{\substack{1\le h\le c/d_0 \\ h\equiv jr(mod\ f)}} \sum_{u=1}^{p-1} \widetilde{B}_1(\frac{ha}{c/d_0} + \frac{u}{p})\widetilde{B}_1(\frac{h}{c/d_0}) \tag{14.7}$$

where the second equality uses the fact that $(d_0 f, p) = 1$ and the last equality uses the assumption $n(d_0, p^{-1}r) = n(d_0, r)$ for all $r \in (\mathbb{Z}/f\mathbb{Z})^\times$.

On the other hand if we use 4) of theorem 6.1 combined with the explicit formula given by equation (12.10) we find

$$\tilde{\mu}_j\{i\infty \to \frac{a}{c}\}(\mathbb{Z}_p \times \mathbb{Z}_p^\times) = -12\sum_{d_0,r} n(d_0,r)(D_{1,1}^{jr(mod\ f)}(a, c/d_0) - D_{1,1}^{jr(mod\ f)}(pa, c/d_0))$$

$$= -12\sum_{d_0,r} n(d_0,r) \sum_{\substack{1\le h\le c/d_0 \\ h\equiv jr(mod\ f)}} \widetilde{B}_1(\frac{h}{c/d_0})\widetilde{B}_1(\frac{ah}{c/d_0})$$

$$+ 12\sum_{d_0,r} n(d_0,r) \sum_{\substack{1\le h\le c/d_0 \\ h\equiv jr(mod\ f)}} \widetilde{B}_1(\frac{h}{c/d_0})\widetilde{B}_1(\frac{pah}{c/d_0}) \tag{14.8}$$

Now using the identity $\widetilde{B}_1(px) = \sum_{j=0}^{p-1} \widetilde{B}_1(x + \frac{j}{p})$ and substituting it in (14.8) we get after simplification the right hand side of (14.7).

15 Stability property of $\widetilde{\mu}_j\{c_1 \to c_2\}$ on balls of decreasing radius marked at an integral center

We want to show in this section that our measures $\widetilde{\mu}_j\{c_1 \to c_2\}$ satisfy some stability property when evaluated on a ball of decreasing radius for which the center if fixed and has integral coordinates.

Proposition 15.1 *Let $(u,v) \in \mathbb{Z}^2 \cap \mathbb{X}$ then there exists a positive integer $C(u,v)$ such that for $s \geq C(u,v)$ one has that*

$$\widetilde{\mu}_j\{c_1 \to c_2\}(U_{u,v,s}) = M \in \mathbb{Z}$$

where M is independent of (u,v). In the special case where $c_1 = i\infty$ and $c_2 = \frac{a}{c}$ one has that

$$\widetilde{\mu}_j\{i\infty \to \frac{a}{c}\}(U_{u,v,s}) = -12 \sum_{d_0|N_0, r\in(\mathbb{Z}/f\mathbb{Z})^\times} n(d_0,r) D_{1,1}^{jr(mod\ f)}(a, c/d_0)$$

for s large enough.

Proof In order to prove the proposition it is enough to show that for s large enough one has

$$\widetilde{\mu}_1\{i\infty \to \frac{a}{c}\}(U_{u,v,s}) = -12 \sum_{d_0,r} n(d_0,r) D_{1,1}^{r(mod\ f)}(a,c). \tag{15.1}$$

Because $n(d_0,r) = n(d_0,pr)$ for all $r \in (\mathbb{Z}/f\mathbb{Z})^\times$ we deduce from proposition 14.1 that

(15.2)

$$\widetilde{\mu}_1\{i\infty \to \frac{a}{c}\}(U_{u,v,s}) =$$

$$-12 \sum_{d_0|N_0} \sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times} \sum_{h=1}^{c/fd_0} n(d_0,r) \widetilde{B}_1\left(\frac{a}{c/fd_0}\left(h + \frac{A_{p^s r}}{fp^s}\right) - \frac{d_0 fu}{p^s}\right) \widetilde{B}_1\left(\frac{1}{c/fd_0}\left(h + \frac{A_{rp^s}}{fp^s}\right)\right).$$

We set $A_{rp^s} = fv + rp^s$. Substituting in the right hand side of (15.2) we obtain

$$-12 \sum_{d_0|N_0} \sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times} \sum_{h=1}^{c/fd_0} n(d_0,r) \widetilde{B}_1\left(\frac{a}{c/fd_0}\left(h + \frac{v}{p^s} + \frac{r}{f}\right) - \frac{d_0 fu}{p^s}\right)$$
$$\cdot \widetilde{B}_1\left(\frac{1}{c/fd_0}\left(h + \frac{v}{p^s} + \frac{r}{f}\right)\right)$$

(15.3)

$$= -12 \sum_{d_0|N_0} \sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times} \sum_{\substack{0\leq h\leq c/d_0-1 \\ h\equiv r(mod\ f)}} n(d_0,r) \widetilde{B}_1\left(\frac{ah}{c/d_0} + \frac{av}{c/fd_0p^s} - \frac{d_0fu}{p^s}\right) \widetilde{B}_1\left(\frac{h}{c/d_0} + \frac{v}{p^s c/fd_0}\right).$$

Now let us choose s such that $\left|\frac{av}{c/fp^s}\right| + \left|\frac{N_0fu}{p^s}\right| < \frac{1}{c}$. The equation (15.3) is then equal to

$$-12 \sum_{d_0|N_0} \sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times} \sum_{\substack{0\le h\le c/d_0-1 \\ h\equiv r(mod\ f)}} n(d_0,r)\left(\left\{\frac{ah}{c/d_0}\right\} + \frac{av}{c/fd_0p^s} - \frac{d_0fu}{p^s}\right)\left(\frac{h}{c/d_0} + \frac{v}{p^sc/fd_0}\right)$$

which can be written as

(15.4)

$$-12 \sum_{d_0|N_0} \sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times} \sum_{\substack{0\le h\le c/d_0-1 \\ h\equiv r(mod\ f)}} n(d_0,r)\left\{\frac{ah}{c/d_0}\right\}\frac{h}{c/d_0} +$$

$$-\frac{12}{p^s} \sum_{d_0|N_0} \sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times} \sum_{\substack{0\le h\le c/d_0-1 \\ h\equiv r(mod\ f)}} n(d_0,r)\left[\left(\frac{av}{c/fd_0} - d_0fu\right)\left(\frac{h}{c/d_0} + \frac{v}{p^sc/fd_0}\right) + \left\{\frac{ah}{c/d_0}\right\}\frac{v}{c/fd_0}\right]$$

The triple summation

(15.5)

$$\sum_{d_0|N_0} \sum_{r\in(\mathbb{Z}/f\mathbb{Z})^\times} \sum_{\substack{0\le h\le c/d_0-1 \\ h\equiv r(mod\ f)}} n(d_0,r)\left[\left(\frac{av}{c/fd_0} - d_0fu\right)\left(\frac{h}{c/d_0} + \frac{v}{p^sc/fd_0}\right) + \left\{\frac{ah}{c/d_0}\right\}\frac{v}{c/fd_0}\right]$$

can be bounded by a positive constant C_0 independent of s. Now choose s large enough such that $\frac{12C_0}{p^s} < \frac{1}{c^2}$. Since (15.4) is an integer and the first term of (15.4) has denominator dividing c^2 we conclude that (15.5) has to be equal to 0. This concludes the proof. □

We make the following definition

Definition 15.1 *Consider the measure $\widetilde{\mu}_j\{c_1 \to c_2\}$ on the space $\mathbb{X}$ where $c_1, c_2 \in \Gamma_0(fN_0)(i\infty)$ and $j \in (\mathbb{Z}/f\mathbb{Z})^\times$. Let $(u,v) \in \mathbb{X}$ then we say that a ball $U_{u,v,r}$ is stable with respect to the marked center (u,v) for the measure $\widetilde{\mu}_j\{c_1 \to c_2\}$ if for all $s \ge r$ one has*

$$\widetilde{\mu}_j\{c_1 \to c_2\}(U_{u,v,s}) = \widetilde{\mu}_j\{c_1 \to c_2\}(U_{u,v,r}).$$

Remark 15.1 Note by proposition 15.1 that every point $(u,v) \in \mathbb{X} \cap \mathbb{Z}^2$ is contained in some stable ball of centre (u,v) for the measure $\widetilde{\mu}_j\{c_1 \to c_2\}$. Indeed take the ball $U_{u,v,s}$ where s is big enough.

We have the following interesting theorem which a priori seems very surprising taking into account the compactness of $\mathbb{X}$.

Theorem 15.1 *Consider the $\mathbb{Z}$-valued measure $\widetilde{\mu}_r\{c_1 \to c_2\}$ on the space $\mathbb{X}$ and assume that the integer M in proposition 15.1 is not equal to 0. Then there exists no cover of $\mathbb{X}$ by stable balls with marked centres in $\mathbb{X} \cap \mathbb{Z}^2$.*

Proof Assume that

$$\bigcup_{i \in I} U_{u_i,v_i,s_i} = \mathbb{X}$$

where I is some indexing set and the U_{u_i,v_i,s_i}'s are stable balls with respect to the centres $(u_i, v_i) \in \mathbb{X} \cap \mathbb{Z}^2$. If $U_{u_i,v_i,s_i} \cap U_{u_j,v_j,s_j} \neq \emptyset$ then either $U_{u_i,v_i,s_i} \subseteq U_{u_j,v_j,s_j}$ or $U_{u_j,v_j,s_j} \subseteq U_{u_i,v_i,s_i}$. We can thus discard the smaller ball and still get a cover of $\mathbb{X}$. By repeating this we can assume without lost of generality that the balls covering $\mathbb{X}$ are disjoint.

By compactness there exists a finite set $J \subseteq I$ such that

$$\bigcup_{j \in J} U_{u_j,v_j,s_j} = \mathbb{X}.$$

By proposition 15.1 we have that

$$\widetilde{\mu}_r\{c_1 \to c_2\}(U_{u_j,v_j,s_j}) = M$$

for some integer M independent of j. By additivity of the measure on compact open sets we deduce that

$$\widetilde{\mu}_r\{c_1 \to c_2\}(\mathbb{X}) = |J|M.$$

On the other hand we have

$$\widetilde{\mu}_r\{c_1 \to c_2\}(\mathbb{X}) = 0,$$

which gives us a contradiction. □

16 Explicit formulas of $\zeta(\delta,(A,\tau),1-k)$ in terms of the measures $\widetilde{\mu}_r\{c_1 \to c_2\}$

In this section we would like to relate the value at $s=0$ of linear combinations of partial zeta functions of K to the measure $\widetilde{\mu}_r\{c_1 \to c_2\}$ evaluated on a certain ball of $\mathbb{X}$. Roughly speaking we would like to relate the value

$$\zeta(\mathfrak{a}, p^n f\infty, 0)$$

to the value

$$\widetilde{\mu}_r\{c_1 \to c_2\}\left((u+p^n\mathbb{Z}_p)\times(v+p^n\mathbb{Z}_p)\right), \tag{16.1}$$

for suitable r, u, v, c_1 and c_2 which depend on $\mathfrak{a}$. Remember by lemma 8.2 that if $\mathfrak{a}$ is an integral ideal coprime to pf then

$$\zeta(\mathfrak{a}^{-1}, p^n f, w_1, s) = f^{-2s}\widehat{\Psi}\left(\frac{\mathfrak{a}}{fp^n\sqrt{D}}, w_1, s\right),$$

and also by equation (7.15) we have

$$4\zeta(\mathfrak{a}^{-1}, p^n f\infty, 0) = \zeta(\mathfrak{a}^{-1}, p^n f, w_1, 0).$$

So instead of relating the value $\zeta(\mathfrak{a}^{-1}, p^n f\infty, 0)$ to (16.1) it is enough to relate the value $\widehat{\Psi}\left(\frac{\mathfrak{a}}{fp^n\sqrt{D}}, w_1, 0\right)$ to (16.1).

Let us start with some explicit formulas obtained by Siegel where he relates special values of a zeta function attached to an indefinite binary quadratic form to Bernoulli polynomials. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ be a hyperbolic matrix with its two real fixed points $\tau > \tau^\sigma$ where $G_{\mathbb{Q}(\tau)/\mathbb{Q}} = \{1, \sigma\}$. Let $Q_\tau(x,y) = A(x-\tau y)(x-\tau^\sigma y) = Ax^2+Bxy+Cy^2$ ($A>0$) be the indefinite primitive quadratic form attached to τ. Let u, v be two rational numbers not both integers. Assume furthermore that $mu+nv \equiv m'u+n'v (mod\,\mathbb{Z})$ for all $m, n \in \mathbb{Z}^2$ where $\gamma\begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} m' \\ n' \end{pmatrix}$. For $s = 2, 3, \ldots$ define

$$\varphi_\gamma((u,v),\tau,s) := \sum_{\langle\gamma\rangle\backslash(m,n)\in\mathbb{Z}^2\backslash(0,0)} \frac{e^{2\pi i(mu+nv)}}{Q_\tau(m,n)^s}.$$

and for $s = 1$ set

$$\varphi_\gamma((u,v),\tau,1) := \lim_{s\to 1^+} \sum_{\langle\gamma\rangle\backslash(m,n)\in\mathbb{Z}^2\backslash(0,0)} \frac{e^{2\pi i(mu+nv)} sign(Q_\tau(m,n))}{|Q_\tau(m,n)|^s}.$$

Note that the limit exists since u and v are not both integers. We also let

$$R_s(z) := \int_{\frac{-d}{c}}^{z} Q_\tau(w,1)^{s-1} dw.$$

Siegel proved the following theorem:

Theorem 16.1 *The quantity*

$$\pi^{-2s} disc(Q_\tau)^{\frac{1}{2}} \varphi_\gamma((u,v),\tau,s)$$

is a rational number that can be expressed using periodic Bernoulli polynomials. More precisely for $s \geq 1$ we have

(16.2)
$$\begin{aligned} &sign(a+d)(s-1)!^2(2\pi)^{-2s} disc(Q_\tau)^{s-\frac{1}{2}} \varphi_\gamma((u,v),\tau,s) \\ &= \sum_{k=0}^{2s-1} \frac{(-1)^k c^{2s-k-1}}{k!(2s-k)} R_s^{(k)}(\tfrac{a}{c}) \sum_{l(mod\ c)} \widetilde{B}_k\left(\frac{a(u+l)}{c} + v\right) \widetilde{B}_{2s-k}\left(\frac{u+l}{c}\right) \end{aligned}$$

where $R_s^{(k)}(z)$ is the k-th derivative of the rational polynomial $R_s(z)$.

Proof This is the main theorem of [Sie68]. □

We obtain the following corollary:

Corollary 16.1 *We have*

(16.3)

$$\zeta^*(\delta_j,(A,\tau),0) = 4 \cdot sign(a+d) \sum_{d_0,r} n(d_0,r) \sum_{l(mod\ c/fd_0)} \widetilde{B}_1\left(a\frac{(\frac{jr}{f}+l)}{c/fd_0}\right) \widetilde{B}_1\left(\frac{(\frac{jr}{f}+l)}{c/fd_0}\right).$$

where $Q_\tau(x,y) = Ax^2 + Bxy + Cy^2$ and $\gamma_\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c\tau + d > 1$.

Proof Using the functional equation between Ψ^* and $\widehat{\Psi}^*$ we get

(16.4)

$$\zeta^*(\delta,(A,\tau),s)$$
$$= -\frac{F^*(1-s)}{F^*(s)}\sum_{d_0,r} n(\frac{N_0}{d_0},r)d_0^s \sum_{\langle\gamma_\tau(fd_0)\rangle\backslash\{(0,0)\neq(m,n)\in\Lambda_{fd_0\tau}\}} \frac{sign(Q_{d_0\tau^*}(m,n))e^{2\pi in/f}}{|Q_{d_0\tau^*}(m,n)|^{1-s}}$$
$$= -\frac{F^*(1-s)}{F^*(s)}\sum_{d_0,r} n(d_0,r)\left(\frac{N_0}{d_0}\right)^s \sum_{\langle\gamma_\tau(fd_0)\rangle\backslash\{(0,0)\neq(m,n)\in\Lambda_{fd_0\tau}\}} \frac{sign(Q_{fd_0\tau})(m,n)e^{2\pi im/f}}{|Q_{fd_0\tau}(m,n)|^{1-s}}.$$

Now evaluate (16.4) at $s=0$. Using the assumption that δ is a good divisor with the explicit formulas in theorem 16.1 we deduce (16.3). $\square$

Remark 16.1 If we compare (16.3) with (9.11) we see that the two formulas coincide since $sign(a+d)=1$ and $sign(c)=1$.

Now let $\lambda_{\alpha,\beta}=\lambda=\alpha+\beta\sqrt{D}$ be an algebraic integer coprime to f where α,β are integers to be specialized later on. Consider the zeta function

$$\Psi\left(\frac{\lambda A\Lambda_\tau}{f\sqrt{D}},w_1,s\right),$$

where $Q_\tau(x,y)=Ax^2+Bxy+Cy^2$ and $\tau=\frac{-B-\sqrt{D}}{2A}$. Using equation (7.1) we find that

(16.5)

$$\Psi\left(\frac{\lambda A\Lambda_\tau}{f\sqrt{D}},w_1,s\right)=w_1(\lambda\sqrt{D})\sum_{\langle\gamma_\tau\rangle\backslash\{(m,n)\in\mathbb{Z}^2\backslash(0,0)\}}\frac{e^{2\pi i\left(m(\frac{2A\beta}{f})+n(\frac{-B\beta-\alpha}{f})\right)}sign(Q_\tau(m,n))}{|Q_\tau(m,n)|^s}$$

Now we want to specialize α and β. Let u be an integer coprime to f and v be any integer. Choose integers α and β such that

$$\beta\equiv(2A)^{-1}u(mod\ f),\ \alpha\equiv-B(2A)^{-1}u-v(mod\ f),\ \alpha+\beta\sqrt{D}\gg 0. \tag{16.6}$$

Note that $(\beta,f)=1$. In general we cannot guarantee that $(\alpha+\beta\sqrt{D},f)=1$. However if we assume that all the primes dividing f are inert in K then we get automatically that $(\alpha+\beta\sqrt{D},f)=1$.

Proposition 16.1 *Let $A\Lambda_\tau$ be the integral ideal associated to $Q_\tau(x,y) = Ax^2 + Bxy + Cy^2$ and $\gamma_\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be as in section 8.2. Let u be integer coprime to f and v be any integer. Let $\lambda_{\alpha,\beta} := \lambda$ be chosen to satisfy (16.6). Assume that the primes dividing f are inert in K. Then if $s \geq 1$ is an odd integer we have*

$$(16.7) \qquad -\frac{sign(a+d)2^{2s}\pi\Gamma(\frac{s+1}{2})^2}{\Gamma(\frac{2-s}{2})^2\Gamma(s)^2}$$
$$\cdot \sum_{k=0}^{2s-1} \frac{(-1)^k c^{2s-k-1}}{k!(2s-k)} R_s^{(k)}(\frac{a}{c}) \sum_{l(mod\ c)} \widetilde{B}_k\left(\frac{a(\frac{u}{f}+l)}{c} + \frac{v}{f}\right) \widetilde{B}_{2s-k}\left(\frac{\frac{u}{f}+l}{c}\right)$$
$$= \widehat{\Psi}\left(\frac{\lambda A\Lambda_\tau}{f\sqrt{D}}, w_1, 1-s\right).$$

Proof When $s \geq 1$ is odd we deduce form (16.5) that

$$\varphi((\frac{u}{f}, \frac{v}{f}), w_1, s) = -\Psi\left(\lambda\frac{A\Lambda_\tau}{f\sqrt{D}}, w_1, s\right).$$

Now using the functional equation of Ψ (see equation (8.8)) combined with the theorem 16.1 we deduce (16.7). $\square$

We would like to relate the special values

$$\widehat{\Psi}\left(\frac{\lambda\mathfrak{a}}{fp^n\sqrt{D}}, w_1, 0\right)$$

where $\mathfrak{a}, \lambda$ are coprime to fp with the evaluated measures $\widetilde{\mu}_j\{i\infty \to \frac{a_n}{c_n}\}(U_{u_n,v_n,n})$ for certain

$$j, u_n, v_n, a_n, c_n$$

depending on $\mathfrak{a}, \lambda$ and n. We will make a simplifying assumption. We will assume that $\epsilon \neq 1(mod\ p)$ where $\langle\epsilon\rangle = \mathcal{O}_K(f\infty)^\times$ and $\epsilon > 1$.

Proposition 16.2 *Let u and v be fixed integers not both divisible by p. Let $\mathfrak{a} = A\Lambda_\tau$ where $Q_\tau(x,y) = Ax^2 + Bxy + Cy^2$ and τ is reduced. Assume that $(\mathfrak{a}, pf) = 1$. Let $\lambda = \alpha + \beta\sqrt{D} \gg 0$ be an algebraic integer coprime to fp. Let $\langle\epsilon\rangle = \mathcal{O}_K(f\infty)^\times$ where $\epsilon > 1$. Assume furthermore that $\epsilon \neq 1(mod\ p)$. Let $\gamma_\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the*

matrix corresponding to the action of ϵ *on the* Λ_τ *with respect to the ordered basis* $\{\tau, 1\}$ *then*

$$(16.8)\quad \widehat{\Psi}\left(\frac{\lambda A\Lambda_\tau}{fp^n\sqrt{D}}, w_1, 0\right) = 4\sum_{h=1}^{c_n} \widetilde{B}_1\left(\frac{a_n}{c_n}\left(\frac{2A\beta}{fp^n}+l\right)-\frac{B\beta+\alpha}{fp^n}\right)\widetilde{B}_1\left(\frac{(\frac{2A\beta}{fp^n}+l}{c_n}\right)$$

where $\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{p^n}$. *If we choose* α *and* β *in such a way that*

$$\beta \equiv (2A)^{-1}u(mod\ fp^n),\ \ \alpha_y \equiv \left(-B(2A)^{-1}u - v(f+y)\right)(mod\ fp^n),\ \ \alpha+\beta\sqrt{D} \gg 0$$

then we have

$$(16.9)\quad -3\sum_{y=1}^{f}\sum_{d_0|N_0, r\in(\mathbb{Z}/f\mathbb{Z})^\times} n(d_0,r)\widehat{\Psi}\left(\frac{\lambda_y rA\Lambda_{d_0\tau}}{fp^n\sqrt{D}}, w_1, 0\right) = \widetilde{\mu}_u\{i\infty \to \frac{a_n}{c_n}\}(U_{\frac{v}{f},\frac{u}{f},n}).$$

where $\lambda_y = \alpha_y + \beta\sqrt{D}$.

Proof The proof of (16.8) follows directly from proposition 16.1. It remains to prove (16.9). We have

$$\begin{aligned}
&-3\sum_{y=1}^{f}\sum_{d_0|N_0, r\in(\mathbb{Z}/f\mathbb{Z})^\times} n(d_0,r)\widehat{\Psi}\left(\frac{\lambda_y rA\Lambda_{d_0\tau}}{fp^n\sqrt{D}}, w_1, 0\right)\\
&= -12\sum_{d_0|N_0, r\in(\mathbb{Z}/f\mathbb{Z})^\times} n(d_0,r)\sum_{y=1}^{f}\sum_{h=1}^{c_n/d_0}\widetilde{B}_1\left(\frac{a_n}{c_n/d_0}\left(\frac{ru}{fp^n}+h\right)-\frac{rv(f+y)}{fp^n}\right)\widetilde{B}_1\left(\frac{(\frac{ru}{fp^n}+l}{c_n/d_0}\right)\\
&= -12\sum_{d_0|N_0, r\in(\mathbb{Z}/f\mathbb{Z})^\times} n(d_0,r)\sum_{h=1}^{c_n/d_0}\widetilde{B}_1\left(\frac{fa_n}{c_n/d_0}\left(\frac{ru}{fp^n}+h\right)-\frac{rv}{p^n}\right)\widetilde{B}_1\left(\frac{(\frac{ru}{fp^n}+h}{c_n/d_0}\right)\\
&= -12\sum_{d_0|N_0, r\in(\mathbb{Z}/f\mathbb{Z})^\times} n(d_0,r)\sum_{h=1}^{c_n/fd_0}\widetilde{B}_1\left(\frac{a_n}{c_n/fd_0}\left(\frac{ru}{fp^n}+h\right)-\frac{rv}{p^n}\right)\widetilde{B}_1\left(\frac{(\frac{ru}{fp^n}+h}{c_n/fd_0}\right)\\
&= \widetilde{\mu}_u\{i\infty \to \frac{a_n}{c_n}\}(U_{\frac{v}{f},\frac{u}{f},n})
\end{aligned}$$

where the third equality follows from the distribution relation of $\widetilde{B}_1(x)$ and the last equality is a consequence of proposition 14.1. □

Remark 16.2 Note the similarity with the formula (14.1) which corresponds to $\widetilde{\mu}_1\{i\infty \to \frac{a}{c}\}(U_{u,v,n})$ for $a = a_0$ and $c = c_0$. The difference with the formula (14.1) is the variation of the cusp $\frac{a_n}{c_n}$ as n vary and the dependence of $\widetilde{\mu}$ with the first coordinate of the centre of the ball $U_{u,v,n}$.

We have the following corollary

Corollary 16.2 *With the same notation as proposition 16.2 we have*

$$-3\sum_{y=1}^{r}\zeta_{fp^n}(\lambda_y \star \delta,(A,\tau),0) = \widetilde{\mu}_u\{i\infty \to \frac{a_n}{c_n}\}(U_{\frac{v}{f},\frac{u}{f},n})$$

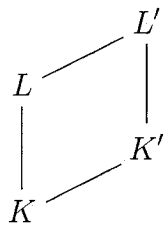
where the index $p^n f$ emphasizes the fact that the conductor is fp^n and not f. In particular, when $n = 0$, the zeta function $\zeta_f(\lambda_y \star \delta,(A,\tau),0)$ is exactly the same as the one appearing in definition 9.2.

17 Some evidence for the algebraicity of the $u(r,\tau)$ invariant

In this section we would like to prove a norm formula for our p-adic elements $u(r,\tau)$. But before this we would like to remind the reader some functorial properties of the reciprocity map and apply it to the number field $K(f\infty)$, i.e. the narrow ray class field of K of conductor f.

17.1 The reciprocity map of Class field theory applied to $K(f\infty)$

Consider the following Hasse diagram



with L/K and L'/K' finite abelian extensions of number fields K and K' where $K \subseteq K'$ and $L \subseteq L'$. Let S and S' be the set of real places of K and K' respectively. Let $cond(L/K) = \mathfrak{f}\infty_T$ be the conductor of the extension of L/K where $\mathfrak{f}$ is an integral ideal of $\mathcal{O}_K$ and ∞_T is a product over all real places of T where $T \subseteq S$. Similarly we let $cond(L'/K') = \mathfrak{f}'\infty_{T'}$ where $T' \subseteq S'$. Class field theory gives us the following commutative diagram:

$$\begin{array}{ccc} G_{L'/K'} & \xrightarrow{\quad res \quad} & G_{L/K} \\ \Big\downarrow{\scriptstyle rec_{L'/K'}} & & \Big\downarrow{\scriptstyle rec_{L/K}} \\ I_{K'}(\mathfrak{f}')/J_{L'/K'} & \xrightarrow{\quad N_{K'/K} \quad} & I_K(\mathfrak{f})/J_{L/K} \end{array}$$

where $I_K(\mathfrak{f})$ is the group of fractional ideals of K which are coprime to $\mathfrak{f}$ and

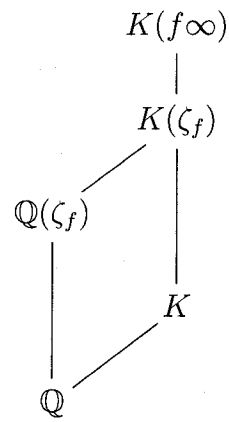
$$J_{L/K} := P_{K,1}(\mathfrak{f}\infty_T)N_{L/K}(I_L(\mathfrak{f}\mathcal{O}_L))$$

where $P_{K,1}(\mathfrak{f}\infty_T)$ is the group of principal fractional ideals of K that can be generated by an element $\lambda \in K$ congruent to 1 modulo $\mathfrak{f}$ such that $\lambda^\sigma > 0$ for all $\sigma \in T$. The vertical arrows of the diagram are isomorphisms given by:

$$\begin{aligned} rec_{L/K}^{-1} : I_K(\mathfrak{f}) &\to G_{L/K} \\ \mathfrak{p} &\mapsto Frob(\wp/\mathfrak{p}) \end{aligned}$$

where $\wp$ is any prime ideal of L above $\mathfrak{p}$ ($\mathfrak{p}$ is assumed to be unramified in L/K) and $Frob(\wp/\mathfrak{p})$ is the relative Frobenius of $\wp$ over $\mathfrak{p}$. It thus follows from the commutativity of diagram that the set of prime ideals of K that split completely in L are precisely the prime ideals inside $J_{L/K}$. In the special case where $K = K'$ we again deduce from the commutativity of the diagram that $J_{L/K}/J_{L'/K} \simeq G_{L'/L}$.

Let $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic number field with $D = disc(K) > 0$ and $Gal(K/\mathbb{Q}) = \{1, \sigma\}$. Let f be a positive integer coprime to D. Let p be a prime number inert in K which is coprime to fD. We denote the two infinite places of K by ∞_1 and ∞_2 and also $\infty = \infty_1\infty_2$. Consider the Hasse diagram



where $K(f\infty) \subseteq \overline{\mathbb{Q}}$ stands for the narrow ray class field of conductor f. By class field theory $K(f\infty)$ corresponds to the maximal abelian extension of K for which a fractional ideal $\mathfrak{a}$ of K splits completely in $K(f\infty)$ iff $\mathfrak{a} = (\pi)$ for some totally positive element $\pi \equiv 1(mod\ f)$. Let $\tau \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ then the extension $K(f\infty)^\tau/K$ is again abelian over K. The ideals of $\mathcal{O}_K$ which split completely in $K(f\infty)^\tau$ are the principal ideals $(\pi^\tau) \subseteq \mathcal{O}_K$ where $\pi \equiv 1(mod\ f)$ and π is totally positive. Since π^τ is totally positive and congruent to 1 modulo f we get, by the maximality of $K(f\infty)$ with respect to the latter property on ideals of $\mathcal{O}_K$ which split completely in it, that $K(f\infty)^\tau \subseteq K(f\infty)$. Since τ was arbitrary it follows that $K(f\infty)$ is normal over $\mathbb{Q}$. We denote by $\mathcal{O}_K(f)^\times$ the group of units of $\mathcal{O}_K$ which are congruent to 1 modulo f and by $\mathcal{O}_K(f\infty)^\times$ the group of totally positive units of $\mathcal{O}_K$ which are congruent to 1 modulo f. In order to have the existence of strong p-units in $K(f\infty)$ we make the following assumption

Assumptions: We assume that

(1) $\mathcal{O}_K(f)^\times = \mathcal{O}_K(f\infty)^\times$,

(2) $\mathcal{O}_K[\frac{1}{p}](f)^\times = \mathcal{O}_K[\frac{1}{p}](f\infty)^\times$.

Obviously (2) implies (1) but in general the converse is false. In the case where $p \equiv 1(mod\, f)$ it is easy to see that (1) and (2) are equivalent. Assumption (1) implies that $K(f\infty)$ is a totally complex Galois extension over $\mathbb{Q}$. Similarly, if we let $\wp = p\mathcal{O}_K$, assumption (2) implies that $K(f\infty)^{\langle Fr_\wp \rangle}$ is a totally complex Galois extension over $\mathbb{Q}$.

We have the following short exact sequence

$$1 \to G_{K(f\infty)/K} \to G_{K(f\infty)/\mathbb{Q}} \to G_{K/\mathbb{Q}} \to 1 \tag{17.1}$$

which splits. Let us denote by $\iota \in G_{K(f\infty)/\mathbb{Q}}$ a lift of σ i.e. $\iota|_K = \sigma|_K$ and $\iota^2 = 1$. One can show that if $g \in G_{K(f\infty)/\mathbb{Q}}$ then $g^\iota := \iota g \iota^{-1} = g^{-1}$. Therefore the conjugation by ι on $G_{K(f\infty)/K}$ corresponds to -1. Let $g \in G_{K(f\infty)/\mathbb{Q}}$ and denote again by $g : K(f\infty) \hookrightarrow \mathbb{C}$, the corresponding complex embedding. We can write g as $g = h\iota$ for some $h \in G_{K(f\infty)/K}$. Let $c \in G_{K(f\infty)/\mathbb{Q}}$ be a complex conjugation induced on $K(f\infty)$ via the embedding $g : K(f\infty) \hookrightarrow \mathbb{C}$, so $c = g^{-1}\tau_\infty g$ where τ_∞ is the complex conjugation on $\mathbb{C}$. Note that $c \in G_{K(f\infty)/K}$ since K is real. Let $g' \in G_{K(f\infty)/\mathbb{Q}}$ and write it as $g' = h'\iota$ for some $h' \in G_{K(f\infty)/K}$. We have

$$g'^{-1}cg' = (h'\iota)^{-1}c(h'\iota) = \iota^{-1}h'^{-1}ch'\iota = \iota^{-1}c\iota = c^{-1} = c.$$

It thus follows that the complex conjugation is independent of the choice of the complex embedding of $K(f\infty)$ in $\mathbb{C}$ and therefore $K(f\infty)$ is a Galois CM field over $\mathbb{Q}$. A similar computation shows that $K(f\infty)^{Fr_\wp}$ is also a Galois CM field over $\mathbb{Q}$. From now on we denote the complex conjugation of $K(f\infty)$ (or $K(f\infty)^{Fr_\wp}$) by τ_∞.

The inverse of the reciprocity map gives us an isomorphism

$$rec^{-1} : I_K(f)/P_{K,1}(f\infty) \to Gal(K(f\infty)/K).$$

The action of $G_{K/\mathbb{Q}}$ on $I_K(f)/P_{K,1}(f\infty)$ is the obvious one. If $[\mathfrak{a}] \in I_K(f)/P_{K,1}(f\infty)$ is an ideal class then $[\mathfrak{a}]^\sigma = [\mathfrak{a}^\sigma]$. We also have another involution in $G_{K(f\infty)/\mathbb{Q}}$ which plays an important role namely the complex conjugation τ_∞. One can show that τ_∞ corresponds to the ideal class $(1 + f\sqrt{D})P_{K,1}(f\infty)$ under *rec*.

17.2 A "norm" formula for $u(r, \tau)$

In this section we want to compute a certain norm of $u(r, \tau)$ in order to relate it to a product of normalized Gauss sums. In order to simplify the notation we set

$$A_{f\infty} := I_K(f)/P_{K,1}(f\infty) \text{ and } G_{f\infty} := Gal(K(f\infty)/K).$$

If $(r, \tau) \in \mathcal{H}_p^{\mathcal{O}_K}(N_0, f)$ then the basis $\{\tau, 1\}$ is oriented i.e. $\tau - \tau^\sigma > 0$. Let $A_r\Lambda_\tau$ be the integral ideal corresponding to (r, τ) then $(A_r\Lambda_\tau)^\sigma = A_r\Lambda_{\tau^\sigma}$ and the basis $\{\tau^\sigma, 1\}$ is no more oriented. Nevertheless we can still define $u(r, \tau^\sigma)$ in the obvious way. If we denote again by σ the non trivial automorphism of $G_{K_p/\mathbb{Q}_p}$ then one readily sees that $u(r, \tau)^\sigma = u(r, \tau^\sigma)$.

To any divisor $\delta = \sum_{a \in A_{f\infty}} n_a a \in \mathbb{Z}[A_{f\infty}]$ and a set of positive integers $\{d_a\}_{a \in A_{f\infty}}$ coprime to p we associate the zeta function

$$\begin{aligned} \zeta(\delta, s) &:= \sum_{a \in A_{f\infty}} n_a d_a^s \widehat{\Psi}\left(\frac{\mathfrak{a}_a}{f\sqrt{D}}, w_1, s\right) \\ &= \sum_{a \in A_{f\infty}} n_a d_a^s f^{2s} \zeta(\mathfrak{a}_a^{-1}, f, w_1, s) \end{aligned} \tag{17.2}$$

where $\mathfrak{a}_a \in a$ is an arbitrary chosen integral ideal and $w_1 = sign \circ N_{K/\mathbb{Q}}$. To any divisor $\widetilde{\delta} = \sum_{d_0, r} n(d_0, r)[d_0, r] \in D(N_0, f)$ and element $(1, \tau) \in \mathcal{H}_p^{\mathcal{O}_K}(N_0, f)$ we can attach a divisor

$$\delta = \sum_{d_0 | N_0, r \in (\mathbb{Z}/f\mathbb{Z})^\times} n(d_0, r)[\Omega(r/d_0, d_0\tau) \cap \mathcal{O}_K] \in \mathbb{Z}[A_{f\infty}]$$

where the map Ω is the map appearing in definition 5.7. Having such a divisor δ allows us to associate a zeta function (similar to the zeta function appearing in 1) of

definition 9.2)

$$\begin{aligned}\zeta(\delta,s) &= \sum_{d_0,r} n(d_0,r) d_0^s \widehat{\Psi}\left(\frac{\Omega(r/d_0,d_0\tau)\cap\mathcal{O}_K}{\sqrt{D}f}, w_1, s\right)\\ &= \sum_{d_0,r} n(d_0,r) d_0^s \widehat{\Psi}\left(\frac{A_{r/d_0}\Lambda_{d_0\tau}}{\sqrt{D}f}, w_1, s\right)\\ &= \sum_{d_0,r} n(d_0,r) d_0^s f^{2s} \zeta((A_{r/d_0}\Lambda_{d_0\tau})^{-1}, f, w_1, s).\end{aligned}$$

where $\Omega(r/d_0,d_0\tau)\cap\mathcal{O}_K = A_{r/d_0}\Lambda_{d_0\tau}$ is an integral ideal of $\mathcal{O}_K$ and A_{r/d_0} is a positive integer such that $A_{r/d_0} \equiv r/d_0 (mod\ f)$. We have suppressed the τ in the notation of $\zeta(\delta,s)$ since it already appears in the writing of δ. In a similar way, one can define a zeta function $\zeta^*(\delta,s)$ (similar to the zeta function appearing in 2) of definition 9.2)

$$\zeta^*(\delta,s) := \sum_{d_0,r} n(d_0,r) d_0^s \widehat{\Psi}^*\left(\frac{\Omega(r/d_0,d_0\tau)\cap\mathcal{O}_K}{\sqrt{D}f}, w_1, s\right). \tag{17.3}$$

For the definitions of Ψ and Ψ^* see definition 8.2.

Remark 17.1 In general the zeta functions $\zeta^*(\delta,s)$ and $\zeta(\delta,s)$ are different. Later on, it will be crucial to be able to compare one to each other. This is the content of proposition 17.3.

In section 9 we have constructed a p-adic zeta function $\zeta_p^*(\delta,s)$ which interpolates special values of $\zeta^*(\delta,s)$ at negative integers congruent to 0 modulo $p-1$. We have proved also a p-adic Kronecker limit formula relating special values of $\zeta_p^*(a\star\delta,s)$ to our p-adic invariant $u(a)\in K_p^\times$. More precisely we have proved that

(1) $3\zeta^*(a\star\delta,0) = v_p(u(a))$

(2) $3(\zeta_p^*)'(a\star\delta,0) = -\log_p N_{K_p/\mathbb{Q}_p}(u(a))$

for any $a\in A_{f\infty}$. Having in mind the theory of CM for imaginary quadratic number fields we have formulated the following conjecture

Conjecture 17.1 *Let $a\in A_{f\infty}$ be an ideal class of the narrow ray class group of conductor f. Then the element $u(a)$ is a strong p-unit in $L := K(f\infty)^{\langle Fr_\wp\rangle} \subseteq K_p$,*

i.e. for all places $\nu \nmid p$ of $K(f\infty)$ (including the infinite one) we have $|u(a)|_\nu = 1$. Furthermore if we let

$$rec^{-1} : I_K(f)/J_{L/K} \to G_{L/K}$$

be the inverse of the reciprocity map given by class field theory then

$$u(a)^{rec^{-1}(a')} = u(aa').$$

Remark 17.2 The field $L = K(f\infty)^{\langle Fr_\wp \rangle}$ is the largest subfield of $K(f\infty)$ for which $\wp = p\mathcal{O}_K$ splits completely.

By construction $u(c)$ lives naturally in $K_p^\times$ so we can write it as

$$u(c) = p^{3\zeta^*(a\star\delta,0)}\epsilon(c)$$

where $\epsilon(c) \in \mathcal{O}_{K_p}^\times$. If conjecture 17.1 is true then the polynomial

$$f(x) := \prod_{b \in I_K(f)/J_{L/K}} (x - u(b)) \tag{17.4}$$

should have coefficients in $\mathcal{O}_K[\frac{1}{p}]$.

Remark 17.3 From conjecture 5.1, since $u(a)$ is a strong p-unit, we have that $u(a)^{\tau_\infty} = u(a)^{-1}$ where τ_∞ corresponds to the complex conjugation in $G_{L/K}$. It thus follow that if α is a root of $f(x)$ then α^{-1} is also a root of $f^{\tau_\infty}(x) = f(x)$. From this we deduce that $f(x) = x^{deg(f)}f(\frac{1}{x})$, i.e. $f(x)$ is a palindrome polynomial with coefficients in $\mathcal{O}_K[\frac{1}{p}]$. Similarly $g(x) = f(x)f^\sigma(x) \in \mathbb{Z}[\frac{1}{p}][x]$ is a palindrome polynomial with coefficients in $\mathbb{Z}[\frac{1}{p}]$.

Let us fix a prime $\mathfrak{p}$ in $L = K(f\infty)^{\langle Fr_\wp \rangle}$ above $\wp$. Conjecture 17.1 tells us that it is possible to take an embedding $L \hookrightarrow K_p$ such that

$$u(c)\mathcal{O}_L = \prod_{b \in I_K(f)/J_{L/K}} \left(\mathfrak{p}^{rec^{-1}(c^{-1}b)}\right)^{3\zeta^*(b\star\delta,0)}$$

where we think of $u(c)$ as a root of $f(x)$. We should point out that up to a root of unity in L the strong p-unit $u(c)$ is completely determined by the set of integers

$$\{3\zeta^*(b \star \delta, 0)\}_{b \in I_K(f)/J_{L/K}},$$

since two such units would differ by an element of norm 1 for all places of L, therefore a root of unity in L.

Constructing a subgroup of p-units of maximal rank in $K(f\infty)$ is a very difficult problem since we don't even know how to construct explicitly the field $K(f\infty)$. However, much is known about the strong p-units of the subfield $K(\zeta_f) \subseteq K(f\infty)$ which is nothing else than a subfield of the cyclotomic field $\mathbb{Q}(\zeta_f, \zeta_D)$. For the cyclotomic field $\mathbb{Q}(\zeta_N)$, with N coprime to p, the construction of a subgroup of the group of strong p-units of maximal rank is provided by normalized Gauss sums. Let r be the smallest integer such that $p^r \equiv 1(mod\ N)$ and set $q = p^r$. Let

$$\omega_q : \mathbb{F}_q^\times \to \mu_{q-1}$$

be the Teichmüller character. Let also ζ_p be a primitive p-th root of unity. We define as usual an additive character of $\mathbb{F}_q$ as

$$\begin{aligned} \psi_q : \mathbb{F}_q &\to \mu_p \\ a &\mapsto \zeta_p^{Tr_{\mathbb{F}_q/\mathbb{F}_p}(a)}. \end{aligned}$$

A Gauss sum with respect to the character ω_q^j is defined as

$$\tau(\omega_q^j) := \sum_{a \in \mathbb{F}_q} \omega_q^j(a)\psi_q(a) \in \mathbb{Q}(\zeta_p, \zeta_{q-1}).$$

Because the Frobenius automorphism of $\mathbb{F}_q$, $x \mapsto x^p$, is bijective and that $\psi_q(x^p) = \psi_q(x)$ we deduce that

$$\tau(\omega_q^{pj}) = \tau(\omega_q^j). \tag{17.5}$$

A normalized Gauss sum is an expression of the form $\frac{\tau(\omega_q^j)}{\tau(\omega_q^{\frac{q-1}{2}})}$. We set

$$g\left(\frac{c}{N}\right) := \frac{\tau(\omega_q^{c\frac{q-1}{N}})}{\tau(\omega_q^{\frac{q-1}{2}})} \in \mathbb{Q}(\zeta_N, \zeta_p)$$

where c is any integer. From (17.5) we deduce that $g(\frac{pc}{N}) = g(\frac{c}{N})$. Normalized Gauss sums are strong p-units and one can compute explicitly their factorization in the

number field $\mathbb{Q}(\zeta_p, \zeta_{q-1})$. An example of a subgroup of strong p-units of maximal rank in $\mathbb{Q}(\zeta_N)$ is provided by

$$U := \langle \left\{ g\left(\frac{j}{N}\right)^{2N} \right\}_{j \in \mathbb{Z}/N\mathbb{Z}/\{\pm 1\}} \rangle \subseteq \mathbb{Q}(\zeta_N)^{\times}. \tag{17.6}$$

Note the presence of the exponent $2N$. Since $g\left(\frac{pc}{N}\right)^{2N} = g\left(\frac{c}{N}\right)^{2N} \in \mathbb{Q}(\zeta_N)$ we deduce that

$$g\left(\frac{c}{N}\right)^{2N} \in \mathbb{Q}(\zeta_N)^{Fr_p} \subseteq \mathbb{Q}_p^{ur}.$$

In particular in the case where $\overline{-1} \in \langle \overline{p} \rangle \leq (\mathbb{Z}/N\mathbb{Z})^{\times}$ we have that $\mathbb{Q}(\zeta_N)^{Fr_p}$ is totally real and therefore $g\left(\frac{c}{N}\right)^{2N} = \pm 1$. This can be proven in purely elemantary way. Let us prove it without appealing to the notion of strong p-units since the computation is instructive. Let s be such that $p^s \equiv -1 (mod\ N)$. Then since $g\left(\frac{c}{N}\right) = g\left(\frac{p^s c}{N}\right)$ we get that

$$g\left(\frac{c}{N}\right) = g\left(\frac{-c}{N}\right).$$

On the other hand a direct computation shows that

$$\overline{g\left(\frac{c}{N}\right)} = g\left(\frac{-c}{N}\right).$$

Combining both we conclude that $g\left(\frac{c}{N}\right)$ is a real number with absolute value 1 so it is equal to $\{\pm 1\}$.

From now on let us assume that $\overline{-1} \notin \langle \overline{p} \rangle \leq (\mathbb{Z}/N\mathbb{Z})^{\times}$. In this case one can show that the $\mathbb{Z}$-rank of U is $\frac{\phi(N)}{2r}$. This is proved essentially by showing that the divisors of Gauss sums give rise to the *universal odd distribution* of degree 0, see Lemma 2.3 of chapter 17 of [Lan94a]. Using proposition 1.1 and the fact that $\overline{-1} \notin \langle \overline{p} \rangle \leq (\mathbb{Z}/N\mathbb{Z})^{\times}$ we deduce that the $\mathbb{Z}$-rank of the group of strong p-units of $\mathbb{Q}(\zeta_N)$ is equal to $\frac{\phi(N)}{2r}$. From this we conclude that U has maximal rank. It follows that one can find $\frac{\phi(N)}{2r}$ elements inside the set $(\mathbb{Z}/N\mathbb{Z})/\{\pm 1\}$ that give rise to $\mathbb{Z}$-linearly independent normalized Gauss sums (inside the multiplicative group $\mathbb{Q}(\zeta_N)^{\times}$). Note that in order to get a subgroup of maximal rank one really needs to go over all $j \in \mathbb{Z}/N\mathbb{Z}/\{\pm 1\}$ and not just over $j \in (\mathbb{Z}/N\mathbb{Z})^{\times}/\{\pm 1\}$. In fact one can give an example of an integer N

(divisible by three distinct primes) such that the group generated by the normalized Gauss sums arising from the indices $j \in (\mathbb{Z}/N\mathbb{Z})^\times/\{\pm 1\}$ has not a maximal rank. For a basis of universal odd distribution of degree 0 see [Kuc92].

Even if we don't know the algebraicity of $u(c)$ the Shimura reciprocity law formulated in conjecture 17.1 allows us to define a pseudo norm of $u(c)$. Note that the p-adic element $u(c) \in K_p^\times$ depends only on the ideal class $c \in A_{f\infty}$ as the notation indicates.

Definition 17.1 *Let M and M' be number fields such that*

$$K \subseteq M \subseteq M' \subseteq K(f\infty).$$

The reciprocity isomorphism gives us canonical isomorphisms

$$rec^{-1}_{M'/K} : I_K(f)/J_{M'/K} \to Gal(M'/K)$$

and

$$rec^{-1}_{M/K} : I_K(f)/J_{M/K} \to Gal(M/K).$$

Therefore $rec_{M'/K}$ induces a canonical isomorphism between $J_{M/K}/J_{M'/K}$ and $Gal(M'/M)$. We define

$$N_{M'/M}(u(c)) := \prod_{b \in J_{M/K}/J_{M'/K}} u(bc).$$

If we have $K \subseteq M \subseteq M' \subseteq M'' \subseteq K(f\infty)$ on can verify that this pseudo norm satisfies the usual transitivity property namely

$$\mathbf{N}_{M'/M} \circ N_{M''/M'} u(c) = N_{M''/M} u(c).$$

From this we deduce that if $u(c)$ is expected to lie in M' i.e. if it is constant on all $c \in J_{M'/K}/J_{M''/K}$ then

$$\mathbf{N}_{M''/M} u(c) = \left(\mathbf{N}_{M'/M} u(c)\right)^{[M'':M']}.$$

Suppose that $\widetilde{M} \subseteq K(f\infty)$ is an abelian extension of $\mathbb{Q}$ contained in M such that $\widetilde{M} \cdot K = M$ and $\widetilde{M} \cap K = \mathbb{Q}$. Let $Gal(K/\mathbb{Q}) = \{1, \sigma\}$. Then there exists a unique

embedding $\widetilde{\sigma} : M \to \mathbb{C}$ *such that* $\widetilde{\sigma}|_K = \sigma|_K$ *and* $\sigma|_{\widetilde{M}} = Id_M$. *Since* $u(c) \in K_p^\times$ *we have a natural action of* $Gal(K/\mathbb{Q}) \simeq Gal(K_p/\mathbb{Q}_p) = \{1, \sigma\}$ *on* $u(c)$. *This allows us to define*

$$\mathbf{N}_{M'/\widetilde{M}}(u(c)) := \mathbf{N}_{K_p/\mathbb{Q}_p} \circ \mathbf{N}_{M'/M}(u(c)).$$

For the rest of the paper we set

$$L := K(f\infty)^{Fr_\wp} \subseteq \mathbb{Q}_p^{ur},$$

$$\widetilde{M} := \mathbb{Q}(\zeta_f)^{Fr_p} \subseteq L,$$

and

$$M := K \cdot \widetilde{M}.$$

Note that $L \cap K(\zeta_f) = K \cdot \mathbb{Q}(\zeta_f)^{\langle Fr_p^2 \rangle} \supseteq M$ since $Fr_\wp(\zeta_f) = Fr_p^2(\zeta_f) = \zeta_f^{p^2}$ where $\wp = p\mathcal{O}_K$. Note also that

$$J_{L/K} = \langle P_{K,1}(f\infty), (p) \rangle,$$

and

$$J_{M/K} = \mathbf{N}_{K(f\infty)/M}(K(f\infty)^\times) \cdot P_{K,1}(f\infty).$$

We want to prove the following theorem which is the main result of this section:

Theorem 17.1 *Let* p, f, N_0 *be chosen as usual and let* $\delta \in D(N_0, f)^{\langle p \rangle}$ *be a good divisor. Assume that all the primes* $q|f$ *are inert in* K *and that* $\overline{-1} \notin \langle \overline{p} \rangle \leq (\mathbb{Z}/f\mathbb{Z})^\times$. *Then one has*

$$\mathbf{N}_{L/\widetilde{M}}(u(r, \tau)) = S (mod\, \mu_F), \tag{17.7}$$

where S *is a product of normalized Gauss sums in* $F := \widetilde{M} \cdot \mathbb{Q}(\zeta_p) \subseteq \mathbb{Q}_p^{ur}(\zeta_p)$.

Corollary 17.1 *The quantity* $\mathbf{N}_{L/\widetilde{M}}(u(r, \tau))$ *lies in* $K_p^\times \cap F = \widetilde{M} = \mathbb{Q}(\zeta_f)^{Fr_p} \subseteq \mathbb{Q}_p^{ur}$. *Note that because of our assumption we have that* $\widetilde{M}$ *is a CM field.*

There are 4 steps in proving the "norm formula" of theorem 17.1.

(1) Calculate $(\zeta_p^*)'(c \star \delta, 0)$ (where $\zeta_p^*(\delta, s)$ is the p-adic zeta function appearing in definition 10.1) and relate it to the p-adic invariant $u(c)$. This is the content of the theorem 10.1.

(2) For the second step one considers a slightly different p-adic zeta function denoted by $\zeta_{p,0}(\delta, s)$ (see the proof of proposition 17.1 for the definition). This step consists in expressing

$$Tr_{K(f\infty)/M}\zeta_{p,0}(c \star \delta, s)$$

as a linear combination of p-adic L-functions that behave well under the base change from $G_{M/K}$ to $G_{\widetilde{M}/\mathbb{Q}}$. More precisely if we let $\widetilde{\chi} \in \widehat{G}_{\widetilde{M}/\mathbb{Q}}$ then the base change translates as a factorization of L-functions of the form

$$L(s, \widetilde{\chi} \circ N_{K/\mathbb{Q}}) = L(s, \widetilde{\chi})L(s, \widetilde{\chi}\left(\frac{D}{\cdot}\right)). \tag{17.8}$$

where $\widetilde{\chi} \circ N_{K/\mathbb{Q}}$ is a character of $G_{M/K}$. We can interpolate special values of (17.8) p-adically and we obtain

$$L_p(s, \widetilde{\chi}\omega_p \circ N_{K/\mathbb{Q}}) = L_p(s, \widetilde{\chi}\omega_p)L_p(s, \widetilde{\chi}\omega_p\left(\frac{D}{\cdot}\right)).$$

The appearance of the Teichmuller character raised to the power 1 is an artifact of p-adic interpolation. Note that only odd characters $\widetilde{\chi}$'s of $Gal(\widetilde{M}/\mathbb{Q})$ will contribute to the p-adic interpolation since we are only interested by the values of $L(\widetilde{\chi}, 1-m)$ for odd integers $m \geq 1$.

(3) Compute $L_p'(0, \widetilde{\chi}\omega_p)$ for odd characters of $G_{\widetilde{M}/\mathbb{Q}}$ (note that $L_p(0, \widetilde{\chi}\omega_p) = 0$ since $\widetilde{\chi}(p) = 1$) and relate it to normalized Gauss sums. This is accomplished by combining a limit formula for $L_p(s, \widetilde{\chi}\omega_p)$ that was proved by Ferrero-Greenberg ([FG78]) with the Gross-Koblitz formula ([GK79]) relating the p-adic gamma function $\Gamma_p(s)$ to Gauss sums.

(4) Relate $\zeta_p^*(\delta, s)$ to $\zeta_{p,0}(\delta, s)$ and $\zeta^*(\delta, 0)$ to $\zeta(\delta, 0)$.

The steps 2 and 3 are proved in the next proposition:

Proposition 17.1 *Let* $\delta = \sum_{a \in A_{f\infty}} n_a[a] \in \mathbb{Z}[A_{f\infty}]$ *and* $\{d_a\}_{a \in A_{f\infty}}$ *be a set of integers coprime to* p. *Set*

$$\zeta(\delta, s) := \sum_{a \in A_{f\infty}} n_a d_a^s \widehat{\Psi}\left(\frac{\mathfrak{a}_a}{f\sqrt{D}}, w_1, s\right).$$

where $\mathfrak{a}_a \in a$ *is chosen to be an integral ideal. Then for every fixed congruence class* i *modulo* $p-1$ *there exists a* p*-adic zeta function denoted by* $\zeta_{p,i}(\delta, s)$ $(s \in \mathbb{Z}_p)$ *such that for all* $n \leq 0$, $n \equiv i (\mathrm{mod}\, p-1)$

(1) $(Tr_{L/M}\zeta_{p,i})(\delta, n) = (1 - p^{-2n}) Tr_{L/M}\zeta(\delta, n)$ *where* $Tr_{L/M}$ *is taken under the natural action of* $Gal(L/M)$ *on* δ *under the reciprocity map. Note that the action is well defined since* $p \star \delta = \delta$.

(2) In the case where $i = 0$ *let*

$$\widetilde{\delta} = \sum_{d_0, r} n(d_0, r)[d_0, r] \in D(N_0, f)^{\langle p \rangle}$$

be a good divisor for the data p, f, N_0 *and let*

$$\delta = \sum_{d_0 | N_0, r \in (\mathbb{Z}/f\mathbb{Z})^\times} n(d_0, r)[A_{r/d_0}\Lambda_{d_0\tau}] \in \mathbb{Z}[A_{f\infty}] \tag{17.9}$$

where $(1, \tau) \in \mathcal{H}_p^{\mathcal{O}_K}(N_0, f)$ *with* τ *reduced. Suppose that* $\overline{-1} \notin \langle \overline{p} \rangle \leq (\mathbb{Z}/f\mathbb{Z})^\times$ *then*

$$3(Tr_{L/M}\zeta_{p,0})'(\delta, 0) = -24 \log_p S, \tag{17.10}$$

where S *is a product of normalized Gauss sums in* $\widetilde{M} \cdot \mathbb{Q}(\zeta_p) \subseteq \overline{\mathbb{Q}}_p$. *Moreover we have*

$$6f Tr_{L/M}\zeta(\delta, 0) = 12 v_p(S^{2f}). \tag{17.11}$$

Note that $S^{2f} \in \widetilde{M} \subseteq \mathbb{Q}_p^{ur}$ *so it makes sense to take its valuation at* p.

Proof of proposition 17.1 Let $\delta = \sum_a n_a[a] \in \mathbb{Z}[A_{f\infty}]$. Recall that

$$\zeta(\delta, s) = \sum_{a \in A_{f\infty}} n_a d_a^s \widehat{\Psi}\left(\frac{\mathfrak{a}_a}{f\sqrt{D}}, w_1, s\right).$$

Since $\widehat{\Psi}$ is essentially a partial zeta function attached to K (see lemma 8.2) then applying the main theorem of [DR80] to every term of $\zeta(\delta, s)$ we get that for a fixed $i(mod\, p-1)$ and a fixed $b \in A_{f\infty}$ that the values

$$(1-p^{-2n})\zeta(b\star\delta, n) \tag{17.12}$$

vary p-adically continuously when n ranges over integers $n \leq 0$ and $n \equiv i(mod\, p)$. By density this gives us a p-adic zeta function

$$\zeta_{p,i}(\delta, s) : \mathbb{Z}_p \to \mathbb{Q}_p$$

which interpolate the values of (17.12) on this fixed congruence class modulo $p-1$. Let

(1) $Tr_{L/M}\zeta(\delta, s) = \sum_{b\in J_{M/K}/J_{L/K}} \zeta(b\star\delta, s)$

(2) $Tr_{L/M}\zeta_{p,i}(\delta, s) = \sum_{b\in J_{M/K}/J_{L/K}} \zeta_{p,i}(b\star\delta, s)$

By the definition of $Tr_{L/M}\zeta_{p,i}(\delta, s)$ the values

$$(1-p^{-2n})Tr_{L/M}\zeta(\delta, n)$$

coincide with $Tr_{L/M}\zeta_{p,i}(\delta, n)$ for integers $n \leq 0$ and $n \equiv i(mod\, p-1)$. Since the values in (17.12) vary p-adically continuously when $n \leq 0$ and $n \equiv i(mod\, p-1)$ this implies that $Tr_{L/M}\zeta(\delta, s)$ varies p-adically continuously on this subset. This proves the first part of the theorem.

It remains to prove the trace formula (equation (17.10)) and the valuation formula (equation (17.11)) for the p-adic zeta function $\zeta_{p,0}(\delta, s)$. Since $\zeta_{p,0}(\delta, 0) = 0$ we have $\frac{d}{ds}(d_a^s\zeta_{p,0}(\delta, s))|_{s=0} = \zeta'_{p,0}(\delta, 0)$. Also we have $(d_a^s\zeta(\delta, s))|_{s=0} = \zeta(\delta, 0)$. Therefore without lost of generality we can assume that all the d_a's are equal to 1. We have that

$$\begin{aligned}\widehat{\Psi}(\frac{A_r\Lambda_\tau}{f\sqrt{D}}, w_1, n) &= f^{2n}\zeta((A_r\Lambda_\tau)^{-1}, f, w_1, n)\\ &= 4f^{2n}\zeta((A_r\Lambda_\tau)^{-1}, f\infty, n)\end{aligned}$$

for integers $n \leq 0$ and $n \equiv 0(mod\ 2)$. The second equality comes from equation (7.13) proved in section 7. From this we deduce that

$$Tr_{L/M}\zeta(\delta, n) = 4f^{2n} \sum_{d_0, r} n(d_0, r) \sum_{a \in J_{M/K}/J_{L/M}} \zeta((A_{r/d_0}\Lambda_{d_0\tau})^{-1} \cdot \mathfrak{a}, f\infty, n)$$

for integers $n \leq 0$ and $n \equiv 0(mod\ 2)$ and $\mathfrak{a} \in a$. We let

$$\begin{aligned} \sigma_{\_} : I_K(f)/J_{M/K} &\to G_{M/K} \\ [\mathfrak{a}] &\to \sigma_{\mathfrak{a}} \end{aligned}$$

be the isomorphism induced by class field theory where $\sigma_{\mathfrak{a}}$ is the Frobenius associated to the ideal class of $\mathfrak{a}$. We have

$$\begin{aligned} Tr_{L/M}(\zeta(\delta, n)) &= 4f^{2n} \sum_{d_0, r} n(d_0, r) \sum_{a \in J_{M/K}/J_{L/K}} \zeta((A_{r/d_0}\Lambda_{d_0\tau})^{-1} \cdot a, K(f\infty)/K, n) \\ &= 4f^{2n} \sum_{d_0, r} n(d_0, r)\zeta(\sigma^{-1}_{I(d_0,r)}, K \cdot \mathbb{Q}(\zeta_f)^{\langle Fr_p \rangle}/K, n), \end{aligned} \tag{17.13}$$

where

$$I(d_0, r) := A_{r/d_0}\Lambda_{d_0\tau} = \Omega(r/d_0, d_0\tau) \cap \mathcal{O}_K.$$

We have a natural isomorphism between $Gal(K(\zeta_f)/K)$ and $Gal(\mathbb{Q}(\zeta_f)/\mathbb{Q})$ induced by the restriction. At the level of the ideals class groups the restriction map corresponds to the norm $\mathbf{N}_{K/\mathbb{Q}}$. Under this natural identification we have

$$\begin{aligned} \left\{\chi \in \widehat{G}_{M/K}\right\} &= \left\{\chi \in \widehat{I_K(f)/P_{K,1}}(f\infty) : \chi|_{J_{M/K}} = 1\right\} \\ &= \left\{\tilde{\chi} \circ N_{K/\mathbb{Q}} : \tilde{\chi} \in \widehat{I_{\mathbb{Q}}(f)/P_{\mathbb{Q},1}}(f\infty), \tilde{\chi}(p) = 1\right\} \end{aligned} \tag{17.14}$$

If we restrict to odd characters of $Gal(M/K)$ then we have

$$\begin{aligned} &\left\{\chi \in \widehat{I_K(f)/P_{K,1}}(f\infty) : \chi|_{J_{M/K}} = 1, \chi_\infty = w_1\right\} \\ &= \left\{\tilde{\chi} \circ N_{K/\mathbb{Q}} : \tilde{\chi} \in \widehat{I_{\mathbb{Q}}(f)/P_{\mathbb{Q},1}}(f\infty), \tilde{\chi}_f(-1) = -1, \tilde{\chi}(p) = 1\right\}. \end{aligned} \tag{17.15}$$

We can write any character χ of $\widehat{I_K(f)/P_{K,1}}(f\infty)$ as $\chi_f\chi_\infty$ where χ_f is the finite part of χ and χ_∞ is its infinite part (see the end of subsection 7.1). We say that χ

is even if $\chi_\infty = w_0$ and odd if $\chi_\infty = w_1 = sign \circ \mathbf{N}_{K/\mathbb{Q}}$. A similar thing holds for characters $\widetilde{\chi} \in \widehat{I_{\mathbb{Q}}(f)/P_{\mathbb{Q},1}(f\infty)}$. One verifies easily that $\widetilde{\chi}_\infty = 1$ if $\widetilde{\chi}_f$ is is an even character of $(\mathbb{Z}/f\mathbb{Z})^\times$ and $\widetilde{\chi}_\infty = sign$ if $\widetilde{\chi}_f$ is odd. It is easy to see that characters $\chi \in I_K(f)/P_{K,1}(f\infty)$ s.t. $\chi|_{J_{M/K}} = 1$ and $\chi_\infty = w_1$ are induced by the norm of an odd character of $(\mathbb{Z}/f\mathbb{Z})^\times \simeq I_{\mathbb{Q}}(f)/P_{\mathbb{Q},1}(f\infty)$.

Let $\sigma \in Gal(M/K)$ then the partial zeta function $\zeta(M/K, \sigma, s) = \zeta(\sigma, s)$ can be written as

$$\zeta(\sigma, s) = \frac{1}{|G_{M/K}|} \sum_{\chi \in \widehat{G}_{M/K}} \overline{\chi}(\sigma) L(s, \chi) \tag{17.16}$$

Substituting this in (17.13) we obtain

$$\begin{aligned} Tr_{L/K}(\zeta(\delta, n)) &= \frac{4f^{2n}}{|G_{M/K}|} \sum_{d_0, r} n(d_0, r) \sum_{\substack{\chi \in I_K(f)/\widehat{P_{K,1}(f\infty)} \\ \chi|_{J_{M/K}} = 1}} \overline{\chi}(\sigma_{I(d_0,r)}^{-1}) L(n, \chi) \\ &= \frac{4f^{2n}}{|G_{M/K}|} \sum_{d_0, r} n(d_0, r) \sum_{\substack{\tilde{\chi} \in \widehat{(\mathbb{Z}/f\mathbb{Z})^\times} \\ \tilde{\chi}(p) = 1}} \overline{\tilde{\chi}} \circ \mathbf{N}_{K/\mathbb{Q}}(I(d_0, r)^{-1}) L(n, \tilde{\chi} \circ \mathbf{N}_{K/\mathbb{Q}}) \\ &= \frac{4f^{2n}}{|G_{M/K}|} \sum_{d_0, r} n(d_0, r) \sum_{\substack{\tilde{\chi} \in \widehat{(\mathbb{Z}/f\mathbb{Z})^\times} \\ \tilde{\chi}(p) = 1}} \overline{\tilde{\chi}} \circ \mathbf{N}_{K/\mathbb{Q}}(I(d_0, r)^{-1}) L(n, \tilde{\chi}) L(n, \tilde{\chi}\left(\frac{D}{*}\right)) \end{aligned}$$

where the second equality uses the identification given by (17.14). We want to interpolate p-adically those special values. For every integer $m \geq 1$ and $m \equiv 1 (mod\ 2)$ we can rewrite the right hand side of the last equality as

$$= \frac{4f^{2(1-m)}}{|G_{M/K}|} \sum_{\substack{\tilde{\chi} \in \widehat{(\mathbb{Z}/f\mathbb{Z})^\times} \\ \tilde{\chi}(p) = 1 \\ \tilde{\chi}(-1) = -1}} \tilde{\chi} \circ N_{K/\mathbb{Q}}(I(d_0, r)) L(1 - m, \tilde{\chi}) L(1 - m, \tilde{\chi}\left(\frac{D}{*}\right)) \tag{17.17}$$

since for $\tilde{\chi}$ an even character $L(1 - m, \tilde{\chi}) = 0$. We have the following well known proposition

Proposition 17.2 *Let χ be a Dirichlet character then there exists a p-adic L-function $L_p(s, \chi\omega_p)$ such that*

$$L_p(1 - n, \chi\omega_p) = -\left(1 - \frac{(\chi\omega_p^{1-n})(p)}{p^{1-n}}\right) \frac{B_{n, \chi\omega_p^{1-n}}}{n}.$$

for all integers $n \geq 1$ *and* $n \equiv 1(mod\, p-1)$.

Proof see theorem 5.11 in [Was87]. □

From the previous proposition we deduce in the case where $\chi \in \widehat{(\mathbb{Z}/f\mathbb{Z})^\times}$ and $\chi(p) = 1$ that for $n \equiv 1(mod\, p-1)$

$$L_p(1-n, \chi\omega_p) = -\left(1 - \frac{1}{p^{1-n}}\right)\frac{B_{n,\chi}}{n}, \quad L_p(1-n, \chi\left(\frac{D}{*}\right)\omega_p) = -\left(1 + \frac{1}{p^{1-n}}\right)\frac{B_{n,\chi(\frac{D}{*})}}{n}.$$

The second equality follows from the fact that $\left(\frac{D}{p}\right) = -1$. From this we deduce that $L_p(1-n, \chi\omega_p) = (1-1/p^{1-n})L(1-n, \chi)$ and $L_p(1-n, \chi\left(\frac{D}{*}\right)\omega_p) = (1+1/p^{1-n})L(1-n, \chi\left(\frac{D}{*}\right))$ for integers $n \geq 1$ and $n \equiv 1(mod\, p-1)$. Substituting in (17.17) and recalling the definition of $(Tr_{L/M}\zeta_p)(\delta, s)$ we get

$$(Tr_{L/M}\zeta_p)(\delta, 1-m) =$$
$$\frac{4f^{2(1-m)}}{|G_{M/K}|}\sum_{d_0|N_0, r\in(\mathbb{Z}/f\mathbb{Z})^\times} n(d_0, r)\sum_{\substack{\tilde{\chi}\in\widehat{(\mathbb{Z}/f\mathbb{Z})^\times}\\ \tilde{\chi}(p)=1\\ \tilde{\chi}(-1)=-1}} \tilde{\chi}\circ\mathbf{N}_{K/\mathbb{Q}}(I(d_0, r))L_p(1-m, \tilde{\chi}\omega_p)L_p(1-m, \tilde{\chi}\left(\frac{D}{*}\right)\omega_p),$$

for $m \geq 1$ and $m \equiv 1(mod\, p-1)$. By density of the set $\{n \geq 1 : n \equiv 1(mod\, p-1)\}$ in $\mathbb{Z}_p$ we obtain

$$(Tr_H\zeta_p)(\delta, s) =$$
$$\frac{4\langle f\rangle^{2s}}{|G_{M/K}|}\sum_{d_0|N_0, r\in(\mathbb{Z}/f\mathbb{Z})^\times} n(d_0, r)\sum_{\substack{\tilde{\chi}\in\widehat{(\mathbb{Z}/f\mathbb{Z})^\times}\\ \tilde{\chi}(p)=1\\ \tilde{\chi}(-1)=-1}} \tilde{\chi}\circ\mathbf{N}_{K/\mathbb{Q}}(I(d_0, r))L_p(s, \tilde{\chi}\omega_p)L_p(s, \tilde{\chi}\left(\frac{D}{*}\right)\omega_p)$$

for all $s \in \mathbb{Z}_p$. Let us define an auxiliary p-adic L-function that will play an important role later on.

Definition 17.2 *We define the Archimedean zeta function* $\Theta(s)$ *and its p-adic counterpart* $\Theta_p(s)$ *as*

$$\Theta(s) :=$$
$$\frac{4f^{2s}}{|G_{M/K}|}\sum_{d_0|N_0, r\in(\mathbb{Z}/f\mathbb{Z})^\times} n(d_0, r)\sum_{\substack{\tilde{\chi}\in\widehat{(\mathbb{Z}/f\mathbb{Z})^\times}\\ \tilde{\chi}(p)=1\\ \tilde{\chi}(-1)=-1}} \tilde{\chi}\circ\mathbf{N}_{K/\mathbb{Q}}(I(d_0, r))L(s, \tilde{\chi})L(0, \tilde{\chi}\left(\frac{D}{*}\right)).$$

and

$$\Theta_p(s) :=$$
$$\frac{4\langle f\rangle^{2s}}{|G_{M/K}|}\sum_{d_0|N_0,r\in(\mathbb{Z}/f\mathbb{Z})^\times} n(d_0,r)\sum_{\substack{\tilde{\chi}\in\widehat{(\mathbb{Z}/f\mathbb{Z})^\times}\\ \tilde{\chi}(p)=1\\ \tilde{\chi}(-1)=-1}} \tilde{\chi}\circ\mathbf{N}_{K/\mathbb{Q}}(I(d_0,r))L_p(s,\tilde{\chi}\omega_p)L(0,\tilde{\chi}\left(\frac{D}{*}\right)).$$

Note that $L(0,\tilde{\chi}\left(\frac{D}{*}\right)) = -B_{1,\tilde{\chi}\left(\frac{D}{*}\right)}$ *and therefore* $2L(0,\tilde{\chi}\left(\frac{D}{*}\right)) = L_p(0,\tilde{\chi}\left(\frac{D}{*}\right)\omega_p)$. *Moreover when* $k\geq 1$ *and* $k\equiv 1(mod\, p-1)$ *then*

$$(1-\frac{1}{p^{1-k}})\Theta(1-k) = \Theta_p(1-k).$$

Now let us take the derivative of $(Tr_{L/M}\zeta_p)(\delta,s)$ at $s=0$. Applying the chain rule and using the observation that $L_p(0,\chi\omega_p)=0$ we get

(17.18)

$$(Tr_{L/M}\zeta_p)'(\delta,0) =$$
$$\frac{4}{|G_{M/K}|}\sum_{d_0|N_0,r\in(\mathbb{Z}/f\mathbb{Z})^\times} n(d_0,r)\sum_{\substack{\tilde{\chi}\in\widehat{(\mathbb{Z}/f\mathbb{Z})^\times}\\ \tilde{\chi}(p)=1\\ \tilde{\chi}(-1)=-1}} \tilde{\chi}\circ\mathbf{N}_{K/\mathbb{Q}}(I(d_0,r))L_p'(0,\tilde{\chi}\omega_p)L_p(0,\tilde{\chi}\left(\frac{D}{*}\right)\omega_p).$$

From this we deduce that

$$(Tr_{L/M}\zeta_p)'(\delta,0) = 2\Theta_p'(0). \tag{17.19}$$

A straight forward calculation also shows that

$$(Tr_{L/M}\zeta)(\delta,0) = \Theta(0). \tag{17.20}$$

Note the dicrepancy of a factor 2 of the two previous formulas.

Now we would like to write the zeta functions Θ and $\Theta_p(s)$ in terms of p-adic partial zeta functions attached to $\mathbb{Q}$. Using the definition of $\Theta_p(s)$ we see that in order to do this it is enough to relate $L_p(s,\widetilde{\chi}\omega_p)$ to p-adic partial zeta functions of $\mathbb{Q}$. The function $L(s,\widetilde{\chi})$ can be rewritten in terms of partial zeta functions of $\mathbb{Q}$ as

$$L(s,\widetilde{\chi}) = \sum_{a\in(\mathbb{Z}/f\mathbb{Z})^\times}\widetilde{\chi}(a)\zeta(a,f\infty,s) \tag{17.21}$$

Note that every partial zeta function $\zeta(b, pf\infty, s)$ (where $(b,p) = 1$) is p-adically continuous when s is restricted to the set of integers $n \leq 0$ and $n \equiv 0(mod\, p-1)$. Therefore the values

$$\sum_{\substack{b(mod\ pf),(b,p)=1 \\ b(mod\ f)\in a\langle\overline{p}\rangle}} \zeta(b, pf\infty, s) = (1-1/p^s)\sum_{i=1}^{r}\zeta(p^i a, f\infty, s) \tag{17.22}$$

are p-adically continuous when s is restricted to the set of integers $n \leq 0$ and $n \equiv 0(mod\, p-1)$. Remember that r was defined to be the order of p modulo f. We define

$$\zeta_p(a, f\infty, s) \tag{17.23}$$

to be the p-adic zeta function which interpolates p-adically (17.22) on the set of integers $\{n \leq 0 : n \equiv 0(mod\, p-1)\}$. It was crucial here to take the sum of the right hand side of (17.22) over all congruence classes of the powers of p modulo f in order to be able to factor out the Euler factor at p. Note that by construction $\zeta_p(a, f\infty, 0) = 0$.

Remark 17.4 The reader should be careful to not confuse the different zeta functions introduced so far. When $a \in \mathbb{Z}$ and $(a,f) = 1$, the notations $\zeta(a, f\infty, s)$ and $\zeta_p(a, f\infty, s)$ correspond to partial zeta functions attached to $\mathbb{Q}$. The partial zeta functions introduced earlier which were involving a divisor $\delta \in \mathbb{Z}[A_{f\infty}]$ were attached to K. Namely:

(1) The Archimedean ones: $\zeta(\delta, s)$ defined by the equation (17.2) and $\zeta^*(\delta, s)$ defined by equation (17.3).

(2) The p-adic ones: $\zeta_p^*(\delta, s)$ which interpolates special values of $\zeta^*(\delta, s)$ and $\zeta_{p,0}(\delta, s)$ which interpolates special values of $\zeta(\delta, s)$.

For a character $\widetilde{\chi}$ which is trivial on $\langle\overline{p}\rangle \leq (\mathbb{Z}/f\mathbb{Z})^\times$ we can rewrite (17.21) as

$$\begin{aligned} L(s,\chi) &= \sum_{i=1}^{r}\sum_{a\in(\mathbb{Z}/f\mathbb{Z})^\times/\langle\overline{p}\rangle}\widetilde{\chi}(p^i a)\zeta(p^i a, f\infty, s) \\ &= \sum_{a\in(\mathbb{Z}/f\mathbb{Z})^\times/\langle\overline{p}\rangle}\widetilde{\chi}(a)\sum_{i=1}^{r}\zeta(p^i a, f\infty, s). \end{aligned} \tag{17.24}$$

From the latter equality and the density of the set of integers $\{n \leq 0 : n \equiv 0(mod\, p-1)\}$ in $\mathbb{Z}_p$ we deduce that

$$L_p(s, \widetilde{\chi}\omega_p) = \sum_{a \in (\mathbb{Z}/f\mathbb{Z})^\times/\langle \overline{p} \rangle} \widetilde{\chi}(a)\zeta_p(a, f\infty, s), \tag{17.25}$$

for all $s \in \mathbb{Z}_p$. We will need the following lemma

Lemma 17.1 *Assume that* $\overline{-1} \notin \langle \overline{p} \rangle \leq (\mathbb{Z}/f\mathbb{Z})^\times$. *The zeta functions* $\Theta(s)$ *and* $\Theta_p(s)$ *can be rewritten as*

$$\Theta(s) = 4f^{2s} \sum_{a \in (\mathbb{Z}/f\mathbb{Z})^\times/\langle \overline{p} \rangle} n_a \sum_{i=1}^{r} \zeta(p^i a, f\infty, s) \tag{17.26}$$

and

$$\Theta_p(s) = 4(f)^{2s} \sum_{a \in (\mathbb{Z}/f\mathbb{Z})^\times/\langle \overline{p} \rangle} n_a \zeta_p(a, f\infty, s) \tag{17.27}$$

where the n_a*'s are elements in* $\frac{1}{2}\mathbb{Z}$ *given by the following formula*

$$n_a := \frac{1}{|G_{M/K}|} \sum_{\substack{d_0 | N_0 \\ r \in (\mathbb{Z}/f\mathbb{Z})^\times}} \sum_{\substack{\tilde{\chi} \in \widehat{(\mathbb{Z}/f\mathbb{Z})^\times} \\ \tilde{\chi}(p)=1 \\ \tilde{\chi}(-1)=-1}} \widetilde{\chi}(a)\mathbf{N}_{K/\mathbb{Q}}(I(d_0, r)) \left(L(0, \tilde{\chi}\left(\frac{D}{*}\right)) \right) \in \frac{1}{2}\mathbb{Z}. \tag{17.28}$$

Moreover we have $n_{ap} = n_a$ *and* $n_{-a} = -n_a$.

Proof The fact that the n_a's are equal to the expression (17.28) follows directly from the definition of $\Theta(s)$, $\Theta_p(s)$ and of equations (17.24) and (17.25). Also the fact that $n_{ap} = n_a$ and $n_{-a} = -n_a$ are straight forward. It is also easy to see that that the n_a's are invariant under $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ therefore they lie in $\mathbb{Q}$. The character $\widetilde{\chi}\left(\frac{D}{*}\right)$ is a character modulo Df. We have

$$\begin{aligned} L(0, \tilde{\chi}\left(\frac{D}{*}\right)) &= -B_{1, \tilde{\chi}\left(\frac{D}{*}\right)} \\ &= \frac{1}{Df} \sum_{\substack{1 \leq a \leq Df \\ (a, Df)=1}} \tilde{\chi}(a)\left(\frac{D}{a}\right) a \end{aligned}$$

therefore $fD|G_{M/K}|n_a \in \mathbb{Z}$. It remains to show that $2n_a \in \mathbb{Z}$. We can rewrite n_a as

$$\frac{1}{Df|G_{M/K}|}\sum_{\substack{d_0|N_0\\ r\in(\mathbb{Z}/f\mathbb{Z})^\times}}\sum_{\substack{\tilde{\chi}\in\widehat{(\mathbb{Z}/f\mathbb{Z})^\times}\\ \tilde{\chi}(p)=1\\ \tilde{\chi}(-1)=-1}}\sum_{\substack{1\le a\le Df\\ (a,Df)=1}} n(d_0,r)\tilde{\chi}\circ\mathbf{N}_{K/\mathbb{Q}}(I(d_0,r))\tilde{\chi}(a)\left(\frac{D}{a}\right)a$$

Let $Q_\tau(x,y)=Ax^2+Bxy+Cy^2$ then we have $\mathbf{N}(\Lambda_{d_0\tau})=\frac{1}{A/d_0}$. Also since $A_{r/d_0}\equiv r/d_0(mod\ f)$ we have $\mathbf{N}(A_{r/d_0}\Lambda_{d_0\tau})=A_{r/d_0}^2\frac{d_0}{A}\equiv\left(\frac{r}{d_0}\right)^2 d_0A^{-1}(mod\ f)$. We can thus rewrite the previous expression as

$$\frac{1}{Df|G_{M/K}|}\sum_{\substack{d_0|N_0\\ r\in(\mathbb{Z}/f\mathbb{Z})^\times}}\sum_{\substack{\tilde{\chi}\in\widehat{(\mathbb{Z}/f\mathbb{Z})^\times}\\ \tilde{\chi}(p)=1\\ \tilde{\chi}(-1)=-1}}\sum_{\substack{1\le a\le Df\\ (a,Df)=1}} n(d_0,r)\tilde{\chi}(ar^2d_0^{-1}A^{-1})\left(\frac{D}{a}\right)a$$

which again can be rewritten as

$$\frac{1}{Df|G_{M/K}|}\sum_{\substack{d_0|N_0\\ r\in(\mathbb{Z}/f\mathbb{Z})^\times}}\sum_{\substack{1\le a\le Df\\ (a,Df)=1}} n(d_0,r)\left(\frac{D}{a}\right)a\sum_{\substack{\tilde{\chi}\in\widehat{(\mathbb{Z}/f\mathbb{Z})^\times}\\ \tilde{\chi}(p)=1\\ \tilde{\chi}(-1)=-1}}\tilde{\chi}(ar^2d_0^{-1}A^{-1}). \tag{17.29}$$

Let $G=(\mathbb{Z}/f\mathbb{Z})^\times/\langle p(mod\ f)\rangle$. By assumption we have $\overline{-1}\notin\langle p\ (mod\ f)\rangle\le(\mathbb{Z}/f\mathbb{Z})^\times$. Therefore there exists a character Ψ of G such that $\Psi(-1)=-1$, i.e. Ψ is odd. Let us denote by $\widehat{G}^{even}$ and $\widehat{G}^{odd}$ the set of even and odd characters of G respectively. Note that $\widehat{G}^{odd}=\Psi\widehat{G}^{even}$. An easy computation shows that for every $a\in(\mathbb{Z}/f\mathbb{Z})^\times$

$$\sum_{\chi\in\widehat{G}^{odd}}\chi(a)=\begin{cases}\frac{|G_{M/K}|}{2} & \text{if } a\in\langle p(mod\ f)\rangle\\ -\frac{|G_{M/K}|}{2} & \text{if } a\in-\langle p(mod\ f)\rangle\\ 0 & \text{otherwise}\end{cases}$$

We can thus rewrite (17.29) as

$$\frac{1}{2Df}\sum_{\substack{d_0|N_0\\ r\in(\mathbb{Z}/f\mathbb{Z})^\times\\ 1\le a\le Df\\ (a,Df)=1\\ ar^2d_0^{-1}A^{-1}\in\pm\langle p(mod\ f)\rangle}} n(d_0,r)\left(\frac{D}{a}\right)a\epsilon(ar^2d_0^{-1}A^{-1}) \tag{17.30}$$

where $\epsilon(a)=1$ if $a(mod\ f)\in\langle p(mod\ f)\rangle$ and $\epsilon(a)=-1$ if $a(mod\ f)\in-\langle p(mod\ f)\rangle$. Every element $0\le a\le Df-1$ can be written as (a_1,a_2) where $a_1\equiv a(mod\ D)$ and

$a_2 \equiv a(mod\ f)$. Every term $a = (a_1, a_2)$ in the sum (17.30) can be paired with the term $a' = (-a_1, a_2)$. Since $\left(\frac{D}{a}\right) = \left(\frac{D}{a'}\right)$ (the quadratic character $\left(\frac{D}{*}\right)$ is associated to a real quadratic field and $\epsilon(a) = \epsilon(a')$) we see that the sum in (17.30) is congruent to 0 modulo D. Now using the fact that δ is a good divisor we have for a fixed $r \in (\mathbb{Z}/f\mathbb{Z})^\times$ that $\sum_{d_0|N_0} n(d_0, r)d_0 = 0$. Let us fix an element $b \in \pm\langle p(mod\ f)\rangle$ and an element $r \in (\mathbb{Z}/f\mathbb{Z})^\times$. Summing over all the elements $\{\frac{d_0bA}{r^2}\}_{d_0|N_0}$ we obtain

$$\sum_{d_0|N_0} n(d_0, r)\left(\frac{D}{\frac{d_0bA}{r^2}}\right)\frac{d_0bA}{r^2}\epsilon(b) \equiv 0(mod\ f).$$

For the latter congruence we have used the fact that all the primes dividing N_0 are split in $\mathbb{Q}(\sqrt{D})$ and also that δ is a good divisor. From this we deduce that (17.30) lies in $\frac{1}{2}\mathbb{Z}$. This completes the proof of the lemma. $\square$

We can now state the key ingredient that allowed us to relate the first the derivative at $s = 0$ of $Tr_{L/M}\zeta_{p,0}(\delta, s)$ with normalized Gauss sums.

Theorem 17.2 *Let $a \in (\mathbb{Z}/f\mathbb{Z})^\times$ and let $\zeta_p(a, f\infty, s)$ be the p-adic zeta function introduced in* (17.25). *Then*

$$\zeta_p'(a, f\infty, 0) = -\log_p g\left(\frac{a}{f}\right)$$

where $g\left(\frac{a}{f}\right) = \frac{\tau(w_q^{a\frac{q-1}{f}})}{\tau(\omega_q^{\frac{q-1}{2}})} \in \mathbb{Q}(\zeta_f, \zeta_p)$, $w_q : \mathbb{F}_q^\times \to \mu_{q-1} \subseteq \overline{\mathbb{Q}}_p$ *is the Teichmüller character and* $q = p^r \equiv 1(mod\ f)$ *where* $r = ord_f(p)$. *Note that* $g\left(\frac{a}{f}\right)^{2f} \in \mathbb{Q}(\zeta_f)^{Fr_p}$. *We have*

$$v_p\left(g\left(\frac{a}{f}\right)^{2f}\right) = 2f\sum_{i=1}^{r}\zeta(p^i a, f\infty, 0) = 2f\sum_{i=1}^{r}\left(\frac{\widetilde{(p^i a)}}{f} - \frac{1}{2}\right),$$

where $\widetilde{x}$ is chosen to be the unique integer between 1 *and* $f-1$ *such that* $\widetilde{x} \equiv x(mod\ f)$.

Proof Combine the results of [FG78] with [GK79] plus standard results about factorization of Gauss sums. $\square$

Using (17.27) we deduce that

$$\begin{aligned}\Theta_p'(s) &= 4 \sum_{a\in(\mathbb{Z}/f\mathbb{Z})^\times/\langle\overline{p}\rangle} n_a\zeta_p'(a, f\infty, 0)\\ &= 4 \sum_{\left\{\substack{1\le a\le f/2\\(a,f)=1}\right\}/\langle\overline{p}\rangle} 2n_a\zeta_p'(a, f\infty, 0)\end{aligned} \tag{17.31}$$

where for the second equality we have used the fact that $n_{-a} = -n_a$, $g\left(\frac{-a}{f}\right) = \pm g\left(\frac{a}{f}\right)^{-1}$ and that $\overline{-1} \notin \langle\overline{p}\rangle$. Now using theorem 17.2 we can rewrite the right hand side of (17.31) as

$$-4 \sum_{\left\{\substack{1\le a\le f/2\\(a,f)=1}\right\}/\langle\overline{p}\rangle} (2n_a)\log_p g\left(\frac{a}{f}\right). \tag{17.32}$$

Now from lemma 17.1 we get that $2n_a \in \mathbb{Z}$. Set

$$S = \prod_{\left\{\substack{1\le a\le f/2\\(a,f)=1}\right\}/\langle\overline{p}\rangle} g\left(\frac{a}{f}\right)^{2n_a}.$$

We thus have by definition that

$$\Theta_p'(s) = -4\log_p S.$$

Now using (17.19) and the last equality we deduce that

$$(Tr_{L/M}\zeta_p)'(\delta, 0) = -8\log_p S.$$

This proves (17.10). It remains to show the valuation formula (17.11). Using theorem 17.2 with the definition of S we get

$$\begin{aligned}12v_p(S^{2f}) &= 12 \sum_{\left\{\substack{1\le a\le f/2\\(a,f)=1}\right\}/\langle\overline{p}\rangle} (2n_a)2f\sum_{i=1}^{r}\zeta(p^i a, f\infty, 0)\\ &= 6f\cdot 4 \sum_{a\in(\mathbb{Z}/f\mathbb{Z})^\times/\langle\overline{p}\rangle} n_a\sum_{i=1}^{r}\zeta(p^i a, f\infty, 0)\\ &= 6f\Theta(0)\\ &= 6f(Tr_{L/M}\zeta)(\delta, 0),\end{aligned}$$

where the second last equality uses (17.26) and the last one uses (17.20). This shows equation (17.11) and therefore concludes the proof of proposition 17.1. □

In order to finish the proof of theorem 17.1 we need to relate $\zeta(\delta,0)$ to $\zeta^*(\delta,0)$ and $\zeta_p^*(\delta,s)$ to $\zeta_{p,0}(\delta,s)$. The next proposition takes care of this.

Proposition 17.3 *Let $\widetilde{\delta}=\sum_{d_0,r}n(d_0,r)[d_0,r]\in D(N_0,f)^{\langle p\rangle}$ be a good divisor. Let $(1,\tau)\in\mathcal{H}_p^{\mathcal{O}_K}(N_0,f)$ with τ reduced and let $\delta=\sum_{d_0,r}n(d_0,r)[A_r\Lambda_{d_0\tau}]\in A_{f\infty}$. Assume furthermore that all primes dividing f are inert in K then*

$$\zeta^*(\delta,0)=\sum_{u=0}^{f-1}\zeta(\lambda_u\star\delta,0),$$

and

$$\zeta_p^*(\delta,s)=\sum_{u=0}^{f-1}\zeta_{p,0}(\lambda_u\star\delta,s).$$

where λ_u is an algebraic integers of K chosen so the $\lambda_u\equiv(\frac{A}{N_0}u+\tau^\sigma)(mod\ f)$, λ_u is coprime to p and totally positive.

Proof Using proposition 9.4 gives us

$$\begin{aligned}\zeta^*(\delta,s)&=\sum_{d_0,r}n(d_0,r)d_0^s\widehat{\Psi}^*(\frac{A_r\Lambda_\tau}{\sqrt{D}f},w_1,s)\\&=\sum_{u=0}^{f-1}\sum_{d_0,r}n(d_0,r)d_0^s\widehat{\Psi}(\lambda_u\frac{A_r\Lambda_\tau}{f\sqrt{D}},w_1,s)\\&=\sum_{u=0}^{f-1}\zeta(\lambda_u\star\delta,s)\end{aligned}\tag{17.33}$$

Using corollary 10.1 we deduce

$$\begin{aligned}(1-p^{-2n})\sum_{u=0}^{f-1}\zeta(\lambda_u\star\delta,,n)&\overset{(17.33)}{=}(1-p^{-2n})\zeta^*(\delta,n)\\&\overset{10.1}{=}\zeta_p^*(\delta,(1,\tau),n)\\&:=\zeta_p^*(\delta,n).\end{aligned}\tag{17.34}$$

for every $n \leq 0$ and $n \equiv 0(mod\, p-1)$. By density of the set of integers $\{n \leq 0 : n \equiv 0(mod\, p-1)\}$ in $\mathbb{Z}_p$ we get

$$\zeta_p^*(\delta, s) = \sum_{u=0}^{f-1} \zeta_{p,0}(\lambda_u \star \delta, s).$$

□

We can now prove theorem 17.1. Using the latter proposition with equations (17.10) and (17.11) of proposition 17.1 we get that

$$6fTr_{L/M}(\zeta_p^*)'(\delta, 0) = -\log_p(S')^{2f} \tag{17.35}$$

and that

$$12fTr_{L/M}\zeta^*(\delta, 0) = v_p((S')^{2f}) \tag{17.36}$$

where S' is a product of normalized Gauss sums inside $\mathbb{Q}(\zeta_f)^{Fr_p} \cdot \mathbb{Q}(\zeta_p)$. Note that $(S')^{2f} \subseteq \mathbb{Q}_p^{ur}$ so it makes sense to take the p-adic valuation. On the other hand in section 17.2 we have proved the existence of an element $u \in K_p^\times$ for which

$$\begin{aligned} 6fTr_{L/M}(\zeta_p^*)'(\delta, 0) &= -2f \log_p \mathbf{N}_{L/M} \circ \mathbf{N}_{K_p/\mathbb{Q}_p}(u) \\ &= -2f \log_p \mathbf{N}_{L/\widetilde{M}}(u), \end{aligned} \tag{17.37}$$

and

$$6fTr_{L/M}(\zeta^*)(\delta, 0) = 2fv_p(\mathbf{N}_{L/M}(u)).$$

Now using the observation that $v_p(u^\sigma) = v_p(u)$ where $Gal(K_p/\mathbb{Q}_p) = \{1, \sigma\}$ we get

$$12fTr_{L/M}(\zeta^*)(\delta, 0) = 2fv_p(\mathbf{N}_{L/\widetilde{M}}(u)). \tag{17.38}$$

Comparing (17.35) with (17.37) we obtain

$$\log_p(S')^{2f} = 2f \log_p \mathbf{N}_{L/\widetilde{M}}(u).$$

Comparing (17.36) with (17.38) we obtain

$$v_p((S')^{2f}) = 2fv_p(\mathbf{N}_{L/\widetilde{M}}(u)).$$

From this we conclude that

$$S' = \mathbf{N}_{L/\widetilde{M}}(u) \ (mod\ \mu_F).$$

This concludes the proof of theorem 17.1. □

We should expect a refinement of theorem 17.1 of the following form

Conjecture 17.2 *The element*

$$N_{L/M}(u(c))$$

is a product of normalized Gauss sums in $M \cdot \mathbb{Q}(\zeta_p)$.

18 Numerical examples

In this section we record the results of some computations that we did using the software Magma in order to test the validity of the conjecture 5.1.

Conjecture 5.1 says something about the existence of strong p-units in abelian extensions of real quadratic number fields. In order to make sure that such units exist one needs to impose a certain number of conditions on the real quadratic field. To fix the idea let us assume that $f = 3$, $N_0 = 4$ and that the good divisor $\delta = 2[1,1] - 3[2,1] + 1[4,1] \in D(N_0, f) = D(4,3)$. It thus follows that the modular unit attached to the data (f, N_0, δ) is

$$\beta_{\delta_1}(\tau) = g_{(\frac{1}{3},0)}(3\cdot\tau)^{2\cdot 12} g_{(\frac{1}{3},0)}(3\cdot 2\tau)^{-3\cdot 12} g_{(\frac{1}{3},0)}(3\cdot 4\tau)^{1\cdot 12}.$$

Le $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic field where $D = discriminant(K)$. In order to have the existence of non trivial strong p-units in $K(3\infty)$ attached to the previous modular unit one needs to have that

(1) $(D, 3) = 1$ (not essential but simplifies the construction)

(2) $D \equiv 1 (mod\ 8)$ (we want 2 to split in K)

(3) $\left(\frac{D}{p}\right) = -1$ (p has to be inert in $\mathbb{Q}(\sqrt{D})$)

(4) $\mathcal{O}_K(3)^\times = \mathcal{O}_K(3\infty)^\times$ ($K(3\infty)$ has to be totally complex otherwise there exists no strong p-units other that $\{\pm 1\}$).

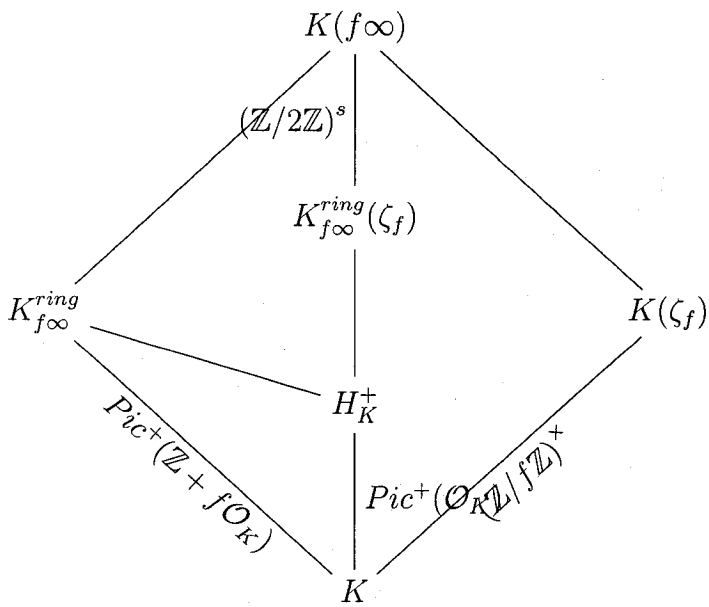
The first thing to notice is that the fourth condition is always satisfied. If $\epsilon \in \mathcal{O}_K^\times$, $\epsilon = u + 3v\sqrt{D} \equiv 1 (mod\ 3)$ and $\mathbf{N}(\epsilon) = -1$ then the equation $x^2 - 3^2 D y^2 = -1$ would admit a non trivial integral solution namely $(x, y) = (u, v)$. A congruence modulo 3 shows that it doesn't have any. Therefore $K(3\infty)$ is always totally complex. Using class field theory one deduces that

$$K(3\infty) \supseteq K(\zeta_3) = K(\sqrt{-3}). \tag{18.1}$$

If we let $\mathfrak{p}$ be a prime ideal of $K(3\infty)$ above $p\mathcal{O}_K$ then our conjecture predicts that the strong p-units lie in $K(3\infty)^{\langle Frob(\mathfrak{p}/p)\rangle}$. Since we would like our strong p-units to be primitive elements of $K(3\infty)$ over K we will impose the additional condition that $Frob(\mathfrak{p}/p) = 1$ which is equivalent by class field theory to the congruence $p \equiv 1 (mod\ 3)$.

By (18.1) it follows that in the case where the narrow ray class group of conductor 3 of K is of order 2 we expect $K(3\infty) = K(\sqrt{-3})$. For a general f and a general real quadratic number field $K = \mathbb{Q}(\sqrt{D})$ such that $\mathcal{O}_K(f)^\times = \mathcal{O}_K(f\infty)^\times$ we have the

following Hasse diagram:



where $K_{f\infty}^{ring}$ (respectively H_K^+) stand for the narrow ring class field of conductor f (respectively the narrow Hilbert class field). Here s is some integer that can be computed explicitly.

The tables in the next pages record some of our results. For every admissible D the narrow class group of conductor 3 is given by $I_K(3)/P_{K,1}(3\infty)$ where $K = \mathbb{Q}(\sqrt{D})$. For every class $C \in I_K(3)/P_{K,1}(3\infty)$ we pick an ideal $\mathfrak{a}_C$ that can be written as $r\Lambda_\tau$ for some $r \in \mathbb{Z}$ coprime to 3 and $\tau \in K$ (reduced with respect to p) such that $\Lambda_{4\tau}$ is again an $\mathcal{O}_K$-ideal. We always choose τ in such a way that $\tau - \tau^\sigma > 0$. Let $\epsilon > 1$ be such that $\mathcal{O}_K(3\infty)^\times = \{\pm 1\} \times \epsilon^{\mathbb{Z}}$ then we let γ_τ be the matrix corresponding to the action of ϵ on Λ_τ with respect to the ordered basis $\{\tau, 1\}$. So we have

$$\gamma_\tau \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \epsilon \begin{pmatrix} \tau \\ 1 \end{pmatrix}. \tag{18.2}$$

If we let $\gamma_\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then one can verify that $c \equiv 0 (mod\ 12)$ and $d \equiv 1 (mod\ 3)$. For this ideal class C we thus have a p-adic invariant $u(C) := u(r\Lambda_\tau) = u(r, \tau) \in K_p^\times$.

If our conjecture is true then we expect

$$P_D(x) := \prod_{C \in I_K(3)/P_{K,1}(3\infty)} (x - u(C)) \in \mathcal{O}_K[\frac{1}{p}][x],$$

where $P_D(x)$ is a *palindrome polynomial.* Because of the presence of the 12-th power in the definition of $\beta_\delta(\tau)$, it turns out that very often our units $u(C)$ are power of smaller units. For every admissible D we define a certain integer $n_D|12$. This allows us to define a polynomial

$$\widetilde{P}_D(x) := \prod_C (x - u(C)^{1/n_D}) \in \mathcal{O}_K[\frac{1}{p}][x]$$

which has smaller height. The column valuation in the tables correspond to the valuation of the distinct roots of $\widetilde{P}_D(x)$.

We want to explain briefly how one can go about the computation of the p-adic integral

$$u(r,\tau) = \int\!\!\!\!\!\!\diagup_{\mathbb{X}} (x - \tau y) d\widetilde{\mu}_r\{i\infty \to \gamma_\tau(i\infty)\}(x,y),$$

where $\gamma_\tau \in \Gamma_0(fN_0)$. We follow the same method that was developed in [Das05]. When τ is reduced, we have an explicit formula for $v_p(u(r,\tau))$ (see equation (9.11)). In practice one doesn't compute directly $v_p(u(r,\tau))$ from the formula (9.11) since the rational number $\gamma_\tau(i\infty) = \frac{A}{C}$ tends to have a too big height. In practice one computes $\pi_{(\frac{r}{f},0)}(\gamma)$ where γ goes over a system of generators of $\Gamma_0(fN_0)$ and then uses the cocycle property that was proved in proposition 5.2 to compute $\pi_{(\frac{r}{f},0)}(\gamma_\tau)$ quite efficiently. Having said this it remains to explain how to compute

$$\int_{\mathbb{X}} \log_p(x - \tau y) d\widetilde{\mu}_r\{i\infty \to \frac{A}{C}\}(x,y), \tag{18.3}$$

where $\log_p$ is the p-adic Iwasawa logarithm and

$$\int_{\mathbb{X}} \log_\zeta(x - \tau y) d\widetilde{\mu}_r\{i\infty \to \frac{A}{C}\}(x,y) \tag{18.4}$$

where $\log_\zeta : \mu_{p^2-1} \to \mathbb{Z}/(p^2-1)\mathbb{Z}$ is the discrete logarithm with respect to the generator ζ of μ_{p^2-1}. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(fN_0)$ and for $m = [c_2] - [c_1] \in$

$Div_0(\Gamma_0(fN_0)(i\infty))$ denote $\widetilde{\mu}_r\{c_1 \to c_2\}$ by $\widetilde{\mu}_r[m]$. We have

$$\int_{\mathbb{X}} \log_p(x - \tau y) d\widetilde{\mu}_r[m](x,y) = \int_{\mathbb{X}} \log_p(x' - \tau y') d\widetilde{\mu}_r[m](x',y')$$

where $\gamma \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then we have

$$\begin{aligned}
\int_{\mathbb{X}} \log_p(x' - \tau y') d\widetilde{\mu}_r[m](x',y') &= \int_{\mathbb{X}} \log_p((ax+by) - \tau(cx+dy)) d\widetilde{\mu}_r[m](x',y') \\
&= \int_{\mathbb{X}} \log_p(x(a-\tau c) - y(-b+\tau d)) d\widetilde{\mu}_r[m](x',y') \\
&= \int_{\mathbb{X}} \log_p \left(x - y\frac{-b+d\tau}{a-c\tau} \right) d\widetilde{\mu}_r[m](x',y') \\
&= \int_{\mathbb{X}} \log_p \left(x - y\gamma^{-1}\tau \right) d\widetilde{\mu}_r[m](x',y')
\end{aligned}$$

where for the third equality we have used the fact that $\widetilde{\mu}_r[m]$ has total measure 0. Using the Γ_0-invariance of $\mu_r[m]$ we deduce from the last equality that

$$\int_{\mathbb{X}} \log_p(x - \tau y) d\widetilde{\mu}_r[m](x,y) = \int_{\mathbb{X}} \log_p \left(x - y\gamma^{-1}\tau \right) d\widetilde{\mu}_{\gamma^{-1}\star r}[\gamma^{-1}m](x,y).$$

Replacing m by γm we get

$$\int_{\mathbb{X}} \log_p(x - \tau y) d\widetilde{\mu}_r[\gamma m](x,y) = \int_{\mathbb{X}} \log_p \left(x - y\gamma^{-1}\tau \right) d\widetilde{\mu}_{\gamma^{-1}\star r}[m](x,y). \tag{18.5}$$

We have a similar computation when $\log_p$ is replaced by $\log_\zeta$. In the appendix of this thesis we show that $Div_0(\Gamma_0(fN_0)(i\infty))$ is a finitely generated $\mathbb{Z}[\Gamma_0(fN_0)]$-module. Let $\{m_1, \ldots, m_s\}$ be a set of generators of $Div_0(\Gamma_0(fN_0)(i\infty))$ over $\mathbb{Z}[\Gamma_0(fN_0)]$. Then given a $m \in Div_0(\Gamma_0(fN_0)(i\infty))$ we can find integers $a_{i,\gamma}$ such that

$$m = \sum_{i=1}^{s} \sum_{\gamma \in \Gamma_0(fN_0)} a_{\gamma,i} \gamma m_i. \tag{18.6}$$

Using (18.5) we find that

$$\int_{\mathbb{X}} \log_p(x - \tau y) d\widetilde{\mu}_r[m](x,y) = \sum_{i=1}^{s} \sum_{\gamma \in \Gamma_0(fN_0)} a_{\gamma,i} \int_{\mathbb{X}} \log_p \left(x - y\gamma^{-1}\tau \right) d\widetilde{\mu}_{\gamma^{-1}\star r}[m_i](x,y).$$

Using the last equality we see that it is sufficient to compute

$$\int_{\mathbb{X}} \log_p (x - y\tau)\, d\widetilde{\mu}_r[m_i](x,y) \tag{18.7}$$

for fixed $\tau \in \mathcal{H}_p \cap K$, $r \in (\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle$ and $i \in \{1,\dots,s\}$. The latter observation is quite important. Essentially it says that the knowledge of the moments of $\widetilde{\mu}_r[m_i]$ for $i = 1,\dots,s$ and $r \in (\mathbb{Z}/f\mathbb{Z})^\times/\langle \overline{p}\rangle$,up to a certain precision, are sufficient to compute (18.3), see equation (18.8). This is crucial since in general the height of $\gamma_\tau(i\infty) = \frac{A}{C}$ is very big. Moreover, observe that the computation of those moments doesn't depend on τ and therefore doesn't depend on the real quadratic number field $K = \mathbb{Q}(\sqrt{D})$. Therefore those moments, which depend only δ, f, N_0 and p, need only to be computed once. Once they are computed one can let the discriminant D vary and reuse them. This is one of the key feature of the method.

The integral (18.7) can be rewritten in the following way

$$\begin{aligned}
&\int_{\mathbb{X}} \log_p (x - y\tau)\, d\widetilde{\mu}_r[m_i](x,y) \\
&= \int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} \log_p (x - y\tau)\, d\widetilde{\mu}_r[m_i](x,y) + \int_{\mathbb{Z}_p^\times\times p\mathbb{Z}_p} \log_p (x - y\tau)\, d\widetilde{\mu}_r[m_i](x,y) \\
&= \int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} \log_p(y) d\widetilde{\mu}_r[m_i](x,y) + \int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} \log_p\left(\frac{x}{y} - \tau\right) d\widetilde{\mu}_r[m_i](x,y) \\
&+ \int_{\mathbb{Z}_p^\times\times p\mathbb{Z}_p} \log_p(x) d\widetilde{\mu}_r[m_i](x,y) + \int_{\mathbb{Z}_p^\times\times p\mathbb{Z}_p} \log_p\left(1 - \tau\frac{y}{x}\right) d\widetilde{\mu}_r[m_i](x,y) \\
&= \int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} \log_p(y) d\widetilde{\mu}_r[m_i](x,y) + \int_{\mathbb{Z}_p^\times\times p\mathbb{Z}_p} \log_p(x) d\widetilde{\mu}_r[m_i](x,y) \\
&+ \int_{\mathbb{Z}_p} \log_p (t - \tau)\, d\mu_r[m_i](t) + \int_{\mathbb{P}^1(\mathbb{Q}_p)\setminus\mathbb{Z}_p} \log_p\left(1 - \frac{\tau}{t}\right) d\mu_r[m_i](t)
\end{aligned} \tag{18.8}$$

where $\pi_*\widetilde{\mu}_r[m] = \mu_r[m]$. One can compute the first term of the last equality by finding a polynomial $f(y) \in \mathbb{Z}_p[y]$ such that $|f(y) - \log_p y|_p$ is very small for all $y \in \mathbb{Z}_p^\times$. Having done this one can then use the explicit formulas of proposition 11.5 to compute

$$\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} f(y) d\widetilde{\mu}_r[m_i](x,y).$$

A similar computation can be done for the second term of (18.8). For the third term of (18.8) one can expand $\log_p(t-\tau)$ around the balls $i + p\mathbb{Z}_p$ (for $i = 0 \dots p-1$).

After a few manipulations and tricks one can use the explicit formulas of proposition 11.5 to compute this term. A similar computation can be done to handle the fourth term of (18.8). Note that all those integrals are computed up to a certain accuracy which depends on the accuracy of the various moments of $\widetilde{\mu}_r[m_i]$ for $i = 1, \ldots, s$, $r \in (\mathbb{Z}/f\mathbb{Z})^\times/\langle\overline{p}\rangle$.

For $p = 7$ and $\delta = 2[1,1] - 3[2,1] + 1[4,1]$

D	$I_K(3)/P_{K,1}(3\infty)$	n_D	valuations	$\widetilde{P}_D(x)$
17	$\mathbb{Z}/2\mathbb{Z}$	12	± 2	$x^2 - \frac{71}{49}x + 1$
41	$\mathbb{Z}/2\mathbb{Z}$	12	± 2	$x^2 - \frac{71}{49}x + 1$
73	$(\mathbb{Z}/2\mathbb{Z})^2$	12	$\pm 2, \pm 2$	$(x^2 - \frac{71}{49}x + 1)^2$
89	$\mathbb{Z}/2\mathbb{Z}$	12	± 2	$x^2 - \frac{71}{49}x + 1$
97	$(\mathbb{Z}/2\mathbb{Z})^2$	12	$\pm 2, \pm 2$	$(x^2 - \frac{71}{49}x + 1)^2$
145	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$	3	$\pm 0, \pm 0, \pm 4, \pm 4, \pm 4, \pm 4, \pm 8, \pm 8$	$x^{16} + \frac{1}{7^8}(-232650\sqrt{D} - 21142198)x^{15}$ $\frac{1}{7^{16}}(-63553176225\sqrt{D} + 345833578130241)x^{14}$ $\frac{1}{7^{20}}(2873907075070350\sqrt{D} - 1633501333699078382)x^{13}$ $+\frac{1}{2\cdot 7^{24}}(-1220402710916392137 75\sqrt{D} + 13994939454565494390367)x^{12}$ $+\frac{1}{7^{28}}(2458797964659562076347 50\sqrt{D} - 2443852592564008493430 8094)x^{11}$ $+\frac{1}{7^{32}}(-1111134115782593132787676350\sqrt{D} + 7934128329756590561549651 3974)x^{10}$ $+\frac{1}{7^{32}}(1257312832261114545316699200\sqrt{D} - 9157124847683319470124441 6496)x^9$ $+\frac{1}{2\cdot 7^{32}}(-3047800972612593555659676375\sqrt{D} + 1975916252375457996792738 46779)x^8$ $+\frac{1}{7^{32}}(1257312832261114545316699200\sqrt{D} - 9157124847683319470124441 6496)x^7$ $+\frac{1}{7^{32}}(-1111134115782593132787676350\sqrt{D} + 7934128329756590561549651 3974)x^6$ $\frac{1}{7^{28}}(2458797964659562076347 50\sqrt{D} - 2443852592564008493430 8094)x^5$ $\frac{1}{2\cdot 7^{24}}(-1220402710916392137 75\sqrt{D} + 13994939454565494390367)x^4$ $\frac{1}{7^{20}}(2873907075070350\sqrt{D} - 1633501333699078382)x^3$ $\frac{1}{7^{16}}(-63553176225\sqrt{D} + 345833578130241)x^2$ $\frac{1}{7^8}(-232650\sqrt{D} - 21142198)x + 1$
185	$(\mathbb{Z}/2\mathbb{Z})^2$	12	$\pm 2, \pm 2$	$(x^2 - \frac{71}{49}x + 1)^2$
209	$(\mathbb{Z}/2\mathbb{Z})^2$	12	$\pm 2, \pm 2$	$(x^2 - \frac{71}{49}x + 1)^2$
241	$(\mathbb{Z}/2\mathbb{Z})^2$	12	$\pm 2, \pm 2$	$(x^2 - \frac{71}{49}x + 1)^2$
257	$\mathbb{Z}/6\mathbb{Z}$	6	$\pm 4 \pm 4, \pm 12$	$x^6 + \frac{1}{2\cdot 7^{12}}(-3861384345\sqrt{D} + 2642736525)x^5$ $+\frac{1}{2\cdot 7^{15}}(-131838694065\sqrt{D} + 38755163079075)x^4$ $+\frac{1}{2\cdot 7^{20}}(-42697160860228875\sqrt{D} + 52795271447651171)x^3$ $+\frac{1}{2\cdot 7^{15}}(-131838694065\sqrt{D} + 38755163079075)x^2$ $+\frac{1}{2\cdot 7^{12}}(-3861384345\sqrt{D} + 2642736525)x + 1$
265	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	12	$\pm 1, \pm 1, \pm 1, \pm 1$	$(x^2 + 13/7x + 1)^4$
313	$(\mathbb{Z}/2\mathbb{Z})^2$	12	$\pm 4, \pm 4$	$(x^2 - 239/2401x + 1)^2$
353	$\mathbb{Z}/2\mathbb{Z}$	12	± 6	$x^2 + 153502/117649x + 1$
377	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$	6	$\pm 0, \pm 0, \pm 0, \pm 4, \pm 4, \pm 4, \pm 4, \pm 8$	$x^{16} + \frac{1}{7^8}(-1760385\sqrt{D} - 7054747)x^{15}$ $+\frac{1}{2\cdot 7^{12}}(9559963245\sqrt{D} + 464126557983)x^{14}$ $+\frac{1}{7^{16}}(-53041186688295\sqrt{D} - 537520756632797)x^{13}$ $+\frac{1}{2\cdot 7^{20}}(192563525818981905\sqrt{D} + 6876802449703149427)x^{12}$ $\frac{1}{7^{24}}(-4854820000499920758 75\sqrt{D} - 4797177871518763359825)x^{11}$ $+\frac{1}{7^{24}}(2435443982040991353 60\sqrt{D} + 9678515302483741595848)x^{10}$ $+\frac{1}{7^{24}}(-5033483261569695553 20\sqrt{D} - 4848335400921890746456)x^9$ $+\frac{1}{2\cdot 7^{24}}(5102287362972620506 35\sqrt{D} + 1973917718746523583 7509)x^8$ $+\frac{1}{7^{24}}(-5033483261569695553 20\sqrt{D} - 4848335400921890746456)x^7$ $+\frac{1}{7^{24}}(2435443982040991353 60\sqrt{D} + 9678515302483741595848)x^6$ $+\frac{1}{7^{24}}(-4854820000499920758 75\sqrt{D} - 4797177871518763359825)x^5$ $+\frac{1}{2\cdot 7^{20}}(192563525818981905\sqrt{D} + 6876802449703149427)x^4$ $+\frac{1}{7^{16}}(-53041186688295\sqrt{D} - 537520756632797)x^3$ $+\frac{1}{2\cdot 7^{12}}(9559963245\sqrt{D} + 464126557983)x^2$ $+\frac{1}{7^8}(-1760385\sqrt{D} - 7054747)x + 1$

D	$I_K(3)/P_{K,1}(3\infty)$	n_D	valuations	$\widetilde{P}_D(x)$
409	$(\mathbb{Z}/2\mathbb{Z})^2$	12	$\pm2, \pm2$	$(x^2 - \frac{71}{49}x + 1)^2$
433	$(\mathbb{Z}/2\mathbb{Z})^2$	12	$\pm2, \pm2$	$(x^2 - 239/2401x + 1)^2$
481	$(\mathbb{Z}/2\mathbb{Z})^3$	6	$\pm4, \pm4, \pm4, \pm4$	$(x^2 - 4034/2401x + 1)^2$ $\cdot(x^2 + 4034/2401x + 1)^2$
521	$\mathbb{Z}/2\mathbb{Z}$	12	±6	$x^2 + 153502/117649x + 1$
545	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$	6	$\pm4,\pm4,\pm4,\pm4$ $\pm8,\pm0,\pm0,\pm0$	$x^{16} + \frac{1}{7^8}(-821748\sqrt{D} - 883036)x^{15}$ $\frac{1}{7^{12}}(-662987514\sqrt{D} + 64572360210)x^{14}$ $\frac{1}{7^{16}}(-4804666374063\sqrt{D} + 168656428624495)x^{13}$ $\frac{1}{7^{20}}(-29687922599132553\sqrt{D} + 14837925027826488 5)x^{12}$ $\frac{1}{7^{24}}(-24730779023499008949\sqrt{D} + 16690257017620443176 85)x^{11}$ $\frac{1}{2\cdot7^{24}}(-101267373093542176521\sqrt{D} + 2291647792063133649065)x^{10}$ $\frac{1}{7^{24}}(-67135410173257826013\sqrt{D} + 6565268024341119418 85)x^9$ $\frac{1}{2\cdot7^{24}}(-37079433752321502423\sqrt{D} + 3398662046679747603795)x^8$ $\frac{1}{7^{24}}(-67135410173257826013\sqrt{D} + 6565268024341119418 85)x^7$ $\frac{1}{2\cdot7^{24}}(-101267373093542176521\sqrt{D} + 2291647792063133649065)x^6$ $\frac{1}{7^{24}}(-24730779023499008949\sqrt{D} + 16690257017620443176 85)x^5$ $\frac{1}{7^{20}}(-29687922599132553\sqrt{D} + 14837925027826488 5)x^4$ $\frac{1}{7^{16}}(-4804666374063\sqrt{D} + 168656428624495)x^3$ $\frac{1}{7^{12}}(-662987514\sqrt{D} + 64572360210)x^2$ $\frac{1}{7^8}(-821748\sqrt{D} - 883036)x + 1$
577	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/14\mathbb{Z}$	3	$\pm0, \pm0, \pm0, \pm0$ $\pm0, \pm0, \pm4, \pm4$ $\pm4, \pm4, \pm8, \pm8$ $\pm16, \pm16$	$x^{28} + \frac{1}{7^{16}}(3072931836030\sqrt{D} - 224019365010010)x^{27}$ $\frac{1}{7^{32}}(-807445277082293675830760385\sqrt{D} + 252506432641750602095051464 59)x^{26}$ $\frac{1}{7^{40}}(163142954974661344770984804066231 45\sqrt{D} - 365561307455028402276987851148978069)x^{25}$ $?x^{24}+?x^{23}+?x^{22}+?x^{21}+?x^{20}+?x^{19}$ $+?x^{18}+?x^{17}+?x^{16}+?x^{15}+?x^{14}$ $?x^{13}+?x^{12}+?x^{11}+?x^{10}+?x^9$ $?x^8+?x^7+?x^6+?x^5+?x^4$ $\frac{1}{7^{40}}(163142954974661344770984804066231 45\sqrt{D} - 365561307455028402276987851148978069)x^3$ $\frac{1}{7^{32}}(-807445277082293675830760385\sqrt{D} + 252506432641750602095051464 59)x^2$ $\frac{1}{7^{16}}(3072931836030\sqrt{D} - 224019365010010)x + 1$
593	$\mathbb{Z}/2\mathbb{Z}$	12	±10	$x^2 + 445987849/7^{10}x + 1$
601	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	12	$\pm4, \pm4$	$(x^2 - 239/2401x + 1)^2$
649	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	12	$\pm4, \pm4$	$(x^2 - 239/2401x + 1)^2$
689	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$	6	$\pm0, \pm0, \pm4, \pm4$ $\pm4, \pm4, \pm8, \pm8$	$x^{16} + \frac{1}{7^8}(-618426\sqrt{D} - 8713894)x^{15}$ $+\frac{1}{7^{12}}(4604823397503\sqrt{D} + 147886093075761)x^{14}$ $+\frac{1}{7^{20}}(-19183045951916226\sqrt{D} - 407733319394678270)x^{13}$ $+\frac{1}{2\cdot7^{24}}(10965995189190302681 7\sqrt{D} + 3910704142483041468031)x^{12}$ $+\frac{1}{7^{28}}(-23225559689501708130 9810\sqrt{D} - 4964477528251534939685550)x^{11}$ $+\frac{1}{7^{32}}(52504047410937954683503869 0\sqrt{D} + 1557677153797028287557212 3638)x^{10}$ $+\frac{1}{7^{32}}(-70609922876177460652464316 8\sqrt{D} - 1554482467503013487202152 7376)x^9$ $+\frac{1}{2\cdot7^{32}}(12670629504360963083207748 09\sqrt{D} + 4300800295142352579507128 7675)x^8$ $+\frac{1}{7^{32}}(-70609922876177460652464316 8\sqrt{D} - 1554482467503013487202152 7376)x^7$ $+\frac{1}{7^{32}}(52504047410937954683503869 0\sqrt{D} + 1557677153797028287557212 3638)x^6$ $+\frac{1}{7^{28}}(-23225559689501708130 9810\sqrt{D} - 4964477528251534939685550)x^5$ $+\frac{1}{2\cdot7^{24}}(10965995189190302681 7\sqrt{D} + 3910704142483041468031)x^4$ $\frac{1}{7^{20}}(-19183045951916226\sqrt{D} - 407733319394678270)x^3$ $\frac{1}{7^{16}}(4604823397503\sqrt{D} + 147886093075761)x^2$ $\frac{1}{7^8}(-618426\sqrt{D} - 8713894)x + 1$

D	$I_K(3)/P_{K,1}(3\infty)$	n_D	valuations	$\widetilde{P}_D(x)$
713	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$	12		can't find good representatives
745	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	12		can't find good representatives
761	$\mathbb{Z}/6\mathbb{Z}$	6	$\pm 4, \pm 12, \pm 20$	$x^6 + \frac{1}{2\cdot 7^{20}}(-326067672535605\sqrt{D} - 15927525578674267 5)x^5$ $\frac{1}{2\cdot 7^{32}}(80320232406070663678321 65\sqrt{D} - 74279172985794451974334433 1)x^4$ $\frac{1}{2\cdot 7^{36}}(-12154109980551447665799417375\sqrt{D} + 45109875498047841894180875154 59)x^3$ $\frac{1}{2\cdot 7^{32}}(80320232406070663678321 65\sqrt{D} - 74279172985794451974334433 1)x^2$ $\frac{1}{2\cdot 7^{20}}(-326067672535605\sqrt{D} - 15927525578674267 5)x + 1$
769	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	12	$\pm 4, \pm 4$	$(x^2 - 239/2401x + 1)^2$
817	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$	12		can't find good representatives
857	$\mathbb{Z}/2\mathbb{Z}$	6	± 28	$x^2 + 59171744646898367649580 6/7^{28}x + 1$
881	$\mathbb{Z}/2\mathbb{Z}$	12		
913	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	12	$\pm 8, \pm 8$	$(x^2 - 4743554/7^8x + 1)^2$
929	$\mathbb{Z}/2\mathbb{Z}$	12		

For $p = 13$ and $\delta = 2[1,1] - 3[2,1] + 1[4,1]$

D	$I_K(3)/P_{K,1}(3\infty)$	n_D	valuations	$\widetilde{P}_D(x)$
41	$\mathbb{Z}/2\mathbb{Z}$	12	± 2	$x^2 - 337/169x + 1$
73	$(\mathbb{Z}/2\mathbb{Z})^2$	12	$\pm 2, \pm 2$	$(x^2 + 337/169x + 1)^2$
89	$\mathbb{Z}/2\mathbb{Z}$	12	± 2	$x^2 + 337/169x + 1$
97	$(\mathbb{Z}/2\mathbb{Z})^2$	12	$\pm 2, \pm 2$	$x^4 + 674/169x^3 + 170691/28561x^2 + 674/169x + 1 =$ $(x^2 + 337/169x + 1)^2$
137	$\mathbb{Z}/2\mathbb{Z}$	12	± 6	$x^2 + 9397582/4826809x + 1$
145	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$	3	$\pm 0, \pm 0, \pm 1, \pm 1$ $\pm 1, \pm 1, \pm 2, \pm 2$	$x^{16} + \frac{1}{13^8}(25064550\sqrt{D} + 1456407962)x^{15}$ $\frac{1}{13^{16}}(-9909170774179425\sqrt{D} + 367233567480055041)x^{14}$ $\frac{1}{13^{20}}(35640369711526913550\sqrt{D} - 45167895449503053818222)x^{13}$ $?x^{12}+?x^{11}+?x^{10}+?x^9+?x^8+?x^7+?x^6+?x^5+?x^4$ $\frac{1}{13^{20}}(35640369711526913550\sqrt{D} - 45167895449503053818222)x^3$ $\frac{1}{13^{16}}(-9909170774179425\sqrt{D} + 367233567480055041)x^2$ $\frac{1}{13^8}(25064550\sqrt{D} + 1456407962)x + 1$
161	$(\mathbb{Z}/2\mathbb{Z})^2$	12	$\pm 2, \pm 2$	$(x^2 + 337/169x + 1)^2$
193	$(\mathbb{Z}/2\mathbb{Z})^2$	12	$\pm 4, \pm 4$	$(x^2 - 56447/28561x + 1)^2$
241	$(\mathbb{Z}/2\mathbb{Z})^2$			
265	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$			
281	$\mathbb{Z}/2\mathbb{Z}$			
305	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$			
353	$\mathbb{Z}/2\mathbb{Z}$			
385	$(\mathbb{Z}/2\mathbb{Z})^3$			
401	$\mathbb{Z}/10\mathbb{Z}$	6	$\pm 4, \pm 4, \pm 4$ $\pm 4, \pm 6$	$x^{10} + \frac{1}{2\cdot 13^{12}}(7954953835725\sqrt{D} - 13563872824361)x^9$ $\frac{1}{13^{16}}(-4235895970542018\sqrt{D} + 433368100400613030 3)x^8$ $\frac{1}{2\cdot 13^{20}}(164111282415722579834 07\sqrt{D} + 279540733246854591156 57)x^7$ $?x^6+?x^5+?x^4+?x^3$ $\frac{1}{13^{16}}(-4235895970542018\sqrt{D} + 433368100400613030 3)x^2$ $\frac{1}{2\cdot 13^{12}}(7954953835725\sqrt{D} - 13563872824361)x + 1$
409	$(\mathbb{Z}/2\mathbb{Z})^2$			
449	$\mathbb{Z}/2\mathbb{Z}$			
457	$(\mathbb{Z}/2\mathbb{Z})^2$			
505	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$			
553	$(\mathbb{Z}/2\mathbb{Z})^3$			
577	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/14\mathbb{Z}$			
593	$\mathbb{Z}/2\mathbb{Z}$			

19 Discussion and future directions

In this thesis we have proposed a conjectural construction of elements lying in totally complex ray class fields of a real quadratic number fields K. Our construction is very much in the spirit of the theory of complex multiplication available for imaginary

quadratic number fields. We have been able to provide some theoretical evidence for the algebraicity of the local elements $u(r,\tau) \in K_p^\times$ (see theorem 17.1). Despite the latter result, it seems that for the moment the proof of the algebraicity of $u(r,\tau)$ is out of reach. Since our units can be related with the first derivative at $s = 0$ of a p-adic zeta function interpolating classical values of partial zeta functions attached to K we see that the strong p-units that we have constructed are nothing else than Gross-Stark p-units that were predicted by the p-adic Gross-Stark conjectures (see [Gro81]). Therefore we are not constructing new units. But instead of proposing only a formula for the logarithm of its norm, we propose a formula for the unit itself, which can be seen as a refinement. The main feature of the approach used in [DD06] and in this thesis resides probably in the fact that we can compute those units p-adically in polynomial time using modular symbols coming from Eisenstein series. With a certain amount of work, the method could be implemented using the mathematical software Magma and allowed ourself to test the truth of conjecture 5.1.

The relative situation $K/\mathbb{Q}$ with K real quadratic admits an obvious generalization namely the case L'/L, were L and L' are totally real number fields and L' is quadratic over L. In this case if the degree of L' over $\mathbb{Q}$ is $2n$ then the group of units of L has rank $n-1$ and the one of L' has rank $2n-1$. Therefore the units in L' which are not coming from L form a lattice of rank n in $\mathcal{O}_{L'}^\times$. In this special situation one can replace the one variable Eisenstein series attached to $\mathbb{Q}$ by the n-variable Eisenstein series of parallel weight k attached to L namely

$$E_k(\frac{\mathfrak{b}}{\mathfrak{f}\mathfrak{d}}, \mathfrak{a}, z) = \mathbf{N}(\frac{\mathfrak{b}}{\mathfrak{f}\mathfrak{d}})^k \sum_{\mathcal{O}_L(\mathfrak{f}\infty)^\times \backslash \{(0,0) \neq (\alpha,\beta) \in \frac{\mathfrak{b}\mathfrak{a}}{\mathfrak{d}\mathfrak{f}} \times \frac{\mathfrak{b}}{\mathfrak{f}\mathfrak{d}}\}} \frac{e^{2\pi i Tr(\beta)}}{\mathbf{N}(\alpha z + \beta)^k} \tag{19.1}$$

where $\mathbf{N}(\alpha z + \beta) = \prod_{i=1}^n (\alpha^{(i)} z_i + \beta^{(i)})$, $\mathfrak{a}, \mathfrak{b}, \mathfrak{f}$ are integral ideals of L such that $(\mathfrak{f}, \mathfrak{b}) = 1$, $\mathfrak{d}$ is the different ideal of the number field L and $\mathcal{O}_L(\mathfrak{f}\infty)^\times$ are the totally positive units of L congruent to 1 modulo $\mathfrak{f}$. The constant term of the q-expansion of (19.1) is a partial zeta function associated to $L(\mathfrak{f}\infty)/L$ where $L(\mathfrak{f}\infty)$ corresponds to the narrow ray class field of conductor $\mathfrak{f}$ of L. The special values of these partial zeta functions were studied in section 7 of the present thesis. A unit $\epsilon \in \mathcal{O}_{L'}^\times$ acts naturally on the $\mathcal{O}_L$-lattice $\mathcal{O}_L + \tau\mathcal{O}_L \subseteq L'$ where $\tau \in L' \backslash L$ and therefore gives rise to a matrix in $SL_2(\mathcal{O}_L)$ having τ as a fixed point. Let $\wp$ be a prime ideal of L which

is inert in L'. As in the one variable case one can probably construct a family of $\mathbb{Z}$-valued measures on $\mathbb{P}^1(L_\wp)$ where $L_\wp$ denotes the completion of L at $\wp$. This family of $\mathbb{Z}$-valued measures can probably be indexed by pairs $(c_1, c_2) \in (\Gamma(i\infty))^n \times (\Gamma(i\infty))^n$ where Γ would be a suitable congruence subgroup of $SL_2(\mathcal{O}_\wp)$. For a pair (c_1, c_2) one could define first a measure $\mu\{c_1 \to c_2\}$ on the distinguished compact open ball $\mathcal{O}_{L_\wp}$ by the rule

$$\mu\{c_1 \to c_2\}(\mathcal{O}_{L_\wp}) = \int_{c_{11}}^{c_{21}} \int_{c_{12}}^{c_{22}} \cdots \int_{c_{1n}}^{c_{2n}} E_{k,\wp}(\frac{\mathfrak{b}}{\mathfrak{f}\mathfrak{d}}, \mathfrak{a}, z) dz_1 \cdots dz_n$$

where $c_1 = (c_{1i})$, $c_2 = (c_{2i})$ and $E_{k,\wp}(\frac{\mathfrak{b}}{\mathfrak{f}\mathfrak{d}}, \mathfrak{a}, z)$ is the $\wp$-stabilization of $E_k(\frac{\mathfrak{b}}{\mathfrak{f}\mathfrak{d}}, \mathfrak{a}, z)$. Using the almost transitive action of Γ on balls of $\mathbb{P}^1(\mathcal{O}_{L_\wp})$ and extending $\mu\{c_1 \to c_2\}$ to all balls of $\mathbb{P}^1(\mathcal{O}_{L_\wp})$, by forcing a Γ-invariance, one obtains a family of measures indexed by pairs $(c_1, c_2) \in (\Gamma(i\infty))^n \times (\Gamma(i\infty))^n$ which are Γ-invariant by construction. As in the one variable case one can probably use this family of measures to construct a (n+1)-cocycle κ in $Z^{n+1}(\Gamma, {L_\wp}^\times)$. We should expect this $n+1$-cocycle to split. One strategy to show the splitting of κ would be to try to lift the family of measures introduced previously to the larger space $\mathbb{X} := (\mathcal{O}_{L_\wp} \times \mathcal{O}_{L_\wp}) \backslash (\wp\mathcal{O}_{L_\wp} \times \wp\mathcal{O}_{L_\wp})$. Most computations that we have done in this thesis can probably be carried over to this setting. The only thing which is missing is an analogue of the Gross-Koblitz formula. Therefore proving an analogue of theorem 17.1 might be out of reach.

For the next discussion we have in mind the recent construction obtained by Dasgupta in [Das06]. Let K be a totally real number field and L a CM abelian extension of K. Let S be a set of places of K containing all the Archimedean places and all the finite primes which ramify in L/K. Consider the group ring $\mathbb{Q}[G_{L/K}]$. Let $\sigma \in Gal(L/K)$ then we define

$$\zeta_S(L/K, \sigma, s) = \zeta_S(\sigma, s) = \sum_{\substack{(\mathfrak{a}, S)=1 \\ \sigma_\mathfrak{a} = \sigma}} \frac{1}{\mathbf{N}(\mathfrak{a})^s}, \quad Re(s) > 1.$$

For every negative integer $k \leq 0$ define the Stickelberger element

$$\Theta_{L/K,S}(k) = \sum_{\sigma \in G_{L/K}} \zeta(L/K, \sigma, k)\sigma^{-1} \in \mathbb{Q}[G_{L/K}].$$

Let $\mathcal{A}(L/K)$ be the annihilator of the $\mathbb{Z}[G_{L/K}]$-module μ_L of roots of unity of L. In [Coa77], Coates shows how the main theorem of [DR80] implies the following result

Theorem 19.1 *Assume the main theorem proved in [DR80]. Let $k \leq 0$ be a negative integer then if $\alpha \in \mathcal{A}(L/K)$ then $\alpha\Theta_{L/K,S}(k) \in \mathbb{Z}[G_{L/K}]$.*

We are now ready to state Brumer's conjecture, which is an attempt to generalize the classical theorem of Stickelberger.

Conjecture 19.1 *Let $C_{L,S}$ be the S-ideal class group of L. Then one has an inclusion of $\mathbb{Z}[G_{L/K}]$-ideals*

$$\mathcal{A}(L/K)\Theta_{L/K,S}(0) \subseteq Ann_{\mathbb{Z}[G_{L/K}]}(C_{L,S}).$$

Moreover when $\alpha = w\Theta_{L/K,S}(0)$, where $w = \#\mu_L$, we have for all ideal $\mathfrak{a}$ of L

$$\mathfrak{a}^{\alpha} = (a)$$

for some $a \in (L)^{-}$.

Note that the generator a is uniquely determined up to a root of unity in L. When S is large enough the first part of the conjecture was proved by Wiles as a consequence of the main conjecture for totally real number fields, see [Wil90].

Let us assume that the data $(L/K, S)$ satisfies the following assumptions

(1) $S = \{\mathfrak{p}\} \cup T$ where T consists exactly of the infinite places of K and finite primes which ramify in L/K

(2) The prime p is inert in K and and $p\mathcal{O}_K = \mathfrak{p}$ splits completely in L.

(3) L is a CM field corresponding to the narrow ray class field of conductor $\mathfrak{f}$ of K where $\mathfrak{f}$ is some ideal of K coprime to p.

Let $L_n = K_{\mathfrak{f}p^n}$ be the ray class field of conductor $\mathfrak{f}p^n$ over K. For every $n \geq 0$ we have a group ring element $\Theta_{L_n/K,S}(0) \in \mathbb{Q}[G_{L_n/K}]$. For every integer $0 \leq m \leq n$ we let res_{nm} be the natural restriction maps $res_{nm} : \mathbb{Q}[G_{L_n/K}] \to \mathbb{Q}[G_{L_m/K}]$. The elements $\Theta_{L_n/K,S}(0)$ satisfy the distribution relations

$$res_{nm}(\Theta_{L_n/K,S}(0)) = \Theta_{L_m/K,S}(0).$$

Let $w_n = \#\mu_{L_n}$. Note that for n large enough one has $w_{n+1} = pw_n$. Let $\mathfrak{p}_n$ be a prime ideal of L_n above p chosen in such a way that $\mathfrak{p}_{n+1}|\mathfrak{p}_n$. Note that this tower of primes depends only on the initial choice $\mathfrak{p}_0$ in $L_0 = K_\mathfrak{f}$ since after the first step all the extensions are totally ramified at $\mathfrak{p}_0$. Using the Brumer-Stark conjecture for every n there exists a unique strong p-unit $u_n \in (L_n)^-$ (up to a root of unity) defined by the relation

$$\mathfrak{p}_n^{w_n \Theta_{L_n/K,S}(0)} = (u_n).$$

For every $0 \leq m \leq n$, those strong p-units are related by the norm in the following way

$$N_{L_n/L_m}(u_n) = (u_m)^{w_n/w_m}.$$

When $n \geq 1$ and $\sigma \in Gal(L_n/K)$ the p-adic zeta function $\zeta_{p,S}(L_n/K, \sigma, s)$ has no zero at $s = 0$ since all the primes $\mathfrak{q}$ of L_n above a prime of S are ramified in L_n/K. Therefore we fall outside our initial setting where the order of vanishing of the p-adic L-function at $s = 0$ was equal to 1. However when $n = 0$, the order of vanishing of $\zeta_{p,S}(L_0/K, \sigma, s)$ at $s = 0$ is equal to 1, and therefore one has a conjectural p-adic formula for the element u_0 viewed as an element of $(L_0)_{\mathfrak{p}_0}$. From this point of view, it seems to be a very natural question to look for a similar formula for the element u_n viewed as an element of $(L_n)_{\mathfrak{p}_n}$. Even though we fall outside our original setting, where the order of vanishing of the partial zeta function at $s = 0$ was assumed to be 1, a formula similar to what Dasgupta is proposing in [Das06] might exist. It would be quite interesting to provide such a conjectural p-adic formula for the strong p-units u_n.

A Partial modular symbols are finitely generated over the group ring

A modular symbol taking value in an abelian group A is a function

$$m : \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) \longrightarrow A$$

denoted by the suggestive notation $m(x,y) := m\{x \to y\}$ such that

(1) $m\{x \to y\} = -m\{y \to x\}$ for all $x, y \in \mathbb{P}^1(\mathbb{Q})$,

(2) $m\{x \to y\} + m\{y \to z\} = m\{x \to z\}$, for all $x, y, z \in \mathbb{P}^1(\mathbb{Q})$.

We have a natural action of $GL_2(\mathbb{Q})$ on modular symbols given by

$$(\gamma m)\{x \to y\} := \{\gamma^{-1}x \to \gamma^{-1}y\}$$

for all $\gamma \in GL_2(\mathbb{Q})$. One can define a universal $\mathbb{Z}$-module X s.t. for any modular symbol $m : \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) \to A$ we have the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\iota} & \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) \\ & \searrow^{\exists ! \widetilde{m}} & \downarrow m \\ & & A \end{array}$$

where $\widetilde{m}$ is a group homomorphism. When $A = \mathbb{C}$ one can show that $X \simeq Div_0(\mathbb{P}^1(\mathbb{Q}))$ as a $\mathbb{Z}$-module. For any $\gamma \in GL_2(\mathbb{Q})$ and a modular symbol m we define

$$(\gamma \star m)\{c_1 \to c_2\} = m\{\gamma^{-1}c_1 \to \gamma^{-1}c_2\}.$$

In practice one is interested to Γ-invariant modular symbols for some subgroup $\Gamma \leq GL_2(\mathbb{Q})$. Very often Γ is discrete but not always.

Definition A.1 *A partial modular m with respect to a subgroup $\Gamma \subseteq GL_2(\mathbb{Q})$ which takes value in an abelian group A is a map*

$$m : S \times S \to A$$

for a certain subset $S \subseteq \mathbb{P}^1(\mathbb{Q})$ which is Γ-invariant and for all $x, y, z \in S$ we require

(1) $m\{x \to y\} = -m\{y \to x\}$

(2) $m\{x \to y\} + m\{y \to z\} = m\{x \to z\}$

Let us prove now prove a very useful theorem.

Theorem A.1 *Let $x \in \mathbb{P}^1(\mathbb{Q})$ and Γ be a finitely generated subgroup of $GL_2(\mathbb{Q})$ then the Γ-module $Div_0(\Gamma x)$ is finitely generated. The number of generators of $Div_0(\Gamma x)$ can be taken to be less than or equal to the number of generators of Γ.*

Proof Let $G = \langle g_1, \ldots, g_n \rangle$. I claim that $\{[x] - [g_i x]\}_{i=1}^n$ is a generating set as a Γ-module of $Div_0(\Gamma x)$. Let $F_n = \langle x_1, \ldots, x_n \rangle$ be the free group of n elements. We have a natural onto group homomorphism $f : F_n \to G$ where $f(x_i) = g_i$. For an element $w \in F_n$ that is reduced we have a well defined notion of length. We define $S_k := \{w \in F_n : length(w) = k\}$. Since $\cup_k S_k = F_n$ we have $\cup_k f(S_k) = G$. We do a proof by induction.

Let $M = \mathbb{Z}[\Gamma](\{[x] - [g_i x]\}_{i=1}^n)$. We need to show that $M = Div_0(\Gamma x)$. Assume that for all $g \in f(S_k)$ and $k \leq m-1$ we have $[x] - [gx] \in M$ then we claim that

$$\text{If } g' \in f(S_m) \text{ then } [x] - [g'x] \in M.$$

Let us prove it. Since $g \in f(S_m)$ there exists a word $w \in F_n$ of length m such that $f(w) = g$. So there exists a x_i s.t $x_i w' = w$ where w' is a word of length $m-1$. By induction we have $[x] - [f(w')x] \in M$. Finally note that $[x] - [gx] = g_i([x] - [f(w')x]) + ([x] - [g_i x]) \in M$. Since the induction hypothesis is true for $k = 1$ it is true for any k by the inductive step. $\square$

Corollary A.1 *Assume that $\mathbb{P}^1(\mathbb{Q})/\Gamma$ is finite and and Γ finitely generated then $Div_0(\mathbb{P}^1(\mathbb{Q}))$ is a finitely generated Γ-module.*

Proof Let $G = \langle g_1, \ldots, g_n \rangle$ and $\mathbb{P}^1(\mathbb{Q}) = \cup_{i=1}^m \Gamma x_i$. Then we claim that

$$M := \mathbb{Z}[\Gamma](\{[x_j] - [g_i x_j]\}_{i,j} \cup \{[x_1] - [x_j]\}_{j=2\ldots n})$$

is equal to $Div_0(\mathbb{P}^1(\mathbb{Q}))$. Let $y_i \in \Gamma x_i$ and $y_j \in \Gamma x_j$. By the previous theorem we have $[x_i] - [y_i] \in M$ and $[x_j] - [y_j] \in M$. Also $[x_i] - [x_j] \in M$. Therefore $[y_i] - [y_j] = ([x_j] - [y_j]) + ([y_i] - [x_i]) + ([x_i] - [x_j]) \in M$. $\square$

So more generally for any finitely generated subgroup $\Gamma \leq GL_2(\mathbb{Q})$ and a subset of cusps $S = \cup_{i=1}^k \Gamma x_i$ (having finitely many Γ-orbits) we find that $Div_0(S)$ is a finitely generated Γ-module.

Corollary A.2 *Since $\Gamma_0(N)$ is finitely generated we have that $Div_0(\Gamma_0(N)(i\infty))$ is a finitely generated $\mathbb{Z}[\Gamma_0(N)]$-module.*

So this gives us a theoretical way of computing a partial modular symbol knowing only the values on a set of generators for $Div_0(S)$ over Γ. Let us work out an explicit example. Consider the modular group $\Gamma_0(N) =< g_1, \ldots, g_r >$. For any element $g \in G$ there exists a reduced word $x_1 x_2 \ldots x_n = g$ where $x_i \in \{g_1, g_1^{-1}, \ldots, g_r, g_r^{-1}\}$ and $x_i \neq x_{i+1}^{-1}$ for all $1 \leq i \leq n-1$. For any integer $k \geq 1$ we let $X_k = \prod_{i=1}^{k} x_i$. A direct computation reveals that

$$[i\infty] - [g(i\infty)] = \sum_{i=1}^{n} X_{n-i} \left([i\infty] - [x_{n-i+1}(i\infty)] \right).$$

Note that if $x_i = g_j^{-1}$ then $\left([i\infty] - [g_j^{-1}(i\infty)] \right) = -g_j^{-1} \left([i\infty] - [g_j(i\infty)] \right)$.

References

[B. 74] B. Schöneberg. *Elliptic modular forms.* Springer-Verlag, New York, 1974.

[Coa77] J. Coates. *On p-adic L-functions and Iwasawa's theory.* Academic Press, 1977.

[Dar01] H. Darmon. Integration on $\mathcal{H}_p \times \mathcal{H}$ and arithmetic applications. *Ann. of Math. (2)*, 154:589–639, 2001.

[Dar04] H. Darmon. *Rational points on modular elliptic curves.* American Mathematical Society, 2004.

[Das05] S. Dasgupta. Computations of elliptic units for real quadratic number fields. *to appear*, 2005.

[Das06] S. Dasgupta. Shintani zeta functions and Gross-Stark units for totally real fields. *to appear*, 2006.

[DD06] H. Darmon and S. Dasgupta. Elliptic units for real quadratic fields. *Annals of Mathematics (2)*, 163:301–346, 2006.

[DK81] S. Lang D.S. Kubert. *Modular units.* Springer-Verlag, New York Heidelberg Berlin, 1981.

[DR80] P. Deligne and K.A. Ribet. Values of abelian L-functions at negative integers over totally real fields. *Inventiones Math.*, 59:227–286, 1980.

[FG78] B. Ferrero and R. Greenberg. On the behavior of p-adic L-functions at s=0. *Inventiones Math.*, 50:91–102, 1978.

[GK79] B. Gross and B. Koblitz. Gauss sums and the p-adic gamma function. *Annals of Math.*, 109:569–581, 1979.

[Gro81] B. Gross. p-adic l-series at s=0. *J. Fac. Sci. Univ. Tokyo*, 28:979–994, 1981.

[Hal85] U. Halbritter. Some new reciprocity formulas for generalized Dedekind sums. *Results Math.*, 8:21–46, 1985.

[Kuc92] R. Kucera. On bases of odd and even universal ordinary distributions. *J. Number Theory*, 40:284–316, 1992.

[Lan94a] S. Lang. *Cyclotomic Fields I and II, Combined Second Edition.* Springer-Verlag, New York, 1994.

[Lan94b] S. Lang. *Elliptic Functions, Second Edition.* Springer-Verlag, New York, 1994.

[Men67] J. Mennike. On ihara's modular group. *Invent. Math*, 4:202–228, 1967.

[Ser70] J.-P. Serre. Le problème des groupes de congruences pour sl_2. *Annals of Math.*, 92:489–527, 1970.

[Sie68] C.L. Siegel. Bernoullische Polynome und quadratische Zahlkörper. *Nach. Akad. Wiss. Göttingen Math.-Phys. K1. II*, pages 7–38, 1968.

[Sie69] C.L. Siegel. Berechnung von zetafunktionen an ganzzahligen stellen. *Nach. Akad. Wiss. Göttingen Math-Phys. Kl. II*, pages 87–102, 1969.

[Sie80] C.L. Siegel. *Advanced analytic number theory, Second Edition.* Tata Institute of Fundamental Research, Bombay, 1980.

[Was87] Lawrence C. Washington. *Introduction to cyclotomic fields, Second Edition.* Springer-Verlag 83, 1987.

[Wil90] A. Wiles. On a conjecture of brumer. *Ann. Math. (2) 131, No.3*, pages 555–565, 1990.

[You01] P.T. Young. Kummer congruences for values of Bernoulli and Euler polynomials. *Acta Arithmitica*, 48:277–288, 2001.