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Peano on Symbolization, Design Principles for Notations, and the Dot Notation

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Résumé : Peano a été l'une des forces motrices dans le développement du formalisme mathématique actuel. Dans cet article, nous étudions son approche particulière de la conception notationnelle et présentons quelques caractéristiques originales de ses notations. Pour motiver l'approche de Peano, nous présentons d'abord sa vision de la logique comme méthode d'analyse et son désir d'un symbolisme rigoureux et concis pour représenter les idées mathématiques. Sur la base à la fois de sa pratique et de ses réflexions explicites sur les notations, nous discutons des principes qui ont guidé Peano dans l'introduction de nouveaux symboles, le choix des caractères et la mise en forme des formules. Enfin, nous examinons de plus près, d'un point de vue systématique et historique, l'une des innovations les plus marquantes de Peano, à savoir l'usage de points pour regrouper des sous-formules.

Abstract: Peano was one of the driving forces behind the development of the current mathematical formalism. In this paper, we study his particular approach to notational design and present some original features of his notations. To explain the motivations underlying Peano's approach, we first present his view of logic as a method of analysis and his desire for a rigorous and concise symbolism to represent mathematical ideas. On the basis of both his practice and his explicit reflections on notations, we discuss the principles that guided Peano's introduction of new symbols, the choice of characters, and the layout of formulas. Finally, we take a closer look, from a systematic and historical perspective, at one of Peano's most striking innovations, his use of dots for the grouping of subformulas.

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1 Introduction

One of the concerns of philosophers of mathematics is to clarify the principles and methods that drive the development of mathematics. In this paper, we shall take a closer look at Giuseppe Peano's (1858-1932) general views about logic as a method of analysis of mathematical ideas and his more practical concerns regarding the presentation of the results of such analyses.[1] As we shall see, these notions are subtly intertwined and motivated by his general aims of striving for rigor and conciseness.

Like other mathematicians and logicians in the 19th century, Peano attributed the lack of satisfying solutions to many questions in the foundations of mathematics to the ambiguities of ordinary language [Peano 1889*a*, III]. In the use of symbolic languages to represent and analyze mathematical ideas and their logical relations, Peano envisaged a way of avoiding such ambiguities. However, Peano also realized that certain restrictions had to be imposed on these symbolisms. For example: to avoid ambiguities, each symbol should have a unique and precise meaning; to avoid errors, the symbols themselves, although arbitrary in principle, should be such that the cognitive effort necessary for their use is reduced to a minimum. Accordingly, Peano considered the development of an appropriate symbolic language, which he called "symbolic writing" (*scrittura simbolica*) or "ideography" [Peano 1896-1897, 202], to be a crucial task for the advancement of mathematics. As a consequence, in addition to formulating his famous axiomatization of arithmetic, Peano also originated many innovations in mathematical symbolism, including the dot notation in logic.

The early development of Peano's logical notation can be easily retraced by considering his publications from 1887 to 1889.[2] Before 1888, Peano's publications (e.g., [Genocchi 1884] and [Peano 1887]) do not contain any specific notations for logic. In the latter, Grassmann is mentioned in the Preface, but no logicians are. Logical notation appears for the first time in Peano's *Calcolo Geometrico secondo l'Ausdehnungslehre di H. Grassmann* [Peano 1888], whose preface is dated February 1, 1888, and which begins with a short chapter on "The operations of deductive logic", based on Schröder's *Der Operationskreis des Logikkalkuls* [Schröder 1877]. However, Peano replaces all of Schröder's symbols "in order to forestall any possible confusion between the symbols of logic and those of mathematics" [Peano 1888, X] (quoted from [Peano 2000, xiv]). A year later, Peano published his famous work on arithmetic, *Arithmetices Principia nova methodo exposita* [Peano 1889*a*],

1. For background on Peano's life and works, see [Kennedy 2002]; for discussions of his philosophy and works, see [Kennedy 1963] and [Skof 2011]. In the following text, all translations are by the author (DS) unless a reference to a published translation is given.

2. For a more detailed discussion, including the background of Peano's development, see [Bottazzini 1985].

in Latin, which begins with a chapter on logical notations in which he also introduces the difference between set membership (symbolized by "ε") and inclusion (replacing the symbol "<" that he used earlier with "Ɔ"). Again, Peano also replaces some of the logical symbols, but, more importantly, he introduces the dot notation for the grouping of subexpressions. From then on, this notation was employed in most of Peano's publications, beginning with *I Principii di Geometria logicamente esposti* [Peano 1889*b*] published later in the same year, as well as articles dedicated explicitly to mathematical logic, such as [Peano 1891*b*] and [Peano 1891*a*], and the various editions of the *Formulario* [Peano 1895*a*, 1897, 1901, 1903, 1905].

How these historical developments are intertwined with Peano's general views about methodology in mathematics is the main topic of this paper. In the following, we begin by discussing Peano's general views on symbolization and his view of logic as a method of analysis (Section 2). In Section 3, we relate these views to Peano's considerations for the design of notations. In particular, we present in detail the principles that guide the introduction of new symbols, the choice of characters, and the layout of mathematical formulas. In the third part of the paper (Section 4), Peano's use of dots for the grouping of subformulas is explained and discussed in the context of its historical development. This particular notation is one of the most striking of Peano's innovations and has been widely popularized by its use in Whitehead and Russell's *Principia Mathematica* [Whitehead & Russell 1910-1913], but it has hardly received any attention in the literature.[3]

2 Logic as method of analysis

2.1 Peano and Frege on logic

Let us begin by comparing and contrasting Peano's general attitude toward logic with Frege's, given that the latter has been studied extensively and is thus widely known. Both share the desire to secure rigorous reasoning in mathematics with the use of a symbolic language with clearly defined, unique meanings [Peano 1890*a*, 186]. However, their conceptions of rigor differ with regard to the level of explicitness of the analysis of logical reasoning. Frege, on the one hand, wanted to avoid any appeal to intuition in mathematical inferences and thus emphasized his use of formal rules of inference. On the other hand, possibly due to the fact that his main influences in logic came from the algebraic tradition of Boole and Schröder, Peano's paradigm of deduction was that of reasoning with algebraic equations [Peano 1889*a*, III] and [Peano 1889*b*, 28–29]. His lack of explicit inference rules was criticized by van Heijenoort as "a grave defect" [van Heijenoort 1967, 84]. In practice,

3. For example, a discussion of the dot notation is conspicuously missing in [Kennedy 2002].

however, Peano's derivations can be construed formally as being based on instances of axioms, the substitution of equalities, and *modus ponens* [von Plato 2017, 55–56]. In short, Peano's system is not a "formal system" in the modern sense, i.e., with a recursively defined language and explicit rules of inference, but, according to von Plato, it could be fairly straightforwardly constructed as one.[4]

Another aspect in which Frege and Peano differ is their attitude toward an investigation into the fundamental principles of logic. While Frege put his theory on a firm axiomatic foundation, Peano did not, although he had done so for arithmetic and geometry, and remarked that "it would be an interesting study" [Peano 1889*b*, 29]. Unlike Boole and Schröder, both Frege and Peano intended their logical formalisms to be applied to mathematics and not be used in isolation, merely for the efficient solution of logical problems. Peano writes:

> I understand how important theoretical studies of logic are; but, given the immensity of such studies, I prefer directing my forces toward application. [Letter from Peano to Couturat, 1 June 1899] (reprinted in [Roero 2011, 87])

Nevertheless, with regard to the aim of applying logic to mathematics, Frege and Peano differed: for Frege, it was a theoretical exercise aimed at clarifying concepts and securing the foundations of mathematics; for Peano, it was a practical matter of actually doing mathematics in a new way. Because of this emphasis on practical use, Peano concentrated his efforts on developing a convenient formalism for the analysis and concise representation of mathematical ideas.

2.2 The *Formulario* project and concise notations

Soon after completing his axiomatizations and symbolic presentations of arithmetic [Peano 1889*a*] and geometry [Peano 1889*b*], Peano envisaged an impressive collaborative project, aimed at publishing a collection of important mathematical results expressed in a symbolic language. The first edition of the *Formulaire de mathématiques*, or *Formulario Mathematico*, as it was later called, appeared in 1895; an Introduction, in which Peano presented his logical notation, had already been published one year earlier [Peano 1894]. Four different editions of the *Formulario* were subsequently published in 1897, 1901, 1903, and 1905, each of which was the result of substantial revisions of the one preceding it. Peano also showed great historical awareness by often listing a theorem together with a reference to where it first occurred. The idea for this project is put forward in print for the first time in 1891 as the concluding note of a paper on the concept of number. Before ending the paper by inviting

4. Deviating from the modern usage, we shall thus refer to Peano's symbolic language as a formalization.

suggestions for theorems to be included in the collection, Peano motivates the project as follows:

> It would also be very useful to collect all the known propositions referring to certain parts of mathematics, and to publish these collections. Limiting ourselves to arithmetic, I do not believe there would be any difficulty in expressing them in logical symbols. Then, besides acquiring precision, they would also be concise, so much so, probably, that the propositions referring to certain subjects in mathematics could be contained in a number of pages not greater than that required for the bibliography. [Peano 1891*c*]; [Peano 1957-1959, III, 109] (quoted from [Kennedy 2002, 63])

Given the sheer volume of the project, a concise form of representation was indispensable. Thus, while Peano writes that "the fundamental utility of the logical symbols is rigor and precision" [Peano 1908, X], he also emphasizes the importance of symbolization for reducing the length of presentations, because in some cases they would be impossible otherwise:

> It turns out that symbolic writing is about ten times shorter than in ordinary language. A publication of the ample present *Formulario* in ordinary language would be almost impossible in practice, as would be the publication of logarithmic tables in ordinary language or using Roman numerals. [Peano 1908, IX][5]

We note that, for Peano, one of the main practical requirements for the design of a notation is the reduction of the length of individual formulas, rather than the number of different signs that are employed. These two desiderata are frequently in tension with each other, as the comparison between binary and decimal place-value notations illustrates: the former uses only two signs instead of ten but results in longer expressions.[6] Given the aim of reducing the length of expressions, Peano's interest in reducing the use of parentheses should not come as a big surprise. We shall return to this in Section 4, when discussing the development of Peano's dot notation.

2.3 Formalization as method of analysis

It is clear from the announcement of the *Formulario* quoted above that the use of a symbolism was an integral part of the project from the beginning, since it allows for both precision and conciseness. Moreover, the process of formalization itself is a method of conceptual analysis that begins with the

5. In fact, Roman numerals for natural numbers (without subtractive notation) are on average 2.6 times longer than Indo-Arabic numerals [Schlimm & Neth 2008, 2101].

6. E.g., "10010011" vs. "147".

following two steps:[7] (1) The identification of the fundamental mathematical ideas, and (2) the representation of these ideas by primitive signs of the symbolism. The first step requires a precise and unambiguous identification of the underlying ideas:

> The reduction of a new theory into symbols requires a profound analysis of the ideas that occur in this branch. Imprecise ideas cannot be represented by symbols. [Peano 1895*a*, iv] (quoted from [Kennedy 2002, 67])

As a consequence, the more thoroughly ideas have been analyzed and expressed in ordinary language (Step 1), the easier it is to translate them into a symbolism (Step 2). Peano writes:

> The transformation into symbols of propositions and proofs expressed in the ordinary form [...] is a very easy thing when treating propositions of the more accurate authors, who have already analyzed their ideas. It is enough to substitute, in the works of these authors, for the words of ordinary language, their equivalent symbols. Other authors present greater difficulty. For them one must completely analyze their ideas and then translate into symbols. Not rarely it is the case that a pompously stated proposition is only a logical identity or a preceding proposition, or a form without substance. [Peano 1891*c*]; [Peano 1957-1959, III, 109] (quoted from [Kennedy 2002, 63])

However, it is not only the use of ambiguous or pompous language that obscures the ideas to be uncovered by logical analysis, but also the ideas' fundamental character and the fact that they do not necessarily correspond to basic expressions in ordinary language. As Peano explains in a textbook of arithmetic and algebra written for use in secondary schools, the logical symbols "$\supset$", "ε", and "Ǝ", which stand for derivation, membership, and existence,

> represent simple ideas and it is precisely their simplicity that prevented them for a long time to be isolated and stripped from the complexity with which they present themselves both in ordinary language and the language of science. [Peano 1902, III]

The surface structure of language can mislead even skilled logicians, such as Schröder. His use of a single sign to denote the ideas represented by "$\supset$" and "ε" is criticized by Peano as a major flaw that prevents Schröder's symbolism from being a proper ideography [Peano 1898*a*, 97–98].

7. The view of logic as analysis was also clearly formulated by Peirce, who wrote: "In logic, our great object is to analyze all the operations of reason and reduce them to their ultimate elements; and to make a calculus of reasoning is a subsidiary object" [Peirce 1880, 21]. This work is referred to in [Peano 1889*a*, IV]. However, in contrast to Peano, Peirce was also interested in theoretical investigations of logic itself.

That Peano indeed considered formalization as a method of analysis can also be seen from the subtitle of Peano's work on the axiomatization of natural numbers, which reads "nova methodo exposita" ("presented by a new method", [van Heijenoort 1967, 83]), and in [Peano 1896-1897, 202], where he speaks of the "analytic instrument"[8] that has been applied by himself and others.

2.4 Formalization as a method for checking an analysis

The utility of a symbolic language is not exhausted once mathematical ideas are expressed in it; the formalization itself can be used to check the adequacy of the analysis. Thus, a third step is added to the method of analysis: (3) Further study of the symbolic expressions to determine consequences and possible simplifications. For this, Peano suggests the following:

> After having written a formula in symbols, it is useful to apply several logical transformations to it. It can thus be seen if it is possible to reduce it to a simpler form, and one can easily recognize if the formula has not been well written. This is because the notations of logic are not just a shorthand way of writing mathematical propositions; they are a powerful instrument for analyzing propositions and theories. [Peano 1895*a*, vi] (adapted from [Kennedy 2002, 68])

With regard to the analysis of theories, i.e., sets of propositions and not just individual ones, a formalization can also be used to impose a logical order on the propositions (i.e., present some as axioms and others as theorems) and to check the definitions. Peano writes:

> It is always difficult to order the propositions of a theory. One can order them according to the signs employed for writing them. This rule yields, in general, good results. [Peano 1895*a*, vi][9]

Once a theory is symbolized, i.e., the primitive ideas are determined and expressed by primitive symbols, the propositions ordered, and symbols for complex ideas introduced by definitions, one can easily verify that all symbols used in the definiens have been properly introduced. This can be done "in a mechanical way", because only the symbols need to be considered and no recourse to the original ideas is necessary. Peano explains:

> The ideography makes evident, in a mechanical way, that definitions are correct and that demonstrations are rigorous.

8. Kennedy translates "strumento analitico" as "analytic method" [Kennedy 1973, 190].

9. See [Peano 1897, 28] for an application of this suggestion; see [Cantù 2014] for a discussion of Peano's views on the order of the primitive ideas of a science.

> For example, it is a fundamental rule of definitions, that the defined symbol must be expressed by previous symbols. Thus, if we consider for example the definition of prime number on p. 58, we see that it is expressed by the symbols $-$, 1, $+$, $\times$, N_i, which were introduced on pages 10, 29, 29, 32, 37, and that several of among these symbols are defined by previous symbols, and so on, until we reach a decomposition into primitive ideas that are determined by primitive propositions. [Peano 1908, X]

2.5 Depth, uniqueness, and arbitrariness of analysis

2.5.1 Depth and uniqueness of analysis

So far, we have learned that formalization yields a "profound" and "complete" analysis of mathematical ideas, but how do we know when this process is complete? In the following passage from a letter to Felix Klein, Peano explains the aim of mathematical logic and mentions an additional goal with regard to the outcome of logical analysis:

> It is the aim of mathematical logic to analyze the ideas and forms of reasoning that occur especially in the mathematical sciences. The analysis of the ideas allows to find the fundamental ideas, with which all other ideas are expressed, and the relations between various ideas, i.e., the logical identities, that are those forms of reasoning. This analysis also leads us to indicate the simplest ideas with conventional symbols, which, when appropriately combined, represent composite ideas. This yields a symbolism or symbolic writing that represents all propositions with the smallest number of signs. [Letter from Peano to Felix Klein, 19 September 1894] (reprinted in [Peano 1990, 124])

Thus, a successful analysis yields a minimal set of fundamental, simple ideas that are represented by symbols, such that through the combinations of these symbols all complex ideas can also be expressed.[10] A small number of symbols is therefore a hallmark of a formalism, because it indicates the depth of the analysis. Indeed, at the beginning of many of his publications, Peano proudly emphasizes the small number of primitive symbols being used and, in his discussion of Frege's work, he takes the diminution of the number of primitive symbols as indicative of a more thorough and deeper analysis. In his review of the first volume of Frege's *Grundgesetze* [Peano 1895*b*], Peano compares his own with Frege's formalism, acknowledging that many of their ideas are analogous. However, among other criticisms, Peano points out the lack of a

10. See also [Peano 1894, 173] for a similar formulation; in [Peano 1890*b*], he sets out to "find the minimum of signs and conventions necessary to express the 25 propositions of the Fifth Book of Euclid".

symbol for set membership in Frege's *Begriffsschrift* as a defect and, since Peano's notation is allegedly built on fewer primitives than Frege's, Peano regards his own analysis as "more penetrating" [Dudman 1971, 30].[11]

With regard to the outcome of different symbolic analyses, Peano makes the following general remark:

> Now if, independently of each other, there arise two systems both capable of representing and analysing the propositions of a theory, one will have to be able to present an absolute formal difference between them; but there will have to exist at bottom a substantial analogy; and if the two systems are equally developed, the relation between them will have to be that of identity. For mathematical logic does not consist of a set of arbitrary conventions, variable according to the author's fancy. It consists rather of the analysis of ideas and propositions into those that are primitive and those that are derivative. And this analysis is unique. [Peano 1895*b*, 123] (quoted from [Dudman 1971, 28])

This claim about the uniqueness of logical analysis, which is repeated again at the end of Peano's review, fits together with the earlier claim about the minimality of the set of simple ideas. In what sense, however, different analyses could result in unique, "substantially analogous" systems is left unclear. Based on the minimality and uniqueness claims, the passages quoted above could be interpreted as expressing some kind of realist view, according to which the structure of the symbolism mirrors the (true) logical structure of the ideas.

2.5.2 Arbitrariness of analysis

The realist interpretation of the representations of mathematical ideas and propositions offered at the end of the previous paragraph is called into question by the intertranslatability of various logical connectives, which Peano discusses in the same text [Peano 1895*b*]. As he is well aware, in propositional logic either implication or disjunction can be taken as primitive (together with negation) and the other as defined. Moreover, in other places, Peano is quite explicit about the difficulties involved in determining which ideas and propositions should be taken as fundamental: he notes that the distinction between primitive and derived ideas is "somewhat arbitrary" or "a little bit arbitrary" on numerous occasions[12] and that "each author can begin with the group that they find most satisfying" [Peano 1898*a*, 100]. To choose

11. Frege vehemently disagreed: "I do not regard the mere counting of primitive symbols as sufficient to substantiate a judgment about the profundity of analysis toward fundamentals" [Dudman 1971, 35]. See also Peano's review of Schröder's formalism, in which he points out that the latter is based on 15 primitive ideas, whereas his own system is built on 8 [Peano 1898*a*].

12. See, e.g., [Peano 1889*b*, 25], [Peano 1891*a*, 25], [Peano 1894, 50–51], and [Peano 1897, 27].

between alternative sets of primitives, Peano frequently invokes a notion of "simplicity".[13] However, this notion is left unspecified, and he notes that "there is arbitrariness in the assessment of simplicity" [Peano 1894, 51].

On the first page of the Preface to the first edition of the *Formulario*, Peano states the independence of mathematics from particular representations even more forcefully:

> The notations are a bit arbitrary, but the propositions are absolute truths, independent of the notations used. [Peano 1895*a*, III]

On the basis of these considerations, Peano's attitude has been frequently characterized as "strictly instrumental" with regard to the role of logic [Segre 1994, 286] and "instrumentalist" with regard to notations [Bellucci, Moktefi *et al.* 2018, 3].

2.5.3 Possible resolution of the tension between uniqueness and arbitrariness

The tension between Peano's claims about the uniqueness of an analysis, which leads to a minimal set of primitives, and his conviction about a certain insurmountable arbitrariness regarding the choice of primitives can be resolved by taking a careful look at what Peano says in the following passage, in the context of whether "point" and "segment", or "point" and "ray", should be chosen as primitives in geometry (as we have seen above, an analogous situation arises in logic):

> It is clear that not all entities can be defined, but it is important in every science to reduce the number of the undefined entities to a minimum. [...] The reduction of the undefined entities to a minimal number can be somewhat arbitrary; so, if by means of a and b we can define c, and by means of a and c we can define b, our choice between a, b and a, c as an irreducible system remains arbitrary. [Peano 1889*b*, 25] ([Peano 1957-1959, II, 78])

Here, both minimality and arbitrariness are considered: the number of primitives should be minimal, but among the possible minimal sets, it is arbitrary which one is chosen. Moreover, if one of these sets is taken as primitive and the other entities are defined in terms of this set, and if the axioms are chosen appropriately, the same theorems will follow. The theories, then, understood as sets of theorems, are indeed identical and the analysis unique, as Peano claimed in his review of Frege's *Grundgesetze* (see the second quote in Section 2.5.1).[14] This interpretation is also

13. See, e.g., [Peano 1891*a*, 25], [Peano 1894, 50–51], and [Peano 1897, 27].

14. This interpretation is also consistent with Peano's remark on the identity of domains on the basis of them satisfying the same propositions [Kennedy 1973, 225].

compatible with all examples that Peano mentions in his discussions of arbitrary choices of primitives.[15]

A prominent case in which the minimality of the set of primitives is frequently given up is propositional logic itself. As Peirce noticed in the 1880s and Sheffer rediscovered some 30 years later [Sheffer 1913], a single binary connective (either Peirce's arrow or the Sheffer stroke) can be used to define all other propositional connectives. Accordingly, we would expect Peano to consider this analysis of propositional logic to be deeper and more profound than his own.[16] Although it is known that Sheffer visited Peano in 1911 and that they corresponded in 1921, I am not aware of any reactions by Peano to Sheffer's discovery.[17]

We have established so far that, given different minimal sets of primitives, Peano saw no theoretical reasons to prefer one over the others.[18] However, for any actual presentation, such a choice has to be made, and for this, practical reasons come into play, even ones that push toward giving up the minimality of the set of primitives.

2.5.4 Practical considerations against minimality

The demand for a notation to be concise was discussed in Section 2.2 in connection with Peano's *Formulario* project. This suggests adopting a set of primitives that is not minimal, as they allow for shorter expressions without having to define derived symbols. Other reasons given by Peano for dropping the requirement of minimality of the set of primitives in logic are related to the readability of formulas and their connection to expressions in ordinary language. For example, after noting that $a \supset b$ is equivalent to $a - b = \Lambda$, so that the sign "Ɔ" could be omitted from the list of primitives, Peano notes:

> We shall keep it, however, for greater variety and for analogy with the common form of expressing the thought. [Peano 1891*b*, 6], [Peano 1957-1959, II, 98] (quoted from [Kennedy 1973, 160])

In sum, it appears that Peano's criteria for deciding which ideas are to be taken as primitive are guided more by considerations of practicality and convenience of use than by some kind of epistemological or metaphysical considerations.

15. See the references in Footnote 12.

16. Such was indeed the reaction of Russell; see the Introduction to the second edition of *Principia Mathematica* [Whitehead & Russell 1925].

17. I thank Juliet Floyd for this information. The extant correspondence between Peano and Sheffer, which is held at the Harvard University Archives, does not mention Sheffer's innovation.

18. This is in contrast to Frege, who invoked a notion of simplicity of content to choose the conditional as a primitive connective in his system; see [Bellucci, Moktefi *et al.* 2018, 6–7] and [Schlimm 2018, 71–73].

3 Design principles for characters and layout

After having discussed Peano's general outlook on logic and formalization, we now take a closer look at his approach to mathematical notations. We have seen above that, for Peano, a symbolism that represents the result of an analysis should represent the basic concepts of a domain of inquiry by individual symbols and more complex concepts by symbols that are defined from them. We have also seen that there are some difficulties in identifying the primitive concepts, but that is not our primary concern here. Rather, it is the question of how to represent them, once we have settled on them.

In general, we can consider a notation to consist of a set of characters (also referred to as signs or symbols)[19], structural rules that determine well-formed expressions, and an interpretation that assigns meanings to (at least some of) the characters and expressions. Although the choice of characters is arbitrary from a theoretical point of view, Peano did formulate some design principles explicitly, while others can be extrapolated from his practice.

As we shall see, the general aims of rigor and conciseness that motivated Peano's use of a symbolic language in the first place also underlie his choice of characters (Section 3.1). In addition, Peano tried to design his notations in such a way that they reduce the cognitive effort necessary for their use, e.g., by linking their shapes to their meanings and by using a layout that facilitates their readability (Section 3.2). Presumably, this would reduce errors and mistakes when using the notation. Finally, Peano also considered factors that influence the horizontal and vertical arrangement of the notation on the printed page (Section 3.3).

3.1 Conciseness and reduction of ambiguity

3.1.1 Uniqueness of meanings and new symbols to avoid ambiguities

In [Peano 1888], where he presents the logical calculus of Schröder, Peano replaces each of the five basic symbols employed by Schröder with his own:

> It seemed useful to substitute the symbols $\cap$, $\cup$, $-$, $\bigcirc$, ● for the logical symbols $\times$, $+$, A_i, 0, 1 used by Schröder, in order to forestall any possible confusion between the symbols of logic and those of mathematics (a thing otherwise advised by Schröder himself). [Peano 1888, X] (quoted from [Peano 2000, xiv])

19. To be clear, we mean here character types or symbol types, not tokens, but this distinction plays no particular role in our discussion.

Because Peano intends to use his logical symbolism in conjunction with the usual mathematical notations, he must introduce new characters, which are not already used in other mathematical domains, to avoid ambiguities. The use of standard arithmetical symbols in logic does not pose a problem for Schröder, since he presents logic as a self-standing theory that is not used in conjunction with other theories, such as ordinary arithmetic.

Peano took the symbols "$\cap$" and "$\cup$" possibly from Grassmann [1844, 5], who uses them for his more abstract theory of magnitudes; the circle and filled circle do not have any obvious previous uses, but the symbol for negation (or set complement) looks very similar to the minus sign. This seems to have bothered Peano as he later recommends the following:

> In the manuscript, it is best to give the sign for 'not' the form $\sim$, so as not to confuse it with $-$ (minus). [Peano 1895*a*, VI] (quoted from [Kennedy 2002, 68])

This comment comes from the beginning of the first edition of the *Formulario*, where Peano included a list of remarks and rules to facilitate future collaborations. The above considerations about the introduction of new symbols are encapsulated for the general case in the third item on the list:

> Every time a new theory is translated into symbols, new signs will be introduced to indicate the new ideas, or the new combinations of preceding ideas, that are met in this theory. [Peano 1895*a*, III–IV] (quoted from [Kennedy 2002, 67])[20]

Behind this principle lies the more fundamental principle that each symbol that stands for an idea should have only a single fixed meaning. An example for application of this design principle is Peano's preference for writing the decimal point:

> I prefer the English notation $1 \cdot 23$ and $\cdot 45$ to $1,23$ and $0,45$ for writing decimal fractions, because the comma has too many meanings. [Peano 1916] (translated from [Peano 1957-1959, I, 448])

An explicit formulation of the principle that each symbol must be assigned a unique meaning is given in the following passage, where Peano extends it also to those symbols that are used for grouping subexpressions, such as parentheses:

> If one wants mathematical formulas to say everything without the need of verbal additions, one cannot give two values to the same sign. The assignment of another function to parentheses, other than the grouping of more signs, is like trying to make a decimal arithmetic in which the numbers 6 and 9 are represented by the same sign. [Peano 1912, 377]

20. See also [Peano 1895*b*] for a similar formulation [Dudman 1971, 28].

Despite his convictions, Peano himself did not always adhere to this principle. For example, he interpreted the symbol "Ɔ" as both deduction and material conditional, for which he was criticized by Frege [1896, 372–374] (quoted from [Frege 1969, 8–9]).

3.1.2 Simplifications of frequently used expressions

Peano not only uses symbols to represent the primitive ideas of a discipline, but also allows for the introduction of new symbols within a theory through definitions. However, he suggests to restrict such additions to the following situations:

> A new notation will be introduced by means of a definition when it brings a notable simplification. A new notation will not be formed when the same ideas can already be simply represented by the preceding notations. [...]
>
> A new notation will be introduced only if the simplification that it brings will be used in the propositions following. Definitions alone do not make a theory. [Peano 1895*a*, IV] (quoted from [Kennedy 2002, 68])

An early example for the application of this principle is Peano's introduction of a symbol to express "every A is B" in [Peano 1888, 3]. After noting that this can be expressed with the primitive symbols of his theory as $A\overline{B} = \bigcirc$, he continues:

> Even though the preceding proposition for indicating that proposition is already quite simple, for greater convenience we will nevertheless also indicate it by the expression $A < B$ or $B > A$ [...]. [Peano 1888, 3] (quoted from [Peano 2000, 2])

This symbol is indeed used very frequently in the further development and it considerably shortens the expression introducing only one symbol in addition to the variables, instead of three ("$^{-}$", "=", "○"). The notion of simplicity appealed to in this principle thus refers to reducing the length of expressions.[21]

Another example for the application of this principle is Peano's introduction of expressions containing an existential quantifier "Ǝ a", which he motivates as follows:

> The proposition $a \sim = \Lambda$, where a is a class, thus signifies "some a exist". Since this relation occurs rather often, some workers in this field hold it useful to indicate it by a single notation, instead of the group $\sim = \Lambda$. [Peano 1896-1897] (quoted from [Kennedy 1973, 203])

21. That expressions can be expressed with fewer symbols is one of Peano's two meanings of "simplicity", according to [Bellucci, Moktefi *et al.* 2018, 3]; the other concerns the number of primitive logical symbols used in a theory.

Again, a single symbol replaces three, and Peano explicitly mentions the frequent use of this expression. We can thus summarize Peano's principle as: symbols that stand for derived ideas or relations should shorten expressions and be used frequently. The particular shape of the new symbol, "Ǝ", was chosen on the basis of semantical considerations, to which we turn next.

3.2 Semantical considerations

3.2.1 Iconicity and mnemonics

Even when taking only a cursory glance at Peano's works, one cannot miss his use of mnemonics when introducing new characters, although he does not discuss it as an explicit principle. To illustrate this practice, let us look at one of the most famous symbols introduced by Peano, the "horseshoe". The symbol for "proves" (or deduces) and "contains" was changed several times in Peano's writings. With an implicit analogy to the less-than relation in algebra, it was first introduced as $a < b$ in [Peano 1888, 3] for the calculus of classes. In later writings, Peano formulated this analogy explicitly:

> Segner in 1740 and Lambert in 1765 used $a < b$ and $a > b$, respectively; because the relation corresponds to the sign $<$ or $>$, or better to $\leqq$ or $\geqq$, of algebra, depending on whether with the class one considers the number of individuals that constitute it, or the number of ideas that determine it. [Peano 1900, 10]

Thus, Peano deliberately chose a symbol that bears some connection to the represented relation. This connection, whereby the intended interpretation is suggested by the particular shape of the symbol, is often called "iconic", following terminology introduced by [Peirce 1885, 181].[22]

Possibly because the less-than symbol is also used in algebra, thus violating the principle that new symbols should be introduced for new ideas (Section 3.1.1), Peano quickly replaced it with an inverted capital letter "C" a year later, writing *a*Ɔ*b* [Peano 1889*a*, viii] and interpreting it as a relation both between classes and propositions.[23] It is described as "the reversed initial letter of the word *contains* [*contiene*]" in [Peano 1889*b*, 6],[24] whereas the symbol

22. Peano himself called such notations "figurative": referring to $a^{-}b$, $a^{\vdash}b$, $a^{\dashv}b$, $a^{\vdash\!\dashv}b$, to indicate whether the endpoints of an interval a and b are excluded or included, he notes that "this figurative notation is very convenient and fairly widespread" [Peano 1916-1917, 455]. That the use of iconic mathematical symbols has indeed cognitive advantages has recently been shown in [Wege, Batchelor *et al.* 2020].

23. In the original publication the symbol C does not appear aligned on the baseline as the text, but somewhat lower, as in: *a* Ɔ *b*.

24. Similarly, with the French word "*contient*" in [Peano 1890*a*, 183]; [Peano 1900, 316] refers to Gergonne for using "C" as the initial of "contains"; Quine [1987, 5] refers to [Gergonne 1816-1817].

"C" is described as the first letter of the word "consequence [*consequenza*]" in [Peano 1891*b*, 100, footnote 5]. In [Peano 1894], the symbol remains an inversed capital "C", but in a smaller font, such that it appears as $a \mathbin{\text{Ɔ}} b$. Finally, the "⊃" symbol appears in Peano's writings in 1898, e.g., [Peano 1898*a*], and is described as "a deformation of Ɔ, the reversed first letter of the word "contains" [*contient*]" in [Peano 1900, 10]. Thus, Peano used the first letter of a word that expresses the meaning of the relation as the sign that represents it. The reversal was probably done to avoid confusing the symbol with the name of a variable.

Other examples of Peano's use of mnemonics to guide the choice of symbols are: "P" for propositions, "Th" for theorems, "M" for maximum, and "D" for divides [Peano 1889*a*, VI]. In some cases Peano chose the first letter of a word in a different language than Italian or French, such as "V" for *verum* (the Latin word for true),[25] or "ε" and "ι", which are the first letters of the Greek words for "is" (ἐστί) and "equal" (ἴσος) [Peano 1894, 7 and 38].

3.2.2 Inverted symbols for inverse relations and operations

Another principle for the choice of characters that Peano frequently employs, and that was already hinted at above, is the introduction of an inverted symbol to express the contrary or inverse of the meaning of the original symbol. This practice is referred to by Quine [1987, 18] as "Peano's strategy of notational inversion".

For example, after introducing the symbol "V" for *verum*, Peano replaced his earlier symbol for absurdity, "○", with "Λ" [Peano 1889*a*, VIII]. What is unusual in this case is that the "V" itself is not used in the further development of the theory, thus violating the principle identified above, according to which only symbols that are actually used should be introduced (Section 3.1.2). The desire for providing a set of symmetric symbols is likely to have motivated him to do so. This becomes clear in a later publication, where, after listing the symbols ε, ᴄ, Ɔ, =, ∪, ∩, −, ᴠ, ʌ, which allow for the expression of all logical relations, he remarks:

> The signs ᴄ and ᴠ are mentioned here for the sake of symmetry, but they have no practical utility. [Peano 1894, 7]

In [Peano 1891*b*, 159], Peano explicitly refuses the use of "V", explaining:

> We shall not introduce the sign V, which corresponds by duality to Λ, because we do not need it.

In other works again, the inverted "V" is introduced without even mentioning the upright letter at all; e.g., in [Peano 1889*b*, 6], where Peano simply explains that "Λ" is the first letter of the word *vero* ("true" in Italian), and in

25. In works written in Italian or French, it is motivated by the words *vero* and *vrai*.

[Peano 1894, 7], where it is introduced as the first letter of the French word for "true", *vrai*.

Although Peano does not give reasons for his frequent choice of inverted symbols, their use arguably reduces the cognitive effort of learning the meaning of new symbols. For example, given the meanings of "M" and "D" as maximum and divisibility, the meanings of "ꟽ" and "ᗡ" as minimum and multiple can easily be inferred [Peano 1889*a*, VI].[26]

Another reason for simply inverting symbols lies in the fact that the printing types are readily available. For example, while Peano uses square brackets as "symbols for inversion", e.g., to write $[x\varepsilon]$ for class abstraction in [Peano 1889*a*, XIV], he changes this to $\overline{x\varepsilon}$ in [Peano 1894, 20] without giving any reasons, possibly to shorten the expression. But in the German translation of [Peano 1896-1897], which appeared as an appendix in [Genocchi 1899], we find the added footnote: "Instead of $\overline{x\varepsilon}p_x$ one can also write $x \ni p_x$ for easier printing" [Peano 1990, 18].[27] Here, the inversion of symbols is explicitly motivated by typographical considerations, a topic we turn to next.

3.3 Horizontal and vertical arrangement

3.3.1 Symbol size and spacing for easier readability

In addition to the choice of characters themselves, Peano's concerns extended also to their arrangement on the page, e.g., their spacing. For the second edition of the *Formulario*, he explicitly designed his symbolism for easier readability:

> In providing this material we tried to combine the clarity of the formulas with the ease of composition. For example, we fixed the length of the signs
>
> $$= \quad > \quad \supset \quad + \quad - \quad \times \quad / \quad \wedge \quad \surd$$
>
> in proportion to the numbers
>
> $$10 \quad 10 \quad 10 \quad 8 \quad 8 \quad 6 \quad 6 \quad 4 \quad 4$$
>
> that measure them in typographical points; these dimensions help to naturally read the formulas according to the common conventions regarding the omission of parentheses. [Peano 1898*b*, 233]

What lies behind this remark is the idea that symbols that are closer together are more readily seen as belonging together, such that the spacing around a

26. Other pairs of symbols that Peano uses are "f" and "ɟ" [Peano 1894, 27–29] and "↑" and "↓" [Peano 1894, 39–40]; the symbol for exponentiation "↾" is described as "the reversed sign for radicals" [Peano 1905, 34].
27. On the use of overlining, see also [Peano 1900, 8].

symbol can support the correct interpretation of its binding strength. Recent work by Landy and Goldstone [2007] has empirically validated this claim with regard to the reading and writing of algebraic equations. Notice in the above quotation how the width of the symbols ("the length of the signs") correlates with their usual binding strength: the less space a symbol occupies, the stronger it binds.[28] In his later reflections on mathematical typography, Peano elaborates on how the spacing can support the correct reading of formulas:

> The spacing of the formulas does not present typographical difficulties; it can facilitate the reading. The formulas $a{+}b\ \times\ c$ and $a\ +\ b{\times}c$ suggest the readings $(a + b) \times c$ and $a + (b \times c)$, where the former is contrary to and the latter in conformity with the conventions in algebra. The spacing $a + b \times c$ has become standard in typography. The reading will be easier if the sign $\times$ is smaller than $+$. [Peano 1915, 403]

In the last remark, Peano not only suggests to tighten the spacing but to actually make the sign smaller to indicate a stronger binding, which accords with the common writing of $a \times b$ as $a \cdot b$ or simply ab. By carefully selecting the size of the characters and the spacing between them, Peano wanted to ensure that the way his formulas appeared on the paper would facilitate their readability.

3.3.2 Printing costs and typographical convenience

Due to his leading position in the *Formulario* project, Peano was more involved with the practical matters of printing than most other mathematicians. In particular, this included being concerned about the cost of publishing mathematical works. In general, whenever a notation requires types that are not readily available by the typesetter or yields expressions that exceed the height of a line, its production becomes more costly. Peano frequently alluded to the cost of printing when discussing notational design, e.g., noting that the typographical realization of the usual notation for fractions, where the numerator appears above a line and the denominator below it, as in "$\frac{a}{b}$", costs three times of that of writing them in a single line, as "a/b" [Peano 1912, 377].[29] Accordingly, Peano notes that from the second edition onward of the *Formulario*, he introduced only notations that could be printed within a single line.

28. This idea is formulated somewhat cryptically in the introduction of the third edition of the *Formulario* as: "To make them stand out better, we will give the signs different dimensions, helping ourselves with typographic spaces" [Peano 1901, 3].

29. The production process and pricing is explained in [Peano 1915, 281]: multiple fractions in a line are five times as expensive as a single line, and with nested fractions the cost increase is "dizzying" (*vertiginosamente*).

In an article dedicated exclusively to typographical issues, Peano offers suggestions for writing formulas without extending the height of a line so as to keep the publishing costs down and to increase the readability of mathematical texts, such as: avoiding large parentheses, large integral and sum signs, stacked symbols such as “$\dot{x}$” and “$\binom{m}{n}$”, and radical signs with a vinculum [Peano 1915]. Here is an example of how such considerations affected Peano’s notations: Peano did not use overlining in his 1889 books to write the inverse of a function f as $\overline{f}$, as Dedekind did in his axiomatic presentation of arithmetic that was published a year earlier in 1888, but rather square brackets, because of “typographical convenience” [Peano 1890*a*, 187].[30]

4 Peano’s dot notation

We now turn to one of the most striking innovations in Peano’s notation, the use of dots to indicate the grouping of subformulas. In the 1894 introduction to the *Formulario*, the dot notation is described as being “equivalent” to the use of parentheses and vincula. To illustrate this point, Peano presents the following three representations,

$$ab\,.\,cd : e\,.\,fg \mathrel{\therefore} hk\,.\,l, \qquad \big\{[(ab)(cd)][e(fg)]\big\}[(hk)l], \qquad \underline{\underline{\underline{ab}\,\underline{cd}}\,\underline{e\,\underline{fg}}}\,\underline{\underline{hk}l},$$

and justifies his choice of using the dots for the grouping of propositions by a brief remark that parentheses render formulas “very complicated” [Peano 1894, 11]. At other occasions, Peano notes that “a convenient system of dots” achieves the same as parentheses, but “with greater simplicity” [Peano 1891*b*, 155], and that “parentheses would be absolutely bulky and cumbersome [*absolument encombrantes*]” [Peano 1897, 22][31]. As we shall see presently, the notions of simplicity and convenience that Peano attributes to the dot notation are closely related to considerations about notations discussed earlier: conciseness (Section 2.2) and vertical arrangement (Section 3.3.2). Compared to those with parentheses, expressions that are grouped using dots are shorter, while at the same time requiring no extra vertical space, as vincula do.

Before addressing the development of the dot notation in Peano’s writings in Section 4.2, let me first briefly explain how it works and use syntax trees to illustrate its relation to the usual linear notation.[32]

30. See also the example discussed at the end of the section 3.2.2.
31. This is repeated in [Peano 1900, 1901].
32. Syntax trees can be seen as a canonical notation for propositional logic and were also employed in [Schlimm 2018] to shed light on Frege’s *Begriffsschrift* notation.

4.1 Syntax trees and the dot notation

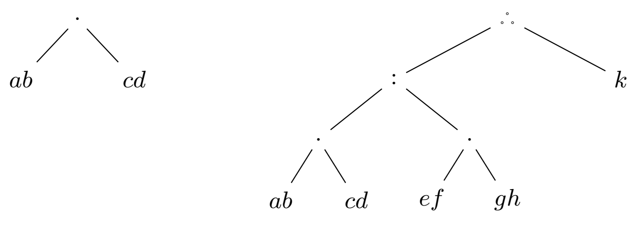
The main idea behind the dot notation is that, instead of *aggregating* elements that belong together by enclosing them within a pair of parentheses, groups of dots are used to *separate* two parts of an expression by marking the position at which the formula is divided. To understand the way in which a formula is partitioned by the dot notation, it is illustrative to consider the grouping of a string of symbols. The following are two examples discussed by Peano when introducing the dot notation for the first time [Peano 1889*a*, 104]. The groupings that are effected by parentheses in the expressions

$$(ab)(cd) \quad \text{and} \quad (((ab)(cd))((ef)(gh)))k$$

are represented in the dot notation by

$$ab \,.\, cd \qquad \text{and} \qquad ab \,.\, cd : ef \,.\, gh \therefore k$$

Notice that in the first example, all four parentheses are replaced by a single dot, while the 14 symbols for parentheses in the second example are replaced by four groups of dots, for a total of seven dots. This economy of symbols is the main reason explicitly stated by Peano for using dots instead of parentheses.[33] In order to deepen our intuitive understanding of the dot notation, it is instructive to look at the following syntax trees, which display the structure of the above groupings in a perspicuous fashion:



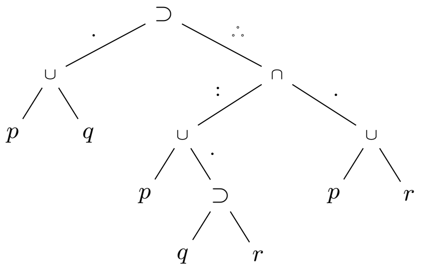
This representation illustrates how the number of dots in Peano's notation corresponds to the level in the syntax tree; more precisely, the number of dots of a node indicates (or is determined by) the length of the longest path from it to a leaf.

To employ the dot notation in logical formulas, we place a group of dots adjacent to a connective (to the left, right, or both) to separate a subformula from the rest of the expression. Consider, for example, the formula

$$(p \cup q) \supset ((p \cup (q \supset r)) \cap (p \cup r)) \qquad (1)$$

33. These comparisons in terms of the number of symbols used raise the question of whether to count either the individual dots or each group of dots (e.g., . , : , ∴ , ::) as symbols. We shall consider the latter as symbols because groups of dots are never broken up and always used as a single unit.

The use of syntax trees allows us to obtain the corresponding formula in the dot notation very easily. We first draw the syntax tree for the formula and label its edges as follows: if the longest path from the following node to a leaf has length n, mark the edge with a group of n dots. For the formula shown above, this yields:



(2)

We can also think of arriving at these labels by starting from the leaves and labeling each edge with an increasing number of dots, while moving upward toward the root, starting with zero. If the edges below a node are labeled with different numbers of dots, say n and m, then the edge immediately above this node is $\max(n, m) + 1$. In other words, the label of an edge above a connective is one more than the greatest label of the edges that are immediately below it. For example, since the edges that extend downward from the "$\cap$" symbol are labeled with groups of 1 and 2 dots, the edge above it must be labeled with 3 dots. If a syntax tree is annotated in this way, the labels contain information about the nesting of the subformulas: we immediately notice that the connective that has the greatest label on one of its downward edges is the main connective; the same also holds for each subtree.

Finally, to represent a formula in the linear dot notation, we parse its syntax tree in the usual (infix) way and write the groups of dots before a connective, if they appear on the left downward edge of the corresponding connective, and after it, if they appear on the right downward edge. In our example, this yields:

$$p \cup q \,.\, \supset \,\therefore\, p \cup \,.\, q \supset r \,:\, \cap \,.\, p \cup r \tag{3}$$

When comparing this formula to its representation with parentheses (1), we see that a group of dots to the right of a connective corresponds to one or more opening parentheses and that a group to the left corresponds to one or more closing parentheses. This close connection between parentheses and the dot notation can also be illustrated by using labeled parentheses. Using a numeric label above a parenthesis to indicate its depth of nesting, we obtain for (1):

$$\overset{1}{(}\, p \cup q \,\overset{1}{)} \supset \overset{3}{(}\overset{2}{(}\, p \cup \overset{1}{(}\, q \supset r \,\overset{1}{)}\overset{2}{)} \cap \overset{1}{(}\, p \cup r \,\overset{1}{)}\overset{3}{)} \tag{4}$$

To arrive at the dot notation starting from this representation, we first have to discard some of the parentheses that are redundant: all outer parentheses, both at the beginning and the end of the formula, are omitted; if two or more parentheses occur consecutively, we only keep the one with the greatest label and discard all others. After these modifications, formula (4) becomes:

$$p \cup q \overset{1}{)} \supset \overset{3}{(} p \cup \overset{1}{(} q \supset r \overset{2}{)} \cap \overset{1}{(} p \cup r \tag{5}$$

Now, simply replacing any labeled parenthesis with a group of as many dots as are indicated by the label yields the dot representation (3) of the formula. If the dot notation is introduced without reference to syntax trees, one often speaks of the *scope* of a group of dots, i.e., the subformula that is determined by that group.[34] The scope extends to the left of the group if the dots are to the left of a connective, and to the right if the dots are to the right of a connective. By looking at the syntax tree, it is easy to see that the scope of a group of dots is a subtree, i.e., it extends beyond all groups that consist of a smaller number of dots.

Because we always need two parentheses to enclose a subformula, but only one group of dots to separate a subformula, the dot notation uses fewer symbols. In our example, Formula (1), which contains 6 connectives, has 10 parentheses, omitting outer parentheses as is convention, but its representation in the dot notation (3) needs only 5 groups of dots. Because of this, the dot representation is more concise, a fact that is frequently used to argue in its favor.

4.2 Peano's use of the dot notation

Peano introduced the dot notation in his 1889 booklet on arithmetic, in which he presented his famous axiomatization of the natural numbers [Peano 1889*a*].[35] His explanation for the notation is surprisingly short; he apparently expected his readers to have no difficulties in using it.[36] Peano writes:

34. See, e.g., the introduction of dots in *Principia Mathematica* [Whitehead & Russell 1910-1913, I, 9–11].

35. Shortly afterwards, Peano published a logical exposition of geometry in which the dot notation is introduced with a very similar wording [Peano 1889*b*, 7].

36. In [Peano 1891*b*], he remarks that dots are already used in analysis, where "one writes *d.uv* and *du.v* instead of $d(uv)$ and $(du)v$" and notes some analogy to a notation used by Leibniz, referring to [Leibniz 1855, 276] and [Leibniz 1863, 55]. In these passages, Leibniz discusses the use of vincula and parentheses to group subexpressions, and he also uses groups of commas for this purpose, but without much explanation; in one example he also uses a combination of a comma with a dot. Whether Peano's dot notation was actually inspired by Leibniz or whether he found the passage from Leibniz only after having developed his own notation remains unclear.

> We shall generally write signs on a single line. To show the order in which they should be taken, we use *parentheses*, as in algebra, or *dots*, ., :, .:., : :, and so on.
>
> To understand a formula divided by dots we first take together the signs that are not separated by any dot, next those separated by one dot, then those separated by two dots, and so on. [Peano 1889*a*, VII] (quoted from [van Heijenoort 1967, 86])

This explanation is followed by the two examples presented at the beginning of Section 4.1, and by a brief remark that the dots can be omitted if different punctuations do not change the meaning of a formula (e.g., both "*ab* . *c*" and "*a* . *bc*" can be rendered simply as "*abc*' if the operation is associative) and if no ambiguities arise. Because groups of dots have also been used in other mathematical contexts (e.g., for multiplication and division), Peano warns:

> To avoid the danger of ambiguity we never use . or : as signs for arithmetic operations. [Peano 1889*a*, VII] (adapted from [van Heijenoort 1967, 87])[37]

Despite the fact that dots make formulas shorter, Peano allows for both dots and parentheses to be used within the same formula, with the convention that parentheses bind stronger than dots [Peano 1889*a*, VII] ([van Heijenoort 1967, 87]).[38] In general, both dots and parentheses occur in the same formula to syntactically mark semantical differences between logical and arithmetical expressions, e.g., in Definition 18 [Peano 1889*a*, 2]:

$$a, b \,\epsilon\, \mathrm{N} \,.\, \supset \,.\, a + (b + 1) = (a + b) + 1.$$

From this example, one might be tempted to surmise that Peano uses parentheses for the grouping of mathematical (or algebraic) expressions and dots for their logical grouping. While this is indeed mostly the case and he praises the use of dots for avoiding "the confusion with parentheses in algebraic formulas" [Peano 1891*b*, 155], Peano's usage is not completely consistent in this regard, sometimes relying simply on good judgement. For example, in [Peano 1889*a*, IX], he uses dots in

$$23. \quad a \cup b \,.=\, .\!:\!. - : - a \,.\, - b$$

but parentheses in the very similar formula

$$25. \quad -(a \cap b) = (-a)(-b)$$

37. See also [Peano 1894, 13]: "When introducing the dots to separate the parts of a proposition, one must discontinue their use for indicating multiplication $a \,.\, b$, which will be written ab or $a \times b$, and division $a : b$, which will be written a/b."

38. The sentence in question is missing in Kennedy's translation [Kennedy 1973, 104].

The two previous formulas also illustrate that, in addition to the symbol for conjunction, "∩", Peano also uses juxtaposition to indicate conjunction. Thus, the dot between "$-\,a$" and "$-\,b$" in Proposition 23, above, separates these two expressions, such that they are rendered as "$(-\,a)(-\,b)$" using parentheses (cf. Proposition 25, above). Citing the conciseness of the resulting expressions as motivation, Peano explains:[39]

> The sign ∩ is read *and*. Let a and b be propositions; then $a \cap b$ is the simultaneous affirmation of the propositions a and b. For the sake of brevity, we ordinarily write ab instead of $a \cap b$. [Peano 1889*a*, VII] (quoted from [van Heijenoort 1967, 87])

In rare cases, Peano even uses both juxtaposition and ∩ within the same formula, as in [Peano 1894, 12]:

$$a \,\text{Ɔ}\, .b \,\text{Ɔ}\, c : d \,\text{Ɔ}\, e \cup f \therefore \text{Ɔ} : h \cap k \,\text{Ɔ}\, l \,.\, \text{Ɔ} \,.\, m \,\text{Ɔ}\, n$$

Here, on the one hand, the conjunction of "$a \,\text{Ɔ}\, .b \,\text{Ɔ}\, c$" and "$d \,\text{Ɔ}\, e \cup f$" is indicated by the first occurrence of ":", which separates the two juxtaposed expressions. The conjunction of h and k, on the other hand, is expressed by "∩".

To determine the number of dots in a group that separates two conjuncts, it is helpful to consider again the representation of formulas in terms of syntax trees. As an example, consider Formula (3) and its corresponding syntax tree (2) on p. 115, above. In this case the conjunction symbol ("∩") is explicit and has a group of two dots on the left and a single dot on the right. To express this formula using juxtaposition for conjunction, we simply have to replace the symbol "∩" by the group next to it with the greatest number of dots and omit the other group. Replacing ": ∩ ." with ":", this results in the formula

$$p \cup q \,.\, \supset \therefore p \cup \,.\, q \supset r : p \cup r$$

Note that, while Whitehead and Russell explicitly introduced a single dot as the symbol for conjunction in *Principia Mathematica* [Whitehead & Russell 1910-1913, I, 6], for Peano both dots and parentheses are *exclusively* marks for grouping.[40] In the third edition of the *Formulario*, Peano is unequivocal about the use of parentheses for this unique purpose:

> The symbols of the *Formulario* always have the same meaning. Using parentheses to group the parts of a formula prevents us from using them in another meaning. We will be able to denote by (a) neither a power of a, as does Girard 1629 [...], nor the

39. This is also consistent with the considerations about readability and spacing (Section 3.3.1).

40. Presumably due to Russell's use, Peano's notation has been misinterpreted in the literature as using dots for conjunction; see, e.g., [Kneale & Kneale 1984, 521] and [Lolli 2011, 57, footnote 27].

> integral part of a, nor the absolute value of a, nor a function of a. In general, a single letter will never be enclosed in parentheses, because it is not grouped. [Peano 1901, 3][41]

Analogously, as Peano uses dots as marks for grouping, the expression "$a \,.\, b$" would not be in accord with his usage, given that each symbol must have a unique meaning and that a single letter cannot be grouped. Accordingly, Peano omits dots on the side of a connective that has only a single variable in its scope, as in $a \supset . a \cup b$ [Peano 1889*a*, IX, Prop. 26].

Peano's use of the dot notation is systematic but not rigid. It is evident from his formulas that he also employs implicit conventions regarding the binding strength of operations that are familiar from algebra, e.g., that juxtaposition binds stronger than any binary connective and that logical connectives bind stronger than the equality symbol. If this were not the case, Propositions 11, 14, and 27 in [Peano 1889*a*, IX],

$$ab \supset a, \qquad aa = a, \qquad \text{and} \qquad a \cup b = b \cup a$$

would have to be written as

$$ab \,.\supset a, \qquad aa \,.= a, \qquad \text{and} \qquad a \cup b \,.=.\, b \cup a$$

In later publications, Peano discusses some of his binding conventions explicitly, e.g., [Peano 1894, 12–13], but also mentions additional ones to further reduce the number of dots needed in a group. Presumably, however, this practice would not pose serious difficulties for a reader familiar with the typical binding conventions used in algebra.

In general, using more dots than are necessary in a group does not alter the structure of a formula, as long as the number of dots in groups with more dots are also increased accordingly. Thus, superfluous dots may be introduced in a formula and some later authors will systematically do so, presumably to enhance readability.[42] Also, larger groups of dots stand out more and are thus easier to see at a glance, which allows readers to faster identify the main connective in a formula, because it is flanked by the greatest number of dots. However, Peano does not add dots in a systematic fashion, but only occasionally, e.g., in Proposition 2 [Peano 1889*a*, VIII], possibly due to considerations of symmetry:

$$a \supset b \,.\, b \supset c :\supset: a \supset c. \tag{6}$$

In fact, this proposition is rendered without the superfluous dot on the right side of the main implication symbol as Proposition 6 in [Peano 1891*b*, 156]:

$$a \supset b \,.\, b \supset c :\supset .\, a \supset c. \tag{7}$$

41. See also [Peano 1897, 54], where Peano insists on writing fx instead of $f(x)$.

42. For example, Landini adds dots such that each connective has always the same number of dots on each side, "for easier reading" [Landini 2012, ix].

As we have seen earlier, Peano's symbolic language is not a formal language in the modern sense, i.e., based on an explicit, recursively defined grammar. Similarly, his dot notation is not defined rigorously, its usage is not entirely uniform, and several conventions are left implicit. This practice is consistent with common practice in algebra and with Peano's general attitude of being more interested in actually using the logical formalism for the representation of mathematics than in giving a rigorous presentation of the symbolism itself. Moreover, his general aims of attaining rigor and clarity while striving at the same time for conciseness can also be seen at play in this use of the dot notation.

5 Conclusion

Although some of Peano's views were shared by other influential logicians at the time, such as Peirce and Frege, his particular outlook and the associated enterprise of the *Formulario* remain unique. Perhaps because of the collaborative nature of the latter, Peano was more explicit than most other mathematicians about the principles that guide the introduction of new symbols, their shapes, and their layout. Moreover, while Frege maintained that "the convenience of the typesetter is not the highest Good" [Frege 1896], Peano was willing to take such practical matters into consideration. Despite his efforts, Peano's symbolism was not widely adopted outside of Italy in the late 19th century. Felix Klein wrote to Pieri in 1897:

> My general experience indicates that articles which are written using this symbolism, at least in Germany, find practically no readers and moreover meet with immediate rejection. [Marchisotto & Smith 2007, 365; see also 383]

Nevertheless, with time, many of Peano's symbols did eventually enter the mathematical canon, and, through its adoption by Whitehead and Russell in their groundbreaking *Principia Mathematica* [Whitehead & Russell 1910-1913], Peano's dot notation also became very popular in logical works in the first half of the 20th century, though it has by now almost completely faded into the background.[43]

The above discussions have given us insight into Peano's views on logic and his motivations for the development of a logical symbolism as a methodological tool for the *analysis* of mathematical ideas and as an indispensable practical tool for the *presentation* of mathematical theories. Accordingly, two main normative ideals underlie Peano's symbolizations: attaining *rigor and clarity*, mainly through the avoidance of ambiguities, which is primarily achieved by ensuring the uniqueness of meanings as well a judicious choice of notation,

43. A more detailed account of the use of the dot notation in early 20th century logic is in preparation by the author.

and *conciseness*. In addition, practical considerations, such as reducing the effort to learn and memorize the meanings of the notation, enhancing the clarity of the presentation and the ease with which it can be read, and, finally, reducing the printing costs, all guided Peano's design of notations. All of these considerations also support Peano's most conspicuous notational innovation, the dot notation for grouping subexpressions.

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