Failure of the finitely generated intersection property for ascending HNN extensions of free groups

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ABSTRACT

The main result of this thesis is a proof of the failure of the finitely generated intersection property of ascending HNN extensions of non-cyclic finite rank free groups. This class of group consists of free-by-cyclic groups and properly ascending HNN extensions of free groups. We also give a sufficient condition for the failure of the FGIP in the context of relative hyperbolicity, we apply this to the free-by-cyclic groups of exponential growth.

ABRÉGÉ

Le résultat principal de cette thèse est une démonstration de l'échec de la propriété Howson dans le cas d'extension HNN de groupes libres non-cyclique de génération finie. Cette classe de groupe consiste des groupes ayant un sous-groupe normal libre dont le quotient est cyclique, ainsi que les groupes d'extension HNN propre. Nous donnons une condition suffisante pour l'échec de la propriété de Howson dans le context d'hyperbolicité relative, nous utilisons ce resultat dans le cas où l'extension HNN est un automorphism de croissance exponentielle.

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Contribution of Authors

Unless otherwise stated, the results presented in this thesis constitutes of joint work with my supervisor, Professor Daniel T. Wise.

Contents

ABSTRACT 2

ABRÉGÉ 2

Acknowledgements 3

Contribution of Authors 3

1. Introduction 5

2. The Finitely Generated intersection property 7

3. Proper ascending HNN extension 8

4. Relative train track maps 11

4.1. Mapping Tori 13

5. Polynomially growing case 13

6. The exponential case 15

7. Failure of the FGIP for certain relatively hyperbolic groups 19

7.1. Ping-pong 19

7.2. Relative hyperbolicity and the FGIP 20

7.3. Exponential growth free-by-cyclic 20

References 21

1. Introduction

Definition 1.1. A group has the *finitely generated intersection property* (FGIP) if the intersection of any two finitely generated subgroups is also finitely generated.

The most famous class of groups having the FGIP are locally quasiconvex word-hyperbolic groups [27]. This generalizes the fact that free groups have the FGIP which was proven by Howson [17], and indeed the FGIP is sometimes referred to as the *Howson property*.

The purpose of this thesis is to examine the FGIP for ascending HNN extensions of finitely generated free groups. We show the failure of the *finitely generated intersection property* (FGIP) for ascending HNN extensions of non-cyclic finite rank free groups.

Definition 1.2 (Ascending HNN extension of a free group)**.** Let $\phi : F \to F$ be a monomorphism from a free group to itself. Its associated *ascending HNN extension* is the group $G = \langle F, t \mid tft^{-1} = \phi(f) \ : \ \forall f \in F \rangle$. If ϕ is surjective then G is *free-by-cyclic* and we denote G by $F \rtimes_\phi \mathbb{Z}$. If ϕ is not surjective then G is a *proper ascending HNN extension*.

Our main goal is the following result which is new for proper HNN extensions:

Theorem 1.3. *Any ascending HNN extension G of a finitely generated non-cyclic free group F fails to have the FGIP.*

The failure of the FGIP for $F \times \mathbb{Z}$ was first proved in [23]. The failure for free-by-cyclic groups was later proved in [8]. The authors were unfortunately oblivious to the work of Burns and Brunner and formulated an argument of the free-by-cyclic case which is separated into the *polynomially growing* and *exponentially growing* cases in Sections 5 & 6. These use Bestvina-Feighn-Handel *relative train track* maps [4, 5] which are recalled in Section 4. We believe our argument to be similar to [8] in nature, but hopefully less oblique as it relies less on combinatorial group theory and more on geometric intuition but at the expense of a powerful tool. In Theorem 3.1, the properly ascending case of Theorem 1.3 is proved using elementary methods.

When G is hyperbolic Theorem 1.3 has a much crisper explanation using the ping pong lemma (see Section 7.1). The idea is that for any $f \in F - \{1_G\}$, the ping pong lemma provides a free subgroup $K = \langle f^m, t^m \rangle$ and the intersection $F \cap K$ is not finitely generated. We develop this idea further in Section 7 and obtain the following sufficient condition for the failure of the FGIP in the relative hyperbolic framework:

Theorem 1.4. *Let G be hyperbolic relative to a collection of proper subgroups. Let $N \subset G$ be a finitely generated subgroup. Suppose $tNt^{-1} \subset N$ for some infinite order $t \notin N$. Suppose there is an infinite order $w \in N$ such that t, w do not lie in the same parabolic or virtually cyclic subgroup. Then G fails to have the FGIP.*

We combine this result with recent powerful statements concerning relative hyperbolicity of free-by-cyclic groups [12, 13, 14] to provide a unified proof of the failure of the FGIP for exponentially growing free-by-cyclic groups. This explanation is complex since it depends upon a constellation of deep results, but it is interesting to see how the exponential case fits into general framework via Theorem 1.4. We expect this unity to prevail as relatively hyperbolic structures are constructed for general ascending HNN extensions. Note that the exponentially growing case is the main case since failure in the polynomially growing case is a simple consequence of the failure for $F \times \mathbb{Z}$.

Finitely generated subgroups of ascending HNN extensions of rank 1 free groups are easily shown to be either trivial, cyclic or of finite index (see e.g. [21]). Intersecting a subgroup H with a finite index subgroup K yields a finite index subgroup of H. Hence if H is finitely generated so is $H \cap K$. We therefore focus on ascending HNN extensions of a free group F with $\text{rank}(F) \geq 2$.

2. The Finitely Generated intersection property

In this section we survey known results on the FGIP [29]. It is also called the *Howson property* since Howson first proved that free groups have the FGIP [17]. Moldavansky later proved that $F \times \mathbb{Z}$ does not have the FGIP and that the Baumslag Solitar group $BS(1,n)$ has the FGIP [23].

Greenberg proved that fundamental groups of compact surfaces have the FGIP [15]. Soma studied the FGIP for geometric 3 manifolds, giving examples of 3-manifolds with and without the FGIP [28]. In particular, hyperbolic 3-manifolds of infinite volume have the FGIP and closed hyperbolic 3-manifolds do not. Polycyclic-by-finite groups have the FGIP [1].

Baumslag proved that the FGIP is preserved under free products [3], this was generalized by Cohen to amalgamated products and HNN-extensions over finite subgroups [9]. Burns proves that amalgamated products of free groups along infinite cyclic subgroups have the FGIP if and only if the cyclic subgroup is maximal cyclic in both factors [7].

A right angled Artin group $A(\Gamma)$ has the FGIP iff every connected component of Γ is complete [26].

Dahmani proved that limit groups have the FGIP [11].

A large class of groups, including many of the above, having the FGIP are locally quasi-convex groups, a proof can be found in [27]. Wise and McCammond used the previous result to prove that many small cancellation groups have the FGIP [22]. Using these techniques, Schupp proved that many Coxeter groups have the FGIP [25]. One relator groups of form $G = \langle x_1, \cdots x_m | r^n = 1\rangle$ where $n \geq |r|$ are locally quasi-convex and therefore also have the FGIP [18]. Kapovich gives examples of one-relator groups that fail the FGIP [19], note that many ascending HNN extensions of free groups are one relator groups, for example $\langle a,\ b,\ t\ :\ tat^{-1} = b, tbt^{-1} = abab\rangle \cong \langle b,\ t\ :\ tbt^{-1} = t^{-1}btbt^{-1}btb\rangle$.

The Grigorchuk group does not have the FGIP [24].

A "generic" m-generator, n-relator group has the FGIP [2].

The wreath product $\mathbb{Z} \wr \mathbb{Z}$ does not have the FGIP [20].

The question of whether $SL(3,\mathbb{Z})$ has the FGIP remains open. Note that $SL(n,\mathbb{Z})$ where $n > 3$ has $F \times \mathbb{Z}$ subgroups where F is free of rank 2 so fails to have the FGIP.

3. Proper ascending HNN extension

In this section we prove the following theorem:

Theorem 3.1. *Any proper ascending HNN extension of a non-cyclic finitely generated free group fails to have the FGIP.*

Lemma 3.2. *Let $H \subset F$ be a finitely generated infinite index subgroup of a free group. There exists f such that $\langle f, H\rangle = \langle f\rangle * H$.*

Proof. Regard F as $\pi_1(B,b)$, where B is a bouquet of circles. Let $(\widehat{B},\hat{b}) \to (B,b)$ be a based covering map with $H = \pi_1(\widehat{B},\hat{b})$. Since H is finitely generated, we can chose a finite based subgraph $(C,\hat{b}) \subset (\widehat{B},\hat{b})$ whose inclusion induces a π_1-isomorphism. We claim that it is possible to add an edge e to C to obtain an immersion $D \to B$. Note that $\pi_1(D,b) = \langle f\rangle * H$ for some nontrivial $f \in \pi_1(B,b)$. We now prove the claim: Since $\widehat{B}$ is not a finite cover of B, there is an edge $\hat{a}$ of $\widehat{B} - C$ that is incident with C at a vertex v. Suppose $\hat{a}$ maps to the edge a of B. Observe that, as it is finite, C has the same total number of incoming and outgoing a-edges at its vertices. So if $\hat{a}$ is incoming/outgoing at v, there is a missing outgoing/incoming a-edge at some vertex u of C. We may thus attach an edge e mapping to a at the vertices u, v. □

Lemma 3.3. *Let $\phi : G \to G$ be group automorphism. The following are equivalent for $w \in G$.*

(1) *$\{\phi^i(w),\ i \in \mathbb{Z}\}$ form a basis for a free subgroup.*

(2) *$\{\phi^i(w),\ i \in \mathbb{N}\}$ form a basis for a free subgroup.*

Proof. (1) $\Rightarrow$ (2) is clear, so we focus on (2) $\Rightarrow$ (1). If a product $\prod_{i=1}^{k} \phi^{n_i}(w)$ where $n_i \in \mathbb{Z}$ represents the identity then $\phi^{max(|n_i|)}(\prod_{i=1}^{k} \phi^{n_i}(w)) = e$, so $\prod_{i=1}^{k} \phi^{m_i}(w)$ where $m_i = max(|n_i|) + n_i \in \mathbb{N}$ represents the identity. $\square$

Corollary 3.4. *Let G be a group and $t, w \in G$. The following are equivalent:*

(1) $\{t^i w t^{-i},\ i \in \mathbb{Z}\}$ *form a basis for a free subgroup.*

(2) $\{t^i w t^{-i},\ i \in \mathbb{N}\}$ *form a basis for a free subgroup.*

Proof. Apply Lemma 3.3 to the inner automorphism consisting of conjugation by the element t. $\square$

Lemma 3.5. *Let H be a group generated by elements w and t, and let $\rho : H \to \mathbb{Z}$ be a homomorphism with $\rho(t) = 1$ and $\rho(w) = 0$. Then* $\ker(\rho) = \langle\langle w \rangle\rangle = \langle t^n w t^{-n} : n \in \mathbb{Z} \rangle$.

We use the notation $\langle\langle w \rangle\rangle$ for the normal closure of w.

Proof. Let J be the free group on w and t. We get a homomorphism $J \to \mathbb{Z}$ by composing $J \to H \to \mathbb{Z}$. Its kernel is the normal closure of $w \in J$, since w is mapped to 0 and $J/\langle\langle w \rangle\rangle \cong \mathbb{Z}$. Realizing J as the fundamental group of a bouquet of circles on w and t and considering the associated cyclic covering space one sees that the normal closure of w in J is generated by conjugates of w by t^n for $n \in \mathbb{Z}$.

The kernel of $J \to \mathbb{Z}$ maps surjectively on the kernel of $H \to \mathbb{Z}$. The image under $J \to H$ of the normal closure of w in J is the normal closure of w in H and the image of conjugates of w by t^n in J are conjugates of w by t^n in H, so $\ker(\rho) = \langle\langle w \rangle\rangle = \langle t^n w t^{-n} : n \in \mathbb{Z} \rangle$. $\square$

Corollary 3.6. *In the setting of Lemma 3.5, if $\{t^i w t^{-i},\ i \in \mathbb{Z}\}$ forms a basis for a free subgroup then H is free of rank* 2.

Proof. If $\{t^i w t^{-i},\ i \in \mathbb{Z}\}$ forms a basis for a free subgroup of H, then $J \to H$ restricts to an isomorphism on the kernels of the homomorphisms to $\mathbb{Z}$. Apply the five lemma to the two short exact sequences to get $J \cong H$. $\square$

Lemma 3.7. *Let $H \subset F$ be a nontrivial subgroup of a free group and let $f \in F$ be such that $fHf^{-1} \subset H$ but $f^n \notin H$ for any $n > 0$. Then H is not finitely generated.*

Proof. Realize F as π_1 of a bouquet of circles (B, p). Let $(\widehat{B}, \hat{p})$ be a based covering space corresponding to H. Considering the covering transformation action of F on the vertices of $\widehat{B}$. If $f^m\hat{p} = f^n\hat{p}$ then $f^{m-n}\hat{p} = \hat{p}$ which implies $f^{m-n} \in H$, which implies $m = n$ by assumption, therefore $\{f^n\hat{p}, n \in \mathbb{Z}\}$ is infinite. By non-triviality of H, let σ be a nontrivial cycle. Using that $f^n \notin H$ for any $n > 0$, we can find a sequence of elements $\{f^{n_i}\hat{p}\}$ such that the lifts of σ starting at $f^{n_i}\hat{p}$ are pairwise disjoint. Hence H is not finitely generated since $\mathsf{H}_1(\widehat{B})$ is not finitely generated. $\square$

The following corollary is used in the free-by-cyclic case.

Corollary 3.8. *Let F be a free group and $H \subset F$ a normal subgroup of infinite index, then H is not finitely generated.*

Proof. Suppose towards a contradiction that H is finitely generated. By Lemma 3.2 there is an element $f \in F - H$ such that $f^n \notin H$ for any $n > 0$. Since H is normal, $fHf^{-1} \subset H$ so H is not finitely generated by Lemma 3.7. $\square$

Proof of Theorem 3.1. We first observe that $[F, \phi_*(F)] = \infty$. Indeed, if $(\widehat{B}, \hat{b})$ is a degree d cover of a bouquet of $r \geq 2$ circles, then $\chi(\widehat{B}) = d\chi(B) = d(1 - r)$. Thus $\chi(H) < \chi(F)$ for any proper finite index subgroup H of F. So F is not isomorphic to a proper finite index subgroup of itself, and therefore $\phi_*(F)$ is of infinite index.

By Lemma 3.2 there exists an $f \in F - \phi(F)$ such that $\langle f, \phi(F)\rangle = \langle f\rangle * \phi(F)$. Therefore $\langle f, \phi(f), \phi^2(F)\rangle = \langle f\rangle * \langle\phi(f), \phi^2(F)\rangle = \langle f\rangle * \langle\phi(f)\rangle * \phi^2(F)$, and thus $\langle\phi^i(f),\ i \in \mathbb{N}\rangle = *_{i\in\mathbb{N}}\langle\phi^i(f)\rangle$. In particular Lemma 3.3 tells us that $\{t^i f t^{-i},\ i \in \mathbb{Z}\}$ forms a basis for a free subgroup. By Corollary 3.6 where $H = \langle f, t\rangle$ and ρ from the HNN structure, H is free of rank 2. To conclude, $H \cap F$ is conjugated into itself by t, and $t^n \notin F$ for any $n > 0$, so $H \cap F$ is not finitely generated by Lemma 3.7. $\square$

4. Relative train track maps

In this section we review the theory of train track maps developed in [5] and [4]. A *marked graph* is a graph V together with a homotopy equivalence to the bouquet R_n of n petals, $\pi_1 V$ can therefore be identified with the free group of rank n up to inner automorphism. A homotopy equivalence $\Phi : V \to V$ of a marked graph is a *topological representative* of the outer automorphism it induces on the free group. Note that every outer automorphism has a topological representative, but that topological representatives are not unique. In [5], particularly nice representatives called *relative train track maps* are defined. An algorithm exists turning a homotopy equivalence into a relative train track map inducing the same outer automorphism ([5], Thm 5.12). Any automorphism therefore has a relative train track map as a topological representative, this means that the homotopy enjoys the following properties:

Definition 4.1. The *tightening* $[\![w]\!]$ of a combinatorial path w in V is the immersed path in V that is path-homotopic to w. By a *reduced* path we mean a path equal to its tightening. By a *cyclically reduced* path we mean a path w such that $[\![w]\!][\![w]\!] = [\![w^2]\!]$.

Definition 4.2. A path $w = e_1 \cdots e_k$ in V is *legal* if $\Phi^n(e_j)$ and $\Phi^n(e_{j+1}^{-1})$ have distinct initial edges for all $n \geq 0$. The path $w = e_1 \cdots e_k$ in V is *r-legal* if for all $n \geq 1$, the paths $\Phi^n(e_j)$ and $\Phi^n(e_{j+1}^{-1})$ have distinct initial edge e unless $e \subset V^{r-1}$.

Definition 4.3 (Relative train track map)**.** Let $\emptyset = V^0 \subsetneq V^1 \subsetneq ... \subsetneq V^k = V$ be a filtration of V by subgraphs and let $S^r = Cl(V^r - V^{r-1})$. Let $\Phi : V \to V$ be a *tight relative train track map* in the sense of [5]. This means that Φ sends vertices to vertices and edges to combinatorial paths, and that each of the following holds:

(1) Each V^r is Φ-invariant.

(2) For each edge e in an exponentially growing stratum $S^r \subset V^r$, the path $\Phi(e)$ starts and ends with an edge of S^r.

(3) For each exponentially growing S^{r+1} and each non-trivial immersed path $P \to V^r$ starting and ending in $S^{r+1} \cap V^r$, the path $\Phi(P)$ is essential, in the sense that $[\![\Phi(P)]\!]$ is non-trivial.

(4) For each exponentially growing S^r and each legal path $P \to S^r$ the path $\Phi(P)$ is r-legal.

If $e_1, e_2, ..., e_m$ is the collection of edges in some stratum S^r, the *transition matrix* of H^r is the non-negative $m \times m$ matrix M_r whose ij-th entry is the number of times the ϕ-image of e_j crosses e_i, regardless of orientation. M_r is *irreducible* if for every tuple $1 \leq i, j \leq m$ there is some exponent $n > 0$ such that the ij-th entry of M_r^n is non-zero. We may assume that the filtration in Definition 4.3 is *maximal* in the sense that every transition matrix is irreducible or the zero matrix since otherwise, a further reduction is possible. In the irreducible case, let $\lambda_r \geq 1$ be the largest eigenvalue of M_r, also known as the *Perron-Frobenius eigenvalue.* If $\lambda_r > 1$ we say S^r is *exponentially growing.* If $\lambda_r = 1$, we say S^r is *polynomially growing.* Otherwise, $M_r = 0$ and we say S^r is a *zero stratum.*

In Section 5, we enjoy some addition properties on topological representatives, namely some properties of improved relative train track map in the sense of [4, Thm 5.1.5.]. Any outer automorphism has an improved relative train track representative. We define the properties of such maps that we use.

Definition 4.4 (Improved relative train track map)**.** The relative train track map $\Phi : V \to V$ is improved if:

(1) If S^r is a zero stratum, then S^{r+1} is exponentially growing.

(2) If S^r is a zero stratum, then it is the union of contractible components of V^r

(3) If S^r is a polynomially growing stratum, then S^r consists of a single edge e, and $\Phi(e) = eP$, where P is a closed path in V^{r-1} whose initial point is fixed by Φ. If P is trivial, then e is a periodic edge.

4.1. **Mapping Tori.** Lurking behind the discussions in Sections 5 and 6 is the following geometric realization whose description is a continuation of Definition 1.2.

Definition 4.5 (Mapping tori)**.** Let $\Phi : V \to V$ be a map from a connected graph to itself. The *mapping torus* M_Φ of Φ is the 2-complex: $M_\Phi = V \times [0,1]/\{(v,0) \sim (\Phi(v),1) \ : \ \forall v \in V\}$. When Φ is basepoint preserving, it induces a homomorphism $\phi : \pi_1 V \to \pi_1 V$, otherwise we identify $\pi_1(V,p)$ with $\pi_1(V,\Phi(p))$ in the usual way to get a homomorphism $\phi : \pi_1 V \to \pi_1 V$. The group G is an ascending HNN extension of a free group if $G = \pi_1(M_\Phi)$ with $\phi : \pi_1 V \to \pi_1 V$ injective. If ϕ is an automorphism then G is *free-by-cyclic* and we denote G by $F \rtimes_\phi \mathbb{Z}$ where $F = \pi_1 V$. If ϕ is not surjective then G is a *proper HNN extension.* (Note that this depends on the choice of the decomposition of G.)

Remark 4.6. If two free group automorphisms ϕ and ϕ' belong to the same outer class (they differ by conjugation by a group element), then $F \rtimes_\phi \mathbb{Z} \cong F \rtimes_{\phi'} \mathbb{Z}$. Therefore Theorem 3.1, Theorem 5.4, and Theorem 6.1 partition Theorem 1.3.

5. Polynomially growing case

In this section we study free-by-cyclic groups that are ascending HNN extensions over train track maps with no exponential stratum. This is usually called the *polynomial case*, since the growth of $|\Phi^n(w)|$ is bounded by a polynomial for each path $w \subset V$. The method used consists of either finding a $F \times \mathbb{Z}$ subgroup which fails the FGIP as proved in [23], or finding a free subgroup of rank$= 2$ surjecting onto the $\mathbb{Z}$ factor.

Theorem 5.1. *$F \times \mathbb{Z}$ fails to have the FGIP, when F is a non-cyclic finitely generated free group.*

Proof. Let $H = \langle (f_0, 0), (f_1, 1)\rangle$ and suppose $J = \langle f_0, f_1 \rangle$ is not a cyclic subgroup of F. Since J is non-cyclic, $[f_0, f_1] \neq id$. Hence $H \cap (F\{0\})$ is nontrivial, since $[(f_0, 0), (f_1, 1)] \in F \times \{0\}$.

However $[H : H \cap F] = \infty$ because $(f_1, 1)$ has image 1 under the homomorphism to $\mathbb{Z}$. The result follows from Corollary 3.8. $\square$

The explanation in the polynomial case is simplified by considering an improved relative train track representative of the automorphism. By Definition 4.4.(1) there can only be a zero stratum at the highest stratum, in which case we may ignore it by Definition 4.4.(2). We therefore assume all of our strata to be of polynomial growth. A hierarchy for the group is obtained by means of Definition 4.4.(3). This gives an increasing sequence of mapping tori M_{Φ_i} where $\Phi_i = \Phi|_{V^i}$: each stage is obtained from the previous by an HNN extension whose stable letter e_n conjugates $P_n t$ to t, as described in Definition 4.4.(3). Each inclusion $M_{\Phi_i} \hookrightarrow M_\Phi$ are π_1-injective, so it suffices to show that M_{Φ_2} fails to have the FGIP.

In fact, M_{Φ_2} has a very simple structure:

Lemma 5.2. $\pi_1(M_{\Phi_2})$ *splits as an HNN extension of* $\mathbb{Z}^2$ *over cyclic subgroups.*

Proof. Each strata is polynomially growing by assumption. Let e_i be the unique edge in S^i, and P_i be the closed path in V^{i-1} such that $\phi(e_i) = e_i P_i$ as in Definition 4.4.(3). Since $V^0 = \emptyset$, we must have $\phi(e_1) = e_1$. So $\pi_1(M_{\phi_1})$ is isomorphic to $\mathbb{Z}^2$. Moreover since V^1 contains only e_1, $P_2 = e_1^n$ for some $n \in \mathbb{Z}$ so $\phi(e_2) = e_2 e_1^n$. Thus $\pi_1(M_{\phi_1})$ is isomorphic to the group $\langle e_1, e_2, t \mid te_1t^{-1}e_1^{-1},\ te_2t^{-1}{e_1}^n e_2\rangle$. This is an HNN extension of the subgroup $\langle e_1, t\rangle \cong \mathbb{Z}^2$ over cyclic subgroups $\langle {e_1}^n t\rangle$ and $\langle t\rangle$ with stable letter e_2 sending generator to generator. $\square$

Remark 5.3. In addition to the existing HNN extension structure on $\pi_1(M_{\Phi_2})$ with stable letter t, the previous lemma provides another HNN extension structure on $\pi_1(M_{\Phi_2})$ with stable letter e_2. Each such structure comes with a homomorphism onto $\mathbb{Z}$, we denote them by homomorphisms to $\langle t\rangle$ and $\langle e_2\rangle$ respectively, induced by the retracts to the stable letters.

We conclude this section with the proof of

Theorem 5.4. *Any ascending HNN extension of a non-cyclic finite rank free group over an improved relative train track map with no exponential stratum fails to have the FGIP.*

Proof of Theorem 5.4. We focus on the subgroup $\pi_1(M_{\Phi_2}) \cong \langle e_1,\ t,\ e_2 \mid [e_1, t],\ e_2{e_1}^n te_2^{-1} = t\rangle$ where $n \in \mathbb{Z}$ as in the proof of Lemma 5.2. If $n = 0$ then $\pi_1(M_{\Phi_2}) \cong F \times \mathbb{Z}$ which was resolved in Lemma 5.1. If $n \neq 0$, we find a nontrivial finitely generated free subgroup K such that $K \cap F$ is not finitely generated. Since $n \neq 0$, words in $\langle e_1,\ e_2 t {e_2}^{-1}\rangle$ do not have subwords of form ${e_2}^{-1} t e_2$ or $e_2 {e_1}^n t {e_2}^{-1}$. By Britton's Lemma $K = \langle e_1,\ e_2 t {e_2}^{-1}\rangle$ is free of rank 2. Moreover $K \cap F$ is nontrivial since $e_1 \in K$, and $[K, K \cap F] = \infty$ because $e_2 t {e_2}^{-1}$ has image t in the homomorphism to $\langle t\rangle$ introduced in Remark 5.3. The conclusion follows from Corollary 3.8. □

6. The exponential case

In this section, we prove the following:

Theorem 6.1. *Any ascending HNN extension of a non-cyclic finite rank free group over a relative train track map with an exponential stratum fails to have the FGIP.*

Remark 6.2. Throughout this section we will possibly raise Φ to a positive exponent Φ^p. We then prove failure of FGIP in $\pi_1(M_{\Phi^p})$, which can be identified with the finite index subgroup obtained as the kernel of the composition of the projection onto the $\mathbb{Z}$ factor with $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$.

Definition 6.3. An edge $e \in S^r$ is an *r-edge*. The *r-length* $|w|_r$ of a path w is the number of r-edges in w.

Lemma 6.4. *Let $\Phi : V \to V$ be a relative train track map. After possibly replacing Φ by Φ^p, for any ℓ-legal path $w \subset V^\ell$ and exponential stratum S^ℓ, we have $2|w|_\ell \leq |\Phi(w)|_\ell$. Consequently, $2|\Phi^n(e)|_\ell \leq |\Phi^{n+1}(e)|_\ell$ for each edge e of S^ℓ and any $n > 0$.*

Proof. Since the transition matrix of the ℓ-stratum is irreducible with eigenvalue strictly larger than 1, for each edge e in S^ℓ, there is a positive integer n_e such that

$|\Phi^{n_e}(e)|_\ell > 1$. Replacing Φ by $\Phi^{\prod_{e\in S^\ell} n_e}$ ensures that $|\Phi(e)|_\ell \geq 2$ for all $e \in S^\ell$. Condition 4.3.(2) further ensures that $2|w|_\ell \leq |\Phi(w)|_\ell$ since w is ℓ-legal. $\square$

Definition 6.5. For a reduced and cyclically reduced closed path w, define $w^{+\infty}$ to be the infinite path containing w^n as an initial subpath for each $n \in \mathbb{N}$.

Definition 6.6. If w, u are path with same initial/terminal vertex v, the *initial/terminal overlap* is the maximal common subpath starting/ending at v.

Lemma 6.7. *Let $\Phi : V \to V$ be a relative train track map. After replacing Φ by a positive power Φ^p, the following holds for any exponential stratum S^ℓ. There is a closed path $w \to V$ such that:*

(1) *w is cyclically reduced; so $w \to V$ is an immersed cycle.*
(2) *$\langle w\rangle$ is a maximal cyclic subgroup.*
(3) *The first edge of w is an ℓ-edge.*
(4) *w^m is ℓ-legal for each $m \in \mathbb{N}$.*
(5) *$[\![w]\!]$ and $[\![\Phi(w)]\!]$ have nontrivial initial overlap and terminal overlap.*
(6) *$[\![\Phi^N(w^{\pm1})]\!]^{+\infty} \neq [\![\Phi^M(w^{\pm1})]\!]^{+\infty}$ if $N \neq M$.*

Proof. Consider Φ from Lemma 6.4 . By the pigeonhole principle, since $|\Phi^n(e)|_\ell$ diverges as $n \to \infty$ there is an n_0 such that $\Phi^{n_0}(e)$ contains two copies of the same ℓ-edge. Let w be the subpath of $\Phi^{n_0}(e)$ starting with the repeating ℓ-edge and ending at the initial vertex of the second occurrence of the repeating ℓ-edge, giving Conditions (1) and (3). Condition (4) is ensured since length 2 subpath of w^∞ appears in $\Phi^{n_0}(e)$, and $\Phi^{n_0}(e)$ is ℓ-legal by Definition 4.3.(4).

Consider $\{\Phi^n(w), n > 0\}$, find n_0 and n_1 such that $\Phi^{n_0}(w)$ and $\Phi^{n_1}(w)$ have nontrivial initial overlap, in particular they are based at the same point p. Change Φ to $\Phi^{n_1-n_0}$ and w to $\Phi^{n_0}(w)$. Do the same for to ensure that w^{-1} and $\Phi(w^{-1})$ have nontrivial initial overlap. This ensures Condition (5) and does not alter Conditions (1)-(4).

Condition (6) is verified by considering the action of $\pi_1(V,p)$ on the universal cover $\widetilde{V}$ associated to a basepoint $\tilde{p} \in \widetilde{V}$. Let $\widetilde{[\![\Phi^N(w)]\!]}^{+\infty}$ be the lift of $[\![\Phi^N(w)]\!]^{+\infty}$ at $\tilde{p}$. Condition (2) ensures $Stab(\widetilde{[\![\Phi^N(w)]\!]}^{\infty}) = \langle \Phi^N(w) \rangle \cong \mathbb{Z}$. So $[\![\Phi^N(w^{\pm 1})]\!]^{+\infty} = [\![\Phi^M(w^{\pm 1})]\!]^{+\infty}$ implies $\langle \Phi^N(w^{\pm 1}) \rangle = \langle \Phi^M(w^{\pm 1}) \rangle$, which itself implies $\Phi^N(w^{\pm 1}) = \Phi^M(w^{\pm 1})$ by maximality of the subgroups. However this is only possible if $N = M$, since otherwise $|\Phi^N(w^{\pm 1})|_\ell \neq |\Phi^M(w^{\pm 1})|_\ell$ by Lemma 6.4. $\square$

Definition 6.8. The *length* $|w|$ of a path w is the number of edges in w.

The final tool we introduce is bounded cancellation, we refer the reader to [10] for a proof of the following:

Definition 6.9. Let a, b be words in a graph V, define the *cancellation between a and b* to be the maximal common initial subword of a^{-1} and b denoted by c_{ab}.

Lemma 6.10 (Bounded cancellation)**.** *Let Φ be a homotopy equivalence of a finite graph. Then there exists a constant C_Φ such that if $|w_1 w_2| = |w_1| + |w_2|$ then $|\Phi(w_1)\Phi(w_2)| - 2c_{\Phi(w_1)\Phi(w_2)} \geq |\Phi(w_1)| + |\Phi(w_2)| - C_\Phi$.*

Corollary 6.11. *Let σz and σr be words in a graph V such that $|\sigma z| = |\sigma| + |z|$ and $|\sigma r| = |\sigma| + |r|$ and $\Phi : V \to V$ a homotopy equivalence with bounded cancellation C_Φ. Write $\Phi(\sigma z)$ as a reduced word $\sigma' z'$ where σ' is the maximal initial subpath of $[\![\Phi(\sigma z)]\!]$ and $[\![\Phi(\sigma r)]\!]$. Then $|z'| > |\Phi(z)| - C_\Phi$.*

Proof. We consider $c_{\Phi(\sigma)\Phi(z)}$, $c_{\Phi(\sigma)\Phi(r)}$ and $c_{\Phi(z)^{-1}\Phi(r)}$, notice that for any choice of words and Φ, at least two of these three words have to be equal. Consider three cases, the first when $|c_{\Phi(\sigma)\Phi(z)}| \geq |c_{\Phi(\sigma)\Phi(r)}| = |c_{\Phi(z)^{-1}\Phi(r)}|$, in which case $z' = [\![c_{\Phi(\sigma)z}\Phi(z)]\!]$. The second when $|c_{\Phi(\sigma)\Phi(r)}| \geq |c_{\Phi(\sigma)\Phi(z)}| = |c_{\Phi(z)^{-1}\Phi(r)}|$, in which case $z' = [\![c_{\Phi(\sigma)r}\Phi(z)]\!]$. The last case when $|c_{\Phi(z)^{-1}\Phi(r)}| \geq |c_{\Phi(\sigma)\Phi(z)}| = |c_{\Phi(\sigma)\Phi(r)}|$, in which case $z' = [\![c_{\Phi(\sigma)r}\Phi(z)]\!]$. In each case we obtain z' by pre-composing $[\![\Phi(z)]\!]$ with a word of length smaller than C_Φ therefore getting the desired inequality. $\square$

We now turn to the main goal of this section:

Proof of Theorem 6.1. Let w and Φ be the path and map provided by Lemma 6.7. We claim there exists $p > 0$ such that for every $i \in \mathbb{N}$, $[\![\Phi^i(w^p)]\!]$ is not a prefix of $[\![\Phi^{i+1}(w^p)]\!]$ but that they share a nontrivial prefix as in Lemma 6.7.(5). Moreover p can be chosen such that the same holds for $[\![\Phi^i(w^{-p})]\!]$ and $[\![\Phi^{i+1}(w^{-p})]\!]$.

As Figure 1 illustrates, this provides an increasing sequence of subgraphs $\theta_0 \subset \theta_1 \subset \theta_2 \subset \cdots$ with $\chi(\theta_n) = -n$, and immersions $\theta_n \to V$, with π_1-image $\langle \Phi^i(w^{2p}) : 0 \leq i \leq n\rangle$. Thus $\langle \Phi^i(w^{2p}) \mid i \in \mathbb{N}\rangle = *_{i\in\mathbb{N}}\langle \Phi^i(w^{2p})\rangle$. By Lemma 3.5 the kernel of $\langle w^{2p}, t\rangle \to \mathbb{Z}$ is $\langle \Phi^i(w^{2p}) \mid i \in \mathbb{Z}\rangle$ which is not finitely generated by Lemma 3.3.

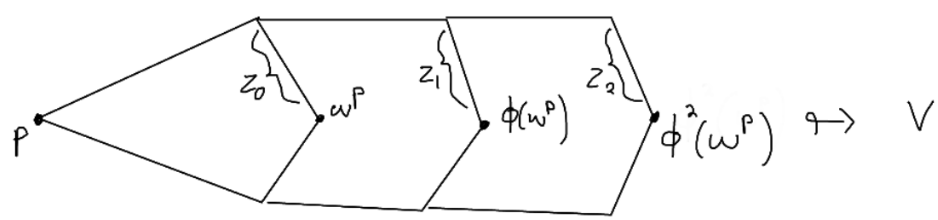

FIGURE 1. Nontrivial z_is in θ_2.

To find p, first find p_0 such that $[\![w]\!]^{p_0}$ is not a prefix of $[\![\Phi(w)]\!]^{p_0}$, this is possible by Lemma 6.7.(6). Write $[\![\Phi^i(w)]\!]^{p_0}$ as a reduced word $\sigma_i z_i$ where σ_i is the maximal common initial subpath of $[\![\Phi^i(w)]\!]^{p_0}$ and $[\![\Phi^{i+1}(w)]\!]^{p_0}$. Our goal is now to show that a choice of p_0 ensures that $|z_i| > 0$ for each $i \geq 0$. We may assume that $|z_0|_\ell > C_\Phi$, since increasing p_0 adds copies of $[\![w]\!]$ to z_0 which increases the ℓ-length by Lemma 6.7.(3)&(4).

The following induction argument shows that our choice of p_0 ensures $|z_i| > 0$ for all i. If $|z_i|_\ell > C_\Phi$, then $|\Phi(z_i)|_\ell > 2C_\Phi$ by Lemma 6.4, and so $|\Phi(z_i)| > 2C_\Phi$. Using Corollary 6.11 on $z' = z_{i+1}$, $u = \sigma_i$ and $r = [\![\sigma_i^{-1}\Phi^{i+1}(w^{p_0})]\!]$ we also have $|z_{i+1}| > |\Phi(z_i)| - C_\Phi$. Combining these, if $|z_i|_\ell > C_\Phi$ then $|z_{i+1}| > C_\Phi$ completing our induction.

Similarly, pick p_1 such that the same holds true for $\Phi^{i+1}(w^{-p_1})$ and $\Phi^i(w^{-p_1})$. We may fix $p = max(p_0, p_1)$ and w^p will be the desired word. $\square$

Remark 6.12. Notice that by Lemma 3.5 the elements w^{2p} and t generate a free subgroup of rank 2 .

7. Failure of the FGIP for certain relatively hyperbolic groups

In this section we propose a short argument explaining the failure of the FGIP for certain relatively hyperbolic groups. We then combine this result with a powerful result on relative hyperbolicity of free-by-cyclic groups with exponentially growing automorphism [13, 14, 12] to give a second explanation of the failure of the FGIP for these groups.

7.1. **Ping-pong.** We recall the relatively hyperbolic generalization of Gromov's application of the ping pong lemma [16] to a hyperbolic group acting as a convergence group on its boundary.

Lemma 7.1. *Let G be a relatively hyperbolic group. Let $w, t \in G$ be infinite order elements with no common fixed point in ∂G. Then there exists $m > 0$ such that $\langle w^m, t^m \rangle$ is free of rank 2.*

We use the action of G as a convergence group on its boundary ∂G. [6]

Proof. Let $\{a, b\}$ and $\{x, y\}$ be the points stabilized respectively by w and t. Let N_a, N_b, N_x, N_y be pairwise disjoint open neighborhoods of a, b, x, y respectively unless $x = y$ or $a = b$ in which case $N_x = N_y$ and $N_a = N_b$. By compactness we may assume that the closure of these neighborhoods are disjoint. Since G acts as a convergence group and w, t have $\{a, b\}$ and $\{x, y\}$ as attracting/repelling points respectively. There is a constant M such that $t^m(\bar{N}_a \cup \bar{N}_b) \subset N_x \cup N_y$ and $w^m(\bar{N}_x \cup \bar{N}_y) \subset N_a \cup N_b$ whenever $|m| > M$. Applying the ping pong lemma on $\bar{N}_a \cup \bar{N}_b$ and $\bar{N}_x \cup \bar{N}_y$ gives us the desired result. $\square$

7.2. **Relative hyperbolicity and the FGIP.**

Theorem 7.2. *Let G be hyperbolic relative to a collection of proper subgroups. Let $N \subset G$ be a finitely generated subgroup. Suppose $tNt^{-1} \subset N$ for some infinite order $t \notin N$. Suppose there is an infinite order $w \in N$ such that t, w do not lie in the same parabolic or virtually cyclic subgroup. Then G fails to have the FGIP.*

Remark 7.3. The hypotheses on w, t are equivalent to being (infinite order) parabolic and/or loxodromic elements without common fixed points on ∂G. In particular this always holds when one is parabolic and the other loxodromic. [6, Lem 2.2]

Proof. By Lemma 7.1, there exists $m \in \mathbb{Z}$ such that w^m, t^m generate a rank 2 free subgroup F. Observe that $K = N \cap F$ is nontrivial since $w^m \in N \cap F$, and $tKt^{-1} \subset K$ but $t^m \notin N$ $\forall$ $m \in \mathbb{Z}$. By Lemma 3.7, K is not finitely generated. □

The hyperbolic case simplifies to the following.

Corollary 7.4. *Let G be hyperbolic, let $N \subset G$ be an finitely generated infinite subgroup, and suppose $tNt^{-1} \subset N$ for some infinite order t with $\langle t \rangle \cap N$ trivial. Then G fails to have the FGIP.*

7.3. **Exponential growth free-by-cyclic.** Combining Theorem 7.2 with recent powerful results on relative hyperbolicity of mapping tori, we obtain an alternate explanation of Theorem 6.1.

Corollary 7.5. *If $\phi : F \to F$ is an exponentially growing automorphism of a finitely generated free group. Then the free-by-cyclic group $F \rtimes_\phi \mathbb{Z}$ fails to have the FGIP.*

Proof. From [12, Thm 3.5] , $F \rtimes_\phi \mathbb{Z}$ is hyperbolic relative to mapping tori of maximal polynomially growing subgroups. These are the conjugacy classes of finitely many subgroups of form $\langle H, gt \rangle$ where $H < F$ is polynomially growing, $g \in F$ and t is the stable letter of the decomposition.

Let $w \in F$ be an exponentially growing element. Observe that w is loxodromic in the above relatively hyperbolic structure. Note that the semi direct structure ensures that w and t do not lie in the same cyclic subgroup. Hence Remark 7.3 ensures that the criterion Theorem 7.2 applies with $N = F$. □

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