# Spectral Geometry of Conformally Covariant Operators

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December 3, 2012.

A thesis submitted to the Faculty of Graduate Studies and Research
in partial fullfilment of the requirements of the degree of Doctor of Philosophy.

*A mis padres con amor.*

# Abstract

In this manuscript we discuss several aspects of conformally covariant operators from a spectral theory point of view. We consider operators acting on sections of a bundle or on scalar functions. For operators that are elliptic and formally self-adjoint we obtain results on the continuity and multiplicity of their eigenvalues. In particular, we prove that the eigenvalues of the GJMS operators are continuous and simple for a residual set of metrics. We also introduce several new conformal invariants that arise from considering eigenvalues and nodal sets of null-eigenvectors. As an application of our results we address a scalar curvature prescription problem.

# Résumé

Dans cette thèse, nous traiterons de divers aspects concernant les opérateurs covariants conformes, et ce du point de vue de la géométrie spectrale. Nous considérons tout au long de cette thèse des opérateurs agissant sur des sections d'un fibré ou encore sur des fonctions scalaires. Dans le cas d'opérateurs elliptiques et formellement auto-adjoints, nous obtenons des résultats sur la continuité et sur la multiplicité de leurs valeurs propres. En particulier, nous prouvons que les valeurs propres des opérateurs de type GJMS sont continues et simples pour un ensemble résiduel de métriques. Nous introduisons également plusieurs nouveaux invariants conformes qui surgissent lorsque nous considérons des valeurs propres et des domaines nodaux de certaines fonctions propres. Nos résultats ont notamment comme application de permettre la résolution d'un certain problème de courbure scalaire prescrite.

# Contents

Abstract . . . . . i
Résumé . . . . . iii
Acknowledgements . . . . . vii
Contributions of the author . . . . . ix

**1 Introduction** **1**
1.A. Convention . . . . . 3
1.B. Statement of the results . . . . . 3

**2 Differential operators on vector bundles** **5**
2.A. Basic concepts . . . . . 5
2.B. Spectral Theory . . . . . 8

**3 Conformally Covariant Operators (CCO)** **11**
3.A. Examples . . . . . 12

**4 Local continuity of eigenvalues** **15**

**5 Multiplicity of eigenvalues of CCO** **21**
5.A. Background on perturbation theory . . . . . 24
5.B. Perturbation theory of CCO . . . . . 26
5.C. Multiplicity for conformal deformations . . . . . 29
5.D. Multiplicity for general deformations . . . . . 32

**6 Conformal invariants from CCO** **35**
6.A. Conformal invariants from eigenvalues . . . . . . . . . . . . . . . 35
6.B. Conformal invariants from Ker$P_g$ . . . . . . . . . . . . . . . . . . . 37

**7 Conformal Laplacian** **39**
7.A. Conformal invariants . . . . . . . . . . . . . . . . . . . . . . . . . . 39
7.B. Scalar curvature prescription problems . . . . . . . . . . . . . . . 41

**Bibliography** **45**

# Acknowledgements

First, I would like to thank my supervisor Dmitry Jakobson. To say that he is wonderful would be an understatement. When I first got to Montreal I mentioned to him that I had no clue of what it meant to work and live as a mathematician. He replied “Don’t worry, I’ll make sure you do by the time you are ready to leave McGill”. And he did. During these past four years Dima took every chance to help me learn from his daily life experience. He involved me in many of his own projects, took me to every conference and introduced me to several people. He spent so much time working with me! Most importantly, he gave me complete freedom to choose what I wanted to do. Dima always cared about me, and made sure I was motivated and happy. I owe Dima so much. I’m infinitely grateful to him.

I would also like to thank John Toth, he played a very important role in who I turned out to be after these PhD years. John taught me a great deal, always being sweet and patient, always challenging me to be better. I not only learned math from him, he also taught me how to explain my ideas and how to express them correctly in english (he still has to teach me a few more lessons on this). Working with John is a great pleasure.

Niky Kamran also deserves a huge “thank you” for he has always been there for me. I think of Niky as my role model. He inspires me a great admiration.

I am thankful to my family in Montreal, Ana, Claudia, Coni, Janine, Layan, Nico B, Nico F, Nelson, and Renato. These last four years with them have been amazing. I thank them for their support and for making my life be so much fun! I also want to thank Guilleaume P, Guilleaume RF, Jan, Layan and Patrick. Being their math sister has been everything but dull, I am very lucky to have them.

Finally, I would like to thank my life companion, Nico. Being with him makes everything make sense.

# Contributions of the author

The main results of this thesis are presented in Chapter 4 and Chapter 5. The content of these chapters is based on the paper *"On the multiplicity of eigenvalues of conformally covariant operators"* written by the author [17].

The results presented in Chapter 6 and Chapter 7 are based on the preprint *"Conformal invariants from nodal sets.I. Negative eigenvalues and curvature prescriptions"*, written by Rod Gover, Dmitry Jakobson, Raphael Ponge and the author [18]. All the results presented in Chapter 6 are a generalization, to the setting of all conformally covariant operators, of some of the results in [18]. Most of the results in Chapter 6 and Chapter 7 were obtained by D. Jakobson and the author.

During my PhD I have also been fortunate to work in the following projects:

- Y. Canzani, D. Jakobson and I. Wigman. *Scalar curvature and Q-curvature of random metrics.*
  To appear in JGEA, Arxiv: 1002.0030.

- Y. Canzani, D. Jakobson and J. Toth. *On the distribution of perturbations of Scrödinger eigenfunctions.*
  Submitted for publication, Arxiv: 1210.4499.

- Y. Canzani and J.Toth. *Counting intersections of nodal sets of Laplace eigenfunctions with real analytic curves on surfaces.*
  In Preparation.

CHAPTER 1

# Introduction

Conformally covariant operators (see Definition 3.1) are known to play a key role in Physics and Spectral Theory. In the past few years, much work has been done on their systematic construction, understanding, and classification [3, 8, 6, 7, 21, 26, 27, 36, 42]. In Physics, the interest for conformally covariant operators started when Bateman [4] discovered that the classical field equations describing massless particles (like Maxwell and Dirac equations) depend only on the conformal structure. These operators are also important tools in String Theory and Quantum Gravity where they are used to relate scattering matrices on conformally compact Einstein manifolds with conformal objects on their boundaries at infinity [28]. In Spectral Geometry, the purpose is to relate global geometry of a manifold to the spectrum of some natural operators. For example, the nice behavior of conformally covariant operators with respect to conformal deformations of a metric yields a closed expression for the conformal variation of the determinants leading to important progress in the lines of [8, 9, 15].

Not many statements can be proved simultaneously for all conformally covariant operators, even if self-adjointness and ellipticity are enforced. Some of these operators act on functions, others act on bundles. For some of them the maximum principle is satisfied, whereas for others, is not. Some of them are bounded below while others are not. We therefore emphasize that most of the techniques in this manuscript work for the whole class of conformally covariant operators.

When it comes to perturbing a metric to deal with any of the problems described above, it is often much simpler to work under the assumption that the eigenvalues of a given operator are a smooth, or even continuous, function of a metric perturbation parameter. But reality is much more complicated, and usually, when possible, one has to find indirect ways of arriving to the desired results without such assumption. However, it is generally believed that eigenvalues of formally self-adjoint operators with positive leading symbol are generically simple. And, as Branson and Ørsted point out in [14, pag 22], since many of the quantities of interest are universal expressions, the generic case is often all that one needs. In many cases, it has been shown that the eigenvalues of *metric dependent*, formally self-adjoint, elliptic operators are generically simple. In 1976, Uhlembeck showed that several families of *second order*, self-adjoint, elliptic operators have generically simple eigenvalues [40]. The main example is the *Laplace* operator on smooth functions on a compact manifold, see [40, 39, 2, 5]. The simplicity of eigenvalues has also been shown, generically, for the *Hodge-Laplace* operator on forms on a compact manifold of dimension 3 (see [22]). Besides, in 2002, Dahl proved such result for the Dirac operator on spinors of a compact spin manifold of dimension 3; see [20]. It seems to be the case that in the class of conformally covariant operators the latter is the only situation for which the simplicity of the eigenvalues has generically been established. The purpose of this thesis is to shed some light in this direction.

Conformally covariant operators also provide the opportunity to generate conformal invariants, and hence better understand the space of Riemannian metrics over a given manifold. In this manuscript we study conformal invariants arising from nodal sets and negative eigenvalues of GJMS operators (probably the most well known ones among the class of conformally covariant operators). We prove that the number of negative eigenvalues is a conformal invariant. We also show that nodal sets of null-eigenfunctions are conformal invariants too. To our knowledge this is the first time that nodal sets have been considered generally in the setting of conformal geometry. We then study in detail all our invariants for the Conformal Laplacian. In addition, we give some applications of our invariants to curvature prescription problems.

## 1.A. Convention

Unless stated otherwise, throughout the manuscript we work under the following assumptions.

- $(M, g)$ is a compact connected Riemannian manifold of dimension $n$.
- $E_g$ is a smooth bundle over $M$ with product on the fibers $(\ ,\ )_x$. We write $\Gamma(E_g)$ for the space of smooth sections of the bundle and denote by $\langle\ ,\ \rangle_g$ the global inner product $\langle u, v\rangle_g = \int_M (u(x), v(x))_x \mathrm{dvol}_g$, for $u, v \in \Gamma(E_g)$.
- $P_g : \Gamma(E_g) \to \Gamma(E_g)$ *is an elliptic, formally self-adjoint, conformally covariant operator of order* $m$.
- The space $\mathcal{M}$ of Riemannian metrics over $M$ is endowed with the $C^\infty$-topology.

## 1.B. Statement of the results

In **Chapter 2** we introduce basic definitions and tools in Spectral Geometry that we shall use in the rest of the manuscript.

In **Chapter 3** we introduce the class of conformally covariant operators and provide multiple examples.

We dedicate **Chapter 4** to discuss continuity results of the eigenvalues of $P_g$ in the $C^\infty$-topology of metrics (Theorem 4.1). In particular, we prove that if $P_g$ is strongly elliptic, then all its eigenvalues are continuous functions in the metric parameter (Corollary 4.2).

Among the main results of **Chapter 5** are:

– Suppose $P_g : \Gamma(E_g) \to \Gamma(E_g)$ *has no rigid eigenspaces* (see Definition 5.2). Then the set of functions $f \in C^\infty(M, \mathbb{R})$ for which $P_{e^f g}$ has only simple non-zero eigenvalues is a residual set in $C^\infty(M, \mathbb{R})$ (Theorem 5.4).

- If $P_g : \Gamma(E_g) \to \Gamma(E_g)$ has no rigid eigenspaces for a dense set of metrics, then all non-zero eigenvalues are simple for a residual set of metrics in $\mathcal{M}$ (Corollary 5.5).

- Suppose $P_g : \Gamma(E_g) \to \Gamma(E_g)$ satisfies the unique continuation principle for a dense set of metrics in $\mathcal{M}$. Then the multiplicity of all non-zero eigenvalues is smaller than the rank of the bundle for a residual set of metrics in $\mathcal{M}$ (Corollary 5.8).

- As an application, if $P_g$ acts on $C^\infty(M)$ (e.g. GJMS operators), then its non-zero eigenvalues are simple for a residual set of metrics in $\mathcal{M}$ (Proposition 5.16 combined with Corollary 5.5).

In **Chapter 6** we look at conformal invariants that arise from eigenvalues and nodal sets of null-eigenfunctions of $P_g$. Among the main results are

- The number of negative eigenvalues is a conformal invariant (Theorem 6.2).

- The sign of the first eigenvalue is a conformal invariant (Theorem 6.3).

- Nodal sets and nodal domains of any null-eigenfunction of a conformally covariant operator are conformal invariants (Proposition 6.4).

The purpose of **Chapter 7** is to extend and apply the results in Chapter 6 to the Conformal Laplacian $P_{1,g} = \Delta_g + \frac{n-2}{4(n-1)} R_g$, where $\Delta_g = \delta_g d$ and $R_g$ is the scalar curvature.

- If $P_{1,g}$ has $m$ negative eigenvalues, then every null-eigenfunction has at most $m+1$ nodal domains (Theorem 7.1).

- The number of negative eigenvalues of $P_{1,g}$ can become arbitrarily large as the conformal class varies (Theorem 7.2).

- Let $u$ be any null-eigenfunction of $P_{1,g}$ and consider any nodal domain $\Omega$ of $u$. If $R_{\hat{g}}$ is the scalar curvature of some metric $\hat{g}$ in the conformal class of $g$, then $R_{\hat{g}}$ must be negative somewhere on $\Omega$ (Corollary 7.8).

CHAPTER 2

# Differential operators on vector bundles

## 2.A. Basic concepts

We start this chapter introducing the notion of a vector bundle. A vector bundle of rank $k$ can be thought of as a family of vector spaces isomorphic to $\mathbb{C}^k$ (or $\mathbb{R}^k$) parametrized by a given manifold in the following sense:

**Definition 2.1. (Vector bundle)** Let $E$ and $M$ be differentiable manifolds and $\pi : E \to M$ a differentiable map. $(E, \pi, M)$ is said to be a *complex* (resp., *real) differential vector bundle* of *rank* $k$ if

1. For all $x \in M$ the *fiber* $E_x := \pi^{-1}(x)$ is a $k$-dimensional $\mathbb{C}$- vector space (resp., $\mathbb{R}$- vector space).

2. For all $x \in M$ there exists an open neighborhood $U$ of $x$ and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times \mathbb{C}^k$ $(resp., U \times \mathbb{R}^k)$ so that for all $y \in U$

$$\varphi|_{E_y} : E_y \to \{y\} \times \mathbb{C}^k \quad (resp., \{y\} \times \mathbb{R}^k)$$

   is an isomorphism.

$E$ is called the *total space*, $M$ the *base space*, $\pi : E \to M$ the *projection map*, and $(\varphi, U)$ as above, is called a *bundle chart*.

A vector bundle is locally a product of base and fibers. When this statement can be made global, that is, when $E$ is isomorphic to $M \times \mathbb{C}^k$ (resp., $M \times \mathbb{R}^k$), the bundle is said to be *trivial*. A simple example of a trivial vector bundle is $E = M \times \mathbb{C}$.

**Definition 2.2. (Bundle homomorphism)** Fix two bundles $(E_1, \pi_1, M)$ and $(E_2, \pi_2, M)$ over $M$. A differentiable map $\kappa : E_1 \to E_2$ is said to be a *bundle homomorphism* if $\pi_2 \circ \kappa = \pi_1$ and the fiber maps $\kappa|_{E_{1x}} : E_{1x} \to E_{2x}$ are homomorphisms for all $x \in M$.

**Definition 2.3. (Section of a bundle)** Let $(E, \pi, M)$ be a vector bundle. A differentiable *section* of $E$ is a differentiable map $u : M \to E$ with $\pi \circ u = id_M$. The space of smooth sections is denoted by $\Gamma(E)$.

For the trivial vector bundle $E = M \times \mathbb{C}$, one has $\Gamma(E) \cong C^\infty(M)$.

Given a manifold $M$ with local coordinates $(x_1, \ldots, x_n)$, and given $\alpha \in \mathbb{Z}_+^n$, we write $\frac{\partial^\alpha}{\partial x_\alpha} := \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}}$ for $|\alpha| = \sum_{i=1}^n \alpha_i$.

**Definition 2.4. (Differential operator)** Let $M$ be a compact manifold and let $E, F$ be differentiable complex (resp. real) vector bundles over $M$. Let $rank(E) = \ell$ and $rank(F) = k$. A *differential operator of order* $m$ on $M$ is a linear map $P : \Gamma(E) \to \Gamma(F)$ that can be written in the form

$$P = \sum_{|\alpha| \le m} A_\alpha \frac{\partial^\alpha}{\partial x_\alpha}$$

where the matrix valued functions $A_\alpha : M \to \mathcal{M}_{k\times\ell}(\mathbb{C})$ (resp., $\mathcal{M}_{k\times\ell}(\mathbb{R})$) are smooth, and $A_\alpha \neq 0$ for some $\alpha$ with $|\alpha| = m$. Here, $\mathcal{M}_{k\times\ell}(\mathbb{C})$ denotes the complex valued $k \times \ell$ matrices.

Let $(x_1, \ldots, x_n)$ be local coordinates on $M$, $x \in M$ and $\xi \in T_x^*M$. Write $\xi = \sum_{i=1}^n \xi_i \, dx_i|_x$, and for $\alpha \in \mathbb{Z}_+^n$ write $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_n^{\alpha_n}$.

**Definition 2.5. (Elliptic operator)** Let $P : \Gamma(E) \to \Gamma(F)$ be a differential operator of order $m$ on $M$. For each $x \in M$ and $\xi \in T_x^*M$ consider

$$\sigma_P(\xi) : E_x \to F_x, \qquad \sigma_P(\xi) = i^m \sum_{|\alpha|=m} \xi^\alpha A_\alpha(x).$$

We say that $P$ is *elliptic* whenever for all $x \in M$ and each non-zero cotangent vector $\xi \in T_x^*M$ the map $\sigma_P(\xi)$ is invertible.

**Definition 2.6. (Bundle metric)** Let $(E, \pi, M)$ be a complex (resp. real) vector bundle. A *bundle metric* is given by a family of hermitian (resp. scalar) inner products $( \, , \, )_x$ on the fibers $E_x$, depending smoothly on $x \in M$.
If $(M, g)$ is a compact Riemannian manifold and $(E, \pi, M)$ is a vector bundle equipped with a bundle metric, then, the smooth sections $\Gamma(E)$ inherit an inner product

$$\langle u, v\rangle_g := \int_M (u(x), v(x))_x \, \mathrm{dvol}_g \qquad u, v \in \Gamma(E).$$

In this case we often write $\Gamma(E_g)$ instead of $\Gamma(E)$. The completion of $\Gamma(E_g)$ with respect to $\langle , \rangle_g$ is a Hilbert space denoted $\Gamma_{L^2}(E_g)$.

**Definition 2.7. (Formally self-adjoint operator)** A differential operator $P : \Gamma(E_g) \to \Gamma(E_g)$ is said to be *formally self-adjoint* provided

$$\langle Pu, v\rangle_g = \langle u, Pv\rangle_g \qquad \forall u, v \in \Gamma(E_g).$$

**Definition 2.8. (Strongly elliptic operator)** Let $P : \Gamma(E_g) \to \Gamma(E_g)$ be a differential operator of order $m$ on $M$. $P$ is said to be *strongly elliptic* provided there exists $\varepsilon > 0$ such that

$$\mathrm{Re}\,(\sigma_P(\xi)\eta, \eta)_x \geq \varepsilon\, |\xi|^m_{g(x)} (\eta, \eta)_x$$

for all $x \in M$, $\xi \in T_x^*M$, $\eta \in (E_g)_x$.

If $P$ is strongly elliptic and formally self-adjoint, then there exists $\beta \in \mathbb{R}$ for which

$$\mathrm{Re}\,\langle Pu, u\rangle_g \geq \beta \|u\|_g \quad \text{for all} \quad u \in \Gamma(E_g).$$

In particular, the eigenvalues of $P$ are real and bounded below.

**Definition 2.9. ($C^m$- topology on the space of metrics)** Given a compact manifold $M$, write $\mathcal{M}$ for the space of Riemannian metrics over $M$. Fix a background metric $g \in \mathcal{M}$, and define the distance $d_g^m$ between two metrics $g_1$, $g_2 \in \mathcal{M}$ by

$$d_g^m(g_1, g_2) := \max_{k=0,\ldots,m} \|\nabla_g^k (g_1 - g_2)\|_\infty.$$

The topology induced on $\mathcal{M}$ by $d_g^m$ is independent of the background metric and it is called the $C^m$-topology of metrics on $M$.

**Definition 2.10. (Continuity of metric dependent coefficients)** Let $M$ be a compact manifold and consider a map $P$ that to every Riemannian metric $g$ over $M$ associates a differential operator $P_g : \Gamma(E_g) \to \Gamma(E_g)$ of order $m$. We say that the coefficients of $P$ are *continuous in the $C^m$-topology of metrics* if for every metric $g_0$ there is an open neighborhood $\mathcal{W}$ of $g_0$ in the $C^m$-topology of metrics, so that for every metric $g \in \mathcal{W}$ there is an isomorphism of vector bundles $\tau_g : E_g \to E_{g_0}$ with the property that the coefficients of the differential operator

$$\tau_g \circ P_g \circ \tau_g^{-1} : \Gamma(E_{g_0}) \to \Gamma(E_{g_0})$$

depend continuously on $g$.

## 2.B. Spectral Theory
### (of formally self-adjoint, elliptic operators)

We start this section by recalling the definition of the Schwartz space. Through this section we assume all functions take values in $\mathbb{C}^k$, but every statement can be restated in terms of $\mathbb{R}^k$. The *Schwartz space* is the space of smooth functions

$$\mathcal{S} = \left\{ u \in C^\infty(\mathbb{R}^n) : \ \forall \alpha, \ell, \ \exists\, C_{\alpha,\ell} \text{ so that } \left| \frac{\partial^\alpha}{\partial x^\alpha} u(x) \right| \le \frac{C_{\alpha,\ell}}{(1+|x|)^\ell} \quad \forall x \in \mathbb{R}^n \right\}$$

If $u \in \mathcal{S}$, its *Fourier transform* is the function $\hat{u} \in \mathcal{S}$ given by

$$\hat{u}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} u(x) dx.$$

If $s \in \mathbb{R}$ and $u \in \mathcal{S}$, the *Sobolev s-norm* of $u$ is defined by

$$\|u\|_s^2 := \int_{\mathbb{R}^n} (1 + |\xi|)^{2s} \, |\hat{u}(\xi)|^2 d\xi.$$

Let $(M, g)$ be a compact Riemannian manifold and let $E_g$ be a vector bundle over $M$ equipped with a bundle metric. Since $M$ is compact, there exist closed coordinate sets $U_j$, $j = 1, \dots, N$ with coordinate charts $y_j : U_j \to \{y : \|y\| \leq 1\} \subset \mathbb{R}^n$ satisfying that $M = \cup_{j=1}^N B_j$ for $B_j := y_j^{-1}(\{y : \|y\|^2 < \frac{1}{2}\})$. For each $j = 1, \dots, N$ set $x_j := \frac{y_j}{\sqrt{1-\|y_j\|^2}}$ and observe that $x_j(\text{int } U_j) = \mathbb{R}^n$ and $x_j(B_j) = \{x : \|x\| < 1\}$. Fix $\{\chi_j\}_{j=1}^N$ a smooth partition of unity subordinate to the covering $\{B_j\}_{j=1}^N$.

Over each set $U_j$ choose a bundle chart $(\varphi_j, U_j)$ possessing a smooth extension to an open neighborhood of $U_j$. Given a smooth section $u \in \Gamma(E_g)$, since $\varphi_j \circ \chi_j u\,|_{U_j} : U_j \to U_j \times \mathbb{C}^k$ is the identity in the first coordinate, we identify it with the function $\varphi_j \circ \chi_j u : U_j \to \mathbb{C}^k$. We further consider the function

$$\tilde{u}_j = \varphi_j \circ \chi_j u \circ x_j^{-1} : \mathbb{R}^n \to \mathbb{C}^k.$$

Observe that by our construction $\tilde{u}_j$ is supported on $\{x \in \mathbb{R}^n : \|x\| < 1\}$, so the function $\left|\frac{\partial^\alpha}{\partial x^\alpha}\tilde{u}_j(x)\right|(1+|x|)^\ell$ is bounded for all $\alpha$ and $\ell$, giving $\tilde{u}_j \in \mathcal{S}$. We define the *Sobolev s-norm* on $\Gamma(E_g)$ by

$$\|u\|_s := \sum_{j=1}^N \|\tilde{u}_j\|_s.$$

The completion of $\Gamma(E_g)$ in this norm is the *Sobolev Space* $\Gamma_{L^2_s}(E_g)$. Note that we have been writing $\Gamma_{L^2}(E_g)$ for the Hilbert space $\Gamma_{L^2_0}(E_g)$.

The following are well known results that we shall use in our arguments. For a reference see Chapter III of the book by Lawson and Michelsohn [33].

**Theorem 2.11. (Sobolev embedding Theorem)** *Let $s, s' \in \mathbb{R}$ and $k \in \mathbb{N}$.*

1. *The inclusion $\Gamma_{L^2_{s'}}(E_g) \subset \Gamma_{L^2_s}(E_g)$ is compact for $s' > s$.*
2. *There is a continuous embedding $\Gamma_{L^2_s}(E_g) \subset \Gamma_{C^k}(E_g)$ for all $s > \frac{n}{2} + k$.*

**Proposition 2.12. (Bounded elliptic extensions)** *Suppose $(M, g)$ is a compact Riemannian manifold. Let $P : \Gamma(E_g) \to \Gamma(E_g)$ be a differential operator of order $m$.*

1. *$P$ extends to a bounded linear operator $P : \Gamma_{L^2_s}(E_g) \to \Gamma_{L^2_{s-m}}(E_g)$ for all $s \in \mathbb{R}$. If $P$ is elliptic, then the extension is a Fredholm map (its kernel and cokernel are finite dimensional, and its range is closed).*

2. *If $P$ is elliptic, then the norms $\|\cdot\|_s$ and $\|\cdot\|_{s-m}+\|P(\cdot)\|_{s-m}$ are equivalent for all $s$. In particular, there exists $c_0 > 0$ such that*

$$\|u\|_m \leq c_0(\|u\|_0 + \|Pu\|_0) \qquad \forall u \in \Gamma_{L^2_m}(E_g).$$

**Theorem 2.13. (Spectral decomposition)** *Let $(M, g)$ be a compact Riemannian manifold. Suppose $P : \Gamma(E_g) \to \Gamma(E_g)$ is a formally self-adjoint elliptic differential operator of order $m$. Then,*

- *The spectrum of $P$, $\mathrm{Spec}(P)$, consists of eigenvalues that are discrete and real.*

- *The multiplicities of the eigenvalues are finite, i.e., $\dim\ker(P-\lambda I) < \infty$ for all $\lambda \in \mathrm{Spec}(P)$.*

- *The eigensections are smooth, i.e., $\ker(P - \lambda I) \subset \Gamma(E_g)$ for all $\lambda \in \mathrm{Spec}(P)$.*

- *The Hilbert space $\Gamma_{L^2}(E_g)$ can be decomposed as*

$$\Gamma_{L^2}(E_g) = \bigoplus_{\lambda\in\mathrm{Spec}(P)} \ker(P - \lambda I).$$

CHAPTER 3

# Conformally Covariant Operators (CCO)

As explained in the Introduction, conformally covariant operators play an important role in Physics and Spectral Geometry. In this Chapter we state their definition and give multiple examples.

Let $g$ be a Riemannian metric over $M$ and $P_g : C^\infty(M) \to C^\infty(M)$ a (metric dependent) differential operator of order $m$. $P_g$ is said to be a *conformally covariant operator of bidegree* $(a, b)$ -for $a, b \in \mathbb{R}$- if for any conformal perturbation of the reference metric, $g \to e^f g$ with $f \in C^\infty(M, \mathbb{R})$, the operators $P_{e^f g}$ and $P_g$ are related by the formula

$$P_{e^f g} = e^{-\frac{bf}{2}} \circ P_g \circ e^{\frac{af}{2}}.$$

When considering operators acting on vector bundles the definition becomes more involved. Let $M$ be a compact manifold and $E_g$ a vector bundle over $M$ equipped with a bundle metric.

**Definition 3.1. (Conformally covariant operator)** Let $a, b \in \mathbb{R}$. A *conformally covariant operator* $P$ *of order* $m$ *and bidegree* $(a, b)$ is a map that to every Riemannian metric $g$ over $M$ associates a differential operator $P_g : \Gamma(E_g) \to \Gamma(E_g)$ of order $m$, in such a way that

1. For any two conformally related metrics, $g$ and $e^f g$ with $f \in C^\infty(M, \mathbb{R})$, there exists a bundle isomorphism

$$\kappa : E_{e^f g} \to E_g$$

that preserves length fiberwise and for which

$$P_{e^f g} = \kappa^{-1} \circ e^{-\frac{bf}{2}} \circ P_g \circ e^{\frac{af}{2}} \circ \kappa.$$

2. The coefficients of $P_g$ depend continuously on $g$ in the $C^\infty$-topology of metrics (see Definition 2.10).

For a more general formulation see [1, pag. 4]. It is well known that for all these operators one always has $a \neq b$.

## 3.A. Examples

We proceed to introduce some examples of operators to which our results can be applied; see [1, pag 5], [13, pag 253], and [41, pag 285] for more.

**Conformal Laplacian.** On surfaces, the Laplace operator $\Delta_g$ is conformally covariant and has bidegree $(0, 2)$. In higher dimensions its generalization is the second order, elliptic operator, named Conformal Laplacian:

$$P_{1,g} = \Delta_g + \frac{n-2}{4(n-1)} R_g \tag{3.1}$$

Here $\Delta_g = \delta_g d$ and $R_g$ is the scalar curvature. $P_{1,g}$ is a conformally covariant operator of bidegree $\left(\frac{n-2}{2}, \frac{n+2}{2}\right)$.

**Paneitz Operator.** On compact 4 dimensional manifolds, Paneitz discovered the 4th order, elliptic operator

$$P_{2,g} = \Delta_g^2 + \delta_g \left( \frac{2}{3} R_g \, g - 2Ric_g \right) d$$

acting on $C^\infty(M)$. Here $Ric_g$ is the Ricci tensor of the metric $g$ and both $Ric_g$ and $g$ are acting as $(1,1)$ tensors on 1-forms. $P_{2,g}$ is a formally self-adjoint, conformally covariant operator of bidegree $(0,4)$. See [36].

**GJMS Operators.** On compact manifolds of dimension $n$ Graham-Jenne-Mason-Sparling constructed formally self-adjoint, elliptic, operators $P_{k,g}$ that act on $C^\infty(M)$. For $k \leq \frac{n}{2}$ when $n$ is even, and for all non-negative integers $k$ when $n$ is odd, there is a conformally invariant operator $P_{k,g}$ of biweight $\left(\frac{n}{2}-k, \frac{n}{2}+k\right)$ such that

$$P_{k,g} = \Delta_g^k + \text{ lower order terms.}$$

$P_{k,g}$ is a conformally covariant operator of order $2k$ that generalizes the Conformal Laplacian, $P_{1,g}$, and the Paneitz operator, $P_{2,g}$, to higher even orders. See [27].

**Dirac Operator.** Let $(M,g)$ be a compact Riemannian spin manifold of dimension $n$, with spinor bundle $E_g$. The Dirac operator acts on $\Gamma(E_g)$, and is formally self-adjoint, conformally covariant, and elliptic. Its order is 1 and it has bidegree $\left(\frac{n-1}{2}, \frac{n+1}{2}\right)$. See [25, pag. 9].

**Rarita-Schwinger Operator.** In the setting of the previous example, let $T_g$ denote the twistor bundle. The Rarita-Schwinger operator acts on $\Gamma(T_g)$ and has order 1. It is elliptic, formally self-adjoint, conformally covariant, and has bidegree $\left(\frac{n-1}{2}, \frac{n+1}{2}\right)$. See [11].

**Conformally Covariant Operators on forms.** In 1982 Branson introduced a general second order conformally covariant operator $D_{(2,k),g}$ on differential forms of arbitrary order $k$. It has leading order term $(n-2k+2)\delta_g d + (n-2k-2)d\delta_g$ for $n \neq 1,2$ being the dimension of the manifold. Later, Branson generalized it to a fourth order operator $D_{(4,k),g}$ with leading order term $(n-2k+4)(\delta_g d)^2 + (n-2k-4)(d\delta_g)^2$ for $n \neq 1,2,4$. Both $D_{(2,k),g}$ and $D_{(4,k),g}$ are formally self-adjoint, conformally covariant operators and their leading symbols are positive provided $k < \frac{n-2}{2}$ and $k < \frac{n-4}{2}$ respectively. On functions, $D_{(2,0),g} = P_{1,g}$ and $D_{(4,k),g} = P_{2,g}$. See [12, pag 276], [13, pag 253]. For recent results and higher order generalizations see [10] for example.

CHAPTER 4

# Local continuity of eigenvalues

It is often simplifying to work under an assumption that the eigenvalues of a given operator are continuous functions of a metric parameter. Most of the time this is not the case and usually one has to find indirect ways of arriving to the desired results without such assumption. However, in this chapter we prove that the eigenvalues of conformally covariant operators, that are strongly elliptic and formally self-adjoint, depend continuously in the metric parameter. *Throughout this chapter we work under the assumptions described in Section 1.A..*

For $c \in \mathbb{R}$, consider the set

$$\mathcal{M}_c := \{g \in \mathcal{M} : \ c \notin \mathrm{Spec}(P_g)\} .$$

For $g \in \mathcal{M}_c$, let

$$\mu_1(g) \leq \mu_2(g) \leq \mu_3(g) \leq \dots$$

be all the eigenvalues of $P_g$ in $(c, +\infty)$ counted with multiplicity. Note that it may happen that there are only finitely many $\mu_i(g)$'s. We prove

**Theorem 4.1.** *The set $\mathcal{M}_c$ is open and the maps $\mu_i : \mathcal{M}_c \to \mathbb{R}$ are continuous in the $C^\infty$-topology of metrics.*

If $P_g$ is strongly elliptic, its spectrum is bounded below. It can be shown (see Lemma 4.5) that for a fixed metric $g_0$ there exists $c \in \mathbb{R}$ and a neighborhood $\mathcal{V}$ of $g_0$ so that $\mathrm{Spec}(P_g) \subset (c, +\infty)$ for all $g \in \mathcal{V}$ . An immediate consequence is

**Corollary 4.2.** *If $P_g : \Gamma(E_g) \to \Gamma(E_g)$ is strongly elliptic, then all its eigenvalues are continuous for $g \in \mathcal{M}$ in the $C^\infty$-topology.*

The arguments we present in this section are an adaptation of the proof of Theorem 2 in [32] by Kodaira and Spencer; they prove similar results to Theorem 4.1 for strongly elliptic operators that have coefficients that depend continuously on a parameter $t \in \mathbb{R}^n$ in the $C^\infty$-topology.

From now on fix a Riemannian metric $g_0$. By the continuity of the coefficients of $P_g$ (see Definition 2.10), there exists $\mathcal{W}_{g_0}$ neighborhood of $g_0$ in the $C^\infty$-topology of metrics, so that for every metric $g \in \mathcal{W}_{g_0}$ there is an isomorphism of vector bundles $\tau_g : E_g \to E_{g_0}$ with the property that the coefficients of the differential operator

$$Q_g := \tau_g \circ P_g \circ \tau_g^{-1} : \Gamma(E_{g_0}) \to \Gamma(E_{g_0}) \tag{4.1}$$

depend continuously on $g \in \mathcal{W}_{g_0}$.
Since $P_g$ is elliptic and formally self-adjoint, its spectrum $\mathrm{Spec}(P_g)$ is real and discrete. Note that the spectrum of $P_g$ and $Q_g$ coincide. Indeed, $u$ is an eigensection of $P_g$ with eigenvalue $\lambda$ if and only if $\tau_g u$ is an eigensection of $Q_g$ with eigenvalue $\lambda$. Fix $\xi \in \mathbb{C}$ and define

$$Q_g(\xi) := Q_g - \xi : \Gamma(E_{g_0}) \to \Gamma(E_{g_0}).$$

We repeatedly use that $Q_g(\xi)$ admits a bounded extension $Q_g(\xi) : \Gamma_{L^2_m}(E_{g_0}) \to \Gamma_{L^2_0}(E_{g_0})$, see Proposition 2.12. In particular, because of the continuity of the coefficients of $Q_g(\xi)$ as functions of $g$ and $\xi$, one obtains that $\|Q_g(\xi)u\|_0$ depends continuously on $g$ and $\xi$ for all $u \in \Gamma_{L^2_m}(E_{g_0})$.

It is well known that $Q_g(\xi)$ is surjective provided $\xi$ belongs to the resolvent set of $Q_g$ (i.e. $\xi \notin spec(P_g)$). For $\xi_0$ in the resolvent set of $P_{g_0}$, set

$$b_{g_0} := \inf_{\lambda \in spec(P_{g_0})} |\lambda - \xi_0|.$$

We then know

$$\|Q_{g_0}(\xi_0)u\|_0 \geq b_{g_0}\|u\|_0 \qquad \forall u \in \Gamma_{L^2_m}(E_{g_0}).$$

**Lemma 4.3.** *There exists $\delta > 0$ and $\mathcal{V} \subset \mathcal{W}_{g_0}$ neighborhood of $g_0$ so that the resolvent operator $R_g(\xi) := Q_g(\xi)^{-1}$ exists for $g \in \mathcal{V}$ and $|\xi - \xi_0| < \delta$. In addition, for every $u \in \Gamma(E_{g_0})$ the section $R_g(\xi)u$ depends continuously on $\xi$ and $g$ in the $\|\cdot\|_0$ norm.*

*Proof.* We first prove the injectivity of $Q_g(\xi)$. It follows from Proposition 2.12 and the continuity of the coefficients of $Q_g(\xi)$ -in both $g$ and $\xi$- that there exists an open neighborhood $\mathcal{W}$ of $g_0$, $\delta_0 > 0$ and $c_0 > 0$, such that for all $g \in \mathcal{W}$ and $|\xi - \xi_0| < \delta_0$,

$$\|u\|_m \leq c_0(\|Q_g(\xi)u\|_0 + \|u\|_0) \qquad \forall u \in \Gamma_{L^2_m}(E_{g_0}). \tag{4.2}$$

To prove the injectivity statement it suffices to show that for all $\varepsilon > 0$ there exists $\delta > 0$ and $\mathcal{V} \subset \mathcal{W}$ such that for all $u \in \Gamma(E_{g_0})$

$$\|Q_g(\xi)u\|_0 \geq (b_{g_0} - \varepsilon)\|u\|_0,$$

for $g \in \mathcal{V}$ and $|\xi - \xi_0| < \delta$. We proceed by contradiction. Suppose there exists $\varepsilon > 0$ together with a sequence $\{(\delta_i, \mathcal{V}_i, u_i, \xi_i)\}_i$, with $\delta_i \xrightarrow{i} 0$, $\mathcal{V}_i$ shrinking around $g_0$, $u_i \in \Gamma(E_{g_0})$, and $|\xi_i - \xi_0| < \delta_i$, such that

$$\|Q_{g_i}(\xi_i)u_i\|_0 < (b_{g_0} - \varepsilon)\|u_i\|_0$$

for $g_i \in \mathcal{V}_i$ and $|\xi_i - \xi_0| < \delta_i$. Without loss of generality assume $\|u_i\|_0 = 1$. From (4.2) we know $\|u_i\|_m \leq c_0(b_{g_0} - \varepsilon + 1)$ for all $i = 1, 2, \ldots$, and by the continuity in $g$ of the coefficients of $Q_g$, it follows that $\|(Q_{g_i}(\xi_i) - Q_{g_0}(\xi_0))u_i\|_0 \to 0$.

On the other hand,

$$\|(Q_{g_i}(\xi_i) - Q_{g_0}(\xi_0))u_i\|_0 \geq \|Q_{g_0}(\xi_0)u_i\|_0 - \|Q_{g_i}(\xi_i)u_i\|_0 \geq b_{g_0} - (b_{g_0} - \varepsilon) = \varepsilon.$$

We obtain the desired contradiction.

To prove the continuity statement notice that

$$\begin{aligned} b_{g_0}\|R_g(\xi)u - R_{g_0}(\xi_0)u\|_0 &\leq \|Q_g(\xi)R_g(\xi)u - Q_g(\xi)R_{g_0}(\xi_0)u\|_0 \\ &= \|u - Q_g(\xi)R_{g_0}(\xi_0)u\|_0 \\ &= \|Q_{g_0}(\xi_0)R_{g_0}(\xi_0)u - Q_g(\xi)R_{g_0}(\xi_0)u\|_0 \\ &\leq \Big(\| \left(Q_{g_0} - Q_g\right) \left(R_{g_0}(\xi_0)u\right)\|_0 + |\xi - \xi_0| \|(R_{g_0}(\xi_0)u)\|_0\Big), \end{aligned}$$

and use the continuity in $g$ of the coefficients of $Q_g$. □

Let $g_0 \in \mathcal{M}$ and continue to write $\mathcal{W}_{g_0}$ for the neighborhood of $g_0$ for which the vector bundle isomorphism $\tau_g : E_g \to E_{g_0}$ is defined for all $g \in \mathcal{W}_{g_0}$. Let $C$ be a differentiable curve with interior domain $D \subset \mathbb{C}$. For $g \in \mathcal{W}_{g_0}$, write $\mathbf{F}_g(C)$ for the linear subspace

$$\mathbf{F}_g(C) := span\left\{\tau_g u : \quad u \in \ker(P_g - \lambda I) \text{ for } \lambda \in D \cap \operatorname{Spec}(P_g)\right\} \subset \Gamma(E_{g_0}).$$

Note that

$$\dim \mathbf{F}_g(C) = \sum_{\lambda \in D \cap \operatorname{Spec}(P_g)} \dim \ker(P_g - \lambda I). \tag{4.3}$$

**Proposition 4.4.** *If $C$ meets none of the eigenvalues of $P_{g_0}$, then there exists a neighborhood $\mathcal{V} \subset \mathcal{W}_{g_0}$ of $g_0$ so that for all $g \in \mathcal{V}$*

$$\dim \mathbf{F}_g(C) = \dim \mathbf{F}_{g_0}(C). \tag{4.4}$$

*Proof.* We divide the proof in three steps.

*Step 1.* For $g \in \mathcal{W}_{g_0}$, define the spectral projection operator $F_g(C) : \Gamma(E_{g_0}) \to \Gamma(E_{g_0})$ to be the projection of $\Gamma(E_{g_0})$ onto $\mathbf{F}_g(C)$. Since $C$ meets none of the eigenvalues of $P_{g_0}$, by Lemma 4.3 there exist a neighborhood $C'$ of the curve $C$ and a neighborhood $\mathcal{V}' \subset \mathcal{W}_{g_0}$ of $g_0$ so that none of the eigenvalues of $P_g$ belong to $C'$ for $g \in \mathcal{V}'$. By holomorphic functional calculus

$$F_g(C)\, u = -\frac{1}{2\pi i}\int_C R_g(\xi)\, u \, d\xi \qquad u \in \Gamma(E_g).$$

By Lemma 4.3 it follows that $F_g(C)\, u$ depends continuously on $g \in \mathcal{V}'$.

*Step 2.* Let $d = \dim \mathbf{F}_{g_0}(C)$ and $u_{\lambda_1(g_0)}, \ldots, u_{\lambda_d(g_0)}$ be the eigenfunctions of $P_{g_0}$ spanning $\mathbf{F}_{g_0}(C)$ with respective eigenvalues $\lambda_1(g_0) \le \cdots \le \lambda_d(g_0)$. Since $F_g(C)u$ depends continuously on $g \in \mathcal{V}'$, for all $u \in \Gamma(E_{g_0})$ we know that

$$\lim_{g \to g_0} \left\| F_g(C)\, [u_{\lambda_j(g_0)}] - u_{\lambda_j(g_0)} \right\|_0 = 0, \quad \text{for } j = 1, \ldots, d,$$

and therefore there exists $\mathcal{V} \subset \mathcal{V}'$ neighborhood of $g_0$ so that

$$F_g(C)\, [u_{\lambda_1(g_0)}], \ldots, F_g(C)\, [u_{\lambda_d(g_0)}]$$

are linearly independent for $g \in \mathcal{V}$. We thereby conclude,

$$\dim \mathbf{F}_g(C) \ge \dim \mathbf{F}_{g_0}(C) \qquad \text{for } \; g \in \mathcal{V}. \tag{4.5}$$

*Step 3.* By possibly shrinking $\mathcal{V}$, the continuity of $F_g(C)$ on $g$ gives that $Id - F_g(C)$ is invertible on $(\ker(Id - F_{g_0}(C)))^c$ for all $g \in \mathcal{V}$. Therefore

$$\dim \mathbf{F}_g(C) = \dim \ker(Id - F_g(C)) \leq \dim \ker(Id - F_{g_0}(C)) = \dim \mathbf{F}_{g_0}(C).$$

Thereby, for $g \in \mathcal{V}$, equality (4.4) follows from the previous inequality and (4.5).

□

## Proof of Theorem 4.1

For $c \in \mathbb{R}$, we continue to write $\mathcal{M}_c = \{g \in \mathcal{M} : \ c \notin \text{Spec}(P_g)\}$. Also, for $g \in \mathcal{M}_c$, we write

$$\mu_1(g) \leq \mu_2(g) \leq \mu_3(g) \leq \dots$$

for the eigenvalues of $P_g$ in $(c, +\infty)$ counted with multiplicity. We recall that it may happen that there are only finitely many $\mu_j(g)$'s.

To see that $\mathcal{M}_c$ is open, fix $g_0 \in \mathcal{M}_c$. Let $\delta > 0$ be so that the circle $C_\delta$ centered at $c$ of radius $\delta$ contains no eigenvalue of $P_{g_0}$. By Proposition 4.4 there exists $\mathcal{V}_1 \subset \mathcal{W}_{g_0}$, a neighborhood of $g_0$, so that for all $g \in \mathcal{V}_1$

$$\dim \mathbf{F}_g(C_\delta) = \dim \mathbf{F}_{g_0}(C_\delta) = 0.$$

It follows that $\mathcal{V}_1 \subset \mathcal{M}_c$.

We proceed to show the continuity of the maps

$$\mu_i : \mathcal{M}_c \to \mathbb{R}, \qquad\qquad g \mapsto \mu_i(g).$$

We first show the continuity of $g \mapsto \mu_1(g)$ at $g_0 \in \mathcal{M}_c$. Fix $\varepsilon_0 > 0$ and consider $0 < \varepsilon < \varepsilon_0$ so that the circle $C_\varepsilon$ centered at $\mu_1(g_0)$ of radius $\varepsilon$ contains only the eigenvalue $\mu_1(g_0)$ among all the eigenvalues of $P_{g_0}$. Let $\delta > 0$ be so that there is no eigenvalue of $P_{g_0}$ in $[c-\delta, c]$. Consider a differentiable curve $C$ that intersects transversally the $x$-axis only at the points $c - \delta$ and $\mu_1(g_0) - \varepsilon/2$. By Proposition 4.4 there exists $\mathcal{V}_2 \subset \mathcal{W}_{g_0}$, a neighborhood of $g_0$, so that for all $g \in \mathcal{V}_2$

$$\dim \mathbf{F}_g(C) = \dim \mathbf{F}_{g_0}(C) \quad \text{and} \quad \dim \mathbf{F}_g(C_\varepsilon) = \dim \mathbf{F}_{g_0}(C_\varepsilon).$$

Since $\dim \mathbf{F}_{g_0}(C) = 0$, it follows that no perturbation $\mu_i(g)$, $i \geq 1$, belongs to $[c, \mu_1(g_0) - \varepsilon/2]$. Also, since the dimension of $\mathbf{F}_g(C_\varepsilon)$ is preserved, it follows that there exists $j$ so that $|\mu_j(g) - \mu_1(g_0)| < \varepsilon$ for $j \neq 1$. Since

$$\mu_1(g_0) - \varepsilon < \mu_1(g) \leq \mu_j(g) \leq \mu_1(g_0) + \varepsilon,$$

it follows that for $g \in \mathcal{V}_2$ we have $|\mu_1(g) - \mu_1(g_0)| < \varepsilon$ as wanted.

The continuity of $g \mapsto \mu_i(g)$, for $i \geq 2$, follows by induction. One should consider a circle of radius $\epsilon$ centered at $\mu_i(g_0)$ and a circle $C$ that intersects transversally the $x$-axis only at the points $c - \delta$ and $\mu_i(g_0) - \varepsilon/2$.

□

As described at the beginning of this chapter, the following lemma is all one needs to establish Corollary 4.2.

**Lemma 4.5.** *Suppose $P_{g_0} : \Gamma(E_{g_0}) \to \Gamma(E_{g_0})$ is strongly elliptic. Then, there exists $c \in \mathbb{R}$ and an open neighborhood $\mathcal{V}$ of $g_0$ such that for all $g \in \mathcal{V}$*

$$\mathrm{Spec}(P_g) \subset (c, +\infty)\,.$$

*Proof.* Since $P_{g_0}$ is strongly elliptic, we can choose $\beta \in \mathbb{R}$ so that $P_{g_0} > \beta$. Then, for $u \in \Gamma(E_{g_0})$ normalized,

$$\begin{aligned} Re\langle Q_g u, u\rangle_{g_0} &= Re\langle P_{g_0} u, u\rangle_{g_0} + Re\langle (Q_g - P_{g_0})u, u\rangle_{g_0} \\ &\geq \beta - \|(Q_g - P_{g_0})u\|_0. \end{aligned}$$

Since the coefficients of $Q_g$ are continuous in the $C^\infty$-topology, given $\varepsilon > 0$ there exists an open neighborhood $\mathcal{V}$ of $g_0$ such that $\|(Q_g - P_{g_0})u\|_0 \leq \varepsilon$ for all $g \in \mathcal{V}$.

Let $u$ be so that $Q_g u = \lambda u$ with $\lambda < 0$ and $\|u\|_0 = 1$. Then $Re\langle Q_g u, u\rangle_g \geq \beta - \varepsilon$, and so $\mathrm{Spec}(Q_g) \subset (\beta - \varepsilon, +\infty)$. Since $P_g$ and $Q_g$ have the same spectrum the result follows. □

CHAPTER 5

# Multiplicity of eigenvalues of CCO

It is believed that eigenvalues of formally self-adjoint, elliptic operators are generically simple. One of the main contributions to this conjecture was done by Uhlembeck [40] in 1976; she showed that for several interesting families of *second order*, self-adjoint, elliptic operators (for example the Laplacian) the spectrum is generically simple. In 2002, Dahl proved such result for the Dirac operator on spinors of a compact spin manifold of dimension 3; see [20]. It seems to be the case that in the class of conformally covariant operators the latter is the only situation for which the simplicity of the eigenvalues has generically been established. In this chapter we study this problem for conformally covariant operators.

There are many ways of splitting the spectrum of an operator. The main ideas presented in this chapter are inspired by the constructive methods of Bleecker and Wilson [5]. *Throughout this chapter we work under the assumptions described in Section 1.A., unless stated otherwise.*

Given a topological space $X$, a subset $A \subset X$ is said to be **residual** if it is the intersection of countably many open sets. On a Fréchet space, a residual set is the complement of a meager set. In particular, on Fréchet spaces residual sets are dense. We take this opportunity to mention that both $C^\infty(M,\mathbb{R})$ and the space of Riemannian metrics $\mathcal{M}$ are Fréchet spaces when endowed with the $C^\infty$-topology. On the other hand, the space $\mathcal{M}$ endowed with the $C^m$-topology is not complete for any $m \in \mathbb{N}$. In what follows the main results of this chapter are stated.

**Theorem 5.1.** *Suppose $P_g$ acts on smooth functions, $P_g : C^\infty(M) \to C^\infty(M)$. Then the set of functions $f \in C^\infty(M, \mathbb{R})$ for which all the non-zero eigenvalues of $P_{e^f g}$ are simple is a residual set in $C^\infty(M, \mathbb{R})$.*

To obtain a generalization of Theorem 5.1 for operators acting on bundles we introduce the following definition.

**Definition 5.2. (Rigid eigenspace)** An eigenspace of $P_g : \Gamma(E_g) \to \Gamma(E_g)$ is said to be a *rigid eigenspace* if it has dimension greater or equal than two, and for any two eigensections $u, v$ with $\|u\|_g = \|v\|_g = 1$ then

$$\|u(x)\|_x = \|v(x)\|_x \qquad \forall x \in M.$$

By the polarization identity this condition is equivalent to the existence of a function $c_g$ on $M$ so that for all $u, v$ in the eigenspace

$$(u(x), v(x))_x = c_g(x)\langle u, v\rangle_g \qquad \forall x \in M.$$

**Remark 5.3.** In Proposition 5.16 we show that operators acting on $C^\infty(M)$ (e.g. GJMS operatos) have no rigid eigenspaces.

We establish the following generalizations to operators acting on bundles.

**Theorem 5.4.** *If $P_g : \Gamma(E_g) \to \Gamma(E_g)$ has* no *rigid eigenspaces, the set of functions $f \in C^\infty(M, \mathbb{R})$ for which all the non-zero eigenvalues of $P_{e^f g}$ are simple is a residual set in $C^\infty(M, \mathbb{R})$.*

As a consequence of Theorem 5.4 (or Theorem 5.1) we obtain the following result.

**Corollary 5.5.** *Suppose $P_g : \Gamma(E_g) \to \Gamma(E_g)$ has no rigid eigenspaces for a dense set of metrics in $\mathcal{M}$, or that it acts on $C^\infty(M)$. Then, the set of metrics*

*$g \in \mathcal{M}$ for which all non-zero eigenvalues of $P_g$ are simple is a residual subset of $\mathcal{M}$.*

Of course, one would like to get rid of the "non rigidity" assumption. Probably, this assumption cannot be dropped if we restrict ourselves to conformal deformations only. However, it is likely that a generic set of deformations should break the rigidity condition. We thereby make the following conjecture.

**Conjecture 5.6.** *$P_g$ has no rigid eigenspaces for $g$ in a dense subset of $\mathcal{M}$.*

If we remove the "non rigidity" condition and require the operator to satisfy the unique continuation principle we obtain

**Theorem 5.7.** *If $P_g : \Gamma(E_g) \to \Gamma(E_g)$ satisfies the unique continuation principle, the set of functions $f \in C^\infty(M, \mathbb{R})$ for which all the non-zero eigenvalues of $P_{e^f g}$ have multiplicity smaller than rank$(E_g)$ is a residual set in $C^\infty(M, \mathbb{R})$.*

In particular for line bundles the unique continuation principle gives simplicity of eigenvalues, for a generic set of conformal deformations.

**Corollary 5.8.** *Suppose $P_g : \Gamma(E_g) \to \Gamma(E_g)$ satisfies the unique continuation principle for a dense set of metrics in $\mathcal{M}$. Then, the set of metrics $g \in \mathcal{M}$ for which all non-zero eigenvalues of $P_g$ have multiplicity smaller than the rank of the bundle is a residual subset of $\mathcal{M}$.*

**Observation**. All the results stated above hold for *non-zero* eigenvalues. Given a non-zero eigenvalue of multiplicity greater than 1, we use conformal transformations of the reference metric to reduce its multiplicity. This cannot be done for zero eigenvalues since their multiplicity, $\dim \ker(P_g)$, is a conformal invariant (see Remark 6.1).

**Acknowledgement**. One of the main results of this chapter, Theorem 5.4, is a generalization to the whole class of conformally covariant operators of the results presented by Dahl in [20], for the Dirac operator on 3-manifolds. In an earlier version of this manuscript, part of the argument in Dahl's paper was being reproduced (namely, the first two lines of the proof of Proposition 3.2 in [20]). In June 2012, Raphaël Ponge visited McGill University and started working with the author on an extension of the results in this paper to the

class of pseudodifferential operators. While doing so, R. Ponge realized there was a mistake in Dahl's argument which was reproduced in the earlier version of this manuscript. The author is grateful to Raphaël Ponge for pointing out the mistake.

## 5.A. Background on perturbation theory

In this section we introduce the definitions and tools we need to prove our main results. We follow the presentation in Rellich's book [38], and a proof for every result stated can be found there.

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\, , \rangle$ and $\mathcal{U}$ a dense subspace of $\mathcal{H}$. A linear operator $A$ on $\mathcal{U}$ is said to be *formally self-adjoint*, if it satisfies $\langle Au, v \rangle = \langle u, Av \rangle$ for all $u, v \in \mathcal{U}$. A formally self-adjoint operator $A$ is said to be *essentially self-adjoint* if the images of $A+i$ and $A-i$ are dense in $\mathcal{H}$; if these images are all of $\mathcal{H}$ we say that $A$ is *self-adjoint*.

If $A$ is a linear operator on $\mathcal{U}$, its *closure* is the operator $\bar{A}$ defined on $\overline{\mathcal{U}}$ as follows: $\overline{\mathcal{U}}$ is the set of elements $u \in \mathcal{H}$ for which there exists a sequence $\{u_n\} \subset \mathcal{U}$ with $\lim_n u_n = u$ and $Au_n$ converges. Then $\bar{A}u := \lim_n Au_n$. We note that if $A$ is formally self-adjoint, so is $\bar{A}$.

A family of linear operators $A(\varepsilon)$ on $\mathcal{U}$ indexed by $\varepsilon \in \mathbb{R}$ is said to be *regular* in a neighborhood of $\varepsilon = 0$ if there exists a bounded bijective operator $U : \mathcal{H} \to \mathcal{U}$ so that for all $v \in \mathcal{H}$, $A(\varepsilon)[U(v)]$ is a regular element, in the sense that it is a power series convergent in a neighborhood of $\varepsilon = 0$. Finding the operator $U$ is usually very difficult. Under certain conditions on the operators, proving regularity turns out to be much simpler. To this end, we introduce the following criterion.

**Criterion 5.9.** *([38, page 78]) Suppose that $A(\varepsilon)$ on $\mathcal{U}$ is a family of formally self-adjoint operators in a neighborhood of $\varepsilon = 0$. Suppose that $A^{(0)} = A(0)$ is essentially self-adjoint, and there exist formally self-adjoint operators $A^{(1)}, A^{(2)}, \dots$ on $\mathcal{U}$ such that for all $u \in \mathcal{U}$*

$$A(\varepsilon)u = A^{(0)}u + \varepsilon A^{(1)}u + \varepsilon^2 A^{(2)}u + \dots$$

*Assume in addition that there exists* $a \geq 0$ *so that*

$$\|A^{(k)}u\| \leq a^k \|A^{(0)}u\|, \quad \textit{for all } k = 1, 2, \ldots$$

*Then, on* $\mathcal{U}$*,* $A(\varepsilon)$ *is essentially self-adjoint and* $\overline{A(\varepsilon)}$ *on* $\overline{\mathcal{U}}$ *is regular in a neighborhood of* $\varepsilon = 0$*.*

For the purpose of splitting non-zero eigenvalues, the next proposition plays a key role.

**Proposition 5.10.** *([38, page 74]) Suppose that* $B(\varepsilon)$ *on* $\mathcal{U}$ *is a family of regular, formally self-adjoint operators in a neighborhood of* $\varepsilon = 0$*. Suppose that* $B^{(0)} = B(0)$ *is self-adjoint. Suppose that* $\lambda$ *is an eigenvalue of finite multiplicity* $\ell$ *of the operator* $B(0)$*, and suppose there are positive numbers* $d_1, d_2$ *such that the spectrum of* $B(0)$ *in* $(\lambda - d_1, \lambda + d_2)$ *consists exactly of the eigenvalue* $\lambda$*.*
*Then, there exist power series* $\lambda_1(\varepsilon), \ldots, \lambda_\ell(\varepsilon)$ *convergent in a neighborhood of* $\varepsilon = 0$ *and power series* $u_1(\varepsilon), \ldots, u_\ell(\varepsilon)$*, satisfying*

1. $u_i(\varepsilon)$ *converges for small* $\varepsilon$ *in the sense that the partial sums converge in* $\mathcal{H}$ *to an element in* $\mathcal{U}$*, for* $i = 1 \ldots \ell$*.*
2. $B(\varepsilon)u_i(\varepsilon) = \lambda_i(\varepsilon)u_i(\varepsilon)$ *and* $\lambda_i(0) = \lambda$*, for* $i = 1, \ldots, \ell$*.*
3. $\langle u_i(\varepsilon), u_j(\varepsilon)\rangle = \delta_{ij}$*, for* $i, j = 1, \ldots, \ell$*.*
4. *For each pair of positive numbers* $d'_1, d'_2$ *with* $d'_1 < d_1$ *and* $d'_2 < d_2$*, there exists a positive number* $\delta$ *such that the spectrum of* $B(\varepsilon)$ *in* $[\lambda - d'_1, \lambda + d'_2]$ *consists exactly of the points* $\lambda_1(\varepsilon), \ldots, \lambda_\ell(\varepsilon)$*, for* $|\varepsilon| < \delta$*.*

We note that since $B(\varepsilon)u_i(\varepsilon) = \lambda_i(\varepsilon)u_i(\varepsilon)$, differentiating with respect to $\varepsilon$ both sides of the equality we obtain

$$\begin{aligned}\langle B^{(1)}(\varepsilon)u_i(\varepsilon), u_j(\varepsilon)\rangle + \langle u'_i(\varepsilon), &B(\varepsilon)u_j(\varepsilon)\rangle = \\ &= \langle \lambda'_i(\varepsilon)u_i(\varepsilon), u_j(\varepsilon)\rangle + \langle u'_i(\varepsilon), \lambda_i(\varepsilon)u_j(\varepsilon)\rangle.\end{aligned}$$

When $i = j$ the above equality translates to

$$\lambda'_i(\varepsilon) = \langle B^{(1)}(\varepsilon)u_i(\varepsilon), u_i(\varepsilon)\rangle. \tag{5.1}$$

Also, evaluating at $\varepsilon = 0$ we get

$$\lambda'_i(0) = \langle B^{(1)}(0)u_i(0), u_i(0). \tag{5.2}$$

## 5.B. Perturbation theory of CCO

Consider a conformal change of the reference metric $g \to e^{\varepsilon f} g$ for $f \in C^\infty(M)$ and $\varepsilon \in \mathbb{R}$. Since $P_g : \Gamma(E_g) \to \Gamma(E_g)$ is a *conformally covariant operator* of bidegree $(a,b)$, there exists $\kappa : E_{e^{\varepsilon f} g} \to E_g$, a bundle isomorphism that preserves the length fiberwise, so that

$$P_{e^{\varepsilon f} g} = \kappa^{-1} \circ e^{-\frac{b\varepsilon f}{2}} \circ P_g \circ e^{\frac{a\varepsilon f}{2}} \circ \kappa. \tag{5.3}$$

We work with a modified version of $P_{e^{\varepsilon f} g}$. For $c := \frac{a+b}{4}$ set

$$\eta := c - \frac{b}{2} = \frac{a}{2} - c$$

and define

$$A_f(\varepsilon) : \Gamma(E_g) \to \Gamma(E_g), \qquad A_f(\varepsilon) := e^{\eta\varepsilon f} \circ P_g \circ e^{\eta\varepsilon f}.$$

The advantage of working with these operators is that, unlike $P_{e^{\varepsilon f} g}$, they are formally self-adjoint with respect to $\langle\, , \rangle_g$. Note that $\eta \neq 0$ for $a \neq b$, and observe that

$$\begin{aligned} A_f(\varepsilon) &= e^{\eta\varepsilon f} \circ P_g \circ e^{\eta\varepsilon f} \\ &= e^{c\varepsilon f} e^{-\frac{b\varepsilon f}{2}} \circ P_g \circ e^{\frac{a\varepsilon f}{2}} e^{-c\varepsilon f} \\ &= \kappa \circ e^{c\varepsilon f} \circ P_{e^{\varepsilon f} g} \circ e^{-c\varepsilon f} \circ \kappa^{-1}. \end{aligned}$$

**Remark 5.11.** $A_f(\varepsilon)$ and $P_{e^{\varepsilon f} g}$ have the same eigenvalues. Indeed, $u(\varepsilon)$ is an eigensection of $P_{e^{\varepsilon f} g}$ with eigenvalue $\lambda(\varepsilon)$ if and only if $\kappa(e^{c\varepsilon f} u(\varepsilon))$ is an eigensection for $A_f(\varepsilon)$ with the same eigenvalue.

$A_f(\varepsilon)$ is a deformation of $P_g = A_f(0)$ that has the same spectrum as $P_{e^{\varepsilon f} g}$ and is formally self-adjoint with respect to $\langle , \rangle_g$. Note also that $A_f(\varepsilon)$ is elliptic so there exists a basis of $\Gamma_{L^2}(E_g)$ of eigensections of $A_f(\varepsilon)$.

**Lemma 5.12.** *The operators* $A_f^{(k)}(\varepsilon) := \frac{1}{k!}\frac{d^k}{d\varepsilon^k} A_f(\varepsilon)$ *are formally self-adjoint and*

$$\left\| A_f^{(k)}(\varepsilon) u \right\|_g \leq \frac{(2\,|\eta|\,\|f\|_\infty)^k}{k!} \left\| A_f(\varepsilon) u \right\|_g \qquad \forall u \in \Gamma(E_g). \tag{5.4}$$

*Proof.* Since $A_f(\varepsilon)$ is formally self-adjoint, so is $A_f^{(k)}(\varepsilon)$. Indeed, for $u, v \in \Gamma(E_g)$, $\langle A_f(\varepsilon)u, v\rangle_g - \langle u, A_f(\varepsilon)v\rangle_g = 0$. Hence,

$$0 = \frac{d^k}{d\varepsilon^k}(\langle A_f(\varepsilon)u, v\rangle_g - \langle u, A_f(\varepsilon)v\rangle_g)\big|_{\varepsilon=0} = k!(\langle A_f^{(k)} u, v\rangle_g - \langle u, A_f^{(k)} v\rangle_g).$$

For the norm bound, observe that

$$\frac{d^k}{d\varepsilon^k}\Big[A_f(\varepsilon)(u)\Big] = \eta^k \sum_{l=0}^{k} \binom{k}{l} f^{k-l} A_f(\varepsilon)(f^l u), \tag{5.5}$$

and notice that from the fact that $A_f(\varepsilon)$ is formally self-adjoint it also follows that $\|A_f(\varepsilon)(hu)\|_g \leq \|h\|_\infty \|A_f(\varepsilon)u\|_g$ for all $h \in C^\infty(M)$. □

In the following Proposition we show how to split the multiple eigenvalues of $P_g$. From now on we write $A_f^{(k)} := A_f^{(k)}(0)$.

**Proposition 5.13.** *Suppose $\lambda$ is a non-zero eigenvalue of $P_g$. Write $V_\lambda$ for the corresponding eigenspace of eigenvalue $\lambda$ and $\Pi$ the orthogonal projection onto it. With the notation of Proposition 5.10, if $\Pi \circ A_f^{(1)}|_{V_\lambda}$ is not a multiple of the identity, there exists $\varepsilon_0 > 0$ and a pair $(i, j)$ for which $\lambda_i(\varepsilon) \neq \lambda_j(\varepsilon)$ for all $0 < \varepsilon < \varepsilon_0$.*

*Proof.* Assume the results of Proposition 5.10 are true for $B(\varepsilon) = \overline{A_f(\varepsilon)}$, and note that since there is a basis of $\Gamma_{L^2}(E_g)$ of eigensections of $A_f(\varepsilon)$, the eigensections of $\overline{A_f(\varepsilon)}$ and $A_f(\varepsilon)$ coincide. By relation (5.2), $\lambda_1'(0), \dots, \lambda_\ell'(0)$ are the eigenvalues of $\Pi \circ A_f^{(1)}|_{V_\lambda}$. Since $\Pi \circ A_f^{(1)}|_{V_\lambda}$ is not a multiple of the identity, there exist $i, j$ with $\lambda_i'(0) \neq \lambda_j'(0)$ and this implies that $\lambda_i(\varepsilon) \neq \lambda_j(\varepsilon)$ for small $\varepsilon$, which by Remark 5.11 is the desired result. We therefore proceed to show that all the assumptions in Proposition 5.10 are satisfied for $B(\varepsilon) = \overline{A_f(\varepsilon)}$, $\mathcal{U} = \overline{\Gamma(E_g)}$ and $\mathcal{H} = \Gamma_{L^2}(E_g)$.

$\overline{A_f(0)} = \overline{P_g}$ *is self-adjoint:* This follows from the fact that $P_g$ is essentially self-adjoint, and the closure of an essentially self-adjoint is a self-adjoint operator. To see that $A_f(0) = P_g$ is essentially self-adjoint, note that since there is a basis of $\Gamma_{L^2}(E_g)$ of eigensections of $P_g$, it is enough to show that for any eigensection $\phi$ of eigenvalue $\lambda$ there exist $u, v \in \Gamma(E_g)$ for which $P_g u + iu = \phi$ and $P_g v - iv = \phi$. Thereby, it suffices to set $u = \frac{1}{\lambda+i}\phi$ and $v = \frac{1}{\lambda-i}\phi$.

$\overline{A_f(\varepsilon)}$ *is regular on* $\Gamma(E_g)$*:* From Lemma 5.12 and Criterion 5.9 applied to $A(\varepsilon) = A_f(\varepsilon)$, we obtain that $A_f(\varepsilon)$ is a family of operators on $\Gamma(E_g)$ which are essentially self-adjoint and their closure $\overline{A_f(\varepsilon)}$ on $\overline{\Gamma(E_g)}$ is regular.

□

## Splitting non-zero eigenvalues

Recall from Definition 5.2 that an eigenspace of $P_g$ is said to be a *rigid eigenspace* if it has dimension greater or equal than two, and for any two eigensections $u, v$ with $\|u\|_g = \|v\|_g = 1$ one has

$$\|u(x)\|_x = \|v(x)\|_x \qquad \forall x \in M.$$

Being an operator with *no* rigid eigenspaces is the condition that allows us to split eigenvalues. For this reason, at the end of this section we show that operators acting on $C^\infty(M)$ have no rigid eigenspaces (see Proposition 5.16). Our main tool is the following proposition.

**Proposition 5.14.** *Suppose $P_g$ has no rigid eigenspaces. Let $\lambda$ be a non-zero eigenvalue of $P_g$ of multiplicity $\ell \geq 2$. Then, there exists a function $f \in C^\infty(M, \mathbb{R})$ and $\varepsilon_0 > 0$ so that among the perturbed eigenvalues $\lambda_1(\varepsilon), \ldots, \lambda_\ell(\varepsilon)$ of $P_{e^{\varepsilon f} g}$ there exists a pair $(i, j)$ for which $\lambda_i(\varepsilon) \neq \lambda_j(\varepsilon)$ for all $0 < \varepsilon < \varepsilon_0$.*

*Proof.* Since $P_g$ has no rigid eigenspaces, there exist $u, v \in \Gamma(E_g)$ linearly independent normalized eigensections in the $\lambda$-eigenspace so that $\|u(x)\|_x^2 \neq \|v(x)\|_x^2$ for some $x \in M$. For such sections there exists $f \in C^\infty(M, \mathbb{R})$ so that $\langle fu, u\rangle_g \neq \langle fv, v\rangle_g$. To prove our result, by Proposition 5.13 it would suffice to show that

$$\langle A_f^{(1)} u, u\rangle_g \neq \langle A_f^{(1)} v, v\rangle_g.$$

Using that $P_g$ is formally self-adjoint and evaluating equation (5.5) at $\varepsilon = 0$ (for $k$=1) we have

$$\langle A_f^{(1)} u, u\rangle_g = \eta\, \langle f P_g(u) + P_g(fu), u\rangle_g = 2\eta\, \lambda \langle fu, u\rangle_g,$$

and similarly, $\langle A_f^{(1)} v, v\rangle_g = 2\eta\, \lambda \langle fv, v\rangle_g$. The result follows. □

A weaker but more general result is the following

**Proposition 5.15.** *Suppose $P_g : \Gamma(E_g) \to \Gamma(E_g)$ satisfies the unique continuation principle. Let $\lambda$ be a non-zero eigenvalue of $P_g$ of multiplicity $\ell >$ rank$(E_g)$. Then, there exists $\varepsilon_0 > 0$ and a function $f \in C^\infty(M, \mathbb{R})$ so that among the perturbed eigenvalues $\lambda_1(\varepsilon), \ldots, \lambda_\ell(\varepsilon)$ of $P_{e^{\varepsilon f} g}$ there is a pair $(i, j)$ for which $\lambda_i(\varepsilon) \neq \lambda_j(\varepsilon)$ for all $0 < \varepsilon < \varepsilon_0$.*

*Proof.* Let $\{u_1, \dots, u_\ell\}$ be an orthonormal basis of the $\lambda$-eigenspace. If for some $i \neq j$ there exists $x \in M$ for which $\|u_i(x)\|_x \neq \|u_j(x)\|_x$ we proceed as in Proposition 5.14 and find $f \in C^\infty(M, \mathbb{R})$ for which $\langle fu_i, u_i\rangle_g \neq \langle fu_j, u_j\rangle_g$. We show that under our assumptions this is the only possible situation.

Suppose that for any two normalized eigensections $u, v \in \Gamma(E_g)$ of eigenvalue $\lambda$ one has $\|u(x)\|_x^2 = \|v(x)\|_x^2$ for all $x \in M$. Then by the polarization identity (see remark in Definition 5.2) we would obtain $(u_i(x), u_j(x))_x = 0$ for all $i \neq j$ and $x \in M$. By the rank condition, for some $i = 1, \dots, \ell$ the section $u_i$ has to vanish on an open set. Indeed, for each $x \in M$ there exists $i \in \{1, \dots, \ell\}$ such that $u_i(x) = 0$ and so $M = \cup_{i=1}^{\ell}\{x : u_i(x) = 0\}$. It follows that there exists $i \in \{1, \dots, \ell\}$ for which $int(\{x : u_i(x) = 0\})$ is nonempty, and so $u_i$ vanishes on an open set. By the unique continuation principle $u_i$ must vanish everywhere, and this is not possible. □

We finish this section translating the previous results to the setting of smooth functions.

**Proposition 5.16.** *Operators acting on $C^\infty(M)$ have no rigid eigenspaces.*

*Proof.* Let $\tilde{u}, \tilde{v}$ be two linearly independent, orthonormal eigenfunctions of $P_g$ with eigenvalue $\lambda$. Set $D := \{x \in M : \tilde{u}(x) \neq \tilde{v}(x)\}$. If there is $x \in D$ with $\tilde{u}(x) \neq -\tilde{v}(x)$, the functions $u = \tilde{u}$ and $v = \tilde{v}$ break the rigidity condition. If for all $x \in D$ we have $\tilde{u}(x) = -\tilde{v}(x)$, the functions $u = \frac{\tilde{u}+\tilde{v}}{\|\tilde{u}+\tilde{v}\|_g}$ and $v = \frac{\tilde{u}-\tilde{v}}{\|\tilde{u}-\tilde{v}\|_g}$ do the job. Indeed, $v = 0$ on $D^c$ and there exists $x \in D^c$ for which $u(x) \neq 0$ because otherwise $\tilde{u} \equiv -\tilde{v}$ and this contradicts the independence. □

## 5.C. Multiplicity for conformal deformations

In this section we address the proofs of Theorems 5.1, 5.4 and 5.7.

### Proof of Theorems 5.1 and 5.4.

Given $\alpha \in \mathbb{N}$, and $g \in \mathcal{M}$ consider the set

$$F_{g,\alpha} := \Big\{ f \in C^\infty(M, \mathbb{R}) : \lambda \text{ is } \textit{simple} \quad \text{for all } \lambda \in \mathrm{Spec}(P_{e^f g}) \cap \big([-\alpha, 0) \cup (0, \alpha]\big) \Big\}.$$

The set of functions $f \in C^\infty(M,\mathbb{R})$ for which all the non-zero eigenvalues of $P_{e^f g}$ are simple coincides with the set $\bigcap_{\alpha\in\mathbb{N}} F_{g,\alpha}$. To show that the latter is a residual subset of $C^\infty(M,\mathbb{R})$, we prove that the sets $F_{g,\alpha}$ are open and dense in $C^\infty(M,\mathbb{R})$. Here $C^\infty(M,\mathbb{R})$ is endowed -as usual- with the $C^\infty$-topology.

**Remark 5.17.** Since the eigenvalues of $P_g$ are continuous on $g$ in the $C^m$-topology of metrics, the sets $F_{g,\alpha}$ are open in the $C^m$-topology on $C^\infty(M)$.

We note that for conformal metric deformations, the multiplicity of the zero eigenvalue remains fixed. Indeed, according to (5.3), for $u \in \Gamma(E_g)$ and $f \in C^\infty(M,\mathbb{R})$, we know

$$P_g(u) = 0 \qquad \text{if and only if} \qquad P_{e^f g}(\kappa^{-1}(e^{-\frac{af}{2}}u)) = 0.$$

*Throughout this subsection we assume the hypothesis of Theorems 5.1 or Theorem 5.4 hold.*

$F_{g,\alpha}$ **is dense in** $C^\infty(M,\mathbb{R})$**.**

Fix $f_0 \notin F_{g,\alpha}$ and let $W$ be an open neighborhood of $f_0$. Since at least one of the eigenvalues in $[-\alpha,0)\cup(0,\alpha]$ has multiplicity greater than two, we proceed to split it. By Proposition 5.14 (and Proposition 5.16 when the operator acts on $C^\infty(M)$) there exists $f_1 \in C^\infty(M,\mathbb{R})$ for which at least two of the eigenvalues of $P_{e^{\varepsilon_1 f_1}(e^{f_0}g)}$ in $[-\alpha,0)\cup(0,\alpha]$ are different as long as $\varepsilon_1$ is small enough. Moreover, those eigenvalues that were simple would remain being simple for such $\varepsilon_1$. Also, for $\varepsilon_1$ small enough, we can assume that none of the eigenvalues that originally belonged to $[-\alpha,\alpha]^c$ have perturbations belonging to $[-\alpha,\alpha]$. Let $\varepsilon_1$ be small as before and so that $\varepsilon_1 f_1 + f_0$ belongs to $W$. If $\varepsilon_1 f_1 + f_0$ belongs to $F_{g,\alpha}$ as well, we are done. If not, in finitely many steps, the repetition of this construction will lead us to a function $\varepsilon_N f_N + \cdots + \varepsilon_1 f_1 + f_0$ in $W \cap F_{g,\alpha}$. Hence, $F_{g,\alpha}$ is dense.

$F_{g,\alpha}$ **is open in** $C^\infty(M,\mathbb{R})$**.**

Fix $f_0 \in F_{g,\alpha}$. In order to show that $F_{g,\alpha}$ is open we need to establish how rapidly the eigenvalues of $A_f(\varepsilon)$ grow with $\varepsilon$. From now on we restrict ourselves to perturbations of the form $e^{\varepsilon f}(e^{f_0}g)$ for $f \in C^\infty(M,\mathbb{R})$ with $\|f\|_\infty < 1$. Let $u(\varepsilon)$ be an eigensection of $A_f(\varepsilon)$ with eigenvalue $\lambda(\varepsilon)$. Equation (5.1) gives

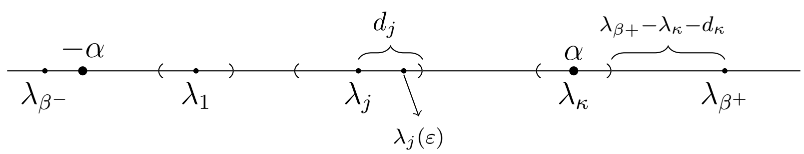
$|\lambda'(\varepsilon)| \leq \|A_f^{(1)}(\varepsilon)u(\varepsilon)\|_g$ for $j = 1, \ldots, \alpha$. Putting this together with inequality (5.4) for $k = 1$ we get

$$|\lambda'(\varepsilon)| \leq 2|\eta| \, \|A_f(\varepsilon)u(\varepsilon)\|_g = 2|\eta| \, |\lambda(\varepsilon)|.$$

The solution of the differential inequality leads to the following bound for the growth of the perturbed eigenvalues:

$$|\lambda(\varepsilon) - \lambda| \leq |\lambda| \left(e^{2|\eta| \, |\varepsilon|} - 1\right), \quad |\varepsilon| < \delta.$$

Write $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_\kappa$ for all the eigenvalues (repeated according to multiplicity) of $P_{e^{f_0}g}$ that belong to $[-\alpha, 0) \cup (0, \alpha]$. Let $d_1, \ldots, d_\kappa$ be so that the intervals $[\lambda_j - d_j, \lambda_j + d_j]$ for $j = 1, \ldots, \kappa$, are disjoint. Write $\lambda_{\beta^-}$ for the biggest eigenvalue in $(-\infty, -\alpha)$ and $\lambda_{\beta^+}$ for the smallest eigenvalue in $(\alpha, +\infty)$. We further assume that $\lambda_{\beta^-} \notin [\lambda_1 - d_1, \lambda_1 + d_1]$ and $\lambda_{\beta^+} \notin [\lambda_\kappa - d_\kappa, \lambda_\kappa + d_\kappa]$.




In order to ensure that for each $j = 1, \ldots, \alpha$ the perturbed eigenvalue $\lambda_j(\varepsilon)$ belongs to $[\lambda_j - d_j, \lambda_j + d_j]$, select $0 < \delta_1 \leq \delta$, so that whenever $|\varepsilon| < \delta_1$ we have that $|\lambda_j(\varepsilon) - \lambda_j| \leq |\lambda_j| \left(e^{2|\eta| \, |\varepsilon|} - 1\right) \leq d_j$ for all $j = 1, \ldots, \kappa$.
To finish our argument, we need to make sure that none of the perturbations of the eigenvalues that initially belonged to $(-\infty, -\alpha) \cup (\alpha, +\infty)$ coincide with the perturbations corresponding to $\lambda_1, \ldots, \lambda_\kappa$. To such end, it is enough to choose $0 < \delta_2 \leq \delta$ so that for $|\varepsilon| < \delta_2$,

$$|\lambda_{\beta^+}| \left(e^{2|\eta| \, |\varepsilon|} - 1\right) < \min\{\lambda_{\beta^+} - \lambda_\kappa - d_\kappa, \, \lambda_{\beta^+} - \alpha\},$$

and

$$|\lambda_{\beta^-}| \left(e^{2|\eta| \, |\varepsilon|} - 1\right) < \min\{\lambda_1 - d_1 - \lambda_{\beta^-}, \, -\alpha - \lambda_{\beta^-}\}.$$

Summing up, if $\|f\|_\infty < 1$ and $|\varepsilon| < \min\{\delta_1, \delta_2\}$, then $\varepsilon f + f_0 \in F_{g,\alpha}$. Or in other words, $\{f_0 + f : \|f\|_\infty < \varepsilon\} \subset F_{g,\alpha}$, so $F_{g,\alpha}$ is open.

### Proof of Theorem 5.7.

The set of functions $f \in C^\infty(M,\mathbb{R})$ for which all the eigenvalues of $P_{e^f g}$ have multiplicity smaller than $rank(E_g)$ can be written as $\cap_{\alpha\in\mathbb{N}}\hat{F}_{g,\alpha}$ where

$$\hat{F}_{g,\alpha} := \Big\{ f \in C^\infty(M,\mathbb{R}) : \ \dim\ker(P_{e^f g} - \lambda) \le rank(E_g) \\ \text{for all } \lambda \in \operatorname{Spec}(P_{e^f g}) \cap \big([-\alpha,0)\cup(0,\alpha]\big)\Big\}.$$

$\hat{F}_{g,\alpha}$ is dense in $C^\infty(M,\mathbb{R})$ by the same argument presented in the proof of Theorems 5.1 and 5.4, using Proposition 5.15 to find the $f_i$'s. The proof that $\hat{F}_{g,\alpha}$ is open in $C^\infty(M,\mathbb{R})$ is analogue to the one for $F_{g,\alpha}$.

## 5.D. Multiplicity for general deformations

In this section we give the proofs of Corollaries 5.5 and 5.8.

### Proof of Corollary 5.5

For $\delta \in (0,1)$ and $\alpha \in (0,+\infty)$ with $\delta < \alpha$, consider the sets

$$\mathcal{G}_{\delta,\alpha} := \Big\{ g \in \mathcal{M} : \ \lambda \text{ is } \textit{simple} \text{ for all } \lambda \in \operatorname{Spec}(P_g) \cap \big([-\alpha,-\delta]\cup[\delta,\alpha]\big)\Big\}. \tag{5.6}$$

Assumming the hypothesis of Theorem 5.4 hold, we prove in Proposition 5.18 that the sets $\mathcal{G}_{\delta,\alpha}$ are open and dense in $\mathcal{M}$ with the $C^\infty$-topology. Let $\{\delta_k\}_{k\in\mathbb{N}}$ be a sequence in $(0,1)$ satisfying $\lim_k \delta_k = 0$, and let $\{\alpha_k\}_{k\in\mathbb{N}}$ be a sequence in $(0,+\infty)$ satisfying $\lim_k \alpha_k = +\infty$ and $\delta_k < \alpha_k$ for all $k$. Then

$$\bigcap_{k=1}^{\infty} \mathcal{G}_{\alpha_k,\delta_k}$$

is a residual set in $\mathcal{M}$ that coincides with the set of all Riemannian metrics for which all non-zero eigenvalues are simple. For the proof of Corollary 5.5 to be complete, it only remains to prove the following result.

**Proposition 5.18.** *Suppose that $P_g : \Gamma(E_g) \to \Gamma(E_g)$ has no rigid eigenspaces for a dense set of metrics. Then, the sets $\mathcal{G}_{\delta,\alpha}$ are open and dense in the $C^\infty$-topology.*

*Proof.* We first show that the sets $\mathcal{G}_{\delta,\alpha}$ are open. Consider $g_0 \in \mathcal{G}_{\delta,\alpha}$ and write $\lambda_1(g_0), \dots, \lambda_d(g_0)$ for all the eigenvalues of $P_{g_0}$ in $[-\alpha,-\delta] \cup [\delta,\alpha]$, which by definition of $\mathcal{G}_{\delta,\alpha}$ are simple. Assume further that the eigenvalues are labeled so that

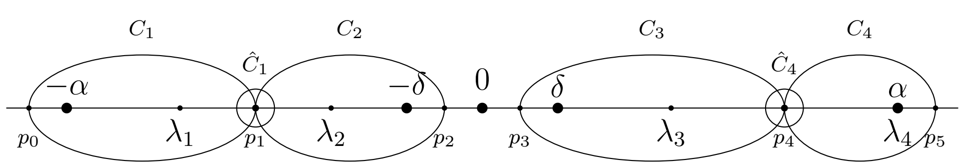
$$-\alpha \le \lambda_1(g_0) < \cdots < \lambda_k(g_0) \le -\delta \quad \text{and} \quad \delta \le \lambda_{k+1}(g_0) < \cdots < \lambda_d(g_0) \le \alpha.$$

Consider $\varepsilon_1 > 0$ small so that no eigenvalue of $P_{g_0}$ belongs to

$$[-\alpha-\varepsilon_1, -\alpha) \cup (-\delta, -\delta+\varepsilon_1] \cup [\delta-\varepsilon_1, \delta) \cup (\alpha, \alpha+\varepsilon_1].$$

For all $1 \le i \le k-1$ let $p_i := \frac{1}{2}(\lambda_i(g_0) + \lambda_{i+1}(g_0))$, and for $k+2 \le i \le d$ let $p_i := \frac{1}{2}(\lambda_{i-1}(g_0)+\lambda_i(g_0))$. We also set $p_0 := -\alpha-\varepsilon_1$, $p_k := \delta+\varepsilon_1$, $p_{k+1} := \delta-\varepsilon_1$ and $p_{d+1} := \alpha+\varepsilon_1$.

For all $1 \le i \le k$ (resp. $k+1 \le i \le d$), let $C_i$ be a differentiable curve that intersects the real axis transversally only at the points $p_{i-1}$ and $p_i$ (resp. $p_i$ and $p_{i+1}$). In addition, let $\varepsilon_2 > 0$ be so that for each $1 \le j \le k-1$ and $k+2 \le j \le d$, the circle $\hat{C}_j$ centered at $p_j$ of radius $\varepsilon_2$ does not contain any eigenvalue of $P_{g_0}$.




By Proposition 4.4 there exists an open neighborhood $\mathcal{V} \subset \mathcal{W}_{g_0}$ of $g_0$ in the $C^\infty$-topology so that for all $g \in \mathcal{V}$ and all $i, j$ for which $C_i$ and $\hat{C}_j$ were defined,

$$\dim \mathbf{F}_g(C_i) = \dim \mathbf{F}_{g_0}(C_i) = 1 \qquad \text{and} \qquad \dim \mathbf{F}_g(\hat{C}_j) = \dim \mathbf{F}_{g_0}(\hat{C}_j) = 0. \tag{5.7}$$

Since $[-\alpha,-\delta] \cup [\delta,\alpha]$ is contained in the union of all $C_i$'s and $\hat{C}_j$'s, it then follows from (4.3) and (5.7) that for all $g \in \mathcal{V}$,

$$\dim \ker(P_g - \lambda I) = 1 \qquad \forall \lambda \in \operatorname{Spec}(P_g) \cap \big([-\alpha,-\delta] \cup [\delta,\alpha]\big).$$

Since $\mathcal{V} \subset \mathcal{G}_{\delta,\alpha}$, it follows that $\mathcal{G}_{\delta,\alpha}$ is open.

We proceed to show that the sets $\mathcal{G}_{\delta,\alpha}$ are dense. Let $g_0 \notin \mathcal{G}_{\delta,\alpha}$ and $\mathcal{O}$ be an open neighborhood of $g_0$. Our assumptions imply that there exists $g \in \mathcal{O}$ so that the hypotheses of Theorem 5.4 are satisfied for $P_g$. It then follows that there exist a function $f \in C^\infty(M)$ so that the metric $e^f g \in \mathcal{O}$ and all non-zero eigenvalues of $P_{e^f g}$ are simple. Therefore, $e^f g \in \mathcal{O} \cap \mathcal{G}_{\delta,\alpha}$.

□

**Remark 5.19.** We note that our proof actually yields that the sets $\mathcal{G}_{\delta,\alpha}$ are dense in the $C^\infty$-topology of metrics in $\mathcal{M}$.

## Proof of Corollary 5.8

For $\delta \in (0,1)$ and $\alpha \in (0,+\infty)$, consider the sets

$$\hat{\mathcal{G}}_{\delta,\alpha} := \Big\{ g \in \mathcal{M} : \quad \dim\ker(P_g - \lambda I) \leq rank(E_g) \\ \text{for all } \; \lambda \in \operatorname{Spec}(P_g) \cap \big([-\alpha,-\delta] \cup [\delta,\alpha]\big) \Big\}$$

Using the same argument in Proposition 5.18 it can be shown that the sets $\hat{\mathcal{G}}_{\delta,\alpha}$ are open. To show that the sets $\hat{\mathcal{G}}_{\delta,\alpha}$ are dense, one carries again the same argument presented in Proposition 5.18, using the hypothesis of Theorem 5.7 to find the metric $g$. Let $\{\delta_k\}_{k\in\mathbb{N}}$ be a sequence in $(0,1)$ satisfying $\lim_k \delta_k = 0$, and let $\{\alpha_k\}_{k\in\mathbb{N}}$ be a sequence in $(0,+\infty)$ satisfying $\lim_k \alpha_k = +\infty$ and $\delta_k < \alpha_k$. Then $\cap_k \hat{\mathcal{G}}_{\alpha_k,\delta_k}$ is a residual set in $\mathcal{M}$ that coincides with the set of all Riemannian metrics for which all non-zero eigenvalues of $P_g$ have multiplicity smaller than the rank of the bundle $E_g$. This completes the proof of Corollary 5.5.

# CHAPTER 6

# Conformal invariants from CCO

In this chapter we discuss several new conformal invariants that arise from eigenvalues and nodal sets of null-eigenfunctions of conformally covariant operators.

## 6.A. Conformal invariants from eigenvalues

*Throughout this section we work under the assumptions described in Section 1.A., and in addition we assume that $P_g : \Gamma(E_g) \to \Gamma(E_g)$ is strongly elliptic.* For such operators the spectrum consists of a sequence of real eigenvalues converging to $\infty$. We thus can order the eigenvalues of $P_g$ as a non-decreasing sequence,

$$\lambda_1(P_g) \leq \lambda_2(P_g) \leq \cdots ,$$

where each eigenvalue is repeated according to multiplicity. We start this section with a natural remark that was already discussed in Chapter 5.

**Remark 6.1.** **(**$\dim(\mathbf{ker} P_g)$ **is a conformal invariant)** We recall that for zero eigenvalues it is easy to see that their number determines a conformal invariant. Indeed, if $u$ belongs to the kernel of $P_g$, and $\hat{g} = e^f g$, then $\kappa^{-1}(e^{-\frac{a}{2}f}u)$ belongs to the kernel of $P_{\hat{g}}$, for $P_g$ of biweight $(a, b)$ and $\kappa : E_{\hat{g}} \to E_g$ the bundle isomorphism in Definition 3.1.

We next show that the number of negative eigenvalues defines a conformal invariat too. For any $g \in \mathcal{M}$, we define

$$\nu(P_g) := \#\{j \in \mathbb{N};\ \lambda_j(P_g) < 0\}.$$

**Theorem 6.2.** **($\nu(P_g)$ is a conformal invariant)**
*For $g \in \mathcal{M}$, $\nu(P_g)$ is an invariant of the conformal class $[g]$.*

*Proof.* Let $g \in \mathcal{M}$, and set $m = \nu(P_g)$ and $l = \dim\ker P_g$. Thus $\lambda_j(P_g) < 0$ for $j \leq m$, and $\lambda_j(P_g) = 0$ for $j = m+1, \cdots, m+l$, and $\lambda_j(P_g) > 0$ for $j \geq m + l + 1$.
Let $\delta > 0$ be so that 0 is the only eigenvalue of $P_g$ in $[-\delta, \delta]$. Then, by Corollary 4.2 there exists an open neighborhood $\mathcal{W}$ of $g$ such that for all $\hat{g} \in \mathcal{W}$ one has

$$\lambda_j(P_{\hat{g}}) \in (-\infty, -\delta) \quad j \leq m, \quad \text{and} \quad \lambda_{m+l+1}(P_{\hat{g}}) \in (\delta, +\infty).$$

By Remark 6.1 $\lambda_{m+j}(P_{\hat{g}}) = 0$ for $j = 1, \ldots, l$. Then, for $\hat{g} \in \mathcal{W}$ it follows that $\nu(P_{\hat{g}}) = m$. All this shows that the map $g \to \nu(P_g)$ is locally constant when restricted to the conformal class $[g]$. As $[g]$ is a connected subset of $\mathcal{M}$ (since $[g]$ is the image of $C^\infty(M, \mathbb{R})$ under $f \to e^f g$), we deduce that $\nu(P_g)$ is actually constant along the conformal class $[g]$. This proves the Theorem. □

A result of Kazdan-Warner [30, Theorem 3.2] asserts that the sign of the first eigenvalue $\lambda_1(P_{1,g})$ is an invariant of the conformal class $[g]$. We generalize it to the following result.

**Theorem 6.3.** **($\mathrm{sgn}(\lambda_1(P_g))$ is a conformal invariant)**
*The sign of $\lambda_1(P_g)$ is an invariant of the conformal class $[g]$.*

*Proof.* Notice that

- $\lambda_1(P_g) < 0$ if and only if $\nu(P_g) \geq 1$.
- $\lambda_1(P_g) = 0$ if and only if $\dim\ker P_g \geq 1$ and $\nu(P_g) = 0$.
- $\lambda_1(P_g) > 0$ if and only if $\dim\ker P_g = \nu(P_g) = 0$.

Therefore, the result follows from the conformal invariance of $\dim\ker P_g$ and $\nu(P_g)$. □

## 6.B. Conformal invariants from Ker$P_g$

*Throughout this section, we assume that $(M,g)$ is a compact, connected, Riemannian manifold of dimension $n$ and that $P_g : C^\infty(M) \to C^\infty(M)$ is a conformally covariant operator of biweight $(a,b)$.*

Let $u_g \in KerP_g$. Note that since $P_g$ is a conformally covariant operator of biweight $(a,b)$, we know that for a conformal change of the metric $g \mapsto e^f g$ then $e^{-\frac{af}{2}} u_g \in KerP_{e^f g}$. Therefore, throughout this section we shall regard $u_g$ as *conformal density of weight* $-a$, that is, a family $(u_{\hat{g}})_{\hat{g}\in[g]} \subset C^\infty(M)$ parametrized by the conformal class $[g]$ in such way that

$$u_{e^f g} = e^{-\frac{af}{2}} u_g \qquad \forall f \in C^\infty(M,\mathbb{R}).$$

We observe that if $(u_{\hat{g}})_{\hat{g}\in[g]}$ is a conformal density, then its nodal set

$$\mathcal{N}(u_{\hat{g}}) := \{x \in M : \ u_{\hat{g}}(x) = 0\}$$

is independent of the metric $\hat{g} \in [g]$, and hence is an invariant of the conformal class $[g]$. Applying this observation to the null-eigenvectors of $P_g$ we then get

**Proposition 6.4. (Nodal sets are conformal invariants)**

1. *If* $\dim\ker P_g \geq 1$, *and* $u_g \in kerP_g \backslash \{0\}$, *then the nodal sets* $\mathcal{N}(u_{\hat{g}})$, $\hat{g} \in [g]$, *are invariants of the conformal class* $[g]$. *Their complements, the nodal domains, are invariants too.*

2. *If* $\dim\ker P_g \geq 2$, *then intersections of nodal sets of null-eigenvectors* $u_g \in kerP_g$ *and their complements are invariants of the conformal class* $[g]$. ∎

**Remark 6.5.** A connected component $N$ of an intersection of $p$ nodal sets should generically be a co-dimension $p$ submanifold of $M$, and in the case it is, the corresponding homology class in $H_{n-p}(M)$ would be a conformal invariant. Further interesting conformal invariants should arise from considering the topology of $M \setminus N$. For example, if $\dim M = 3$ and $\dim\ker P_g = 2$, and $u_g, v_g \in \ker P_g$, then $\mathcal{N}(u_g) \cap \mathcal{N}(v_g)$ defines a "generalized link" in $M$, and all topological invariants of that set and its complement in $M$ would be conformal invariants.

The next remark is then natural.

**Remark 6.6.** The kernel of $P_g$ contains a nowhere vanishing eigenfunction if and only if there is a metric $\hat{g}$ in the conformal class $[g]$ such that $P_{\hat{g}}(1) = 0$. Indeed, if $u \in \ker P_g$ and $u(x) > 0$ for all $x \in M$, then for $\hat{g} = e^{\frac{2}{a}\ln u}$ we have $P_{\hat{g}}(1) = e^{-b\ln u} P_g(e^{\ln u}) = 0$. On the other hand, if $P_{\hat{g}}(1) = 0$ for some $\hat{g} = e^f g \in [g]$, then $P_g(e^{-\frac{af}{2}}1) = 0$ and so $u(x) = e^{-\frac{af(x)}{2}}$ does the job.

If $P_g$ is a conformally covariant operator of biweight $(0, b)$, then all the level sets $\{x \in M;\ u_g(x) = \lambda\}$, $\lambda \in \mathbb{C}$, are independent of the representative metric $g$:

**Proposition 6.7.** *If $a = 0$, all the level sets of any non-constant eigenvector in $\ker P_g$ are invariants of the conformal class $[g]$.*

We continue to assume that $P_g$ is a conformally covariant operator of biweight $(a, b)$. When $a \neq 0$ we prove the following result.

**Proposition 6.8. (Invariants from norms)** *Suppose $a \neq 0$. Let $u_g \in \ker P_g$ and let us regard it as a conformal density of weight $-a$. Then the integral*

$$\int_M |u_g(x)|^{\frac{n}{a}}\, \mathrm{dvol}_g(x)$$

*is an invariant of the conformal class $[g]$.*

*Proof.* Let $\hat{g} = e^f g$, $f \in C^\infty(M, \mathbb{R})$, be a metric in the conformal class $[g]$. Then

$$\begin{aligned}\int_M |u_{\hat{g}}(x)|^{\frac{n}{a}}\, \mathrm{dvol}_{\hat{g}}(x) &= \int_M \left| e^{-\frac{af}{2}(x)} u_g(x) \right|^{\frac{n}{a}} e^{\frac{nf}{2}(x)}\, \mathrm{dvol}_g(x) \\ &= \int_M |u_g(x)|^{\frac{n}{a}}\, \mathrm{dvol}_g(x).\end{aligned}$$

This proves the result. □

CHAPTER 7

# Conformal Laplacian

In this chapter we focus on the Conformal Laplacian. We translate most of the results in Chapter 6 to this setting. Let $(M, g)$ be a compact, connected, Riemannian manifold of dimension $n$. We start by recalling from (3.1) that the Conformal Laplacian is defined by

$$P_{1,g} = \Delta_g + \frac{n-2}{4(n-1)} R_g,$$

where $\Delta_g = \delta_g d$ and $R_g$ is the scalar curvature. $P_{1,g}$ is a conformally covariant operator of bidegree $\left(\frac{n-2}{2}, \frac{n+2}{2}\right)$. We continue to order the eigenvalues of $P_g$ as a non-decreasing sequence,

$$\lambda_1(P_{1,g}) \leq \lambda_2(P_{1,g}) \leq \cdots ,$$

where each eigenvalue is repeated according to multiplicity.

## 7.A. Conformal invariants

The results in the previous chapter can all be applied to the Conformal Laplacian. In particular, the sign of the first eigenvalue, the number of negative eigenvalues, and the nodal sets of null-eigenvectors give rise to conformal invariants. In this section we discuss in more detail some of their features.

**Sign of first eigenvalue.** Let $g_0$ be a metric of constant scalar curvature in the conformal class of a reference metric $g$. As the nullspace of the Laplacian consists of constant functions, $\lambda_1(P_{1,g_0}) = \frac{n-2}{4(n-1)} R_{g_0}$. Therefore, the sign of $\lambda_1(P_{1,g})$ agrees with that of the constant scalar curvature $R_{g_0}$. We also see that $\lambda_1(P_{k,g}) = 0$ if and only if $R_{g_0} = 0$. Furthermore, in that case $\ker P_{1,g_0}$ consists of constant functions and $\ker P_{1,g}$ is spanned by a single positive function.

**Nodal sets of null-eigenfunctions.** Since nodal sets of null-eigenfunctions of conformally covariant operators are conformal invariants, it seems natural to introduce the following version of Courant's Nodal Domain Theorem.

**Proposition 7.1.** *Assume that the Conformal Laplacian, $P_{1,g}$, has $m$ negative eigenvalues, $m \geq 1$. Then any null eigenfunction of $P_{1,g}$ has at most $m+1$ nodal domains.*

*Proof.* By Proposition 6.4 the nodal domains of $P_{1,g}$ are conformal invariants. Therefore, without any loss of generality we may assume that the scalar curvature $R_g$ is constant. Then the eigenvalues of $P_{1,g}$ are obtained by adding $\frac{n-2}{4(n-1)} R_g$ to the eigenvalues of the Laplacian $\Delta_g$ and the corresponding eigenfunctions agree.

Let $u \in \ker P_{1,g}$. By assumption $P_{1,g}$ has $m$ negative eigenvalues, so the eigenvalue $\lambda = 0$ is the $j$-th eigenvalue of $P_{1,g}$ for some $j \geq m$. It then follows that $u$ is an eigenfunction of $\Delta_g$ for its $j$-th eigenvalue. Applying Courant's Nodal Domain Theorem then shows that $u$ has at most $m+1$ nodal domains and completes the proof. □

**Large number of negative eigenvalues.** The question that naturally arises is whether the number of negative eigenvalues of $P_{1,g}$ can be arbitrary large as $g$ ranges over metrics on $M$.

**Theorem 7.2.** *For every $m \in \mathbb{N}$, there is a metric $g$ on $M$ for which $P_{1,g}$ has at least $m$ negative eigenvalues counted with multiplicity.*

*Proof.* By a result of Lohkamp [34, Theorem 2], given $\lambda > 0$, there is a metric $g$ on $M$ such that

(i) The $m$ first positive eigenvalues of the Laplacian $\Delta_g$ counted with multiplicity are equal to $\lambda$.

(ii) The volume of $(M, g)$ is equal to 1.

(iii) The Ricci curvature of $g$ is $\leq -m^2$.

The condition (iii) implies that $R_g \leq -nm^2$. Combining this with (ii) shows that, for all $u \in C^\infty(M)$, we have

$$\begin{aligned} \langle P_{1,g}u, u\rangle &= \langle \Delta_g u, u\rangle + \frac{(n-2)}{4(n-1)} \int_M R_g(x)|u(x)|^2 \, \mathrm{dvol}_g(x) \\ &\leq \langle \Delta_g u, u\rangle - \frac{(n-2)}{4(n-1)} nm^2 \|u\|_2^2. \end{aligned} \tag{7.1}$$

Assume $\lambda < \frac{(n-2)}{4(n-1)}nm^2$ and denote by $V_\lambda$ the eigenspace of $\Delta_g$ associated to $\lambda$. Notice that $V_\lambda$ is a subspace of $C^\infty(M)$ and has dimension $k \geq m$. Moreover, if $u$ a unit vector in $V_\lambda$, then (7.1) shows that $\langle P_{1,g}u, u\rangle \leq \lambda - \frac{(n-2)}{4(n-1)}nm^2 < 0$. Combining this with the min-max principle we see that $\lambda_m(P_{1,g}) \leq \lambda_k(P_{1,g}) < 0$. Thus, $P_{1,g}$ has at least $m$ negative eigenvalues counted with multiplicity. The proof is complete. □

## 7.B. Scalar curvature prescription problems

The problem of prescribing the curvature (Gaussian or scalar) of a given compact manifold is very classical and is known as the *Kazdan-Warner problem* (see [30] and the references therein). In this chapter we apply the results on the invariance of the nodal sets of eigenfunctions in the kernel of the Conformal Laplacian to Scalar curvature prescription problems.

We next make some observations about $\int_M R_g u \, \mathrm{dvol}_g$ for $u \in \ker(P_{1,g})$.

**Proposition 7.3.** *Assume that the scalar curvature $R_g$ is constant and let us regard $u_g \in \ker P_{1,g}$ as a conformal density of weight $-\frac{n}{2}+1$ Then*

$$\langle R_{\hat{g}}, u_{\hat{g}}\rangle_{\hat{g}} = 0 \qquad \forall \hat{g} \in [g].$$

*Proof.* Note that if the curvature $R_g$ is constant, then $u$ is an eigenfunction of the (positive) Laplacian $\Delta_g$ with eigenvalue $\lambda = -\frac{n-2}{4(n-1)}R_g$.

- If $R_g = 0$, then the integral vanishes.
- If $R_g \neq 0$ and $R_g$ is constant, then $u$ is orthogonal to the constant eigenfunction, and so again we find that $\int_M R_g u \, \mathrm{dvol}_g = 0$.

- Assume $R_g$ is constant and $R_{\hat{g}} \not\equiv R_g$, for $\hat{g} = e^f g$. Using the formula for the transformation of the scalar curvature we obtain

$$\int_M R_{\hat{g}} e^{-\frac{n-2}{4}f} u \,\mathrm{dvol}_{\hat{g}} = \int_M \left( \tfrac{4(n-1)}{n-2} e^{-\frac{n+2}{4}f} P_{1,g}(e^{\frac{n-2}{4}f}) \right) e^{-\frac{n-2}{4}f} u \, e^{\frac{n}{2}f} \mathrm{dvol}_g.$$

Since $P_{1,g}$ is formally self-adjoint, we can rewrite the right hand side as

$$\tfrac{4(n-1)}{n-2} \int_M e^{\frac{n-2}{4}f} P_{1,g}(u) \,\mathrm{dvol}_g = 0.$$

We conclude that

$$\int_M R_{\hat{g}} e^{-\frac{n-2}{4}f} u \,\mathrm{dvol}_{\hat{g}} = 0, \qquad u \in \ker(P_{1,g}). \tag{7.2}$$

□

Next, we consider the scalar curvature restricted to nodal domains.

**Theorem 7.4.** *Let $u \in \ker(P_{1,g})$ and let $\Omega$ be a nodal domain of $u$. Then, for all $v \in C^\infty(M)$,*

$$\int_\Omega |u| \, P_{1,g}(v) \,\mathrm{dvol}_g = - \int_{\partial\Omega} v \, \|\nabla_g u\|_g \, d\sigma_g,$$

*where $\sigma_g$ is the surface-area measure of $\partial\Omega$.*

**Remark 7.5.** The intersection of the critical and nodal sets of $u$ has locally finite (n-2)-Hausdorff dimension (see [29]). Therefore, $\partial\Omega$ admits a normal vector almost everywhere, and hence the surface measure $d\sigma_g$ is well-defined.

*Proof.* Observe that $u$ has constant sign on $\Omega$. In addition, let $\nu$ be the outward unit normal vector to the hypersurface $\partial\Omega$. Then $\partial_\nu u$ agrees with $-\|\nabla_g u\|_g$ (resp., $\|\nabla_g u\|_g$) on $\partial\Omega$ in case $u$ is positive (resp., negative) on $\Omega$. Therefore, upon replacing $u$ by $-u$ if needed, we may assume that $u$ is positive on $\Omega$.

Let $v \in C^\infty(M)$. As $P_{1,g} u = 0$ and the Conformal Laplacian agrees with the Laplacian $\Delta_g$ up to a the multiplication by a function, we have

$$\int_\Omega |u| \, P_{1,g}(v) \,\mathrm{dvol}_g = \int_\Omega \left( u P_{1,g} v - v P_{1,g} u \right) \,\mathrm{dvol}_g = \int_\Omega \left( u \Delta_g v - v \Delta_g u \right) \,\mathrm{dvol}_g.$$

Using the divergence theorem we see that

$$\int_\Omega |u| \, P_{1,g}(v) \,\mathrm{dvol}_g = - \int_{\partial\Omega} \left( u \, \partial_\nu v - v \, \partial_\nu u \right) \, d\sigma_g = - \int_{\partial\Omega} v \, \|\nabla_g u\|_g \, d\sigma_g,$$

where we have used the fact that $u = 0$ and $\partial_\nu u = -\|\nabla_g u\|_g$ on $\partial\Omega$. The proof is complete. $\square$

Decomposing the manifold into a disjoint union of positive nodal domains, negative nodal domains and the nodal set of $u$, and applying Theorem 7.4 we obtain

**Corollary 7.6.** *For all $u \in \ker P_{1,g}$ and $v \in C^\infty(M)$,*

$$\int_M |u|\, P_{1,g}(v)\, \mathrm{dvol}_g = -2 \int_{\mathcal{N}(u)} v\, \|\nabla_g u\|_g \, d\sigma_g,$$

*where $\mathcal{N}(u)$ is the nodal set of $u$.*

**Theorem 7.7.** *Let $R \in C^\infty(M)$ be the scalar curvature of some metric in the conformal class $[g]$. Then, there is a positive function $\omega \in C^\infty(M)$, such that, for any $u \in \ker(P_{1,g})$ with non-empty nodal set and any nodal domain $\Omega$ of $u$,*

$$\int_\Omega R\,|u|\,\omega\,\mathrm{dvol}_g < 0.$$

*Proof.* By assumption $R = R_{\hat{g}}$ for some metric $\hat{g} = e^f g$ with $f \in C^\infty(M, \mathbb{R})$. Thus $P_{1,\hat{g}}(1) = \frac{n-2}{4(n-1)} R_{\hat{g}} = \frac{n-2}{4(n-1)} R$. Let $u \in \ker(P_{1,g})$ and let $\Omega$ be nodal domain of $u$. In addition, set $\omega = \frac{n-2}{4(n-1)} e^{\frac{n+2}{4} f}$ and $\hat{u} = e^{\frac{2-n}{4} f} u$. Then

$$\int_\Omega R|u|\omega\,\mathrm{dvol}_g = \tfrac{n-2}{4(n-1)} \int_\Omega |\hat{u}|\, R\,\mathrm{dvol}_{\hat{g}} = \int_\Omega |\hat{u}| P_{1,\hat{g}}(1)\,\mathrm{dvol}_{\hat{g}}. \tag{7.3}$$

Since $\hat{u} \in ker P_{1,\hat{g}}$ and $\Omega$ is a nodal domain for $\hat{u}$ we can apply Theorem 7.4 to $\hat{u}$ and $v = 1$ and using (7.3) get

$$\int_\Omega R|u|\,\omega\,\mathrm{dvol}_g = \int_\Omega |\hat{u}| P_{1,\hat{g}}(1)\,\mathrm{dvol}_{\hat{g}} = -\int_{\partial\Omega} \|\nabla_{\hat{g}} u\|_{\hat{g}}\, d\sigma_{\hat{g}}.$$

As the intersection of the critical and nodal sets of $u$ has locally finite $(n-2)$-Hausdorff dimension, the integral $\int_{\partial\Omega} \|\nabla_{\hat{g}} u\|_{\hat{g}}\, d\sigma_{\hat{g}}$ must be positive. $\square$

Theorem 7.7 seems to be new. Since $1 \in Ker P_{1,\hat{g}}$ for some $\hat{g} \in [g]$, there is always a non vanishing eigenfunction in $Ker P_{1,\hat{g}}$. Therefore, we remark that when $\dim \ker(P_{1,g}) \geq 2$ Theorem 7.7 gives infinitely many constraints on $R_{\hat{g}}$.

**Corollary 7.8.** *Let $u \in \ker(P_{1,g})$ with non-empty nodal set and let $\Omega$ be a non-empty nodal domain of $u$. Then, for any metric $\hat{g}$ in the conformal class $[g]$, the scalar curvature $R_{\hat{g}}$ cannot be everywhere positive on $\Omega$.*

Let $u \in \ker(P_{1,g})$ and let $\Omega$ be nodal domain of $u$. Given any metric $\hat{g} = e^f g$, $f \in C^\infty(M, \mathbb{R})$, in the conformal class $[g]$ we define

$$T(u, \Omega, \hat{g}) := -\tfrac{4(n-1)}{n-2} \int_{\partial\Omega} e^{\frac{2-n}{4}f} \|\nabla_{\hat{g}} \hat{u}\|_{\hat{g}} \, d\sigma_{\hat{g}},$$

where we have set $\hat{u} = e^{\frac{2-n}{4}f} u$.

**Proposition 7.9.** *For all metrics $\hat{g}$ in the conformal class $[g]$,*

$$T(u, \Omega, \hat{g}) = \int_\Omega |u| R_g \, \mathrm{dvol}_g.$$

*Proof.* Let $\hat{g} = e^f g$, $f \in C^\infty(M, \mathbb{R})$, be a metric in the conformal class $[g]$. Set $\hat{u} = e^{\frac{2-n}{4}f} u$ and $v = e^{\frac{2-n}{4}f}$. As pointed out in the proof of Theorem 7.7, $\hat{u}$ lies in $\ker P_{1,\hat{g}}$ and $\Omega$ is a nodal domain. Applying Theorem 7.4 to $\hat{u}$ and $v$ then gives

$$\tfrac{n-2}{4(n-1)} T(u, \Omega, \hat{g}) = -\int_{\partial\Omega} v \|\nabla_{\hat{g}} \hat{u}\|_{\hat{g}} \, d\sigma_{\hat{g}} = \int_\Omega |\hat{u}| \, P_{1,\hat{g}} v \, \mathrm{dvol}_{\hat{g}}.$$

As $P_{1,\hat{g}} v = e^{-\frac{n+2}{4}f} P\left(e^{\frac{n-2}{4}f} \cdot e^{\frac{2-n}{4}f}\right) = e^{-\frac{n+2}{4}f} P_{1,g}(1) = \frac{n-2}{4(n-1)} e^{-\frac{n+2}{4}f} R_g$, we get

$$\begin{aligned} T(u, \Omega, \hat{g}) &= \int_\Omega |\hat{u}| e^{-\frac{n+2}{4}f} R_g \, \mathrm{dvol}_{\hat{g}} \\ &= \int_\Omega e^{\frac{2-n}{4}f} |u| e^{-\frac{n+2}{4}f} R_g \, e^{\frac{nf}{2}} \, \mathrm{dvol}_g \\ &= \int_\Omega |u| R_g \, \mathrm{dvol}_g. \end{aligned}$$

The result is proved. □

Proposition 7.9 provides us with some conserved quantities for the conformal class. In particular, if $R_g$ is constant, we obtain

$$T(u, \Omega, \hat{g}) = R_g ||u||_{L^1(\Omega)} \qquad \forall \hat{g} \in [g].$$

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