A distinguished, nontrivial class in the Chevalley cohomology associated with the Lie algebra of infinitesimal automorphisms of contact structures

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Abstract

A distinguished class of cocycles in the local Chevalley cohomology associated with the representation of the Lie algebra of vector fields on the vector space of differential 2-forms is defined and proven to be non-zero. Two prerequisites to this proof are a theorem that local operators must be locally differential and a characterization of invariant tensors under the representation of $gl(\mathbb{R}^n)$ on the tensor algebra. Finally, the constructed cocycles are shown to be non-trivial even if the cochains of the cohomology are restricted to the Lie algebra of infinitesimal automorphisms of a higher order contact structure.

Résumé

Nous construisons une classe prévilégiée de cocycles dans la cohomologie de Chevalley associée à la représentation de l'algèbre de Lie de champs de vecteurs sur l'espace vectoriel des 2-formes différentielles, et prouvons qu'elle n'est jamais nulle. Pour ce faire nous démontrons qu'un opérateur local doit étre locallement différentiel et nous donnons une caractérisation des tenseurs invariants par la représentation de $gl(\mathbb{R}^n)$ sur l'algèbre tensorielle. Finalement, nous démontrons que ces cocycles demeurent non-triviaux même lorsque la cohomologie est restreinte à l'algèbre de Lie des automorphismes infinitésimaux d'une structure de contact d'ordre supérieur.

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1 Introduction

What is this thesis about? An examination of the table of contents would seem to indicate that it is an unlikely mixture of several diverse topics. Section 2 is analysis: operators, function spaces, norms and convergence, and a theorem relating two different operator properties. Section 3 is pure algebra; it is really a mixture of finite dimensional linear algebra and basic representation theory. Sections 4 and 5 are differential geometry: the Chevalley cohomology of vector field operators, the Lie derivative of a connection, contact structures on the jet bundle, and lots of local coordinate computations.

Actually, only in Section 5 do we come to the issue that motivates this entire dissertation; the other sections are just preparatory diversions (lengthy and tedious diversions if one is solely interested in the results of Section 5, but useful diversions if one is trying to learn some mathematics). The origin of this issue is [6], a paper by Lichnerowicz and Pereira DaSilva, in which the authors demonstrate that there is a natural, non-trivial way to extend the Lie algebra of vector fields by using the space of differential 2-forms. The extension is based on the construction of a distinguished, non-zero cohomology class in the Chevalley cohomology of the Lie algebra of vector fields. The extension by the 2-forms turns out to be non-trivial precisely because the cohomology class is non-trivial, i.e. cannot be represented as a coboundary. The non-triviality of the cohomology class in question had been previously demonstrated by A. Lichnerowicz in [3] and by DeWilde and Lecomte in [10]. The authors of [6] then consider manifolds with quite a number of different geometrical structures: foliations, unimodular structures, Poisson and Jacobi brackets. In each case, they look at the reduced Lie algebra of those vector fields which are infinitesimal automorphisms of the structure in question. Remarkably, they show that the cocycles in the above-mentioned cohomology class remain non-trivial when restricted to these Lie subalgebras, and thereby show that these Lie algebras can be extended as well in a non-trivial way.

The present work carries out this program for the Lie algebra of infinitesimal automorphisms of first and higher order contact structures. Now, specifying a Jacobi bracket on a manifold is equivalent, in a certain way, to specifying a first order contact structure on the manifold (see for example [5]); and so [6] implicitly resolves the case of the infinitesimal automorphisms of first order contact structures. Thus, the only original part of this dissertation is the observation that the above-mentioned class of cocycles remains non-trivial when restricted to the Lie algebra of infinitesimal automorphisms of higher order contact structures. The key to proving this observation ends up being the fact that the infinitesimal automorphisms of higher order contact structures all arise from the "prolongation" of infinitesimal automorphisms of first order contact structures, thereby relating the situation under discussion to the above-mentioned case of the Jacobi bracket.

Sections 2,3, and 4 are an exposition of some of the issues related to the cocycles of section 5. Section 4 introduces the Chevalley cohomology (in which cochains are operators that take vector field arguments and give 2-form results), constructs the cocycle in question, and shows that the cohomology class is, in general, non-trivial. Section 2 develops the result that a local cochain is necessarily differential, thereby permitting a less restrictive criterion for which cochains can be included in the above-mentioned cohomology. Section 3 discusses the algebraic prerequisites for the proof that the cocycles constructed in section 4 are not coboundaries. The theme of Section 3 is the representation of the permutation group and the representation of the Lie algebra gl(n) on various tensor spaces; the seminal ideas come from the treatment of this subject by H. Weyl in [9]. The central result of section 3 is the proof that the actions of the permutation group on tensor space generate the commutator algebra of the actions of gl(n).

The most important result of section 4 is the proof that the cocycles of the above-mentioned class are not coboundaries. Actually, three different proofs are presented. The second of these proofs is a corollary of the central theorem of section 3; the other two proofs are self-contained. The third version of the proof requires being able to express the given cocycle as a differential operator; The first version of the proof only needs the assumption that the cocycle is a local operator. Thus, by restricting oneself to version 1 of the non-triviality proof, one can eliminate the need for sections 2 and 3.

2 Local Operators

It is not difficult to prove that an operator $L : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ which is continuous with respect to a certain topology on $C^\infty(\mathbb{R}^n)$ is local if and only if it is differential (see for instance Dieudonné [1]). Actually, this result is true even without the continuity assumption. This was shown by J. Peetre in [7]. The approach taken by that work is rather abstract; the author uses the language of sheaf and distribution theory to state and prove the result. A generalization of the result to multi-linear operators is given by DeWilde and Lecomte in [10]. Their approach is less sophisticated, but still requires Peetre's result as a prerequisite as well as relying on the use of the Baire Category Theorem. In the present section we will give an original (to the author's best knowledge), self-contained proof that a local multi-linear L must be differential. The resulting theorem generalizes easily to local operators whose arguments and value are tensor fields on a finite-dimensional manifold, rather than just functions on $\mathbb{R}^n$. We will need this extended result about the equivalence of local and differential operators in the sections that follow.

The assumption that the operators in question have C^∞ arguments, rather than analytic ones, is essential for the above mentioned proof. The reason, roughly speaking, is that there exist C^∞ "plateau" functions: these are functions with compact support that are identically equal to 1 on some open set. Given a function, f, and a point, p, if we multiply f by a properly scaled and translated plateau function, the result will be a function with arbitrarily small support, but with the same behaviour as f in some neighborhood around p. In other words, it is possible to extend a given function germ to a globally defined function with arbitrarily small support. Such a construction is not possible when working in the "analytic category" because the germ of an analytic function fully determines global behaviour. It is the C^∞ extension property that makes possible the proof that local operators must actually be differential. We will show that if a given linear operator, L, does not act like a differential operator at sufficiently many points, then it is possible to use the extention property to construct a C^∞ function, f, such that $L(f)$ is not continuous. Such an L cannot, therefore, be a local operator. With a little more work, we will then be able show that a local operator must act like a differential one in a neighborhood around each point of $\mathbb{R}^n$.

2.1 Preliminaries

Let $C^\infty(\mathbb{R}^n)$ denote the space of infinitely differentiable, real-valued functions on $\mathbb{R}^n$. By a multi-index, α, of order d we will mean a list of n non-negative integers, $(\alpha^1, \ldots, \alpha^n)$, such that $\sum_i \alpha^i = d$. We will use $\alpha!$ to denote $\alpha^1! \cdot \ldots \cdot \alpha^n!$ and use the symbol $\leq$ to refer to the lexicographic partial order relation on the space of multi-indices. For multi-indices α, β of orders a and b, respectively, and for $1 \leq i \leq n$, we will use $[i]$ to denote the multi-index $(0, \ldots, 0, 1 \text{ (i-th position)}, 0, \ldots, 0)$; $[\alpha, i]$ will denote the $(a+1)$-st order multi-index $(\alpha^1, \ldots, \alpha^i + 1, \ldots, \alpha^n)$; and $\alpha + \beta$ will denote the $(a+b)$-th order multi-index $(\alpha^1 + \beta^1, \ldots, \alpha^n + \beta^n)$. We will use multi-indices to specify basic polynomials functions, x^α, and basic multi-differential operators, ∂_α. Using $x^1, \ldots, x^n$ to denote the coordinate functions on $\mathbb{R}^n$ we define

$$x^\alpha = (x^1)^{\alpha^1} \cdot \ldots \cdot (x^n)^{\alpha^n},$$

and

$$\partial_\alpha = \left(\frac{\partial}{\partial x^1}\right)^{\alpha^1} \cdots \left(\frac{\partial}{\partial x^n}\right)^{\alpha^n}.$$

Let U be a subset of $\mathbb{R}^n$ and d a non-negative integer. For $f \in C^\infty(\mathbb{R}^n)$ put

$$\|f\|_{U,d} = \sup_{\substack{p \in U \\ |\alpha| \leq d}} |\partial_\alpha f|(p).$$

For $U = \mathbb{R}^n$ we will simply write $\|f\|_d$. It is not difficult to verify that if U is bounded, then the above defines a norm on $C^\infty(\mathbb{R}^n)$. This norm is useful because it "measures" the maximum variation of f and all of its derivatives up to order d on the set U.

2.2 Linear Local Operators

The proof that multi-linear local operators are differential combines several different ideas. For the sake of clarity, we will first consider the simpler case of linear operators, i.e. operators with only one argument. This will permit us to highlight certain essential ideas without involving the full range of technical details needed for the proof of the general case.

Let $L : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ be a linear operator. When we say that L is local, we mean that for a given function, f, the value of Lf at a point is completely determined by the behaviour of f around that point. As a rigorous definition, this can be expressed in two ways.

Definition 2.1 *We call L a local operator if it satisfies the following two conditions, for every open $U \subset \mathbb{R}^n$ and $f, g \in C^\infty(\mathbb{R}^n)$*

(i) $f\,|_U = 0$ *implies that* $(Lf)\,|_U = 0$.

(ii) $f\,|_U = g\,|_U$ *implies that* $(Lf)\,|_U = (Lg)\,|_U$.

Proposition 2.2 *Conditions* (i) *and* (ii) *are equivalent*

Proof. This result is an immediate consequence of the linearity of L. □

There are several ways to interpret the notion of *differential operator*: we could be speaking about a global, a local or a point-wise condition. A formal, infinite sum $\sum_\alpha a_\alpha \partial_\alpha$, where the coefficients, a_α, are arbitrary functions will specify a well-defined local operator as long as only finitely many of the coefficients are non-zero at each point. The range of such an operator will not in general lie in the space of continuous functions: for that it is sufficient to assume that the coefficients are themselves continuous. If such is the case, a standard compactness argument shows that in a given bounded set only finitely many coefficients are not identically zero. It would be fitting to call such an operator locally differential. Of course, just because there is a local bound on the number of the non-zero a_α's does not imply that a global bound exists as well. Indeed, one can easily choose continuous coefficients so that no a_α is identically zero, but so that only finitely many a_α's are non-zero when restricted to a bounded domain. In other words, just because an operator is locally equivalent to a finite sum of ∂_α's does not mean that it can be expressed that way globally. We will not be dealing with the global differential condition. Rather, what we will be trying to show, is that a local operator must necessarily be locally differential.

Definition 2.3 *We will call L locally differential of order d or less if for every bounded $U \subset \mathbb{R}^n$ we can choose C^∞ functions a_α ($|\alpha| \le d$) such that for every $f \in C^\infty(\mathbb{R}^n)$*

$$Lf = \sum_{|\alpha| \le d} a_\alpha \, \partial_\alpha f \,,$$

at all points of U.

Definition 2.4 *L is said to be point-wise differential of order d or less at a point, p, if it satisfies the following three conditions:*

(i) *Given any f, if we have that $\partial_\alpha f(p) = 0$ for all $|\alpha| \le d$, then we also have*

$$(Lf)(p) = 0 \quad .$$

(ii) *Given any f and g, if we have that $\partial_\alpha f(p) = \partial_\alpha g(p)$ for all $|\alpha| \leq d$, then we also have*

$$(Lf)(p) = (Lg)(p) \quad .$$

(iii) *There are real constants, $a_{\alpha;p}$ ($|\alpha| \leq d$), such that for every f we have*

$$(Lf)(p) = \sum_{|\alpha|\leq d} a_{\alpha;p}\, \partial_\alpha f(p)\,. \tag{1}$$

We will shortly show that these three conditions are actually equivalent. Why define the same concept in three different ways? The term *point-wise differential* is an apt description for an operator that satisfies (iii); after all, (iii) simply says that the action of L at p is equivalent to the action of some multi-differential operator. We will see, however, that being able to express this notion in terms of conditions (i) and (ii) is essential in showing that a non-differential operator cannot be local.

Proposition 2.5 *Each of the three conditions of Definition 2.4 implies the other two.*

Proof. (i)$\Rightarrow$(ii) Suppose that L obeys (i) and that f and g satisfy the premise of (ii). Thus, all derivatives of $f - g$ of order d or less are zero at p and hence, $(L(f-g))(p) = 0$. The conclusion of (ii) follows by the linearity of L.

(ii)$\Rightarrow$(iii) Suppose that L obeys (ii). For a given function, f, we can always choose a polynomial, g, such that $\partial_\alpha f(p) = \partial_\alpha g(p)$ for all $|\alpha| \leq d$ and hence, it is enough to choose the constants $a_{\alpha,p}$ so that (1) holds for all $f = x^\alpha$ ($|\alpha| \leq d$). Since

$$\partial_\alpha x^\beta = \frac{\beta!}{(\beta-\alpha)!}\, x^{\beta-\alpha},$$

if $\alpha \leq \beta$, and equals 0 otherwise we need to choose the constants so that

$$(Lx^\beta)(p) = \sum_{\alpha\leq\beta} a_{\alpha;p}\, \frac{\beta!}{(\beta-\alpha)!}\, x^{\beta-\alpha}(p), \tag{2}$$

for every possible $|\beta| \leq d$. This can be accomplished with the following inductive definition. For a given $|\beta| \leq d$, after having defined all $a_{\alpha;p}$ ($\alpha < \beta$), put

$$a_{\beta;p} = \frac{1}{\beta!}\left((Lx^{\beta})(p) - \sum_{\alpha<\beta} a_{\alpha;p}\,\frac{\beta!}{(\beta-\alpha)!}\,x^{\beta-\alpha}(p)\right) .$$

(iii)⇒(i) Trivial. □

The reason for the awkward phrase "of order d or less" in Definition 2.4 is that if the conditions of the definition hold for a certain $d = N$, then they will also hold for $d = N + 1$.

Definition 2.6 *We call N the differential order of L at p if the conditions of Definition 2.4 hold for $d = N$, but do not hold for any smaller d. If L is such that these conditions cannot be satisfied for any d, then we will say that the differential degree of L at p is ∞.*

Having given a rigorous definition of *local* and *differential operator* we can now state the primary result of this section.

Theorem 2.7 *If L is local, then it is also locally differential.*

The proof of this theorem requires a number of Lemmas. First, we need to construct a "canonical" plateau function.

Lemma 2.8 *There exists a $C^{\infty}(\mathbb{R}^n)$ function, φ, with compact support such that $\varphi \equiv 1$ in some neighborhood of the origin.*

Proof. We construct φ from a 1 dimensional bump function, i.e. a positive C^{∞} function $\varphi_1 : \mathbb{R} \longrightarrow \mathbb{R}$ with compact support. We use the following standard technique to define φ_1. For $x \in \mathbb{R}$ put

$$\varphi_1(x) = \begin{cases} \exp(\frac{1}{x^2-1}) & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} .$$

Next, we define a C^∞ function $\varphi_2 : \mathbb{R} \longrightarrow \mathbb{R}$ whose graph has a plateau of height 1, but a plateau that extends indefinitely to the right. For $x \in \mathbb{R}$, put

$$\varphi_2(x) = \frac{\int_{-\infty}^{x} \varphi_1(s)ds}{\int_{-\infty}^{\infty} \varphi_1(s)ds} .$$

We can make a 1 dimensional "plateau" function, φ_3, by shifting φ_2 to the left and combining the resulting function with its reflection. For $x \in \mathbb{R}$, put

$$\varphi_3(x) = \begin{cases} \varphi_2(2+x) & x \leq 0 \\ \varphi_2(-(2+x)) & x > 0 \end{cases} .$$

The resulting, $\varphi_3(x)$, has support in $-3 \leq x \leq 3$ and is identically 1 for $-1 \leq x \leq 1$.

The desired n-dimensional plateau function, φ, can now be constructed by using the standard norm on $\mathbb{R}^n$. For $p \in \mathbb{R}^n$ put

$$\varphi(p) = \varphi_3(|p|) .$$

□

An essential technique that will be used in the proof of Theorem 2.7 is the construction of a function by the superposition of a countable number of C^∞ functions with disjoint supports. The following lemma gives conditions that are sufficient to guarantee that the constructed function will itself be C^∞. Indeed, let $f_k \in C^\infty(\mathbb{R}^n)$ be functions with disjoint supports and put $f = \sum_k f_k$.

Lemma 2.9 *In order for f to be C^∞, it is sufficient that $\lim_{k\to\infty} \|f_k\|_d = 0$ for every d.*

Proof. Let us denote the support of f_k by O_k; that is O_k is the set of points, p, such that f_k is not identically zero on every neighborhood of p. Let $1 \leq i \leq n$ be given. Since we are assuming that $\lim_{k\to\infty} \|f_k\|_d = 0$ for all d, and since $\|f_k\|_{d+1} \geq \|\partial_i f_k\|_d$, we also have that $\lim_{k\to\infty} \|\partial_i f_k\|_d = 0$ for all d. Since the support of $\partial_i f_k$ is contained in O_k, the premise of the lemma remains true if we replace each f_k by $\partial_i f_k$. Therefore, we can prove the lemma by showing that $\partial_i f$ exists and equals $\sum_k \partial_i f_k$, and then using induction. Since each O_k is open and the O_k's are disjoint, f is C^∞ at all points of $\cup_k O_k$. Thus, it is enough to show that $\partial_i f(p)$ exists and equals 0 for every $p \notin \cup_k O_k$. Let such a p be given. We have to show that $\lim_{h\to 0} f(p + he_i)/h = 0$[1]. Whatever $h > 0$ is, the point $p + he_i$ is either in some O_k or it is in none of them. In the first case, $f(p + he_i)/h = f_k(p + he_i)/h$; and in the second case, $f(p + he_i)/h = 0$. Now, for a fixed k, we certainly have that $\lim_{h\to 0} f_k(p + he_i)/h = 0$; we must show that this limit is uniform over k. Let $\epsilon > 0$ be given. Since $\lim_{k\to\infty} \|f_k\|_1 = 0$, there will be only finitely many k such that $|f_k(p+he_i)/h|$ is not always less than ϵ. Therefore, by making h small enough we will have that $|f_k(p + he_i)/h| \leq \epsilon$ for all k. □

The following lemma is another technical result needed for the proof of Theorem 2.7. It shows how to extend a function germ to a globally defined function with arbitrarily small support, but extend it in such a way that the resulting function is "just as flat" as the given germ.

Lemma 2.10 *Suppose that $f \in C^\infty(\mathbb{R}^n)$ is such that $\partial_\alpha f(p) = 0$ for all $|\alpha| \leq d$. Then, for every $\epsilon > 0$ and every neighborhood O of p, there exists a $g \in C^\infty(\mathbb{R}^n)$ such that*

[1]We are using $e_1, \ldots, e_n$ to denote the canonical basis of $\mathbb{R}^n$

(i) *the support of g is contained in O.*

(ii) *$f \equiv g$ in some neighborhood of p.*

(iii) $\|g\|_d \leq c$.

Proof. Let φ be the "plateau" function constructed in Lemma 2.8, scaled in such a way that the support of φ is contained in the unit ball. For $t > 0$, define $\varphi_t : \mathbb{R}^n \longrightarrow \mathbb{R}$ to be the C^∞ function $\varphi_t(q) = \varphi((q-p)/t)$. Thus, φ_t is a "plateau" function whose support is contained in a p-ball of radius t, and furthermore, $\varphi_t \equiv 1$ in some neighborhood of p. For each $t > 0$ put $g_t(q) = f(q)\varphi_t(q)$. The resulting C^∞ function, g_t, satisfies (ii) and has its support contained in the p-ball of radius t. Thus, if t is sufficiently small, g_t will also satisfy (i). What isn't obvious is that by making t small we can also get g_t to satisfy (iii); in other words we are going to show that $\lim_{t\to 0} \|g_t\|_d = 0$. For $|\alpha| \leq d$, we have by the Leibnitz rule that

$$\partial_\alpha g_t = \sum_{\beta+\gamma=\alpha} C_\beta \, \partial_\beta f \, \partial_\gamma \varphi_t \quad , \tag{3}$$

where the C_β's ($\beta \leq \alpha$) are positive integer constants which we do not need to compute here. In finding an upper-bound for g_t we need only be concerned with points whose distance from p is less than t. As we decrease t, such points lie closer and closer to p, and hence, $\partial_\beta f$ at these points goes to zero for all $|\beta| \leq d$. However,

$$(\partial_\gamma \varphi_t)(q) = \frac{1}{t^{|\gamma|}} \, \partial_\gamma \varphi_1\Big(\frac{q-p}{t}\Big) \quad , \tag{4}$$

and hence, as t goes to zero, the derivatives of φ_t grow like a $|\gamma|$-degree polynomial in $1/t$. We must show that as t goes to zero the derivatives of f decrease "faster" than the derivates of φ_t grow. The constant $K = \|\varphi_1\|_d$ is an upper bound for $|\partial_\gamma \varphi_1(q)|$ and thus (3) and (4) imply that

$$|\partial_\alpha g_t(q)| \leq \sum_{\beta+\gamma=\alpha} C_\beta K \, \frac{|\partial_\beta f(q)|}{t^{|\gamma|}} \quad . \tag{5}$$

The assumptions about f imply that $f(q)$ is $o(|q-p|^d)$ and that $\partial_\beta f(q)$ is $o(|q-p|^{d-|\beta|})$. Since for every $\beta+\gamma=\alpha$, we have that $|\gamma| = |\alpha| - |\beta| \leq d - |\beta|$, and since the sum in (5) is over a finite number of terms, the preceding remark implies that

$$\lim_{t\to 0} \sup\{|\partial_\alpha g_t(q)| : |q-p| \leq 1/t\} = 0 \quad .$$

Since the support of g_t is contained in $\{q : |q-p| \leq 1/t\}$ and since the above holds for all $|\alpha| \leq d$ we can conclude that $\lim_{t\to 0} \|g_t\|_d = 0$. □

The purpose of the next lemma is not to furnish a rigorous demonstration of yet another technical detail. Rather, in it we give the essential idea needed for the proof of Theorem 2.7. Let $\{p_k\}$ be an infinite sequence of distinct points contained in a bounded subset of $\mathbb{R}^n$. For each k, let N_k denote the differential order of L at p_k.

Lemma 2.11 *If L is local, then* $\limsup_{k\to\infty} N_k$ *is finite.*

Proof. Suppose, on the contrary that

$$\limsup_{k\to\infty} N_k = \infty \quad .$$

Thus, we can assume without loss of generality that $N_k > k$ for each k . The fact that the p_k's come from a compact set will guarantee the existence of an accumulation point of the set $\{p_k\}$, and hence a convergent subsequence of $\{p_k\}$ can be extracted. Thus, we may without loss of generality assume that the p_k's are distinct, that they converge to some point, p, and that $N_k > k$ for each k. According to Defition 2.4, for each k, we can choose a function f_k such that $\partial_\alpha f_k(p_k) = 0$ for all $|\alpha| \leq d$ and yet such that $(Lf_k)(p_k) \neq 0$. Multilplying each f_k by a sufficiently large constant, we can assume without loss of generality that the sequence $(Lf_k)(p_k)$ increases without bound. Let us choose an open neighborhood, O_k, around each p_k in such a way that the resulting O_k's are disjoint. Using Lemma 2.10, we can replace each f_k by a function g_k that has the same germ at p_k, has its support in O_k, and such that $\|g_k\|_k < 1/k$. We have chosen the g_k's to be sufficiently "flat" so that

$$\lim_{k\to\infty} \|g_k\|_d = 0 \quad ,$$

for all d and hence, using Lemma 2.9, we can superimpose the g_k's to get a C^∞ function

$$g = \sum_k g_k \quad .$$

The restriction of g to O_k is just g_k. Since L is local, the preceeding fact means that $(Lg)(p_k) = (Lg_k)(p_k)$ and hence, $\{(Lg)(p_k)\}$ is unbounded. And yet, since we are assuming that Lg is continuous, we must have that

$$\lim_{k\to\infty} (Lg)(p_k) = (Lg)(p) \quad .$$

This is a contradiction. □

We need one final technical lemma before giving the proof of Theorem 2.7.

Lemma 2.12 *Let O be an open, bounded subset of $\mathbb{R}^n$. Suppose that the differential order of L is less than or equal to N at all but a finite number of points of O Then, the differential order of L is less than or equal to N at all points of O, and furthermore, L is a local differential operator of order N or less on O.*

Proof. For each $|\beta| \le N$, inductively define C^∞ functions a_β by first defining all a_α for $\alpha < \beta$ and then putting

$$a_\beta = \frac{1}{\beta!}\left(Lx^\beta - \sum_{\alpha<\beta} a_\alpha \frac{\beta!}{(\beta-\alpha)!}\, x^{\beta-\alpha}\right) \quad .$$

This definition is a global analogue of (2). Indeed, having defined the a_β's this way we can be sure by (iii) of Definition 2.4, that for every $f \in C^\infty(\mathbb{R}^n)$

$$Lf = \sum_{|\beta|\le N} a_\beta\, \partial_\beta f \quad ,$$

at all but finitely many points of O. But, both sides of the above equation are assumed to be continuous and therefore we can conclude that the above relation holds everywhere on O. □

We are now ready to give the proof of Theorem 2.7. It is a direct consequence of Lemmas 2.11 and 2.12.

Proof of Theorem 2.7. Assume that L is a local operator, let O be an open, bounded subset of $\mathbb{R}^n$, and let S denote the set of points of O where the differential order of L is infinite. Lemma 2.11 tells us two things: there are at most finitely many points in S, and there exists an upper bound, N, for the differential order of L at points of $O \backslash S$. Thus, we can use Lemma 2.12 to conclude that $S = \emptyset$ and that L is a local differential operator of order N or less on all of O. □

2.3 Multi-linear local operators

We now turn to the case of multi-linear operators. We will follow the treatment of the preceding section, extending definitions and proofs where necessary.

Let $L : C^\infty(\mathbb{R}^n) \times \ldots (j \text{ times}) \ldots \times C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(\mathbb{R}^n)$ be a j-linear operator.

Definition 2.13 *We call L a local operator if it satisfies the following two conditions: for every open $U \subset \mathbb{R}^n$ and $f_1, \ldots, f_j, g_1, \ldots, g_j \in C^\infty(\mathbb{R}^n)$*

(i) $f_i \mid_U = 0$ *for any i implies that* $L(f_1, \ldots, f_j) \mid_U = 0$

(ii) $f_i \mid_U = g_i \mid_U$ *for every i implies that* $L(f_1, \ldots, f_j) \mid_U = L(g_1, \ldots, g_j) \mid_U$

Proposition 2.14 *Conditions* (i) *and* (ii) *are equivalent.*

Proof. Again, this is a direct consequence of the j-linearity of L. □

In our notation for multi-linear differential operators we will make use of lists of multi-indices. Such a list will be written $\alpha_1 \dots \alpha_j$ and called a *j-place multi-index.* We will use $\leq$ to denote the lexicographic partial order on the space of j-place multi-indices. In other words, we will use $\alpha_1 \dots \alpha_j \leq \beta_1 \dots \beta_j$ to mean that $\alpha_i \leq \beta_i$ for every i.

Definition 2.15 *We will call L locally differential of order d or less, if for every bounded $U \subset \mathbb{R}^n$ we can choose C^∞ functions $a_{\alpha_1 \dots \alpha_j}$ ($|\alpha_i| \leq d$) such that for every $f_1, \dots, f_j \in C^\infty(\mathbb{R}^n)$,*

$$L(f_1, \dots, f_j) = \sum_{|\alpha_i| \leq d} a_{\alpha_1 \dots \alpha_j} \, \partial_{\alpha_1} f_1 \cdot \dots \cdot \partial_{\alpha_j} f_j \quad ,$$

at all points of U.

Definition 2.16 *L is said to be point-wise differential of order d or less at a point, $p \in \mathbb{R}^n$ if it satisfies the following three conditions: for $f_1, \dots, f_j, g_1, \dots, g_j \in C^\infty(\mathbb{R}^n)$*

(i) *the existence of an i such that $\partial_\alpha f_i(p) = 0$ for all $|\alpha| \leq d$ implies that*

$$L(f_1, \dots, f_j)(p) = 0 \quad .$$

(ii) *$\partial_\alpha f_i(p) = \partial_\alpha g_i(p)$ for all i and $|\alpha| \leq d$ implies that*

$$L(f_1, \dots, f_j)(p) = L(g_1, \dots, g_j)(p) \quad .$$

(iii) *There exists real constants $a_{\alpha_1 \dots \alpha_j ; p}$ ($|\alpha_i| \leq d$) such that*

$$L(f_1, \dots, f_j)(p) = \sum_{|\alpha_i| \leq d} a_{\alpha_1 \dots \alpha_j ; p} \, \partial_{\alpha_1} f_1(p) \cdot \dots \cdot \partial_{\alpha_j} f_j(p) \quad , \tag{6}$$

for every $f_1, \dots, f_j \in C^\infty(\mathbb{R}^n)$.

Proposition 2.17 *The three conditions of the above definition are equivalent.*

Proof. (i)⇒(ii) Suppose that L obeys (i) and that $f_1, \ldots, f_j, g_1, \ldots, g_j$ satisfy the premise of (ii). For every $|\alpha| \leq d$ we have $\partial_\alpha(f_i - g_i) = 0$ and hence, since L is multi-linear, we have

$$\begin{aligned} L(f_1, f_2, \ldots, f_j)(p) &= L(g_1, f_2, \ldots, f_j)(p) \\ &= L(g_1, g_2, \ldots, f_j)(p) \\ &\ldots \\ &= L(g_1, g_2, \ldots, g_j)(p) \quad . \end{aligned}$$

(ii)⇒(iii) We proceed analogously to the proof of Proposition 2.5. Just as before, we need only choose the constants so that (6) holds for all basic polynomial functions of degree d or less. This can be accomplished inductively by defining $a_{\alpha_1 \ldots \alpha_j;p}$ for every $\alpha_1 \ldots \alpha_j < \beta_1 \ldots \beta_j$ and then putting

$$a_{\beta_1 \ldots \beta_j;p} = \frac{1}{\beta_1! \ldots \beta_j!} \left(L(x^{\beta_1}, \ldots, x^{\beta_j})(p) - \sum_{\alpha_1 \ldots \alpha_j < \beta_1 \ldots \beta_j} a_{\alpha_1 \ldots \alpha_j}(p) \prod_i \frac{\beta_i!}{(\beta_i - \alpha_i)!} x^{\beta_i - \alpha_i}(p) \right) \tag{7}$$

(iii)⇒(i) Trivial □

Definition 2.18 *We call N the differential order of L at p if the conditions of Definition 2.16 hold for $d = N$, but do not hold for any smaller d. If L is such that these conditions cannot be satisfied for any d, then we will say that the differential degre of L at p is ∞.*

The following lemma is the "multi-linear" analogue of Lemma 2.11. The statement of the Lemma does not change, but the proof is complicated by the fact that L takes multiple arguments. As before, let $\{p_k\}$ be a sequence of points contained in a bounded subset of $\mathbb{R}^n$. For each k, let N_k denote the differential order of L at p_k.

Lemma 2.19 *If L is local, then* $\limsup_{k\to\infty} N_k$ *is finite.*

Proof. Suppose, on the contrary, that

$$\limsup_{k\to\infty} N_k = \infty \quad .$$

We may assume without loss of generality that the sequence $\{p_k\}$ converges to a point, p, and that each $N_k > k$. According to Definition 2.16, for each k, we can choose functions $f_1^{(k)},\ldots,f_j^{(k)}$ such that for one of these functions, say $f_{a_k}^{(k)}$, we have $\partial_\alpha f_{a_k}^{(k)}(p_k) = 0$ for all $|\alpha| \le d$ and yet such that

$$L(f_1^{(k)},\ldots,f_j^{(k)})(p) \neq 0 \quad .$$

The sequence $a_1, a_2, \ldots$ consists of numbers from 1 to j and hence, one of these numbers, say a, will be repeated infinitely many times. We can therefore, assume without loss of generality that $a_k = a$ for all k. As before, choose disjoint, open neighborhoods, O_k, around each p_k. For each k and each $i \neq a$, multiply $f_i^{(k)}$ by an appropriate "plateau" function so that the resulting function, let us call it $g_i^{(k)}$, has its support contained in O_k and so that $f_i^{(k)} \equiv g_i^{(k)}$ in some neighborhood of p_k. We know nothing about how big $\|g_i^{(k)}\|_d$ is, and thus cannot be sure that $\sum_k g_i^{(k)}$ will be continuous, much less C^∞. We can, however, choose constants $C_i^{(k)}$ so that $\|C_i^{(k)} g_i^{(k)}\|_k < 1/k$ and put

$$g_i = \sum_k C_i^{(k)} g_i^{(k)} \quad .$$

Having done so, we see that

$$\lim_{k\to\infty} \|C_i^{(k)} g_i^{(k)}\|_d = 0 \quad ,$$

for all d and hence, by Lemma 2.9, the g_i's are of class C^∞. Now, choose constants $C_a^{(k)}$ so that the numbers

$$C_a^{(k)} \left(\prod_{i\neq a} C_i^{(k)}\right) \; L(f_1^{(k)},\ldots,f_j^{(k)})(p_k) \quad ,$$

increase without bound as $k \to \infty$. Since $\partial_\alpha f_a^{(k)}(p_k) = 0$ for all $|\alpha| \le d$, we can use Lemma 2.10 to choose $g_a^{(k)}$ so that the support of each $g_a^{(k)}$ is contained in O_k, so that $g_a^{(k)} \equiv f_a^{(k)}$ in some neighborhood of p_k, and so that

$$\|g_a^{(k)}\|_k < \frac{1}{kC_a^{(k)}} \quad .$$

We've chosen the $g_a^{(k)}$'s in such way that

$$\lim_{k\to\infty} \|C_a^{(k)} g_a^{(k)}\|_d = 0 \quad ,$$

for every d. That will allow us to put

$$g_i = \sum_k C_a^{(k)} g_a^{(k)}$$

and have g_a be of class C^∞. Since L is local and multi-linear, we also have for each k that

$$\begin{aligned} L(g_1, \dots, g_j)(p_k) &= L(C_1^{(k)} f_1^{(k)}, \dots, C_j^{(k)} f_j^{(k)})(p_k) \\ &= \left(\prod_i C_i^{(k)}\right) L(f_1^{(k)}, \dots, f_j^{(k)})(p_k) \quad , \end{aligned}$$

thereby implying that

$$\lim_{k\to\infty} L(g_1, \dots, g_j)(p_k) = \infty \quad ,$$

which is impossible because we had assumed that $L(g_1, \dots, g_j)$ is continuous and that $p_k \to p$. □

We also need to give a multi-linear analogue of Lemma 2.12.

Lemma 2.20 *Let O be an open, bounded subset of $\mathbb{R}^n$. Suppose that the differential order of L is less than or equal to N at all but a finite number of points of O. Then, the differential order of L is less than or equal to N at all points of O, and furthermore, L is a local differential operator of order N or less on O.*

Proof. For each choice of a j-place multi-index $\beta_1 \ldots \beta_j$ ($|\beta| \leq N$), inductively define C^∞ functions $a_{\beta_1 \ldots \beta_j}$ by first defining all $a_{\alpha_1 \ldots \alpha_j}$ for $\alpha_1 \ldots \alpha_j < \beta_1 \ldots \beta_j$ and then putting

$$a_{\beta_1 \ldots \beta_j} = \frac{1}{\beta_1! \ldots \beta_j!} \left(L(x^{\beta_1}, \ldots, x^{\beta_j}) - \sum_{\alpha_1 \ldots \alpha_j < \beta_1 \ldots \beta_j} a_{\alpha_1 \ldots \alpha_j} \prod_i \frac{\beta_i!}{(\beta_i - \alpha_i)!} x^{\beta_i - \alpha_i} \right) \quad .$$

Let p be a point where we know L to be point-wise differential. Recalling how the coefficients $a_{\beta_1 \ldots \beta_j, p}$ were defined in (7) we can see that $a_{\beta_1 \ldots \beta_j, p} = a_{\beta_1 \ldots \beta_j}(p)$. In other words,

$$L(f_1, \ldots, f_j) = \sum_{|\beta_i| \leq N} a_{\beta_1 \ldots \beta_j} \partial_{\beta_1} f_1 \cdot \ldots \cdot \partial_{\beta_j} f_j \quad ,$$

at all but finitely many points of O. But, both sides of the above equation are assumed to be continuous and therefore we can conclude that the above relation holds everywhere on O. □

Theorem 2.7 remains true for a multi-linear L; there is no point in restating either the theorem or its proof. We should only note that the proof of the present version should use Lemmas 2.19 and 2.20 rather than Lemmas 2.11 and 2.12.

2.4 Operators with Tensor Valued Arguments

It is not difficult to generalize the preceding results to operators whose arguments are tensor fields on a given C^∞ manifold, M. As a canonical example we will consider a local multi-linear operator

$$L : V(M) \times \ldots \times V(M) \text{ (k times)} \longrightarrow C^\infty(M)$$

wherer $V(M)$ denotes the vector space of all C^∞ vector fields on M. Let us fix a system of local coordinates $(x_1, \ldots, x_n)$ on M, thereby identifying an open subset of M with an open $U \subset \mathbb{R}^n$. Accordingly, L induces n^k different multi-linear operators

$$L^{j_1 \ldots j_k} : C^\infty(U) \times \ldots \times C^\infty(U) \text{ (k times)} \longrightarrow C^\infty(U) \quad 1 \leq j_1, \ldots, j_k \leq n$$

which are given by

$$L^{j_1 \dots j_k}(f_1, \dots, f_n) = L\left(f_1 \frac{\partial}{\partial x_{j_1}}, \dots, f_k \frac{\partial}{\partial x_{j_k}}\right)$$

The above expression is meaningful even though the vector fields $f_i \partial/\partial x_{j_i}$ are only locally defined. This is because L is local and because a locally defined vector field can always be extended to a global one [2]. It is easy to see that the assumption that L is local implies that each $L^{j_1 \dots j_k}$ is local as well. Therefore, we can use the Theorem 2.7 to show that each $L^{j_1 \dots j_k}$ is locally differential, i.e.

$$L^{j_1 \dots j_k}(f_1, \dots, f_k) = \sum_{\alpha_1 \dots \alpha_k} \partial_{\alpha_1} f_1 \cdot \dots \cdot \partial_{\alpha_k} f_k \; a^{\alpha_1 \dots \alpha_k}_{j_1 \dots j_k}$$

where the sum is formally infinite, although only finitely many of the coefficient functions $a^{\alpha_1 \dots \alpha_k}_{j_1 \dots j_k}$ are not identically zero on any given bounded subset of U. Since $X = \sum_i X^i \partial/\partial x_i$, the action of L can be expressed as

$$L(X) = \sum_{\alpha_1 \dots \alpha_k} \sum_{j_1 \dots j_k} \partial_{\alpha_1} X^{j_1} \cdot \dots \cdot \partial_{\alpha_k} X^{j_k} \; a^{\alpha_1 \dots \alpha_k}_{j_1 \dots j_k}$$

Therefore, L is a locally differential operator.

[2] One can, for instance, multiply the given vector field by an appropriate "plateau" function

3 Invariant Tensors

In the present section we are going to study a certain naturally defined representation of gl(U), where U is a finite-dimensional real vector space, on the tensor algebra of U and give a result that characterizes the invariant tensors. The methods and ideas for doing so come from chapter 3 of H. Weyl's book [9], although we are going to work without coordinates and use Chapter 1 of [8] as the source of our multi-linear algebra notation. The characterization of tensors which are invariant under the actions of gl(U) will provide us with a proof of an important theorem in Section 4.

In this section we will only consider real vector spaces. $\mathrm{Hom}(U, V)$ will denote the vector space of linear maps from vector space U to vector space V, and $\mathrm{End}(U)$ will denote $\mathrm{Hom}(U, U)$, the linear maps of U into itself. U^* will denote $\mathrm{Hom}(U, \mathbb{R})$, the dual space of U. $C^p(U)$ will denote $U \otimes \ldots \otimes U$ (p times), the vector space of tensors of type $(p, 0)$. $S^p(U) \subset C^p(U)$ will denote the space of symmetric tensors and $\Lambda^p(U) \subset C^p(U)$ will denote the space of skew-symmetric tensors. $1_U \in \mathrm{End}(U)$ will denote the identity endomorphism of U. For $L \in \mathrm{End}(U)$, we will use $L^t \in \mathrm{End}(U^*)$ to denote the transpose of L.

3.1 Lie Algebra Representations

We begin by recalling the basic facts about Lie algebra representations. Let U be a real vector space. The Lie algebra, gl(U) is defined as the vector space $\mathrm{End}(U)$ together with the bracket operation

$$[a, b] = ab - ba, \quad a, b \in \mathrm{End}(U) \quad .$$

Suppose that $\mathfrak{g}$ is a Lie algebra over the reals. A Lie algebra homomorphism

$$\mathcal{L} : \mathfrak{g} \longrightarrow \mathrm{gl}(U)$$

is called a representation of $\mathfrak{g}$ on U. In other words, $\mathcal{L}$ is a linear map such that

$$\mathcal{L}(a)\mathcal{L}(b) - \mathcal{L}(b)\mathcal{L}(a) = \mathcal{L}([a,b]) \quad ,$$

for all $a, b \in \mathfrak{g}$. We call $\mathcal{L}(a) \in \mathrm{End}(U)$ the action of a on U and for notational convenience write it as $\mathcal{L}_a$. Let us remark that throughout Section 3 we will be using the symbol $\mathcal{L}$ to denote representations of Lie algebras. Ambiguity can arise when we will be considering several representations at once. In these circumstances we will write $\mathcal{L}$ with a superscript so as to indicate exactly the space on which $\mathfrak{g}$ is operating.

Definition 3.21 *We call $u \in U$* **invariant under the action of** $\mathfrak{g}$ *if $\mathcal{L}_a u = 0$ for all $a \in \mathfrak{g}$.*

Suppose that we have representations of $\mathfrak{g}$ on real vector spaces U and V. There is a natural way to define representations of $\mathfrak{g}$ on $U \otimes V$ and $\mathrm{Hom}(U, V)$. Let us first recall the "universal" property that characterizes the tensor products of vector spaces.

Proposition 3.22 *Let U_i $(i = 1, 2, \ldots, k)$ be real vector spaces and suppose that φ is multi-linear mapping of $U_1 \times \ldots \times U_k$ into another real vector space, V. Then, there exists a unique linear map $\tilde{\varphi} : U_1 \otimes \ldots \otimes U_K \longrightarrow V$ such that for all $u_i \in U_i$ we have*

$$\varphi(u_1, \ldots, u_k) = \tilde{\varphi}(u_1 \otimes \ldots \otimes u_k) \quad .$$

Corollary 3.23 *Let U_i's be as above and let $L_i \in \mathrm{End}(U_i)$ be given endomorphisms. Then there is a unique endomorphism of $U_1 \otimes \ldots \otimes U_k$, which we denote by $L_1 \otimes \ldots \otimes L_k$, whose action is given by*

$$u_1 \otimes \ldots \otimes u_k \mapsto L_1 u_1 \otimes \ldots \otimes L_k u_k, \quad u_i \in U_i,\ i = 1, 2, \ldots, k \quad .$$

Now, we can show how to construct a representation of $\mathfrak{g}$ on $U \otimes V$. For $a \in \mathfrak{g}$ put

$$\mathcal{L}_a^{U\otimes V} = \mathcal{L}_a^U \otimes 1_V + 1_U \otimes \mathcal{L}_a^V \quad .$$

Proposition 3.24 *The above formula defines a representation, $\mathcal{L}^{U\otimes V}$, of $\mathfrak{g}$ on $U\otimes V$.*

Proof. For $a, b \in \mathfrak{g}$ we have

$$\mathcal{L}_a^{U\otimes V}\mathcal{L}_b^{U\otimes V} = \mathcal{L}_a^U\mathcal{L}_b^U \otimes 1_V + \mathcal{L}_b^U \otimes \mathcal{L}_a^V + \mathcal{L}_a^U \otimes \mathcal{L}_b^V + 1_U \otimes \mathcal{L}_a^V\mathcal{L}_b^V \quad .$$

$\mathcal{L}_b^{U\otimes V}\mathcal{L}_a^{U\otimes V}$ has the same expression, save that a and b are switched. Taking the difference of the two expressions we see that the middle two terms of each cancel and thus we have

$$\begin{aligned}[\mathcal{L}_a^{U\otimes V}, \mathcal{L}_b^{U\otimes V}] &= \mathcal{L}_a^U\mathcal{L}_b^U \otimes 1_V + 1_U \otimes \mathcal{L}_a^V\mathcal{L}_b^V - \mathcal{L}_b^U\mathcal{L}_a^U \otimes 1_V + 1_U \otimes \mathcal{L}_b^V\mathcal{L}_a^V \\ &= \mathcal{L}_{[a,b]}^{U\otimes V} \quad .\end{aligned}$$

□

Suppose that we have representations of $\mathfrak{g}$ on vector spaces $U_1, \ldots, U_k$. The above construction of a representation of $\mathfrak{g}$ on a tensor product of two spaces easily generalizes to a method for constructing a representation of $\mathfrak{g}$ on $P = U_1 \otimes \ldots \otimes U_k$. For $a \in \mathfrak{g}$ put

$$\mathcal{L}_a^P = \mathcal{L}_a^{U_1} \otimes 1_{U_2} \otimes \ldots \otimes 1_{U_k} + 1_{U_1} \otimes \mathcal{L}_a^{U_2} \otimes \ldots \otimes 1_{U_k} + 1_{U_1} \otimes \ldots \otimes 1_{U_{k-1}} \otimes \mathcal{L}_a^{U_k} \quad . \tag{8}$$

Proposition 3.25 *$\mathcal{L}^P$, as defined above, is a representation of $\mathfrak{g}$ on $U_1 \otimes \ldots \otimes U_k$.*

Proof. Put $P' = U_2 \otimes \ldots \otimes U_3$ and note that

$$\mathcal{L}_a^P = \mathcal{L}_a^{U_1} \otimes 1_{P'} + 1_{U_1} \otimes \mathcal{L}_a^{P'} \quad .$$

Since $P = U_1 \otimes P'$ we see that $\mathcal{L}^P$ is a representation, if $\mathcal{L}^{P'}$ is one too. We can now use induction on k to conclude that $\mathcal{L}^P$ is a representation of $\mathfrak{g}$. □

Now we turn to the extension of a representation of $\mathfrak{g}$ on U and V to a representation of $\mathfrak{g}$ on $H = \mathrm{Hom}(U, V)$. For $a \in \mathfrak{g}$ and $\varphi \in \mathrm{Hom}(U, V)$ put

$$\mathcal{L}_a^H \varphi = \mathcal{L}_a^V \varphi - \varphi \mathcal{L}_a^U \quad .$$

Proposition 3.26 $\mathcal{L}^H$, *as defined above, is a representation of* $\mathfrak{g}$ *on* $\mathrm{Hom}(U, V)$.

Proof. It is easy to verify that $\mathcal{L}_a^H$, as given above, is an endomorphism of $\mathrm{Hom}(U, V)$. For $a, b \in \mathfrak{g}$ we have

$$\begin{aligned} \mathcal{L}_a^H \mathcal{L}_b^H \varphi &= \mathcal{L}_a^V(\mathcal{L}_b^H \varphi) - (\mathcal{L}_b^H \varphi)\mathcal{L}_a^U \\ &= \mathcal{L}_a^V \mathcal{L}_b^V \varphi - \mathcal{L}_a^V \varphi \mathcal{L}_b^U - \mathcal{L}_b^V \varphi \mathcal{L}_a^U + \varphi \mathcal{L}_b^U \mathcal{L}_a^U \quad . \end{aligned}$$

The expression for $\mathcal{L}_b^H \mathcal{L}_a^H \varphi$ is the same, save that a and b are switched. Taking the difference of the two expressions and cancelling the middle two terms from each we have

$$\begin{aligned} [\mathcal{L}_a^H, \mathcal{L}_b^H]\varphi &= \mathcal{L}_a^V \mathcal{L}_b^V \varphi + \varphi \mathcal{L}_b^U \mathcal{L}_a^U - \mathcal{L}_b^V \mathcal{L}_a^V \varphi - \varphi \mathcal{L}_a^U \mathcal{L}_b^U \\ &= \mathcal{L}_{[a,b]}^H \varphi \quad . \end{aligned}$$

□

Let us assume that the only one-dimensional representation of $\mathfrak{g}$ we shall use is the trivial representation. In other words we define $\mathcal{L}_a^{\mathbf{R}}$ to be multiplication by zero for all $a \in \mathfrak{g}$. The preceding discussion gives us a natural way to define a representation of $\mathfrak{g}$ on $U^* = \mathrm{Hom}(U, \mathbb{R}^n)$. We define $\mathcal{L}_a^{U^*}$ to be the element of $\mathrm{End}(U^*)$ such that

$$(\mathcal{L}_a^{U^*} \alpha)u = -\alpha(\mathcal{L}_a^U u) \quad , \tag{9}$$

for all $\alpha \in U^*$ and $u \in U$. In other words, $\mathcal{L}_a^{U^*}$ is the negative of $(\mathcal{L}_a^U)^t$.

If one deals with finite-dimensional vector spaces, then there are several naturally occurring isomorphisms between the spaces constructed with the $\otimes$, Hom, and the duality operations. The representations of $\mathfrak{g}$ constructed above will coincide under such identifications.

Let U and V be finite dimensional vector spaces. There is a natural isomorphism between $V \otimes U^*$ and $H = \mathrm{Hom}(U, V)$ that identifies $v \otimes \alpha \in V \otimes U^*$ with the homomorphism

$$u \mapsto \alpha(u)v, \quad u \in U .$$

Proposition 3.27 *With the above identification we have,* $\mathcal{L}^H = \mathcal{L}^{V \otimes U^*}$.

Proof. Let $v \otimes \alpha \in V \otimes U^*$ be given and let us identify it with an element of $\mathrm{Hom}(U, V)$ as per above. For $L^U \in \mathrm{End}(U)$ and $L^V \in \mathrm{End}(V)$ it is easy to verify that

$$L^V(v \otimes \alpha) = L^V v \otimes \alpha \quad ,$$

and that

$$(v \otimes \alpha)L^U = v \otimes (L^U)^t \alpha \quad .$$

Of course, $\mathcal{L}_a^U$ is an endomorphism of U and $\mathcal{L}_a^V$ is an endomorphism of V and thus,

$$\begin{aligned} \mathcal{L}_a^H(v \otimes \alpha) &= \mathcal{L}_a^V(v \otimes \alpha) - (v \otimes \alpha)\mathcal{L}_a^U \\ &= \mathcal{L}_a^V v \otimes \alpha - v \otimes (\mathcal{L}_a^U)^t \alpha \\ &= \mathcal{L}_a^V v \otimes \alpha + v \otimes \mathcal{L}_a^{U^*} \alpha \quad . \end{aligned}$$

□

For finite dimensional vector spaces U_i $(i = 1, 2, \ldots, k)$ there is a natural isomorphism between $U_1^* \otimes \ldots \otimes U_k^*$ and $(U_1 \otimes \ldots \otimes U_k)^*$ that identifies $\alpha_1 \otimes \ldots \otimes \alpha_k$ $(\alpha_i \in U_i^*)$

with the linear form

$$u_1 \otimes \ldots \otimes u_k \mapsto \prod_i \alpha_i(u_i) \quad (u_i \in U_i) \quad .$$

Proposition 3.28 *With the above identification the representation of* $\mathfrak{g}$ *on* $(U_1 \otimes \ldots \otimes U_k)^*$ *coincides with the representation on* $U_1^* \otimes \ldots \otimes U_k^*$.

Proof. Let $L_i \in \mathrm{End}(U_i)$ be given. It is easy to verify that with the above isomorphism

$$(L_1 \otimes \ldots \otimes L_k)^t \in \mathrm{End}((U_1 \otimes \ldots \otimes U_k)^*)$$

corresponds to

$$L_1^t \otimes \ldots \otimes L_k^t \in \mathrm{End}(U_1^* \otimes \ldots \otimes U_k^*) \quad .$$

Hence, the action of $a \in \mathrm{gl}(U)$ on $(U_1 \otimes \ldots \otimes U_k)^*$ corresponds to the following endomorphism of $U_1^* \otimes \ldots \otimes U_k^*$:

$$-\left(\mathcal{L}_a^{U_1}\right)^t \otimes 1_{U_2}^t \otimes \ldots \otimes 1_{U_k}^t - 1_{U_1}^t \otimes \left(\mathcal{L}_a^{U_2}\right)^t \otimes \ldots \otimes 1_{U_k}^t - 1_{U_1}^t \otimes \ldots \otimes 1_{U_{k-1}}^t \otimes \left(\mathcal{L}_a^{U_k}\right)^t \quad .$$

The latter expression is equal to the action of a on $U_1^* \otimes \ldots \otimes U_k^*$ because $1_{U_i}^t = 1_{U_i^*}$ and because $-\left(\mathcal{L}_a^{U_i}\right)^t = \mathcal{L}_a^{U_i^*}$. □

3.2 The Representation of gl(U) on the Tensor Algebra of U

Let U be a finite-dimensional real vector space. In this section we will be interested in the representation of gl(U) on the tensor algebra of U. As per (9), we can define a representation of gl(U) on U^* by defining $\mathcal{L}^{U^*}$ to be the negative of the transpose operation. Since $T(U)$, the tensor algebra of U, is a direct sum of the various spaces obtained by tensoring together copies of U and U^*, we can use (8) to define a representation of gl(U) on all of $T(U)$.

The question of whether a tensor is invariant under the action of $\mathrm{gl}(U)$ is easy to answer if the contravariant degree of the tensor is not equal to its covariant degree. Indeed, let us consider the representation of $\mathrm{gl}(U)$ on $U_1 \otimes \ldots \otimes U_k$, where each U_i $(i = 1, 2, \ldots, k)$ is either U or U^*. Let r be the number of times that $U_i = U$ and s the number of times that $U_i = U^*$.

Theorem 3.29 *If $r \neq s$ then 0 is the only element of $U_1 \otimes \ldots \otimes U_k$ that is invariant under the action of* $\mathrm{gl}(U)$.

Proof. Consider how $1_U \in \mathrm{End}(U)$ acts on $T(U)$. From (9) we see that the action of 1_U on U^* is -1_{U^*} and hence, by (8) the action of 1_U on $U_1 \otimes \ldots \otimes U_k$ must be

$$(r - s)1_{U_1} \otimes \ldots \otimes 1_{U_k} = (r - s)1_{U_1 \otimes \ldots \otimes U_k} \quad .$$

Therefore the only element of the latter space annihilated by the action of 1_U is 0.

□

Characterizing invariant tensors of equal covariant and contravariant degree is considerably more difficult. Let us fix p and put $T = \bigotimes^p(U)$. The space $\mathrm{End}(T)$ is identified with $T \otimes T^*$, and since T^* is identified with $\bigotimes^p(U^*)$ we can view $\mathrm{End}(T)$ as a space of tensors of type (p, p). Propositions 3.27 and 3.28 tell us that we get the same representation of $\mathrm{gl}(U)$ on $\mathrm{End}(T)$ regardless of whether we view $\mathrm{End}(T)$ as a space of endomorphisms or as the tensor product of p copies of U and p copies of U^*. Therefore, a tensor, φ, of type (p, p) is invariant under the action of $\mathrm{gl}(U)$ if and only if φ, when viewed as an element of $\mathrm{End}(T)$, satisfies

$$\mathcal{L}_a^T \varphi - \varphi \mathcal{L}_a^T = 0 \quad ,$$

for all $a \in \mathrm{gl}(U)$. Thus we have to find the $\varphi \in \mathrm{End}(T)$ that commute with all $\mathcal{L}_a^T$. In order to do so we have to introduce the permutation group on p letters, Π_p.

The group of permutations acts on the space T in the following way: for a permutation $\pi \in \Pi_p$ we define

$$\pi(u_1 \otimes \ldots \otimes u_p) = u_{\pi_1} \otimes \ldots \otimes u_{\pi_p} , \quad u_i \in U ,$$

thereby associating π with an element of $\mathrm{End}(T)$.

Proposition 3.30 *Let $a \in \mathrm{gl}(U)$ and $\pi \in \Pi_p$ be given. Then, $\mathcal{L}_a^T$ commutes with the action of π on T.*

Proof. It is a matter of direct verification that both $\mathcal{L}_a^T \pi$ and $\pi \mathcal{L}_a^T$ map $u_1 \otimes \ldots \otimes u_p$ ($u_i \in U$) to

$$a u_{\pi_1} \otimes u_{\pi_2} \otimes \ldots \otimes u_{\pi_p} + u_{\pi_1} \otimes a u_{\pi_2} \otimes \ldots \otimes u_{\pi_p} + \ldots + u_{\pi_1} \otimes u_{\pi_2} \otimes \ldots \otimes a u_{\pi_p} \quad .$$

□

Thus, the action of a permutation on T corresponds to a type (p, p) tensor that is invariant under the action of $\mathrm{gl}(U)$. The relation of the permutation group to the invariant tensors is actually even stronger; we can classify all invariant type (p, p) tensors in terms of the action of Π_p on T.

Theorem 3.31 *Suppose that $\varphi \in \mathrm{End}(T)$ commutes with $\mathcal{L}_a^T$ for every $a \in \mathrm{gl}(U)$. Then φ can be given as a linear combination of permutation actions; i.e. there are real coefficients c_π ($\pi \in \Pi_p$) such that*

$$\varphi(t) = \sum_{\pi} c_\pi \pi(t) , \quad t \in T .$$

We will need to develop some preliminary results before proving this theorem. Recall that $S^p(U)$ is generated by elements of the form

$$\mathcal{S}(u_1 \otimes \ldots \otimes u_p) , \quad u_i \in U .$$

where $\mathcal{S}$ is the symmetrization operator

$$\mathcal{S}(t) = \frac{1}{p!} \sum_{\pi \in \Pi_p} \pi(t) \ . \quad t \in T \ .$$

Since $\mathrm{End}(U)$ is isomorphic to $U \otimes U^*$ and $\mathrm{End}(T)$ is isomorphic to $C^p(U) \otimes C^p(U^*)$, we can identify $\mathrm{End}(T)$ with

$$C^p(\mathrm{End}(U)) = (U \otimes U^*) \otimes \ldots \otimes (U \otimes U^*) \ (p \text{ times}) \ .$$

Π_p acts on $\mathrm{End}(T)$ in a manner analogous to the way it acts on T. For $\pi \in \Pi_p$ and $a_i \in \mathrm{End}(U)$,

$$\pi(a_1 \otimes \ldots \otimes a_p) = a_{\pi_1} \otimes \ldots \otimes a_{\pi_p} \quad .$$

Thus, the symmetrization operator on $C^p(\mathrm{End}(T))$ can be expressed as

$$\mathcal{S}(a_1 \otimes \ldots \otimes a_p) = \frac{1}{p!} \sum_{\pi} a_{\pi_1} \otimes \ldots \otimes a_{\pi_p} \quad .$$

In the subsequent discussion we will need a formula for the composition of two elements of $S^p(\mathrm{End}(T))$. For $a_i, b_i \in \mathrm{End}(U)$, the composition of $a_1 \otimes \ldots \otimes a_p$ and $b_1 \otimes \ldots \otimes b_p$ is just $a_1 b_1 \otimes \ldots \otimes a_p b_p$. Using this identity, it is not hard to verify that

$$\mathcal{S}(a_1 \otimes \ldots \otimes a_p)\mathcal{S}(b_1 \otimes \ldots \otimes b_p) = \frac{1}{p!} \sum_{\pi} \mathcal{S}(a_1 b_{\pi_1} \otimes \ldots \otimes a_p b_{\pi_p}) \quad . \tag{10}$$

We should also note that there is a close relation between the action of Π_p on T and the action of Π_p on $\mathrm{End}(T)$.

Proposition 3.32 *Let $\varphi \in \mathrm{End}(T)$ and $\pi \in \Pi_p$ be given. Then,*

$$\pi(\varphi) = \pi \varphi \pi^{-1} \quad ,$$

where on the right hand side π and π^{-1} denote the elements of $\mathrm{End}(T)$ that are associated with these two permutations.

Proof. The proof is a matter of verifying the above for $\varphi = a_1 \otimes \ldots \otimes a_p$. □

In view of this proposition, the symmetrization operator on $C^p(\mathrm{End}(T))$ can be expressed in terms of the action of Π_p on T. Indeed, for $\varphi \in \mathrm{End}(T)$ we have

$$\mathcal{S}(\varphi) = \frac{1}{p!} \sum_{\pi} \pi \varphi \pi^{-1} \quad .$$

We will need the following definition in order to prove an upcoming result.

Definition 3.33 *For $k = 1, 2, \ldots, p$, let us call elements of $S^p(\mathrm{End}(U)) \subset \mathrm{End}(T)$ that have the form*

$$\mathcal{S}(a_1 \otimes \ldots \otimes a_k \otimes 1_U \otimes \ldots \otimes 1_U), \quad a_i \in \mathrm{End}(U) \ ,$$

symmetric endomorphisms of order k.

Lemma 3.34 $\varphi \in \mathrm{End}(T)$ *commutes with all $\mathcal{L}_a^T$ if and only if φ commutes with all elements of $S^p(\mathrm{End}(U))$.*

Proof. Let us note that

$$\mathcal{L}_a^T = p\,\mathcal{S}(a \otimes 1_U \otimes \ldots \otimes 1_U) \quad ;$$

in other words, $\{\mathcal{L}_a^T : a \in \mathrm{gl}(U)\}$ is just the set of symmetric endomorphisms of order 1. Therefore, if φ commutes with all symmetric endomorphisms, it commutes in particular with all elements of $\{\mathcal{L}_a^T : a \in \mathrm{gl}(U)\}$.

Next, we will show that a symmetric endomorphism of any order (i.e. any element of $S^p(\mathrm{End}(U))$) can be generated by symmetric elements of order 1 (i.e. elements of $\{\mathcal{L}_a^T : a \in \mathrm{gl}(U)\}$). This will prove that if φ commutes with all $\mathcal{L}_a^T$ then it must commute with all of $S^p(\mathrm{End}(U))$. A calculation of the composition of two symmetric endomorphisms of order 1 reveals the key to the proof. Using (10) we see that $\mathcal{S}(a \otimes 1_U \otimes \ldots \otimes 1_U)$ composed with $\mathcal{S}(b \otimes 1_U \otimes \ldots \otimes 1_U)$ is

$$\frac{p-1}{p}\,\mathcal{S}(a \otimes b \otimes 1_U \otimes \ldots \otimes 1_U) + \frac{1}{p}\,\mathcal{S}(ab \otimes 1_U \otimes \ldots \otimes 1_U) \quad .$$

The second term of the right hand side is just a symmetric endomorphism of order 1, and therefore, symmetric endomorphism of order 1 can generate ones of order 2. More generally, a symmetric endomorphism of order 1 composed with one of order k gives

$$\begin{aligned}\mathcal{S}(a \otimes 1_U \otimes \ldots \otimes 1_U)\mathcal{S}(b_1 \otimes \ldots \otimes b_k \otimes 1_U \ldots \otimes 1_U) = \\ \frac{p-k}{p}\,\mathcal{S}(a \otimes b_1 \otimes \ldots \otimes b_k \otimes 1_U \otimes \ldots \otimes 1_U) + \\ \frac{1}{p}\sum_{j=1}^{k} \mathcal{S}(ab_j \otimes b_1 \otimes \ldots \widehat{b_j} \ldots \otimes b_k \otimes 1_U \otimes \ldots \otimes 1_U) \quad ,\end{aligned}$$

and therefore, if symmetric endomorphisms of order 1 can generate ones of order k, then they can also generate ones of order $k+1$. Proceeding by induction on k we can conclude that symmetric endomorphisms of order 1 generate symmetric endomorphisms of orders $2, 3, \ldots, p$. □

Let V be a finite dimensional vector space and let $\langle \cdot, \cdot \rangle$ be a bilinear form on $V \times V$. Recall that $\langle \cdot, \cdot \rangle$ is called symmetric if for all $u, v \in V$ we have $\langle u, v \rangle = \langle v, u \rangle$ and that it is called positive semi-definite if for every $u \in V$ we have $\langle u, u \rangle \geq 0$. A sequence u_i in V is called almost orthonormal with respect to $\langle \cdot, \cdot \rangle$ if $\langle u_i, u_i \rangle$ is either 0 or 1 and if $\langle u_i, u_j \rangle = 0$ whenever $i \neq j$.

Lemma 3.35 *Suppose that $\langle \cdot, \cdot \rangle$ is a bilinear, symmetric, positive semi-definite form on $V \times V$. Then, there exists an almost orthonormal basis, u_i, of V with respect to $\langle \cdot, \cdot \rangle$.*

Proof. We will first prove that if for some $u \in V$ we have $\langle u, u \rangle = 0$, then $\langle u, v \rangle = 0$ for all $v \in V$. Suppose that $\langle u, u \rangle = 0$. Then for every $v \in V$ and every real number $x \neq 0$, we have

$$\langle u + xv, u + xv \rangle = 2x\langle u, v\rangle + x^2\langle v, v\rangle \geq 0 \quad ,$$

and hence, $2/x\langle u, v\rangle + \langle v, v\rangle \geq 0$. Since x can be chosen arbitrarily, the latter is possible only if $\langle u, v\rangle = 0$.

Let u_i be an almost orthonormal, linearly independent sequence that doesn't span all of V. Choose a $v \in V$ which is not a linear combination of the u_i and put $u = v - \sum_i \langle v, u_i\rangle u_i$. Because of the way v was chosen u cannot be a linear combination of the u_i's either. Note that $\langle u, u_j\rangle = \langle v, u_j\rangle(1 - \langle u_j, u_j\rangle)$. Certainly, if $\langle u_j, u_j\rangle = 1$ we must have $\langle u, u_j\rangle = 0$. If on the other hand $\langle u_j, u_j\rangle = 0$, we remarked earlier that $\langle u, u_j\rangle = 0$. Therefore, $\langle u, u_j\rangle = 0$ for every j. By multiplying u by a constant, if necessary, we can assure that $\langle u, u\rangle$ is either 0 or 1. Appending u to the sequence u_i creates a longer, almost orthonormal, linearly independent sequence. A almost orthonormal basis of V can be found by using the above argument in an inductive construction . □

Lemma 3.36 *Let* $\omega \in \mathrm{End}(V)$ *be such that* $\mathrm{tr}(\omega\varphi) = \mathrm{tr}(\varphi)$ *for all* $\varphi \in \mathrm{End}(V)$. *Then,* $\omega = 1_V$.

Proof. For every $v \in V$ and every $\alpha \in V^*$ we have

$$\alpha(\omega v) = \mathrm{tr}(\omega(v \otimes \alpha)) = \mathrm{tr}(v \otimes \alpha) = \alpha(v) \quad .$$

This is only possible if $\omega v = v$. □

Lemma 3.37 *There exists a* $\psi \in \mathrm{End}(T)$ *such that*

$$\sum_\pi \mathrm{tr}(\psi\pi^{-1})\pi = 1_T \quad .$$

Proof. Choose a basis, $e_1, \ldots, e_n$, of U and let $\{e_{i_1 \ldots i_p}^{j_1 \ldots j_p}\}$ be the corresponding basis for End(T). Let π be a permutation. It is easy to verify the following three facts:

$$\begin{aligned} \pi e_{i_1 \ldots i_p}^{j_1 \ldots j_p} &= e_{i_1 \ldots i_p}^{j_{\pi_1} \ldots j_{\pi_p}} \quad , \\ e_{i_1 \ldots i_p}^{j_1 \ldots j_p} \pi^{-1} &= e_{i_{\pi_1} \ldots i_{\pi_p}}^{j_1 \ldots j_p} \quad , \\ \operatorname{tr}(e_{i_1 \ldots i_p}^{j_1 \ldots j_p}) &= \delta_{i_1}^{j_1} \cdot \ldots \cdot \delta_{i_p}^{j_p} \quad . \end{aligned}$$

A very simple proof of the lemma is available in the case when $p \leq n$, where n is the dimension of the underlying vector space, U. The general case requires a technique of some sophistication, and so it seems worthwhile to furnish a separate proof for the simple case.

Suppose that $p \leq n$. Put $\psi = e_{12 \ldots p}^{12 \ldots p}$. For a permutation, π, the preceding identities show that $\operatorname{tr}(\psi\pi^{-1})$ is equal to one if π is the identity permutation and is equal to zero otherwise. Therefore $\psi = \sum_\pi \operatorname{tr}(\psi\pi^{-1})\pi$ must be the identity map.

Let us now give the general proof. Define the transpose operation on End(T) with respect to the above choice of basis, i.e put $(e_{i_1 \ldots i_p}^{j_1 \ldots j_p})^t = e_{j_1 \ldots j_p}^{i_1 \ldots i_p}$. Using the above identities, it is easy to verify that for $\varphi \in$ End(T) and $\pi \in \Pi_p$, we have

$$\operatorname{tr}(\varphi\pi^{-1}) = \operatorname{tr}(\pi\varphi^t) \quad .$$

For $\varphi, \theta \in$ End(T), put

$$\langle \varphi, \theta \rangle = \sum_\pi \operatorname{tr}(\pi\varphi)\operatorname{tr}(\pi\theta) \quad .$$

It is easy to check that $\langle \cdot, \cdot \rangle$ is a bilinear, symmetric, positive semi-definite form on End(T). For $\varphi \in$ End(T) note that $\langle \varphi, \varphi \rangle = \sum_\pi \operatorname{tr}(\pi\varphi)^2$ and hence $\langle \varphi, \varphi \rangle = 0$ implies that $\operatorname{tr}(\varphi) = 0$. Using Lemma 3.35, we can choose an almost orthonormal basis, ψ_i, of End(T) with respect to $\langle \cdot, \cdot \rangle$. Having done so, put

$$\omega = \sum_i \sum_\pi \operatorname{tr}(\psi_i)\operatorname{tr}(\pi\psi_i)\pi \quad .$$

Hence,

$$\mathrm{tr}(\omega\psi_j) = \mathrm{tr}(\psi_j)\langle\psi_j, \psi_j\rangle = \mathrm{tr}(\psi_j) \quad .$$

The second equality is justified because $\langle\psi_j, \psi_j\rangle$ is either 1 or 0 and because if the second case holds, we showed earlier that $\mathrm{tr}(\psi_j) = 0$. Since ψ_i is a basis for $\mathrm{End}(T)$, we must have $\mathrm{tr}(\omega\varphi) = \mathrm{tr}(\varphi)$ for all $\varphi \in \mathrm{End}(T)$ and therefore ω must be 1_T. Put

$$\psi = \sum_i \mathrm{tr}(\psi_i)\psi_i^t \quad ,$$

and note that

$$\begin{aligned}
\sum_\pi \mathrm{tr}(\psi\pi^{-1})\pi &= \sum_\pi \mathrm{tr}(\pi\psi^t)\pi \\
&= \sum_i \sum_\pi \mathrm{tr}(\psi_i)\mathrm{tr}(\pi\psi_i)\pi \\
&= \omega = 1_T \quad .
\end{aligned}$$

□

Proof of Theorem 3.31. Suppose that $\varphi \in \mathrm{End}(T)$ commutes with $\mathcal{L}_a^T$ for all $a \in \mathrm{gl}(U)$. By Lemma 3.34, φ must commute with all elements of $S^p(\mathrm{End}(U))$ as well. By Lemma 3.37 we can choose a $\psi \in \mathrm{End}(T)$ such that

$$\sum_\pi \mathrm{tr}(\psi\pi^{-1})\pi = 1_T \quad .$$

Since $\mathrm{End}(T)$ is isomorphic to $T \otimes T^*$, we can choose $u_i \in T$ and $\alpha_i \in T^*$ so that

$$\psi = \sum_i u_i \otimes \alpha_i \quad .$$

Hence, for any $v \in T$ we have

$$\begin{aligned}
v &= \sum_\pi \mathrm{tr}(\psi\pi^{-1})\pi v \\
&= \sum_\pi \sum_i (\alpha_i \pi^{-1} u_i)\pi v \\
&= \sum_i \sum_\pi \pi(v \otimes \alpha_i)\pi^{-1} u_i \\
&= \sum_i p!\, \mathcal{S}(v \otimes \alpha_i) u_i \quad .
\end{aligned}$$

φ commutes with $\mathcal{S}(v \otimes \alpha_i)$ because the latter is an element of $S^p(\mathrm{End}(U))$, and hence,

$$\begin{aligned} \varphi v &= \varphi \sum_i p!\, \mathcal{S}(v \otimes \alpha_i) u_i \\ &= \sum_i p!\, \mathcal{S}(v \otimes \alpha_i) \varphi u_i \\ &= \sum_i \sum_\pi (\alpha_i \pi^{-1} \varphi u_i) \pi v \\ &= \sum_\pi \mathrm{tr}(\varphi \psi \pi^{-1}) \pi v \quad . \end{aligned}$$

Therefore φ can be expressed as a linear combination of permutation actions, as was to be shown. □

4 A certain interesting cocycle

Let M be an n-dimensional, C^∞, paracompact manifold. We will use $V(M)$ to denote the vector space of C^∞ vector fields on M and $\mathcal{G}$ to denote the Lie algebra which consists of $V(M)$ together with the Lie bracket operation. $\Phi_k(M)$ will denote the vector space of C^∞ differential k-forms on M. For $X \in V(M)$, let

$$i(X) : \Phi_{k+1}(M) \longrightarrow \Phi_k(M)$$

denote interior multiplication with respect to X and let

$$\mathcal{L}_X : \Phi_k(M) \longrightarrow \Phi_k(M)$$

denote the Lie derivative operator with respect to X. Recall that $X \mapsto \mathcal{L}_X$ defines a representation of $\mathcal{G}$ on $\Phi_k(M)$.

Let $\mathfrak{C}^k$ denote the vector space of local, multi-linear, skew-symmetric operators

$$C : V(M) \times \ldots \times V(M) \ (k \text{ times}) \longrightarrow \Phi_2(M) \quad .$$

For $C \in \mathfrak{C}^k$ and $X_i \in V(M)$ $(1 \leq i \leq k+1)$ put

$$\begin{aligned} \partial C(X_1, \ldots, X_{k+1}) &= \sum_i (-1)^{i-1} \mathcal{L}_{X_i} C(X_1, \ldots \widehat{X_i} \ldots, X_{k+1}) + \\ &\quad \sum_{i<j} (-1)^{i+j} C([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{k+1}) \quad (11) \end{aligned}$$

A straightforward, but laborious computation shows that $\partial\partial C = 0$ and hence, that with

$$\partial : \mathfrak{C}^k \longrightarrow \mathfrak{C}^{k+1} \quad ,$$

as the coboundary operator, the $\mathfrak{C}^k$'s form a cochain complex. The resulting cohomology is referred to as the **local Chevalley cohomology** [10] associated with the representation of $\mathcal{G}$ on $\Phi_2(M)$ and denoted by $\mathrm{H}^*_{\mathrm{loc}}(\mathcal{G}, \Phi_2(M))$. We will denote this cohomology simply by $\mathrm{H}^*_{\mathrm{loc}}$.

The primary objective of this section is to describe a certain distinguished, non-trivial element S of H^2_{loc}. This cohomology class will be defined in the following way. First, there is a natural way to assign to each connection, Γ, on M a certain 2-cocycle S^Γ. Second, any two such cocycles are related by a coboundary. Since M is paracompact, a connection on M can always be found and thus S can be defined as the class of all cocycles of the type S^Γ. The latter part of this section will be devoted to the demonstration that S is not trivial, i.e. that no coboundary can be a cocycle of the type S^Γ.

4.1 The Lie derivative of a connection

Before defining a cocycle of the type S^Γ we will need to define the Lie derivative of a connection, Γ, with respect to a vector field, X. We will do so by considering the covariant derivative operator, ∇, associated with the given connection.

Recall that a covariant derivative operator associated with Γ is a multi-linear differential operator

$$\nabla : V(M) \times V(M) \longrightarrow V(M) \quad ,$$

such that for all $A, B \in V(M)$ and all $f \in C^\infty(M)$

$$\begin{aligned} \nabla_{fA}B &= f\nabla_A B \quad , \\ \nabla_A fB &= f\nabla_A B + (Af)B \quad , \end{aligned} \qquad (12)$$

and such that in terms of local coordinates we have

$$(\nabla_{\partial_i}\partial_j)^k = \Gamma^k_{ij} \quad 1 \leq i, j, k \leq n \; .$$

In fact, one can show that a differential operator ∇ which satisfies the preceding three conditions completely characterizes the connection, Γ (see [4]).

Like a connection, a type (1,2) tensor field, T, also has an associated multi-linear differential operator

$$T : V(M) \times V(M) \longrightarrow V(M) \quad ,$$

that satisfies slightly different conditions. For all $A, B \in V(M)$ and $f \in C^\infty(M)$, the differential operator, T, must satisfy

$$fT(A, B) = T(fA, B) = T(A, fB) \quad , \tag{13}$$

and

$$(T(\partial_i, \partial_j))^k = T^k_{ij} \quad .$$

It is also possible to show that a differential operator which satisfies the preceding two conditions completely characterizes the tensor field, T (see [4]).

The Lie derivative of a type (1,2) tensor field, T, with respect to vector field X can be defined (see [4]) as the type (1,2) tensor field, $\mathcal{L}_X T$, such that

$$(\mathcal{L}_X T)(A, B) = [X, T(A, B)] - T([X, A], B) - T(A, [X, B]) \quad A, B \in V(M) \quad . \tag{14}$$

The Lie derivative of a connection, Γ, with respect to X is defined analogously; i.e.

$$\mathcal{L}_X \Gamma : V(M) \times V(M) \longrightarrow V(M)$$

is the differential operator given by

$$(\mathcal{L}_X \Gamma)(A, B) = [X, \nabla_A B] - \nabla_{[X,A]} B - \nabla_A [X, B] \quad . \tag{15}$$

Proposition 4.38 *The differential operator* $\mathcal{L}_X \Gamma$ *satisfies* (13) *and is therefore, associated with a type (1,2) tensor which is given by*

$$(\mathcal{L}_X \Gamma)^k_{ij} = \partial_{ij} X^k + \sum_l X^l \partial_l \Gamma^k_{ij} - \Gamma^l_{ij} \partial_l X^k + \Gamma^k_{li} \partial_i X^l + \Gamma^k_{jl} \partial_j X^l \quad . \tag{16}$$

Proof. The proposition is proven by performing a straightforward calculation using (15) and (12). □

Since the Lie derivative operation defines a representation of $\mathcal{G}$ on the vector space of type (1,2) tensor fields on M (see [4]) we have

$$\mathcal{L}_{[X,Y]}T = \mathcal{L}_X(\mathcal{L}_Y T) - \mathcal{L}_Y(\mathcal{L}_X T) \quad ,$$

for all $X, Y \in V(M)$. An analogous identity can be proven for the Lie derivative of a connection, Γ:

$$\mathcal{L}_{[X,Y]}\Gamma = \mathcal{L}_X(\mathcal{L}_Y \Gamma) - \mathcal{L}_Y(\mathcal{L}_X \Gamma) \quad . \tag{17}$$

One merely has to perform the necessary calculation using (15) and (14).

4.2 Definition of S^Γ and S

For a real vector space, U, let

$$\phi : U \otimes U^* \otimes U^* \otimes U \otimes U^* \otimes U^* \longrightarrow U^* \otimes U^*$$

denote the following contraction

$$u_1 \otimes \alpha_1 \otimes \beta_1 \otimes u_2 \otimes \alpha_2 \otimes \beta_2 \mapsto \beta_2(u_1)\beta_1(u_2)\, \alpha_1 \otimes \alpha_2 \quad u_i \in U,\ \alpha_i, \beta_i \in U^*,\ i = 1, 2 \ .$$

Now suppose that A and B are type (1,2) tensor fields on M. We define $A \diamond B$ to be the element of $\Phi_2(M)$ given by

$$A \diamond B = \phi(A \otimes B) - \phi(B \otimes A) \quad .$$

It is obvious from this definition that $A \diamond B = -B \diamond A$. In terms of local coordinates we have

$$(A \diamond B)_{ij} = \sum_{k,l} A^l_{ik} B^k_{jl} - B^l_{ik} A^k_{jl} \quad .$$

Proposition 4.39 *For $X \in V(M)$ and type (1,2) tensor fields A, B, we have*

$$\mathcal{L}_X(A \diamond B) = (\mathcal{L}_X A) \diamond B + A \diamond (\mathcal{L}_X B)$$

Proof. Recall that $\mathcal{L}_X$ commutes with contractions and that

$$\mathcal{L}_X(A \otimes B) = (\mathcal{L}_X A) \otimes B + A \otimes (\mathcal{L}_X B) \quad .$$

(see, for instance, II.8 of [8]) □

Definition 4.40 *For a connection Γ and $X, Y \in V(M)$, define the cochain S^Γ of $\mathfrak{C}^2$ according to the formula:*

$$S^\Gamma(X, Y) = \mathcal{L}_X \Gamma \diamond \mathcal{L}_Y \Gamma \quad . \tag{18}$$

Proposition 4.41 $\partial S^\Gamma = 0$, *i.e.* S^Γ *is a cocycle.*

Proof. Let X, Y, Z be vector fields. According to the definition of the coboundary operator,

$$\partial S^\Gamma(X, Y, Z) = \mathcal{S}\{\mathcal{L}_X S^\Gamma(Y, Z) - S^\Gamma([X, Y], Z)\} \quad ,$$

where $\mathcal{S}$ denotes summation of the formulas which result after the three circular rearrangements of X, Y, and Z. By the definition of S^Γ and Proposition 4.39 one has

$$\mathcal{L}_X S^\Gamma(Y, Z) = \mathcal{L}_X(\mathcal{L}_Y \Gamma) \diamond \mathcal{L}_Z \Gamma - \mathcal{L}_X(\mathcal{L}_Z \Gamma) \diamond \mathcal{L}_Y \Gamma \quad ,$$

and hence by (17)

$$\begin{aligned}
\mathcal{S}\{\mathcal{L}_X S^\Gamma(Y, Z)\} &= \mathcal{L}_X(\mathcal{L}_Y \Gamma) \diamond \mathcal{L}_Z \Gamma - \mathcal{L}_X(\mathcal{L}_Z \Gamma) \diamond \mathcal{L}_Y \Gamma + \\
&\quad \mathcal{L}_Y(\mathcal{L}_Z \Gamma) \diamond \mathcal{L}_X \Gamma - \mathcal{L}_Y(\mathcal{L}_X \Gamma) \diamond \mathcal{L}_Z \Gamma + \\
&\quad \mathcal{L}_Z(\mathcal{L}_X \Gamma) \diamond \mathcal{L}_Y \Gamma - \mathcal{L}_Z(\mathcal{L}_Y \Gamma) \diamond \mathcal{L}_X \Gamma \quad . \\
&= \mathcal{L}_{[X,Y]} \Gamma \diamond \mathcal{L}_Z \Gamma + \mathcal{L}_{[Y,Z]} \Gamma \diamond \mathcal{L}_X \Gamma + \mathcal{L}_{[Z,X]} \Gamma \diamond \mathcal{L}_Y \Gamma \\
&= \mathcal{S}\{S^\Gamma([X, Y], Z)\}
\end{aligned}$$

□

Proposition 4.42 *For any two connections, Γ and Γ', the corresponding cocycles, S^Γ and $S^{\Gamma'}$, differ by the coboundary of a certain one-cochain.*

Proof. Let ∇ and ∇' be the corresponding covariant derivatives. Recall that the action of $\nabla' - \nabla$ is given by an operator associated with a certain (1,2) tensor field, T. To be more precise, one has for any vector fields A, B

$$\nabla'_A B = \nabla_A B + T(A, B) \quad .$$

Hence, the definitions given earlier of $\mathcal{L}_X\Gamma$ and $\mathcal{L}_X T$ make it easy to see that for a vector field X,

$$\mathcal{L}_X\Gamma' = \mathcal{L}_X\Gamma + \mathcal{L}_X T.$$

Harking back to the definition of S^Γ one can now compute the difference of $S^{\Gamma'}$ and S^Γ. For any vector fields X, Y one has

$$S^{\Gamma'}(X,Y) = S^\Gamma(X,Y) + (\mathcal{L}_X\Gamma)\diamond(\mathcal{L}_Y T) + (\mathcal{L}_X T)\diamond(\mathcal{L}_Y\Gamma) + (\mathcal{L}_X T)\diamond(\mathcal{L}_Y T) \quad .$$

The desired one-cochain, call it R, can defined by the following formula:

$$R(X) = T\diamond(\mathcal{L}_X\Gamma) + 1/2\, T\diamond(\mathcal{L}_X T) \quad .$$

Using Proposition 4.39, by (17), and the formula for the coboundary of R,

$$\partial R(X,Y) = \mathcal{L}_X R(Y) - \mathcal{L}_Y R(X) - R([X,Y]) \quad .$$

one verifies that

$$\partial R(X,Y) = (\mathcal{L}_X\Gamma)\diamond(\mathcal{L}_Y T) + (\mathcal{L}_X T)\diamond(\mathcal{L}_Y\Gamma) + (\mathcal{L}_X T)\diamond(\mathcal{L}_Y T) \quad .$$

□

The paracompactness of M implies existence of a connection (see [4]) and hence the existence of at least one cocycle of the type S^Γ. This fact and the preceding two propositions make it possible to define $S \in H^2_{loc}$ as the equivalence class of cocycles of type S^Γ.

4.3 Three proofs of the nontriviality of the cohomology class S

Theorem 4.43 *A cocycle of the type, S^Γ, can never be a coboundary.*

We have already proved that all cocycles of this type are related by a coboundary; therefore it is enough to choose some connection, Γ and to prove that for every 1-cochain, T, we can never have $\partial T = S^\Gamma$. We will consider two approaches to the proof. One strategy is to look for specific vector fields so that the results obtained by operating on them with ∂T would be different than the results obtained by operating with S^Γ. The other strategy is to use local coordinates to express the actions of the two operators. Now, if ∂T really did equal S^Γ, then the corresponding expressions in local coordinates would be the same for both operators. Thus, the second approach consists of showing that there is no T for which the preceding is true.

First proof of Theorem 4.43. Let T be a given 1-cochain. Choose a system of coordinates, $(x_1, \ldots, x_n)$, so that some point, p, of the base manifold is mapped to the origin. Define A, B, C, D to be vector fields on the coordinate domain which are given by

$$A = (x_2)^2\partial_1 \ , \quad B = (x_1)^2\partial_2 \ ,$$
$$C = x_1x_2\partial_1 \ , \quad D = x_1x_2\partial_2 \ .$$

Extend these vector fields to the rest of the manifold by multiplying them by a "plateau function" that has the "plateau" around p. [3] Doing so does not change the above local coordinate expression of A, B, C, D in some neighborhood of p. Note that in this neighborhood

$$[A, B] = 2(x_2)^2x_1\partial_2 - 2(x_1)^2x_2\partial_1 \quad ,$$
$$[C, D] = (x_2)^2x_1\partial_2 - (x_1)^2x_2\partial_1 \quad ,$$

[3]There is a description of plateau functions in Section 2

and hence $[A, B] = 2[C, D]$ there. Since this is a local rather than just a point-wise relation and since T is a local operator we actually have $T([A, B]) = 2T([C, D])$ at all points of the neighborhood in question.

The other significant property of vector fields A, B, C, D is the fact that all of their components and all the first derivatives of their components are zero at p. As we are about to see, this fact means that the formula for ∂T and S^Γ operating on these vector fields at p has a particularly simple form. Recall the following identity for the Lie derivative operator on $\Phi_k(M)$. For $f \in C^\infty(M)$, $X \in V(M)$

$$\mathcal{L}_{fX} = f\mathcal{L}_X + df \wedge \imath(X) \quad , \tag{19}$$

Thus, using local coordinates to represent X as $\sum_i X^i \partial_i$ we have

$$\mathcal{L}_X = \sum_k X^k \mathcal{L}_{\partial_k} + dX^k \wedge i(\partial_k) \quad .$$

Therefore at any point where the components of X and their first derivatives are all zero, the result of operating with $\mathcal{L}_X$ must give a zero result. Recalling the definition of ∂T in (11) we can therefore conclude that

$$\partial T(A, B)(p) = -T([A, B])(p) \quad ,$$
$$\partial T(C, D)(p) = -T([C, D])(p) \quad ,$$

and hence that

$$\partial T(A, B)(p) = 2\partial T(C, D)(p) \quad .$$

Now, let us compute $S^\Gamma(A, B)(p)$ and $S^\Gamma(C, D)(p)$. By (16) we have that

$$(\mathcal{L}_A\Gamma)(p) = (2dx_2 \otimes dx_2 \otimes \partial_1)(p) \quad ,$$
$$(\mathcal{L}_B\Gamma)(p) = (2dx_1 \otimes dx_1 \otimes \partial_2)(p) \quad ,$$
$$(\mathcal{L}_C\Gamma)(p) = (dx_1 \otimes dx_2 \otimes \partial_1 + dx_2 \otimes dx_1 \otimes \partial_1)(p) \quad ,$$
$$(\mathcal{L}_D\Gamma)(p) = (dx_1 \otimes dx_2 \otimes \partial_2 + dx_2 \otimes dx_1 \otimes \partial_2)(p) \quad ,$$

and hence by the definition of S^Γ

$$S^\Gamma(A,B)(p) = (\mathcal{L}_A\Gamma \diamond \mathcal{L}_B\Gamma)(p) = 4(dx_2 \wedge dx_1) \quad ,$$

$$S^\Gamma(C,D)(p) = (\mathcal{L}_C\Gamma \diamond \mathcal{L}_D\Gamma)(p) = dx_1 \wedge dx_2 \quad .$$

Therefore, S^Γ cannot be equal to ∂T, since unlike the case of the latter, we have

$$S^\Gamma(A,B)(p) = -4S^\Gamma(C,D)(p) \quad ,$$

and since neither the left or the right hand sides are zero. □

In Section 2 we showed that a local, multi-linear operator with C^∞ arguments and C^∞ values must necessarily be locally differential. That means that the cochains of $\mathfrak{C}^k$ are, at least locally, equivalent to a certain multi-differential operator with coefficients that take values in $\Phi_2(M)$. Thus, for $C \in \mathfrak{C}^k$ and $X_1, \ldots, X_k \in V(M)$ we have

$$C(X_1,\ldots,X_k) = \sum_{\alpha_1\ldots\alpha_k} \sum_{\substack{j_1 \ldots j_k \\ ab}} \partial_{\alpha_1}X_1^{j_1} \cdot \ldots \cdot \partial_{\alpha_k}X_k^{j_k}\, \sigma^{\alpha_1\ldots\alpha_k}_{j_1\ldots j_k,ab}\, dx_a \wedge dx_b \quad ,$$

where $1 \leq j_1,\ldots,j_k,a,b \leq n$ and where the sum over the α_i's is formally infinite, although only a finite number of the coefficient functions, $\sigma^{\alpha_1\ldots\alpha_k}_{j_1\ldots j_k,a,b}$ are not identically zero on any compact subset of the coordinate domain.

For fixed orders of $\alpha_1,\ldots,\alpha_k$, $|\alpha_i| = d_i$, the notation $\sigma^{\alpha_1\ldots\alpha_k}_{j_1\ldots j_k,ab}$ is reminiscent of the notation for the components of a tensor in the vector space

$$S^{d_1}(\mathbb{R}^n) \otimes \ldots \otimes S^{d_k}(\mathbb{R}^n) \otimes C^k(\mathbb{R}^{n*}) \otimes \Lambda^2(\mathbb{R}^{n*}) \quad . \tag{20}$$

This observation suggests the following definition.

Definition 4.44 *For $p \in M$ and non-negative integers $d_1,\ldots,d_k$ we define the* **associated tensor of C of order $d_1\ldots d_k$ at** *p to be the element of the vector space in (20) with components $\sigma^{\alpha_1\ldots\alpha_k}_{j_1\ldots j_k,ab}(p)$ [4] and denote this tensor by $\sigma^{d_1\ldots d_k}_{C,p}$.*

[4]The associated tensor as it is given here is a construction relative to a *fixed* system of coordinates in $\mathbb{R}^n$. It is tempting to try and define the associated tensor in terms of the tangent space at p,

It will be more convenient to regard the vector space in (20) as

$$\mathrm{Hom}\Big(S^{d_1}(\mathbb{R}^{n^*}) \otimes \ldots \otimes S^{d_k}(\mathbb{R}^{n^*}) \otimes C^k(\mathbb{R}^n)\,,\ \Lambda^2(\mathbb{R}^{n^*}) \Big)$$

and identify a tensor, σ, from such a vector space with a mapping

$$\sigma : \overbrace{\mathbb{R}^{n^*} \times \ldots \times \mathbb{R}^{n^*}}^{k \text{ times}} \times \overbrace{\mathbb{R}^n \times \ldots \mathbb{R}^n}^{k \text{ times}} \longrightarrow \Lambda^2(\mathbb{R}^{n^*}) \quad , \tag{21}$$

which is homogenous of degree $d_1, \ldots, d_k$ in the first k variables, and linear in the last k variables. For $\xi_i \in \mathbb{R}^{n^*}$ and $u_i \in \mathbb{R}^n$, the action of this mapping is given by

$$\begin{aligned}&\sigma(\xi_1, \ldots, \xi_k; u_1, \ldots, u_k) = \\ &\quad \sigma(\overbrace{\xi_1 \otimes \ldots \otimes \xi_1}^{d_1 \text{ times}} \otimes \overbrace{\xi_2 \otimes \ldots \otimes \xi_2}^{d_2 \text{ times}} \otimes \ldots \otimes \overbrace{\xi_k \odot \ldots \odot \xi_k}^{d_k \text{ times}} \odot u_1 \odot \ldots \odot u_k) \quad .\end{aligned}$$

Using coordinates we have

$$\sigma^{d_1 \ldots d_k}_{C;p}(\xi_1, \ldots, \xi_k; u_1, \ldots, u_k) = \sum_{\substack{\alpha_1 \ \alpha_k \\ |\alpha_i| = d_i}} \sum_{\substack{j_1 \ j_k \\ ab}} \prod_{i=1}^{k} (\xi_i)^{\alpha_i} \prod_{i=1}^{k} u_i^{j_i}\, \sigma^{\alpha_1 \cdots \alpha_k}_{j_1 \cdots j_k, ab}(p) \epsilon_a \wedge \epsilon_b \quad , \tag{22}$$

where $\epsilon_1, \ldots, \epsilon_n$ is the canonical basis of $\mathbb{R}^{n^*}$, and where for $\xi_i \in \mathbb{R}^{n^*}$ and a multi-index α_i, we use $(\xi_i)^{\alpha_i}$ to denote the homogenous product $(\xi_i^1)^{\alpha_i^1}(\xi_i^2)^{\alpha_i^2} \ldots (\xi_i^n)^{\alpha_i^n}$.

Using associated tensors to describe cochains turns out to be a useful idea; we can use them to prove Theorem 4.43. First, we must make some remarks about the representation of $\mathrm{gl}(\mathbb{R}^n)$ on the tensor algebra of $\mathbb{R}^n$ and then we need to prove a lemma about invariance of certain types of tensors under this representation.

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rather than $\mathbb{R}^n$; this would be possible if the components of the associated tensor transformed properly under a change of coordinates. Unfortunately, only the components of the associated tensor with maximal orders transform properly under arbitrary changes of coordinates. The sum of the components with maximal order is usually referred to as the *total symbol* and gives coordinate independent information about the operator. It is not used in the present work

Let us define the representation of $\mathrm{gl}(\mathbb{R}^n)$ on the tensor algebra of $\mathbb{R}^n$ as it was done in Section 3. We need to state some explicit formulas for expressing the action of $\mathrm{gl}(\mathbb{R}^n)$ on associate tensors.

Lemma 4.45 *Let $\sigma \in S^d(\mathbb{R}^n)$ be a type $(d,0)$, symmetric tensor identified with the degree d, homogenous mapping from $\mathbb{R}^{n^*}$ to $\mathbb{R}$,*

$$\sigma(\xi) = \sigma(\overbrace{\xi \otimes \ldots \otimes \xi}^{d\ times}) , \quad \xi \in \mathbb{R}^{n^*} \quad .$$

Then, for $\eta \in \mathbb{R}^{n^*}$ and $v \in \mathbb{R}^n$ we have

$$(\mathcal{L}_{v\otimes\eta}\sigma)(\xi) = \xi(v) \left.\frac{d\sigma(\xi + t\eta)}{dt}\right|_{t=0} , \quad \xi \in \mathbb{R}^{n^*} \quad .$$

Lemma 4.46 *For $\omega \in \Lambda^k(\mathbb{R}^{n^*})$, for $\eta \in \mathbb{R}^{n^*}$, and $v \in \mathbb{R}^n$ we have*

$$\mathcal{L}_{v\otimes\eta}\omega = \eta \wedge \ i(v)\omega \quad .$$

Lemma 4.47 *For $\alpha \in \mathbb{R}^{n^*}$, for $\eta \in \mathbb{R}^{n^*}$, and $v \in \mathbb{R}^n$ we have*

$$(\mathcal{L}_{v\otimes\eta}\alpha)(u) = -\eta(u)\alpha(v) , \quad u \in \mathbb{R}^n \quad .$$

The preceding three lemmas allow us to develop a formula for the action of $\mathrm{gl}(U)$ on an associated tensor.

Lemma 4.48 *Let σ be a type $(d_1+\ldots+d_k, k+2)$ tensor from the vector space in (20) identified with a mapping of the type given in (21). For $\eta, \xi_j \in \mathbb{R}^{n^*}$ and $v, u_j \in \mathbb{R}^n$ we have*

$$\begin{aligned}(\mathcal{L}_{v\otimes\eta}\sigma)(\xi_1, \ldots, \xi_k; u_1, \ldots, u_k) \;=\; & \sum_{j=1}^{k} \xi_j(v) \left.\frac{d\sigma(\xi_1, \ldots, \xi_{j-1}, \xi_j + t\eta, \xi_{j+1}, \ldots, \xi_k; u_1, \ldots, u_k)}{dt}\right|_{t=0} \\ & - \sum_{j=1}^{k} \eta(u_j)\ \sigma(\xi_1, \ldots, \xi_k; u_1, \ldots, u_{j-1}, v, u_{j+1}, \ldots, u_k) \\ & - \eta \wedge i(v)\sigma(\xi_1, \ldots, \xi_k; u_1, \ldots, u_k)\end{aligned}$$

Lemma 4.49 *Let $\sigma \in S^3(\mathbb{R}^n) \otimes \mathbb{R}^{n^*} \otimes C^2(\mathbb{R}^{n^*})$ be a type (3,3) tensor which is invariant under the action of* $\mathrm{gl}(\mathbb{R}^n)$. *Then, identifying σ with an element of*

$$\mathrm{Hom}(C^3(\mathbb{R}^{n^*}) \otimes \mathbb{R}^n, C^2(\mathbb{R}^{n^*})) \quad ,$$

σ must be of the form

$$\sigma(\xi_1 \otimes \xi_2 \otimes \xi_3 \otimes u) = c \sum_{\pi \in \Pi_3} \xi_{\pi_1}(u)\, \xi_{\pi_2} \otimes \xi_{\pi_3}\,, \quad \xi_1, \xi_2, \xi_3 \in \mathbb{R}^{n^*} \quad u \in \mathbb{R}^n \quad ,$$

where c is some real constant.

Proof. By Theorem 3.31, σ must be of the form

$$\sigma(\xi_1 \otimes \xi_2 \otimes \xi_3 \otimes u) = \sum_{\pi \in \Pi_3} c_\pi\, \xi_{\pi_1}(u)\, \xi_{\pi_2} \otimes \xi_{\pi_3} \quad .$$

for some real constants c_π ($\pi \in \Pi_3$). By assumption, interchanging the order of ξ_1, ξ_2, ξ_3 does not change the value of σ and hence, for any $\rho \in \Pi_3$ we also have

$$\sigma(\xi_1 \otimes \xi_2 \otimes \xi_3 \otimes u) = \sum_\pi c_\pi\, \xi_{\rho\pi_1}(u)\, \xi_{\rho\pi_2} \otimes \xi_{\rho\pi_3} \quad .$$

Therefore, by "averaging" over all $\rho \in \Pi_3$, and using the substitution $\mu = \rho\pi$, we conclude that

$$\begin{aligned}
\sigma(\xi_1 \otimes \xi_2 \otimes \xi_3 \otimes u) &= \frac{1}{6} \sum_{\rho,\pi} c_\pi\, \xi_{\rho\pi_1}(u)\, \xi_{\rho\pi_2} \otimes \xi_{\rho\pi_3} \\
&= \frac{1}{6} \sum_\mu \left(\sum_\rho c_{\rho^{-1}\mu} \right) \xi_{\mu_1}(u)\, \xi_{\mu_2} \otimes \xi_{\mu_3} \\
&= c \sum_\mu \xi_{\mu_1}(u)\, \xi_{\mu_2} \otimes \xi_{\mu_3} \quad ,
\end{aligned}$$

where $c = 1/6 \sum_\pi c_\pi$. □

Second proof of Theorem 4.43. Let $T \in \mathfrak{C}^1$ be given and choose some $p \in M$. We showed in Section 2 that T is a locally differential operator and hence that in some neighborhood around p,

$$T(X) = \sum_\alpha \sum_{jab} \partial_\alpha X^j \sigma^\alpha_{jab} dx_a \wedge dx_b \quad , \tag{23}$$

where only finitely many of the functions σ^{α}_{jab} are not identically zero in every compact neighborhood of p. We are going to use associated tensors to compare the local coordinate expressions of ∂T and of S^{Γ} and thereby show that $\partial T \neq S^{\Gamma}$.

Now, let us compute the expression of ∂T in local coordinates. For vector fields X,Y, we have by (19)

$$\mathcal{L}_X T(Y) = \sum_r dX^r \wedge i(\partial_r)T(Y) + \sum_i X^i \mathcal{L}_{\partial_i} T(Y) \tag{24}$$

$$\mathcal{L}_Y T(X) = \sum_r dY^r \wedge i(\partial_r)T(X) + \sum_i Y^i \mathcal{L}_{\partial_i} T(X) \tag{25}$$

Expressing $[X, Y]$ in local coordinates we have

$$T([X, Y]) = \sum_{\alpha} \sum_{jab} \sum_{r} \partial_{\alpha}(X^r \partial_r Y^j - Y^r \partial_r X^j) \sigma^{\alpha}_{jab} dx_a \wedge dx_b \quad . \tag{26}$$

Let d denote the differential order of T at p; i.e. the largest integer, d, such that $\sigma^{\alpha}_{jab}(p) \neq 0$ for some choice of $|\alpha| = d$ and $1 \leq j, a, b \leq n$. Since

$$\partial T(X, Y) = \mathcal{L}_X T(Y) - \mathcal{L}_Y T(X) - T([X, Y]) \quad ,$$

we can see from the above equations that ∂T cannot have terms at p of order higher than $(d+1, 0)$, $(d, 1)$, $(d-1, 2)$, $\ldots$, $(1, d)$, $(0, d+1)$. Now, let us consider the local coordinate expression of S^{Γ}. Referring to (16) and (18) we see that for $X, Y \in V(M)$ we have

$$S^{\Gamma}(X, Y) = \sum_{ijkl} \partial_{ik} X^l \partial_{jl} Y^k dx_i \wedge dx_j + \ldots \quad , \tag{27}$$

where the ellipsis denotes terms of order (1,1) or less. If $d = 1, 2$, it is therefore impossible for ∂T to equal S^{Γ} becase the latter has a non-zero term of order $(2, 2)$. Thus, we may assume without loss of generality that $d \geq 3$. Equation (27) also shows that S^{Γ} has no terms of order $(d, 1)$ at p and therefore, we can prove that $\partial T \neq S^{\Gamma}$, if we can show that the associated tensor of order $(d, 1)$ of ∂T is not zero. Another strategy for showing that $\partial T \neq S^{\Gamma}$ would be to prove that the $(2, 2)$ term of ∂T

must be different from the $(2,2)$ term of S^r. This approach will be pursued below, in Proof 3.

From (25) and (26) we see that only $\mathcal{L}_Y T(X)$ and $T([X,Y])$ can furnish terms of order $(d,1)$ for ∂T. These terms are

$$-\sum_{rs}\sum_{jab}\sum_{|\alpha|=d} \partial_\alpha X^j \, \partial_s Y^r \, \sigma^\alpha_{jab} \, dx_s \wedge i(\partial_r) dx_a \wedge dx_b \quad ,$$

$$-\sum_{r}\sum_{jab}\sum_{|\alpha|=d} \partial_\alpha X^r \, \partial_r Y^j \, \sigma^\alpha_{jab} \, dx_a \wedge dx_b \quad ,$$

$$\sum_{r}\sum_{jab}\sum_{|\alpha|=d}\sum_{[\gamma,k]=\alpha} \alpha^k \, \partial_{[\gamma,r]} X^j \, \partial_k Y^r \, \sigma^\alpha_{jab} \, dx_a \wedge dx_b \quad .$$

The summation condition $[\gamma,k]=\alpha$ says to sum over all k for which $\alpha^k \neq 0$ and that for each value of k, γ is the multi-index obtained from α by decrementing α^k. Using (22) to compute $\sigma^{d,1}_{\partial T;p}$ and recalling Lemma 4.48 we can see that for $u, v \in \mathbb{R}^n$ and for $\xi, \eta \in \mathbb{R}^{n^*}$ we have

$$\sigma^{d,1}_{\partial T;p}(\xi,\eta;u,v) = (\mathcal{L}_{v\otimes\eta}\sigma^d_{T,p})(\xi;u) \quad .$$

Thus, in order of $\sigma^{d,1}_{\partial T;p}$ to be zero, it is necessary that $\sigma^d_{T,p}$ is invariant under the action of $\mathrm{gl}(\mathbb{R}^n)$. Using Lemma 4.49 we see that a type $(3,3)$ tensor

$$\sigma \in \mathrm{Hom}(S^3(\mathbb{R}^{n^*}) \otimes \mathbb{R}^n, \Lambda^2(\mathbb{R}^{n^*})) \quad ,$$

which is invariant under the action of $\mathrm{gl}(\mathbb{R}^n)$ must satisfy

$$\sigma(\xi;u) = \sigma(\xi \odot \xi \otimes \xi \otimes u) = 6c\,\xi(u)\,\xi \otimes \xi \quad ,$$

for all $\xi \in \mathbb{R}^{n^*}$ and $u \in \mathbb{R}^n$. Thus, the values of σ must be both symmetric and skew-symmetric, which means that σ must be the zero tensor. $\sigma^d_{T;p}$ cannot be the zero tensor because of the assumption that the order of T at p is d. Therefore, $\sigma^d_{T;p}$ cannot be invariant under the action of $\mathrm{gl}(\mathbb{R}^n)$, which means that $\sigma^{d,1}_{\partial T,p}$ is not zero. □

Third proof of Theorem 4.43 Let T be a given 1-cochain . In this proof we will compute $\sigma^{2,2}_{\partial T}$ and show that it cannot be equal to $\sigma^{2,2}_{S^\Gamma}$. Let the expression for T in local coordinates be as in (23). We have already mentioned that $\mathcal{L}_X T(Y)$ contributes terms of order 1 and 0 in X and that $\mathcal{L}_Y T(X)$ contributes terms of order 1 and 0 in Y. If ∂T is to have terms of order (2,2) they must come from $T([X,Y])$ and be generated by those terms of T that have order 3. Thus, the (2,2) terms of ∂T are given by

$$\sum_{|\alpha|=3} \sum_{jab} \sum_{[\gamma,k]=\alpha} \alpha^k(\partial_\gamma X^r \partial_{kr} Y^j - \partial_\gamma Y^r \partial_{kr} X^j)\sigma^\alpha_{jab} dx_a \wedge dx_b$$

Therefore, we have

$$\sigma^{2,2}_{\partial T}(\xi,\eta;u,v) = \eta(u) \left.\frac{d\sigma^3_T(\xi+t\eta;v)}{dt}\right|_{t=0} - \xi(v) \left.\frac{d\sigma^3_T(\eta+t\xi;u)}{dt}\right|_{t=0} \tag{28}$$

Now, let us compute $\sigma^{2,2}_{S^\Gamma}$. Recalling the local coordinate form of S^Γ, given in (27), we see that

$$\sigma^{2,2}_{S^\Gamma}(\xi,\eta;u,v) = \sum_{ijkl} \xi^i \xi^k u^l \eta^j \eta^l v^k \varepsilon_k \wedge \varepsilon_l = \xi(v)\eta(u)\xi \wedge \eta \tag{29}$$

Let us assume that

$$\sigma^{2,2}_{\partial T} = \sigma^{2,2}_{S^\Gamma}$$

and argue by contradiction. Let $\xi \in \mathbb{R}^{n^*}$ be given. Suppose that $u \in \mathbb{R}^n$ is such that $\xi(u) = 0$. Choose a $v \in \mathbb{R}^n$ so that $\xi(v) = 1$. By (29) we see that

$$\sigma^{2,2}_{S^\Gamma}(\xi,\xi;u,v) = 0$$

and hence, using (28), that

$$-3\sigma^3_T(\xi;u) = 0$$

On the other hand, suppose that $u \in \mathbb{R}^n$ is such that $\xi(u) \neq 0$. Again by (29) we have that

$$\sigma^{2,2}_{S^\Gamma}(\xi,2\xi,u,u) = 0$$

and hence, using (28), that

$$9\sigma_T^3(\xi, u) = 0$$

Therefore, for every $u \in \mathbb{R}^n$,

$$\sigma_T^3(\xi, u) = 0$$

and this is absurd because it means that both $\sigma_{\partial T}^{2,2}$, and $\sigma_{ST}^{2,2}$, are zero. □

5 Infinitesimal automorphisms of contact structures

In the previous section we considered cochains that were local operators on the Lie algebra of all smooth vector fields. In this section we will suppose that the base manifold is $J^k(\mathbb{R}^n, \mathbb{R})$, the k-th order jet bundle of smooth maps from $\mathbb{R}^n$ to $\mathbb{R}$ and focus our attention on the Lie algebra of infinitesimal automorphisms of the contact system which is attached to the jet bundle. The cochains of the associated Chevalley cohomology are thus local operators, but operators whose choice of arguments is restricted to those vector fields that are infinitesimal automorphisms of the contact system. The preceding proofs that S is non-trivial consisted of showing that there is no T such that S^Γ and ∂T are the same differential operator. It is conceivable that distinct differential operators can act identically on a restricted choice of arguments and therefore there may be a differential operator T such that ∂T and S^Γ specify the same co-chain. There is another obstruction: since our proof that local operators are locally differential (see Section 2) relies on the fact that any smooth vector field can be taken as an argument we cannot exclude (at least not without doing more work) the existence of local cochains of a non-differential nature. That means that the second proof of the non-triviality of S^Γ given in the preceding section cannot be adopted to the present situation; i.e. we cannot use local coordinates to express a 1-cochain, T, as a differential operator and then compare the local coordinate expression of ∂T to that of S^Γ. Recall, however, that the first proof of the non-triviality of S^Γ did not rely on T being differential, only local. We could carry this proof over to the present circumstances if we could choose the four vector fields in question so that they are infinitesimal automorphishms of the contact system. This turns out to be possible.

5.1 The contact structure on the bundle of 1-jets

Let us begin by considering the bundle $J^1(\mathbb{R}^n, \mathbb{R})$, a manifold diffeomorphic to $\mathbb{R}^{2n+1}$. Using (x_j, y, y_j), where $1 \leq j \leq n$, to denote the local jet coordinates, we have that the contact system, $\Omega^{(1)}$, on $J^1(\mathbb{R}^n, \mathbb{R})$ is generated by the differential 1-form

$$\omega = dy - \sum_j y_j dx_j \quad .$$

Thus, an infinitesimal automorphism of $\Omega^{(1)}$ is a vector field of $J^1(\mathbb{R}^n, \mathbb{R})$ such that $\mathcal{L}_X\omega$ is some multiple of ω. We are going to show that there is an isomorphism between the vector space of infinitesimal automorphisms of $\Omega^{(1)}$ and the vector space of real valued functions on $J^1(\mathbb{R}^n, \mathbb{R})$.

Suppose that A, B are vector fields and π is a differential 1-form. The following identities are a basic computational tool for the proofs that follow.

$$i([A, B])\pi = i(A)\mathcal{L}_B\pi - \mathcal{L}_B i(A)\pi \quad . \tag{30}$$

$$i(A)\mathcal{L}_B\pi - \mathcal{L}_B i(A)\pi = -i(B)\mathcal{L}_A\pi + \mathcal{L}_A i(B)\pi \quad . \tag{31}$$

The first is a standard (see for instance [4]) and the second identity follows immediately because $[A, B] = -[B, A]$.

For a function, f, on $J^1(\mathbb{R}^n, \mathbb{R})$ put

$$X_f = \left(f - \sum_j y_j \frac{\partial f}{\partial y_j}\right)\frac{\partial}{\partial y} + \sum_j \left(\frac{\partial f}{\partial x_j} + y_j \frac{\partial f}{\partial y}\right)\frac{\partial}{\partial y_j} - \sum_j \frac{\partial f}{\partial y_j}\frac{\partial}{\partial x_j} \quad . \tag{32}$$

Notice that $i(X_f)\omega = f$. Indeed, we have

Proposition 5.50 *For all functions f, the vector field X_f is an infinitesimal automorphism of $\Omega^{(1)}$ and furthermore, every infinitesimal automorphism, X, is of the form X_f, where $f = i(X)\omega$.*

Proof. For $1 \leq j \leq n$, put

$$Z_j = \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial y} \quad .$$

Note that the $2n$ vector fields Z_j and $\partial/\partial y_j$ span the vector space of vector fields that annihilate ω. Therefore, X is an infinitesimal automorphism of the contact structure if and only if $i(Z_j)\mathcal{L}_X\omega$ and $i(\partial/\partial y_j)\mathcal{L}_X\omega$ are zero for all j.

Let X be a vector field on $J^1(\mathbb{R}^n, \mathbb{R})$ and put $f = i(X)\omega$. A simple calculation shows that

$$\mathcal{L}_{Z_j}\omega = dy_j \quad ,$$

and that

$$\mathcal{L}_{\partial/\partial y_j}\omega = -dx_j \quad .$$

Using identity (30) with π repaced by ω, A replaced by X, and B replaced by alternately Z_j and $\partial/\partial y_j$, we see that X is an infinitesimal automorphism of the contact structure if and only if

$$-i(X)dx_j = \partial f/\partial y_j \quad ,$$

and

$$i(X)dy_j = Z_j f \quad ,$$

for every j. Thus, if $X = X_g$, for some function, g, we may conclude that X is an infinitesimal automorphism because

$$f = i(X_g)\omega = g \quad ,$$

$$-i(X_g)dx_j = \partial g/\partial y_j \quad ,$$

and because

$$i(X_g)dy_j = Z_j g \quad .$$

On the other hand, if X is an infinitesimal automorphism we may conclude that the x_j and y_j components of X are equal to the respective components of X_f. To conclude that $X = X_f$ we need to show that

$$i(X)dy = f - \sum_j y_j \partial f/\partial y_j \quad .$$

But, this is true because by definition,

$$f = i(X)dy - \sum_j y_j i(X)dx_j \quad ,$$

and because as we have already shown,

$$\partial f/\partial y_j = -i(X)dx_j \quad .$$

□

Proposition 5.51 *Suppose that $X = X_f$ is an infinitesimal automorphism of $\Omega^{(1)}$. Then, $\mathcal{L}_X\omega = (\partial f/\partial y)\omega$.*

Proof. We are assuming that $\mathcal{L}_X\omega = c\omega$. By considering the local coordinate form for ω we see that the multiplying function is given by

$$c = i(\partial/\partial y)\mathcal{L}_X\omega \quad .$$

Using identity (31) with ω, $\partial/\partial y$, and X in place of π, A, and B we have

$$c - \mathcal{L}_X 1 = -i(X)\mathcal{L}_{\partial/\partial y}\omega + \partial f/\partial y \quad .$$

A calculation in local coordinates shows that

$$\mathcal{L}_{\partial/\partial y}\omega = 0 \quad ,$$

and hence,

$$c = \partial f/\partial y \quad .$$

□

For functions f,g on $J^1(\mathbb{R}^n, \mathbb{R})$, put

$$\{f,g\} = i([X_f, X_g])\omega \quad . \tag{33}$$

It is clear that the bracket $\{,\}$ obeys the Jacobi identity because $X_{\{f,g\}} = [X_f, X_g]$ and because the Jacobi identity holds for the Lie bracket of vector fields.

Proposition 5.52 *In local coordinates we have*

$$\{f,g\} = \sum_j y^j \left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial y_j} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial y_j}\right) + \sum_j \left(\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial y_j} - \frac{\partial g}{\partial x_j}\frac{\partial f}{\partial y_j}\right) + f\frac{\partial g}{\partial y} - g\frac{\partial f}{\partial y} \quad . \tag{34}$$

Proof. Using identity (30) with ω, X_f, X_g in place of π, A, B we have

$$i([X_f, X_g])\omega = X_f(g) - \frac{\partial g}{\partial y}f \quad .$$

and this is equivalent to what we are trying to conclude. □

It is interesting to note that if f and g are functions which do not depend on y then $\{f,g\}$ is just a Poisson bracket of the two functions. The case of infinitesimal automorphisms of Poisson brackets was treated by Lichnerowitz and Flato in [3]. We will follow their proof in showing that S^Γ restricted to infinitesimal automorphisms of $\Omega^{(1)}$ is not equal to a co-boundary.

Theorem 5.53 *The cohomology class S remains non-trivial in the local Chevalley cohomology of infinitesimal automorphisms of $\Omega^{(1)}$.*

Proof. Let T be any local 1 co-chain. We will show that $\partial T \neq S^\Gamma$. Put

$$f_1 = (x_1)^2 y_1 \ , \quad g_1 = x_1 (y_1)^2 \ ,$$
$$f_2 = (x_1)^3 \ , \qquad g_2 = (y_1)^3 \ .$$

According to (34) we have

$$\{f_1, g_1\} = 3(x_1)^2(y_1)^2 \quad ,$$

$$\{f_2, g_2\} = 9(x_1)^2(y_1)^2 \quad ,$$

and thus

$$3\{f_1, g_1\} = \{f_2, g_2\} \quad ,$$

everywhere. Hence,

$$3[X_{f_1}, X_{g_1}] = [X_{f_2}, X_{g_2}],$$

everywhere as well. Since the four functions in question all have zeros of degree 3 at the origin, we can see by looking at (32) that the infinitesimal automorphisms generated by these functions will have zeros of at least order 2 at the origin. Thus we can use the same argument as in Theorem 4.43, to show that

$$3T(X_{f_1}, X_{g_1})(0) = T(X_{f_2}, X_{g_2})(0)^5 \quad .$$

Let us use (32) to compute the four infinitesimal automorphisms in question and then equation (16) to compute the respective Lie derivatives of the connection, Γ, at the origin.

$$X_{f_1} = 2x_1y_1\frac{\partial}{\partial y_1} - (x_1)^2\frac{\partial}{\partial x_1} \quad .$$

$$X_{g_1} = -x_1(y_1)^2\frac{\partial}{\partial y} + (y_1)^2\frac{\partial}{\partial y_1} - 2x_1y_1\frac{\partial}{\partial x_1} \quad .$$

$$X_{f_2} = (x_1)^3\frac{\partial}{\partial y} + 3(x_1)^2\frac{\partial}{\partial y_1} \quad .$$

$$X_{g_2} = -2(y_1)^3\frac{\partial}{\partial y} - 3(y_1)^2\frac{\partial}{\partial x_1} \quad .$$

$$\mathcal{L}_{X_{f_1}}\Gamma(0) = 2(dx_1 \odot dy_1 + dy_1 \odot dx_1) \otimes \frac{\partial}{\partial y_1} - 2dx_1 \otimes dx_1 \otimes \frac{\partial}{\partial x_1} \quad .$$

[5]We are using 0 to denote the origin of $\mathbb{R}^n$

$$\mathcal{L}_{X_{g_1}}\Gamma(0) = 2dy_1 \otimes dy_1 \otimes \frac{\partial}{\partial y_1} - 2(dx_1 \otimes dy_1 + dy_1 \otimes dx_1) \otimes \frac{\partial}{\partial x_1} \quad .$$

$$\mathcal{L}_{X_{f_2}}\Gamma(0) = 6dx_1 \otimes dx_1 \otimes \frac{\partial}{\partial y_1} \quad .$$

$$\mathcal{L}_{X_{g_2}}\Gamma(0) = -6dy_1 \otimes dy_1 \otimes \frac{\partial}{\partial x_1} \quad .$$

Hence, according to the definition of S^Γ we have

$$S^\Gamma(X_{f_1}, X_{g_1})(0) = \mathcal{L}_{X_{f_1}}\Gamma \diamond \mathcal{L}_{X_{g_1}}\Gamma(0) = 8dx_1 \wedge dy_1 \quad ,$$

$$S^\Gamma(X_{f_1}, X_{g_1})(0) = \mathcal{L}_{X_{f_2}}\Gamma \diamond \mathcal{L}_{X_{g_2}}\Gamma(0) = -36dx_1 \wedge dy_1 \quad ,$$

and therefore ∂T cannot equal S^Γ because unlike the former,

$$-36 S^\Gamma(X_{f_1}, X_{g_1})(0) = 8 S^\Gamma(X_{f_2}, X_{g_2})(0) \quad .$$

and because the preceding two expressions are not zero. □

5.2 Higher order contact systems

Now, let us work with the k-th order jet bundle, $J^k(\mathbb{R}^n, \mathbb{R})$, a manifold diffeomorphic to $\mathbb{R}^{n+C_k}$, where

$$C_k = \binom{n+k-1}{k}$$

is the number of multi-differential indices of order k or less. Let us use (x_j, y_α), where j is an index with range $1 \leq j \leq n$ and α is a multi-index with range $|\alpha| \leq k$, to denote coordinates on the jet bundle. We will express a vector field, X, on $J^k(\mathbb{R}^n, \mathbb{R})$ in local coordinates by writing

$$X = \sum_j X^j \frac{\partial}{\partial x_j} + \sum_{|\alpha| \leq k} X^\alpha \frac{\partial}{\partial y_\alpha} \quad .$$

In other words, $X^j = i(X)dx_j$ and $X^\alpha = i(X)dy_\alpha$. The contact system, $\Omega^{(k)}$, on the k-order jet bundle is a pfaffian system generated by the C_{k-1} differential 1-forms

$$\omega_\alpha^{(k)} = dy_\alpha - \sum_j y_{[\alpha,j]} dx_j, \quad |\alpha| \leq k-1 \quad ,$$

An infinitesimal automorphism, X, of the contact system is any vector field on $J^k(\mathbb{R}^n, \mathbb{R})$ such that $\mathcal{L}_X \omega_\alpha^{(k)} \in \Omega^{(k)}$ for every α.

Before proceeding with an equivalent of Theorem 5.53 for $J^k(\mathbb{R}^n, \mathbb{R})$ we need to study how the infinitesimal automorphisms of $\Omega^{(k)}$ are related to the infinitesimal automorphisms of $\Omega^{(1)}$. There is a natural way to define a projection from a higher order jet bundle to one of a lower order. For $1 \le l < k$, the projection in question is the map

$$\pi_l^k : J^k(\mathbb{R}^n, \mathbb{R}) \longrightarrow J^l(\mathbb{R}^n, \mathbb{R})$$

that acts by "forgetting" coordinates y_β $(l < |\beta| \le k)$, i.e. for $p \in J^k(\mathbb{R}^n, \mathbb{R})$, we define $q = \pi_l^k(p)$ to be the point such that

$$x_j(q) = x_j(p) \ (1 \le j \le n) \quad ,$$

$$y_\alpha(q) = y_\alpha(p) \ (|\alpha| \le l) \quad .$$

Recall that whenever there is a projection from one manifold to another, there are acompanying notions of a vector field prolongation. Indeed, for a vector field, X, on $J^l(\mathbb{R}^n, \mathbb{R})$ and a vector field, $\tilde{X}$, on $J^k(\mathbb{R}^n, \mathbb{R})$ we say that $\tilde{X}$ is a prolongation of X if $(\pi_l^k)_* \tilde{X} = X \circ \pi_l^k$. This rather abstract definition has a simpler equivalent in terms of local coordinates.

Proposition 5.54 *$\tilde{X}$ is a prolongation of some vector field on $J^l(\mathbb{R}^n, \mathbb{R})$ if and only if for every $1 \le j \le n$ and $|\alpha| \le l$, $\tilde{X}^j$, and $\tilde{X}^\alpha$ are functions which are independent of coordinates y_β, $l < |\beta| \le k$.*

We can see from the preceding proposition that a given vector field, X, can have many different prolongations and that there is no "natural" way to distinguish one of them. But, if we assume that X is an infinitesimal automorphism of $\Omega^{(l)}$, we will be able to show that there exists a unique prolongation of X which is also an infinitesimal automorphism of $\Omega^{(k)}$. Surprisingly, the converse is also true, i.e. we will also show that every infinitesimal automorphism of $\Omega^{(k)}$ is a prolongation of an infinitesimal automorphism of $\Omega^{(l)}$. This result is due to Bäcklund and is typically referred to as Bäcklund's Theorem.

For every $1 \leq a \leq n$, put

$$Z_a^{(k)} = \frac{\partial}{\partial x_j} + \sum_{|\alpha| \leq k-1} y_{[\alpha,a]} \frac{\partial}{\partial y_\alpha} \quad .$$

The vector fields $Z_a^{(k)}$ will be a crucial aid in developing our results about the infinitesimal automorphisms of $\Omega^{(k)}$.

Proposition 5.55 *A differential 1-form is in $\Omega^{(k)}$ if and only if it is annihilated by every $Z_a^{(k)}$ and every $\partial/\partial y_\beta$ ($|\beta| = k$).*

Proof. The $n + C_k - C_{k-1}$ vector fields $Z_a^{(k)}$, $\partial/\partial y_\beta$ are linearly independent and annihilate every $\omega_\alpha^{(k)}$. Since the $\omega_\alpha^{(k)}$'s are linearly independent themselves, the space of vector fields that annihilates them must have dimension $n + C_k - C_{k-1}$ and therefore $Z_a^{(k)}$'s and $\partial/\partial y_\beta$'s span that space. □

Proposition 5.56 *A vector field X is an infinitesimal automorphism of $\Omega^{(k)}$ if and only if both $[Z_a^{(k)}, X]$ and $[\frac{\partial}{\partial y_\beta}, X]$ annihilate $\omega_\alpha^{(k)}$ for all a, $|\beta| = k$, and $|\alpha| \leq k-1$.*

Proof. This proposition is a corollary of the preceding Proposition and of identity (30).

□

Proposition 5.57 $[Z_a^{(k)}, Z_b^{(k)}] = 0$.

Proof. If the action of $Z_a^{(k)}Z_b^{(k)} - Z_b^{(k)}Z_a^{(k)}$ annihilates all the coordinate functions, then the vector field $[Z_a^{(k)}, Z_b^{(k)}]$ must be zero. Since $Z_a^{(k)}(x_j)$ is a constant (either 0 or 1) and $Z_a^{(k)}(y_\beta) = 0$ for $|\beta| = k$, we only need to consider the case of y_α, where $|\alpha| \leq k-1$. For such an α we have $Z_a^{(k)}(y_\alpha) = y_{[\alpha,a]}$ and therefore $Z_a^{(k)}Z_b^{(k)}(y_\alpha)$ is 0 when $|\alpha| = k-1$, and is $y_{[\alpha,b,a]}$, otherwise. The conclusion follows by noting that $y_{[\alpha,b,a]} = y_{[\alpha,a,b]}$. □

Let α be a given multi-index. Choose $a_1, \ldots, a_j$ so that $\alpha = [a_1 \ldots a_j]$ and put

$$Z_\alpha^{(k)} = Z_{a_1}^{(k)} \ldots Z_{a_j}^{(k)} \quad .$$

Since the actions of the differential operators $Z_a^{(k)}$ commute, the action of the differential operator $Z_\alpha^{(k)}$ is determined by α and not by the order of the a_j's. In order to avoid any possible confusion, we should note that $Z_a^{(k)}$ refers to a either a vector field or a differential operator of order 1, while $Z_\alpha^{(k)}$ refers to a multi-differential operator of order $|\alpha|$.

Following a course similar to the one we took when working with $J^1(\mathbb{R}^n, \mathbb{R})$, we should now try to define an infinitesimal automorphism of $\Omega^{(k)}$ in terms of a generating function. Let f be a function on $J^k(\mathbb{R}^n, \mathbb{R})$ and put

$$X_f^{(k)} = -\sum_j \frac{\partial f}{\partial y_{[j]}}\frac{\partial}{\partial x_j} + \sum_{0\leq|\alpha|\leq k-1}\left(Z_\alpha^{(k)}(f) - \sum_j y_{[\alpha,j]}\frac{\partial f}{\partial y_{[j]}}\right)\frac{\partial}{\partial y_\alpha} + \sum_{|\beta|=k} Z_\beta^{(k)}(f)\frac{\partial}{\partial y_\beta} \quad . \tag{35}$$

The next proposition recasts this definition into a another, perhaps more useful form.

Proposition 5.58 *The following conditions characterize a vector field* $X = X_f^{(k)}$.

$$i(X)\omega_\alpha^{(k)} = Z_\alpha^{(k)}(f) \quad , \tag{36}$$

$$X^\beta = Z_\beta^{(k)}(f) \quad , \tag{37}$$

$$X^j = -\frac{\partial f}{\partial y_{[j]}} \quad , \tag{38}$$

for all indices with ranges $0 \leq |\alpha| \leq k-1$, $|\beta| = k$, $1 \leq j \leq n$.

Our next goal is to prove the following.

Theorem 5.59 *Suppose that f is a function of variables x_j, y_0 and $y_{[a]}$ only. Then, $X = X_f^{(k)}$ is a prolongation of every $X_f^{(l)}$, $1 \leq l < k$ and is, furthermore, an infinitesimal automorphism of $\Omega^{(k)}$.*

We will proceed via the following three lemmas. The first two lemmas are verified by simple local coordinate computations.

Lemma 5.60 *For every a and $|\beta| = k$ we have*

$$\mathcal{L}_{Z_a^{(k)}}\omega_\alpha^{(k)} = \begin{cases} \omega_{[\alpha,a]}^{(k)} & \text{if } |\alpha| \leq k-2 \\ dy_{[\alpha,a]} & \text{if } |\alpha| = k-1 \end{cases},$$

$$\mathcal{L}_{\frac{\partial}{\partial y_\beta}}\omega_\alpha^{(k)} = \begin{cases} -dx_j & \text{if } \beta = [\alpha, j] \text{ for some } j \\ 0 & \text{otherwise} \end{cases}.$$

Lemma 5.61 *For every a and $|\beta| \leq k$ we have*

$$\left[\frac{\partial}{\partial y_\beta}, Z_a^{(k)}\right] = \begin{cases} \frac{\partial}{\partial y_\alpha} & \text{if } \beta = [\alpha, a] \text{ for some } \alpha \\ 0 & \text{otherwise} \end{cases}.$$

Lemma 5.62 *If f and X are as in the condition of Theorem 5.59, then for all $|\beta| = k$ and $|\alpha| \leq k-1$ we have*

$$\frac{\partial Z_\alpha^{(k)}(f)}{\partial y_\beta} = \begin{cases} \frac{\partial f}{\partial y_{[j]}} & \text{if } \beta = [\alpha, j] \text{ for some } j \\ 0 & \text{otherwise} \end{cases}. \tag{39}$$

Proof. The proof is by induction on k. The case $k = 1$ is trivially true. So, let us suppose that the lemma has been shown to be true for a certain $k = N$ and show that the lemma is also true for $k = N+1$. Let $|\beta| = N+1$ be given. If we consider the definition of $Z_\alpha^{(k)}$, we can see that $Z_\alpha^{(k)}(f)$ is a function of variables x_j and y_γ ($|\gamma| \le |\alpha|+1$), only. Hence, (39) is true for $|\alpha| \le N-1$. So, suppose that $|\alpha| = N$. If there is a j such that $\beta = [\alpha, j]$, we can choose a $|\gamma| = N-1$ and an a, such that $\alpha = [\gamma, a]$ and $\beta = [\gamma, a, j]$. Using Lemma 5.61 and the induction hypothesis we have

$$\begin{aligned}
\frac{\partial Z_\alpha^{(k)}(f)}{\partial y_\beta} &= \frac{\partial Z_{[a]}^{(k)}(Z_\gamma^{(k)}(f))}{\partial y_{[\gamma,a,j]}} \\
&= \left[\frac{\partial}{\partial y_{[\gamma,a,j]}}, Z_{[a]}^{(k)}\right](Z_\gamma^{(k)}(f)) + Z_{[a]}^{(k)}\left(\frac{\partial Z_\gamma^{(k)}(f)}{\partial y_{[\gamma,a,j]}}\right) \\
&= \frac{\partial Z_\gamma^{(k)}(f)}{\partial y_{[\gamma,j]}} + Z_{[a]}^{(k)}\left(\frac{\partial Z_\gamma^{(k)}(f)}{\partial y_{[\gamma,a,j]}}\right) \\
&= \frac{\partial f}{\partial y_{[j]}} \quad .
\end{aligned}$$

If no such j exists, then it is possible to choose an a and $|\gamma| = N-1$ so that $\alpha = [\gamma, a]$ and $\beta^a = 0$. Thus, using Lemma 5.61 we have

$$\begin{aligned}
\frac{\partial Z_\alpha^{(k)}(f)}{\partial y_\beta} &= \frac{\partial Z_{[a]}^{(k)}(Z_\gamma^{(k)}(f))}{\partial y_\beta} \\
&= \left[\frac{\partial}{\partial y_\beta}, Z_{[a]}^{(k)}\right](Z_\gamma^{(k)}(f)) + Z_{[a]}^{(k)}\left(\frac{\partial Z_\gamma^{(k)}(f)}{\partial y_\beta}\right) \\
&= 0 \quad .
\end{aligned}$$

□

Proof of Theorem 5.59. We first show that $X_f^{(k)}$ is a prolongation of $X_f^{(l)}$ for every $1 \le l < k$. We do so by comparing the local coordinate expressions for these two vector fields (see (35)). Since $Z_\alpha^{(k)}(f)$ is a function of variables x_j and y_γ ($|\gamma| \le |\alpha|+1$) only, we can see that $Z_\alpha^{(l)}(f) = Z_\alpha^{(k)}(f)$ for $|\alpha| \le j-1$. Thus, in order to show that $X_f^{(k)}$ is a prolongation of $X_f^{(l)}$ we need only show that for $|\alpha| = j$

$$Z_\alpha^{(l)}(f) = Z_\alpha^{(k)}(f) - \sum_j y_{[\alpha,j]} \frac{\partial f}{\partial y_{[j]}} \quad . \tag{40}$$

Choose a j such that $\alpha^j \ne 0$ and write $\alpha = [\gamma, j]$. Let us consider the difference of $Z_j^{(l)}(Z_\gamma^{(l)}(f))$ and $Z_j^{(k)}(Z_\gamma^{(k)}(f))$. Since $Z_\gamma^{(l)}(f) = Z_\gamma^{(k)}(f)$ and since these do not depend on variables y_β ($|\beta| > l$), the difference is one extra group of terms that are present in $Z_j^{(k)}(Z_\gamma^{(k)}(f))$, namely

$$\sum_{|\alpha|=l} y_{[\alpha,j]} \frac{\partial Z_\gamma^{(k)}(f)}{\partial y_\alpha} \quad .$$

By Lemma 5.62 this is equal to

$$\sum_j y_{[\alpha,j]} \frac{\partial f}{\partial y_{[j]}} \quad .$$

thereby showing that (40) is true.

Now we turn to the proof that X is an infinitesimal automorphism. Since X satisfies conditions (36) and (37) of Proposition 5.58, we have

$$i(X)\omega_{[\alpha,a]}^{(k)} - Z_a^{(k)}(i(X)\omega_\alpha^{(k)}) = 0 \quad ,$$

$$i(X)dy_{[\beta,a]} - Z_a^{(k)}(i(X)\omega_\beta^{(k)}) = 0 \quad ,$$

for all a, $|\alpha| \leq k-2$, and $|\beta| = k-1$. Hence, using Lemma 5.60 and identity (30), we have

$$i([Z_a^{(k)}, X])\omega_\alpha^{(k)} = 0 \quad .$$

Again, using identity (30) and Lemma 5.60 we have for $|\beta| = k$ and $|\alpha| \leq k-1$

$$i\left(\left[\frac{\partial}{\partial y_\beta}, X\right]\right)\omega_\alpha^{(k)} = \frac{\partial Z_\alpha^{(k)}(f)}{\partial y_\beta} - i(X)\mathcal{L}_{\frac{\partial}{\partial y_\beta}}\omega_\alpha^{(k)} \quad .$$

If we suppose that there is an j such that $\beta = [\alpha, j]$, then by Lemmas 5.60 and 5.62, both of the terms of the right side are $-X^j$ and hence, the above expression is 0. If we suppose that no such j exists then both of the terms in the right side are zero by the same two Lemmas, and therefore

$$i\left(\left[\frac{\partial}{\partial y_\beta}, X\right]\right)\omega_\alpha^{(k)} = 0$$

for all β and α. Proposition 5.55 allows us to conclude that X must be an infinitesimal automorphism of $\Omega^{(k)}$. □

We now prove the converse of the preceding theorem.

Theorem 5.63 (Bäcklund) *Suppose that X is an infinitesimal automorphism of $\Omega^{(k)}$. Then, $f = i(X)\omega_0^{(k)}$ is a function that depends only on variables x_j, y_0, $y_{[a]}$; and, furthermore, $X = X_f^{(k)}$, thereby implying that X is a prolongation of an infinitesimal automorphism of $\Omega^{(1)}$ (namely $X_f^{(1)}$).*

We first need to prove the following converse of Lemma 5.62.

Lemma 5.64 *Suppose that (39) holds for all $|\beta| = k$ and $|\alpha| \leq k-1$. Then, f only depends on x_j, y_0, $y_{[a]}$; and $-X^j = \frac{\partial f}{\partial y_{[j]}}$.*

Proof. Let $|\beta| = k-1$ and $|\alpha| \leq k-2$ be given. Choose any a and use Lemma 5.61 to see that

$$\begin{aligned} \frac{\partial Z^{(k)}_{[\alpha,a]}(f)}{\partial y_{[\beta,a]}} &= \frac{\partial Z^{(k)}_{[a]}(Z^{(k)}_{\alpha}(f))}{\partial y_{[\beta,a]}} \\ &= \left[\frac{\partial}{\partial y_{[\beta,a]}}, Z^{(k)}_{a}\right](Z^{(k)}_{\alpha}(f)) + Z^{(k)}_{[a]}\left(\frac{\partial Z^{(k)}_{\alpha}(f)}{\partial y_{[\beta,a]}}\right) \\ &= \frac{\partial Z^{(k)}_{\alpha}(f)}{\partial y_{[\beta]}} \quad . \end{aligned}$$

Note that the condition that there exists an j such that $\beta = [\alpha, j]$ is true if and only if the condition that there exists an j such that $[\beta, a] = [\alpha, a, j]$ is true as well. Thus, the preceding calculation shows that the premise of the lemma remains true if we replace k by $k-1$. Proceeding inductively we see that the premise of the lemma is true if we replace k by any of $1, 2, \ldots, k-1$. In particular, we have

$$\frac{\partial f}{\partial y_{[j]}} = \frac{\partial Z^{(k)}_0(f)}{\partial y_{[j]}} = -X^j \quad ,$$

and for $2 \leq |\beta| \leq k$ we have

$$\frac{\partial f}{\partial y_\beta} = \frac{\partial Z^{(k)}_0(f)}{\partial y_\beta} = 0 \quad .$$

□

Proof of Theorem 5.63. By Proposition (5.55) we have

$$i([Z_a^{(k)}, X])\omega_\alpha^{(k)} = 0 \quad ,$$

and hence, (36) and (37) are true by identity (30) and by Lemma (5.60). We also have

$$i\left(\left[\frac{\partial}{\partial y_\beta}, X\right]\right)\omega_\alpha^{(k)} = 0 \quad ,$$

for all $|\beta| = k$ and all $|\alpha| \leq k-1$. Using Lemma 5.64, we conclude that X satisfies all conditions of Proposition 5.58 and that f is a function of variables x_j, y_0, and $y_{[\alpha]}$ only. □

We are now in a position to prove an anologue of Theorem 5.53 for higher order jet bundles. For functions, f, g on $\mathrm{J}^k(\mathbb{R}^n, \mathbb{R})$, put

$$\{f, g\}^{(k)} = i([X_f^{(k)}, X_g^{(k)}])\omega_0^{(k)} \quad .$$

Using basic propoperties of prolongations we have that $[X_f^{(k)}, X_g^{(k)}]$ is a prolongation of of $[X_f, X_g]$, because $X_f^{(k)}$ and $X_g^{(k)}$ are prlongations of X_f and X_g. Thus

$$\{f, g\}^{(k)} = i([X_f^{(k)}, X_g^{(k)}])\omega_0^{(k)} = (i([X_f, X_g])\omega_0^{(1)} = \{f, g\} \quad .$$

Therefore we have defined exactly the same bracket operation as in (33) and just as before we have

$$X_{\{f,g\}}^{(k)} = [X_f^{(k)}, X_g^{(k)}] \quad .$$

Theorem 5.65 *The cohomology class S remains non-trivial in the local Chevalley cohomology of infinitesimal automorphisms of $\Omega^{(k)}$.*

Proof. We will follow the proof of Theorem 5.53. Let T be any local 1 cochain. Taking

$$f_1 = (x_1)^2 y_{[1]} \quad , \quad g_1 = x_1(y_{[1]})^2 \quad ,$$
$$f_2 = (x_1)^3 \quad , \qquad g_2 = (y_{[1]})^3 \quad .$$

we have, as before, that

$$3T(X_{f_1}^{(k)}, X_{g_1}^{(k)})(0) = T(X_{f_2}^{(k)}, X_{g_2}^{(k)})(0) \quad .$$

The computation of $S^{\Gamma}(X_f^{(k)}, X_g^{(k)})$ is quite a bit more complicated in the present case, because the local coordinate expression of $X_f^{(k)}$ is considerably more complex than the expression of X_f. Since we can work with any connection, Γ, let us work with the flat connection on $J^k(\mathbb{R}^n, \mathbb{R})$. Doing so gives a simpler local coordinate expression for $\mathcal{L}_X\Gamma$, because for a flat Γ we have by (16) that $(\mathcal{L}_X\Gamma)^c_{ab} = \partial_{ab}X^c$. Restricting our attention to computing the $dx_1 \wedge dy_{[1]}$ component of $S^{\Gamma}(X_{f_1}^{(k)}, X_{g_1}^{(k)})(0)$ and of $S^{\Gamma}(X_{f_2}^{(k)}, X_{g_2}^{(k)})(0)$ we see that $S^{\Gamma}(X_{f_1}^{(k)}, X_{g_1}^{(k)})_{1[1]}$ at the origin is positive and that $S^{\Gamma}(X_{f_2}^{(k)}, X_{g_2}^{(k)})_{1[1]}$ at the origin is negative; and hence, that

$$3S^{\Gamma}(X_{f_1}^{(k)}, X_{g_1}^{(k)})(0) \neq S^{\Gamma}(X_{f_2}^{(k)}, X_{g_2}^{(k)})(0) \quad .$$

This means, of course, that ∂T and S^{Γ} cannot be the same operator. □

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