Map Enumeration and Phase Transitions of Random Graphs

Mary Elaine Herrera

Department of Mathematics & Statistics

McGill University, Montreal, Canada

elaine.herrera@mail.mcgill.ca

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Abstract

In this thesis, we explore probabilistic and enumerative aspects of graphs, primarily studying the Erdős-Rényi random graph, minor-closed classes of graphs, and graphs on surfaces. In particular, we present a proof of the phase transition in connected components of the Erdős-Rényi random graph, introduce and implement Tutte's recursive method, as well as discuss conditions that guarantee the algebraicity of functional equations obtained from Tutte's recursive method. We will also present a conjecture on phase transitions of random graphs sampled from minor-closed classes. This is supplemented with examples where the conjecture is known to be true, namely in uniform random graphs, random planar graphs, and random forests.

Résumé

Dans cette thèse, nous explorons les aspects probabilistes et énumératifs de graphes. Nous étudions principalement des graphes Erdős-Rényi, des graphes de classes closes par mineur et des graphes sur des surfaces. En particulier, nous présentons une preuve de la transition de phase des composantes connexes des graphes Erdős-Rényi. De plus, nous introduisons et implémentons la méthode récurrente de Tutte et nous étudions quelles conditions garantissent l'algebraïcité des équations fonctionnelles résultantes de la méthode récurrente de Tutte. Nous présentons également une conjecture sur les transitions de phase de graphes aléatoires échantillonnés de classes closes par mineur. Ceci est complété par des examples pour lesquels la conjecture est vraie: les graphes uniformes aléatoires, les graphes planaires aléatoires et les forêts aléatoires.

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Chapter 1

Introduction

Random graph models have been heavily studied, particularly on how their behaviour changes with incremental shifts of its parameters. The earliest models are the *uniform* and *binomial random graph*, both are referred to as *Erdős-Rényi random graphs*, which were introduced in 1959 [6, 11]. It is well known that both models exhibit a *phase transition* in the sizes of their connected components. After imposing planarity and acyclicity restrictions on the Erdős-Rényi random graphs, such phase transitions are still present [14, 20].

In the uniform random graph model, as well as the resulting models after imposing planarity and acyclicity restrictions, we can consider them as uniformly selecting a graph from a certain family of graphs. It turns out in each case, the family of graphs we are sampled from is a (subclass of a) *minor-closed class* [17, 24, 26]. Furthermore, as graphs on surfaces can be characterized as minor-closed classes of graphs, and as the phase transition results for the connected components in these classes have been proved using enumerative methods, then studying *maps* and methods for their enumeration is a natural prerequisite step to studying phase transitions of random graph models sampled from other minor-closed classes [3, 4, 23, 29].

This chapter will cover background knowledge about phase transitions for the connected components of random graphs, introductions to the notions of graph minor theory, and will discuss embeddability of graphs on surfaces.

We lead with some preliminary notation and probabilistic bounds.

1.1 Probabilistic Notation and Commonly Used Bounds

We leave this section as a collection of probabilistic terminology, notation for asymptotics, as well as bounds that may be referenced throughout this thesis. This section will be used to supplement the next where we discuss probabilistic aspects of random graphs and is based on [10].

1.1.1 Probability and Landau Notation

It will be useful to recall some probabilistic and asymptotic terminology. We will use $\mathbb{P}, \mathbb{E}$, and **Var** to denote probability, expectation, and variance respectively.

Definition 1.1.1. An event E is said to occur *almost surely (a.s.)* if $\mathbb{P}(E) = 1$. We say that a sequence of events $(E_n)_{n\geq 1}$ occur *with high probability (w.h.p.)* if

$$\lim_{n\to\infty} \mathbb{P}(E_n) = 1.$$

We frequently use indicator functions as well.

Definition 1.1.2. Let E be an event and define the *indicator function* of E by

$$\mathbf{1}_{\{E\}} = \begin{cases} 1 & \text{if } E \text{ occurs,} \\ 0 & \text{otherwise.} \end{cases}$$

We note that for an event E,

$$\mathbb{E}\left(\mathbf{1}_{\{E\}}\right) = \mathbb{P}(E).$$

The following is commonly used Landau notation:

- Little-o:

  For functions $f(n), g(n)$ where $g(n) > 0$, we write $f(n) = o(g(n))$ as $n \to \infty$ when

$$\lim_{n\to\infty} \frac{|f(n)|}{g(n)} = 0.$$

- Big-O:

  For functions $f(n), g(n)$ where $g(n) > 0$, we write $f(n) = O(g(n))$ as $n \to \infty$ when

$$\limsup_{n\to\infty} \frac{|f(n)|}{g(n)} < \infty.$$

- Big Theta:

  For functions $f(n), g(n) > 0$, we write $f(n) = \Theta(g(n))$ as $n \to \infty$ when $f(n) = O(g(n))$ and $g(n) = O(f(n))$.

1.1.2 Some Useful Bounds

In this section, we will provide some bounds as well as recall common probabilistic tools, such as the first moment and second moment methods.

Lemma 1.1.3.

(a) For all $x \in \mathbb{R}$, $1 + x \leq e^x$.

(b) For $x \in [0, 1)$, $1 - x \geq e^{-x/(1-x)}$.

(c) For all $n, k \in \mathbb{N}$, $\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$.

(d) For all n, k, $\binom{n}{k} \leq \frac{n^k}{k!}\left(1 - \frac{k}{2n}\right)^{k-1}$.

(e) *For all* n, k, $\binom{n}{k} \leq \frac{n^k}{k!} e^{-k(k-1)/(2n)}$.

(f) *For all non-negative* n, $\frac{1}{n!} \leq \left(\frac{e}{n}\right)^n$.

(g) *For* $x \in [0, 1]$, $e^x \leq 1 + 2x$.

The proof of this Lemma can be found in the Appendix A. A commonly used approximation is the following, given without proof.

Theorem 1.1.4 (Stirling's Formula)**.** *For* $n \in \mathbb{N}$,

$$n! = (1 + o(1)) \left(\frac{n}{e}\right)^n \sqrt{2\pi k},$$

and moreover,

$$n! \leq \left(\frac{n}{e}\right)^n \sqrt{2\pi k}.$$

We end this section with some well known probabilistic inequalities.

Theorem 1.1.5 (Markov's Inequality)**.** *Let* X *be a non-negative random variable. Then for all* $t > 0$,

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}(X)}{t}.$$

Proof. Let X be a non-negative random variable and $t > 0$. Note that,

$$\begin{aligned} X &= X \cdot \mathbf{1}_{\{X \geq t\}} + X \cdot \mathbf{1}_{\{X < t\}} \\ &\geq X \cdot \mathbf{1}_{\{X \geq t\}} \\ &\geq t \cdot \mathbf{1}_{\{X \geq t\}}. \end{aligned}$$

Then as expectation is monotonic,

$$\mathbb{E}(X) \geq t\mathbb{E}\left(\mathbf{1}_{\{X \geq t\}}\right) = t\mathbb{P}(X \geq t). \qquad \square$$

The following corollary is immediate by taking $t = 0$.

Corollary 1.1.6 (First Moment Method)**.** *Let X be a non-negative integer valued random variable. Then,*

$$\mathbb{P}(X > 0) \leq \mathbb{E}(X).$$

The first moment method is a powerful tool used to show that a random variable must be equal to 0, with high probability. In contrast, the second moment method can be used to show that a random variable is positive, with high probability. This is also a consequence of Markov's inequality.

Theorem 1.1.7 (Chebyshev's Inequality)**.** *Let X be a random variable with finite mean and variance. Then for all $t > 0$,*

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq t) \leq \frac{\mathbf{Var}(X)}{t^2}.$$

Proof. Let X be as above and $t > 0$. Note that $|X - \mathbb{E}(X)| \geq t$ if and only if $(X - \mathbb{E}(X))^2 \geq t^2$. Thus,

$$\begin{aligned} \mathbb{P}(|X - \mathbb{E}(X)| \geq t) &= \mathbb{P}\left((X - \mathbb{E}(X))^2 \geq t^2\right) \\ &\leq \frac{\mathbb{E}((X - \mathbb{E}(X))^2)}{t^2} && \text{(Theorem 1.1.5)} \\ &= \frac{\mathbf{Var}(X)}{t^2}. \end{aligned}$$

□

The following is immediate by taking $t = \mathbb{E}(X)$.

Corollary 1.1.8 (Second Moment Method)**.** *Let X be a non-negative integer valued random variable, then*

$$\mathbb{P}(X = 0) \leq \frac{\mathbf{Var}(X)}{\mathbb{E}(X)^2} = \frac{\mathbb{E}(X^2)}{\mathbb{E}(X)^2} - 1.$$

1.2 Component Sizes in Various Families of Random Graphs

For a positive integer n, denote $[n] := \{1, \ldots, n\}$ and for a set S, denote $\binom{S}{2} = \{S' \subseteq S : |S| = 2\}$. Recall that a *simple graph* is an ordered pair $G = (V, E)$ where V is a finite set and $E \subseteq \binom{V}{2}$. Unless otherwise stated, a *graph* will refer to a simple graph. Furthermore given a graph G, we let $V(G)$ denote its vertex set and $E(G)$ denote its edge set. Since the vertex set of a graph is finite, we may assume $V(G) = [n]$. When we refer to the *size* or *order* of graph G, we are simply referring to the number of vertices in G, denoted by $|G|$.

There are two very closely related random graph models, both commonly referred to as *Erdős-Rényi random graph models.*

Definition 1.2.1. For $n, m \in \mathbb{N}$, let $\mathcal{G}_{n,m}$ be the collection of graphs with vertex set $[n]$ and with exactly m edges. We say $G_{n,m}$ is a *uniform random graph* with parameters n, m which is uniformly distributed over $\mathcal{G}_{n,m}$.

This model was introduced by Erdős and Rényi in 1959 [6] and the following model was introduced by Gilbert in the same year [11].

Definition 1.2.2. For $p \in (0, 1)$ and $n \in \mathbb{N}$, $G_{n,p}$ is a *binomial random graph* with parameters n and p so that $V = V(G_{n,p}) = [n]$ and each possible edge $e \in \binom{V}{2}$ appears independently in $G_{n,p}$ with probability p.

When n is very large and taking $m = \binom{n}{2}p$, then $G_{n,p}$ and $G_{n,m}$ behave very similarly. A correspondence between the two models can be found in Chapter 1 of [10]. We will be interested in the connected components of $G_{n,m}$ and $G_{n,p}$ and it will be useful to have notation to refer to such components.

Definition 1.2.3. Let $G = (V, E)$ be a graph with $V = [n]$. For a vertex $v \in V$, the *component of* v is the connected component in G that contains v, and is denoted by $\mathscr{C}_G(v)$.

When it is clear we are referring to a particular graph G, then we may write $\mathscr{C}(v) := \mathscr{C}_G(v)$. Furthermore, we consider an ordering on the components in terms of size.

Definition 1.2.4. Let $G = (V, E)$ be a graph with $V = [n]$. Let $C^{(i)} = C^{(i)}(G)$ be the components of G ordered in terms of size, $\left|C^{(1)}\right| \geq \left|C^{(2)}\right| \geq \cdots$. To break ties, if C, C' are distinct components so that $|C| = |C'|$, then list C, C' in increasing order based off of their smallest vertex label. We say that $C^{(i)}$ is the *i-th largest component* of G. Furthermore, let $\mathrm{Comp}(G)$ denote the collection of all the components of G.

In the uniform random graph model, it is seen that minor changes to the parameter m in $G_{n,m}$ can yield very different behaviours; this phenomenon is referred to as a *phase transition*. In particular, $G_{n,m}$ undergoes a phase transition in the sizes of its connected components when m is around $\frac{n}{2}$. When $m < \frac{n}{2}$, it is said that the $G_{n,m}$ model is in the *subcritical regime* and the model is referred to be in the *supercritical regime* when $m > \frac{n}{2}$. We say that $G_{n,m}$ is in the *critical regime* when $m = \frac{n}{2} + O(n^{2/3})$. This discovery was initially published in [7] by Erdős and Rényi with analogous results in the $G_{n,p}$ model by Gilbert in [11]. Several improvements on precision were later developed, the following results for the $G_{n,m}$ model are due to the work of Łuczak [18, 19] as well as Łuczak, Pittel, and Wierman [21].

Theorem 1.2.5. *Take $s = s(n)$ and set $m = \frac{n}{2} + s$. Consider the $G_{n,m}$ model with $C^{(i)} = C^{(i)}(G_{n,m})$ for all i. Then with high probability,*

$$\left|C^{(i)}\right| = \begin{cases} \left(\frac{1}{2} + o(1)\right) \frac{n^2}{s^2} \log \frac{|s^3|}{n^2} & \text{if } \frac{s^3}{n^2} \to -\infty, \\ \Theta(n^{2/3}) & \text{if } \frac{s^3}{n^2} \to c \in \mathbb{R}, \\ (4 + o(1))s & \text{if } \frac{s^3}{n^2} \to \infty \text{ and } i = 1, \\ \left(\frac{1}{2} + o(1)\right) \frac{n^2}{s^2} \log \frac{|s^3|}{n^2} & \text{if } \frac{s^3}{n^2} \to \infty \text{ and } i \geq 2. \end{cases}$$

The phenomenon of obtaining the linear order component after surpassing $m = \frac{n}{2}$ is known as the *emergence of the giant component* and the largest component of $G_{n,m}$ for $m > \frac{n}{2}$ is often referred to as the *giant component*.

In Chapter 2, we provide a proof for this phase transition in terms of the $G_{n,p}$ model where the phase transition is in terms of p. In particular the critical regime occurs at $p = \frac{1}{n}$, at which value $G_{n,p}$ has around $\frac{n}{2}$ edges with high probability. The goal of the proof is to show the existence of the phase transition as well as the uniqueness of the giant component, and hence will not show the results with same amount of precision as stated in Theorem 1.2.5. It will be done by assessing cyclic components and tree components separately. In fact, we will see that in the subcritical regime, with high probability the largest component is a tree.

In the uniform random graph model, $G_{n,m}$ uniformly picks a graph from the collection $\mathcal{G}_{n,m} = \{G = (V, E) \ : \ V = [n], |E| = m\}$. One can look at a similar model for more restrictive families of random graphs, for example conditioning that a graph is planar or is acyclic. In these settings, we can likewise study the existence of such phase transitions for component sizes as a function of the number of edges. In this case, we will be imposing conditions on the $G_{n,m}$ model. The following two examples gives results for the phase transitions in these two more restrictive cases, these results will be given without proof.

Example 1.2.6 (Random Forests)**.** Let $\mathcal{F}_{n,m}$ be the collection of forests on vertex set $[n]$ and having m edges. Let $F_{n,m}$ be a uniform element taken from $\mathcal{F}_{n,m}$, we call $F_{n,m}$ a *random forest*. The limiting behaviour of component sizes is described in [20] showing that random forests too exhibit a phase transition when $m = \frac{n}{2}$.

Theorem 1.2.7. *Take $s = s(n)$ and set $m = \frac{n}{2} + s$. In the random forest model $F_{n,m}$, set $C^{(i)} = C^{(i)}(F_{n,m})$ for all i. Then with high probability,*

$$\left|C^{(i)}\right| = \begin{cases} \left(\frac{1}{2}+o(1)\right)\frac{n^2}{s^2}\log\frac{|s^3|}{n^2} & \textit{if } \frac{s^3}{n^2}\to-\infty, \\ \Theta\left(n^{2/3}\right) & \textit{if } \frac{s^3}{n^2}\to c\in\mathbb{R}, \\ (4+o(1))s & \textit{if } \frac{s^3}{n^2}\to\infty, n-s\to\infty, \textit{ and } i=1, \\ O\left((n-s)^{2/3}\right) & \textit{if } \frac{s^3}{n^2}\to\infty, n-s\to\infty \textit{ and } i\geq 2. \end{cases}$$

In the subcritical regime, the component sizes are identical to the bounds found in Theorem 1.2.5 for the subcritical regime. This is consistent with the fact that most of the components in the uniform random graph are trees with high probability. Furthermore, random forests are similar to the uniform random graph in the critical regime. In the critical regime of the uniform random graph, the ratio $\frac{|C^{(1)}(G_{n,m})|}{|C^{(i)}(G_{n,m})|}$ is bounded in probability for any i, and in the case of random forests in the critical regime, [20] shows that ratio $\frac{|C^{(1)}(F_{n,m})|}{|C^{(i)}(F_{n,m})|}$ is likewise bounded in probability.

Example 1.2.8 (Random Planar Graphs)**.** Let $\mathcal{P}_{n,m}$ be the collection of planar graphs on vertex set $[n]$ and having m edges. Let $P_{n,m}$ be a uniform element taken from $\mathcal{P}_{n,m}$, we call $P_{n,m}$ a *random planar graph*. Kang and Łuczak proved that $P_{n,m}$ exhibits a similar phases transition as $G_{n,m}$ with the emergence of a giant component [14]. The critical value for this case is $m=\frac{n}{2}+O\left(n^{2/3}\right)$.

Theorem 1.2.9. *Take $s=s(n)$ and let $m=\frac{n}{2}+s$. In the random planar graph model, set $C^{(i)}=C^{(i)}\left(P_{n,m}\right)$ being the i-th largest component of $P_{n,m}$. Then with high probability,*

$$\left|C^{(i)}\right| = \begin{cases} \left(\frac{1}{2}+o(1)\right)\frac{n^2}{s^2}\log\frac{|s^3|}{n^2} & \textit{if } \frac{s^3}{n^2}\to-\infty, s=o(n), \\ \Theta(n^{2/3}) & \textit{if } \frac{s^3}{n^2}\to c\in\mathbb{R}, \\ (2+o(1))s & \textit{if } \frac{s^3}{n^2}\to\infty, n-s\to\infty, \textit{ and } i=1, \\ \Theta\left(n^{2/3}\right) & \textit{if } \frac{s^3}{n^2}\to\infty, n-s\to\infty \textit{ and } i\geq 2. \end{cases}$$

1.3 Graph Minors and the Graph Minors Theorem

This section will cover introductory concepts about graph minor theory coming from [17, 24, 26]. We begin with some definitions.

Definition 1.3.1. Let $G = (V, E)$ be graph. A *contraction* of an edge uv in G is the replacement of u and v by a new vertex w which is adjacent to all of the neighbours of u and v. The graph obtained after contracting the edge uv is denoted G/uv.

Definition 1.3.2. Let $G = (V, E)$ be a graph. A *subdivision* of an edge $uv \in E$ is the deletion of uv in G and the addition of a new vertex w along with two new edges uw and vw. A graph which has been derived from G by a sequence of edge subdivisions is called a *subdivision* of G.

An example of an edge contraction and edge subdivision can be seen in Figure 1.1.

Definition 1.3.3. A graph H is said to be a *minor* of a graph G if H can be obtained from G via a (possibly empty) sequence of edge deletions, vertex deletions, and edge contractions. If H is a minor of G, we write $H \preccurlyeq G$. We say H is a *proper* minor of G, and write $H \prec G$, if $H \preccurlyeq G$ and $H \neq G$.

Remark 1.3.4. We use the convention that the minor relation is considered up to graph isomorphisms. That is, if H is a minor of a graph G and H, H' are isomorphic graphs, then we also say that H' is a minor of G.

Note that if $H \prec G$ then H has either fewer edges than G or fewer vertices than G, so $H \neq G$. One shall notice that the minor relation between graphs forms a poset.

Proposition 1.3.5. *Let $\mathcal{G}$ be the collection of all graphs and $\preccurlyeq$ be the graph minor relation, then $(\mathcal{G}, \preccurlyeq)$ is a partially ordered set.*

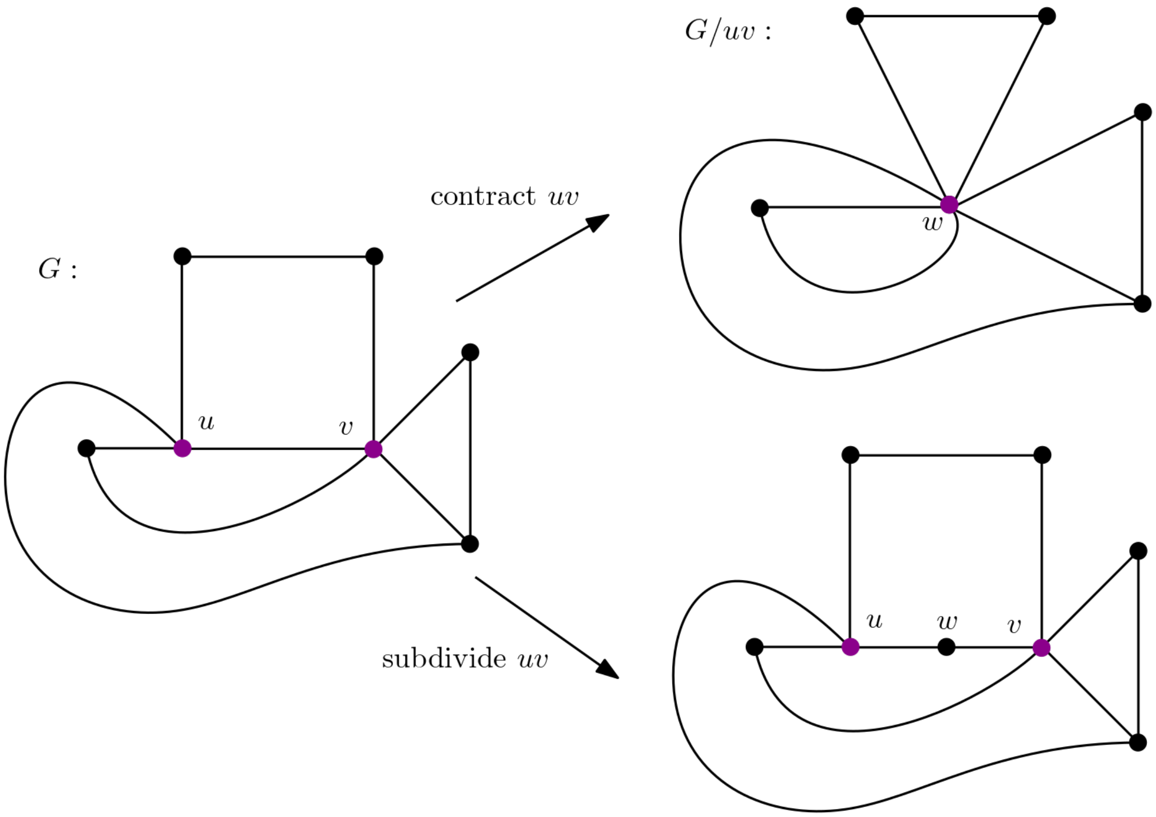


Figure 1.1: Example of an edge contraction and edge subdivision.

Proof. Consider the pair $(\mathcal{G}, \preccurlyeq)$ and note that every graph is a minor of itself, so $\preccurlyeq$ is reflexive.

Let $G_1, G_2, G_3 \in \mathcal{G}$ so that $G_1 \preccurlyeq G_2$ and $G_2 \preccurlyeq G_3$. We can see that $G_1 \preccurlyeq G_3$ by concatenating a sequence of deletions and contractions to obtain G_2 from G_3 with a sequence of deletions and contractions to obtain G_1 from G_2. Thus, $\preccurlyeq$ is transitive.

For antisymmetry, suppose $G, H \in \mathcal{G}$ so that $G \preccurlyeq H$ and $H \preccurlyeq G$. For the sake of contradiction, suppose $G \neq H$ where without loss of generality $H \prec G$. Then $G \preccurlyeq H \prec G$, so by transitivity $G \prec G$, a contradiction. $\square$

Before proceeding with an example, we state two well-known planarity theorems. The first is a characterization of planar graphs via subdivisions, established by Kuratowski in

1930 [16].

Theorem 1.3.6 (Kuratowski's Theorem)**.** *A finite graph G is planar if and only if G contains no subdivision of the complete graph K_5 or the complete bipartite graph $K_{3,3}$.*

In the same decade due to Wagner, another characterization of planar graphs was established, this time using graph minors [31].

Theorem 1.3.7 (Wagner's Theorem)**.** *A finite graph G is planar if and only if neither the complete graph K_5 nor the complete bipartite graph $K_{3,3}$ are minors of G.*

Definition 1.3.8. A family of graphs $\mathcal{G}$ is *minor-closed* if for any $G \in \mathcal{G}$ and $H \preccurlyeq G$, then $H \in \mathcal{G}$.

In this definition, we may assume that the family of graphs $\mathcal{G}$ is closed under graph isomorphisms since the minor relation is considered up to graph isomorphisms, as mentioned in Remark 1.3.4. Theorem 1.3.7 in particular implies that the family of planar graphs is minor-closed. The following shows an example of a minor, along with a non-example that makes use of Theorem 1.3.6.

Example 1.3.9. Let G be the the graph in Figure 1.2. The complete graph on four vertices

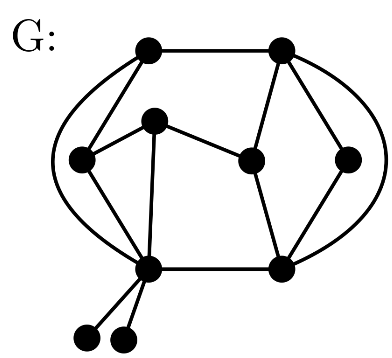


Figure 1.2: The graph G in Example 1.3.9.

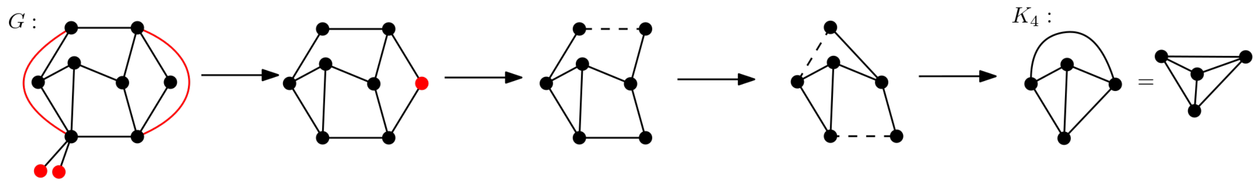


Figure 1.3: An example of a sequence to obtain K_4 from G in Example 1.3.9.

K_4 is a graph minor of G. A sequence of vertex deletions, edge deletions and edge contractions is depicted in Figure 1.3 to show how K_4 can be obtained from G.

In this figure, multiple vertex deletions, edge deletions, and edge contractions may be done all at once for efficiency. Let the coloured vertices and coloured edges depict which vertex or edge is deleted in the following step of the sequence. The dashed edges depict which edge is contracted in the following step in the sequence.

In the case of infinite graphs, the sequence of vertex deletions, edge deletions, and edge contractions in the definition may be infinite. However, we will restrict our attention to finite graphs. We want to discuss possible ways to characterize a minor-closed family.

Definition 1.3.10. Let $\mathcal{G}$ be the collection of finite simple graphs. Let I be a not necessarily finite index set and take a collection of graphs $\{G_i \ : \ i \in I\}$. Then

$$\operatorname{Forb}\{G_i \ : \ i \in I\} = \{G \in \mathcal{G} \ : \ G_i \not\preccurlyeq G \ \forall i \in I\}$$

is the collection of graphs that forbids any of $\{G_i \ : \ i \in I\}$ as a minor.

Some very well known families of graphs can be thought of in terms of the set $\operatorname{Forb}\{\cdot\}$. For example, take $\mathcal{G}$ to be the collection of all finite graphs, we see that $\operatorname{Forb}\{\emptyset\} = \mathcal{G}$. Take $\mathcal{F}$ to be the collection of all finite forests, then as a forest is characterized as an acyclic graph, we can write $\mathcal{F} = \operatorname{Forb}\{C_3\}$ where C_3 is the cycle of length three. One more example is the collection of all finite planar graphs, call it $\mathcal{P}$. By Wagner's Theorem, it can be seen that

Forb $\{K_{3,3}, K_5\} = \mathcal{P}$.

A property we hope for in Forb $\{\cdot\}$ is that it is itself minor-closed. This indeed is the case.

Proposition 1.3.11. *For a collection of graphs $\{G_i \;:\; i \in I\}$, the set Forb $\{G_i \;:\; i \in I\}$ is minor-closed.*

Proof. Let $G \in$ Forb $\{G_i \;:\; i \in I\}$ and suppose $H \preccurlyeq G$. Suppose for a contradiction that $H \notin$ Forb $\{G_i \;:\; i \in I\}$, so there is some $i \in I$ so that $G_i \preccurlyeq H$. Then $G_i \preccurlyeq H \preccurlyeq G$, and by transitivity, $G_i \preccurlyeq G$. This contradicts the fact that $G \in$ Forb $\{G_i \;:\; i \in I\}$ and so it must be that $H \in$ Forb $\{G_i \;:\; i \in I\}$. As $G \in$ Forb $\{G_i \;:\; i \in I\}$ and $H \preccurlyeq G$ were arbitrary, then it follows that Forb $\{G_i \;:\; i \in I\}$ is minor-closed. $\square$

Any collection of graphs defined by Forb $\{\cdot\}$ is minor-closed. If every minor-closed family of graphs can be written in terms of Forb $\{\cdot\}$, then this provides a characterization of minor-closed families of graphs in terms of which graphs they exclude.

Definition 1.3.12. For a family of graphs $\mathcal{G}$, we say a graph H is an *excluded minor* of $\mathcal{G}$ if H is not a minor of G for every $G \in \mathcal{G}$. In this case, we also call H a *forbidden minor*. For an excluded minor H of G, we say H is *minimal* if every minor of H is a minor of some graph in $\mathcal{G}$.

Note that if $\mathcal{G}$ is a minor-closed family and $H \notin \mathcal{G}$, then H is automatically an excluded minor of G. It follows that taking $\mathcal{F} = \{H \;:\; H \text{ is a graph}, H \notin \mathcal{G}\}$, then $\mathcal{G} =$ Forb $\{\mathcal{F}\}$. Thus, all minor-closed families have a forbidden minor characterization. In fact, every minor-closed family can be characterized by a finite set of excluded minors. This was conjectured by Wagner [32] and later proved by Robertson and Seymour [27].

Definition 1.3.13. A partially ordered set $(P, \leq)$ is a *well-quasi ordering* if for every infinite sequence $(x_i)_{i\geq 1}$ of elements from P we can find two indices i, j so that $i < j$ and $x_i \leq x_j$.

Robertson and Seymour proved that any infinite sequence of finite graphs must have one graph that is a proper minor of another. The Graph Minors Theorem is a direct consequence.

Theorem 1.3.14 (Robertson, Seymour)**.** *Let $\mathcal{G}$ be the collection of all finite graphs, then $(\mathcal{G}, \preccurlyeq)$ is a well-quasi ordering.*

Corollary 1.3.15 (Graph Minors Theorem)**.** *A minor-closed family $\mathcal{G}$ can be characterized by a finite list of forbidden minors.*

Proof. Let $\mathcal{G}$ be a minor-closed family and $\mathcal{H}$ be the collection of minimal excluded minors. Note that $\mathcal{H}$ characterizes $\mathcal{G}$ as every graph not in $\mathcal{G}$ is either in $\mathcal{H}$ or has a minor in $\mathcal{H}$ so $\mathcal{G} = \text{Forb}\{\mathcal{H}\}$. If $\mathcal{H}$ were infinite, then by the Robertson-Seymour Theorem there would be $G, H \in \mathcal{H}$ so that $G \prec H$. In this case, H is not a minimal minor, so this contradicts $\mathcal{H}$ being a collection of minimal minors. Thus, $\mathcal{H}$ is finite. $\square$

The examples in Section 1.2 looked at the phase transitions for the largest connected component for each of the binomial random graph $G_{n,m}$, random forests $F_{n,m}$, and random planar graphs $P_{n,m}$. Furthermore, recall that

$$\mathcal{G}_{n,m} = \{G = (V, E) \ : \ V = [n], |E| = m\},$$

where $G_{n,m}$ was uniformly selected from $\mathcal{G}_{n,m}$. Similarly, $F_{n,m}$ and $P_{n,m}$ were uniformly selected from $\mathcal{F}_{n,m}$ and $\mathcal{P}_{n,m}$, respectively. Each of $\mathcal{G}_{n,m}, \mathcal{F}_{n,m}, \mathcal{P}_{n,m}$ are subsets of minor-closed families of graphs. Precisely,

$$\begin{aligned}
\mathcal{G}_{n,m} &= \mathcal{G}_{n,m} \cap \text{Forb}\{\emptyset\}, \\
\mathcal{F}_{n,m} &= \mathcal{G}_{n,m} \cap \mathcal{F} = \mathcal{G}_{n,m} = \mathcal{G}_{n,m} \cap \text{Forb}\{C_3\}, \\
\mathcal{P}_{n,m} &= \mathcal{G}_{n,m} \cap \mathcal{P} = \mathcal{G}_{n,m} = \mathcal{G}_{n,m} \cap \text{Forb}\{K_{3,3}, K_5\}.
\end{aligned}$$

So in the previous examples, we can consider each random graph to be sampled from a subset of a minor-closed class. A natural direction to look into is to check if such a phase transition exists for any minor-closed family. We conclude this section with a conjecture.

Conjecture 1. *Let $\widetilde{\mathcal{G}}$ be a minor-closed family of graphs. For all $n, m \in \mathbb{N}$, set*

$$\widetilde{\mathcal{G}}_{n,m} = \left\{ \widetilde{G} \in \widetilde{\mathcal{G}} \ : \ V\left(\widetilde{G}\right) = [n], \left| E\left(\widetilde{G}\right) \right| = m \right\}$$

and take $\widetilde{G}_{n,m}$ to a random graph uniformly selected from $\widetilde{\mathcal{G}}_{n,m}$. Then there is a constant $c > 0$ so that:

(i) If $a < c$ then for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\left| C^{(1)}\left(\widetilde{G}_{n,an}\right) \right| > \varepsilon n \right) = 0.$$

(ii) If $a > c$ then there exists $\varepsilon > 0$ so that,

$$\lim_{n\to\infty} \mathbb{P}\left(\left| C^{(1)}\left(\widetilde{G}_{n,an}\right) \right| > \varepsilon n \right) = 1.$$

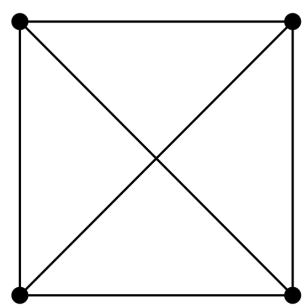
We conjecture that for any random graph model $\widetilde{G}_{n,m}$, that is defined by uniformly selecting an element from a subset of a minor-closed class of graphs, there is a critical value cn so that w.h.p. $\widetilde{G}_{n,m}$ has a component of linear size when $m > an$ for $a > c$. Otherwise, w.h.p. every component of $\widetilde{G}_{n,m}$ is sub-linear when $m < an$ for $a < c$.

1.4 Embeddability of Maps and Graphs

The collection of terminology is taken from [1, 22, 29]. We discuss the notions of what it means for a graph and map to be embeddable on a surface. We also discuss the distinction between planar graphs and planar maps. Whenever we refer to a *surface*, we are always considering an oriented, compact, connected 2-manifold without boundary.

Definition 1.4.1. Let G be a graph and S be a surface. Then G is said to be *embeddable* on S if there exists a drawing of G on S without any edge crossings.

The definition of embeddability requires the existence of a drawing without edge crossings, although many drawings could exist with edge crossings. A particular case we will study is when S is the sphere. In this case, a graph that is embeddable on S is called a *planar graph.* We will often view planar graphs drawn on the plane instead of the sphere. One common example of a planar graph is the complete graph K_4, which is the largest complete planar graph. Figure 1.4 shows two drawings of K_4, the drawing on the right is an embedding into the plane without edge crossings


Non-planar drawing of K_4

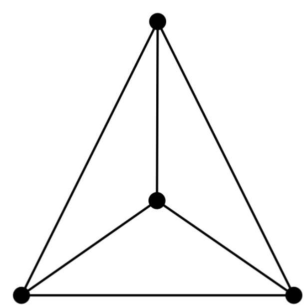

Planar drawing of K_4

Figure 1.4: A non-planar and planar drawing of K_4.

Another example to consider is K_5, which by Kuratowski's Theorem is non-planar. However, we see in Figure 1.5 that K_5 is embeddable on the torus.

Definition 1.4.2. Let S be a surface and G be a graph embeddable on S. Embed G in S, then $S - G$ is obtained my removing the images in S of all the vertices and edges in G. A *face* of G is a connected component of $S - G$.

An example of $S - G$ for a graph G embedded in the sphere is shown in Figure 1.6.

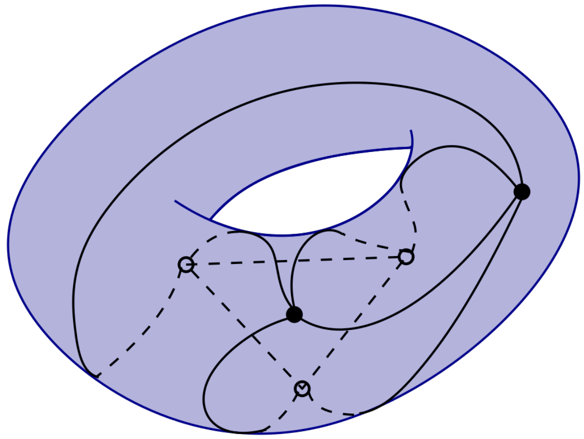


Figure 1.5: An embedding of K_5 on the torus.

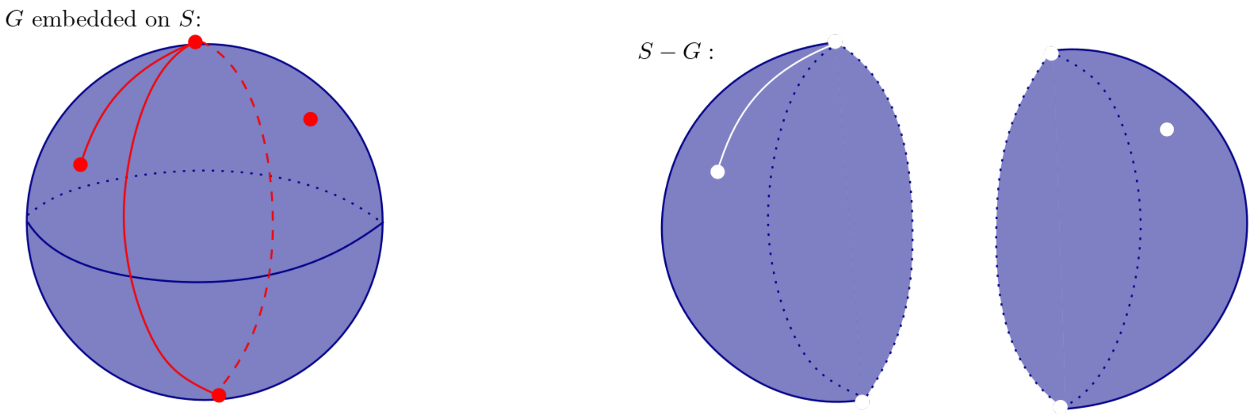


Figure 1.6: $S - G$ where G is a graph embeddable on the sphere.

Definition 1.4.3. Let S be a surface, then a *map* in S is an embedding of a connected graph G in S drawn so that each face is homeomorphic to an open disk. That is, each connected component of $S - G$ is simply connected. In this case, call G the *underlying graph* of M and denote it by $G(M)$.

For notation, take a map M and its underlying graph $G(M) = (V, E)$. Then, we take $V(M)$ to be the vertices V under the embedding M and $E(M)$ to be the set of edges E under the embedding M. When it is clear we are referring to the vertices and edges under the embedding, we simply say M has vertex set V and edge set E.

In Figure 1.6, we see that the embedding of G is not a map as the removal of the isolated vertex in the right hemisphere does not result in a face that is homeomorphic to a disk. However, if we took H to be the graph obtained from G by removing the singleton, then under the induced embedding, we obtain a map on S.

From this definition, a map M is an embedding of $G(M)$. Note that this underlying graph is unique. Furthermore, the definition of a map implies that for any map M in any surface S, the underlying graph $G(M)$ is connected. An observation to note is that embeddability in S is closed under taking minors.

Proposition 1.4.4. *Let S be a surface and G be a graph embeddable in S, then any minor of G is also embeddable in S.*

Proof. Since minors are obtained through vertex deletions, edge deletions, and edge contractions, and as embeddability is clearly preserved under taking subgraphs, we only need to check that edge contractions preserve embeddability in S. This is clear by viewing S as a polygon. By the classification theorem of surfaces [5], S can be viewed as a polygon with its sides labelled and oriented. Fix a graph G that is embeddable in S and embed G so that all the vertices are on the face of the polygon. For an edge uv, draw G/uv by fixing u and draw all the edges of v to be incident with u by following near to the path from$u$ to v given by the embedding of edge uv.

A depiction of this process can be seen in Figure 1.7. Two examples are shown: one when the edge being contracted is completely on the face of the polygon, and one where the edge being contracting passes through one of the polygon sides. Then G/uv is drawn on S with no edges crossing, which shows that embeddability is preserved under edge contractions. □

For a graph G that is embeddable in S, then any embedding of G in S corresponds to the same graph. However, different embeddings of G could correspond to different maps. The notion of map isomorphism becomes pertinent.

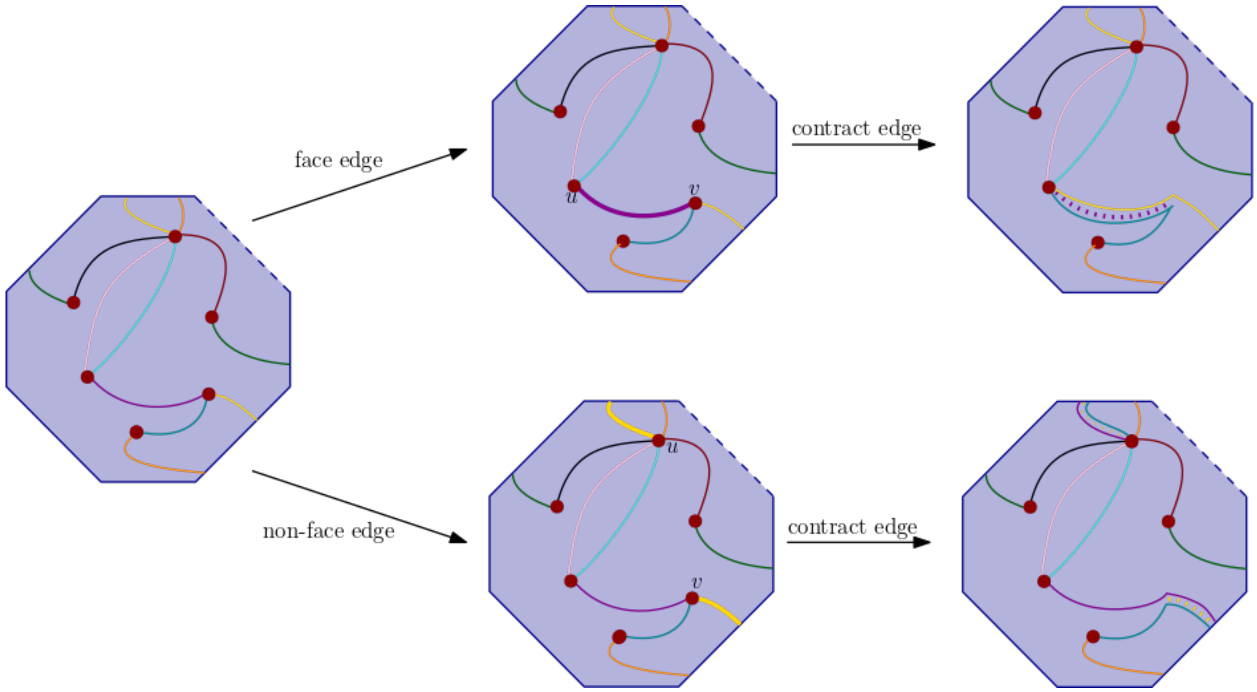


Figure 1.7: Two examples of contracting edges on a surface for Proposition 1.4.4. The edge labels and orientations of the polygon are omitted and the different edge colours are used to distinguish distinct edges. The edges with both endpoints labelled will be contracted.

Definition 1.4.5. Let M, M' be maps on surfaces S, S' respectively. Then M is *isomorphic* to M' if there is an orientation preserving homeomorphism $\varphi : S \longrightarrow S'$ so that $\varphi(V(M)) = V(M')$ and $\varphi(E(M)) = E(M')$. In this case, we call φ an *map isomorphism.*

In this paper, our focus will be in the case where $S = S'$. So maps are only viewed as equivalent up to orientation preserving homeomorphisms of S. Figure 1.8 provides an example where the first two embeddings on the plane are equivalent but the last is not.

From the notion of map isomorphism, we see that maps not only rely on the structure of their underlying graph but also on the structure of the surface they are embedded onto. Due to these factors, there are many more maps than graphs that are embeddable on a fixed surface S.

Enumerating graphs embeddable on a fixed surface becomes difficult due to overcounting

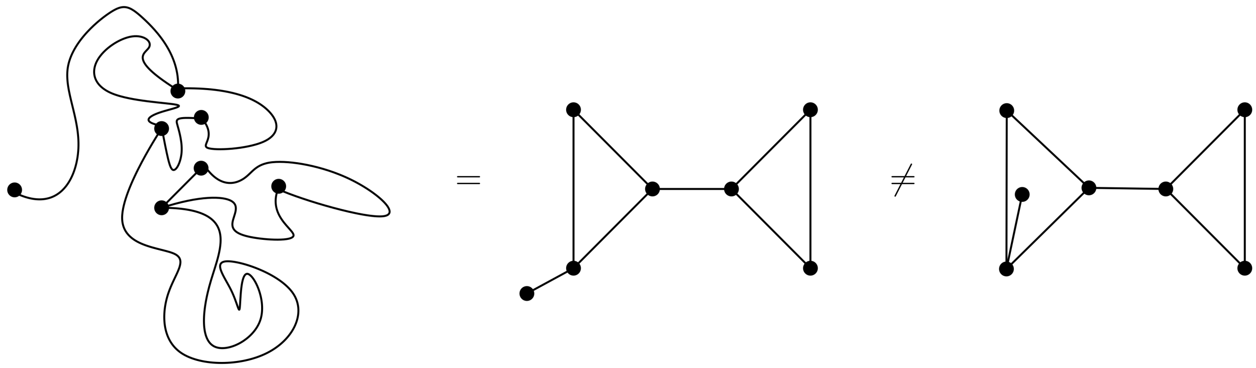


Figure 1.8: Equivalent and non-equivalent embeddings on the plane.

embeddable graphs. This is particularly seen in the planar case where enumerative formulas for planar maps were established almost 50 years prior to their analogues for planar graphs. Tutte established a combinatorial argument for enumerating planar maps in [29] and in this same paper provided an enumeration for 3-connected planar maps. This formulation for 3-connected planar maps exactly counted 3-connected planar graphs as Whitney showed that embeddings of of 3-connected planar graphs are equivalent [33]. Tutte's recursive method was used for these two enumerations, however once this connectivity restraint is removed, the combinatorics of counting planar graphs with 2-connectivity or less must resort to a different technique. Bender, Gao, and Wormald [2] use singularity analysis to count 2-connected planar graphs. This was extended to counting general planar maps by Giménez and Noy [12] in 2008, also using singularity analysis. Such singularity analysis techniques will not be covered but can be seen in [9].

1.5 Outline

The uniform random graph, random planar graph, and random forest models all share a similar phase transition for the largest connected component. Further noticing that these models are all uniformly sampled from subsets of minor-closed classes is the underlying motivation for Conjecture 1. Although we were not able to obtain the result in this conjecture, there were many natural avenues to attempt. We display two in this thesis.

Chapter 2 contains a combinatorial proof of the existence of the phase transition for the binomial random graph model. The proofs of the phase transition results in the examples from Section 1.2 rely on enumerative methods, and as minor-closed families are closely related to graphs on surfaces [15, 22], another approach to attempt would be to develop the enumerative theory of graphs on surfaces, and then extend it to more general minor-closed families. In Chapter 3, we enumerate different classes of planar maps as well as provide a general strategy in establishing the algebraicity of systems of functional equations which can be used to study the generating functions of such classes.

Chapter 2

Phase Transition in $G_{n,p}$

We follow the proof given in [10] which is combinatorial in nature. Here, we provide approximate bounds on the sizes of the components in the subcritical and supercritical regime as well as the uniqueness of the giant component. Proofs with more precise constants and results on the critical regime can be found in [18, 19, 21]. Another proof can be found in Chapter 5 of [30] that utilizes branching processes.

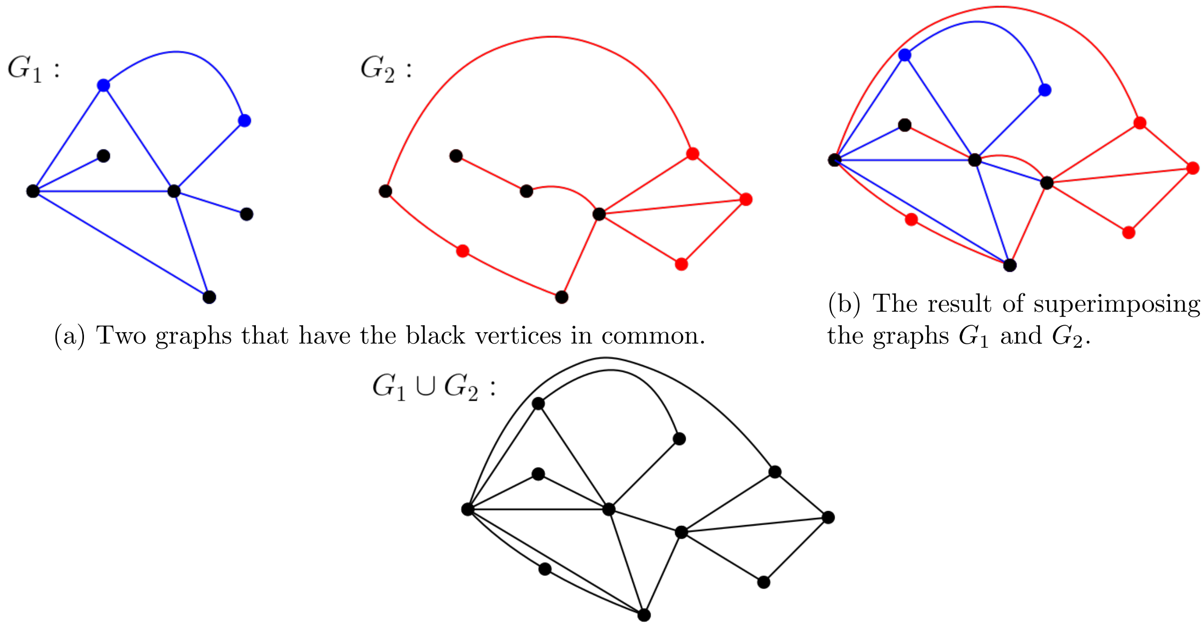
We first provide definitions and notation that will be used in the proofs of both regimes.

Definition 2.0.1. Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. Then the *union graph* of G and G' is the graph

$$G \cup G' = (V \cup V', E \cup E').$$

Furthermore, a realization of $G \cup G'$ can be obtained by taking G and superimposing G'. In this case, any double edges would be replaced by a single edge; see Figure 2.1. Given graphs G and H, we write $H \subset G$ if H is a subgraph of G, and write $H \in \mathrm{Comp}(G)$ if H is a connected component of G.

Lastly before continuing on to the proof of the sizes of the connected components in the $G_{n,p}$ model, we note that in many instances, we will be taking either the floor or the ceiling



(a) Two graphs that have the black vertices in common.

(b) The result of superimposing the graphs G_1 and G_2.

(c) The graph $G_1 \cup G_2$

Figure 2.1: An example of superimposing graphs to get their union.

when referring to component sizes. For the purpose of exposition, these operations will be omitted.

2.1 Subcritical Regime

When $p < \frac{1}{n}$, the random graph model $G_{n,p}$ falls into the subcritical regime. We expect every component to have order $O(\log n)$ and furthermore, we see through the course of the proof that w.h.p. each component is either a tree or unicyclic.

Theorem 2.1.1. *Let $p = \frac{c}{n}$ where $c \in (0, 1)$ is a constant. Then with high probability, the order of the largest component of a random graph $G_{n,p}$ is $\Theta(\log n)$.*

The proof will be given in a sequence of lemmas.

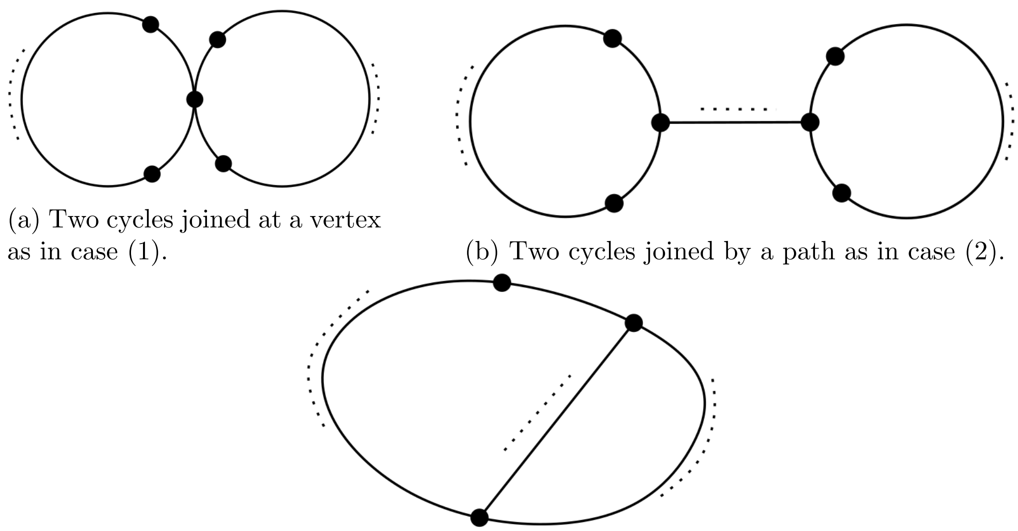
Lemma 2.1.2. *Let $p = \frac{c}{n}$ where $c \in (0, 1)$ is a constant. Then with probability $1 - O\left(\frac{1}{n}\right)$, every component of $G_{n,p}$ has at most one cycle.*

Proof. In a graph G, we call a pair of cycles C_1, C_2 of G *minimal* if C_1, C_2 lie in the same component of G and one of the following hold, which is depicted in Figure 2.2:

(1) C_1, C_2 are joined at a vertex

(2) C_1, C_2 are joined by a path P (in this case, there could be several paths connecting C_1 and C_2, fix one of them to be the path P)

(3) $C_1 \cup C_2$ form a cycle with a diagonal path



(a) Two cycles joined at a vertex as in case (1).

(b) Two cycles joined by a path as in case (2).

(c) Union of two cycles that create a cycle with a diagonal path as in case (3).

Figure 2.2: Conditions of minimal cycles in Lemma 2.1.2.

Now define $H \subset G$ by setting $H = C_1 \cup C_2$ in cases (1) or (3), or $H = C_1 \cup C_2 \cup P$ in case (2), and call H a *minimal subgraph.* Then we see that H is a union $H = P_1 \cup P_2 \cup P_3$ where P_1 is a path and P_2, P_3 are additional distinct edges attached to the endpoints of P_1; refer to Figure 2.3. If H is a subgraph on k vertices, then H consists of $k+1$ edges. Further, if we insist that $V(H)$ is a fixed set of k labelled vertices, then we bound the total number of

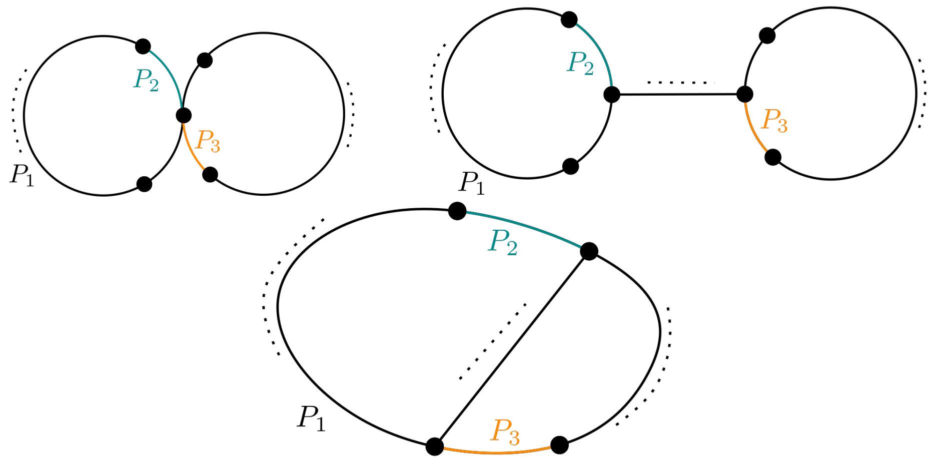


Figure 2.3: Showing three paths whose union is H in Lemma 2.1.2.

minimal subgraphs H by

$$|\{H \subset G \ : \ V(H) = [k], \ H = P_1 \cup P_2 \cup P_3 \text{ for a minimal pair of cycles}\}| \leq k!k^2, \quad (2.1.1)$$

where $k!$ counts the number of options for P_1 and k^2 counts the second attachment locations for P_2 and P_3.

Now in the $G_{n,p}$ model for any pair of cycles C_1, C_2 in the same component, either they are minimal or a minimal pair exists in the same component. We define a random variable that counts the number of minimal subgraphs in $G_{n,p}$,

$$\begin{aligned} X &= |\{H \subset K_n \ : \ H \text{ is a minimal subgraph in } G_{n,p}\}| \\ &= \sum_{k=4}^{n} \sum_{\substack{H \subset K_n, \\ |H|=k}} \mathbf{1}_{\{H \subset G_{n,p} \text{ is minimal}\}} \\ &= \sum_{k=4}^{n} \sum_{\substack{H \subset K_n, \\ V(H)=[k]}} \binom{n}{k} \mathbf{1}_{\{H \subset G_{n,p} \text{ is minimal}\}}. \end{aligned}$$

But since $c \in (0,1)$, then there is some constant $M < \infty$ so that

$$\sum_{k\geq 1} k^2 c^{k+1} = M. \tag{2.1.2}$$

Then applying the first moment method,

$$\begin{aligned}
\mathbb{P}\left(X > 0\right) &\leq \sum_{k=4}^{n} \sum_{\substack{H\subset K_n,\\ V(H)=[k]}} \binom{n}{k} \mathbb{E}\left(\mathbf{1}_{\{H\subset G_{n,p} \text{ is minimal}\}}\right) && \text{(Corollary 1.1.6)}\\
&\leq \sum_{k=4}^{n} \binom{n}{k} k! k^2 p^{k+1} && \text{(equation (2.1.1))}\\
&\leq \sum_{k=4}^{n} \frac{n^k}{k!} k! k^2 \frac{c^{k+1}}{n^{k+1}}\\
&\leq \frac{1}{n} \sum_{k\geq 1} k^2 c^{k+1}\\
&= \frac{M}{n}, && \text{(equation (2.1.2))}
\end{aligned}$$

where p^{k+1} in the second line is the probability of the $k+1$ edges being present. Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}\left(G_{n,p} \text{ has at most one cycle in each component}\right) &= \lim_{n\to\infty} \left(1 - \mathbb{P}\left(X > 0\right)\right)\\
&\geq \lim_{n\to\infty} \left(1 - \frac{M}{n}\right)\\
&= 1.
\end{aligned}$$

□

From this, the contenders for the largest component would be unicyclic components and isolated trees. The next lemma says that w.h.p. any uncyclic component has order $O(\omega)$ for any function $\omega = \omega(n)$ that tends to infinity. In particular, w.h.p. every unicylic component has order $O(\log n)$.

Lemma 2.1.3. *Let $p = \frac{c}{n}$ where $c \in (0,1)$ is a constant and $\omega = \omega(n)$ be any function* *that tends to infinity. Then with high probability, the number of vertices in components with exactly one cycle is $O(\omega)$.*

Proof. Take $p = \frac{c}{n}$ with constant $c \in (0,1)$ and consider the $G_{n,p}$ model. We approximate the number of vertices in unicyclic components. For fixed k define a random variable

$$\begin{aligned} X_k &= |\{H \subset K_n \ : \ |H| = k, \ H \text{ is a unicyclic component in } G_{n,p}\}| \\ &= \sum_{\substack{H \subset K_n \text{ unicyclic}, \\ |H|=k}} k \cdot \mathbf{1}_{\{H \in \mathrm{Comp}(G_{n,p})\}}. \end{aligned}$$

Note that a unicyclic graph on k vertices is precisely a tree with an additional distinct edge. Thus, we may bound the number of unicyclic graphs on vertex set $[k]$ by

$$\begin{aligned} |\{H \ : \ H \text{ is a unicyclic graph}, \ V(H) = [k]\}| &\leq \binom{k}{2} \cdot |\{T \ : \ T \text{ is a tree}, V(T) = [k]\}| \\ &= \binom{k}{2} k^{k-2}, \end{aligned} \tag{2.1.3}$$

where the factor k^{k-2} comes from Cayley's formula for labelled trees. Furthermore, for a fixed unicyclic graph H on vertex set $[k]$ then H has k edges and observe

$$\mathbb{P}\left(H \in \mathrm{Comp}(G_{n,p})\right) = p^k (1-p)^{\binom{k}{2} - k + k(n-k)}, \tag{2.1.4}$$

where the factor p^k is for the k edges of H being present, the factor $(1-p)^{\binom{k}{2}-k}$ corresponds to no other edges among the vertex set $[k]$ being present, and the factor $(1-p)^{k(n-k)}$ accounts for excluding the edges between $[k]$ and $[n] \setminus [k]$. Estimating the first moment of X_k,

$$
\begin{aligned}
&\mathbb{E}\left(X_k\right)\\
&= \sum_{\substack{H\subset K_n \text{ unicyclic,}\\ |H|=k}} k\mathbb{P}\left(H\in \mathrm{Comp}(G_{n,p})\right)\\
&= k\binom{n}{k}\sum_{\substack{H\subset K_n \text{ unicyclic}\\ V(H)=[k]}} \mathbb{P}\left(H\in \mathrm{Comp}(G_{n,p})\right)\\
&\le k\binom{n}{k}\binom{k}{2}k^{k-2}p^k(1-p)^{\binom{k}{2}-k+k(n-k)} && \text{(equations (2.1.3), (2.1.4))}\\
&\le \frac{n^k}{k!}\exp\left\{-\frac{k(k-1)}{2n}\right\}k^{k+1}\cdot\frac{c^k}{n^k}\exp\left\{-ck\left(\binom{k}{2}-k+k(n-k)\right)\right\} && \text{(Lemma 1.1.3 (a), (e))}\\
&\le \frac{n^k}{k!}\exp\left\{-\frac{k(k-1)}{2n}\right\}k^{k+1}\cdot\frac{c^k}{n^k}\exp\left\{-ck+\frac{ck(k-1)}{2n}+\frac{k}{n}\left(\frac{3c}{2}\right)\right\}\\
&\le \frac{e^k}{k^k}k^{k+1}c^k\exp\left\{-\frac{k(k-1)}{2n}-ck+\frac{k(k-1)}{2n}+\frac{3c}{2}\right\} && \text{(Lemma 1.1.3 (f))}\\
&= kc^k\exp\left\{k(1-c)\right\}e^{\frac{3c}{2}}\\
&= k(ce^{1-c})^k e^{\frac{3c}{2}}.
\end{aligned}
$$

Note that $ce^{1-c}<1$ for $c\neq 1$ and so,

$$\sum_{k\ge 1} k^2c^{k+1}=M, \tag{2.1.5}$$

for some constant $M<\infty$. Computing the expectation for varying k,

$$\mathbb{E}\left(\sum_{k=3}^{n} X_k\right)\le \sum_{k=3}^{n} k(ce^{1-c})^k e^{\frac{3c}{2}}\le e^{\frac{3c}{2}}\sum_{k\ge 1} k(ce^{1-c})^k = Me^{\frac{3c}{2}}=O(1), \tag{2.1.6}$$

where the second last equality is due to equation (2.1.5). It follows that for any function

$\omega = \omega(n)$ so that $\omega(n) \to \infty$ as $n \to \infty$,

$$\begin{aligned} \mathbb{P}\left(\sum_{k=3}^{n} X_k \geq \omega\right) &\leq \frac{\mathbb{E}\left(\sum_{k=3}^{n} X_k\right)}{\omega} && \text{(Theorem 1.1.5)} \\ &= O\left(\frac{1}{\omega}\right) \\ &= o(1). \end{aligned}$$

That is, for any growing function $\omega = \omega(n)$, w.h.p. the number of vertices lying in unicyclic components is bounded by ω. □

This lemma will be useful in the proofs which address the supercritical regime. The remaining candidate for the largest component in the subcritical case is an isolated tree. Part (i) of the following lemma provides the lower bound of Theorem 2.1.1 and part (ii) finishes the upper bound. Furthermore, the following lemma will also be used in the later proofs that address the supercritical regime and will note now that the proof of (i) holds for general $c \neq 1$.

Lemma 2.1.4. *Let $p = \frac{c}{n}$ for a constant $c < 1$. Set $\alpha = c - 1 - \log c$ and take $\omega = \omega(n)$ to be a growing function so that $\omega = o(\log \log n)$. Then:*

(i) with high probability, there exists an isolated tree of order

$$k_- = \frac{1}{\alpha}\left(\log n - \frac{5}{2}\log\log n\right) - \omega,$$

(ii) with high probability, there is no isolated tree of order at least

$$k_+ = \frac{1}{\alpha}\left(\log n - \frac{5}{2}\log\log n\right) + \omega.$$

Proof. Note that as $c \neq 1$ is a positive constant, then $\alpha = c - 1 - \log c$ is a positive constant as well. Considering the random graph $G_{n,p}$ with $p = \frac{c}{n}$, we count the number of isolated trees. For $k \in \mathbb{N}$, let

$$\begin{aligned} X_k &= |\{T \subset K_n \ : \ T \text{ is an isolated tree in } G_{n,p}, \ |T| = k\}| \\ &= \sum_{\substack{T \subset K_n \text{ tree}, \\ |T|=k}} \mathbf{1}_{\{T \in \mathrm{Comp}(G_{n,p})\}}. \end{aligned}$$

Observe that for a fixed tree T on vertex set $[k]$ then T has $k-1$ edges, so

$$\mathbb{P}\left(T \in \mathrm{Comp}(G_{n,p})\right) = p^{k-1}(1-p)^{\binom{k}{2}-k+1+k(n-k)}, \tag{2.1.7}$$

where the factor p^{k-1} is for the $k-1$ edges of T being present, the factor $(1-p)^{\binom{k}{2}-k+1}$ corresponds to no other edges among the vertex set $[k]$ being present, and the factor $(1-p)^{k(n-k)}$ accounts for excluding the edges between $[k]$ and $[n] \setminus [k]$. Computing the first moment,

$$\begin{aligned} \mathbb{E}\left(X_k\right) &= \sum_{\substack{T \subset K_n \text{ tree}, \\ |T|=k}} \mathbb{P}\left(T \in \mathrm{Comp}(G_{n,p})\right) \\ &= \binom{n}{k} \sum_{\substack{\text{trees } T, \\ V(T)=[k]}} \mathbb{P}\left(T \in \mathrm{Comp}(G_{n,p})\right) \\ &= \binom{n}{k} \sum_{\substack{\text{trees } H, \\ V(T)=[k]}} p^{k-1}(1-p)^{\binom{k}{2}-k+1+k(n-k)} && \text{(equation (2.1.7))} \\ &= \binom{n}{k} k^{k-2} \left(\frac{c}{n}\right)^{k-1} \left(1 - \frac{c}{n}\right)^{k(n-k)+\binom{k}{2}-k+1}. && (2.1.8) \end{aligned}$$

To prove (i), suppose $k = O(\log n)$ and notice,

$$
\begin{aligned}
&\mathbb{E}\left(X_k\right) \\
&= \binom{n}{k} k^{k-2}\left(\frac{c}{n}\right)^{k-1}\left(1-\frac{c}{n}\right)^{k(n-k)+\binom{k}{2}-k+1} \\
&= \binom{n}{k} k^{k-2}\frac{c^{k-1}}{n^{k-1}}(1+o(1))\exp\left\{-\frac{c}{n}\left\{k(n-k)+\binom{k}{2}-k+1\right\}\right\} && \text{(Lemma 1.1.3 (a), (b))} \\
&= (1+o(1))\frac{n}{c}\cdot\frac{k^{k-2}}{k!}c^k\exp\{-ck+o(1)\} && (\text{since } k=O(\log n)) \\
&= (1+o(1))\frac{n}{c}\cdot\frac{k^{k-2}}{k!}(ce^{-c})^k && (2.1.9) \\
&= \frac{(1+o(1))}{c\sqrt{2\pi}}\cdot\frac{n}{k^{5/2}}(ce^{1-c})^k && \text{(Theorem 1.1.4)} \\
&= \frac{(1+o(1))}{c\sqrt{2\pi}}\cdot\frac{n}{k^{5/2}}\exp\{(\log c+1-c)k\} \\
&= \frac{(1+o(1))}{c\sqrt{2\pi}}\cdot\frac{n}{k^{5/2}}e^{-\alpha k}. && (2.1.10)
\end{aligned}
$$

Taking $k = k_- := \frac{1}{\alpha}\left(\log n - \frac{5}{2}\log\log n\right) - \omega$,

$$
\begin{aligned}
\mathbb{E}\left(X_k\right) &= \frac{(1+o(1))}{c\sqrt{2\pi}}\cdot\frac{n}{k^{5/2}}\exp\left\{-\log n+\frac{5}{2}\log\log n+\alpha\omega\right\} \\
&= \frac{(1+o(1))}{c\sqrt{2\pi}}\cdot\frac{1}{k^{5/2}}(\log n)^{5/2}e^{\alpha\omega} \\
&\geq Ae^{\alpha\omega}, && (2.1.11)
\end{aligned}
$$

for some $A > 0$. Assessing the second moment of X_k,

$$
\mathbb{E}\left(X_k^2\right) = \sum_{\substack{\text{trees } T_1,T_2\subset K_n,\\ |T_1|=|T_2|=k}} \mathbb{P}\left(T_1\in\mathrm{Comp}(G_{n,p}),\ T_2\in\mathrm{Comp}(G_{n,p})\right)
$$

$$= \sum_{\substack{\text{trees } T_1 \subset K_n, \\ |T_1|=k}} \mathbb{P}\left(T_1 \in \mathrm{Comp}(G_{n,p})\right) + \sum_{\substack{\text{trees } T_1, T_2 \subset K_n, \\ |T_1|=|T_2|=k \\ T_1 \neq T_2}} \mathbb{P}\left(T_1 \in \mathrm{Comp}(G_{n,p}),\ T_2 \in \mathrm{Comp}(G_{n,p})\right) \tag{2.1.12}$$

The first sum in equation (2.1.12) is simply $\mathbb{E}\left(X_k\right)$. For the second, note that if $V(T_1) \cap V(T_2) = \emptyset$ and $|T_1| = k$ then,

$$\mathbb{P}\left\{T_2 \in \mathrm{Comp}(G_{n,p}) \mid T_1 \in \mathrm{Comp}(G_{n,p})\right\} = \mathbb{P}\left(T_2 \in \mathrm{Comp}(G_{n,p})\right)(1-p)^{-k^2},$$

since the conditioning tells us that the k^2 potential edges between $V(T_1)$ and $V(T_2)$ are absent. Also, if $V(T_1) \cap V(T_2) = \emptyset$ but $T_1 \neq T_2$ then,

$$\mathbb{P}\left\{T_2 \in \mathrm{Comp}(G_{n,p}) \mid T_1 \in \mathrm{Comp}(G_{n,p})\right\} = 0.$$

So equation (2.1.12) gives that

$$\begin{aligned}
\mathbb{E}\left(X_k^2\right) &= \mathbb{E}\left(X_k\right) + \sum_{\substack{\text{trees } T_1, T_2 \subset K_n, \\ |T_1|=|T_2|=k \\ V(T_1)\cap V(T_2)=\emptyset}} \mathbb{P}\left(T_1 \in \mathrm{Comp}(G_{n,p})\right)\mathbb{P}\left(T_2 \in \mathrm{Comp}(G_{n,p})\right)(1-p)^{-k^2} \\
&= \mathbb{E}\left(X_k\right) + (1-p)^{-k^2} \sum_{\substack{\text{trees } T_1 \subset K_n, \\ |T_1|=k}} \mathbb{P}\left(T_1 \in \mathrm{Comp}(G_{n,p})\right) \sum_{\substack{\text{trees } T_2 \subset K_n, \\ |T_1|=k \\ V(T_1)\cap V(T_2)=\emptyset}} \mathbb{P}\left(T_2 \in \mathrm{Comp}(G_{n,p})\right) \\
&= \mathbb{E}\left(X_k\right) + (1-p)^{-k^2}\mathbb{E}\left(X_k\right) \sum_{\substack{\text{trees } T_2 \subset K_n, \\ |T_1|=k \\ V(T_1)\cap[k]=\emptyset}} \mathbb{P}\left(T_2 \in \mathrm{Comp}(G_{n,p})\right) \\
&\leq \mathbb{E}\left(X_k\right) + (1-p)^{-k^2}\mathbb{E}\left(X_k\right) \sum_{\text{trees } T_2 \subset K_n} \mathbb{P}\left(T_2 \in \mathrm{Comp}(G_{n,p})\right) \\
&= \mathbb{E}\left(X_k\right) + (1-p)^{-k^2}\mathbb{E}\left(X_k\right)^2.
\end{aligned}$$

As the focus is on the asymptotic behaviour of $G_{n,p}$, we may assume that n is large enough so that $\frac{ck^2}{n-c} < 1$. Then note that,

$$(1-p)^{-k^2} = \left(\frac{n}{n-c}\right)^{k^2} = \left(1 + \frac{1}{n-c}\right)^{k^2} \leq \exp\left\{\frac{c}{n-c}k^2\right\} \leq 1 + 2\frac{ck^2}{n-c},$$

the last bound holding when $\frac{ck^2}{n-c} < 1$ by Lemma 1.1.3 (g). Computing the variance of X_k, we obtain the bound

$$\begin{aligned}
\mathbf{Var}\left(X_k\right) &= \mathbb{E}\left(X_k^2\right) - \mathbb{E}\left(X_k\right)^2 \\
&\leq \mathbb{E}\left(X_k\right)\left(1 + (1-p)^{-k^2}\mathbb{E}\left(X_k\right)\right) - \mathbb{E}\left(X_k\right)^2 \\
&= \mathbb{E}\left(X_k\right) + \mathbb{E}\left(X_k\right)^2\left((1-p)^{-k^2} - 1\right) \\
&\leq \mathbb{E}\left(X_k\right) + \frac{2ck^2}{n-c}\mathbb{E}\left(X_k\right)^2.
\end{aligned}$$

For fixed $\varepsilon > 0$, Chebyshev's inequality (Theorem 1.1.7) then implies that

$$\begin{aligned}
\mathbb{P}\left(\left|X_k - \mathbb{E}\left(X_k\right)\right| \geq \varepsilon\mathbb{E}\left(X_k\right)\right) &\leq \frac{\mathbf{Var}\left(X_k\right)}{\varepsilon^2\mathbb{E}\left(X_k\right)^2} \\
&\leq \frac{\mathbb{E}\left(X_k\right)}{\varepsilon^2\mathbb{E}\left(X_k\right)^2} + \frac{2ck^2\mathbb{E}\left(X_k\right)^2}{(n-c)\varepsilon^2\mathbb{E}\left(X_k\right)^2} \qquad (2.1.13) \\
&\leq \frac{1}{\varepsilon^2 Ae^{\alpha\omega}} + o(1) \\
&= o(1),
\end{aligned}$$

where the last equality is due to ω being a growing function. Therefore, w.h.p.

$$X_k = \mathbb{E}\left(X_k\right) \geq Ae^{\alpha\omega} > 0,$$

where $k = k_-$, and this completes the proof of (i).

Now for (ii), we first estimate an upper bound on $\mathbb{E}\left(X_k\right)$,

$$\begin{aligned}
\mathbb{E}\left(X_k\right) &= \binom{n}{k} k^{k-2}\left(\frac{c}{n}\right)^{k-1}\left(1-\frac{c}{n}\right)^{k(n-k)+\binom{k}{2}-k+1} \\
&\leq \frac{n^k}{k!}\left(1-\frac{k}{2n}\right)^{k-1} k^{k-2}\left(\frac{c}{n}\right)^{k-1}\left(1-\frac{c}{n}\right)^{kn-k^2+\frac{k^2}{2}-\frac{k}{2}-k+1} \\
&\leq \frac{n^k}{\sqrt{2\pi k}}\left(\frac{e}{k}\right)^k\left(1-\frac{k}{2n}\right)^{k-1} k^{k-2}\left(\frac{c}{n}\right)^{k-1}\exp\left\{-ck+\frac{ck^2}{2n}+\frac{ck}{2n}+\frac{ck}{n}-\frac{c}{n}\right\},
\end{aligned}$$

where the penultimate inequality is an application of Lemma 1.1.3 (d) and the last inequality uses Theorem 1.1.4. Note that $\frac{ck}{2n}+\frac{ck}{n}-\frac{c}{n}=O(1)$ is bounded as n varies, so collecting constants, there is some $B>0$ so that

$$\begin{aligned}
\mathbb{E}\left(X_k\right) &\leq \frac{B}{\sqrt{k}}\left(\frac{ne}{k}\right)^k k^{k-2}\left(1-\frac{k}{2n}\right)^{k-1}\left(\frac{c}{n}\right)^{k-1}\exp\left\{-ck+\frac{ck^2}{2n}\right\} \\
&= \frac{Bn}{k^{5/2}} e^{k\left(1-c\left(1-\frac{k}{2n}\right)\right)}\frac{c\left(1-\frac{k}{2n}\right)^k}{c\left(1-\frac{k}{2n}\right)} \\
&= \frac{2Bn}{k^{5/2}c}\left(\widehat{c_k}e^{1-\widehat{c_k}}\right)^k,
\end{aligned}$$

where $\widehat{c_k}=c\left(1-\frac{k}{2n}\right)$, and $\frac{1}{1-\frac{k}{2n}}\leq 2$ follows from the fact that $k\leq n$ and $c>0$. If $c<1$, then it follows that

$$\begin{aligned}
\widehat{c_k}e^{1-\widehat{c_k}} &= c\left(1-\frac{k}{2n}\right)e^{1-c\left(1-\frac{k}{2n}\right)} \\
&\leq ce^{1-c}\left(1-\frac{k}{2n}\right)e^{ck/2n} \\
&\leq ce^{1-c}\exp\left\{\frac{ck}{2n}-\frac{k}{2n}\right\} \\
&\leq ce^{1-c}.
\end{aligned}$$

So recalling that $\alpha = c - 1 - \log c$,

$$
\begin{aligned}
\sum_{k=k_+}^{n} \mathbb{E}\left(X_k\right) &\leq \sum_{k=k_+}^{n} \frac{2Bn}{k^{5/2}c} \left(\widehat{c}_k e^{1-\widehat{c}_k}\right)^k \\
&\leq \frac{2Bn}{k_+^{5/2}c} \sum_{k=k_+}^{n} \left(ce^{1-c}\right)^k \\
&= \frac{2Bn}{k_+^{5/2}c} \sum_{k=k_+}^{\infty} e^{-\alpha k} \\
&= \frac{2Bn}{k_+^{5/2}c} \cdot \frac{e^{-\alpha k_+}}{1-e^{-\alpha}} \\
&= \frac{2Bn}{c\left(1-e^{-\alpha}\right)} \cdot \frac{\exp\left\{-\log n + \frac{5}{2}\log\log n - \alpha\omega\right\}}{\left(\frac{1}{\alpha}\left(\log n - \frac{5}{2}\log\log n\right) + \omega\right)^{5/2}} \quad \text{(plugging in value of } k_+\text{)} \\
&= O(1)e^{-\alpha\omega} \\
&= o(1). \qquad (2.1.14)
\end{aligned}
$$

It follows from Corollary 1.1.6 that

$$\mathbb{P}\left(\sum_{k\geq k_+} X_k \geq 0\right) \leq \mathbb{E}\left(\sum_{k\geq k_+} X_k\right) = o(1).$$

Therefore, w.h.p. all tree components have order less than k_+. $\square$

Part (ii) of this lemma says that with high probability, every tree component has order less than $k_+ = O(\log n)$. As this was the last contender for the largest component of $G_{n,p}$ then every component of $G_{n,p}$ has order $O(\log n)$, with high probability. Part (i) guarantees w.h.p. the existence of some component having order at least $O(\log n)$, giving the result that w.h.p. $|C^{(1)}\left(G_{n,p}\right)| = \Theta(\log n)$.

2.2 Supercritical Regime

Before proving the emergence of the giant component, we provide two tools that will be used in the proof. The following observation allows one to generate a uniform random graph $G_{n,p}$ in two independent steps.

Observation 2.2.1. Let n be a positive integer and $p \in (0,1)$. Take $p_1 \in (0,p)$ and $p_2 \in (0,1)$ defined by

$$1 - p = (1 - p_1)(1 - p_2).$$

Take the random graphs G_{n,p_1}, G_{n,p_2} and consider the random graph $G = G_{n,p_1} \cup G_{n,p_2}$ obtained from G_{n,p_1} and superimposing it with G_{n,p_2}. Then note that $V(G) = [n]$ and any edge $e \in \binom{[n]}{2}$ is not present in G if and only if e is not present in G_{n,p_1} and G_{n,p_2}. That is, each edge $e \in \binom{[n]}{2}$ has independent probability $(1-p_1)(1-p_2) = 1-p$ of not being present in G. So in fact, $G_{n,p} = G_{n,p_1} \cup G_{n,p_2}$.

The second tool is the following identity.

Lemma 2.2.2. *Let $c > 0$ so that $c \neq 1$ is a constant and define $x = x(c)$ by*

$$x = \begin{cases} c & c \leq 1, \\ \text{the solution in } (0,1) \text{ to } xe^{-x} = ce^{-c} & c > 1. \end{cases}$$

Then,

$$\frac{1}{x} \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} \left(ce^{-c}\right)^k = 1.$$

Proof. First suppose $c < 1$. Consider the $G_{n,p}$ model with $p = \frac{c}{n}$ and define following random variables

$$X = \sum_{v \in [n]} \mathbf{1}_{\{\mathscr{C}_{G_{n,p}}(v) \text{ is not a tree}\}},$$

and

$$X_k = \sum_{\substack{\text{trees } T \subset K_n, \\ |V(T)|=k}} \mathbf{1}_{\{T \in \mathrm{Comp}(G_{n,p})\}}.$$

So X counts the number of vertices in non-tree components and X_k is as in Lemma 2.1.4 which counts the number of tree components of order k. Then,

$$n = \sum_{k=1}^{n} kX_k + X,$$

and taking expectations,

$$n = \mathbb{E}\left(\sum_{k=1}^{n} kX_k + X\right) = \sum_{k=1}^{n} k\mathbb{E}\left(X_k\right) + \mathbb{E}\left(X\right).$$

From Lemma 2.1.2, the contribution to $\mathbb{E}(X)$ from components with more than one cycle is $O(1)$. Then by equation (2.1.6) of Lemma 2.1.3,

$$\mathbb{E}\left(X\right) = O(1).$$

Furthermore, taking $k_+ = \frac{1}{\alpha}\left(\log n - \frac{5}{2}\log\log n\right) + \omega$ as in Lemma 2.1.4, then equation (2.1.9) gives

$$\mathbb{E}\left(X_k\right) = (1+o(1))\frac{n}{ck!}k^{k-2}\left(ce^{1-c}\right)^k,$$

for $k < k_+$. So applying these equations and bounds, we get

$$\begin{aligned} n &= \sum_{k=1}^{k_+} k\mathbb{E}\left(X_k\right) + \sum_{k=k_++1}^{n} k\mathbb{E}\left(X_k\right) + O(1) \\ &= \frac{n(1+o(1))}{c}\sum_{k=1}^{k_+}\frac{k^{k-1}}{k!}\left(ce^{1-c}\right)^k + \sum_{k=k_++1}^{n} k\mathbb{E}\left(X_k\right) + O(1) \end{aligned}$$

$$
\begin{aligned}
&= \frac{n(1+o(1))}{c} \sum_{k=1}^{k_+} \frac{k^{k-1}}{k!} \left(ce^{1-c}\right)^k + o(n) && \text{(equation (2.1.14))} \\
&= \frac{n(1+o(1))}{c} \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} \left(ce^{1-c}\right)^k + o(n),
\end{aligned}
$$

where the last equality is due to the fact that $\sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} \left(ce^{1-c}\right)^k$ converges, so the tail

$$
\frac{n(1+o(1))}{c} \sum_{k=k_+ +1}^{\infty} \frac{k^{k-1}}{k!} \left(ce^{1-c}\right)^k = o(n).
$$

Dividing through by n, it follows that

$$
\frac{1}{c} \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} \left(ce^{1-c}\right)^k = 1 + o(1).
$$

Taking $n \to \infty$ and recalling that $x = c$ when $c < 1$, we obtain the desired identity

$$
1 = \frac{1}{c} \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} \left(ce^{1-c}\right)^k.
$$

Finally, for $c > 1$, since $x = x(c) \in (0,1)$ is a solution to $xe^{-x} = ce^{-c}$, we have

$$
\begin{aligned}
1 &= \frac{1}{x} \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} \left(xe^{1-x}\right)^k && (x < 1) \\
&= \frac{1}{x} \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} \left(ce^{1-c}\right)^k. && \left(xe^{-x} = ce^{-c}\right) \qquad \square
\end{aligned}
$$

We are now in position to prove the existence and uniqueness of the giant component as well as provide bounds on the component sizes.

Theorem 2.2.3. *Let $p = \frac{c}{n}$ with constant $c > 1$. Take $x = x(c)$ to be the solution of $ce^{-c} = xe^{-x}$ in $(0,1)$. Then with high probability, $G_{n,p}$ has a unique giant component with order $\left(1 - (1+o(1))\frac{x}{c} + o(1)\right) n$. Furthermore, all other components have order at most*

$O(\log n)$, *with high probability.*

Proof. Take $p = \frac{c}{n}$ with $c > 1$ a constant and consider $G_{n,p}$. Let Z_k be the random variable that counts the number of components in $G_{n,p}$ of order k. Note that if H is a component in $G_{n,p}$ then there exists a spanning tree T_H of H such that the edges of T_H are present in $G_{n,p}$. Furthermore, no edges between $V(T_H) = V(H)$ and $[n] \setminus V(T_H)$ are present in $G_{n,p}$. Then we may bound Z_k by

$$Z_k \leq \sum_{\substack{\text{tree } T \subset K_n, \\ |T|=k}} \mathbf{1}_{\{T \subset G_{n,p}\}} \mathbf{1}_{\{\text{no edges present between } V(T) \text{ and } [n] \setminus V(T)\}}.$$

Fixing a tree T_k on vertex set $[k]$,

$$\begin{aligned}
\mathbb{E}(Z_k) &\leq \sum_{\substack{\text{tree } T \subset K_n, \\ |T|=k}} \mathbb{P}\left(T \subset G_{n,p}, \text{ no edges present between } V(T) \text{ and } [n] \setminus V(T)\right) \\
&= \binom{n}{k} \sum_{\substack{\text{tree } T \subset K_n, \\ V(T)=[k]}} \mathbb{P}\left(T \subset G_{n,p}, \text{ no edges present between } V(T) \text{ and } [n] \setminus V(T)\right) \\
&= \binom{n}{k} \sum_{\substack{\text{tree } T \subset K_n, \\ V(T)=[k]}} \mathbb{P}\left(T_k \subset G_{n,p}, \text{ no edges present between } V(T_k) \text{ and } [n] \setminus V(T)\right) \\
&= \binom{n}{k} k^{k-2} p^{k-1} (1-p)^{k(n-k)} \\
&\leq \frac{n^k}{k!} k^{k-2} \left(\frac{c}{n}\right)^{k-1} \exp\left\{-\frac{c}{n} \cdot k(n-k)\right\} \\
&\leq \frac{n}{k^2} \cdot c^k \exp\left\{k - ck + \frac{ck^2}{n}\right\} && \text{(Lemma 1.1.3 (f), } c > 1) \\
&= \frac{n}{k^2} \left(ce^{1-c+\frac{ck}{n}}\right)^k.
\end{aligned}$$

Take $\beta_1 = \beta_1(c)$ to be a constant small enough so that

$$\delta := ce^{1-c+c\beta_1} < 1,$$

which exists as $1-c<0$. Also take $\beta_0 = \beta_0(c)$ to be a constant large enough so that

$$\left(ce^{1-c+\frac{c}{n}\beta_0 \log n}\right)^{\beta_0 \log n} \leq \frac{1}{n^2}.$$

Such a constant β_0 exists as

$$\left(ce^{1-c+\frac{c}{n}\beta_0 \log n}\right)^{\beta_0 \log n} = c^{\beta_0 \log n}\left(e^{\log n}\right)^{(1-c)\beta_0} e^{\frac{c}{n}\beta_0^2 \log^2 n} = \left(\frac{c^{\log n}}{n^{c-1}}\right)^{\beta_0}\left(e^{\frac{c\log^2 n}{n}}\right)^{\beta_0^2},$$

where we observe that $\frac{c^{\log n}}{n^{c-1}} < 1$ and $\lim_{n\to\infty} e^{\frac{c\log^2 n}{n}} = 1$. Focusing on components of order $k \in [\beta_0 \log n, \beta_1 n]$, we then have

$$\begin{aligned}
\mathbb{P}\left(\sum_{k=\beta_0 \log n}^{\beta_1 n} Z_k > 0\right) &\leq \sum_{k=\beta_0 \log n}^{\beta_1 n} \mathbb{E}\left(Z_k\right) && \text{(Corollary 1.1.6)}\\
&\leq \sum_{k=\beta_0 \log n}^{\beta_1 n} \frac{n}{k^2}\left(ce^{1-c}+\frac{ck}{n}\right)^k\\
&\leq (\beta_0 \log n + \beta_1 n)\left(\frac{n}{\beta_0^2 \log^2 n}\cdot\frac{1}{n^2}+\frac{n\delta^{n\beta_1}}{\beta_1^2 n^2}\right) && \text{(choice of } \beta_0,\ \beta_1)\\
&= \frac{1}{\beta_0 n \log n} + \frac{\beta_0 \delta^{n\beta_1}}{\beta_1^2} + \frac{\beta_1}{n\beta_0^2 \log^2 n} + \frac{\delta^{n\beta_1}}{\beta_1}\\
&= o(1).
\end{aligned}$$

Hence, with high probability, no component in $G_{n,p}$ is of order $k \in [\beta_0 \log n, \beta_1 n]$. Next, we will estimate the number of vertices that lie in components of size at most $\beta_0 \log n$. By doing this, we may infer that the remaining vertices will lie in giant components. That is, the components of order at least $\beta_1 n$. The estimate of the vertices in 'small' components will be done in two claims.

Claim 1: The number of vertices in tree components of order at most $\beta_0 \log n$ is $(1+$

$o(1))\frac{xn}{c} + o(n)$.

As in Lemma 2.1.4, define

$$X_k = \sum_{\substack{\text{tree } T \subset K_n,\\ |V(T)|=k}} \mathbf{1}_{\{T \in \mathrm{Comp}(G_{n,p})\}},$$

to be the number of isolated trees of order k. We consider the value of k in two parts. First consider when $1 \le k \le k_0 = \left\lfloor \frac{1}{2\alpha} \log n \right\rfloor$, where $\alpha = c - 1 - \log c > 0$ as in Lemma 2.1.4. So by equation (2.1.9),

$$\begin{aligned} \mathbb{E}\left(\sum_{k=1}^{k_0} kX_k\right) &= (1+o(1))\frac{n}{c}\sum_{k=1}^{k_0} \frac{k^{k-1}}{k!}(ce^{-c})^k \\ &= (1+o(1))\frac{n}{c}\sum_{k=1}^{\infty} \frac{k^{k-1}}{k!}(ce^{-c})^k, \end{aligned} \tag{2.2.1}$$

where we may extend the summation as $\sum_{k=1}^{\infty} \frac{k^{k-1}}{k!}(ce^{-c})^k$ converges. Then taking $\varepsilon = \frac{1}{\log n}$,

$$\begin{aligned} &\mathbb{P}\left(|X_k - \mathbb{E}\left(X_k\right)| \ge \varepsilon \mathbb{E}\left(X_k\right) \text{ for some } k \in [k_0]\right) \\ &\le \sum_{k=1}^{k_0} \mathbb{P}\left(|X_k - \mathbb{E}\left(X_k\right)| \ge \varepsilon \mathbb{E}\left(X_k\right)\right) \\ &\le \sum_{k=1}^{k_0} \left(\frac{\log^2 n}{\mathbb{E}\left(X_k\right)} + \frac{2ck^2 \log^2 n}{n-c}\right) && \text{(equation (2.1.13))} \\ &\le \sum_{k=1}^{k_0} \left(\frac{O(\log^2 n)e^{\alpha k}k^{5/2}}{(1+o(1))n} + \frac{2ck_0^2 \log^2 n}{n-c}\right) && \text{(equation (2.1.10))} \\ &\le k_0 \left(\frac{O(\log^2 n)e^{\alpha k_0}k_0^{5/2}}{(1+o(1))n} + \frac{O(\log^5 n)}{n-c}\right) \\ &\le \frac{O(\log^{11/2} n)}{n^{1/2}} + \frac{O(\log^6 n)}{n-c} \\ &= o(1). \end{aligned}$$

So w.h.p., $kX_k \in [(1-\varepsilon)\mathbb{E}(X_k), (1+\varepsilon)\mathbb{E}(X_k)]$. Then as $\varepsilon = \frac{1}{\log n} = o(1)$, w.h.p.,

$$\begin{aligned}
\sum_{k=1}^{k_0} kX_k &= (1+o(1)) \sum_{k=1}^{k_0} \mathbb{E}(X_k) \\
&= (1+o(1)) \frac{n}{c} \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} (ce^{-c})^k && \text{(equation (2.2.1))} \\
&= (1+o(1)) \frac{n}{c} \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} (xe^{-x})^k \\
&= (1+o(1)) \frac{nx}{c}. && \text{(Lemma 2.2)}
\end{aligned}$$

Now for $k_0 \le k \le \beta_0 \log n$, we have

$$\begin{aligned}
&\mathbb{E}\left(\sum_{k=k_0+1}^{\beta_0 \log n} kX_K \right) \\
&= \sum_{k=k_0+1}^{\beta_0 \log n} k\mathbb{E}(X_K) \\
&= \sum_{k=k_0+1}^{\beta_0 \log n} \binom{n}{k} k^{k-1} \left(\frac{c}{n}\right)^{k-1} \left(1-\frac{c}{n}\right)^{k(n-k)+\binom{k}{2}-k+1} && \text{(equation (2.1.8))} \\
&\le \sum_{k=k_0+1}^{\beta_0 \log n} \frac{n^k}{k!} k^{k-1} \left(\frac{c}{n}\right)^{k-1} \exp\left\{ -\frac{c}{n}\left(k(n-k) + \binom{k}{2} - k + 1 \right) \right\} && \text{(Lemma 1.1.3 (a))} \\
&\le \frac{n}{c} \sum_{k=k_0+1}^{\beta_0 \log n} \frac{e^k}{k^k} k^{k-1} c^k \exp\left\{ -ck + \frac{ck^2}{n} - \frac{ck^2}{2n} + \frac{ck}{2n} + \frac{ck}{n} - \frac{c}{n} \right\} && \text{(Lemma 1.1.3 (f))} \\
&\le \frac{n}{c} \sum_{k=k_0+1}^{\beta_0 \log n} \frac{c^k}{k} \exp\{k - ck\} \exp\left\{ \frac{ck^2}{n} + \frac{ck}{2n} \right\} \\
&\le \frac{n}{c} \sum_{k=k_0+1}^{\beta_0 \log n} c^k \exp\{k - ck\} \exp\left\{ \frac{c\beta_0^2 \log^2 n}{n} + \frac{c\beta_0 \log n}{2n} \right\} \\
&= \frac{O(1)n}{c} \sum_{k=k_0+1}^{\beta_0 \log n} \left(ce^{1-c}\right)^k
\end{aligned}$$

$$
\begin{aligned}
&= \frac{O(1)n}{c} \cdot \left(ce^{1-c}\right)^{k_0} \sum_{k=k_0+1}^{\beta_0 \log n} \left(ce^{1-c}\right)^{k-k_0} \\
&= O\left(n \left(ce^{1-c}\right)^{k_0}\right),
\end{aligned}
$$

where the last equality follows from the fact that $\sum_{k\geq 1}(ce^{1-c})^k$ converges. Then noticing that

$$
\left(ce^{1-c}\right)^{k_0} = O(1) \cdot \left(\exp\left\{\log c + 1 - c\right\}\right)^{\frac{\log n}{2\alpha}} = O(1) \cdot \exp\left\{-\alpha \cdot \frac{\log n}{2\alpha}\right\} = O\left(n^{-1/2}\right),
$$

gives

$$
\mathbb{E}\left(\sum_{k=k_0+1}^{\beta_0 \log n} kX_k\right) \leq O\left(n^{1/2}\right).
$$

Now for any function $\lambda = \lambda(n)$ so that $\lambda = o(n)$ and $n^{1/2} = o(\lambda)$, we have

$$
\begin{aligned}
\mathbb{P}\left(\sum_{k=k_0+1}^{\beta_0 \log n} kX_K \geq \lambda\right) &\leq \frac{\mathbb{E}\left(\sum_{k=k_0+1}^{\beta_0 \log n} kX_K\right)}{\lambda} \qquad \text{(Theorem 1.1.5)} \\
&\leq \frac{O\left(n^{1/2}\right)}{\lambda} \\
&= o(1).
\end{aligned}
$$

Therefore, w.h.p.

$$
\sum_{k=k_0+1}^{\beta_0 \log n} kX_k = o(n).
$$

Thus w.h.p., the number of vertices in tree components of order at most $\beta_0 \log n$ is

$$
\sum_{k=1}^{\beta_0 \log n} kX_k = (1 + o(1))\frac{nx}{c} + o(n),
$$

and this completes Claim 1.

Claim 2: The number of vertices in non-tree components of order at most $\beta_0 \log n$ is $o(n)$ with high probability.

Consider the random variable Y_k that counts the number of non-tree components in $G_{n,p}$ of order k. Note that if H is a non-tree component of $G_{n,p}$ then there is a (not necessarily unique) unicyclic subgraph U_H that spans H. Then the edges of U_H are present in $G_{n,p}$, and as H is a component, then there are no edges between $V(U_H)$ and $[n] \setminus U_H$. Then we may bound Y_k by

$$Y_k \leq \sum_{\substack{\text{unicyclic } U \subset K_n, \\ |U|=k}} \mathbf{1}_{\{U \subset G_{n,p}\}} \mathbf{1}_{\{\text{no edges present between } V(U) \text{ and } [n] \setminus V(U)\}}.$$

Take U_k to be a fixed unicyclic graph on vertex set $[k]$ and note that U_k has precisely k edges. Then following a similar argument as in Lemma 2.1.3 and using equation (2.1.3),

$$\begin{aligned}
&\mathbb{E}\left(\sum_{k=1}^{\beta_0 \log n} kY_k\right) \\
&= \sum_{k=1}^{\beta_0 \log n} k\mathbb{E}\left(Y_k\right) \\
&\leq \sum_{k=1}^{\beta_0 \log n} k \sum_{\substack{\text{unicyclic } U \subset K_n, \\ |U|=k}} \mathbb{P}\left(U \subset G_{n,p},\ \text{no edges present between } V(U) \text{ and } [n] \setminus V(U)\right) \\
&= \sum_{k=1}^{\beta_0 \log n} k\binom{n}{k} \sum_{\substack{\text{unicyclic} \\ U \subset K_n, \\ V(U)=[k]}} \mathbb{P}\left(U \subset G_{n,p},\ \text{no edges present between } V(U) \text{ and } [n] \setminus V(U)\right) \\
&= \sum_{k=1}^{\beta_0 \log n} k\binom{n}{k} \sum_{\substack{\text{unicyclic} \\ U \subset K_n, \\ V(U)=[k]}} \mathbb{P}\left(U_k \subset G_{n,p},\ \text{no edges present between } V(U_k) \text{ and } [n] \setminus V(U)\right) \\
&= \sum_{k=1}^{\beta_0 \log n} \binom{n}{k} k^{k-1} \binom{k}{2} \left(\frac{c}{n}\right)^k \left(1 - \frac{c}{n}\right)^{k(n-k)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{k=1}^{\beta_0 \log n} n^k \left(\frac{e}{k}\right)^k k^{k-1} \frac{k^2}{2} \cdot \frac{c^k}{n^k} \exp\left\{-\frac{c}{n} \cdot k(k-k)\right\} \\
&= \sum_{k=1}^{\beta_0 \log n} k \left(ce^{1-c}\right)^k e^{ck^2/n} \\
&\leq O(1) \sum_{k=1}^{\infty} \left(ce^{1-c}\right)^k \\
&= O(1),
\end{aligned}
$$

where the last equality follows from the fact that $\sum_{k\geq 1} \left(ce^{1-c}\right)^k$ converges. As before, take a growing function $\lambda = \lambda(n)$ so that $\lambda = o(n)$. Then,

$$
\begin{aligned}
\mathbb{P}\left(\sum_{k=1}^{\beta_0 \log n} kY_k \geq \lambda\right) &\leq \frac{\mathbb{E}\left(\sum_{k=1}^{\beta_0 \log n} kY_k\right)}{\lambda} && \text{(Theorem 1.1.5)} \\
&\leq \frac{O(1)}{\lambda} \\
&= o(1).
\end{aligned}
$$

Therefore, w.h.p.

$$
\sum_{k=1}^{\beta_0 \log n} kY_k = o(n),
$$

completing Claim 2.

From Claim 1 and Claim 2, w.h.p. , there are $(1 + o(1))\frac{nx}{c} + o(n)$ vertices that lie in components of order at most $\beta_0 \log n$. Thus, the remaining vertices will fall into giant components, showing the existence of a giant component. What remains to show is that the giant component is unique.

The proof of uniqueness of the giant component will use Observation 2.2.1. Take

$$c_1 = c - \frac{\log n}{n} \text{ and } p_1 = \frac{c_1}{n},$$

and note that $0 < p_1 < p$. Define p_2 to satisfy

$$1 - p = (1 - p_1)(1 - p_2).$$

Then,

$$p_2 = 1 - \frac{1-p}{1-p_1} = \frac{\log n}{\log n - n + nc} \geq \frac{\log n}{n^2},$$

for n large. By the choice of p_1, p_2, we see that

$$G_{n,p} = G_{n,p_1} \cup G_{n,p_2}.$$

Take x_1 to be the unique solution of $x_1 e^{-x_1} = c_1 e^{-c_1}$ in $(0,1)$. We may assume n is large and so $x_1 = (1 + o(1))x$. Then by the same analysis of Z_k as performed above for X_k, then w.h.p. G_{n,p_1} has no components non-tree of order $k \in [\beta_0 \log n, \beta_1 n]$.

Suppose $C_1, C_2, \ldots, C_l$ are all the components of G_{n,p_1} for which $|V(C_i)| > \beta_1 n$, where $l \leq \frac{1}{\beta_1}$. Recall that $G_{n,p}$ is obtained from G_{n,p_1} and superimposing G_{n,p_2}. Then the giant component of $G_{n,p}$ is unique if for any pair of components among $C_1, \ldots, C_l$, there is an edge in G_{n,p_2} that connects them. We compute the probability that there are two components that do not get joined by G_{n,p_2}:

$$
\begin{aligned}
&\mathbb{P}\left(\text{no edge in } G_{n,p_2} \text{ joins } C_i, C_j \text{ for some } i,j\right)\\
&= \mathbb{P}\left(\sum_{i,j} \mathbf{1}_{\{C_i \text{ is not joined to } C_j \text{ in } G_{n,p_2}\}} > 0\right)\\
&\leq \mathbb{E}\left(\sum_{i,j} \mathbf{1}_{\{C_i \text{ is not joined to } C_j \text{ in } G_{n,p_2}\}}\right) && \text{(Theorem 1.1.5)}\\
&= \sum_{i,j} \mathbb{P}\left(C_i \text{ is not joined to } C_j \text{ in } G_{n,p_2}\right)\\
&= \sum_{i,j} \mathbb{P}\left(C_1 \text{ is not joined to } C_2 \text{ in } G_{n,p_2}\right)\\
&= \binom{l}{2}\left(1-p_2\right)^{(\beta_1 n)^2}\\
&\leq l^2\left(1-\frac{\log n}{n^2}\right)^{(\beta_1 n)^2}\\
&\leq l^2 \exp\left\{-\beta_1^2 \log n\right\}\\
&o(1).
\end{aligned}
$$

Therefore, with high probability, every component of order greater than $\beta_1 n$ in G_{n,p_1} gets connected after superimposing G_{n,p_2}. And so with high probability, the giant component of $G_{n,p}$ is unique. □

Chapter 3

Graphs on Surfaces

This chapter will cover some techniques seen in [3, 4, 23]. We will see examples of a method for enumerating maps, namely Tutte's Recursive Method, to count planar maps in terms of a functional equation, as well as a strategy to solve these functional equations.

3.1 Tutte's Recursive Method

We introduced graphs on surfaces in Chapter 1. In this section we will focus on the case where the surface S is the sphere S^2 so unless otherwise stated, a map is synonymous to a planar map. Furthermore, as maps are viewed as equivalent up to orientation preserving homeomorphisms, the labels in the figures throughout are for expository purposes only.

3.1.1 Maps and Rooted Maps

The main elements of a planar map are its vertices, edges, and faces. For a fixed map M with vertex set V and edge set E, there is some added terminology to it.

Definition 3.1.1. Let M be a map with vertex set V and edge set E. An incidence between an edge $e \in E$ and a vertex $v \in V$ is called a *half edge* in M. An incidence of e with a face of M is called a *side* of e. An incidence between a face of M and vertex $v \in V$ is a *corner* of M. For a vertex $v \in V$, the *degree* of v is the number of half edges incident to v, denoted $\deg(v)$. For a face F in M, the *degree* of F is the number of edge sides incident to F denoted $\deg(F)$. We say that an edge of a map M is a *bridge* if its removal disconnects the underlying graph $G(M)$.

Example 3.1.2 exhibits each definition with a particular example.

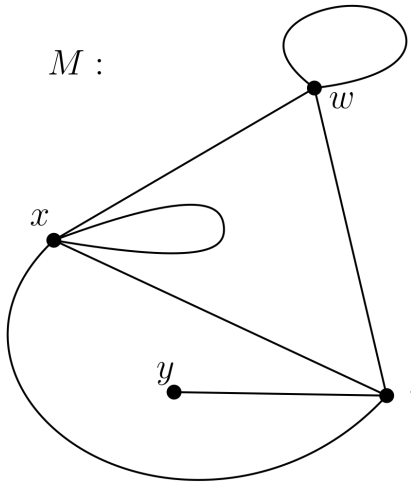
Example 3.1.2. Figure 3.1 (a) gives an example of a map M. Consider the surface S to be the sphere with the embedding of map M drawn on the plane.

Since the degree of a vertex counts half edges, we see that a loop is counted twice. Looking at Figure 3.1 (b), we see that $\deg(w) = 4$ as it is incident to a loop and two other edges. Counting the degrees of the other vertices, we have $\deg(v) = 4$, $\deg(x) = 5$ and $\deg(y) = 1$.

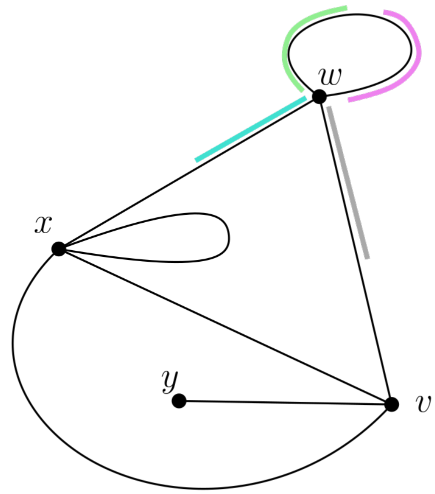
The degree of a face counts edge sides, so a bridge would contribute twice. This map has five faces, labelled $F_1, \ldots, F_5$. In Figure 3.1 (c), the face F_5 has a bridge and so combining with the two other edges, we have $\deg(F_5) = 4$. Counting the remaining face degrees, $\deg(F_1) = 4$, $\deg(F_2) = 1$, $\deg(F_3) = 4$ and $\deg(F_4) = 1$.

Lastly, note that every vertex is incident to at least one corner. Furthermore, it is possible for a vertex and a face to share several corners. For example in part (d) of Figure 3.1, vertex x and face F_3 share two corners.

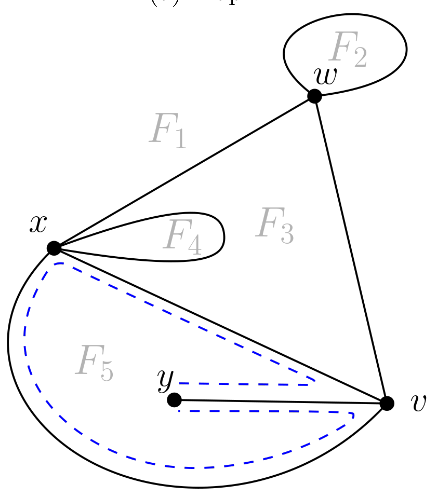
From these definitions, we see that for a map M with underlying graph $G(M) = (V, E)$, there are $2 \cdot |E|$ half edges in M, each edge has two sides, and each side is incident to a unique face. Note that both sides of an edge could be incident to the same face. For example, in Figure 3.1, edge yv has both edge sides incident to face F_5 whereas the edge xw is incident to faces F_1 and F_3.



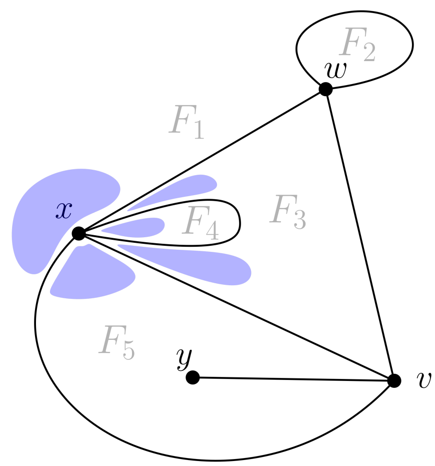
(a) Map M.



(b) Distinct half edges incident to w are marked with different colours.



(c) The dotted path shows the edge sides bounding the face F_5.



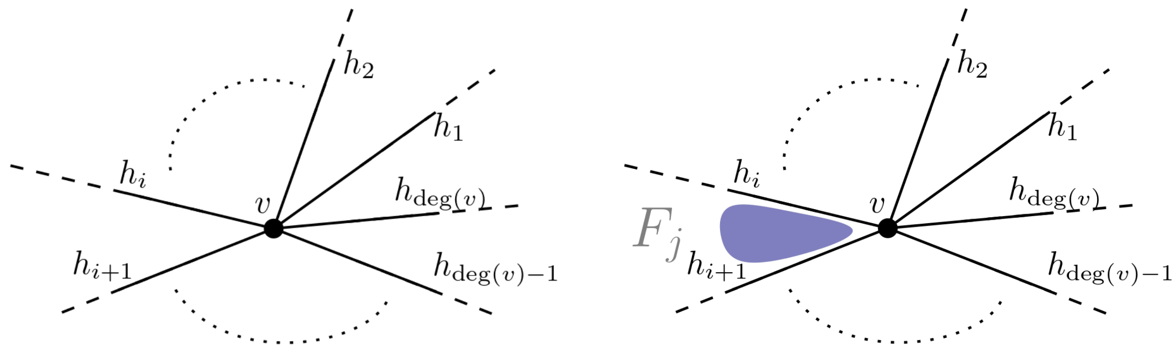
(d) The five corners incident to vertex x are highlighted.

Figure 3.1: The corresponding map in Example 3.1.2.

Furthermore, we see that for each vertex $v \in V$, there are $\deg(v)$ many corners incident to v. This can easily be seen in Example 3.1.2. In general, fix vertex $v \in V$. Label the $\deg(v)$ half edges incident to v in counterclockwise order around v by $h_1, h_2, \ldots, h_{\deg(v)}$. Then each of the pairs

$$(h_1, h_2), (h_2, h_3), \ldots, (h_{\deg(v)-1}, h_{\deg(v)}), (h_{\deg(v)}, h_1)$$

uniquely determines a face incident to v. So in particular, each such pair of half edges uniquely determines a distinct corner of M that is incident to v. This is depicted in Figure 3.2, where (b) shows that the pair (h_i, h_{i+1}) determines a particular corner of v with some face F_j.



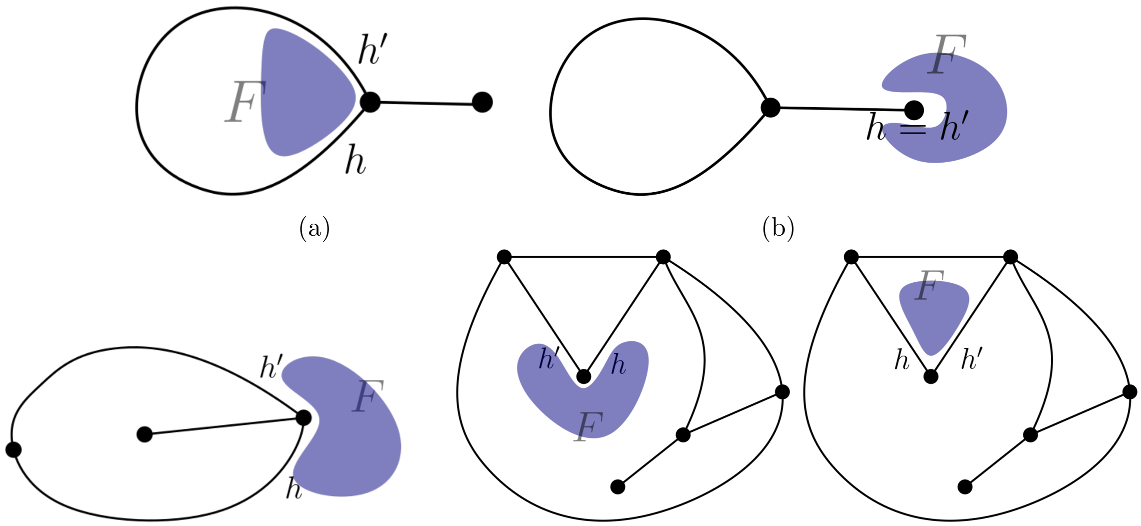
(a) Labelled half edges of vertex v. (b) The corner determined by the pair (h_i, h_{i+1}).

Figure 3.2: This shows that a vertex v is incident to $\deg(v)$ corners.

It is often useful to specify corners in terms of an ordered pair of half edges. Fixing a face F, then given (h, h') of half edges which are counterclockwise consecutive around F and share a common vertex, then this specifies a corner κ. Figure 3.3 provides a few examples of a corner determined by the pair of half edges (h, h') that are counter clockwise consecutive around a face F. In particular, note that half edges h, h' need not be distinct. The order pair (h, h) determine a corner κ precisely when the vertex of κ is a leaf, for example, as in Figure 3.3 (b).

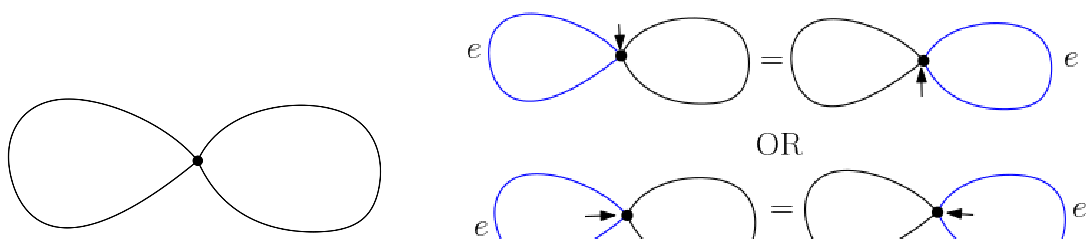
We will usually specify a corner κ by giving an ordered pair of counterclockwise consecutive half edges (h, h'), and will even refer to such a pair as a corner itself.

We noted that maps are considered as equivalent up to orientation preserving homeomorphisms. However, other symmetries are still possible. For example in Figure 3.4 (a), without marking the edges there is no way to tell the two edges apart. To deal with this, we will focus on maps with a marked corner.



(a) (b) (c) (d) (e)

Figure 3.3: Examples of corners determined by pairs (h, h').



(a) There is no way to distinguish the two edges.

(b) Marking a corner gives two distinct rooted maps.

Figure 3.4: Unrooted and rooted versions of a symmetric map.

Definition 3.1.3. A *rooted map* is a map is a pair (M, κ) where M is a map and $\kappa = (h, h')$ is a corner of M that is determined by the ordered pair of half edges (h, h') which are counterclockwise consecutive around a face F which share the same vertex. We refer to κ as the *root corner*. The trivial case is when $G(M)$ is a single vertex, then we call (M, κ) the *atomic map*, where we take $\kappa = \emptyset$.

Once a map M is rooted at corner κ, this automatically fixes the *root vertex*, *root face* and *root edge*. The root vertex and root face are the respective vertex and face at the marked corner κ. The root edge is the oriented edge with h' at its tail. In other words, to define the root edge, start at the root corner and move in a counterclockwise manner; the root edge is the first edge we encounter. In the case where (M, κ) is the atomic map then there is no root edge.

Further note that for a map M and given the root face, root vertex and root edge, then we can automatically determine which corner to mark to obtain a rooted planar map (M, κ) due to this counterclockwise convention.

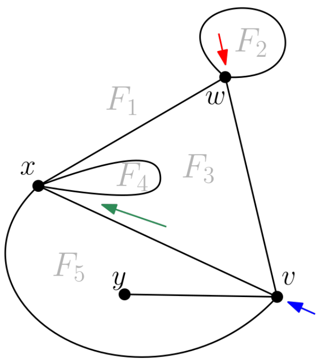
In a planar drawing, the root corner is marked by an arrow pointing at a vertex, called the *rooting arrow*. From this, the root vertex is the vertex which the rooting arrow is pointing to and the root face is the face that the rooting arrow lies in. Following the the counterclockwise construction of the root edge, if the root face is the outer face then the root edge lies to the right of the head of the rooting arrow, otherwise the root edge lies to the left of the head of the rooting arrow. This notion of rooting provides a way to remove map symmetries, for example Figure 3.4 (b) shows the different ways to root the map in Figure 3.4 (a), which distinguishes the two edges. Taking the map M from Example 3.1.2, Figure 3.5 shows three distinct rooted maps $(M, \kappa_1), (M, \kappa_2), (M, \kappa_3)$.

3.1.2 A Canonical Embedding and Isomorphisms

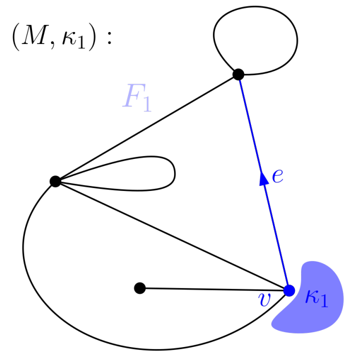
Canonical Embedding:

The methods we will see will always deal with rooted planar maps. By convention, we draw rooted planar maps so that the root face F is the outer face. This provides a canonical way to draw a planar map as drawn in the plane. We assume that a rooted planar map is viewed in this canonical embedding, unless otherwise stated.

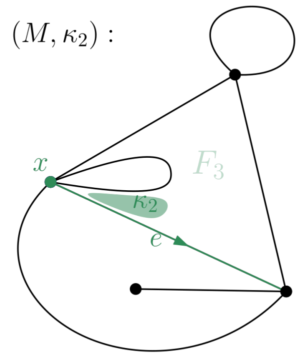
Figure 3.6 provides an intuitive visualization as to why we can always take the outer face as the root face and how this canonical embedding indeed does not affect the combinatorics of rooted planar maps. In this visualization, the rooting arrow determines our marking



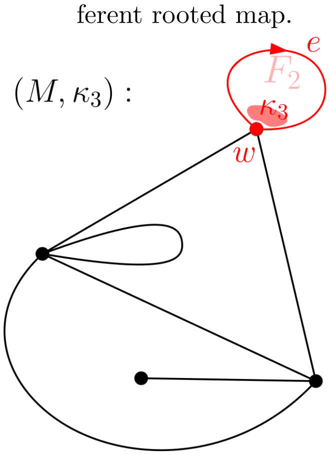
(a) Each coloured rooting arrow corresponds to a different rooted map.



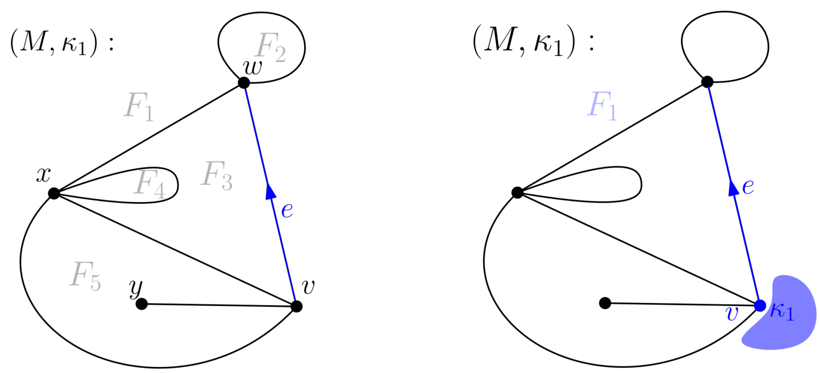
(b) (M, κ_1) is determined by the blue rooting arrow. The root vertex, root face, and root edge are distinguished.



(c) (M, κ_2) is determined by the green rooting arrow. The root vertex, root face, and root edge are distinguished.



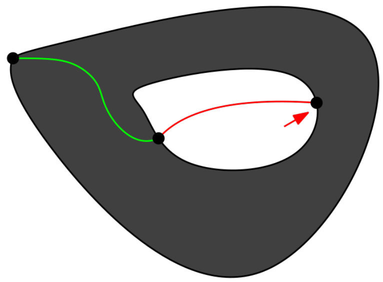
(d) (M, κ_3) is determined by the red rooting arrow. The root vertex, root face, and root edge are distinguished.



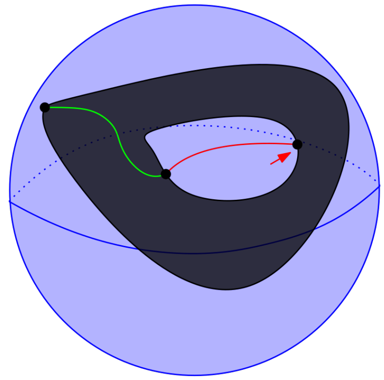
(e) Under the canonical embedding, the root vertex, root face and root edge is known given the oriented edge e.

Figure 3.5: Three different ways to root M from Example 3.1.2

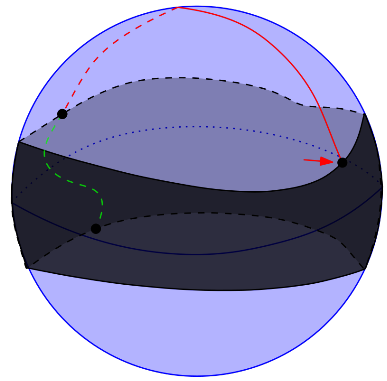
corner and the root edge is also distinguished. By moving a map around the sphere so that the north pole is inside the root face, taking the stereographic projection yields a planar drawing with the root edge on the boundary of the outer face. Note that the sphere minus the north pole $S^2 \setminus (0, 0, 1)$ is homeomorphic to $\mathbb{R}^2$. Then as the stereographic projection is an orientation preserving homeomorphism from $S^2 \setminus (0, 0, 1)$ to $\mathbb{R}^2$, then this justifies our canonical embedding of planar maps.



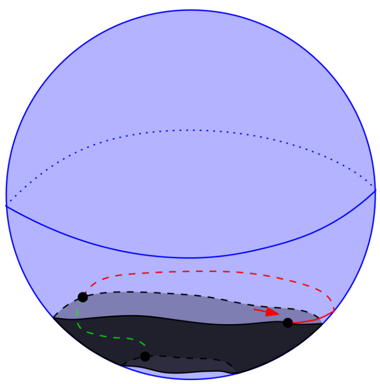
(a) A rooted map drawn on the plane so that the root face is not the outer face. The root corner and root edge are distinguished, with the remainder of the map encapsulated in the black blob. A green edge is added as another reference point, for visualization.



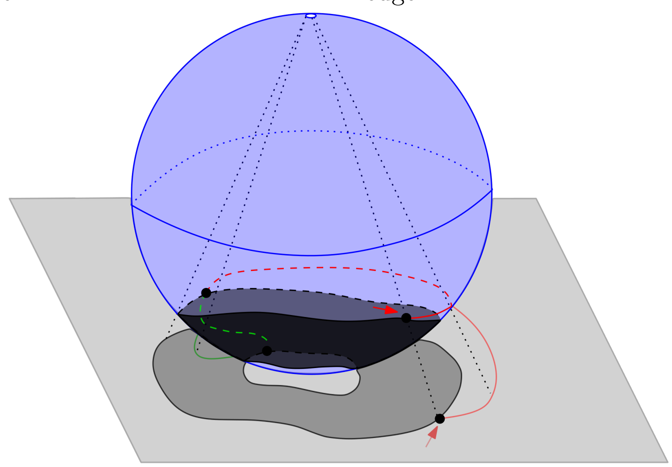
(b) Embed the map onto the sphere.



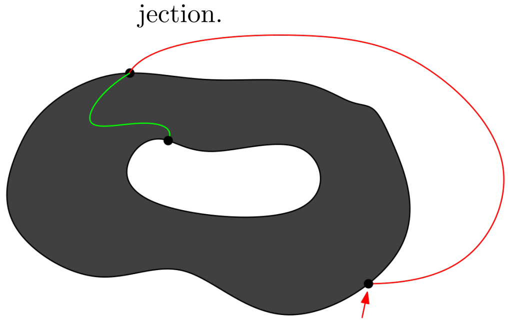
(c) Stretch the map around the sphere, lengthening the root edge.



(d) Pull the root edge to the other side of the sphere.



(e) Project the map onto the plane via stereographic projection.



(f) A map obtained by an orientation preserving homeomorphism shows the root edge is on the outer face and the counterclockwise orientation is preserved.

Figure 3.6: Visual depiction to show the canonical embedding.

Due to this convention, we refer to the degree of the root face of a rooted planar map (M, κ) as the outer degree and denote it by $\mathrm{od}(M)$. This convention provides one more visual simplification as to how we mark the root corner. Given a rooted planar map (M, κ) we have that $\kappa = (h, h')$ where h, h' are counterclockwise consecutive around the outer face. Therefore, a planar embedding with a counterclockwise oriented edge on the boundary of the outer face provides all the information about the rooting corner κ. In particular, this oriented edge orients away from h' and the root vertex. The root face is the outer face, the root edge is oriented edge, and the root vertex is the vertex at the tail of the oriented edge. Refer to Figure 3.5 (e), this shows how a rooting of the map M from Example 3.1.2 is given by an oriented edge.

Isomorphisms:

We defined a map isomorphism in Definition 1.4.5, one can similarly define this notion for rooted maps. One can view [23] as a reference for maps of higher genus.

Definition 3.1.4. Let $(M, \kappa), (M', \kappa')$ be rooted maps on surfaces S, S' respectively. Furthermore, take e, e' to be the root edges of $(M, \kappa), (M', \kappa')$ respectively. A map isomorphism $\varphi : S \longrightarrow S'$ of (M, κ) and (M', κ') is said to be a *root-preserving isomorphism* if additionally, $\varphi(e) = e'$. In this case, (M, κ) and (M', κ') are *isomorphic* rooted maps.

We introduc rooted maps to deal with map symmetries caused by automorphisms from a planar map M to itself. In fact if (M, κ) is a rooted planar map, then the only root-preserving automorphism of (M, κ) to itself is the identity map.

Proposition 3.1.5. *Let (M, κ) be a rooted planar map with root edge e and S be the sphere. If $\varphi : S \longrightarrow S$ is a map automorphism so that $\varphi(e) = e$, then φ is the identity map.*

This proposition is Proposition 3.2 in [29].

3.1.3 Tutte's Recursive Method

We now introduce generating functions of combinatorial classes before restricting our attention to enumerating rooted maps.

Definition 3.1.6. Let $\mathcal{S}$ be a set of objects equipped with a size function $|\cdot| : \mathcal{S} \longrightarrow \mathbb{N}$ so that each $s \in \mathcal{S}$ has an associated size $|s| \in \mathbb{N}$. Further denote for each $n \in \mathbb{N}$

$$\mathcal{S}_n = \{s \in \mathcal{S} \ : \ |s| = n\} \text{ and } s(n) = |\mathcal{S}_n|.$$

Then the *generating function* of $\mathcal{S}$ counted by size is

$$S(t) = \sum_{s \in \mathcal{S}} t^{|s|} = \sum_{n \geq 0} t^n s(n),$$

and each coefficient $s(n)$ of $S(t)$ can be written as

$$[t^n]S(t) := s(n).$$

Given a rooted planar map (M, κ), write $|M| = |(M, \kappa)|$ to be the number of edges of (M, κ). Given a family $\mathcal{M}$ of rooted planar maps, recall that $\mathcal{M}_n = \{(M, \kappa) \in \mathcal{M} \ : \ |M| = n\}$. Then denote the generating function of $\mathcal{M}$ by

$$M(t) = \sum_{n \geq 0} t^n |\mathcal{M}_n|.$$

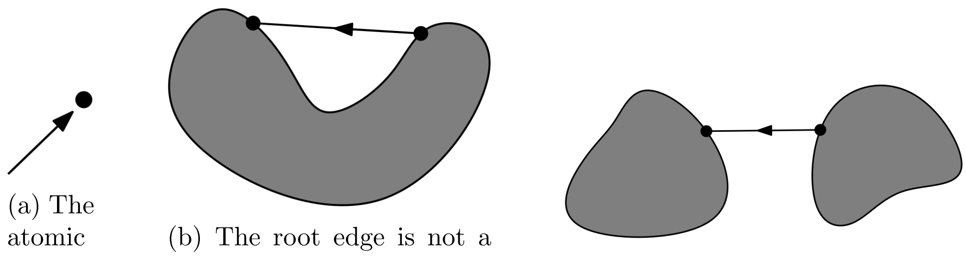
Take $(M, \kappa) \in \mathcal{M}$ and remove the root edge e; then Tutte's method allows us to recursively create a functional equation for $M(t)$ based off of whether e was a bridge or not. There are typically three cases:

1. (M, κ) is the atomic map.

2. The root edge e is not a bridge.

3. The root edge e is a bridge.

Figure 3.7 illustrates the above three cases.



(a) The atomic map.

(b) The root edge is not a bridge.

(c) The root edge is a bridge.

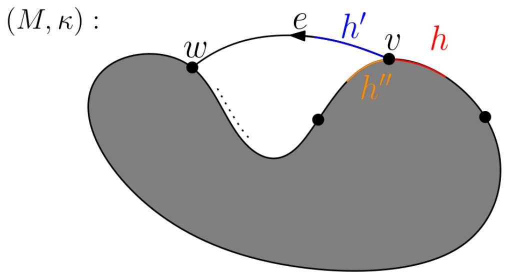
Figure 3.7: The three cases in Tutte's method.

The case for the atomic map is trivial as there is no root edge. The second and third cases will need more care. Take $G := G(M) = (V, E)$ and denote $G - e = (V, E \setminus \{e\})$. In order to provide a recursion for computing $M(t)$, we must create rooted planar maps using the components of $G - e$.

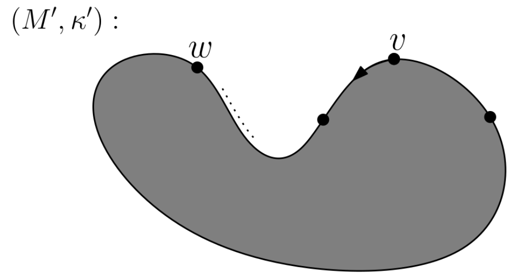
Construction 3.1.7. Let (M, κ) be a map with $\kappa = (h, h')$, v the root vertex and e the root edge. Take h'' to be the half edge preceding h in the counterclockwise order around v and let w be the other endpoint of e.

- In the case where e is not a bridge, then $G - e$ is connected. Take M' to be the embedding of $G - e$ induced by M, so M' is a map. Mark the corner $\kappa' = (h, h'')$ of M' by the incidence of the root vertex v with the outer face determined by the ordered pair of half edges (h, h''). Then (M', κ') is a rooted planar map. This construction can be seen in Figure 3.8.

- For the case where e is a bridge, label the other half edge of e by f', take f and f'' to be the half-edges preceding and following f' around w in a counterclockwise manner; as in



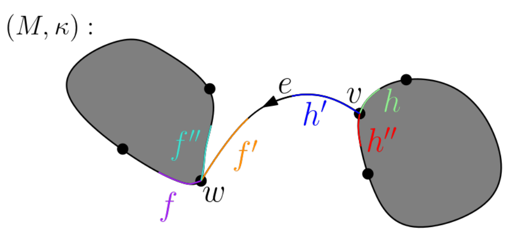
(a) A general rooted map (M, κ) where root edge e is not a bridge.



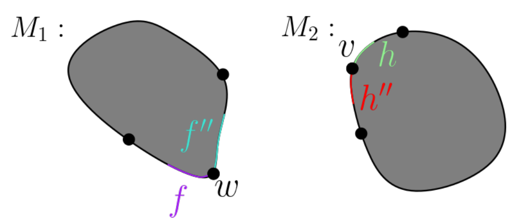
(b) The resulting rooted map (M', κ') after removing e and rerooting.

Figure 3.8: The case when e is not a bridge in Construction 3.1.7.

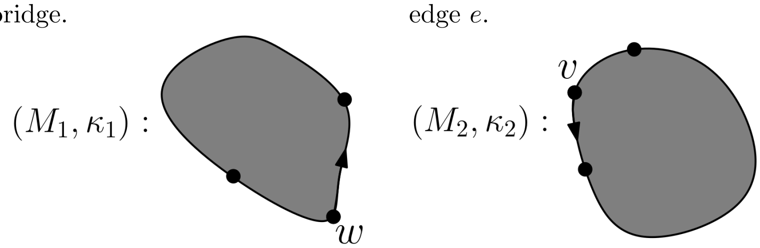
Figure 3.9 (a). We have that $G-e$ consists of two connected components G_1, G_2, where G_2 contains v and G_1 contains w. Take M_1 and M_2 to be the respective embeddings of G_1 and G_2 induced by M, so M_1 and M_2 are maps. Mark their respective corners κ_1, κ_2 where $\kappa_1 = (h, h'')$ and $\kappa_2 = (f, f'')$. So (M_1, κ_1) and (M_2, κ_2) are rooted planar maps. Refer to Figure 3.9 for this construction.



(a) A general rooted map (M, κ) where root edge e is a bridge.



(b) Maps M_1, M_2 as a result of removing edge e.



(c) Root M_1, M_2 to obtain $(M_1, \kappa_1), (M_2, \kappa_2)$.

Figure 3.9: When e is a bridge in Construction 3.1.7.

These constructions of obtaining rooted planar maps upon removing the root edge will be used throughout. It will always be the case that in the bridge setting, the root vertex of (M, κ) will fall into the map M_2.

Here, I will emphasize that our size parameter counts the number of edges as opposed to vertices. We start with a basic application of Tutte's Recursive Method where the underlying family of graphs are trees.

Example 3.1.8 (Rooted Trees)**.**

Let $\mathcal{A}$ be the family of rooted planar maps with one face, so the corresponding graph of each map is a tree. Such maps are called *plane trees*. Let $a(n)$ count the number of plane trees with n edges and $A(t)$ be its generating function,

$$A(t) = \sum_{n \geq 0} a(n) t^n.$$

Analysing the coefficients:

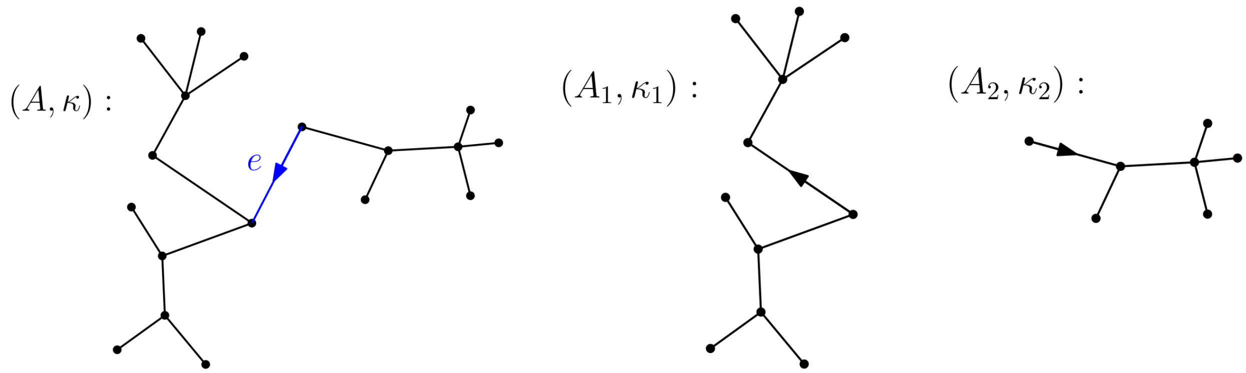
- As there is only one way to root the atomic map, then $a(0) = 1$.
- For $n \geq 1$, note that each edge is a bridge. Consider a new set

$$\mathcal{B} = \{((B_1, \kappa_1), (B_2, \kappa_2)) \ : \ (B_1, \kappa_1) \in \mathcal{A}_i, \ (B_2, \kappa_2) \in \mathcal{A}_j, \ i + j = n - 1\},$$

  consisting of pairs of plane trees so that together, the total number of edges is $n - 1$. Note that

$$|\mathcal{B}| = \sum_{\substack{i,j \geq 0: \\ i+j=n-1}} a(i)a(j).$$

  Now by Tutte's method and following our Construction 3.1.7, for each $(A, \kappa) \in \mathcal{A}_n$ the removal of the root edge e and rerooting gives two rooted maps $(A_1, \kappa_1), (A_2, \kappa_2)$. Furthermore, since $(A, \kappa) \in \mathcal{A}_n$, then $|A_1| + |A_2| = n - 1$. By definition, the root vertex of (A_2, κ_2) is the same root vertex as (A, κ). We thus obtain a function $f : \mathcal{A}_n \to \mathcal{B}$ where for each $(A, \kappa) \in \mathcal{A}_n$, $f(A, \kappa)$ is the ordered pair $((A_1, \kappa_1), (A_2, \kappa_2))$ which is an element of $\mathcal{B}$. A depiction this construction can be seen in Figure 3.10.



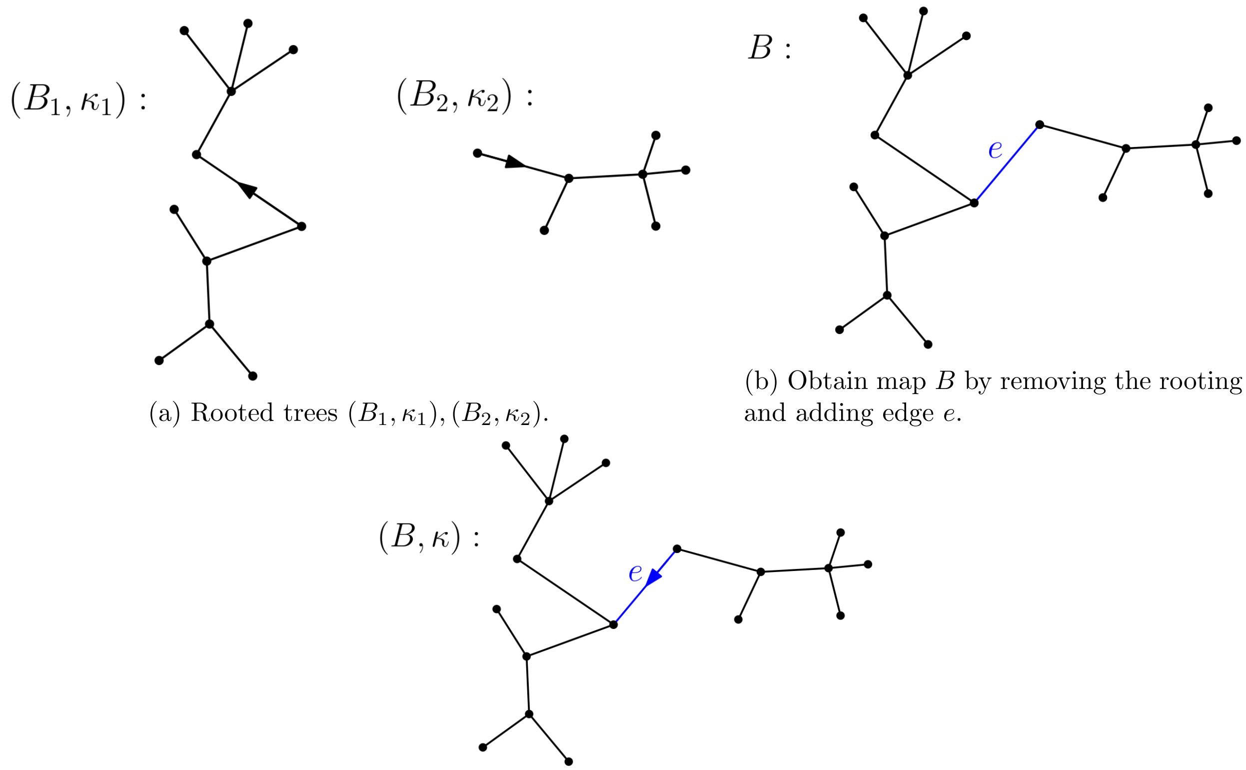
(a) Rooted tree (A, κ). (b) Resulting rooted trees after applying Construction 3.1.7.

Figure 3.10: Obtaining two rooted trees from one rooted tree in Example 3.1.8.

The inverse function f^{-1} can be described as follows. For each pair $((B_1, \kappa_1), (B_2, \kappa_2)) \in \mathcal{B}$, take B to be the map by adding a new edge e connecting B_1 and B_2 by their root vertices of $(B_1, \kappa_1), (B_2, \kappa_2)$. So B is a planar map with one face. Root B by taking the edge e and take the root vertex to be the same root vertex as (B_2, κ_2). Since the root corner κ is uniquely determined by the root vertex together with the root edge, this uniquely defines (B, κ) as a rooted planar map with one face. Furthermore, $|B| = |B_1| + |B_2| + 1 = n$, then $(B, \kappa) \in \mathcal{A}_n$. This is the above construction working in reverse, refer to Figure 3.11.

From the constructions of f and f^{-1}, it follows that

$$a(n) = |\mathcal{A}_n| = |\mathcal{B}| = \sum_{\substack{i,j \geq 0: \\ i+j=n-1}} a(i)a(j).$$



(a) Rooted trees $(B_1, \kappa_1), (B_2, \kappa_2)$.

(b) Obtain map B by removing the rooting and adding edge e.

(c) Root B to obtain (B, κ).

Figure 3.11: Obtaining one rooted tree from two rooted trees in Example 3.1.8.

Therefore,

$$
\begin{aligned}
A(t) &= \sum_{n \geq 0} a(n) t^n \\
&= 1 + t \sum_{n \geq 1} \sum_{\substack{i,j \geq 0: \\ i+j=n-1}} a(i) a(j) t^{n-1} \\
&= 1 + t \sum_{n \geq 0} \sum_{\substack{i,j \geq 0: \\ i+j=n}} a(i) a(j) t^{i+j} \\
&= 1 + t \left(\sum_{i \geq 0} a(i) t^i \right) \left(\sum_{j \geq 0} a(j) t^j \right) \\
&= 1 + t A(t)^2. \qquad (3.1.1)
\end{aligned}
$$

In fact, referring to [9, 13], one can find explicitly the value of each coefficient of $A(t)$. For $n \in \mathbb{N}$,

$$[t^n]\, A(t) = \frac{1}{n}\binom{2n-2}{n-1},$$

and note that this is the $(n-1)$-th *Catalan number*.

In this application of Tutte's method, we obtained a recurrence relation and simplified it to be expressed as a functional equation. In the following examples, we will omit the intermediate computations for the recurrences as in equation (3.1.1) and immediately write the functional equation. Referring back to Example 3.1.8, a terser explanation of the logic leading to the functional equation, more in line with that of the subsequent examples, is as follows.

The atomic map contributes $t^0 = 1$ to the expression of $A(t)$ as it has no edges. In the bridge case, we see that every rooted tree (T, κ) can be uniquely constructed from two rooted trees $(T_1, \kappa_1), (T_2, \kappa_2)$ which satisfy that $|T_1| + |T_2| = |T| - 1$ as follows. Add an edge between the root vertices of T_1 and T_2, attached in the root corners of both maps. Then mark a corner κ by taking the root vertex of T to be the root vertex of T_2 and the root edge of T to be the new edge added. Thus, this case contributes $tA(t)^2$ where the factor t comes from the new root edge and each of $(T_1, \kappa_1), (T_2, \kappa_2)$ contributed a factor of $A(t)$. Therefore, $A(t) = 1 + tA(t)^2$.

Such simplifications are valuable due to the complexity of the recurrence relations we will obtain in forthcoming examples. In particular, we will see that the next application of Tutte's method will require a multivariate generating function and hence a multivariate recurrence relation. First, we define a multivariate version of a generating function.

Definition 3.1.9. Let $\mathcal{S}$ be a set of objects equipped with $r+1$ size functions,

$$|\cdot|, f_1, f_2, \ldots, f_r : \mathcal{S} \longrightarrow \mathbb{N},$$

with each of $r+1$ functions corresponding to different parameters. The *multivariate generating function* of $\mathcal{S}$ counted by its $r+1$ size parameters is

$$\begin{aligned} S(t; x_1, \ldots, x_r) &= \sum_{s \in \mathcal{S}} t^{|s|} x_1^{f_1(s)} \cdots x_r^{f_r(s)} \\ &= \sum_{n \geq 0} t^n p_n(x_1, \ldots, x_r), \end{aligned}$$

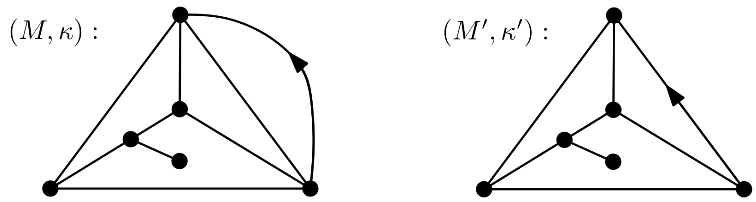
where $p_n(x_1, \ldots, x_r) = \sum_{s \in \mathcal{S},\ |s|=n} x_1^{f_1(s)} \cdots x_r^{f_r(s)}$ is a polynomial in n variables $x_1, \ldots, x_r$. For the second identity to hold, we require that $\{s \in \mathcal{S} \ : \ |s| = n\}$ is finite for all $n \in \mathbb{N}$.

This definition becomes useful in our next example, where we will count planar maps in terms of the number of edges and the outer degree.

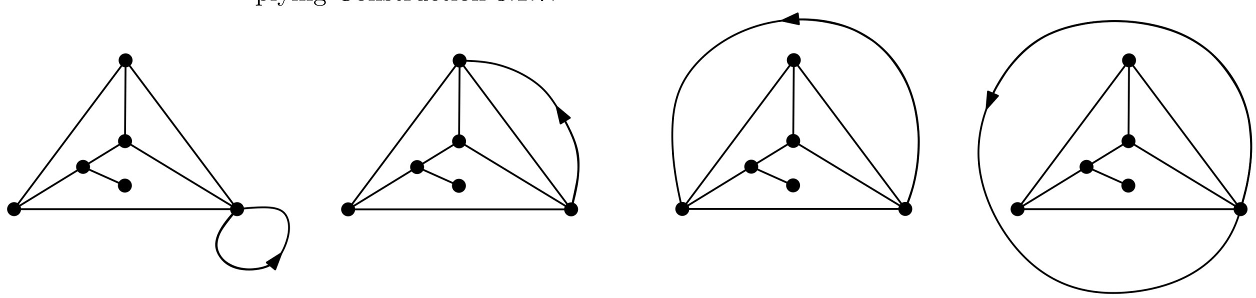
Example 3.1.10 (Rooted Planar Maps)**.**
Let $\mathcal{M}$ be the family of rooted planar maps. Let $(M, \kappa) \in \mathcal{M}$ with root edge e, we recall the possible cases from Tutte's method.

The computations for the atomic map case and the bridge case can follow an identical argument as in Example 3.1.8. However, when e is not a bridge, an additional parameter will be needed. The reason for this can be understood from the example in Figure 3.12. We see that when $\mathrm{od}(M') > 0$, then we no longer have a one-to-one correspondence as in our previous case.

To obtain a functional equation which handles the case when the root edge e is not a bridge, we will need to look at a multivariate generating function of $\mathcal{M}$. Take $|\cdot| : \mathcal{M} \to \mathbb{N}$ to be the size function counting the number of edges and recall that $\mathrm{od}(\cdot)$ counts the degree



(a) The rooted map (M', κ') that results from (M, κ) after applying Construction 3.1.7.



(b) The different possible rooted maps obtained after adding a new edge to (M', κ').

Figure 3.12: Showing that adding a new root edge to (M', κ') in Example 3.1.10 is not unique.

of the outer face. We shall obtain a functional equation for

$$M(t; y) = \sum_{(M,\kappa) \in \mathcal{M}} t^{|M|} y^{\mathrm{od}(M)}.$$

It will be useful to consider a separate generating function for rooted planar maps with specified outer degree $d \in \mathbb{N}$. For $d \in \mathbb{N}$ define

$$M_d(t) = \sum_{\substack{(M,\kappa) \in \mathcal{M}, \\ \mathrm{od}(M) = d}} t^{|M|},$$

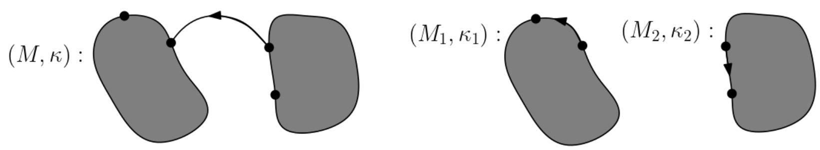
to be the generating function for rooted planar maps with outer degree d. Note that $M_d(t)$ is independent of the variable y and observe that

$$M(t; y) = \sum_{d \geq 0} y^d M_d(t).$$

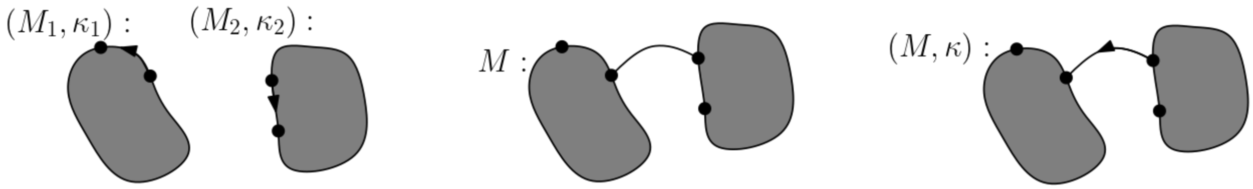
Again using Tutte's method, we assess the contributions of each case to $M(t;y)$.

- There is exactly one way to root the atomic map. As there are no edges and hence no edges sides, then this contributes $t^0y^0 = 1$ in the expression of $M(t;y)$.

- Let $(M,\kappa) \in \mathcal{M}_n$ and suppose that the root edge e is a bridge. Then the removal of e and rerooting as in Construction 3.1.7 yields two rooted planar maps $(M_1,\kappa_1),(M_2,\kappa_2)$, as in Figure 3.13 (a), and together (M_1,κ_1) and (M_2,κ_2) have $n-1$ edges.



(a) Obtaining $(M_1,\kappa_1),(M_2,\kappa_2)$ from (M,κ) as in Construction 3.1.7.



(b) Starting with two rooted planar maps to obtain one rooted planar map.

Figure 3.13: The bridge case in Example 3.1.10.

On the other hand, starting with any two rooted planar maps $(M_1,\kappa_1),(M_2,\kappa_2)$. Add an edge e connecting the root vertices of (M_1,κ_1) and (M_2,κ_2) and take M to be the resulting map. Note that M is planar as M_1 and M_2 are. Let v be the root vertex of M_2, then v and the new edge e determines a corner κ in M, refer to Figure 3.13 (b). As the root corner is uniquely determined by the root vertex together with the root edge, then this construction uniquely defines a rooted planar map (M,κ). This construction results in a one-to-one correspondence between rooted planar maps where the root edge is a bridge and ordered pairs of rooted planar maps and contributes $ty^2M(t;y)^2$. The factor ty^2 comes from the root edge and its two edge sides, and the ordered pair of planar maps contributes the factor $M(t;y)^2$.

- Suppose $(M, \kappa) \in \mathcal{M}$ so that its root edge e is not a bridge. The removal of e and rerooting as in Construction 3.1.7 gives a planar map (M', κ'). An example can be seen in Figure 3.15.

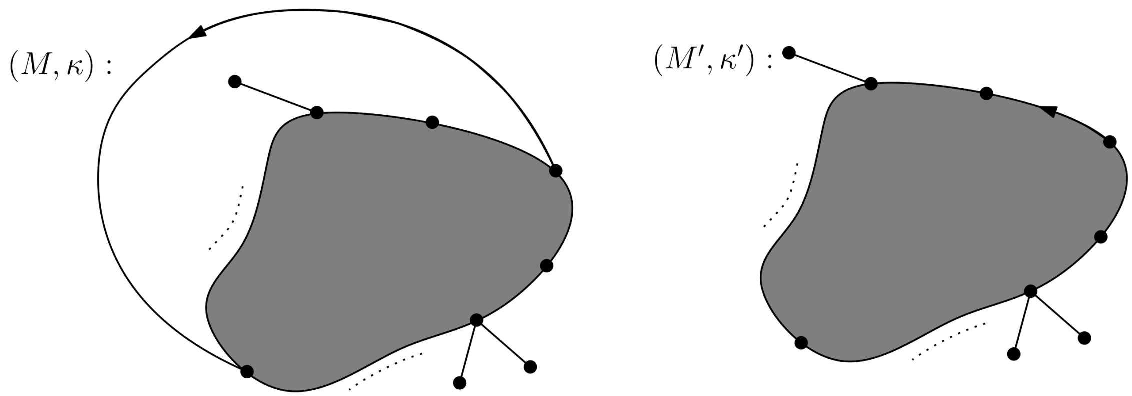
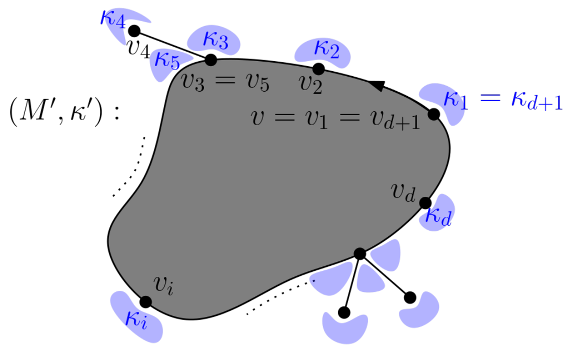


Figure 3.14: Obtaining (M', κ') from (M, κ) as in Construction 3.1.7 for the non-bridge case in Example 3.1.10.

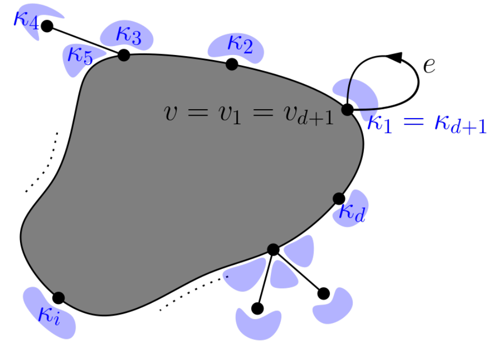
Conversely, if $\mathrm{od}(M') = d$, then we show that there are $d+1$ distinct maps (M, κ) which yield (M', κ') after applying Construction 3.1.7.

Take M' and the vertex v that is determined by κ'. The outer face of M' consists of d edge sides, so moving around the outer face in a counterclockwise order starting and ending at v, we encounter $d+1$ not necessarily distinct vertices. List these vertices in order as $v_1, v_2, \ldots, v_{d+1}$ and note that $v_1 = v_{d+1} = v$. This sequence likewise defines a sequence of corners $\kappa_1, \ldots, \kappa_{d+1}$ where κ_i is incident to v_i for each $1 \leq i \leq d+1$, as in Figure 3.15 (a). We can add a new edge e with one endpoint being v_1 and the second endpoint in any of $\kappa_1, \ldots, \kappa_{d+1}$.

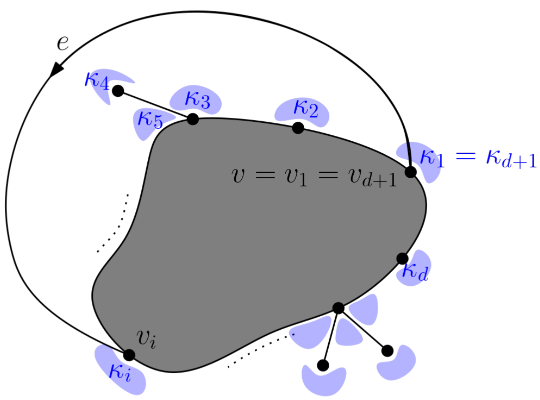
If e joins v to v_1, then embed e so that it is a loop that does not bound any of the corners κ_i, for $i = 1, \ldots, d+1$. If e joins v to v_2, embed e so that it bounds only the root edge of (M', κ'). For $i = 3, \ldots, d+1$, let e join v to v_i in such a way so that $\kappa_2, \ldots, \kappa_{i-1}$ falls into the bounded face created, as seen in Figure 3.15 (c). So there are $d+1$ possible ways to add a new edge e where v is at least one of the endpoints.



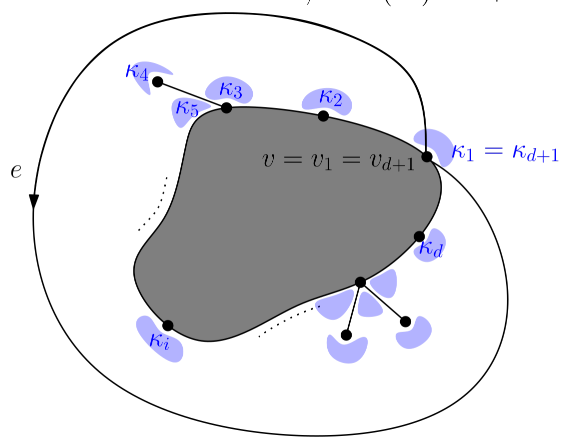
(a) (M', κ') with vertices and corners labelled.



(b) Adding a root edge to M' with v being both endpoints. This does not bound corners of the outer face, so $\text{od}(M) = d + 1$.



(c) Adding a root edge to M' with v_i being one endpoint. The edge e bounds $\kappa_2, \ldots, \kappa_{i-1}$.



(d) Adding a root edge to M' with v_{d+1} being one endpoint. The edge e bounds all corners except for $\kappa_1 = \kappa_{d+1}$.

Figure 3.15: Different possible (M, κ) from (M', κ') based off of $\text{od}(M')$ for the non-bridge case in Example 3.1.10.

Note that each possible way to add a new root edge results in a different value of $\text{od}(M)$. Take (M, κ) to be the rooted map obtained from adding the edge e to M' and taking κ to be determined by e and v. Recall that κ_i is the i-th corner of the outer face of M', moving in a counterclockwise order. If e joined v to v_1 the vertex in κ_1, then the outer face of M contains all edge side of the outer face of M' as well as the new edge side from e so $\text{od}(M) = \text{od}(M') + 1 = d + 1$. For $i \geq 2$, if e joined v to v_i the vertex in κ_i then e bounds the corners $\kappa_2, \ldots, \kappa_{i-1}$ and hence bounds $i - 1$ edge sides of M'. So e contributes a new edge side to the outer face of M and removes $i - 1$ edges sides of M' from the outer face of M. Thus, $\mathrm{od}(M) = \mathrm{od}(M') - (i-1) + 1 = d + 2 - i$. These outer degrees can be depicted in Figure 3.15 (b)–(d).

In the contribution to $M(t; y)$, the root edge accounts for a factor of t. Summing over maps (M', κ') with $\mathrm{od}(M') = d$, contributes an $M_d(t)$ factor and the possible outer degrees of the resulting map (M, κ) provides a factor of $(y^{d+1} + \cdots + y^1)$. Thus, the non-bridge case gives a contribution of $t \sum_{d \geq 0} M_d(t) \left(y^{d+1} + y^d + \cdots + y^1\right)$.

Summing the contributions of all three cases, we obtain the following functional equation for $M(t; y)$:

$$\begin{aligned}
M(t;y) &= 1 + ty^2 M(t;y)^2 + t \sum_{d \geq 0} M_d(t) \left(y^{d+1} + y^d + \cdots + y^1\right) \\
&= 1 + ty^2 M(t;y)^2 + t \sum_{d \geq 0} M_d(t) \frac{y(y^{d+1} - 1)}{y - 1} && \text{(geometric series)} \\
&= 1 + ty^2 M(t;y)^2 + \frac{ty}{y-1} \left(y \sum_{d \geq 0} y^d M_d(t) - \sum_{d \geq 0} M_d(t) \right) \\
&= 1 + ty^2 M(t;y)^2 + \frac{ty}{y-1} \left(yM(t;y) - M(t;1) \right). && (3.1.2)
\end{aligned}$$

When $y = 1$, we obtain explicit values for the coefficients of $M(t; 1)$. For each $n \in \mathbb{N}$,

$$[t^n]\, M(t;1) = 2 \cdot \frac{(2n)! 3^n}{n!(n+2)!} \sim \frac{2}{\sqrt{\pi n^5}} \cdot 12^n.$$

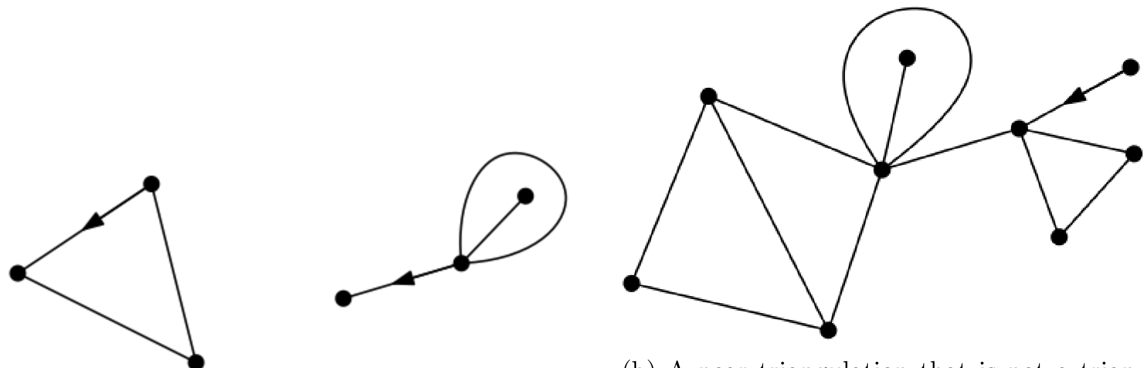
For the computations, refer to Proposition VII.II in [9].

We conclude this section with a more restrictive family of rooted maps, namely the triangulations of a polygon.

Definition 3.1.11. A rooted planar map is a *triangulation* if every face has degree 3. A *near triangulation* is a rooted planar map so that every non-root face has degree 3.

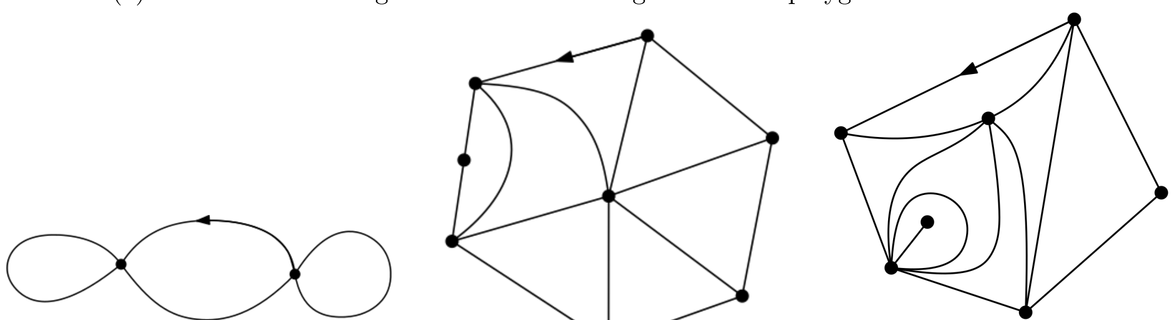
Let M be a planar map and take $\{v_1, \ldots, v_m\}$ to be the collection of vertices of some face F. A face F is *simple* if no two corners incident to F are incident to the same vertex. Lastly, a *triangulation of a polygon* is a near triangulation so that the root face is simple.

Examples of the above definitions are given in Figure 3.16.



(a) Two different triangulations.

(b) A near triangulation that is not a triangulation of a polygon.



(c) A rooted map that is not a near triangulation nor is the outer face simple.

(d) A triangulation of a polygon.

(e) Another triangulation of a polygon. Note the bounded faces need not be simple.

Figure 3.16: Examples for Definition 3.1.11.

Take (M, κ) to be a triangulation of a polygon and e to be any non-bridge edge. Since e is not a bridge, then e is incident to at least one face F. Since (M, κ) is a triangulation of a polygon, then $\deg(F) = 3$. The observation that any non-bridge edge forms a triangle with an interior face is invoked throughout the next example.

By convention, say that the atomic map is not a triangulation of a polygon. Next, we apply Tutte's method to triangulations of a polygon.

Example 3.1.12 (Triangulations of a Polygon)**.**

Let $\mathcal{P}$ be the family of triangulations of a polygon. We will enumerate such maps using the number of edges and the outer degree, consider the bivariate generating function

$$P(t;y) = \sum_{(P,\kappa)\in\mathcal{P}} t^{|P|} y^{\mathrm{od}(P)}.$$

As before, it will be useful to consider a separate generating function for triangulations of a polygon with specified outer degree $d \in \mathbb{N}$. Define

$$P_d(t) = \sum_{\substack{(P,\kappa)\in\mathcal{P},\\ \mathrm{od}(P)=d}} t^{|P|},$$

to be the generating function for triangulations of a polygon with outer degree d. Note that $P_d(t)$ is independent of the variable y and observe that

$$P(t;y) = \sum_{d\geq 0} y^d P_d(t).$$

In our analysis of the contributions of each case of Tutte's method to $P(t;y)$, we take the bridge case as our initial case and omit the atomic map.

Suppose $(P,\kappa) \in \mathcal{P}$ with root edge e. If e is a bridge, then as the the root face is simple, the only possible map P is a single edge. Refer to Figure 3.17 and note that this case contributes the term ty^2 to $P(t;y)$, t from the edge and y^2 for the two sides of e.

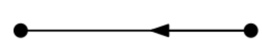
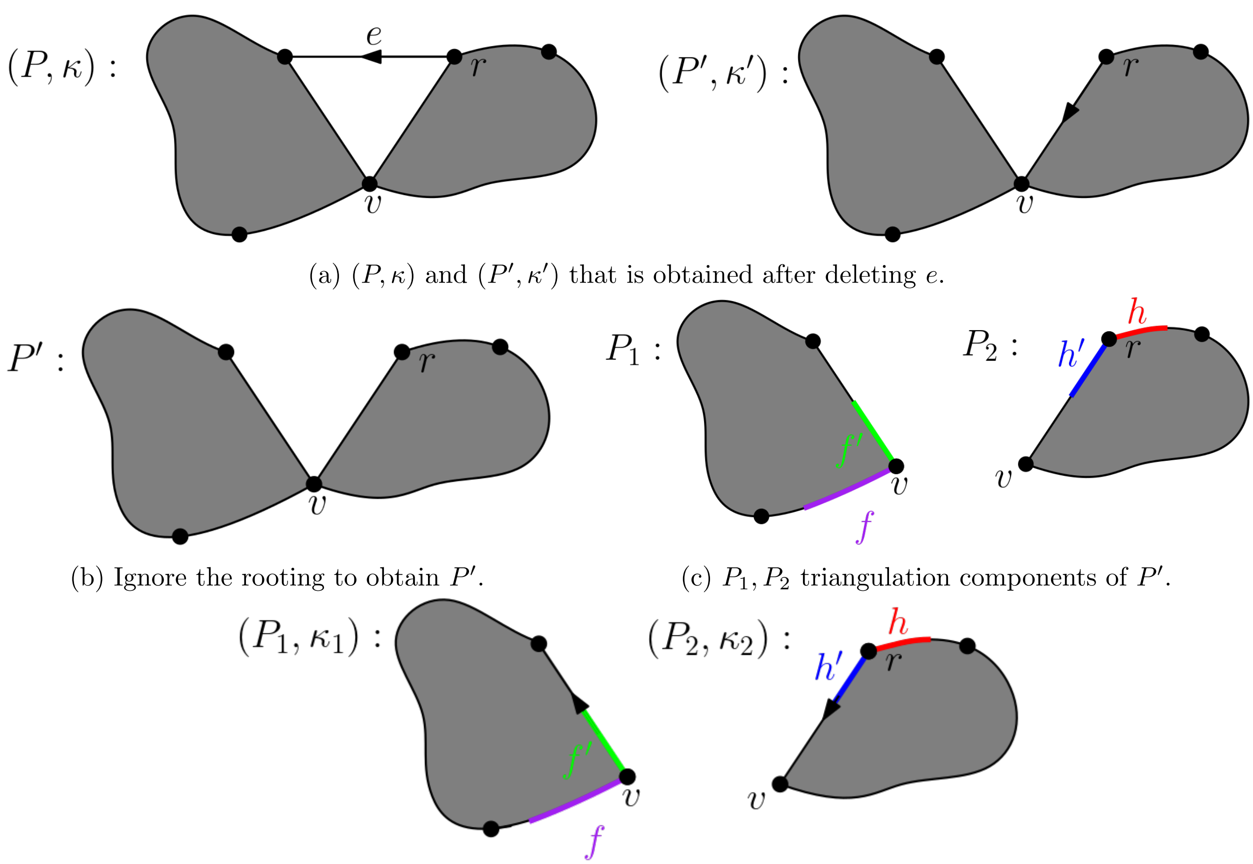
(P,κ) : 

Figure 3.17: Initial bridge case for Example 3.1.12.

We now consider the case where e is not a bridge, so $G(P) - e$ is connected. Taking P' to be the embedding of $G(P) - e$ induced by P, then P' is a map. Since the outer face of

P is simple, then every edge on the outer face belongs to a triangle. Thus, the removal of e yields (P', κ') as a near triangulation. However, we must break our analysis into cases of whether the outer face of P' is simple or not. Particularly, note that the root edge e forms part of a triangle where the two vertices of e lie on the boundary of the root face. We must consider whether the third vertex of the triangle is a boundary vertex or not.

- Suppose the third vertex is a boundary vertex, so the root face of P' is not simple. Let r be the root vertex of (P, κ) and v be the third vertex of the triangle that is on the boundary. Take (P', κ') as the rooted map obtained by removing root edge e and taking rv to be the new root edge, as seen in Figure 3.18 (a).



(a) (P, κ) and (P', κ') that is obtained after deleting e.

(b) Ignore the rooting to obtain P'.

(c) P_1, P_2 triangulation components of P'.

(d) Root P_1, P_2 to yield $(P_1, \kappa_1), (P_2, \kappa_2)$.

Figure 3.18: The non-bridge case when the third vertex is on the boundary of Example 3.1.12. Obtaining $(P_1, \kappa_1), (P_2, \kappa_2)$ from (P, κ).

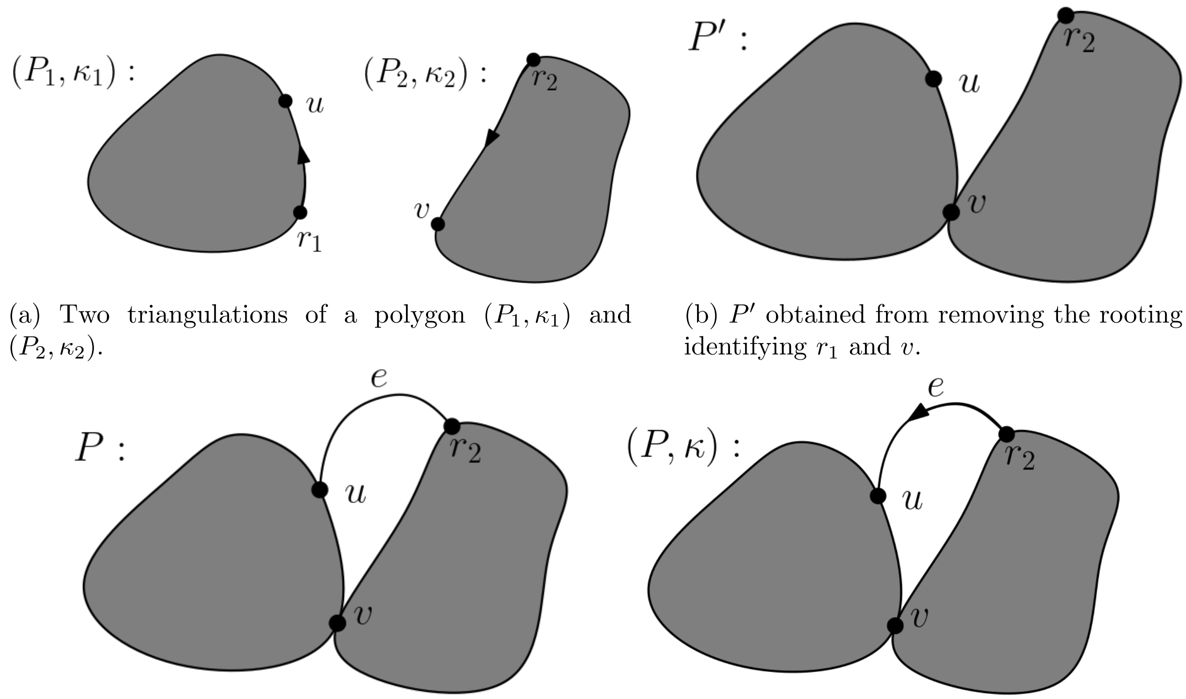
In the following arguments, we will refer to *triangulation components*. These are embedded subsets of $G(P')$ under the induced embedding of P, which form a triangulation of a polygon after some choice of root corner.

Figure 3.18 (b) and (c) shows two maps P_1, P_2 obtained by taking P_1 as the maximal triangulated component of P' that does not contain the vertex r and P_2 to be the maximal triangulated component of P' that contains r. Explore the outer face of P_1 in a counterclockwise manner and take f and f' to be the respective half edges encountered before and after vertex v. Similarly in P_2, let h, h' be the respective half edges encountered before and after vertex r upon visiting the outer face of P_2 in a counterclockwise fashion. Take $\kappa_1 = (f, f')$ and $\kappa_2 = (h, h')$ to be corners in P_1 and P_2 respectively to obtain $(P_1, \kappa_1), (P_2, \kappa_2) \in \mathcal{P}$. This is depicted in Figure 3.18 (d).

Conversely, given $(P_1, \kappa_1), (P_2, \kappa_2) \in \mathcal{P}$, let r_1, r_2 be the root vertices of $(P_1, \kappa_1), (P_2, \kappa_2) \in \mathcal{P}$ respectively. Let u, v be the non-root endpoints of the root edges of $(P_1, \kappa_1), (P_2, \kappa_2) \in \mathcal{P}$ respectively. Combine the maps P_1 and P_2 into a single map P' by identifying vertices r_1 and v as in Figure 3.19 (a) and (b). Under this identification, the outer face of P' is not simple.

To obtain a triangulation of a polygon, take P' and add an edge e incident to r_2 and u so that the triple (r_2, u, v) is a triangle as in Figure 3.19 (c), and call this resulting map P. Note that the outer face of P is simple and every bounded face is a triangle as $(P_1, \kappa_1), (P_2, \kappa_2)$ are triangulations of a polygon and (r_2, u, v) forms a triangle. Note that the outer face together with vertex r_2 and edge e determines a corner κ in P which is seen in Figure 3.19 (d), then $(P, \kappa) \in \mathcal{P}$.

This construction uniquely recovers (P, κ) from (P_1, κ_1) and (P_2, κ_2) and and we can conclude that this case contributes $\frac{t}{y}P(t; y)^2$. The ordered pair of maps $(P_1, \kappa_1), (P_2, \kappa_2)$ contribute the factor $P(t; y)^2$. The root edge contributes a factor ty, but by adding



(a) Two triangulations of a polygon (P_1, κ_1) and (P_2, κ_2).

(b) P' obtained from removing the rooting identifying r_1 and v.

(c) P obtained from P' by adding edge e.

(d) Rooting to get (P, κ).

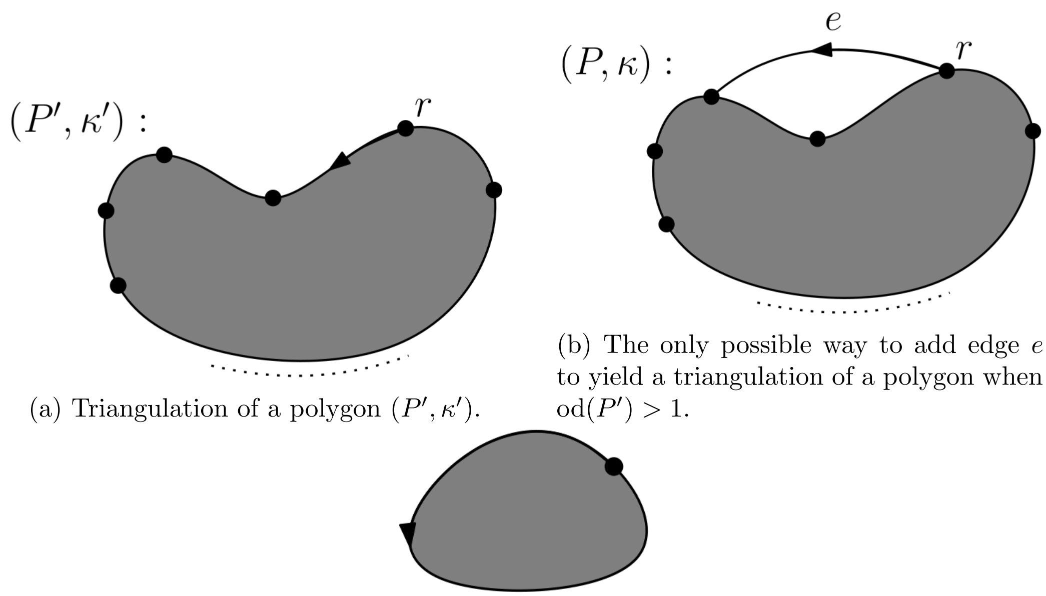
Figure 3.19: The non-bridge case when the third vertex is on the boundary of Example 3.1.12. Obtaining (P, κ) from (P_1, κ_1) and (P_2, κ_2).

the root edge e to P', we remove the sides r_2v and vu from the outer face so we get the factor $ty \cdot \frac{1}{y^2} = \frac{t}{y}$.

- Suspose the third vertex is not a boundary vertex. As every boundary edge is in a triangle, then the removal of the root edge and rerooting as in Construction 3.1.7 yields the outer face of P' to be simple, as seen in Figure 3.20 (a). Thus, $(P', \kappa') \in \mathcal{P}$.

  Conversely, take a map $(P', \kappa') \in \mathcal{P}$. We must consider if the addition of an edge to P' and rerooting gives a new map in $\mathcal{P}$. If $\mathrm{od}(P') \geq 2$, there is exactly one way to add the root edge since the root edge must be in a triangle. This is seen in Figure 3.20 (b). The case $\mathrm{od}(P') = 1$ is depicted in Figure 3.20 (c), in this instance there is no way to add a root edge so that it forms a triangle, so this case must be subtracted.

  This contributes $\frac{t}{y}(P(t; y) - y \cdot P_1(t))$ to the expression of $P(t; y)$. The terms $P(t; y) -$



(a) Triangulation of a polygon (P', κ').

(b) The only possible way to add edge e to yield a triangulation of a polygon when $\mathrm{od}(P') > 1$.

(c) The forbidden case of $\mathrm{od}(P') = 1$.

Figure 3.20: Non-bridge case when the third vertex is not on the boundary of Example 3.1.12.

$yP_1(t)$ are due to the construction above being unique while omitting the case where $\mathrm{od}(P') = 1$, in which case, provides a factor y for the edge side. The addition of the root edge provides the factor $\frac{t}{y}$, as the total number of edges increases by one and the degree of the outer face decreases by one.

Summing the contributions of all three cases, we obtain the functional equation,

$$P(t;y) = ty^2 + \frac{t}{y}P(t;y)^2 + \frac{t}{y}\left(P(t;y) - y \cdot P_1(t)\right). \tag{3.1.3}$$

Like the previous two examples, explicit formulations of the coefficients were obtained, given some conditions [25, 28]. For each $n \in \mathbb{N}$, the number or rooted triangulations with $2n$ triangles, $3n$ edges, and $n+2$ vertices is

$$T_n = \frac{2(4n-3)!}{n!(3n-1)!}.$$

Note that with these conditions on the number of triangles, edges, and vertices, then triangulations like in Figure 3.16 (e) are not counted. Specifically, triangles on two vertices with two edges are not included in the computations of T_n.

Remark 3.1.13. For our convention, we do not consider an atomic map as a triangulation of a polygon. If it were, then our computations in Example 3.1.12 would be a bit different.

The atomic map would contribute a term of $y^0t^0 = 1$ to the functional equation. For the non-boundary case of the non-bridge case, we would need to consider the possibility of $\mathrm{od}(P') = 0$. Here we would instead obtain the difference $P(t; y) - yP_1(t) - y^0P_0(t) = P(t; y) - yP_1(t) - 1$ instead of $P(t; y) - yP_1(t)$. Lastly, for the boundary case of the non-bridge case, then we must omit the case when $(P_1, \kappa_1), (P_2, \kappa_2)$ are atomic maps themselves. This results in $(P(t; y) - P_0(t))^2 = (P(t; y) - 1)^2$ instead of $P(t; y)^2$. Thus, our functional equation would instead be

$$P(t; y) = 1 + ty^2 + \frac{t}{y}(P(t; y) - 1)^2 + \frac{t}{y}(P(t; y) - y \cdot P_1(t) - 1).$$

We see that including the atomic case is redundant as the atomic case gets systematically removed in the analysis.

Another indicator to support why we may want to omit the atomic map as our convention is that the bridge case is a natural initial condition for triangulations of a polygon. When we are considering the boundary vertex case, we see that the smallest possible P_1, P_2 are bridges, so it makes sense to take bridges as the initial map and omit the atomic map.

In the examples above, we've seen Tutte's method applied to families of graphs with constraints on connectivity and face degree. Connectivity is restricted in the sense that the removal of an edge or vertex would disconnect the map, we can see this in action when analysing the bridge cases or even talking about boundary vertices as in Example 3.1.12. In general, Tutte's approach can be applied to a large variety of families of maps to obtain a functional equation, particularly in the cases of families that are characterized by face degree and connectivity constraints.

3.2 Solving Functional Equations

This section is modelled after [4]. We discussed in the previous section Tutte's approach to obtaining functional equations for generating functions of rooted maps. Here, we will follow Bousquet-Mélou and Jahanne's general strategy in solving these functional equations.

3.2.1 Motivation and Algebraic Notation

In Example 3.1.10, if we view $M(t;y) \equiv M(y)$ as a power series in y with coefficients being power series in t, then rewriting equation (3.1.2) gives the functional equation

$$M(y) = 1 + ty^2M(y)^2 + ty \cdot \frac{yM(y) - M(1)}{y-1},$$

where $M(1) = M(t;1)$ is a power series.

Similarly in Example 3.1.12, take $P(t;y) \equiv P(y)$ and $P_d(t) \equiv P_d$ for each $d \in \mathbb{N}$, then rewriting equation (3.1.3) gives the functional equation

$$P(y) = ty^2 + \frac{t}{y}P(y)^2 + \frac{t}{y}(P(y) - y \cdot P_1).$$

In the respective examples, we obtained a polynomial equation in the main series $M(y), P(y)$ with some specialization in y, namely $M(1)$ and P_1.

As mentioned before, Tutte's method is robust in its applications and can be used in more restrictive cases, which would result in more complicated functional equations. Consider the following example.

Example 3.2.1. Similar to the triangulations of a polygon, if we can instead consider simple rooted planer maps where we insist that each finite face has degree 7. Denote the collection of such rooted maps by $\mathcal{F}$ with bivariate generating function $F(t; y) \equiv F(y)$ counted by number of edges and outer degree respectively. For each $d \in \mathbb{N}$, let $F_d(t) \equiv F_d$ be the generating function where we insist that the outer degree is d.

In the application of Tutte's method, we will consider a map $F' \in \mathcal{F}$ and see how many ways we can add a new edge e and reroot to obtain a rooted planar map $F \in \mathcal{F}$. In order to guarantee that the new edge we are adding bounds a face of degree 7, we must require that $\mathrm{od}(F') \geq 6$. So in our computation, we obtain a term that has a factor of

$$F(y) - y^0 F_0 - \cdots - y^5 F_5.$$

In this case, Tutte's method yields a polynomial equation

$$\mathrm{Pol}\left(F(y), F_1, \ldots, F_5, t, y\right).$$

In fair generality, the functional equations obtained from Tutte's method, upon rearranging, yield a polynomial equation of the form

$$\mathrm{Pol}\left(F(y), F_1, \ldots, F_k, t, y\right) = 0,$$

where $F(y) = F(t; y)$ is the main series and $F_1, \ldots, F_k$ are auxiliary series independent of y. Using the terminology from [34], this equation is called a *polynomial equation with one catalytic variable* y.

Bousquet-Mélou and Jehanne present a general strategy, under some conditions, to show that the main series and auxiliary series in a polynomial equation with one catalytic variable are all algebraic and furthermore, provide a polynomial equation for them. It is indeed the case that functional equations obtained from Tutte's method satisfy these conditions.

When we are dealing with functional equations obtained from Tutte's recursive method, note that in the expression

$$\operatorname{Pol}\left(F(y), F_1, \ldots, F_k, t, y\right) = 0,$$

$F(y)$ is the main series and $F_1, \ldots, F_k$ are specializations in the catalytic variable. The idea behind the general strategy, is to obtain a system of equations to show each $F_1, \ldots, F_k$ are algebraic. Then it follows that $F(y)$ is algebraic.

We will provide a general method in this setting, specifically generating functions of families of rooted planar maps counted by number of edges and outer degree, but first we provide notation for various power series.

If $\mathbb{K}$ is a commutative ring, then $\mathbb{K}[t]$ denotes the set of polynomials with coefficients in $\mathbb{K}$. If further $\mathbb{K}$ is a field, then:

- $\mathbb{K}(t)$ is the field of fractions of $\mathbb{K}[t]$.
- $\mathbb{K}[[t]]$ is the set of formal power series in t with coefficients in $\mathbb{K}$. For $A(t) \in \mathbb{K}[[t]]$, then $A(t)$ has the form
$$A(t) = \sum_{n \geq 0} a_n t^n,$$
where $a_n \in \mathbb{K}$.
- $\mathbb{K}^{\mathrm{fr}}[[t]]$ is the set of fractional power series in t with coefficients in $\mathbb{K}$. For $A(t) \in \mathbb{K}^{\mathrm{fr}}[[t]]$, then $A(t)$ has the form
$$A(t) = \sum_{n \geq 0} a_n t^{n/d},$$
where $a_n \in \mathbb{K}^{\mathrm{fr}}[[t]]$ and $d \in \mathbb{N}$.

Lastly, take $\mathbb{K}[y][[t]]$ to be the set of formal power series in t with coefficients in $\mathbb{K}[y]$.

3.2.2 Strategy for Solving Functional Equations

We now provide the general strategy obtained in Section 2 of [4] in terms of the functional equations obtained from Tutte's recursive method.

Let $F(t,y) \in \mathbb{C}[y][[t]]$ and $F_1(t), \ldots, F_k(t) \in \mathbb{C}[[t]]$. Write $F(y) \equiv F(t,y)$ and $F_i \equiv F_i(t)$ for each $i = 1, \ldots, k$. Suppose that the $(k+1)$-tuple $(F(y), F_1, \ldots, F_k)$ of formal power series in t is completely determined by a polynomial equation

$$\operatorname{Pol}(F(y), F_1, \ldots, F_k, t, y) = 0, \tag{3.2.1}$$

where the polynomial $\operatorname{Pol}(x_0, x_1, \ldots, x_k, t, y)$ is a non-trivial polynomial in $k+3$ variables with coefficients in $\mathbb{C}$.

Further assume that equation (3.2.1) defines $(F(y), F_1, \ldots, F_k)$ uniquely in $\mathbb{C}[y][[t]] \times \mathbb{C}[[t]]^k$. The solution of an equation of the form (3.2.1) relies on the following observation.

Differentiate equation (3.2.1) with respect to y, then by the multivariate chain rule,

$$\begin{aligned}
&\frac{\partial}{\partial y}\operatorname{Pol}(F(y), F_1, \ldots, F_k, t, y) \\
&= F'(y) \cdot \frac{\partial}{\partial x_0}\operatorname{Pol}(F(y), F_1, \ldots, F_k, t, y) + \frac{\partial}{\partial y}\operatorname{Pol}(F(y), F_1, \ldots, F_k, t, y) \\
&= 0.
\end{aligned} \tag{3.2.2}$$

Let $Y(t) \equiv Y \in \mathbb{C}^{\mathrm{fr}}[[t]]$, then $F(Y) \equiv F(t,Y)$ is a well-defined power series in t. Further suppose that Y satisfies

$$\frac{\partial}{\partial x_0}\operatorname{Pol}(F(Y), F_1, \ldots, F_k, t, Y) = 0, \tag{3.2.3}$$

then we obtain from equation (3.2.2) the identity

$$\frac{\partial}{\partial y}\mathrm{Pol}\left(F(Y), F_1, \ldots, F_k, t, Y\right) = 0.$$

Now, if we can show that there are k distinct fractional power series $Y_1, \ldots, Y_k \in \mathbb{C}^{\mathrm{fr}}[[t]]$ satisfying equation (3.2.3), then applying this observation to each $Y_1, \ldots, Y_k$ yields a system of $3k$ polynomial equations

$$\mathrm{Pol}\left(F(Y_i), F_1, \ldots, F_k, t, Y_i\right) = 0 \tag{3.2.4}$$

$$\frac{\partial}{\partial x_0}\mathrm{Pol}\left(F(Y_i), F_1, \ldots, F_k, t, Y_i\right) = 0 \tag{3.2.5}$$

$$\frac{\partial}{\partial y}\mathrm{Pol}\left(F(Y_i), F_1, \ldots, F_k, t, Y_i\right) = 0 \tag{3.2.6}$$

for $i = 1, \ldots, k$.

The system of $3k$ polynomial equations relate $t, Y_1, \ldots, Y_k, F(Y_1), \ldots, F(Y_k), F_1, \ldots, F_k$. We hope to be able to eliminate the Y_i's and $F(Y_i)$'s to obtain k polynomial equations

$$\mathrm{Pol}_1\left(t, F_1\right) = 0, \ldots, \mathrm{Pol}_k\left(t, F_k\right) = 0.$$

If that is attained, then the series $F_1, \ldots, F_k$ are algebraic and hence $F(y)$ will be algebraic.

Remark 3.2.2. There is quite a bit of optimism to hope that everything works out; we require that there are enough distinct Y_i's to obtain a system of equations with as many equations as there are unknowns. And furthermore, we want this system to completely characterize the $3k$ unknown series so we can do algebraic eliminations to obtain the algebraicity of the F_i's.

We will apply this strategy to our examples from Tutte's method, before discussing more general cases.

Example 3.2.3 (Rooted Planar Maps)**.** From Example 3.1.10, applying Tutte's method yields the functional equation

$$M(y) = 1 + ty^2M(y)^2 + ty \cdot \frac{yM(y) - M(1)}{y - 1}.$$

Multiplying through by $y - 1$,

$$(y - 1)M(y) = (y - 1)(1 + ty^2M(y)^2) + ty \cdot (yM(y) - M(1)).$$

Then, the tuple $(M(y), M(1))$ is related by an equation

$$\mathrm{Pol}\,(M(y), M(1), t, y) = 0,$$

where $\mathrm{Pol}\,(x_0, x_1, t, y) = (1-y)x_0 + (y-1)(1+ty^2x_0^2) + ty \cdot (yx_0 - x_1)$. Taking the derivative with respect to x_0 gives

$$\frac{\partial}{\partial x_0}\mathrm{Pol}\,(x_0, x_1, t, y) = (1 - y) + (y - 1)(2ty^2x_0) + ty_2 = 0.$$

Applying this recipe, we want to check if there is some $Y \in \mathbb{C}^{\mathrm{fr}}[[t]]$ so that

$$\frac{\partial}{\partial x_0}\mathrm{Pol}\,(M(Y), M(1)t, Y) = (1 - Y) + 2tY^2(Y - 1)M(Y) + tY^2 = 0.$$

Upon rearranging, we want to find $Y \in \mathbb{C}^{\mathrm{fr}}[[t]]$ so that

$$Y = 1 + 2tY^2(Y - 1)M(Y) + tY^2. \tag{3.2.7}$$

Claim: There is a unique $Y \in \mathbb{C}^{\mathrm{fr}}[[t]]$ so that equation (3.2.7) holds.

We will work with the right hand side of equation (3.2.7) to construct $Y = \sum_{n\geq 0} y_n t^n$. Extracting coefficients, we see that

$$\begin{aligned}
[t^0]Y &= [t^0]\{1 + 2tY^2(Y-1)M(Y) + tY^2\} \\
&= 0
\end{aligned}$$

as all other terms of the right hand side have a factor of t. Thus, if Y exists then $Y = 1 + \sum_{n\geq 1} y_t^n$. Now,

$$\begin{aligned}
[t]Y &= [t]\{1 + 2tY^2(Y-1)M(Y) + tY^2\} \\
&= [t]\{2tY^2(Y-1)M(Y)\} + [t]tY^2 \\
&= [t]2tY^2\left(\sum_{n\geq 1} y_n t^n\right) M(Y) + [t]\left(1 + \sum_{n\geq 1} y_t^n\right)\left(1 + \sum_{n\geq 1} y_t^n\right) \\
&= 0 + 1 \\
&= 1.
\end{aligned}$$

Proceeding inductively, we see that $[t^n]\{1 + 2tY^2(Y-1)M(Y) + tY^2\}$ will depend on y_k for $k \leq n-1$. So we will be able to obtain Y uniquely, concluding the claim.

Now as $\mathrm{Pol}\,(x_0, x_1, t, y) = (1-y)x_0 + (y-1)(1 + ty^2x_0^2) + ty \cdot (yx_0 - x_1)$, we have the following derivatives:

$$\frac{\partial}{\partial x_0}\mathrm{Pol}\,(x_0, x_1, t, y) = 1 - y + (y-1)2x_0ty^2 + ty^2$$

and

$$\frac{\partial}{\partial y}\mathrm{Pol}\,(x_0, x_1, t, y) = -x_0 + (1_t y^2 x_0^2) + (y-1)(2tyx_0^2) + t(yx_0 - x_1) + tyx_0.$$

Letting Y be the unique value such that (3.2.7) holds, by the above derivatives it follows like in (3.2.5) and (3.2.6) that $\frac{\partial}{\partial x_0}\text{Pol}\,(x_0, x_1, t, Y) = 0$ and $\frac{\partial}{\partial y}\text{Pol}\,(x_0, x_1, t, Y) = 0$.

Evaluating these derivatives at $(M(Y), M(1), t, Y)$ and by equations (3.2.4), (3.2.5), (3.2.6), we obtain a system of three equations relating $t, Y, M(Y), M(1)$:

$$\begin{cases} (Y-1)M(Y) = Y - 1 + tY^2(Y-1)M(Y)^2 + tY^2M(Y) - tYM(1) \\ Y - 1 = 2tY^2(Y-1)M(Y) + yY^2 \\ M(Y) = 1 + tY(3Y-2)M(Y)^2 + 2tYM(Y) - tM(1) \end{cases},$$

from which one can eliminate Y and $M(Y)$ by hand or using a computer to obtain an algebraic equation for $M(1)$ which has three factors. Only one of these factors, corresponding to

$$M(1) = 1 - 16t + 18tM(1) - 27t^2M(1)$$

has the correct behaviour in t,

$$M(1) = M(t;1) = 1 + 2t + O(t^2).$$

So this uniquely identifies $M(1)$ and in particular shows that $M(1)$ is algebraic.

Example 3.2.4 (Triangulations of a Polygon)**.** From Example 3.1.12, applying Tutte's method yields the functional equation

$$P(y) = ty^2 + \frac{t}{y}P(y)^2 + \frac{t}{y}(P(y) - y \cdot P_1).$$

Multiplying through by y,

$$yP(y) = ty^3 + tP(y)^2 + t(P(y) - y \cdot P_1).$$

So the pair $(P(y), P_1)$ is related by an equation

$$\mathrm{Pol}\,(P(y), P_1, t, y) = 0,$$

where $\mathrm{Pol}\,(x_0, x_1, t, y) = x_0(t-y) + ty^3 + tx_0^2 - tyx_1$. Taking the derivative with respect to x_0 gives

$$\frac{\partial}{\partial x_0}\mathrm{Pol}\,(x_0, x_1, t, y) = t - y + 2tx_0.$$

Applying this recipe, we want to check if there is some $Y \in \mathbb{C}^{\mathrm{fr}}[[t]]$ so that

$$\frac{\partial}{\partial x_0}\mathrm{Pol}\,(P(Y), P_1 t, Y) = t - Y + 2tM(Y) = 0.$$

Upon rearranging, we want to find $Y \in \mathbb{C}^{\mathrm{fr}}[[t]]$ so that

$$Y = t + 2tP(Y). \tag{3.2.8}$$

As in the the previous example, we can show a unique such Y exists by extracting coefficients. Now since $\mathrm{Pol}\,(x_0, x_1, t, y) = x_0(t-y)+ty^3+tx_0^2-tyx_1$, then we have the following derivatives:

$$\frac{\partial}{\partial x_0}\mathrm{Pol}\,(x_0, x_1, t, y) = t - y + 2tx_0$$

and

$$\frac{\partial}{\partial y}\mathrm{Pol}\,(x_0, x_1, t, y) = -x_0 + 3ty^2 - tx_1.$$

Evaluating these derivatives at $(P(Y), P_1, t, Y)$ and by equations (3.2.4), (3.2.5), (3.2.6), we obtain a system of three equations relating $t, Y, P(Y), P_1$:

$$\begin{cases} tYP_1 = P(Y)(t-Y) + tY^3 + tP(Y)^2 \\ Y = t + 2tP(Y) \\ P(Y) = 3tY^2 - tP_1 \end{cases}.$$

Similarly, we can eliminate $Y, P(Y)$ from the system of equations to obtain a polynomial identity $\mathrm{Pol}\,(t, P_1) = 0$ and so P_1 is algebraic, implying that $P(y)$ is algebraic. Furthermore, by matching rates of growth in t to find the correct factor, we can show that P_1 satisfies

$$64t^5P_1^3 - t(96t^3-1)P_1^2 + (30t^3-1)P_1 + t^2 - 27t^5 = 0.$$

This strategy allows us to ensure algebraicity of functional equations obtained from Tutte's method and additionally, provides a polynomial equation for each of the series in equation (3.2.1). One reason why algebraicity is of interest is due to the fact that if a class of combinatorial objects is counted by an algebraic series, then this is a good indicator that one may be able to construct objects recursively, particularly through concatenation. For examples, refer to [8].

3.2.3 A General Algebraicity Theorem

The examples with the rooted planar maps and triangulations of a polygon fall into the framework of a general algebraicity theorem. We will define the necessary terms and state the theorem but omit the proof, which can be seen in Section 4 of [4].

Definition 3.2.5. Given a function $f(y)$, define the *divided difference* or *discrete derivative* to the operator Δ given by

$$\Delta f(y) = \frac{f(y) - f(0)}{y}.$$

Note that taking the limit as $y \to 0$ gives the derivative of f evaluated at 0,

$$\lim_{y \to 0} \Delta f(y) = f'(0).$$

Then one may define the operator $\Delta^{(i)}$ as the result of applying the Δ operator i times,

$$\Delta^{(i)} f(y) = \frac{f(y) - f(0) - yf'(0) - \cdots - \frac{y^{i-1}}{(i-1)!} f^{(i-1)}(0)}{y^i}.$$

Take $\mathbb{K}$ to be a field, in the previous examples we had $\mathbb{K} = \mathbb{C}$. Let $\operatorname{Pol}(x_0, x_1, \ldots, x_k, t, y)$ be a polynomial in $k+3$ indeterminates, with coefficients in $\mathbb{K}$. Take a functional equation

$$F(y) \equiv F(t, y) = F_0(y) + t \cdot \operatorname{Pol}\left(F(y), \Delta F(y), \Delta^{(2)} F(y), \ldots, \Delta^{(k)} F(y), t, y\right), \qquad (3.2.9)$$

so that $F_0(y) \in \mathbb{K}[y]$ is given explicitly.

Note that equation (3.2.9) has a unique solution $F(t, y) \in \mathbb{K}[y][[t]]$, which can be seen by extracting coefficients $[t^n]F(t, y)$ from (3.2.9), in the same manner as in Examples 3.2.3 and 3.2.4. A generalization of functional equations that can be solved by the strategy given in Chapter 3.2.2 follows.

Theorem 3.2.6 ([4], Theorem 3)**.** *For a field $\mathbb{K}$, a formal power series $F(t, y)$ defined by* (3.2.9) *is algebraic over $\mathbb{K}(t, y)$.*

Appendix A

Appendix

For completeness, we provide the proof of Lemma 1.1.3.

Proof of Lemma 1.1.3.

(a) Consider the function $f(x) = e^x - 1 - x$ on $\mathbb{R}$. Taking derivatives, we obtain

$$f'(x) = e^x - 1,$$

and

$$f''(x) = e^x.$$

Note that $f'(x) = 0$ if and only if $x = 0$. As $f''(0) = 1 > 0$, then $x = 0$ is a local minimum of $f(x)$. However, since $f''(x) = e^x > 0$ on $\mathbb{R}$, the second derivatives test says that $f(x)$ is concave up on $\mathbb{R}$. Therefore, $x = 0$ is a global minimum and $f \geq 0$. Hence all $x \in \mathbb{R}$, $1 + x \leq e^x$.

(b) Suppose $x \in [0, 1)$. By part (a) we have

$$\frac{1 - x + x}{1 - x} = 1 + \frac{x}{1 - x} \leq e^{x/(1-x)}.$$

Then as $0 \leq x < 1$ and rearranging,

$$\frac{1-x+x}{e^{x/(1-x)}} \leq 1-x.$$

Thus,

$$e^{-x/(1-x)} \leq 1-x.$$

(c) First note that

$$e^k = \sum_{i \geq 0} \frac{k^i}{i!} \geq \frac{k^k}{k!},$$

so we have that $\frac{1}{k!} \leq \left(\frac{e}{k}\right)^k$. Then,

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \leq n^k \cdot \left(\frac{e}{k}\right)^k,$$

as desired.

(d) It suffices to show that

$$\frac{n(n-1)\cdot(n-k+1)}{n^k} \leq \left(1-\frac{k}{2n}\right)^{k-1}.$$

First note that

$$\frac{n(n-1)\cdot(n-k+1)}{n^k} = \prod_{i=1}^{k-1}\left(1-\frac{i}{n}\right),$$

and we compare $\left(1-\frac{i}{n}\right)\left(1-\frac{k-i}{n}\right)$ with $\left(1-\frac{k}{2n}\right)^2$.

Claim: For $i \leq \frac{k}{2}$, $\left(1-\frac{i}{n}\right)\left(1-\frac{k-i}{n}\right) \leq \left(1-\frac{k}{2n}\right)^2$.

Expanding, we obtain,

$$\left(1-\frac{k}{2n}\right)^2 = 1-\frac{k}{n}+\frac{k^2}{4n^2},$$

and

$$\left(1-\frac{i}{n}\right)\left(1-\frac{k-i}{n}\right)=1-\frac{k}{n}+\frac{ki}{n^2}-\frac{i^2}{n^2}.$$

To obtain the claim, left to show is

$$\frac{ki}{n^2}-\frac{i^2}{n^2}\leq\frac{k^2}{4n^2},$$

or equivalently,

$$4ki-4i^2\leq k^2.$$

The latter follows from the fact that

$$0\leq(k-2i)^2=k^2-4ki+4i^2,$$

and this completes the claim. Now if k is odd, then

$$\begin{aligned}
\frac{n(n-1)\cdots(n-k+1)}{n^k} &= \prod_{i=1}^{k-1}\left(1-\frac{i}{n}\right)\\
&= \prod_{i=1}^{\frac{k-1}{2}}\left(1-\frac{i}{n}\right)\left(1-\frac{k-i}{n}\right)\\
&\leq \prod_{i=1}^{\frac{k-1}{2}}\left(1-\frac{k}{2n}\right)^2 && \text{(by the claim)}\\
&= \left(1-\frac{k}{2n}\right)^{2\cdot\frac{k-1}{2}}\\
&= \left(1-\frac{k}{2n}\right)^{k-1}.
\end{aligned}$$

Otherwise if k is even,

$$\begin{aligned}
\frac{n(n-1)\cdots(n-k+1)}{n^k} &= \prod_{i=1}^{k-1}\left(1-\frac{i}{n}\right) \\
&= \left(1-\frac{k}{2n}\right)\cdot\prod_{i=1}^{\frac{k-2}{2}}\left(1-\frac{i}{n}\right)\left(1-\frac{k-i}{n}\right) \\
&\leq \left(1-\frac{k}{2n}\right)\cdot\prod_{i=1}^{\frac{k-1}{2}}\left(1-\frac{k}{2n}\right)^2 && \text{(by the claim)} \\
&= \left(1-\frac{k}{2n}\right)\cdot\left(1-\frac{k}{2n}\right)^{2\cdot\frac{k-2}{2}} \\
&= \left(1-\frac{k}{2n}\right)^{k-1}.
\end{aligned}$$

(e) It suffices to show that $\frac{n(n-1)\cdots(n-k+1)}{n^k} \leq e^{-k(k-1)/2n}$. Observe that

$$\begin{aligned}
\frac{n(n-1)\cdots(n-k+1)}{n^k} &= \prod_{i=0}^{k-1}\left(1-\frac{i}{n}\right) \\
&\leq \prod_{i=0}^{i=1}\exp\left\{-\frac{i}{n}\right\} \\
&= \exp\left\{-\sum_{i=0}^{i=1}\frac{i}{n}\right\} \\
&= \exp\left\{-\frac{k(k-1)}{2n}\right\}.
\end{aligned}$$

In both cases,

$$\frac{n(n-1)\cdot(n-k+1)}{n^k} \leq \left(1-\frac{k}{2n}\right)^{k-1},$$

as desired.

(f) From the Taylor expansion of e^n,

$$e^n = \sum_{k\geq 0} \frac{n^k}{k!} \geq \frac{n^n}{n!}.$$

Rearranging,

$$\frac{1}{n!} \leq \frac{e^n}{n^n}.$$

(g) This follows from the fact that e^x is convex. Take $g(x) = 1 + 2x$ and note that $g(0) = 1 = e^0$. Furthermore, $g(1) = 3 > e^1$. As g is linear, intersects e^x at $x = 0$ and is strictly greater than e^x at $x = 1$ then convexity of e^x guarantees $g(x) \geq e^x$ for $x \in [0, 1]$.

$\square$

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