PERTURBATION THEORY OF CHARGED SCALAR SOLITONS WITH ELECTROMAGNETIC INTERACTION

by

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Résumé

Une classe générale de potentiels non-linéaires satisfaisant le théorème de localisation est définie. Une équation de Klein-Gordon complexe et non-linéaire incorporant l'interaction électromagnétique est dérivée pour les états stationnaires du soliton. Posant l'hypothèse d'une petite perturbation dépendante du temps, les équations d'onde sont dérivées pour les ordres zéro et premier de la perturbation et il est montré que le tout respecte les principales lois de conservation. On prend la limite non-relativiste de l'équation de Klein-Gordon non-linéaire pour ensuite deduire une équation intégrale, décrivant un atome d'hydrogène, valable pour la région de haute densité de charge. Une expression formelle pour le coefficient d'Einstein décrivant la transition spontanée d'un état excité a l'état stable de l'atome est finalement atteinte.

Abstract

A general class of nonlinear potentials satisfying the localization theorem is given. A complex nonlinear Klein-Gordon equation with an electromagnetic interaction is derived for the stationary states of the soliton. Assuming a small time-dependent perturbation, wave equations are found for zeroth and first order of the perturbation and important conservation laws are proved to hold. The nonrelativistic limit of the nonlinear Klein-Gordon equation is carried out and an integral equation describing the hydrogen atom is obtained for the region of high charge density. An expression for the Einstein's coefficient describing spontaneous transition from an excited state to the ground state of the atom is finally reached.

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Introduction

I-1 History

Solitons have been known now for more than a century. Their history goes back to 1834 when they were first observed on water by J. Scott Russell in Scotland. The interpretation of the phenomenon caused many discussions among physicists until a complete solution was found in 1895. The soliton was the solution of a nonlinear hydrodynamical equation . Reproduction of original papers can be found in a book by T.D. Lee [1] . The soliton was defined as a stable and nondispersive wave-packet which was regular everywhere.

Mie [2] was the first to try to describe electrons field theoretically as extended structures in opposition to the idea of the point-electron of classical electrodynamics. The latter gives rise to infinite self-energy and implies that the field equations are not valid at these points. However, Mie's theory involved serious problems. The dependence of his lagrangian on the potential implies that there is no gauge invariance. Later, Born and Infeld [3,4] tried to remove this difficulty by assuming that the lagrangian was a function of only the electromagnetic tensor $F_{\mu\nu}$. But this theory still retained a sin-

gularity and was therefore rejected. Rosen [5] , in 1938, tried to build a theory with a potential-dependent lagragian, to avoid singularities, to which he added new terms to make it gauge invariant. The major drawback of this last attempt was that electrons had a negative mass. The main interest in this early history of the description of matter by solitons is the idea to replace the point-particle by extended structures, the so-called energy-knots of Weyl [6] .

For a while, there were just a few attempts to describe particles as solitons. One can think of the paper of Finkelstein et al. about nonlinear spinor fields [7] ,or of the paper of Schiff about bosons [8], or of the work by Anderson and Derrick [9,10]. However, these models were all having problems with stability [11] .

Later on, Lee and collaborators found classicaly stable extended structures which, in principle, could describe elementary particles. This was achieved by requiring the conservation of an additive quantum number, called charge, and by assuming the presence of a neutral scalar field [12,13]. Coleman [14] worked on Lee's model and suggested that the energy density of the soliton has to be bounded away from zero at all time in a bounded domain by stating that a dis-

sipative solution is such that:

$$\lim_{t\to\infty} \max_{\underline{x}} T_{00}(\underline{x}, t) = 0$$

where T_{00} is the energy density.

At about the same time, Bialynicki-Birula and Mycielski [15,16] worked on a class of nonlinear Schrödinger-type equations. By an appropriate choice of the domain of a free parameter, the energy is bounded away from zero and their solutions are stable and localized in any number of dimensions. Furthermore, Planck's relation is valid in this picture. They also pointed out that similar solutions existed for nonlinear Klein-Gordon equations with complex scalar fields.

Following this, Werle [17] found explicit solutions for a particular class of these Klein-Gordon equations which exhibit confinement inside a finite volume. Then, Morris began working on this problem and developed the localization theorem [17,18] which permits to determine if a solution is really a soliton and found a class of nonlinear potentials respecting this theorem. He worked on the case of logarithmic nonlinearities [19] but mostly on the case of the fractional potential [20,21,22,23,24]. Of these, two [20,22] deal with the electromagnetic potential. In the first, it is proved that it is still possible to have a stable state of a soliton despite

self-repulsion. In the second, the De Broglie relation and the fine-structure constant are determined from a multisoliton stability condition.

At the same time, Simonov and Tjon [25] worked on a model similar to that of Morris. They studied collisions of solitons by computer simulation in one space dimension. They came out with very interesting results, especially the creation of small entities which they called breathers. There is still much to be done along these lines and work is in progress on this subject by Valin and Morris [26].

This historical introduction is by no means complete, especially for the last ten years when research on solitons became intensive. The papers to which the reader is referred relate mainly to classical solitons and more specifically to nontopological classical solitons. The quantum solitons and the topological solitons form by themselves very large areas of research and it would be beyond the scope of this work to try to review everything that has been published about them.

For particle physicists, the study of solitons is important because solitons present a basic solution to the problem of the classical confinement of energy without the introduction of singularities.

I-2 Work Outline

In this work, a classical nontopological soliton is studied when it is perturbed by the presence of an electromagnetic field. In the first chapters, the field is left as general as possible but in the last one, it is restricted to the case of a proton at the center of the soliton. The soliton is assumed to have a charge different from zero. As the soliton is used in this work to describe the electron, this last assumption is quite natural.

In the first chapter are gathered a number of concepts upon which further developments are based. First, the relevant equations of classical field theory for complex scalar fields are set forth. The second part of this chapter consists of an introduction to solitons: its definition, types and examples. Third, a class of confining potentials are defined and, in the last part, the localization theorem is introduced and the proof is given that the solutions obtained with the potentials defined in the third section are indeed confined for all time.

In the second chapter, a perturbed wave function is used with the nonlinear Klein-Gordon equation. Equations for currents and charges are also derived in this first section. In

the second, the continuity equation is shown to hold up to the first order of perturbation. Finally, in the last part, the conservation of energy is studied up to first order also.

In the third chapter, the nonrelativistic limit of the nonlinear Klein-Gordon equation is obtained and compared with the linear Schrödinger equation for the hydrogen atom. Using the fact that the soliton is highly concentrated at the center of its distribution, a time-dependent integral equation describing the perturbed system is derived. In the last section, the Einstein's coefficient for spontaneous transitions in the atom is found in the form of an integral.

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Chapter 1

Useful Concepts

In this chapter, the general concepts that will be used throughout this work will be developed. In the first section, some basic equations of classical field theory will be recalled. The interaction between a complex scalar field and the electromagnetic field will be introduced in a gauge invariant manner. In the second section, a description of the soliton will be given. This includes its definition, the distinction between different kinds of solitons, as well as an example of nontopological solitons. In the third part, a specific class of self-confining potentials is introduced, and finally, in the last section, a proof is given that these potentials give rise to solitons by means of the localization theorem.

1-1 Field Theory

As only complex scalar fields are studied in this work, only the relevant equations of classical field theory are developed.

The coupling between the electromagnetic and complex fields has to be gauge-invariant. This is done in the now usual way developed in the 1930's by Pauli and Weisskopf [28]. The lagrangian density is

$$(1.1)\quad \mathcal{L} = (D_\mu \vec{\phi})^* (D^\mu \vec{\phi}) - U(\vec{\phi}^* \vec{\phi}) - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where $U(\vec{\phi}^* \vec{\phi})$ is the nonlinear potential term and $F_{\mu\nu}$ is the electromagnetic tensor:

$$(1.2)\quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \qquad \mu, \nu = 0, 1, 2, 3$$

The operator D is the covariant derivative and is defined by

$$(1.3)\quad D_\mu = \partial_\mu + i\varepsilon \vec{A}_\mu$$

In the two last expressions, A_μ represents the electromagnetic potential with A_0 being the scalar potential . One can easily obtain the equations of motion from equation (1.1) by using the variation principle. This gives the following nonlinear Klein-Gordon equation:

$$(1.4)\quad D_\mu D^\mu \vec{\phi} + U'(\vec{\phi}^* \vec{\phi}) \vec{\phi} = 0$$

as well as a similar equation for $\vec{\phi}^*$. The canonical momenta are given by

$$(1.5)\quad \pi = \frac{\partial \mathcal{L}}{\partial \dot{\vec{\phi}}} = (D_0 \vec{\phi})^* \qquad \text{where} \quad \dot{\vec{\phi}} = \partial_0 \vec{\phi}$$

$$(1.6)\quad \pi^* = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} = D_0 \phi$$

$$(1.7)\quad E^\mu = \frac{\partial \mathcal{L}}{\partial A_\mu} = F^{\mu 0}$$

and therefore, the hamiltonian density can be written

$$(1.9)\quad H = \pi \dot{\phi} + \pi^* \dot{\phi}^* + E^\mu \dot{A}_\mu - \mathcal{L}$$

and the total energy is

$$(1.10)\quad E = \int H\, d^3x = \int \left[\tfrac{1}{2} F^{ij} F_{ij} + \tfrac{1}{2} E^i E^i + \dot{\phi}\dot{\phi}^* + i\varepsilon A_0 (\phi \dot{\phi}^* - \dot{\phi}\phi^*) + \nabla\phi^* \cdot \nabla\phi + i\varepsilon \underline{A} \cdot (\phi \nabla \phi^* - \phi^* \nabla \phi) + U(\phi^* \phi) \right] d^3x$$

The three-momentum $\underline{P}$ is defined as

$$(1.11)\quad \underline{P} = -\int \left[\frac{\partial \mathcal{L}}{\partial \dot{\phi}^*} \nabla \phi^* + \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \nabla \phi + \frac{\partial \mathcal{L}}{\partial \dot{A}^i} \nabla A^i \right] d^3x$$

Some straightforward calculations lead to

$$(1.12)\quad \underline{P} = -\int \left[\dot{\phi} \nabla \phi^* + \dot{\phi}^* \nabla \phi - i\varepsilon A_0 (\phi^* \nabla \phi - \phi \nabla \phi^*) + (\partial^0 A^i - \partial^i A^0) \nabla A_i \right] d^3x$$

The conserved four-current is

$$(1.13)\quad J_\alpha = i\varepsilon (\phi^* D_\alpha \phi - \phi (D_\alpha \phi)^*)$$

and therefore, the charge q is

$$(1.14)\quad q = \int J_0 d^3x = i\varepsilon \int (\phi^* D_0 \phi - \phi (D_0 \phi)^*)\, d^3x = \varepsilon Q$$

As the current must be conserved, one has the following conservation rule :

(1.15) $\partial^{\mu} J_{\mu} = 0$

Repeated indices imply a summation except when otherwise stated.

In this work, a metric with signature (+,-,-,-) is used as well as the natural system of units in which c=1. Unless otherwise specified, a primed function is the derivative of this function with respect to its argument. Greek indices run from 0 to 3 and the latin ones , from 1 to 3. The ordinary space vector is written $\underline{x}$.

1-2 Solitons

A soliton could be defined as a solitary wave that propagates without dissipating towards the vacuum and recovers its original shape after a collision. But, if one wants to describe elementary particles as being solitons, this definition is too narrow because, for example, it cannot describe reactions in which new particles are created. Lee [1,2] defined a classical soliton as a solution of a nonlinear local

field equation confined in a finite region of space and possessing a finite non-zero rest mass. The main idea is the localization of the field in space to which we shall return later in this chapter.

Solitons can be formed in two ways. One of them is to impose special boundary conditions at infinity for the field that will differ from the physical vacuum [3]. This implies that the vacuum state must be degenerate. These are topological solitons. As examples of models giving rise to such solitons, one can quote the Φ^4 model [4] or the sine-Gordon model [5]. But there are difficulties with this kind of soliton as pointed out by Loo [6]: first, spinless solitons would be stable only in one dimension as stated by Derrick's theorem [7] and second, some models would force alternation of solitons and antisolitons. Both these facts are irreconciliable with the aim to describe particles as solitons. But, these restrictions hold only for static and spinless fields.

The introduction of spin and gauge fields can get rid of the problems stated above. These theories lead to gauge monopoles [8,9] and the euclidian instantons [10]. These instantons are indeed solitons in four-dimensional space which means they have a limited extension in time. An example of a gauge mono-

pole can be studied [11] using the Georgi-Glashow model which is a gauge theory of the symmetry group SO(3). 't Hooft and Polyakov [8,9] have studied a special case of this model (n=1) which gives rise to a magnetic monopole of charge $g = \frac{4\pi}{e}$, a result similar to what Dirac obtained by other means [12]. However, this is the only point in common between the two as emphasized by Coleman [31]. These monopoles have been proved to be very heavy which explains why they haven't been discovered yet, if indeed they exist.

Topological solitons have very interesting properties for particle physicists but there is still a lot to be done about stability and multisoliton systems. Nontopological solitons, the ones used in the rest of this work, form the other class. It is held together in a dynamical way by a nonlinear potential. Its boundary condition is the same as the vacuum. Therefore, the vacuum does not have to be degenerate as in the previous case. However, one needs an additive conservation law to ensure the existence of such a soliton. It is interesting to note that these solitons can exist in any space dimension [13].

Lee [14] worked the following simple example in one space dimension using a complex scalar field, ϕ. The lagrangian

density is given by

$$(1.16)\quad \mathcal{L} = \frac{\partial \vec{\phi}^*}{\partial x_\mu} \frac{\partial \vec{\phi}}{\partial x_\mu} - U(\vec{\phi}^*\vec{\phi}) \qquad \mu = 0,1$$

Using the variation principle, the equation of motion is found to be

$$(1.17)\quad \frac{\partial^2 \vec{\phi}}{\partial x_\mu^2} + U'(\vec{\phi}^*\vec{\phi})\vec{\phi} = 0$$

If one chooses the minimum of the potential to be zero, U can be expanded in a power series of $\vec{\phi}^*\vec{\phi}$

$$(1.18)\quad U(\vec{\phi}^*\vec{\phi}) = m^2\vec{\phi}^*\vec{\phi} + \vartheta(\vec{\phi}^3) + \vartheta(\vec{\phi}^4) \quad .....$$

The lagrangian is invariant under gauge transformation as

$$(1.19)\quad \vec{\phi} \rightarrow e^{i\theta}\vec{\phi}$$

This implies [15] that the current is given by

$$(1.20)\quad j_\mu = i \frac{\partial \mathcal{L}}{\partial(\partial^\nu \vec{\phi}^*)} \vec{\phi}^* - i \frac{\partial \mathcal{L}}{\partial(\partial^\nu \vec{\phi})} \vec{\phi}$$

and that it satisfies

$$(1.21)\quad \frac{\partial j_\mu}{\partial x_\mu} = \partial^\mu j_\mu = 0$$

The time component of j_μ gives the charge density

$$(1.22)\quad \rho = i(\vec{\phi}^*\dot{\vec{\phi}} - \dot{\vec{\phi}}^*\vec{\phi})$$

The space integral of the equation (1.22) gives the total

charge Q

$$(1.23) \quad Q = \int \rho \, dx$$

which is a conserved quantity in time because of (1.21)

$$(1.24) \quad \dot{Q} = 0$$

Therefore, for Q different from zero, ϕ must be a variable quantity with respect to time. To find the time dependence of the lowest energy state, it suffices to do the following steps [16]. The charge Q must obey the following inequality.

$$(1.25) \quad Q = i \int (\phi^* \dot{\phi} - \dot{\phi}^* \phi) dx < 2 \left| \int \phi^* \dot{\phi} \, dx \right|$$

Using the Schwarz inequality for the last term, one gets

$$(1.26) \quad \left| \int \phi^* \dot{\phi} \, dx \right| \leq \left[\int \phi^* \phi \, dx \right]^{\frac{1}{2}} \left[\int \dot{\phi}^* \dot{\phi} \, dx \right]^{\frac{1}{2}}$$

The equality in equation (1.26) holds only if ϕ and $\dot{\phi}$ are linearly dependent. From this, one can deduce that the equality in (1.25) holds only if

$$(1.27) \quad \dot{\phi} + i\omega\phi = 0$$

with

$$(1.271) \quad \omega = \frac{Q}{2I} \qquad \text{where} \quad I = \int \phi^* \phi \, dx$$

Following arguments due to Morris [29], one can say that E and

$\hat{E}$ are instantaneously equal for Cauchy data that satisfies both equations (1.27) and (1.271) where

$$(1.272)\quad E \geq \hat{E}(Q,\underline{\phi},\underline{\phi}^*) = \frac{Q^2}{4I} + \int \left(|\underline{\nabla}\underline{\phi}|^2 + U \right) dx$$

Therefore, for Cauchy data which gives the minimum value of the functional $\hat{E}$, the equality must be true for all time in (1.272). Thus, equation (1.27) is also valid for all time and $\underline{\phi}$ is a stationary state. Therefore, $\underline{\phi}$ has the form

$$(1.28)\quad \underline{\phi}(x,t) = \phi(x)\ e^{-i\omega t}$$

with $\phi(x)$ a real quantity. Then, (1.17) can be written

$$(1.29)\quad \frac{d^2\phi}{dx^2} + \omega^2\phi - \phi \frac{d}{d\phi^2} U(\phi^2) = 0$$

Multiplying by $\frac{d\phi}{dx}$ and integrating, one obtains

$$(1.30)\quad \tfrac{1}{2}\left(\frac{d\phi}{dx} \right)^2 + \tfrac{1}{2}\omega^2\phi^2 - \tfrac{1}{2} U(\phi^2) = 0$$

The potential must have the form given in figure 1.1 for the case of polynomial potentials [13] in order to have non-topological solitons as verified thereafter.

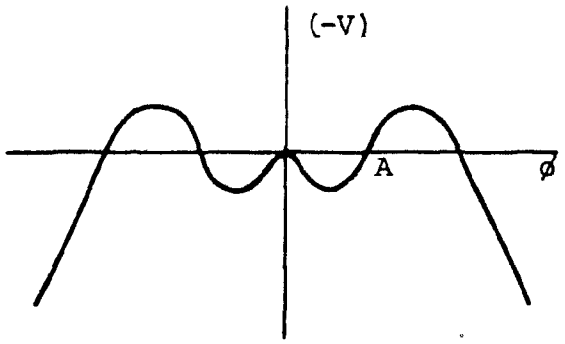

Fig. 1.1: Sketch of the potential $V(\phi^2) = \frac{1}{2}(U - \omega^2\phi^2)$

Using the mechanical analog of a particle moving in a potential (-V), one finds

$$(1.31)\quad x-a = \int_{A}^{\phi} \left(2\, V(\phi^2) \right)^{-\frac{1}{2}} d\phi$$

where a is the integration constant (x=a when $\phi^2 = A$).

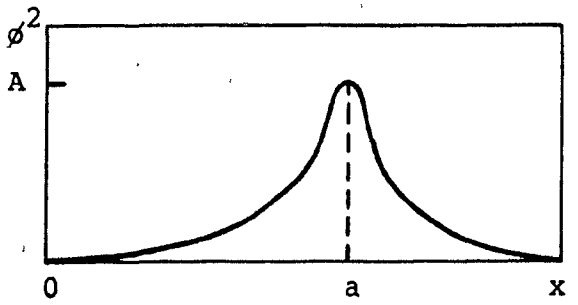

Fig. 1.2: A nontopological soliton

It satisfies the boundary condition for a nontopological soliton which is

$$\phi^2 \to 0 \quad \text{when} \quad x \to \pm\infty$$

All of this can be seen on figure 1.1 with the mechanical analogy. Suppose there is a particle at $\phi^2 = 0$ at time $x = -\infty$. If the particle is displaced just off its equilibrium point, this particle is going to move in the potential well (-V) up to point A and come back to $\phi^2 = 0$ at time $x = +\infty$ which is the behaviour described on figure 1.2 for a soliton. Thus, figure 1.1 represents the kind of potential needed to describe nontopological solitons. Furthermore, in the limit $\phi^2 \to 0$, using equation (1.18), the potential is

$$(1.32)\quad V \to \tfrac{1}{2}(m^2 - \omega^2)\, \phi^2$$

which implies that

(1.33) $\omega^2 < m^2$

to ensure this is a soliton solution by forcing the upwards concavity of $V(\phi^2)$ at $\phi^2 = 0$. Otherwise, the solution is a plane-wave as one can check with the same analogy that has been used above. So, there is a limitation on the values ω can take. Similar limitations in two or three space dimensions can be derived [27] . Figure 1.3 gives a graphical representation of this condition in three dimensions where $\omega = \frac{dE}{dQ}$

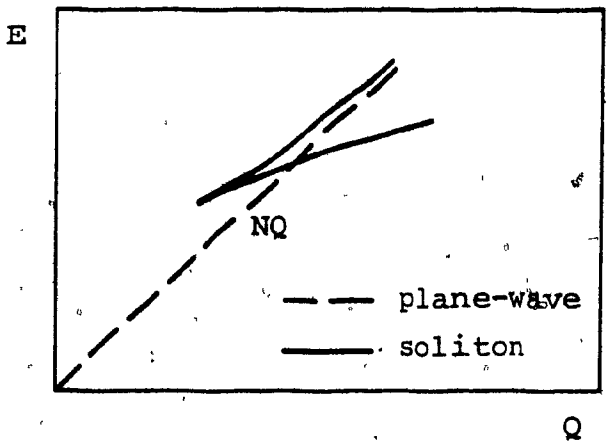

Fig. 1.3: E(Q) for three-dimensional model studied by Lee and collaborators [13]

The existence of plane-wave solutions allows decays of solitons in plane-waves as can be seen in figure 1.3. In order to discard this possibility, a new class of potentials is now introduced upon which are imposed certain restrictions to obtain the desired behaviour.

1-3 Confining Potentials

In cases like the one represented by figure 1.3[2] , the confinement is caused by the action of a field on another. Instead, self-confining potentials will be used in this paper, i.e. fields coupled nonlinearly with themselves. These potentials should not allow any plane-wave states. In this section is defined this particular class of self-confining potentials.

The self-confining potential $U(\phi^*\phi)$ has to satisfy conditions that have been established by Morris [17,18]. It is a non-negative function of $\phi^*\phi$ such that $U(0)=0$ and $U'(0)=\infty$. It is assumed that the vacuum is non-degenerate so that U has just one zero and that the vacuum state is a solution of the wave equation. Furthermore, with $\phi=0$ being the vacuum, equation (1.4) limits the singularity of $U'(0)$:

$$(1.34) \quad \lim_{\phi \to 0} U'(\phi^*\phi)\phi = 0$$

For further convenience, some other conditions are added that will limit the number of possible potentials but will simplify the construction of proofs in the next section. Let $U(\phi^*\phi)$ be bounded from below by a function $u(\phi^*\phi)$ which satisfies the following conditions [19] :

a) $u(\phi^*\phi)$ is concave for $0 < \phi^*\phi < \infty$

b) $u(\phi^*\phi) \geq \mu^2 \phi^*\phi$ for $0 < \phi^*\phi < \infty$ and with $\mu^2 > 0$

c) $u(0) = 0$, $u'(0) = \infty$ and $u'(\phi^*\phi)\phi \to 0$ as $\phi \to 0$

d) $u'(\phi^*\phi)$ is convex for $0 < \phi^*\phi < \infty$

Condition (a) is obvious as U has to be a positive-definite and non-decreasing function with an infinite slope at the origin. Condition (c) is just a repetition of the conditions that had been imposed on the potential U. Condition (d) is introduced to simplify the proofs of different results in the rest of this chapter. Finally, condition (b) was prescribed to ensure that U would be non-decreasing and therefore, never negative. This permits to write

$$(1.421) \quad E = \int (\underline{\nabla}\phi^* . \underline{\nabla}\phi + \mu^2 \phi^*\phi) \, d^3x + \text{positive quantity}$$

The integral defines a Hilbert space of Sobolev type and this justifies the use of the Sobolev inequality to come later in this work. Even if these properties eliminate some self-confining potentials, there is still a large field to work in.

An example of such a potential is given by the fractional potential [20] :

$$(1.35) \quad U(\phi^*\phi) = b(\phi^*\phi)^a + \mu^2 \phi^*\phi \qquad \text{with } \tfrac{1}{2} < a < 1$$

One can easily check that (1.35) fulfills all the conditions given previously.

Now, it can be verified that the plane-wave solution does not exist for this class of potential. Let A_μ be equal to zero in equation (1.10) as it won't change the result. A plane-wave solution is given by

$$(1.36) \quad \phi = A \exp(-ik_\nu x^\nu)$$

Putting this in equation (1.4), one gets for a given frequency k_0

$$(1.37) \quad k_i^2 = k_0^2 - U'(A^*A)$$

Therefore, k_i^2 becomes negative as U' increases which means that the wave is reflected back at a certain point. It can also be shown that a plane-wave solution would contribute an infinite amount of energy. Introducing (1.36) in equation (1.10), one gets

$$(1.38) \quad E = \int [(k_0^2 + k_i^2)A^*A + U(A^*A)] \, d^3x$$

Using a box normalization, $A = \frac{D}{V^{\frac{1}{2}}}$ where D is a constant and V is the volume of the box, one easily obtains by integrating over the volume of the box

$$(1.39) \quad E = (k_0^2 + k_i^2) D^*D + U\left(\frac{D^*D}{V}\right) V$$

Now, as $U(\phi^*\phi) \geq u(\phi^*\phi)$ and $u(0)=0$ and using property (a) of u, the following inequalities are found :

(1.40) $U(\phi^*\phi) \geq u(\phi^*\phi) \geq u'(\phi^*\phi)\phi^*\phi$

Using this in (1.39) gives

(1.41) $E \geq (k_0^2 + k_i^2)D^*D + D^*D\, u'(\frac{D^*D}{V})$

For $V \to \infty$, $u'(\frac{D^*D}{V}) \to \infty$ by property (c) and therefore the energy E is infinite. From these two last points, it can be seen that the plane-wave solution cannot exist for this class of potential. From now on, this is the only class that will be considered. To finish this section, an example for a one-dimensional case is given.

Simonov and Tjon [30] studied solvable models where they used that kind of potential. For a fractional potential such as

(1.411) $U = \mu^2 |\phi|^2 + \lambda |\phi|^{2-\alpha}$

with $\lambda > 0$ and $0 \geq \alpha > 1$. The time-independent wave equation for stationary states is

(1.412) $- \nabla^2\phi + \lambda\phi \frac{d}{d|\phi|^2} |\phi|^{2-\alpha} = -k^2\phi = (\omega^2 - \mu^2)\phi$

They worked out a solution in the same manner as has been done in the previous section:

(1.413) $(\frac{d\phi}{dx})^2 - (\lambda |\phi|^{-\alpha} + k^2) |\phi|^2 = 0$

which has for solution

$$(1.414)\quad \phi = \left(\frac{\lambda}{k^2}\right)^{\frac{1}{2k}} \left(\cos\frac{\alpha k x}{2}\right)^{\frac{2}{k}} \qquad \text{for } |x| < \frac{\pi}{\alpha k} = x_0$$

$$\phi = 0 \qquad \text{for } |x| > x_0$$

The effective potential in (1.413) can be written under the form

$$(1.415)\quad V = U(|\phi|^2) - \omega^2 \phi^2$$

and whose shape is shown on figure (1.4).

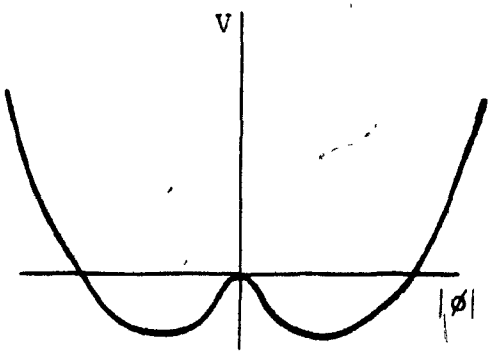

Fig. 1.4: Effective self-confining potential as a function of $|\phi|$

1-4 Localization Theorem

In this last section, a criterion will be introduced in order to be able to determine if a soliton solution will re-

main localized for all time starting from some imposed initial conditions. This is of the greatest importance to have such a criterion to be sure that the solutions which are obtained behave like solitons.

Coleman [21] defined a dissipative solution to be such that

$$(1.42) \quad \lim_{t\to\infty} \sup_{\underline{x}} T_{00}(\underline{x},t) = 0$$

as mentioned earlier. T_{00} is the energy density ($\equiv$ H of section 1-1) and is positive definite ($T_{00} > 0$). So, a solution contradicting (1.42) was stated to be non-dissipative, and therefore, to be a soliton. However, in 1978, Morris [18] showed that there were some particular distributions which violated the above condition. His work showed that contradiction of (1.42) was necessary but not sufficient to ensure the existence of solitons. That lead him to define a localized field as one that obeys the following conditions [18] :

A) $T_{00} \geq 0$

B) $E = \int T_{00} d^3x$ is finite

C) For some fixed $\delta > 0$, $T_{00} > \delta$ for all time throughout some set of finite volume $V > V_0 > 0$

It can be seen from these conditions that $V\delta$ is a lower bound

for the energy E and therefore $V_0 < V < \frac{E}{\delta}$ which is a necessary geometrical condition for an extended localized structure. These conditions are not too restrictive and they allow electromagnetic radiation as well as a flux of particles from the soliton .

Let's write equation (1.10) under another form

$$(1.43)\quad E = \int T^{00}\, d^3x = \int \Big\{ \tfrac{1}{4} F^{ij} F_{ij} + \tfrac{1}{2} E^i E^i + |\pi|^2 + \sum_j |D_j \phi|^2 + U(\phi^* \phi) \Big] d^3x$$

where use has been made of equations (1.5) and (1.6), and the fact

$$(1.44)\quad a^* a = |a|^2$$

Therefore, the energy density is defined as

$$(1.45)\quad T^{00} = \tfrac{1}{4} F^{ij} F_{ij} + \tfrac{1}{2} E^i E^i + |\pi|^2 + \sum_j |D_j \phi|^2 + U(\phi^* \phi)$$

This quantity is positive definite if U is a positive quantity. This is the case for the class of nonlinear potentials defined in the previous section because of condition (b). U and also T^{00} are zero only for the vacuum state.

Now, to prove the localization of the field with our particular class of potentials, one has to proceed in three

steps [22]:

1) A number of inequalities are proved

(1.46) $E \geq \frac{1}{2}\left[\int |\phi|^6 \, d^3x\right]^{1/3}$

(1.47) $a_2 > \int |\phi|^2 \, d^3x \equiv I(t) > a_1$

(1.48) $b_2 > \int |\phi|^4 \, d^3x \equiv B(t) > b_1$

where the positive constants a_1, a_2, b_1, b_2 depend on the initial conditions

2) It is shown that volume measure zero cannot contribute to integrals in (1.47) and (1.48).

3) The following inequality is established over some set of finite volume measure

(1.49) $\phi^*\phi > \frac{b_1}{a_2}$

Smooth enough initial conditions are assigned in order to have finite values for the constants of motion as well as for E and Q. The evolution of the field has to conform with these given bounds . From equation (1.43), one can establish the following inequalities :

(1.50) $E \geq \int \pi^* \pi \, d^3x$

(1.51) $E \geq \mu^2 \int \phi^* \phi \, d^3x$

where one of the conditions on U given in the last section has been used for the second relation. An expression for the

charge Q is obtained by working out equation (1.14) :

$$(1.52)\quad Q = i\int (\phi^*\pi^* - \phi\pi)d^3x$$

and therefore

$$(1.53)\quad Q^2 \leq 4\left|\int \phi\pi \, d^3x\right|^2 \leq 4\int|\phi|^2 d^3x \int|\pi|^2 d^3x$$

where the last inequality is just an application of the Schwarz inequality. Combining equations (1.50) and (1.53), one gets

$$(1.54)\quad I(t) = \int|\phi|^2 \, d^3x \geq \frac{Q^2}{4E} > a_1$$

and from (1.51), the upper bound is found to be

$$(1.55)\quad a_2 > \frac{E}{\mu^2} \geq I(t)$$

These two last equations give the relation (1.47). The two other relations can be proved by using a generalization of Sobolev's inequality which can be found in an appendix of a paper by Morris [23]. This inequality is the following :

$$(1.56)\quad \int (D_j\phi)^* \, (D_j\phi) \, d^3x \geq \tfrac{1}{4}\left[\int|\phi|^6 \, d^3x\right]^{1/3}$$

and, as the energy E is greater or equal to the left-hand side of the previous inequality as can be seen in equation (1.43), the proof of the first of the relations (equ. 1.46) is completed. For relation (1.48), one first uses the Schwarz

inequality

$$(1.57)\quad \int |\phi|^4 \, d^3x = \int |\phi|^3 |\phi| \, d^3x \leq \left[\int |\phi|^6 \, d^3x \right]^{\frac{1}{2}} \left[\int |\phi|^2 \, d^3x \right]^{\frac{1}{2}}$$

which, combined with (1.46) and (1.51), gives

$$(1.58)\quad \left(\frac{4}{3} \right)^{\frac{3}{2}} \frac{E^2}{\mu} \geq \int |\phi|^4 \, d^3x \equiv B(t)$$

Therefore, the constant b_2 has to be chosen such that

$$(1.59)\quad b_2 > \left(\frac{4}{3} \right)^{\frac{3}{2}} \frac{E^2}{\mu}$$

To prove the second part of equation (1.48), (1.43) is used and this yields

$$(1.60)\quad E \geq \int \pi^* \pi \, d^3x + \int U(\phi^*\phi) \, d^3x$$

Using (1.50) to eliminate the first term from the right-hand side of the previous equation, one gets

$$(1.61)\quad E \geq \left[\tfrac{1}{2} Q^2 + I \int U(\phi^*\phi) \, d^3x \right] / \, I$$

where I is defined in relation (1.47). Furthermore, as $(a-b)^2 \geq 0$, one has $a^2+b^2 \geq 2ab$. Applying this trivial inequality to the bracket in (1.61), the following relation is obtained :

$$(1.62)\quad \tfrac{1}{2} Q^2 + I \int U \, d^3x \geq Q \left[I \int U \, d^3x \right]^{\frac{1}{2}}$$

$$(1.63)\quad E \geq Q \left[\int U \, d^3x \, / \, I \right]^{\frac{1}{2}}$$

U is now replaced by its lower bound u which, remembering conditions (a) and (b) stated in the previous section, satisfies because of its concavity [24] :

$$(1.64)\quad u(\phi^*\phi) \geq u'(\phi^*\phi)\phi^*\phi$$

With (1.63) and (1.64) follows a new inequality

$$(1.65)\quad \frac{E^2}{Q^2} \geq \frac{1}{I}\int u'(\phi^*\phi)\phi^*\phi \; d^3x$$

As u' is a convex function by property (d) of last section, one can use the Jensen's integral theorem on convex functions [25] :

> If $p(\underline{x})$ and $q(\underline{x})$ are two functions on a segment $[a,b]$ such that $\gamma \leq p(\underline{x}) \leq \beta$, $q(\underline{x}) \geq 0$, and $\Psi(u)$ is a convex function defined on $[\gamma,\beta]$, then
>
> $$(1.66)\quad \frac{\int_a^b \Psi(p)\, q(\underline{x})\, d^3x}{\int_a^b q(\underline{x})\, d^3x} \geq \Psi\left(\frac{\int_a^b p(\underline{x})\, q(\underline{x})\, d^3x}{\int_a^b q(\underline{x})\, d^3x}\right)$$

Putting $\Psi = u'$ and $p = q = \phi^*\phi$, one obtains

$$(1.67)\quad \frac{1}{I}\int u'(\phi^*\phi)\phi^*\phi \; d^3x \geq u'\left(\frac{\int |\phi|^4 \, d^3x}{\int |\phi|^2 \, d^3x}\right)$$

Now, using (1.47), (1.48) and (1.65), one gets

$$(1.68)\quad \frac{E^2}{Q^2} \geq u'\left(\frac{B(t)}{I(t)}\right)$$

As E is finite and Q is non-zero, the argument of u' cannot go to zero because $u'(0)=\infty$. Hence, B/I is bounded away from zero and, as $I(t) > a_1$, B(t) is itself bounded by some constant, b_1. That completes the proof for this first part.

It must now be proven that sets of volume zero cannot contribute to the integrals (1.47) and (1.48) because it would imply that the $|\phi|^2$ and $|\phi|^4$ distributions could become singular in the limit of infinite time. Noting the fact that $(|\phi|^4)^{\frac{3}{2}}$ is a convex function, one can use Jensen's inequality with $\Psi(u) = u^{\frac{3}{2}}$, $p = |\phi|^4$, $q = 1$ inside a volume V and $q = 0$ elsewhere to find

$$(1.69)\quad \frac{1}{V}\int_V (|\phi|^4)^{\frac{3}{2}}\, d^3x > \frac{\left[\int_V |\phi|^4\, d^3x\right]^{3/2}}{V^{\frac{3}{2}}}$$

and hence

$$(1.70)\quad \int_V |\phi|^6\, d^3x \geq \frac{\left[\int_V \phi^4 d^3x\right]^{\frac{3}{2}}}{V^{\frac{3}{2}}} \geq \frac{b_1^{\frac{3}{2}}}{V^{\frac{3}{2}}}$$

So, as the volume shrinks to zero, $\int |\phi|^6 \, d^3x$ goes to infinity which is a contradiction with equation (1.46). Therefore, the $|\phi|^4$ distribution cannot collapse. It can be proven that this is also the case for $|\phi|^2$ using similar arguments. These results can also be obtained from Hölder's inequality :

$$\left|\int fg \, d^3x\right| \leq \left[\int |f|^p \, d^3x\right]^{p^{-1}} \left[\int |g|^q \, d^3x\right]^{q^{-1}}$$

$$\text{with} \quad p^{-1} + q^{-1} = 1$$

However, this inequality can be derived from Jensen's inequality by the use of the convexity property. If ϕ converges in a nonuniform manner to some discontinuous limit function, the above results show that the discontinuity must be mild enough to be excluded from the domain of integration [26].

To prove the last condition for localization, one can use the following identity

$$(1.71) \quad \int |\phi|^4 \, d^3x = \left(\frac{B(t)}{I(t)}\right) \int |\phi|^2 \, d^3x$$

in combination with inequalities of part (1) which yields

$$(1.72) \quad \int |\phi|^2 \left(|\phi|^2 - \frac{b_1}{a_2}\right) d^3x > 0$$

This implies that $|\phi|^2 > b_1/a_2$ for a non-zero volume. Therefore, $T^{00} > \mu^2 |\phi|^2 > \frac{\mu^2 b_1}{a_2}$ and the last condition for localiza-

tion, condition 3, is satisfied.

Now that it has been verified that the class of nonlinear potentials studied in section three satisfy the localization theorem and give rise to solitons, we can move to the next chapter where a perturbation will be introduced in the nonlinear Klein-Gordon equation given at the beginning of this chapter.

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Chapter 2

Perturbation of a Charged Scalar Soliton

In this chapter, a small time-dependent perturbation will be introduced in the Klein-Gordon equation (eq. 1.4) and the basic formulae will be developed. Then, different conservation laws will be verified for the zeroth and first order of the perturbation. First, the continuity equation will be studied. Then, in integral form, charge conservation will lead to a condition on the perturbed wave-function. Finally, it will be shown that the divergence of the energy-momentum tensor is zero.

2-1 Basic Formulae

Starting with equation (1.4), expanding the different terms and separating the time components from the space components, one obtains

$$\ddot{\phi} - \nabla^2\phi + i\varepsilon(2A_0\dot{\phi} + \dot{A}_0\phi) + i\varepsilon(2\underline{A}\cdot\underline{\nabla}\phi + \phi\,\underline{\nabla}\cdot\underline{A}) + \varepsilon^2(|\underline{A}|^2 - A_0^2)\phi + U'(\phi^*\phi)\phi = 0 \tag{2.1}$$

together with a similar equation for $\dot{\phi}^*$. In the limit $\varepsilon \to 0$, the ε^2 term

could be neglected in eq(2.1) as well as in the following calculations. If the existence of stationary states is supposed, then

$$(2.2)\quad \Phi = \phi\, e^{-i\omega t}$$

and the substitution is made in the first equation, which leads to

$$(2.3)\quad \ddot{\phi} - 2i\omega\dot{\phi} - \omega^2\phi - \nabla^2\phi + i\varepsilon(2 A_0\dot{\phi} + \dot{A}_0\phi + 2 \underline{A}\cdot\underline{\nabla}\phi + \phi\,\underline{\nabla}\cdot\underline{A}) + 2\varepsilon\omega A_0\phi + U'(\phi^*\phi)\phi + \varepsilon^2 (|\underline{A}|^2 - A_0^2)\phi = 0$$

If the supposition is made that ϕ is a perturbed state, i.e. $\phi=\phi(t)$, it can be written

$$(2.4)\quad \phi(t) = \phi_0 + \eta(t)$$

where ϕ_0 is time-independent and real. It is the unpertürbed state of equation (2.3) and $\eta(t)$ is the perturbation that is considered as being small in comparison with ϕ_0.

Without any perturbation, the system is spherically symmetric and time-independent. Therefore, $\underline{A}$ and $\dot{A}_0$ can be set to zero. Here, A_0 is the sum of the self-field of the soliton and the external field and will now be denoted by $\bar{A}_0$. This gives, using eq.(1.4),

$$(2.5)\quad (\partial_0 + i\varepsilon\bar{A}_0)(\partial^0 + i\varepsilon\bar{A}^0)(\phi_0 e^{-i\omega t}) + U'(\phi_0^2)\phi_0\, e^{-i\omega t} = 0$$

which leads to

$$(2.6)\quad [- \omega^2 - \nabla^2 + 2\varepsilon\omega\bar{A}^0 - \varepsilon^2\bar{A}_0^2 + U'(\phi_0^2)]\ \phi_0 = 0$$

If one now considers the presence of a time-dependent perturbation, the vector potential term has to be put into the equations to take into account the possible currents. Furthermore, A_0 is now composed of three terms : the two previous terms contained in $\bar{A}_0$ and the field a_0 arising from the perturbation $\eta(t)$. This last term is the only one of the three which is time-dependent. In the notation used here, the state ϕ_0 is left unaffected by the perturbation and is described by eq.(2.6). To determine the equation describing the perturbation, eq.(2.4) is used in eq.(2.3) and the result is simplified by the use of eq.(2.6) and the Lorentz condition,

$$(2.7) \quad \partial_\alpha A^\alpha = 0$$

This yields, to first order in η,

$$(2.8) \quad \ddot{\eta} - 2i(\omega - \varepsilon \bar{A}_0)\dot{\eta} - \left[\omega^2 + \nabla^2 - 2\varepsilon\omega\bar{A}_0 + \varepsilon^2\bar{A}_0^2 - U'(\phi^2)\right]\eta + U''(\phi^2)\phi^2(\eta + \eta^*) + 2\varepsilon(\omega - \varepsilon\bar{A}_0)a_0\phi_0 + 2i\varepsilon\,\underline{a}\cdot\underline{\nabla}\phi_0 = 0$$

Here, it should be remarked that the perturbed four-potential a_μ is considered to be of first order in η, and that terms in $a_\mu\eta$ have therefore been dropped. The complex conjugate of eq.(2.8) is

$$(2.9) \quad \ddot{\eta}^* + 2i(\omega - \varepsilon\bar{A}_0)\dot{\eta}^* - \left[\omega^2 + \nabla^2 - 2\varepsilon\omega\bar{A}_0 + \varepsilon^2\bar{A}_0^2 - U'(\phi_0^2)\right]\eta^* + U''(\phi_0^2)\phi_0^2(\eta + \eta^*) + 2\varepsilon(\omega - \varepsilon\bar{A}_0)a_0\phi_0 - 2i\,\underline{a}\cdot\underline{\nabla}\phi_0 = 0$$

With similar equations but without the presence of the electromagnetic field, Loo [1] has shown that η was bounded by some maximum value η_{max} by proving that, if

$$(2.10) \quad \eta(\underline{x},t) = \psi_1(\underline{x})\, e^{i\Omega t} + \psi^*_2(\underline{x})\, e^{-i\Omega^* t} \quad ,$$

Ω was real and therefore that η was oscillating between two maxima. In the case studied here, it is not that simple because η could decay if the system emits radiation. Then, Ω would not be real. Thus, it will be assumed that η is bounded by a maximum value and that it can only decreases in time if the system is isolated. If one wants to use eq.(2.10) for η , the above assumption will mean approximately that

$$(2.11) \quad \Omega = \xi + i\gamma \qquad \xi \text{ and } \gamma : \text{real constants}, \ \gamma \geq 0$$

Furthermore, if η is a bound state as it is the case in chapter three, there should be a large number of oscillations before the system decays which is equivalent to the requirement $\gamma \ll \xi$.

Now that the wave equations have been found for η and η^*, the same will be done for the electromagnetic fields :

$$(2.12) \quad \partial_\mu \partial^\mu A_\mu = J_\mu$$

where J_μ is given by eq.(1.13) plus a term J^e_0 due to the ex-

ternal charge which is supposed fixed ($\underline{J}^e = 0$). Introducing eq.(2.2) and (2.4) in eq.(1.13) yields for J_0

$$(2.13)\quad J_0 = 2\varepsilon(\omega - \varepsilon\bar{A}_0)\,\phi_0^2 + 2\varepsilon(\omega - \varepsilon\bar{A}_0)\phi_0(\eta + \eta^*) + i\varepsilon\phi_0(\dot{\eta} - \dot{\eta}^*) - 2\varepsilon^2 a_0\phi_0(\eta + \eta^*) + J_0^e + \mathcal{O}(\eta^2)$$

and for the current $\underline{J}$,

$$(2.14)\quad \underline{J} = -i\varepsilon[\phi_0\underline{\nabla}(\eta - \eta^*) - (\eta - \eta^*)\underline{\nabla}\phi_0] - 2\varepsilon^2\underline{a}\phi_0 + \mathcal{O}(\eta^2)$$

These terms can be separated in zeroth and first order in η:

$$(2.15)\quad \bar{J}_0 = 2\varepsilon(\omega - \varepsilon\bar{A}_0)\phi_0^2 + J_0^e$$

$$(2.16)\quad j_0 = 2\varepsilon(\omega - \varepsilon\bar{A}_0)\phi_0(\eta + \eta^*) + i\varepsilon\phi_0(\dot{\eta} - \dot{\eta}^*) - 2\varepsilon^2 a_0\phi_0^2$$

$$(2.17)\quad \underline{\bar{J}} = 0$$

$$(2.18)\quad \underline{j} = -i\varepsilon[\phi_0\underline{\nabla}(\eta - \eta^*) - (\eta - \eta^*)\underline{\nabla}\phi_0] - 2\varepsilon^2\underline{a}\phi_0^2$$

These currents can be introduced in eq.(2.12) to give the following differential equations :

$$(2.19)\quad -\nabla^2\bar{A}_0 = 2\varepsilon(\omega - \varepsilon\bar{A}_0)\phi_0^2 + J_0^e$$

$$(2.20)\quad \ddot{a}_0 - \nabla^2 a_0 = 2\varepsilon(\omega - \varepsilon\bar{A}_0)\phi_0(\eta + \eta^*) + i\varepsilon\phi_0(\dot{\eta} - \dot{\eta}^*) - 2\varepsilon^2 a_0\phi_0^2$$

$$(2.21)\quad \ddot{\underline{a}} - \nabla^2\underline{a} = -i\varepsilon[\phi_0\underline{\nabla}(\eta - \eta^*) - (\eta - \eta^*)\underline{\nabla}\phi_0] - 2\varepsilon^2\underline{a}\phi_0^2$$

In summary, the complete system of equations describing a perturbed soliton in the presence of an external electrostatic field is composed of the equations (2.6), (2.8), (2.9), and (2.19) to (2.21).

2-2 Continuity Equation and Charge Conservation

In the preceding section, a set of differential equations has been derived to describe the system in zeroth and first order of perturbation. In this section, it will be proved that the equation of continuity is respected at each of these orders and that the first order contribution of the perturbation of the charge is zero. The equation of continuity, which is included in Maxwell's equations, is written

$$\frac{d}{dt} J_0 + \underline{\nabla}\cdot\underline{J} = 0 \qquad (2.22)$$

and must be true to all orders of the perturbation. At zeroth order, it is obvious that eq.(2.22) holds as $\bar{J}_0$ is time-independent and $\bar{\underline{J}}$ is zero. For the first order, one has first to calculate

$$\frac{d}{dt} j_0 = 2\varepsilon(\omega - \varepsilon\bar{A}_0)\phi_0(\dot{\eta} + \dot{\eta}^*) + i\varepsilon\phi_0(\ddot{\eta} - \ddot{\eta}^*) - 2\varepsilon^2\dot{a}_0\phi_0^2 \qquad (2.23)$$

and

$$\nabla\cdot\underline{j} = -i\varepsilon\left[\phi_0\nabla^2(\eta-\eta^*) - (\eta-\eta^*)\nabla^2\phi_0\right] - 2\varepsilon^2\phi_0\nabla\cdot\underline{a} - 4\varepsilon^2\phi_0\underline{a}\cdot\nabla\phi_0 \tag{2.24}$$

Adding these two equations, one gets

$$\frac{d}{dt}j_0 + \nabla\cdot\underline{j} = i\varepsilon\phi_0(\ddot{\eta}-\ddot{\eta}^*) + 2\varepsilon(\omega-\varepsilon\bar{A}_0)\phi_0(\dot{\eta}+\dot{\eta}^*) - i\varepsilon\phi_0\nabla^2(\eta-\eta^*) - 4\varepsilon^2\phi_0\underline{a}\cdot\nabla\phi_0 + i\varepsilon(\eta-\eta^*)\nabla^2\phi_0 \tag{2.25}$$

where the Lorentz condition has been used. To prove that these terms cancel, one has to substract eq.(2.9) from eq.(2.8), multiply the result by $i\varepsilon\phi_0$ and make the substitution for the first four terms in eq.(2.25). The first two steps give

$$i\varepsilon\phi_0\left[(\ddot{\eta}-\ddot{\eta}^*) - 2i(\omega-\varepsilon\bar{A}_0)(\dot{\eta}+\dot{\eta}^*) - \nabla^2(\eta-\eta^*) - \omega^2(\eta-\eta^*) + 2\varepsilon\omega\bar{A}_0(\eta-\eta^*) + 4i\varepsilon\underline{a}\cdot\nabla\phi_0 - \varepsilon^2\bar{A}_0^2(\eta-\eta^*) + U'(\phi_0^2)(\eta-\eta^*)\right] = 0 \tag{2.26}$$

Making the substitution gives

$$\frac{d}{dt}j_0 + \nabla\cdot\underline{j} = i\varepsilon\left\{\left[\omega^2 - 2\varepsilon\omega\bar{A}_0 + \varepsilon^2\bar{A}_0^2 - U'(\phi_0^2) + \nabla^2\right]\phi_0\right\}(\eta-\eta^*) = 0 \tag{2.27}$$

because of eq.(2.6). This completes the proof that the equa-

tion of continuity holds for the first two orders. Now, it will be showed that the integral of j_0 over all space does not add anything to the total charge q of the unperturbed system. To do so, one integrates eq.(2.26) over all space and uses Green's theorem,

$$(2.28)\quad \int \phi_0 \nabla^2 (\eta - \eta^*)\, d^3x = \int (\eta - \eta^*) \nabla^2 \phi_0\, d^3x$$

plus a vanishing surface integral. This leads to

$$(2.29)\quad i\varepsilon \int \{(\eta - \eta^*)\,[-\omega^2 - \nabla^2 + 2\varepsilon\omega \bar{A}_0 - \varepsilon^2 \bar{A}_0^2 + U'(\phi_0^2)]\,\phi_0 + [(\ddot{\eta} - \ddot{\eta}^*) - 2i(\omega - \varepsilon \bar{A}_0)(\dot{\eta} + \dot{\eta}^*) + 4i\varepsilon\phi_0 \underline{a}\cdot\underline{\nabla}]\,\phi_0\}\, d^3x = 0$$

Again with the help of eq.(2.6) to get rid of the first bracket and the fact that

$$(2.30)\quad \int 4\phi_0 \underline{a}\cdot\underline{\nabla}\phi_0 d^3x = 2\int \underline{a}\cdot\underline{\nabla}\phi_0^2\, d^3x = -2\int \phi_0^2 \underline{\nabla}\cdot\underline{a}\, d^3x = 2\int \dot{a}_0 \phi^2\, d^3x$$

one ends up with

$$(2.31)\quad \frac{d}{dt}\int [i\varepsilon(\dot{\eta} - \dot{\eta}^*)\phi_0 + 2\varepsilon\phi_0(\omega - \varepsilon \bar{A}_0)(\eta + \eta^*) - 2\varepsilon^2 a_0 \phi_0^2]\, d^3x = 0$$

or, using eq.(2.16),

$$\frac{d}{dt}\int j_0\, d^3x = 0$$

This implies that this integral is a constant with respect to time. Before the perturbation happened, this integral was zero and hence is always zero :

$$(2.32) \quad \int j_0 \, d^3x = 0$$

This could have also been proven simply by integrating eq. (2.27) :

$$(2.33) \quad \frac{d}{dt}\int j_0 \, d^3x = -\int \underline{\nabla}\cdot\underline{j} \, d^3x = \int_S \underline{j}\cdot d\underline{s} = 0$$

It should be remarked before going on to the next section that eq. (2.32) imposes a restriction on the perturbation (η, η^*, a_0) that must be respected at all time.

2-3 Conservation of Energy

Finally, in this last section, it will be shown that the energy-momentum tensor respects the conservation law :

$$(2.34) \quad \partial_\nu T^{\mu\nu} = 0$$

More specifically, it will be proven to first two orders

in η that

$$(2.35)\quad \partial_0 T^{00} + \partial_k T^{0k} = 0$$

where T^{00} is given by the expression for H (= T^{00}) in eq. (1.10).

$$(2.36)\quad T^{00} = \tfrac{1}{4}(\partial^i A^j - \partial^j A^i)(\partial_i A_j - \partial_j A_i) + \tfrac{1}{2}(\partial^i A^0 - \partial^0 A^i) (\partial^i A^0 - \partial^0 A^i) + \dot{\phi}^*\dot{\phi} + i\varepsilon A_0(\phi\dot{\phi}^* - \dot{\phi}\phi^*) + \underline{\nabla}\phi\cdot\underline{\nabla}\phi^* + i\varepsilon\underline{A}\cdot(\phi\underline{\nabla}\phi^* - \phi^*\underline{\nabla}\phi) + \varepsilon^2 A_0^2\phi^*\phi + U(\phi^*\phi)$$

However, expression (1.12) for the momentum $\underline{P}$ will not be used for T^{0k} because it is not symmetric. Instead, the expression given by Wentzel [2] for the scalar part and the one by Barut [3] for the electromagnetic part will be used.

$$(2.37)\quad T^{0k} = (\partial^j A^0 - \partial^0 A^j)(\partial^k A_j - \partial_j A^k) + \dot{\phi}\partial^k\phi^* + \dot{\phi}^*\partial^k\phi - i\varepsilon A_0(\phi^*\partial^k\phi - \phi\partial^k\phi^*) - i\varepsilon A^k(\dot{\phi}\phi^* - \dot{\phi}^*\phi) + 2\varepsilon^2 A^0 A^k\phi^*\phi$$

Now, using eq.(2.2) and eq.(2.4) for ϕ, separating A_μ in its different components and keeping only the terms of zeroth and first order in η, the two last equations become

$$(2.38)\quad T^{0k} = \partial^j\bar{A}_0(\partial^k a_j - \partial_j a^k) + (\dot{\eta} + \dot{\eta}^*)\partial^k\phi_0 - 2\varepsilon(\omega - \varepsilon\bar{A}_0)a^k\phi_0^2 + i(\omega - \varepsilon\bar{A}_0)[(\eta^* - \eta)\partial^k\phi_0 - \phi_0\partial^k(\eta^* - \eta)]$$

and

$$(2.39)\quad T^{00} = \tfrac{1}{2}\,|\partial^i A_0| + \partial^i \bar{A}_0\,(\partial^i a^0 - \dot{a}^i) + \omega^2 \phi_0(\eta+\eta^*) + \omega^2\phi_0^2 + |\nabla\phi_0|^2 + i(\omega - \varepsilon\bar{A}_0)\phi_0(\dot{\eta}-\dot{\eta}^*) - 2\varepsilon\omega\bar{A}_0\phi_0^2 - 2\varepsilon\omega a_0\phi_0^2 - 2\varepsilon\omega\bar{A}_0\phi_0(\eta+\eta^*) + \underline{\nabla}\phi_0\cdot\underline{\nabla}(\eta+\eta^*) + \varepsilon^2\bar{A}_0^2\phi_0^2 + 2\varepsilon^2\bar{A}_0 a_0\phi_0^2 + \varepsilon^2\bar{A}_0^2\phi_0(\eta+\eta^*) + U(\phi_0^2) + U'(\phi_0^2)\phi_0(\eta+\eta^*)$$

Putting these two last expressions in eq.(2.35) yields for zeroth order

$$(2.40)\quad \partial_0 T^{00} + \partial_k T^{0k} = \partial_0\left[\tfrac{1}{2}\,|\partial^i A_0|^2 + (\omega - \varepsilon\bar{A}_0)^2\phi_0^2 + |\underline{\nabla}\phi_0|^2 + U(\phi_0^2)\right] + \partial_k(0)$$

which clearly gives

$$(2.41)\quad \partial_0 T^{00}_{(0)} + \partial_k T^{0k}_{(0)} = 0$$

because all the terms are time-independent. At first order, the proof is more involved. It is easy to obtain

$$(2.42)\quad \partial_\nu T^{0\nu}_{(1)} = \{-\ddot{a}^i\partial^i\bar{A}_0 + \partial^i\bar{A}_0\partial^i a_0 + [\omega^2 - 2\varepsilon\omega\bar{A}_0 + \varepsilon^2\bar{A}_0^2 + U'(\phi_0)]\phi_0(\dot{\eta}+\dot{\eta}^*) - i(\omega-\varepsilon\bar{A}_0)\phi_0(\ddot{\eta}-\dot{\eta}^*) - 2\varepsilon\omega\dot{a}_0\phi_0^2 + \underline{\nabla}\phi_0\cdot\underline{\nabla}(\dot{\eta}+\dot{\eta}^*) + 2\varepsilon^2\bar{A}_0\dot{a}_0\phi_0^2\} + \{\partial^j\bar{A}_0(-\nabla^2 a_j - \partial_j\underline{\nabla}\cdot\underline{a}) + \partial^j(\partial_k\bar{A}_0)(\partial^k a_j - \partial_j a^k) - (\dot{\eta}+\dot{\eta}^*)\nabla^2\phi_0 - \underline{\nabla}(\dot{\eta}+\dot{\eta}^*)\cdot\underline{\nabla}\phi_0 - 2\varepsilon(\omega-\varepsilon\bar{A}_0)\underline{a}\cdot\underline{\nabla}\phi^2 - i(\omega-\varepsilon\bar{A}_0)[(\eta^*-\eta)\nabla^2\phi_0 - \phi_0\nabla^2(\eta^*-\eta)] + i\varepsilon(\underline{\nabla}\bar{A}_0)\cdot[(\eta^*-\eta)\underline{\nabla}\phi_0 - \phi_0\underline{\nabla}(\eta^*-\eta)] - 2\varepsilon(\omega-\varepsilon\bar{A}_0)\phi_0^2\underline{\nabla}\cdot\underline{a} + 2\varepsilon^2\phi_0^2\underline{a}\cdot\underline{\nabla}\bar{A}_0\}$$

Obviously, the second term of the second bracket is identically zero because they are antisymmetric in j and k and that these two indices run over the same values. It can be seen that the terms in $\dot{a}_o$ and $\underline{\nabla}\cdot\underline{a}$ will add up to zero by means of the Lorentz condition. Then, eq.(2.42) can be simplified to

$$
\begin{aligned}
(2.43)\quad \partial_\nu T^{0\nu}_{(1)} = {} & (\dot{\eta}+\dot{\eta}^*)(\omega^2 - 2\varepsilon\omega\bar{A}_0 + \varepsilon^2\bar{A}_0^2 + U'(\phi_0^2) - \nabla^2)\phi_0 \\
& -i(\omega-\varepsilon\bar{A}_0)(\ddot{\eta}^*-\ddot{\eta})\phi_0 - i(\omega-\varepsilon\bar{A}_0) \\
& [(\eta^*-\eta)\nabla^2\phi_0 - \phi_0\nabla^2(\eta^*-\eta)] - 2\varepsilon(\omega-\varepsilon\bar{A}_0)\underline{a}\cdot\underline{\nabla}\phi_0^2 \\
& + i\varepsilon(\underline{\nabla}\bar{A}_0)\cdot[(\eta^*-\eta)\underline{\nabla}\phi_0 - \phi_0\underline{\nabla}(\eta^*-\eta)] \\
& + 2\varepsilon^2\phi_0^2\underline{a}\cdot\underline{\nabla}\bar{A}_0 + (\ddot{\underline{a}} + \underline{\nabla}\dot{a}_0)\cdot(\underline{\nabla}\bar{A}_0) \\
& + (\underline{\nabla}\bar{A}_0)\cdot(-\nabla^2\underline{a} + \underline{\nabla}\underline{\nabla}\cdot\underline{a})
\end{aligned}
$$

To solve this, one can replace $(-\nabla^2\phi_0 + U'(\phi_0^2))$ by using eq.(2.6) and can use equations (2.18) and (2.24) to replace the third and fourth terms. After some manipulations, this yields

$$
\begin{aligned}
(2.44)\quad \partial_\nu T^{0\nu}_{(1)} = {} & 2(\dot{\eta}+\dot{\eta}^*)(\omega-\varepsilon\bar{A}_0)^2\phi_0 - i(\omega-\varepsilon\bar{A}_0)\phi_0(\ddot{\eta}^*-\ddot{\eta}) \\
& +\varepsilon^{-1}(\omega-\varepsilon\bar{A}_0)\underline{\nabla}\cdot\underline{j} + 2\varepsilon(\omega-\varepsilon\bar{A}_0)\underline{a}\cdot\underline{\nabla}\phi_0^2 \\
& - 2\varepsilon(\omega-\varepsilon\bar{A}_0)\dot{a}_0\phi_0^2 - (\underline{\nabla}\bar{A}_0)\cdot\underline{j} - 2\varepsilon^2\phi_0^2\underline{a}\cdot\underline{\nabla}\bar{A}_0 \\
& - 2\varepsilon(\omega-\varepsilon\bar{A}_0)\dot{\underline{a}}\cdot\underline{\nabla}\phi_0^2 + 2\varepsilon^2\phi_0^2\underline{a}\cdot\underline{\nabla}\bar{A}_0 + (\ddot{\underline{a}} + \underline{\nabla}\dot{a}_0)\cdot \\
& (\underline{\nabla}A_0) + (\underline{\nabla}A_0)\cdot(-\underline{\nabla}\cdot\underline{\nabla}\underline{a} + \underline{\nabla}\underline{\nabla}\cdot\underline{a})
\end{aligned}
$$

which simplifies to

$$(2.44) \quad \partial_\nu T^{0\nu}_{(1)} = 2(\dot\eta + \dot\eta^*)(\omega - \varepsilon\bar A_0)^2\phi_0 + \varepsilon^{-1}(\omega - \varepsilon\bar A_0)\nabla\cdot\underline{j} + (\omega - \varepsilon\bar A_0)\left[i\phi_0(\ddot\eta - \ddot\eta^*) - 2\varepsilon\dot a_0\phi_0^2 \right] - (\nabla\bar A_0)\cdot\underline{j} + (\nabla\bar A_0)\cdot(-\underline{\dot e} + \nabla\times\underline{b})$$

The last term comes from the relations

$$(2.45) \quad -\underline{e} = \dot{\underline{a}} + \nabla a_0$$

and

$$(2.46) \quad \nabla\times\underline{b} = \nabla\times\nabla\times\underline{a} = \nabla\nabla\cdot\underline{a} - \nabla\cdot\nabla\underline{a}$$

To simplify eq.(2.44) further, one can use eq.(2.23) to replace the third term and get after some simplifications

$$(2.47) \quad \partial_\nu T^{0\nu} = \varepsilon^{-1}(\omega - \varepsilon\bar A_0)\left(\nabla\cdot\underline{j} + \frac{d}{dt} j_0\right) - (\nabla\bar A_0)\cdot(\underline{j} + \underline{\dot e} - \nabla\times\underline{b})$$

The first term is zero because of eq.(2.27) and eq.(2.47) becomes

$$(2.48) \quad \partial_\nu T^{0\nu} = -(\nabla\bar A_0)\cdot(\underline{j} + \underline{\dot e} - \nabla\times\underline{b})$$

which is equal to zero because the second parenthesis is just one of the Maxwell's equations :

$$(2.49) \quad \nabla\times\underline{B} = \underline{J} + \underline{\dot E}$$

Hence, it has been proven that the system respects the principal conservation laws. That will allow us to go on a new approximation level by taking the nonrelativistic limit of eq.(2.8) and to study the hydrogen atom. But before, it should be noticed that the momentum would not be conserved if a Coulomb-like potential was used to represent the external potential.

1- D. Loo, " Small Oscillation Dynamics of Special Models of Charged Scalar Solitons ", Master's thesis pres. at McGill Univ., Sept. 1982, pp. 50-59

2- G. Wentzel, " Quantum Theory of Fields ", Interscience Publ. Inc., NY, 1949, p. 68

3- A.O. Barut, " Electrodynamics and Classical Theory of Fields and Particles ", Dover, NY, 1980, p. 118

Chapter 3

Nonrelativistic Limit of the Nonlinear Klein-Gordon Equation and Spontaneous Emission

In this chapter, the nonrelativistic limit of the nonlinear Klein-Gordon equation in η (eq.(2.8)) will be derived and compared with the linear case. Then, the special example of hydrogen will be taken for the rest of the chapter. Using a specific nonlinear potential and a trial function for ϕ_0, one is able to derive an integral equation describing the atom in the high charge density region. Finally, an expression for the Einstein's coefficient for spontaneous emission is worked out, the atom radiating because of the time-dependent terms.

3-1 Nonrelativistic Limit of the Nonlinear Klein-Gordon Equation

It would be good to first rewrite the nonlinear Klein-Gordon equation for the perturbed η, eq.(2.8), keeping the

terms in ε^2 for completeness.

$$(3.1)\quad \ddot{\eta} - 2i(\omega - \varepsilon\bar{A}_0)\dot{\eta} - [\omega^2 + \nabla^2 - 2\varepsilon\omega\bar{A}_0 + \varepsilon^2\bar{A}_0^2 - U'(\phi_0^2)]\eta + U''(\phi_0^2)\phi_0^2(\eta + \eta^*) + 2\varepsilon(\omega - \varepsilon\bar{A}_0)a_0\phi_0 + 2i\varepsilon\underline{a}\cdot\underline{\nabla}\phi_0 = 0$$

The usual way [1] to reach the nonrelativistic limit of an equation such as eq.(3.1) is to discard $\ddot{\eta}$ as negligible because it is not multiplied by a factor ω or ω^2. To make sure that this method can be applied, the order of magnitude of Ω will be found ($\eta = \psi e^{-i\Omega t}$). For this, only the following terms are kept, because of their superior order of magnitude,

$$(3.2)\quad \ddot{\eta} - 2i\omega\dot{\eta} - (\omega^2 - \mu^2)\eta = 0$$

where the μ^2 term is the mass term included in U'[See appendix 2]. The frequency ω has the following form for a soliton defined with a confining potential as in section 1-3 :

$$(3.3)\quad \omega = \mu + \delta \qquad \text{where} \qquad 0 < \delta \ll \mu$$

Using eq.(3.3) in eq.(3.2), one gets a quadratic equation in Ω whose roots are

$$(3.301)\quad \Omega_1 = -\delta \quad \text{or} \quad \Omega_2 = -(2\mu + \delta)$$

Therefore, the rest of this derivation will be done

for the case $\Omega = -\delta$ which permits to use the usual procedure for the nonrelativistic limit. Neglecting $\varepsilon \bar{A}_0$ compared to ω in the term multiplying $\dot{\eta}$ and U" in front of U'[See appendix 2], equation (3.1) becomes :

$$(3.4)\quad i\dot{\eta} = \left[-\frac{\nabla^2}{2\omega} - \frac{\omega}{2} + \varepsilon\bar{A}_0 - \frac{\varepsilon^2\bar{A}_0^2}{2\omega} + \frac{U'(\phi_0^2)}{2\omega} \right] \eta + \varepsilon a_0 \phi_0 + \frac{i\varepsilon}{\omega}\, \underline{a}\cdot\nabla\phi_0$$

One can compare this equation with the linear Schrodinger equation for an electromagnetic potential which is [2] :

$$(3.5)\quad i\dot{\xi} = \left[-\frac{\nabla^2}{2\mu} + eA_0 - \frac{e^2A_0^2}{2\mu} + \frac{ie}{\mu}\,\underline{A}\cdot\underline{\nabla} + \frac{e^2}{2\mu}\,\underline{A}^2 \right]\xi$$

where the Lorentz gauge has been applied to the original equation instead of dropping the $\dot{A}_0$ term as it is done in the reference [2]. These two equations are quite similar but this will be more evident after some manipulations. First, The $\varepsilon^2\underline{A}^2$ term can be neglected in eq.(3.5). Second, the nonlinear potential U' can be written under the form [See app. 2]

$$(3.6)\quad U'(\phi_0^2) = \mu^2 + V'(\phi_0^2)$$

where $\quad V'(\phi_0^2) = ab(\phi_0^2)^{a-1} - \mu^2 \xrightarrow[|x|\ll R]{} 2\mu^2 \frac{|x|^2}{R^2}$

Therefore, adding the ω term in eq.(3.4) to this μ term

yields, using eq. (3.3),

$$(3.7) \quad -\frac{\omega}{2} + \frac{\mu^2}{2\omega} = -\delta$$

Now, eq. (3.4) reads

$$(3.8) \quad i\dot{\eta} = \left[-\frac{\nabla^2}{2\omega} - \delta + \varepsilon \bar{A}_0 - \frac{\varepsilon^2 \bar{A}_0^2}{2\omega} + V'(\phi_0^2)\right]\eta + \varepsilon a_0 \phi_0 + \frac{i\varepsilon}{\omega} \underline{a}\cdot\underline{\nabla}\phi_0$$

In the linear limit, i.e. δ going to zero as well as V', the two equations would look alike except for the two last terms of eq. (3.8). The first, $\varepsilon a_0 \phi_0$, represents the effect of the perturbed scalar potential on the unperturbed charge distribution and the second, $\underline{a}\cdot\underline{\nabla}\phi_0$, the effect of the vector potential on the unperturbed charge distribution. No such things exist in the linear theory as the electron is considered a point particle and, therefore, has no inner structure. The extra $\underline{A}\cdot\underline{\nabla}\xi$ term in eq. (3.5) can be considered negligible as it describes the effect of the vector potential, due to the movement of the electron, on the electron itself. This is obviously of second order and this term can be neglected as $\underline{a}\cdot\underline{\nabla}\eta$ was in equations (3.4) and (3.8).

3-2 Integral Equation for the Stationary States of the Hydrogen Atom

It is possible to find a formal solution of eq.(3.8) by using Green's functions. If the first bracket of the right-hand side of eq.(3.8) is considered as an unperturbed Hamiltonian, there will be a discrete spectrum of eigenstates.

$$(3.81)\quad H\eta_n = \left[\frac{-\nabla^2}{2\omega} - \delta + \mathcal{E}\bar{A}_0 - \frac{\mathcal{E}^2\bar{A}_\theta^2}{2\omega} + V'(\phi_0^2)\right]\eta_n = \Omega_n\eta_n$$

where Ω_n is the energy of the nth eigenstate and is a real quantity. If the function G is given by the equation

$$(3.82)\quad i\frac{\partial G}{\partial t} - HG = \delta^3(\underline{x}-\underline{x}')\,\delta(t-t')\quad ,$$

it can be written as

$$(3.83)\quad G(\underline{x}',t';\underline{x},t) = -i\theta(t'-t)\sum_n \psi_n(\underline{x}')\psi_n^*(\underline{x})\exp(-i\Omega_n(t'-t))$$

with

$$(3.84)\quad \eta_n(\underline{x},t) = \psi_n(x)\exp(-i\Omega_n t)$$

$$(3.85)\quad \theta(t'-t) = \begin{cases} 0 & t' < t \\ 1 & t' > t \end{cases}$$

The effect of the two other terms on the right-hand

side of eq.(3.8) will be to induce transitions between the eigenstates of H. Therefore, the wave-function describing the whole system is simply given by the following expression [3]:

$$(3.9)\quad \eta(\underline{x}',t') = \eta_i(\underline{x}',t') + \varepsilon \int G(\underline{x}',t';\underline{x},t)\left[(a_0(\underline{x},t) + \frac{i}{\omega}\,\underline{a}(\underline{x},t)\cdot\underline{\nabla}\right]_i \phi_0(\underline{x})\, d^3x\, dt$$

where $\eta_i(\underline{x},t)$ is the particular excited state in which the system is at the beginning and the index on the bracket means that the fields a_0 and $\underline{a}$ are dependent on η_i for the first order of approximation. The scalar potential term a_0 is the solution of

$$(3.10)\quad \ddot{a}_0 - \nabla^2 a_0 = j_0$$

where the charge j_0 is

$$(3.11)\quad j_0 = \varepsilon\left[2(\omega - \varepsilon A_0)(\eta+\eta^*) + i(\dot{\eta}-\dot{\eta}^*) - 2\varepsilon\phi_0 a_0\right]\phi_0$$

From this, it can be seen that a_0 depends upon η as given by eq.(3.9). It should therefore be calculated for each state η_n and the summation would have to be carried out. This summation would introduce an additional time-dependence other than $e^{-i\Omega t}$. But, in the order of approximation mentioned above, $a_0(\eta)$ will be replaced by $a_0(\eta_n)$ in eq.(3.9), which means that a_0 can be separated in the

following manner

$$(3.13)\quad a_0(n,\underline{x},t) = a_0'(\underline{x})\, e^{-i\Omega_n t} + a_0'^*(\underline{x})\, e^{i\Omega_n t}$$

The term j_0 can then be written under the simplified form

$$(3.14)\quad j_0(n,\underline{x},t) = \varepsilon\left[2(\omega - \varepsilon \bar{A}_0)\Psi + \Omega_n\Psi - 2\varepsilon\phi_0\, a_0' \right] \phi_0 e^{-i\Omega_n t} + c.c.$$

$$= j_0'\, e^{-i\Omega_n t} + c.c.$$

Hence, eq.(3.10) becomes, for each state η_n,

$$(3.15)\quad -\Omega^2 a_0' - \nabla^2 a_0' = j_0'$$

$$(3.16)\quad -\Omega^2 a_0'^* - \nabla^2 a_0'^* = j_0'^*$$

This looks like the inhomogeneous Helmholtz equation but it should be remarked that j_0 depends upon a_0. If one defines

$$(3.17)\quad g_n(\underline{x}) = j_0'(\underline{x}) + 2\varepsilon^2\phi_0^2 a_0' = \varepsilon\phi_0(\underline{x})\left[2(\omega - \varepsilon \bar{A}_0) + \Omega_n \right] \Psi(\underline{x}),$$

eq.(3.15) reads

$$(3.18)\quad \left[\nabla^2 + \Omega^2 - 2\varepsilon^2\phi_0^2\right] a_0' = -g(\underline{x})$$

A similar equation for the vector potential is reached from eq.(2.21).

$$(3.181)\quad \left[\nabla^2 + \Omega^2 - 2\varepsilon^2\phi_0^2\right] \underline{a}' = -\underline{h}(\underline{x})$$

where

$$(3.182)\quad \underline{a}(n,\underline{x},t) = \underline{a}'(\underline{x})\ e^{-i\Omega_n t} + \underline{a}'^{*}(\underline{x})\ e^{i\Omega_n t}$$

and

$$(3.183)\quad \underline{h}_n(\underline{x}) = -i\mathcal{E}\ (\phi_0 \underline{\nabla}\Psi_n - \Psi_n \underline{\nabla}\phi_0) + c.c.$$

These last equations cannot easily be solved except if $\phi_0^2(\underline{x})$ is a constant, which is not the case. A trial function for ϕ_0 respecting all boundary conditions is given in appendix 1 by the equation (A1.8) :

$$(3.184)\quad \phi_0(\underline{x}) = \lambda\ (\ 1 - r^2 R^{-2})^{m/2}$$

with

$$(3.19)\quad m = 2(1-a)^{-1}$$

The form of this trial function, squared, is given in fig.(3.1). It can be seen that the distribution falls sharply as, at one hundredth of the total radius, the charge density ϕ_0^2 is equal to about 1.4×10^{-4} of its value at the center.

At this point, it is possible to continue at one of many different approximation levels. First, if one wants

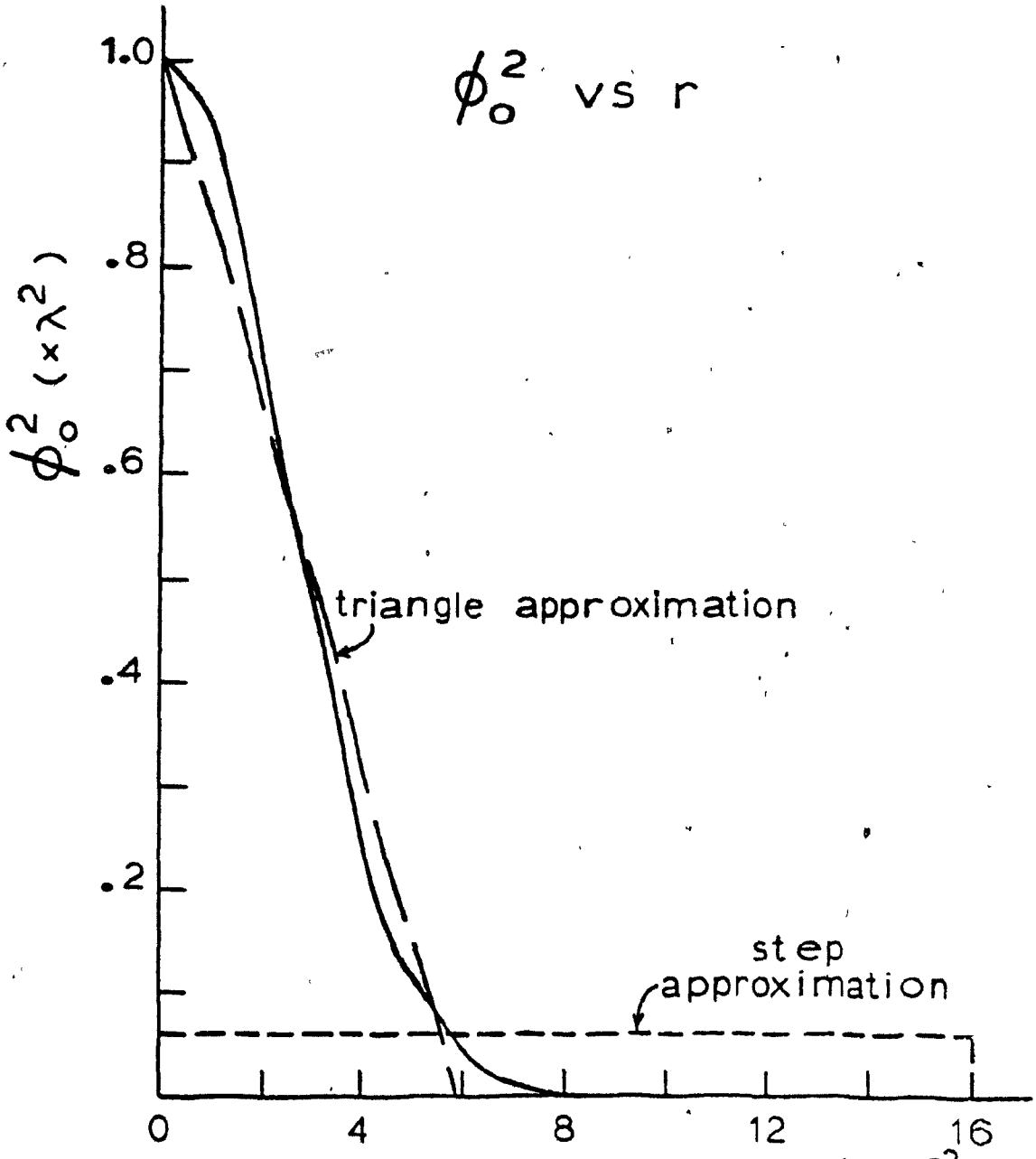

Fig. 3.1

This figure represents the trial function for ϕ_0^2 (full line) and different steps in the procedure of approximation.

to solve the general case, a computer should be used to solve the system of equations :

(3.20) a) $-\nabla^2 A_0 + 2\varepsilon^2(1- r^2R^{-2})^m \bar{A}_0 = 2\varepsilon\omega(1 - r^2R^{-2})^m + J_0^e$

b) eq. (3.18)

c) eq. (3.181)

d) eq. (3.9)

where J_o^e is the proton charge density. Another way would be to work out a solution for the region around the center where the charge is highly concentrated. Writing the equation for ϕ_0, using the approximation of the potential for $r \ll R$ [App. 2], one gets

$$(3.21)\quad -\nabla^2\phi_0 - (\omega - \varepsilon\bar{A}_0)^2\phi_0 + (\mu^2 + 2\mu^2\frac{r^2}{R^2})\phi_0 = 0$$

Now, if it is supposed that $\varepsilon\bar{A}_0$ is negligible compared with ω, one gets an harmonic oscillator and ϕ_0 has the form

$$(3.22)\quad \phi_0(r) = K \exp\left(-\frac{\mu}{2^{\frac{1}{2}}R} r^2\right) \qquad r = |\underline{x}|$$

K : constant

Using the expression for R given in appendix 1, one obtains

$$(3.23)\quad \phi_0 = K \exp\left(-54164\ r^2R^{-2}\right)$$

This last expression could then replace $\lambda(1-r^2R^{-2})^{m/2}$ in the system of equations (3.20) but the system would still have to be solved with a computer.

To be able to solve this problem analytically, a rougher approximation has to be made. One can think of a triangle to describe ϕ_0^2 [See fig. 3.1] :

$$(3.24)\quad \phi_0^2(r) = \lambda^2\left(1 - k\frac{r}{R}\right)$$

where k is a positive constant such that $\phi_0^2=0$ at about six thousandth of the total radius of the soliton. So, all the terms depend only on the variable r and if one puts

$$(3.25)\quad a_0' = a_0''\, r^{-1}$$

and chooses to work out the case l=0, the wave equation takes the form

$$(3.26)\quad \frac{d^2}{dr^2}\, a_0'' + (\Omega^2 - 2\epsilon^2\lambda^2)a_0'' + 2\epsilon^2\lambda^2 k\frac{r}{R}\, a_0'' = -g(r)$$

Trying first to solve the homogeneous equation, one puts

$$(3.27)\quad a_0'' = \sum_{n=0}^{\infty} c_n x^n$$

which leads to

$$(3.28)\quad c_2 = \frac{\Omega^2 - 2\epsilon^2\lambda^2}{2}\, c_0 = k_1 c_0$$

$$(3.29)\quad c_m = \frac{(\Omega^2 - 2\epsilon^2\lambda^2)}{m(m-1)} c_{m-2} + \frac{2\epsilon^2 k}{m(m-1)R}\, c_{m-3}$$

or

$$(3.30)\quad c_m = \frac{2k_1}{m(m-1)}\, c_{m-2} + \frac{k_2}{m(m-1)}\, c_{m-3}$$

where

$$(3.31)\quad k_1 = \frac{\Omega^2 - 2\varepsilon^2\lambda^2}{2}$$

and

$$(3.32)\quad k_2 = \frac{2\varepsilon^2 k}{R}$$

The result for the homogeneous equation is then

$$(3.33)\quad (a_0'')_H = c_0\left[1 + k_1x^2 + \frac{1}{6}k_2x^3 + \frac{1}{6}k_1x^4 + \ldots\right] + c_1\left[x + \frac{1}{3}k_1x^3 + \frac{1}{12}k_2x^4 + \frac{1}{10}k_1x^5 + \ldots\right]$$

$$(3.34)\quad (a_0'')_H = c_0\left\{\sum_{m=1}^{\infty}\left[\frac{k_1}{m(2m-1)}x^{2m} + \frac{k_2}{2m(2m+1)}x^{2m+1}\right] + 1\right\} + c_1\left\{\sum_{m=1}^{\infty}\left[\frac{k_1}{m(2m+1)}x^{2m+1} + \frac{k_2}{2(m+1)(2m+1)}x^{2(m+1)}\right] + x\right\}$$

The last equation gives the solution of the homogeneous wave equation for a_0'' :putting c_0 equal to 1 and c_1 to zero in eq.(3.34) for the first independent solution $(a_0'')_{H1}$ and conversely for the second solution $(a_0'')_{H2}$. The use of the method of variation of parameters will yield a particular solution of the inhomogeneous equation (3.26) under the form :

$$(3.35)\quad (a_0'')_P = \int^{x} \frac{\left\{(a_0''(s))_{H1}(a_0''(r))_{H2} - (a_0''(r))_{H1}(a_0''(s))_{H2}\right\}g(s)\,ds}{(a_0''(s))_{H1}\frac{d}{ds}(a''(s))_{H2} - (a''(s))_{H2}\frac{d}{ds}(a''(s))_{H1}}$$

Finally, this leads to a complete analytical solution for a_0 of the form

$$(3.36)\quad a_0(r,t) = \frac{e^{-i\Omega_n t}}{r}\left((a_0''(r))_{H1} + (a_0''(r))_{H2} + (a_0''(r))_P \right) + c.c$$

if eq.(3.35) is solvable. Even in that case, the integral involves so many terms that it becomes difficult to handle. So, one has to go to a next step of approximation to find an expression that could be used in eq.(3.9). The only way the added term in ϕ_0^2 could let the equation (3.18) have a usable answer would be that ϕ_0^2 equal a constant over some domain. Then, the idea would be to use a step function for ϕ_0^2. There are many ways to define this step function. First, one may try

$$(3.37)\quad \phi_0^2(\underline{x}) = \langle \phi_0^2 \rangle \qquad 0 < r < R$$
$$= 0 \qquad r > R$$

The result of eq.(3.18) with ϕ_0^2 given by the preceding expression is

$$(3.38)\quad a_0'(x) = \frac{1}{4\pi}\int_0^R \frac{\exp(\, ik_n |\underline{x}-\underline{x}'|\,)}{|\underline{x} - \underline{x}|}\, g(\underline{x})\, d^3x \quad + c.c.$$

where

$$(3.39)\quad k_n^2 = \Omega_n^2 - 2\varepsilon^2 \langle \phi_0^2 \rangle \qquad 0 < r < R$$

$$k_n^2 = \Omega_n^2 \qquad r > R$$

$$g(\underline{x}) = \varepsilon(\langle \phi_0^2 \rangle)^{\frac{1}{2}} (2(\omega - \varepsilon \bar{A}_0) + \Omega_n)\Psi_n \qquad 0 < r < R$$

$$g(\underline{x}) = 0 \qquad r \geq R$$

One can calculate the height of the step by using the values given in the first appendix and find

$$(3.40) \quad \langle \phi_0^2 \rangle = \frac{3}{4\pi R^3} \int \phi_0^2 d^3x = \frac{3\lambda^2}{R^3} \int (1 - r^2R^{-2}) r^2 dr = \frac{6.9\times10^{-6}}{R^2}$$

This will give

$$(3.41) \quad \varepsilon^2 \langle \phi_0^2 \rangle = \frac{4\pi\alpha}{R^3} (6.9\times10^{-6}) = 2.8\times10^{-5} \text{ (eV)}^2$$

where

$$(3.42) \quad \varepsilon^2 = \frac{4\pi\alpha}{\bar{Q}} = \frac{4\pi\alpha}{\hbar} \xrightarrow[\hbar=1]{} \varepsilon^2 = 4\pi\alpha = 0.09$$

and

$$(3.43) \quad R = \frac{7.66\times10^4}{\mu} = \frac{7.66\times10^4}{5.11\times10^5} \text{ (eV)}^{-1} = 0.15 \text{ (eV)}^{-1}$$

As Ω^2 is of the order of 100(eV)^2, it is obvious that the discontinuity in the charge distribution is negligible. But this choice of step function does not represent very well the distribution as one can check.

(3.44) $$\phi_0^2 = 1.25\ 10^{-7} \times \phi_0^2(r=0)$$

$$\phi_0^2 = 4.8\ 10^{27} \times \phi_0^2(r=.03R)$$

The soliton is so spread out that the calculation of $\langle\phi_0^2\rangle$ includes a large part of space where, practically speaking, nothings exists. A more realistic approach would be to confine the charge in the region up to about four mean square radii r_c or sixteen thousandth of the total radius. The field a_0' would still be described by equations (3.38) and (3.39) except for the upper limit of integration and the value of $\langle\phi_0^2\rangle$ which would be changed for

(3.45) $$\langle\phi_0^2\rangle_r = 0.06\,\lambda^2 = \frac{1.83}{R^2} = 7.3\ (eV)^2$$

From now on, the calculations that will be made in this section will be for the region mentioned in the precedent paragraph ($r < 0.016R$). For more precise calculations, the reader is referred to the set of equations (3.20). So, the solution of the problem in this approximation takes the form :

(3.46) a) $$a_0'(\underline{x}) = \frac{1}{4\pi} \int \frac{\exp(ik_n |\underline{x}-\underline{x}'|)}{|\underline{x}-\underline{x}'|}\, g(\underline{x}')\, d^3x' + c.c.$$

b) $$\underline{a}(\underline{x}) = \frac{1}{4\pi} \int_0^{.016R} \frac{\exp(ik_n |\underline{x}-\underline{x}'|)}{|\underline{x}-\underline{x}'|}\, \underline{h}(\underline{x}')\, d^3x' + c.c.$$

c) $$\bar{A}_0(\underline{x}) = \frac{1}{4\pi} \int \frac{\exp(-1 |\underline{x}-\underline{x}'|)}{|\underline{x}-\underline{x}'|}\, (2\varepsilon\omega\phi_0^2 + J_0^e(\underline{x}'))\, d^3x'$$

$$d)\eta(\underline{x},t) = \eta_i(\underline{x},t) + \frac{\varepsilon}{4\pi}\int G(\underline{x},t;\underline{x}',t')\, e^{-i\Omega_i t'}$$

$$\left[\int_0^{\infty} \frac{\exp(ik_i|\underline{x}'-\underline{x}''|)}{|\underline{x}'-\underline{x}''|}\,(g(\underline{x}'') + \frac{i}{\omega}\,\underline{h}(\underline{x}'')\cdot\underline{\nabla}')\,d^3x''\right]$$

$$\phi_0(\underline{x}')\, d^3x'\, dt'$$

where $(\langle\phi_0^2\rangle)^{\frac{1}{2}}$ has replaced ϕ_0 in g and $\underline{h}$ and

$$(3.47)\quad k_n^2 = \Omega_n^2 - 2\varepsilon^2\langle\phi_0^2\rangle_r$$

and

$$(3.48)\quad 1^2 = 2\varepsilon^2\langle\phi_0^2\rangle_r$$

The equation (3.46d) is the integral equation describing the perturbed system. It can still be worked out by using eq. (3.83) for G in eq.(3.46d),which yields

$$(3.49)\quad \eta(\underline{x},t) = \eta_i(\underline{x},t) - \frac{i\varepsilon}{4\pi}\sum_n \eta_n(\underline{x},t)\int\psi_n(x')e^{i(\Omega_n-\Omega_i)t'}$$

$$\left[\int_0^{\infty} \frac{\exp(ik_i|\underline{x}'-\underline{x}''|)}{|\underline{x}'-\underline{x}''|}\,(g(\underline{x}'') + \frac{i}{\omega}\,\underline{h}(\underline{x}'')\cdot\underline{\nabla}')\,d^3x''\right]$$

$$\phi_0(\underline{x}')\;\theta(t-t')\, d^3x'\, dt'$$

The time integration simply gives

$$(3.50)\quad \int e^{-i\Omega_{ni}t'}\,(t-t')dt' = 2\Omega_{ni}^{-1}\, e^{-i\Omega_{ni}t/2}\,\sin\tfrac{1}{2}\Omega_{ni}t \qquad i \neq n$$

$$= t \qquad i = n$$

where

$$(3.501)\quad \Omega_{ni} = \Omega_i - \Omega_n$$

Finally, using equations (3.17), (3.183), (3.46c) and (3.50) in eq. (3.49) yield a final expression for .

$$(3.51)\quad \eta(\underline{x},t) = \eta_i(\underline{x},t) - \frac{i\varepsilon^2}{2\pi\Omega_{ni}} \sum_{n\neq i} \left[d_n e^{-i\Omega_n t/2} \sin \tfrac{1}{2}\Omega_{ni} t \right] \eta_n - \frac{i\varepsilon^2 d_i t}{4\pi} \eta_i$$

where the coefficients of the expansion d_n are given by

$$(3.52)\quad d_n = \int \psi_n^*(\underline{x}') \left\{ \int_0^{\text{out}} \frac{\exp(ik_n|\underline{x}'-\underline{x}''|)}{|\underline{x}'-\underline{x}''|} \left[\left\{ \bar{\phi}_0(\underline{x}'') \left[2\omega + \Omega_n - \frac{\varepsilon}{2\pi} \int \frac{\exp(-1|\underline{x}''-\underline{x}'''|)}{|\underline{x}''-\underline{x}'''|} \left(2\varepsilon\omega\bar{\phi}_0^2(\underline{x}''') + J_0^e(\underline{x}''') \right) d^3x''' \right] \psi_n(\underline{x}'') \right\} + \frac{1}{\omega} \left\{ \left(\phi_0(\underline{x}'') \underline{\nabla}'' \psi_n(\underline{x}'') - \psi_n(\underline{x}'') \underline{\nabla}'' \phi_0(\underline{x}'') \cdot \underline{\nabla}' \right\} d^3x'' \right\} \phi_0(\underline{x}')\, d^3x'$$

with

$$(3.53)\quad \bar{\phi}_0 = \left(\langle \phi_0^2 \rangle_r \right)^{\frac{1}{2}}$$

These two last equations form together an integral equation which describes the perturbed system of a soliton with a proton at its center. All this exhibits clearly the mutual feedback between the field a_μ and the wave-function ψ, which is a purely a nonlinear phenomenon. The time-dependence of the terms in eq.(3.51) clearly indicates that transitions are possible from one state to another without the presence of any external electromagnetic field.

3-3 Einstein's Coefficient for Spontaneous Transitions

The explanation of spontaneous transitions in atoms with the usual linear theory requires the quantization of the electromagnetic field. In the scheme developed here, it is possible to find an expression for such transitions without quantization. The time-dependent wave-function ψ gives rise to a current in the atom that can emit radiation. As one is interested to study the fields in the radiation zone, some simplifications will be made possible. The step approximation for the charge distribution of the soliton will be kept in order

to be able to continue these calculations analytically. It will be supposed that the hydrogen atom has a radius of one sixteen thousandth (.016) of the total radius and that the rest of the soliton is empty space. The vector potential is given by eq.(3.46b) in that approximation. It will also be assumed that, over a small region around the border of the step, there a function connecting the upper to the lower step of the ϕ_0-distribution in order to avoid discontinuities. The effect of this fringe on $\underline{a}$ will be neglected.

In the radiation zone, $|\underline{x}'|$ becomes negligible compared to $|\underline{x}|$, the point of observation. Thus, $\underline{a}$ can be written, for an atom in the nth state,

$$(3.54)\quad \underline{a} = -\frac{i\varepsilon}{4\pi}\frac{e^{-ikr}}{r}\int_{0}^{0\&R} \underline{d}(\underline{x}')\, e^{i\underline{k}\cdot\underline{r}'}\, d^3\underline{x}' \quad e^{-i\Omega t} + \text{c.c.}$$

where the dependence on is understood and where $\underline{k}$ is in the direction of $\underline{x}$. The following definitions have been used :

$$(3.55)\quad \underline{d}(\underline{x}') = \bar{\phi}_0 \underline{\nabla}\psi_n - \psi_n \underline{\nabla}\bar{\phi}_0$$

$$(3.56)\quad r = |\underline{x}| \qquad r' = |\underline{x}'|$$

It should be remarked that, apart from a factor, $\underline{d}(\underline{x})$ is the transition current from the state ψ_n to the ground state ϕ_0 . In the radiation zone, the B-field is given by [4] :

$$(3.57)\quad \underline{B} = \nabla \times \underline{a} = \frac{ik_n \underline{x} \times \underline{a}}{r} = ik_n |\underline{a}| \sin\theta$$

where θ is the angle between $\underline{a}$ and $\underline{x}$. To calculate the radiated power, one must find the time-average of $|\underline{B}|^2$ which is simply

$$(3.58)\quad \langle |\underline{B}|^2 \rangle = \tfrac{1}{2} |\underline{B}|^2 = \tfrac{1}{2} k^2 |\underline{a}|^2 \sin^2\theta$$

because of the simple time dependence of $\underline{a}$. The Poynting vector is given by

$$(3.59)\quad \langle \underline{S} \rangle = |\underline{B}|^2 \frac{\underline{x}}{r}$$

and the power emitted by unit solid angle,

$$(3.60)\quad \left\langle \frac{dP}{d\alpha} \right\rangle = r\, \underline{x} \cdot \langle \underline{S} \rangle = \tfrac{1}{2} k^2 r^2 |\underline{a}|^2 \sin^2\theta$$

Using eq.(3.54) in eq.(3.60) gives the expression

$$(3.61)\quad \left\langle \frac{dP}{d\alpha} \right\rangle = \frac{\varepsilon^2 k^2 \sin^2\theta}{32\pi^2} \left| \int_0^{\text{DIAR}} (\underline{d}(\underline{x}')\, e^{i\underline{k}\cdot\underline{x}'} + \text{c.c.})\, d^3x' \right|^2$$

Therefore, the total power emitted by the atoms is

$$(3.62)\quad P = \int \left\langle \frac{dP}{d\alpha} \right\rangle d\alpha = \int \frac{\varepsilon^2 k^2}{32\pi^2} \left| \int_0^{\text{DIAR}} (\underline{d}(\underline{x}')\, e^{i\underline{k}\cdot\underline{x}'} + \text{c.c.})\, d\, x' \right|^2 \sin^2\theta\, d\alpha$$

If all the atoms were in the same state of energy Ω_i

and fell to the ground state ϕ_0, the total energy radiated by the system is given by

$$(3.63) \quad E = N\Omega_i$$

where N is the total number of atoms . From the two last equations, one can deduce something very similar to the Einstein's coefficient for spontaneous emission which gives the number of transitions per unit time per atom :

$$(3.64) \quad A_{fi} = \frac{P}{E} = \frac{\varepsilon^2 k^2}{32\pi^2\Omega_i} \left| \int_0^{\text{over}} (\underline{\alpha}(x')\, e^{i\underline{k}\cdot\underline{x}'} + \text{c.c.}) d^3x' \right|^2 \sin^2\theta \, d\alpha$$

where N has been put equal to 1. Using eq.(3.55) yields a more detailed expression.

$$(3.65) \quad A_{fi} = \frac{\varepsilon^2 k^2}{32\pi^2\Omega_i} \int \left| \int_0^{\text{over}} [(\bar{\phi}_0 \underline{\nabla} \psi_n - \psi_n \underline{\nabla} \bar{\phi}_0) e^{i\underline{k}\cdot\underline{x}'} + \text{c.c.}] \, d^3x \right|^2 \sin^2\theta \, d\alpha$$

Carrying out the angle integral leads to

$$(3.66) \quad A_{fi} = \frac{\varepsilon^2 k^2}{12\pi\Omega_i} \left| \int_0^{\text{over}} [(\bar{\phi}_0 \underline{\nabla} \psi_n - \psi_n \underline{\nabla} \bar{\phi}_0) e^{i\underline{k}\cdot\underline{x}'} + \text{c.c.}] \, d^3x' \right|^2$$

To get any further with this formula would need to solve eq.(3.51) in ψ. However, this equation does not seem to be solvable analytically. Therefore, equation (3.65) is the

final step one can reach to express Einstein's coefficient before having to do some more approximations or to do computer work. It must be remembered that this expression was obtained with a crude approximation for the unperturbed charge distribution and that it should be checked by numerical calculations if the results obtained from eq.(3.66) would be similar to those obtained without doing any approximation.

Another point must be mentioned about calculations that could be done from eq.(3.66). One cannot simply replace Ψ by known wave-functions of the hydrogen atom but must normalize because of the following. The unperturbed charge of a soliton is given, neglecting $\bar{A}_0$, by

$$(3.67) \qquad \bar{J}_0 d^3x = 2\varepsilon\omega \int \phi_0^2 \; d^3x = e$$

where e is the charge of the electron. Therefore, the current is

$$(3.68) \qquad \underline{j}_n = \left[-i\varepsilon(\phi_0 \underline{\nabla} \psi_n - \psi_n \underline{\nabla}\phi_0) - 2\varepsilon^2 \underline{a}' \, \phi_0^2\right] e^{-i\Omega_n t} + \text{c.c.}$$

$$= \left[\frac{-ie(\phi_0 \underline{\nabla} \psi_n - \psi_n \underline{\nabla}\phi_0)}{2\omega \int \phi_0^2 \, d^3x} - \frac{e^2 \underline{a}' \, \phi_0^2}{2\omega^2 \left(\int \phi_0^2 d^3x\right)^2} \right] e^{-i\Omega t} + \text{c.c.}$$

which means that k_n and $\underline{h}_n(\underline{x})$ must expressed in terms of e

in eq. (3.66) before doing any calculations.

$$(3.69)\quad k_n^2 = \Omega_n^2 - \frac{e^2 \langle \phi_0^2 \rangle_r}{2\omega^2 \left(\int \phi_0^2 \, d^3x \right)^2}$$

$$(3.70)\quad \underline{h}_n(\underline{x}) = \frac{-ie\left(\bar{\phi}_0 \underline{\nabla} \psi_n - \psi_n \underline{\nabla} \bar{\phi}_0 \right)}{2\omega \int \phi_0^2 \, d^3x}$$

This should permit to calculate numerically the coefficient for spontaneous emission from an excited state to the ground state. This formula should be good especially for transitions starting from a p-state, the other ones being suppressed by an additional factor α^2 for each higher multipole. To calculate transitions from an excited state to another, it would be necessary to keep the second order terms in η to have currentsof the form $\psi_n \underline{\nabla} \psi_m$. To finish this chapter, let's compare the expression for the Einstein's coefficient found by semi-classical treatment in quantum mechanics [5] for transitions to the ground state in the dipole approximation ($e^{i\underline{k}\cdot\underline{x}} = 1$) and in rationalized units

$$(3.71)\quad A_{fi} = \frac{e^2 \Omega_i}{3\pi} \left| \int \phi_0 \underline{\nabla} \psi_n \, d^3x \right|^2$$

with ours,

$$(3.72)\quad A_{fi} = \frac{e^2 k_i^2}{12\pi\Omega_i} \left| \int_0^{.016R} \left[(\phi_0 \underline{\nabla} \psi_n - \psi_n \underline{\nabla} \bar{\phi}_0) + \text{c.c.} \right] d^3x \right|^2$$

where equations (A2.15) and (A2.16) have been used to replace the integral of ϕ_0^2 and $\hbar$ has been put equal to 1. Taking the real part of the bracket in eq.(3.72) and putting $k_n \approx \Omega_n$ gives

$$(3.73)\quad A_{fi} = \frac{e^2 \Omega_i}{3\pi} \left| \int_0^{.016R} \mathrm{Re}\left(\bar{\phi}_0 \underline{\nabla} \psi_n - \psi_n \underline{\nabla} \bar{\phi}_0 \right) d^3x \right|^2$$

As $\underline{\nabla}\bar{\phi}_0$ is zero between 0 and .016R, one finally obtains

$$(3.74)\quad A_{fi} = \frac{e^2 \Omega_i}{3\pi} \left| \int_0^{.016R} \mathrm{Re}\ \left(\bar{\phi}_0 \underline{\nabla} \psi_n \right) d^3x \right|^2$$

1- L.I. Schiff, "Quantum Mechanics", McGraw-Hill, NY, 1968, pp. 468-469

2- Idem, p. 179

3- Bjorken, Drell, "Relativistic Quantum Mechanics", McGraw-Hill, NY, 1964, p.88

4- J.D. Jackson, "Classical Electrodynamics", Wiley, NY, 1975, p. 395

5- L.I. Schiff, Ibid.,pp. 413-414

Conclusion

In the first chapter, general concepts about solitons were given as well as a general theorem, the theorem of localization, which permits to determine if a given model really describes solitons. Also, a general class of nonlinear potentials giving rise to solitons as verified with the localization theorem was defined.

Once this was done, a Klein-Gordon equation with a nonlinear potential was worked out for a stationary state of the system. Then, a small perturbation was introduced and the equation was divided in zeroth and first order differential equations, the second order terms being neglected. Relevant quantities such as unperturbed and perturbed charges and currents were derived. This allowed to verify that the continuity equation and the conservation of energy were respected for the first two orders of the perturbation. Also, charge conservation permitted one to express a condition to be satisfied by the perturbed wave-function.

Then, the nonrelativistic limit of the nonlinear Klein-

Gordon equation was reached after some simplifications and the comparison with the linear Schrödinger equation showed that they differ only by the presence of the added nonlinear term and by two terms due to the effect of the perturbed electromagnetic fielf on the unperturbed wave-function.

Using the fractional potential model and a specific trial function for the unperturbed state leads to an integral equation which described, in the nonrelativistic limit, the behaviour of the soliton in its central region. As almost the whole charge distribution is concentrated in this region, this equation can be considered as describing the atom.

It was possible to derive an expression which gives the probability of transition from an excited state to the ground state in absence of an external field. This expression looked quite like the Einstein's coefficient for spontaneous emission as was seen at the end of the third chapter. Similar expressions could be derived for transitions from an excited state to another excited state by keeping the second order terms. This is a very interesting result because the spontaneous emission of light in an atom cannot be explained in linear quantum

mechanics except if the electromagnetic field is quantized. Here, the transitions take place because of the dependence of some of the potential terms on the wave-function itself. These terms arise because of the nonlinearity of the equations.

In conclusion, this last result indicates that the description of electrons by solitons deserves more study, as experimental facts can be explained by this theory and as unpleasant things like infinite self-energy are removed from the theory. However, one can remark that the spin is not explained by this model. The two most evident ways to solve this problem is to add the spin in the same manner it was done for the Schrödinger equation, or to work on a nonlinear Dirac equation. Along that last line, one can refer to the work of Rañada [1] or to the work of P. Mathieu and T.F. Morris that should be published in the near future. Another drawback of the model developed here is the size of the soliton which is rather large. One can hope that this is due to the choice of the trial function and this is not a fundamental problem. Nevertheless, the model appears to have some validity in the low energy limit.

1- Rañada, Int. J. of Theor. Phys., 1977, vol. 16, p. 795

Appendix 1

Important Relations and Quantities

In this work, there is no derivation of the important quantities or relations that describe a soliton as its total radius, its root mean square radius, its self-energy, etc. All of this was done by Morris in one of his papers [1] with the same starting Lagrangian (eq. 1.1). The potential he used was the fractional one :

$$(A1.1) \quad U(\phi^*\phi) = b(\phi^*\phi)^a \qquad \tfrac{1}{2} < a < 1$$

In order to be able to use his results, the same potential is used in the third chapter.

The results derived in his paper are based on the stability of a multiparticle system. This condition gives the De Broglie relation between rest mass and frequency and leads to a formula for the fine-structure constant. Considering N_i solitons of charge Q_i with a small interaction, Morris found

$$(A1.2) \quad E(Q_i) = Q_i \omega(Q_i)$$

for which only one solution exists. This implies that all

charges are equal :

(A1.3) $Q_i = \bar{Q}$

If this value of $\bar{Q}$ is associated with Planck's constant, the relation (A1.2) becomes the De Broglie equation

(A1.4) $E(\bar{Q}) = \bar{Q}\,\omega(\bar{Q})$

As the electrical charge is $\varepsilon\bar{Q}$, the fine-structure constant is given by

(A1.5) $$\alpha = \frac{(\varepsilon\bar{Q})^2}{4\pi\bar{Q}}$$

Using extensively virial theorems to eliminate as many parameters as possible, Morris established some basic relations which can be used to determine fundamental quantities of the system. But in order to get results under an usable form, one has to use a trial function for ϕ_0 which respects the boundary conditions

(A1.6) $\phi_0 \to 0$ and $\frac{d}{dr}\phi_0 \to 0$ as $r \to R$

where R is the total radius of the soliton, and also

(A1.7) $\frac{d}{dr}\phi_0 = 0$ at $r = 0$

The trial function is found to be

(A1.8) $\phi_0 = \lambda(1 - y^2)^{1/1-a}$ with $y = r/R$

This gives the following interesting results :

(A1.9) $a = 1 - \frac{4\alpha^2}{3\pi}$ $\qquad b = \mu^2 (\mu^2 \bar{Q})^{1-a}$

(A1.10) Radius of the soliton

$$R = \left(\frac{3}{2} \right)^{\frac{1}{2}} \frac{\pi}{2\mu\alpha^2} \approx 560\ r_B \qquad (r_B : \text{Bohr radius})$$

(A1.11) RMS radius

$$r_c = \langle r^2 \rangle \quad \frac{3(3\pi)^{\frac{1}{2}}}{4\mu\alpha} \approx \frac{\alpha}{\sqrt{\pi}} R \approx 2.3\ r_B$$

(A1.12) Electrostatic energy of the self-field

$$E_{SF} = \frac{1}{2} \int E^i E_i d^3x = \frac{2\mu\alpha}{3\pi} \quad \frac{(\varepsilon \bar{Q})^2}{4\pi} \approx 5.3 \text{ eV}$$

where μ is identified as the inverse Compton wavelength as it can be seen in

(A1.13) $E \approx \mu \bar{Q}$

1- T.F. Morris, Hadronic Journal , 1980, vol. 3, p. 1360

Appendix 2

Fractional Potential Model

The fractional potential is written under the form

(A2.1) $U(\phi^*\phi) = b(\phi^*\phi)^a$

where a and b are parameters whose values are given in the first appendix. It should be remarked that a differs from 1 by a quantity proportional to α^2 which is very small (order of 10^{-5}). Therefore, for the unperturbed state ϕ_0, one has

(A2.3) $U(\phi_0^2) = b\phi_0^{2a}$

(A2.4) $U'(\phi_0^2) = ab\phi_0^{2(a-1)}$

(A2.5) $U''(\phi_0^2)\phi_0^2 = a(a-1)b\phi_0^{2(a-1)}$

It is obvious that the last term is negligible in front of the others because of the factor (a-1).

As the soliton is highly condensed at the center of its own distribution, it is of much interest to describe the system near the origin. To do so, the trial function introduced

in the preceding appendix will be used.

$$(A2.6)\quad \phi_0 = (1 - y^2)^{1/1-a} \qquad \text{for} \quad r < R$$

$$= 0 \qquad \text{for} \quad r > R$$

with

$$(A2.7)\quad y = \frac{r}{R}$$

The constant λ is found to be [1]

$$(A2.8)\quad \lambda = \left[\frac{\bar{Q}}{2^{\frac{1}{2}} \pi \alpha R^2} \right]^{\frac{1}{2}}$$

or with $\bar{Q}=\hbar=1$,

$$(A2.9)\quad \lambda = (2^{\frac{1}{2}} \pi \alpha R^2)^{-\frac{1}{2}}$$

and the units of λ are $[1]^{-1}$ or $[M]$.

From equations (A2.4), (A2.6) and (A2.9), the expression for $U'(\phi_0^2)$ can be writen

$$(A2.10)\quad U'(\phi_0^2) = ab(2^{\frac{1}{2}} \pi \alpha R^2)^{1-a} (1 - y^2)^{-2}$$

Using eq.(A1.9), this last expression becomes

$$(A2.11)\quad U'(\phi_0^2) = \mu^2 (2^{\frac{1}{2}} \pi \alpha \mu^2 R^2)^{1-a} (1 - y^2)^{-2}$$

As (1-a) is very small, the quantity in the first could be put equal to 1 as long as it does not carry any units .

The quantity in the first parenthesis does not carry units ($[M] \leftrightarrow [L]^{-1}$ when $\hbar = 1$) and hence, one can further simplify

$$(A2.12) \quad U'(\phi_0^2) = \mu^2 \ (1 - y^2)^{-2}$$

For the region of few Bohr radii around the origin, where more than 99.9% of the charge of the electron is concentrated , the following relation is verified

$$(A2.13) \quad y^2 \ll 1 \quad \text{for} \quad r \leq 10 \ r_B$$

This permits to keep only the first two terms of the development in series of the parenthesis in (A2.12) :

$$(A2.14) \quad U'(\phi_0^2) = \mu^2 \ (1 + 2y^2)$$

Therefore, the potential is felt by most of the electron as an harmonic oscillator potential shifted by a constant quantity μ^2.

It can be remarked that the expressions for ϕ_0 and U' are the same for both cases, i.e. soliton free or bound in an atom. The difference will be in the value of R that should be less in the second case. One should also note that ϕ_0 is not normalized to unity :

$$(A2.15) \quad \int \phi_0^2(\underline{x}) \ d^3x = \lambda^2 R^3 \int_0^1 (1-y^2)^m \ d^3y = K(1)$$

where K(1) represents a constant with units of length :

$$(A2.16) \quad K(1) = \left(\frac{\pi(1-a)}{2} \right)^{1.5} \frac{R}{2^{\frac{1}{2}} \pi \alpha} = \frac{1}{2\mu}$$

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