Systems of equations over free groups: Structures and Complexity

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May 2009

A thesis submitted to McGill University in partial fulfilment of the requirements of
the degree of Doctor of Philosophy

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DEDICATION

This thesis is dedicated to the variables x and y. These variables, arguably the hardest working of them all, have left an indelible mark throughout all branches of mathematics since the invention of the lower case Roman alphabet.

ACKNOWLEDGEMENTS

First and foremost I wish to thank my supervisor Olga Kharlampovich for all the support she has given me over the years. Olga taught me the importance of tenacity, open-mindedness and, if need be, a philosophical outlook in pursuing research. From her I learned how to be a mathematician. Not only I was given a great amount of freedom in the pursuit of my projects, but Olga was also always there to listen to and comment on my ideas and often gave insightful suggestions. I am thankful for her confidence in me.

I am also thankful to Alexei Miasnikov for all the advanced group theory classes he gave and the various seminars he organized. Over the years he has given me many ideas and tools for my research, and his vision and energy are truly inspirational.

And even though he sometimes made fun of me, I also want to thank Dani Wise, who also taught me a great deal of geometric group theory.

I am grateful to the FQRNT and NSERC for financial support and grateful to my friends for all other kinds of support. A special thanks to Charlène for making me happy. I also want to thank my geometric group theory friends: Andrey, Dave, Emily, Ilya, James, Jeremy, Jithendra, Joe, Lynne, Montse, and Svetla who throughout the years were a lot of fun to be around.

Finally I want to thank my parents for their love, and also for giving me the opportunity to attend university in the first place.

ABSTRACT

In the first part of the thesis, we give a description of the fully residually F quotients of $F * \langle x, y \rangle$. The techniques we use rely extensively on the structure results of fully residually free groups developed by O. Kharlampovich, A. Miasnikov, and by Z. Sela. We also use the now classical theory of H. Bass and J.-P. Serre of the actions of groups on trees. As an application we completely recover the descriptions of the solutions sets of systems of equations in two variables given by Hmelevskiĭ and Ozhigov, the most precise to date, using completely algebraic methods. We also construct some examples to illustrate the richness of the theory.

In the second part of the thesis, we prove some results on algorithmic complexity. First we show that the the problem of deciding the solvability of an arbitrary quadratic equation over a free group is NP-complete. We also give an algorithm for Stallings' Folding Process; a fundamental technique in geometric group theory; which, for a fixed free group, runs in worst case time $O(n \log^*(n))$.

ABRÉGÉ

Dans la première partie de cette thèse, nous décrivons les quotients de $F * \langle x, y \rangle$ qui sont des $F-$groupes limites. Les techniques que nous utilisons proviennent des résultats sur la structure des groupes limites développées par O. Kharlampovich, A. Miasnikov et indépendamment par Z. Sela. Nous utilisons également la théorie maintenant classique de H. Bass et de J.-P. Serre sur les actions des groupes sur les arbres. Comme application nous redérivons les descriptions des solutions des systèmes d'équations à deux variables données par Hmelevskiĭ et Ozhigov, qui jusqu'à présent demeurent les plus précis, mais cette fois-ci en utilisant des méthodes complètement algébriques. Nous construisons également des exemples qui illustrent la richesse de la théorie.

Dans la deuxième partie de cette thèse, nous prouvons des résultats de complexité algorithmique. Tout d'abord nous démontrons que le problème de décider si une équation quadratique arbitraire sur un groupe libre a une solution est NP-complet. Nous donnons finalement un algorithme pour le processus du *Stallings' Folding* qui, donné un groupe libre fixe, opère en temps au plus $O(n \log^*(n))$.

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CHAPTER 1
Introduction

1.1 A slightly biased history of the theory of equations over free groups

Although solving equations is a central theme in mathematics it is in general a hard thing to do without the right tools. In the next two sections we will review previous results that lead up to and contextualize my work on systems of equations in two variables.

1.1.1 Systems of equations over free groups with one unknown

In the 1950's Vaught conjectured that the solutions for x, y, z in the equation

$$x^2y^2z^2 = 1$$

over a free group commuted pairwise. This was answered positively by Lyndon in 1959 [38] using methods from combinatorial group theory. The next year he published a paper [40] giving a description of the solution set of an equation over a free group in a single variable in terms of collections of *parametric words* . This gave a method to attack the problem. Appel and Lorenc [1, 33] announced more precise descriptions of the solution set, but the proofs were incomplete.

In 1989 Remeslennikov observed in [49] that finitely generated fully residually free groups are exactly the models of the existential theory of free groups. Baumslag, Kharlampovich, Miasnikov and Remeslennikov in [29, 4] set up an analogue the basic notions of classical (i.e. commutative) algebraic geometry for the class of torsion free

CSA groups, which includes free groups and torsion-free hyperbolic groups. From this work we have a characterization of fully residually free groups in terms of coordinate groups of irreducible varieties. This characterization of fully residually free groups puts them at the forefront of the study of solutions of equations: indeed every solution to a system of equations over a free group corresponds to a homomorphism from a fully residually free group to a free group.

This gave a new approach to study solutions of equations: classify the corresponding fully residually free groups. In 2000 Chiswell and Remeslennikov in [7] using techniques involving length functions and Lyndon's group $F^{\mathbb{Z}[t]}$, developed by Alperin, Bass, Lyndon, Miasnikov, Remeslennikov, Chiswell, Promislow and Wilkens, showed that the fully residually free groups that arose fell into only three categories (see Theorem 4.1.2.) In this way they were finally able to give a proof of the result claimed by Appel and Lorenc.

1.1.2 From algorithms to the algebraic structure of fully residually free groups

In 1969 Appel proved that parametric words were insufficient to describe the solutions to Malc'ev's equation $[x, y] = [a, b]$. This foreshadowed the fact that the two variable case was going to be considerably more complicated. In the early 1970's Hmelevskiĭ and Wicks worked independently on systems of equations in two variables over free groups [22, 62]. They both obtained an algorithm to decide solvability of equations of the form $w(x, y) = u$ where x, y are unknowns and u lies in a free group F and gave descriptions of solution sets. Hmelevskiĭ also had a result for another specific type equation in two variables.

In 1983 Ozhigov in [46] obtained an algorithm that outputs the complete solution set of a system of equations in two variables over free groups. He moreover gave a description of possible solutions sets in terms of so-called *forms.* These forms are like parametric words that also involve something called a Hmelevskiĭ function, which it turns out corresponds to automorphisms of *QH subgroups.* In 1982 Makanin [42] constructed an algorithm which decided the solvability of an arbitrary system of equations over a free group.

So far these results were essentially algorithmic, this changed with the work of Razborov [48]. In his PhD thesis Razborov improved on Makanin's result by giving an algorithm which produced the complete solution set of a system of equations over a free group. It should be noted that although Razborov's result in a sense supersedes Ozhigov's work, Ozhigov's detailed description of solutions of systems of equations in two variables doesn't follow from Razborov's machinery.

Razborov's other big insight was to be more algebraic: his description is given in terms of a diagram consisting of a collection of epimorphisms of groups associated to *generalized equations* . To each of these groups there is an attached group of *canonical automorphisms.*

The next step in this direction came from the work of Kharlampovich and Miasnikov. In their papers [27, 28] they develop the so-called *Elimination Process* (EP) which constructs Hom diagrams which are variations of Razborov's diagrams and proved that all solutions of a system of equations can be obtained from solutions of a finite number of *NTQ* systems. These groups are shown to embed into chains of extensions of centralizers and therefore are fully residually free. This work actually implies that a finitely generated group is fully residually free if and only if it embeds into chain of extensions of centralizers.

Finitely generated fully residually free groups were already known to have some nice properties following directly from their definition, but this embedding theorem has far reaching consequences. Firstly that finitely generated fully residually free groups are finitely presented, secondly that if they are not free abelian then they have non trivial cyclic splittings, and finally that they admit a *hierarchy* in which the vertex groups can be studied via so called *strict epimorphisms.*

Sela in [52], using techniques for groups acting on $\mathbb{R}$-trees (which were also in part inspired by the work of Makanin), arrived at essentially the same structural results.

1.1.3 My contribution

In the first part of the thesis I generalize Chiswell and Remeslennikov's approach to systems of equation in two variables over free groups. Although the main idea remains the same i.e. classify the arising coordinate groups, the methods are different. Specifically we must make extensive use of the structure theorems of Kharlampovich, Miasnokov, and Sela for fully residually free groups. Another difference is the extensive use of Bass-Serre theory, which is useful for dealing with more complicated graphs of groups.

As a result we obtain Theorem 4.1.6: a classification of the coordinate groups of the fully residually free groups arising in systems of equations with two unknowns over free groups, extending the classification given in [7]. We are also able to recover the descriptions of solutions of systems of equations in two variables given by Hmelevskiĭ and Ozhigov (see Theorems 3.1.27 and 4.1.8.)

We also give some concrete examples of Hom diagrams and of fully residually free groups that illustrate the extent of the richness of the theory that can arise even when considering only two variables (see Figure 3–1, Theorem 3.2.1, and Section 4.1.2.)

1.2 Beyond Decidability: Complexity

Much of the theory of equations over free groups was devoted to algorithmic decidability results, without mention of running time. For example it is shown in [32] that Makanin's original algorithm isn't even primitive recursive. Nonetheless it seems that there are more and more results which indicate that this problem might be algorithmically tractable. The strongest result in this direction to date is an algorithm in PSPACE due to Plandowski [47] which can be used to decide if a system of equations over a free group has a solution. Another interesting example is a result due to Ciobanu given in [8]: there is a polynomial time algorithm which decides if an equation of the form $w(x, y) = u$ (i.e. like the one considered by Hmelevskiĭ and Wicks) has a solution.

1.2.1 Quadratic equations

The class of quadratic equations (every variable occurs exactly twice) is somewhat special in this theory. Indeed in 1962 Malc'ev gave a complete description of the solutions to the equation $[x, y] = [a, b]$ over the free group $F(a, b)$ [43]. In the 1980's Comerford and Edmunds in [10] and Grigorchuk and Kurchanov in [21] gave complete and detailed descriptions of solutions of arbitrary quadratic equations over free groups. The case of quadratic equations is significantly easier than the general case of equations over free groups. One of the main reasons for this is that Nielsen-like techniques and the topology of surfaces provide relatively simple yet extremely powerful tools for dealing with these equations.

In 1989 Ol'shanskiĭ and Grigorchuk, Kurchanov gave explicit polynomial time complexity upper bounds for deciding if a quadratic equation over a free group has a solution [45, 21]. These upper bounds however are not uniform and depend on the *genus* of the equation. This result implies a uniform exponential time upper bound on the complexity of deciding if a quadratic equation has a solution. NP-hardness of quadratic word equations was also established by Diekert and Robson [16].

We improve on this in Chapter 6 by showing that the uniform problem for deciding if a quadratic equation over a free group has a solution is in fact in NP. We then show a NP-hard lower bound for the complexity of this problem and therefore establish that the problem of deciding if a quadratic equation over a free group has a solution is NP-complete. This result is also significant because it gives a lower bound for the complexity of the problem of deciding if an arbitrary system of equations over a free group has a solution.

1.2.2 Stallings' Foldings

Stallings' foldings were originally used to study subgroups of free groups [54]. Specifically it is a process which takes a set of generators of a subgroup of a free group and constructs a labeled graph which canonically represents the subgroup. This graph turns out to be a very useful algorithmic tool which enables us for example to solve the membership problem or compute the index of a subgroup of a free group. See [23, 53, 56] for details.

The technique has also been generalized to Bass-Serre theory in the works of Stallings, Dunwoody, Feighn, Bestvina, Kapovich, Miasnikov, and Weidmann [55, 5, 17, 25]. There is also a generalization of this to subgroups $F^{\mathbb{Z}[t]}$ developed by Kharlampovich, Miasnikov, Serbin, and Remeslennikov in [31], which for example allows us to easily solve the conjugacy problem (the equation $xgx^{-1} = h$, where x is the unknown).

In Chapter 7 using a data structure due to Tarjan [58] we give an algorithm to perform Stallings' Folding process in worst case time $O(n \log^*(n))$. As a corollary we have that the uniform membership problem over a fixed free groups runs in almost linear time.

1.3 Statement of Originality

The material presented in Chapters 3, 4, 6, and 7 of this thesis is new and constitutes original scholarship in mathematics. Some auxiliary results, included in this thesis to make it reasonably self contained, are clearly identified as previously known, and the reader is referred to the original sources.

1.4 Contribution of co-authors

Chapter 6 is based on my joint paper [26] with I.G. Lysënok, O. Kharlampovich, and A. Miasnikov. The idea of using tiling problems to study the complexity of quadratic equations is due to A. Miasnikov and I.G. Lysënok. The reduction to the bin packing problem is due to O. Kharlampovich, the technical details, however, were my responsibility. The proof that the problem is in NP is also my own.

Part I

Structures

CHAPTER 2
Fully residually F groups and equations

In this chapter we present some preliminary results and definitions that will be used in Chapters 3 and 4. The material presented here is the starting point of my work.

2.1 F-groups and Algebraic Geometry

A complete account of the material in this section can be found in [4]. Fix a free group F. An equation in variables $x_1, \ldots, x_n$ over F is an expression of the form

$$E(x_1, \ldots, x_n) = 1$$

where $E(x_1, \ldots, x_n) = f_1 {z_1}^{m_1} \ldots {z_n}^{m_n} f_{n+1};\ f_i \in F, z_j \in \{x_1, \ldots, x_n\}$ and $m_k \in \mathbb{Z}$.

We view an equation as an element of the group

$$F[x_1, \ldots, x_n] = F * F(x_1, \ldots, x_n)$$

. A *solution* of an equation is a substitution

$$x_i \mapsto g_i;\ i = 1, \ldots n;\ g_i \in F \tag{2.1}$$

so that in F the product $E(g_1, \ldots, g_n) =_F 1$. A *system of equations* in variables $x_1, \ldots, x_n;\ S(x_1, \ldots, x_n) = 1$; is a subset of $F[x_1, \ldots, x_n]$ and a solution of $S(x_1, \ldots, x_n)$ is a substitution as in (2.1) so that all the elements of $S(x_1, \ldots, x_n)$ vanish in F.

Definition 2.1.1. A group G equipped with a distinguished monomorphism

$$i : F \hookrightarrow G$$

is called an *F-group*, we denote this by (G, i). Given F-groups (G_1, i_1) and (G_2, i_2), we define an $F-$homomorphism to be a homomorphism of groups f such that the following diagram commutes:

$$\begin{tikzcd}
G_1 \arrow[r, "f"] & G_2 \\
F \arrow[u, "i_1"] \arrow[ur, "i_2"'] &
\end{tikzcd}$$

We denote by $\mathrm{Hom}_F(G_1, G_2)$ the set of F-homomorphisms from (G_1, i_1) to (G_2, i_2).

In the remainder the distinguished monomorphisms will in general be obvious and not explicitly mentioned. It is clear that every mapping of the form (2.1) induces an F-homomorphism $\phi(g_1, \ldots, g_n) : F[x_1, \ldots, x_n] \to F$, it is also clear that every $f \in \mathrm{Hom}_F(F[x_1, \ldots, x_n], F)$ is induced from such a mapping. It follows that we have a natural bijective correspondence

$$\mathrm{Hom}_F(F[x_1, \ldots, x_n], F) \leftrightarrow \{(g_1, \ldots, g_n) | g_i \in F\}$$

Definition 2.1.2. Let $S = S(x_1, \ldots, x_n)$ be a system of equations. The subset

$$V(S) = \{(g_1, \ldots, g_n) \in \underbrace{F \times \ldots \times F}_{n \text{ times}} \mid x_i \mapsto g_i \text{ is a solution of } S\}$$

is called the *algebraic variety* of S.

We have a natural bijective correspondence

$$\mathrm{Hom}_F(F[x_1, \ldots, x_n]/ncl(S), F) \leftrightarrow V(S)$$

Definition 2.1.3. The *radical* of S is the normal subgroup

$$Rad(S) = \bigcap_{f \in \mathrm{Hom}_F(F[x_1,\ldots,x_n]/ncl(S),F)} \ker(f)$$

and we denote the *coordinate group* of S

$$F_{R(S)} = F[x_1, \ldots, x_n]/Rad(S)$$

It follows that there is a natural bijective correspondence

$$\mathrm{Hom}_F(F[x_1, \ldots, x_n]/ncl(S), F) \leftrightarrow \mathrm{Hom}_F(F_{R(S)}, F)$$

so that $V(S) = V(Rad(S))$. We say that $V(S)$ or S is *reducible* if it is a union

$$V(S) = V(S_1) \cup V(S_2); V(S_1) \subsetneq \cup V(S) \supsetneq V(S_2)$$

of algebraic varieties.

Definition 2.1.4. An F-group G is said to be *fully residually* F if for every finite subset $P \subset G$ there is some $f_P \in \mathrm{Hom}_F(G, F)$ such that the restriction of f_P to P is injective.

Theorem 2.1.5 ([4])**.** *S is irreducible if and only if $F_{R(S)}$ is fully residually F.*

Theorem 2.1.6 ([4])**.** *Either $F_{R(S)}$ is fully residually F or*

$$V(S) = V(S_1) \cup \ldots \cup V(S_n)$$

where the $V(S_i)$ are irreducible and there are canonical epimorphisms $\pi_i : F_{R(S)} \to F_{R(S_i)}$ such that each $f \in Hom_F(F_{R(S)}, F)$ factors through some π_i.

Corollary 2.1.7. *If* $F[x_1, \ldots, x_n]/ncl(S)$ *is fully residually* F *then* $F_{R(S)} = F[x_1, \ldots, x_n]/ncl(S)$.

Theorem 2.1.8 ([4])**.** *Let* $F_{R(S)}$ *be fully residually* F*, then in particular it has the following properties.*

- *it is torsion free,*
- *it satisfies the CSA property (maximal abelian subgroups are malnormal,) which implies commutation transitivity and 2-acylindricity of any almost-reduced abelian splitting (see Definition 2.2.12.)*
- *elements* $g, h \in F_{R(S)}$ *either commute or freely generate a free group of rank 2.*

2.1.1 Rational Equivalence

Definition 2.1.9. A map $g : F[x_1, \ldots, x_n] \to F[x_1, \ldots, x_n]$ induced by the mapping

$$\begin{array}{rcl} f & \mapsto & f; \text{ if } f \in F \\ x_i & \mapsto & X_i(F, x_1, \ldots, x_n) \end{array}$$

is called a polynomial map.

Proposition 2.1.10. *Let* g *be a polynomial map such that* $g(x_i) = X_i(F, x_1, \ldots, x_n)$ *and* $S_1(F, x_!, \ldots, x_n)$ *be a system of equations. Let* $S_2(F, x_1, \ldots, x_n) = S_1(F, g(x_1), \ldots, g(x_n))$*. Then the map* $\widetilde{g} : \underbrace{F \times \ldots \times F}_{n \text{ times}} \to \underbrace{F \times \ldots \times F}_{n \text{ times}}$ *given by*

$$(a_1, \ldots, a_n) \mapsto (X_1(F, a_1, \ldots, a_n), \ldots, X_n(F, a_1, \ldots, a_n))$$

restricts to a mapping $\widetilde{g} : V(S_2) \to V(S_1)$ *called a* morphism of varieties.

Definition 2.1.11. If there are polynomial maps $f, g : F[x_1, \ldots, x_n] \to F[x_1, \ldots, x_n]$ such that $\widetilde{f} \circ \widetilde{g} = 1_{V(S_1)}$ and $\widetilde{g} \circ \widetilde{f} = 1_{V(S_2)}$ then we say that the systems of equations

S_1 is *rationally equivalent* to S_2, moreover we say the varieties $V(S_1)$ and $V(S_2)$ are *isomorphic.*

Theorem 2.1.12 ([4] Corollary 9)**.** *$F_{R(S)}$ and $F_{R(S_1)}$ are F-isomorphic if and only if S and S_1 are rationally equivalent.*

We have an application of this:

Proposition 2.1.13. *(i) $Aut_F(F[x_1, \ldots, x_n])$ is generated by the elementary Nielsen transformations on the basis $\{F, x_1, \ldots, x_n\}$ that fix F elementwise.*

(ii) If S, T are rationally equivalent via $\phi \in Aut_F(F[x_1, \ldots, x_n])$, then the natural map $\widetilde{\phi}$ in the commutative diagram below is an isomorphism.

$$\begin{tikzcd}
F[x_1, \ldots, x_n] \arrow[r, "\phi"] \arrow[d, "\pi"] & F[x_1, \ldots, x_n] \arrow[d, "\pi"] \\
F_{R(S)} \arrow[r, "\widetilde{\phi}"] & F_{R(T)}
\end{tikzcd}$$

Since this thesis deals specifically with systems of equations in two variables, these next two facts are formulated only in the two variable case.

Proposition 2.1.14. *Suppose $w(x, y)$ is a primitive (by* primitive *we mean an element that belongs to some basis) element of $F(x, y)$, then there exist words $X(u, z), Y(u, z)$ such that the set of solutions of $w(x, y) = u$ corresponds to the set of pairs*

$$(x, y) = (X(u, z), Y(u, z))$$

where z takes arbitrary values in F.

Proof. Let $S = \{w(x, y)u\}$. By assumption there is $\phi \in Aut_F(F[x, y])$ that sends $w(x, y)$ to x and ϕ extends to an F-automorphism of $F[x, y]$. This means that S is rationally equivalent to $T = \{xu^{-1}\}$. The first thing to note is that $F_{R(T)}$ is a free group, hence so is $F_{R(S)}$. $\mathrm{Hom}_F(F_{R(T)}, F)$ is given by

$$V(T) = \{(x, y) \in F \times F | x = u, y \in F\}$$

the result now follows by precomposing with $\widetilde{\phi}^{-1}$, as defined in Proposition 2.1.13. $\square$

Lemma 2.1.15. *Suppose the free group $F(x, y)$ on generators $\{x, y\}$ admits a presentation*

$$F(x, y) = \langle \xi, \zeta, p | [\xi, \zeta] p^{-1} \rangle$$

where $\xi, \zeta, p \in F(x, y)$. Then the mapping $\phi(\xi) = x, \phi(\zeta) = y, \phi(p) = [x, y]$, extends to an automorphism $\phi : F(x, y) \to F(x, y)$.

Proof. Notice that the basis elements x, y of $[x, y]$ obviously satisfy the identity $[x, y][x, y]^{-1} = 1$, so the mapping ϕ gives an automorphism. $\square$

2.2 Splittings

We present the basics of Bass-Serre theory.

Definition 2.2.1. A *graph of groups* $\mathcal{G}(A)$ consists of a connected directed graph A with vertex set VA and edges EA. A is directed in the sense that to each $e \in EA$ there are functions $i : EA \to VA, t : EA \to VA$ corresponding to the *initial and terminal* vertices of edges. To A we associate the following:

- To each $v \in VA$ we assign a *vertex group* A_v.
- To each $e \in EA$ we assign an *edge group* A_e.
- For each edge $e \in EA$ we have monomorphisms

$$i_e : A_e \to A_{i(e)},\ t_e : A_e \to A_{t(e)}$$

we call the maps i_e, t_e *boundary monomorphisms* and the images of these maps *boundary subgroups*.

We also formally define the following expressions: for each $e \in EA$

$$(e^{-1})^{-1} = e,\ i(e^{-1}) = t(e),\ t(e^{-1}) = i(e),\ i_{e^{-1}} = t_e,\ t_{e^{-1}} = i_e$$

A graph of groups has a fundamental group denoted $\pi_1(\mathcal{G}(A))$. We say that a group *splits* as the fundamental group of a graph of groups if $G = \pi_1(\mathcal{G}(A))$ and refer to the data $D = (G, \mathcal{G}(A))$ as a *splitting*.

Definition 2.2.2. A sequence of the form

$$a_0, e_1^{\epsilon_1}, a_1, e_2^{\epsilon_2}, \ldots e_n^{\epsilon_n}, a_n$$

where $e_1^{\epsilon_1}, \ldots e_n^{\epsilon_n}$ is an edge path of A and where $a_i \in A_{i(e_{i+1}^{\epsilon_i+1})} = A_{t(e_1^{\epsilon_i})}$ is called a $\mathcal{G}(A)$-path.

Definition 2.2.3. We denote by $\pi_1(G(A), u)$ the group generated by $\mathcal{G}(A)$ paths whose underlying edge path is a loop at u.

We have in particular that $\pi_1(\mathcal{G}(A), u) \approx \pi_1(\mathcal{G}(A))$.

Convention 2.2.4. If $F_{R(S)}$ is the fundamental group of $\mathcal{G}(A)$, then we will always assume that the basepoint v is the vertex $v \in VA$ such that $F \leq A_v$.

Definition 2.2.5 (Moves on $\mathcal{G}(A)$)**.** We have the following moves on $\mathcal{G}(A)$ that do not change the fundamental group.

- *Change the orientation of edges* in $\mathcal{G}(A)$, and relabel the boundary monomorphisms.

- *Conjugate boundary monomorphisms*, i.e. replace i_e by $\gamma_g \circ i_e$ where γ_g denotes conjugation by g and $g \in A_{i(e)}$.
- *Slide*, i.e. if there are edges e, f such that $i_e(A_e) = i_f(A_f)$ then we change X by setting $i(f) = t(e)$ and replacing i_f by $t_e \circ i_e^{-1} \circ i_f$.
- *Folding*, i.e. if $i_e(A_e) \leq A \leq A_{i(e)}$, then replace $A_{t(e)}$ by $A_{t(e)} *_{t_e(A_e)} A$, replace A_e by a copy of A and change the boundary monomorphism accordingly.
- *Collapse an edge* e, i.e. for some edge $e \in EA$, take the subgraph $star(e) = \{i(e), e, t(e)\}$ and consider the quotient of the graph A, subject to the relation $\sim$ that collapses $star(e)$ to a point. The resulting graph $A' = A/\sim$ is again a directed graph. Denote the equivalence class $v' = [star(e)] \in A'$, then we have $A'_{v'} = A_{i(e)} *_{A_e} G_{t(e)}$ or $A_{i(e)} *_{A_e}$ depending whether $i(e) = t(e)$ or not. For each edge f of A incident to either $i(e)$ or $t(e)$, we have boundary monomorphisms $A_f \to A'_{v'}$ given by $i'_f = j \circ i_f$ or $t'_f = j \circ t_f$, where j is the one of the inclusion $A_{t(e)} \subset A'_{v'}$ or $A_{i(e)} \subset A_{v'}$.
- *Conjugation*, i.e. for some $g \in G$ replace all the vertex groups A_v by A_v^g and postcompose boundary monomorphisms with γ_g (which denotes conjugation by g).

2.2.1 Relative presentations

Although the use of graphs of groups is critical, they are notationally cumbersome. We therefore recall how to obtain relative presentations from graphs of groups, and explain the graphical notation we will use.

Let $G_1, \dots G_n$ be groups with presentations $\langle X_1 \mid R_1 \rangle, \dots, \langle X_n \mid R_n \rangle$ (resp.) and $t_1, \dots t_k$ a set of letters. Let R denote a set of words in $\bigcup X_i^{\pm 1} \cup \{t_1, \dots t_k\}^{\pm 1}$ then we will define the *relative presentation*

$$\langle G_1, \ldots, G_n, t_1, \ldots t_k \mid R \rangle$$

to be the group defined by the presentation

$$\langle X_1, \ldots, X_n, t_1, \ldots t_k \mid R_1, \ldots R_n, R \rangle$$

If G is the fundamental group of a graph of groups $\mathcal{G}(A)$ with vertex groups $G_1, \ldots, G_n$ and cyclic edge groups we can give G a relative presentation as follows:

(A) Take a spanning tree T of the underlying graph A.

(B) For each edge e of T we can assume that $G_{i(e)} \cap G_{t(e)} = G_e$, and therefore can form an iterated amalgam.

(C) For each edge f not in T we add a "stable letter" t_f and the relation $t_f = (t_f)^{-1} \alpha t_f = \alpha'$; where α and α' are the images in $G_{i(f)}$ and $G_{e(f)}$ resp. of a generator of G_f via boundary monomorphisms.

The resulting presentation gives a group isomorphic to G, although it depends on the choice of spanning tree. We can take the underlying graph A and encode this relative presentation by labelling as follows:

- Vertices are labelled by the vertex groups.
- Edges in the spanning tree T are represented by undirected edges labelled by a generator of the edge group.
- Edges f not in the spanning tree T are directed, labelled by the corresponding stable letter t_f, moreover the endpoints are decorated by the elements α, α' as defined in (C) above.

We will switch freely between words represented as $G(A)$-paths and words in generators of relative presentations.

2.2.2 The cyclic JSJ decomposition

Definition 2.2.6. An *elementary cyclic splitting* D of G is a splitting of G as either a free product with amalgamation or an HNN extension over a cyclic subgroup. We define the *Dehn twist along* D, δ_D, as follows.

- If $G = A *_{\langle\gamma\rangle} B$ then

$$\delta_D(x) = \begin{cases} x & \text{if } x \in A \\ x^\gamma & \text{if } x \in B \end{cases}$$

- If $G = \langle A, t | t^{-1}\gamma t = \beta\rangle, \gamma, \beta \in A$ then

$$\delta_D(x) = \begin{cases} x & \text{if } x \in A \\ t\beta & \text{if } x = t \end{cases}$$

A Dehn twist generates a cyclic subgroup of $Aut(G)$. A splitting such that all the edge groups are nontrivial and cyclic is called a *cyclic splitting*.

We can generalize the notion of a Dehn twist to arbitrary cyclic splittings.

Definition 2.2.7. let D be a cyclic splitting of G with underlying graph A and let e be an edge of of A. Then a *Dehn twist* along e is an automorphism that can be obtained by collapsing all the other edges in A to get a splitting D' of G with only the edge e and applying one of the applicable automorphisms of Definition 2.2.6

Definition 2.2.8. (i) A subgroup $H \leq G$ is elliptic in a splitting D if H is conjugable into a vertex group of D, otherwise we say it is hyperbolic.

(ii) Let D and D' be two elementary cyclic splittings of a group G with boundary subgroups C and C', respectively. We say that D' is *elliptic in* D if C' is elliptic in D. Otherwise D' is *hyperbolic* in D

A splitting D of an F-group is said to be *modulo* F if the subgroup F is contained in a vertex group.

The following is proved in [50]:

Theorem 2.2.9. *(i) Let G be freely indecomposable (modulo F) and let D', D be two elementary cyclic splittings of G (modulo F). D' is elliptic in D if and only if D is elliptic in D'.*

(ii) Moreover if D' is hyperbolic in D then G admits a splitting E such that one of its vertex groups is the fundamental group $Q = \pi_1(S)$ of a punctured surface S such that the boundary subgroups of Q are puncture subgroups. Moreover the cyclic subgroups $\langle d \rangle, \langle d' \rangle$ corresponding to D, D' respectively are both conjugate into Q.

Definition 2.2.10. A subgroup $Q \leq G$ is a quadratically hanging (QH) subgroup if for some cyclic splitting D of G, Q is a vertex group that arises as in item (ii) of Theorem 2.2.9.

Not every surface with punctures can yield a QH subgroup. By Theorem 3 of [27], the projective plane with puncture(s) and the Klein bottle with puncture(s) cannot give QH subgroups.

Definition 2.2.11. (i) A QH subgroup Q of G is a *maximal QH* (MQH) subgroup if for any other QH subgroup Q' of G, if $Q \leq Q'$ then $Q = Q'$.

(ii) Let D be a splitting of G with Q be a QH vertex subgroup and let C be a splitting of Q with boundary subgroup $\langle c \rangle$ then there is a splitting D' of G called a *refinement of D along C* such that D is obtained from a collapse of D' along an edge whose corresponding group is $\langle c \rangle$.

Definition 2.2.12. (i) A splitting D is *almost reduced* if vertices of valence one and two properly contain the images of edge groups, except vertices between two MQH subgroups that may coincide with one of the edge groups.

(ii) A splitting D of G is *unfolded* if D can not be obtained from another splitting D' via a folding move (See Definition 2.2.5).

Theorem 2.2.13 (Proposition 2.15 of [30])**.** *Let H be a freely indecomposable modulo F f.g. fully residually F group. Then there exists an almost reduced unfolded cyclic splitting D called the* cyclic JSJ splitting of H modulo F *with the following properties:*

(1) Every MQH subgroup of H can be conjugated into a vertex group in D; every QH subgroup of H can be conjugated into one of the MQH subgroups of H; non-MQH [vertex] subgroups in D are of two types: maximal abelian and non-abelian [rigid], every non-MQH vertex group in D is elliptic in every cyclic splitting of H modulo F.

(2) If an elementary cyclic splitting $H = A *_C B$ or $H = A*_C$ is hyperbolic in another elementary cyclic splitting, then C can be conjugated into some MQH subgroup.

(3) Every elementary cyclic splitting $H = A *_C B$ or $H = A*_C$ modulo F which is elliptic with respect to any other elementary cyclic splitting modulo F of H can be obtained from D by a sequence of moves given in Definition 2.2.5.

(4) If D_1 is another cyclic splitting of H modulo F that has properties (1)-(2) then D_1 can be obtained from D by a sequence of slidings, conjugations, and modifying boundary monomorphisms by conjugation (see Definition 2.2.5.)

Definition 2.2.14. Suppose first that G is freely indecomposable. Given D, a cyclic JSJ decomposition of $F_{R(S)}$ modulo F, we define the group Δ of *canonical* $F-$*automorphisms with respect to* D of $F_{R(S)}$ to be generated by the following:

- Dehn twists along edges of D that fox F pointwise;
- automorphisms of the MQH groups that fix edge groups pointwise;
- automorphisms of the abelian vertex groups that fix edge groups point wise.

If G is freely decomposable modulo F then Δ is generated by the extensions of the canonical $F-$automorphisms of its freely indecomposable free factors.

Convention 2.2.15. Unless stated otherwise, instead of saying the *cyclic JSJ decomposition of* $F_{R(S)}$ *modulo* F, we will simply say the *JSJ of* $F_{R(S)}$.

Convention 2.2.16. Unless stated otherwise, instead of saying the *canonical* $F-$*automorphisms with respect to* D where D is a JSJ, we will simply say the *canonical automorphisms of* $F_{R(S)}$.

The following Theorem is proved in [28].

Theorem 2.2.17. *If $F_{R(S)} \neq F$ is fully residually free and is freely indecomposable (modulo F) then it admits a non trivial cyclic splitting modulo F.*

Corollary 2.2.18. *If $F_{R(S)} \neq F$ then it has a nontrivial JSJ.*

2.3 The Structure of $\mathbf{Hom}_F(F_{R(S)}, F)$

Definition 2.3.1. *A Hom diagram* for $\mathrm{Hom}_F(G, F)$, denoted $\mathrm{Diag}(G, F)$, consists of a finite directed rooted tree T with root v_0, along with the following data:

- To each vertex, except the root, v of T we associate a fully residually F group $F_{R(S_v)}$.
- The group associated to each leaf of T is a free product $F * F(Y)$ where Y is some set of variables.
- To each edge e with initial vertex v_i and terminal vertex v_t we have a proper F-epimorphism $\pi_e : F_{R(S_{v_1})} \to F_{R(S_{v_2})}$

We point out that in the work of Sela, the *Hom* diagram is called a *Makanin-Razborov* diagram (relative to F) and that our fully residually F groups are *limit groups* (relative to F). The following theorem gives a finite parametrisation of the solutions of systems of equations over a free group.

Theorem 2.3.2 ([30, 52])**.** *For any system of equations $S(x_1, \dots, x_n)$ there exists a Hom diagram Diag($F_{R(S)}, F$) such that for every $f \in Hom_F(F_{R(S)}, F)$ there is a path*

$$v_0, e_1, v_1, e_2, \dots, e_{m+1}, v_{m+1}$$

from the root v_0 to a leaf v_{m+1} such that

$$f = \rho \circ \pi_{v_{m+1}} \circ \sigma_{v_m} \circ \dots \circ \sigma_{v_1} \circ \pi_{e_1}$$

where the σ_{v_j} are canonical *F-automorphisms of $F_{R(S_{v_j})}$, the π_j are epimorphisms $\pi_j : F_{R(S_{v_j})} \to F_{R(S_{v_{j+1}})}$ inside Diag($F_{R(S)}, F$), and ρ is any F-homomorphism $\rho : F_{R(S_{v_{m+1}})} \to F$ from the free group $F_{R(S_{v_{m+1}})}$ to F.*

Definition 2.3.3. Let D be a cyclic splitting of $F_{R(S)}$. If v is a valence 1 vertex of A, the graph underlying D, and A_v is cyclic, then it is called a *hair*.

Definition 2.3.4. Let D be the JSJ of $F_{R(S)}$ and let D' be splitting of $F_{R(S)}$ obtained by collapsing hairs into the adjacent vertex groups. Then D' is called the *hairless JSJ of* $F_{R(S)}$.

Lemma 2.3.5. *Theorem 2.3.2 holds if we replace the canonical automorphisms w.r.t. the JSJ by the canonical automorphisms w.r.t. the hairless JSJ.*

Proof. Since the hairless JSJ is a collapse of the JSJ, we have that the canonical automorphisms w.r.t. the hairless JSJ are a subset of the canonical automorphisms. On the other hand all the Dehn twists associated to hairs are trivial so the sets of automorphisms are equal. ☐

Convention 2.3.6. Unless stated otherwise, we will always replace the JSJ by the hairless JSJ.

2.3.1 Resolutions

An extremely useful tool for studying the vertex groups and getting structural information is resolutions.

Definition 2.3.7. An epimorphism of $\rho : F_{R(S)} \to F_{R(S')}$ of fully residually F groups is called *strict* if it satisfies the following conditions on the cyclic JSJ splitting modulo F.

- For each abelian vertex group A, ρ is injective on the subgroup $A_1 \leq A$ generated by the boundary subgroups in A.
- ρ is injective on edge groups.
- The image of QH subgroups is nonabelian.
- For every rigid subgroup R, ρ is injective on the *envelope* $\widetilde{R}$ of R, defined by first replacing each abelian vertex group with its boundary subgroups and

letting $\widetilde{R}$ be the subgroup of the resulting group generated by R and by the centralizers of incident edge groups.

Definition 2.3.8. A resolution of $F_{R(S)}$

$$\mathcal{R}: \ F_{R(S)} \xrightarrow[\pi_1]{} \cdots \xrightarrow[\pi_p]{} F_{R(S_p)} \xrightarrow[\pi_{p+1}]{} F$$

is a sequence of proper epimorphisms of fully residually F groups.[1] A *strict* resolution is a resolution such that all the epimorphisms are strict.

The next result follow immediately from Theorem 2.3.2.

Theorem 2.3.9. *[52, 28] If $F_{R(S)}$ is fully residually F then it admits a strict resolution.*

This next result, however, requires more work:

Lemma 2.3.10. *[52, 28] The subset $\Phi_{\mathcal{R}} \leq \hom_F(F_{R(S)}, F)$ of F-morphisms that factor through a strict resolution $\mathcal{R}$ F-discriminate $F_{R(S)}$.*

This next definition illustrates what is going on with non-free rigid vertex groups and why they eventually split along the strict resolution.

Definition 2.3.11. Let $R_1 \leq F_{R(S)}$ be a non free rigid subgroup and let $\beta \in R_1$ generate a boundary subgroup. We say that β *obstructs* R_1 if there is a splitting of R_1 in which β is hyperbolic. We say that an element of an abelian vertex group $A \leq F_{R(S)}$ is *exposed* if it does not lie in a boundary subgroup.

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[1] In [28] this is called a *fundamental sequence*

2.3.2 Parametric words

Parametric words were first used by Lyndon to describe the solution sets of one variable equations. Although collections of parametric words alone are insufficient to describe solutions of systems of equations over free groups in general, even for only two variables (Appel [2],) they appear very often. Given the description of solutions given in Theorem 2.3.2 it should be clear how these parametric words arise naturally.

Definition 2.3.12. Let F be a free group. An expression of the form

$$f_1 p_1^{n_1} f_2 \dots f_n p_2^{n_2} f_{n_1}$$

where the f_i and p_i lie in F and the n_i are variables in $\mathbb{Z}$ is called a *one level parametric word in* F. Inductively we define an *n-level parametric word* to be an expression of the form

$$f_1 p_1^{n_1} f_2 \dots f_n p_2^{n_2} f_{n_1}$$

where $f_i \in F, n_i$ are variables and the p_i are m_i-level parametric with

$$\max(m_i) = n - 1$$

Formally, a parametric word defines a subset of F.

CHAPTER 3

The equation $w(x, y) = u$ over free groups

In this chapter we will study equations of the simple form $w(x, y) = u$ over F, where $u \in F$, $w(x, y)$ is a word in $\{x, y\}^{\pm 1}$ and x, y are the unknowns. This is the class of equations was considered by Hmelevskiĭ and Wicks in [22, 62]. Not only will we give a complete description of the possible arising coordinate groups, but we will also describe all the possible Hom diagrams, see Theorem 3.1.27. This enables us to give a complete and explicit description of the solutions sets of these equations. We will then give an example of a single equation not of the type above whose solution set does not fall into the previous description.

The content of this chapter is based on the published article [60].

3.1 The system of equations $S = \{w(x, y)u^{-1}\}$

Definition 3.1.1. Let ϕ be a solution of S, then the *rank* of ϕ is the rank of the subgroup $\langle \phi(x), \phi(y) \rangle \leq F$.

If all solutions of S are of rank 1, then $V(S)$ is easy to describe and is given in Section 3.1.1. If S has solutions of rank 2, then there will be infinitely many such solutions. For this case we will prove that $\mathrm{Diag}(F_{R(S)}, F)$ correspond to one of three cases (see Figure 3–1.) We will moreover describe the possible splittings of $F_{R(S)}$ and the associated canonical automorphisms. This description along with Theorem 2.3.2, will enable us to describe $V(S)$ as a set of pairs of words in F (see Theorem 3.1.27).

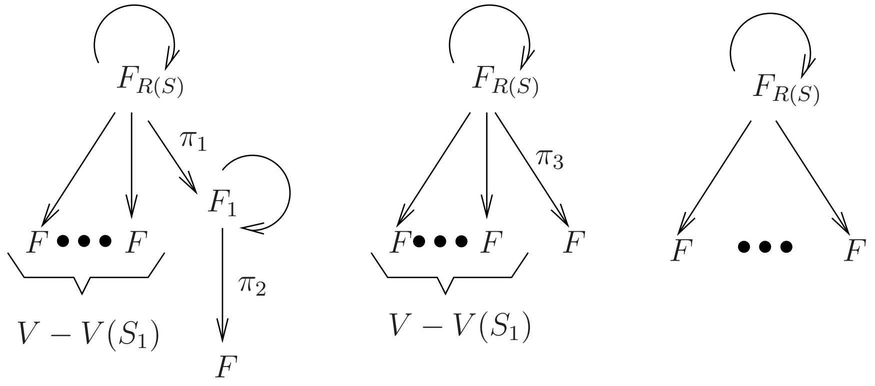

Figure 3–1: Hom diagrams corresponding to cases 1., 2., and 3. of Corollary 3.1.11, π_1, π_2, π_3 are given in Proposition 3.1.13.

3.1.1 Easy Cases and Reductions

By Proposition 2.1.14 we need only concern ourselves with the case where $w(x, y)$ is not primitive. We state some results that enable us to simplify matters:

Lemma 3.1.2. *The equation* $w(x, y) = 1$ *doesn't admit any rank 2 solutions.*

Let $\sigma_x(w)$ and $\sigma_y(w)$ be the exponents sums of x and y respectively in the word $w(x, y)$. Then it is easy to see that

$$V(S) = \{(r^{n_1}, r^{n_2}) \in F \times F | r \in F;\ n_1\sigma_x(w) + n_2\sigma_y(w) = 0\} \tag{3.1}$$

In this case we have that $F_{R(S)} \approx F * < t >$ and the mapping $F[x, y]/\mathrm{ncl}(S) \to F_{R(S)}$ is given by the mapping

$$\begin{cases} f \mapsto f, f \in F \\ x \mapsto t^{r_x} \\ y \mapsto t^{r_y} \end{cases}$$

where (r_x, r_y) is a generator of the subgroup $\{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} | a\sigma_x(w) + b\sigma_y(w) = 0\}$.

Lemma 3.1.3. *If $w(x, y) = v(x, y)^n, n > 1$ then either the variety $V(\{w(x, y)u^{-1}\})$ is empty or $u = r^n$ for some $r \in F$ and we have the equality $V(\{w(x, y)u^{-1}\}) = V(\{v(x, y)r^{-1}\})$.*

We will always assume that $w(x, y)$ is not a proper power. Although this may seem somewhat contrived, our reason for doing so is twofold: firstly, requiring that an element is primitive is not enough; in our theorems we want to exclude the case where $w(x, y)$ is a proper power of a primitive element as, again, solutions are easy to describe. Secondly, if $w(x, y) = v(x, y)^n$ with n maximal, then in the cyclic JSJ splitting of $F_{R(S)}$ modulo F, the edge group will be generated by $v(x, y)$ and not $w(x, y)$. For the next result we need the following theorem:

Theorem 3.1.4 (Main Theorem of [3]). *Let $w = w(x_1, x_2, \dots, x_n)$ be an element of a free group F freely generated by $x_1, x_2, \dots, x_n$ which is neither a proper power nor a primitive. If $g_2, g_2, \dots, g_n, g$ are elements of a free group connected by the relation*

$$w(g_1, g_2, \dots, g_n) = g^m \quad (m > 1)$$

then the rank of the group generated by $g_1, g_2, \dots, g_n$ is at most $n - 1$.

Corollary 3.1.5. *Suppose that $w(x, y)$ is neither primitive nor a proper power. If $u = r^n, n > 1$ is a proper power then the equation $w(x, y) = u$ doesn't have any rank 2 solutions.*

Proof. Suppose not then there is a solution $\phi : F_{R(S)} \to F$ such that $\overline{x} = \phi x, \overline{y} = \phi y$ and $[\overline{x}, \overline{y}] \neq 1$ which means that $\langle \overline{x}, \overline{y} \rangle$ is free group of rank two. But we have the identity $w(\overline{x}, \overline{y}) = r^n$, which by Theorem 3.1.4 implies that rank of $\langle \overline{x}, \overline{y} \rangle$ is at most one –contradiction. $\square$

3.1.2 Possible cyclic JSJ splittings of $F_{R(S)}$ and canonical automorphisms

Lemma 3.1.6. *Suppose that $w(x,y)$ is neither primitive nor a proper power. If $w(x,y) = u$ has a rank 2 solution then the group*

$$F[x,y]/ncl(S) \approx F *_{u=w(x,y)} \langle x, y \rangle$$

is fully residually F and, in particular, we have that

$$F_{R(S)} = F *_{u=w(x,y)} \langle x, y \rangle$$

Proof. Let $(\overline{x}, \overline{y})$ be a rank 2 solution. Let $F_1 = \langle F, t | t^{-1}ut = u \rangle$, F_1 is a rank one free extension of a centralizer of F, and therefore is fully residually F. By definition $F-$subgroups are also fully residually F. Let $H = \langle \overline{x}, \overline{y} \rangle \leq F$ and let $H' = t^{-1}Ht$. By Britton's Lemma we see that $H' \cap F = \langle u \rangle$ and that

$$\langle F, H \rangle \approx F_{u=w(\overline{x}^t,\overline{y}^t)} H' \approx F *_{u=w(x,y)} \langle x, y \rangle$$

so this gives an $F-$embedding $F *_{u=w(x,y)} \langle x, y \rangle \hookrightarrow F_1$ so $F *_{u=w(x,y)} \langle x, y \rangle$ is fully residually F. By Corollary 2.1.7 we obtain the equality

$$F_{R(S)} = F[x,y]/\mathrm{ncl}(S)$$

$\square$

Lemma 3.1.7. *If $w(x,y)$ is not primitive nor a proper power then $F_{R(S)} = F *_{u=w(x,y)} \langle x, y \rangle$ is freely indecomposable modulo F.*

Proof. Suppose not. Since $\langle x, y \rangle$ is a free group of rank 2, if it splits freely with nontrivial factors, then it must split as a free product of two cyclic groups. Since any splitting of $F_{R(S)}$ modulo F must also be modulo $w(x,y)$ we have that $w(x,y)$ must lie in one of these free cyclic factors, contradicting the hypotheses of the lemma. $\square$

Given this first decomposition as an amalgam, we wish to see how it can be refined to a cyclic JSJ decomposition modulo F. By the Freiheitssatz, the subgroup $\langle \overline{x}, \overline{y} \rangle \leq F_{R(S)}$ is free of rank 2. So to investigate cyclic JSJ decomposition modulo F, we must first look at the possible cyclic splitting of $\langle \overline{x}, \overline{y} \rangle$. Our main tool will be the following theorem of Swarup:

Theorem 3.1.8 (Theorem 1 of [57])**.** *(A) Let $G = G_1 *_H G_2$ be an amalgamated free product decomposition of a free group G with H finitely generated. Then, there is a non-trivial free factor H' of H such that H' is a free factor of either G_1 or G_2.*

(B) Let $G = J*_{H,t}$ be an HNN decomposition of a free group G with H finitely generated. Then there are decompositions $H = H_1 * H_2, J = J_1 * J_2$ with H_1 non trivial such that H_1 is a free factor of J_1 and $t^{-1}H_1t$ is conjugate in J to a subgroup of J_2.

Corollary 3.1.9. *If $G = G_1 *_{\langle\gamma\rangle} G_2$ is an amalgamated free product decomposition of a free group over a nontrivial cyclic subgroup, then $Rank(G) = Rank(G_1) + Rank(G_2) - 1$.*

Lemma 3.1.10. *Let G be a free group of rank 2 and let $w \in G$ be non primitive, and not a proper power. Then the only possible almost reduced (see Definition 2.2.12) nontrivial cyclic splittings of G as the fundamental group of a graph of groups with w elliptic are as*

(i) a star of groups, specifically a graph of groups whose underlying graph is simply connected, consisting of a center vertex v_c and a collection of peripheral vertices

$v_1, \ldots, v_m$ connected to v_c by an edge. The group associated to v_c, called the central group*, is free of rank 2 and each edge group is nontrivial, cyclic and is a proper finite index subgroup of the associated "peripheral" vertex group*[1] *; or*

(ii) as an HNN extension

$$G = \langle H, t | t^{-1} p t = q \rangle; p, q \in H - \{1\}$$

where $w \in H$ and H is another free group of rank 2. Moreover we have that $H = \langle p \rangle * \langle q \rangle$ i.e. $G = \langle p, t \rangle$.

Proof. Let D be a splitting of G. If G splits as a free product with amalgamation $G = G_1 *_{\langle \gamma \rangle} G_2$ then if γ is not trivial, Corollary 3.1.9 forces one of the factors to be cyclic. Since we are assuming almost reducedness we must have that the edge group is a finite index subgroup of one of the cyclic factors. Suppose G_2 is a cyclic factor and let z be a generator of G_2. Then the free group G is obtained by adjoining the n^{th} root z of the element $\gamma \in G_1$, which is a free group of rank 2. It is however impossible to have a further splitting $G_1 *_{\langle \gamma \rangle} G_2 *_{\langle \gamma' \rangle} *G_3$ with G_2 and G_3 cyclic and with $\langle \gamma \rangle, \langle \gamma' \rangle$ proper finite index subgroups of G_2, G_3 (resp.) since then, by an easy computation using normal forms, it would be possible to get a counter example to commutation transitivity, which must hold in a free group. The general star case follows.

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[1] The vertex groups $v_1, \ldots, v_m$ are in fact hairs as in Definition 2.3.3

If the underlying graph of D is simply connected and one of the edge groups is trivial, then we can collapse D to a free product $G_1 * G_2$ with nontrivial factors, and with w lying in one of the vertex groups, by Grushko's Theorem we must have $\text{Rank}(G_1) = \text{Rank}(G_2) = 1$ and our assumption that w is elliptic in D and not a proper power forces w to be primitive –contradiction. We have therefore covered the case where the underlying graph is simply connected.

If the underlying graphs has two cycles (and a nontrivial vertex group), then we would have a proper epimorphism $G \to F(a, b)$ which contradicts the Hopf property. *Claim: If $G = \langle H, t | t^{-1}pt = q\rangle$, then H is a free group of rank 2.* By Theorem 3.1.8 (B) and conjugating boundary monomorphisms we can arrange so that

$$H = H_1 * H_2 \text{ with } p \in H_1 \text{ and } q \in H_2 \tag{3.2}$$

Theorem 3.1.8 (B) moreover gives us that without loss of generality we can assume that $\langle q\rangle$ is a free factor of H_2. This means that

$$H_2 = H_2' * \langle q\rangle \tag{3.3}$$

Letting $H' = H_1 * H_2'$ we get that $H = H' * \langle q\rangle$ so combining (3.2) and (3.3) gives us a presentation $G = \langle H', t, q | t^{-1}pt = q\rangle$ which via a Tietze transformation gives us

$$G = \langle H', t | \emptyset\rangle \tag{3.4}$$

which forces H' to be cyclic which means that H has rank 2. Moreover, we see immediately that $H = \langle p\rangle * \langle q\rangle$. $\square$

Recall that by Lemma 2.3.5 we can ingore case (i) above. We denote by Δ the group of canonical $F-$automorphisms of $F_{R(S)}$ (see Definition 2.2.14.)

Corollary 3.1.11. *There are three possible classes of cyclic JSJ decomposition modulo F of $F_{R(S)}$:*

1. $F_{R(S)} \approx F *_{u=w(x,y)} \langle x, y\rangle$ *and* $\Delta = \langle \gamma_w \rangle$, *where* γ_w *is the automorphism that extend the mapping:*

$$\gamma_w : \begin{cases} f \mapsto f; & f \in F \\ z \mapsto w^{-1}zw; & z \in \langle \overline{x}, \overline{y}\rangle \end{cases}$$

2. *The subgroup* $\langle x, y\rangle$ *splits as a cyclic HNN-extension:*

$$\langle x, y\rangle = \langle H, t | t^{-1}pt = q\rangle$$

with $w(x, y) \in H$ *so that* $F_{R(S)} \approx F *_{u=w(x,y)} \langle H, t|t^{-1}pt = q\rangle$ *and* $\Delta = \langle \gamma_w, \tau\rangle$ *where these are the automorphisms that extend the mappings:*

$$\gamma_w : \begin{cases} f \mapsto f; & f \in F \\ z \mapsto w^{-1}zw; & z \in \langle \overline{x}, \overline{y}\rangle \end{cases} ; \tau : \begin{cases} z \mapsto z; & z \in \langle F, H\rangle \\ t \mapsto tq \end{cases}$$

3. $F_{R(S)} \approx F *_{u=w(x,y)} Q$ *where* Q *is a QH subgroup and, up to rational equivalence,* $Q = \langle x, y, w|[x, y]w^{-1}\rangle$. Δ *is generated by the automorphisms extending the mappings:*

$$\gamma_w;\ \delta_x : \begin{cases} x \mapsto yx \\ \textit{identity on } F \cup \{y\} \end{cases} ;\ \delta_y : \begin{cases} y \mapsto xy \\ \textit{identity on } F \cup \{x\} \end{cases}$$

Proof. Suppose first that the cyclic JSJ decomposition of $F_{R(S)}$ modulo F has a QH subgroup Q. Then Q must be a subgroup of $\langle \overline{x}, \overline{y} \rangle$, in particular there must be a splitting of $\langle \overline{x}, \overline{y} \rangle$ modulo w such that Q is one of its vertex groups. By Lemma 3.1.10 we must either have that $Q = \langle \overline{x}, \overline{y} \rangle$, or $\langle \overline{x}, \overline{y} \rangle$ is an HNN extension of Q. Either way we must have that Q is a free group of rank 2. The possible punctured surfaces S such that $\pi_1(S)$ is a free group of rank 2 are the once punctured torus or the once punctured Klein bottle, the latter is not allowed (see Theorem 3 of [27].) Moreover, we see that if $\langle \overline{x}, \overline{y} \rangle$ is an HNN extension of Q then the associated subgroups must be conjugate in Q, which would imply that $\langle \overline{x}, \overline{y} \rangle$ contains an abelian free group of rank 2 –contradiction. It follows from Corollary 2.1.15 that, up to rational equivalence, the only possibility is as in case 3. of the statement.

The rest of the statement follows immediately from Lemma 3.1.10 and Definition 2.2.14. ☐

3.1.3 Solutions of rank 1

We now consider solutions of rank 1. Although everything can easily be described in terms of linear algebra, it is instructive to explain this in terms of Hom diagrams and canonical automorphisms, because as we shall see these provide examples of canonical epimorphisms that are not *strict* (recall Definition 2.3.7.)

As we saw earlier, rank 1 solutions occur when we are solving $w(x, y) = 1$. More generally a rank 1 solutions occurs if and only if $w(x, y) = u = v^d$ where $d = gcd(\sigma_x(w), \sigma_y(w))$; $\sigma_x(w), \sigma_y(w)$ denote the exponent sums of x, y in $w(x, y)$. Corollary 3.1.5 states that if $d > 1$, but $w(x, y)$ not primitive and not a proper power, then all solution of $w(x,y)=u$ have rank 1. If $d=1$ then $w(x,y)=u$ may have both rank 1 and rank 2 solutions.

Let $S_1 = \{w(x,y)u^{-1}, [x,y]\}$, then all rank 1 solutions must factor through $F_{R(S_1)}$. If $d>1$ then, since all solutions are rank 1, we must have we in fact have $Rad(\{w(x,y)u^{-1}\}) = ncl(\{w(x,y)u^{-1}, [x,y]\})$. As a set, these solutions are easy to describe:

$$V(S_1) = \{(u^{n_1}, u^{n_2}) \in F \times F | n_1\sigma_x(w) + n_2\sigma_y(w) = d\} \tag{3.5}$$

Let p, q be integers such that

$$p\sigma_x(w) + q\sigma_y(w) = d \tag{3.6}$$

then doing some linear algebra we have that n_1, n_2 in (3.5) are given by

$$(n_1, n_2) = (p, q) + m(\sigma_y(w), -\sigma_x(w));\ m \in \mathbb{Z} \tag{3.7}$$

We now investigate the situation where $w(x,y)=u$ has rank 1 and rank 2 solutions, i.e $V(S) \supsetneq V(S_1)$. We first want to understand $F_{R(S_1)}$.

Lemma 3.1.12. *Suppose that $w(x,y)$ is not primitive nor a proper power and suppose moreover that $w(x,y)=u$ admits rank 1 and rank 2 solutions. Then there $F_{R(S_1)}$ is isomorphic to $\langle F, s | [u,s]=1\rangle = F_1$. The $F-$morphism $\pi_1 : F_{R(S_1)} \to F_1$ given by*

$$\pi_1(x) = u^p s^{\sigma_y(w)} = \overline{x};\ \pi_1(y) = u^q s^{-\sigma_x(w)} = \overline{y} \tag{3.8}$$

where p, q are as in equation (3.6), realizes this isomorphism.

Proof. Consider the $F-$epimorphism $\pi_1 : F_{R(S_1)} \to \langle F, s|[u,s]=1\rangle = F_1$ given by (3.8) On one hand we see that π_1 is surjective which gives an injection

$$\mathrm{Hom}_F(F_1, F) \hookrightarrow \mathrm{Hom}_F(F_{R(S_1)}, F) \tag{3.9}$$

via pullbacks $f \mapsto f \circ \pi_1$. On the other hand F_1, a free rank 1 extension of a centralizer, is fully residually free. On the third hand the group Δ_1 of canonical F automorphisms of F_1 is generated by the automorphism given by:

$$\delta : \begin{cases} s \mapsto su \\ f \mapsto f \quad f \in F \end{cases}$$

and if we consider the $F-$epimorphism $\pi_2 : F_1 \to F$ given by $\pi_2(s) = u$ then we immediately see that the set

$$V = \{(\pi_2(\sigma^m(\overline{x})), \pi_2(\sigma^m(\overline{y})) \in F \times F | \sigma \in \Delta_1\}$$

of images of (x,y) via the mappings $\pi_2 \circ \sigma \circ \pi_1, \sigma \in \Delta_1$ coincides with $V(S_1)$. And since $\mathrm{Hom}_F(F_1, F) = \{\pi_2 \circ \sigma | \sigma \in \Delta_1\}$ we get that the correspondence (3.9) is in fact a bijective correspondence. It follows that $F_{R(S_1)} \approx_F F_1$. $\square$

Proposition 3.1.13. *Let $w(x,y)$ be non primitive and not a proper power. Suppose moreover that $w(x,y)=u$ has rank 1 and rank 2 solutions. Then*

(i) if $F_{R(S)}$ is as in 1. *in Corollary 3.1.11 , then $V(S_1)$ is represented by the following branch in Diag$(F_{R(S)}, F)$:*

$$\begin{tikzcd} F_{R(S)} \arrow[r, "\pi_1"] & F_1 \arrow[r, "\pi_2"] \arrow[loop above, "\sigma"] & F \end{tikzcd} \tag{3.10}$$

where $\sigma \in \Delta_1$.

(ii) If $F_{R(S)}$ *is as in* 2. *in Corollary 3.1.11 , then* $V(S_1)$ *is represented by the following branch in Diag*$(F_{R(S)}, F)$*:*

$$\begin{tikzcd} F_{R(S)} \arrow[loop above, "\sigma"] \arrow[r, "\pi_3"] & F \end{tikzcd} \tag{3.11}$$

where $\sigma \in \Delta$ *and* $\pi_3 = \pi_2 \circ \pi_1$

Where π_1, π_2 *and* Δ_1 *were defined in the previous proof.*

Proof. We first note that if $F_{R(S)}$ corresponds to case 3. of Corollary 3.1.11, then the equality (3.6) is impossible. In both possible cases we have epimorphisms

$$\begin{tikzcd} F_{R(S)} \arrow[r, "\pi_1"] & F_1 \arrow[r, "\pi_2"] & F \end{tikzcd} \tag{3.12}$$

We saw that all solutions rank 1 solutions factor through π_1. If $F_{R(S)}$ is as in 1. in Corollary 3.1.11 then Δ is generated by γ_w, now since $\pi_1 \circ \gamma_w = \pi_1$ we have that solutions in $V(S_1)$ must factor through F_1 and are parameterized by Δ_1.

If $F_{R(S)}$ is as in 2. in Corollary 3.1.11, then $\langle \overline{x}, \overline{y} \rangle$ splits as

$$\langle H, t | t^{-1}pt = q \rangle; p, q \in H$$

moreover by Lemma 3.1.10 we have that $\langle \overline{x}, \overline{y} \rangle = \langle p, t \rangle$. We consider this basis of $\langle \overline{x}, \overline{y} \rangle$. Let $\pi_1(t) = \overline{t}, \pi_1(p) = \overline{p}$, then the subgroup $\mathbb{Z} \oplus \mathbb{Z} \approx A = \langle u, s \rangle \leq F_1$ is generated by $\overline{p}, \overline{t}$. We note that in $F_{R(S)}$, as written as a word in $\{p, t\}^{\pm 1}$, $w(x, y) = w'(p, t) = u$ has exponent sum zero in the letter t. Since A is the abelianization of $\langle \overline{x}, \overline{y} \rangle$, we have that in $A, u = 0\overline{t} + n\overline{p}$ and since u lies in a minimal generating set of

A we must have $n = \pm 1$. It therefore follows that for the Dehn twist τ, which sends $t \mapsto tq$, we have $\pi_1 \circ \tau = \delta \circ \pi_1$, where δ is the generator of Δ_1. It follows that the canonical F-automorphisms of F_1 in (3.12) can be "lifted" to $F_{R(S)}$ and the branch (3.11) gives us a parameterization of $V(S_1)$. $\square$

3.1.4 Solutions of rank 2

Before being able to make our finiteness arguments we need some preliminary setup. We will study more closely mappings $F(x,y) \to F$.

Definition 3.1.14. (i) Let (f_1, f_2) be a pair of words in a free group, then an *elementary Nielsen move* (e.N.m.) is a mapping of the form

$$(f_1, f_2) \mapsto (f_1, (f_2^{\epsilon_1} f_1^{\epsilon_2})^{\epsilon_3}) \text{ or } (f_1, f_2) \mapsto ((f_1^{\epsilon_1} f_2^{\epsilon_2})^{\epsilon_3}), f_2)$$

with $\epsilon_1, \epsilon_3 \in \{-1, 1\}$ and $\epsilon_2 \in \{-1, 0, 1\}$.

(ii) For $F(x,y)$, the free group on the basis $\{x, y\}$, an *elementary Nielsen transformation* (e.N.t.) is an element of $Aut(F(x,y))$ that is defined by the mappings:

$$\begin{cases} x \mapsto (x^{\epsilon_1} y^{\epsilon_2})^{\epsilon_3} \\ y \mapsto y \end{cases} \text{or} \begin{cases} x \mapsto x \\ y \mapsto (y^{\epsilon_1} x^{\epsilon_2})^{\epsilon_3} \end{cases}$$

with $\epsilon_1, \epsilon_3 \in \{-1, 1\}$ and $\epsilon_2 \in \{-1, 0, 1\}$.

Lemma 3.1.15. *Suppose ϕ, given by $(x_0, y_0) \in F \times F$, is a rank 2 solution of $w(x,y) = u$, let*

$$(x_0, y_0) \xrightarrow[m_1]{} \cdots \xrightarrow[m_n]{} (x_n, y_n)$$

be a sequence of e.N.m. then

(i) there is a corresponding sequence of e.N.t $t_1, \ldots, t_n$ *such that letting* $w_0(x,y) = w(x,y)$ *and* $w_{j+1}(x,y) = t_{j+1}(w_j(x,y))$ *we have the equalities*

$$u = w_0(x_0, y_0) = \ldots = w_n(x_n, y_n) \tag{3.13}$$

(ii) Let

$$\alpha = t_n \circ \ldots \circ t_1 \in Aut(F(x,y)) \tag{3.14}$$

then the mapping $\phi' = \phi \circ \alpha^{-1} : F(x,y) \to F$ *is given by the pair* (x_n, y_n)

sketch of proof. Noting that a rank 2 solution isomorphically identifies the subgroup $\langle \overline{x}, \overline{y} \rangle \leq F_{R(S)}$ with a rank 2 subgroup of a free group, the proof is essentially the same as the proof that elementary Nielsen transformations generate the automorphisms of a f.g. free group (See Proposition I.4.1. of [39]). □

The reader can look at Section I.2 of [39] for the necessary background for the next lemma.

Lemma 3.1.16. *Fix a basis* X *of* F*, then to any subgroup* $H \leq F$ *of rank* n *we can canonically associate an ordered set of Nielsen reduced generators* $(j_1, \ldots, j_n)$*, moreover this ordered set can be obtained from any ordered* n*-tuple of generators* $(h_1, \ldots, h_n)$ *of* H *via a sequence of e.N.m.*

We now give names to all of these:

Definition 3.1.17. Let ϕ, given by (x_0, y_0), be a solution of $w(x,y) = u$. Let

$$(x_0, y_0) \xrightarrow[m_1]{} \cdots \xrightarrow[m_n]{} (x_n, y_n)$$

be the sequence of e.N.m. that brings the pair (x_0, y_0) to the canonical pair (x_n, y_n) of generators of $\langle x_0, y_0\rangle$ guaranteed by Lemma 3.1.16. Then we have:

- The pair (x_n, y_n) is called the *terminal pair* of ϕ (denoted $tp(\phi)$.)
- The word $w_n(x, y) \in \langle \overline{x}, \overline{y}\rangle$ in (3.13) is called the *terminal word* of ϕ (denoted $tw(\phi)$.)
- The automorphism $\alpha \in Aut(F(x, y))$, is the *automorphism associated* to ϕ (denoted α_ϕ.)

Proposition 3.1.18. *Let $S = \{w(x, y) = u\}$ and let $U \subset V(S)$ be the open subvariety of rank 2 solutions, then there are only finitely many possible terminal pairs and terminal words that can be associated to solutions $\phi \in U$.*

Proof. Fix a basis X of F, we first show finiteness of possible terminal pairs.

Let ϕ be a solution, given by (x_0, y_0) and let $H = \langle x_0, y_0\rangle \leq F$ and let Γ be the Stallings graph for H (See, for instance, [54].) Then there is a path in Γ with label u. We also have that Nielsen generators can be read directly off Γ (see [23]) as labels of simple closed paths. If we define the *radius* of Γ to be the distance between the basepoint of Γ and the "farthest" vertex, then we see that the length of the Nielsen generators (x_m, y_m) is bounded by two times the radius. Moreover since $w(x, y)$ is neither primitive nor a proper power in $F(x, y) \approx H$, u is not primitive nor a proper power in H. It follows that the reduced path in Γ labeled u must cover the whole graph which means $|u|$ is at least twice the radius, hence

$$|x_m|, |y_m| \leq |u|$$

so the number of possible terminal pairs is bounded.

Consider now the terminal word $w_n(x,y)$. Since $(x_m, y_m) \in F \times F$ is a Nielsen reduced pair we have that

$$|w_n(x,y)|_{\{x,y\}} \leq |w_n(x_n,y_n)|_X = |u|_X$$

which bounds the number of terminal words. □

We now connect all these ideas to solutions of equations. The next observation is obvious but critical.

Lemma 3.1.19. *Let $F_{R(S)}$ be the coordinate group of $w(x,y) = u$, with $w(x,y)$ not primitive, not a proper power and such that $w(x,y)$ has a rank 2 solution. Then the group of F-automorphisms of $F_{R(S)}$ are induced by the automorphisms of the free subgroup $\langle x, y\rangle$ that fix $w(x,y)$.*

Proposition 3.1.20. *Suppose that ϕ and ϕ' are solutions $F_{R(S)} \to F$ of $w(x,y) = u$. And suppose moreover that $tp(\phi) = tp(\phi')$ and $tw(\phi) = tw(\phi')$, then there is an automorphism $\beta \in Aut_F(F_{R(S)})$ such that $\phi' = \phi \circ \beta$.*

Proof. Let ϕ be given by (x_0, y_0) and let ϕ' be given by (x_0', y_0'). Then we have a sequence of e.N.m.

$$\begin{tikzcd}
(x_0,y_0) \arrow[r, "m_1"'] & \cdots \arrow[r, "m_n"'] & tp(\phi) = tp(\phi') & \cdots \arrow[l, "m_r'"'] & (x_0', y_0') \arrow[l, "m_1'"']
\end{tikzcd}$$

And we have automorphisms $\alpha_\phi, \alpha_{\phi'}$ such that $\alpha_\phi(w(x,y)) = \alpha_{\phi'}(w(x,y)) = tw(\phi)$. On one hand we have that $\beta = \alpha_\phi^{-1} \circ \alpha_{\phi'} \in \text{stab}(w)$, so by Lemma 3.1.19, $\beta \in Aut(F(x,y))$ extends to an automorphism of $F_{R(S)}$. We moreover have by Lemma

3.1.15 we have that the mappings $F(x,y) \to F$, $\phi' \circ \alpha_{\phi'}^{-1} = \phi \circ \alpha_{\phi}^{-1}$ which means that

$$\phi' = \phi \circ \alpha_{\phi}^{-1} \circ \alpha_{\phi'} = \phi \circ \beta$$

$\square$

So we have proved that all rank 2 solutions are obtained from a finite family $\phi_1, \ldots \phi_N$ of solutions and precomposition with $F-$automorphisms of $F_{R(S)}$. Nothing so far has been said about canonical automorphisms.

Definition 3.1.21. Let $\Delta \leq Aut(F_{R(S)})$ be the group of canonical $F-$automorphisms of $F_{R(S)}$ associated to a cyclic JSJ decomposition modulo F. Let $\phi, \phi' \in \mathrm{Hom}_F(F_{R(S)}, F)$, we say $\phi \sim_\Delta \phi'$ if there is a $\sigma \in \Delta$ such that $\phi \circ \sigma = \phi'$. $\phi \in \mathrm{Hom}_F(F_{R(S)}, F)$ is *minimal* if after fixing a basis X of F the quantity $l_f = |\phi(x)| + |\phi(y)|$ is minimal among all F-morphisms in ϕ's $\sim_\Delta$ equivalence class.

We wish to show that there are only finitely many Δ-*minimal* rank 2 solutions to $w(x,y) = u$. In light of Proposition 3.1.20, this is equivalent to the statement $[\mathrm{stab}(w) : \Delta] < \infty$.

Proving finite index

In [6], it is proved that for freely indecomposable fully residually free groups, the subgroup canonical automorphism is of finite finite index in the group of outer automorphisms. Unfortunately, the result as formulated does not cover the case involving only automorphisms modulo F. We therefore prove this fact directly. What we will essentially show is that *the internal F-automorphisms are of finite index in the whole group of F-automorphisms.* The main pillars of the argument are that the JSJ decomposition is *canonical* in the sense of (4) of Theorem 2.2.13 and the following Theorem:

Theorem 3.1.22 (Corollary 15.2 of [30])**.** *Let G be a nonabelian fully residually free group, and let $\mathcal{A} = \{A_1, \dots, A_n\}$ be a finite set of maximal abelian subgroups of G. Denote by $Out(G; \mathcal{A})$ the set of those outer automorphisms of G which map each $A_i \in \mathcal{A}$ onto a conjugate of itself. If $Out(G; \mathcal{A})$ is infinite, then G has a nontrivial abelian splitting, where each subgroup in $\mathcal{A}$ is elliptic. There is an algorithm to decide whether $Out(G; \mathcal{A})$ is finite or infinite. If $Out(G; \mathcal{A})$ is infinite, the algorithm finds the splitting. If $Out(G; \mathcal{A})$ is finite, the algorithm finds all its elements.*

This next lemma follows immediately from the fact that in free groups n^{th} roots are unique and centralizers of elements are cyclic.

Lemma 3.1.23. *Let $\langle \overline{x}, \overline{y} \rangle$ be a free group and suppose*

$$\langle \overline{x}, \overline{y} \rangle = \langle H, t | t^{-1} p t = q \rangle ; p, q \in H - \{1\}$$

Suppose that for some $g \in \langle \overline{x}, \overline{y} \rangle$ we have the equality

$$g^{-1} p g = q$$

then $g = tq^j$ for some $j \in \mathbb{Z}$.

Proposition 3.1.24. *$\Delta \leq Aut(F(x, y))$ is of finite index in $stab(w)$.*

Proof. If w is conjugate to either $[x, y]$ or $[y, x]$ then the result follows immediately since the stab(w) coincides with the automorphisms given in Corollary 3.1.11. (See, for instance, [43].) We first concentrate on the case where the JSJ of $F_{R(S)}$ is as in case 2. of Corollary 3.1.11.

Suppose the induced splitting of $\langle \overline{x}, \overline{y} \rangle$ is of the form

$$\langle \overline{x}, \overline{y} \rangle = \langle H, t | t^{-1} p t = q \rangle p, q \in H - \{1\}$$

Let $\alpha \in \mathrm{stab}(w) \leq Aut(\langle \overline{x}, \overline{y} \rangle)$, then we can extend α to $\widehat{\alpha} : F_{R(S)} \to F_{R(S)}$. We wish to understand the action of $\widehat{\alpha}$ on $F_{R(S)}$. First note that $\widehat{\alpha}$ restricted to F is the identity and $\widehat{\alpha}(\langle \overline{x}, \overline{y} \rangle) = \langle \overline{x}, \overline{y} \rangle$ On the other hand, $\widehat{\alpha}$ gives another cyclic JSJ decomposition D_1 modulo F:

$$F_{R(S)} = F *_{u=w(x,y)} \langle \widehat{\alpha}(H), \widehat{\alpha}(t) | \widehat{\alpha}(t)^{-1} \widehat{\alpha}(p) \widehat{\alpha}(t) = \widehat{\alpha}(q) \rangle \tag{3.15}$$

with $w \in \widehat{\alpha}(H)$. By Theorem 2.2.13 (4), D_1 can be obtained from D by a sequence of slidings, conjugations and modifying boundary monomorphisms.

$\widehat{\alpha}(H) \cap F = \langle w \rangle$, and H must be obtained from $\widehat{\alpha}(H)$ as in (4) of Theorem 2.2.13, i.e. by slidings, conjugating boundary monomorphisms and conjugations. The only inner automorphism of $F_{R(S)}$ that fixes w is conjugation by $w^k; k \in \mathbb{Z}$; (use Bass-Serre theory and properties of free groups) and since $\widehat{\alpha}(H)$ and H are attached to F at $\langle w \rangle$, slidings will have no effect. It follows that $\widehat{\alpha}(H) = H$. Applying Theorem 2.2.13 again forces p, q to be conjugate in H to $\widehat{\alpha}(p), \widehat{\alpha}(q)$ [respectively or in the other order]. We now have strong information enough on the dynamics of $\mathrm{stab}(w)$ to apply Theorem 3.1.22.

Indeed since $\widehat{\alpha}(H) = H$, we have a natural homomorphism $\rho : \mathrm{stab}(w) \to \widetilde{\mathrm{stab}(w)} \leq Aut(H)$ given by the restriction $\alpha \mapsto \alpha|_H$. Moreover we see that any almost reduced cyclic splitting of H modulo $\{\langle w \rangle, \langle p \rangle, \langle q \rangle\}$ must be trivial, otherwise contradicting Lemma 3.1.10. Let $\pi : Aut(H) \to Out(H)$ be the canonical map (i.e.

quotient out by $Inn(G)$, the subgroup of inner automorphisms). It therefore follows from Theorem 3.1.22 that the image $\pi \circ \rho(\mathrm{stab}(w)) = \overline{\mathrm{stab}(w)}$ must be finite.

First note that $Inn(H) \cap \widetilde{\mathrm{stab}(w)} = \langle \gamma_w \rangle$ which means that

$$\overline{\mathrm{stab}(w)} \approx \widetilde{\mathrm{stab}(w)}/\langle \gamma_w \rangle$$

and this isomorphism is natural. Let $\alpha \in \ker \rho$ then we must have that $\alpha|_H = 1$. In particular we have

$$\alpha(t)^{-1} p \alpha(t) = q$$

which by Lemma 3.1.23 implies that $\alpha(t) = tq^j$ it follows that $\ker(\rho) \leq \langle \tau \rangle$. The other inclusion is obvious so

$$\ker(\rho) = \langle \tau \rangle$$

There is a bijective correspondence between subgroups K of $\widetilde{\mathrm{stab}(w)}$ and subgroups of $\mathrm{stab}(w)$ that contain $\langle \tau \rangle$ given by $K \mapsto \rho^{-1}(K)$. Moreover this correspondence sends normal subgroups to normal subgroups. It follows that $ker(\pi \circ \rho) = \langle \tau, \gamma_w \rangle$ and so we get:

$$\mathrm{stab}(w)/\langle \tau, \gamma_w \rangle \approx \overline{\mathrm{stab}(w)}$$

which is finite. It follows that $[\mathrm{stab}(w) : \langle \tau, \gamma_w \rangle] < \infty$.

In the case where D, the cyclic JSJ of $F_{R(S)}$ modulo F is as in case 1. of Corollary 3.1.11 then again elements of $\alpha \in \mathrm{stab}(w)$ will give new splittings $F_{R(S)} = F *_{u=w(x,y)} \widehat{\alpha}(H)$. Arguing as before, we get that $\widehat{\alpha}(H) = H$ and we can apply Theorem 3.1.22 with $\mathcal{A} = \{\langle w \rangle\}$. We get that $Out(H; \mathcal{A}) \approx \mathrm{stab}(w)/\langle \gamma_w \rangle$ must be finite, otherwise H could split further, contradicting the fact that D was a JSJ splitting, and the result follows. $\square$

By Lemma 3.1.19, Propositions 3.1.18, 3.1.20, and 3.1.24 we get the second half of our main result:

Proposition 3.1.25. *Suppose that $w(x, y)$ is not a proper power, nor is it primitive. Then there are finitely many $\Delta-$minimal rank 2 solutions to the equation $w(x, y) = u$.*

3.1.5 The description of $V(\{w(x, y)u^{-1}\})$

These next two results now follow immediately from Proposition 3.1.25, 3.1.13, Corollary 3.1.11, Lemma 3.1.10 and Theorem 2.3.2.

Theorem 3.1.26. *Suppose that $w(x, y) = u$ has rank 2 solutions and that $w(x, y)$ is not a power of a primitive element. Then the possible Hom diagrams are given in Figure 3–1.*

Theorem 3.1.27. *Suppose that $w(x, y) = u$ has rank 2 solutions and that $w(x, y)$ is neither primitive nor a proper power. Let $\{\phi_i | i \in I\}$ be the collection of $\Delta-$minimal solutions. Then $V(S) = V(S_1) \cup V'$, where $V' = V(S) - V(S_1)$,is given by the following:*

1. *$F_{R(S)} \approx F *_{u=w(x,y)} \langle x, y\rangle$, let $\phi_i(x) = x_i, \phi_i(y) = y_i$ then $V(S) = V(S_1) \cup V'$ where*

   $$V' = \{(u^{-n}x_iu^n, u^{-n}y_iu^n) | i \in I \text{ and } n \in \mathbb{Z}\}$$

   and if the exponent sums $\sigma_x(w), \sigma_y(w)$ of x, y respectively in w are relatively prime, then $V(S_1)$ is non empty and is given by (3.5).

2. $F_{R(S)} \approx F *_{u=w(x,y)} \langle H, t | t^{-1}pt = q \rangle$, $H = \langle p, q \rangle$ *and we can write* $x, y \in \langle \overline{x}, \overline{y} \rangle$ *as words* $x = X(p,q,t), y = Y(p,q,t)$*. Let* $\phi_i(p) = p_i, \phi_i(q) = q_i, \phi_i(t) = t_i$ *then we have that* $V(S) = V(S_1) \cup V'$ *where*

$$\begin{aligned} V' =& \{(X(u^{-n}p_iu^n, u^{-n}q_iu^n, u^{-n}t_iq_i^m u^n), \\ & Y(u^{-n}p_iu^n, u^{-n}q_iu^n, u^{-n}t_iq_i^m u^n)) \mid i \in I, n, m \in \mathbb{Z}\} \end{aligned}$$

and if the exponent sums $\sigma_x(w), \sigma_y(w)$ *of* x, y *respectively in* w *are relatively prime, then* $V(S_1)$ *is non empty and is given by (3.5).*

3. $F_{R(S)} \approx F *_{u=w(x,y)} Q$ *where* Q *is a QH subgroup and, up to rational equivalence,* $Q = \langle x, y, w | [x,y]w^{-1} \rangle$*. Then* $V(S_1)$ *is empty. Let* $\phi_i(x) = x_i, \phi_i(y) = y_i$ *then*

$$V(S) = \{(X_\sigma(x_i, y_i), Y_\sigma(x_i, y_i)) | \sigma \in \Delta\}$$

where the words $\sigma(x) = X_\sigma(x,y), \sigma(y) = Y_\sigma(x,y) \in \langle \overline{x}, \overline{y} \rangle$*.*

We finally note that unless $w(x,y) = u$ is quadratic, then solutions are given by "one level parametric" words (see Definition 2.3.12.)

3.2 An Interesting Example

The Hom diagrams given for $w(x,y) = u$ were very simple. In particular, modulo the slight technicalities of Theorem 3.1.27 item 1, we can say that; unless $w(x,y)$ is a power of a primitive element; there are only finitely many minimal solutions to $w(x,y) = u$ with respect to a group of canonical automorphisms. This translates as the Hom diagram having only one "level". This also means that all *fundamental sequences* or *strict resolutions* of $F_{R(S)}$ have length 1 (see [28] or [52], respectively for definitions.) It is natural to ask this holds true for general equations in two variables. We answer this negatively:

Theorem 3.2.1. *Let $F = F(a,b)$ then the Hom diagram associated to the equation with variables x, y*

$$[a^{-1}ba[b,a][x,y]^2x, a] = 1 \tag{3.16}$$

has branches corresponding to rank 2 solutions that have length at least 2*.*

Proof. First note that via Tietze transformations, we have the following isomorphism:

$$\begin{aligned}&\langle F, x, y | [a^{-1}ba[b,a][x,y]^2x, a] = 1\rangle\\ &\approx \langle F, x, y, t | [x,y]^2x = [a,b]a^{-1}b^{-1}at; [t,a] = 1\rangle\end{aligned}$$

Let $w(x,y) = [x,y]^2x$ and let $u = [a,b]a^{-1}b^{-1}at$. We now embed $G = \langle F, x, y, t | w(x,y) = u, [t,a] = 1\rangle$ into a chain of extensions of centralizers. Let $F_1 = \langle F, t | [t,a] = 1\rangle$ and let $F_2 = \langle F_1, s | [u,s] = 1\rangle$. Let $\overline{x} = b^{-1}t$ and $\overline{y} = b^{-1}ab$. First note that

$$[\overline{x}, \overline{y}]^2\overline{x} = ((t^{-1}b)(b^{-1}a^{-1}b)(b^{-1}t)(b^{-1}ab))^2(b^{-1}t) = [a,b]a^{-1}b^{-1}at = u$$

We now form a double, i.e. we set $x = \overline{x}^s, y = \overline{y}^s$ and let $H = \langle \overline{x}, \overline{y}\rangle = \langle \overline{x}, \overline{y}\rangle^s$. By Britton's Lemma we have that $H \cap \widetilde{F_1} = \langle u\rangle$ and it follows that $\langle F, x, y\rangle$ is isomorphic to the amalgam $F_1 *_{\langle u\rangle} H = G$. Since chains of extensions of centralizers of F are fully residually F, we have that our equation (3.16) is an irreducible system of equations, we write $F_{R(S)} = G$. We note that we have the nontrivial cyclic splitting

$$D: \ F_{R(S)} \approx F_1 *_{\langle u = w(x,y)\rangle} \langle \overline{x}, \overline{y}\rangle$$

moreover since $w(x, y) = [x, y]^2 x$ cannot belong to a basis (see [9]) of $\langle \overline{x}, \overline{y} \rangle$ we have that $F_{R(S)}$ is freely indecomposable modulo F_1. On the other hand, if we take the Grushko decomposition of $F_{R(S)}$ modulo F

$$F_{R(S)} = \widetilde{F} * K_1 * \ldots K_n;\ F \leq \widetilde{F}$$

we see that we must have $F_1 \leq \widetilde{F}$ since $[t, a] = 1 \Rightarrow t \in \widetilde{F}$. It follows that $F_{R(S)}$ is actually freely indecomposable modulo F. It follows that D can be refined to a cyclic JSJ decomposition modulo F.

Suppose towards a contradiction that all branches of the Hom diagram for $\mathrm{Hom}_F(F_{R(S)}, F)$ corresponding to rank 2 solutions had length 1. This means that there are finitely many minimal rank 2 solutions $\phi : F_{R(S)} \to F$. On one hand the element t must be sent to arbitrarily high powers of a, since $F_{R(S)}$ is fully residually F. On the other hand, for there to be a canonical automorphism of $F_{R(S)}$ that sends $t \mapsto ta^n$, there must be a splitting D' of $F_{R(S)}$ with some conjugate of $\langle a \rangle$ as a boundary subgroup, but u would have to be hyperbolic in such a splitting, and since $\langle a \rangle$ is elliptic in D, we would have an elliptic-hyperbolic splitting which by Theorem 2.2.9 would contradict free indecomposability modulo F. □

We now provide some illustration. We determined that $F_{R(S)} = F_1 *_{\langle u = w(x,y) \rangle} \langle \overline{x}, \overline{y} \rangle$ with $u = [a, b]ab^{-1}a^{-1}t$. Now the mapping $x \mapsto x^u$ and $y \mapsto y^u$ extends to a canonical automorphism of $F_{R(S)}$ and along some branch there must be another canonical automorphism that maps $t \mapsto ta^r$. By checking directly we see that $\phi : F_{R(S)} \mapsto F$ given by $x = b^{-1}a, y = b^{-1}ab$ is a solution, so we can get the family of solutions:

$$x = ([a,b]ab^{-1}a^{-1}a^n)^m(b^{-1}a)([a,b]ab^{-1}a^{-1}a^n)^{-m}$$

$$y = ([a,b]ab^{-1}a^{-1}a^n)^m(b^{-1}ab)([a,b]ab^{-1}a^{-1}a^n)^{-m}$$

with n, m in $\mathbb{Z}$. Notice that no precomposition by a canonical automorphism of $F_{R(S)}$ can affect the n parameter. It follows that the set of solution of (3.16) can not be given by precomposing a finite collection of maps $\phi_1, \ldots, \phi_n : F_{R(S)} \to F$ with canonical automorphisms.

CHAPTER 4
The fully residually F quotients of $F * \langle x, y \rangle$

In this chapter we classify the fully residually F quotients of $F[x, y]$. We give a list containing all possible coordinate groups of systems of equations in two variables over F in terms of their cyclic JSJ splittings modulo F. Specifically we give the underlying graphs of groups and the possible vertex groups (see Theorem 4.1.6.) We also give examples showing that many of the entries in this list actually occur. Combining this with the results of Chapter 3 we are able to recover the description of the solutions of systems of equations in two variables given by Ozhigov in [46].

We shall first state the main results before presenting the additional necessary machinery and proof.

4.1 The Classification Theorem

So far the only comprehensive classification theorems of fully residually free groups in terms of the number of generators are the following:

Theorem 4.1.1. *[19] If G is fully residually free group, then*

1. *if Rank$(G) = 1$ then G is infinite cyclic.*
2. *if Rank$(G) = 2$ then G is free or free abelian of rank 2.*
3. *if Rank$(G) = 3$ then G is either free or free abelian of rank 3 or, G is isomorphic to a free rank one extension of a centralizer of a free group of rank 2.*

This next result follows from the results of Appel and Lorenc, although their proofs contained gaps. A correct proof was provided by Chiswell and Remeslennikov.

Theorem 4.1.2. *[1, 33, 7] If $S(x)$ is a an irreducible system of equations over F in one variable then*

$$F_{R(S)} \approx \begin{cases} F \\ F * \langle t \rangle \\ F *_u Ab(u,t) \end{cases} \tag{4.1}$$

where $Ab(u,t)$ denotes the free abelian group with basis $\{u,t\}$.

The class of coordinate groups of irreducible systems of equations in two variables over F is much more varied. The classification we give will be in terms of cyclic JSJ decompositions modulo F. Specifically we will describe the cyclic graphs of groups in terms of the underlying graph, vertex groups, and edge groups. We will also indicate where, up to rational equivalence, the variables x, y are sent.

The groups will be organized as follows: (A) will be the freely decomposable modulo F groups; (B) will be the groups whose JSJ has only one vertex; (C) will be the groups whose JSJ has more than one vertex group, but such that only one of them is nonabelian; (D),(E), and (F) will be the remaining cases. This classification closely follows the proof.

Definition 4.1.3. If $F_{R(S)}$ has a JSJ D with more than 2 nonabelian vertex groups, then we will call a *cyclic collapse* of D a graph of groups obtained by performing a maximal number of edge collapses (see Definition 2.2.5) while ensuring that the resulting graph of groups has at least two nonabelian vertex groups.

Definition 4.1.4. The *first Betti number* of G denoted $b_1(G)$ is the rank of the torsion-free summand of the abelianization of G.

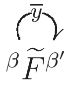
Convention 4.1.5. Throughout the paper whenever $F_{R(S)}$ is given as the fundamental group of a graph of groups modulo F, we shall denote by $\widetilde{F}$ the vertex group that contains F.

Theorem 4.1.6 (The classification theorem)**.** *Let F be a free group of rank $N \geq 2$. If $S = S(x, y)$ is an irreducible system of equations over F in two variables then $F_{R(S)}$ must be one of the following.*

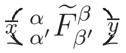
(A) If $F_{R(S)}$ is freely decomposable then,

$$F_{R(S)} \approx \begin{cases} F * \langle t \rangle \\ F * H; \text{ where } H \text{ is fully residually free of rank 2} \\ (F *_u Ab(u, t)) * \langle s \rangle \end{cases}$$

(B) If the JSJ of $F_{R(S)}$ modulo F has one vertex group and one edge group then, up to rational equivalence, we can arrange to have a relative presentation



such that $\widetilde{F} = \langle F, \overline{x}, \beta' \rangle$ and $\beta \in \langle F, \overline{x} \rangle$. If the JSJ has two edges, then, up to rational equivalence, we can arrange to have a relative presentation



such that $\widetilde{F} = \langle F, \alpha', \beta' \rangle$ and $\alpha \in F$. In both cases the vertex group $\widetilde{F}$ is either $F * \langle z \rangle$ or a cyclic HNN extension of $F * \langle z \rangle$ and we must have $b_1(\widetilde{F}) = N + 1$.

(C) *If the cyclic JSJ decomposition of $F_{R(S)}$ modulo F has only the nonabelian vertex group $\widetilde{F}$ and all other vertex groups are abelian, then the possible graphs of groups are:*

$$\mathcal{G}(A) = \ u\bullet\!\!\longrightarrow\!\!\bullet v\ ,\ v\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet u\!\!-\!\!\!-\!\!\!-\!\!\bullet w\ ,\ \text{or}\ \circlearrowleft u\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet v$$

with $A_u = \widetilde{F}$ and the other vertex groups free abelian. Moreover we have $b_1(\widetilde{F}) \leq N+1$ if there are two vertex group and $\widetilde{F} = F$ if there are three vertex groups. In the case where we have two edges and two vertices, then exactly one of the boundary subgroups must be conjugate into F itself.

Otherwise, the cyclic JSJ decomposition modulo F of $F_{R(S)}$ is given by a graph of groups $\mathcal{G}(A)$ that has at least two non abelian vertex groups, in which case either:

(D) *The JSJ of $F_{R(S)}$ has the cyclic collapse $u\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet v$. In all cases we have two nonabelian vertex groups. Up to rational equivalence we can arrange to have the following:*

  (I) *The JSJ of $F_{R(S)}$ has one edge and we have the relative presentation $F_{R(S)} = \widetilde{F} *_p H$. Moreover either:*

    1. *$\widetilde{F} \neq F$ then $\widetilde{F} = F *_u Ab(u,t)$, $p \notin F$ and H is free of rank 2 and generated by $\overline{x}$ and $\overline{y}$; or*
    2. *$\widetilde{F} = F$ and H is free of rank 2 and generated by $\overline{x}$,$\overline{y}$ and $p \in F$; moreover*
    3. *H is a QH subgroup then we can arrange so that*

$$F_{R(S)} = F *_{u=p} \langle x, y, p | [x,y] = p \rangle.$$

(II) The JSJ of $F_{R(S)}$ has two edges and either:

1. *there are only two vertex groups, then the JSJ is*

$$\begin{tikzcd} F_{R(S)} = \widetilde{F} \arrow[r, dash, "u"'] & H'^{q}_{q'} \arrow[loop right] \end{tikzcd}$$

   where the subgroups u and q are not conjugate in $F_{R(S)}$. The subgroup $H'^{q}_{q'} = H$ is free of rank 2 and $\widetilde{F}$ and H are as in (I) above; or

2. *there are three vertices and we have a presentation*

$$F_{R(S)} = F *_u H *_q Ab(q,t)$$

   H is free of rank 2 and the subgroup $H *_u Ab(q,t)$ is generated by $\overline{x}, \overline{y}$ and $u \in F$. Moreover u and q may be conjugate.

(III) The JSJ of $F_{R(S)}$ has three edge groups, then the only possibility is

$$\begin{tikzcd} F_{R(S)} = F \arrow[r, dash, "u"'] & {}^{\alpha}H^{\alpha'} \arrow[loop above, "t"] \arrow[r, dash, "p"'] & Ab(u,s) \end{tikzcd}$$

moreover α may be conjugate to either u or p, but not both.

(E) The JSJ of $F_{R(S)}$ has the cyclic collapse $u\bullet \overset{\frown}{\rule{3em}{0.4pt}} \bullet v$ and again there are two nonabelian vertex groups. Up to rational equivalence we can arrange to have the following:

(I) The JSJ of $F_{R(S)}$ has two edges and we have a relative presentation:

$$\begin{tikzcd} F_{R(S)} = \widetilde{F}^{\beta} \arrow[r, dash, "\alpha"'] \arrow[r, bend left, "\overline{y}"] & {}^{\gamma}H \end{tikzcd}$$

moreover either:

1. *$\widetilde{F} = F$ and H is a free group of rank 2 generated by $\overline{x}, \alpha, \gamma$. Where $\gamma = \overline{y}^{-1}\beta\overline{y}$, with $\beta \in F$.*
2. *$\widetilde{F} = F *_u Ab(u, s)$ and H is a free group of rank 2 generated by $\overline{x}$ and $\alpha \in F$.*

(II) The JSJ of $F_{R(S)}$ has three edges and either:

1. *We have the relative presentation*

$$\begin{tikzcd} F_{R(S)} \approx F^{\gamma} \arrow[r, bend left, "\overline{y}"] \arrow[r, no head, "\alpha"'] & {}^{\gamma'}H'^{q}_{q'} \arrow[loop right] \end{tikzcd}$$

where $H = {}^{\gamma}H'^{q}_{q'} \circlearrowright$ is free of rank 2, moreover α and γ are not conjugate, but it is possible for q to be conjugate to one of them. The vertex groups are generated as in (II).1. above.

2. *$F_{R(S)} = F^{\beta} \underset{\alpha}{\overset{\overline{y}}{\rightrightarrows}} {}^{\gamma}H \underset{u}{\rule{2em}{0.4pt}} Ab(u, s)$ where H is a free group of rank 2 and the rank 1 free extension of a centralizer $H *_u Ab(u, s)$ is generated by $\alpha, \gamma, \overline{x}$. Again γ and α cannot be conjugate, but u may be conjugate to one of them.*

(F) The JSJ $F_{R(S)}$ has cyclic collapse $v\bullet \equiv \bullet u$ and up to rational equivalence we can get the relative presentation

$$\begin{tikzcd} F_{R(S)} \approx F^{\beta}_{\gamma} \arrow[r, bend left, "\overline{x}"] \arrow[r, no head, "\alpha"] \arrow[r, bend right, "\overline{y}"] & {}^{\delta}_{\epsilon}H \end{tikzcd}$$

where $\overline{x}$ and $\overline{y}$ are sent to stable letters and H is a free group of rank 2 generated by α, δ, ϵ.

In all cases the cyclic edge groups can be taken to be maximal cyclic in the vertex groups.

We finally note that this description in terms of relative presentations also gives a description of what *irreducible systems of equations* over F look like since these systems of equations are the relations of the groups. We make some remarks that could be seen as corollaries of this theorem.

Remark 1. If the JSJ of $F_{R(S)}$ has three vertices or three edges then one of the vertex groups is F and the other vertex groups are free of rank 2 or free abelian of rank 2.

Remark 2. If the JSJ of $F_{R(S)}$ has at least two nonabelian vertex groups, then one of them is free of rank 2.

Remark 3. If $\text{Rank}(F) = N$ and $b_1(F_{R(S)}) = N + 1$ then using Proposition 4.2.8 and looking at abelianized relative presentations we see that either $F_{R(S)}$ is a rank 1 free extension of a centralizer of F or $F_{R(S)}$ doesn't have any noncyclic abelian subgroups.

Direct inspection shows us that:

Corollary 4.1.7. *The height of the canonical analysis lattice relative to F (see Section 4 of [52]) for $F_{R(S)}$ is at most 3, i.e. the lattice terminates at level L^2.*

4.1.1 A description of Solutions

In [46], Ozhigov gives a description of the solution set of a system of equations in two variables over F. For each system of equations, his algorithm produces a finite collection of *forms*. Theorem 4.1.8 below gives us exactly the same description.

Denote by $\Phi = Stab_{Aut(F(x,y))}([x, y])$. We have $\phi(x) = W(x, y), \phi(y) = V(x, y)$. For a pair $(u, v) \in F \times F$ and $\phi \in \Phi$ as before we denote

$$\phi_x(u) = W(u, v), \phi_y(v) = V(u, v)$$

Theorem 4.1.8. *Let $S(x, y) = 1$ be any system of equations over F. Then the solutions of $S(x, y)$ in $F \times F$ are described by a finite number of families of pairs of the following form:*

(a) (f, g) *where* f, g *take arbitrary values in* F.

(b) $(P(F, r), Q(F, r))$ *where* P, Q *are parametric words in* $F * \langle r \rangle$ *and* r *takes arbitrary values in* F.

(c) $(P(F), Q(F))$ *where* P, Q *are parametric words in* F

(d) $(P(F, \phi_x(u), \phi_y(v)), Q(F, \phi_x(u), \phi_y(v)))$ *where* $u, v \in F$ *are fixed and* P, Q *are fixed parametric words in* $F \cup \{\phi_x(u), \phi_y(v)\}^{\pm 1}$, *where* ϕ *takes arbitrary values in* Φ.

Proof. Apply Theorem 2.3.2. Each family that is given corresponds to the solutions that factor through a branch of the Hom diagram. We also know what all the possible canonical automorphisms look like, in particular and non-surface automorphism will affect the variables in the parametric words.

From Theorem 4.1.6. it is easy to see the what $Hom_F(F_{R(S)}, F)$ is when $F_{R(S)}$ is freely decomposable, in particular all such Hom diagrams have one level. In particular they fall into category (a) and (b).

There is only one type fully residually F quotient of $F[x, y]$ that contains a QH subgroup, moreover by the description of Hom diagrams for such fully residually F groups given in Chapter 3 (also deducible from [45, 21, 41]) we see that any branch in the Hom diagram that contains a group with a QH subgroup will give a family of solutions describable as (d). □

4.1.2 Examples and Questions

We give some examples of the groups given in Theorem 4.1.6. We first note that all the freely decomposable groups are easy to construct as quotients of $F[x, y]$. Examples of Theorem 4.1.6 (C) are easy to construct by taking extensions of centralizers of F or $F * \langle r \rangle$ or by taking a chain of extension of centralizers of height 2. The next few examples are more delicate.

Example 4.1.9 (Example of (D).I.1 of Theorem 4.1.6)**.** Let $F = F(a, b)$. Recall from Section 3.2 that the group

$$\begin{aligned} G &= \langle F, x, y | [a^{-1}ba[b, a][x, y]^2 x, a] = 1\rangle \\ &\approx \langle F, x, y, t | [x, y]^2 x = [a, b]a^{-1}b^{-1}at; [t, a] = 1\rangle \end{aligned}$$

is freely indecomposable modulo F, that

$$G \approx F_1 *_{\langle u = w(x,y)\rangle} \langle x, y\rangle$$

where F_1 is a rank 1-extension of a centralizer of F. Moreover G is shown to be fully residually F by the F-embedding into the chain of extensions of centralizers

$$F_2 = \langle F, t, s \mid [t, a] = 1, [s, u] = 1\rangle$$

where $u = [a, b]a^{-1}b^{-1}at$ via the mapping, $x \mapsto s^{-1}(b^{-1}t)s$ and $y \mapsto s^{-1}(b^{-1}ab)s$

Example 4.1.10 (Example of (E).I.2 of Theorem 4.1.6)**.** We modify Example 4.1.9. Let $F = F(a, b)$ and let

$$F_1 = \langle F, s, t, r | [t, a] = 1, [s, b^{-1}ab] = 1, [u, r] = 1\rangle$$

where $u = [a,b]a^{-1}b^{-1}at$. F_1 is a chain of extensions of centralizers. Let $x' = b^{-1}t, y' = b^{-1}ab$ and let $G = \langle F, r^{-1}x'r, sr\rangle$. Let $H = r^{-1}\langle x', y'\rangle r$ and consider $G \cap H$. We see that $(sr)^{-1}b^{-1}ab(sr) = r^{-1}b^{-1}abr$ so $H \leq G \cap H$, on the other hand letting $\overline{y} = (sr)$ and $\overline{x} = r^{-1}x'r$ and by Britton's lemma we have a splitting:

$$\begin{tikzcd}
G = \widetilde{F}^{b^{-1}ab} \arrow[r, dash, "u"'] \arrow[r, bend left, "\overline{y}"] & H
\end{tikzcd}$$

With $\widetilde{F} = \langle F, t\rangle$ and H free of rank 2, not freely decomposable modulo edge groups.

Example 4.1.11 (Example of (D).III of Theorem 4.1.6)**.** Let $F = F(a,b)$ and consider the chain of extensions of centralizers

$$F_2 = \langle F, s, t \mid [s,a] = 1, [t, (a^2(b^{-1}ab)^2)^s] = 1\rangle$$

One can check that the subgroup $K \leq \langle F, s^{-1}bs, t\rangle$ has induced splitting:

$$\begin{tikzcd}
F *_a & {}^{\gamma}H^{\gamma'} \arrow[loop above, "r"] & *_p Ab(p,t)
\end{tikzcd}$$

where $H = s^{-1}\langle a, b^{-1}ab\rangle s$, $\gamma = s^{-1}as$, $\gamma' = s^{-1}b^{-1}abs$, $r = s^{-1}bs$, and $p = (a^2(b^{-1}ab)^2)^s$. Moreover it is freely indecomposable, fully residually F and generated by two elements modulo F.

Compared to these, constructing an example of (F) of Theorem 4.1.6 is relatively easy.

Questions

Conspicuously absent from the list is an example of Theorem 4.1.6 (B). In fact I have been unable to construct examples of such groups which leads me to the following conjecture:

Conjecture 4.1.12. There are no fully residually F groups generated by two elements modulo F such that their JSJ has only one vertex group, i.e. where the HNN extensions are *separated*.

Also absent from the examples are groups from Theorem 4.1.6 (C) such that the vertex group is not simply an extension of a centralizer or groups like in case (D).II.2 where u and q are *conjugate*. This motivates the following question:

Question 4.1.13. Consider the free group $F(a, b)$, is there a non primitive element $u \in F(a, b)$ and some element $w \in F(a, b)$ such that $\langle u, w\rangle \lneq F(a, b)$ but for some conjugate pup^{-1} of u in $F(a, b)$ we have $\langle u, pup^{-1}, w\rangle = F(a, b)$?

A positive answer would enable the construction of such groups, whereas a negative answer would probably exclude a few possibilities.

4.2 Tools for fully residually F groups:

We first give some extra machinery that will be needed later.

Definition 4.2.1. A subgroup $K \leq G$ is said to have property CC (conjugacy closed) if for $k, k' \in K$

$$\exists g \in G \text{ such that } k^g = k' \Rightarrow \exists \widetilde{k} \in K \text{ such that } k^{\widetilde{k}} = k'$$

Lemma 4.2.2. $F \leq F_{R(S)}$ *has property* CC

Proof. Let $f, f' \in F$ we have a retraction $r : F_{R(S)} \to F$, then

$$f^g = f' = r(f') = r(f^g) = f^{r(g)}$$

□

We give some first tools for analyzing fully residually F groups.

4.2.1 Some notions of complexity

We present two measures of complexity. The first one gives us bounds on the complexity of underlying graphs of groups:

Definition 4.2.3. Let $\mathcal{G}(X)$ be a graph of groups modulo F. We define the complexity $B(\mathcal{G}(X))$ to be following

$$B(\mathcal{G}(X) = \text{ Rank}(\pi_1(X)) + \sum_{v \in VX} \chi(v)$$

where

- $\chi(v) = 0$ if $F \leq G_v$.
- $\chi(v) = (\text{ Rank}(G_v) - \sum \text{ Rank}(B_i))$ if G_v is abelian and the B_i are the boundary subgroups in G_v.
- $\chi(v) = \max\{0, 2-k\}$ if G_v is rigid and k is the number of incident edge groups.
- $\chi(v) = 2g$ if G_v is a QH subgroup and where g is the genus of the underlying surface.

Lemma 4.2.4. *Let H be a fully residually free group of rank two or greater, and let $\gamma \in H$ then the "one relator" quotient $H/ncl(\gamma)$ is non trivial.*

Proof. If H is abelian, the result is clear. If H is nonabelian then H admits an epimorphism ψ onto a free group K of rank at least two, we have the following commutative diagram:

$$\begin{tikzcd}
H \arrow[r, "\psi"] \arrow[d] & K \arrow[d] \\
H/ncl(\gamma) \arrow[r, "\overline{\psi}"] & K/ncl(\psi(\gamma)) = \overline{K}
\end{tikzcd} \tag{4.2}$$

and the one relator group $\overline{K}$ is seen to be nontrivial by abelianizing. ☐

Proposition 4.2.5. *The complexity $B(\mathcal{G}(X))$ gives a lower bound for the rank of $\pi_1(\mathcal{G}(X))$. Specifically, if $F_{R(S)} = \pi_1(\mathcal{G}(X))$ then $B(\mathcal{G}(X)$ cannot exceed the number of variables in S.*

Proof. Consider a quotient $\overline{G}$ of $G = \pi_1(\mathcal{G}(X))$ obtained by killing $\widetilde{F}$, the vertex group containing F, and killing all the edge groups. $\overline{G}$ is therefore a free product, and Lemma 4.2.4 implies that $\chi(v)$ indeed gives a lower bound for the rank of the image of G_v in $\overline{G}$. The lower bound for Rank(G) now follows from Grushko's Theorem.

If G is a quotient of $F[x_1, \dots, x_n]$ then $\overline{G}$ is generated by $\langle x_1, \dots, x_n \rangle$, this proves the second part of the claim. ☐

Corollary 4.2.6. *If $S = S(x, y)$ and $F_{R(S)} = \pi_1(\mathcal{G}(X))$ then $B(\mathcal{G}(X) \leq 2$.*

Our second measure of complexity is the *first Betti number.*

Since fully residually free groups are finitely presented it is easy to compute b_1 directly from the presentation. The fundamental group of a graph of groups $G = \pi_1(\mathcal{G}(X))$ with cyclic edge groups we have the following lower bound: for $T \subset X$ a maximal spanning tree we have

$$b_1(G) \geq \sum_{v \in T} b_1(G_v) - E \tag{4.3}$$

where E is the number of edges in T. If there is an epimorphism $G \to H$ then $b_1(G) \geq b_1(H)$. The following useful fact is obvious from a relative presentation:

Lemma 4.2.7. *Let $H < G$ be a rank n extension of a centralizer of H, then $b_1(G) = b_1(H) + n$.*

Proposition 4.2.8. *Let F be a free group of rank N and let G be a fully residually F $F-$group. Then $b_1(G) = N$ if and only if $G = F$.*

Proof. Suppose towards a contradiction that $G \neq F$ but $b_1(G) = N$. Then, by being fully residually F, there is a retraction $G \to F$. It follows that $b_1(G) \geq N$. By Corollary 2.2.18, G has D, a nontrivial JSJ decomposition. Let $F \leq \widetilde{F} \leq G$ be the vertex group containing F, obviously $\widetilde{F}$ is also fully residually F. By formula (4.3), $b_1(G) \geq b_1(\widetilde{F})$ and if D has more than one vertex group then the inequality is proper which forces $b_1(G) > N$ – contradiction. We must therefore have that D is a bouquet of circles with a single vertex group $\widetilde{F}$. By Lemma 4.2.2 if $\widetilde{F} = F$ then the stable letters of the splitting D in fact extend centralizers of elements of F, so by Lemma 4.2.7 we have that $b_1(G) > b_1(F)$ – contradiction.

It follows that we cannot have $\widetilde{F} = F$. We therefore look at the JSJ of $\widetilde{F}$. Again we find that it must have a unique vertex group $\widetilde{F}^1$. Combining Theorem 2.2.17, finite presentability of G, and the hierarchical accessibility result in [15] over the class of cyclic splittings gives a finite sequence of inclusions

$$\widetilde{F} > \widetilde{F}^1 > \ldots \widetilde{F}^r > \widetilde{F}^{r+1} = F$$

where $\widetilde{F}^{i+1}$ is the unique vertex group of the JSJ of $\widetilde{F}^i$. Now, by assumption, we must have $N = b_1(\widetilde{F}) \geq b_1(\widetilde{F}^1) \ldots \geq b_1(\widetilde{F}^{r+1}) = N$ but we must have that $\widetilde{F}^r$ has a splitting D^r that is a bouquet of circles with vertex group F, we saw that in this case we must have $b_1(\widetilde{F}^r) > b_1(F)$ –contradiction. □

4.3 Bass-Serre Theory Techniques

We first establish some notation; we will denote the commutator $x^{-1}y^{-1}xy = [x, y]$. For conjugation we will use the following convention:

$$\begin{aligned} x^w &= w^{-1}xw \\ {}^w x &= wxw^{-1} \end{aligned}$$

We use this convention since ${}^x({}^y w) = {}^{xy}w$. Our group will always act on a tree T from the left i.e. for all $g, h \in F_{R(S)}$ and for all $v \in T$ we have

$$ghv = g(hv)$$

It follows that if for any point $v \in T$, and for any $g \in F_{R(S)}$ we will have $\text{stab}(gv) = g(\text{stab}(v))g^{-1} = {}^g\text{stab}(x)$.

We will write $w = w(F, X)$, for example, to denote a *word*; i.e. an explicit product of symbols from F and $X^{\pm 1}$.

Definition 4.3.1. Let G act on a simplicial tree T without edge inversions and let $K \leq G$ act on T via the normal restriction. We will say that two vertices v, w of T are K-equivalent or of the same K-*type*, denoted $tp_K(v) = tp_K(w)$, if there is a $k \in K$ such that $kv = w$. We similarly define have K-types for edges of $T(K)$.

Theorem 4.3.2 (The Fundamental Theorem of Bass-Serre Theory)**.** *If G acts on a simplicial tree T without edge inversions and the quotient $A = G\backslash T$ is a finite graph, then G splits as a the fundamental group of a graph of groups with underlying graph A. Vertex and edge groups correspond to stabilizers of vertices and edges of T (respectively) via the action of G.*

Conversely, if G splits as a graph of groups, then it acts on a simplicial tree T without inversions called the Bass-Serre tree*.*

Definition 4.3.3. An action of G on a tree T is said to be *$k-$acylindrical* if the largest subset of T fixed by an element of G has diameter k. A *splitting* of G is said to be k-acylindrical if the action of G on the induced Bass-Serre tree is k-acylindrical.

We note that if we take a splitting of $F_{R(S)}$ whose edge groups are maximal abelian in the vertex groups, then the splitting will be $1-$acylindrical.

4.3.1 Induced splittings and $\mathcal{G}(A)$-graphs

Suppose that G has a splitting D as the fundamental group of a graph of groups and let H be a subgroup of G. Then G acts on a tree T and H acts on the minimal H-invariant subtree $T(H) \subset T$ in such a way that H also splits as a graph of groups. We call this splitting D_H the *induced splitting of H.*

We now present the folding machinery developed in [25], which is a more combinatorial version of the Stallings-Bestvina-Feighn-Dunwoody folding techniques in Bass-Serre theory, used to compute induced splittings. Although this machinery is technically demanding, it gives an alternative to normal forms when dealing with fundamental groups of graphs of groups which greatly simplifies the arguments of Sections 4.4 and 4.6.

Basic definitions

We follow [25].

Definition 4.3.4. A $\mathcal{G}(A)$-graph $\mathcal{B}$ consists of an underlying graph B with the following data:

- A graph morphism $[.] : B \to A$

- For each $u \in VB$ there is a group B_u with $B_u \leq A_{[u]}$, called a vertex group.
- To each edge $e \in EB$ there are two associated elements $e_i \in A_{[i(f)]}$ and $e_t \in A_{[t(f)}$ such that $(e^{-1})_i = (e_t)^{-1}$ for all $e \in EB$.

Convention 4.3.5. We will usually denote $G(A)-$graphs by $\mathcal{B}$ and will assume that the underlying graph of $\mathcal{B}$ is some graph B.

When drawing these graphs we give vertices v the label $(B_v, [v])$. We give an edge e the label $(e_i, [e], e_t)$. We will say that an edge e is of *type* $[e]$.

Definition 4.3.6. Let $\mathcal{B}$ be a $\mathcal{G}(A)$-graph and suppose that $e_1^{\epsilon_1}, \ldots, e_n^{\epsilon_n}$; where $e_j \in EB, \epsilon_j \in \{\pm 1\}$; is an edge path of B. A sequence of the form

$$b_0, e_1^{\epsilon_1}, b_1, e_2^{\epsilon_2}, \ldots, e_n^{\epsilon_n}, b_n$$

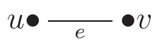
where $b_j \in B_{t(e_j^{\epsilon_j})}$ is called a $\mathcal{B}$-path. To each b path we associate a label

$$\mu(p) = a_0 [e_1]^{\epsilon_1} a_1 [e_2]^{\epsilon_2} \ldots [e_n]^{\epsilon_n} a_n$$

where $a_0 = b_0 (e_1^{\epsilon_1})_i, a_j = (e_j^{\epsilon_1})_t b_j (e_{j+1}^{\epsilon_1})_i$ and $a_n = (e_n^{\epsilon_n})_t b_n$ which is a $\mathcal{G}(A)$-path.

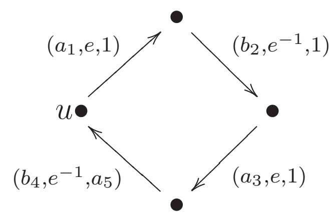
Definition 4.3.7. Let $\mathcal{B}$ be a $\mathcal{G}(A)$-graph with a basepoint u. Then we define the subgroup $\pi_1(\mathcal{B}, u) \leq \pi_1(\mathcal{G}(A), [u])$ to be the subgroup generated by the $\mu(p)$ where p is a $\mathcal{B}$-loop based at u.

Example 4.3.8. Let $G = \pi_1(\mathcal{G}(X), u)) = A *_C B$. The underlying graph is



with $X_u = A, X_v = B$ and $X_e = C$. Let $g = a_1 b_2 a_3 b_4 a_5$ where $a_i \in A$ and $b_j \in B$. Then the $\mathcal{G}(X)$-graph

$\mathcal{B} =$



whose vertex groups are all trivial is such that $\pi_1(\mathcal{B}, u) = \langle g \rangle$

This example motivates a definition.

Definition 4.3.9. Let $g = b_0, e_1^{\epsilon_1}, b_1, e_2^{\epsilon_2}, \ldots, e_n^{\epsilon_n}, b_n$ be some element of $\pi_1(\mathcal{G}(X), u)$. Then we call the based $\mathcal{G}(X)$-graph $\mathcal{L}(g; u)$ a *g-loop* if $\mathcal{L}(g; u)$ consists of a cycle starting at u whose edges are all coherently oriented and have labels

$$(b_0, e_1^{\epsilon_1}, 1), (b_1, e_2^{\epsilon_2}, 1), \ldots, (b_{n-2}, e_{n-1}^{\epsilon_{n-1}}, 1), (b_{n-1}, e_n^{\epsilon_n}, b_n)$$

respectively.

Definition 4.3.10. $(\mathcal{G}(A), v_0)$ be be a graph of groups decomposition of $F_{R(S)}$. Let the *$\overline{x}, \overline{y}$-wedge*, $\mathcal{W}(F, \overline{x}, \overline{y}; u)$, be the based $\mathcal{G}(A)$-graph formed from a vertex v with label (F, v_0) and two attaching the loops $\mathcal{L}(\overline{x}; v_0)$ and $\mathcal{L}(\overline{y}; v_0)$.

It is clear that $\pi_1(\mathcal{W}(F, \overline{x}, \overline{y}; u), u) = \langle F, \overline{x}, \overline{y} \rangle = F_{R(S)}$. *Usually we will omit the mention of the vertex u.*

Moves on $\mathcal{G}(A)$-graphs

Let $\mathcal{B}$ be a $\mathcal{G}(A)$ graph, with underlying graph B. We now briefly define the moves on $\mathcal{B}$ given in [25] that we will use, we will sometimes replace an edge e by e^{-1} to shorten the descriptions:

A0: **Conjugation at v:** For some vertex v, assume w.l.o.g. that for each edge incident to v, $i(e) = v$. For some some $g \in A_{[v]}$ do the following: replace B_v by gB_vg^{-1}, for each edge such that $i(e) = v$ replace e_i by ge_i, and for each edge such that $t(e) = v$ replace e_t by e_tg^{-1}.

A1: **Bass-Serre Move at e:** For some edge e, replace its label $(a, [e], b)$ by $(ai_e(c), [e], t_e(c^{-1})b)$ for some c in $A_{[e]}$.

A2: **Simple adjustment at u on e:** For some vertex u and some edge e such that w.l.o.g $i(e) = v$, we replace the label $(a, [e], b)$ by $(ga, [e], b)$ where $g \in B_u$

F1: **Simple fold of e_1 and e_2 at the vertex u:** For a vertex u and edges e_1, e_2 such that w.l.o.g. $i(e_1) = i(e_2) = u$ but $t(e_1) = v_1 \neq t(e_2) = v_2$ but $[v_1] = [v_2]$, if e_1 and e_2 have the same label, then identify the edges e_1 and e_2. The resulting edge has the same label as e_1 and the vertex resulting from the identification of v_1 and v_2 has label $\langle B_{v_1}, B_{v_2}\rangle$.

F4: **Double edge fold (or collapse) of e_1 and e_2 at the vertex u:** For edges e_1, e_2 such that w.l.o.g. $i(e_1) = i(e_2) = u$, $t(e_2) = t(e_2) = v$, and $[e_1] = [e_2] = f$ if they have labels (a, f, b_1) and (a, f, b_2) respectively, and moreover $[e_1] = [e_2] = f$, then we can identify the edges e_1 and e_2, the image of these edges has label (a, f, b_1) and the the group B_v is replaced by $\langle B_v, b_1^{-1}b_2\rangle$. *We will also call such a fold a collapse from u towards v.*

The moves F2 and F3 in [25] are analogous to F1 and F4, respectively only they involve simple loops. However, because these moves only show up in the proof of Lemma 4.7.1, we do not describe them explicitly. We also introduce four new moves:

T1: **Transmission from u to v through e:** For an edge e such that $i(e) = u$ and $t(e) = v$ with label $(a, [e], b)$, let $g \in A_{[e]}$ be such that $ai_{[e]}(g)a^{-1} = c \in B_u$, then replace B_v by $\langle B_v, b^{-1}t_{[e]}(g)b \rangle$.

L1: **Long range adjustment:** Perform a sequence of *transmissions* through edges $e_1, \dots e_n$ followed by a simple adjustment that changes the label of an edge f which is not one of the $e_1, \dots e_n$ and leaves unchanged the labels of the edges $e_1, \dots e_n$. Finally replace all the modified vertex groups by what they were before the sequence of transmissions.

N1: **Nielsen move:** Replace a wedge $\mathcal{W}(F, \overline{x}, \overline{y})$ by $\mathcal{W}(F, \overline{x}', \overline{y}')$, where the triple $(F, \overline{x}', \overline{y}')$ is obtained from $(F, \overline{x}, \overline{y})$ via a Nielsen move modulo F.

S1: **Shaving move:** Suppose that u is a vertex of valence 1 such that $u = t(e)$ and $v = i(e)$, e has label $(a, [e], b)$ and $B_u = b^{-1}(t_{[e]}(C))b$, where $C \leq G_{[e]}$. Then delete the vertex u and the edge e and replace B_v by $\langle B_v, a(i_{[e]}(C))a^{-1} \rangle$.

Convention 4.3.11. Although formally applying a move to a $\mathcal{G}(A)$-graph $\mathcal{B}$ gives a new graph $\mathcal{B}'$ unless noted otherwise we will denote this new $\mathcal{G}(A)$-graph as $\mathcal{B}$ as well.

We regard the transmission above as the group B_u sending the element c to B_v through the edge e. When edge groups are *slender* in the sense of [18], multiple transmissions can be used instead of the edge equalizing moves F5-F6 in [25]. N1 type moves are products of A0-A3, L1 and F1 type moves. We also notice that the moves T1, L1, S1 and N1 do not change the group $\pi_1(\mathcal{B}, u)$. Note that vertices of valence 1 with trivial group can always be shaved off.

The Folding Process

Definition 4.3.12. A $\mathcal{G}(A)$-graph such that it is impossible to apply any moves other than A0-A2 is called *folded.*

This next important result is essentially a combination of Proposition 4.3, Lemma 4.16 and Proposition 4.15 of [25].

Theorem 4.3.13. *[25] Applying the moves A0-A2, F1-F4, and T1 to* $\mathcal{B}$ *do not change* $H = \pi_1(\mathcal{B}, u)$*, moreover if* $\mathcal{B}$ *is folded, then the associated data (see Definition 4.3.4) gives the groups decomposition of* H *induced by* $H \leq \pi_1(\mathcal{G}(A))$

This theorem implies the existence of a folding process. Consider the three following moves:

- **Adjustment:** Apply a sequence of moves A0-A2, L1,S1, and N1.
- **Folding:** Apply moves of type F1 and F4.
- **Transmission:** Apply a move of type T1.

First note that each folding decreases the number of edges in the graph, and that adjustments (except for shavings) are essentially reversible. In the folding process there is therefore a finite number of foldings and between foldings there are adjustments and transmissions. Transmissions enlarge the vertex groups and so in a sense increase the complexity of the $\mathcal{G}(A)$-graph.

The goal of the next section is to find conditions that enable us to perform a maximal number of foldings without having to resort to transmissions. In a sense it will give us a folding process where each step decreases the complexity of the $\mathcal{G}(A)$-graph, making it more tractable.

Transmission graphs and 1-acylindrical splittings

Suppose that $\mathcal{G}(A)$ is 1-acylindrical and the edge groups are maximal abelian. Let $\mathcal{B}$ be a $\mathcal{G}(A)$-graph.

Definition 4.3.14. We define the *transmission graph* B_0 of $\mathcal{B}$ to be the the underlying graph B of $\mathcal{B}$ equipped with the following coloring:

- A vertex v such that B_v is nonabelian is *green*.
- A vertex v such that B_v is abelian is *yellow*.
- Otherwise edges and vertices are black.

Suppose that we performed a sequence of transmission $t_1, \ldots, t_n$ on a $\mathcal{G}(A)$ graph $\mathcal{B}$. We will produce a analogous sequence of colored graphs $B_1, \ldots, B_n$. As follows for B_{i+1} start with B_i and make the following changes:

1. If for a vertex v, B_v is abelian and nontrivial, then color it *yellow*.
2. If for a vertex v, B_v is non-abelian, then color it *green*.
3. If the transmission move t_{i+1} is through the edge e, then color e *red*.

Lemma 4.3.15. *Let $\mathcal{G}(A)$ be 1-acylindrical and let $\mathcal{B}$ be a $\mathcal{G}(A)-$graph and let B_n be a transmission graph. Then after performing the corresponding sequence of transmission moves we have that for each black vertex u, B_u is trivial, for each yellow vertex v, such that w.l.o.g. $v = t(e)$, and the incoming edge e is labeled $(a, [e], b)$ we have $B_u \leq b^{-1}(t_{[e]}(G_{[e]}))b$.*

Proof. This follows immediately from 1-acylindricity of $\mathcal{G}(A)$ and the construction. □

The usefulness of the next lemmas is that they enables us to control the vertex groups during the folding process.

Definition 4.3.16. A path in a transmission graph that consists of two red edges and a yellow vertex u is called a *cancellable path centered at* u.

Lemma 4.3.17. *Let* $\mathcal{G}(A)$ *be 1-acylindrical and let* $\mathcal{B}$ *be a* $\mathcal{G}(A)-$*graph and suppose that after a sequence of transmissions* $t_1, \dots, t_n$ *the transmission graph* B_n *has a cancellable path centered at* u*. Then it is possible to perform a folding (F1 or F4) at* u *in* $\mathcal{B}$ *after using only a Bass-Serre A1 move, and maybe a conjugation A0 move.*

Proof. If u is yellow, but has two (or more) adjacent red edges, then w.l.o.g. we have $i(e) = i(e') = u$ and that e and e' have labels $(a, [e], b)$ and $(a', [e], b')$ respectively. Let A denote the image of $G_{[e]}$ in $A_{[e]}$. The fact that both edges e' and e are red in particular imply that $aAa^{-1} \cap a'A(a')^{-1} \neq \{1\}$ which implies that $(a')^{-1}(aAa^{-1})a' \neq \{1\}$. Now 1-acylindricity implies malnormality of the edge groups, it therefore follows that

$$a'^{-1}a = \alpha \in A \Rightarrow a = a'\alpha$$

This means that in $\mathcal{B}$ all we need is to do a Bass-Serre move A1 so that the label of e' becomes $(a'\alpha, [e], \alpha^{-1}b')$ and then (for move F1) a conjugation move A0 so that e' has the same label $(a'\alpha, [e], b) = (a, [e], b)$ as e. □

Lemma 4.3.18. *Let* $\mathcal{G}(A)$ *be 1-acylindrical and let* $\mathcal{B}$ *be a* $\mathcal{G}(A)-$*graph. Suppose that after a series of transmissions and adjustments it is possible to perform either move F1 or F4 at a yellow or black vertex* u*, of the corresponding transmission graph, then it is also possible to perform a folding move F1 or F4 at* v *after only a sequence of adjustments.*

Proof. Let $\mathcal{B}'$ be the result of applying all the transmission moves to $\mathcal{B}$. Suppose that it was impossible to perform the move F1 or F4 at u in $\mathcal{B}$ but that it is possible to to it in $\mathcal{B}'$. Then it follows that $B'_u > B_u$ and that there is an A2 move that can now be performed at u so that F1 or F4 is now possible.

Suppose first that u is black then $B_u = B'_u = \{1\}$, which in particular implies that the move F1 or F4 could already have been applied to u in $\mathcal{B}$.

Suppose now that u is yellow and that in $\mathcal{B}'$ we can perform an A2 move changing the label of e, where $i(e) = u$ and then after performing moves A1,A0 at $t(e)$ we can do either move F1 or F4 identifying e and e'. In particular e and e' had labels $(a, [e], b)$ and $(a', [e], b')$ respectively and after doing move A2 the labels became $(a', [e], b)$ and $(a', [e], b')$ respectively. We wish to show that instead we can change the label of e in the same way by performing a long range adjustment in $\mathcal{B}$ or moves A0,A1.

The only obstruction to long range adjustments is if both edges e and e' are either directed or red. By the construction of the transmission graph, a yellow vertex cannot have two incoming directed edges or an incoming edge and a red edge because that implies that the vertex is green. It follows that the only obstruction is if both edges e, e' are red, in which case Lemma 4.3.17 applies. ☐

Definition 4.3.19. The transmission graphs obtained after performing all possible transmissions is called an *terminal transmission graph.*

4.3.2 When $F_{R(S)}$ splits as a graph of groups with two cycles

We suppose our group $F_{R(S)} = \langle F, \overline{x}, \overline{y} \rangle$ can be collapsed to a cyclic graph of groups modulo F with one vertex and two edges, i.e.

$$F_{R(S)} = {}^{t}\circlearrowleft \widetilde{F} \circlearrowright^{s} \tag{4.4}$$

where $F \leq \widetilde{F}$. We denote by s and t the corresponding stable letters.

It should be noted that the three stable letter case cannot occur, since killing the vertex group $F_{R(S)}$ yields a quotient that is a free group of rank 3 that is generated by two elements –contradiction.

Definition 4.3.20. For a generating set X and a word $W = W(X)$ in X, for a letter $x \in X$ we denote the exponent sum of x in W by $\sigma_x(W)$.

Definition 4.3.21. For a stable letter t from a splitting of G and a word an element $g \in G$, let $\langle X|R \rangle$ be a presentation that represents the splitting, i.e. $t \in X$ and we have the appropriate relation. Let $g = W(X)$ where $W(X)$ is in normal form. We define the *exponent sum* $\sigma_t(g)$ of t in g as

$$\sigma_t(g) = \sigma_t(W(X))$$

It follows from Britton's Lemma that this quantity doesn't depend the choice of word chosen to represent g.

Definition 4.3.22. For $g \in G$, we define the (t, s)-*signature* $\operatorname{sgn}_{(t,s)}(g)$ of g to be the be the pair of integers

$$\operatorname{sgn}_{(t,s)}(g) = (\sigma_t(g), \sigma_s(g))$$

It is customary to drop the subscripts.

We must be able to write $t = T(F, \overline{x}, \overline{y})$ and $s = S(F, \overline{x}, \overline{y})$ as words in our generators. On the other hand, by Britton's Lemma, for any word $V = V(F, \overline{x}, \overline{y})$ we have for $u = t, s$

$$\sigma_u(V) = \sigma_{\overline{x}}(V)\sigma_u(\overline{x}) + \sigma_{\overline{y}}(V)\sigma_u(\overline{y}) \tag{4.5}$$

Which means that we should be able to express vectors $(1, 0)$ and $(0, 1)$ as linear combinations of $\mathrm{sgn}(\overline{x})$ and $\mathrm{sgn}(\overline{y})$ in $\mathbb{Z}^2$, *in particular they are linearly independent as vectors.*

Lemma 4.3.23. *Suppose $F_{R(S)}$ split as (4.4), then if some word $W(F, \overline{x}, \overline{y})$ lies in $\widetilde{F}$ then it must have exponent sum 0 in both $\overline{x}$ and $\overline{y}$.*

Proof. If $w = W(F, \overline{x}, \overline{y}) \in \widetilde{F}$, then $\mathrm{sgn}(w) = (0, 0)$. The result now follows immediately from equation (4.5) and the fact that $\mathrm{sgn}(\overline{x})$ and $\mathrm{sgn}(\overline{y})$ are linearly independent. ☐

Lemma 4.3.24. *Suppose $A \leq F_{R(S)}$ is a noncyclic abelian subgroup of $F_{R(S)}$. Then A cannot be generated by words $W_i(F, \overline{x}, \overline{y})$ such that for each W_i the exponent sums in $\overline{x}$ and $\overline{y}$ are zero.*

Proof. Let $\mathcal{R}$ be a strict resolution with fully residually F groups $\{F_{R(S_i)}\}_{i \in I}$. Then for some $j \in I$ we must have that a summand $\langle r \rangle \leq A$ is *exposed* in the JSJ of $F_{R(S_j)}$. Without loss of generality $\pi(A) = A' \oplus \langle r \rangle$, where π is the composition of maps $\pi : F_{R(S)} \to F_{R(S_j)}$. We can now view $F_{R(S_j)}$ as an HNN extension

$$F_{R(S_j)} = \langle G, r | \forall a \in A', [r, a] = 1 \rangle$$

On one hand $\overline{x}$ and $\overline{y}$ are sent to elements with normal forms $\overline{x}'(H,r), \overline{y}'(H,r)$ on the other hand, r is in the image of A and by hypothesis we can write $r = R(F, \overline{x}'(H,r), \overline{y}'(H,r))$ where R has exponent sum zero in $\overline{x}'(H,r)$ and $\overline{y}'(H,r)$, but by Britton's Lemma $R(F, \overline{x}'(H,r), \overline{y}'(H,r)) = r$ must have exponent sum zero in r which is a contradiction. ☐

Corollary 4.3.25. *If $F_{R(S)}$ has a cyclic splitting modulo F with two cycles, then none of the vertex groups can contain noncyclic abelian subgroups.*

4.3.3 The Nielsen Weidmann Technique for Groups acting on Trees

We present some of the techniques developed by Richard Weidmann in [61]. Let G be a group and let T be a simplicial tree on which G acts.

Definition 4.3.26. Let $M \subset G$ be partitioned as

$$M = S_1 \sqcup \ldots S_p \sqcup H$$

where $H = \{h_1, \ldots, h_s\}$. We say that M has *markings* $(S_1, \ldots, S_p; H)$. We now have elementary Nielsen transformations on *marked sets*:

$T1$: Replace some S_i by S_i^g where $g \in M - S_i$.

$T2$: Replace some element $h \in H$ by $g_1 h g_2$ where $g_1, g_2 \in M - \{h\}$

Let

$$T_i = \{x \in T | \exists g \in \langle S_i \rangle \text{ such that } gx = x\} \cup \text{the minimal } \langle S_i \rangle\text{-invariant subtree}$$

An important subcase will be when the groups $\langle S_i \rangle$ have global fixed points. It is convenient to note that there is no difference if we replace an element a subset S_i by the subgroup $\langle S_i \rangle$. We can now formulate the main results in [61].

Theorem 4.3.27. *Let M be a set with markings $(S_1, \ldots, S_p; H)$. Then either*

$$\langle M \rangle = \langle S_1 \rangle * \ldots * \langle S_p \rangle * F(H)$$

or by successively applying transformations $T1$ and $T2$ we can bring $(S_1, \ldots, S_p; H)$ to a normalized *marked set $\widetilde{M} = \langle \widetilde{S_1}, \ldots, \widetilde{S_p}, \widetilde{H} \}$ such that one of the following must hold:*

1. *$\widetilde{T_i} \cap \widetilde{T_j} \neq \emptyset$, for some $i \neq j$.*
2. *$\exists h \in \widetilde{H}, h\widetilde{T_i} \cap \widetilde{T_i} \neq \emptyset$*
3. *There is some $h \in \widetilde{H}$ that has a fixed point.*

This is especially useful to us if we let $F \leq S_1$. Notice that replacing transformation T_1 by T_1' given by:

- For S_1, do nothing.
- If $g \notin S_i$ replace S_i by $S_i^{g^{-1}}$. If $g \in S_i$ do nothing.
- If $g \neq h$ replace h by $h^{g^{-1}}$. If $g = H$ do nothing.

We can arrange so that in Theorem 4.3.27 $\widetilde{S_1} = S_1$ and that the results still holds. We call these *Nielsen moves on marked sets modulo F*.

4.3.4 Hyperbolic elements with small translation lengths

Lemma 4.3.28. *Let $F_{R(S)}$ act on a simplicial tree T with maximal abelian edge stabilizers. Then $F \leq F_{R(S)}$ acts transitively on the edges of T_F (see Definition 4.3.26). In particular if we have that A is an edge stabilizer and that $F \cap A = \langle \alpha \rangle$ and there are $g \in F_{R(S)}, u \in F$ such that $u^g = \alpha^n$ then there is some $g' \in F$ such that $u^{g'} = \alpha^n$, this means that $g^{-1}g' \in A$*

Proof. Let e_1, e_2 be edges of T_F. Let $\mathrm{stab}(e_1) \cap F = \langle \gamma \rangle, \mathrm{stab}(e_2) \cap F = \langle \beta \rangle$. There is some $g \in F_{R(S)}$ such that $ge_1 = e_2$. It follows that

$$g\gamma g^{-1} e_2 = (g\gamma g^{-1})e_1 = e_2$$

which means that

$$g\gamma g^{-1} \in stab(e_2) \Rightarrow [g\gamma g^{-1}, \beta] = 1$$

Suppose first that $g\gamma g^{-1} \notin \langle \beta \rangle$, then $\langle \beta, g\gamma g^{-1} \rangle$ forms a free abelian group of rank 2, but since $\gamma \in F$ we see that Lemma 4.3.24 applies and yields a contradiction. It therefore follows that $g\gamma g^{-1} = \beta^n$ for some $n \in \mathbb{Z} - \{0\}$. Since F has property CC it follows that there exists $g' \in F$ such that $g'\gamma g'^{-1} = \beta^n$, it follows that $(g')^{-1}g$ fixes e_1. Since e_1 and e_2 were arbitrary, the result follows. $\square$

We now focus on the situation where the action of $F_{R(S)}$ on its Bass-Serre tree T is 1-acylindrical. Of particular interest to us is the situation in Theorem 4.3.27 when $T_F \cap \rho T_F \neq \emptyset$ and where $d(v_0, \rho v_0) \leq 2$ where v_0 is the vertex of T fixed by F. To simplify notation we will use relative presentations. The path from v_0 to ρv_0 contains either one or two edge types in which case the splitting of $F_{R(S)}$ contains the "subgraphs"

$$\widetilde{F}^R \overset{\overset{t}{\curvearrowright}}{—C—} {}^S H \text{ or } \widetilde{F} —C— H \tag{4.6}$$

with $F \leq \widetilde{F}$.

Lemma 4.3.29. *Suppose that we have $F \leq \widetilde{F} \leq F_{R(S)}$ and T as above, assume moreover that the action of $F_{R(S)}$ on T is 1-acylindrical. If there is some ρ such that $T_F \cap \rho T_F \neq \emptyset$, with $v_0 = fix(F)$, then there are $f_1, f_2 \in F$ such that either*

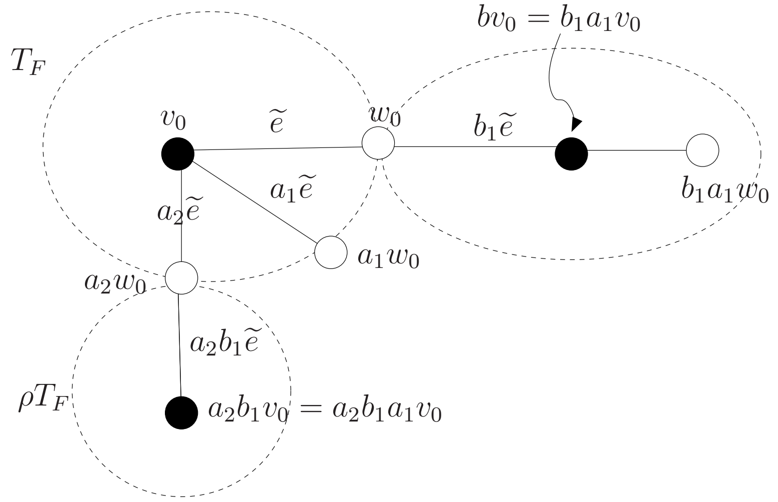


Figure 4–1: The path from v_0 to ρv_0 only contains 1 edge G-type

1. *the path $[v_0, \rho v_0]$ contains one edge $G-$type and $f_2\rho f_1 = h \in H$; or*
2. *the path $[v_0, \rho v_0]$ contains two edge $G-$types and $f_2\rho f_1 = th$ for some $h \in H$; which could be assumed to be the stable letter t if we change the presentation by conjugating a boundary monomorphism.*

Proof. We first prove case 1. We first must have that T_F is not just a point. If necessary, we conjugate boundary monomorphisms so that the lift $\widetilde{e}$ of the edge e in T is fixed by some element of F. Suppose we have that there is only one edge $G-$type traversed on the path from v_0 to ρv_0, as in Figure 4–1. Then we must have

$$\rho = a_2b_1a_1; a_i \in \widetilde{F}, b_1 \in H$$

and looking at Figure 4–1 we see that there must be some $\kappa \in F$ that fixes the edge $a_2\widetilde{e}$. Let $C = \text{stab}_G(\widetilde{e})$ be the subgroup stabilizer of the edge $\widetilde{e}$. Then there is some $g \in F_{R(S)}$ such that ${}^g\kappa \in C$. By Lemma 4.3.28 we find that there is some $g' \in F$

such that ${}^{g}\kappa = {}^{g'}\kappa \in C \cap F$. Letting $g' = f_2 \in F$ we have ${}^{f_2}\kappa = \gamma \in \mathrm{stab}(\widetilde{e})$ so

$$\gamma^{f_2} \in \mathrm{stab}(a_2\widetilde{e}) \Rightarrow f_2^{-1}\widetilde{e} = a_2\widetilde{e} \Rightarrow f_2a_2\widetilde{e} = \widetilde{e}$$

which means $f_2a_2 = b' \in \mathrm{stab}(te) \leq H$. This means that

$$f_2\rho = f_2a_2b_1a_1 = b'b_1a_1;\ b'b_1 \in H$$

so we may assume from now on that $a_2 = 1$.

We look again at Figure 4–1 and we see that the edge $b_1\widetilde{e}$ must lie in ρT_F and since $\rho = b_1a_1$ we have:

$$\begin{aligned} &{}^{b_1}\mathrm{stab}(e) \cap {}^{b_1a_1}F \neq 1 \\ &\Rightarrow \mathrm{stab}(e) \cap {}^{a_1}F \neq 1 \\ &\Rightarrow {}^{a_1^{-1}}\mathrm{stab}(e) \cap F \neq 1 \end{aligned}$$

which is like saying that $\mathrm{stab}(a_1^{-1}e) = {}^{a_1^{-1}}C \cap F \neq 1$. Reusing the argument to get rid of a_2 we can find some $f_1 \in F$ such that $a_1f_1 \in \mathrm{stab}(\widetilde{e}) \leq H$ which means that $\rho f_1 = b \in H$. This completes case 1 of the Lemma.

We now tackle case 2 of the Lemma. Consider Figure 4–2. Then we see that since we have $T_F \cap \rho T_F \neq \emptyset$ and that T_K has radius at most 1, we must have that $d(v_0, \rho v_0) \leq 2$ which means that

$$\rho = a_2tb_1a_1,\ a_i \in \widetilde{F}, b_1 \in H$$

and letting $R = \mathrm{stab}_{\widetilde{F}}(t), C = \mathrm{stab}(\widetilde{e})$ we can arrange, perhaps after conjugating boundary monomorphisms, that $F \cap C \neq 1 \neq F \cap R$. We can then find f_1, f_2 such

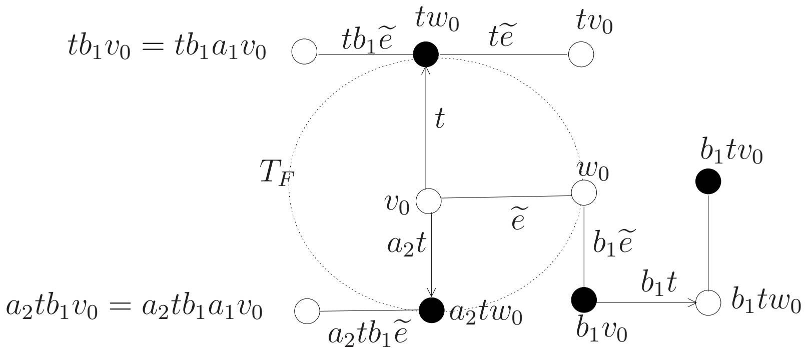


Figure 4–2: The path from v_0 to ρv_0 contains 2 edge G-types

that

$$f_2\rho f_1 = f_2a_2tb_1a_1f_1 = tb'$$

as in case 1. The proof is almost exactly as in case 1. The only difference is that we get $f_2a_2 = r \in R$ which means that we can "commute" it through the stable letter t to get the result. ☐

4.4 Coordinate Groups that are freely decomposable modulo F

Proposition 4.4.1. *Suppose that* $F_{R(S)} = \widetilde{F} * H$, *then*

$$Rank(\widetilde{F}) + \ Rank(H) \leq N + 2$$

Proof. First note that the underlying graph of the splitting $F_{R(S)} = \widetilde{F} * H$ consists of an edge and two distinct vertices. Let $\mathcal{G}(A)$ denote this graph of groups and let $\mathcal{B}$ be any $\mathcal{G}(A)$-graph. Only the moves A0-A3,F1,and F4 can be applied.

Take $\mathcal{W}$ to be the wedge $\mathcal{W}(F, \overline{x}, \overline{y})$. Since $\pi_1(\mathcal{W}) = F_{R(S)}$ we have by Theorem 4.3.13 that $\mathcal{W}$ can be brought to a graph with a single edge and two distinct vertices. The underlying graph of $\mathcal{W}$ has two cycles and A has no cycles, which means that two collapses must occur. Moreover each collapse, maybe after applying F1 moves, either contributes a generator to H or to $\widetilde{F}$. The result now follows. $\square$

Corollary 4.4.2. *If $F_{R(S)}$ is freely decomposable modulo F then either it is one generated modulo F or*

$$F_{R(S)} \approx \begin{cases} F * \langle x, y \rangle \\ F *_u Ab(u,t) * \langle x \rangle \\ F * Ab(x,y) \end{cases}$$

Proof. Apply Proposition 4.4.1 and Theorems 4.1.1 and 4.1.2. $\square$

4.5 Splittings with one nonabelian vertex group and at least one abelian vertex group

We prove item (C) of Theorem 4.1.6, i.e. we consider the case where the JSJ of $F_{R(S)}$ has abelian vertex groups but only one nonabelian vertex group $\widetilde{F} \geq F$.

Lemma 4.5.1. *If the cyclic JSJ decomposition of $F_{R(S)}$ modulo F contains only one nonabelian vertex group then we need only consider the following possibilities*

$$F_{R(S)} = \begin{cases} A_1 *_u \widetilde{F} *_v A_2 \\ \widetilde{F} *_u A \\ \circlearrowright \widetilde{F} \underset{u}{\overline{\qquad}} A \\ A_1{}^{\beta_1}_{\delta_1} \overset{\frown}{\overline{\qquad}} {}^{\alpha_1}_{\gamma_1}\widetilde{F}^{\alpha}_{\gamma} \overset{\frown}{\overline{\qquad}} {}^{\beta}_{\delta}A \\ t \circlearrowright {}^{\mu}_{\eta}\widetilde{F}^{\alpha}_{\gamma} \overset{\frown}{\overline{\qquad}} {}^{\beta}_{\delta}A \\ \widetilde{F} \overset{\frown}{\underset{\smile}{\overline{\qquad}}} A \end{cases}$$

where the vertex groups A, A_1, A_2 are abelian.

Proof. By the complexity bound given in Corollary 4.2.6, the only missing possibility is

$$\begin{tikzcd} F_{R(S)} = A_1 *_u \widetilde{F}^{\alpha}_{\gamma} \arrow[r, dash, bend left] \arrow[r, dash] & {}^{\beta}_{\delta}A \end{tikzcd}$$

But by Corollary 4.2.6 we see that the $\text{Rank}(A_1) = 2$, which means that $A_1 *_u \widetilde{F}$ can be seen as an extension of a centralizer, which is an HNN extension and therefore falls in one of the mentioned cases. $\square$

Proposition 4.5.2. *If the cyclic JSJ decomposition modulo* F *of* $F_{R(S)}$ *contains at least two vertex groups but only one of them,* $\widetilde{F}$*, is non-abelian, then the only possibilities for the splitting are*

$$\begin{tikzcd} F_{R(S)} = A_1 *_u \widetilde{F} *_v A_2,\ \widetilde{F} *_u A,\ \textit{or}\ \widetilde{F} \arrow[loop left] \arrow[r, dash, "u"'] & A \end{tikzcd}$$

with A, A_1, A_2 *abelian.*

Proof. First suppose that the underlying graph of D, the cyclic JSJ decomposition modulo F, contains two cycles. Then we can collapse D to a double HNN extension of some group H, say with stable letters s, t. By Corollary 4.3.25, H cannot contain any non-cyclic abelian subgroups – contradiction. The underlying graph of D therefore contains at most one cycle.

The only case left to check is when $\begin{tikzcd} F_{R(S)} = \widetilde{F}^{\alpha}_{\gamma} \arrow[r, dash, bend left] \arrow[r, dash] & {}^{\beta}_{\delta}A \end{tikzcd}$. Since $\widetilde{F} \neq F$ we have that α, γ obstruct $\widetilde{F}$. Let $\mathcal{R} :\ F_{R(S)} \xrightarrow[\pi_1]{} \cdots \xrightarrow[\pi_p]{} F_{R(S_p)}$ be a strict resolution. As long as β, α are unexposed (which means, $\widetilde{F}$ can't split) we have that $\widetilde{F}$ and A' are mapped monomorphically. On one hand the images of β and δ must be sent to powers of some common element in F since they commute. By Lemma 2.3.10 the associated $\Phi_{\mathcal{R}}$-morphisms F-discriminate $F_{R(S)}$, β and δ must be sent to arbitrarily high powers via $\Phi_{\mathcal{R}}$-morphisms. It follows that for some $F_{R(S_i)}$ in $\mathcal{R}$, say, β is exposed. At this point either $\widetilde{F}$ splits or it doesn't, if it doesn't split $\widetilde{F}$ lies in a nonabelian vertex group and α therefore fixes a vertex of nonabelian type, whereas β doesn't contradicting the fact that α and β are conjugate in $F_{R(S_i)}$. If $\widetilde{F}$ splits in $F_{R(S_i)}$, since γ and δ are conjugate, γ must be elliptic which means that α obstructed $\widetilde{F}$ and therefore is hyperbolic in $F_{R(S_i)}$, but β is elliptic –contradiction. ☐

Combining Proposition 4.5.2 and the following proposition proves most of item (C) of Theorem 4.1.6.

Proposition 4.5.3. *If* $F_{R(S)} = \widetilde{F} *_\alpha A_1$ *or* $F_{R(S)} = A_2 *_\beta \widetilde{F} *_\gamma A_3$ *and* $Rank(A_1) \geq 3$ *then* $\widetilde{F} = F$.

Proof. This follows immediately from computing $b_1(F_{R(S)})$. ☐

Case (C) of Theorem 4.1.6 is therefore proved.

4.6 Splittings with two or more nonabelian vertex groups

4.6.1 Maximal Abelian Collapses

Suppose that that JSJ decomposition of $F_{R(S)}$ contains at least two nonabelian vertex groups. The complexity bounds given in Section 4.2.1 do not give us bounds on the actual number of vertices in D. To have something more tractable to work with we take D and do the following (see Definition 2.2.5):

(i) If any boundary subgroup $\langle\alpha\rangle = i_e(G_e)$ is a proper subgroup of a maximal abelian subgroup A, then do a folding move where we replace G_e by a copy of A.

(ii) Ensuring that the resulting graph of groups always has at least two non abelian vertex groups perform sliding and collapsing moves until it is no longer possible to decrease the number of vertices or edges

In the end the resulting graph of groups $\mathcal{G}(A)$ will have one of three possible forms:

$$\begin{tikzcd} \widetilde{F} \arrow[r, dash] & H \end{tikzcd}, \quad \begin{tikzcd} \widetilde{F} \arrow[r, dash] \arrow[r, dash, bend left] & H \end{tikzcd}, \text{ or } \begin{tikzcd} \widetilde{F} \arrow[r, dash] \arrow[r, dash, bend left] \arrow[r, dash, bend right] & H \end{tikzcd} \tag{4.7}$$

where the vertex group $\widetilde{F}$ contains F and boundary subgroups are maximal abelian in their vertex groups. Moreover, by item (ii) of our construction, distinct edges have distinct boundary subgroups. It follows that this splitting is 1-acylindrical. It also follows that only the moves given in Section 4.3.1 are applicable to $\mathcal{G}(A)$-graphs.

Definition 4.6.1. We will call such a splitting a *maximal abelian collapse of* $F_{R(S)}$.

The strategy

To analyze the possibilities for $F_{R(S)}$, for each possibility given in (4.7) we will do the following:

1. Get the abelian collapse of the JSJ of $F_{R(S)}$, and use this splitting as the underlying graph of groups.
2. Start with the wedge $\mathcal{W} = \mathcal{W}(F, \overline{x}, \overline{y})$. Then using Theorem 4.3.27 and Lemma 4.3.29 we will simplify $\mathcal{W}$ by N1 moves (see Section 4.3.1) so that the loop $\mathcal{L}(\overline{x})$ is somehow simple.
3. We will then apply moves F1,F4, and L1 to simplify the graph as much as possible. It will turn out that the resulting $\mathcal{G}(A)$-graph $\mathcal{B}$ will have the same underlying graph as $\mathcal{G}(A)$.

4. All that will remain to get a folded graph is to make some transmission moves, keeping track of these will tell us how the vertex groups are generated.
5. Finally, by arguing algebraically we will recover the original cyclic JSJ decomposition modulo F.

4.6.2 The one edge case

We consider the case where $F_{R(S)}$ splits as

$$F_{R(S)} = \widetilde{F} *_A H \tag{4.8}$$

with $F \leq \widetilde{F}$, A maximal abelian in both factors and H non abelian. Throughout this section $\widetilde{F}, A, H$ will denote these groups.

Lemma 4.6.2. *Let $F_{R(S)}$ split as in (4.8). Using Nielsen moves on $(F, \overline{x}, \overline{y})$ modulo F we can arrange, conjugating boundary monomorphisms if necessary, that $\overline{x}$ lies in either $\widetilde{F}$ or H.*

Proof. Since we are assuming free indecomposability of $F_{R(S)}$ modulo F, we can apply Theorem 4.3.27. Let T be the Bass-Serre tree induced from the splitting (4.8). Let v_0 be the vertex fixed by $F \leq F_{R(S)}$. We start by looking at the marked generating set $(F; \{\overline{x}, \overline{y}\})$. We consider different cases.[1]

Case 1) T_F is a point: Since $F_{R(S)}$ isn't freely decomposable by Theorem 4.3.27 without loss of generality $\overline{x}$ must be elliptic. $T_{\overline{x}}$ is either a vertex or an edge

---

[1] Our notation is such that e.g. Case 2.3.4 is a subcase of Case 2.3.

Case 1.1) $T_F \cap T_{\overline{x}} = \emptyset$: Consider the the marked generating set $(F, \langle \overline{x} \rangle; \{\overline{y}\})$ and apply Theorem 4.3.27 again. We now find that either, without loss of generality, $\overline{y}T_{\overline{x}} \cap T_{\overline{x}} \neq \emptyset$ or $\overline{y}$ is also elliptic.

The first case is impossible. Indeed, by 1-acylindricity $T_{\overline{x}}$ is either an edge or a point so for $\overline{y}T_{\overline{x}} \cap T_{\overline{x}} \neq \emptyset$ we must have that $\overline{x}$ fixes one of the endpoints of $T_{\overline{x}}$ which implies that $\overline{y}$ is elliptic.

If $\overline{y}$ is also elliptic then the trees $T_F, T_{\overline{x}}, T_{\overline{y}}$ cannot all be disjoint, otherwise $F_{R(S)}$ would be freely decomposable. If $T_{\overline{y}} \cap \ T_F \neq \emptyset$ then we can switch $\overline{x}$ and $\overline{y}$ and pass to Case 1.2. Otherwise $T_{\overline{y}} \cap T_{\overline{x}} \neq \emptyset$ then the tree $T_{\langle \overline{y}, \overline{x} \rangle}$ is either an edge or has radius 1. Passing to the marking $(F, \langle \overline{x}, \overline{y} \rangle; \emptyset)$ and applying Theorem 4.3.27 implies that $T_{\langle \overline{y}, \overline{x} \rangle}$ can be taken so that $T_F \cap T_{\langle \overline{y}, \overline{x} \rangle} \neq \emptyset$. Which means that both $\overline{x}$ and $\overline{y}$ can be brought into H.

Case 1.2) $T_F \cap T_{\overline{x}} \neq \emptyset$: Since we have the splitting (4.8) and by our assumptions on edge stabilizers we have that if $\overline{x}$ fixes v_0 then $\overline{x} \in \widetilde{F}$ and we are done.

Case 2) T_F is not a point: Conjugating boundary monomorphism, we can arrange for some generator α of A to lie in F. We apply Theorem 4.3.27 and find that either $\overline{x}T_F \cap T_F \neq \emptyset$ or $\overline{x}$ is elliptic. In the former case, since T_F has radius 1 and $F \leq F_{R(S)}$ has property CC we can apply Lemma 4.3.29 to make $\overline{x} \in H$ and we are done. Otherwise $\overline{x}$ is elliptic and we consider the next case.

Case 2.1) $T_F \cap T_{\overline{x}} = \emptyset$: We consider the marked set $(F, \overline{x}; \{\overline{y}\})$ and we see that applying Theorem 4.3.27 we can either arrange for $\overline{y}$ to be elliptic or get that either $T_F \cap \overline{y}T_F \neq \emptyset$ or $T_{\overline{x}} \cap \overline{y}T_{\overline{x}} \neq \emptyset$. In Case 1.1 the latter possibility was seen to be impossible unless $\overline{y}$ was elliptic. If $\overline{y}T_F \cap T_F \neq \emptyset$, then we can apply Lemma 4.3.29 as for the previous case and obtain that $\overline{y} \in H$ and we are done.

The remaining possibility in this case is that $\overline{y}$ is elliptic and $T_{\overline{y}} \cap T_F = \emptyset$. For our group not to be freely decomposable we must have that $T_{\overline{y}} \cap T_{\overline{x}} \neq \emptyset$, moreover since both $\overline{x}, \overline{y}$ are elliptic the tree $T_{\langle \overline{x}, \overline{y} \rangle}$ must have radius 1. This is dealt with exactly as in the end of Case 1.1.

Case 2.2) $T_F \cap T_{\overline{x}} \neq \emptyset$: Then we have, after perhaps changing the boundary monomorphisms, that $\overline{x} \in H$ or $\overline{x} \in \widetilde{F}$ and we are done. ☐

Lemma 4.6.3. *Suppose $F_{R(S)}$ splits as (4.8) maximal abelian group A with $F \leq \widetilde{F}$ and H non abelian. If $\overline{x}$ can be brought into the subgroup $\widetilde{F}$ via Nielsen moves modulo F, then $F_{R(S)}$ is freely decomposable.*

Proof. Let $\mathcal{G}(X)$ be the graph of groups representing the splitting (4.8). We start with the wedge $\mathcal{W}(F, \overline{x}, \overline{y})$, and suppose that after a sequence of N1 moves, the element $\overline{x}$ lies in $\widetilde{F}$. Then $F_{R(S)}$ can be represented with a $\mathcal{G}(X)$-graph graph $(\mathcal{B}, v)$, which consists of a vertex v and a $\overline{y}$-loop $\mathcal{L}(\overline{y}, v)$, where the vertex v has label $\langle F, \overline{x} \rangle$.

Since $\pi_1(\mathcal{B}, v) = F_{R(S)}$, by Theorem 4.3.13 we should be able to bring $\mathcal{B}$ to $\mathcal{G}(X)$. Now only the vertex group B_v of $\mathcal{B}$ is non trivial. We do our folding process, but only doing adjustments and foldings, in particular, doing moves only A0-A3,F1,F4,L1 and S1. If a collapse occurs then $\mathcal{B}$ doesn't have any cycles and B_0 has an extra yellow vertex. By doing S1 shaving moves we can assume that B_0 has at most one yellow vertex. By Lemma 4.3.18, as long as the terminal transmission graph has at most one green vertex or a cancellable path, we will always be able to perform one of the moves F1, F4, S1 after a some sequence of L1 and A0-A3 moves.

We see that each step of the way the terminal transmission graph of $\mathcal{B}$ has at most one green vertex or cancellable path, unless one of three possibilities occurs, let v be the vertex such that $B_v \geq \langle F, \overline{x} \rangle$:

(a) $\mathcal{B}$ has two vertices v, u and one edge e and after a transmission B_u is generated by at most two elements. The graph is then folded, but we see that $H = A * H'$ which implies free decomposability.

(b) $\mathcal{B}$ has no cycles, three vertices and two edges. We assume that all shaving moves were performed. Then the only possibility for $\mathcal{B}$ is that it has endpoints v and u, and B_u is cyclic. But then if the transmission graph had any red edges then the vertex u could have been shaved off. It follows that it is impossible for the transmission graph to have two green vertices.

(c) $\mathcal{B}$ consists of a cycle of length 2, with vertices v and u. This means that the vertex group B_u is trivial. For $\mathcal{B}$ to be folded, after a transmission we must be able to perform a F4 collapse move.

    If the collapse is towards u then no transmissions are needed and B_u will be generated by some element and the edge group, which implies free decomposability. If the collapse is towards v then if the collapse can be done without doing transmissions first, we reduce to case (a).

    Otherwise, before the collapse, do a conjugation move at u and two transmissions so that $B_u = \langle b_1 A' b_1^{-1}, A'' \rangle$ where $A', A'' \subset A$ and $b_1 \notin A$, we now do a simple adjustment and collapse, in the end we get that $B_u = H = \langle b_1 A' b_1^{-1}, A \rangle$. By Lemma 4.6.14 this implies free decomposability.

□

Lemma 4.6.4. *Suppose $F_{R(S)}$ splits as (4.8) and via Nielsen moves we were able to bring $\overline{x} \in H$. Then $F_{R(S)}$ is generated by $\langle F, \overline{x}, \overline{y}' \rangle$ where $\overline{y}'$ also lies in H.*

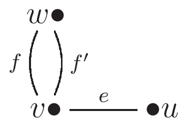
Proof. We start with the graph with one edge e and two vertices v, u with $B_v = F$ and $B_u = \langle \overline{x} \rangle$ and then at v attach the $\overline{y}$-loop $\mathcal{L}(\overline{y}, v)$. Start our adjustment-folding process applying only moves A0-A3,F1,F4,L1,S1, as much as possible. By Lemma 4.3.18, as long as the transmission graph has at most two green vertices, we will always be able to perform one of the moves F1, F4, S1 after a some sequence of L1 and A0-A3 moves.

Suppose first that along the process avoiding transmissions, a collapse occurred, then $\mathcal{B}$ doesn't have any cycles, either $\mathcal{B}$ can be brought to a graph with one edge and two vertices in which case, applying Lemma 4.6.3 if $B_u = \langle \overline{x} \rangle$, the result follows.

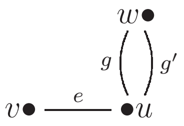
Otherwise, after shaving, we may assume that B has three vertices u, v, w and two edges. We already have an edge e between u and v which means that the other edge f must be either between v and w; in which case we can perform F1 at v without doing any transmissions; or f is between w and u; in which case transmission from w to u implies that w can be shaved off, so we can do an F1 move at u without transmissions. Either way, by Lemma 4.6.3 we must have $B_v = F$ and $B_u = \langle \overline{x}, \overline{y}' \rangle$ and all that is needed to get a folded graph is transmissions, so the result holds.

Suppose now that no collapses occurred, but it is impossible to perform any moves of type F1 or F4 without doing transmissions. Then by Lemma 4.3.18, the transmission graph must have at least three green vertices, this can occur only if one of these cases occur:

Case I



Case II



Case III

We first tackle Case I. If after transmissions a move of type F1 is possible, then it is easy to see that such a move could have been done without transmission and this reduces to Case III. It therefore follows that the edges f, f' have to be collapsed. If the collapse is towards w, then no transmission moves are necessary, we get a graph with three vertices with $B_w = \langle \overline{y}' \rangle, B_v = F$ and $B_u = \langle \overline{x} \rangle$, we note that if any transmissions from w or u to v are possible, then we can shave off these vertices and end up as in the case of Lemma 4.6.3. It therefore follows that we can perform a F1 folding move at v and replacing $\overline{y}'$ by a conjugate if necessary we get a graph with two vertices v, u with $B_v = F$ and $B_u = \langle \overline{x}, \overline{y}' \rangle$, all that remains to get a folded graph are transmission moves so the result follows.

Suppose now that the collapse was towards v then first we transmit towards w through both edges f and f' (if we only needed to transmit through one edge, we could have used a long range adjustment contradicting the assumption.) Now suppose $i(f) = i(f') = w$ and after a conjugation move we have that f and f' have labels $(c, [f], b)$ and $(1, [f], b')$ respectively. Moreover $c \notin A$, since otherwise we could have used a Bass-Serre move A1, and then collapsed. Then after the transmissions $B_w = \langle \alpha, c\alpha c^{-1} \rangle$ where $\langle \alpha \rangle = F \cap A$. To be able to make an F4 move, we must be able to make a simple adjustment which implies that $c \in \langle \alpha, c\alpha c^{-1} \rangle$. Let $c\alpha c^{-1} = \beta$, we have that $\langle \alpha, \beta \rangle$ freely generate a free group of rank 2. But if $c = W(\alpha, \beta)$, then we would have the relation

$$W(\alpha, \beta)\alpha W(\alpha, \beta)^{-1} = \beta$$

which is impossible.

We now consider Case II, as before it may reduce to Case III. Let g and g' be the edges between v and w. Suppose first that the collapse is towards u, but there had to be transmissions to w first, then as in the previous paragraph we can derive a contradiction. It follows that the collapse must be towards w, but in this case long range adjustments can be used instead of transmissions and simple adjustments and we revert to the no cycle case.

Finally, we consider Case III, since there can be no transmissions from u to v, we must have that the collapse is from v to u. Note that since F has property CC and that if it was possible to transmit through both edges towards v, instead of doing so one can make a simple adjustment A2 at F and collapse towards u and the result holds. This means that only one edge can transmit, so it follows that the F4 move can simply be preceded by long range adjustments and A0,A2 moves. But then we have that $B_v = \langle \overline{x} \rangle$ and the only moves left are transmissions which imply that $H = \langle \overline{x} \rangle * A$ which implies free decomposability of $F_{R(S)}$. $\square$

From the previous lemmas we get.

Proposition 4.6.5. *If $F_{R(S)}$ is freely indecomposable and splits as in (4.8) with $F \leq \widetilde{F}$. Then $F_{R(S)}$ can be generated by F and two elements $\overline{x}, \overline{y}' \in H$. Therefore, up to rational equivalence, we can assume that $\overline{x}, \overline{y}$ are sent into H.*

This next proposition enables us to revert back to a cyclic splitting.

Proposition 4.6.6. *Suppose that $F_{R(S)}$ has a splitting as in (4.8) then $F_{R(S)}$ admits a cyclic splitting*

$$\widetilde{F}' *_{\langle p \rangle} H'$$

where either:

1. *$\widetilde{F}' = F$ and H is generated by $\langle p, \overline{x}, \overline{y} \rangle, p \in F$ i.e. H is a three generated fully residually free group (see Theorem 4.1.1).*
2. *$\widetilde{F} = \langle F, p \rangle$ and $H' = \langle \overline{x}, \overline{y} \rangle$ with $p \in H'$ i.e. H' is free of rank 2.*

Proof. We first consider when the amalgamating maximal abelian subgroup A in (4.8) is cyclic. We write $A = \langle p \rangle$ and $\widetilde{F}' = \widetilde{F}, H' = H$, then we apply Proposition 4.6.5 and we get that $\overline{x}, \overline{y} \in H$. Looking at normal forms we have that $\widetilde{F} = \langle F, p \rangle$ and $H = \langle p, \overline{x}, \overline{y} \rangle$ if $p \in F$ then $\widetilde{F} = F$ and H is three generated fully residually free. If $p \notin F$ then we must have by normal forms that $p \in \langle \overline{x}, \overline{y} \rangle$ and it follows that $H = \langle \overline{x}, \overline{y} \rangle$.

We now consider the case where A in (4.8) is not cyclic. By Proposition 4.6.5 we can assume that $\overline{x}, \overline{y} \in H$ and it follows that $\widetilde{F} = \langle F, A \rangle$. $\widetilde{F}$ is fully residually F but not equal to F so $\widetilde{F}$ has a nontrivial cyclic splitting modulo F and A must be elliptic. By Theorem 4.1.2, the only possibility for $\langle F, A \rangle$ is

$$\widetilde{F} = F *_p A$$

which means that $F_{R(S)}$ admits the splitting:

$$F_{R(S)} = F *_p (A *_A H)$$

and as before we get that $H = \langle \overline{x}, \overline{y}, p \rangle$ is a three generated fully residually free group. $\square$

Lemma 4.6.7. *Suppose $F_{R(S)}$ splits as*

$$\widetilde{F} *_u (\langle u \rangle \oplus \langle \gamma \rangle) *_\gamma \langle \overline{x}, \overline{y} \rangle$$

then $F_{R(S)}$ is freely decomposable, in particular, $\langle \gamma \rangle$ must be a free factor of $F(\overline{x}, \overline{y})$.

Proof. There are solutions $f : F_{R(S)} \to F$ such that the restriction of f to $F(\overline{x}, \overline{y})$ is a monomorphism since $F(\overline{x}, \overline{y})$ is a two generated nonabelian subgroup of a fully residually free group. We fix a solution f and denote by $H \leq F$ the image of $F(\overline{x}, \overline{y})$. H is free of rank 2 but we see that in the ambient group $\overline{\gamma} = f(\gamma)$ is a proper power, i.e. $\overline{\gamma} = u^n$ and n can be made arbitrarily large via Dehn twists. Theorem 3.1.4 applies and the assumption that $\gamma \in F(\overline{x}, \overline{y})$ is not a proper power (by malnormality of boundary subgroups) and not primitive (by free indecomposability modulo F) force the image of $F(\overline{x}, \overline{y}) \to F$ to by cyclic contradicting injectivity. $\square$

Proposition 4.6.8 (Theorem 4.1.6 (D) item 1)**.** *If $F_{R(S)}$ has a QH subgroup then the only possibility is*

$$F_{R(S)} = F *_{u=p} \langle x, y, p | [x, y] = p \rangle$$

Proof. Note that by Proposition 4.2.5 and Corollary 4.2.6 there can be at most one MQH subgroup which must have genus at most 1, and the underlying graph of the graph of groups must be simply connected. The only suitable surface is the torus with one puncture. Again using the complexity bound in Corollary 4.2.6 the only possible cyclic JSJ splittings modulo F are of the form:

$$F_{R(S)} = \widetilde{F}' \underbrace{*_q H_1 *_{a_2} \ldots *_{a_k} H_k *_p}_{H} Q$$

where Q is the MQH subgroup and $F \leq \widetilde{F}'$. Consider the splitting

$$(\widetilde{F}' *_q H) *_p Q$$

where p lies inside a maximal abelian subgroup A, then we fold if necessary to get

$$(\widetilde{F}' *_q H) *_A (A *_p Q)$$

we now apply Proposition 4.6.5 to get $\overline{x}, \overline{y} \in (A *_p Q)$. First suppose that A is noncyclic abelian, then we get $\widetilde{F} = \widetilde{F}' *_q H = \langle F, A \rangle$ which as in the proof of Proposition 4.6.6 must be $\widetilde{F} = F *_q A$ so we get that $F_{R(S)} = F *_q A *_p Q$. If $p \neq q$ then, by Lemma 4.6.7, $F_{R(S)}$ is freely decomposable modulo F. If $p = q$ then if $\mathcal{G}(X)$ is the corresponding graph of groups with vertex groups F, A, Q then we find that $B(\mathcal{G}(X)) \geq 3$ which contradicts Corollary 4.2.6.

The possibility $A = \langle p \rangle$ remains. Proposition 4.6.5 gives us that $F_{R(S)}$ is generated by F and by $\overline{x}, \overline{y} \in Q$ and by Proposition 4.6.6 we have either $p \in F$ or $p \notin F$, in the former case the result follows i.e. we have that $Q = \langle \overline{x}, \overline{y}, p \rangle)$ and $\widetilde{F} = F$. In the latter case we have $\widetilde{F} = \langle F, p \rangle$ and we must have $p \in \langle \overline{x}, \overline{y} \rangle$. By Theorem 4.1.2 and by free indecomposability modulo F we must have that $\langle F, p \rangle = \widetilde{F} \approx \langle F, t | t^{-1}ut = u \rangle$. I.e. $\widetilde{F}$ contains a free abelian group of rank 2. On the other hand p has exponent sum 0 in both $\overline{x}$ and $\overline{y}$, so by Lemma 4.3.24 we have a contradiction. $\square$

Proposition 4.6.9. *Suppose that $F_{R(S)}$ splits as*

$$F *_{\langle p \rangle} H \tag{4.9}$$

and $H = G *_q Ab(q,t)$ is a rank 1 free extension of a centralizer of a free group of rank 2. Then (4.9) refines to

$$F *_{\langle p \rangle} G *_q Ab(q,t) \tag{4.10}$$

Proof. This situations corresponds to item 1 of Proposition 4.6.6. We need to show that p is conjugable into G. Suppose towards a contradiction that this was impossible.

We first consider the possible cyclic JSJ splittings of $F_{R(S)}$, since q must be elliptic, any induced cyclic splitting of H must induced a cyclic splitting of G modulo q. G is a free group of rank two so, by Lemma 3.1.10, G may only split further as an HNN extension $G = \langle G', s \mid s^{-1}us = u' \rangle$ modulo q.

Now consider the cyclic JSJ splitting of H modulo p, by Lemma 4.6.7 p is not conjugate to a power of t. By our assumption and, if necessary, by replacing G with a vertex group G' containing p and q we have that H has no nontrivial cyclic splittings modulo p. On the other hand H is not free and there is an H-discriminating family of F-homomorphisms that send p to some fixed element of F, we must therefore have that H has a cyclic splitting modulo p –contradiction. ☐

We note that this proof did not exclude the case where p=q.

Proposition 4.6.10. *Suppose that $F_{R(S)}$ splits as*

$$\widetilde{F} *_{\langle p \rangle} H \text{ or}$$

where H is free of rank 2 and suppose moreover that H splits further as an HNN extension

$$H = \langle G, t \mid t^{-1}\mu t = \mu' \rangle$$

modulo p. Then we have that p and μ cannot be conjugate in H.

Proof. By Lemma 3.1.10, we must have that $H = \langle \mu, t \rangle$, if α and μ are conjugate in H then by conjugating boundary monomorphisms and doing Tietze transformations we see that $F_{R(S)} = \widetilde{F} * \langle t \rangle$. □

The same argument yields:

Proposition 4.6.11. *Suppose that $F_{R(S')}$ splits as*

$$F *_{\langle p \rangle} H *_u Ab(u, t)$$

with H free of rank 2 and suppose moreover that H splits further as an HNN extension

$$H = \langle G, t \mid t^{-1}\mu t = \mu' \rangle$$

modulo p and u, then we cannot have that both p and u are conjugate to μ

Corollary 4.6.12. *If $F_{R(S)}$ is freely indecomposable and the maximal abelian collapse of its cyclic JSJ decomposition modulo F is a free product with amalgamation. Then all the possibilities for the JSJ of $F_{R(S)}$ are given in item (D) of Theorem 4.1.6*

4.6.3 The two edge case

We now consider the case where, after "folding and sliding", the splitting of $F_{R(S)}$ has underlying graph

$$\begin{tikzcd} X = v\bullet \arrow[r, bend left, dash, "e"] \arrow[r, bend right, dash, "f"'] & \bullet u \end{tikzcd} \tag{4.11}$$

and to which we give the relative presentation:

$$\widetilde{F}^{B}\xrightarrow[A]{t}{}^{C}H \tag{4.12}$$

where $F \leq \widetilde{F} = X_u, H = X_v$ and A, B, C are maximal abelian and conjugacy separated (in their vertex groups). In particular we have that the action of $F_{R(S)}$ on the corresponding Bass-Serre tree is 1-acylindrical. We now prove the lemmas that will enable us to prove item (E) of Theorem 4.1.6.

Lemma 4.6.13. *If $F_{R(S)}$ splits as in (4.12), then using Nielsen moves on $(F, \overline{x}, \overline{y})$ modulo F we can arrange, conjugating boundary monomorphisms if necessary, that $\overline{x}$ either lies in $\widetilde{F} \cup H$ or $\overline{x} = t$.*

Proof. We first observe that $F_{R(S)}$ cannot be generated by elliptic elements w.r.t. the splitting (4.12). We assume free indecomposability modulo F and apply Theorem 4.3.27 to the marked generating set $(F; \{\overline{x}, \overline{y}\})$. Let T be the Bass-Serre tree of the splitting. Let $v_0 \in T$ be the vertex fixed by $F \leq F_{R(S)}$.

Suppose that T_F is a point. Then by Theorem 4.3.27 we are forced to have that $\overline{x}$ can be brought to an elliptic element and that we can arrange $T_{\overline{x}} \cap T_F \neq \emptyset$ or $\overline{y}T_{\overline{x}} \cap T_F \neq \emptyset$. It follows that we can bring $\overline{x} \in \widetilde{F}$. Suppose now that T_F is not a point. Either

1. $T_F \cap \overline{x}T_F \neq \emptyset$. Then we can apply Lemma 4.3.29 and get that $\overline{x}$ is either in H or $\overline{x} = th, h \in H$; or
2. $\overline{x}$ is elliptic. Then we use Theorem 4.3.27 on the marked set $(F, \overline{x}; \{\overline{y}\})$. $\overline{y}$ cannot also be elliptic.

If $T_F \cap T_{\overline{x}} \neq \emptyset$ then if $v_0 \in T_{\overline{x}}$ then we can assume that $\overline{x} \in \widetilde{F}$. If $\overline{x}$ fixes a vertex w' adjacent to v_0, then we can assume that $\overline{x} \in \mathrm{stab}(w')$. By Lemma 4.3.28 we either have that $\overline{x}$ can be brought into H or tHt^{-1}, after possibly changing the spanning tree, the result follows.

If $T_F \cap \overline{y} T_F \neq \emptyset$ then as before we can arrange to that $\overline{y} = th$ and interchanging $\overline{x}$ and $\overline{y}$ the result will follow. The remaining case is $T_{\overline{x}} \cap \overline{y} T_{\overline{x}} \neq \emptyset$. We note, however, that $T_{\overline{x}}$ has at most one $F_{R(S)}-$type of edge, but since $\overline{y}$ has exponent sum 1 in the stable letter, we find that the path ρ connecting $T_{\overline{x}}$ and $\overline{y} T_{\overline{x}}$ must contain both edge types. It follows that $T_{\overline{x}} \cap \overline{y} T_{\overline{x}} = \emptyset$. So we must have $T_F \cap \overline{y} T_{\overline{x}} \neq \emptyset$ which reduces to an earlier possibility.

□

Lemma 4.6.14. *Let $A, B \leq G$ be two abelian subgroups of a fully residually free group G such that for some $a \in A, b \in B$ we have $[a, b] \neq 1$. Then we have*

$$\langle A, B \rangle = A * B$$

Proof. Let $w = a_1 b_2 a_3 \dots b_n$ be product of non trivial factors $a_i \in A$ and $b_j \in B$ with perhaps the exception that a_1 or b_n are trivial. Since G is fully residually free there exists a map of G into F such that all the nontrivial a_i, b_j as well as some commutator $[a, b], a \in A, b \in B$ do not vanish. We have that the a_i are sent to powers of some element $u \in F$ and the b_j are sent to powers of some $v \in F$. It follows that the homomorphic image of w is sent to a freely reduced word in u and v, and since $u, v \in F$ do not commute they freely generate a free subgroup of F. It follows that w is not sent to a trivial element. □

Lemma 4.6.15. *If the vertex group H in the splitting (4.12) is generated by conjugates of its boundary subgroups, i.e. $H = \langle A^{h_1}, C^{h_2} \rangle$, $h_i \in H$, then $F_{R(S)}$ is freely decomposable modulo F.*

Proof. Without loss of generality, by conjugating boundary subgroups we may assume, $h_1, h_2 = 1$. Take the presentation of $F_{R(S)}$ from the splitting, since by Lemma 4.6.14 $H = A * C$ we can use a Tietze transformation to get rid of C. The resulting group will have a relative presentation

$$\langle \widetilde{F}, t | \emptyset \rangle = \widetilde{F} * \langle t \rangle$$

$\square$

Lemma 4.6.16. *Suppose that $F_{R(S)}$ splits an in (4.12) and that $\overline{x} \in \widetilde{F}$, then $F_{R(S)}$ is freely decomposable modulo F.*

Proof. It is clear that, perhaps after Nielsen moves modulo $\langle F, \overline{x} \rangle, \overline{y}$ can be sent to an element with exactly one occurrence of the stable letter t. A and C are not conjugate, which implies that $B \cap \langle F, \overline{x} \rangle = B' \neq 1$ and that the occurrences of t cancel in some product

$$\zeta^{-1} \beta \zeta$$

for some $\beta \in B'$. Let $C' = (B')^{\overline{y}}$, we have that C' is conjugable into H. Let $A' = \langle F, \overline{x} \rangle \cap A$.

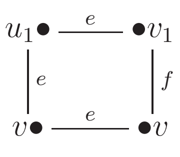
To derive free decomposability it is enough to assume that $C' \leq H$. $C' \cap A = 1$ which means that $\langle C', A \rangle = C' * A$ and in particular that $\langle C, A' \rangle \cap A = A'$ which means that $H \cap \langle F, \overline{x}, \overline{y} \rangle = \langle A, C' \rangle$, which by Lemma 4.6.15 implies that $F_{R(S)}$ is freely decomposable. $\square$

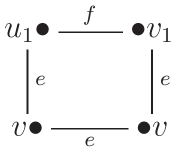
Lemma 4.6.17. *Suppose that* $\overline{x} \in H$*, then we can arrange so that* $F_{R(S)}$ *is generated by* F *and elements* $\overline{x} \in H$ *and some* $\overline{y}' = th, h \in H$*.*

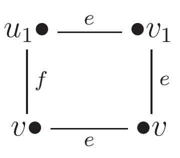
Proof. We first note that if $\overline{x}$ is conjugate into an edge group, then we can assume $\overline{x} \in \widetilde{F}$, which leads to a contradiction. The hypotheses imply that $F_{R(S)}$ is the fundamental group of a $\mathcal{G}(X)-$ graph $\mathcal{B}$ obtained by taking an edge labelled, say $(1, e, 1)$, with endpoints v and u, attaching the $\overline{y}-$loop $\mathcal{L}(\overline{y}, v)$, and setting $B_v = F, B_u = \langle \overline{x} \rangle$.

The transmission graph B_0 has the green vertex v and the yellow vertex u. We start our folding process using only moves A0-A3,F1,F4,L1,S1. Note that if a F4 collapse move occurs, then the underlying graph will be simply connected, which is impossible. By Lemma 4.3.18, this will be able to continue folding and adjusting as long as the terminal transmission graph has at most two green vertices or a cancellable path, which must be the case if, after shaving, there are more than four vertices.

Suppose that $\mathcal{B}$ has only four vertices, noting that this must fold to a graph like (4.11) using F1 moves we see (exchanging the labels e and f, if necessary) that the only possibilities after doing S1 moves are:

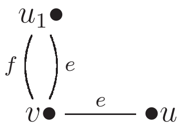


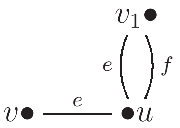




where here the edges are labelled by their type. If we look more closely at the possible transmissions, we see that the vertices u_1, v_1 will always be yellow in transmission graphs, so we can continue folding without using transmissions.

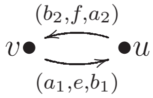
Suppose now that $\mathcal{B}$ has only three vertices, then the only possibilities after doing S1 moves are





Since the only folding that can occur is a F4 move at v or u, we see that we can use moves A0-A4,L1 and then make an F1 move.

It follows that $\mathcal{B}$ can be brought to a graph of the form:



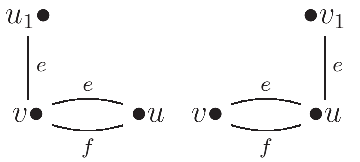
with $B_v = F$ and $\mathcal{B}_u = \langle \overline{x} \rangle$. Moreover our assumption that $\overline{x} \in H$ meant that $a_1 = b_1 = 1$. $F_{R(S)}$ is therefore generated by $F, \overline{x} \in H$ and $\overline{y}' = f, b_2, e, a_2$ with $a_2 \in \widetilde{F}$ and $b_2 \in H$, by conjugating boundary monomorphisms we can assume that $\overline{y}' = t$. □

Lemma 4.6.18. *Suppose that $\overline{x} = t$, then we can arrange so that $F_{R(S)}$ is generated by F, $\overline{x}$ and some $\overline{y}' \in H$.*

Proof. The hypotheses imply that $F_{R(S)}$ is the fundamental group of a $\mathcal{G}(X)$-graph $\mathcal{B}$ obtained by taking two edges with labels $(1, e, 1)$ and $(1, f, 1)$ with common endpoints v and u, setting $B_v = F, B_u = \{1\}$, and attaching the $\overline{y}$-loop $\mathcal{L}(\overline{y}, v)$.

The transmission graph B_0 has only the green vertex v, but the terminal transmission graph has green vertices u and v. We start our folding process using only moves A0-A3,F1,F4,L1,S1 as much as possible. By Lemma 4.3.18, this will be able to continue as long as the terminal transmission graph has at most two green vertices or a cancellable path.

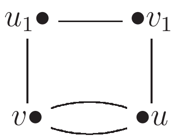
Suppose first that a F4 collapse occurred. Then $\mathcal{B}$ has one cycle. After shaving, the only possibilities (interchanging labels e and f, if necessary) so that the terminal transmission graph has more than two green vertices are



(4.13)

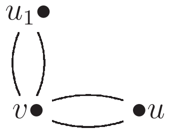
We note that in this case, on can always use moves A0-A3,and L1 so that F1 can be performed. In these cases we either get a graph with $B_v = \langle F, \overline{y}' \rangle, B_u = \{1\}$ or $B_v = F, B_u = \{\overline{y}'\}$. To get a folded graph, all that remains are transmissions, but note that in the former case, Lemma 4.6.15 will imply free decomposability modulo F, in the latter case the result follows.

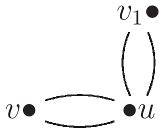
Suppose that no collapses occurred, but the terminal transmission graph has more than two green vertices and no cancellable paths. Then $\mathcal{B}$ has at most 4 vertices and the only possibility with 4 vertices is something of the form:



Either an F1 type fold can be performed at u or v, in which case we need only make moves A0-A3,L1 before, or the fold occurs at v_1 or u_1. In these latter cases (noting that all vertex groups except B_v are trivial) we see that if either u_1 or v_1 become green then before doing the F1 type fold, we will only need to use moves A0-A3. It follows that we can continue without transmissions.

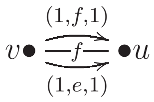
Suppose now that $\mathcal{B}$ has three vertices, then the possibilities are:

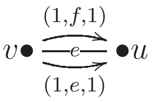




If the next fold will be of type F1, then it will occur at either v or u and moves A0-A3,L1, will be sufficient, and we get a graph with two vertices and three edges. If the next fold is a collapse, first note that, by the exact same arguments as in Lemma 4.6.4 Case I, if it is possible to get to collapse towards either v or u, then this must be possible without using transmissions first. Either collapse will reduce the situation to (4.13).

Suppose now that $\mathcal{B}$ has two distinct vertices and three edges, the possibilities are





where the remaining edge is marked only by its type. Note that these cases are symmetric. Suppose the third edge has label (a, e, b) with $a \in \widetilde{F}$ and $b \in H$. First note that if it is possible to transmit through both e-type edges from v to u, because F has property CC this means that there is some $a' \in F$ such that $a'a \in X_e$ which means that after an A1 Bass-Serre move we can make a F4 collapse towards u.

The remaining case is that one cannot transmit through both e-type edges from v to u, but then this means that we can make a long range adjustment to either bring (a, e, b) to $(a, e, 1)$ or bring $(1, e, 1)$ to $(1, e, b)$, and collapse.

If the collapse is towards v, then $B_v = \langle F, \overline{y}' \rangle$ and $B_u = \{1\}$, or $B_v = F$ and $B_u = \langle \overline{y}' \rangle$. All that remains to be done to get a folded graph is transmissions, in the latter case the result follows, in the former case we can derive free decomposability modulo F. □

All these lemmas combine to give:

Proposition 4.6.19. *If we have the splitting described in (4.12) then we have that* $F_{R(S)}$ *is generated by the stable letter* t *and some element* $\overline{y}' \in H$ *. Therefore, up to rational equivalence, we can assume that* $\overline{x} = t$ *and* $\overline{y}$ *us sent into* H*.*

The next proposition enables us to revert to a cyclic splitting.

Proposition 4.6.20. *Suppose that* $F_{R(S)}$ *has a splitting as in (4.12), then* $F_{R(S)}$ *admits one of the two possible cyclic splittings:*

1. $F_{R(S)} = F^\beta \xrightarrow[\alpha]{\overline{x}} H'$ *where* $H' = \langle \alpha, \overline{x}^{-1}\beta\overline{x}, \overline{y} \rangle$ *is a three generated fully residually free group.*
2. $F_{R(S)} = \widetilde{F}^\beta \xrightarrow[\alpha]{\overline{x}} H'$ *where* $\widetilde{F}$ *is a rank 1 free extension of a centralizer of* F *and* H' *is generated by* $\alpha, \overline{y}$*. Moreover* H' *may not split further as an HNN extension.*

Proof. Let $\widetilde{F}, A, B, C, H, t$ be as in (4.12). By Proposition 4.6.19 we can assume that $\overline{x} = t$ and $\overline{y} \in H$. We can always assume that either $F \cap A \neq \{1\}$ (otherwise we could derive free decomposability modulo F.)

Suppose first that $F \cap A = \langle \alpha \rangle$ *and* $F \cap B = \{1\}$. To ensure free indecomposability modulo F we need there to be some $\gamma \in \langle \alpha, \overline{y} \rangle$ such that $\overline{x}\gamma\overline{x}^{-1} = \beta \in \widetilde{F} \cap B$. Now by Theorem 4.1.2 if $\langle F, \beta \rangle \neq F * \langle \beta \rangle$ then we must have $\langle F, \beta \rangle = \langle F, t | [u, t] = 1 \rangle$ for some $u \in F$. If u is not conjugate to α in F then $F_{R(S)}$ has a cyclic splitting as in item 2. If u and α are conjugate in F, then we can assume that $\alpha = u$, so then the group A in (4.12) is noncyclic abelian of rank 2. We study the maximal abelian subgroup $C \leq H$. We already had that $\gamma(\overline{y}, \alpha) \in C$. If $C \leq H$ is not cyclic then there must be some $\gamma_1 \in \langle \overline{y}, A \rangle$ such that γ and γ_1 do not lie in a common cyclic subgroup and which satisfies the relation

$$[\gamma, \gamma_1] = 1 \tag{4.14}$$

however by Lemma 4.6.14 we have

$$\langle A, \overline{y} \rangle = A * \langle \overline{y} \rangle \tag{4.15}$$

which means that (4.14) is impossible. It follows that $\widetilde{F} = \langle F, \beta \rangle$. This gives the cyclic splitting:

$$\begin{tikzcd} F_{R(S)} = \langle F, t \rangle \arrow[r, bend left, "\overline{x}"] \arrow[r, dash, "\alpha"'] & {}^{\gamma}\langle \alpha, \overline{y} \rangle \end{tikzcd}$$

Suppose now towards a contradiction that $\langle \alpha, \overline{y} \rangle$ split further as an HNN extension:

$$\langle \alpha, \overline{y} \rangle = \langle K | p^t = q \rangle; p, q \in K$$

then we have

$$\begin{tikzcd} F_{R(S)} = \widetilde{F}^{\beta}_{\alpha} \arrow[r, bend left] \arrow[r, bend right] & {}^{\beta'}_{\alpha'}K^{p}_{q} \arrow[loop right, "t"] \end{tikzcd}$$

Then we can collapse this splitting to a double HNN extension, and applying Lemmas 4.3.23 and 4.3.24 we see that $\widetilde{F}$ cannot contain any noncyclic abelian subgroups – contradiction.

Suppose now that $F \cap A = \langle \alpha \rangle$ *and* $F \cap B = \langle \beta \rangle$*:* Then we first consider $H'' = \langle F, \alpha, \gamma \rangle$ where $\gamma = \overline{x}^{-1}\beta\overline{x}$. Suppose that A are B are non cyclic, then we see that

$$\langle F, A, B \rangle = A *_{\alpha} F *_{\beta} B$$

since $\langle F, A, B \rangle$ is fully residually F. which means that letting $H' = \langle H'', A, B \rangle$ we get the cyclic splitting

$$F_{R(S)} \approx F^{\beta} \xrightarrow[\alpha]{\overline{x}} {}^{\gamma}H'$$

and looking at normal forms and words in $\{F, \overline{x}, \overline{y}\}$ we see that $H' = \langle \alpha, \gamma, \overline{y} \rangle$. In particular we have a splitting as in item 1. *Having exhausted all the possibilities, the result holds* ☐

Proposition 4.6.21. *Suppose that* $F_{R(S)}$ *splits as*

$$F^{\beta} \xrightarrow[\alpha]{} {}^{\gamma}H$$

and suppose moreover that H *splits as an HNN extension*

$$H = \langle G, s \mid s^{-1}\mu s = \mu' \rangle$$

modulo α, γ*. Then* α *and* γ *cannot both be conjugate to* μ *in* H*.*

Proof. After conjugating boundary monomorphisms and applying Lemma 3.1.10, it is easy to get Tietze transformations that exhibit a free decomposition. ☐

Proposition 4.6.22. *Suppose that* $F_{R(S)}$ *splits as* $F_{R(S)} = F^{\beta} \underset{\alpha}{\overset{\xrightarrow{\overline{y}}}{\rule{2em}{0.4pt}}} H'$ *where* H' *is a rank 1 free extension of a centralizer of a free group* H *of rank 2. Then this splitting can be refined to*

$$F_{R(S)} = F^{\beta} \underset{\alpha}{\overset{\xrightarrow{\overline{y}}}{\rule{2em}{0.4pt}}} {}^{\gamma}H \underset{u}{\rule{2em}{0.4pt}} Ab(u, s)$$

such that α *and* γ *are not conjugate and such that* u *is not conjugate to both* α, γ. *This cyclic splitting cannot be further refined.*

Proof. Noting that the elements α and $\gamma = \overline{x}^{-1}\beta\overline{x}$ of H' are conjugate into F enables us to use the arguments of Proposition 4.6.9 to show that the elements α and γ must be conjugable into H'.

If u is conjugate to both α, γ, then it is clear that the group is freely decomposable modulo F.

Finally, if this splitting could be refined, then the only possibility by Lemma 3.1.10 is that H refines to an HNN extension, but then we could apply Lemma 4.3.24, contradicting the existence of a non-cyclic abelian subgroup. ☐

All these Propositions imply:

Corollary 4.6.23. *If* $F_{R(S)}$ *is freely indecomposable and the maximal abelian collapse of its cyclic JSJ decomposition modulo* F *has two edges. Then all the possibilities for the JSJ of* $F_{R(S)}$ *are given in item* (E) *of Theorem 4.1.6*

4.6.4 The three edge case

We now consider the case where, after "folding, sliding, and collapsing", the splitting of $F_{R(S)}$ has underlying graph

$$X = \begin{tikzcd} v\bullet \arrow[r, bend left, "e"] \arrow[r, "f" description] \arrow[r, bend right, "g"'] & \bullet u \end{tikzcd} \tag{4.16}$$

and to which we give the relative presentation:

$$\begin{tikzcd} \widetilde{F}^{A}_{D} \arrow[r, bend left, "s"] \arrow[r, "C" description] \arrow[r, bend right, "t"'] & {}^{B}_{E}H \end{tikzcd} \tag{4.17}$$

Where $F \leq \widetilde{F} = X_v, H = X_u$ and A, B, C, D, E are maximal abelian in their vertex groups, hence the splitting is 1-acylindrical. Note that by Corollary 4.3.25, $F_{R(S)}$ cannot contain any noncyclic abelian subgroups, in particular the subgroups A, B, C, D, E must all be cyclic.

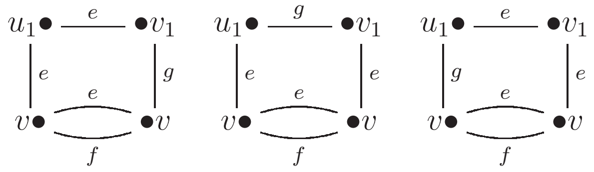
Lemma 4.6.24. *Let $F_{R(S)}$ split as in (4.17). Using Nielsen moves on $(F, \overline{x}, \overline{y})$ modulo F we can arrange; conjugating boundary monomorphisms if necessary so that $\overline{x} = t$.*

Proof. Since we are assuming free indecomposability of $F_{R(S)}$ modulo F, we can apply Theorem 4.3.27. Let T be the Bass-Serre tree corresponding to the splitting (4.16). We note that neither $\overline{x}$ nor $\overline{y}$ can be brought to elliptic elements w.r.t. the splitting (4.16). Let v_0 be the vertex fixed by F. W.l.o.g. we have $T_F \cap \overline{x}T_f \neq \emptyset$. 1-acylindricity implies that $d(v_0, \overline{x}v_0) = 2$. It follows w.l.o.g. that $\overline{x}$ is of the form a_1, f, b_1, e^{-1}, a_2 where $b_1 \in H, a_1, a_2 \in \widetilde{F}$, by Lemma 4.3.29 we can arrange so that $\overline{x} = f, b, e^{-1}$, and conjugating boundary monomorphisms enables us to assume that $\overline{x} = f, e^{-1} = s^{-1}$ in terms of the relative presentation (4.17). ☐

Lemma 4.6.25. *Suppose that $F_{R(S)}$ splits as in (4.17) and that $\overline{x} = t$, then $F_{R(S)}$ is generated by F, t, and s*

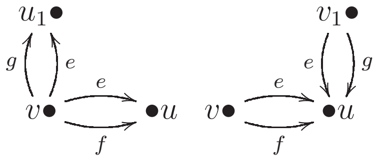
Proof. The hypotheses imply that $F_{R(S)}$ is the fundamental group of a $\mathcal{G}(X)$-graph $\mathcal{B}$ obtained by taking two edges with labels $(1, e, 1)$ and $(1, f, 1)$ with common endpoints v and u, setting $B_v = F, B_u = \{1\}$, and attaching the $\overline{y}$-loop $\mathcal{L}(\overline{y}, v)$.

Again we start our adjustment-folding process, using only moves A0-A2,L1,F1, and F4. Moreover we see that F4 collapses are forbidden since they reduce the number of cycles in the underlying graph. As long as there are strictly more than 4 vertices, the terminal transmission graph will only have two green vertices or a cancellable path. Suppose that the terminal transmission graph has only 4 vertices then, interchanging e and f if necessary, and noting that $\mathrm{sgn}_s(\overline{y}) = 1$, the only possibilities are:

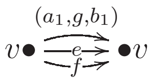


where the edges are marked by their type. In all three cases, though, we see that the vertices w_1, v_1 in the terminal transmission graph must be yellow, so we can continue our adjustment-folding process.

If there are only three vertices then we only the have possibilities:



and the last fold is of type F1 at either u or v and in particular no transmissions are needed. We get that B is given by



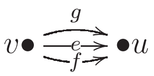
where the edges labelled e and f have labels $(1, e, 1)$ and $(1, f, 1)$ respectively. In the end we have that $F_{R(S)}$ is generated by F, t and some element $\overline{y}' = a_1, g, b_1, e^{-1}$ where $b_1 \in H$ and $a_1 \in \widetilde{F}$. After conjugating boundary monomorphisms, we may assume that $\overline{y}' = s$. $\square$

Corollary 4.6.26. *If $F_{R(S)}$ is freely indecomposable and the maximal abelian collapse of its cyclic JSJ decomposition modulo F has three edges, then the JSJ of $F_{R(S)}$ is what is described in item (F) of Theorem 4.1.6*

Proof. All we need to show is that the vertex groups are F and a free group of rank 2.

By the two previous lemmas we have that $F_{R(S)}$ is the fundamental group of the $\mathcal{G}(X)$-graph

$\mathcal{B} =$



with $B_v = F$ and $B_u = \{1\}$. To get a folded graph, all that are needed are transmissions. We also saw that the edge groups are cyclic. Suppose first that the only possible transmission is from u to v through e, then by 1-acylindricity, it is impossible for there to be any further transmissions from u back to v through the other edges.

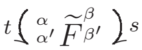
Suppose that now there were transmissions possible only from v to u through edges e and f. So as not to have free decomposability modulo F, we must have a transmission from u to v through g. We note that the boundary subgroups associated to the edges e, f must be maximal cyclic because they lie in F, it then follows that there are no further possible transmissions and the graph is folded. In particular we find that $B_u = \widetilde{F} = \langle F, \alpha \rangle$ where α is the element transmitted from H to $\widetilde{F}$. $F_{R(S)}$ is freely indecomposable only if $\widetilde{F} \neq F * \langle \alpha \rangle$, but by Theorem 4.1.2 the only other possibility for $\langle F, \alpha \rangle$ is $F *_u Ab(u, t)$, which is impossible since $F_{R(S)}$ has no noncyclic abelian subgroups.

It therefore follows that $\widetilde{F} = F$ and H is a free group of rank 2 generated by its boundary subgroups. $\square$

4.7 Splittings with one vertex group

We now consider the situation where $F_{R(S)}$ has a cyclic JSJ decompositions modulo F, which yields relative presentations:

(4.18)



(4.19)

4.7.1 First Classifications

We give a first description of the vertex group. Note that if $\widetilde{F} = F$ then the splittings (4.18) (4.19) must be extensions of centralizers of F.

Lemma 4.7.1. *Suppose $F_{R(S)}$ has the cyclic JSJ decomposition modulo F (4.19) then either $\widetilde{F} = F$, or the following hold:*

1. *One of the vertex groups, say* $\langle\alpha\rangle$*, is conjugate into* F*.*
2. $\widetilde{F}$ *is two generated modulo* F*.*
3. *The elements* α *and* β *are not conjugate in* $F_{R(S)}$
4. $\widetilde{F}$ *has no abelian subgroups*
5. *The splitting is 1-acylindrical.*
6. *Up to rational equivalence we can assume* $\overline{x} = t$ *and* $\overline{y} = t$
7. *If* $\widetilde{F}$ *is freely indecomposable, then its JSJ has two vertices and at most two edges. Moreover one of the vertex groups is* F*.*

Proof. The fact that $\widetilde{F}$ cannot contain any abelian subgroups follows immediately from Corollary 4.3.25. We are assuming that $\widetilde{F} \neq F$ which means that $\widetilde{F}$ has a nontrivial cyclic $D_{\widetilde{F}}$ splitting modulo F. It follows that α and β must obstruct this splitting. On the other hand we have free indecomposability of $F_{R(S)}$, and it is impossible for both $\overline{x}$ and $\overline{y}$ to be elliptic in this splitting (linear algebra on exponent sums of stable letters.) It follows that Theorem 4.3.27 forces T_F to have edges, i.e. w.l.o.g. $\alpha \in F$.

Suppose now that α and β were conjugate. $\alpha \in F$ would imply that α', β, β' are elliptic in $D_{\widetilde{F}}$, contradicting the fact that α, β obstruct the splitting $\langle\delta\rangle_{\widetilde{F}}$. We have thus shown that the edge groups are maximal abelian and that α, β are not conjugate, it follows that the splitting (4.19) is 1-acylindrical.

We apply Theorem 4.3.27 to the marked generating set $(F; \{\overline{x}, \overline{y}\})$. We can arrange via Nielsen moves so that $T_F \cap \overline{x}T_F \neq \emptyset$, by 1-acylindricity we have $d(v_0, \overline{x}v_0) < 2$ where $v_0 = \mathrm{fix}(F)$. Since β is not conjugate into F we have that path from v_0 to $\overline{x}v_0$ has only t-type edges, and since $\mathrm{sgn}_{(t,s)}(\overline{x}), \mathrm{sgn}_{(t,s)}(\overline{y})$ are a basis of $\mathbb{Z}^2$ (see Definition 4.3.22) we have that $\overline{x} = f_1 t f_2, f_i \in \widetilde{F}$, so conjugating boundary monomorphisms we can assume that $\overline{x} = t$.

$F_{R(S)}$ is given by the $\mathcal{G}(X)-$graph $\mathcal{B}$ with one vertex labelled v with label $(\langle \widetilde{F}, \alpha' \rangle, [v])$, an edge e with label $(1, [e], 1)$, i.e. $\mathcal{L}(\overline{x}, v)$; and the $\overline{y}-$loop $\mathcal{L}(\overline{y}, v)$. By 1-acylindricity we see that we can bring $\mathcal{B}$ to a graph with one vertex and two edges without any transmissions. In order to get a folded graph, we must finally transmit through the other edge so that the new label of v is $(\langle F, \alpha', \beta' \rangle, [v])$. It follows that $F_{R(S)}$ is generated by F, s, t and $\widetilde{F} = \langle F, \alpha', \beta' \rangle$.

Finally, suppose that the JSJ of $F_{R(S)}$ had only one vertex. Then since $\widetilde{F} \neq F$ the JSJ is nontrivial. Eventually the monomorphic image of $\widetilde{F}$ will split in some term $F_{R(S')}$ of a strict resolution of $F_{R(S)}$. In particular, $\widetilde{F}$ is generated by elliptic elements F and α'. Moreover since the induced splitting of $\widetilde{F}$ is as an HNN extension, then $F_{R(S')}$ must also split as an HNN extension, say with stable letter z. Since F and α' are elliptic, $\beta \in \langle F, \alpha' \rangle$ must have exponent sum 0 in z. On the other hand since $\widetilde{F} = F, \alpha', \beta'$ splits as an HNN extension, β' is forced to have exponent sum 1 in z, which is impossible since β' and β are conjugate in $F_{R(S')}$.

Since $\widetilde{F}$ cannot have any abelian subgroups, previous classifications imply that the JSJ of $F_{R(S)}$ has exactly two vertex groups one of which is F. Moreover in the $\widetilde{F}$ is generated by F and α' which are elliptic and some β' which may or may not be elliptic. Now a simple exponent sum argument excludes the possibility of a JSJ with more than two edges. $\square$

Corollary 4.7.2. *Suppose that $F_{R(S)}$ has cyclic JSJ modulo F:*

$$\begin{tikzcd}\widetilde{F} \arrow[loop left, "\alpha", "\alpha'"'] \arrow[r, dash, "u"'] & A\end{tikzcd}$$

with A abelian, then α and u are not conjugate and moreover either α or u are conjugate into F.

Lemma 4.7.3. *If $F_{R(S)}$ splits as (4.18) and $\widetilde{F} \neq F$ then $\beta, \beta' \notin F \leq \widetilde{F}$ and $\widetilde{F}$ is 2 generated modulo F, in particular $\widetilde{F} = \langle F, \overline{x}, \beta' \rangle$. Moreover the splitting is 1-acylindrical and, up to rational equivalence, we can assume that $\overline{y} = t$.*

Proof. Suppose towards a contradiction that $\beta \in F$, then since $\widetilde{F} \neq F$, $\widetilde{F}$ has a nontrivial cyclic splitting modulo F $D_{\widetilde{F}}$ It follows that β' must also be elliptic in $D_{\widetilde{F}}$, in which case we can "refine" the splitting (4.18), contradicting the fact that it's a cyclic JSJ modulo F. 1-acylindricity now follows from the fact that β, β' are hyperbolic in a cyclic splitting of $\widetilde{F}$ modulo F, which means that β, β' cannot lie in noncyclic abelian subgroups.

We now let $F_{R(S)}$ act on the Bass-Serre tree corresponding to (4.18) and consider the marked generating set $(F; \{\overline{x}, \overline{y}\})$. We have that T_F must be a point, so by Theorem 4.3.27 $\overline{x}$ can be sent to an elliptic element and can be sent into $\widetilde{F}$ via Nielsen moves modulo F. It follows that we must have $\beta \in \langle F, \overline{x} \rangle$. And since we must have $T_{\langle F, \overline{x} \rangle} \cap \overline{y} T_{\langle F, \overline{x} \rangle} \neq \emptyset$ by 1-acylindricity we easily conclude $\overline{y} = f_1 t^{\pm 1} f_2$ for $f_1, f_2 \in \widetilde{F}$, conjugating boundary monomorphisms, we may therefore assume that $\overline{y} = s$ and we have $\widetilde{F} = \langle F, \overline{x}, \beta' \rangle$. □

4.7.2 The case where $\widetilde{F}$ is not freely decomposable modulo F

In order to be able to say more about $\widetilde{F}$ we pass to a quotient of $F_{R(S)}$. In a strict quotient $\pi : F_{R(S)} \to F_{R(S')}$, the restriction $\pi|_{\widetilde{F}}$ is injective but the image $\pi(\widetilde{F})$ will have a nontrivial induced cyclic splitting modulo F. *We will identify $\widetilde{F}$ with its image in $F_{R(S')}$.* The author wishes to acknowledge that many ideas in this section, in particular Lemma 4.7.10 as well as its effect on the first Betti number come from the proof of Proposition 4.3 of [52], i.e. the construction of the so-called cyclic analysis lattice.

Convention 4.7.4. We will first consider the case where where the JSJ of $F_{R(S)}$ has one vertex and one edge. Hence from now until Section 4.7.2 $F_{R(S)}$ will be assumed to have a JSJ with one edge and one vertex.

Lemma 4.7.5. *Let $\pi : F_{R(S)} \to F_{R(S')}$ be a strict quotient. Then there are no cyclic splittings of $F_{R(S)}$ modulo $\widetilde{F}$.*

Proof. Suppose towards a contradiction that was not the case. Then we can obtain a cyclic splitting of $F_{R(S')}$

$$\begin{tikzcd} A *_C B \text{ or } {}^{C}A_{C'} \arrow[loop above, "r"] \end{tikzcd}$$

such that $\widetilde{F} \leq A$. π is surjective. If $F_{R(S')}$ split as an amalgam then we would have

$$\pi(\overline{y})^{-1}\beta\pi(\overline{y}) = \beta \tag{4.20}$$

where β and lie in A, and $\pi(y)$ must not lie in A. If B is abelian of rank 3 then $A = F$ and we must have that $\widetilde{F} = F$. If B is abelian of rank 2 then we will consider this splitting as an HNN extension and tackle it in the next case. We can therefore assume that B is nonabelian and by malnormality of the centralizer of C in both factors we see that (4.20) is impossible.

Suppose now that $F_{R(S')}$ split as an HNN extension then $\overline{y}$ must have exponent sum 1 in the stable letter r. In particular, by Britton's lemma we see that β and β' must be conjugate into C and C', that $A = F_{R(S')} \cap \langle F, \pi(\overline{x}), \pi(\overline{y})\rangle$ must be equal to $\widetilde{F}$ and so that that conjugating boundary monomorphisms we can bring $\pi(\overline{y})$ to r, contradicting the fact that π is a proper surjection. ☐

Corollary 4.7.6. *Any cyclic splitting of* $F_{R(S')}$ *induces a cyclic splitting of* $\widetilde{F}$.

Corollary 4.7.7. *If* $\widetilde{F}$ *is freely indecomposable modulo* F*, then so must be* $F_{R(S')}$.

Definition 4.7.8. Let e be an edge in the cyclic JSJ splitting of $\widetilde{F}$ modulo F. We say that e is *visible* in a one edge splitting D of $F_{R(S')}$ if the edge group associated to e is conjugate into the edge group of D.

Definition 4.7.9. We will call the conjugacy classes of a boundary subgroup boundary an *edge class.*

Controlling induced splittings

The proofs in this section are very technical. Their goal is to obtain Corollary 4.7.14. Using this corollary we will be able to apply the following Lemma:

Lemma 4.7.10. *If* $F_{R(S')}$ *has a one edge cyclic* D *splitting and if* $\widetilde{F}$ *has an induced one edge cyclic splitting* $D_{\widetilde{F}}$*, then* $F_{R(S')}$ *can be obtained from* $\widetilde{F}$ *by adding an element* $\sqrt[n]{\eta}$*: an* n^{th} *root of the generator* η *of an edge group of* $D_{\widetilde{F}}$.

Proof. Consider the action of $F_{R(S')}$ and $\widetilde{F}$ on the Bass-Serre tree T corresponding to the splitting D. Abusing notation we also use $\overline{y}$ to denote the image of $\overline{y}$ in $F_{R(S')}$.

We have hyperbolic elements $\beta, \beta' \in \widetilde{F}$ such that $\beta^{\overline{y}} = \beta'$. This means that $\overline{y}\mathrm{Axis}(\beta') = \mathrm{Axis}(\beta)$. Which means that there is are edges $e', e \in \mathrm{Axis}(\beta'), \mathrm{Axis}(\beta)$ respectively such that $\overline{y}e' = e$. On the other hand $\mathrm{Axis}(\beta), \mathrm{Axis}(\beta') \subset T(\widetilde{F})$, the minimal $\widetilde{F}-$invariant subtree.

We also have that $D_{\widetilde{F}}$ has only one edge, which means that $\widetilde{F}$ is transitive on the set of edges in $T(\widetilde{F})$. Let $g \in \widetilde{F}$ be such that $ge = e'$. Then we have that $\overline{y}g \in \mathrm{stab}_{\widetilde{F}}(e)$. Let $\mathrm{stab}_{F_{R(S')}}(e) \cap \widetilde{F} = \langle \eta \rangle$, then we have that $\langle \eta, \overline{y}g \rangle \leq \mathrm{stab}_{F_{R(S')}}(e)$ which is cyclic, so $\langle \eta, \overline{y}g \rangle = \langle \sqrt[n]{\eta} \rangle$. By $\widetilde{F}$-transitivity on the edges of $T(\widetilde{F})$, $\mathrm{stab}_{\widetilde{F}}(e)$ for each $e \in T(\widetilde{F})$ are conjugate in $\widetilde{F}$ so the result follows. ☐

It should be noted that the many ideas from the proof of Proposition 4.3 of [52] are used. We now prove the lemmas.

Lemma 4.7.11. *Let F be a free group and let $\mu, \mu' \in F$ be two conjugate elements. Let $K = \langle \mu, \mu' \rangle \leq F$ be noncyclic and let $\gamma \in K$ be some element that is conjugate to μ in F. Then γ must be conjugate to either μ or μ' in K.*

Proof. If H is cyclic the result follows. Fix a basis of F so that μ is cyclically reduced. Suppose towards a contradiction that the result did not hold. Let $\Gamma(H)$ be the Stallings graph over F for H with basepoint v (see [23],[54]). Let $\mu' = f^{-1}\mu f$. Then $\Gamma(H)$ is obtained by connecting two loops labelled μ by a path labelled f and folding.

Topologically, the resulting folded graph will always composed of two vertices and three arcs α, β, I. Since the graph must contain two cycles, we have two topological possibilities: either the graph can be broken into two connected components by removal of an arc; in which case we say it is *separable*; or there is no such arc.

We first consider the separable case. We have that the two cycles c_1, c_2, as point sets, are contained in the arcs α, β. Let $u \in \Gamma(H)$ be the basepoint, w.l.o.g. it is contained in the arc α. Consider the covering space $\widetilde{\Gamma(H)}$ corresponding to the subgroup $\langle\gamma\rangle$. $\widetilde{\Gamma(H)}$ is a union of arcs that map either on to α, β or I. The core of $\widetilde{\Gamma(H)}$ must be a cycle c, moreover the cyclic word (in the basis of F) read along this arc is reduced in fact it is the cyclic word of γ. If c consists only of an arc of type α or β then we see that γ is conjugate to either μ or μ' in H. On the other hand if c contains an I-type arc, then it's length must automatically exceed that of μ, which is impossible since μ and γ are conjugate and hence must have equal cyclic words.

We now consider the non separating case. $\Gamma(H)$ consists of three arcs, moreover up to changing basis of F, we can arrange so that the path labelled μ in $\Gamma(H)$ starting at v is obtained by traversing I and then α (in particular the basepoint v is a topological vertex of $\Gamma(H)$). Since we are assuming that μ is cyclically reduced and not a proper power we have that the label of α is different from the label of I. We moreover see that, up to conjugating μ' by μ^n for some n that $(\mu')^{\pm 1}$ is the label read, starting at the basepoint v, around the cycle going first through the arc I and then through the arc β. We consider I, α, β as subwords so, abusing notation we write $\mu' = (I\beta)^{\pm 1}$ and $\mu = I\alpha$.

We have $|\alpha| = |\beta|$. Again we take the covering space $\widetilde{\Gamma(H)}$ corresponding to the subgroup $\langle\gamma\rangle$, and look at the core c. c is a cycle composed of arcs that map onto α, β, I. If c contains an arc of type I, then it must contain an arc of type either α or β. Since c has the same length as μ, c can only consist of those two arcs, which immediately implies that γ is conjugate to either μ or μ' in H. It follows that c can only consist of arcs that map onto α and β.

If $|I| \leq |\mu|/2$ then $|c| \geq |\alpha| + |\beta| > |\mu|$, which is impossible.

Suppose finally that $|I| \geq |\mu|/2$. We first make an observation about cyclically reduced words over free groups. Let $w, \widetilde{w}$ be cyclically reduced words that are conjugate over the alphabet $X^{\pm 1}$ then the biinfinite words

$$\begin{gathered} \ldots \quad www \quad \ldots \\ \ldots \quad \widetilde{w}\widetilde{w}\widetilde{w} \quad \ldots \end{gathered}$$

viewed as maps $\mathbb{Z} \to X^{\pm 1}$ differ only by precomposition by some translation $n \mapsto n+a$ with $a < |w|$. This means that when considering the biinfinite words given by μ and μ' the subwords labelling the segment I must overlap, hence they are coherently oriented. We must therefore have that μ' is the label of the path starting at v, going first through I and then through β. We have that γ labels a path $(\alpha^{-1}\beta)^n$ for some nonzero integer n. w.l.o.g. $\gamma = (\alpha^{-1}\beta)^n$ with $n > 0$. Since γ and μ are conjugate we must have that up to a shift the biinfinite words

$$\begin{gathered} \ldots \quad \alpha^{-1}\beta\alpha^{-1}\beta\alpha^{-1}\beta \quad \ldots \\ \ldots \qquad I\alpha I\alpha I\alpha \end{gathered}$$

are equal. There can be no overlap between the α and α^{-1} segments, on one hand this forces $|\beta| = |I|$ which implies $|I| = |\mu|/2$. We must then have $I = \alpha^{-1}$ implying triviality of μ–contradiction. Having exhausted all possibilities the result follows. □

Lemma 4.7.12. *If the JSJ of $\widetilde{F}$ does not contain an abelian vertex group then we can always find a one edged cyclic splitting of $F_{R(S')}$ modulo F such that the induced splitting of $\widetilde{F}$ also has only one edge.*

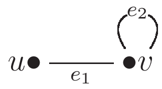
Proof. Since $\widetilde{F}$ is 2 generated modulo F and is freely indecomposable modulo F, it must a group of the types previously described.

If the cyclic JSJ decomposition of $\widetilde{F}$ modulo F has only one edge and one vertex, then there is nothing to show. We now consider the cyclic JSJ splittings modulo F with two edges.

If the splitting of $\widetilde{F}$ has two edges and one vertex then, as we saw, either both edge classes are conjugate into F and they are conjugacy separated in F or only one of the edge classes lies in F. In both these cases only one of the edges can be visible in a one edged splitting of $F_{R(S')}$.

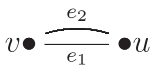
If the splitting of $\widetilde{F}$ has two edges we have possibilities for $\widetilde{F} = \mathcal{G}(A)$:

2-2-A



2-2-B

2-2-C



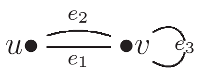
In case 2-2-C either the edges classes associated to e_1 and e_2 are both conjugate into F and are conjugacy separated or only one of them is conjugate into F, either way we can always find a one edge splitting of $F_{R(S')}$ such that only one edge is visible.

By what has been proved so far in Theorem 4.1.6, we know that case 2-2-A is either an amalgam of $\widetilde{F}^1 \geq F$ and a free group H of rank 2 that splits as an HNN extension; or $A_v \geq F$ and A_u is abelian. In the latter case there is nothing to show. By Proposition 4.6.10 the edge classes associated to e_1 and e_2 are not conjugate in $\widetilde{F}$, but if both edges were visible in a one edged splitting of $F_{R(S')}$ then the corresponding edge classes would be conjugate in $F_{R(S')}$. It follows that we could find a map $\phi : \widetilde{F} \to F$ that is injective on H, and such that the corresponding boundary subgroups were sent to conjugates. By Lemma 3.1.10 and Lemma 4.7.11 we get that the generators of the boundary subgroups are conjugate in $\phi(H)$, and hence also in H and hence in $F_{R(S')}$–contradiction.

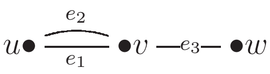
In case 2-2-B we can assume that A_w is abelian so there is nothing to show.

The splittings with three edges that occur are:

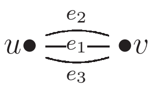
2-3-A



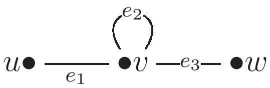
2-3-B



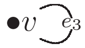
2-3-C



2-3-D



In cases 2-3-B and 2-3-D, one of the vertex groups, say A_w, must be abelian so there is nothing to show.

Consider case 2-3-A. Then by the classification and Lemma 3.1.10, the subgroup  is free of rank 2 and A_v is generated by the boundary subgroups associated to e_3. Moreover by Proposition 4.6.21 we know that there must be at least two distinct edge classes. We moreover know that $A_u = F$. Consider first the case where the edge class associated to e_1 (or symmetrically e_2) is conjugate to the edge class associated to e_3. Since the edges e_1 and e_2 are never both visible in a one-edged splitting of $F_{R(S')}$, the only possible obstruction is that, say, both e_1 and e_3 are visible in a one edged splitting of $F_{R(S')}$. If the cyclic JSJ splitting D of $F_{R(S')}$ modulo F has more than one edge then by Corollary 4.7.6 we can get another splitting of $F_{R(S')}$ such that only e_2 is visible. D must therefore have only one edge, and this edge must be conjugate into F. The only possibilities by Lemma 4.7.3 and our previous classification is that $F_{R(S')}$ is of the form $F *_u H$ where H is either free of rank 2, free abelian of rank 2 or 3, or a rank 1 extension of a centralizer with only one edge class. We look at possible induced splittings. Since the edge e_2 is not visible the subgroup $u\bullet \!-\!{\scriptstyle e_2}\!-\! \bullet v$ must be elliptic. Since it contains F it must be a subgroup of F which is impossible.

In case 2-3-C either there are only two edge classes, and we find ourselves as in case of 2-3-A or all edge classes are conjugate into F and $F-$conjugacy separated, so that only one edge will be visible in a 1 edge splitting of $F_{R(S')}$. We have exhausted all the possibilities and the result follows. □

We now tackle the remaining case.

Lemma 4.7.13. *Suppose that the JSJ of* $\widetilde{F}$ *has an abelian vertex group then we can find a one edge splitting of* $F_{R(S')}$ *modulo* F *such that the induced splitting of* $\widetilde{F}$ *also has one edge.*

Proof. In this case the possible JSJs of $F_{R(S)}$ are 2-2-A, 2-2-B, 2-3-B, and 2-3-D of Lemma 4.7.12. Moreover obstructions only occur when the JSJ of $F_{R(S')}$ has only one edge class. In particular, if the edge class is conjugate into F, then the possibilities for $F_{R(S')}$ are

$$F *_u A,\ F *_u H,\ \text{or}\ F *_u H *_u A$$

where H is free of rank 2 and A is free abelian of rank 2 or 3. We note moreover that none of the groups under consideration can be embedded into $F *_H$ since this group has no noncyclic abelian subgroups.

In case 2-2-A, Corollary 4.7.2 applies and we have that the edge groups associated to e_1 and e_2 cannot be conjugate.

In case 2-3-B, the vertex group $A_u = F$. The edge classes associated to e_1 and e_2 are either equal, in which case after sliding we reduce to case 2-3-D; or they are distinct in which case the edges cannot be simultaneously visible in a one edged splitting of $F_{R(S')}$. The only obstruction is that, say, the edges e_1 and e_3 are visible in a one edged splitting of $F_{R(S')}$, then we must have that the JSJ of $F_{R(S')}$ has only one edge class which forces (as in case 2-3-A of the previous proof) $F *_{A_{e_2}} A_u$ to lie in F.

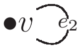
In case 2-3-D we can assume that $A_u = F$. Moreover the subgroup  is free of rank 2. By Proposition 4.6.11 there must be at least two edge classes associated to e_1, e_2, e_3 in $\widetilde{F}$. By Lemmas 3.1.10 and 4.7.11 arguing as in case 2-3-A in the previous proof, it follows that all three edges cannot be visible in a one edged splitting of $F_{R(S')}$.

Suppose first that the edges e_2 and e_3 were visible in a one edged splitting of $F_{R(S')}$, then we would have that $F *_{A_{e_1}} A_v$ is elliptic in $F_{R(S')}$. If one of the vertex groups of the JSJ of $F_{R(S')}$ is F itself, then we immediately get a contradiction. If the JSJ of $F_{R(S')}$ has two vertices and one edge, then it must be of the form

$$\widetilde{F}^1 *_q H$$

where $\widetilde{F}^1$ is rank 1 extension of a centralizer of F and H is free, but to get the desired induced splitting, we must have that A_w, which is free abelian of rank 2, is a subgroup of a conjugate of H which is impossible. The remaining possibility is that $F_{R(S')}$ has one edge and one vertex, but to get the induced splitting of $\widetilde{F}$ in question we would need the one of the boundary subgroups of $F_{R(S')}$ to lie in a non cyclic free abelian subgroup, which by Lemma 4.7.3, forces $F_{R(S')} = F *_u A$ with A free abelian, and we can easily derive a contradiction.

Suppose now that only the edges e_1 and e_2 were visible in a one edge splitting of $F_{R(S')}$, then by earlier arguments we must have that the edge classes associated to e_1 and e_2 coincide in $\widetilde{F}$. Then the edge class of the JSJ of $F_{R(S')}$ is conjugate into F. We also have that the subgroup $A_v *_{A_{e_3}} A_w$ is elliptic. This is clearly impossible unless $F_{R(S')} = F *_u H *_u A$ with H free and A abelian. In which case we have that $A_v *_{A_{e_3}} A_w$ is conjugate into $H *_u A$, which is only possible if if u is conjugate to A_{e_3} in $F_{R(S')}$. This means that boundary subgroups associated to e_1, e_2 and e_3 are all conjugate in $F_{R(S')}$, using Lemma 4.7.11 and some earlier arguments we can show that the boundary subgroups associated to e_1, e_2 and e_3 are conjugate in $\widetilde{F}$ –contradiction.

The remaining possibility is that only the edges e_1 and e_3 are visible in the one edged splitting. Since  is free of rank 2, we can reduce this to case 2-2-B.

We now tackle case 2-2-B. We can assume that $A_u = F, A_v$ is free of rank 2, and A_w is free abelian of rank 2. We moreover have two cases the *separated* case, where $\widetilde{F}$ has two edge classes; and the *unseparated* case, where $\widetilde{F}$ has one edge class.

We first consider the unseparated subcase. Since the edge class is conjugate into F, any one edged splitting of $F_{R(S')}$ must have F as a vertex group. The possibilities for $F_{R(S')}$ are

$$F *_u A \text{ or } F *_u H *_u A$$

where A is abelian and H is free. By basic Bass-Serre theory we see that $\widetilde{F}$ cannot be embedded into $F *_u A$ in such a way that both edges are visible. If $F_{R(S')} = F *_u H *_u A$, then either A_v is conjugate into F or into H. If A_v is conjugate into F, then it must be embedded into some subgroup of $faHa^{-1}f^{-1}$ with $f \in F$ and $a \in A$, then we can collapse the splitting of $F_{R(S')}$ to $(F *_u A) *_u H$ and we get a splitting of $F_{R(S')}$ modulo $\widetilde{F}$ – contradiction. We therefore have that A_v is conjugable into H and A_w is conjugable into A, it follows that the splitting of $\widetilde{F}$ induced by $(F *_u A) *_u H$ has only one edge.

We now consider the separated case. There are only two problematic cases for $F_{R(S')} = \pi_1(\mathcal{G}(X))$

$$\begin{aligned} \mathcal{G}(X) &= u\bullet \xrightarrow{e} \bullet v \\ \mathcal{G}(X) &= \bullet v \xleftarrow{e_1} u\bullet \xrightarrow{e_2} \bullet w \end{aligned}$$

Where $X_u = F, X_v$ is noncyclic abelian and X_w is free of rank 2. Moreover all edge groups are conjugate. Consider first the case where $F_{R(S')}$ has two edges, and consider the induced splitting of $\widetilde{F}$ in $F_{R(S')}$. If we collapse the edge e_2 then if the splitting of $\widetilde{F}$ in this new induced splitting has only one edge then we are done, otherwise the induced splitting of $\widetilde{F}$ still has two edges.

We can therefore reduce to the case where $F_{R(S')}$ is the fundamental group of

$$\mathcal{G}(X) = \; u\bullet \xrightarrow{e} \bullet v \tag{4.21}$$

with X_u non abelian and X_v noncyclic abelian. We also have $\widetilde{F} = F *_p H *_q A$ which gives the graph of groups

$$\mathcal{G}(Y) = \; r\bullet \xrightarrow[f_1]{} s\bullet \xrightarrow[f_2]{} \bullet t \tag{4.22}$$

and this must be the splitting induced from $\widetilde{F} \leq F_{R(S')}$ with the splitting (4.21). There is only one way to embed $\widetilde{F}$ into $F_{R(S')}$, the embedding is

$$\widetilde{F} \approx F *_p [a(\overline{H} *_{{}^g p} {}^g\overline{A})a^{-1}] \tag{4.23}$$

where $a \in X_v, g \in F = X_u, \overline{A} \leq X_v$, and $\overline{H} \leq F$. Moreover, we have that $p, {}^g p \in \overline{H}$ but $g \notin \overline{H}$. It follows that if we construct the induced $G(X)$-graph $\mathcal{B}$ we have

$$(F,u)\bullet \xrightarrow[(1,e,1)]{} \bullet(\langle p\rangle, v) \xrightarrow[(a,e^{-1},1)]{} \bullet(\overline{H}, u) \xrightarrow[(g,e,1)]{} \bullet(\overline{A}, v) \tag{4.24}$$

in particular the edge f_1 has length 2 and the edge f_2 has length 1. Consider the action of $\widetilde{F}$ on the Bass-Serre tree T corresponding to the splitting (4.21) of $F_{R(S')}$.

We have elements β, β' of $\widetilde{F}$ that are conjugate via the image $\overline{t}$ of the stable letter t from the splittings (4.18),(4.19) in $F_{R(S')}$. In particular we have $\overline{t}\mathrm{Axis}(\beta') = \mathrm{Axis}(\beta)$. If t sends an edge $\epsilon' \in \mathrm{Axis}(\beta')$ to an edge $\epsilon \in \mathrm{Axis}(\beta)$ such that there is some $h \in \widetilde{F}$ such that $h\epsilon' = \epsilon$ then as in the proof Lemma 4.7.10 we get that $F_{R(S')}$ is obtained from $\widetilde{F}$ by adjoining a root to generator of an edge group, which here implies $\widetilde{F} = F_{R(S')}$, and the result follows.

Otherwise let $T' = T(\widetilde{F}) \leq T$ be the minimal $\widetilde{F}$-invariant subtree. Looking at (4.24) we see that there are three $\widetilde{F}$-orbits E_1, E_2, and E_3 of edges in $T' \cap (\text{Axis}(\beta) \cup \text{Axis}(\beta'))$ we assume that E_2 corresponds to the edge f_2 in (4.22). Let $S_1, S_2, S_3 \leq \widetilde{F}$ each stabilize an edge of in E_1, E_2, E_3 resp.

Consider an edge ϵ' in $\text{Axis}(\beta') \cap E_2$ that is fixed by some conjugate of the edge group $E_{f_2} \leq \widetilde{F}$ of f_2 in the splitting (4.22). Then since $t\text{Axis}(\beta') = \text{Axis}(\beta)$. We have that tS_2t^{-1} fixes some edge say in $\epsilon \in \text{Axis}(\beta) \cap S_1$. The possibilities are limited and it follows that by conjugating boundary monomorphisms, i.e. replacing t by $f_1tf_2; f_1, f_2 \in \widetilde{F}$, we can arrange so that $\bar{t} = aga^{-1}$, where a, g are as in (4.23). This means that $\bar{t}$ interchanges edges in E_1 and E_2. It follows that $\bar{t}$ must map edges in E_3 to edges in E_3, i.e. there is some $\epsilon \in E_3$ and some $h \in \widetilde{F}$ such that $h\epsilon = \bar{t}\epsilon$ and the result follows. ☐

We combine these two parts:

Corollary 4.7.14. *We can find a one edge splitting of* $F_{R(S')}$ *such that the induced splitting of* $\widetilde{F}$ *also has one edge.*

Controlling the Strict Quotients

Proposition 4.7.15. *If the JSJ of* $F_{R(S)}$ *has only one edge and one vertex group* $\widetilde{F}$ *that is not freely decomposable modulo* F *and the JSJ of* $\widetilde{F}$ *has more than one edge and one vertex group, then* $F_{R(S)}$ *is an extension of a centralizer of* $\widetilde{F}$*.*

Proof. Let $F < \widetilde{F}'$ be the vertex group of the JSJ of $\widetilde{F}$ that contains F. Suppose first that $\widetilde{F}' = F$ or was an extension of a centralizer and that the root $\sqrt[n]{\eta}$ obtained in Lemma 4.7.10 was added to an edge group $\langle \eta \rangle$ lying in $\widetilde{F}$. $\eta \in F$ then $\sqrt[n]{\eta} = \eta$ and $F_{R(S')} = \widetilde{F}$ so the result follows. If $\widetilde{F}'$ was an extension of a centralizer $F *_u Ab(u, s)$

then by our classification since either $\eta \in F$ or $\widetilde{F}' = \langle F, \eta \rangle$ which means that $\eta^{\pm 1} = f_1 s f_2$ with $f_i \in F - \langle u \rangle$ we have that there is no $F-$morphisms $F_{R(S)} \to F$ that sends η to a proper power in F so again $\sqrt[n]{\eta} = \eta$

Otherwise we see that $F_{R(S')}$ is obtained from $\widetilde{F}$ by adding elements to its vertex groups. Consider the graph of groups obtained from the JSJ of $\widetilde{F}$ and adding these elements to the vertex groups. This gives a cyclic splitting D' of $F_{R(S')}$ with the same underlying graph as the JSJ of $F_{R(S)}$. It follows from our classification so far that we can in fact collapse D' to a one edge splitting so that the induced splitting of $\widetilde{F}$ either has an edge group that lies in F or in $\widetilde{F}$ in the way described in the previous paragraph.

It follows that $F_{R(S')} = \widetilde{F}$ so $F_{R(S)}$ is an extension of a centralizer of $\widetilde{F}$. ☐

Definition 4.7.16. We say that two elements in $x, y \in G$ are *pseudo-conjugate* if the difference $[x] - [y]$ of their images in the abelianization of G lies in the torsion subgroup.

Proposition 4.7.17. *If the JSJ of $F_{R(S)}$ has only one edge and one vertex group $\widetilde{F}$ which is freely indecomposable and $\pi : F_{R(S)} \to F_{R(S')}$ is a proper strict quotient then $b_1(F_{R(S')}) < b_1(F_{R(S)})$.*

Proof. If $\widetilde{F}$ and $F_{R(S)}$ satisfied the hypotheses of Proposition 4.7.15 then the result follows immediately. We consider simultaneously the cases where the JSJ of $F_{R(S)}$ has one or two edges (i.e. (4.18) or (4.19)) replacing $\widetilde{F}$ by $\begin{tikzcd} {}^{\alpha}_{\alpha}\widetilde{F} \arrow[loop left, "t"] \end{tikzcd}$ if need be.

Denote by $Ab(\widetilde{F})$ the abelianization of $\widetilde{F}$. We see that $F_{R(S')}$ is obtained from $\widetilde{F}$ by adding a the root of an element it follows that $b_1(F_{R(S')}) \leq b_1(\widetilde{F})$. We consider two the elements β, β' given in (4.18) and (4.19) and consider two cases.

Suppose first that β and β' are pseudo-conjugate in $\widetilde{F}$, then we see that $b_1(F_{R(S)}) > b_1(\widetilde{F})$ by abelianizing a relative presentation. The result now follows.

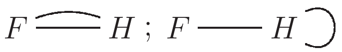
Suppose now that β and β' are not pseudo-conjugate in $\widetilde{F}$. Then in particular the element $[\beta] - [\beta']$ does not lie in the torsion subgroup. $Ab(F_{R(S')})$ can be obtained from $Ab(\widetilde{F})$ by first adding the root $\sqrt[n]{\eta}$ of some element, which does not change b_1 and then identifying the images $[\beta]$ and $[\beta']$ which sends $[\beta] - [\beta']$ to zero, thus dropping the rank of the torsion free summand. The result now follows. ☐

The two edge case

Proposition 4.7.18. *If the JSJ of $F_{R(S)}$ has two edges and one vertex then $\widetilde{F}$ must be be freely decomposable modulo F.*

Proof. Suppose towards a contradiction that $\widetilde{F}$ was freely indecomposable modulo F. Let $F_{R(S')}$ be the first term in a strict resolution of $F_{R(S)}$ where $\widetilde{F}$ splits. Then the elements β, β' must be hyperbolic elements of $F_{R(S')}$. Consider first the case where the JSJ of $\widetilde{F}$ has only one edge. Then the corresponding edge group must lie in F and this splitting of $\widetilde{F}$ must be induced by a one edged splitting of $F_{R(S')}$ which means that Lemma 4.7.10 applies, and in fact we must have that β and β' are conjugate in $\widetilde{F}$, contradicting the choice of JSJ.

Suppose now that the JSJ of $\widetilde{F}$ had two edges, then the possibilities are



Suppose first that $F_{R(S')}$ split as an HNN extension with stable letter z and that the induced splitting of $\widetilde{F}$ also had one edge. The one of the corresponding edge group either lies in F, which is impossible, or we are adding a proper root to elements of the vertex group H of $\widetilde{F}$.

In this latter case, we have that β must have exponent sum 0 in z, but that β', because $\widetilde{F} = \langle F, \alpha', \beta' \rangle$ has an induced splitting as an HNN extension, must have exponent sum 1 in z, which is impossible since they are conjugate in $F_{R(S')}$.

If $F_{R(S')}$ splits as an amalgam, and the induced splitting of $\widetilde{F}$ has only one edge, then $\widetilde{F}$ must also split as an amalgam and the corresponding edge group must lie in F.

If the induced splitting of $\widetilde{F}$ has two edges, then either conjugation my $\bar{t}$, the image of the stable letter t, permutes the edge classes or fixes them. First note that the edge classes must be conjugacy separated in $\widetilde{F}$. If conjugation by $\bar{t}$ fixes them, we can again argue that $\bar{t}$ contributes a root of an edge group lying in F, which implies that $\langle \widetilde{F}, \bar{t} \rangle = \widetilde{F}$.

Otherwise $\bar{t}$ permutes the edge classes which is impossible by Lemma 4.7.11, or by the fact that both have corresponding subgroups lying in F. $\square$

4.7.3 The case where $\widetilde{F}$ is freely decomposable modulo F

Proposition 4.7.19. *Suppose that the JSJ of $F_{R(S)}$ has only one vertex group $\widetilde{F}$ and suppose that $\widetilde{F}$ is freely decomposable modulo F. Then the vertex group $\widetilde{F}$ can only be $F * \langle z \rangle$*

Proof. By Lemmas 4.7.3 and 4.7.1 and Theorem 4.1.6 (A). $\widetilde{F}$ must be either $F * \langle z \rangle, F *_u Ab(u, r) * \langle w \rangle, F * \langle z, w \rangle$. If $\widetilde{F} = F * \langle z, w \rangle$ then Lemmas 4.7.3 and 4.7.1 would imply that $F_{R(S)}$ is freely decomposable. Applying Corollary 4.3.25 to the two edged case excludes $F *_u Ab(u, z) * \langle w\rangle$ as a vertex group. We now prove that $F *_u Ab(u, z) * \langle w\rangle$ cannot be a vertex group in a one edged splitting.

Suppose towards a contradiction that $F *_u Ab(u, z) * \langle w\rangle$ was the vertex group of a one edged, one vertex JSJ of $F_{R(S)}$. By Lemma 4.7.3 we may assume that $\widetilde{F}$ is generated by $\overline{x}$ and some element β'. We know moreover that the exponent sums of z and w of words in $F \cup \{z, w\}^{\pm 1}$ representing $\overline{x}, \beta, \beta'$ do not depend on the choice of word.

$\beta \in \langle F, \overline{x}\rangle$ which means that writing β as a word in $F \cup \{\overline{x}\}^{\pm 1}$ gives equalities of exponent sums $\sigma_z(\beta) = \sigma_{\overline{x}}(\beta) * \sigma_z(\overline{x})$ and $\sigma_w(\beta) = \sigma_{\overline{x}}(\beta) * \sigma_w(\overline{x})$. Which means that we have an equality of vectors:

$$(\sigma_u(\beta), \sigma_w(\beta)) = \sigma_{\overline{x}}(\beta)(\sigma_z(\overline{x}), \sigma_w(\overline{x})) \tag{4.25}$$

i.e. they are linearly dependent. Now $b_1((F *_u Ab(u, z) * \langle w\rangle) = N + 2$ which means that if $b_1(F_{R(S)}) \leq N+2$ then β and β' must lie in the same one dimensional subgroup in the abelianization of $F \cup \{\overline{x}\}^{\pm 1}$, which means that

$$l(\sigma_z(\beta), \sigma_w(\beta)) = k(\sigma_z(\beta'), \sigma_w(\beta'))$$

for some $l, k \in \mathbb{Z}$, but this and (4.25) imply that $\overline{x}$ and β' cannot generate $\widetilde{F}$ modulo F –contradiction. □

Lemma 4.7.20. *Suppose the JSJ of $F_{R(S)}$ has one vertex group $\widetilde{F}$ that is freely indecomposable, but does not satisfy the hypotheses of Proposition 4.7.15. Then the* *image of a strict epimorphism* $F_{R(S)} \to F_{R(S')}$ *must be*

$$F_{R(S')} = {}_{s}\circlearrowleft^{\delta}_{\delta'} \widetilde{F}' \tag{4.26}$$

where $\widetilde{F}' = F * \langle r \rangle$. *Moreover* $\widetilde{F}$ *must be of the form*

$$\widetilde{F} = {}_{t}\circlearrowleft^{\gamma}_{\gamma'} \widetilde{F}_1 \tag{4.27}$$

where $\widetilde{F}_1 \approx F * \langle z \rangle \leq \widetilde{F}'$.

Proof. By the hypotheses, we must have that the JSJ of $\widetilde{F}$ has one vertex group $\widetilde{F}_1$ and one edge. By Corollary 4.7.7 $F_{R(S')}$ must be freely indecomposable.

By Proposition 4.7.17, $b_1(F_{R(S')}) \leq N + 1$ where N is the rank of F. If $b_1(F_{R(S')}) = N$ then by Proposition 4.2.8 $F_{R(S')} = F$ which would force $\widetilde{F} = F$ contradicting our hypotheses.

Let $\widetilde{F} \leq \widetilde{F}'$ be the vertex group of the JSJ of $F_{R(S')}$ that contains F. We have that $\widetilde{F}_1 \leq \widetilde{F}'$. If $\widetilde{F}'$ is freely indecomposable then we could take a strict quotient of $F_{R(S')}$, apply Proposition 4.7.17 we would get that the image of this quotient is F, which means that $\widetilde{F}' \leq F$. It would then follow that $\widetilde{F}_1 = F$ and $\widetilde{F}$ would have to be a rank 1 free extension of a centralizer of F. But then $\widetilde{F}$ would satisfy the hypotheses of Proposition 4.7.15, which is a contradiction.

It therefore follows that $\widetilde{F}'$ must be freely decomposable modulo F. Looking at the possibilities given in Proposition 4.7.19 and recalling that $b_1(F_{R(S')}) \leq N + 1$ we have that the only possibility of $\widetilde{F}'$ is $\widetilde{F}' \approx F * \langle r \rangle$. Since $\widetilde{F}_1$ is a subgroup of $\widetilde{F}'$ the only possibility is $\widetilde{F}_1 \approx F * \langle z \rangle$. ∎

Proposition 4.7.21. *If the JSJ of $F_{R(S)}$ has only one vertex group $\widetilde{F}$, then $b_1(\widetilde{F}) = N+1$.*

Proof. In light of Proposition 4.7.19 and Lemma 4.7.20 we need only verify the case in Lemma 4.7.20.

Since $b_1(F_{R(S')}) \leq N+1$ we must have that the elements δ, δ' given in (4.26). By Lemma 4.7.10, $F_{R(S')}$ is obtained from $\widetilde{F}$ by the adjunction of roots of γ and γ given in (4.27). $\widetilde{F}_1$ is a free subgroup of the free group $\widetilde{F}'$, obtained by adjoining roots. On one hand both $\widetilde{F}'$ and $\widetilde{F}_1$ have the same rank. On the other hand, by Theorem 3.1.4, $\text{Rank}(\widetilde{F}') = \text{Rank}(\widetilde{F}_1)$ only if γ, γ' are primitive elements of $\widetilde{F}'$. Now consider the map of free abelian groups $j : A \to A'$ obtained by adding a proper root to one basis element and then adding a proper root to another basis element. Suppose that $b_1(\widetilde{F}) = N+2$. Then we must have that γ', γ are pseudo conjugate, but if that were the case, since the abelianization of $\widetilde{F}$ embeds naturally into the abelianization of $\widetilde{F}_1$ we must have that δ, δ' are also pseudo conjugate, which is a contradiction. □

Collecting all these results gives:

Corollary 4.7.22. *If the JSJ of $F_{R(S)}$ has one vertex group, then all the possibilities for $F_{R(S)}$ are given by item (B) of Theorem 4.1.6*

Part II

Complexity

CHAPTER 5
Problems, algorithms and complexity

We now briefly give notions of complexity theory that will be used in this part. We refer the reader to [12] for a more comprehensive treatment.

Formally, a *decision problem* $\mathcal{P}$ is a mapping from a usually infinite set of inputs to a set of outputs {"yes","no"} corresponding to a question. For example $\mathcal{P}(x, y)$ might be "is x greater than y?" and we would have $\mathcal{P}(3, 2) =$"yes". An *algorithm* $\mathcal{A}$ is a procedure which takes an input and after finitely many steps produces an output and terminates. The field of complexity theory is devoted to how difficult a decision problem is to solve. One measure is *time complexity,* or for this discussion simply *complexity.*

The *worst case complexity* of an algorithm $\mathcal{A}$ is a function $c_{\mathcal{A}} : \mathbb{N} \to \mathbb{N}$. Noting that the input of a problem or algorithm is always a string of symbols over a fixed alphabet we define $c_{\mathcal{A}}(n) =$ "The maximum number, over all inputs i of length $\leq n$, of basic steps needed by $\mathcal{A}$ to terminate on the input i."

We now define big O notation, which is used to give upper bounds for complexity. Given two functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ we say that f *is* $O(g)$ if and only if there is a positive C and $k \in \mathbb{N}$ such that for all $n > k$ we have such that

$$f(n) \leq Cg(n)$$

For a given function $f : \mathbb{N} \to \mathbb{N}$ we say that an algorithm runs in time $O(f(n))$ if $c_{\mathcal{A}}(n)$ is $O(f(n))$. In particular we will say that an algorithm $\mathcal{A}$ runs in *polynomial time* if for some $m \in \mathbb{N}$ we have $c_{\mathcal{A}}(n)$ is $= O(n^m)$.

We say a problem $\mathcal{P}$ has complexity bounded above by $f : \mathbb{N} \to \mathbb{N}$ if there is an algorithm solving $\mathcal{P}$ that runs in time $O(f)$. We say that a problem $\mathcal{P}$ has complexity bounded below by $f : \mathbb{N} \to \mathbb{N}$ if for every algorithm $\mathcal{A}$ solving $\mathcal{P}$ we have $f(n)$ is $O(c_{\mathcal{A}}(n))$. This gives us an intrinsic definition of the complexity of a decision problem. It goes without saying that the complexity of a problem gives us a lot of information about how feasible it is in real life to solve it.

We say that a problem $\mathcal{P}$ is in *NP* if there is a polynomial time algorithm $\mathcal{A}$ and a a fixed $m \in \mathbb{N}$ such that for every input i, $\mathcal{P}(i) =$"yes" *if and only if* there exists *certificate* $c(i)$ of length less than $\big(\text{length}(i)\big)^m$ such that $\mathcal{A}(c(i)) =$"yes". NP stands for "non-deterministic polynomial". The significance of this complexity class comes from a result of Cook [11] in which he gives a problem $\mathcal{P}$ in NP such that for any other problem $\mathcal{P}'$ that is in NP there exists a polynomial time algorithm $\mathcal{A}_{\mathcal{P},\mathcal{P}'}$ such that

$$\mathcal{P}'(i) = \mathcal{P}(\mathcal{A}_{\mathcal{P},\mathcal{P}'}(i))$$

In other words he gave a problem that was in NP that was, up to polynomial time reduction, at least as hard as any other problem in NP. Such a problem is said to be *NP-complete* . NP-complete is therefore an upper and lower bound for the complexity of a problem which means that it is as hard as possible within the class NP.

Many examples of NP-complete problems were soon discovered afterwards, we refer the reader to [20] for additional details and examples. These problems are particularly tantalizing because showing that an NP-complete problem either has a polynomial upper bound or has a superpolynomial lower bound for complexity guarantees instant fame, since it would answer the question “does polynomial coincide with NP?” which at the time of writing is the most outstanding open problem in computer science.

CHAPTER 6

The solvability problem for quadratic equations over free groups is NP-complete

We will now prove (see Theorems 6.2.11 and 6.1.2) that deciding if a quadratic equations over a free group has a solution is an NP-complete problem. Our proofs are geometric, relying on the topological results of [45] and disc diagram techniques. The content of this chapter is based on the published article [26] and is reproduced here with kind permission of Springer Science and Business Media.

6.1 The Solvability Problem for Quadratic equations over free groups is in NP

Let A be a finite alphabet and let A^{-1} be a set of formal inverses of elements of A. We denote by $(A \cup A^{-1})^*$ the free monoid with involution with basis A and for $w \in (A \cup A^{-1})^*$, we denote by w^{-1} its involution. We denote by $F(A)$ the free group on A.

6.1.1 Standard form

A quadratic equation E with variables $\{x_i, y_i, z_j\}$ and non-trivial coefficients $\{w_i, d\} \in F(A)$ is said to be in *standard form* if its coefficients are expressed as freely and cyclically reduced words in A^* and E has either the form:

$$\left(\prod_{i=1}^{g}[x_i, y_i]\right)\left(\prod_{j=1}^{m-1} z_j^{-1} w_j z_j\right) d = 1 \text{ or } \left(\prod_{i=1}^{g}[x_i, y_i]\right) d = 1 \tag{6.1}$$

where $[x, y] = x^{-1}y^{-1}xy$, in which case we say it is *orientable* or it has the form

$$\left(\prod_{i=1}^{g} x_i^2\right)\left(\prod_{j=1}^{m-1} z_j^{-1} w_j z_j\right) d = 1 \text{ or } \left(\prod_{i=1}^{g} x_i^2\right) d = 1 \tag{6.2}$$

in which case we say it is non-orientable. The *genus* of a quadratic equation is the number g in (6.1) and (6.2) and m is the number of coefficients. If $g = 0$ then we will define E to be orientable. If E is a quadratic equation we define its *reduced Euler characteristic*, $\overline{\chi}$ as follows:

$$\overline{\chi}(E) = \begin{cases} 2 - 2g \text{ if } E \text{ is orientable} \\ 2 - g \text{ if } E \text{ is not orientable} \end{cases}$$

We finally define the *length* of a quadratic equation E to be

$$\text{length}(E) = |w_1| + \ldots + |w_{n-1}| + d + 2(\text{number of variables})$$

It is a well known fact that an arbitrary quadratic equation over a free group can be brought to a standard form in time polynomial in its length.

6.1.2 Ol'shanskii's result

The following is proved in [45].

Theorem 6.1.1. *Let E be a quadratic equation over $F(A)$ in standard form. If $g = 0, m = 2$, or E is not orientable and $g = 1, m = 1$ then we set $N = 1$. Otherwise we set $N = 3(m - \overline{\chi}(E))$. E has a solution if and only if for some $n \leq N$;*

(i) there is a set $P = \{p_1, \ldots p_n\}$ of variables and a collection of m discs $D_1, \ldots, D_m$ such that,

(ii) *the boundaries of these discs are circular 1-complexes with directed and labeled edges such that each edge has a label in* P *and each* $p_j \in P$ *occurs exactly twice in the union of boundaries;*

(iii) *if we glue the discs together by edges with the same label, respecting the edge orientations, then we will have a collection* $\Sigma_0, \ldots, \Sigma_l$ *of closed surfaces and the following inequalities: if* E *is orientable then each* Σ_i *is orientable and*

$$\left(\sum_{i=0}^{l} \chi(\Sigma_i)\right) - 2l \geq \overline{\chi}(E)$$

if E *is non-orientable either at least one* Σ_i *is non-orientable and*

$$\left(\sum_{i=0}^{l} \chi(\Sigma_i)\right) - 2l \geq \overline{\chi}(E)$$

or, each Σ_i *is orientable and*

$$\left(\sum_{i=0}^{l} \chi(\Sigma_i)\right) - 2l \geq \overline{\chi}(E) + 2$$

and

(iv) *there is a mapping* $\overline{\psi} : P \to (A \cup A^{-1})^*$ *such that upon substitution, the coefficients* $w_1, \ldots, w_{m-1}$ *and* d *can be read without cancellations around the boundaries of* $D_1, \ldots, D_{m-1}$ *and* D_m*, respectively; and finally that*

(v) *if* E *is orientable the discs* $D_1, \ldots, D_m$ *can be oriented so that* w_i *is read clockwise around* ∂D_i *and* d *is read clockwise around* ∂D_m*, moreover all these orientations must be compatible with the gluings.*

Proof. It is shown in Sections 2.4 [45] that the solvability of a quadratic equation over $F(A)$ coincides with the existence of a *diagram* Δ *over* $F(A)$ on the appropriate surface Σ with boundary. This diagram may not be *simple*, so via surgeries we produce from Σ a finite collection of surfaces $\Sigma_1, \ldots, \Sigma_l$ with induced simple diagrams $\Delta_1, \ldots \Delta_l$ which we can recombine to get back Σ and Δ. So existence of a diagram Δ on Σ is equivalent to existence of a collection of simple diagrams Δ_i on surfaces Σ_i such that the inequalities involving Euler characteristics given in the statement of the Theorem are satisfied.

In Section 2.3 of [45] the bounds on n are proved. It is also shown in that section that if one can glue discs together as described in the statement of the Theorem with the condition on the boundaries , then there exist simple diagrams Δ_i on surfaces Σ_i. □

6.1.3 The certificate

Theorem 6.1.1 enables us to construct a good certificate.

Theorem 6.1.2. *There exists a polynomial time algorithm $\mathscr{A}$ such that a quadratic equation E over $F(A)$ in standard form has a solution if and only if there is a certificate c of size bounded by*

$$2(|w_1| + \ldots + |w_m - 1| + |d| + 3(2g + m)) \leq 8 * \mathit{length}(E)$$

such that $\mathscr{A}$ answers "yes" on the input (E, c).

Proof. The certificate will consist of the following:

1. A collection of variables $P = \{p_1, \ldots, p_n\}$ where $n \leq \max\{3(2g + m), 1\}$
2. A collection of substitutions $\overline{\psi} = \{p_i \mapsto a_i, i = 1 \ldots n\}$ where $a_i \in (A \cup A^{-1})^*$.

3. A collection of words in P^*

$$\mathcal{C} = \begin{cases} C_1 = p_{11}^{\epsilon_{11}} \dots p_{1l}^{\epsilon_{1j(l)}} \\ \dots \\ C_m = p_{m1}^{\epsilon_{m1}} \dots p_{mj(m)}^{\epsilon_{mj(m)}} \end{cases}$$

with $p_{ij} \in P, \epsilon_{ij} \in \{-1, 1\}$ and each $p_i \in P$ occurring exactly twice.

The $C_i' s$ represent the labels of the boundaries of the discs $D_1, \dots D_l$. It follows that checking conditions (i) and (ii) of Theorem 6.1.1 can be done quickly, moreover we see that the size of $\mathcal{C}$ is at most $2n \leq 6(2g + m)$.

$\overline{\psi}$ extends to a monoid homomorphism $\psi : (P \cup P^{-1})^* \to (A \cup A^{-1})^*$. (iv) can also be verified quickly since for $i = 1, \dots m - 1$ we just need to check that some cyclic permutation of $\psi(C_i)$ is equal to w_i and some cyclic permutation of $\psi(C_m)$ is equal do d. Moreover, since the equality is graphical we have that

$$|a_1| + \dots |a_n| \leq |w_1| + \dots + |w_m| + |d|$$

Therefore the size of the certificate is bounded as advertised. All that is left is to determine the topology of the glued together discs. We describe the algorithm without too much detail.

Step 1: *Built a forest of discs:* We make a graph Γ such that each vertex $v_i \in V(\Gamma)$ corresponds the disc D_i and each edge $e_j \in E(\Gamma)$ corresponds to the variable $p_j \in P$. The edge e_k goes from v_i to v_j if and only if the variable p_k occurs in the boundary of D_i and in the boundary of D_j or if $i = j$ then there are two different occurrences of the variable p_k. We construct a spanning forest $\mathcal{F}$. This enables us to count the number of connected components $\Sigma_0, \dots, \Sigma_l$.

Step 2: *Determine orientability:* For each maximal tree $T_r \subset \mathcal{F}$ we get a "tree of discs" by gluing together only the pairs of edges whose labels correspond to elements of $E(T_r)$. The resulting tree of discs is a simply connected topological space that can be embedded in the plane and we can read a cyclic word $c(T_r)$ in P^* along its boundary. The surface Σ_r obtained by gluing together the remaining paired edges of the tree of discs will be orientable only if whenever $p_j^{\pm 1}$ occurs in $c(T_r)$ then $p_j^{\mp 1}$ also occurs. We can also check (v) at this point.

Step 3: *Compute Euler characteristic:* The identification of the boundary of the discs with graphs, enables us to think of the discs as polygons. If a disc D_i has N_i sides then we give each corner of D_i an angle of $\pi(N_i - 2)/N_i$. Then for each tree of discs produced in the previous step, we identify the remaining pairs of edges to get the surfaces $\Sigma_0, \ldots \Sigma_l$, which now have an extra angular structure. To each Σ_i, we can apply the Combinatorial Gauss-Bonnet Theorem (see Section 4 of [44]) which states that for an angled two-complex X,

$$2\pi\chi(X) = \sum_{f \in X^{(2)}} \kappa(f) + \sum_{v \in X^{(0)}} \kappa(v)$$

where $X^{(2)}$ is the set of faces and $X^{(0)}$ is the set of vertices. This angle assignment gives each face f a curvature $\kappa(f) = 0$ and each vertex has curvature

$$\kappa(v) = 2\pi - \left(\sum_{c \in link(v)} \measuredangle(c) \right)$$

i.e. $\kappa(v)$ is 2π minus the sum of the angles that meet at v.

With an appropriate data structure one can perform steps 1-3 (not necessarily in sequential order) in at most quadratic time in the size of $\mathcal{C}$. Once all that is done, verifying the inequalities of (*iii*) is easy and we are finished. $\square$

6.2 The Solvability Problem for Quadratic equations over free groups is NP-hard

We will present the bin packing problem which is known to be NP-complete and show that it is equivalent to deciding if a certain type of quadratic equation has a solution.

6.2.1 Bin Packing

Problem 6.2.1 (Bin Packing)**.**

- INPUT: A $k-$tuple of positive integers $(r_1, \ldots, r_k)$ and positive integers B, N.
- QUESTION: Is there a partition of $\{1, \ldots, k\}$ into N subsets

$$\{1, \ldots, k\} = S_1 \sqcup \ldots \sqcup S_N$$

such that for each $i = 1, \ldots, N$ we have

$$\sum_{j \in S_i} r_j \leq B \tag{6.3}$$

This problem is NP-hard in the strong sense (see [20] p.226), i.e. there are NP-hard instances of this problem when both B and the r_j are bounded by a polynomial function of k.

Let $t = NB - \sum_{i=1}^{k} r_i$. Then by replacing $(r_1, \ldots, r_k)$ by the $k + t$-tuple $(r_1, \ldots, r_k, \ldots, 1, \ldots, 1)$ we can assume that the inequalities (6.3) are actually equalities. This modified version is still NP hard in the strong sense. We state it explicitly:

Problem 6.2.2 (Exact Bin Packing)**.**

- INPUT: A $k-$tuple of positive integers $(r_1, \ldots, r_k)$ and positive integers B, N.
- QUESTION: Is there a partition of $\{1, \ldots, k\}$ into N subsets

  $$\{1, \ldots, k\} = S_1 \sqcup \ldots \sqcup S_N$$

  such that for each $i = 1, \ldots, N$ we have

  $$\sum_{j \in S_i} r_j = B \tag{6.4}$$

The authors warmly thank Laszlo Babai for drawing their attention to this problem in connection to tiling problems.

6.2.2 Tiling discs

Throughout this section we will consider the discs to be embedded in the Euclidean plane $\mathbb{E}^2$ and will always read clockwise around closed curves.

Definition 6.2.3. An $[a, b^n]$-*disc* is a disc as in section 6.1.2 equipped with an orientation along whose boundary one can read the cyclic word $[a, b^n]$ in the clockwise direction. *We will always assume that* $n \geq 1$*.*

Definition 6.2.4. An $[a, b^n]$-*ribbon* is a rectangular cell complex embedded in $\mathbb{E}^2$ obtained by attaching $[a, b^j]$-discs by their a-labeled edges, while respecting orientation, such that we can read $[a, b^n]$ along its boundary. The *top* of an $[a, b^n]$ ribbon is the boundary subpath along which we can read the word b^{-n}, the *bottom* is the boundary subpath along which we can read the word b^n.

Definition 6.2.5. Let D be a disc embedded in $\mathbb{E}^2$ tiled by coherently oriented $[a, b^n]$-discs. We define the $a-$*pattern* of D to be the graph defined as follows:

1. In the middle of each a-labeled edge put a vertex.
2. Between any two vertices contained in the same $[a, b^n]$-disc draw an edge.

Connected components of $a-$patterns are called $a-$*tracks*

Lemma 6.2.6. *A disc D embedded in $\mathbb{E}^2$ tiled by finitely many coherently oriented $[a, b^n]$-discs cannot have any circular $a-$tracks.*

Proof. It is clear that every $a-$track is a graph whose vertices have valence at most 2. If an $a-$track t has vertices of valence 1 then they must lie on ∂D.

Suppose towards a contradiction that D has a circular a-track c. Then c divides D into two components: an interior and an exterior. If we examine the interior we see that it is a planar union of discs with only the letter b occurring on its boundary, it follows that the interior contains a disc D' with circular a-track. Repeating the argument we find that D must have infinitely many cells which is a contradiction. ☐

Corollary 6.2.7. *Let D be a disc embedded in $\mathbb{E}^2$ tiled by finitely many coherently oriented $[a, b^n]$-discs. Then it is impossible for an a-track t to start and end inside a segment $\alpha \subset \partial D$ labeled a^m for some $m \geq 1$.*

Proof. Suppose towards a contradiction that this was not true. Then for some D and some a-track t in D, we have that t starts and ends in some arc $\alpha \subset \partial D$ labeled a^m. Without loss of generality α lies on the $x-$axis of $\mathbb{E}^2$ and consider the reflection about the $x - axis$, then we have a resulting disc D', and reversing the orientations of all the $b-$labeled edges, makes D' another disc tiled by finitely many $[a, b^n]$-discs. Attaching D to D' along α gives a new disc D'' that has a circular a-track, contradicting Lemma 6.2.6 ☐

Corollary 6.2.8. *We cannot tile a sphere S with finitely many coherently oriented $[a, b^n]$-discs.*

Proof. Suppose towards a contradiction that this was possible. Then in particular all the a-tracks are closed and compact and therefore circles. If S contains only one a-track t, then S is obtained as some topological quotient of an annulus A such that A is obtained by gluing the edges labeled a in the boundary of some $[a, b^N]$-ribbon. Now ∂A consists of two circles c_1, c_2 with label b^N. Since t separates S into two discs, we see that the images of c_1, c_2 are disjoint via the quotient map $\pi : A \to S$. It therefore follows that π must continuously map c_1, which has label b^N, to something simply connected, i.e. a simplicial tree, while respecting the orientations of the edges, which is impossible.

Otherwise S has at at least two a-tracks, if we remove from S some $[a.b^n]$-disc D not lying in some track t. Then $S - D$ embeds into E^2, has a circular a-track, and therefore contradicts Lemma 6.2.6 ☐

Lemma 6.2.9. *Let R be some $[a, b^N]$-ribbon. Suppose there is a continuous map $\psi : R \to D$ where D is a disc embedded in $\mathbb{E}^2$ tiled by finitely many coherently oriented $[a, b^n]$-discs, such that ψ is injective on the interior of R and sends edges to edges, labels to labels and preserves edge orientations. Then ψ is an embedding.*

Proof. Let $t \subset R$ be the unique a-track and let t_R and b_R be the top and bottom of R respectively. By Lemma 6.2.6 the edges labeled a of ∂R have disjoint images. We can remove $[a, b^n]$-discs from D to get a smaller disc D' such that $\psi(t)$ separates D' into two pieces. From this it is clear that the images $\psi(t_R) \cap \psi(b_R)$ are disjoint.

It follows that the only possible failures of injectivity are in restrictions to t_R or b_R. Suppose ψ is not injective on, say, t_R. Then if $\psi(t_R)$ bounds a sub-disc in $D'' \leq D$, then we see that D'' must have a circular a-track –contradiction. It follows that $\psi(t_R)$ maps onto a tree of edges labeled b, but this would contradict the fact that ψ preserved edge orientations. ☐

Proposition 6.2.10. *Suppose that D is a disc embedded in $\mathbb{E}^2$ with boundary label $[a^N, b^B]$ that is the result of gluings of $[a, b^n]$-discs respecting the orientation, then it is obtained from a collection of M $[a, b^B]$-ribbons $R_1, \ldots R_M$ such that the bottom of R_{i+1} is glued to the top of R_i, $i = 1, \ldots M$.*

Proof. We divide ∂D into four arcs l_a, t_b, r_a, b_b that have labels a^{-N}, b^{-B}, a^N, b^B respectively, i.e. the left, top, right and bottom sides. By Lemma 6.2.6 and Corollary 6.2.7 each a-track starts in l_a and ends in r_a. By Lemma 6.2.9 each a-track lies in an embedded ribbon. Since each $[a, b^n]$ disc lies in one of these ribbons, it follows that D is obtained by gluing together N-ribbons as stated in the Proposition. Now b_b must lie in the bottom-most ribbon R_1 which means that R_1 is an $[a, b^B]$-ribbon. It follows that all the ribbons are $[a, b^B]$-ribbons. ☐

6.2.3 A special genus zero quadratic equation

Equipped with Proposition 6.2.10 we shall deduce NP hardness of the following equation:

$$\prod_{j=1}^{k} z_j^{-1}[a, b^{n_j}]z_j = [a^N, b^B] \tag{6.5}$$

By Theorem 6.1.1, (6.5) has a solution if and only if there is a collection of discs D_j with boundary labels $[a, b^{n_j}]$ for $j = 1 \ldots k$ respectively and a disc D_m with boundary label $[a^N, b^B]$ such that, glued together in a way that respect labels and orientation of edges, form a union of spheres (this is forced by the first inequality in (iii), Theorem 6.1.1.

Theorem 6.2.11. *Deciding if the quadratic equation (6.5) with coefficients*

$$[a, b^{n_1}], \ldots, [a, b^{n_k}] \text{ and } [a^N, b^B]$$

has a solution is equivalent to deciding if problem 6.2.2; with input $(n_1, \ldots, n_m)$ *and positive integers* B, N*; has a positive answer.*

Proof. "Bin packing ⇒ solution." Suppose that Problem 6.2.2 has a positive answer on the specified inputs. For each subset S_i of the given partition of $\{1, \ldots, k\}$ we form a $[a, b^B]$-ribbon R_i by gluing together the $[a, b^{n_j}]$-discs for $j \in S_i$, this is possible by (iv) of Theorem 6.1.1 and equation (6.4). We then construct one hemisphere by gluing the ribbons $R_1, \ldots, R_N$. The other hemisphere is the remaining disc with boundary label $[a^N, b^B]^{-1}$, the resulting sphere proves the solvability of (6.5) with the given coefficients.

"Solution ⇒ bin packing." If (6.5) has a solution then there is a union of spheres tiled with $[a, b^{n_i}]$-discs and one $[a^N, b^B]^{-1}$-disc, moreover these discs are coherently oriented. By condition (v) and Corollary 6.2.8 there can only be one sphere: the sphere S_0 containing the unique $[a^N, b^B]^{-1}$-disc. If we remove this $[a^N, b^B]^{-1}$-disc from S_0 what remains will be a disc D with boundary label $[a^N, b^B]$ tiled with $[a, b^{n_i}]$-discs. Applying Proposition 6.2.10 divides D into ribbons $R_1, \ldots R_N$ and we immediately see that these ribbons provide a partition of $\{n_1, \ldots n_k\}$, showing that Problem 6.2.2 has a positive solution on the given input. □

CHAPTER 7
A fast algorithm for Stallings' folding process

We now give an algorithm that quickly performs Stallings' Folding algorithm for finitely generated subgroups of a free group. The content of this chapter is based on the published article [59].

Let Γ be a directed labeled graph with the labels lying in some alphabet $X = \{x_1, x_2, \ldots, x_n\}$. Such a graph is said to be *folded* if at each vertex v there is at most one edge with a given label and incidence starting (or terminating) at v. We now state the following topologically flavored definition.

Definition 7.0.12. An *elementary folding* of a directed labeled graph Γ is a (continuous) quotient map $\pi : \Gamma \to \Delta$, where Δ is another directed labeled graph, that is obtained by identifying two edges e_1 and e_2, which at some vertex v, have the same incidence and label at v and if e_1 and e_2 are edges between vertices v, w and v, w' respectively then the vertices w and w' are also identified.

A *folding process* takes as input reduced words in $J_1, \ldots, J_m$ in $X^{\pm 1}$, makes a graph with m loops with labels $J_1, \ldots, J_m$ and attaches them all at some vertex v_0 to make a graph Γ_0 which is a bouquet of m circles with labels $J_1, \ldots, J_m$ if read starting at v_0 and following the obvious convention with respect to incidence and inverses. The algorithm then consists of a sequence of elementary foldings until it is

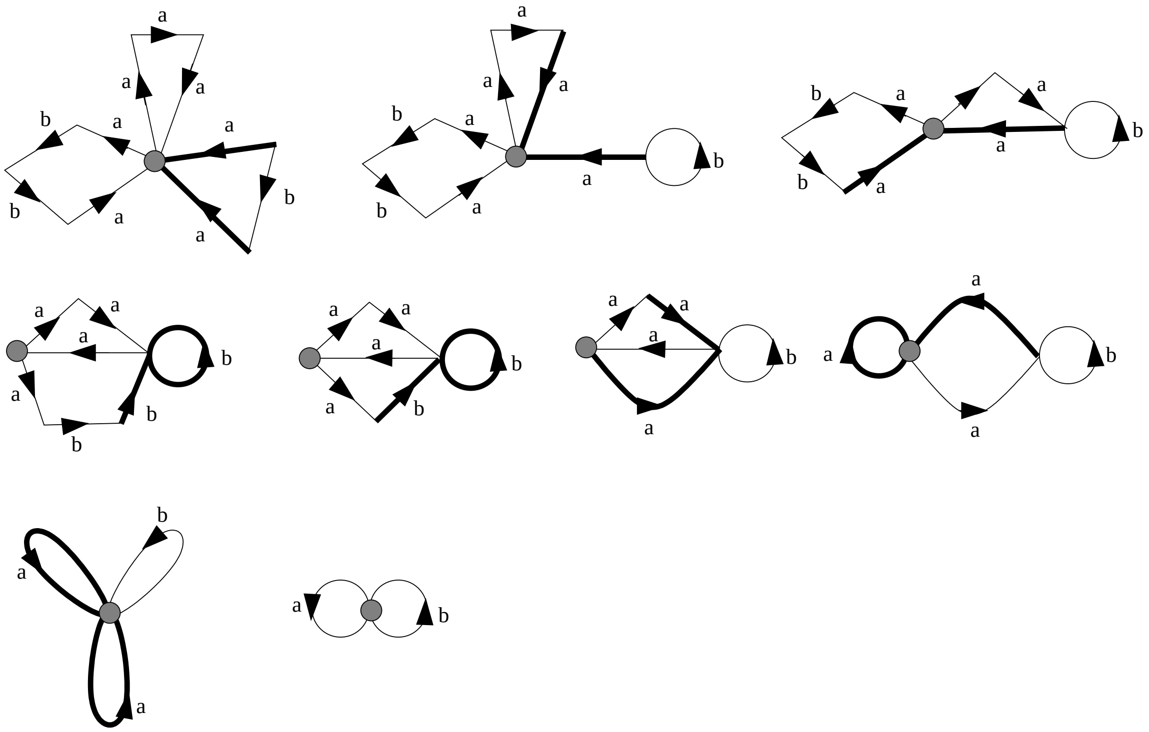


Figure 7–1: A Stallings' folding process

impossible to fold any further:

$$\Gamma_0 \to \Gamma_1 \to \ldots \to \Gamma_M = \Gamma$$

The process terminates because Γ_0 has finitely many edges and each elementary folding decreases the number of edges by 1. The output will be the folded graph $\Gamma = \Gamma(J_1, \ldots, J_m)$ which is independent of the sequence of foldings (see [23]).

Example 7.0.13. Figure 7–1 is a Stallings' folding process for inputs:$J_1 = abba, J_2 = a^{-1}ba, J_3 = aaa$. The thickened edges represent the elementary foldings. The progression is to be read left to right, top to bottom.

When we get to a point where we can no longer fold and so we stop. From this, we can now infer that $H = \langle J_1, J_2, J_3 \rangle = F(a, b)$

This folded graph gives us a picture of the subgroup $H = \langle J_1, \ldots J_m \rangle \leq F(X)$. Topologically, if we view $F(X)$ as the fundamental group $\pi_1(B, x_0)$ of a bouquet of n circles B, then constructing Γ amounts to constructing the "core" of the covering space $\widetilde{B}$ of B corresponding to the subgroup H. [54]

What is also of great interest are the "computational" properties of Γ. One can immediately verify that $w \in H$ by checking that w is the label of a loop based at v_0. It follows that once Γ is constructed the *membership problem* for the word w and the subgroup H is solvable in linear time. If we take a spanning tree of Γ using the breadth first method, which takes time linear in the number of vertices of the graph, we can obtain a *Nielsen Basis* for H. We can also compute the index of H in $F(X)$: if Γ is *regular*, i.e. at each vertex v for each $x \in X$ there are edges with label x with both incidences, then the index is the number of vertices in Γ otherwise, $[F(X) : H] = \infty$. There is also a bijective correspondence between spanning trees of Γ and *Schreier systems of coset representatives* (given a spanning tree T, take labels of subtrees of T rooted at v_0 that do not have any vertices of valency more than two). These systems of coset representatives are very important in the theory of rewriting systems. We now state the main result [53, 56]:

Definition 7.0.14. The function $\log^* : \mathbb{N} \to \mathbb{N}$ assigns to each natural number n the least natural number k such that:

$$\underbrace{\log \circ \log \circ \ldots \circ \log}_{k \text{ times}}(n) \leq 1$$

where we are using the base 2 logarithm. Equivalently $log^*(2^n) = log^*(n) + 1$.

Notice that:

$$\log^*(2^{2^{2^{2^2}}}) = \log^*(2 \cdot 10^{19728}) = 5$$

It follows that for most practical purposes, $\log^*$ grows so slowly that it can be considered a constant.

Theorem 7.0.15. *Let $F(X)$ be the free group over the generators $x_1, \ldots, x_n$, let $J_1, \ldots, J_m$ be words in $X^{\pm 1}$ and let $N = \sum |J_i|$. Then there is an algorithm for the folding process that given the input $J_1, \ldots, J_m$ will terminate in time at most $O(N \cdot log^*(N))$.*

Corollary 7.0.16. *Given generators $J_1, \ldots, J_m$ as before and the subgroup $H = \langle J_1, \ldots, J_m \rangle \leq F(X)$ we can:*

1. *Compute the index of H.*
2. *Obtain a Nielsen Basis for H.*
3. *Get a Schreier Transversal*

In time $O(N \cdot log^*(N))$. And once Γ is constructed we can solve the membership problem for a word w in time O(m) where m is the length of w.

We can also slightly generalize the algorithm to obtain the following very useful fact:

Theorem 7.0.17. *Let Δ be any connected directed labeled graph. Suppose it has V vertices and E edges, then there is an algorithm that will fold Δ in time at most $O(E + (V + E)log^*(V))$.*

We first present the data structures that will be used in our algorithm and state results pertaining to running times of various operations. All this could then be coded using object oriented languages like Java or C++.

7.1 Data Structures

The terminology I will use is non-standard in computer science, but hopefully more comprehensible to mathematicians. The details in this section are only given for completeness, all that is really important here are the theorems on running times.

For our purposes, a *data type* is a tuple $(X, f_1, \ldots, f_m)$ where X is a set and $f_1, \ldots, f_m$ are n-ary functions, i.e. functions with n arguments such that for each i and a fixed Y_i:

$$f_i : \underbrace{X \times \ldots \times X}_{n_i \text{ times}} \rightarrow Y_i$$

Moreover we allow the functions to be undefined and allow ourselves to change their values. These functions will be called *operations*. For example $X = \mathcal{P}(\mathbb{N})$ is the collection of sets of natural numbers, with binary operations, *union, intersection* and the unary operation *least element* (which a set to to a natural number). We will also want to allow different *instances* of a data type, e.g. the data type is math students with the function *grade*: $\{\text{students}\} \rightarrow \mathbb{R}$ and we have two instances: calculus students and linear algebra students. Maybe some students will be taking both classes so they will have two grades, one for calculus and one for linear algebra it follows that there will be two instances of the grade function defined on different (though maybe not disjoint) sets of students.

So far nothing can be said about running times. To this end we have to flesh out our construction, we give the actual algorithms that perform our operations. *Primitive operations* are unary operations (or simply functions) that either correspond to variable assignment or so-called *pointers* used in object oriented programming. We will directly invoke primitive operations in algorithms. We assume that the *operating time cost* of either evaluating a primitive operation or changing its value on one entry will be 1.

Some operations will not be primitive, so to calculate them we will provide a *method* which is basically an algorithm which, using primitive operations, enables one to perform a more complicated operation. Once a method is given, it will be possible to calculate the running time of the associated operation. The reason the word *method* is used instead of simply algorithm is that we want to stress that it is at a lower level of abstraction, once it's given we want to forget everything about it other than its running time and the fact it works. As will be seen, there will also be a certain structure to the way the elements in our data type are interrelated which motivates the terminology *data structure*.

The reason for this rather artificial formalism is mainly to ease analysis. The main algorithm that will be given in Section 7.2.3 will be given in terms of operations whose semantic meaning is clear and it will be obvious that the algorithm actually works. The explicit methods presented in this section will give us running times for our operations and we'll be able to compute the running time of the algorithm.

As a remark to computer scientists, the definition of data type given here resembles an *interface* and combined with the methods, what we're actually describing is an *abstract data structure*.

7.1.1 Ordered Sets

This data structure actually is actually made of two interdependent components *lists* and *list nodes*. List nodes have two primitive operations:

1. next : {list nodes} → {list nodes}
2. prev : {list nodes} → {list nodes}

And two operations that will require methods.

1. list : {list nodes} → {lists}
2. remove : {list nodes} → {lists}

For lists we have the two primitive operations:

1. head : {lists} → {list nodes}
2. tail : {lists} → {list nodes}

As well as the binary operation:

1. concatenate:{lists} × {lists} → {lists}

Finally, we need an operation to add a node to a list:

1. addnode : {list nodes} × {lists} → {lists}

So far we have two types of objects and some functions. An ordered set will be encoded as a *doubly linked list*. It can be thought of as a chain of list nodes.

Example 7.1.1. Here we have a list L, and list nodes a,b and c. We encode this as shown in Table 7–1. We visualize this as Figure 7–2.

Table 7–1: Encoding a doubly linked list

list node n	next(n)	prev(n)	list(n)
a	b	undefined	L
b	c	a	undefined
c	undefined	b	L

List$X$	head(X)	tail(X)
L	a	c

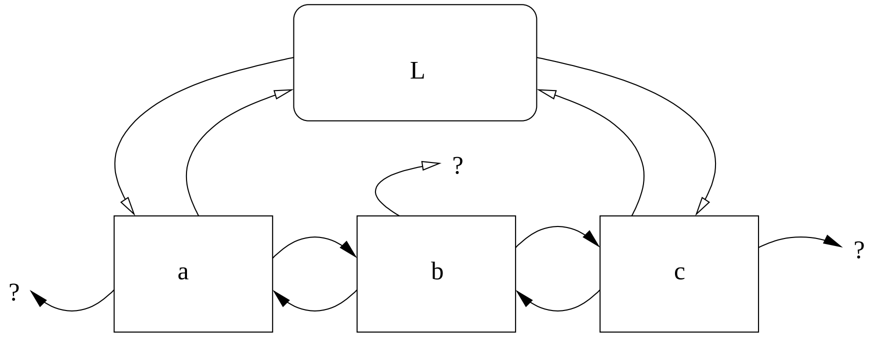


Figure 7–2: A well formed list

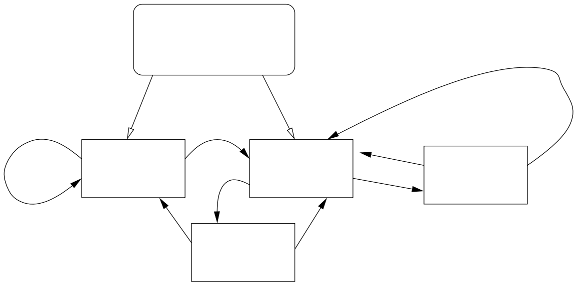


Figure 7–3: A badly formed list

A priori, there are no restrictions on what values functions can take, but if we're not careful our list will not be *well formed*, for example see Figure 7–3

We can ensure that our structures will be well formed if we make sure that our methods keep structures well formed and only use these methods. We now give the methods associated to operations on ordered sets. When invoking a method we will use the `typewriter` font. The method associated to the function remove will be called `remove` and we will denote "performing the `remove` method on a list node n" by `remove`(n). This method does not return anything, it simply removes the list node n from a list while keeping it well formed

`remove`(n):

1. Get the variables h=head(list(n)) and t=tail(list(n)).
2. If $h = t = n$ then make head(list(n)) and tail(list(n)) undefined.
3. If $h = n \neq t$ then set head(list(n))=next(n), set list(next(n))=list(n), set prev(next(n))=undefined, and set next(n)=prev(n)=undefined.
4. If $h \neq n = t$ then do the same as the previous with prev and next interchanged.
5. If $h \neq n \neq t$ then set next(prev(n))=next(n), set prev(next(n))= prev(n) and set next(n)=prev(n)=undefined.

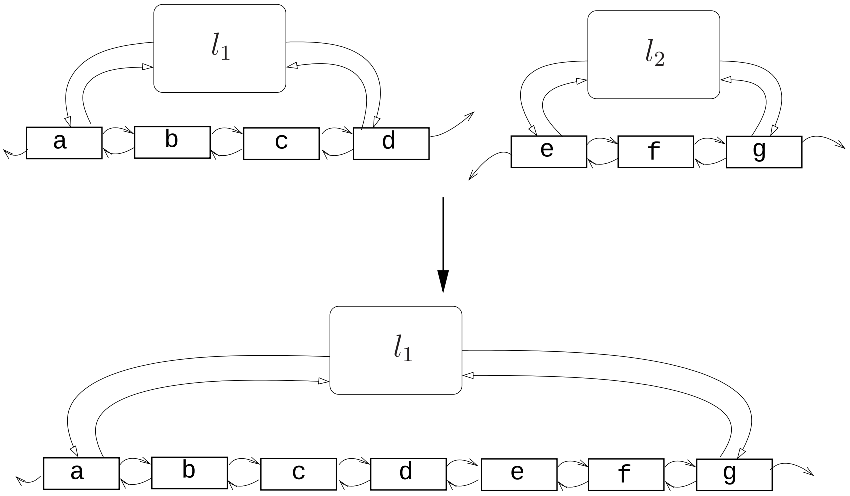


Figure 7–4: Concatenating two lists

The next method is for the concatenate operation for two lists l_1, l_2. We call the method `concatenate`, it appends the list nodes of l_2 to those of l_1 and leaves the list l_2 empty.

`concatenate`(l_1, l_2):

1. If head(l_2) is undefined (l_2 is empty) then do nothing.
2. If head(l_1) is undefined, then set head(l_1)=head(l_2), set list(head(l_2))=l_1, set tail(l_1)=tail(l_2), set list(tail(l_2))=l_1 and set head(l_2)=tail(l_2)=undefined.
3. Else set next(tail(l_1))=head(l_2), set prev(head(l_2))=tail(l_1), set tail(l_1)=tail(l_2) and set list(tail(l_2))=l_1.

The Figure 7–4 illustrates the `concatenate` operation. The method `addnode` for the addnode operation will not be given, but it is quite obvious. The following theorem holds.

Theorem 7.1.2. *There exists methods of the operations remove, concatenate and addnode that take a constant amount of time.*

Proof. The associated methods `remove`, `concatenate` and `addnode` involve only a bounded number of primitive operations. ☐

We can also enumerate a list l_1, indeed take head(l_1) then repeatedly perform "next" operations, once the value "undefined" is reached, the list is exhausted.

7.1.2 Disjoint Sets

In our case we have a sequence of elementary foldings:

$$\Gamma_0 \to \Gamma_1 \to \ldots \to \Gamma_M = \Gamma$$

The composition, $\pi = \pi_M \circ \pi_{M-1} \ldots \circ \pi_1$ of all the quotient maps $\pi_i : \Gamma_i \to \Gamma_{i+1}$ gives a quotient map $\pi : \Gamma_0 \to \Gamma$. This map π, in turn, induces an equivalence relation on the vertices of of Γ_0, i.e $v \sim w \iff \pi(v) = \pi(w)$. In fact one can consider the vertices of Γ as equivalence classes of vertices of Γ_0. These equivalence classes are "built" from smaller disjoint sets by successively merging them in each elementary folding. For example if the vertices v, w in Γ_i correspond to equivalence classes $\{v_1, \ldots, v_r\}, \{w_1, \ldots w_s\}$ respectively and if $\pi_i(v) = \pi_i(w) = \bar{u}$, then the vertex $\bar{u}$ of Γ_{i+1} will correspond to the set of vertices $\{v_1, \ldots, v_r, w_1, \ldots, w_s\} \subset$ Vertices(Γ_0). Though this doesn't fully motivate our interest in the following data structure and it's clever methods it does give an example of how they are going to be used.

The Disjoint Set Forest data structure has an underlying set of nodes. On the set of nodes we have the following primitive operations:

1. rank:{nodes} $\to \mathbb{N}$

2. parent:{nodes} → {nodes}.

From this it is seen that nodes can be organized into rooted trees. We have the following non-primitive operations:

1. root:{nodes} → {nodes}
2. merge: {nodes} × {nodes} → {trees}

Some explanations are in order. We have a set X of nodes and we want to build equivalence classes out of them. An equivalence class will be encoded as a rooted directed tree. We shall identify the trees by their root nodes, i.e. the unique node in the tree that has itself as a parent. If we want to know to which equivalence class a node n belongs we use the function root(n) which returns the root of n's tree, similarly we can check if two nodes are "congruent" by checking if they have the same root. We will use the merge(u, v) operation to form the union of the equivalence classes containing u and v. It is clear that here too some care must be taken to avoid "malformed" trees.

Example 7.1.3. Figure 7–5 shows a set partitioned into two equivalence classes. Notice that the nodes pointing to themselves are roots or equivalence class representatives.

`Initialization:` When a node n is created we need the to set following initial values so that everything works:

1. set parent(n)=n
2. set rank(n)=0.

This is like putting n into an equivalence class with only itself in it.

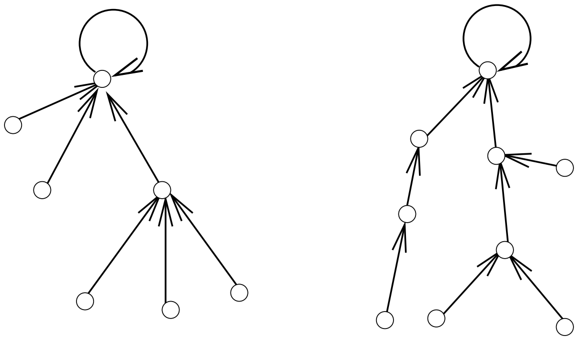


Figure 7–5: Trees encoding sets

To perform the root(n) operation we use a method called `Find-set`(n) which takes a node n and returns the node r which is the root of its tree. It is given recursively:

```
Find-set(n)
  1. If parent(n)=n, return n.
  2. Else set parent(n)=Find-set(parent(n)) and return parent(n).
```

Proposition 7.1.4. *This method actually works.*

Proof. We basically do this by induction on the *depth* of n i.e. the least integer M such that:

$$\underbrace{\text{parent} \circ \ldots \circ \text{parent}}_{M-1 \text{ times}}(n) = \underbrace{\text{parent} \circ \ldots \circ \text{parent}}_{M \text{ times}}(n)$$

If the depth is 0, i.e. n is a root, then it works. If it works for all nodes of depth M or less and n has depth $M+1$ then `Find-set`(parent(n)) will return the root of n's tree and all is well. □

Clearly this is not the most expedient way to get the root node (which in this case would simply consist of successively evaluating parents until we hit a "fixed

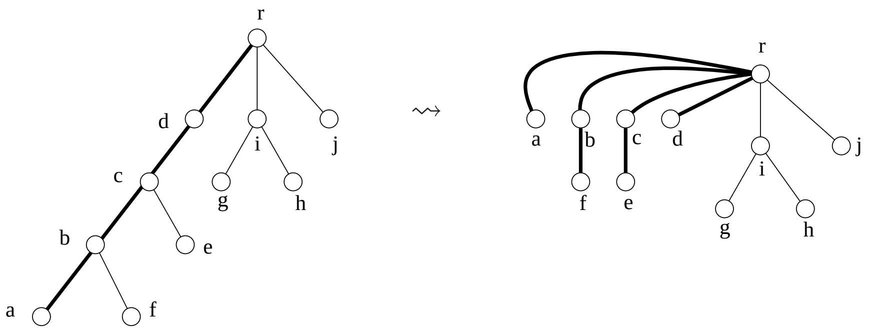


Figure 7–6: Path compression

point"). However something interesting happens, instead of working your way up to the tree root r, you work your way up to the root and then back down again and at each step on the way back you set the values of parent functions to r. This is called *path compression* and it makes the tree "bushier" and will make successive root operations faster. Figure 7–6 shows what arises after performing root(a):

Though tree itself changes, the mathematical object it represents is the same: we still have the same nodes and the same equivalence classes. The tree, however, has been partially optimized.

The last operation, merge, should takes two nodes x, y and make the union of of the equivalence classes containing x and y respectively. Here we use the rank, which is basically an upper bound on the depth of the tree. It is used to determine which node will be the new parent. We call the associated method `Merge`(x, y):

1. Get $r_1 =$`Find-set`(x), $r_2 =$`Find-set`(y).
2. If rank(r_1)>rank(r_2) then set parent(r_2)=parent(r_1).
3. If rank(r_2)>rank(r_1) then set parent(r_1)=parent(r_2).
4. Else set parent(r_2)=r_1 and set rank(r_1)=rank(r_1)+1

We now come to a truly amazing result due to Tarjan whose proof can be found in [12]. This proof uses the methods we just described. This result, however, is not obvious to prove. An *amortized* running time is the combined running time of a sequence of operations.

Theorem 7.1.5. *Suppose we perform* n *Disjoint Set operations, i.e. root and merge operations, on a Disjoint Set forest containing* N *nodes. Then there exist methods for the root and merge operations such that the* amortized *running time devoted to these operations will be at most* $O((n+N)\cdot\log^*(N))$.[1]

7.1.3 Directed Labeled Graphs

We now encode a graph. We assume that we are working over $F = F(a, b)$ the free group on the alphabet $\{a, b\}$. A graph will have two underlying sets consisting of *vertex* objects and *edge* objects. The idea is that there are functions assigning to edges their terminal and initial vertices and each vertex has list of adjacent edges. It follows that each edge will be a node in two lists. We will also want to organize vertices into Disjoint Set forests and put them in a list called UNFOLDED. We have the following primitive operations:

1. edgelist:{vertices} → {lists}
2. initial:{edges} → {vertices}
3. terminal:{edges} → {vertices}
4. label:{edges} → $\{a, b\}$

---

[1] The result in [12] actually gives an even better bound: instead of $\log^*$ it's an inverse Ackerman function.

We also want to make lists of edges so we define two instances of the list node operations on the set of edges. One instance for the list at an edge's initial vertex and one instance for the list at an edge's terminal vertex. Hopefully the nomenclature will be self-explanatory:

1. next-initial:$\{\text{edges}\} \rightarrow \{\text{edges}\}$
2. next-terminal:$\{\text{edges}\} \rightarrow \{\text{edges}\}$
3. prev-initial:$\{\text{edges}\} \rightarrow \{\text{edges}\}$
4. prev-terminal:$\{\text{edges}\} \rightarrow \{\text{edges}\}$
5. remove-initial:$\{\text{edges}\} \rightarrow \{\text{lists}\}$
6. remove-terminal:$\{\text{edges}\} \rightarrow \{\text{lists}\}$
7. addnode-initial:$\{\text{edges}\} \times \{\text{lists}\} \rightarrow \{\text{lists}\}$
8. addnode-terminal:$\{\text{edges}\} \times \{\text{lists}\} \rightarrow \{\text{lists}\}$

And for vertices we have the following additional operations:

1. next-UNFOLDED:$\{\text{vertices}\} \rightarrow \{\text{vertices}\}$
2. prev-UNFOLDED:$\{\text{vertices}\} \rightarrow \{\text{vertices}\}$
3. remove-UNFOLDED:$\{\text{vertices}\} \rightarrow \{\text{lists}\}$
4. addnode-UNFOLDED:$\{\text{vertices}\} \times \{\text{lists}\} \rightarrow \{\text{lists}\}$
5. root:$\{\text{vertices}\} \rightarrow \{\text{vertices}\}$
6. rank:$\{\text{vertices}\} \rightarrow \mathbb{N}$
7. merge:$\{\text{vertices}\} \times \{\text{vertices}\} \rightarrow \{trees\}$

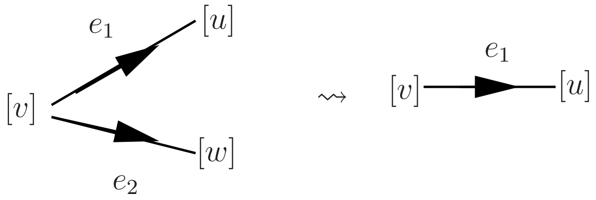


Figure 7–7: Elementary Folding

7.2 Ideas and the Algorithm

7.2.1 Elementary Foldings

Recall that in the sequence of elementary foldings

$$\Gamma_0 \rightarrow \Gamma_1 \rightarrow \ldots \rightarrow \Gamma_M = \Gamma$$

The vertices of Γ_i could be seen as equivalence classes of vertices of Γ_0. For this reason we will denote vertices of Γ_i as $[v]$, i.e. "the equivalence class in the set of vertices of Γ_0 with representative v."

Definition 7.2.1. A vertex $[v]$ is said to be *folded* if there are no edges with same label and incidence an $[v]$. Otherwise we say $[v]$ is *unfolded*.

Consider the following identification of the edges e_1 and e_2 via an elementary folding shown in Figure 7–7

We see that that the vertices $[u]$ and $[w]$ get identified so that in the next graph in our sequence the equivalence class represented by u will consist of the union $[u] \cup [w]$ we shall denote this by $[u]'$. In our computer program such an elementary folding would be accomplished by performing the operation merge(u, v) (in the example rank$(u) \geq$ rank(w)), removing the edge e_2 from the edge lists at w and v (essentially deleting it) and finally performing concatenate(edgelist(u),edgelist(w)). Recall that after an elementary folding the edges at $[u]'$ will be the edges at $[u]$ plus the edges at $[w]$ minus the deleted edge. This is reflected by concatenating the edgelists and though none of the edges in $[w]$'s old edgelist are set to point to u yet (edges go between vertices, not equivalence classes) it is possible to update them. However if we completely update all the edges at each folding we'll end up having something that runs in *quadratic* time! Some care is therefore needed. The updating of edges only occurs when checking whether a vertex is folded (see Observation 1 in Section 7.2.2) and in the second step of the loop in the algorithm in Section 7.2.3 and when either case happens, we only update at most five edges at a time. This is the trick to get the algorithm to run in almost linear time.

Consider the Figure 7–8. The figure on the top is the graph Γ_i as a topological object with vertices corresponding to equivalence classes of vertices of Γ_0. We see that the edges outgoing from $[u]$ labeled a will be identified in some elementary folding. The figure on the bottom is at a lower level of abstraction, it shows what is encoded in the computer. The circles represent "vertex" objects, notice that the vertices parent pointers as well as graph edges coming out of (going into) them:

We see that the equivalence class $[u]$ contains eight elements, that the v's edgelist has four entries but that there is only one edge "actually" at v, i.e. some edge e with label(e)=b and initial(e)=v.

7.2.2 Detecting Unfolded Vertices

The only other difficulty is figuring out *where* to fold. Three observations tell us that we can easily keep track of the unfolded vertices and when we know that there are none left, then we're done.

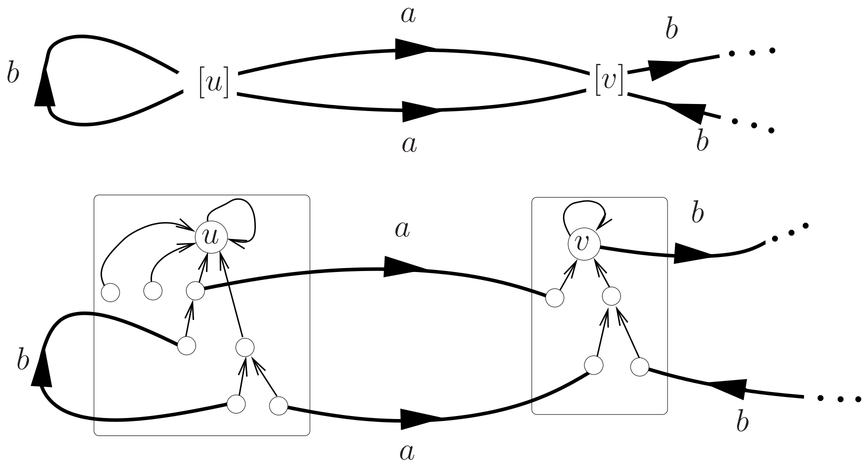


Figure 7–8: A view "under the hood"

Observation 1. To check whether or not a vertex $[v]$ is folded takes a bounded number of operations. Indeed, we need only go through the edge list of $[v]$ and check the labels and incidences of the edges.

To find the incidence of an edge e in $[v]$'s edge list, find u =initial(e) and w =terminal(e) and perform the operations root(u) and root(w) to find equivalence class representatives. If for example root(u)=v then e is outgoing at $[v]$. Similarly we can determine if e is incoming or forms a simple loop at $[v]$. At this point we could also update the edges i.e. set initial(e)=root(initial(e)) and set terminal(e)=root(terminal(e)) for an extra two operations.

Now go through the edge list of v. Either you find two edges with same label and incidence so $[v]$ is unfolded or you exhaust the edgelist without finding edges with the same incidence and label so $[v]$ is folded. Since we are assuming that we are working over $F(a, b)$ it is clear that an edgelist with five or more entries must result in unfoldedness. It follows that *we never check more than 5 edges at a time.*

Observation 2. An elementary folding is an essentially local operation. That is, whenever two edges get identified we need only to check for three vertices whether they have gone from being folded to unfolded or vice-versa. Any vertex that is not the initial or terminal vertex of some edge being identified with another edge at that elementary folding will have the same number of incoming and outgoing edges after the elementary folding.

Observation 3. At the beginning there is exactly one unfolded vertex, i.e. where we initially attach our loops, and the algorithm terminates when there are no unfolded vertices left.

These three observations tell that we can have a list called UNFOLDED which contains exactly the unfolded vertices and that at each elementary folding we need perform a bounded number of primitive, ordered set and disjoint set operations to keep it updated.

7.2.3 The Algorithm

We will make a distinction between ordered set operations and disjoint set operations. We will call primitive operations and ordered set operations simply "operations" and mention disjoint set operations explicitly.

Initialization:

We are given an input $(J_1, \dots, J_n)$ of reduced words in $F(a, b)$. For each J_i we make a directed labeled loop l_i with label J_i starting at v_0 and initialize each vertex as in Section 7.1.2 we call the resulting graph Γ_0 (Figure 7–9.) At this point there is only one unfolded vertex: v_0. We also create the list UNFOLDED containing the single vertex v_0.

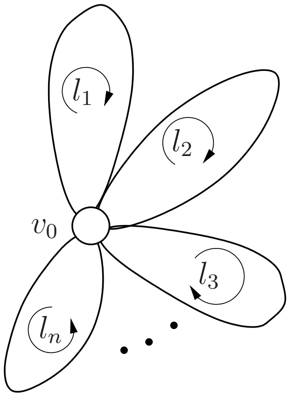


Figure 7–9: The bouquet of generators

All this takes time $O(N)$.

Folding:

While UNFOLDED is not empty do the following:

1. Get v=head(UNFOLDED) to get an unfolded vertex. This costs 1 operation.
2. Get L=edgelist(v). Get e_1=head(L) get u_1=root(initial(e_1)), v_1=root(terminal(e_1)) and label(e_1) to get the label and incidence of e_1 at $[v]$. Then set initial(e_1))=u_1 and set terminal(e_1)=v_1 to "update" the edge. Take e_2= either next-initial(e_1) or next-terminal(e_1) (depending on the incidence of e_1) and again get the incidence, get the label and update the edge. Keep performing "next" operations until you get two edges with the same label and incidence and can fold. This costs 1+1 operations + $\leq 5 \cdot$ (6 operations + 2 disjoint set operation + some constant amount of time)

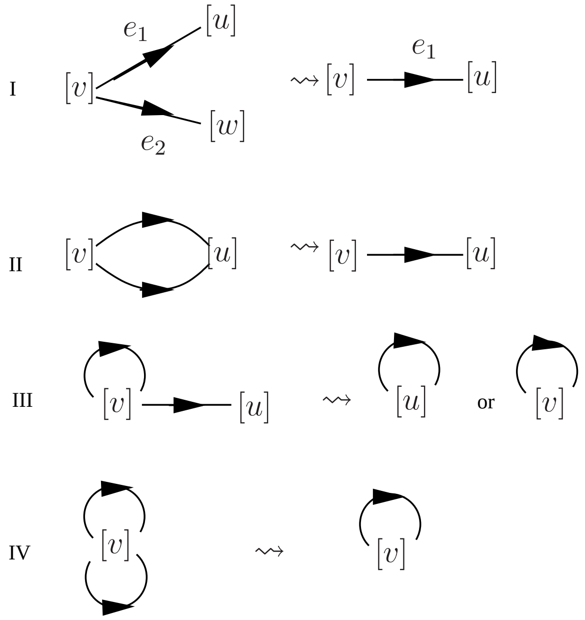


Figure 7–10: Different folding situations

At this point we have found 2 edges e_{i_1}, e_{i_2} (without loss of generality e_1, e_2) with same incidence and label. We have the four possible local situations given by Figure 7–10.

From Step 2 we know the the endpoints of e_2 and e_1 and can therefore establish which case we are dealing with (this takes constant time).

Case I:

I.1 merge(u, w) (assume the new representative is u.) This costs 1 disjoint set operation.

I.2 if necessary, remove the non representative vertex w from UNFOLDED. This costs 1 operation.

I.3 concatenate(edgelist(u),edgelist(w)). This costs 1 operation.

I.4 We assume that e_2 is the edge going from v to w. Then we do remove-initial(e_2) and remove-terminal(e_2). At this point we can assume that e_2 is deleted. This costs is 2 operations.

I.5 Check whether the remaining vertices $[u]$ and $[v]$ are folded and add or remove them from UNFOLDED accordingly. By Observation 1 this again takes a bounded number of disjoint set and "normal" operations.

How to handle cases II-IV is similar and will not be given. When we exit the "while" loop, i.e. UNFOLDED is empty, the algorithm terminates. All the remaining edges point to their representative vertices and no vertex is unfolded, so we have a usable folded graph.

7.2.4 Analysis

Each time the "while" loop executes an edge gets deleted so the loop runs at most N times i.e. the total length of the input. Each run through the loop in fact corresponds to an elementary folding. Each time the loop runs, a constant bounded number of "standard" and disjoint-set operations are executed so applying Theorem 7.1.5 this runs in time $O(N) + O(N \log^*(N)) = O(N \log^*(N))$. This proves the main result, Theorem 7.0.15. We can also give the following:

Proof of Theorem 7.0.17. We do a search through our graph and check at each vertex v if it is folded. If not then we add v to UNFOLDED. The search takes time $O(E)$. We then proceed as usual. □

CHAPTER 8
Conclusion

The first part of this thesis gives the most extensive account of the fully residually F groups arising from systems of equations in two variables over free groups to date. This enabled us in particular to recover all previously known qualitative descriptions of the solutions of this class of systems of equations. Nonetheless, I still do not think that the given treatment is completely satisfactory. Some questions still remain, such as those brought up in Section 4.1.2. What would also be really nice is a complete description of the Hom diagrams, like the one given in Chapter 3, but for arbitrary systems of equations in two variables over free groups. Given the announced proof that fully residually free groups have finite Krull dimensions by Louder [34, 35, 36, 37], such a finite description exists. Moreover I am optimistic that the Hom diagrams will be much smaller than the upper bounds given in Louder's work.

However I believe that the methods we have used so far are insufficient for this task. We need a better understanding of the possible epimorphisms between fully residually free groups and possibly a better understanding of how to combine fully residually free groups. In fact a good characterization of when a free product of two free groups with amalgamation over a cyclic group is fully residually free is still unknown. There is still much more work to be done in this field and the special case of two variable equations will remain a useful testing ground.

Another direction is the study of systems of equations in two variables over torsion free hyperbolic groups. The structure theorems for fully residually torsion free hyperbolic groups are very similar, the main difficulties will come from the study of two generated subgroups of torsion free hyperbolic groups. A good starting point for this would be the paper of Kapovich and Weidmann [24] where they obtain a result that is not unlike Lemma 3.1.10 and the work of Delzant [14] which bounds the number of conjugacy classes of non-free two generated subgroups of word hyperbolic groups.

The complexity of solving equations is also a very exciting topic these days. Indeed it was recently announced by I.G. Lysënok that the general problem of solving a system of equations over a free group is in fact in NP, thus by our result NP-complete. This result is not only interesting in its own right, but also because solving equations over groups plays an important role for many algorithmic applications, for example in the solutions to the isomorphisms problems for torsion free hyperbolic groups that do not have a small essential action on an $\mathbb{R}$-tree by Sela [51], for fully residually free groups by Bumagin, Kharlampovich and Miasnikov [6], and for toral relatively hyperbolic groups by Dahmani and Groves [13]. NP is still pretty fast, and algorithmic tractability results in this direction are therefore very encouraging for the feasibility of these algorithms in real life.

The fast algorithm for Stallings' folding process is a good concrete result. Many basic algorithmic problems involving subgroups of free groups can be now solved extremely quickly. In my opinion this demonstrates some of the power of geometric reasoning. The previously known best running time was quadratic, which isn't that bad. However for practical applications, say for example computing the image of a subgroup of a free group under successive endomorphisms, the advantage of the almost linear running time becomes quite apparent. Hopefully this algorithm will become standard in computer algebra packages.

The next step is to try to obtain an upper bound for the complexity of the analogue of Stalling's folding process for subgroups of $F^{\mathbb{Z}[t]}$.

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