Coarse-Grained Holography

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Contributions of Author

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This project was initiated by Alex Maloney back in April 2014 — shortly after the announcement that BICEP 2 had claimed to have detected B-modes from primordial gravitational waves — as an attempt to characterize the quantum field theory whose renormalization group flow is putatively dual to inflation. It then evolved into a study of the dilaton effective action induced by holographic RG flows, and an attempt to establish a general map between parameters encoding the breaking of the bulk spacetime symmetry and those of the boundary RG flow. I performed all computations in this project, often in tandem with or simultanenously as Gim Seng Ng. It was Alex Maloney who noticed the analogy with the method of variation of parameters in solving the Goldstone boson's equation of motion, leading to the Dyson series solution for the coefficients describing the mixing within the basis of instantaneous AdS solutions.

ABSTRACT

In this thesis we study domain-wall geometries in which the AdS isometries are weakly broken and the renormalization group flows of their field theory duals. First, we introduce the AdS/CFT correspondence and review the computation of CFT_d correlation functions from AdS_{d+1} gravity. We then show how to compute renormalized correlation functions from the bulk, introduce domain-wall geometries as the bulk duals of field theories with broken conformal invariance and demonstrate the holographic realization of conformal anomalies. Finally, we present a new approximation scheme that is useful for computing the on-shell action in an effective description of AdS domain-wall geometries (or equivalently, the dilaton effective action of the dual field theory). This framework is general in principle but is particularly suited to the case that the AdS isometries are weakly broken, corresponding to a 'slow flow' interpolating between AdS asymptotic regions. We demonstrate the utility of this approximation scheme with two applications. First, we compute the dilaton effective action for a flow driven by a weakly relevant operator from the dual bulk effective theory, and show that it reproduces the correct form of the RG-improved two point function. Secondly, we compute the gravitational on-shell action in a generalized 'slow-flow' setup in four dimensions, and show that upon Wick rotation we recover the inflationary power spectrum and spectral index of curvature perturbations at horizon-crossing to second-order in slow-roll.

ABRÉGÉ

Dans cette thèse, nous étudions les géométries 'mur-de-domaine' dans lesquelles les isométries de l'éspace-temps anti-de Sitter sont faiblement cassées et aussi le flux du groupe de renormalisation de leurs théories des champs doubles. Tout d'abord, nous introduisons la correspondance AdS/CFT et révisons la dérivation des fonctions de corrélation dans la théorie conforme des champs de la gravité dans l'éspace anti-de Sitter. Nous montrons ensuite comment calculer les fonctions de corrélation renormalisées des théories de la gravité de l'espace-temps AdS; nous introduisons les géométries mur-de-domaine qui sont les doubles des théories des champs avec la symétrie conforme cassée; et nous démontrons la réalisation holographique des anomalies conformes. Enfin, nous présentons un nouveau schéma d'approximation qui est utile pour évaluer l'action gravitationnelle avec la solution des équations du mouvement dans une description effective des géométries 'mur-de-domaine' (ou de manière équivalente, l'action effective du dilaton de la théorie des champs double). En principe cette méthode est générale, mais elle est particulièrement adaptée au cas que les isométries de l'espace-temps AdS sont faiblement cassées, correspondant à un 'flux lent' qui interpole entre les géométries AdS asymptotiques. Nous démontrons l'utilité de ce schéma d'approximation avec deux applications. Premièrement, nous calculons l'action effective du dilaton pour un flux du groupe de renormalisation causé par un opérateur faiblement-essentiel de la théorie effective gravitationnelle, et montrons qu'il reproduit la forme correcte de la fonction de correlation à deux points 'RG-amélioré.' Deuxièmement, nous évaluons l'action gravitationnelle avec une solution des équations du mouvmement dans une configuration généralisée 'flux lent' en quatre dimensions, et montrons que lors de la rotation Wick nous récupérons le spectre de puissance et l'indice spectral des perturbations de courbure de l'inflation cosmique après le passage de l'horizon cosmologique au deuxième ordre dans le schéma d'approximation 'flux lent.'

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Chapter 1
Introduction

The quantization of gravity is one of the most daunting problems in modern theoretical physics — a systematic treatment of the quantum theory of gravity remains elusive. However, a quantum theory of gravity is required to understand its place in a unified description of the fundamental interactions; to provide a top-down understanding of cosmology rooted in fundamental physics, as the physics of the very early universe is sensitive to scales well beyond where classical general relativity is known to break down; and to make sense of the fact that black holes seem to obey the laws of thermodynamics.

In dimensions greater than two, the naive quantization of the classical theory of general relativity is obstructed by the fact that it is non-renormalizable by simple power-counting. At the Planck-scale, the theory is strongly coupled and so divergences that appear in the usual Feynman diagram expansion of scattering matrix elements require an infinite number of counterterms to tame, so the usual devices of quantum field theory are of no use. To see this, consider for concreteness the case of four dimensions. Expanding the metric around flat space

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_{\rm Pl}} h_{\mu\nu}, \tag{1.1}$$

and applying the rules of effective field theory to general relativity, one arrives at the following form for the effective gravitational action [1]

$$S \sim \int \mathrm{d}^4x \left[(\partial h)^2 + \frac{1}{M_{\rm Pl}} h (\partial h)^2 + \ldots + \frac{1}{M_s^2} \left((\partial^2 h)^2 + \frac{1}{M_{\rm Pl}} h(\partial^2 h)^2 + \ldots \right) \right] \tag{1.2}$$

where the first term corresponds to the usual Einstein-Hilbert term and the second comes from higher-derivative terms such as R^2 which become important at some scale $M_s \leq M_{\rm Pl}$. It is clear that at energies of order the Planck mass (or smaller, depending on the value of M_s), loop diagrams become more important than tree-level diagrams. In the ultraviolet (UV), one must include the infinite cascade of higher derivative curvature terms, leading to an infinite number of free parameters and a loss of predictive power. One proposed UV completion of general relativity is string theory, where the new UV degrees of freedom are made up of vibrational modes of fundamental open and closed strings and divergences are tamed by smoothing out interactions over the two-dimensional worldsheet of the strings. However, this comes at the cost of additional fields in the spectrum and the requirement of additional compact dimensions. As a result, a complete understanding of string theory evades us.

In negatively curved spacetimes, the anti-de Sitter space/conformal field theory (AdS/CFT) correspondence [2, 3, 4] in its strongest form offers a non-perturbative 'definition' of quantum gravity in terms of a conformal field theory defined on the boundary of the spacetime. Furthermore, the correspondence provides a novel strategy for the study of strongly-coupled conformal field theories via classical supergravity in asymptotically AdS spacetimes. We review the central components of the AdS/CFT correspondence, including the derivation of CFT_d correlation functions from AdS_{d+1} gravity, in chapter 2 of this thesis. This strategy can even be extended to study field theories that break conformal invariance explicitly (by the addition of a relevant operator deformation to the action) or spontaneously (by an operator acquiring a non-trivial vacuum expectation value), in which case the gravitational dual is a so-called domain-wall solution. There is a sense in which radial evolution in these geometries is dual to the renormalization group flow of the boundary field theory, as will be discussed in chapter 3. In this thesis we study a general effective bulk theory [5] describing the universal features of holographic renormalization group flows, with particular attention paid to the case where the AdS isometries corresponding to dilatations in the boundary theory are weakly broken.

This thesis is organized as follows. Chapters 2 and 3 are aimed at providing evidence for the lore that plagues the gauge/gravity duality literature, often making use of toy models to provide concrete realizations and simplified explanations of the physical mechanisms behind the lore. Chapter 2 provides a cursory introduction to the AdS/CFT correspondence. In particular, we review the geometry of anti-de Sitter space in section 2.1, then briefly discuss the Klein-Gordon equation in AdS_{d+1} in section 2.2 in order to demonstrate the relationship between the masses of bulk scalar fields and the scaling dimensions of the dual field theory operators. In section 2.3 we review the derivation of CFT_d correlation functions from AdS_{d+1} gravity; we introduce the bulk-to-bulk and bulk-to-boundary propagators, then provide the canonical derivation of CFT_d 2- and 3-point functions, and discuss the derivation of arbitrary n-point functions via Witten diagrams. In chapter 3 we provide an introduction to the program of holographic renormalization. In section 3.1 we show how to obtain renormalized CFT_d correlation functions by the introduction of covariant local counterterms. In section 3.2 we show how to move beyond conformal invariance and discuss RG flows holographically by introducing domain-wall geometries. In section 3.3 we show that the first-order form for the domain-wall equations of motion follows from the Hamilton-Jacobi equation for the gravity-scalar system, which in turn implies the Callan-Symanzik equation for renormalized correlation functions in the boundary theory. Finally, in section 3.4 we review the holographic derivation of the conformal anomaly. In chapter 4 we describe a new approximation scheme useful for computing renormalized field theory correlation functions from domain-wall geometries which weakly break the AdS isometries.

Chapter 2
Holography for dummies: a utilitarian's guide to AdS/CFT

In this chapter we provide a lightning review of the most basic elements of the AdS/CFT correspondence, with particular emphasis on those most relevant to the understanding of holographic renormalization group flows. We make no attempt to provide a comprehensive review of the correspondence,[1] and lean heavily on several canonical papers and existing reviews of the subject [6, 7, 3, 4, 8, 9].

The AdS/CFT correspondence as put forward by Maldacena [2] provides an explicit realization of the 'holographic principle' first imagined by 't Hooft and Susskind. Taking inspiration from the universal observation that the entropy of a black hole scales with its area rather than its volume, the holographic principle dictates that the *entirety* of the physics of a bulk gravitational theory is in some way captured by the boundary of the spacetime. The original duality proposed by Maldacena was the equivalence of $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang-Mills theory in $3+1$ dimensions and type IIB string theory on $\mathrm{AdS}_5 \times S^5$, and was observed by taking the near-horizon limit of a stack of N $D3$-branes.

However, gauge/gravity duality is expected to hold more generally — that is, quantum gravity in anti-de Sitter space (AdS_{d+1}) (perhaps times some compact manifold) is believed to be dynamically equivalent to a conformally-invariant quantum field theory (CFT_d) defined on the boundary of the bulk spacetime. The statement of the duality is that the CFT partition function is equal to the generating functional of the bulk gravitational theory — schematically

$$Z_{\mathrm{grav}}[\bar{\phi}] = \int_{\Phi|_{\partial M} \sim \bar{\phi}} [\mathcal{D}\Phi \cdots] e^{-S_{\mathrm{grav}}[\Phi,\ldots]} = \langle e^{-\int_{\partial M} \mathrm{d}^d\vec{x}\bar{\phi}\mathcal{O}(\vec{x})} \rangle = Z_{\mathrm{CFT}}[\bar{\phi}], \tag{2.1}$$

[1] As the literature is in no need of any more of these.

where $\bar{\phi}$ characterizes the near-boundary behaviour of the bulk field Φ and, we will see, acts as a source for an operator $\mathcal{O}$ in the dual field theory. One should then be able to match correlation functions on both sides of the correspondence.

On its face, this is an absurd assertion. Further, the gauge/gravity correspondence is a strong/weak duality, so in principle one can perform tractable computations entirely on one side of the duality and infer conclusions about the other. Naively it seems bizarre that vanilla quantum field theories are secretly higher dimensional string theories in disguise, and that computations in classical gravity could teach us about otherwise intractable strongly-coupled field theories.

One clue about the emergence of the extra dimension in the bulk is the fact that, as a consequence of spacetime locality, the renormalization group equation for running of a field theory coupling is local in the RG scale μ [7]

$$\mu \frac{\mathrm{d}g(\mu)}{\mathrm{d}\mu} = \beta(g(\mu)). \tag{2.2}$$

That is, one only needs to know the coupling at some scale μ to determine its evolution as a function of scale. The renormalization group then suggests a relationship between the emergent dimension in the bulk and the RG scale of the boundary field theory. In this thesis we study much simpler instances of gauge/gravity duality than the one involved in the original Maldacena conjecture — including those with broken AdS isometries in the bulk corresponding to broken conformal invariance in the boundary field theory — in an attempt to make this relationship more precise.

2.1 The geometry of (asymptotically) anti-de Sitter space

Anti-de Sitter space in $d+1$ dimensions is a maximally-symmetric solution to Einstein's equations with negative cosmological constant $\Lambda = -\frac{d(d-1)}{L^2_{\mathrm{AdS}}}$. It is otherwise known as the Lorentzian version of hyperbolic space $\mathbb{H}^{d+1}$, and can be thought of as a hyperboloid embedded in $\mathbb{R}^{d,2}$

$$-X_0^2 - X_{d+1}^2 + X_1^2 + \ldots + X_d^2 = -L^2_{\mathrm{Ads}}. \tag{2.3}$$

In this formulation, it is clear that the symmetry group of AdS_{d+1} is $SO(d,2)$ — precisely the conformal group in d-dimensional Minkowski space.[2] One can introduce *global coordinates* by parameterizing the embedding coordinates in the following way

$$\begin{aligned} X_0 &= L_{\mathrm{AdS}} \cosh\rho \cos\tau \\ X_{d+1} &= L_{\mathrm{AdS}} \cosh\rho \sin\tau \\ X_i &= L_{\mathrm{AdS}} \sinh\rho \ \Omega_i, \end{aligned} \tag{2.4}$$

where $\sum_{i=1}^{d} \Omega_i^2 = 1$ and L_{AdS} is the 'AdS length,' leading to the following line element[3]

$$\mathrm{d}s^2 = L_{\mathrm{AdS}}^2 \left(-\cosh^2\rho \ \mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sinh^2\rho \ \mathrm{d}\Omega_{d-1}^2 \right). \tag{2.5}$$

By introducing the coordinate $\theta \in [0, \pi/2)$ through $\sinh\rho = \tan\theta$, the metric becomes

$$\mathrm{d}s^2 = \frac{L_{\mathrm{AdS}}^2}{\cos^2\theta}(-\mathrm{d}\tau^2 + \mathrm{d}\theta^2 + \sin^2\theta \ \mathrm{d}\Omega_{d-1}^2), \tag{2.6}$$

and it is manifest that AdS_{d+1} is conformally equivalent to half of an Einstein static universe ($\mathbb{R} \times S^d$), for which $\theta \in [0, \pi]$. Crucially, we see that the boundary (approached as $\theta \to \pi/2$) of the conformal compactification of AdS_{d+1} is equivalent to the conformal compactification of Minkowski space in one less dimension. Indeed, we define a spacetime to be asymptotically AdS (AAdS) if it admits a conformally compact Einstein metric.

By now it should be clear that there are some curious features of general relativity in AdS_{d+1}. It is often said that one can think of anti-de Sitter space as a box.[4]

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[2] There is a subtlety in the case of $d = 2$, where the asymptotic symmetry group of AdS_3 is enlarged to two copies of the Virasoro algebra [10], corresponding to the infinite-dimensional conformal group in two dimensions.

[3] In order to preserve a sensible causal structure and avoid closed timelike curves, one works with the manifold's covering space where τ is not periodically identified.

[4] Thanks to Alex Maloney for repeatedly emphasizing this point.

Indeed, the curvature of the spacetime acts like an effective negative potential for massive observers, such that massive geodesics are bounded away from the boundary of the spacetime. Furthermore, the introduction of the AdS length L_{AdS} as a scale inherent to the spacetime means that anti-de Sitter space provides a natural arena for the realization of the holographic principle as imagined by 't Hooft and Susskind. It is straightforward to show that in a fixed-τ slice of the global coordinates (2.5) the volume inside a near-boundary cutoff region $\rho \leq \rho_0$ scales as the area of that region, that is

$$\lim_{\rho_0 \to \infty} \frac{V(\rho_0)}{A(\rho_0)} = \frac{L_{\text{AdS}}}{2^{d-2}(d-1)} < \infty. \tag{2.7}$$

We conclude this section by introducing yet another set of coordinates that will prove especially useful in this thesis. The *Poincaré* coordinates ($z > 0$, $(t, x^i) \in \mathbb{R}^{d-1,1}$) are defined through the embedding coordinates

$$\begin{aligned} X_0 &= \frac{z}{2}\left(1 + \frac{L_{\text{AdS}}^2 + \vec{x}^2 - t^2}{z^2}\right) \\ X_{d+1} &= \frac{L_{\text{AdS}} t}{z} \\ X_d &= \frac{z}{2}\left(1 + \frac{L_{\text{AdS}}^2 - \vec{x}^2 + t^2}{z^2}\right) \\ X_i &= \frac{L_{\text{AdS}} x^i}{z} \quad (i \in \{1, \ldots, d-1\}), \end{aligned} \tag{2.8}$$

and cover one half of the hyperboloid (2.3). The line element in these coordinates is given by

$$\mathrm{d}s^2 = \frac{L_{\text{AdS}}^2}{z^2}\left(\mathrm{d}z^2 - \mathrm{d}t^2 + \mathrm{d}\vec{x}^2\right) = \frac{L_{\text{AdS}}^2}{z^2}(\mathrm{d}z^2 + \mathrm{d}x^\nu \mathrm{d}x_\nu), \tag{2.9}$$

such that the boundary of the spacetime is approached as $z \to 0$. These coordinates make manifest the invariance under d-dimensional Poincaré transformations of the boundary coordinates x^μ and transformations $(z, t, x^i) \to \lambda(z, t, x^i)$ $(\lambda > 0)$ corresponding to dilatations of the conformal symmetry group of the boundary. It is easy to check that the metric (2.9) also enjoys the discrete isometry of inversion

$$z \to L_{\text{AdS}}^2 \frac{z}{z^2 + x^\mu x_\mu}, \quad x^\mu \to L_{\text{AdS}}^2 \frac{x^\mu}{z^2 + x^\nu x_\nu}. \tag{2.10}$$

This fact will prove quite convenient when we compute CFT_d correlation functions via AdS_{d+1} gravity.

2.2 The wave equation in AdS

We consider the following toy model of a scalar field ϕ in a Euclidean AdS_{d+1} background, with action

$$S = \frac{1}{8\pi G_N} \int \mathrm{d}z \mathrm{d}^d x \sqrt{g} \left(\frac{1}{2} g^{ab} \partial_a \phi \partial_b \phi + \frac{1}{2} m^2 \phi^2 \right), \tag{2.11}$$

leading to the usual Klein-Gordon equation of motion

$$\left(\frac{1}{\sqrt{g}} \partial_a (\sqrt{g} g^{ab} \partial_b \phi) - m^2 \right) \phi = 0. \tag{2.12}$$

Fourier transforming all but the radial coordinate (that is, setting $\phi(z,x) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} e^{ik\cdot x} \phi_k(z)$), leads to the following differential equation for the Fourier coefficients

$$\left(z^2 \partial_z^2 - (d-1) z \partial_z - (m^2 L_{\mathrm{AdS}}^2 + k^2 z^2) \right) \phi_k(z) = 0. \tag{2.13}$$

The two independent solutions to this equation are distinguished by their asymptotic behaviour as $z \to 0$,

$$\phi \to \phi_{(-)} z^{\Delta_-} + \phi_{(+)} z^{\Delta_+} \tag{2.14}$$

where $\Delta_\pm$ are the solutions to

$$\Delta(\Delta - d) = m^2 L_{\mathrm{AdS}}^2 \to \Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2 L_{\mathrm{AdS}}^2}. \tag{2.15}$$

The solutions corresponding to Δ_+ and Δ_- are often referred to as the 'normalizable' and 'non-normalizable' modes respectively; indeed it is straightforward to check that the action (2.11) evaluated on a solution with asymptotic behaviour $\phi \sim z^\Delta$ is only finite for $\Delta \geq \frac{d}{2}$. However, scalar fields with negative mass-squared are allowed so long as the mass satisfies the 'Breitenlohner-Freedman' (BF) bound

[11, 12]: $m^2 L_{\rm AdS}^2 \geq -\frac{d^2}{4}$.[5] In fact, integrating the original action (2.11) by parts, one finds a boundary term that is finite for $\Delta \leq \frac{d}{2}$. In removing this boundary term, one changes the original action by a term that leaves the equations of motion intact. In this case, solutions with asymptotic behaviour $\phi \sim z^{\Delta}$ are normalizable for

$$\Delta \geq \frac{d-2}{2}. \tag{2.16}$$

Note that this is exactly the unitarity bound for the dimension of a scalar operator in quantum field theory!

Indeed, the correspondence (2.1) implies that the bulk scalar ϕ is dual to a scalar primary operator $\mathcal{O}$ of dimension Δ in the spectrum of the dual conformal field theory. In d dimensions, such operators transform under conformal transformations[6]

$$\vec{x} \to \vec{x}'(\vec{x}), \quad g_{ab}(\vec{x}) \to g'_{ab}(\vec{x}') = \Omega(\vec{x}) g_{ab}(\vec{x}) \tag{2.17}$$

as

$$\mathcal{O}(\vec{x}) \to \mathcal{O}'(\vec{x}') = \left| \frac{\partial \vec{x}'}{\partial \vec{x}} \right|^{-\Delta/d} \mathcal{O}(\vec{x}) \tag{2.18}$$

where the Jacobian of the transformation is related to the scale factor via $\left| \frac{\partial \vec{x}'}{\partial \vec{x}} \right| = \Omega^{-d/2}$. For $\frac{d-2}{2} \leq \Delta < \frac{d}{2}$ we identify $\phi_{(+)}$ as the source for the dual operator, $\phi_{(-)}$ as the source for dual operators of dimension $\Delta > \frac{d}{2}$. From here onwards, unless otherwise stated we will assume $\Delta > \frac{d}{2}$; we refer to Δ as the dimension of the dual operator and $\bar{\phi}$ as the source, so that $\phi \to \bar{\phi} z^{d-\Delta}$ as $z \to 0$. These elements of the operator-field correspondence will be elucidated further in the following section.

[5] One can think of this in the following way: the curvature of the spacetime contributes a potential such that a scalar field of mass m in AdS behaves like a scalar field in flat space with mass-squared $m^2 + \frac{d^2}{4L_{\rm AdS}^2}$.

[6] Here g_{ab} is the metric of the dual conformal field theory.

2.3 CFT correlation functions from gravity

2.3.1 A tale of two propagators

We are now in a position to present a comprehensible derivation of CFT_d 2- and 3-point functions from AdS_{d+1} gravity. It will be convenient to change conventions slightly so that the Euclidean Poincaré metric is given by

$$\mathrm{d}s^2 = \frac{L_{\text{AdS}}^2}{x_0^2}(\mathrm{d}x_0^2 + \mathrm{d}\vec{x}^2), \tag{2.19}$$

where $\vec{x} \in \mathbb{R}^d$. We seek a Green function that allows us to write the bulk field $\phi(x)$ in terms of its boundary value $\phi(x) \to \bar{\phi}(\vec{x})x_0^{d-\Delta}$, that is

$$\phi(x) = \int_{\partial M} \mathrm{d}^d y K_\Delta(x, \vec{y})\bar{\phi}(\vec{y}). \tag{2.20}$$

Obviously, such a *bulk-to-boundary propagator* must satisfy the Klein-Gordon equation

$$\left(\Box_x - m^2\right) K_\Delta(x, \vec{y}) = 0, \tag{2.21}$$

and reduce to a delta-function on the boundary

$$\lim_{x_0 \to 0} \frac{K_\Delta(x, \vec{y})}{x_0^{d-\Delta}} = \delta^d(\vec{x} - \vec{y}). \tag{2.22}$$

Witten's insight [3] was to apply inversion invariance (2.10). We begin by solving for K'_Δ with delta-function support at the boundary point P: $x_0 \to \infty$. Since both the boundary conditions and the metric are invariant under translations of $\vec{x}$, we know that K'_Δ is a function of x_0 only, and so the solution to (2.21) that vanishes at the boundary $x_0 = 0$ is given by $K'_\Delta(x_0) = c' x_0^\Delta$ for some constant c'. The inversion (2.10) maps the point P to the origin, and takes K'_Δ to

$$K'_\Delta \to K_\Delta(x, 0) = c\left(\frac{x_0}{x_0^2 + |\vec{x}|^2}\right)^\Delta, \tag{2.23}$$

for some other constant c. By integrating $\frac{K_\Delta}{x_0^{d-\Delta}}$ over the boundary, it is easy to check that the bulk-to-boundary propagator indeed has delta-function support on the boundary, with $c = c_\Delta = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta-\frac{d}{2})}$. By translation-invariance, the full bulk-to-boundary propagator is then given by

$$K_\Delta(x,\vec{y}) = c_\Delta \left(\frac{x_0}{x_0^2 + |\vec{x}-\vec{y}|^2}\right)^\Delta, \tag{2.24}$$

so that the solution to the homogeneous Klein-Gordon equation has the form

$$\phi_{(0)}(x,\vec{y}) = c_\Delta \int_{\partial M} \mathrm{d}^d y \left(\frac{x_0}{x_0^2 + |\vec{x}-\vec{y}|^2}\right)^\Delta \bar{\phi}(\vec{y}). \tag{2.25}$$

In the presence of interactions in the bulk, one must solve the sourced Klein-Gordon equation, that is

$$\left(\Box_x - m^2\right)\phi(x) = J(x), \tag{2.26}$$

where, for instance, for an interaction of the form $\frac{\lambda}{3}\phi^3$, $J(x) = \lambda\phi^2$. One can solve (2.26) by employing a *bulk-to-bulk propagator* $G_\Delta(x,y)$, where

$$\left(\Box_x - m^2\right) G_\Delta(x,y) = \frac{\delta^{d+1}(x-y)}{\sqrt{g}}, \tag{2.27}$$

via

$$\phi(x) = \int_M \mathrm{d}^{d+1}y\sqrt{g}G_\Delta(x,y)J(y). \tag{2.28}$$

Clearly G_Δ must respect the symmetries of anti-de Sitter space, and indeed the solution for G_Δ is given by a hypergeometric function of the 'chordal distance' ξ,

$$\begin{aligned} G_\Delta(x,y) &= \frac{c_\Delta}{2^\Delta(2\Delta-d)}\xi^\Delta {}_2F_1\left(\frac{\Delta}{2},\frac{\Delta+1}{2};\Delta-\frac{d}{2}+1;\xi^2\right) \\ \xi(x,y) &= \frac{2x_0y_0}{x_0^2+y_0^2+|\vec{x}-\vec{y}|^2}, \end{aligned} \tag{2.29}$$

where ${}_2F_1$ is a hypergeometric function. Furthermore, as one would expect, the boundary limit of the bulk-to-bulk propagator reproduces the bulk-to-boundary propagator

$$K_\Delta(x,\vec{y}) = \lim_{y_0\to 0}\frac{2\Delta-d}{y_0^\Delta}G_\Delta(x,y). \tag{2.30}$$

This allows us to organize our solution for the bulk field as a perturbative expansion in the interaction parameter. In the case of a $\frac{\lambda}{3}\phi^3$ interaction, this yields

$$\phi(x) = \phi_{(0)}(x) + \lambda \int \mathrm{d}^{d+1}y \sqrt{g} G_\Delta(x,y)\phi_{(0)}(y)^2 + \mathcal{O}(\lambda^2). \tag{2.31}$$

This should be very familiar from standard field theory, and indeed will lead to a diagrammatic perturbation theory analogous to Feynman diagrams in flat-space quantum field theory.

2.3.2 2- and 3-point functions

Approximating the gravitational partition function by the leading classical saddle

$$Z_{\text{grav}}\big|_{\phi \to \bar{\phi} x_0^{d-\Delta}} \approx e^{-S_{\text{grav}}[\bar{\phi}]}, \tag{2.32}$$

where S_{grav} is the action evaluated on a solution to the supergravity equations of motion (2.31), the correspondence (2.1) dictates that the on-shell action S_{grav} is the generating functional for connected correlation functions of the dual operator $\mathcal{O}(x)$ in the boundary conformal field theory; that is

$$S_{\text{grav}}[\bar{\phi}] = W[\bar{\phi}] = -\log\langle e^{-\int \mathrm{d}^d\vec{x}\bar{\phi}\mathcal{O}(\vec{x})}\rangle_{\text{CFT}}. \tag{2.33}$$

Equivalently, one may think of the boundary values of the bulk fields as sources for dual primary operators in the boundary CFT. In particular, this implies that the two-point function of the dual operator should be given simply in terms of the on-shell gravitational action by

$$\langle \mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle = -\left.\frac{\delta^2 S_{\text{grav}}[\bar{\phi}]}{\delta\bar{\phi}(\vec{x})\delta\bar{\phi}(\vec{y})}\right|_{\bar{\phi}=0}. \tag{2.34}$$

So we simply plug the solution (2.31) into the action and take derivatives with respect to the source — in this way we can obtain any boundary n-point function. Now, by construction, the bulk solution satisfies the equations of motion, so the action reduces to a boundary term, and we obtain

$$S_{\text{grav}}[\bar{\phi}] = -\frac{1}{2\kappa}\int \mathrm{d}^d x\sqrt{g}g^{00}\phi(x)\partial_0\phi(x)\bigg|_{x_0=\epsilon}, \tag{2.35}$$

where $x_0 = \epsilon$ is a regulator surface and $\kappa = 8\pi G_N$. The cutoff surface is necessary to regularize the divergence coming from the infinite volume of spacetime; we will see that it plays the role of a UV regulator in the dual field-theory. Plugging in the solution to the Klein-Gordon equation, this reduces to

$$\begin{aligned} S_{\text{grav}}[\bar{\phi}] &= -\frac{L_{\text{AdS}}^{d-1}}{2\kappa}\int \mathrm{d}^d y\int \mathrm{d}^d w\bar{\phi}(\vec{y})\bar{\phi}(\vec{w})\left(\int \mathrm{d}^d x\frac{K_\Delta(x,\vec{y})x_0\partial_0 K_\Delta(x,\vec{w})}{x_0^d}\bigg|_{x_0=\epsilon}\right) \\ &\equiv -\frac{L_{\text{AdS}}^{d-1}}{2\kappa}\int \mathrm{d}^d y\int \mathrm{d}^d w\bar{\phi}(\vec{y})\bar{\phi}(\vec{w})\mathcal{G}_{2,\epsilon}(\vec{y},\vec{w}). \end{aligned} \tag{2.36}$$

Let us first consider the case where $|\vec{x}-\vec{y}| > 0$. Making use of the known form of the bulk-to-boundary propagator (2.24), it is then easy to see that the boundary two-point function is given by

$$\begin{aligned} \langle\mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle &= \frac{1}{\kappa}L_{\text{AdS}}^{d-1}\mathcal{G}_{2,\epsilon}(\vec{x},\vec{y}) \\ &= \frac{1}{\kappa}L_{\text{AdS}}^{d-1}\int \mathrm{d}^d z\frac{1}{z_0^d}\left(z_0^{d-\Delta}\delta^d(\vec{z}-\vec{x})+\dots\right)\left((d-\Delta)z_0^{d-\Delta}\delta^d(\vec{z}-\vec{y})+\frac{\Delta c_\Delta z_0^\Delta}{|\vec{z}-\vec{y}|^{2\Delta}}+\dots\right)\bigg|_{z_0=\epsilon} \\ &= \frac{L_{\text{AdS}}^{d-1}\Delta c_\Delta}{\kappa|\vec{x}-\vec{y}|^{2\Delta}}+\mathcal{O}(\epsilon^2). \end{aligned} \tag{2.37}$$

This is precisely the form, entirely fixed by conformal symmetry, of a two-point function of a CFT primary operator of dimension Δ! This corroborates our intuition about the operator-field correspondence. However, when $|\vec{x}-\vec{y}| \to 0$ there is a subtlety. In this case we have[7]

$$\langle \mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle = \frac{1}{\kappa}L_{\text{AdS}}^{d-1}\left((d-\Delta)\epsilon^{d-2\Delta}\delta^d(\vec{x}-\vec{y}) + \frac{c_\Delta \Delta}{|\vec{x}-\vec{y}|^{2\Delta}} + \dots\right). \tag{2.38}$$

While the second term represents the familiar power-law divergence of the two-point function at vanishing separation and we have suppressed terms that vanish as $\epsilon \to 0$, the first term diverges as $\epsilon \to 0$ and is called a *divergent contact term*. Such terms are scheme-dependent and must be cancelled by the addition of covariant boundary counterterms to the action; they leave the equations of motion and finite-separation two-point function invariant. This is the program of *holographic renormalization* and will be discussed in more detail in chapter 3.

In fact, there are some subtleties with the position-space derivation of the two-point function in AdS with a cutoff; the propagators are determined by the symmetries of AdS (in particular, recall that invariance under inversion was used to derive K_Δ), however the cutoff spacetime does not inherit all the symmetries of AdS. This causes the result (2.37) to differ from what we will obtain via holographic renormalization (or would have gotten by finding the exact solution to the equations of motion in momentum space and treating the $\epsilon \to 0$ limit carefully as is done in [13] see also [8, 6]) by a factor of $\frac{\Delta}{2\Delta - d}$.

To illustrate how to compute CFT n-point functions from a bulk theory with interactions, consider a theory of three scalars $\{\phi_1, \phi_2, \phi_3\}$ in AdS_{d+1} with interaction term $\lambda\phi_1\phi_2\phi_3$ dual to scalar operators $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ of conformal dimensions Δ_1, Δ_2 and Δ_3 respectively. In this case, the relevant functional derivative of the on-shell action

[7] We note that we could have reached the same conclusion with regards to boundary divergences in the form of contact terms by working in momentum space and Fourier transforming back to position space; a careful treatment is given in [13].

yields

$$
\begin{aligned}
\langle \mathcal{O}_1(\vec{x})\mathcal{O}_2(\vec{y})\mathcal{O}_3(\vec{z})\rangle &= \left.\frac{\delta^3 S_{\text{grav}}[\bar{\phi}]}{\delta\bar{\phi}_1(x)\delta\bar{\phi}_2(y)\delta\bar{\phi}_3(z)}\right|_{\bar{\phi}=0} \\
&= \frac{\lambda}{\kappa}\int \mathrm{d}^{d+1}w\sqrt{g}K_{\Delta_1}(w,\vec{x})K_{\Delta_2}(w,\vec{y})K_{\Delta_3}(w,\vec{z}) + \mathcal{O}(\lambda^2) \\
&= \frac{\lambda L_{\text{AdS}}^{d+1}c_3}{\kappa|\vec{x}-\vec{y}|^{\Delta_1+\Delta_2-\Delta_3}|\vec{y}-\vec{z}|^{\Delta_2+\Delta_3-\Delta_1}|\vec{z}-\vec{x}|^{\Delta_3+\Delta_1-\Delta_2}} + \mathcal{O}(\lambda^2).
\end{aligned}
\tag{2.39}
$$

This has precisely the form of a three-point function of primary operators of dimensions Δ_1, Δ_2 and Δ_3 in a conformal field theory; like the two-point function, the form of the three-point function is also determined up to a constant by conformal symmetry. Here, c_3 is a constant that can be computed using simple symmetry arguments and by performing integrals over Feynman parameters [13, 8]

$$
c_3 = \frac{1}{2\pi^d}\frac{\Gamma\left(\frac{\Delta_1+\Delta_2-\Delta_3}{2}\right)\Gamma\left(\frac{\Delta_2+\Delta_3-\Delta_1}{2}\right)\Gamma\left(\frac{\Delta_3+\Delta_1-\Delta_2}{2}\right)\Gamma\left(\frac{\Delta_1+\Delta_2+\Delta_3-d}{2}\right)}{\Gamma(\Delta_1-\frac{d}{2})\Gamma(\Delta_2-\frac{d}{2})\Gamma(\Delta_3-\frac{d}{2})}. \tag{2.40}
$$

2.3.3 Witten diagrams and a final bit of lore explained

The examples given above of the computation of the 2- and 3-point functions were meant to illustrate the general approach one takes to compute correlation functions of operators in the dual conformal field theory from gravity in anti-de Sitter space. Evaluating the on-shell action $S_{\text{grav}}[\bar{\phi}]$ using the perturbative solution to the supergravity equations of motion analogous to (2.31) and expanding in powers of $\bar{\phi}$ leads to a diagrammatic framework for the computation of CFT correlation functions. The rules for drawing the relevant *Witten diagrams* and evaluating CFT n-point functions are the following:

- Draw a disc with n insertions on the boundary. These boundary points $\{\vec{x}_1, \ldots, \vec{x}_n\}$ are the locations of the dual CFT operators in $\mathbb{R}^d$.
- Connect the boundary insertions with all the possible vertices consistent with the bulk interactions as in flat-space field theory Feynman diagrams.

- Each line connecting a boundary point with a bulk vertex is accompanied by a bulk-to-boundary propagator K_Δ, where Δ is the dimension of the operator inserted on the boundary.
- Each bulk line connecting vertices is accompanied by a bulk-to-bulk propagator G_Δ, where Δ is the dimension of the operator dual to the bulk field being exchanged.
- Each bulk vertex is accompanied by the coupling constant and combinatorial factor as dictated by the corresponding bulk interaction as in Feynman diagrams.
- Integrate all bulk vertices over AdS_{d+1} as $\int \mathrm{d}^{d+1}x\sqrt{g(x)}$.

It is often said that the coefficient of the subleading ('normalizable') solution for $\phi(x)$ as $x_0 \to 0$ gives the one-point function of the dual operator $\mathcal{O}$ in the presence of the source. Here we review evidence for this claim, due to [14]. The near boundary behaviour of ϕ is given by

$$\phi(x) \to x_0^{d-\Delta}(\bar{\phi}(\vec{x}) + \mathcal{O}(x_0^2)) + x_0^{\Delta}(F(\vec{x}) + \mathcal{O}(x_0^2)). \tag{2.41}$$

The claim is that

$$\langle \mathcal{O}(\vec{x})\rangle_{\bar{\phi}} = \langle \mathcal{O}(\vec{x})e^{-\int \mathrm{d}^d z \bar{\phi}\mathcal{O}(z)}\rangle_{\mathrm{CFT}} = -(2\Delta - d)F(\vec{x}). \tag{2.42}$$

Note that knowledge of the one-point function circumvents the need for the on-shell action; by taking appropriate functional derivatives of $\langle \mathcal{O}(\vec{x})\rangle_{\bar{\phi}}$ with respect to the source $\bar{\phi}$, one obtains all higher-point functions of $\mathcal{O}$. The one-point function of the operator $\mathcal{O}(\vec{x})$ in the presence of the source $\bar{\phi}$ is given by the sum over all Witten diagrams where a bulk-to-boundary propagator K_Δ connects the boundary point $\vec{x}$ to the rest of the diagram with all sources placed on the boundary and integrated over:[8]

$$\langle \mathcal{O}(\vec{x})\rangle_{\bar{\phi}} = -\int \mathrm{d}^d y \bar{\phi}(\vec{y})\langle \mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\rangle_{\mathrm{CFT}} + \frac{1}{2}\int \mathrm{d}^d y \bar{\phi}(\vec{y})\int \mathrm{d}^d z \bar{\phi}(\vec{z})\langle \mathcal{O}(\vec{x})\mathcal{O}(\vec{y})\mathcal{O}(\vec{z})\rangle_{\mathrm{CFT}} + \dots$$
$$\equiv -\int \mathrm{d}^{d+1} y \sqrt{g} K_\Delta(y, \vec{x})\mathcal{J}(y), \tag{2.43}$$

where $\mathcal{J}(y)$ captures the sum of such diagrams. Similarly, the classical bulk field is given by a sum over the exact same diagrams, except one replaces the bulk-to-boundary propagator with a bulk-to-bulk propagator G_Δ ending at the location of the bulk field. So, making use of the relationship between the two kinds of propagators (2.30), we find

$$\begin{aligned} \lim_{x_0 \to 0} \frac{\phi(x)}{x_0^\Delta} &= \lim_{x_0 \to 0} \frac{1}{x_0^\Delta}\int \mathrm{d}^{d+1} y \sqrt{g} G_\Delta(x, y)\mathcal{J}(y) \\ &= \frac{1}{2\Delta - d}\int \mathrm{d}^{d+1} y \sqrt{g} K_\Delta(y, \vec{x})\mathcal{J}(y) \\ &= -\frac{1}{2\Delta - d}\langle \mathcal{O}(\vec{x})\rangle_{\bar{\phi}}, \end{aligned} \tag{2.44}$$

as advertised. Note that in the expression above, we must be careful to take $x_0 \to 0$ away from the source insertions implicit on the right-hand side.

[8] In what follows we have suppressed all factors of L_{AdS}.

Chapter 3
How to renormalize holographically

We have seen in the previous chapter that in computing CFT$_d$ n-point functions from AdS$_{d+1}$ gravity, one generally encounters short-distance divergences that must be regulated; as in standard quantum field theory, one must understand how to renormalize the theory. A general feature of the gauge/gravity correspondence is the relationship between ultraviolet (UV) divergences in the boundary field theory and infrared (IR) divergences in the bulk gravitational theory [15]. In fact, we have already seen an example of this; in computing the CFT two-point function, the near-boundary regulator surface $x_0 = \epsilon$ ended up playing the role of a short-distance regulator in the boundary field theory. In general, in computing correlation functions holographically

$$\langle \mathcal{O}(x_1) \ldots \mathcal{O}(x_n) \rangle = (-1)^{n-1} \frac{\delta^n S_{\rm grav}[\bar{\phi}]}{\delta\bar{\phi}(x_1) \cdots \delta\bar{\phi}(x_n)}, \tag{3.1}$$

both sides of the dictionary require regularization; field theory correlation functions suffer from short-distance divergences, and the right-hand side also diverges due to the infinite volume of spacetime. In field theory, the IR physics has no bearing on the cancellation of UV divergences; furthermore, if the structure of the counterterms used to render correlation functions UV-finite respects some symmetry of the action, then the Ward identity corresponding to that symmetry is non-anomalous. In the bulk theory, this lends the intuition that a *near-boundary analysis* should determine holographic Ward identities and would therefore be sensitive to anomalies. As in field theory, we will cancel the IR divergences in the bulk via covariant counterterms.

In this chapter we show how to obtain *renormalized* field theory correlation functions by regularizing the IR divergences in the bulk gravity theory. This program was actually initiated in a holographic computation of the field theory Weyl anomaly

[16] (a computation we review in §3.4), and was further developed and systematized in [17, 18, 19]. Useful lecture notes are collected in [20, 8], which we follow closely here. Later on, we will introduce the dual geometries of field theories obtained by deformation of a CFT that admits a gravitational dual. We'll see that radial evolution in these spacetimes is related to the renormalization group flow of the boundary theory; in particular, the Hamilton-Jacobi equation for the gravity-scalar system will lead to the Callan-Symanzik equation for renormalized boundary correlation functions.

3.1 Holographic renormalization via counterterms

3.1.1 General formalism

We begin by recalling that an asymptotically anti-de Sitter spacetime admits a conformally compact Einstein metric. In this case, we can take the defining function (that is, the conformal factor relating the AAdS line element to that of the Einstein static universe) to be the radial coordinate, so that the AAdS line element can be written as

$$\mathrm{d}s^2 = \frac{L_{\mathrm{AdS}}^2}{x_0^2}(\mathrm{d}x_0^2 + g_{ij}(x)\mathrm{d}x^i\mathrm{d}x^j), \tag{3.2}$$

where from the above definition and a theorem from differential geometry due to Fefferman and Graham [21], g_{ij} is smooth as we approach the boundary $x_0 \to 0$

$$g_{ij}(x) = g_{(0)ij}(\vec{x}) + x_0 g_{(1)ij}(\vec{x}) + x_0^2 g_{(2)ij}(\vec{x}) + \ldots \tag{3.3}$$

Solving the Einstein equations order-by-order in x_0 fixes the higher $g_{(k)}$ in terms of $g_{(0)}$, and it is straightforward to see that all $g_{(n)}$ vanish for n odd up to $\mathcal{O}(x_0^d)$. For this reason it is convenient to introduce the coordinate $\rho = x_0^2$ so that the metric has the form

$$\begin{aligned} \mathrm{d}s^2 &= L_{\mathrm{AdS}}^2\left(\frac{\mathrm{d}\rho^2}{4\rho^2} + \frac{1}{\rho}g_{ij}(\rho,\vec{x})\mathrm{d}x^i\mathrm{d}x^j\right) \\ g_{ij}(\rho,\vec{x}) &= g_{(0)} + \rho g_{(2)} + \ldots + \rho^{\frac{d}{2}}g_{(d)} + h_{(d)}\rho^{\frac{d}{2}}\log\rho + \ldots, \end{aligned} \tag{3.4}$$

where $h_{(d)}$ only appears in even dimensions and is related to the variation of the conformal anomaly. In a near-boundary-analysis, ρ/L_{AdS}^2 is to be thought of as a small parameter that is taken to zero. Such metrics are asymptotically AdS in the sense that the curvature satisfies

$$R_{\mu\nu\rho\sigma}[g] = g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho} + \mathcal{O}(\rho). \tag{3.5}$$

In fact, the Einstein equations determine $\{g_{(2)}, \ldots, g_{(d-2)}, h_{(d)}\}$ in terms of $g_{(0)}$, but only determine the trace and divergence of $g_{(d)}$.

Following [20], we designate all bulk fields as $\mathcal{F}(\rho, \vec{x})$. The field's equations of motion are second-order in ρ, and so there are two independent solutions with asymptotic behaviour as $\rho \to 0$ of ρ^m and ρ^{m+n} — we have seen this explicitly in the case of a massive scalar in AdS in section 2.2 (where $m = \frac{d-\Delta}{2}$ and $n = \Delta - \frac{d}{2}$). The bulk field behaves asymptotically as

$$\mathcal{F}(\rho, \vec{x}) = \rho^m \left(f_{(0)}(\vec{x}) + \rho f_{(2)}(\vec{x}) + \ldots + \rho^n (f_{(2n)}(\vec{x}) + \tilde{f}_{(2n)}(\vec{x}) \log \rho) + \ldots \right), \tag{3.6}$$

where the logarithmic term is only present if n is integral. We have seen that the coefficient of the leading behaviour as $\rho \to 0$, $f_{(0)}$, is to be thought of as the source for the dual operator on the boundary, while $f_{(2n)}$ is related to the one-point function of the dual operator in the presence of the source $\langle \mathcal{O} \rangle_{f_{(0)}}$. Furthermore, the bulk equations of motion determine $\{f_{(2)}, f_{(4)}, \ldots, f_{(2n-2)}\}$ in terms of $f_{(0)}$, leaving $f_{(2n)}$ undetermined.[1] We will also see that $\tilde{f}_{(2n)}$ is related to conformal anomalies in the boundary theory.

We define the regularized on-shell action by restricting the region of integration to $\rho \in [\epsilon, \infty)$, evaluating boundary terms on the regulator slice $\rho = \epsilon$. As we've seen, a finite number of terms will diverge as we remove the cutoff $\epsilon \to 0$, so the on-shell

[1] This is to be expected, given that $f_{(2n)}$ is the coefficient of the leading asymptotic behaviour of a solution independent from that whose leading behaviour is given by $f_{(0)}$.

action can be written as

$$S_{\text{reg},\epsilon}[f_{(0)}] = \int \mathrm{d}^d x \sqrt{g_{(0)}} \left[\epsilon^{-\nu} \left(a_{(0)} + \epsilon a_{(2)} + \ldots - \epsilon^{\nu} \log \epsilon a_{(2\nu)} \right) + \mathcal{O}(\epsilon^0)) \right] \Big|_{\rho=\epsilon}, \quad (3.7)$$

where ν is some positive number that characterizes the leading divergence and the $\{a_{(k)}\}$ can be determined in terms of $f_{(0)}$. From this the counterterm action can immediately be defined to subtract the divergent terms in $S_{\text{reg},\epsilon}$; to do so in terms of the bulk fields requires that we solve for the source in terms of the bulk field from (3.6). The *renormalized* on-shell action is then defined as the finite part of the regularized action as ϵ is sent to zero:

$$S_{\text{ren}}[f_{(0)}] = \lim_{\epsilon \to 0} \left(S_{\text{reg},\epsilon}[f_{(0)}] + S_{\text{ct},\epsilon}[\mathcal{F}(\epsilon, \vec{x})] \right). \quad (3.8)$$

We then define the renormalized one-point function in the presence of the source $f_{(0)}$ of the operator $\mathcal{O}$ dual to the bulk field $\mathcal{F}$ in the obvious way

$$\langle \mathcal{O}(\vec{x}) \rangle_{f_{(0)}} = \frac{1}{\sqrt{g_{(0)}}} \frac{\delta S_{\text{ren}}[f_{(0)}]}{\delta f_{(0)}(\vec{x})}. \quad (3.9)$$

Note that we do not take the source to zero after taking the derivative. Equivalently, we can functionally differentiate the subtracted action with respect to the value of the bulk-field evaluated at the regulator slice *and then* take $\epsilon \to 0$

$$\langle \mathcal{O}(\vec{x}) \rangle_{f_{(0)}} = \lim_{\epsilon \to 0} \left(\epsilon^{m - \frac{d}{2}} \frac{L_{\text{AdS}}^d}{\sqrt{\gamma}} \frac{\delta(S_{\text{reg},\epsilon}[f_{(0)}] + S_{\text{ct},\epsilon}[\mathcal{F}(\epsilon, \vec{x})])}{\delta \mathcal{F}(\epsilon, \vec{x})} \right), \quad (3.10)$$

where $\gamma_{ij} = L_{\text{AdS}}^2 \frac{g_{ij}(\epsilon, \vec{x})}{\epsilon}$ is the induced metric on the regulator surface. As our earlier intuition suggests, $\langle \mathcal{O}(\vec{x}) \rangle_{f_{(0)}}$ is indeed related to the coefficient $f_{(2n)}$. In particular,

$$\langle \mathcal{O}(\vec{x}) \rangle_{f_{(0)}} = \alpha_{\mathcal{O}} f_{(2n)} + F_{\mathcal{O}}(f_{(0)}), \quad (3.11)$$

where $\alpha_{\mathcal{O}}$ is some theory-dependent coefficient and $F_{\mathcal{O}}$ is some local, scheme-dependent function of the source $f_{(0)}$.

Once one obtains the renormalized one-point functions, it is straightforward to establish the holographic Ward identities. We are particularly interested in the renormalized stress-tensor one-point function

$$\begin{aligned}\langle T(\vec{x})_{ij}\rangle_{g_{(0)}} &= \frac{2}{\sqrt{g_{(0)}}}\frac{\delta S_{\rm ren}}{\delta g^{ij}_{(0)}} \\ &= \lim_{\epsilon\to 0}\left(\frac{L^{d-2}_{\rm AdS}}{\epsilon^{\frac{d}{2}-1}}T_{ij}[\gamma]\right),\end{aligned} \tag{3.12}$$

where $T_{ij}[\gamma]$ is the stress tensor on the regulator slice $\rho=\epsilon$. A computation in exact analogy with the one performed above yields

$$\langle T(\vec{x})_{ij}\rangle_{g_{(0)}} = \alpha_T g_{(d)ij} + F_T(g_{(0)ij}). \tag{3.13}$$

Since the Einstein equations determine the trace and divergence of $g_{(d)ij}$, this is enough to establish holographic Ward identities.

3.1.2 Concrete example: massive scalar

For the moment we provide an explicit example of the formalism described above in the case of a massive scalar in AdS. Recall that AdS corresponds to $g_{(0)ij}=\delta_{ij}$, $g_{(k)ij}=0\ \forall\ k>0$; this choice of coordinates for AdS is referred to as *Fefferman-Graham* coordinates. We've seen that the bulk equations of motion in this case can be solved by an ansatz of the form

$$\begin{aligned}\phi(\rho,\vec{x}) &= \rho^{\frac{d-\Delta}{2}}\left(\phi_{(0)}(\vec{x})+\rho\phi_{(2)}(\vec{x})+\ldots+\rho^{\Delta-\frac{d}{2}}\phi_{(2\Delta-d)}(\vec{x})+\ldots\right) \\ &\equiv \rho^{\frac{d-\Delta}{2}}\varphi(\rho,\vec{x}).\end{aligned} \tag{3.14}$$

The idea is to plug this ansatz into the equations of motion and then solve for $\phi_{(n)}$ recursively, leading to a formula in terms of $\phi_{(0)}$ and derivatives. The equation of motion then reads

$$\left\{m^2-\Delta(\Delta-d)-\rho\left[\Box_0+2(d-2\Delta+2)\partial_\rho\right]-4\rho^2\partial_\rho^2\right\}\varphi(\rho,\vec{x})=0, \tag{3.15}$$

where $\Box_0$ is just the usual Laplacian on the boundary, ie. $\delta^{ij}\partial_i\partial_j$. Setting ρ to zero just yields the usual relationship between m^2 and Δ. Solving the Klein-Gordon equation at $\mathcal{O}(\rho)$ requires

$$\phi_{(2)}(\vec{x}) = \frac{1}{2\Delta - d - 2}\Box_0\phi_{(0)}. \tag{3.16}$$

Continuing in this fashion and requiring that the coefficient at $\mathcal{O}(\rho^{n-1})$ vanishes leads to

$$\begin{aligned}\phi_{(2n)} &= \frac{1}{2n(2\Delta - d - 2n)}\Box_0\phi_{(2n-2)} \\ &= \frac{\Gamma(2\Delta - \frac{d}{2} - n)}{2^{2n}n!\Gamma(2\Delta - \frac{d}{2})}(\Box_0)^n\phi_{(0)}.\end{aligned} \tag{3.17}$$

Obviously, we must be careful if $\Delta = \frac{d}{2} + k$, where $k \in \mathbb{Z}$. In this case, one must take care to include a logarithmic term in the asymptotic expansion (3.14), that is

$$\varphi(\rho, \vec{x}) = \phi_{(0)}(\vec{x}) + \rho\phi_{(2)}(\vec{x}) + \ldots + \rho^k\phi_{(2k)}(\phi_{(2k)} + \psi_{(2k)}\log\rho) + \ldots \tag{3.18}$$

In this case, $\phi_{(2k)} = \phi_{(2\Delta-d)}$ is no longer determined by the equations of motion. It is straightforward to show that $\psi_{(2k)}$ is determined in terms of the source as

$$\begin{aligned}\psi_{(2k)} &= -\frac{1}{4k}\Box_0\phi_{(2k-2)} \\ &= -\frac{1}{2^k k!(k-1)!}\Box_0^k\phi_{(0)}.\end{aligned} \tag{3.19}$$

We now evaluate the on-shell action using this asymptotic solution to the equations of motion, again evaluating the boundary term on a regulator surface $\rho = \epsilon$:

$$\begin{aligned}S_{\text{reg}} &= -\frac{1}{2\kappa}\int \mathrm{d}^d x\sqrt{g}g^{\rho\rho}\phi\partial_\rho\phi\Big|_{\rho=\epsilon} \\ &= -\frac{L_{\text{AdS}}^{d-1}}{\kappa}\int \mathrm{d}^d x\ \epsilon^{\frac{d}{2}-\Delta}\left(\frac{d-\Delta}{2}\varphi^2 + \epsilon\varphi\partial_\rho\varphi\right)\Big|_{\rho=\epsilon} \\ &\equiv \frac{L_{\text{AdS}}^{d-1}}{\kappa}\int \mathrm{d}^d x\ \epsilon^{\frac{d}{2}-\Delta}\left(a_{(0)} + \epsilon a_{(2)} + \ldots - \epsilon^{\Delta-\frac{d}{2}}\log\epsilon\ a_{(2\Delta-d)} + \ldots\right),\end{aligned} \tag{3.20}$$

where the $a_{(k)}$ are local functions of $\phi_{(0)}$ that are easily computable from the asymptotic form of the bulk scalar (3.14) and the order-by-order solutions to the equations of motion (3.17). We find

$$\begin{aligned}
a_{(0)} &= -\frac{d-\Delta}{2}\phi_{(0)}^2 \\
a_{(2)} &= -(d-\Delta+1)\phi_{(0)}\phi_{(2)} = -\frac{d-\Delta+1}{2(d-\Delta-2)}\phi_{(0)}\Box_0\phi_{(0)} \\
&\dots \\
a_{(2\Delta-d)} &= \frac{d}{2}\phi_{(0)}\psi_{(2\Delta-d)} = -\frac{d}{2^{2k+1}\Gamma(k+1)\Gamma(k)}\phi_{(0)}(\Box_0)^k\phi_{(0)},
\end{aligned} \tag{3.21}$$

recalling that we have set $\Delta = \frac{d}{2}+k$. Obviously, the action diverges as $\epsilon \to 0$, so we must add diffeomorphism-invariant counterterms to renormalize the action. Doing so requires that we invert the series (3.14), that is, solve for the source $\phi_{(0)}$ in terms of the bulk field (since *this* is the object that transforms as a scalar under bulk diffeomorphisms) and its derivatives on the regulator surface. To second order in ϵ, we find (for $\Delta \neq \frac{d}{2}+1$)

$$\begin{aligned}
\phi_{(0)}(\vec{x}) &= \frac{1}{\epsilon^{\frac{d-\Delta}{2}}}\left(\phi(\epsilon,\vec{x}) - \frac{1}{2(d-\Delta-2)}\Box_\gamma\phi(\epsilon,\vec{x})\right) \\
\phi_{(2)}(\vec{x}) &= \frac{1}{\epsilon^{\frac{d-\Delta-2}{2}}}\frac{1}{2(d-\Delta-2)}\Box_\gamma\phi(\epsilon,\vec{x}) \\
&\dots,
\end{aligned} \tag{3.22}$$

where $\Box_\gamma = \gamma^{ij}\partial_i\partial_j$ is the Laplacian on the regulator surface. We then construct the counterterm action in order to subtract the divergent terms in the regulated action, that is

$$S_{\text{ct}} = \frac{L_{\text{AdS}}^{-1}}{\kappa}\int \mathrm{d}^d x\sqrt{\gamma}\left(\frac{d-\Delta}{2}\phi(\epsilon,\vec{x})^2 + \frac{1}{2(d-\Delta-2)}\phi(\epsilon,\vec{x})\Box_\gamma\phi(\epsilon,\vec{x}) + \dots\right). \tag{3.23}$$

Obviously, if $\Delta \in (\frac{d}{2}, \frac{d}{2}+1)$, then the two terms included above are the only ones required to renormalize the action — otherwise one must include higher derivative terms corresponding to the solutions for the higher-order $\{\phi_{(n)}\}$ which we have implicitly suppressed above. Furthermore, if $\Delta = \frac{d}{2}+k$, $k \in \mathbb{Z}$, then the coefficient of the $\phi(\Box_\gamma)^k\phi$ term will be proportional to $\log\epsilon$. Note that we are always free to add counterterms that are *finite* as $\epsilon \to 0$ — this corresponds to the scheme dependence of the finite part of correlation functions in the boundary field theory. We will not discuss the issue of scheme-dependence in any further detail here.

It is straightforward to show that the part of the regularized action that is finite as $\epsilon \to 0$ is given by

$$S_{\text{reg,finite}} = -\frac{L_{\text{AdS}}^{d-1} d}{2\kappa} \int \mathrm{d}^d x \; \phi_{(0)}(\vec{x}) \phi_{(2\Delta-d)}(\vec{x}). \tag{3.24}$$

Meanwhile, the finite part of the counterterm action is easily seen to be

$$S_{\text{ct,finite}} = \frac{L_{\text{AdS}}^{d-1}}{\kappa}(d-\Delta) \int \mathrm{d}^d x \; \phi_{(0)}(\vec{x}) \phi_{(2\Delta-d)}(\vec{x}). \tag{3.25}$$

We can then make use of the fact that the solution to the *exact* free equations of motion using the bulk-to-boundary propagator (2.25) allows us to write $\phi_{(2\Delta-d)}$ in terms of $\phi_{(0)}$ as

$$\phi_{(2\Delta-d)} = \lim_{x_0 \to 0} \frac{\phi(x)}{x_0^{\Delta}} = c_\Delta \int \mathrm{d}^d y \frac{\phi_{(0)}(\vec{y})}{|\vec{x}-\vec{y}|^{2\Delta}}. \tag{3.26}$$

This allows us to write the renormalized on-shell action as

$$S_{\text{ren}} = \kappa L_{\text{AdS}}^{d-1} c_\Delta \left(\frac{d}{2} - \Delta\right) \int \mathrm{d}^d x \int \mathrm{d}^d y \frac{\phi_{(0)}(\vec{x})\phi_{(0)}(\vec{y})}{|\vec{x}-\vec{y}|^{2\Delta}} + \ldots, \tag{3.27}$$

where the omitted terms are scheme-dependent and can be cancelled upon the addition of additional finite counterterms. Of course, this leads to the correct form of the CFT two-point function

$$\langle \mathcal{O}(\vec{x}) \mathcal{O}(\vec{y}) \rangle_{\phi_{(0)}} = \frac{\kappa L_{\text{AdS}}^{d-1}(2\Delta - d) c_\Delta}{|\vec{x}-\vec{y}|^{2\Delta}}. \tag{3.28}$$

Furthermore, the form of the renormalized action confirms the suspicion that the renormalized one-point function is given (up to a constant) by the coefficient of the asymptotic falloff of the normalizable solution in the bulk via (3.9)

$$\langle \mathcal{O}(\vec{x}) \rangle_{\phi_{(0)}} = -\frac{L_{\text{AdS}}^{d-1}}{\kappa}(2\Delta - d)\phi_{(2\Delta-d)} + F_{\mathcal{O}}(\phi_{(0)}), \tag{3.29}$$

where $F_{\mathcal{O}}$ is a local, scheme-dependent function of the source $\phi_{(0)}$. In the case that $\Delta - \frac{d}{2} \in \mathbb{Z}$ there will also be a term in the one-point function proportional to the coefficient of the logarithmic term $\psi_{(2\Delta-d)}$; however we've seen that the equations of motion allow us to relate $\psi_{(2\Delta-d)}$ in terms of $\phi_{(0)}$, so it can be absorbed into the definition of $F_{\mathcal{O}}$. Indeed, by adding a term of the form

$$\delta S_{\text{ct,finite}} \propto -\frac{1}{2^{2k-1}\Gamma(k)^2} \int \mathrm{d}^d x \phi_{(0)} (\Box_0)^k \phi_{(0)} = -\int \mathrm{d}^d x \sqrt{\gamma} \mathcal{A}_{\text{matter}} \tag{3.30}$$

where $\mathcal{A}_{\text{matter}}$ is the matter conformal anomaly, we cancel the contribution of $\psi_{(2\Delta-d)}$ to the one-point function.

We note that in the presence of interactions, one can solve for $\phi_{(2\Delta-d)}$ perturbatively in the interaction strength, making use of the relation $G_\Delta(x,y) = \frac{x_0^\Delta}{2\Delta-d} K_\Delta(y_0, \vec{x}-\vec{y}) + \mathcal{O}(x_0^{\Delta+2})$. One can thereby determine all higher n-point functions as

$$\langle \mathcal{O}(\vec{x}_1) \cdots \mathcal{O}(\vec{x}_n) \rangle_{\phi_{(0)}} = (-1)^n \frac{L_{\text{AdS}}^{d-1}(2\Delta - d)}{\kappa} \frac{\delta^{n-1} \phi_{(2\Delta-d)}(\vec{x}_1)}{\delta\phi_{(0)}(\vec{x}_2) \cdots \delta\phi_{(0)}(\vec{x}_n)}. \tag{3.31}$$

Finally, we point out that the fact that $\phi(\rho, \vec{x})$ is a scalar under bulk diffeomorphisms requires a particular transformation of the coefficients $\phi_{(k)}$ under the RG transformation $\rho \to \lambda^2 \rho'$, $\vec{x} \to \lambda \vec{x}'$. We find

$$\begin{aligned}
\phi'_{(0)}(\vec{x}') &= \lambda^{d-\Delta} \phi_{(0)}(\lambda \vec{x}') \\
&\cdots \\
\psi'_{(2\Delta-d)}(\vec{x}') &= \lambda^{\Delta} \psi_{(2\Delta-d)}(\lambda \vec{x}') \\
\phi'_{(2\Delta-d)}(\vec{x}') &= \lambda^{\Delta} \left(\phi_{(2\Delta-d)}(\lambda \vec{x}') + \log \mu^2 \psi_{(2\Delta-d)(\lambda \vec{x}')} \right).
\end{aligned} \tag{3.32}$$

In particular, this implies that the source $\phi_{(0)}$ transforms exactly as the coupling constant of an operator of dimension Δ should, that is

$$\lambda \frac{\partial}{\partial \lambda} \phi_{(0)}(\lambda \vec{x}') = (\Delta - d) \phi_{(0)}(\lambda \vec{x}'). \tag{3.33}$$

We've seen that $\phi_{(0)}$ plays the role of a source or coupling for an operator in the boundary field theory — indeed, the above equation, determined by a near-boundary analysis, has precisely the form of the leading term of a renormalization group equation describing the scale dependence of a coupling of an operator of dimension Δ. This is an idea that will be made more precise in §3.3. Furthermore, the one-point function, which serves to generate all higher n-point functions, transforms almost trivially

$$\langle\mathcal{O}(\vec{x}')\rangle'_{\phi_{(0)}} = \lambda^{\Delta}\left(\langle\mathcal{O}(\lambda\vec{x}')\rangle_{\phi_{(0)}} - (2\Delta-d)\log\mu^2\psi_{(2\Delta-d)}\right). \tag{3.34}$$

We've seen that we can cancel terms proportional to $\psi_{(2\Delta-d)}$ in the one-point function by adding counterterms proportional to the matter conformal anomaly to the bulk action. So we see that the one-point function (and thereby all higher n-point functions) transforms trivially under RG transformations (as dictated by the dimension Δ of the operator $\mathcal{O}$), up to the existence of a conformal anomaly.

The variation of the renormalized on-shell action follows from its definition

$$\delta S_{\rm ren}[g_{(0)ij},\phi_{(0)}] = \int \mathrm{d}^d x\sqrt{g_{(0)}}\left(\frac{1}{2}\langle T_{ij}(\vec{x})\rangle_{g_{(0)}}\delta g_{(0)ij} + \langle\mathcal{O}(\vec{x})\rangle_{\phi_{(0)}}\delta\phi_{(0)}\right). \tag{3.35}$$

Under diffeomorphisms of the d transverse coordinates, the sources transform as

$$\delta g^{ij}_{(0)} = -(\nabla^i\xi^j + \nabla^j\xi^i), \quad \delta\phi_{(0)} = \xi^i\nabla_i\phi_{(0)}, \tag{3.36}$$

while under diffeomorphisms corresponding to boundary Weyl transformations[2] they transform as

$$\delta g^{ij}_{(0)} = -2\sigma g^{ij}_{(0)}, \quad \delta\phi_{(0)} = (\Delta-d)\sigma\phi_{(0)}. \tag{3.37}$$

We've seen that in general (in even dimensions), the counterterms that render the on-shell action finite are generically anomalous under boundary Weyl transformations. Substitution of these variations into the variation of the renormalized action yields the following Ward identities for the covariant divergence and trace of the boundary

[2] These are known as *Penrose-Brown-Henneaux* diffeomorphisms and will be discussed in more detail in section 3.4.

stress tensor

$$\nabla^i \langle T_{ij}(\vec{x})\rangle_{g_{(0)}} = -\langle \mathcal{O}(\vec{x})\rangle_{\phi_{(0)}} \nabla_j \phi_{(0)}(\vec{x})$$
$$\langle T^i_i(\vec{x})\rangle_{g_{(0)}} = (\Delta - d)\phi_{(0)}(\vec{x})\langle \mathcal{O}(\vec{x})\rangle_{\phi_{(0)}} + \mathcal{A}, \tag{3.38}$$

where we've denoted by $\mathcal{A}$ the conformal anomaly.

3.2 RG flows & domain-wall geometries

So far we've worked in exact anti-de Sitter space and have understood how to compute conformal field theory correlation functions holographically. Here, following [8], we will see that a similar method will aid in the computation of correlation functions in quantum field theories obtained by deformation of a CFT that admits a holographic dual. We begin by considering the following toy model of Euclidean gravity interacting with a bulk scalar ϕ

$$S = \int \mathrm{d}^{d+1}x\sqrt{g}\left[-\frac{1}{4}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V(\phi)\right], \tag{3.39}$$

where we have set $\kappa = 2$ so that ϕ carries no dimension and $V(\phi)$ has local extrema at ϕ_i satisfying $V(\phi_i) < 0$. The equations of motion read

$$\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\nu}\partial_\nu\phi\right) = V'(\phi)$$
$$2\left\{\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left[\frac{1}{2}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + V(\phi)\right]\right\} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}. \tag{3.40}$$

Note that for each extremum of the potential there is a trivial solution, that is

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -2g_{\mu\nu}V(\phi_i). \tag{3.41}$$

In fact, these are identical to Einstein's equations for pure gravity in anti-de Sitter space with

$$\Lambda_i = 4V(\phi_i) = -\frac{d(d-1)}{L_i^2}, \tag{3.42}$$

where L_i is the AdS length. That is to say, the 'critical points' of exact AdS_{d+1} with AdS length L_i are solutions of this toy model.

Of course, in order to study the gravitational dual of RG flows, we will need to find less trivial solutions to the equations of motion. We will work in the following background

$$ds^2 = dr^2 + e^{2A(r)}\delta_{ij}dx^i dx^j. \tag{3.43}$$

This *domain-wall geometry*, divided into a 'radial' coordinate $r \in (-\infty, \infty)$ and transverse coordinates $\vec{x} \in \mathbb{R}^d$, is the most general configuration with (manifest!) d-dimensional Poincaré symmetry. A direct consequence of the Einstein equations is the following fact about the function $A(r)$:

$$\ddot{A}(r) = \frac{2}{d-1}(T^i_i - T^r_r) = -\frac{2}{d-1}\dot{\phi}^2, \tag{3.44}$$

where a dot denotes a radial derivative. In particular, this implies that $\ddot{A}(r) < 0$; in fact, this is a consequence of the null-energy theorem and will turn out to have important consequences. The full equations of motion are given by

$$\begin{aligned}\dot{A}^2 &= \frac{2}{d(d-1)}(\dot{\phi}^2 - 2V(\phi)) \\ \ddot{\phi} + d\dot{A}\dot{\phi} &= V'(\phi),\end{aligned} \tag{3.45}$$

from which one can obtain (3.44). In exact AdS (ie. at a critical point $\phi = \phi_i$) the above equations give

$$\dot{A}^2 = L_i^{-2} \tag{3.46}$$

so that $A(r) = \pm\frac{r}{L_i} + c$ for some constant c that we will take to be zero without loss of generality. This is then equivalent to the description of AdS_{d+1} in the Poincaré patch under the change of variables $x_0 = L_i e^{-\frac{r}{L_i}}$. In this section we will be interested in solutions to the bulk equations of motion that interpolate between two asymptotic critical regions; that is, the spacetime is asymptotically AdS with AdS length L_1 as $r \to \infty$ (near the boundary) and flows to another asymptotically AdS region with AdS length L_2 as $r \to -\infty$ (corresponding to the Poincaré horizon).

Now, near a critical point, we are free to approximate the potential in the following way

$$V(\phi) = V(\phi_i) + \frac{1}{2}\frac{m_i^2}{L_i^2}\varphi^2 + \mathcal{O}(\varphi^3), \tag{3.47}$$

where m_i is defined in the obvious way and $\varphi = \phi - \phi_i$. Recall from e.g. chapter 2 that in AdS/CFT, we view the asymptotic boundary value of the bulk scalar as a source for the dual operator $\mathcal{O}$. We then expect the following asymptotic behaviour for the fluctuation $\varphi(r, \vec{x})$

$$\begin{aligned}\varphi(r, \vec{x}) \rightarrow & e^{(\Delta-d)r/L_i}\bar{\varphi}(\vec{x}) \\ = & e^{(\Delta-d)r/L_i}(\phi_{(0)} + \varphi_{(0)}(\vec{x})),\end{aligned} \tag{3.48}$$

where $m_i^2 L_i^2 = \Delta(\Delta - d)$ and in the last line we have separated the boundary contributions of the domain-wall profile $\phi_{(0)}$ from the fluctuation. We can then make use of the AdS/CFT dictionary to compute correlation functions of $\mathcal{O}$ in the usual way, taking functional derivatives of the on-shell action $S_{\text{grav}}[\phi_{(0)} + \varphi_{(0)}]$ with respect to $\varphi_{(0)}(\vec{x})$ and then setting $\varphi_{(0)} = 0$. However, the key difference is that now one thinks of the CFT action as being *deformed* by the term $\phi_{(0)} \int \mathrm{d}^d x \mathcal{O}(\vec{x})$; $\phi_{(0)}$ is *not* set to zero after taking the functional derivatives.

There are three classes of deformations, characterized by the value of $m_i^2 L_i^2$. For $m_i^2 L_i^2 \in (-\frac{d^2}{4}, 0)$, $\Delta < d$ and so the operator $\mathcal{O}$ represents a *relevant* deformation of a UV CFT; such a deformation triggers a renormalization group flow to some different theory in the IR. For $m_i^2 = 0$, $\Delta = d$ and the deformation is classically *marginal*, so that any deviation from conformal invariance is only visible at the quantum level. Finally, for $m_i^2 L_i^2 > 0$, $\Delta > d$ so that the dual operator is *irrelevant*; that is, its effects become invisible at long distances as one flows to a CFT in the IR.

In what follows we consider *interpolating flows*, for which the geometry has two asymptotically AdS regions; the the bulk scalar approaches a local maximum ϕ_1 of the potential near the boundary and and a local minimum ϕ_2 in the deep interior. The non-linear equations of motion (3.45) are in generally very difficult to solve exactly, so we will instead proceed under the approximation of linearized theory; the assumption being that there exists a solution to the full equations of motion that asymptotes to the linearized solutions near the critical points. The linearized equations of motion for φ are given by

$$L_i^2\ddot{\varphi} + dL_i\dot{\varphi} - m^2\varphi = 0 \tag{3.49}$$

with solution

$$\varphi(r,\vec{x}) = \varphi_i^{(-)}(\vec{x})e^{(\Delta_i-d)r/L_i} + \varphi_i^{(+)}e^{-\Delta_i r/L_i}. \tag{3.50}$$

In general one might have expected that we could linearize the scale factor $\dot{A} = L_i^{-1} + f(r)$, however it turns out that the equations of motion demand $f \sim \mathcal{O}(\varphi^2)$, so f is absent in the linearized theory. In what follows we will assume $\Delta_i > \frac{d}{2}$. The assumption of linearization dictates that near the boundary, the exact solution to the full nonlinear equations of motion can be approximated as follows

$$\phi(r,\vec{x}) \to \phi_1 + \varphi_1^{(-)}(\vec{x})e^{(\Delta_1-d)r/L_1} + \varphi_1^{(+)}e^{-\Delta_1 r/L_1}. \tag{3.51}$$

Indeed, we see that consistency of the solution requires $\Delta_1 < d$ (consistent with $m_1^2L_1^2 < 0$ representing a local maximum of the potential), and so the dual field theory is to be interpreted as a UV CFT perturbed by a relevant operator deformation. However, in the finely tuned case that $\varphi_1^{(-)} = 0$, the leading falloff as one approaches the boundary $r \to \infty$ is $\varphi \sim \varphi_2^{(+)}e^{-\Delta_1 r/L_1}$. As we have seen previously, in this case the dual description is a UV CFT deformed by an operator that develops a nontrivial vacuum expectation value; $\langle\mathcal{O}\rangle \sim (2\Delta_1 - d)\varphi_1^{(+)}$. In this case it is the vacuum that spontaneously breaks conformal invariance. In the deep interior, we assume the full solution is well-approximated by

$$\phi(r,\vec{x}) \to \phi_2 + \varphi_2^{(-)}(\vec{x})e^{(\Delta_2-d)r/L_2} + \varphi_2^{(+)}e^{-\Delta_2 r/L_2}. \tag{3.52}$$

Now, since the critical point ϕ_2 is associated with a minimum of the potential (ie. $m_2^2L_2^2 > 0$), then $\Delta_2 > d$; furthermore, regularity in the deep interior requires $\varphi_2^{(+)} = 0$. Therefore, the dimension of the dual operator as the deep interior of the asymptotically AdS space is approached is characteristic of an irrelevant operator; the dual description in the deep interior is that of an IR fixed point.

We previously observed that a very general feature of domain-wall geometries is that away from a critical point, $\ddot{A} < 0$. In fact, this implies the following inequality between the AdS lengths of the asymptotic regions

$$L_1^{-1} < L_2^{-1}. \tag{3.53}$$

From the form of the potential at the critical points (3.42) it then follows that

$$V(\phi_2) < V(\phi_1). \tag{3.54}$$

That is to say, one always flows to a deeper well of the potential; as a result, we conclude that the RG flows dual to Poincaré-invariant domain walls interpolating between asymptotically AdS geometries are in a sense *irreversible*. We will revisit this when discussing the holographic derivation of the conformal anomaly in section 3.4.

3.2.1 The superpotential method

Here we review a method that has proven extremely useful for extracting exact solutions from the full nonlinear domain-wall equations of motion (3.45). Taking inspiration from the BPS condition for supersymmetric domain wall solutions in supergravity, we consider a function $W(\phi)$ — the *superpotential* — determined in terms of the potential $V(\phi)$ through the following differential equation

$$\frac{1}{2}(W'(\phi))^2 - \frac{d}{d-1}W^2(\phi) = V(\phi). \tag{3.55}$$

Taking for simplicity a homogeneous scalar profile $\phi = \phi(r)$, the action (3.39) can be written simply in terms of the superpotential as

$$\begin{aligned} S = &\int \mathrm{d}^d x \mathrm{d}r \; e^{dA} \left(-\frac{d}{4}(d-1)\left(\dot{A} + \frac{2}{d-1}W\right)^2 + \frac{1}{2}(\dot{\phi} - W'(\phi))^2 \right) \\ &+ \int \mathrm{d}^d x \mathrm{d}r \frac{\mathrm{d}}{\mathrm{d}r}\left[e^{dA}(\dot{A} + W) \right]. \end{aligned} \tag{3.56}$$

Now suppose $W(\phi)$ satisfies the following *first-order* equations

$$\begin{aligned}\dot{\phi}(r,\vec{x}) &= W'(\phi)\\ \dot{A}(r) &= -\frac{2}{d-1}W(\phi).\end{aligned} \tag{3.57}$$

In this case the action reduces to a surface term and is proportional to the difference of the values of the superpotential at the critical points

$$S \propto \frac{d-3}{d-1}e^{dA(r)}W(\phi)\Big|_{r=r_2}^{r=r_1}. \tag{3.58}$$

Remarkably, it turns out that any solution to these first-order equations then constitutes a solution to the full second-order nonlinear equations of motion (3.45). Furthermore, it is easily seen that any critical point of $W(\phi)$ is also a critical point of $V(\phi)$; however, the converse does not hold. This method even generalizes to the case of multiple bulk scalars, in which case one just replaces $(W'(\phi))^2 \to \sum_i \left(\frac{\mathrm{d}W(\{\phi^j\})}{\mathrm{d}\phi^i}\right)^2$ and the first equation in (3.57) separates into an equation for each bulk scalar $\dot{\phi}^i(r,\vec{x}) = \frac{\mathrm{d}W(\{\phi^j\})}{\mathrm{d}\phi^i}$. In fact, it is possible for the first-order flow equations to reproduce the physics of the full second-order equations of motion because the flow equations arise as the *Hamilton-Jacobi* equations for the gravity-scalar system [22, 23].

3.3 The Hamilton-Jacobi formalism

Here we follow [22, 23] and show that the first-order equations that govern holographic RG flows in fact follow from the Hamilton-Jacobi (HJ) equation as the Hamiltonian constraint for gravity in its canonical form. This is an explicit realization of the relationship between the classical evolution equations for the bulk fields and the renormalization group (RG) equations that the boundary couplings must obey. In fact, we will show that the first-order flow equations for the bulk on-shell action (themselves a consequence of the Hamilton-Jacobi equation) can be cast as the Callan-Symanzik equations for boundary correlation functions. This supports the AdS/CFT lore that the bulk on-shell action is the quantum effective action for the boundary field theory.

Let us briefly review Hamilton-Jacobi theory. Given a classical action

$$S[x_1, x_2] = \int_{x(t_1)=x_1}^{x(t_2)=x_2} \mathrm{d}t \ \mathcal{L}[q, \dot{q}], \tag{3.59}$$

we consider perturbing the boundary condition $x_2 \to x_2 + \delta x_2$, inducing a corresponding change in the solution to the equations of motion

$$q_0(t) \to q_0(t) + \delta q_0(t). \tag{3.60}$$

The variation of the action then yields

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} \left(\delta q_0 \frac{\partial \mathcal{L}}{\partial q} + \delta \dot{q}_0 \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) \\ &= \int_{t_1}^{t_2} \delta q_0 \left(\frac{\partial \mathcal{L}}{\partial q} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) + \delta q_0 \frac{\partial \mathcal{L}}{\partial \dot{q}} \bigg|_{t_1}^{t_2} \\ &= \delta x_2 p(t_2). \end{aligned} \tag{3.61}$$

So an x_2 derivative of the action yields the conjugate momentum $p(t_2)$. This immediately yields the Hamilton-Jacobi equation, namely

$$H\left(\{q_i\}, \left\{ \frac{\partial S}{\partial q_j} \right\} \right) = -\frac{\partial S}{\partial t}, \tag{3.62}$$

where H is the Hamiltonian of the system.

In particular this applies to gravity in its canonical form. We are always free to write the metric as

$$\mathrm{d}s^2 = N^2 \mathrm{d}r^2 + g_{ab}(r, \vec{x})(\mathrm{d}x^a + N^a \mathrm{d}r)(\mathrm{d}x^b + N^b \mathrm{d}r), \tag{3.63}$$

where N and N^a are the lapse and shift functions respectively. From this point forward we will work in a gauge where $N^a = 0$ and $N = 1$, however the equations of motions for N and N^a must still be imposed as the Hamiltonian and diffeomorphism constraints, respectively. Treating r as the analogue of time in standard Hamilton-Jacobi theory, the Hamilton-Jacobi equations will describe radial flows (generated by the ADM Hamiltonian) in AAdS_{d+1}, which we expect to be dual to renormalization group flows in CFT$_d$. The role of the action in the above discussion with be played by the supergravity on-shell action $S[\phi_{(0)}, g_{(0)}]$, evaluated on solutions to the classical equations of motion on a fixed-r slice near the boundary with boundary data $\phi_{(0)}$ and $g_{(0)}$. The Hamilton-Jacobi equation will be realized by the Hamiltonian constraint.

For concreteness, we now specialize to $d = 4$. We consider gravity minimally coupled to bulk scalars in the (five-dimensional) Einstein frame, with Lagrangian

$$\mathcal{L}[\phi, g] = \frac{1}{2} G_{IJ} \partial^a \phi^I \partial_a \phi^J + V(\phi) + R, \tag{3.64}$$

where G_{IJ} is a metric in field space. The Hamiltonian and diffeomorphism constraints read

$$\begin{aligned} \mathcal{H} = 0 &= \left(\pi^{ab} \pi_{ab} - \frac{1}{3} \pi^a_a \pi^b_b \right) + \frac{1}{2} \pi_I G^{IJ} \pi_J + \mathcal{L}[\phi, g] \\ 0 &= \nabla^a \pi_{ab} + \pi_I \nabla_b \phi^I. \end{aligned} \tag{3.65}$$

The conjugate momenta π_I and π_{ab} are defined in the usual way

$$\pi_I = \frac{1}{\sqrt{g}} \frac{\delta S}{\delta \phi^I}, \quad \pi_{ab} = \frac{1}{\sqrt{g}} \frac{\delta S}{\delta g^{ab}}. \tag{3.66}$$

The diffeomorphism constraint yields the usual invariance of the on-shell action under four-dimensional diffeomorphisms. However, the Hamiltonian constraint is non-trivial and yields the Hamilton-Jacobi equation:

$$\frac{1}{\sqrt{g}} \left[\frac{1}{3} \left(g^{ab} \frac{\delta S}{\delta g^{ab}} \right)^2 - g^{ac} g^{bd} \frac{\delta S}{\delta g^{ab}} \frac{\delta S}{\delta g^{cd}} - \frac{1}{2} G^{IJ} \frac{\delta S}{\delta \phi^I} \frac{\delta S}{\delta \phi^J} \right] = \sqrt{g} \mathcal{L}[\phi, g]. \tag{3.67}$$

This should be thought of as a differential equation determining the form of the on-shell action. With a solution to the HJ in hand, the usual Hamiltonian equations of motion tell us how to compute radial derivatives of the fields

$$\dot{\phi}^I = G^{IJ} \pi_J, \quad \dot{g}_{ab} = -2\pi_{ab} + \frac{2}{3} \pi^c_c g_{ab}. \tag{3.68}$$

We've seen very explicitly in §3.1 that to compute correlation functions in AdS/CFT, one must evaluate the gravitational on-shell action at a cutoff surface and add covariant counterterms to remove divergences that arise as the cutoff is taken to the boundary. The counterterms are local expressions in terms of the sources, and so we decompose the on-shell action as

$$S[\phi, g] = S_{\text{loc}}[\phi, g] + \Gamma[\phi, g]. \tag{3.69}$$

Γ is to be thought of as the generating functional in the boundary field theory; we've seen this differs from the on-shell action by local, divergent counterterms. We choose to parameterize the local part of the action as

$$S_{\text{loc}}[\phi, g] = \int \mathrm{d}^d x \sqrt{g} \left(U(\phi) + \Phi(\phi) R + \frac{1}{2} M_{IJ} \partial^a \phi^I \partial_a \phi^J \right), \tag{3.70}$$

where U, Φ and M_{IJ} are taken to be local functions of the sources, as in §3.1. The generating functional Γ then contains all higher derivative and non-local terms. The idea is to insert the decomposition (3.69) into the HJ equation and require that the terms of different scaling degree individually vanish. With this goal in mind, we separate terms in S_{loc} and $\mathcal{L}$ by their scaling dimension

$$\begin{aligned} S^{(0)}_{\text{loc}} &= \int \mathrm{d}^4 x \sqrt{g} U(\phi), \quad \mathcal{L}^{(0)} = \sqrt{g} V(\phi) \\ S^{(2)}_{\text{loc}} &= \int \mathrm{d}^4 x \sqrt{g} \left(\Phi(\phi) R + \frac{1}{2} M_{IJ} \partial^a \phi^I \partial_a \phi^J \right), \quad \mathcal{L}^{(2)} = \sqrt{g} \left(R + \frac{1}{2} G_{IJ} \partial^a \phi^I \partial_a \phi^J \right). \end{aligned} \tag{3.71}$$

The HJ equation (3.67) can then be written compactly as

$$\{S, S\} = \mathcal{L}^{(0)} + \mathcal{L}^{(2)}, \tag{3.72}$$

where the bracket $\{S, S\}$ is defined as

$$\{S, S\} = \frac{1}{\sqrt{g}} \left[\frac{1}{3} \left(g^{ab} \frac{\delta S}{\delta g^{ab}} \right)^2 - g^{ac} g^{bd} \frac{\delta S}{\delta g^{ab}} \frac{\delta S}{\delta g^{cd}} - \frac{1}{2} G^{IJ} \frac{\delta S}{\delta \phi^I} \frac{\delta S}{\delta \phi^J} \right]. \tag{3.73}$$

Separating the HJ equation into terms of different scaling degree, we get the decomposition

$$\begin{aligned}\{S_{\text{loc}}^{(0)}, S_{\text{loc}}^{(0)}\} &= \mathcal{L}^{(0)} \\ 2\{S_{\text{loc}}^{(0)}, S_{\text{loc}}^{(2)}\} &= \mathcal{L}^{(2)} \\ 2\{S_{\text{loc}}^{(0)}, \Gamma\} + \{S_{\text{loc}}^{(2)}, S_{\text{loc}}^{(2)}\} &= 0.\end{aligned} \tag{3.74}$$

Let us examine the first term in (3.74). After some algebra, it becomes

$$V(\phi) = \frac{1}{3}U^2(\phi) - \frac{1}{2}G^{IJ}\partial_I U \partial_J U. \tag{3.75}$$

Up to an overall constant, this is in fact exactly (3.55) - so it is in fact the Hamilton-Jacobi equation that allows us to write the potential in terms of a 'superpotential,' thereby reducing the second order equations of motion to first-order flow equations. The solutions to the equations of motion (3.57) are then equivalent to

$$\dot{\phi}^I = G^{IJ}\partial_J U, \quad \dot{g}_{ab} = -\frac{1}{3}U(\phi)g_{ab}. \tag{3.76}$$

These equations of motion can easily be solved by an ansatz of the form

$$g_{ab} = a^2 \hat{g}_{ab}, \tag{3.77}$$

where $\hat{g}$ is some fiducial metric that does not depend on the radial coordinate and a must satisfy

$$\dot{a} = -\frac{1}{6}U(\phi)a. \tag{3.78}$$

Since a encodes all scale-dependence of the metric, we now replace all radial derivatives with a derivatives. In this way, we may define 'beta functions'

$$\beta^I(\phi) = -\frac{6}{U(\phi)}G^{IJ}\partial_J U(\phi), \tag{3.79}$$

describing the evolution of the bulk scalars with scale as

$$a\frac{\mathrm{d}}{\mathrm{d}a}\phi^I = \beta^I(\phi). \tag{3.80}$$

Near the boundary, the beta functions behave asymptotically as

$$\beta^I(\phi) \to (\Delta_I - 4)\phi^I, \tag{3.81}$$

where Δ_I encodes the asymptotic falloff of the bulk scalars $\phi^I \to e^{-r(\Delta_I-4)}\bar{\phi}^I$ and $\bar{\phi}^I$ is the source for the dual operator $\mathcal{O}_I$.[3] Note that this is precisely the asymptotic scale dependence we observed earlier in (3.33).

The question then becomes what kinds of potentials can be written in the form (3.75). Taking $G_{IJ} = \delta_{IJ}$ for simplicity of exposition, we expand the bulk potential near one of its critical points[4]

$$V(\phi) = 12 - \frac{1}{2}m_I\phi^I\phi^I + g_{IJK}\phi^I\phi^J\phi^K + \dots \tag{3.82}$$

Taking a similar ansatz for the expansion of the 4d 'local superpotential' $U(\phi)$ near the same critical point

$$U(\phi) = 6 + \frac{1}{2}\lambda_I\phi^I\phi^I + \lambda_{IJK}\phi^I\phi^J\phi^K, \tag{3.83}$$

leads to the following form of the beta functions

$$\beta^I(\phi) = (\Delta_I - 4)\phi^I - c^I_{JK}\phi^J\phi^K, \tag{3.84}$$

where we've defined

$$\Delta_I = 4 - \lambda_I, \quad c_{IJK} = 3\lambda_{IJK}. \tag{3.85}$$

Recalling that Δ_I parameterizes the scaling dimension of the operator dual to the boundary value of ϕ^I, we see that λ_I can be thought of as the 'deviation from

[3] For simplicity we have set $L_{\text{AdS}} = 1$ in this section.

[4] Note the relative factor of -4 between the R term and the potential in the Lagrangian of this section compared to the previous section.

marginality' of the operator $\mathcal{O}_I$; for λ_I positive, the operator is a relevant deformation. Indeed, plugging the expansions of V and U into the constraint equation (3.75) yields

$$\begin{aligned} m_I^2 &= \lambda_I(\lambda_I - 4) \\ g_{IJK} &= \lambda_{IJK}\left(4 - \lambda_I - \lambda_J - \lambda_K\right). \end{aligned} \tag{3.86}$$

The first equation represents the usual relationship between bulk field masses and dual operator dimensions, as well as the BF bound $m_I^2 \geq -4$ [11, 12] required for stability of the bulk solution; the second represents a relationship between the OPE coefficient c_{IJK} that characterizes the quadratic term in the beta function and the cubic term g_{IJK} in the bulk potential. But what terms in U does the HJ equation constrain? Perturbing the 4d potential, we find that the relationship (3.75) is preserved only for perturbations that satisfy

$$\frac{2}{3} U \delta U - (\partial_I U)(\partial_I \delta U) = 0 \to (4 + \beta^I \partial_I)\delta U = 0. \tag{3.87}$$

That is to say, terms in U with total scaling dimension 4 are unconstrained by the HJ equation. As we've seen in section 3.1, these are exactly the terms that remain finite when we send the cutoff to the boundary. Finally, while we will not use them here, we note that the second equation in (3.74) leads to constraints relating the (divergent!) local terms $\{\Phi, M_{IJ}\}$ in $S^{(2)}_{\text{loc}}$ to the parameters (in particular, the beta functions) describing the RG flow of the couplings of the boundary theory.

As should be familiar by now from the discussion in the previous two chapters, it is the effective action Γ — obtained from the supergravity on-shell action on a near-boundary cutoff surface after the subtraction of divergent local counterterms — that contains information about CFT correlation functions. In fact, the Callan-Symanzik equation for the scale-dependence of these correlation functions can be derived from the final equation in (3.74). One finds

$$\frac{1}{\sqrt{g}}\left(2g^{ab}\frac{\delta}{\delta g^{ab}} - \beta^I \frac{\delta}{\delta \phi^I}\right)\Gamma[\phi, g] = (\text{Terms with four derivatives}). \tag{3.88}$$

The Callan-Symanzik equation is obtained from this functional differentiation with respect to the fields $\phi^I(x)$, after which the fields are set to their background values (ie. the couplings in the field theory); furthermore, we take $g_{ab} = a^2(\vec{x})\delta_{ab}$, integrate, and replace the integrated functional derivatives with the appropriate ordinary derivatives with respect to a and ϕ^I. Applying this procedure to the previous equation results in the Callan-Symanzik equation

$$\left(a\frac{\partial}{\partial a} + \beta^I \partial_I\right) \langle \mathcal{O}_{I_1}(\vec{x}_1) \cdots \mathcal{O}_{I_n}(\vec{x}_n)\rangle + \sum_{i=1}^{n} \gamma_{I_i}^{J_i} \langle \mathcal{O}_{I_1}(\vec{x}_1) \cdots \mathcal{O}_{J_i}(\vec{x}_i) \cdots \mathcal{O}_{I_n}(\vec{x}_n)\rangle = 0, \tag{3.89}$$

where the 'anomalous dimensions' are defined through[5]

$$\gamma_I^J = \nabla_I \beta^J. \tag{3.90}$$

The result (3.89) represents the Callan-Symanzik equation for correlation functions of bare operators, derived from an effective action Γ defined on a cutoff surface which has yet to be taken to the boundary. To recover the Callan-Symanzik equation for the renormalized correlation functions, one must define the renormalized metric and couplings via

$$g_{ab} = \epsilon^{-2} g_{ab}^R, \quad \phi^I = \phi^I(\phi_r, \epsilon), \tag{3.91}$$

so that the boundary is approached as $\epsilon \to 0$. The bare couplings are related to the renormalized couplings via integration of the beta functions, along with an appropriate renormalization condition

$$\beta^I(\phi) = -\epsilon \frac{\partial \phi^I}{\partial \epsilon}, \quad \phi^I(\phi_r, \epsilon = 1) = \phi_r. \tag{3.92}$$

We've seen in the discussion of holographic renormalization that S_{loc} will generically diverge as we take $\epsilon \to 0$ (for instance, U is quartically divergent while M and Φ are quadratically divergent, while Γ will generally contain a logarithmic divergence

[5] The derivatives ∇_I are covariant derivatives compatible with the metric G_{IJ}.

in even d). Taming these divergences by the addition of local counterterms gives the renormalized effective action, and it is through functional differentiation of the analogue of (3.88) for the renormalized effective action that one obtains the Callan-Symanzik equations for the renormalized operators

$$\mathcal{O}_I^R = \mathcal{O}_J \frac{\partial \phi^J}{\partial \phi_R^I}. \tag{3.93}$$

3.4 The holographic conformal anomaly

One of the earliest checks of the AdS/CFT correspondence was the holographic computation of the conformal anomaly, due originally to Henningson and Skenderis [16]; we'll see that this is intimately related to the observation made in section 3.2.1 about the irreversibility of domain-wall flows. The lecture notes [8] and textbook [9] both contain illuminating expositions of this derivation and will be followed here.

The setup is very similar to the discussion in section (3.1.2), except we consider pure gravity in AAdS_{d+1}

$$S = -\frac{1}{2\kappa}\int \mathrm{d}^{d+1}x\sqrt{g}\left(R^{(d+1)} + \frac{d(d-1)}{L_{\mathrm{AdS}}^2}\right) - \frac{1}{\kappa}\int \mathrm{d}^d x\sqrt{\gamma}K\bigg|_{\rho=\epsilon}, \tag{3.94}$$

where K is the trace of the extrinsic curvature and the last term is required for the well-posedness of the variational principle. Recall the form of the AAdS_{d+1} line element

$$L_{\mathrm{AdS}}^2\left(\frac{\mathrm{d}\rho^2}{4\rho^2} + \frac{1}{\rho}g_{ij}(\rho,\vec{x})\mathrm{d}x^i\mathrm{d}x^j\right), \tag{3.95}$$

along with the Fefferman-Graham expansion (3.3) for g_{ij}

$$g_{ij}(\rho,\vec{x}) = g_{(0)ij}(\vec{x}) + \rho g_{(2)ij}(\vec{x}) + \ldots + \rho^{d/2}\log\rho\ h_{(d)ij} + \ldots, \tag{3.96}$$

where the logarithmic term appears only for even d. As is familiar from the discussion in section (3.1.2), the coefficient $g_{(k)}$ can be computed by solving the Einstein equations order by order in ρ. For instance, in $d > 2$ one finds

$$g_{(2)ij} = \frac{L_{\mathrm{AdS}}^2}{d-2}\left(R_{ij} - \frac{1}{2(d-1)}Rg_{(0)ij}\right), \tag{3.97}$$

where all curvature invariants are built out of the boundary metric $g_{(0)}$. Proceeding as in §3.1 and evaluating the on-shell action at the surface $\rho = \epsilon$ leads to

$$S_{\mathrm{reg},\epsilon} = -\frac{1}{2\kappa}\int \mathrm{d}^d x\sqrt{g_{(0)}}\epsilon^{-\frac{d}{2}}\left(a_{(0)} + \epsilon a_{(2)} + \ldots - \epsilon^{\frac{d}{2}}\log\epsilon a_{(d)}\right)\Big|_{\rho=\epsilon} + \mathcal{O}(\epsilon^0), \quad (3.98)$$

where the $a_{(k)}$ are determined in terms of the boundary metric $g_{(0)}$ and there is no logarithm for d odd. For instance, the first few are given by [17]

$$\begin{aligned} a_{(0)} &= \frac{2(d-1)}{L_{\mathrm{AdS}}} \\ a_{(2)} &= -\frac{L_{\mathrm{AdS}}}{(d-2)}R \\ a_{(4)} &= \frac{L^3_{\mathrm{AdS}}}{(d-4)(d-2)^2}\left(R^{ij}R_{ij} - \frac{d}{4(d-1)}R^2\right). \end{aligned} \quad (3.99)$$

As before, these divergent contributions are cancelled by the addition of covariant counterterms to the on-shell action. For instance, it is clear that adding a term proportional to $\int \mathrm{d}^d x\sqrt{\gamma}$ will cancel the divergence due to the leading term $a_{(0)}$.

In quantum field theory, a conformal anomaly is signalled by a Weyl transformation of the metric leading to a non-vanishing trace of the stress tensor. To demonstrate this property from the bulk, we seek a $(d+1)$-dimensional diffeomorphism that scales the boundary metric. The required diffeomorphism is called the Penrose-Brown-Henneaux (PBH) transformation [10] and is given by

$$\rho \to \rho'(1 + 2\sigma(\vec{x}')), \quad x^i \to x'^i + a^i(\rho,\vec{x}). \quad (3.100)$$

We demand that the line element (3.95) transforms covariantly under this diffeomorphism, ie. $g_{\rho\rho}$ and $g_{\rho j}$ are left invariant. This requires

$$a^i(\vec{x}) = \frac{L^2_{\mathrm{AdS}}}{2}\int_0^\rho \mathrm{d}\rho' g^{ij}(\rho',\vec{x})\partial_j\sigma(\vec{x}), \quad (3.101)$$

and induces the following transformation in g_{ij}:

$$g_{ij} \to g'_{ij} = g_{ij} - 2\sigma(\vec{x})(1 - \rho\partial_\rho)g_{ij} + \nabla_i a_j(\vec{x}) + \nabla_j a_i(\vec{x}). \quad (3.102)$$

As the boundary is approached, $a_i, \rho\partial_\rho g_{ij} \to 0$ so that indeed we recover the correct Weyl scaling of the boundary metric

$$g_{(0)ij} \to g'_{(0)ij} = (1 - 2\sigma(\vec{x}))g_{(0)ij} \tag{3.103}$$

as claimed in (3.37). While the original action (3.94) and the ensuing equations of motion are invariant under bulk diffeomorphisms, the counterterms required to render the on-shell action finite spoil the invariance under the PBH diffeomorphisms corresponding to Weyl transformations on the boundary. This is of course a signal of the conformal anomaly in the boundary field theory.

For concreteness we now specialize to the case of $d = 4$. In this case, one can show that the covariant counterterms needed to cancel the divergences arising in (3.98) are given by [17, 8]

$$S_{\mathrm{ct},\epsilon}[g_{(0)}] = \frac{1}{\kappa} \int \mathrm{d}^d x \sqrt{\gamma} \left(\frac{3}{L_{\mathrm{AdS}}^2} - \frac{R[\gamma]}{4} - \frac{L_{\mathrm{AdS}}^2 \log \epsilon}{16} (R^{ij}[\gamma] R_{ij}[\gamma] - \frac{1}{3} R^2[\gamma]) \right) \Bigg|_{\rho=\epsilon} . \tag{3.104}$$

Here, all curvature invariants are formed out of the induced metric at $\rho = \epsilon$, so that the first two terms are power-law divergent and the last is logarithmically divergent as $\epsilon \to 0$. We are interested in computing the trace of the stress tensor

$$\langle T_i^i(\vec{x}) \rangle_{g_{(0)}} = g_{(0)}^{ij} \langle T_{ij}(\vec{x}) \rangle_{g_{(0)}} = -\frac{\delta S_{\mathrm{ren}}[g_{(0)}]}{\delta\sigma} = -\lim_{\epsilon \to 0} \frac{\delta(S_{\mathrm{reg},\epsilon}[g_{(0)}] + S_{\mathrm{ct},\epsilon}[g_{(0)}])}{\delta\sigma}, \tag{3.105}$$

where we've used that near the boundary, $\delta g_{(0)ij} = -2g_{(0)ij}\delta\sigma$. Now, by construction, all terms in $S_{\mathrm{reg},\epsilon}$ are invariant under the combined change in coordinates and redefinition of the regulator surface via $\delta\epsilon = 2\epsilon\delta\sigma$. It is straightforward to see that the first two terms in the counterterm action (3.104) are also invariant under the Weyl transformation; however, the logarithmic term spoils the invariance, as $\delta \log \epsilon = 2\delta\sigma$. Thus we find the following as the holographic conformal anomaly

$$\langle T_i^i(\vec{x}) \rangle_{g_{(0)}} = \frac{L_{\mathrm{AdS}}^3}{8\kappa} \left(R^{ij} R_{ij} - \frac{R^3}{3} \right) . \tag{3.106}$$

In fact, in the original formulation of AdS/CFT, it turns out that one can see that this agrees *exactly* with the anomaly of the boundary supersymmetric Yang-Mills theory in the $N \to \infty$ limit, upon noticing that $G_N^{(5)} = \frac{G_N^{(10)}}{\mathrm{vol}(S^5)} = \frac{\pi L_{\mathrm{AdS}}^3}{2N^2}$. Note that in odd d there is no logarithmic term in the regularized on-shell action; correspondingly, there is no need for a logarithmic counterterm and thus no conformal anomaly as one would expect from field theory. Furthermore, the general form of the conformal anomaly in four dimensions is given by

$$\langle T_i^i(\vec{x})\rangle = cW^{ijkl}W_{ijkl} + aE^{ijkl}E_{ijkl} + \dots, \tag{3.107}$$

where W is the Weyl tensor, E^2 is the Euler density and we have neglected overall variations of local terms. In the special case that $c = a$, then the anomaly has the form

$$\langle T_i^i(\vec{x})\rangle = \frac{a}{8\pi^2}(R^{ij}R_{ij} - \frac{1}{3}R^2). \tag{3.108}$$

The computation of the holographic anomaly would then imply that the four-dimensional field theories dual to gravity in AAdS_5 have $a = c = \frac{\pi^2 L_{\mathrm{AdS}}^3}{\kappa}$.

This procedure can obviously be extended to compute the anomalies of the field theories dual to the endpoints of the holographic renormalization group flows discussed in §3.2. As in that section, we consider domain wall geometries interpolating between an asymptotically AdS_5 region near the boundary with length L_1 and another asymptotically AdS_5 region in the deep interior with length L_2.[6] It is clear that the anomalies at the critical points are given by

$$a_1 = \frac{\pi^2 L_1^3}{\kappa}, \quad a_2 = \frac{\pi^2 L_2^3}{\kappa}. \tag{3.109}$$

[6] For the sake of getting the coefficients correct we specialize to $d = 4$ here, but it should be clear that the following discussion generalizes to arbitrary d. Indeed, for arbitrary even d is straightforward to see that the conformal anomaly is proportional to L_{AdS}^{d-1}.

From this, one can construct an 'anomaly function' $a(r)$ that extends into the bulk via

$$
\begin{aligned}
a(r) &= \frac{\pi^2}{\kappa}\left(\frac{1}{\dot{A}(r)}\right)^3 \\
\dot{a}(r) &= \frac{-3\pi^2}{\kappa}\left(\frac{\ddot{A}(r)}{\dot{A}(r)^4}\right).
\end{aligned} \tag{3.110}
$$

This function satisfies all the necessary conditions of an anomaly- or 'c-function' as originally proposed by Zamolodchikov [24] in the proof that all RG flows in two-dimensions are irreversible, making precise the notion that RG flow corresponds to integrating out massive degrees of freedom. That is:

- It takes the values of the anomaly coefficients at the critical points.
- It monotonically decreases along the RG flow of the dual field theory; recall that a very general property of the domain-wall system was $\ddot{A} \leq 0$.
- Furthermore, for flows satisfying the simplified first-order equations of motion (3.57), $a(r)$ is *only* stationary at the critical points (since in this case $\ddot{A} = 0 \rightarrow W'(\phi) = 0$).

Following the proof that such a function exists in two-dimensional quantum field theories [24] (interpolating between the central charges of the CFTs at the fixed points of the flow), the analogous theorem in four-dimensions — the a-theorem, a proposal of Cardy's [25] in which the anomaly functional interpolates between the values of the a anomaly in (3.107) — went unproven for ~ 25 years, until the heroic work of Komargodski and Schwimmer [26, 27]. In these works it was analyticity and unitarity of the scattering amplitudes of the conformal compensator field that proved the a-theorem. However, we see that for field theories whose RG flow is dual to a domain wall geometry, the proof is trivial! Furthermore, as mentioned before, this construction extends trivially to higher even d, implying the a-theorem for field theories in arbitrary even dimensions with gravitational duals.

This concludes the literature review section of the thesis; we now proceed to the presentation of original research.

Chapter 4
Holographic renormalization group flows in the adiabatic limit

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Abstract

We follow [5] in studying the dilaton effective action induced by RG flows between holographic conformal fixed points. In doing so, we present a new approximation scheme that is useful for computing the on-shell action in an effective description of AdS domain-wall geometries (or equivalently, the dilaton effective action of the dual field theory). This framework is particularly useful in the case that the AdS isometries are weakly broken, corresponding to a 'slow flow' interpolating between AdS asymptotic regions. We demonstrate the utility of this approximation scheme with two applications. First, we compute the dilaton effective action for a flow driven by a weakly relevant operator from the dual bulk effective theory, and show that it reproduces the correct form of the RG-improved two point function. Secondly, we compute the gravitational on-shell action in a generalized 'slow-flow' setup in four dimensions, and show that upon Wick rotation we recover the inflationary power spectrum and spectral index of curvature perturbations at horizon-crossing to second-order in slow-roll.

4.1 Introduction

It is useful to think of (renormalizable) quantum field theories in terms of renormalization group (RG) flows between conformal fixed points. The high energy physics can be characterized by a UV conformal field theory (CFT) plus some relevant operator deformations $\{\mathcal{O}\}$ with couplings $\{\phi^{(0)}\}$. These perturbations typically break the conformal symmetry of the high energy theory, triggering a renormalization group flow to some different theory in the IR. However, by promoting the couplings to fields with nontrivial spacetime dependence and demanding a particular transformation of the couplings under scale transformations, one can restore the conformal symmetry. This is done by promoting all mass scales upon which couplings depend to spacetime-dependent functions $M \to Me^{-\tau(x)}$; conformal symmetry then places constraints [26, 27, 28] on the effective action for the field $\tau(x)$.[1] Particular terms in the dilaton action can be uniquely determined by the matching of conformal anomalies in the IR and the UV; this is how the a-theorem was proven in four dimensions.

The construction of a function that decreases monotonically across RG flows in two and four dimensions naturally leads to the question of whether one can similarly show that RG flows in arbitrary dimensions are irreversible. However, it has been shown that the generalization of the Komargodski-Schwimmer construction [26] to 6 and 8 dimensions is in fact nontrivial [28, 29]. Holography provides a useful framework in which to study RG flows [22, 30, 18, 19, 31, 32, 33, 34, 35], often via the bulk dual of a relevant operator deformation of a UV CFT. Indeed, there are holographic arguments related to entanglement entropy [36, 37] for the generalization of the a-theorem to arbitrary dimensions; in even boundary dimensions it is the coefficient of the 'a-type' anomaly that flows, while in odd dimensions it is expected that the finite part of the free energy on S^d decreases along RG trajectories [38, 39, 40, 41].

In this note we take a bottom-up approach to the study of holographic renormalization group flows by making use of the general holographic effective field theory constructed by [5] for π, the Goldstone boson for the broken spacetime symmetry corresponding to boundary dilatations. This effective field theory is described in terms of a general bulk action designed to capture the universal properties of holographic RG flows; the free parameters of this theory capture the space of different quantum field theories. This effective framework for holographic RG flows took inspiration and borrowed heavily from the effective field theory of inflation [42, 43], the general effective theory of broken de Sitter time translations.

[1] Variously known in the literature as the spurion or conformal compensator field in the case of explicit breaking of conformal symmetry, or the dilaton in the case of spontaneous breaking.

By solving the bulk equations of motion for the Goldstone boson π, one can determine universal properties of the dilaton effective action. In [5], the effective field theory was used to holographically obtain the two-dimensional dilaton effective action as well as the corresponding UV and IR conformal anomalies in full generality. In higher dimensions, computations were only tractable in a 'slow-flow' limit — analogous to slow-roll inflation — corresponding to extremely thick domain wall geometries interpolating between AdS regions of radius L_{UV} near the boundary and L_{IR} in the deep interior. In particular, in such a setup where the spacetime symmetry is appropriately softly broken, one is free to neglect mixing with AdS gravity. In this paper we study this regime in more detail.

We start by specifying a bulk metric which is close to AdS_{d+1} everywhere. As a result of breaking the $SO(d+1,1)$ symmetry the bulk now contains a scalar excitation: the goldstone boson π. Having deformed the AdS background, we write down the lowest derivative terms in the action for the the goldstone boson in terms of the deformations from AdS_{d+1} (characterized by geometric quantities analogous to slow-roll parameters familiar from inflation).

Furthermore, for the class of theories of interest, since all geometric quantities can be expanded in some small parameters (characterizing the deviation from exact AdS), we will present an adiabatic approximation scheme to solve the Goldstone's equations of motion as a perturbative series. This allows us to compute the on-shell boundary action for π. In principle, we have a series representation for the solution, which if solved to all-orders, gives the exact result. We shall see how this works in the example of slow-flow geometries dual to weakly relevant flows, and will show how one can resum all terms to obtain the RG-improved two point function as computed in conformal perturbation theory. In a more general setup, however, this adiabatic scheme provides the solution as a perturbative series in parameters encoding the deviation from AdS. We believe that our setup is useful to illustrate various aspects and results of holographic RG flows in a concrete and computationally tractable context.

This paper is organized as follows. In section 4.2.1 we review the formalism of restoring conformal invariance via the background dilaton field and the constraints of conformal symmetry. In section 4.2.2 we describe the class of bulk theories we are interested in, including a review of the effective field theory of holographic RG flows as presented in [5], as well as a discussion of the demixing limit. In section 4.3 we present the adiabatic approximation scheme we use to solve the Goldstone boson's equation of motion; in the demixing limit, we give a formal exact solution for the coefficients describing the mixing between the 'instantaneous AdS' solutions in the bulk, which is useful in practice when the spacetime is nearly AdS everywhere. In section 4.4 we provide the first application of our approximation scheme, and compute the dilaton effective action for a weakly-relevant flow holographically; in particular, the Goldstone boson's on-shell action reproduces the RG-improved two-point function as derived via conformal perturbation theory. In section 4.5 we compute the on-shell action in a generalized slow-flow approximation in four dimensions; upon Wick rotation, we recover the usual inflationary power spectrum and spectral index to second-order in slow-roll. In an appendix we show how to extend this computation to the case with a running speed of sound. We work in Euclidean signature throughout this paper.

4.2 The general setup

4.2.1 Boundary

Here we briefly review the background dilaton formalism for renormalization group flows triggered by explicitly broken conformal symmetry. The basic idea is to introduce the background dilaton field to treat such flows as if they were triggered by a spontaneous breaking of conformal symmetry. Conformal symmetry is generically broken both by the existence of trace anomalies (in curved even-dimensional space) and due to the deformation by the relevant operators. However, one can eliminate the latter by allowing the couplings of the relevant operators to transform under Weyl transformations. To do this, one replaces every mass scale in the theory (upon which couplings depend either explicitly or implicitly, ie. as a cutoff) by $M \to Me^{-\tau(x)}$, and then demands that the dilaton $\tau(x)$ transforms nonlinearly realizes invariance under Weyl transformations. Under Weyl transformations, the background metric transforms as

$$g_{\mu\nu} \to e^{2\sigma(x)} g_{\mu\nu}, \tag{4.1}$$

while the dilaton shifts in such a way as to restore conformal symmetry

$$\tau(x) \to \tau(x) - \sigma(x). \tag{4.2}$$

It should be clear that to linear order, the dilaton always appears in the Lagrangian by coupling to the trace of the stress tensor $\int \mathrm{d}^d x \sqrt{g} \tau(x) T^\mu_\mu(x)$.

The effective action is then rendered invariant under the combined Weyl transformation and shift of the dilaton — up to the trace anomaly of the UV CFT, which should be reproduced by the Weyl variation of the effective action. Much can be learned by studying the dilaton effective action in the infrared, after integrating out massive degrees of freedom. By construction, the (possibly anomalous) conformal symmetry of the UV theory has been restored by the introduction of the dilaton, so the low-energy theory should reproduce the UV anomaly. The conformal field theory that governs the IR physics will contribute to the anomaly, but generically with different coefficients than the UV CFT; as a result, the dilaton functional must transform precisely to compensate for the difference between the UV and IR anomalies under Weyl transformations.

The effective action should inherit the symmetries of the UV theory; that is, it should be diff×Weyl invariant. Clearly, the 'dressed' metric $\hat{g}_{\mu\nu} = e^{-2\tau} g_{\mu\nu}$ is invariant under Weyl transformations, so the effective action should be built out of curvature invariants of $\hat{g}_{\mu\nu}$ plus a term (the Wess-Zumino term) whose Weyl variation reproduces the difference between the UV and IR anomalies. The Wess-Zumino term results in a d-derivative action for the dilaton that persists even after the background metric is taken to be flat. The coefficient of this term is universally determined by the difference between the UV and IR anomalies and thus is blind to the details of the flow. This is the observation that led to proofs of the monotonicity of RG flows in 2 and 4 dimensions in [26, 27]. However, there are obstacles to the naive extension of the proof to higher dimensions. For instance, in $d = 6$ the relevant dilaton four-point scattering matrix element vanishes in the forward-scattering limit [29]; in $d = 8$ there is a contribution to the 8-derivative action from the Weyl-invariants that does not vanish on shell and thus contaminates the anomaly flow [28]. One might then ask how the dilaton is realized holographically — this is answered in the next section.

4.2.2 Bulk

We consider the class of spacetimes that interpolate between AdS_{d+1} near the boundary and another AdS_{d+1} in the deep interior and preserve $ISO(d-1,1)$ isometry. For example, in Einstein-Scalar theory such a domain wall spacetime corresponds to a background scalar profile which interpolates between a maximum and minimum of a potential. However the effective field theory framework in which we work does not rely on the specifics of how this spacetime is realized. We work in the semiclassical $N=\infty$ limit. We consider the case where the boundary field theory lives in flat Euclidean space and the bulk metric has the usual domain wall form

$$\mathrm{d}s^2 = \mathrm{d}r^2 + a^2(r)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j. \tag{4.3}$$

The statement that the geometry interpolates between asymptotic AdS regions is the statement that

$$\lim_{r\to r_{\mathrm{UV}}} a(r) = e^{r/L_{\mathrm{UV}}}, \quad \lim_{r\to -\infty} a(r) = e^{r/L_{\mathrm{IR}}}, \tag{4.4}$$

where r_{UV} is a near-boundary regulator surface. We define the 'Hubble function' $H(r)$ and slow-flow parameters in analogy with the corresponding variables in cosmology

$$H(r) = \frac{\dot{a}}{a}, \quad \epsilon = -\frac{\dot{H}}{H^2}, \quad \eta = \frac{\dot{\epsilon}}{H\epsilon}, \quad \kappa = \frac{\dot{\eta}}{H\eta}, \tag{4.5}$$

where the dot denotes a radial derivative. Note that in exact AdS

$$H(r) = L_{\mathrm{AdS}}^{-1}, \quad \epsilon, \eta, \kappa = 0. \tag{4.6}$$

Given such a spacetime we will study the two point function of a field propagating on this geometry. The procedure applies for the two point function of any field, all we need is the linearized equation of motion. However for concreteness we will focus on the two point function of the scalar fluctuation associated with the broken AdS isometry. The construction of the action for the Goldstone boson is familiar from the construction of the effective field theory of single-field inflation in [42]. The action for the Goldstone boson is constructed via the 'Stuckelberg trick' [42]: π is introduced as the Goldstone boson for the diffeomorphism $L_{\rm AdS}\partial_r + x^i\partial_i$[2] broken by the nontrivial radial dependence of bulk fields and the background metric to restore the full bulk diff invariance. The bulk action is then made up of terms that are invariant under spatial diffeomorphisms as well as under nonlinearly-realized radial diffs: $r \to r + \xi^r(x,r)$, $\pi \to \pi - \xi^r(x,r)$. Performing a broken radial diff, one obtains the most general action for the Goldstone boson.

It was shown in [5] that to leading order in the derivative expansion[3] , the bulk action is made up of contributions from a gravitational sector, a matter sector, and a counterterm sector

$$\begin{aligned} S_\pi =& S_{\rm grav} + S_{\rm m} + S_{\rm ct} \\ S_{\rm m} =& \sum_{n=0}^{\infty} \frac{1}{n!} \int {\rm d}r{\rm d}^d x \sqrt{g} M_n(r+\pi(r,\vec{x})) \left[\frac{\partial(r+\pi(r,\vec{x}))}{\partial x^a} \frac{\partial(r+\pi(r,\vec{x}))}{\partial x^b} g^{ab}(r,\vec{x}) \right], \end{aligned} \tag{4.7}$$

where the $\{M_n\}$ are non-universal, QFT-dependent parameters. We note that the case where the matter sector consists of a single scalar field with a potential corresponds to setting $M_n = 0,\ n \geq 2$. However, for the purposes of computing the quadratic part of the on-shell action, one need only know M_2, which turns out to be related to the speed of sound in cosmological models after Wick rotation. We describe this scenario in detail in appendix D. We take the gravitational action to

[2] Since the broken AdS scale symmetry non-linearly realized by the Goldstone mode π induces a Weyl transformation in the boundary theory, the boundary value of the Goldstone boson is indeed related to the dilaton.

[3] In particular, here we are neglecting terms involving derivatives of the extrinsic curvature.

be Einstein gravity

$$S_{\text{grav}} = -\frac{M_{\text{Pl,d+1}}^{d-1}}{2}\int_{r\geq r_{\text{UV}}} \mathrm{d}r\mathrm{d}^d x\sqrt{g}R^{(d+1)} - M_{\text{Pl,d+1}}^{d-1}\int \mathrm{d}^d x\sqrt{\gamma}K\bigg|_{r=r_{\text{UV}}}, \tag{4.8}$$

where γ is the induced metric on the regulator surface $r = r_{\text{UV}}$, K is the trace of the extrinsic curvature and the last term is the Gibbons-Hawking boundary term required for the well-posedness of the variational principle. One could easily include a Gauss-Bonnet term to distinguish between the A-type anomalies and the central charge of the dual CFT, as is done in [5], however for simplicity of exposition we omit it here. The counterterm action S_{ct} is included to cancel divergences that arise as the regulator surface is taken to the boundary $r_{\text{UV}} \to \infty$. Finally, the background Einstein equations determine the leading parameters M_0 and M_1 in terms of geometric parameters [5]

$$\begin{aligned} M_0 &= -(d-1)M_{\text{Pl,d+1}}^{d-1}\left[\dot{H} + \frac{d}{2}H^2\right] \\ M_1 &= -\frac{d-1}{2}M_{\text{Pl,d+1}}^{d-1}\dot{H}. \end{aligned} \tag{4.9}$$

We further restrict to the *demixing limit* where the Goldstone mode is weakly coupled and the mixing with the gravity transverse-traceless modes can be neglected. It was shown in [42] that in the simplest case where only the lowest-derivative operators are kept, the leading mixing terms with gravity could be neglected for energies above $E_{\text{mix}} \sim \sqrt{-\dot{H}} = \sqrt{\epsilon}H$. This situation corresponds to a single bulk scalar field with a slowly-varying potential (and in the case of spontaneously broken dS symmetry, aligns with standard single-field slow-roll inflation) — this is the general class of theories we intend to study. Thus in this 'slow-flow' approximation, we are free to study the physics of the Goldstone mode while neglecting its mixing with metric fluctuations. For energies much larger than the demixing scale, the action reduces significantly to the following [42, 5]

$$S_\pi = -\frac{(d-1)M_{\text{Pl},d+1}^2}{2}\int \mathrm{d}r\mathrm{d}^d x\ a^d\dot{H}\left(\dot{\pi}^2 + \frac{(\partial_i\pi)^2}{a^2}\right) + \ldots, \tag{4.10}$$

where $(\dots)$ correspond to higher-derivative terms. We will suppress the $(\dots)$ parts for the rest of the paper. The statement that AdS isometries are weakly broken is imposed by the assumption that the slow-flow parameter ϵ and all derivatives thereof that appear in the Goldstone boson's equation of motion (for instance, $\eta = \frac{\dot{\epsilon}}{H\epsilon}$) are small everywhere in the spacetime. Following [5], this is what we refer to as the 'slow-flow' approximation.

AdS/CFT dictates that partition function of the bulk gravitational theory is equal to the generating functional of a boundary quantum field theory. In particular, we expect that the on-shell action for the Goldstone boson of the broken spacetime symmetry in the bulk should be related to the effective action for the dilaton τ encoding the broken conformal symmetry on the boundary

$$W_{\mathrm{QFT}}[\tau] = S_{\pi}^{\text{on-shell}}[\pi|_{r=r_{\mathrm{UV}}}]. \tag{4.11}$$

Anticipating the relationship between the boundary value of the Goldstone mode and the boundary field theory's dilaton field τ, which is dimensionless, it is useful to introduce

$$\hat{\pi} = -H\pi \tag{4.12}$$

such that the action in terms of $\hat{\pi}$ is given by

$$S_{\hat{\pi}} = \frac{(d-1)M_{\mathrm{Pl},d+1}^{d-1}}{2} \int \mathrm{d}r \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d} a^d \epsilon \left(\dot{\hat{\pi}}_{\vec{k}} \dot{\hat{\pi}}_{-\vec{k}} + \frac{k^2}{a^2} \hat{\pi}_{\vec{k}} \hat{\pi}_{-\vec{k}} + \dots \right), \tag{4.13}$$

where we have Fourier-transformed $\hat{\pi}(r, \vec{x}) \equiv \int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d} e^{i\vec{k}\cdot\vec{x}} \hat{\pi}_{\vec{k}}(r)$ and the omitted terms are sub-leading in the derivative expansion and thus drop out in the demixing limit.

The equation of motion for the Goldstone boson then reads

$$\ddot{\hat{\pi}}_{\vec{k}} + H(d+\eta)\dot{\hat{\pi}}_{\vec{k}} - \left(\frac{k}{a}\right)^2 \hat{\pi}_{\vec{k}} = 0, \tag{4.14}$$

so that, assuming regularity of the Goldstone boson in the deep interior, the on-shell action reduces to a boundary term:

$$S_{\hat{\pi}}^{\text{on-shell}} = \frac{(d-1)M_{\mathrm{Pl},d+1}^{d-1}}{2} \int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d} a^d \epsilon \hat{\pi}_{-\vec{k}} \partial_r \hat{\pi}_{\vec{k}} \bigg|_{r=r_{\mathrm{UV}}} . \tag{4.15}$$

4.3 A generalized adiabatic approximation scheme

It is useful to rewrite the equation of motion in Schrodinger form. This can be done by defining

$$\psi(r) \equiv \sqrt{\epsilon a^d}\hat{\pi}(r) \tag{4.16}$$

to cast the equation of motion into

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} - p^2(r,k)\right]\psi = 0, \tag{4.17}$$

where

$$p^2(r,k) \equiv \left(\frac{k}{a}\right)^2 + H^2\left[\left(\frac{d}{2}\right)^2 + \delta\right]$$
$$\delta = -\frac{d}{2}\epsilon + \frac{d}{2}\eta - \frac{1}{2}\epsilon\eta + \frac{1}{4}\eta^2 + \frac{1}{2}\kappa\eta\,. \tag{4.18}$$

As δ encodes the deviation from AdS spacetime and accordingly is at least linear in the 'slow-flow' parameters, it provides a useful perturbative handle when we assume that the AdS isometry corresponding to boundary dilatations is weakly broken.

Here we develop a systematic method for solving the Goldstone boson's equation of motion in Eq (4.17). We are interested in solving this equation in the case where the Hubble function $H(r)$ is slowly varying so that the spacetime symmetry is weakly broken. The solution is known exactly in the case of exact AdS, where the Hubble function $H(r) = L_{\mathrm{AdS}}^{-1}$ is constant, $a = e^{Hr}$ and $\delta = 0$. The solutions in exact AdS are denoted by ϕ_i and solve the following differential equation

$$\begin{aligned}\ddot{\phi}_i &= p_0^2(r,k,H)\phi_i \\ &= \left[e^{-2Hr}k^2 + \frac{d^2}{4}H^2\right]\phi_i,\end{aligned} \tag{4.19}$$

with the following functional form

$$\phi_1[r,k,H] = -\sqrt{2\pi}I_{\frac{d}{2}}(ky_0) + i\sqrt{\frac{2}{\pi}}e^{i\pi d/2}K_{\frac{d}{2}}(ky_0), \quad \phi_2[r,k,H] = i\sqrt{\frac{2}{\pi}}e^{i\pi d/2}K_{\frac{d}{2}}(ky_0), \tag{4.20}$$

where the I and K are the modified Bessel functions and $y_0 = e^{-Hr}/H$. The choice of the normalization and the relative factors between the I and K are chosen so that ϕ_1 is purely e^{+ky_0} and ϕ_2 is purely e^{-ky_0} as $ky_0 \to \infty$. This renders imposing the boundary condition more convenient. Furthermore, the Wronskian is

$$W_{x_0}[\phi_1,\phi_2] = \frac{2}{x_0}ie^{i\pi d/2}\,. \tag{4.21}$$

where we've defined $x_0 = ky_0$.

In general, we can write a solution to Eq (4.17) as

$$\psi(r,k) = c_i(r)\phi_i[r,k,H(r)], \tag{4.22}$$

where H has been promoted to a function of r and the sum on i is implicit. For constant H, of course $c_i =$ constant will be a solution; for H slowly varying, the $c_i(r)$ will be nearly constant, describing how the basis of instantaneous AdS solutions mix with each other as r varies. To be clear, the instantaneous AdS solutions have the following form

$$\phi_1[r,k,H] = -\sqrt{2\pi}I_{\frac{d}{2}}(ky) + i\sqrt{\frac{2}{\pi}}e^{i\pi d/2}K_{\frac{d}{2}}(ky), \quad \phi_2[r,k,H(r)] = i\sqrt{\frac{2}{\pi}}e^{i\pi d/2}K_{\frac{d}{2}}(ky), \tag{4.23}$$

with

$$y \equiv \frac{1}{a(r)H(r)} \equiv \frac{e^{-A(r)}}{H(r)}. \tag{4.24}$$

In particular, $H(r) = \frac{\dot{a}}{a} = \dot{A}$ is no longer constant. Note that

$$\frac{\mathrm{d}y}{\mathrm{d}r} = -yH[1-\epsilon]. \tag{4.25}$$

Substitution of this ansatz into the Schrodinger equation Eq (4.17) yields a second-order differential equation for the $c_i(r)$. However, at this point the $c_i(r)$ are still under-determined. For instance, even when H is constant there will be solutions with c_i non-constant. Inspired by the method of variation of parameters, we choose to impose the following condition

$$\dot{c}_i\phi_i = 0. \tag{4.26}$$

This imposes the condition that as r is varied, $\dot{c}$ is orthogonal to the chosen basis ϕ. With this choice, the differential equation that the coefficients must satisfy simplifies significantly:

$$\dot{c}_i(r)\dot{\phi}_i = -c_i(r)H^2(r)\left(-\epsilon(y\partial_y + 2y^2\partial_y^2) + \epsilon\eta y\partial_y + \epsilon^2 y^2\partial_y^2 - \delta\right)\phi_i, \tag{4.27}$$

where we've used the fact that the ϕ_i satisfy the instantaneous AdS wave equation[4] . This can be written compactly in matrix form

$$\begin{aligned} &\dot{c} = B\cdot c, \\ &B = -H^2W^{-1}\begin{pmatrix} 0 & 0 \\ \alpha_1 & \alpha_2 \end{pmatrix} = -\frac{H^2}{\det W}\begin{pmatrix} -\alpha_1\phi_2 & -\alpha_2\phi_2 \\ \alpha_1\phi_1 & \alpha_2\phi_1 \end{pmatrix}, \end{aligned} \tag{4.28}$$

where

$$\alpha_i = \left[-\epsilon(y\partial_y + 2y^2\partial_y^2) + \epsilon\eta y\partial_y + \epsilon^2y^2\partial_y^2 - \delta\right]\phi_i, \quad W \equiv \begin{pmatrix} \phi_1 & \phi_2 \\ \dot{\phi}_1 & \dot{\phi}_2 \end{pmatrix}. \tag{4.29}$$

[4] That is, $(y\partial_y + y^2\partial_y^2)\phi_i = (k^2 + \frac{d^2}{4})\phi_i$.

The α_i are dimensionless and perturbative in a slow-flow. It is straightforward to see that

$$\det W = -2ie^{i\pi d/2}H(r)(1-\epsilon), \tag{4.30}$$

so that

$$B(r)\mathrm{d}r = -\frac{1}{2ie^{i\pi d/2}(1-\epsilon)^2}\frac{\mathrm{d}y}{y}\begin{pmatrix} -\alpha_1\phi_2 & -\alpha_2\phi_2 \\ \alpha_1\phi_1 & \alpha_2\phi_1 \end{pmatrix}. \tag{4.31}$$

Since the α_i are dimensionless, and ϕ is a function of ky (which is dimensionless), we define $x \equiv ky$ and rewrite

$$\int B(r)\mathrm{d}r = -\frac{1}{2ie^{i\pi d/2}}\int \frac{\mathrm{d}x}{x(1-\epsilon)^2}\begin{pmatrix} -\alpha_1\phi_2 & -\alpha_2\phi_2 \\ \alpha_1\phi_1 & \alpha_2\phi_1 \end{pmatrix}. \tag{4.32}$$

We will make frequent use of the dimensionless coordinate x.

This differential equation for the coefficients is readily solved by a radially-ordered exponential

$$c(r) = \mathcal{R}\exp\left\{\int_{r_0}^{r}\mathrm{d}r_1 B(r_1)\right\}c(r_0). \tag{4.33}$$

This is the main result of this paper. Note that $B(r)$ vanishes when H is constant, so that $c(r)$ will be constant as well. It is straightforward to check that this solution satisfies the condition (4.26). To find approximate solutions when $H(r)$ is slowly varying, we implement a Dyson series (as in time-dependent perturbation theory)

$$\mathcal{R}\exp\left\{\int_{r_0}^{r}\mathrm{d}r_1 B(r_1)\right\} = 1 + \int_{r_0}^{r}\mathrm{d}r_1 B(r_1) + \int_{r_0}^{r}\mathrm{d}r_1\int_{r_0}^{r_1}\mathrm{d}r_2 B(r_1)B(r_2) + \dots \tag{4.34}$$

The upshot is that this provides a systematic perturbative expansion as a solution for the coefficients describing the mixing in the basis of 'instantaneous' AdS solutions, since the α_i are cleanly organized in powers of the slow-flow parameters. Let us rewrite the Dyson series for $c(r)$ in terms of V defined through

$$c(r) = [1 + V(r, r_0)]\cdot c(r_0). \tag{4.35}$$

4.3.1 Imposing boundary conditions

Returning to our problem, we now impose the boundary conditions. We first impose regularity in the deep interior, i.e.

$$\lim_{y_{\rm IR}\to\infty} \hat{\pi}(y_{\rm IR}) = 0 \quad \Rightarrow \quad \lim_{y_{\rm IR}\to\infty} c_1(y_{\rm IR}) = 0 \tag{4.36}$$

such that

$$\begin{aligned} c_1(y) &= V_{12}(y, y_{\rm IR})c_2(y_{\rm IR}) \\ c_2(y) &= [1 + V_{22}(y, y_{\rm IR})]c_2(y_{\rm IR})\,, \end{aligned} \tag{4.37}$$

where

$$V(y, y_{\rm IR}) \equiv \begin{pmatrix} V_{11}(y, y_{\rm IR}) & V_{12}(y, y_{\rm IR}) \\ V_{21}(y, y_{\rm IR}) & V_{22}(y, y_{\rm IR}) \end{pmatrix} . \tag{4.38}$$

Following [5], the UV boundary condition relates the Goldstone scalar to the dilaton

$$\hat{\pi}(y_{\rm UV}) = \tau, \tag{4.39}$$

which translates into

$$\sqrt{(y_{\rm UV}H_{UV})^{-d}\epsilon_{\rm UV}}\tau \;=\; \{\phi_1(y_{\rm UV})V_{12}(y_{\rm UV}, y_{\rm IR}) + \phi_2(y_{\rm UV})[1 + V_{22}(y_{\rm UV}, y_{\rm IR})]\}\, c_2(y_{\rm IR})\,. \tag{4.40}$$

Thus, the c_1 and c_2 that satisfy both the IR and UV boundary conditions are

$$\begin{aligned} c_1(y) &= \left(\frac{\sqrt{(y_{\rm UV}H_{UV})^{-d}\epsilon_{\rm UV}}\tau}{\phi_2(y_{\rm UV})}\right) \frac{V_{12}(y, y_{\rm IR})}{1 + \frac{\phi_1(y_{\rm UV})}{\phi_2(y_{\rm UV})}V_{12}(y_{\rm UV}, y_{\rm IR}) + V_{22}(y_{\rm UV}, y_{\rm IR})} \\ &\equiv \left(\frac{\sqrt{(y_{\rm UV}H_{UV})^{-d}\epsilon_{\rm UV}}\tau}{\phi_2(y_{\rm UV})}\right) \tilde{c}_1(y) \\ c_2(y) &= \left(\frac{\sqrt{(y_{\rm UV}H_{UV})^{-d}\epsilon_{\rm UV}}\tau}{\phi_2(y_{\rm UV})}\right) \frac{1 + V_{22}(y, y_{\rm IR})}{1 + \frac{\phi_1(y_{\rm UV})}{\phi_2(y_{\rm UV})}V_{12}(y_{\rm UV}, y_{\rm IR}) + V_{22}(y_{\rm UV}, y_{\rm IR})} \\ &\equiv \left(\frac{\sqrt{(y_{\rm UV}H_{UV})^{-d}\epsilon_{\rm UV}}\tau}{\phi_2(y_{\rm UV})}\right) \tilde{c}_2(y)\,. \end{aligned} \tag{4.41}$$

Defining

$$S_\pi|_{\text{on-shell}} \equiv \frac{(d-1)M_{\text{Pl,d+1}}^{d-1}}{2} \int \frac{\mathrm{d}^d\vec{k}}{(2\pi)^d} I_{\text{bndy}}, \tag{4.42}$$

the on-shell action can be written given in terms of the coefficient $\tilde{c}_1$ and the basis of solutions $\{\phi_i\}$ as

$$I_{\text{bndy}} = -\frac{\epsilon k^d}{H^{d-1}x^d}\left[x(1-\epsilon)\left(\tilde{c}_1 \frac{\phi_2\partial_x\phi_1 - \phi_1\partial_x\phi_2}{\phi_2^2}\right) + \frac{\partial_x\phi_2}{\phi_2} + \left(\frac{d}{2}+\frac{\eta}{2}\right)\right]\tau_{\vec{k}}\tau_{-\vec{k}}\Big|_{x=x_{\text{UV}}} \tag{4.43}$$

To recapitulate, we have imposed both the IR and the UV boundary conditions without making any further approximations beyond the demixing limit. The remaining task is to compute the $\tilde{c}_i$ via our adiabatic approximation scheme, making use of the assumption that the AdS symmetry is weakly broken.

4.4 Application 1: holographic RG-improved two-point functions in weakly-relevant flows

In this section, we will study holographic duals of weakly-relevant flows obtained by deforming a CFT$_d$ by a nearly marginal operator $\mathcal{O}$ with dimension $\Delta = d - \lambda$ where $0 < \lambda \ll 1$. These flows have been studied in various contexts [44, 25, 45]. For some relevant recent work in the context of a-theorem or holography, see [27, 40, 46, 47, 41].

4.4.1 Review of the field theory setup and results

The action of the perturbed theory is given by

$$S = S_{CFT} + \phi \int d^d x\ \mathcal{O}(x). \tag{4.44}$$

The bare two and three-point functions of $\mathcal{O}$ in the CFT are given by

$$\begin{aligned} \langle O(x)O(y)\rangle_{CFT} &= \frac{G_{\mathcal{O}\mathcal{O}}}{|x-y|^{2(d-\lambda)}} \\ \langle O(x)O(y)O(z)\rangle_{CFT} &= \frac{G_{\mathcal{O}\mathcal{O}}C}{|x-y|^{d-\lambda}|x-z|^{d-\lambda}|y-z|^{d-\lambda}}, \end{aligned} \tag{4.45}$$

where $G_{\mathcal{O}\mathcal{O}}$ is some normalization and C is the OPE coefficient. The beta function of the renormalized coupling g is

$$\beta(g) = \mu \frac{dg(\mu)}{d\mu} = -\lambda g + \frac{1}{2}\Omega_{d-1} C g^2 + \mathcal{O}(g^3)\,, \tag{4.46}$$

where $\Omega_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$. Since $\lambda \ll 1$, there exists a perturbatively controlled IR fixed point at

$$g_* \equiv \frac{2\lambda}{C\Omega_{d-1}}. \tag{4.47}$$

There existence of such a fixed point ensures that the coupling is perturbative throughout the entire flow. Integrating the beta function with the boundary condition

$$\lim_{\mu\to\infty} g(\mu) = \phi\mu^{-\lambda} + \ldots \tag{4.48}$$

gives

$$g(\mu) = \frac{g_*}{1 + g_*/(\phi\mu^{-\lambda})}\,. \tag{4.49}$$

The particular two-point function that we are after is the RG-improved/resummed two-point function in a weakly relevant flow. One can either think of it as the solution to the Callan-Symanzik equation (such as discussed in [48, 47]) or a particular resummation (akin to the leading-log resummation in QFT) of perturbation theory. The latter was carried out in detail in [46, 47]. The resultant two-point function

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle \equiv \langle \mathcal{O}(x)\mathcal{O}(0) e^{\phi \int d^d z\ \mathcal{O}(z)}\rangle_{\mathrm{CFT}} \tag{4.50}$$

is given in momentum space by[5]

$$
\begin{aligned}
\text{odd d} \;:\; \langle \mathcal{O}(k)\mathcal{O}(-k)\rangle' &= G_{\mathcal{O}\mathcal{O}} \frac{2^{-d}\Gamma(-\frac{d}{2})}{\Gamma(d)} \pi^{d/2} \times \frac{k^{d-2\lambda}}{(1+\xi)^4}\,, \\
\text{even d} \;:\; \langle \mathcal{O}(k)\mathcal{O}(-k)\rangle' &= G_{\mathcal{O}\mathcal{O}} \frac{\left(\frac{i}{2}\right)^d \pi^{d/2}}{\lambda\Gamma(d)\Gamma(\frac{d}{2}+1)} \times k^{d-2\lambda} \frac{3+\xi}{3\,(1+\xi)^3}\,,
\end{aligned}
\tag{4.51}
$$

where $\xi \equiv k^{-\lambda}\phi/g_*$. We will show how to reproduce these correlators in the bulk using the adiabatic scheme that we described in the previous sections to compute the on-shell action of the Goldstone boson. Fourier-transforming the above correlation functions gives the position-space RG-improved two-point function

$$
\text{all d} \;:\; \langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \frac{G_{\mathcal{O}\mathcal{O}}}{x^{2(d-\lambda)}} \left[1 + \frac{\phi x^{\lambda}}{g_*}\right]^{-4}. \tag{4.52}
$$

4.4.2 The bulk setup: slow-roll backgrounds in Einstein-scalar theories

Following [46, 47, 33, 34], here we will focus on the class of theories where higher derivative operators are switched off in the Goldstone boson action. These are theories with the Einstein gravity action and a single bulk scalar field with a potential. The (Euclidean) action is

$$
S = M_{\mathrm{Pl,d+1}}^{d-1} \int d^{d+1}x \sqrt{g} \left[-R + \frac{1}{2}(\partial\Phi)^2 + V(\Phi)\right]. \tag{4.53}
$$

In the case of a domain-wall geometry

$$
ds^2 = dr^2 + a(r)^2 \delta_{ij} \mathrm{d}x^i \mathrm{d}x^j, \tag{4.54}
$$

and assuming Φ only carries radial dependence, the Einstein equations reduce to

$$
2(d-1)\dot{H} + \dot{\Phi}^2 = 0, \quad (d-1)\dot{H} + d(d-1)H^2 = V. \tag{4.55}
$$

[5] Here the prime denotes that we have stripped the delta function and accompanying factors of 2π from these momentum space correlators.

It is convenient to view H as a functional of Φ and define

$$W(\Phi) \equiv -2(d-1)H(\Phi) \tag{4.56}$$

to reduce the equations of motion to be of first order form

$$\begin{aligned} W' &= \dot{\Phi} \\ -\frac{1}{2}(W')^2 + \frac{d}{4(d-1)}W^2 &= V, \end{aligned} \tag{4.57}$$

where $W' = \frac{\mathrm{d}W}{\mathrm{d}\Phi}$. For example, in the case of an exact AdS$_{d+1}$ spacetime with AdS length L_{AdS}, we obtain

$$W = -2(d-1)/L_{\mathrm{AdS}}. \tag{4.58}$$

The assumption of 'slow-flow' or softly-broken AdS isometry is to restrict ourselves the potentials where we are allowed to expand $W(\Phi)$ near an AdS extremum (with AdS length L_{UV}):

$$L_{UV}W = -2(d-1) - \frac{1}{2}a_2\Phi^2 + \frac{1}{3!}a_3\Phi^3 + \mathcal{O}(\Phi^4), \tag{4.59}$$

where we have conveniently shifted Φ such that this minimum occurs at $\Phi = 0$. This implies an expansion of V in Φ:

$$-L_{UV}^2 V = -d(d-1) + \frac{1}{2}a_2\,(a_2-d)\,\Phi^2 + \frac{1}{3!}a_3\,(d-3a_2)\,\Phi^3 + \mathcal{O}(\Phi^4)\,. \tag{4.60}$$

Now, from the standard AdS/CFT correspondence, one should identify

$$a_2\,(a_2-d) = \Delta(\Delta-d) \quad \Rightarrow \quad \lambda = d - \Delta_+ = a_2\,. \tag{4.61}$$

The statement that $\lambda \ll 1$ is equivalent to the boundary RG flow being driven by a weakly relevant scalar operator $\mathcal{O}$ of dimension $\Delta = d-\lambda$. Such flows were reviewed on the field theory side in section 4.4.1.

Now, standard AdS/CFT tells us that the OPE coefficient $C_{\mathcal{O}\mathcal{O}}^{\mathcal{O}} \equiv C$ is related to the cubic coupling in the bulk potential; one sees this by taking the ratio of the coefficient of the three-point function to that of the two-point function [13]. In our case, with $\Delta = d - \lambda$, this yields

$$C = \frac{a_3}{\Omega_{d-1}} + \mathcal{O}(\lambda)\,. \tag{4.62}$$

Thus

$$L_{UV} W = -2(d-1) - \frac{1}{2}\lambda\Phi^2 + \frac{1}{3!}\Omega_{d-1} C \Phi^3 + \mathcal{O}(\Phi^4)\,. \tag{4.63}$$

Now using $W' = \dot{\Phi}$, we obtain[6]

$$L_{UV}\dot{\Phi} = -\lambda\Phi + \lambda\Phi^2/\Phi_* + \mathcal{O}(\Phi^3) \tag{4.64}$$

where we have defined

$$\Phi_* \equiv \frac{2\lambda}{C\Omega_{d-1}}. \tag{4.65}$$

One recognizes that the other AdS extremum is at $\Phi = \Phi_*$ with AdS length

$$L_{IR} = H_{IR}^{-1} = -2(d-1)/W(\Phi_*) \approx L_{UV}\left[1 - \frac{\lambda}{12(d-1)}\Phi_*^2\right] + \mathcal{O}(\lambda^4)\,. \tag{4.66}$$

So in this setup we see explicitly that $\frac{\Delta L}{L_{\rm UV}} \sim \mathcal{O}(\lambda^3)$ is suppressed — an assumption that was made independently in establishing the slow-flow limit in the discussion of the dilaton effective action in $d \geq 4$ in [5]. In terms of the bulk scalar, the definitions of the slow-flow parameters imply

$$\begin{aligned}
\epsilon &= 2(d-1)\left(\frac{W'}{W}\right)^2 = \frac{\lambda^2\Phi_*^2}{2(d-1)}\left[\frac{\Phi}{\Phi_*}\left(1 - \frac{\Phi}{\Phi_*}\right)\right]^2 + \mathcal{O}(\lambda^7) \\
\eta &= -2\lambda\left(1 - 2\frac{\Phi}{\Phi_*}\right) + \mathcal{O}(\lambda^4) \\
\eta\kappa &= -4\lambda^2\frac{\Phi}{\Phi_*}\left(1 - \frac{\Phi}{\Phi_*}\right) + \mathcal{O}(\lambda^5).
\end{aligned} \tag{4.67}$$

---

[6] The above equation is the dual of the field theory beta-function equation in Eq. (4.46). The AdS fixed point Φ_* is exactly the same as the IR fixed point g_* in the field theory.

Notice that $\eta \sim \mathcal{O}(\lambda)$ while $\epsilon \sim \mathcal{O}(\lambda^4)$. The slow roll parameters ϵ and κ vanish at both the UV and IR fixed points, while η takes on the asymptotic values of -2λ and $2\lambda + \mathcal{O}(\lambda^4)$, respectively.

Solving Eq. (4.64) with boundary condition $\Phi \to \phi y^\lambda$ as $y \to 0$, we obtain Φ as an expansion in the bare coupling ϕ:

$$\Phi/\Phi_* = \frac{1}{1 + y^{-\lambda}\Phi_*/\phi} = \phi \frac{y^\lambda}{\Phi_*} - \phi^2 \frac{y^{2\lambda}}{\Phi_*^2} + \mathcal{O}(\phi^3). \tag{4.68}$$

From the definition of η, this gives

$$\eta = -2\lambda \left[1 - \frac{2}{1 + y^{-\lambda}\Phi_*/\phi} + \mathcal{O}(\lambda^4)\right] = -2\lambda \left[1 - 2\phi \frac{y^\lambda}{\Phi_*} + 2\phi^2 \frac{y^{2\lambda}}{\Phi_*^2} + \dots\right]. \tag{4.69}$$

From now on we will only keep the leading-λ contributions and not explicitly write out the higher $\mathcal{O}(\lambda^p)$ that have been dropped.

4.4.3 Setting up the adiabatic scheme

For the bulk setup, we consider the Einstein-scalar theories as discussed in Sec. 4.4.2. It turns out to be convenient to choose the basis of instantaneous AdS solutions $\{\phi_i\}$ in the adiabatic scheme to be the modified Bessel function solutions

$$\phi_2[r, k, H(r)] \quad = \quad K_{d/2-\lambda}(ky), \quad \phi_1[r, k, H(r)] = I_{d/2-\lambda}(ky)\,. \tag{4.70}$$

The reason to do so is that these are the exact solutions of the equations of motion near the boundary, so that in the $\phi \to 0$ limit, we automatically recover the CFT two-point function $k^{d-2\lambda}$. To see this, note that the equation of motion (4.18) can be written near the boundary as

$$\begin{aligned} \left\{\frac{\mathrm{d}^2}{\mathrm{d}r^2} - H^2\left[\left(\frac{k}{aH}\right)^2 + \left(\frac{d}{2} - \lambda\right)^2 + \mathcal{O}\left(\frac{\phi y^\lambda}{\Phi_*}\right)\right]\right\}\psi = 0 \\ \left\{x^2\partial_x^2 + x\partial_x - \left[x^2 + \left(\frac{d}{2} - \lambda\right)^2 + \mathcal{O}\left(\frac{\phi y^\lambda}{\Phi_*}\right)\right]\right\}\psi = 0. \end{aligned} \tag{4.71}$$

This choice will trivially modify some of the equations in the previous sections. Since we are interested in the leading result in conformal perturbation theory, we choose to neglect terms that are subleading as $\lambda \to 0$. For instance, we will take

$$\alpha_i = \frac{d}{2}\eta\phi_i + \mathcal{O}(\lambda^2) = d\lambda\left[1 - \frac{2}{1+y^{-\lambda}\Phi_*/\phi}\right]\phi_i + \mathcal{O}(\lambda^2)\,. \tag{4.72}$$

At the end of the day, this ends up modifying the Dyson series for c through the expression of B as in the following way:

$$\int_{r_{IR}} B(r)\mathrm{d}r = \int_\infty \frac{\mathrm{d}y}{y}\alpha\begin{pmatrix} -\phi_1\phi_2 & -\phi_2\phi_2 \\ \phi_1\phi_1 & \phi_2\phi_1 \end{pmatrix} + \mathcal{O}(\lambda^2), \tag{4.73}$$

where

$$\alpha \equiv -2d\lambda\left[1 + y^{-\lambda}\Phi_*/\phi\right]^{-1}. \tag{4.74}$$

4.4.4 The on-shell action

We are now well positioned to evaluate the on-shell action in Eq. (4.15). Recalling the definition (4.42), we write the on-shell action in terms of the coefficient $\tilde{c}_1$ and the basis of solutions $\{\phi_i\}$

$$I_{\text{bndy}} = -\frac{k^d\epsilon}{H^{d-1}}\left\{\frac{1}{x^{d-1}}(1-\epsilon)\left[\frac{\partial_x\phi_2}{\phi_2} + \frac{\tilde{c}_1}{x\phi_2^2}\right] + \frac{1}{x^d}\left(\frac{d}{2} + \frac{\eta}{2}\right)\right\}\tau_{\vec{k}}\tau_{-\vec{k}}\bigg|_{x=x_{\text{UV}}} \tag{4.75}$$

where we have used the UV boundary condition

$$\tilde{c}_2(x_{\text{UV}}) = 1 - \tilde{c}_1(x_{\text{UV}})\frac{\phi_1(x_{\text{UV}})}{\phi_2(x_{\text{UV}})} \tag{4.76}$$

together with the Wronskian of ϕ_2 and ϕ_1.

Denoting $\tilde{c}_i^{(p)}$ as the contribution from the $\mathcal{O}(\phi^p)$ term, collecting up to $\tilde{c}_i^{(2)}$, we obtain

$$\frac{I_{\text{bndy}}}{k^d H_{\text{UV}}^{-d+1}\epsilon_{\text{UV}}\tau_{\vec{k}}\tau_{-\vec{k}}} = -\frac{d+\eta_{UV}}{2x_{\text{UV}}^d} - \frac{1}{x_{\text{UV}}^{d-1}}\frac{\partial_x\phi_2}{\phi_2} - \frac{1}{x_{\text{UV}}^d}\frac{1}{\phi_2^2}(\tilde{c}_1^{(1)} + \tilde{c}_1^{(2)} + \ldots) \tag{4.77}$$

The first term is analytic in x_{UV}, so we shall ignore it since we are looking for non-analytic x_{UV} behavior which is appropriate for a two point function; the terms analytic in $x_{\rm UV} = ky_{\rm UV}$ either vanish as the boundary is approached or diverge and are to be cancelled by the addition of local covariant counterterms to the action. We implicitly drop analytic terms in the following. The leading non-analytic piece of the second term is[7]

$$-\frac{1}{x_{\rm UV}^{d-1}}\frac{\partial_x\phi_2}{\phi_2} \approx -d\frac{2^{-\frac{d}{2}+\lambda-1}\Gamma\left(\lambda-\frac{d}{2}\right)}{2^{\frac{d}{2}-\lambda-1}\Gamma\left(\frac{d}{2}-\lambda\right)}(ky_{\rm UV})^{-2\lambda}. \tag{4.79}$$

Depending on the whether d is even or odd, the expansion in small λ is different, and is given by

$$\begin{aligned}
d \text{ odd} \quad &: \quad -\frac{1}{x_{\rm UV}^{d-1}}\frac{\partial_x\phi_2}{\phi_2} = \frac{2^{-d+1}\Gamma\left(1-\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(ky_{\rm UV})^{-2\lambda} \\
d \text{ even} \quad &: \quad -\frac{1}{x_{\rm UV}^{d-1}}\frac{\partial_x\phi_2}{\phi_2} = -\frac{2\left(\frac{i}{2}\right)^d}{\lambda\Gamma\left(\frac{d}{2}\right)^2}(ky_{\rm UV})^{-2\lambda}
\end{aligned} \tag{4.80}$$

which together with the overall k^d gives $k^{d-2\lambda}$ behavior.

The rest of the terms

$$-\frac{1}{x_{\rm UV}^d}\frac{1}{\phi_2^2}(\tilde{c}_1^{(1)}+\tilde{c}_1^{(2)}+\ldots) \tag{4.81}$$

involve performing the nested integrals. The computation is non-trivial, however, the resummation of all-orders can be done and is described in Appendix A. The result is

---

[7] We make use of the small x expansion of the modified Bessel functions

$$\phi_2(x) \approx x^{\lambda-\frac{d}{2}}\left[2^{\frac{d}{2}-\lambda-1}\Gamma\left(\frac{d}{2}-\lambda\right)+\ldots\right]+x^{\frac{d}{2}-\lambda}\left[2^{-\frac{d}{2}+\lambda-1}\Gamma\left(\lambda-\frac{d}{2}\right)+\ldots\right] \tag{4.78}$$

where $\ldots$ are terms of x^2 higher.

the *all-order* $\sum_{m=1}^{\infty}\tilde{c}_1^{(m)}$ obtained in Eq. (4.123) and Eq. (4.134) which resums into

$$\begin{aligned}
\text{odd d} \quad &: \quad \tilde{c}_1(y_{\rm UV}) = -\frac{1}{2}\Gamma\left(\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right)\left[(1+\xi)^{-4}-1\right] \\
\text{even d} \quad &: \quad \tilde{c}_1(y_{\rm UV}) = \frac{i^d}{2\lambda}\left[\frac{\xi+3}{3(\xi+1)^3}-1\right] ,
\end{aligned} \tag{4.82}$$

where $\xi \equiv k^{-\lambda}\phi/\Phi_*$, and leads to

$$\begin{aligned}
\text{odd } d \quad &: \quad -\frac{1}{x_{\rm UV}^{d-1}}\frac{\partial_x\phi_2}{\phi_2}-\frac{1}{x_{\rm UV}^{d}}\frac{1}{\phi_2^2}\left(\tilde{c}_1^{(1)}+\tilde{c}_1^{(2)}\right) \approx \frac{2^{-d+1}\Gamma\left(1-\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}(kz_{\rm UV})^{-2\lambda}\times(1+\xi)^{-4} \\
\text{even } d \quad &: \quad -\frac{1}{x_{\rm UV}^{d-1}}\frac{\partial_x\phi_2}{\phi_2}-\frac{1}{x_{\rm UV}^{d}}\frac{1}{\phi_2^2}\left(\tilde{c}_1^{(1)}+\tilde{c}_1^{(2)}\right) \approx -\frac{2\left(\frac{i}{2}\right)^d}{\lambda\Gamma\left(\frac{d}{2}\right)^2}(kz_{\rm UV})^{-2\lambda}\times\frac{3+\xi}{3(1+\xi)^3} .
\end{aligned} \tag{4.83}$$

This implies the contributions to the on-shell action

$$\begin{aligned}
\text{odd } d \quad &: \quad I_{\rm bndy} = -\left(\frac{\epsilon_{\rm UV}y_{\rm UV}^{-2\lambda}}{H_{\rm UV}^{d-1}}\right)\tau_{\vec{k}}\tau_{-\vec{k}}\times\frac{d}{\pi^{d/2}}\times\frac{2^{-d}\Gamma\left(-\frac{d}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\pi^{d/2}\times\frac{k^{d-2\lambda}}{(1+\xi)^4} \\
\text{even } d \quad &: \quad I_{\rm bndy} = -\left(\frac{\epsilon_{\rm UV}y_{\rm UV}^{-2\lambda}}{H_{\rm UV}^{d-1}}\right)\tau_{\vec{k}}\tau_{-\vec{k}}\times\frac{2\left(\frac{i}{2}\right)^d}{\lambda\Gamma\left(\frac{d}{2}\right)^2}\times k^{d-2\lambda}\frac{3+\xi}{3(1+\xi)^3} .
\end{aligned} \tag{4.84}$$

These match with Eq. (4.51), so the computation of the Goldstone boson's on-shell action indeed reproduces the RG-improved two-point function as computed in conformal perturbation theory. This is the main result of this section.

4.4.5 Reading off beta functions and anomalous dimensions

Having obtained the RG-improved two point functions from the Goldstone boson's on-shell action, we now proceed as in standard field theory to derive the beta function and the anomalous dimension implied by the Callan-Symanzik equations that RG-improved correlation functions satisfy.

In fact, the two-point functions obtained above are the *bare* two-point functions. Let us denote their position-space version as $\langle\mathcal{O}_{\rm bare}(x)\mathcal{O}_{\rm bare}(0)\rangle$. In any dimension, the relevant bare two-point function is given by

$$\text{all d} \quad : \quad \langle\mathcal{O}_{\rm bare}(x)\mathcal{O}_{\rm bare}(0)\rangle = \frac{G_{\mathcal{O}\mathcal{O}}}{x^{2(d-\lambda)}}\left[1+\frac{\phi x^\lambda}{\Phi_*}\right]^{-4} . \tag{4.85}$$

It satisfies the Callan-Symanzik equation

$$[x\partial_x - \lambda\phi\partial_\phi + 2(d-\lambda)]\,\langle\mathcal{O}_{\rm bare}(x)\mathcal{O}_{\rm bare}(0)\rangle = 0\,. \tag{4.86}$$

This implies the following beta function and anomalous dimension for the bare operator $\mathcal{O}_{\rm bare}$ and the bare coupling ϕ:

$$\beta_\phi = -\lambda\phi; \quad \gamma_\phi = -\lambda\,, \tag{4.87}$$

which are all fixed by the classical scaling dimension of $\mathcal{O}_{\rm bare}$ which is $\Delta = d-\lambda$. The trace of the stress tensor is then

$$T = \beta_\phi\mathcal{O}_{\rm bare} = -\lambda\phi\mathcal{O}_{\rm bare}. \tag{4.88}$$

Using the fact the stress tensor does not renormalize we deduce that

$$T = \beta_{g(\mu)}\mathcal{O}_{\rm ren} = -\lambda\phi\mathcal{O}_{\rm bare} \tag{4.89}$$

where $g(\mu)$ is the renormalized coupling at some scale μ, corresponding to the renormalized operator

$$\mathcal{O}_{\rm ren} \equiv \mathcal{O}_{\rm bare}/\sqrt{Z(g)}. \tag{4.90}$$

We choose the renormalization condition to be

$$\langle\mathcal{O}_{\rm ren}(x)\mathcal{O}_{\rm ren}(0)\rangle|_{x=\mu^{-1}} = G_{\mathcal{O}\mathcal{O}}\mu^{2d} \tag{4.91}$$

with $\langle\mathcal{O}_{\rm bare}(k)\mathcal{O}_{\rm bare}(-k)\rangle|_{\xi=0} = G_{\mathcal{O}\mathcal{O}}k^{d-2\lambda}$. From the explicit form of the two-point function, this implies

$$\sqrt{Z} = \mu^{-\lambda}\left(1+\frac{\phi\mu^{-\lambda}}{\Phi_*}\right)^{-2}. \tag{4.92}$$

On the other hand, using Eq (4.90) and Eq (4.89),

$$\beta_g = -\lambda\phi\sqrt{Z} \tag{4.93}$$

yielding

$$\beta_g = -\lambda(\phi\mu^{-\lambda})\left(1 + \frac{\phi\mu^{-\lambda}}{\Phi_*}\right)^{-2}. \tag{4.94}$$

From the definition of the beta function, we integrate to get

$$g(\mu) = \frac{\Phi_*}{1 + \Phi_*/(\phi\mu^{-\lambda})}\,, \tag{4.95}$$

setting the boundary condition $g \to \phi\mu^{-\lambda}$ as $\mu \to \infty$. Inverting the relation between g and ϕ

$$\phi\mu^{-\lambda} = \frac{g}{1 - \frac{g}{\Phi_*}}\,, \tag{4.96}$$

and rewriting β_g in terms of the renormalized coupling g, we obtain

$$\beta_g = -\lambda g + \frac{\lambda}{\Phi_*}g^2 + \mathcal{O}(g^3) \tag{4.97}$$

which is the familiar result in conformal perturbation theory for weakly relevant flows.

All together, in terms of the renormalized coupling g, the renormalized two point function is given by

$$\langle \mathcal{O}_{ren}(x)\mathcal{O}_{ren}(0)\rangle = G_{\mathcal{O}\mathcal{O}}\mu^{2d}\frac{1}{(\mu x)^{2(d-\lambda)}}\left[1 - \frac{g}{\Phi_*}\left(1 - (\mu x)^{\lambda}\right)\right]^{-4} \tag{4.98}$$

which is what is found in [48, 47]. It satisfies the Callan-Symanzik equations

$$\left[\mu\partial_\mu + \beta(g)\partial_g + 2\gamma\right]\langle \mathcal{O}_{\rm ren}(k)\mathcal{O}_{\rm ren}(-k)\rangle = 0\,, \tag{4.99}$$

where

$$\gamma = -\lambda + 2\lambda\frac{g}{\Phi_*} + \mathcal{O}(\lambda^3) = \frac{d\beta}{dg}. \tag{4.100}$$

In this parameterization it is manifest that the anomalous dimension γ of the dual operator flows between $-\lambda$ in the UV to $\lambda + \mathcal{O}(\lambda^4)$ in the IR.

Upon identifying the RG scale μ with the bulk radial direction $\mu \sim aH$[8] and the running coupling with the bulk scalar field, one can read off the following relationships between the parameters encoding the breaking of spacetime symmetry in the bulk and the RG parameters on the boundary theory

$$\begin{aligned}\beta^2(\Phi) &= \left(\mu\frac{\mathrm{d}\Phi}{\mathrm{d}\mu}\right)^2 = 2(d-1)\frac{\epsilon}{(1-\epsilon)^2} = 2(d-1)\epsilon + \mathcal{O}(\lambda^7)\\ \gamma_\Phi &= \frac{\mathrm{d}\beta(\Phi)}{\mathrm{d}\Phi} = \frac{1}{2}\eta + \mathcal{O}(\lambda^4).\end{aligned} \tag{4.101}$$

In fact, it is easy to see that these relationships are explicitly realized in this setup by comparing (4.67) with (4.97) and (4.100).

4.5 Application 2: reproducing the inflationary power spectrum

4.5.1 The on-shell action

In this section, we reproduce the power-spectrum in slow-roll inflation by making use of our adiabatic approximation scheme and performing a Wick rotation to that the background is weakly-broken de Sitter. The perturbation series is organized in powers of the asymptotic *boundary values* (or, after Wick rotation, super-horizon values) of the slow-roll functions (ϵ, η, κ, etc.).[9]

With this in mind, and restricting to $d = 3$, we are well positioned to evaluate the on-shell action in Eq. (4.15). In this case, the basis of solutions is given by

$$\phi_1(x) = \frac{e^x(1-x)}{x^{3/2}}, \quad \phi_2(x) = \frac{e^{-x}(1+x)}{x^{3/2}}, \tag{4.102}$$

[8] This amounts to a choice of scheme on the boundary

[9] In this bulk computation without a specific boundary dual in mind, we take the inflationary approach and treat all slow-roll parameters as of the same order in perturbation theory.

leading to

$$I_{\text{bndy}} = k^3 \left\{ (1-\epsilon) \left[\frac{2e^{2x_{\text{UV}}}}{(1+x_{\text{UV}})^2} \tilde{c}_1(x_{\text{UV}}) + \frac{1}{x_{\text{UV}}(1+x_{\text{UV}})} \right] - \left(\frac{3\epsilon+\eta}{2x_{\text{UV}}^3} \right) \right\} \frac{\epsilon \tau_{\vec{k}} \tau_{-\vec{k}}}{H_{\text{UV}}^2} \bigg|_{x=x_{\text{UV}}} \tag{4.103}$$

Note that we still have not taken the $x_{\text{UV}} \to 0$ limit. Furthermore, no slow-flow expansion has yet been made beyond imposing the demixing limit. As shown previously, to compute the coefficients $\tilde{c}_i$, one employs the Dyson series expansion - see Appendix C for some explicit calculations.

Employing our perturbative solutions for the coefficients and expanding near $x_{\text{UV}} \to 0$, we find the following as the on-shell action for the Goldstone scalar

$$\begin{aligned} I_{\text{bndy}} = \frac{\epsilon \tau_{\vec{k}} \tau_{-\vec{k}}}{H_{\text{UV}}^2} k^3 \Bigg\{ & \frac{1}{x_{\text{UV}}} \left[1 - \eta - \epsilon + \eta\kappa + \eta^2 + \epsilon^2 + 3\epsilon\eta \right] \\ & - 1 + \eta \left[b - \log x_{\text{UV}} \right] + \epsilon \left[2(b-1) - 2\log x_{\text{UV}} \right] \\ & + \eta\kappa \left[-\frac{\pi^2}{12} - \frac{1}{2}(b - \log x_{\text{UV}})^2 \right] \\ & + \eta^2 \left[-1 - \frac{1}{2}(b - \log x_{\text{UV}})^2 \right] \\ & + \epsilon^2 \left[-9 + 6(b - \log x_{\text{UV}}) - 2(b - \log x_{\text{UV}})^2 \right] \\ & + \epsilon\eta \left[-7 - \frac{\pi^2}{6} + 5(b - \log x_{\text{UV}}) - 3(b - \log x_{\text{UV}})^2 \right] + \mathcal{O}(x_{\text{UV}}) + \dots \Bigg\}, \end{aligned} \tag{4.104}$$

where $b = 2 - \log 2 - \gamma$ and the omitted terms are at least cubic in slow-roll. We note that all slow-roll parameters in the equation are in fact their values as $x_{\text{UV}} \equiv k y_{\text{UV}} \to 0$. In the inflationary setting, we fix y_{UV} and interpret this as the contribution from superhorizon modes; while in AdS/CFT, we fix k and regard this as an evaluation of the on-shell action on a cutoff surface near the boundary.

4.5.2 The power spectrum

The on-shell action (4.104) can be cast in the following form

$$I_{\rm bndy} = \frac{\epsilon \tau_{\vec{k}} \tau_{-\vec{k}}}{H_{\rm UV}^2} k^3 \left\{ -1 + b\eta + 2(b-1)\epsilon + \eta\kappa \left[-\frac{\pi^2}{12} - \frac{1}{2}b^2 \right] + \eta^2 \left[-1 - \frac{1}{2}b^2 \right] \right.$$
$$\left. + \epsilon^2 \left[-9 + 6b - 2b^2 \right] + \epsilon\eta \left[-7 - \frac{\pi^2}{6} + 5b - 3b^2 \right] + \dots \right\}$$
$$\times (k y_{\rm UV})^{-(n_s-1) - \frac{1}{2}\alpha_s \log k y_{\rm UV}} + (\text{divergent as } y_{\rm UV} \to 0), \qquad (4.105)$$

where

$$n_s - 1 = -\eta - 2\epsilon + b\eta\kappa - 2\epsilon^2 + (2b-3)\epsilon\eta,$$
$$\alpha_s = -2\epsilon\eta - \eta\kappa. \qquad (4.106)$$

Following Maldacena's prescription [49] and performing the usual Wick rotation $x \to -ix_{\rm dS}, H \to iH_{\rm dS}$, we recover the following as the classical action for the solution in a weakly-broken de Sitter background

$$i\Gamma_{\rm dS} = -M_{\rm Pl,4}^2 \int \frac{{\rm d}^3 k}{(2\pi)^3} I_{\rm bndy}(-ix_{\rm dS}, iH_{\rm dS}). \qquad (4.107)$$

dS/CFT tells us that the wavefunction of an asymptotically-de Sitter universe Ψ is equal to the partition function Z of a CFT. Approximating the former by the classical saddle, we have

$$\langle T_{\vec{k}} T_{\vec{k}'} \rangle = \left. \frac{\delta^2 Z}{\delta\tau_{\vec{k}} \delta\tau_{\vec{k}'}} \right|_{\tau=0} \sim -2(2\pi)^3 \delta(\vec{k} + \vec{k}') \mathcal{I}_{\rm bndy}(-ix_{\rm dS}, iH_{\rm dS}), \qquad (4.108)$$

where $I_{\rm bndy} = \tau_{\vec{k}} \tau_{-\vec{k}} \mathcal{I}_{\rm bndy}$. The gravitational wavefunction is expected to take the form

$$\Psi = \exp \left\{ \frac{1}{2} \int {\rm d}^3 x_1 {\rm d}^3 x_2 \langle T(x_1) T(x_2) \rangle \tau(x_1) \tau(x_2) + \right.$$
$$\left. + \frac{1}{6} \int {\rm d}^3 x_1 {\rm d}^3 x_2 {\rm d}^3 x_3 \langle T(x_1) T(x_2) T(x_3) \rangle \tau(x_1) \tau(x_2) \tau(x_3) + \dots \right\}, \qquad (4.109)$$

where T is the trace of the stress tensor of the UV CFT. Of course, correlation functions are obtained by path integrating over the wavefunction, for instance $\langle\tau\tau\rangle = \frac{\int \mathcal{D}\tau\tau^2|\Psi[\tau]|^2}{\int \mathcal{D}\tau|\Psi[\tau]|^2}$. We see that only the real part of $i\Gamma_{\rm dS}$ can contribute, and so there is a natural mechanism by which the divergent terms proportional to $\frac{1}{x_{\rm UV}}$ (which would have been cancelled by the addition of local covariant counterterms to the action in AdS) in $I_{\rm bndy}(-ix_{\rm dS}, iH_{\rm dS})$ drop out. In particular, we see that

$$\langle \tau_{\vec{k}}\tau_{-\vec{k}}\rangle' = -\frac{1}{2{\rm Re}\langle T_{\vec{k}}T_{-\vec{k}}\rangle'}, \tag{4.110}$$

where again the $'$ indicates that $(2\pi)^3\delta(\vec{k}+\vec{k'})$ has been omitted.

Now consider the following suggestively-named quantity

$$\begin{aligned}
\Delta_S^2 &\equiv \frac{k^3}{2\pi^2}\langle \tau_{\vec{k}}\tau_{-\vec{k}}\rangle' \\
&= \frac{H_{\rm UV}^2}{8\pi^2 M_{\rm Pl,4}^2\epsilon}\left\{1 + b\eta + 2(b-1)\epsilon + \eta\kappa\left(\frac{\pi^2}{24} - \frac{1}{2}b^2\right) + \eta^2\left(\frac{\pi^2}{8} + \frac{1}{2}b^2 - 1\right)\right. \\
&\quad \left. + \epsilon^2\left(\frac{\pi^2}{2} + 2b^2 - 2b - 5\right) + \epsilon\eta\left(\frac{7\pi^2}{12} + b^2 + b - 7\right) + \dots\right\} \times (ky_{\rm dS})^{(n_s-1)+\frac{1}{2}\alpha_s\log ky_{\rm dS}} \\
&\equiv \mathcal{A}_S(ky_{\rm dS})^{(n_s-1)+\frac{1}{2}\alpha_s\log ky_{\rm dS}}.
\end{aligned} \tag{4.111}$$

As remarked under (4.104), in the inflationary context this result is valid only in the deep infrared. The amplitude $\mathcal{A}_S$ is in exact agreement with the single-field inflationary power spectrum of curvature perturbations evaluated at horizon crossing to second order in slow-roll, while the power-law index $n_s - 1$ (4.106) is consistent with the inflationary spectral index to second order [50].

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Appendix A: Computations of $\tilde{c}_1$ for bulk duals of weakly-relevant flows

Odd d

To extract the leading contribution to the coefficients describing the mixing within the basis of instantaneous AdS solutions as $\lambda \to 0$, it turns out that one only needs to evaluate the $x \to 0$ piece of $(\phi_i\phi_j)(x)$ in the integral of B. The fact that the leading solution for the coefficients describing the mixing between instantaneous AdS solutions come from integrals over the bulk that localize near the boundary is similar to the observation made in [51] in evaluating four-point functions using Witten diagrams in a particular kinematic limit enabling a double OPE expansion. With this simplification,

$$\int B(r)\mathrm{d}r = C\int_{\infty}^{x}\frac{\mathrm{d}x'}{x'}\alpha(x') \tag{4.112}$$

where the *constant* matrix C is defined as

$$C \equiv \begin{pmatrix} -\phi_1\phi_2 & -\phi_2\phi_2 \\ \phi_1\phi_1 & \phi_2\phi_1 \end{pmatrix}\Bigg|_{x=0} = \frac{1}{d}\begin{pmatrix} -1 & a_{22} \\ 0 & 1 \end{pmatrix}; \quad a_{22} \equiv \Gamma\left(\frac{d}{2}\right)\Gamma\left(1-\frac{d}{2}\right). \tag{4.113}$$

The matrix C has the property that $[C^{2q}]_{12} = 0$ and

$$[C^{2q+1}]_{12} = \frac{a_{22}}{d^{2q+1}} \quad ; \quad \left[C^k\right]_{22} = d^{-k}. \tag{4.114}$$

The nested integral is now easy to evaluate:

$$\int_{\infty}^{0}\frac{dx_1}{x_1}\alpha(x_1)\int_{\infty}^{x_1}\frac{dx_2}{x_2}\alpha(x_2)\ldots\int_{\infty}^{x_{2k}}\frac{dx_{2k+1}}{x_{2k+1}}\alpha(x_{2k+1})\,. \tag{4.115}$$

Substituting the expansion of α in power series of ξ, we have[10]

$$
\begin{aligned}
&(2d\lambda)^k \sum_{i_1,\ldots,i_k=1}^{\infty} (-\xi)^{i_{tot}} \int_1^0 \frac{dx_1}{x_1} x_1^{\lambda i_1} \int_1^{x_1} \frac{dx_2}{x_2} x_2^{\lambda i_2} \ldots \int_1^{x_{k-1}} \frac{dx_k}{x_k} x_k^{\lambda i_k} \\
= \quad & (-2d)^k \sum_{i_1,\ldots,i_k=1}^{\infty} \frac{(-\xi)^{i_{tot}}}{i_1(i_1+i_2)(i_1+i_2+i_3)\ldots(i_{tot})} = \frac{(2d)^k}{k!} \log[1+\xi]^k
\end{aligned} \tag{4.119}
$$

where $i_{tot} = \sum_{j=1}^{k} i_j$. We have set the $\lambda = 0$ after the integration. Summing over all odd $k = 2q+1$ gives

$$
V_{12} = a_{22} \sum_{q=0}^{\infty} \frac{2^{2q+1}}{(2q+1)!} \log\left[1+\xi\right]^{2q+1} = \frac{a_{22}}{2} \left[(1+\xi)^2 - \frac{1}{(1+\xi)^2} \right] . \tag{4.120}
$$

---

[10] In practice, in extracting the $1/\lambda$ term in each integral in the nested integral, we encounter integrals of the form

$$
\int_{\infty}^{x_{m-1}} dx_m \; x_m^{p\lambda-1} f(x_m) \tag{4.116}
$$

nested within

$$
\int_{\infty}^{0} dx_1(\ldots). \tag{4.117}
$$

To isolate the leading $(1/\lambda)$ piece, it turns out that one only needs to care about the region of the integration close to $x = 0$. This is very similar to how one isolates the leading pole of, for instance, the gamma function $\Gamma(n)$ as $n \to 0$ from the integral representation of the gamma function. This implies that we can introduce $\varepsilon \ll 1$ and split the outer-most integral $\int_{\infty}^{0} dx_1$ into $-\int_{\varepsilon}^{\infty} - \int_{0}^{\varepsilon}$ and drop the $\int_{\varepsilon}^{\infty}$ piece since this term will only contain subleading terms as $\lambda \to 0$. For the rest of the nested integrals, we similarly replace each integrand by

$$
\int_{\varepsilon}^{x_{m-1}} dx_m \; x_m^{p\lambda-1} f(x_m) \tag{4.118}
$$

since $0 < x_{m-1} < \varepsilon$. Within each integrand, since $x_m \ll 1$, we can Taylor expand the $f(x_m)$ as a power-series in x_m.

Similarly, summing over all k for V_{22} from $k=1$ to ∞ yields

$$V_{22} = \sum_{k=1}^{\infty} \frac{2^k}{k!} \log\left[1+\xi\right]^k = \xi(\xi+2)\,. \tag{4.121}$$

Using

$$\tilde{c}_1(y) = \frac{V_{12}(y, y_{\rm IR})}{1 + \frac{\phi_1(y_{\rm UV})}{\phi_2(y_{\rm UV})} V_{12}(y_{\rm UV}, y_{\rm IR}) + V_{22}(y_{\rm UV}, y_{\rm IR})} = \frac{V_{12}(y, y_{\rm IR})}{1 + V_{22}(y_{\rm UV}, y_{\rm IR})} \tag{4.122}$$

we obtain

$$\tilde{c}_1(y_{\rm UV}) = -\frac{a_{22}}{2}\left[(1+\xi)^{-4} - 1\right] \tag{4.123}$$

as implied by conformal perturbation theory.

Even d

In even d, some care needs to be taken while evaluating $\phi_2^2(x)$ near $x=0$. It turns out that there's already a $x^{2\lambda}/\lambda$ leading term (without integration) in this expansion, and that's exactly what one wants in the even d case. Since the ϕ_2^2 term only occurs *once* in the nested integral, thus this will give the right powers of λ eventually. The other $\phi_i\phi_j(x)$ term has the same behavior near $x=0$ as the odd d case.

Similarly to the odd d case,

$$\int B(r)\mathrm{d}r = \int_{\infty}^{x} \frac{\mathrm{d}x'}{x'} C(x')\alpha(x') \tag{4.124}$$

but now the matrix $C(x')$ is no longer constant, and is given by

$$C(x) \equiv \left.\begin{pmatrix} -\phi_1\phi_2 & -\phi_2\phi_2 \\ \phi_1\phi_1 & \phi_2\phi_1 \end{pmatrix}\right|_{x=0} = \frac{1}{d}\begin{pmatrix} -1 & a_{22}(x) \\ 0 & 1 \end{pmatrix} \quad ; \quad a_{22} \equiv \frac{i^d\left(x^{2\lambda}-1\right)}{\lambda}\,. \tag{4.125}$$

The matrix C has the property that

$$[C^k(x)]_{22} = d^{-k}, \tag{4.126}$$

while

$$[C(x_1).C(x_2)\dots C(x_{2q+1})]_{12} \;=\; -\frac{i^d}{d^{2q+1}\lambda}\left[1+\sum_{j=1}^{2q+1}(-1)^j x_j^{2\lambda}\right] \tag{4.127}$$

$$[C(x_1).C(x_2)\dots C(x_{2q+2})]_{12} \;=\; -\frac{i^d}{d^{2q+1}\lambda}\left[\sum_{j=1}^{2q+2}(-1)^j x_j^{2\lambda}\right]. \tag{4.128}$$

This implies that the computation of V_{22} is as before and gives

$$V_{22} = \sum_{k=1}^{\infty}\frac{2^k}{k!}\log\left[1+\xi\right]^k = \xi(\xi+2)\,. \tag{4.129}$$

We first carry out the nested integral for the x_j-independent term (the first term in Eq. (4.127)). This is easily obtained by the techniques of the previous subsection to yield

$$-\frac{i^d}{\lambda}\frac{2^{2q+1}}{(2q+1)!}\log\left[1+\xi\right]^{2q+1}, \quad q=0,1,2,\dots \tag{4.130}$$

The rest of the terms in Eq. (4.127)) and Eq. (4.127)) and Eq. (4.128) combine into a general integer powers of C. The calculation for these terms is slightly modified from the previous sections to include an extra sum.

$$\begin{aligned}
&(2d\lambda)^k\sum_{j=1}^{k}(-1)^j\sum_{i_1,\dots,i_k=1}^{\infty}(-\xi)^{i_{tot}}\int_1^0\frac{dx_1}{x_1}x_1^{i_1\lambda}\dots\int_1^{x_{j-1}}\frac{dx_j}{x_j}x_2^{(i_j+2)\lambda}\dots\int_1^{x_{k-1}}\frac{dx_k}{x_k}x_k^{i_k\lambda}\\
=\;&(-2d)^k\sum_{j=1}^{k}(-1)^j\sum_{i_1,\dots,i_k=1}^{\infty}\frac{(-\xi)^{i_{tot}}}{i_1(i_1+i_2)(i_1+i_2+i_3)\dots(i_{tot})\,|_{i_j\to i_j+2}}
\end{aligned} \tag{4.131}$$

where $i_{tot}=\sum_{j=1}^{k} i_j$. We have set the $\lambda=0$ after the integration. After some work and collecting all the pieces, these become

$$d^k\left\{\sum_{k=1}^{\infty}\left(\frac{2^{k-1}}{k!}-\frac{(-1)^k}{3k!}\right)\log^k(1+\xi)-\sum_{q=0}^{\infty}\left[\frac{2^{2q+1}}{(2q+1)!}\log^{2q+1}(1+\xi)\right]\right\}. \tag{4.132}$$

Together with the constant term (and with all prefactors), the leading result as $\lambda \to 0$ resums into

$$V_{12} = -\frac{i^d}{\lambda} \sum_{k=1}^{\infty} \left(\frac{2^{k-1}}{k!} - \frac{(-1)^k}{3k!} \right) \log^k(1+\xi) = \frac{i^d}{2\lambda} \left[1 + (\xi+2)\xi\right] \left[\frac{\xi+3}{3(\xi+1)^3} - 1 \right] . \tag{4.133}$$

This implies

$$\tilde{c}_1(y_{\mathrm{UV}}) = \frac{i^d}{2\lambda} \left[\frac{\xi+3}{3(\xi+1)^3} - 1 \right] , \tag{4.134}$$

as required by the conformal perturbation theory result.

Appendix B: The dilaton effective action for weakly relevant flows

Here we review the computations in [27] relating to the dilaton effective action of weakly-relevant flows and show that making use of the RG-improved two-point function is equivalent to integrating over momentum shells as is done in that paper. Once the RG-improved two-point function is known, we will see below that it is elementary to compute the coefficient of $\int \mathrm{d}^d x \tau \Box^{\frac{d}{2}} \tau$ in the dilaton effective action. In even dimensions, this establishes the a-theorem for weakly relevant flows.

Following [27, 26], in a general QFT, since the coupling of τ to matter to linear order is τT^μ_μ, the quadratic term (with derivatives) in the τ-effective action contains a quadratic term:

$$\langle e^{\int d^d x \tau T^\mu_\mu} \rangle_{\mathrm{QFT}} = \frac{1}{2} \int \int \tau(x)\tau(y) \langle T^\mu_\mu(x) T^\mu_\mu(y) \rangle_{\mathrm{QFT}} d^d x d^d y + \ldots \tag{4.135}$$

In turn, the action contains $2n$-derivative terms:

$$\frac{1}{2(2n)!} \int \tau(x) \partial_{\mu_1} \ldots \partial_{\mu_{2n}} \tau(x) \ \left[\int (y-x)^{\mu_1} \ldots (y-x)^{\mu_{2n}} \langle T^\mu_\mu(x) T^\mu_\mu(y) \rangle_{\mathrm{QFT}} d^d y \right] d^d x, \tag{4.136}$$

which reduce to the following by symmetry

$$\frac{\Gamma\left(\frac{d}{2}\right)}{2^{2n+1}\Gamma(n+1)\Gamma\left(\frac{d}{2}+n\right)} \int d^d y |y|^{2n} \langle T^\mu_\mu(x) T^\mu_\mu(y) \rangle_{\mathrm{QFT}} \times \left[\int \tau(x) \Box^n \tau(x) \right] . \tag{4.137}$$

Now suppose the QFT is deformed from a CFT by a relevant operator $\phi \mathcal{O}_{\mathrm{bare}}$. Then the trace of the stress tensor is given by

$$T^\mu_\mu = \beta_\phi \mathcal{O}_{\mathrm{bare}} = -\lambda \phi \mathcal{O}_{\mathrm{bare}} \, . \tag{4.138}$$

The coefficient that we need to compute is then

$$\frac{\Gamma\left(\frac{d}{2}\right)}{2^{2n+1}\Gamma(n+1)\Gamma\left(\frac{d}{2}+n\right)} (-\lambda\phi)^2 \int d^d y |y|^{2n} \langle \mathcal{O}_{\mathrm{bare}}(y) \mathcal{O}_{\mathrm{bare}}(0) e^{\phi \int d^d z \mathcal{O}_{\mathrm{bare}}(z)} \rangle_{\mathrm{CFT}} \, . \tag{4.139}$$

In position-space, the RG-improved two-point function is given by

$$\langle \mathcal{O}_{\text{bare}}(y)\mathcal{O}_{\text{bare}}(0)e^{\phi\int d^dz\mathcal{O}_{\text{bare}}(z)}\rangle_{\text{CFT}} = \frac{1}{|y|^{2(d-\lambda)}}\left[1+\frac{\phi}{g_*}|y|^\lambda\right]^{-4}. \tag{4.140}$$

where we have chosen to normalize the Zamolodchikov metric in a particular way (ie. we've set $G_{\mathcal{O}\mathcal{O}}=1$) and $g_*=\frac{2\lambda}{C\Omega_{d-1}}$. The integral can be be evaluated via analytic continuation in λ, yielding

$$\begin{aligned}&(-\lambda\phi)^2\int d^dy|y|^{2n}\langle \mathcal{O}_{\text{bare}}(y)\mathcal{O}_{\text{bare}}(0)e^{\phi\int d^dz\mathcal{O}_{\text{bare}}(z)}\rangle_{\text{CFT}}\\ =\;& -\frac{\pi g_*^2(d-2n)\Omega_{d-1}\left((d-2n)^2-\lambda^2\right)\csc\left(\frac{\pi(d-2n)}{\lambda}\right)\left(\frac{\phi}{g_*}\right)^{\frac{d-2n}{\lambda}}}{6\lambda^2},\end{aligned} \tag{4.141}$$

We are particularly interested in d-derivative terms in the dilaton effective action. For even d and $n=d/2$, we have

$$\beta_\phi^2\int d^dy|y|^d\langle \mathcal{O}_{\text{bare}}(y)\mathcal{O}_{\text{bare}}(0)e^{\phi\int d^dz\mathcal{O}_{\text{bare}}(z)}\rangle_{CFT} = \frac{2}{3C^2\Omega_{d-1}^2}\lambda^3 \tag{4.142}$$

where we have used

$$g_* = (2\lambda)/(C\Omega_{d-1}). \tag{4.143}$$

All together, for even d, the d-derivative term in the effective action can be written as:

$$\left[\frac{2^{2-d}}{3d\Gamma(d+1)}\frac{\lambda^3}{C^2\Omega_{d-1}}\right]\times\left[\frac{d}{2}\int d^dx\ \tau(x)\Box^{d/2}\tau(x)\right]. \tag{4.144}$$

In weakly-relevant flows, this represents the leading d-derivative term in the dilaton effective action - terms with d derivatives and additional factors of the dilaton are suppressed as the deviation from marginality λ is taken to zero [27]. As a result, this coefficient represents the leading contribution to the flow of the a-type anomaly for RG flows induced by a deformation by a weakly relevant operator. For example, we have

$$\begin{aligned}
d=2 \quad &: \quad \left[\frac{\lambda^3}{12C^2\Omega_1}\right] \times \left[\int d^dx\ \tau(x)\Box\tau(x)\right] \\
d=4 \quad &: \quad \left[\frac{\lambda^3}{1152C^2\Omega_3}\right] \times \left[2\int d^dx\ \tau(x)\Box^2\tau(x)\right] \\
d=6 \quad &: \quad \left[\frac{\lambda^3}{207360C^2\Omega_5}\right] \times \left[3\int d^dx\ \tau(x)\Box^3\tau(x)\right].
\end{aligned} \tag{4.145}$$

Appendix C: Computations of $\tilde{c}_i^{(k,l)}$ in slow-roll inflation

Computing $\mathbf{\tilde{c}_i^{(0,0)}}$

Here we are denoting by $\tilde{c}_i^{(m,n)}$ the contribution to $\tilde{c}_i$ at $\mathcal{O}(\epsilon^m \eta^n)$, where by ϵ and η we mean their boundary values. Since each V is at least linear in the slow-roll parameters, at this order, we have

$$\tilde{c}_1^{(0,0)}(x) \;\; = \;\; 0 \quad , \quad \tilde{c}_2^{(0,0)}(x) = 1 \, . \tag{4.146}$$

Computing $\mathbf{\tilde{c}_i^{(1,0)}}$

At this order

$$\begin{aligned} \tilde{c}_1^{(1,0)}(x_{\rm UV}) &= V_{12}^{(1,0)}(x_{\rm UV}, x_{\rm IR}) \\ \tilde{c}_2^{(1,0)}(x_{\rm UV}) &= -\frac{\phi_1(x_{\rm UV})}{\phi_2(x_{\rm UV})} V_{12}^{(1,0)}(x_{\rm UV}, x_{\rm IR}) \, , \end{aligned} \tag{4.147}$$

where

$$V_{12}^{(1,0)}(x, x_0) \;\; = \;\; \frac{1}{2} \int_{x_0}^{x} \frac{dx_1}{x_1} \epsilon \left(-(2x_1^2 \partial_{x_1} + x_1 \partial_{x_1}) \phi_2 + \frac{3}{2} \phi_2 \right) \phi_2 \, . \tag{4.148}$$

Now, in evaluating an integral such as

$$I(x) \equiv \int_{x_0}^{x} dx_1 \epsilon(x_1) \frac{G(x_1)}{x_1} \, ,$$

we can integrate by parts

$$I(x) = \epsilon(x') \int^{x'} dx_1 \frac{G(x_1)}{x_1} \bigg|_{x_0}^{x} - \int_{z_0}^{z} dz_1 \epsilon'(x_1) \int_{x_0}^{x_1} dx_2 \frac{G(x_2)}{x_2} = \epsilon(x) \int_{x_0}^{x} dx_1 \frac{G(x_1)}{x_1} + \mathcal{O}(\epsilon\eta), \tag{4.149}$$

where we have used

$$\epsilon'(x) = -\frac{\epsilon\eta}{1-\epsilon} \frac{1}{x} \sim \mathcal{O}(\epsilon\eta) \, . \tag{4.150}$$

Of course, this term contributes to $\tilde{c}_1^{(1,1)}$, however we will drop the $\mathcal{O}(\epsilon\eta)$ contributions for the rest of this section. Thus we have

$$I(x) = \epsilon(x') \int^{x'} dx_1 \frac{G(x_1)}{x_1}\Big|_{x_0}^{x} + \dots \tag{4.151}$$

and obtain

$$\begin{aligned} V_{12}^{(1,0)}(x_{\rm UV}, x_{\rm IR}) &= \frac{1}{2}\epsilon(x) \int^x \frac{dx_1}{x_1}\Big|_{x_{\rm IR}}^{x_{\rm UV}} \left(-(2x_1^2\partial_{x_1} + x_1\partial_{x_1})\phi_2 + \frac{3}{2}\phi_2\right)\phi_2 \\ &= \epsilon_{\rm UV}\left[\frac{1}{4}e^{-2x_{UV}}\left(\frac{3}{x_{UV}^3} + \frac{6}{x_{UV}^2} + \frac{3}{x_{UV}} + 2\right) - {\rm Ei}(-2x_{\rm UV})\right] \\ &\equiv \frac{1}{4}\epsilon_{\rm UV} f_1(x_{\rm UV}), \end{aligned} \tag{4.152}$$

where in the second line we have taken $x_{\rm IR} \to \infty$.

Thus, the solution for the coefficients at this order is

$$\begin{aligned} \tilde{c}_1^{(1,0)}(x_{\rm UV}) &= \frac{1}{4}\epsilon_{\rm UV} f_1(x_{\rm UV}) \\ \tilde{c}_2^{(1,0)}(x_{\rm UV}) &= -\frac{\phi_1(x_{\rm UV})}{\phi_2(x_{\rm UV})}\tilde{c}_1^{(1,0)}(x_{\rm UV}) = -e^{2x_{\rm UV}}\left[\frac{1-x_{UV}}{1+x_{UV}}\right]\tilde{c}_1^{(1,0)}(x_{\rm UV})\,. \end{aligned} \tag{4.153}$$

Computing $\tilde{\mathbf{c}}_{\mathbf{i}}^{(\mathbf{0},\mathbf{1})}$

At the next order

$$\begin{aligned} \tilde{c}_1^{(0,1)}(x) &= V_{12}^{(0,1)}(x, x_{\rm IR}) \\ \tilde{c}_2^{(0,1)}(x) &= -\frac{\phi_1(x_{\rm UV})}{\phi_2(x_{\rm UV})}V_{12}^{(0,1)}(x_{\rm UV}, x_{\rm IR}) + V_{22}^{(0,1)}(x, x_{\rm IR}) - V_{22}^{(0,1)}(x_{\rm UV}, x_{\rm IR})\,. \end{aligned} \tag{4.154}$$

where upon defining $r_0 \equiv r(x_0)$, we have

$$V^{(k,l)}(x, x_0) = \int_{r_0}^{r(x)} dr' B^{(k,l)}(r')\,. \tag{4.155}$$

We see that $V_{12}^{(0,1)}(x, x_{\rm IR})$ is IR-finite and that $+V_{22}^{(0,1)}(x, x_{\rm IR}) - V_{22}^{(0,1)}(x_{\rm UV}, x_{\rm IR}) = \int_{r_{UV}}^{r} B^{(0,1)}(r')$ and hence it's independent of r_{IR}, so it is IR-finite.

To evaluate the on-shell action, we will need the $\{\tilde{c}_i\}$ at the boundary $x = x_{\rm UV}$:

$$\begin{aligned} \tilde{c}_1^{(0,1)}(x_{\rm UV}) &= V_{12}^{(0,1)}(x_{\rm UV}, x_{\rm IR}) \\ \tilde{c}_2^{(0,1)}(x_{\rm UV}) &= -\frac{\phi_1(x_{\rm UV})}{\phi_2(x_{\rm UV})} V_{12}^{(0,1)}(x_{\rm UV}, x_{\rm IR})\,. \end{aligned} \tag{4.156}$$

Explicitly,

$$V_{12}^{(0,1)}(x, x_0) = -\frac{3}{4}\eta(x) \int^{x'} \frac{{\rm d}x_1}{x_1} (\phi_2(x_1))^2 \Bigg|_{x_0}^{x} \tag{4.157}$$

where we have integrated by parts and dropped a term proportional to $\dot{\eta} = H\eta\kappa$ since it is quadratic in slow-roll.

The integral is easily evaluated to be:

$$V_{12}^{(0,1)}(x, x_0) = \frac{1}{4}\eta(x) \left[-2{\rm Ei}(-2x) + \frac{e^{-2x}(1-(x-2)x)}{x^3} \right] \Bigg|_{x_0}^{x} . \tag{4.158}$$

Now we set $x_0 = x_{\rm IR}$ and take $x_{\rm IR} \to \infty$ to obtain

$$\lim_{x_{\rm IR}\to\infty} V_{12}^{(0,1)}(x, x_{\rm IR}) = \frac{1}{4}\eta \left[-2{\rm Ei}(-2x) + \frac{e^{-2x}(1-(x-2)x)}{x^3} \right] \equiv \frac{\eta}{4} f_2(x)\,, \tag{4.159}$$

giving

$$\begin{aligned} \tilde{c}_1^{(0,1)}(x_{\rm UV}) &= \frac{\eta}{4} f_2(x_{\rm UV}) \\ \tilde{c}_2^{(0,1)}(x_{\rm UV}) &= -\frac{\phi_1(r_{UV})}{\phi_2(r_{UV})} \tilde{c}_1^{(0,1)}(x_{\rm UV}) = -e^{2x_{\rm UV}} \left[\frac{1 - x_{UV}}{1 + x_{UV}} \right] \tilde{c}_1^{(0,1)}(x_{\rm UV}). \end{aligned} \tag{4.160}$$

It is straightforward to generalize this procedure to higher order in slow-flow; one must just be careful to collect all the terms that contribute at a given order. Unfortunately, the matrix B is of mixed order in slow-flow, so at a given order, there will be contributions from different terms in the Dyson series. We have worked out the solutions up to quadratic order in slow-flow.

Appendix D: Leading corrections due to running speed of sound

Here we compute the leading corrections to the power spectrum and spectral index due to running speed of sound c_s by making use of our adiabatic approximation scheme. The speed of sound is related to the parameter M_2 in the general form of the Goldstone action (4.7) via $c_s^{-2} = 1 - \frac{4M_2}{(d-1)M_{\mathrm{Pl,d+1}}^{d-1}\dot{H}}$. In particular, allowing for a running speed of sound, the Goldstone boson action reduces in the demixing limit to

$$S_\pi = \frac{(d-1)M_{\mathrm{Pl,d+1}}^{d-1}}{2}\int \mathrm{d}r\mathrm{d}^d x\, a^d \left\{ \frac{H^2\epsilon}{c_s^2}\left(\dot{\pi}^2 + c_s^2\frac{(\partial_i\pi)^2}{a^2}\right) - H^2\epsilon(1-c_s^{-2})\left(\dot{\pi}^3 + \dot{\pi}\frac{(\partial_i\pi)^2}{a^2}\right) + \dots \right\}. \tag{4.161}$$

Since the power spectrum can be computed from the quadratic part of the on-shell action, for the moment we will be focusing on solutions to the homogeneous equation of motion, namely

$$\ddot{\hat{\pi}}_{\vec{k}} + H(d+\eta-2s)\dot{\hat{\pi}}_{\vec{k}} - \left(\frac{c_s k}{a}\right)^2\hat{\pi}_{\vec{k}} = 0, \tag{4.162}$$

where we've defined $\hat{\pi} = -H\pi$ as before, ignored terms that are subleading in the demixing limit[11] and defined

$$s \equiv \frac{\dot{c}_s}{Hc_s}. \tag{4.163}$$

To cast the equation of motion into Schrodinger form we now define

$$\psi = f\hat{\pi} = \frac{\sqrt{\epsilon a^d}}{c_s}\hat{\pi}. \tag{4.164}$$

The Schrodinger equation of motion is then given by

$$\ddot{\psi} - \left(\frac{\ddot{f}}{f} + \left(\frac{c_s k}{a}\right)^2\right)\psi = 0, \tag{4.165}$$

[11] Which is now valid for energies much larger than $E_{\mathrm{mix}} \sim H\sqrt{\epsilon(c_s^{-2}-1)}$.

where

$$\frac{\ddot{f}}{f} = H^2\left(\frac{d^2}{4}+\frac{d}{2}\eta-\frac{d}{2}\epsilon-\frac{1}{2}\epsilon\eta+\frac{1}{4}\eta^2+\frac{1}{2}\eta\kappa-ds+s^2-\eta s+\epsilon s-ss_2\right)$$
$$s_2 \equiv \frac{\dot{s}}{Hs}. \tag{4.166}$$

This is the general form of the wave equation described in terms of the parameters characterizing the deviation from AdS and the running of the speed of sound. From here on we focus on the case $d=3$. We will write the solutions to the equation of motion (4.165) as before

$$\psi(r) = c_i(r)\phi_i(r), \tag{4.167}$$

where now

$$\phi_{1/2}(r) = \frac{e^{\pm k\tilde{y}}(1\mp k\tilde{y})}{(k\tilde{y})^{3/2}},$$
$$\tilde{y} \equiv \frac{c_s(r)}{a(r)H(r)}. \tag{4.168}$$

Note that now

$$\frac{\mathrm{d}\tilde{y}}{\mathrm{d}r} = -H\tilde{y}(1-\epsilon-s). \tag{4.169}$$

The differential equation satisfied by the coefficients is then modified to the following

$$\dot{c} = B'\cdot c, \tag{4.170}$$

which is solved by a radially-ordered exponential as before. Now, we have that

$$\int B'_{ij}(r)\mathrm{d}r = \int \frac{\mathrm{d}x}{2x(1-\epsilon-s)^2}\alpha_j(x)\epsilon_{ik}\phi_k(x), \tag{4.171}$$

where $x = k\tilde{y} = kc_s/aH$ and

$$\alpha_i(x) = \left[\epsilon\left(-x\partial_x-2x^2\partial_x^2+\frac{3}{2}\right)-\frac{3}{2}\eta+\epsilon\eta\left(x\partial_x+\frac{1}{2}\right)+\epsilon^2x^2\partial_x^2-\frac{1}{4}\eta^2-\frac{1}{2}\eta\kappa\right.$$
$$+s(-2x\partial_x-2x^2\partial_x^2+3)+\eta s+\epsilon s(x\partial_x+2x^2\partial_x^2-1)+s^2(x\partial_x+x^2\partial_x^2-1)$$
$$\left.+ss_2(x\partial_x+1)\right]\phi_i(x). \tag{4.172}$$

We are now in a position to compute the Goldstone boson's on-shell action to leading order in s. We note that the boundary conditions are modified so that

$$c_1(\tilde{y}_{\rm UV})\phi_1(\tilde{y}_{\rm UV}) + c_2(\tilde{y}_{\rm UV})\phi_2(\tilde{y}_{\rm UV}) = \sqrt{\frac{\epsilon c_s}{(\tilde{y}_{\rm UV} H_{\rm UV})^3}}\tau. \tag{4.173}$$

We then find the following as the contribution of the homogeneous solution to the *quadratic* part of the on-shell action

$$\begin{aligned} I^{(2)}_{\rm bndy} &= \frac{a^3\epsilon}{c_s^2}\hat{\pi}\partial_r\hat{\pi}\Big|_{r=r_{\rm UV}} \\ &= \left\{\psi(r)\partial_r\psi(r) - H(r)\left(\frac{3}{2}+\frac{1}{2}\eta - s\right)\psi^2(r)\right\}\Big|_{r=r_{\rm UV}} \\ &= \left\{-xH(1-\epsilon-s)(c_1\phi_1+c_2\phi_2)(c_1\partial_x\phi_1+c_2\partial_x\phi_2) - H\left(\frac{3}{2}+\frac{\eta}{2}-s\right)(c_1\phi_1+c_2\phi_2)^2\right\}\Big|_{x=x_{\rm UV}}. \end{aligned} \tag{4.174}$$

Employing our perturbative solutions for the coefficients and expanding near $x_{\rm UV} \to 0$, we find the following as the on-shell action for the Goldstone scalar

$$\begin{aligned} I^{(2)}_{\rm bndy} = \frac{\epsilon c_s \tau_{\vec{k}}\tau_{-\vec{k}}}{H^2_{\rm UV}}k^3\Big\{&\frac{1}{x_{\rm UV}}[1-\eta-\epsilon] - 1 + \eta[b - \log x_{\rm UV}] + \epsilon[2(b-\log x_{\rm UV} - 1)] \\ &+ s[b - \log x_{\rm UV} - 2] + \mathcal{O}(x_{\rm UV}) + \ldots\Big\}. \end{aligned} \tag{4.175}$$

It is then straightforward to apply the methods of §4.5.2 to read off the correlation function to next-to-leading order in slow-roll

$$\langle\tau_{\vec{k}}\tau_{-\vec{k}}\rangle' = \frac{H^2_{\rm UV}}{4\epsilon c_s k^3}\left\{1 + b\eta + 2(b-1)\epsilon + (b-2)s + \ldots\right\}(k\tilde{y}_{\rm dS})^{n_s-1}. \tag{4.176}$$

The amplitude corresponds (up to the usual factor of $\frac{k^3}{2\pi^2}$) to the leading correction to the inflationary power spectrum due to the running speed of sound, and agrees with what was found in [52]. Furthermore,

$$n_s - 1 = -\eta - 2\epsilon - s + \ldots \tag{4.177}$$

which agrees with the leading correction in s to the inflationary spectral index found in [43].

Chapter 5
Outlook

In this thesis we have recapitulated the relationship between the emergent radial dimension in asymptotically AdS_{d+1} spacetimes and the renormalization group flow of the dual QFT_d. In particular, we have reviewed the derivation of renormalized field theory correlation functions from gravity in asymptotically anti-de Sitter space and introduced domain-wall geometries as the holographic realization of renormalization group flows in the boundary field theory. We also presented a new method for computing the on-shell action in an effective description of AdS domain-wall geometries, particularly useful in the case of weakly broken AdS isometries. We demonstrated the use of this approximation scheme by considering RG flows induced by deformation by a weakly-relevant operator, and showed that we could compute the RG-improved two-point function of the dual operator entirely from the bulk effective theory. We also computed the on-shell action of the bulk effective theory in a generalized 'slow-flow' setup in four dimensions, and showed that the naive Wick rotation reproduced the inflationary power spectrum and spectral index at horizon-crossing to second-order in slow-roll.

In chapter 4 we focused on the computation of the quadratic part of the on-shell action for the Goldstone boson of the bulk effective theory in the interest of clarity of exposition — this was sufficient to establish the a-theorem in the special case of weakly-relevant flows and to reproduce the inflationary power spectrum. However, it would be interesting to see if, by allowing for more nontrivial interactions of the Goldstone boson (ie. generalizing the analysis beyond the case of a single scalar with potential by accounting for the field-theory dependent coefficients $\{M_n\}$ in (4.7) for $n \geq 2$), our method for the computation of the on-shell action in the bulk effective theory could reproduce the classification of terms that appear in the dilaton effective action [28] at all orders in derivatives. In particular, such an analysis might lend insight as to how to distinguish between non-vanishing on-shell Weyl-invariants and terms corresponding to the flow of the conformal anomaly holographically. This might lead to a more physical principle in the holographic description of field theory anomalies, without having to make reference to a particular domain-wall geometry or assume slow-flow.

As a result of focusing on the computation of the quadratic part of the Goldstone boson's on-shell action, we never strayed beyond the RG-improved two-point function of the dual operator. By including the non-trivial bulk interactions mentioned above, it would be interesting to use our method to compute higher-point RG-improved correlation functions of the dual operator via this general effective framework. At the level of soft-limits of the three-point function, it would be interesting to study the constraints placed by the conformal Ward identity on the effective field theory of holographic RG flows, directly analogous to the Maldacena consistency relations in inflation [49, 43, 53]. At the level of the four-point function, it would be interesting to see the holographic manifestation of the conformal block decomposition in the context of the bulk effective theory [51, 54].

Finally, it is worth pointing out that our method for solving the Goldstone boson's equation of motion and evaluating the on-shell action applies equally well to the effective field theory of inflation [42]. In fact, the computation in section 4.5 is effectively a check of the effective field theory of inflation. Although it is doubtful that there is any desire for such a result in the cosmology community, our method could obviously be used to compute the inflationary observables such as the power spectrum to arbitrary order in slow-roll. The example of the bulk dual of a weakly-relevant flow given in section 4.4 does not quite correspond to standard slow-roll inflation upon Wick rotation. In particular, we saw that $\epsilon \sim \mathcal{O}(\lambda^4)$ while $\eta \sim \mathcal{O}(\lambda)$, as observed in previous studies of holographic cosmology [46, 47]. However, in standard slow-roll inflation, ϵ and η are taken to be of the same order in perturbation theory. It would be very interesting if we could use this effective framework to put more explicit constraints on the field theory whose RG flow is the dual realization of slow-roll inflation.

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