AN EXACT APPROACH TO THE POLLING SYSTEM

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ABSTRACT

A polling system consists of N terminals each with a buffer of unrestricted length, where customers arrive according to a Poisson process. The terminals are served by a single server in a fixed cyclic order, the service time of a customer and the walking time of the server between terminals being independent random variables. Two different service procedures are analysed : (a) The gating model, where at each terminal only the customers that are present at the moment of the server's arrival are served, (b) the exhaustive model, where each terminal is served until it is empty.

By introducing a basic random variable, the Terminal Service Time, we are able to obtain exact and explicit expressions for various moments of the cycle time, intervisit time, buffer size, and the waiting time. As well, results concerning the transient and steady state behaviours are obtained. It is shown that in certain cases the exhaustive model is superior to the gating model. However, for a symmetric multi-terminal system, both models are virtually the same.

SOMMAIRE

Un système de mise en commun varié est fait de N terminaux, chacun ayant un tampon de longueur illimitée où les arrivées des clients sont déterminées par le procédé Poisson. Les terminaux sont servis par un seul serviteur dans un ordre cyclique fixe. Le temps de service du client et le temps de randonnée entre terminaux sont des variables indépendantes aléatoires. Deux méthodes de service différentes sont analysées : (a) Le modèle de la barrière où à chaque terminal seulement les clients qui sont présents à l'arrivée du serviteur sont servis et (b) le modèle exhaustif où chaque terminal est servi jusqu'à ce que le tampon soit vidé.

En introduisant une variable aléatoire de base, le temps de service d'un terminal, nous pouvons obtenir des expressions exactes et explicites pour divers moments, soit le temps de cyclage, le temps d'intervisite, la longueur du tampon et le temps d'attente. De plus, on obtient des résultats concernant les comportements transitoires et permanents.

On démontre que dans certains cas le modèle exhaustif est supérieur au modèle de la barrière. Par contre, ces modèles sont virtuellement identiques pour le cas d'un système symétrique de plusieurs terminaux.

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LIST OF SYMBOLS

T_i	Terminal number i.
N	The number of terminals in the system.
λ_i	The rate of the Poisson new arrival process at T_i.
$\lambda_0 =$	$\sum_{i=1}^{N} \lambda_i$.
ρ_i	Traffic intensity of T_i.
$\rho_0 =$	$\sum_{i=1}^{N} \rho_i$ Total traffic intensity of the system.
G	Subscript denoting the gating model.
E	Subscript denoting the exhaustive model.
*	The convolution operation.
$\delta(\cdot)$	Dirac function.
t_{G_k}	The time instant the server reaches T_{k+1} (in the gating model).
t_{E_k}	The time instant the server leaves T_k (in the exhaustive model).
Prob (a)	The probability density of a, where a is the value of some previously mentioned random variable.
Prob ($^a/b$)	The probability density of a given b.

For "a" a general (continuous or discrete) random variable we denote :

a_i	The random variable associated with T_i.
$P_a(\cdot)$	The probability density function of a.

$A(\cdot)$	The Laplace transform of $P_a(\cdot)$.
$\dot{A}(\cdot), \ddot{A}(\cdot), \ldots,$	The first, second, etc., derivatives of $A(\cdot)$ w.r.t. its whole argument, respectively.
$\overline{a}^n$	the nth moment of a.
σ_a^2	The second central moment (variance) of a.
δ_a^3	The third central moment of a.

The following are random variables :

b	The busy period length.
c	The cycle time.
h	The number of customers served during a busy period.
n	The number of customers that are served in a terminal during a service cycle.
$\hat{n}_T$	Total number of customers that are served in the system during a service cycle.
l_i =	n_i + [the number of customers in T_i at the moment the server leaves it]. Used in the gating model. Or, in Chapter VI, used for the exhaustive model, as the number of customers that are served in T_i after some "customer".
m_i =	The number of customers in a terminal at the moment the server reaches it. Used for the exhaustive model only.

m_{x_i}		The maximum number of customers that exist, at the same time, in T_i during a service cycle.
q_i		Customer waiting time in T_i.
q_0		Customer waiting time in the system.
s		Customer service time.
v		Intervisit time.
w_i		Server walking time from T_i to T_{i+1}.
d	=	$\sum_{i=1}^{N} w_i$.
TST_i		"Terminal Service Time" of T_i, denoted by θ_i.
$\underline{\theta}_k$		N dimensional random vector $(\theta_k, \theta_{k+1}, \ldots, \theta_{k+N-1})$.
$P_k(\underline{\cdot})$		The joint probability density function of $\underline{\theta}_k$.
${}_kG(\underline{\cdot})$		The Laplace transform of $P_k(\underline{\cdot})$.
${}_kF(\underline{\cdot})$	=	$\ln[{}_kG(\underline{\cdot})]$.
${}_kF_i(\underline{\cdot})$, ${}_kF_{i,j}(\underline{\cdot})$, ...	=	The partial derivatives of ${}_kF(\cdot)$ w.r.t. its ith, ith and jth, etc., arguments respectively.
$R(i, j)$	=	$\overline{(\theta_i - \bar{\theta}_i)(\theta_j - \bar{\theta}_j)}$, where T_j is visited before T_i.
$R(j)$	=	$R(i + j, i)$ in the symmetric case.

$R_\theta(j) = \frac{1}{\sigma_\theta^2} \cdot R(j)$ The normalized cross correlation between T_i and T_{i+j}.

$R_c(\cdot)$ The normalized cross correlation between cycles.

$R_v(\cdot)$ The normalized cross correlation between intervisits.

$R_n(\cdot)$ The normalized cross correlation between number of customers that are served.

$u, x, y, z, \tau, i, j, k, K, l, m, p$ are used as working parameters.

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CHAPTER I

INTRODUCTION

Many data communication networks require the connecting of a large number of terminals to each other and to a central processor. Because these terminals are usually characterized by data which bursts at high rates followed by long periods of quiet, a single dedicated communication line to other terminals and the central processor is rather inefficient. This has resulted in the study and building of multiplexors which allow terminals to share a communication channel at, hopefully, no reduction in performance but a substantial reduction in cost. This thesis obtains, for the first time, analytical results for some of the important data multiplexing schemes.

Examples of systems of this sort are starting to become common place. Automatic reading of credit cards by retail outlets, time sharing terminals, point of sale systems, remote-data base inquiry-response, and banking teller terminals are just a few. Unfortunately with the explosive technology, a number of the systems are being built before they are really understood.

A recent literature survey by Chu and Konheim (1972) concludes that present data multiplexing techniques, such as frequency devision multiplexing (FDM) (each terminal is assigned a fixed frequency band in the total channel band) and synchronous time devision multiplexing (STDM) (each terminal is assigned a fixed time period on the communication channel) are very inefficient in channel utilization and response time. The simple "star system" (also Hassing et al (1973)) (each terminal is connected directly and independently to the central unit) is economically in-efficient in handling the busty intermittent kind of traffic involved in those computer communication networks.

The inherent disadvantages of these multiplexing techniques initiated studies on statistical multiplexing or asychronous time devision multiplexing (ATDM), which is not only a feasible technique for data communication but also greatly improves the transmission efficiency and system organization (Chu and Konheim (1972)). The two main ATDM systems are closely related ; they are the LOOP (also known as RING) and the POLLING (also known as MULTIDROP or HUBPOLLING) systems.

In this thesis, an exact analysis of a general polling system is performed. The main results obtained (especially those which concern message waiting times, buffers sizes, and inter-terminal behaviour) are new and important in understanding and designing polling systems.

In both the loop and the polling systems, the N terminals and the central data processor are connected by one loop, as shown in Fig. 1-1. Data, i.e, messages from the terminals to the central data processor, may flow in one direction only (say clockwise as in Fig. 1-1). A message from terminal j (T_j), to the central data processor must pass through terminals j + 1, ... , N and in each a certain delay occurs to enable the terminal to check the message's address, control symbol, and the possibility of transmission.

In the loop system, all messages are divided into units of constant time length, called a time slot. Each terminal can transmit its message (or part of it of one time slot length) in any empty time slot. Clearly, to achieve this, all terminals have to be mutually synchronized to their different beginning of time slots. Hence, in the loop system, T_1 can use any time slot while T_j may transmit during a time slot only if it is not occupied previously with messages from the previous $j - 1$ terminals. The loop system has inherent, fixed, preassigned priorities among the terminals. T_1 experiences the highest service priority and T_N the lowest.

In the polling system, a control symbol (called the server and originating from the central data processor) cycles continuously. Each terminal, when it encounters the control symbol, inserts its buffer's content (all its waiting messages) before it. After the service of a terminal is completed, control is transferred to the next terminal on the loop. Hence, in the polling system there are no inherent service priorities among the terminals.

Using "Queueing Theory" terms, both systems are nonpreemptive, multi-queues, single server models, whereas in the loop system there is a fixed priority service procedure according to the terminal index (to be precise, all messages in this case are required to be of one time slot). In the polling system, we have an alternating priority procedure in cyclic order, depending upon the server's present location.

Another approach to data communication is suggested by the ALOHA system, Abramson (1973), which is at present under development at the University of

Hawaii. In this system, all terminals transmit independently using the same channel and there are no priorities among them. Having a new message, a terminal transmits it immediately but keeps holding it for retransmission until an acknowledgement "message detected" is received from the addressee. If during a fixed waiting time, no such acknowledgement is received (due to interferences with other messages from "independent" terminals), the message is retransmitted within a fixed time according to some probabilistic rule. Retransmission is repeated until an acknowledgement is received. In a sense, this is a control-free system. Computer simulations of the system with traffic intensities much below 1/2 e, the theoretical bound for such multiterminal system, yield attractive results pertaining to waiting times, transmission efficiency, and channel utilization. Analytical results for this ATDM system are few.

For the two other ATDM systems described, the loop system is analytically easier to evaluate than the polling system. An exact explicit solution of average terminal buffer length and waiting time was derived by Spragins (1972a). Variations of these quantities were derived by Spragins (1972b) and a summary of those results, as well as the simple derivation technique, may be found in Spragins (1971). Analysis of the same loop system model studied by Spragins, but from a different approach, was performed by Konheim (1972). His study yields the same buffer length results, but a virtual (instead of an actual) waiting time. Virtual waiting time is the time an imaginary single-time-slot message would have to wait had it been inserted at the top of the buffer at some fixed time. Konheim and

Meister (1973) extended Konheim's (1972) results by including traffic from the central data processor to the terminals, i.e., two way traffic where traffic from the terminals have preemptive priority over traffic from the central data processor. Hayes and Sherman (1971) attempted to estimate the behaviour of such multiloop systems.

While basic results, such as buffer size and waiting time for the loop system are quite abundant, they are very limited for the polling system and the common tendency is to use approximations.

The general model of the communication polling system, we analyze in detail in this thesis, is identical to a general multiqueue single server model studied in operations research. For example, imagine a transportation system, Fig. 1-2 where a single bus services N bus stops in a fixed cyclic order. At each bus stop potential passengers might accumulate, according to some probabilistic process. Each passenger requires some service time while getting into the bus and the riding time of the bus between the terminals is a nonzero quantity, governed by some stochastic process. Important parameters in such a system are the quality of service (passenger waiting time), the required waiting room capacity at each bus stop, and the cycle time of the bus. Another example might be described as follows : a physician who visits N clinics in a cyclic fixed order, at each clinic clients may be awaiting his treatment. Still another example might be a repair man visiting N different factories in a fixed cyclic order, at each some machines might require some service.

The principal mathematical model of the communication polling system is identical to those of the above three systems. Using general expressions from queueing theory we refer to the central data processor (or more exactly to its control symbol), the bus, the physician, and the repair man as the "server". The data terminals, bus stops, clinics, and factories are generally called "terminals" (or as in queueing theory – "queues") . The messages, passengers, clients, and out of order machines are called "customers". The time required to transmit a message, or to serve a passenger, a client, or a machine is called the "service time" . The propagation and synchronization delay between two successive data terminals, the riding time of the bus between two successive bus stops, or that of the physician or the repair man between two successive terminals are referred as the "walking time" . An important parameter in this model, as well as in any other queueing model , is the "traffic intensity" of a terminal defined by the ratio of the average customer service time in the terminal to the average interarrival time of new customers in the terminal. A well known fact from queueing theory is that unless the total traffic intensity in the system (in our model – the sum of the N traffic intensities) is strictly less than one, the system is unstable (saturated) .

Using these expressions, we now give an exact description of the polling model we shall study in this thesis. A schematic figure of it is given in Fig. 1-2.

N terminals of unrestricted length (infinite buffer or waiting room capacity) are attended by a single server in a fixed cyclic order. Arrivals of new

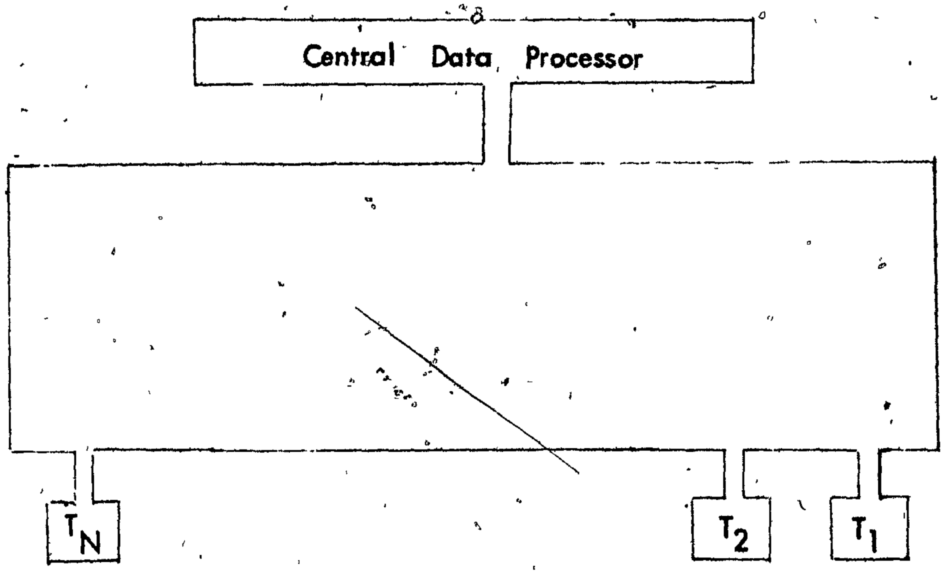

Fig. 1-1 : The communication Loop / Polling system.

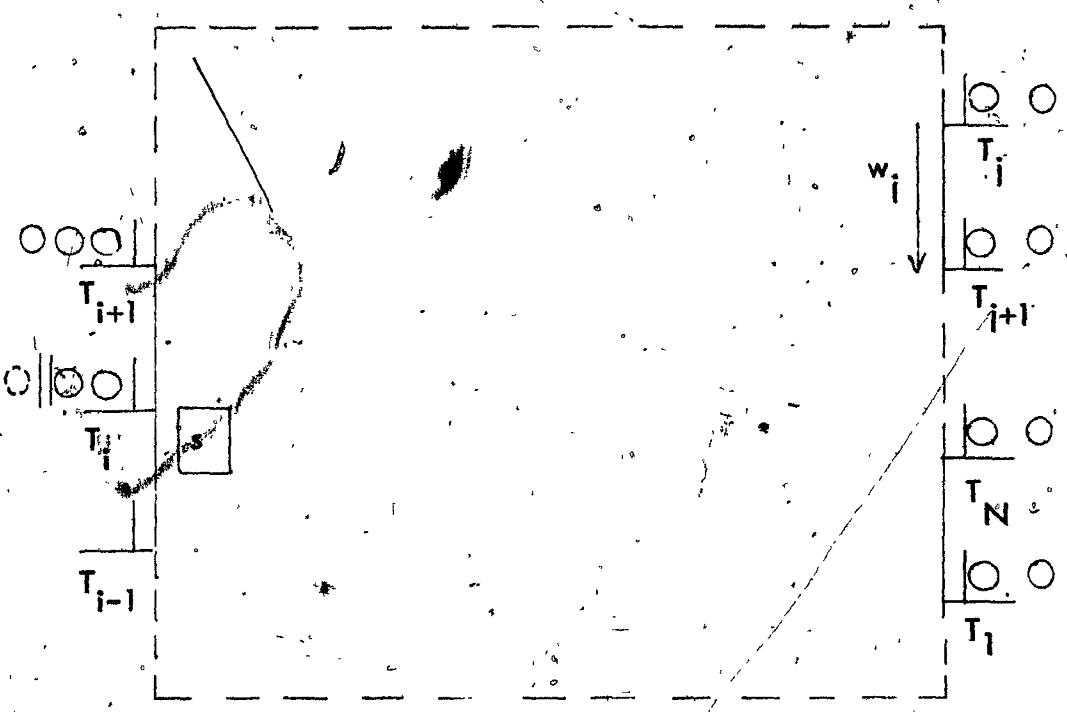

Fig. 1-2 : The general polling model at the moment the server reaches Terminal i (T_i).

○ a customer

[s] the server

Terminal j (T_j)

The "gate"

customers to each terminal are independent and governed by a Poisson process with rate λ_i, the average number of customers per second, and $i = 1, 2, \ldots, N$ being the terminal numbers. The customer service time is a random variable with the probability density function $P_{s_i}(\cdot)$. The server walking time from terminal i to $i+1$ (from N back to 1) is an independent random variable with the probability density function $P_{w_i}(\cdot)$.

We study two variations of the service procedure for this model. In the first, called the GATING model, the server attends only to the customers who are present at each terminal at the moment it arrives ; customers entering a terminal which is being served must wait until the server departs and then returns to it (in the following cycle). In this model, at the instant the server reaches a terminal a "gate" closes behind the waiting customers and only those customers waiting in front of the gate are served during that cycle. This situation is described in Fig. 1-2 at T_i. In the second variation of the service procedure, called the EXHAUSTIVE model, the server attends each terminal until it is empty, i.e., the server advances to a terminal only when the previous one has no waiting customers in it.

In both models, the service procedure within each terminal is on a "first come first served" basis. The order in which the terminals are served is according to their indices, from terminal i to $i + 1$ and from N back to 1. If the server reaches an empty terminal, it walks immediately to the following one. A sequence of N consecutive terminal visits is a cycle, in which, each terminal is visited once by the server.

The intervisit time of a terminal is the time that separates the server's departure from the terminal to the server subsequent return to the same terminal.

Special cases of these models are the symmetric case, where λ_i, $P_{s_i}(\cdot)$, $P_{w_i}(\cdot)$ are independent of i, and the nonrandom case, where the service and walking times are constant.

Though the exhaustive procedure is apparently a more reasonable one, the gating procedure, in the communication polling system, might be cheaper to implement. (The buffer might be a shift register which cannot be loaded while messages are being transmitted, and the new messages are stored on a different register). Another advantage of the gating procedure is that every terminal, while transmitting, does not seize control of the server.

Analytical results for these models are very few and explicit expressions for basic parameters such as first moments of the waiting time, buffer size, and cycle time are almost nonexistent and the general approach is to use approximations.

A buffer size analysis of the nonrandom symmetric system (the exhaustive procedure) was done by Konheim and Meister (1971) . Their result for the average virtual waiting time in such a system is quoted by Chu and Konheim (1972). An approximation for the nonrandom case (both the gating and the exhaustive models) was done by Kruskal (1969) motivated by : "Better have approximations than no analysis at all". Based on an important analytical result obtained by Mack et al

(1957), a symmetric, nonrandom, single buffer system was analyzed by Kaye (1972) and the first two moments of the number of terminals that are served in a cycle were derived. An estimate for the average waiting time in such a system was obtained by Kaye and Richardson (1973). A model related to that used by Kaye, (in each terminal only one message can be served in a cycle, but buffers length are unrestricted), was approximated and simulated by Yuen et al (1972). Hayes and Sherman (1972) used approximations obtained by Leibowitz (1961), for the symmetric gating model, in order to compute delays in the nonrandom symmetric case of the exhaustive model.

Most of the existing analytical results were derived by operations research scientists. A symmetric, nonrandom, single buffer system (known as the repair man problem) was studied by Mack et al (1957), who obtained the steady state probability density function of the number of services per cycle. However, assuming random service time, Mack (1957) derived intractable equations. Leibowitz (1961) approximated the symmetric gating model by assuming independence among the terminals, since an exact solution seemed extremely difficult. In a later article (Leibowitz (1968)), this problem was cited as an example of an important queueing problem to which no exact solution has been obtained. Therefore, the terminals' independence approximation is unavoidable. The general polling model, with the restriction of zero walking time (mainly for the exhaustive model) was analyzed by Cooper and Murray (1969) with unwieldly equations. Cooper (1970) extended the analysis to compute the waiting time and again met with intractable computations. The special case of the exhaustive model, where N = 2 (two terminals), and with zero walking time, was solved by

Avi-Itzhak et al (1965), Takacs (1968), and Jaiswal (1968), and an explicit expression of the average waiting time was derived. The same two-queues exhaustive model, but with zero walking time, was analyzed by Sykes (1970) and an explicit expression for the average waiting time was obtained.

The only complete analytical solution of the general model (mainly the exhaustive case) was derived by Eisenberg (1972). However, the mathematical representation of the model and its basic random variables yield such great numbers of equations that, in concluding, he suggested using Leibowitz's (1961) approximation method. It should be mentioned that even though Hayes and Sherman (1972) were aware of Eisenberg's (1972) exact results, they preferred Leibowitz's (1961) approximation.

For various special models, such as the symmetric and / or the non-random model, commonly used to describe communication polling networks, we obtain exact, and surprisingly simple, explicit solutions for the first moments of the waiting time, buffer size, cycle and intervisit times. These expressions can be applied immediately instead of the approximations used in past technical papers. Moreover, these expressions are of fundamental importance in predicting, understanding, and designing polling systems.

The peculiar choice of the models' basic random variables, as well as most of the mathematical analysis which follows, and the explicit expressions obtained are original. The choice of the "Non classical" basic random variables might be regarded as the main reason for the success of this study.

The main contributions in this study are :

1. Simple mathematical description of the polling system, for both the gating and the exhaustive service procedures, which holds in the transient and the steady states.

2. A study of the transient behaviour of the average cycle and intervisit times.

3. Explicit formulas in the symmetric case for the steady state variances of cycle time, intervisit time, and the number of customers in a terminal that are served in one cycle.

4. Explicit formulas in the symmetric nonrandom case for the steady state correlation between terminals, cycles, and intervisits.

5. Explicit formulas in the symmetric nonrandom case for the third moment of the cycle time for $N = 1, 2, \infty$ for the gating model, and the third moment of the intervisit time for $N = 1, 2, 3, \infty$ for the exhaustive model.

6. An explicit formula in the symmetric case for the average waiting time.

7. Exact results for the asymmetric nonrandom system concerning average waiting time for $N = 2$ in the gating model, and for $N = 3$ in the exhaustive model.

8. Explicit formulas for a general case, for the average system waiting time.

9. Explicit formulas in the symmetric nonrandom case for the variances of the waiting time for the gating model with $N = 2, \infty$ and for the exhaustive model with $N = 2, 3, \infty$. Using Chebyshev's inequality we can then find bound on the probability that the actual waiting time is less than some value T.

In Chapter II, we give an exact description of the model, its basic random variables, and find, in a recursive form, the Laplace transform of the probability density function of the basic random variables. The first three moments of the cycle and intervisit time, for various cases, are derived in Chapter III. In Chapter IV we derive a general relation between the number of customers that are served in a terminal in one cycle and other quantities obtained before. Explicit expressions for the first two moments of this quantity are found. Buffer size requirements are analyzed and bounds for those quantities are obtained. In Chapter V, we analyze the transient behaviour of the average cycle and intervisit times, and the steady state correlation between terminals, cycles, and intervisits. In Chapter VI, we derive the Laplace transform of the probability density function of the waiting time. General expressions for the first two moments of the waiting time are obtained. Analytical and numerical results for various cases are derived. Suggestions for further research and conclusions are given in Chapter VII. All detailed and technical computations are performed in the Appendices.

CHAPTER II

THE MATHEMATICAL ANALYSIS OF THE MODELS

2.1 Mathematical Description of the Models

A detailed description of the models is given in Chapter I. We repeat it briefly here for completeness.

N terminals, $T_1, \ldots, T_i, \ldots, T_N$ each with buffers of unrestricted length and having messages (customers) coming with a Poisson arrival process having parameter λ_i, are served by a single server in a fixed cyclic order. The terminals are served in the order of their indices (as shown in Fig. 1-2). At every terminal, say T_i, the server serves some of the customers, using independent identically distributed random serivce times having a probability density function $P_{s_i}(\cdot)$. Then the server walks to the next terminal (T_{i+1}), where the walking time is a random variable having a probability density function $P_{w_i}(\cdot)$ and is independent of the service time. We study two different service procedures referred to as the gating and the exhaustive models. In the gating model, at each terminal the server serves only those customers that are present at the moment of its arrival. In the exhaustive model the server serves each terminal until it is empty.

In Queueing Theory terms (Jaiswal (1968)), these models are $M_1, \ldots, M_N / G_1, \ldots, G_N / 1$ multiqueue having nonpreemptive cyclic alternating priorities and nonzero changeover (walking) times. The classical way of analyzing such a model, as well as the simple $M / G / 1$ model, is by defining a continuous time N- dimensional discrete state random process, in which the elements —

of the N - dimensional random vector are the number of customers in each queue. The next step is to imbed the random process at its regenerating points, for example the set of customer service-beginning instants in its queue, customer service-completion instants, queue service-beginning instants, or queue service-completion instants. The property of these regenerating points is that a new random process, derived from the above continuous time process, at these time instants, is a Markov process and is called : the imbedded Markov process. Applying basic steady state equations to these processes, we finally obtain discrete time, discrete state, imbedded Markov chains. Details concerning the above techniques can be found in basic books on Markov chains or Queueing theory like Takacs (1962), Jaiswal (1968), and Karlin (1969). Each of these Markov chains is described by an N - dimensional random vector whose elements are the number of customers in each queue at the appropriate regenerating point.

Four characteristics of the above approach are :

1. The basic random variables have discrete states.

2. The model is analyzed in the steady state.

3. A large number of equations is required (for example, Eisenberg (1972) dealt with 4N N - dimensional different imbedded Markov chains).

4. The basic random vector is a "picture" of all the queues (the number of customers in each one) at particular time instants.

The goal of researches of this type is to find the probability density functions of the server cycle times, terminal buffer sizes, and customer waiting times. Having these probability density functions, or explicit or recursive forms of their Laplace transform, the next step is to derive explicit expressions for the averages and variances of their arguments. These quantities, which are essential in designing and understanding polling systems, are obtained by differentiating the Laplace transforms of those probability density functions and solving some set of linear equations. Examples of this procedure can be found in Leibowitz (1961), Takacs (1968), Cooper and Murray (1969), Cooper (1970), and Eisenberg (1972). The basic disadvantage of the above approach is the intractability in deriving explicit expressions for the first moments of the cycle and intervisit times, the buffer sizes (or the number of customers in a terminal that are served in a cycle), and the waiting times. This is due to the fact that a large number of linear equations are involved in deriving these quantities. Cooper (1970), who analyzed the N - terminal exhaustive model with zero walking time, encountered an intractable set of $(N+1) \cdot N$ linear equations for the explicit expression of the average waiting time in a terminal. Eisenberg (1972), who is the only one to perform an exact analysis of the general N terminal exhaustive model, obtained a set of $N(N-1)$ linear equations for the explicit expressions of the second moment of the number of customers to be served in one cycle. Neither Cooper nor Eisenberg obtained their goal. Even assuming the symmetric case, their infractable sets of equations do not get more simple. Beside the average cycle and intervisit times, and the average number of customers to be served in one cycle, no explicit expressions for higher moments and any moment of the waiting time were obtained.

In order to avoid this large set of linear equations, we chose an inherently different basic random vector which has no direct relation to the classical derivation technique but possesses the Markovian property. The choice of a different basic N dimensional random vector, which generally has continuous states, enables us to derive a transient state model, to end up with the smallest set yet of linear equations, and most important, to solve them and obtain exact explicit expressions for some first moments of the cycle and intervisit times, buffer sizes, and waiting times. Our basic random variables do not exist at the same time. Therefore the basic random vector is not a "picture" of the system at some time instants.

We define the new random variable, θ, as "Terminal Service Time" (TST). To avoid unnecessary complexity in future computations, the TSTs are chosen in a slightly different manner for the gating and the exhaustive models.

In the gating moded : TST is the sum of the times required to serve the customers in a terminal and to walk to the next one.

In the exhaustive model : TST is the sum of the times required to walk to a terminal (from the previous one) and to serve the customers there.

We deal with an N dimensional random vector, N successive TSTs, which describes how a cycle time is divided among the N terminals. It should be mentioned that similar concepts in defining a vector of random variables which do not exist at the same time were used by Mack et al (1957) and by Konheim and Meister (1971).

In the following, we construct the basic random variables for both service procedures. To distinguish between the gating and the exhaustive models we use the subscripts G and E respectively.

For the gating model, suppose at the time $t = t_{G_o}$ the server reaches T_1. This time instant is defined as the beginning of the first cycle.

For the exhaustive model, suppose at the time $t = t_{E_o}$ the server leaves T_N. This time instant is defined as the beginning of the first cycle.

For $t \geq t_o$, in each model, we record the time instants when TSTs are completed. To distinguish between the same terminal in different cycles, define terminal i ($1 \leq i \leq N$) in cycle j ($j \geq 1$) as terminal k (T_k) where :

$$k \triangleq (j - 1) N + i \qquad 2.1.1$$

For $k \geq 1$ define the following random variables :

For both models :

$n_k \triangleq$ The number of customers that are served at T_k. 2.1.2

For the gating model :

$t_{G_k} \triangleq$ The time server reaches T_{k+1}

$G\theta_k \triangleq {}_G TST_k \triangleq$ Terminal k service time $= t_{G_k} - t_{G_{k-1}}$

$G^c_{k+N} \triangleq$ The cycle time of $T_{k+N} = \sum_{i=0}^{N-1} G\theta_{k+i}$ 2.1.3

For the exhaustive model :

$t_{E_k} \triangleq$ The time server leaves T_k

$E^{\theta}{}_k \triangleq {}_E TST_k \triangleq$ Terminal k service time $= t_{E_k} - t_{E_{k-1}}$,

$E^{c}{}_{k+N} \triangleq$ The cycle time of $T_{k+N} = \sum_{i=1}^{N} E^{\theta}{}_{k+i}$

$E^{v}{}_{k+N} \triangleq$ The intervisit time of $T_{k+N} = w_{k+N-1} + \sum_{i=1}^{N-1} E^{\theta}{}_{k+i}$

$$\equiv E^{c}{}_{k+N} - E^{\theta}{}_{k+N} + w_{k+N-1} \qquad 2.1.4$$

where w_{k+N-1} is the walk times from T_{k+N-1} to T_{k+N} .

In Fig. 2-1 a schematic representation of the models and their basic random variables is given. We consider the terminals $k, k+1, \ldots, k+N$ where T_k is actually T_{k+N} in the previous cycle. Whenever possible, we omit the indices G and E from the basic random variables.

The basic N dimensional random vector in our models is $\underline{\theta}_k$

$$\underline{\theta}_k \triangleq (\theta_k, \theta_{k+1}, \ldots, \theta_{k+N-1}) \qquad k \geq 1 \qquad 2.1.5$$

An important property of these random vectors is that the conditional joint probability of $\underline{\theta}_k$ given any set of previous time vectors, depends on the latest vector of this set, i. e. :

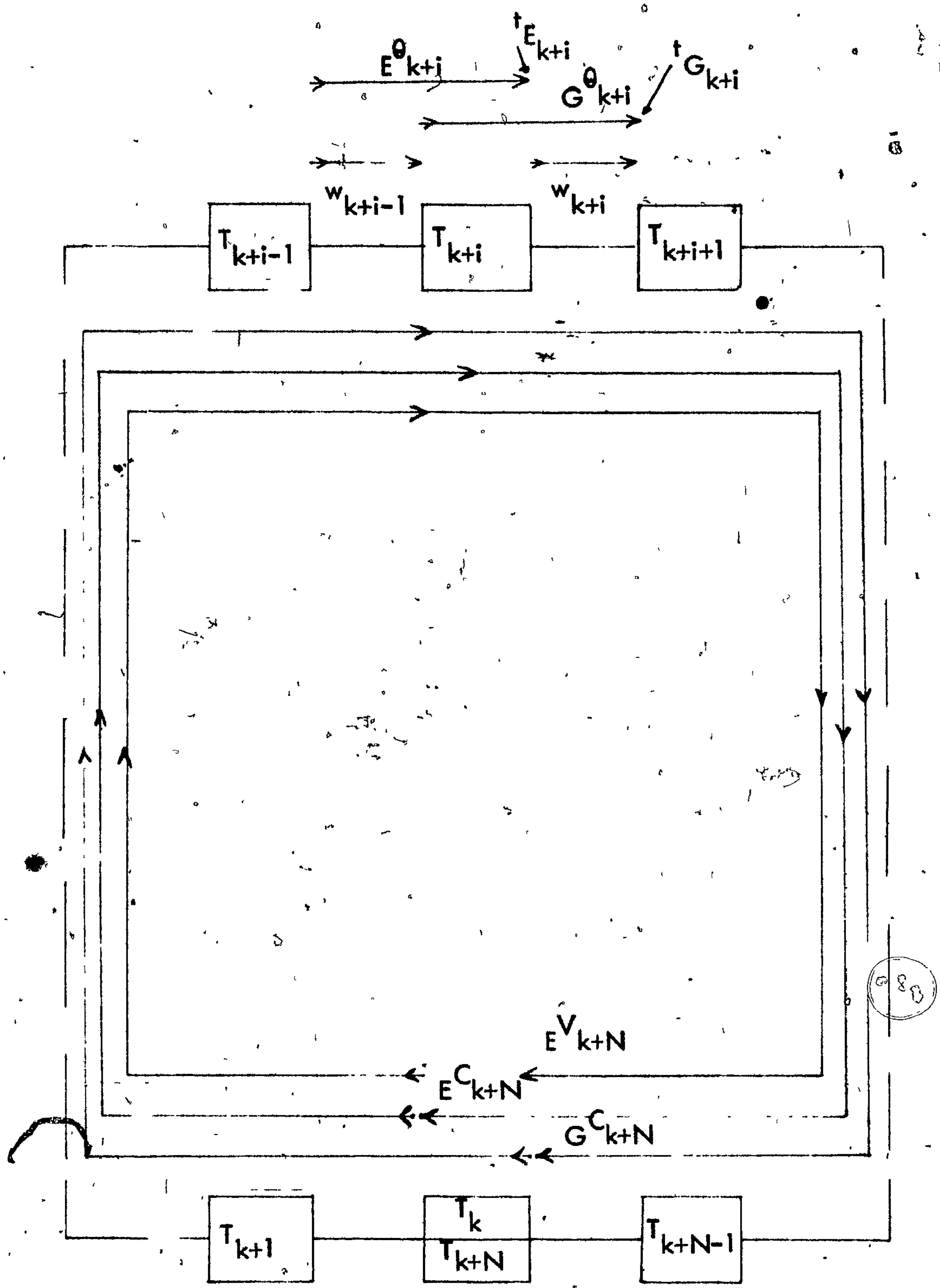

Fig. 2-1 : A schematic representation of the models and their basic random variables.

$$\text{Prob}\,(\underline{\theta}_k / \underline{\theta}_{k-1}, \underline{\theta}_{k-2}, \ldots, \underline{\theta}_{k-l}) = \text{Prob}\,(\underline{\theta}_k / \underline{\theta}_{k-1}) \qquad 2.1.6$$

The above property is due to the fact that the first N-1 elements of $\underline{\theta}_k$ are identical to the last N-1 elements of the given $\underline{\theta}_{k-1}$, and the remaining last element of $\underline{\theta}_k$, θ_{k+N-1}, is the sum of the walking time (w_{k+N-2} or w_{k+N-1} for the exhaustive or gating models respectively) which does not depend on any $\underline{\theta}_{k-i}$ ($i \geq 1$) and the service time of the customers in T_{k+N-1}. This service time depends on the number of customers accumulated in T_{k+N-1} due to a Poisson arrival process. The time length of this process is given by $\underline{\theta}_{k-1}$ (and also w_{k+N-2} for the exhaustive model), as shown in Fig. 2-1. Hence, given $\underline{\theta}_{k-1}$, all $\underline{\theta}_{k-l}$ ($l \geq 2$) are irrelevant to the conditional probability density function of $\underline{\theta}_k$. This property is known as Markov property.

The following are definitions of quantities we use in the next sections of this chapter.

The joint probability density function of $\underline{\theta}_k$ is

$$P_k(\underline{\theta}_k) \stackrel{\Delta}{=} P_k(\theta_k, \theta_{k+1}, \ldots, \theta_{k+N-1}) = \text{Prob}\,(\text{for } i=0,\ldots,N-1,\ TST_{k+i} \doteq \theta_{k+i}) \qquad 2.1.7$$

The Laplace transform of $P_k(\underline{\theta}_k)$ is

$$_kG(\underline{x}) = \int_0^\infty \cdots \int_0^\infty \exp\Big(-\sum_{i=0}^{N-1} x_{i+1}\theta_{k+i}\Big) P_k(\underline{\theta}_k) \prod_{i=0}^{N-1} d\theta_{k+i} \qquad 2.1.8$$

for all x_i such that $\text{Re}\,[x_i] \geq 0$, and

$$_kF(\underline{x}) = \ln_k G(\underline{x}) \tag{2.1.9}$$

The Laplace transforms of the probability density functions of customer service time in T_i and server walking time from T_i to T_{i+1} are respectively

$$S_i(x) = \int_0^\infty \exp(-xt)\, P_{s_i}(t)\, dt$$

$$W_i(x) = \int_0^\infty \exp(-xt)\, P_{w_i}(t)\, dt \tag{2.1.10}$$

for $i = 1, .., N$ and $\mathrm{Re}\,[x] \geq 0$

When no confusion is possible, we omit the dummy variable x in the expressions of the Laplace transforms.

In the following two sections of this chapter we find $_kG(\underline{x})$ in a recursive form for both the gating and the exhaustive models.

2.2 The Gating Model : Solution of the Basic Equation

In this section we find $_kG(\underline{x})$ in a recursive form for the gating model. $_kG(\underline{x})$ is the Laplace transform of the joint probability density function of N successive TSTs, $P_k(\theta_k, \theta_{k+1}, .., \theta_{k+N-1})$.

By the law of total probability :

$$P_{k+1}(\theta_{k+1}, \ldots, \theta_{k+N}) \triangleq P_{k+1}(\underline{\theta}_{k+1}) = \int_0^\infty \sum_{n_{k+N}=0}^\infty \text{Prob}(\theta_{k+N}, n_{k+N}, \underline{\theta}_k)\, d\theta_k$$

$$= \int_0^\infty \sum_{n_{k+N}=0}^\infty \text{Prob}(\theta_{k+N} / n_{k+N}, \underline{\theta}_k)\, \text{Prob}(n_{k+N} / \underline{\theta}_k)\, P_k(\underline{\theta}_k)\, d\theta_k \qquad 2.2.1$$

Since θ_{k+N} is the time required to serve the n_{k+N} customers (which the server finds at the moment he reaches T_{k+N} and are the only ones to be served in this terminal) and to walk to T_{k+N+1}, Prob (θ_{k+N} / n_{k+N}, any "past" TSTs) does not depend on the past TSTs . We have :

$$\text{Prob}(\theta_{k+N} / n_{k+N}, \underline{\theta}_k) = \text{Prob}(\theta_{k+N} / n_{k+N}) \qquad 2.2.2$$

and because θ_{k+N} is the independent sum of n_{k+N} service times and one walking time, we obtain :

$$\text{Prob}(\theta_{k+N} / n_{k+N}) = P_{s_{k+N}}^{(n_{k+N})*}(\theta_{k+N}) * P_{w_{k+N}}(\theta_{k+N}) \qquad 2.2.3$$

where '*' symbolizes the convolution operator.

Since n_{k+N} is the result of a Poisson arrival process which starts at $t_{G_{k-1}}$ (refer to Fig. 2-1) , when no customers are present after the gate, and ends at $t_{G_{k+N-1}}$ and does not depend on $t_{G_{k+i}}$ $0 \le i \le N-2$. We have :

$$\text{Prob}\,(n_{k+N}/\underline{\theta}_k) = \text{Prob}\,(n_{k+N}/\sum_{i=0}^{N-1}\theta_{k+i})$$

$$= \frac{1}{n_{k+N}!}(\lambda_{k+N}\sum_{i=0}^{N-1}\theta_{k+i})^{n_{k+N}}\exp(-\lambda_{k+N}\sum_{i=0}^{N-1}\theta_{k+i}) \qquad 2.2.4$$

We identify $\sum_{i=0}^{N-1}\theta_{k+i}$ as c_{k+N}, the cycle time of T_{k+N}.

Substitution of 2.2.3 and 2.2.4 in 2.2.1 yields

$$P_{k+1}(\underline{\theta}_{k+1}) = \int_0^\infty \sum_{n_{k+N}=0}^{\infty} P_{s_{k+N}}^{*(n_{k+N})}(\theta_{k+N}) * P_{w_{k+N}}(\theta_{k+N})\frac{1}{n_{k+N}!}(\lambda_{k+N}\sum_{i=0}^{N-1}\theta_{k+i})^{n_{k+N}}$$

$$\cdot\exp(-\lambda_{k+N}\sum_{i=0}^{N-1}\theta_{k+i})\quad P_k(\underline{\theta}_k)\,d\theta_k \qquad 2.2.5$$

Equation 2.2.5 is a recursive formula for obtaining $P_{k+1}(\underline{\theta}_{k+1})$ from $P_k(\underline{\theta}_k)$. Given $P_1(\underline{\theta}_1)$ we can obtain $P_k(\underline{\theta}_k)$ for all $k \geq 1$. As $k \to \infty$, and total traffic intensity is less than one, $P_k(\underline{\theta}_k)$ converges to $P_{k\ \mathrm{Mod}\ (N)}(\underline{\theta})$ - the steady state joint probability density function which does not depend on $P_1(\underline{\theta}_1)$.

Applying the Laplace transform to Eq. 2.2.5 we have :

$${}_{k+1}G(\underline{x}) = \int_0^\infty \cdots \int_0^\infty \sum_{n_{k+N}=0}^{\infty} \exp(-\sum_{i=1}^{N} x_i\,\theta_{k+i})\quad P_{s_{k+N}}^{*(n_{k+N})}(\theta_{k+N}) * P_{w_{k+N}}(\theta_{k+N})\frac{1}{n_{k+N}!}$$

$$\cdot(\lambda_{k+N}\sum_{i=0}^{N-1}\theta_{k+i})^{n_{k+N}}\exp(-\lambda_{k+N}\sum_{i=0}^{N-1}\theta_{k+i})\quad P_k(\underline{\theta}_k)\prod_{i=0}^{N} d\theta_{k+i} \qquad 2.2.6$$

To obtain the desired result, ${}_{k+1}G(\underline{x})$ as a function of ${}_{k}G(\underline{\cdot})$, we first integrate over θ_{k+N}, then sum over n_{k+N}. For convenience, define

$$x_o \triangleq 0 \qquad 2.2.7$$

Integration over θ_{k+N} yields

$$ {}_{k+1}G(\underline{x}) = W_{k+N}(x_N) \int_0^\infty \cdots \int_0^\infty \sum_{n_{k+N}=0}^{\infty} \frac{1}{n_{k+N}!} \left(\lambda_{k+N} S_{k+N}(x_N) \sum_{i=0}^{N-1} \theta_{k+i}\right)^{n_{k+N}} \qquad 2.2.8$$

$$\cdot \exp\left(-\sum_{i=0}^{N-1} (x_i + \lambda_{k+N})\theta_{k+i}\right) \cdot P_k(\underline{\theta}_k) \prod_{i=0}^{N-1} d\theta_{k+i}$$

Finally, summation over n_{k+N} yields

$$ {}_{k+1}G(\underline{x}) = W_{k+N}(x_N) \int_0^\infty \cdots \int_0^\infty \exp\left[-\sum_{i=0}^{N-1} (x_i + \lambda_{k+N}(1 - S_{k+N}(x_N)))\theta_{k+i}\right] \qquad 2.2.9$$

$$\cdot P_k(\underline{\theta}_k) \prod_{i=0}^{N-1} d\theta_{k+i}$$

Using definition 2.1.8, the right hand side of Eq. 2.2.9 is ${}_{k}G(\underline{z})$ where :

$$z_i \triangleq x_{i-1} + \lambda_{k+N}(1 - S_{k+N}(x_N)) \quad i = 1, \ldots, N, \quad x_o \triangleq 0 \qquad 2.2.10$$

Since T_k and T_{k+N} are physically the same terminal

$$\lambda_{k+N} \triangleq \lambda_k$$

$$W_{k+N}(\cdot) \triangleq W_k(\cdot)$$

$$S_{k+N}(\cdot) \triangleq S_k(\cdot) \qquad 2.2.11$$

Defining $y_k(x_N) \triangleq \lambda_k(1 - S_k(x_N))$ 2.2.12

we have $z_i \triangleq x_{i-1} + y_k(x_N)$ 2.2.13

Using all above definitions in Eq. 2.2.9 yields

$$_{k+1}G(\underline{x}) = W_k(x_N) \quad _kG(\underline{z}) \qquad 2.2.14$$

or explicitly

$$_{k+1}G(x_1, x_2, \ldots, x_N) = W_k(x_N) \quad _kG[y_k(x_N), x_1 + y_k(x_N), \ldots, x_{N-1} + y_k(x_N)] \qquad 2.2.15$$

The recursive form of Eq. 2.2.15 is the main result of this section. An important feature of this equation is

At $\underline{x} = \underline{0}$, we have

$$W_k(x_N) = S_k(x_N) = 1$$

$$y_k(x_N) = 0$$

and hence $\underline{z} = \underline{0}$ 2.2.16

Therefore, at $\underline{x} = \underline{0}$, Eq. 2.2.15 simply states the expected and consistent identity $1 \triangleq 1$.

This behaviour at $\underline{x} = \underline{0}$ enables us to obtain all the moments of the TSTs. A more convenient form of Eq. 2.2.14, which we use later, is obtained by using definition 2.1.9 in Eq. 2.2.14 to obtain

$$_{k+1}F(\underline{x}) = \ln W_k(x_N) + {}_kF(\underline{z}) \qquad 2.2.17$$

In order to use Eq. 2.2.14 recursively more than one time, define a set of coordinate transformations functions. Expressing Equations 2.2.12 and 2.2.13 in the form

$$\underline{x} = f^{(0)}(\underline{x})$$

$$\underline{z} = f^{(1)}(\underline{x}) \qquad 2.2.18$$

is regarded as one transformation. The result of two successive transformations is :

$$f^{(1)}(\underline{z}) = f^{(2)}(\underline{x})$$

and so on, after j successive transformations, the N dimensional vector $f^{(j)}(\underline{x})$ is obtained. Its last element is defined as $(f^{(j)}(\underline{x}))_N$.

Adopting the above notation, while using Eq. 2.2.14 recursively N times, we obtain

$$_{k+1}G(\underline{x}) = \left[\prod_{i=0}^{N-1} W_{k-i}\left((f^{(i)}(\underline{x}))_N\right)\right] {}_{k+1-N}G(f^{(N)}(\underline{x})) \qquad 2.2.19$$

The above equation ties together two successive cycles, which, in the steady state, have the same joint probability density function.

Using Eq. 2.2.14 recursively k times we have

$$ {}_{k+1}G(\underline{x}) = \left[\prod_{i=0}^{k-1} W_{k-i}\left((f^{(i)}(\underline{x}))_N\right)\right] {}_1G(f^{(k)}(\underline{x})) \qquad 2.2.20 $$

For, Re [$\underline{x}$] ≥ $\underline{0}$, Equations 2.2.12 and 2.2.13 yield:

$$ \mathrm{Re}\,[f^{(i)}(\underline{x})] \geq \underline{0} \text{ for all } i \leq k $$

Therefore, for all $i \leq k$,

$$ \underline{0} \leq {}_{k+1-i}G[f^{(i)}(\underline{x})] \leq \underline{1} $$

$$ 0 \leq W_{k-i}[(f^{(i)}(\underline{x}))_N] \leq 1 \qquad 2.2.21 $$

When total traffic intensity is less than one and Re [$\underline{x}$] ≥ $\underline{0}$ are finite, as $k \to \infty$, we have :

$$ \underline{0} < \lim_{k\to\infty} {}_{k+1}G(\underline{x}) \leq \underline{1} . \qquad 2.2.22 $$

By 2.2.20 and 2.2.21, the only way to satisfy 2.2.22 is when

$$ \lim_{\substack{k,i\to\infty \\ i\leq k}} {}_{k+1-i}G[f^{(i)}(\underline{x})] = \underline{1} . $$

$$\lim_{\substack{k,i \to \infty \\ i \le k}} W_{k-i} \left[(f^{(i)}(\underline{x}))_N \right] \neq 1$$

and we have :

$$\lim_{k \to \infty} {}_{k+1}G(\underline{x}) = \lim_{k \to \infty} \left[\prod_{i=0}^{k} W_{k-i} ((f^{(i)}(\underline{x}))_N) \right] \; {}_1G(f^{(k)}(\underline{x}))$$

$$= \lim_{k \to \infty} \prod_{i=0}^{k} W_{k-i} ((f^{(i)}(\underline{x}))_N)$$

Using steady state notation, we obtain for each T_i, $i = 1, .., N$:

$${}_iG(\underline{x}) = \prod_{j=0}^{\infty} W_{i-j} \left[(f^{(j)}(\underline{x}))_N \right] \qquad 2.2.24$$

where $i-j$ is computed modulo N .

The above equation is a direct solution of the basic equation in the steady state. However this equation presents computational difficulties which precludes its practical use. It should be mentioned that a similar approach to that which led to Eq. 2.2.24 was used by Cooper and Murray (1969) and Eisenberg (1972). However, since they used different basic random variables, their results are different from Eq. 2.2.24.

2.3 The Exhaustive Model : Solution of the Basic Equation

As in the previous section we now derive ${}_kG(\underline{x})$ for the exhaustive model, in a similar recursive form. Here however, we need a new random variable, the "busy period" length in an $M/G/1$ queue.

Define :

$P_{b_i}(t)$ $\triangleq$ The probability density function of the time, t, spent by the server in queue i ($i = 1, .., N$) given that exactly one customer was present in the queue at the moment the server reached it.

In the exhaustive model, although one customer may exist at T_i when the server reaches it, during the service time of this single customer, new ones may arrive, and according to the exhaustive service procedure, the server serves them too, leaving T_i only when it is empty.

In the first part of Appendix A, the relation between the Laplace transform of the busy period probability density function – $B_i(x)$, and other parameters of T_i is developed. This relation is well known in queueing theory and can be found in Takacs (1962). However, because the method leading to it is helping in understanding this section, we derive it in complete detail in Appendix A. The basic expression obtained is :

$$B_i(x) = S_i[x + \lambda_i(1 - B_i(x))] \qquad 2.3.1$$

A simple probabilistic argument shows that given m_i customers present at T_i when the server reaches it, the probability density function of the total service time of this queue is :

$$\text{Prob}(b_i = t / m_i) = P_{b_i} *^{(m_i)}(t) \qquad 2.3.2$$

The reason : Since the total service time of a queue does not depend on the order of service, we can imagine each of the m_i customers initiating an independent busy period. The length of the total busy period is the sum of m_i independent random variables each having the same probability density function.

Regarding the joint probability density function of the basic random vector, by the law of total probability, we have :

$$P_{k+1}(\underline{\theta}_{k+1}) = \int_0^\infty \int_0^{\theta_{k+N}} \sum_{m_{k+N}=0}^{\infty} \text{Prob}(\theta_{k+N}, m_{k+N}, t, \underline{\theta}_k)\, dt\, d\underline{\theta}_k \qquad 2.3.3$$

where m_{k+N} = the number of customers present in T_{k+N} at the moment the server reaches it,

t = the walking time from T_{k+N-1} to T_{k+N}.

It should be mentioned that for the exhaustive model we can deal with an N-1 dimensional random vector. However, for the sake of symmetry with the previous model, we use the N dimensional random vector and as a result are left with some redundancy.

Using Bayes chain rule in 2.3.3, we obtain :

$$P_{k+1}(\underline{\theta}_{k+1}) = \int_0^\infty \int_0^{\theta_{k+N}} \sum_{m_{k+N}=0}^{\infty} \text{Prob}(\theta_{k+N}/m_{k+N}, t, \underline{\theta}_k)\ \text{Prob}(m_{k+N}/t, \underline{\theta}_k)$$

$$\cdot\ \text{Prob}(t/\underline{\theta}_k)\ \ P_k(\underline{\theta}_k)\, dt\ d\,\theta_k \qquad 2.3.4$$

Since θ_{k+N} is the time required to walk from T_{k+N-1} to T_{k+N} ($\triangleq t$) and the time required to serve all customers in T_{k+N} (m_{k+N} of them are present when the server arrives).

$$\text{Prob}(\theta_{k+N} / m_{k+N}, t, \text{ any past TSTs}) = \text{Prob}(\theta_{k+N} / m_{k+N}, t) \qquad 2.3.5$$

Using Eq. 3.3.2 in Eq. 2.3.5, we obtain :

$$\text{Prob}(\theta_{k+N} / m_{k+N}, t, \underline{\theta}_k) = \text{Prob}(\theta_{k+N} / m_{k+N}, t) = P_{b_{k+N}}^{*(m_{k+N})}(\theta_{k+N} - t) \qquad 2.3.6$$

Since m_{k+N} is the result of Poisson arrival process which starts at t_{E_k} and ends at the moment the server reaches T_{k+N}, we have :

$$\text{Prob}(m_{k+N} / \text{time length of the process, any past TSTs}) =$$

$$= \text{Prob}(m_{k+N} / \text{ time length of the process})$$

and we obtain, with the help of of Fig. 2-1 :

$$\text{Prob}(m_{k+N} / t, \underline{\theta}_k) = \text{Prob}(m_{k+N} / t + \sum_{i=1}^{N-1} \theta_{k+i})$$

$$= \frac{1}{m_{k+N}!} [\lambda_{k+N} (t + \sum_{i=1}^{N-1} \theta_{k+i})]^{m_{k+N}} \exp[-\lambda_{k+N} (t + \sum_{i=1}^{N-1} \theta_{k+i})] \qquad 2.3.7$$

where $t + \sum_{i=1}^{N-1} \theta_{k+i}$ is identified as v_{k+N}, the intervisit time of T_{k+N}.

Since future walking time does not depend on past TSTs, we have :

$$\text{Prob}\,(t/\underline{\theta}_k) = P_{w_{k+N-1}}(t) \qquad 2.3.8$$

Substitution of Eqs. 2.3.6 - 7 - 8 in Eq. 2.3.4 yields

$$P_{k+1}(\underline{\theta}_{k+1}) = \int_0^{\infty}\int_0^{\theta_{k+N}} \sum_{m_{k+N}=0}^{\infty} P^{*\,(m_{k+N})}_{b_{k+N}}(\theta_{k+N}-t)\ \frac{1}{m_{k+N}!}\left[\lambda_{k+N}\left(t+\sum_{i=1}^{N-1}\theta_{k+i}\right)\right]^{m_{k+N}} \cdot \qquad 2.3.9$$

$$\cdot \exp\left[-\lambda_{k+N}\left(t+\sum_{i=1}^{N-1}\theta_{k+i}\right)\right] P_{w_{k+N-1}}(t)\quad P_k(\underline{\theta}_k)\,dt\,d\theta_k$$

As with the gating model (Eq. 2.2.5), the above is a recursive form of obtaining $P_{k+1}(\underline{\theta}_{k+1})$ from $P_k(\underline{\theta}_k)$. Applying the Laplace transform to Eq. 2.3.9, we obtain :

$${}_{k+1}G(\underline{x}) = \int_0^{\infty}\cdots\int_0^{\infty}\int_0^{\theta_{k+N}} \sum_{m_{k+N}=0}^{\infty} \exp\left(-\sum_{i=1}^{N} x_i\,\theta_{k+i}\right) P^{*\,(m_{k+N})}_{b_{k+N}}(\theta_{k+N}-t)$$

$$\cdot\ \frac{1}{m_{k+N}!}\left[\lambda_{k+N}\left(t+\sum_{i=1}^{N-1}\theta_{k+i}\right)\right]^{m_{k+N}} \exp\left[-\lambda_{k+N}\left(t+\sum_{i=1}^{N-1}\theta_{k+i}\right)\right] \qquad 2.3.10$$

$$\cdot\ P_{w_{k+N-1}}(t)\quad P_k(\underline{\theta}_k)\,dt\,\prod_{i=0}^{N} d\theta_{k+i}$$

Since $P_{b_i}(t) = 0$ for $t < 0$, the upper limit of the integral over t can be ∞ instead of θ_{k+N}. To obtain the desired recursive form of the Laplace transform, we first integrate over θ_{k+N}, then sum over m_{k+N}, and finally integrate over t.

In order, integration over θ_{k+N} yields :

$$_{k+1}G(\underline{x}) = \int_0^\infty \cdot\cdot \int_0^\infty \sum_{m_{k+N}=0}^\infty \frac{1}{m_{k+N}!} [\lambda_{k+N} B_{k+N}(x_N)(t + \sum_{i=1}^{N-1} \theta_{k+i})]^{m_{k+N}}$$

$$\cdot \exp[-(\lambda_{k+N} + x_N)t - \sum_{i=1}^{N-1} (\lambda_{k+N} + x_i)\theta_{k+i}] \qquad 2.3.11$$

$$\cdot P_{w_{k+N-1}}(t) \quad P_k(\underline{\theta}_k)\, dt \prod_{i=0}^{N-1} d\theta_{k+i}$$

Then summation over m_{k+N} yields :

$$_{k+1}G(\underline{x}) = \int_0^\infty \cdot\cdot \int_0^\infty \exp[-(x_N + \lambda_{k+N}(1 - B_{k+N}(x_N)))t - \sum_{i=1}^{N-1} (x_i + \lambda_{k+N}(1 - B_{k+N}(x_N)))\theta_{k+i}]$$

$$\cdot P_{w_{k+N-1}}(t) \quad P_k(\underline{\theta}_k)\, dt \prod_{i=0}^{N-1} d\theta_{k+i}$$

finally, integration over t yields :

$$_{k+1}G(\underline{x}) = W_{k+N-1}[x_N + \lambda_{k+N}(1 - B_{k+N}(x_N))] \int_0^\infty \cdot\cdot \int_0^\infty \exp[-\sum_{i=1}^{N-1} (x_i + \lambda_{k+N}(1 - B_{k+N}(x_N)))\theta_{k+i}] \qquad 2.3.12$$

$$\cdot P_k(\underline{\theta}_k) \prod_{i=0}^{N-1} d\theta_{k+i}$$

We obtain the recursive structure by noting that T_k and T_{k+N} are physically the same terminal, hence

$$\lambda_{k+N} \triangleq \lambda_k$$
$$W_{k+N-1}(\cdot) \triangleq W_{k-1}(\cdot) \qquad 2.3.13$$
$$B_{k+N}(\cdot) \triangleq B_k(\cdot)$$

and defining

$$y_k(x_N) \triangleq \lambda_k(1 - B_k(x_N)) \qquad 2.3.14$$

and

$$z_i = \begin{cases} 0 & i = 1 \\ x_{i-1} + y_k(x_N) & i = 2, \ldots, N \end{cases} \qquad 2.3.15$$

giving

$${}_{k+1}G(\underline{x}) = W_{k-1}(x_N + y_k(x_N)) \cdot {}_kG(\underline{z}) \qquad 2.3.16$$

or explicitly

$${}_{k+1}G(x_1, x_2, \ldots, x_N) = W_{k-1}(x_N + y_k(x_N))$$
$$\cdot {}_kG(0, x_1 + y_k(x_N), \ldots, x_{N-1} + y_k(x_N)) \qquad 2.3.17$$

as in the gating model, the important feature of 2.3.17 is :

At $\underline{x} = \underline{0}$;

$$y_k(x_N) = 0$$

and hence

$$x_N + y_k(x_N) = 0 \quad , \quad \underline{z} = \underline{0} \qquad 2.3.18$$

Using the definition 2.1.9 we obtain :

$$_{k+1}F(\underline{x}) = \ln W_{k-1}(x_N + y_k(x_N)) + {}_kF(\underline{z}) \qquad 2.3.19$$

Obviously, we can use Eq. 2.3.16 recursively N or ∞ times. Because of its minor importance to the course of the study and the similarity with the gating model (where this development is performed) we will not repeat it here.

2.4 Summary

In this chapter we defined our basic random variables, TST , for both the gating and the exhaustive models (2.1.3 and 2.1.4 respectively) . From them, we defined the basic N - dimensional random vector, $\underline{\theta}_k$, (Eq. 2.1.5) and its joint probability density function, Eq. 2.1.7, and its Laplace transform, Eq. 2.1.8. The main results of this chapter are the recursive forms of the Laplace transforms of the joint probability density function of $\underline{\theta}_k$, (Eq. 2.2.14 for the gating model and

Eq. 2.3.16 for the exhaustive model). In the next chapter we use this recursive structure to derive explicit expressions for the first moments of the TST, the cycle time, and the intervisit time.

CHAPTER III

DERIVATION OF THE FIRST THREE MOMENTS OF TERMINAL SERVICE TIME, CYCLE TIME, AND INTERVISIT TIME

3.1 Technique of Derivation

In this chapter we derive explicit expressions for basic quantities used in the remaining chapters. Explicit expressions for the first three moments of the Terminal Service Time (T S T), θ, the cycle time, c, and the intervisit time, v, are essential in the study of the fundamental behaviour of the polling system. These expressions enable us to study the basic behaviour of the polling system in the transient and the steady states, as well as derive the first two moments of the waiting times and the buffer sizes of the terminals.

In the course of the derivation we obtain a certain set of intractable equations involving the second and third moments of these random variables. When this happens, we confine our study to the symmetric and/or nonrandom cases.

Let l represent any of the following random variables

$$\theta_i \ , \ c_i \ , \ v_i \ , \ s_i \ , \ w_i \ , \ n_i$$

we define

$\overline{l^n}$ $\triangleq$ the nth moment of l,

σ_l^2 $\triangleq$ the second central moment (variance) of l,

δ_l^3 $\triangleq$ the third central moment of l,

$P_l(\cdot)$ $\triangleq$ the probability density function of l,

$L(\cdot)$ $\triangleq$ the Laplace transform of $P_l(\cdot)$,

$\dot{L}(\cdot)$, $\ddot{L}(\cdot)$, $\dddot{L}(\cdot)$ $\triangleq$ the first, second, and third derivatives of $L(\cdot)$ w.r.t. its argument. 3.1.1

In the general (asymmetric) case, each terminal has different λ_i, $P_{s_i}(\cdot)$, and $P_{w_i}(\cdot)$.

In the symmetric case, all λ_i and $P_{s_i}(\cdot)$ are the same, and all central moments of the walking times are the same. Thus, in the symmetric case we have for $i = 1, \ldots, N$:

$$\lambda_i = \lambda$$

$$P_{s_i}(t) = P_s(t)$$

$$P_{w_i}(t) = P_w(t - \overline{w}_i) \qquad 3.1.2$$

In the symmetric case, the nature of the walking time probability density function is closely related to that of practical polling system, where the average changeover times between the terminals are different. However, all the central moments of the changeover times, which are due to the achievement of synchronization between the terminals and the server (the central data processor), are the same.

In the nonrandom case, all service and walking times are fixed (nonrandom) and we have for $i = 1, \ldots, N$:

$$P_{s_i}(t) = \delta(t - \bar{s}_i)$$

$$P_{w_i}(t) = \delta(t - \bar{w}_i), \qquad 3.1.3$$

where $\delta(\cdot)$ is the Dirac function.

In the symmetric nonrandom case, both Equations 3.1.2 and 3.1.3 are satisfied and we have for $i = 1, \ldots, N$:

$$\lambda_i = \lambda,$$

$$P_{s_i}(t) = \delta(t - \bar{s})$$

$$P_{w_i}(t) = \delta(t - \bar{w}_i) \qquad 3.1.4$$

In Chapter IV, we study the socalled discrete case which is a variant of the symmetric nonrandom case where all λ_i are different.

For all cases, using queueing terminology we define :

$\rho_i \triangleq$ Traffic intensity in $T_i = \lambda_i \bar{s}_i$,

$\rho_o \triangleq$ total traffic intensity in the system $= \sum_{i=1}^{N} \rho_i$,

$\bar{d} \triangleq$ average cycle walking time $= \sum_{i=1}^{N} \bar{w}_i$

$$\lambda_o \triangleq \sum_{i=1}^{N} \lambda_i \qquad 3.1.5$$

ρ_i and ρ_o are basic parameters in the analysis of Polling systems. As expected, the condition for stability is :

$$\rho_o < 1 \qquad 3.1.6$$

In Section 3.2 we derive formulas for the following parameters in the gating model :

1. $\bar{\theta}_i$ and $\bar{c}_i$ in the general case.
2. Equations for $\sigma^2_{\theta_i}$, $\overline{(\theta_i - \bar{\theta}_i)(\theta_j - \bar{\theta}_j)}$, and $\sigma^2_{c_i}$ for the general case. From them we obtain :
   2.1. Explicit expressions for the symmetric and the symmetric non-random cases.
   2.2. Explicit expressions for $\sigma^2_{c_i}$, for N=1, 2, in the general case.
3. Explicit expressions for δ^3_c , for $N = 1, 2, \infty$, in the symmetric non-random case.

In Section 3.3 we derive formulas for the following parameters, in the exhaustive model :

1. $\bar{\theta}_i$, $\bar{c}_i$, and $\bar{v}_i$ in the general case.

2. Equations for $\sigma^2_{\theta_i}$, $\overline{(\theta_i - \bar{\theta}_i)(\theta_j - \bar{\theta}_j)}$, $\sigma^2_{c_i}$, and $\sigma^2_{v_i}$ for the general case. From them we obtain :

   2.1 Explicit expressions for the symmetric and the symmetric nonrandom cases.

   2.2 Explicit expressions for $\sigma^2_{v_i}$, for $N = 1, 2, 3,$ in the general case.

3. Explicit expressions for δ^3_v, for $N = 1, 2, 3, \infty,$ in the symmetric nonrandom case.

In deriving these moments for the gating and exhaustive models, we use Equations 2.2.17 and 2.3.19, respectively.

$$_{k+1}F(\underline{x}) = \ln W_k(x_N) + {}_kF(\underline{z})$$

$$_{k+1}F(\underline{x}) = \ln W_{k-1}(x_N + y_k(x_N)) + {}_kF(\underline{z}) \quad 3.1.7$$

where, for $k \geq 1$, by definitions 2.1.9 and 2.1.8 :

$$_{k+1}F(\underline{x}) = \ln {}_{k+1}G(\underline{x})$$

$$_{k+1}G(\underline{x}) = \int_0^{\infty} \cdots \int_0^{\infty} \exp\left(-\sum_{i=1}^{N} x_i \, \theta_{k+i}\right) P_k(\underline{\theta}_k) \prod_{i=1}^{N} d\theta_{k+i}$$

Define $F_i(\underline{0})$, $F_{i,j}(\underline{0})$, etc., as the partial derivatives of $F(\underline{\cdot})$ w. r. t. its ith, ith and jth, etc., arguments at $\underline{0}$. In Appendix B 1 we show that for $i, j, l = 1, \ldots, N$:

$$_{k+1}F_i(\underline{0}) = -\bar{\theta}_{k+i}$$

$$_{k+1}F_{i,j}(\underline{0}) = \overline{(\theta_{k+i} - \bar{\theta}_{k+i})(\theta_{k+j} - \bar{\theta}_{k+j})}$$

$$_{k+1}F_{i,j,l}(\underline{0}) = -\overline{(\theta_{k+i} - \bar{\theta}_{k+i})(\theta_{k+j} - \bar{\theta}_{k+j})(\theta_{k+l} - \bar{\theta}_{k+l})} \qquad 3.1.8$$

Another feature of Equations 3.1.7 is that differentiating each of them w.r.t. x_i, x_i and x_j, etc., where $i, j = 1, \ldots N-1$ at $\underline{x} = \underline{0}$ yields.

$$_{k+1}F_i(\underline{0}) \triangleq {}_kF_{i+1}(\underline{0}) = -\bar{\theta}_{k+i}$$

$$_{k+1}F_{i,j}(\underline{0}) \triangleq {}_kF_{i+1,j+1}(\underline{0}) = \overline{(\theta_{k+i} - \bar{\theta}_{k+i})(\theta_{k+j} - \bar{\theta}_{k+j})} \qquad 3.1.9$$

which are identities and contain no useful information.

Useful information is obtained from the derivatives of 3.1.7 when at least one of the arguments we differentiate with respect to is x_N. This method equations connecting moments of various terminals. Explicit expressions for the moments are obtained by solving these equations.

To derive the moments in the steady state, we use the fact that when $\rho_o < 1$, i.e., stability condition exists, we have :

$$\lim_{k \to \infty} P_{k+i} \left(\underline{\theta}_{k+i} \right) = \lim_{k \to \infty} P_{k+N+i} \left(\underline{\theta}_{k+N+i} \right) \qquad 3.1.10$$

both sides of the above equation are the steady state probability density function of the same random vector.

The details for the Algebraic manipulations are in the various sections of Appendix B .

3.2 The Moments of the Gating Model

The basic equation we deal with is Equation 2.2.17.

For $k \geq 1$

$$_{k+1}F\left(\underline{x}\right) = \ln W_k\left(x_N\right) + {}_kF\left(\underline{z}\right) \qquad 2.2.17$$

$$z_i = \begin{cases} \lambda_k \left(1 - S_k\left(x_N\right)\right) & i = 1 \\ x_{i-1} + \lambda_k \left(1 - S_k\left(x_N\right)\right) & i = 2, \ldots, N \end{cases} \qquad 3.2.1$$

$$\text{Re}\ [\underline{x}] \geq \underline{0}$$

where from definitions 2.1.5, 2.1.8, and 2.1.9

$$\underline{\theta}_k = (\theta_k, \ldots, \theta_{k+N-1}) \qquad 2.1.5$$

$$_kG(\underline{x}) = \int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{i=0}^{N-1} x_{i+1}\theta_{k+i}\right) P_k(\underline{\theta}_k) \prod_{i=0}^{N-1} d\theta_{k+i} \qquad 2.1.8$$

$$_kF(\underline{x}) = \ln {_kG}(\underline{x}) \qquad 2.1.9$$

Differentiation of Equation 2.2.17 w.r.t. x_N yields

$$_{k+1}F_N(\underline{x}) = \frac{\dot{W}_k(x_N)}{W_k(x_N)} - \lambda_k \dot{S}_k(x_N) \sum_{i=1}^{N} {_kF_i}(\underline{z}) \qquad 3.2.2$$

At $\underline{x} = \underline{0}$, using Equation 2.2.16 we find

$$\bar{\theta}_{k+N} = \bar{w}_k + \lambda_k \bar{s}_k \sum_{i=0}^{N-1} \bar{\theta}_{k+i} \qquad 3.2.3$$

$$k \geq 1$$

The above equation enables us to study the transient behaviour of the average T S T. Thus, given $\bar{\theta}_1, \ldots, \bar{\theta}_N$ we may find $\bar{\theta}_{k+N}$ for all $k \geq 1$ recursively. We use this equation in Chapter V while studying the transient behaviour of the average cycle time. In the steady state, by virtue of Equation 3.1.10, we have :

$$\lim_{k\to\infty} \bar{\theta}_{k+N} = \lim_{k\to\infty} \bar{\theta}_k \qquad 3.2.4$$

By definition 2.1.3

$$\bar{c}_{k+N} = \sum_{i=0}^{N-1} \bar{\theta}_{k+i} \qquad 3.2.5$$

In the steady state, define for $i = 1, \ldots, N$

$$\theta_i = \text{Terminal } i \text{ Service Time } (TST_i) \text{ in the steady state} \qquad 3.2.6$$

and we have from Equation 3.2.5

$$\bar{c}_i \overset{\Delta}{=} \bar{c} = \sum_{i=1}^{N} \bar{\theta}_i \,, \qquad 3.2.7$$

where $\bar{c}$ is the average steady state cycle time and is the same for all terminals.

Using Equations 3.2.6 and 3.2.7 in 3.2.3, we obtain for $i = 1, \ldots, N$

$$\bar{\theta}_i = \bar{w}_i + \lambda_i \bar{s}_i \bar{c} \qquad 3.2.8$$

The above is a set of N linear equations with N unknowns. In order to solve it, we sum all the N equations

$$\sum_{i=1}^{N} \bar{\theta}_i \overset{\Delta}{=} \bar{c} = \sum_{i=1}^{N} \bar{w}_i + \bar{c} \cdot \sum_{i=1}^{N} \lambda_i \bar{s}_i \qquad 3.2.9$$

Noting the definition 3.1.5 for average walking time $\bar{d}$, in 3.2.9 we obtain

$$\bar{c} = \bar{d} + \rho_0 \bar{c}$$

which implies that

$$\bar{c} = \frac{\bar{d}}{1 - \rho_0} \qquad 3.2.10$$

Using the above Equation 3.2.8, we have for $i = 1, \ldots, N$

$$\bar{\theta}_i = \bar{w}_i + \frac{\rho_i \bar{d}}{1 - \rho_0} \qquad 3.2.11$$

Equation 3.2.10 was also obtained by Eisenberg (1972). It should be emphasized that this equation holds only for total traffic intensity, ρ_0, strictly less than 1. Otherwise $\bar{c} = \infty$.

To derive the second moments, we differentiate Equation 3.2.2 w.r.t. x_i, $i = 1, \ldots, N$, to obtain

$$_{k+1}F_{N,N}(\underline{x}) = \frac{W_k(x_N)\ddot{W}_k(x_N) - (\dot{W}_k(x_N))^2}{(W_k(x_N))^2} - \lambda_k \ddot{S}_k(x_N) \sum_{i=1}^{N} {}_kF_i(\underline{z})$$

$$+ (\lambda_k \dot{S}_k(x_N))^2 \sum_{i=1}^{N} \sum_{l=1}^{N} {}_kF_{i,l}(\underline{z})$$

and for $J = 1, \ldots, N-1$

$$_{k+1}F_{N,J}(\underline{x}) = -\lambda_k \dot{S}_k(x_N) \sum_{i=1}^{N} {}_kF_{i,J+1}(\underline{z}) \qquad 3.2.12$$

At $\underline{x} = \underline{0}$, after some manipulation shown in Appendix B 2, we obtain :-

$$\overline{(\theta_{k+N} - \bar{\theta}_{k+N})^2} = \sigma^2_{w_k} + \lambda_k \bar{s}^2_k \bar{c}_{k+N} + (\lambda_k \bar{s}_k)^2 \sigma^2_{c_{k+N}}$$

and for $j = 1, \ldots, N-1$

$$\overline{(\theta_{k+N} - \bar{\theta}_{k+N})(\theta_{k+j} - \bar{\theta}_{k+j})} = \lambda_k \bar{s}_k \sum_{i=0}^{N-1} \overline{(\theta_{k+j} - \bar{\theta}_{k+j})(\theta_{k+i} - \bar{\theta}_{k+i})} \qquad 3.2.13$$

where

$$\sigma^2_{c_{k+N}} = \sum_{i=0}^{N-1} \sum_{l=0}^{N-1} \overline{(\theta_{k+i} - \bar{\theta}_{k+i})(\theta_{k+l} - \bar{\theta}_{k+l})}$$

The above equation, with the help of Equation 3.2.3, enables us to study the transient behaviour of the second moments of θ and c . Given all $\bar{\theta}_i$, $\overline{(\theta_i - \bar{\theta}_i)(\theta_j - \bar{\theta}_j)}$ for $i, j = 1, \ldots N$, i.e., second moments in the first cycle, we can recursively find all the second moments in any future cycle.

As $k \to \infty$ and $\rho_0 < 1$, we reach a steady state. Using 3.2.6, we define the steady state cross correlation between T_i and T_j as :

$$R(i, j) \triangleq \overline{(\theta_i - \bar{\theta}_i)(\theta_j - \bar{\theta}_j)} \text{ where } T_j \text{ is visited } \underline{\text{before}} \; T_i \qquad 3.2.14$$

Clearly for $i \geq j$, $R(i, j)$ is the cross correlation between T_j and T_i where from T_j the server continues to $T_{j+1}, \ldots T_{i-1}, T_i$, but $R(j, i)$ is the cross correlation between T_i and T_j where from T_i the server continues to $T_{i+1}, \ldots, T_N, T_1, \ldots T_{j-1}, T_j$.

Using definition 3.2.14 the steady state representation of Equation 3.2.13 is, for $i, j = 1, \ldots, N$, given by

$$R(i,j) = \begin{cases} \rho_i \left[\sum_{l=j}^{i-1} R(l,j) + \sum_{l=1}^{j-1} R(j,l) + \sum_{l=i}^{N} R(j,l) \right] & \text{for } j < i \\ \sigma^2_{w_i} + \lambda_i \overline{s_i^2} \, \bar{c} + \rho_i^2 \, \sigma^2_{c_i} & \text{for } j = i \\ \rho_i \left[\sum_{l=j}^{N} R(l,j) + \sum_{l=1}^{i-1} R(l,j) + \sum_{l=i}^{j-1} R(j,l) \right] & \text{for } j > i \end{cases}$$

where

$$\sigma^2_{c_i} = \sum_{l=1}^{i-1} \left[\sum_{m=i}^{N} R(l,m) + \sum_{m=1}^{l} R(l,m) + \sum_{m=l+1}^{i-1} R(m,l) \right] + \sum_{l=i}^{N} \left[\sum_{m=i}^{l} R(l,m) + \sum_{m=l+1}^{N} R(m,l) + \sum_{m=1}^{i-1} R(m,l) \right] \quad 3.2.15$$

Equation 3.2.15 is a set of N^2 linear equations with N^2 unknowns. Obtaining an explicit solution would seem complicated. However, as we shall show later, the solution for the symmetric case, defined by Equation 3.1.2, is surprisingly simple.

For the general (asymmetric) case we find an explicit solution of 3.2.15 for the case $N = 2$ where there are 4 linear equations. (The solution for $N = 1$ is a special case of the symmetric case, which we solve later). The quantities we are looking for are $\sigma^2_{c_1}$ and $\sigma^2_{c_2}$.

Solution of 3.2.15 for N = 2

Direct substitution of N = 2 in 3.2.15 yields 4 linear equations.

$$R(1,1) = \sigma_{\theta_1}^2 = \sigma_{w_1}^2 + \lambda_1 \overline{s_1^2}\, \bar{c} + \rho_1^2\, \sigma_{c_1}^2$$

$$R(1,2) = \rho_1 [R(2,2) + R(2,1)]$$

$$R(2,1) = \rho_2 [R(1,1) + R(1,2)] \qquad 3.2.16$$

$$R(2,2) = \sigma_{\theta_2}^2 = \sigma_{w_2}^2 + \lambda_2 \overline{s_2^2}\, \bar{c} + \rho_2^2\, \sigma_{c_2}^2$$

where

$$\sigma_{c_1}^2 = R(1,1) + R(2,2) + 2R(2,1)$$

$$\sigma_{c_2}^2 = R(1,1) + R(2,2) + 2R(1,2)$$

In Appendix B 3 the above equation is solved and we obtain

$$\sigma_{c_1}^2 = \frac{[(1-\rho_1\rho_2)(1+2\rho_2) - 2\rho_2^3][\lambda_1 \overline{s_1^2}\,\bar{c} + \sigma_{w_1}^2] + (1+\rho_1\rho_2)(\lambda_2 \overline{s_2^2}\,\bar{c} + \sigma_{w_2}^2)}{(1-\rho_1-\rho_2)(1+\rho_1+\rho_2+2\rho_1\rho_2)(1-\rho_1\rho_2)} \qquad 3.2.17$$

$\sigma_{c_2}^2$ is obtained from 3.2.17 by interchanging the indices 1 and 2.

Solution of Equation 3.2.15 for the Symmetric Case

In the symmetric case the steady state cross correlation between T_j and T_i, $R(i, j)$, does not depend on i and j separately but on their difference. This property is due to the complete symmetry of the system; we have for $i, j = 1, .., N$:

$$R(i, j) = \begin{cases} R(i-j) & \text{for } i \geq j \\ R(N+i-j) & \text{for } i < j \end{cases} \qquad 3.2.18$$

Substitution of 3.2.18 with $i = N$ into 3.2.15 yields

$$R(0) \triangleq \sigma_\theta^2 = \sigma_w^2 + \lambda \overline{s^2}\, \overline{c} + \rho^2 \sigma_c^2$$

and for $J = 1, .., N-1$

$$R(N-J) \triangleq \overline{(\theta_N - \overline{\theta}_N)(\theta_J - \overline{\theta}_J)} = \rho \left[\sum_{i=0}^{J} R(i) + \sum_{i=1}^{N-J-1} R(i) \right]$$

where

$$\sigma_c^2 = N R(0) + 2 \sum_{i=1}^{N-1} (N-i) R(i) \qquad 3.2.19$$

3.2.19 is a set of N linear equations with N unknowns. Define :

$$\sigma_d^2 = \sum_{i=1}^{N} \sigma_{w_i}^2 = N \sigma_w^2 \qquad 3.2.20$$

In Appendix B 4 we show that the unique solution of 3.2.19 is :

$$R(0) \triangleq \sigma_\theta^2 = \frac{1}{N} \cdot \frac{\left(1 - \rho_0 + \frac{\rho_0}{N}\right)\left(\sigma_d^2 + \lambda_0 \overline{s^2}\,\bar{c}\right)}{\left(1 - \rho_0\right)\left(1 + \frac{\rho_0}{N}\right)}$$

and for $l = 1, \ldots, N-1$

$$R(l) = \overline{(\theta_N - \bar{\theta}_N)(\theta_{N-l} - \bar{\theta}_{N-l})} = \frac{\frac{\rho_0}{N}}{1 - \rho_0 + \frac{\rho_0}{N}} \cdot \sigma_\theta^2$$

$$\sigma_c^2 = \frac{\sigma_d^2 + \lambda_0 \overline{s^2}\,\bar{c}}{(1 - \rho_0)\left(1 + \frac{\rho_0}{N}\right)} \qquad 3.2.21$$

A surprising consequence of Equation 3.2.21 is that the cross correlation between any two different terminals is the same.

Define the normalized cross correlation between T_N and T_{N-l} as :

$$R_\theta(l) = \frac{R(l)}{\sigma_\theta^2} \qquad 3.2.22$$

Substituting 3.2.22 into 3.2.21 we find

$$R_\theta(l) = \begin{cases} 1 & l = 0 \\ \dfrac{\frac{\rho_0}{N}}{\left(1 - \rho_0 + \frac{\rho_0}{N}\right)} & l = 1, \ldots, N-1 \end{cases} \qquad 3.2.23$$

As ρ_0 increases from 0 to 1, $R_\theta(l)$, for $0 < l < N$, increases from 0 to 1. Thus the normalized cross correlation is always non-negative and increases as the total traffic intensity increases.

In Chapter V we study the steady state normalized cross correlation between different cycles.

The Second Moments for the Symmetric Nonrandom Case

In the symmetric nonrandom case, defined by 3.1.4, we obtain by using 3.2.10 in 3.2.21 and 3.2.23

$$\sigma_\theta^2 = \frac{1}{N} \cdot \frac{(1 - \rho_0 + \frac{\rho_0}{N})\ \rho_0\ \overline{s}\ \overline{c}}{(1 - \rho_0)\ (1 + \frac{\rho_0}{N})} = \frac{1}{N}\ \frac{(1 - \rho_0 + \frac{\rho_0}{N})\ \rho_0}{(1 - \rho_0)^2\ (1 + \frac{\rho_0}{N})} \cdot \overline{s} \cdot \overline{d}$$

for $l = 1, \ldots, N-1$

$$R_\theta(l) = \frac{\frac{\rho_0}{N}}{1 - \rho_0 + \frac{\rho_0}{N}}$$

$$\sigma_c^2 = \frac{\rho_0\ \overline{s}\ \overline{c}}{(1 - \rho_0)(1 + \frac{\rho_0}{N})} = \frac{\rho_0}{(1 - \rho_0)^2\ (1 + \frac{\rho_0}{N})} \cdot \overline{s} \cdot \overline{d} \qquad 3.2.24$$

The Third Central Moment of c

The expressions we are going to deal with are very involved. For simplicity we restrict the development to the symmetric nonrandom case defined by 3.1.4.

For the symmetric case, differentiating 3.2.12 w.r.t. x_p, $p = 1, \ldots, N$, and defining $W_k(x_N) = W(x_N) \triangleq W$, $S_k(x_N) = S(x_N) \triangleq S$, we obtain

$$ {}_{k+1}F_{N,N,N}(\underline{x}) = \frac{W^2 \dddot{W} - 3 W \dot{W} \ddot{W} + 2 (\dot{W})^3}{(W)^3} - \lambda \dddot{S} \sum_{i=1}^{N} {}_kF_i(\underline{z}) $$

$$ + 3 \lambda^2 \dot{S} \ddot{S} \sum_{i=1}^{N} \sum_{l=1}^{N} {}_kF_{i,l}(\underline{z}) - (\lambda \dot{S})^3 \sum_{i=1}^{N} \sum_{l=1}^{N} \sum_{m=1}^{N} {}_kF_{i,l,m}(\underline{z}) $$

for $J, P = 1, \ldots, N-1$

$$ {}_{k+1}F_{N,N,J}(\underline{x}) = -\lambda \ddot{S} \sum_{i=1}^{N} {}_kF_{i,J+1}(\underline{z}) + (\lambda \dot{S})^2 \sum_{i=1}^{N} \sum_{l=1}^{N} {}_kF_{i,l,J+1}(\underline{z}) $$

$$ {}_{k+1}F_{N,J,P}(\underline{x}) = -\lambda \dot{S} \sum_{i=1}^{N} {}_kF_{i,J+1,P+1}(\underline{z}) \qquad 3.2.25 $$

In Appendix B5, we find the steady state representation of Equation 3.2.25 for the symmetric nonrandom case, for $k = K \cdot N$ such that $K \to \infty$, the left hand side of Equation 3.2.25 are the steady state quantities of T_N, T_J, and T_P. Quoting the result from Appendix B5, we have

$$\overline{(\theta_N - \bar{\theta}_N)^3} \doteq \frac{1-(N-1)\,\rho + 2N\rho^2}{N}\,\bar{s}\,\sigma_c^2 + \rho^3 \delta_c^3$$

for J, P = 1, .., N-1

$$\overline{(\theta_N - \bar{\theta}_N)^2 (\theta_J - \bar{\theta}_J)} = \frac{\rho}{N}\bar{s}\,\sigma_c^2 + \rho^2 \sum_{i=0}^{N-1}\sum_{l=0}^{N-1} \overline{(\theta_J - \bar{\theta}_J)(\theta_i - \bar{\theta}_i)(\theta_l - \bar{\theta}_l)}$$

$$\overline{(\theta_N - \bar{\theta}_N)(\theta_J - \bar{\theta}_J)(\theta_P - \bar{\theta}_P)} = \rho \sum_{i=0}^{N-1} \overline{(\theta_J - \bar{\theta}_J)(\theta_P - \bar{\theta}_P)(\theta_i - \bar{\theta}_i)}$$

where

$$\delta_c^3 = \sum_{i=0}^{N-1}\sum_{l=0}^{N-1}\sum_{m=0}^{N-1} \overline{(\theta_i - \bar{\theta}_i)(\theta_l - \bar{\theta}_l)(\theta_m - \bar{\theta}_m)} \qquad 3.2.26$$

and T_0 is T_N in the previous cycle.

Define, for $1 \leq P \leq J \leq N$

$$R(N-J,\, J-P) = \overline{(\theta_N - \bar{\theta}_N)(\theta_J - \bar{\theta}_J)(\theta_P - \bar{\theta}_P)} \qquad 3.2.27$$

Substitution of 3.2.27 into 3.2.26 yields :

$$R(0,0) = \frac{1-(N-1)\rho+2N\rho^2}{N}\,\bar{s}\,\sigma_c^2 + \rho^3\delta_c^3$$

for $1 \le J < N$:

$$R(0, N-J) = \frac{\rho}{N}\bar{s}\,\sigma_c^2 + \rho^2\Big[\sum_{i=1}^{J+1}\Big(\sum_{l=1}^{i} R(J+1-i, i-l) + \sum_{l=i+1}^{J+1} R(J+1-l, l-i) + \sum_{l=J+2}^{N} R(l-J-1, J+1-i)\Big)$$

$$+ \sum_{i=J+2}^{N}\Big(\sum_{l=1}^{J+1} R(i-J-1, J+1-l) + \sum_{l=J+2}^{i} R(i-l, l-J-1) + \sum_{l=i+1}^{N} R(l-i, i-J-1)\Big)\Big]$$

for $1 \le P \le J < N$:

$$R(N-J, J-P) = \rho\Big[\sum_{i=1}^{P+1} R(J-P, P+1-i) + \sum_{i=P+2}^{J+1} R(J+1-i, i-P-1) + \sum_{i=J+2}^{N} R(i-J-1, J-P)\Big]$$

where

$$\delta_c^3 = \sum_{i=1}^{N}\Big[\sum_{l=1}^{i}\Big(\sum_{m=1}^{l} R(i-l, l-m) + \sum_{m=l+1}^{i} R(i-m, m-l) + \sum_{m=i+1}^{N} R(m-i, i-l)\Big)$$

$$+ \sum_{l=i+1}^{N}\Big(\sum_{m=1}^{i} R(l-i, i-m) + \sum_{m=i+1}^{l} R(l-m, m-i) + \sum_{m=l+1}^{N} R(m-l, l-i)\Big)\Big] \qquad 3.2.28$$

The above equation is a set of $\frac{N(N+1)}{2}$ linear equations with the same number of unknowns. The goal is to find δ_c^3 for all $N \ge 1$. However, an explicit expression for δ_c^3 for all $N \ge 1$ seems difficult to obtain. In the following we find explicit expressions for δ_c^3 in the symmetric nonrandom case for $N = 1, 2, \infty$.

δ_c^3 for $N = 1$

Using 3.2.28 with $N = 1$ ($\rho_0 = \rho$) yields

$$R(0, 0) = \delta_\theta^3 = \delta_c^3 = (1 - 2\rho_0^2)\,\bar{s}\,\sigma_c^2 + \rho_0^3\,\delta_c^3$$

which implies

$$\delta_c^3 = \frac{1 - 2\rho_0^2}{1 - \rho_0^3}\,.\,\bar{s}\,\sigma_c^2$$

Using 3.2.24 with $N = 1$, we obtain

$$\delta_c^3 = \frac{\rho_0\,(1 - 2\rho_0^2)}{(1 - \rho_0)^2\,(1 + \rho_0)\,(1 + \rho_0 + \rho_0^2)}\,.\,\bar{s}^2\,\bar{c} \qquad 3.2.29$$

δ_c^3 for $N = 2$

Using 3.2.28 with $N = 2$ ($\rho_0 = 2\rho$) we obtain

$$\sigma_c^2 = \frac{\rho_0\,\bar{s}.\,\bar{c}}{(1 - \rho_0)\,(1 + \frac{\rho_0}{2})}$$

We obtain 3 linear equations with 3 unknowns :

$$R(0,0) = \frac{\rho_0\left(1 - \frac{\rho_0}{2} + \rho_0^2\right)}{2(1-\rho_0)\left(1+\frac{\rho_0}{2}\right)}\,\overline{s^2}\,\overline{c} + \frac{\rho_0^3}{8}\,\delta_c^3$$

$$R(0,1) = \frac{\rho_0^2}{4(1-\rho_0)\left(1+\frac{\rho_0}{2}\right)}\,\overline{s^2}\,\overline{c} + \frac{\rho_0^2}{4}\left(R(0,0) + 2R(0,1) + R(1,0)\right)$$

$$R(1,0) = \frac{\rho_0}{2}\left(R(0,1) + R(0,0)\right) \qquad 3.2.30$$

where

$$\delta_c^3 = 2R(0,0) + 3R(0,1) + 3R(1,0)$$

In Appendix B6, we derive the solution

$$\delta_c^3 = \frac{\rho_0\left(1 + \frac{1}{2}\rho_0 + \frac{5}{8}\rho_0^2 + \frac{3}{8}\rho_0^3 - \frac{1}{4}\rho_0^4\right)\overline{s^2}\,\overline{c}}{(1-\rho_0)^2\left(1+\frac{1}{2}\rho_0\right)^2\left(1 + \frac{1}{4}\rho_0^2\left(1 - \frac{1}{2}\rho_0\right)\right)} \qquad 3.2.31$$

<u>δ_c^3 for $N = \infty$</u>

For $\overline{s}$, $\overline{d}$ finite positive quantities, as $N \to \infty$ and $\rho_0 < 1$, Equation 3.15 yields

$$\lambda = \frac{\rho_0}{N\overline{s}} \to 0 \qquad \text{as } N \to \infty \qquad 3.2.32$$

and from Equations 3.2.10 and 3.2.21 we find that

$$\bar{c} = \frac{\bar{d}}{1 - \rho_0} \qquad 3.2,10$$

$$\sigma_c^2 = \frac{\rho_0 \bar{s} \bar{c}}{1 - \rho_0} \qquad 3.2.33$$

where $\bar{c}$ and σ_c^2 are finite.

The probability of having n customers in a terminal, given that the terminal cycle time was a finite time c is, according to the Poisson arrival process

$$\text{Prob}(n / c) = \frac{(\lambda c)^n}{n!} \exp(-\lambda c) \qquad 3.2.34$$

Since 3.2.32 is satisfied,

$$\lambda c = \frac{\rho_0 c}{N \bar{s}} \to 0 \qquad \text{as } N \to \infty \qquad 3.2.35$$

we have

$$\begin{aligned} \text{Prob}(n = o / c) &= \exp(-\lambda c) \\ \text{Prob}(n = 1 / c) &= 1 - \exp(-\lambda c) - 0((\lambda c)^2) \\ \text{Prob}(n \geq 2 / c) &= 0((\lambda c)^2). \end{aligned} \qquad 3.2.36$$

The above equation indicates that with probability one the server finds in a terminal at most one customer. The new arrival process here is the negative exponential.

An equivalent symmetric nonrandom model with single-buffer terminals, a negative exponential new arrival process, for all $N \geq 1$, was studied by Mack et al (1957). They showed that the probability that some specific set of I terminals have customers during a service cycle, given that exactly I terminals have customers in this cycle, is independent of the specific set of terminals, i.e., all $\binom{N}{I}$ possible combinations have equal probability. It should be mentioned that Mack (1957) tried to extend this model to the case of random service time and was unable to because this geometric independence property does not hold.

Using this geometric independence property that all possible combinations of I nonempty terminals out of N (as $N \to \infty$) have equal probability, We have from Equation 3.2.27

$$\begin{aligned} R(0,0) &\triangleq R(0) \\ R(0,i) &= R(i,0) \triangleq R(1) \qquad \text{all } i > 0 \\ R(i,j) &\triangleq R(2) \qquad \text{all } i, j > 0 \end{aligned} \tag{3.2.37}$$

Substituting 3.2.37 into 3.2.28, we obtain for $N \to \infty$

$$R(0) = \frac{(1-\rho_0) + \frac{\rho_0}{N}(1+2\rho_0)}{N} \bar{s}\, \sigma_c^2 + \frac{\rho_0^3}{N^3} \delta_c^3$$

$$R(1) = \frac{\rho_0}{N^2} \bar{s}\, \sigma_c^2 + \frac{\rho_0^2}{N} (R(0) + 3(N-1) R(1) + (N-1)(N-2) R(2))$$

$$R(1) = \frac{\rho_0}{N} (R(0) + (N-1) R(1))$$

$$R(2) = \frac{\rho_0}{N} (2R(1) + (N-2) R(2))$$

where

$$\delta_c^3 = N R(0) + 3N(N-1) R(1) + N(N-1)(N-2) R(2) \qquad 3.2.38$$

We have 4 linear equations with 3 unknowns.

Define, for $N \to \infty$

$$r(0) = N R(0)$$

$$r(1) = 3N(N-1) R(1)$$

$$r(2) = N(N-1)(N-2) R(2) \qquad 3.2.39a$$

Equation 3.2.26 implies

$$\delta_c^3 = r(0) + r(1) + r(2) \qquad 3.2.39b$$

Since δ_c^3 is finite nonzero, each of $r(i)$, $i = 0, 1, 2$ is a finite nonzero quantity.

Substituting 3.2.27 into 3.2.26 yields for $N \to \infty$

$$r(0) = [(1-\rho_0) + \frac{\rho_0}{N}(1+2\rho_0)]\,\bar{s}\,\sigma_c^2 + \frac{\rho_0^3}{N^2}\,(r(0)+r(1)+r(2))$$

$$r(1) = \frac{N-1}{N}\cdot 3\rho_0\,\bar{s}\,\sigma_c^2 + \frac{N-1}{N^2}\cdot 3\rho_0^2\,(r(0)+r(1)+r(2))$$

$$r(1) = \frac{N-1}{N}\cdot 3\rho_0\,(r(0) + \frac{1}{3}r(1))$$

$$r(2) = \frac{N-2}{N}\,\rho_0\,(\frac{2}{3}r(1) + r(2)) \qquad 3.2.40$$

Performing the limit $N \to \infty$ and replacing σ_c^2 by Equation 3.2.33 yield

$$r(0) = \rho_0\,\overline{s}^{\,2}\,\overline{c} \qquad 3.2.41a$$

$$r(1) = \frac{3\rho_0^2\,\overline{s}^{\,2}\,\overline{c}}{1-\rho_0} \qquad 3.2.41b$$

$$r(1) = 3\rho_0\,(r(0) + \frac{1}{3}r(1)) \qquad 3.2.41c$$

$$r(2) = \rho_0\,(\frac{2}{3}r(1) + r(2)) \qquad 3.2.41d$$

Equation 3.2.41c is satisfied by expressions 3.2.41 a – b.

The final result is :

$$r(0) = \rho_0 \overline{s^2}\,\bar{c}$$

$$r(1) = \rho_0 \overline{s^2}\,\bar{c}\,\frac{3\rho_0}{1-\rho_0}$$

$$r(2) = \rho_0 \overline{s^2}\,\bar{c}\,\frac{2\rho_0^2}{(1-\rho_0)^2} \qquad 3.2.42$$

Substituting 3.2.42 into 3.2.39b yields

$$\delta_c^3 = \frac{\rho_0 (1+\rho_0)}{(1-\rho_0)^2}\,\overline{s^2}\,\bar{c} \qquad 3.2.43$$

To find the third moment of c for the symmetric nonrandom case with $N = 1, 2, \infty$ we have

$$\overline{c^3} = \overline{((c-\bar{c}) + \bar{c})^3} = \delta_c^3 + 3\bar{c}\,\sigma_c^2 + \bar{c}^3 \qquad 3.2.44$$

Using Equation 3.2.24 and each of Equations 3.2.29, 3.2.31 and 3.2.43, respectively, in 3.2.44, we obtain

<u>For N = 1</u>

$$\overline{c^3} = \bar{c}^3 + \bar{c}^2\,\bar{s}\cdot\frac{3\rho_0}{(1-\rho_0)(1+\rho_0)} + \bar{c}\,\overline{s^2}\,\frac{\rho_0(1+2\rho_0^2)}{(1-\rho_0)^2(1+\rho_0)(1+\rho_0+\rho_0^2)}$$

For N = 2

$$\overline{c^3} = \overline{c}^3 + \overline{c}^2 \overline{s} \cdot \frac{3\rho_0}{(1-\rho_0)(1+\frac{\rho_0}{2})} + \overline{c}\,\overline{s}^2 \frac{\rho_0 (1 + \frac{1}{2}\rho_0 + \frac{5}{8}\rho_0^2 + \frac{3}{8}\rho_0^3 - \frac{1}{4}\rho_0^4)}{(1-\rho_0)^2 (1+\frac{\rho_0}{2})^2 (1 + \frac{\rho_0^2}{4}(1-\frac{\rho_0}{2}))}$$

For N = ∞

$$\overline{c^3} = \overline{c}^3 + \overline{c}^2 \overline{s} \cdot \frac{3\rho_0}{1-\rho_0} + \overline{c}\,\overline{s}^2 \frac{\rho_0 (1+\rho_0)}{(1-\rho_0)^2} \qquad 3.2.45$$

3.3 The Moments of the Exhaustive Model

The derivation technique for this model is closely related to that of the gating model. To avoid repeating similar arguments and derivations we refer, whenever possible, to Section 3.2.

The basic equation we deal with is 2.3.19. For $k \geq 1$

$$_{k+1}F(\underline{x}) = \ln W_{k-1}(x_N + y_k(x_N)) + {}_kF(\underline{z}) \qquad 2.3.19$$

where from Equations 2.3 14 and 2.3.15

$$y_k(x_N) = \lambda_k (1 - B_k(x_N)) \qquad 2.3.14$$

$$z_i = \begin{cases} 0 & i = 1 \\ x_{i-1} + y_k(x_N) & i = 2, .., N \end{cases} \quad 2.3.15$$

Re $[\underline{x}] \geq \underline{0}$.

In order to avoid lengthy subscripts, define

$$y_k \triangleq y_k(x_N)$$

$$u_k \triangleq x_N + y_k(x_N) \quad 3.3.1$$

Using the above definition, Equation 2.3.19 is rewritten as :

$$_{k+1}F(\underline{x}) = \ln W_{k-1}(u_k) + {}_kF(\underline{z}) \quad 3.3.2$$

Differentiating Equation 3.3.2 w.r.t. x_N yields

$$_{k+1}F_N(\underline{x}) = \frac{\dot{u}_k \dot{W}_{k-1}(u_k)}{W_{k-1}(u_k)} - \lambda_k \dot{B}_k(x_N) \sum_{i=2}^{N} {}_kF_i(\underline{z}) \quad 3.3.3$$

At $\underline{x} = \underline{0}$, using 2.3.18, we find

$$\overline{\theta}_{k+N} = (1 + \lambda_k \overline{b}_k)\, \overline{w}_{k-1} + \lambda_k \overline{b}_k \sum_{i=1}^{N-1} \overline{\theta}_{k+i} \quad 3.3.4$$

Replacing $\bar{b}_k$ by Equation A-10, we obtain

$$\bar{\theta}_{k+N} = \frac{1}{1-\rho_k}\left(\bar{w}_{k-1} + \rho_k \sum_{i=1}^{N-1} \bar{\theta}_{k+i}\right) \qquad 3.3.5$$

or

$$\bar{\theta}_{k+N} = \bar{w}_{k-1} + \rho_k \sum_{i=1}^{N} \bar{\theta}_{k+i} \qquad 3.3.6$$

$k \geq 1$

As in Equation 3.2.3, Equation 3.3.5 enables us to study the transient behaviour of the average TST .

Using definition 2.1.4 we obtain

$$\bar{c}_{k+N} = \sum_{i=1}^{N} \bar{\theta}_{k+i} \qquad 3.3.7$$

In the steady state, we have the random vector $\underline{\theta}$ defined by Equation 3.2.6. Substituting 3.2.7 into 3.3.6 yields for $i = 1, \ldots, N$:

$$\bar{\theta}_i = \bar{w}_{i-1} + \rho_i \bar{c} \qquad 3.3.8$$

where

$$\bar{c} = \sum_{i=1}^{N} \bar{\theta}_i$$

Using the same technique employed in the solution of 3.2.8 we have

$$\bar{\theta}_i = \bar{w}_{i-1} + \frac{\rho_i \bar{d}}{1 - \rho_0} \qquad 3.3.9$$

$$\bar{c} = \frac{\bar{d}}{1 - \rho_0} \qquad 3.3.10$$

Equations 3.2.10 and 3.3.10 are identical indicating that the average cycle is the same for both the gating and the exhaustive models.

The average intervisit time is obtained from definition 2.1.4

$$\bar{v}_{k+N} = \bar{w}_{k-1} + \sum_{i=1}^{N-1} \theta_{k+i} = \bar{c}_{k+N} - \bar{\theta}_{k+N} + \bar{w}_{k-1} \qquad 3.3.11$$

and in the steady state

$$\bar{v}_i = \bar{c} - \bar{\theta}_i + \bar{w}_{i-1} \qquad 3.3.12$$

Substituting 3.3.9 into 3.3.12 yields

$$\bar{v}_i = \frac{(1 - \rho_i) \cdot \bar{d}}{(1 - \rho_0)} \qquad 3.3.13$$

$$\bar{v}_i = (1 - \rho_i)\, \bar{c} \qquad 3.3.14$$

The above equation, as well as other relations between v_i and c, are derived also in Appendix C .

To derive the second moments, we differentiate Equation 3.3.3 w.r.t. x_j $j = 1, \ldots, N$ and obtain

$$ {}_{k+1}F_{N,N}(\underline{x}) = \frac{W_{k-1}(u_k)\,[\ddot{u}_k \dot{W}_{k-1}(u_k) + (\dot{u}_k)^2 \ddot{W}_{k-1}(u_k)] - (\dot{u}_k)^2 (\dot{W}_{k-1}(u_k))^2}{(W_{k-1}(u_k))^2} $$

$$ - \lambda_k \ddot{B}_k(x_N) \sum_{i=2}^{N} {}_kF_i(\underline{z}) + (\lambda_k \dot{B}_k(x_N))^2 \sum_{i=2}^{N} \sum_{l=2}^{N} {}_kF_{i,l}(\underline{z}) $$

for $J = 1, \ldots, N-1$; 3.3.15

$$ {}_{k+1}F_{N,J}(\underline{x}) = -\lambda_k \dot{B}_k(x_N) \sum_{i=2}^{N} {}_kF_{i,J+1}(\underline{z}) $$

In the analogy between the gating and the exhaustive models, Equation 3.3.15 and Equation 3.2.12 are related.

In Appendix B 7 we show that at $\underline{x} = \underline{0}$ Equation 3.3.15 is represented by :

$$\overline{(\theta_{k+N} - \bar{\theta}_{K+N})^2} = \frac{1+\rho_k}{1-\rho_k}\,\sigma^2_{w_{k-1}} + \frac{\lambda_k \overline{s_k^2}}{(1-\rho_k)^3}\,\bar{v}_{k+N} + \frac{\rho_k^2}{(1-\rho_k)^2}\,\sigma^2_{v_{k+N}}$$

$$\overline{(\theta_{k+N} - \bar{\theta}_{k+N})(\theta_{k+J} - \bar{\theta}_{k+J})} = \frac{\rho_k}{1-\rho_k} \cdot \sum_{i=1}^{N-1} \overline{(\theta_{k+J} - \bar{\theta}_{k+J})(\theta_{k+i} - \bar{\theta}_{k+i})} \qquad 3.3.16$$

where

$$\sigma^2_{v_{k+N}} = \sigma^2_{w_{k-1}} + \sum_{i=1}^{N-1} \sum_{l=1}^{N-1} \overline{(\theta_{k+i} - \bar{\theta}_{k+i})(\theta_{k+l} - \bar{\theta}_{k+l})}$$

As in Equation 3.2.13, the above equation holds for the transient state. For the steady state, as $k \to \infty$ and $\rho_0 < 1$, we have the random vector $\underline{\theta}$, defined by Equation 3.2.6, and the steady state cross correlation between T_i and T_j, $R(i, j)$, defined by 3.2.14.

From that, the steady state representation of 3.3.16 is

for $i, j = 1, \ldots, N-1$:

$$R(i,j) = \begin{cases} \dfrac{\rho_i}{1-\rho_i}\left[\sum_{l=j}^{i-1} R(l, j) + \sum_{l=1}^{j-1} R(j, l) + \sum_{l=i+1}^{N} R(j, l)\right] & \text{for } j < i \\[2ex] \dfrac{\lambda_i \overline{s_i^2}}{(1-\rho_i)^3}\,\bar{v}_i + \dfrac{1+\rho_i}{1-\rho_i}\,\sigma^2_{w_{i-1}} + \dfrac{\rho_i^2}{(1-\rho_i)^2} \cdot \sigma^2_{v_i} & \text{for } j = i \\[2ex] \dfrac{\rho_i}{1-\rho_i}\left[\sum_{l=j}^{N} R(l, j) + \sum_{l=1}^{i-1} R(l, j) + \sum_{l=i+1}^{j-1} R(j, l)\right] & \text{for } j > i \end{cases} \qquad 3.3.17$$

where

$$\sigma^2_{v_i} = \sigma^2_{w_{i-1}} + \sum_{l=1}^{i-1} [\sum_{m=i+1}^{N} R(l, m) + \sum_{m=1}^{l} R(l, m) + \sum_{m=l+1}^{i-1} R(m, l)]$$

$$+ \sum_{l=i+1}^{N} [\sum_{m=i+1}^{l} R(l, m) + \sum_{m=l+1}^{N} R(m, l) + \sum_{m=1}^{i-1} R(m, l)]$$

Equation 3.3.17 is analogous to Equation 3.2.15. Apparently we have a set of N^2 linear equations with N^2 unknowns. However, all the N unknowns $R(l, l+1)$ $l = 1, .., N$ do not appear in the R.H.S. of 3.3.17. In fact for $i, j = 1, .., N$ and $j \neq i+1$, Equation 3.3.17 is really a set of $N(N-1)$ equations with $N(N-1)$ unknowns.

Solution of Equation 3.3.17 yields all $\sigma^2_{v_i}$ $i = 1, .., N$.

Eisenberg (1972) needed $N(N-1)$ equations for each of the $\sigma^2_{v_i}$, for a total of $N^2(N-1)$ equations. The redundancy in the number of equations is due to the fact that θ_{k+N} does not depend on θ_k. (This is not true in the gating model). This independence caused $z_1 = 0$ in 2.3.15.

To obtain explicit solution of Equation 3.3.17 for the general asymmetric case and for all $N \geq 1$ is again rather complicated. In the following, we derive $\sigma^2_{v_i}$ for $N = 2, 3$ (with 2 and 6 linear equations respectively).

Solution of 3.3.17 for $N = 2$

Substitution of $N = 2$ into 3.3.17 yields 2 linear equations,

for $i = 1, 2$ and $j = 3 - i$

$$R(i, i) = \frac{\lambda_i \overline{s_i^2}}{(1-\rho_i)^3} \bar{v}_i + \frac{1+\rho_i}{1-\rho_i} \sigma^2_{w_i} + \frac{\rho_i^2}{(1-\rho_i)^2} \sigma^2_{v_i} \qquad 3.3.18$$

where

$$\sigma^2_{v_i} = \sigma^2_{w_j} + R(j, j)$$

In Appendix B8 we show that the solution of 3.3.18 is :

$$\sigma^2_{v_1} = \sigma^2_{w_2} + \frac{\rho_2^2 (\lambda_1 \overline{s_1^2}\, \bar{c} + \sigma^2_{w_2}) + (1-\rho_1)^2 (\lambda_2 \overline{s_2^2}\, \bar{c} + \sigma^2_{w_1})}{(1-\rho_1-\rho_2)(1-\rho_1-\rho_2+2\rho_1\rho_2)} \qquad 3.3.19$$

$\sigma^2_{v_2}$ is obtained by interchanging the indices 1 and 2 in 3.3.19.

Eisenberg (1972) computed $\overline{v_1^2} = \bar{v}_1^2 + \sigma^2_{v_1}$, and found a lengthy expression (Equation 55 there). However, simple algebraic manipulations of his equations yield Equation 3.3.19. It seems that his lenghty expressions and equations prevented his seeing the basic fact that all central moments depend on $\bar{d} = \sum_{i=1}^{N} \bar{w}_i$ and not on some general function $f(\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_N)$.

Solution of 3.3.17 for N = 3

Substituting N = 3 into 3.3.17 yields 6 linear equations. For $i = 1, 2, 3$ we define T_j as the terminal which immediately follows T_i and T_l as the terminal prior to T_i. Thus for $i = 1, 2, 3$ the sets (i, j, l) are (1, 2, 3), (2, 3, 1), and (3, 1, 2) respectively. Using this notation and Equation 3.3.14 in Equation 3.3.17, we have for $i = 1, 2, 3$:

$$R(i, i) = \frac{\lambda_i \overline{s_i^2} \bar{c} + (1 - \rho_i^2)\, \sigma^2_{w_l}}{(1-\rho_i)^2} + \frac{\rho_i^2}{(1-\rho_i)^2} \sigma^2_{v_i}$$

$$R(j, i) = \frac{\rho_j}{1-\rho_j} \left(R(i, i) + R(i, l) \right) \qquad 3.3.20$$

where

$$\sigma^2_{v_i} = \sigma^2_{w_l} + R(j, j) + R(l, l) + 2R(l, j)$$

An explicit expression for all $\sigma^2_{v_i}$ is prohibitively lengthy. In Appendix B9, we derive the solution in determinant form. We have

for $i = 1, 2, 3$.

$$\sigma^2_{v_i} = \sigma^2_{w_l} + \frac{T(i, j, l)}{A} \qquad 3.3.21$$

where

$$A(i) = (1-\rho_j)^2 (1-\rho_l)^2 [(1-\rho_i)^2 (1-\rho_j)(1-\rho_l) - (1+\rho_i)\rho_i \rho_j \rho_l]$$

$$B(i) = (1-\rho_i)(1-\rho_l)^2 [(1-\rho_i)(1-\rho_j)(1+\rho_l) - \rho_i \rho_j \rho_l]$$

$$C(i) = (1-\rho_i)(1-\rho_j)^2 [(1-\rho_i)(1-\rho_j)(1-\rho_l) + \rho_i \rho_j \rho_l]$$

$$D(i) = 2\rho_j \rho_l (1-\rho_j)^2 (1-\rho_l)^2$$

$$F(i) = D(i)(\lambda_i \overline{s_i^2}\, \overline{c} + \sigma^2_{w_i}) + B(i)(\lambda_j \overline{s_j^2}\, \overline{c} + \sigma^2_{w_j}) + C(i)(\lambda_l^2 \overline{s_l^2}\, \overline{c} + \sigma^2_{w_l})$$

$$A = \det \begin{vmatrix} A(1) & -\rho_2^2 B(1) & -\rho_3^2 C(1) \\ -\rho_1^2 C(2) & A(2) & -\rho_3^2 B(2) \\ -\rho_1^2 B(3) & -\rho_2^2 C(3) & A(3) \end{vmatrix} \qquad T(i,j,l) = \det \begin{vmatrix} F(i) & -\rho_j^2 B(i) & -\rho_l^2 C(i) \\ F(j) & A(j) & -\rho_l^2 B(j) \\ F(l) & -\rho_j^2 C(l) & A(l) \end{vmatrix}$$

Solution of 3.3.17 for the Symmetric Case

Applying the same technique as for the solution of Equation 3.2.15 in the gating model, substituting 3.2.18 with $i = N$ into 3.3.17 yields

$$R(0) \triangleq \sigma_\theta^2 = \frac{1}{(1-\rho)^2} (\lambda \overline{s^2}\, \overline{c} + (1-\rho^2)\sigma_w^2 + \rho^2 \sigma_v^2)$$

for $J = 1, .., N-1$;

$$R(N-J) = \frac{\rho}{1-\rho} \left[\sum_{i=0}^{J-1} R(i) + \sum_{i=1}^{N-J-1} R(i) \right], \qquad 3.3.22$$

where

$$\sigma_v^2 = \sigma_w^2 + (N-1)\,R(0) + 2\sum_{i=1}^{N-2} (N-1-i)\,R(i)$$

and as in 3.2.19

$$\sigma_c^2 = N\,R(0) + 2\sum_{i=1}^{N-1} (N-i)\,R(i)$$

The above is a set of N-1 linear equations with N-1 unknowns.

In Appendix B10 we show that the unique solution of Equation 3.3.22 is :

$$R(0) \triangleq \sigma_\theta^2 = \frac{1}{N} \cdot \frac{(1-\rho_0+\frac{\rho_0}{N})(\sigma_d^2+\lambda_0 \overline{s^2}\,\overline{c})}{(1-\rho_0)(1-\frac{\rho_0}{N})}$$

and for $l = 1, \ldots, N-1$:

$$R(l) = \overline{(\theta_N-\overline{\theta}_N)(\theta_{N-l}-\overline{\theta}_{N-l})} = \frac{\frac{\rho_0}{N}}{1-\rho_0+\frac{\rho_0}{N}}\,\sigma_\theta^2$$

$$\sigma_c^2 = \frac{\sigma_d^2+\lambda_0\overline{s^2}\,\overline{c}}{(1-\rho_0)(1-\frac{\rho_0}{N})} \qquad 3.3.23$$

$$\sigma_v^2 = \frac{\sigma_d^2}{N} + \frac{N-1}{N} \cdot \frac{\sigma_d^2+\lambda_0\overline{s^2}\,\overline{c}}{(1-\rho_0)}$$

or, as we show in Appendix C

$$\sigma_v^2 = (1 - \frac{\rho_0}{N})^2 \sigma_c^2 - \lambda \bar{s}^2 \bar{c}$$

where

$$\sigma_d^2 = N \sigma_w^2 , \rho_0 = N \rho , \lambda_0 = N \lambda$$

The above is analogous to Equation 3.2.21.

The normalized cross correlation between T_{N-l} and T_N, $R_\theta(l)$, defined by Equation 3.2.22 is the same as for the gating model and is given by Equation 3.2.23.

$$R_\theta(l) = \frac{R(l)}{\sigma_\theta^2} = \begin{cases} 1 & l = 0 \\ \dfrac{\frac{\rho_0}{N}}{1 - \rho_0 + \frac{\rho_0}{N}} & l = 1, \ldots, N-1 \end{cases} \qquad 3,2.23$$

In Chapter V we find $R_\theta(l)$ between different cycles.

The Second Moments for the Symmetric Nonrandom Case

In the symmetric nonrandom case, defined by 3.1.4, we obtain by using 3.3.10 and 3.3.14 in 3.3.23 and 3.2.23 :

$$\sigma_\theta^2 = \frac{1}{N} \frac{(1 - \rho_0 + \frac{\rho_0}{N})\ \rho_0 \bar{s}\ \bar{c}}{(1 - \rho_0)\ (1 - \frac{\rho_0}{N})} = \frac{1}{N} \frac{(1 - \rho_0 + \frac{\rho_0}{N})\ \rho_0}{(1 - \rho_0)^2 (1 - \frac{\rho_0}{N})} \cdot \bar{s}\ \bar{d}$$

and for $l = 1, \ldots, N-1$,

$$R_\theta (l) = \frac{\frac{\rho_0}{N}}{1 - \rho_0 + \frac{\rho_0}{N}}$$

$$\sigma_c^2 = \frac{\rho_0 \bar{s}\ \bar{c}}{(1 - \rho_0)\ (1 - \frac{\rho_0}{N})} = \frac{\rho_0}{(1 - \rho_0)^2 (1 - \frac{\rho_0}{N})} \bar{s}\ \bar{d}$$

$$\sigma_v^2 = \frac{N-1}{N} \frac{\rho_0 \bar{s}\ \bar{c}}{(1 - \rho_0)} = \frac{N-1}{N} \frac{\rho_0 \bar{s}\ \bar{v}}{(1 - \rho_0)(1 - \frac{\rho_0}{N})} = \frac{N-1}{N} \frac{\rho_0}{(1 - \rho_0)^2} \bar{s}\ \bar{d} \qquad 3.3.24$$

It should be mentioned that just as Equation 3.3.22 is reduced from a set of (N-1) linear equation to only 2, Eisenberg (1972) had, even for the symmetric case, a set of $\frac{N(N-1)}{2}$ equations and had to solve all of them, if he had tried, to find σ_v^2 explicitly. The enormous number of equations prevented him from reaching the surprisingly simple solution, Equation 3.3.23. In conclusion, he suggested Leibowitz's (1961), (1968), approximation method which set all $R(i)$, $i = 1, \ldots, N-1$, (all $R(i, j)$, $i \neq j$) to be zero. As we show (by Equation 3.2.23 and in Chapter V), $R(i) > 0$ for all i. This kind of approximation always under estimates σ_v^2 or σ_c^2. For the symmetric nonrandom case, and $N >> 1$, the "approximated" σ_v^2

or σ_c^2 are $(1 - \rho_0)$ times the exact expressions given in 3.3.23. When, for example, $\rho_0 = 0.9$, the approximated value is 0.1 of the exact one. Hayes and Sherman (1972) used these approximations in order to evaluate the average customer waiting time in the symmetric nonrandom case. As $\rho_0 \to 1$ their result yields a substantially smaller value than the exact one which we will derive in Chapter VI.

The Third Central Moment of v

As in the gating model, we restrict the development of the exhaustive model to the symmetric nonrandom case.

For the symmetric case we define here

$$W_k(x_N) = W(x_N) \triangleq W$$

$$B_k(x_N) = B(x_N) \triangleq B^*$$

$$u_k = u$$

Differentiation of 3.3.15 w.r.t. x_p, $P = 1, \ldots, N$, yields

$$ {}_{k+1}F_{N,N,N}(\underline{x}) = [W[\dot{u}\dot{W}(\ddot{u}\dot{W}+(\dot{u})^2\ddot{W}) + W\cdot(\dddot{u}\dot{W}+3\dot{u}\ddot{u}\ddot{W}+(\dot{u})^3\dddot{W}) - 2\dot{u}\dot{W}\cdot(\ddot{u}\dot{W}+(\dot{u})^2\ddot{W})]]\frac{1}{W^3} $$

$$ + \frac{1}{W^2}[2\dot{u}\dot{W}[(\dot{u}\dot{W})^2 - W\cdot(\ddot{u}\dot{W}+(\dot{u})^2\ddot{W}]] $$

$$ - \lambda\dddot{B}\sum_{i=2}^{N} {}_kF_i(\underline{z}) + 3\lambda^2\dot{B}\ddot{B}\sum_{i=2}^{N}\sum_{l=2}^{N} {}_kF_{i,l}(\underline{z}) $$

$$ - \lambda^3\dddot{B}\sum_{i=2}^{N}\sum_{l=2}^{N}\sum_{m=2}^{N} {}_kF_{i,l,m}(\underline{z}) \qquad 3.3.25 $$

for $J, P = 1, \ldots, N-1$

$$ {}_{k+1}F_{N,N,J}(\underline{x}) = -\lambda\ddot{B}\sum_{i=2}^{N} {}_kF_{i,J+1}(\underline{z}) + (\lambda\dot{B})^2\sum_{i=2}^{N}\sum_{l=2}^{N} {}_kF_{i,l,J+1}(\underline{z}) $$

$$ {}_{k+1}F_{N,J,P}(\underline{x}) = -\lambda\dot{B}\sum_{i=2}^{N} {}_kF_{i,J+1,P+1}(\underline{z}) $$

We derive in Appendix B 11 the steady state representation of Equation 3.3.25. (In the same way we dealt with Equation 3.2.25 for the gating model). For the symmetric nonrandom case we obtain for $N \geq 2$:

$$\overline{(\theta_N - \bar{\theta}_N)^3} = \frac{\rho(1-(N-2)\rho + (N-3)\rho^2)}{(1-\rho)^5 (1-N\rho)} \bar{s}^2 \bar{v} + \frac{\rho^3}{(1-\rho)^3} \delta_v^3$$

for J, P = 1, . . , N-1

$$\overline{(\theta_N - \bar{\theta}_N)^2 (\theta_J - \bar{\theta}_J)} = \frac{\rho^2}{(1-\rho)^4 (1-N\rho)} \bar{s}^2 \bar{v} + \frac{\rho^2}{(1-\rho)^2} \sum_{i=1}^{N-1} \sum_{l=1}^{N-1} \overline{(\theta_J - \bar{\theta}_J)(\theta_i - \bar{\theta}_i)(\theta_l - \bar{\theta}_l)}$$

$$\overline{(\theta_N - \bar{\theta}_N)(\theta_J - \bar{\theta}_J)(\theta_P - \bar{\theta}_P)} = \frac{\rho}{1-\rho} \sum_{i=1}^{N-1} \overline{(\theta_J - \bar{\theta}_J)(\theta_P - \bar{\theta}_P)(\theta_i - \bar{\theta}_i)} \qquad 3.3.26$$

where

$$\delta_v^3 = \sum_{i=1}^{N-1} \sum_{l=1}^{N-1} \sum_{m=1}^{N-1} \overline{(\theta_i - \bar{\theta}_i)(\theta_l - \bar{\theta}_l)(\theta_m - \bar{\theta}_m)}$$

Using definition 3.2.27 in 3.3.26 we obtain for $N \geq 2$:

$$R(0, 0) = \frac{\rho(1-(N-2)\rho + (N-3)\rho^2)}{(1-\rho)^5 (1-N\rho)} \bar{s}^2 \bar{v} + \frac{\rho^3}{(1-\rho)^3} \delta_v^3$$

and for $1 \leq J < N$:

$$R(0,N-J) = \frac{\rho^2}{(1-\rho)^4(1-N\rho)}\bar{s}^2\bar{v} + \frac{\rho^2}{(1-\rho)^2}\left[\sum_{i=1}^{J}\left(\sum_{l=1}^{i} R(J-i,\ i-l) + \sum_{l=i+1}^{J} R(J-l,l-i) + \sum_{l=J+1}^{N-1} R(l-J,J-i)\right)\right.$$

$$\left. + \sum_{i=J+1}^{N-1}\left(\sum_{l=1}^{J} R(i-J,J-l) + \sum_{l=J+1}^{i} R(i-l,l-J) + \sum_{l=i+1}^{N-1} R(l-i,\ i-J)\right)\right]$$

and for, $1 \le P \le J < N$:

$$R(N-J,J-P) = \frac{\rho}{1-\rho}\left[\sum_{i=1}^{P} R(J-P,\ P-i) + \sum_{i=P+1}^{J} R(J-i,\ i-P) + \sum_{i=J+1}^{N-1} R(i-J,\ J-P)\right] \qquad 3.3.27$$

where

$$\delta_v^3 = \sum_{i=1}^{N-1}\left[\sum_{l=1}^{i}\left(\sum_{m=1}^{l} R(i-l,l-m) + \sum_{m=l+1}^{i} R(i-m,m-l) + \sum_{m=i+1}^{N-1} R(m-i,i-l)\right)\right.$$

$$\left. + \sum_{l=i+1}^{N-1}\left(\sum_{m=1}^{i} R(l-i,i-m) + \sum_{m=i+1}^{l} R(l-m,m-i) + \sum_{m=l+1}^{N-1} R(m-l,l-i)\right)\right]$$

Apparently Equation 3.3.27 is a set of $\frac{(N+1)N}{2}$ linear equations with the same number of unknowns :

$R(i,j)$; $0 \le i, j \le N-1$ and $i+j \le N-1$.

However, in the R.H.S. of 3.3.27 an expression $R(i,j)$ where $i+j = N-1$ never appears. Hence, what we really have is a set of $\frac{(N-1)N}{2}$ linear equations for :

$R(i, j) : 0 \leq i, j \leq N-2$ and $i+j \leq N-2$.

An explicit expression for δ_v^3 for all $N \geq 1$ seems very complicated. Below we derive explicit expressions for δ_v^3 in the symmetric nonrandom case for $N = 1, 2, 3, \infty$.

δ_v^3 for $N = 1$

For $N = 1$ Equation 3.3.27 does not hold but we have instead :

$$\bar{v} = \bar{w}$$
$$\sigma_v^2 = 0$$
$$\delta_v^3 = 0 \qquad 3.3.28$$

δ_v^3 for $N = 2$

Substitution of $N = 2$ into Equation 3.3.27 ($\rho_0 = 2\rho$) yields

$$R(0, 0) \stackrel{\Delta}{=} \delta_\theta^3 \stackrel{\Delta}{=} \delta_v^3 = \frac{\frac{\rho_0}{2}\left(1 + \frac{\rho_0}{2}\right)}{\left(1 - \frac{\rho_0}{2}\right)^4 (1 - \rho_0)} \frac{2}{s}\,\bar{v} + \frac{\left(\frac{\rho_0}{2}\right)^3}{\left(1 - \frac{\rho_0}{2}\right)^3} \cdot \delta_v^3$$

Hence,

$$\delta_v^3 = \frac{\left(\frac{\rho_0}{2}\right)\left(1+\frac{\rho_0}{2}\right)}{(1-\rho_0)^2\left(1-\frac{\rho_0}{2}\right)\left(1-\frac{\rho_0}{2}\left(1-\frac{\rho_0}{2}\right)\right)} \cdot \overline{s}^2\,\overline{v} \qquad 3.3.29$$

<u>δ_v^3 for $N = 3$</u>

Substitution of $N = 3$ into Equation 3.3.27 ($\rho_0 = 3\rho$) yields 3 equations :

$$R(0,0) = \frac{\left(\frac{\rho_0}{3}\right)}{(1-\rho_0)\left(1-\frac{\rho_0}{3}\right)^4}\,\overline{s}^2\,\overline{v} + \frac{\left(\frac{\rho_0}{3}\right)^3}{\left(1-\frac{\rho_0}{3}\right)^3}\,\delta_v^3$$

$$R(0,1) = \frac{\left(\frac{\rho_0}{3}\right)^2}{(1-\rho_0)\left(1-\frac{\rho_0}{3}\right)^4}\,\overline{s}^2\,\overline{v} + \frac{\left(\frac{\rho_0}{3}\right)^2}{\left(1-\frac{\rho_0}{3}\right)^2}\left(R(0,0) + 2R(0,1) + R(1,0)\right)$$

$$R(1,0) = \frac{\frac{\rho_0}{3}}{1-\frac{\rho_0}{3}}\left(R(0,0) + R(0,1)\right) \qquad 3.3.30$$

where

$$\delta_v^3 = 2R(0,0) + 3R(0,1) + 3R(1,0)$$

In Appendix B12 we show that the solution of the above equation is :

$$\delta_v^3 = \frac{2\left(\frac{\rho_0}{3}\right)\left(1 - \frac{7}{18}\rho_0^2 + \frac{1}{18}\rho_0^3\right)}{(1-\rho_0)^2\left(1-\frac{\rho_0}{3}\right)\left(1-\rho_0+\frac{4}{9}\rho_0^2+\frac{1}{9}\rho_0^3\right)} \bar{s}^2 \bar{v} \qquad 3.3.31$$

δ_v^3 for $N = \infty$

For $\bar{s}$, $\bar{d}$ finite positive quantities and $\rho_0 < 1$, as $N \to \infty$ we show in Section 3.2 that each terminal might have at most one customer. Thus, the gating and the exhaustive model are identical and we have from Equations 3.3.10, 3.3.23 and 3.2.43 (since $\frac{\rho_0}{N} \to 0$) :

$$\bar{v} = \bar{c} = \frac{\bar{d}}{1-\rho_0}$$

$$\sigma_v^2 = \sigma_c^2 = \frac{\rho_0 \bar{s} \bar{c}}{1-\rho_0}$$

$$\delta_v^3 = \delta_c^3 = \frac{\rho_0 (1+\rho_0)}{(1-\rho_0)^2} \bar{s}^2 \bar{v} \qquad 3.3.32$$

As in Equation 3.2.43, we have

$$\overline{v^3} = \bar{v}^3 + 3\bar{v}\sigma_v^2 + \delta_v^3 \qquad 3.3.33$$

Using Equation 3.3.24 and each of Equations 3.3.28, 3.3.29, 3.3.31 and 3.3.32 in 3.3.33, we obtain, respectively :

For N = 1

$$\overline{v^3} = \bar{v}^3$$

For N = 2

$$\overline{v^3} = \bar{v}^3 + \frac{3\rho_0}{2(1-\rho_0)(1-\frac{\rho_0}{2})}\,\bar{s}\,\bar{v}^2 + \frac{\rho_0(1+\frac{\rho_0}{2})}{2(1-\rho_0)^2(1-\frac{\rho_0}{2})(1-\frac{\rho_0}{2}(1-\frac{\rho_0}{2}))}\,\bar{s}^2\,\bar{v}$$

For N = 3

$$\overline{v^3} = \bar{v}^3 + \frac{2\rho_0}{(1-\rho_0)(1-\frac{\rho_0}{3})}\,\bar{s}\,\bar{v}^2 + \frac{2\rho_0(1-\frac{7}{18}\rho_0^2+\frac{1}{18}\rho_0^3)}{3(1-\rho_0)^2(1-\frac{\rho_0}{3})(1-\rho_0+\frac{4}{9}\rho_0^2+\frac{1}{9}\rho_0^3)}\,\bar{s}^2\,\bar{v}$$

For N = ∞

$$\overline{v^3} = \bar{v}^3 + \frac{3\rho_0}{(1-\rho_0)}\,\bar{s}\,\bar{v}^2 + \frac{\rho_0(1+\rho_0)}{(1-\rho_0)^2}\,\bar{s}^2\,\bar{v} \qquad 3.3.34$$

3.4 Summary

In this chapter we derived explicit expressions for some moments of the Terminal Service Time (TST), θ, the terminal cycle time, c, and the terminal intervisit time, v, in various cases. The derivation involved somewhat tedious computation, the expressions obtained are essential in understanding the polling system and are used in all the following chapters.

The main results of this chapter are :

For the Gating Model

1. $\bar{\theta}_i$ in the transient state (Equation 3.2.3).

2. $\bar{c}$, and $\bar{\theta}_i$ in the steady state (Equation 3.2.10 - 11) .

3. $\sigma^2_{\theta_i}$, $\sigma^2_{c_i}$ in the transient state (Equation 3.2.13) .

4. General expressions for $\sigma^2_{c_i}$, $\sigma^2_{\theta_i}$ in the steady state (Equation 3.2.15) .

From it we obtained :

4.1 $\sigma^2_{c_i}$, $\sigma^2_{\theta_i}$, for $N = 2$ (Equation 3.2.17) .

4.2 σ^2_c , σ^2_θ , $R(l)$ $l = 1, .., N-1$ for the symmetric case (Equation 3.2.21).

4.3 The same quantities for the symmetric nonrandom case (Equation 3.2.24) .

5. General expressions for δ^3_c , δ^3_θ in the steady state (Equation 3.2.28).

From it we obtained :

5.1 δ_c^3 for $N = 1, 2, \infty$ for the symmetrical nonrandom case (Equations 3.2.29, 3.2.31 and 3.2.43 respectively).

5.2 $\bar{c}^3$ for the same cases (Equation 3.2.45).

For the Exhaustive Model

1. $\bar{\theta}_i$ in the transient state (Equation 3.3.6).

2. $\bar{\theta}_i$, $\bar{c}$, $\bar{v}_i$ in the steady state (Equations 3.3.9, 3.3.10, 3.3.13 respectively).

3. $\sigma^2_{\theta_i}$, $\sigma^2_{v_i}$ in the transient state (Equation 3.3.16).

4. General expressions for $\sigma^2_{v_i}$, $\sigma^2_{\theta_i}$ in the steady state (Equation 3.3.17).

   From it we obtained :

   4.1 $\sigma^2_{\theta_i}$, $\sigma^2_{v_i}$, for $N = 2, 3$ (Equations 3.3.19 and 3.3.21 respectively).

   4.2 σ^2_v , σ^2_c , σ^2_θ , $R(l)$ $l = 1, \ldots, N-1$ for the symmetric case (Equation 3.3.23).

   4.3 The same quantities for the symmetric nonrandom case (Equation 3.3.24).

5. General expressions for δ^3_v , δ^3_θ in the steady state (Equation 3.3.27).

   From it we obtained :

5.1 δ_v^3 for $N = 1, 2, 3, \infty$ for the symmetric nonrandom case (Equations 3.3.28, 3.3.29, 3.3.31 and 3.3.32 respectively).

5.2 $\overline{v^3}$ for the same cases (Equation 3.3.34).

It should be noted that beside $\overline{\theta}_i$, $\overline{c}_i$, $\overline{v}_i$ for both models and $\sigma^2_{v_i}$ for the exhaustive model with $N = 2$, all the above results are new. The first moments $\overline{\theta}_i$, $\overline{c}_i$ and $\overline{v}_i$ for both models can be obtained by simple probabilistic approach. However, the higher moments require the tedious computational manipulations.

CHAPTER IV

BUFFER SIZES FOR BOTH MODELS

4.1 Introduction

In deriving the basic equations for the gating and exhaustive models of the polling system, we assumed terminals with unrestricted buffer sizes. This assumption of unlimited storage capacity or unlimited waiting-room facility is invalid in most practical polling systems.

The goal of this chapter is to derive a practically sufficient buffer size such that the probability of customer rejection is arbitrarily small. Rejection occurs when a new customer arrives at a fully occupied buffer and cannot enter the system.

Unlike the $M/G/1$ queue with zero walking time, the number of waiting customers in a terminal buffer is changing periodically. During a terminal intervisit time, the number of customers in the terminal buffer attains the minimum at the time the server leaves it and the maximum at the time the server reaches it.

For both the gating and the exhaustive models we have, from definition 2.1.2, for some terminal:

$n \triangleq$ the number of customers that are served at the terminal in one steady state cycle.

For the gating model, n is the number of customers that exist at the terminal at the moment the server reaches it. For the exhaustive model, n is the total number of customers that are served in the terminal ; this number is greater than or equal to the number of customers that exist at the terminal at the moment the server reaches it. For the gating model, we define l to be the sum of n and the number of customers in the terminal at the moment the server leaves it, i.e., finishes the service of the n customers.

$$l \overset{\Delta}{=} n + \{ \text{ the number of customers in the terminal at the moment the server leaves it } \} \qquad 4.1.1$$

For the exhaustive model, we define m as the number of customers in the terminal at the moment the server reaches it.

$$m \overset{\Delta}{=} \text{ the number of customers in the terminal at the moment the server reaches it.} \qquad 4.1.2$$

For both models, we define m_x as the maximum number of customers that exist at the terminal during the service cycle.

For the gating model, we have :

$$n \le m_x \le l \qquad 4.1.3$$

For the exhaustive model, we have :

$$m \le m_x \le n \qquad 4.1.4$$

Using Chebyshev's inequality, which states that for any random variable u and $a > 0$

$$\text{Prob}\ (\ |u - \bar{u}| \geq a\ \sigma_u) \leq \frac{1}{a^2}$$

and since

$$\text{Prob}\ (\ |u - \bar{u}| \geq a\ \sigma_u) \geq \text{Prob}\ (u \geq \bar{u} + a\ \sigma_u)$$

we have :

$$\text{Prob}\ (m_x \geq \bar{m}_x + a\ \sigma_{m_x}) \leq \frac{1}{a^2} \qquad 4.1.5$$

From the above inequality it follows that from the knowledge of the quantities $\bar{m}_x$ and σ_{m_x}, a buffer size of $\bar{m}_x + a\ \sigma_{m_x}$ guarantees a probability of rejection not greater than $1/a^2$.

Let j represent one of the random variables n, m, l ; we define

$P_j(\cdot) \triangleq$ the probability density function of j,

$J(\cdot) \triangleq$ the Laplace transform of $P_j(\cdot)$, i.e.,

$$J(x) \triangleq \sum_{i=0}^{\infty} \exp\ (-\,i\,x)\ P_j(i)\ . \qquad 4.1.6$$

Let c and v represent the terminal cycle time and the terminal inter-visit time, respectively. We define

$P_c(\cdot) \triangleq$ the terminal cycle time probability density function,

$C(\cdot) \triangleq$ the Laplace transform of $P_c(\cdot)$,

$P_v(\cdot) \triangleq$ the terminal intervisit time probability density function,

$V(\cdot) \triangleq$ the Laplace transform of $P_v(\cdot)$.

For the gating model, Section 4.2, we derive the relationship between $N(\cdot)$, $L(\cdot)$ and $C(\cdot)$. From it we derive $\bar{n}$, $\bar{l}$, σ_n^2, and σ_l^2. Using results obtained in Chapter III.2, we study the behaviour of these quantities. For the exhaustive model, Section 4.3, we derive the relationship between $N(\cdot)$, $M(\cdot)$ and $V(\cdot)$, evaluate $\bar{n}$, $\bar{m}$, σ_n^2, and σ_m^2 and, using results of Chapter III.3, study the behaviour of these quantities. A comparison between the models from the buffer size point of view, is done in Section 4.4.

For both models in the nonrandom case we have, for each T_i, $i = 1, \ldots, N$, a linear relation between n_i and θ_i:

$$\theta_i = \bar{w}_j + n_i \bar{s}_i \qquad 4.1.7$$

where $j = \begin{cases} i & \text{for the gating model,} \\ i-1 & \text{for the exhaustive model.} \end{cases}$

Equation 4.1.7 implies:

$$\bar{n}_i = \frac{\bar{\theta}_i - \bar{w}_i}{\bar{s}_i}$$

$$\sigma^2_{n_i} = \frac{1}{\bar{s}_i^2} \cdot \sigma^2_{\theta_i} \qquad 4.1.8$$

and the normalized cross correlation between n_i and n_j is equal to that between θ_i and θ_j. Therefore, for the symmetric nonrandom case we have, from Equation 3.2.22

$$R_n(l) \triangleq \frac{\overline{(n_i - \bar{n}_i)(n_{i+l} - \bar{n}_{i+l})}}{\sigma^2_n} = R_\theta(l) \qquad 4.1.9$$

4.2 The Gating Model

Derivation of N (·)

By the law of total probability, we have :

$$P_n(k) = \int_0^\infty \text{Prob}(k, t)\,dt = \int_0^\infty \text{Prob}(k/t)\,P_c(t)\,dt \qquad 4.2.1$$

where

k = the number of customers that are served in the terminal,

t = the cycle time of the terminal .

Using Equation 2.2.4, we obtain

$$P_n(k) = \int_0^\infty [(\lambda t)^k / k!] \exp(-\lambda t) P_c(t)\, dt \qquad 4.2.2$$

Applying the Laplace transform, we have

$$N(x) = \sum_{k=0}^{\infty} \int_0^\infty [(\lambda t \exp(-x))^k / k!] \exp(-\lambda t) P_c(t)\, dt \qquad 4.2.3$$

Summing first over k, we obtain

$$N(x) = \int_0^\infty \exp[-\lambda (1 - \exp(-x)) t] P_c(t)\, dt \qquad 4.2.4$$

and finally, integration over t, yields

$$N(x) = C(z) \qquad 4.2.5$$

where

$$z = \lambda (1 - \exp(-x))$$

At $x = 0$ we have :

$$z = 0$$

$$\dot{z} = \lambda \exp(-x) = \lambda \qquad 4.2.6$$

$$\ddot{z} = -\lambda \exp(-x) = -\lambda$$

Differentiating Equation 4.2.5 w.r.t. x yields

$$\dot{N}(x) = \dot{z}\,\dot{C}(z)$$

$$\ddot{N}(x) = \ddot{z}\,\dot{C}(z) + (\dot{z})^2\,\ddot{C}(z) \qquad 4.2.7$$

At $x = 0$, applying Equation 4.2.6 in 4.2.7 we obtain

$$\bar{n} = \lambda\,\bar{c}$$

$$\overline{n^2} = \lambda\,\bar{c} + \lambda^2\,\overline{c^2} \qquad 4.2.8$$

Since $\overline{c^2} = \bar{c}^2 + \sigma_c^2$, using 3.2.10, we obtain

$$\bar{n} = \frac{\lambda\,\bar{d}}{1 - \rho_0}$$

$$\sigma_n^2 = \bar{n} + \lambda^2\,\sigma_c^2 \qquad 4.2.9$$

All explicit expressions for σ_c^2 which were derived in Section 3.2 may be applied directly.

For the symmetric case, using Equation 3.2.21 we have

$$\sigma_n^2 = n + \lambda^2 \cdot \frac{\sigma_d^2 + N\,\bar{n}\,\bar{s}^2}{(1 - \rho_0)\,(1 + \frac{\rho_0}{N})} \qquad 4.2.10$$

It should be mentioned that Leibowitz (1961) was unable to derive the above equation and suggested an approximation method. However, for the special case, $N = 2$, using a set of 3 equations with three unknowns he found the result in a very complicated form (Equation (44) in Leibowitz (1961)) . We simplified his results by straight forward algebraic computation and obtained a simplified expression which is identical to Equation 4.2.10 with $N = 2$.

For the symmetric nonrandom case, substituting Equation 3.2.24 in 4.2.9 or 4.1.8, yields :

$$\bar{n} = \frac{\lambda \bar{d}}{(1 - \rho_0)}$$

$$\sigma_n^2 = \bar{n} \cdot \frac{(1 - \rho_0 + \frac{\rho_0}{N})}{(1 - \rho_0)(1 + \frac{\rho_0}{N})} \qquad 4.2.11$$

In this case, for $N >> 1$ such that $\frac{\rho_0}{N} << 1$ we have $\sigma_n^2 = \bar{n} = \frac{\lambda \bar{d}}{(1 - \rho_0)}$.

<u>Derivation of L (x)</u>

By the law of total probability, we have :

$$P_L(k) = \int_0^\infty \int_0^\infty \sum_{n=0}^{k} \text{Prob}(k, \tau, n, t) \, d\tau \, dt$$

$$= \int_0^\infty \int_0^\infty \sum_{n=0}^{k} \text{Prob}(^k/\tau, n, t) \, \text{Prob}(^\tau/n, t) \, \text{Prob}(^n/t) \, P_c(t) \, d\tau \, dt$$

where

n = the number of customers that are served in the terminal,

k = n + the number of new customers that exist in the terminal at the moment the service of the terminal (its n customers) is over.

τ = the service time of the n customers,

t = the cycle time of the terminal .

Since given τ and n, the probability density function of k does not depend on t, and is governed by a Poisson arrival process, we have :

$$\text{Prob}\,(k/\tau, n, t) = \text{Prob}\,(k/\tau, n) = [\,(\lambda\tau)^{k-n}/(k-n)!\,]\exp(-\lambda\tau) \tag{4.2.12a}$$

Since, once n is given, the probability density function of τ does not depend on t, and is the sum of n independent services. We have :

$$\text{Prob}\,(\tau/n, t) = \text{Prob}\,(\tau/n) = P_s^{(n)*}(\tau) \tag{4.2.12b}$$

Using Equations 4.2.12 and 2.2.4, we obtain

$$P_l(k) = \int_0^{\infty}\int_0^{\infty}\sum_{n=0}^{k} [\,(\lambda\tau)^{k-n}/(k-n)!\,]\exp(-\lambda\tau)\; P_s^{(n)*}(\tau)$$

$$\cdot\,[\,(\lambda t)^{n}/n!\,]\exp(-\lambda t)\; P_c(t)\; d\tau\, dt \tag{4.2.13}$$

Applying the Laplace transform to 4.2.13 and rearranging the integration, the summation order, and the limits, we obtain :

$$L(x) = \int_0^\infty \sum_{n=0}^\infty \int_0^\infty \sum_{k=n}^\infty [(\lambda \tau \exp(-x))^{k-n} / (k-n)!] \exp(-\lambda \tau) \quad 4.2.14$$

$$\cdot P_s^{(n)*}(\tau) [(\lambda t \exp(-x))^n / n!] (\exp(-\lambda t)) P_c(t) \, d\tau \, dt$$

To evaluate 4.2.14, we first sum over k, integrate over τ, then sum over n and finally integrate over t .

$$L(x) = \int_0^\infty \sum_{n=0}^\infty \int_0^\infty \exp[-\lambda(1-\exp(-x))\tau] P_s^{(n)*}(\tau) [(\lambda t \exp(-x))^n / n!] \exp(-\lambda t) P_c(t) \, d\tau \, dt$$

$$= \int_0^\infty \sum_{n=0}^\infty [(S[\lambda(1-\exp(-x))] \lambda t \exp(-x))^n / n!] \exp(-\lambda t) P_c(t) \, dt$$

$$= \int_0^\infty \exp[-\lambda(1-\exp(-x) S(\lambda(1-\exp(-x)))) t] P_c(t) \, dt$$

$$L(x) = C(u), \quad 4.2.15$$

where

$$u = \lambda(1 - \exp(-x) S(z))$$

z and its derivatives are defined by Equations 4.2.5 and 4.2.6.

At $x = 0$, we have, $u = 0$, $z = 0$ and

$$\dot{u} = \lambda \exp(-x) \, (s(z) + \dot{z} \, \dot{S}(z)) = \lambda (1 + \rho) \qquad 4.2.16$$

$$\ddot{u} = -\lambda \exp(-x) \, (S(z) + 2 \, \dot{z} \, \dot{S}(z) - \ddot{z} \, \dot{S}(z) - (\dot{z})^2 \, \ddot{S}(z)) = -\lambda (1 + 3\rho + \lambda^2 \overline{s^2})$$

As in Equation 4.2.7 we have

$$\dot{L} = \dot{u} \, \dot{C}(u) \qquad 4.2.17$$

$$\ddot{L} = \ddot{u} \, \dot{C}(u) + (\dot{u})^2 \, \ddot{C}(u)$$

At $x = 0$ substitution of 4.2.16 into 4.2.17 we obtain

$$\overline{l} = \lambda \, \overline{c} \, (1 + \rho) \qquad 4.2.18$$

$$\overline{l^2} = \lambda \overline{c} \, (1 + 3\rho + \lambda^2 \overline{s^2}) + \lambda^2 (1 + \rho)^2 (\overline{c}^2 + \sigma_c^2)$$

which implies :

$$\overline{l} = (1 + \rho) \cdot \frac{\lambda \overline{d}}{1 - \rho_0} \qquad 4.2.19$$

$$\sigma_l^2 = \overline{l} \cdot \frac{(1 + 3\rho + \lambda^2 \overline{s^2})}{(1 + \rho)} + \lambda^2 (1 + \rho)^2 \sigma_c^2$$

All explicit expressions for σ_c^2 obtained in Chapter III.2 are applicable immediately.

Comparing 4.2.19 and 4.2.9, we find for all $0 < \rho < 1$

$$\overline{n} < \overline{l}$$

$$\sigma_n^2 < \sigma_l^2$$

For the symmetric case, using 3.2.21 we obtain

$$\sigma_l^2 = \bar{l} \cdot \frac{\left(1 + 3\left(\frac{\rho_0}{N}\right) + \left(\frac{\rho_0}{N}\right)^2\right)}{1 + \frac{\rho_0}{N}} + \lambda^2 \cdot \frac{\sigma_d^2 \left(1 + \frac{\rho_0}{N}\right) + N \bar{l} \bar{s}^2}{1 - \rho_0} \qquad 4.2.20$$

For the symmetric nonrandom case

$$\bar{l} = \left(1 + \frac{\rho_0}{N}\right) \cdot \frac{\lambda \bar{d}}{1 - \rho_0}$$

$$\sigma_l^2 = \bar{l} \cdot \frac{\left(1 - \rho_0 + 3\left(\frac{\rho_0}{N}\right) - (2N-1)\left(\frac{\rho_0}{N}\right)^2\right)}{(1 - \rho_0)\left(1 + \frac{\rho_0}{N}\right)} \qquad 4.2.21$$

In this case, for $N >> 1$ such that $\frac{\rho_0}{N} << 1$ we have :

$$\bar{l} = \bar{n} = \frac{\lambda \bar{d}}{1 - \rho_0}$$

$$\sigma_l^2 = \bar{l} \qquad 4.2.22$$

4.3 The Exhaustive Model

Derivation of M (·)

By the law of total probability, we have :

$$P_m(k) = \int_0^{\infty} \text{Prob}(k, t)\, dt = \int_0^{\infty} \text{Prob}(k/t)\, P_v(t)\, dt \qquad 4.3.1$$

where

k = the number of customers that exist in the terminal at the moment the server reaches it,

t = the intervisit time of the terminal.

Equation 4.3.1 is analogous to Equation 4.2.1, the difference is that here we have $P_v(t)$ instead of $P_c(t)$. Prob $({}^k/t)$ is identical to the expression used in Equation 4.3.2. An identical development of 4.3.1 yields the analogue of Equation 4.2.5.

$$M(x) = V(z) \qquad 4.3.2$$

where

$$z = \lambda(1 - \exp(-x))$$

As in Equation 4.2.8 we have :

$$\bar{m} = \lambda \bar{v}$$
$$\overline{m^2} = \lambda \bar{v} + \lambda^2 \overline{v^2} \qquad 4.3.3$$

Therefore

$$\bar{m} = \lambda C(1 - \rho) = \frac{\lambda(1-\rho)\bar{d}}{1 - \rho_0}$$
$$\sigma_m^2 = \bar{m} + \lambda^2 \sigma_v^2 \qquad 4.3.4$$

All explicit expressions for σ_v^2 which were derived in Section 3.3 may be applied directly.

For the symmetric case, using Equation 3.3.23, we have :

$$\sigma_m^2 = \bar{m} + \lambda^2 \left[\frac{\sigma_d^2}{N} + \frac{N-1}{N} \cdot \frac{(1 - \frac{\rho_0}{N}) \sigma_d^2 + N \bar{m} \bar{s}^2}{(1 - \rho_0)(1 - \frac{\rho_0}{N})} \right] \qquad 4.3.5$$

For the symmetric nonrandom case, using Equation 3.3.24 in 4.3.4 we obtain :

$$\bar{m} = \frac{\lambda (1 - \frac{\rho_0}{N}) \bar{d}}{(1 - \rho_0)}$$

$$\sigma_m^2 = \bar{m} \left[\frac{1 - (N-1)(\frac{\rho_0}{N}) + (N-1)(\frac{\rho_0}{N})^2}{(1 - \rho_0)(1 - \frac{\rho_0}{N})} \right] \qquad 4.3.6$$

The above equation was also obtained by Konheim and Meister (1971) (quoted by Chu and Konheim (1972) as Equations (47) and (48)) .

For the symmetric nonrandom case, for $N >> 1$ such that $\frac{\rho_0}{N} << 1$, Equation 4.3.6 is identical to 4.2.21,

$$\bar{m} = \frac{\lambda \bar{d}}{1 - \rho_0} \qquad 4.3.7$$

$$\sigma_m^2 = \bar{m} \qquad 4.3.8$$

Derivation of $N(\cdot)$

In order to derive $N(\cdot)$, we first derive the probability density function of the number of customers that are served in a busy period in $M/G/1$ queue.

Define the random variable h :

h = the number of customers in a busy period in an $M/G/1$ queue,

$P_h(\cdot)$ = the probability density function of h,

$H(\cdot)$ = the Laplace transform of $P_h(\cdot)$.

A busy period is initiated by the arrival of the first (initiating) customer, and clearly $h \geq 1$.

In Appendix A we derive $H(\cdot)$ and obtained in Equation A.17

$$H(x) = \exp(-x) \quad S[\lambda(1 - H(x))], \qquad 4.3.9$$

where $S(\cdot)$ is the Laplace transform of the probability density function of customer service time in the queue.

To derive $N(\cdot)$, by the law of total probability we have :

$$P_n(k) = \int_0^\infty \sum_{m=0}^{k} \text{Prob}(k, m, t)\, dt$$

$$= \int_0^\infty \sum_{m=0}^{k} \text{Prob}(k/m, t)\ \text{Prob}(m/t)\, P_v(t)\, dt, \qquad 4.3.10$$

where

k = the number of customers that are served in the terminal,

m = the number of customers that exist in the terminal at the moment the server reaches it,

t = the intervisit time of the terminal.

Since each of the m customer initiates an independent busy period process, and given m, k does not depend on t, we have :

$$\text{Prob}\,(^{k}/m,\ t) = \text{Prob}\,(^{k}/m) = P_h^{(m)*}(k) \qquad 4.3.11$$

Note that $P_h^{(m)*}(k)$ is identically zero for $k < m$.

Using Equations 2.3 7 and 4.3.11 in 4.3.10, we obtain :

$$P_n(k) = \int_0^{\infty} \sum_{m=0}^{\infty} P_h^{(m)*}(k)\ [(\lambda t)^m/m!]\ \exp(-\lambda t)\ P_v(t)\ dt \qquad 4.3.12$$

Applying the Laplace transform we obtain :

$$N(x) = \int_0^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \exp(-kx)\ P_h^{(m)*}(k)\ [(\lambda t)^m/m!]\ \exp(-\lambda t)\ P_v(t)\ dt \qquad 4.3.13$$

To evaluate Equation 4.3.13 we first sum over k, then over m, and finally integrate over t .

$$N(x) = \int_0^{\infty} \sum_{m=0}^{\infty} [(\lambda t H(x))^m / m!] \exp(-\lambda t) P_v(t) \, dt$$

$$N(x) = \int_0^{\infty} \exp[-\lambda (1 - H(x)) t] P_v(t) \, dt \qquad 4.3.14$$

We obtain :

$$N(x) = V(u) \qquad 4.3.15$$

where

$$u = \lambda (1 - H(x))$$

At $x = 0$, $u = 0$. Differentiating u w.r.t. x and using the results of Equation A.20, we obtain at $x = 0$:

$$\dot{u} = -\lambda \dot{H}(x) = \frac{\lambda}{1 - \rho}$$

$$\ddot{u} = -\lambda \ddot{H}(x) = -\frac{\lambda (1 - \rho^2 + \lambda^2 s^2)}{(1 - \rho)^3} \qquad 4.3.16$$

Differentiating 4.3.15 w.r.t. x we obtian

$$\dot{N} = \dot{u} \dot{V}(u)$$

$$\ddot{N} = \ddot{u} \dot{V}(u) + (\dot{u})^2 \ddot{V}(u)$$

and at $x = 0$

$$\bar{n} = \frac{\lambda \bar{v}}{1 - \rho} = \lambda \bar{c}$$

$$\bar{n^2} = \frac{\lambda (1 - \rho^2 + \lambda^2 \bar{s^2})}{(1 - \rho)^3} \bar{v} + \frac{\lambda^2}{(1 - \rho)^2} (\bar{v}^2 + \sigma_v^2) \qquad 4.3.17$$

Rearranging the above, we obtain :

$$\bar{n} = \frac{\lambda \bar{d}}{1 - \rho_0}$$

$$\sigma_n^2 = \frac{1}{(1 - \rho)^2} [\bar{n} (1 - \rho^2 + \lambda^2 \bar{s^2}) + \lambda^2 \sigma_v^2] \qquad 4.3.18$$

All explicit expressions for σ_v^2 which were derived in Section 3.3 may be applied directly.

For the symmetric case, using Equation 3.3.23 we obtain :

$$\sigma_n^2 = \frac{1}{(1 - \frac{\rho_0}{N})^2} [\bar{n} (1 - (\frac{\rho_0}{N})^2 + \lambda^2 \bar{s^2}) + \lambda^2 (\frac{\sigma_d^2}{N} + \frac{N-1}{N} \cdot \frac{\sigma_d^2 + N \bar{n} \bar{s^2}}{(1 - \rho_0)})] \qquad 4.3.19$$

For the symmetric nonrandom case, using Equation 3.3.24 in 4.3.18 or 4.1.8, we obtain :

$$\bar{n} = \frac{\lambda \bar{d}}{1 - \rho_0}$$

$$\sigma_n^2 = \bar{n} \cdot \frac{1 - \rho_0 + \frac{\rho_0}{N}}{(1 - \rho_0)\left(1 - \frac{\rho_0}{N}\right)} \qquad 4.3.20$$

In this case, for $N >> 1$ such that $\frac{\rho_0}{N} << 1$ we obtain the same result as in Equation 4.3.8 (for the gating model)

$$\bar{n} = \frac{\lambda \bar{d}}{1 - \rho_0}$$

$$\sigma_n^2 = \bar{n} \qquad 4.3.21$$

4.4 Comparison between the Models

For the symmetric nonrandom case, we found in Equations 4.2.11, 4.2.21, 4.3.6, 4.3.20:

For the Gating Model :

$$\bar{n} = \frac{\lambda \bar{d}}{1 - \rho_0}$$

$$\sigma_n^2 = \frac{\lambda \bar{d}}{1 - \rho_0} \cdot \frac{1 - (N-1)\frac{\rho_0}{N}}{(1 - \rho_0)\left(1 + \frac{\rho_0}{N}\right)}$$

$$\bar{l} = \left(1 + \frac{\rho_0}{N}\right) \frac{\lambda \bar{d}}{1 - \rho_0}$$

$$\sigma_l^2 = \frac{\lambda \bar{d}}{(1 - \rho_0)} \cdot \frac{1 - (N-3)\frac{\rho_0}{N} - (2N-1)\left(\frac{\rho_0}{N}\right)^2}{(1 - \rho_0)}$$

For the exhaustive model :

$$\bar{m} = \left(1 - \frac{\rho_0}{N}\right) \frac{\lambda \bar{d}}{1 - \rho_0}$$

$$\sigma_m^2 = \frac{\lambda \bar{d}}{1 - \rho_0} \cdot \frac{1 - (N+1)\frac{\rho_0}{N} + (2N-1)\left(\frac{\rho_0}{N}\right)^2}{(1 - \rho_0)}$$

$$\bar{n} = \frac{\lambda \bar{d}}{1 - \rho_0}$$

$$\sigma_n^2 = \frac{\lambda \bar{d}}{1 - \rho_0} \cdot \frac{1 - (N-1)\frac{\rho_0}{N}}{(1 - \rho_0)\left(1 - \frac{\rho_0}{N}\right)} \qquad 4.4.1$$

For $N >> 1$ such that $\frac{\rho_0}{N} << 1$ all averages and all variances converge to the same quantity $\frac{\lambda \bar{d}}{1 - \rho_0} = \lambda \bar{c}$ i.e., a buffer of length $\lambda \bar{c} + a\sqrt{\lambda \bar{c}}$ guarantees that the probability of rejection is not greater than $\frac{1}{a^2}$. For this case, there is no difference between buffer requirements in the gating and the exhaustive models. Let us now compare the number of customers in the terminal at the moment the server reaches it ; n in the gating model and m in the exhaustive model. It can be easily verified by Equation 4.4.1 that for all $N \geq 1$ and $\rho_0 < 1$ we have :

$$\bar{n} > \bar{m}$$

$$\frac{\bar{m}}{\bar{n}} = 1 - \frac{\rho_0}{N}$$

$$\sigma_n^2 > \sigma_m^2$$

$$\frac{\sigma_m^2}{\sigma_n^2} = \frac{1 - \rho_0 + (N-2)\left(\frac{\rho_0}{N}\right)^2 + (2N-1)\left(\frac{\rho_0}{N}\right)^3}{1 - (N-1)\frac{\rho_0}{N}} = 1 - \left(\frac{\rho_0}{N}\right) - \left(\frac{\rho_0}{N}\right)^2 + \frac{N \cdot \left(\frac{\rho_0}{N}\right)^3}{1 - (N-1)\left(\frac{\rho_0}{N}\right)} \qquad 4.4.2$$

From the above we conclude that the exhaustive model requires slightly less buffer capacity than that required by the gating model. However, for $\frac{\rho_0}{N} << 1$ the buffer requirements are the same.

CHAPTER V

THE TRANSIENT AND STEADY STATE BEHAVIOUR OF THE SYSTEM

5.1 Introduction

In this chapter we study two basic features of the polling system.

In Section 5.2 we examine the transient behaviour of the average cycle time, $\bar{c}_k$, for the gating model, and the average intervisit time, $\bar{v}_k$, for the exhaustive model. Given any initial vector $\underline{\bar{\theta}}_1 = (\bar{\theta}_1, \ldots, \bar{\theta}_N)$ which describes the first cycle, we refer to one terminal and examine how these quantities ($\bar{c}_k$ for the gating model and $\bar{v}_k$ for the exhaustive model) change from cycle to cycle until steady state values ($\bar{c}$ or $\bar{v}$) are obtained. As an example we suppose the system has just been activated with all buffers empty. In the first cycle, the server visits the terminal to announce the "system is operating". For this case, the elements of $\underline{\bar{\theta}}_1$ are the average walking times. In this case we examine the transient time of the system. As another example, we suppose that for some time the server has either a malfunction or is occupied in different tasks. During this time new customers are arriving. Hence, when service is resumed we expect $\underline{\bar{\theta}}_1$ to be greater than the steady state value. In this case, we examine the recovery time of the system.

In Section 5.3 we examine both the gating and the exhaustive models in the symmetric case under steady state conditions. For both models we derive recursive formulas for the normalized cross correlation between two terminals, as defined by Equation 3.2.22, in different cycles. Using these formulas, we derive the normalized cross correlation between two different cycles of the same terminal. For the exhaustive model we also derive the normalized cross correlation between two different intervisits of the same terminal.

In the steady state, we define "T_{lN+i}" as "T_i" in the lth following cycle. For the symmetric case $\sigma^2_{\theta_l}$, $\sigma^2_{c_l}$, and $\sigma^2_{v_l}$ for all l are independent of l. As in Equation 3.2.22, we define the normalized cross correlation between T_i and T_{i+l} for all $l \geq 0$ as :

$$R_\theta(l) = \frac{\overline{(\theta_i - \bar{\theta}_i)(\theta_{i+l} - \bar{\theta}_{i+l})}}{\sigma^2_\theta} \qquad l \geq 0 \tag{5.1.1}$$

According to Equation 2.1.3 for the gating model :

$$c_{kN+i} = \sum_{j=0}^{N-1} \theta_{(k-1)N+i+j} \tag{5.1.2}$$

We define the normalized cross correlation between c_{kN+i} and $c_{(k+l)N+i}$ [i,e, two cycle times at the same terminal, T_i, l cycles apart, where $c_{(k+l)N+i}$ is the lth successive cycle time after c_{kN+i}] as :

$$R_c(l) = \frac{\overline{(c_{kN+i} - \bar{c}_{kN+i})(c_{(k+l)N+i} - \bar{c}_{(k+l)N+i})}}{\sigma^2_c} \qquad l \geq 0 \tag{5.1.3}$$

According to Equation 2.1.4, for the exhaustive model :

$$\left.\begin{aligned} c_{kN+i} &= \sum_{j=1}^{N} \theta_{(k-1)N+i+j} \\ v_{kN+i} &= w_{kN+i-1} + \sum_{j=1}^{N-1} \theta_{(k-1)N+i+j} \end{aligned}\right\} \quad 5.1.4$$

The normalized cross correlation between c_{kN+i} and $c_{(k+l)N+i}$ is defined by Equation 5.1.3. The normalized cross correlation between v_{kN+i} and $v_{(k+l)N+i}$ is defined in a similar way as :

$$R_v(l) = \frac{\overline{(v_{kN+i} - \bar{v}_{kN+i})(v_{(k+l)N+i} - \bar{v}_{(k+l)N+i})}}{\sigma_v^2} \qquad 5.1.5$$

$$l \geq 0$$

For the symmetric nonrandom case, referring to Equations 4.1.7 and 4.1.8, in both models, the normalized cross correlation of the number of customers that are served in two terminals is identical to the normalized cross correlation between the two terminals, i.e., :

$$R_n(l) = \frac{\overline{(n_i - \bar{n}_i)(n_{i+l} - \bar{n}_{i+l})}}{\sigma_n^2} = R_\theta(l) , \qquad 5.1.6$$

$$l \geq 0$$

where, by virtue of Equation 4.4.1, for all i, $\bar{n}_i = \bar{n} = \lambda \bar{c}$.

Define n_T as the total number of customers from the N terminals that are served in one cycle. In the symmetric nonrandom case we have a linear relation between c and n_T

$$c = \bar{d} + \bar{s}\, n_T \qquad 5.1.7$$

Hence, for both models, the normalized cross correlation between two n_T's of different cycles equal that of the cycle times themselves

$$R_{n_T}(l) = R_c(l) \qquad 5.1.8$$

The knowledge of the quantities $R_\theta(l)$, $R_c(l)$, $R_v(l)$ is essential for optimal linear mean square error predictions of θ, c, and v. For any terminal, T_i, $i = 1, \ldots, N$, given that its last values of the cycle time and the intervisit time where c_i and v_i respectively, the optimal linear mean square error predictions of, c_{kN+i} and v_{kN+i} (defined as c^*_{kN+i}, v^*_{kN+i} respectively) are :

$$c^*_{kN+i} = \bar{c}_{kN+i} + R_c(k)\,[\,c_i - \bar{c}_i\,]$$

$$v^*_{kN+1} = \bar{v}_{kN+i} + R_v(k)\,[\,v_i - \bar{v}_i\,] \qquad 5.1.9$$

where : $\bar{c}_{kN+i} = \bar{c}_i = \bar{c} = \dfrac{\bar{d}}{1-\rho_0}$

$\bar{v}_{kN+i} = \bar{v}_i = (1-\rho)\dfrac{\bar{d}}{1-\rho_0}$

The errors of the predictions are :

$$ER_c(k) = c_{KN+i} - c^*_{kN+i}$$

$$ER_v(k) = v_{kN+i} - v^*_{kN+i} \qquad 5.1.10$$

These errors have zero mean and their second moments are :

$$ER_c^2(k) = (1 - R_c^2(k))\,\sigma_c^2$$

$$ER_v^2(k) = (1 - R_v^2(k))\,\sigma_v^2 \qquad 5.1.11$$

Using Equation 5.1.9 each terminal can predict the values of any future c or v based on the value of its last c or v.

From the server's point of view, the optimal linear mean square error prediction of θ_l (defined as θ^*_l) given all θ_{l-i}, $i = 1, \ldots, N-1$, might be of interest (with the values of the last N-1 TSTs the server can estimate the expected value of the next TST). The expression of the optimal linear mean square prediction is :

$$\theta^*_l = \bar{\theta}_l + \sum_{i=1}^{N-1} z_i\,(\theta_{l-i} - \bar{\theta}_{l-i}) \qquad 5.1.12$$

Using the fact that for both models we have from Equations 3.2.21 and 3.2.23

$$R_\theta(l) = \begin{cases} 1 & l = 0 \\ \dfrac{\frac{\rho_0}{N}}{1 - \rho_0 + \frac{\rho_0}{N}} & l = 1, \ldots, N-1 \end{cases} \qquad 5.1.13$$

The value of all z_i such that $\overline{(\theta_l - \theta^*_l)^2}$ (the mean square error) is minimized is found to be :

$$z_i = \frac{\frac{\rho_0}{N}}{1 - \frac{\rho_0}{N}} \qquad i = 1, \ldots, N-1 \qquad 5.1.14$$

Substituting 5.1.14 into 5.1.12

$$\theta^*_l = \bar{\theta}_l + \frac{\frac{\rho_0}{N}}{1 - \frac{\rho_0}{N}} \sum_{i=1}^{N-1} (\theta_{l-i} - \bar{\theta}_{l-i})$$

and 5.1.15

$$\overline{(\theta_l - \theta^*_l)^2} = \frac{1 - \rho_0}{(1 - \frac{\rho_0}{N})(1 - (N-1)\frac{\rho_0}{N})} \sigma_\theta^2$$

Where confusion might occur, we add the subscript G or E such as ${}_G R_\theta(l)$ and ${}_G R_c(l)$ to indicate those quantities in the gating model, and ${}_E R_\theta(l)$ and ${}_E R_c(l)$ are those in the exhaustive model.

5.2 The Transient Behaviour of $\bar{c}$ and $\bar{v}$

From Equations 3.2.3 and 3.2.5 we have in the gating model :

$$\bar{\theta}_{k+N} = \bar{w}_k + \rho_k \sum_{i=0}^{N-1} \bar{\theta}_{k+i} \qquad 5.2.1$$

$$\bar{c}_{k+N} = \sum_{i=0}^{N-1} \bar{\theta}_{k+i}$$

From Equations 3.3.5 and 3.3.11 we have in the exhaustive model :

$$\bar{\theta}_{k+N} = \frac{1}{1-\rho_k} \left(\bar{w}_{k-1} + \rho_k \sum_{i=1}^{N-1} \bar{\theta}_{k+i} \right) \qquad 5.2.2$$

$$\bar{v}_{k+N} = \bar{w}_{k-1} + \sum_{i=1}^{N-1} \bar{\theta}_{k+i}$$

Given any initial $\underline{\bar{\theta}}_1 = (\bar{\theta}_1 , \ldots , \bar{\theta}_N)$ which represents the first cycle, we can solve the equations above to find $\bar{\theta}_i$, $\bar{c}_{N+i}$, $\bar{v}_{N+i}$, for all $i \geq 1$.

It should be mentioned that because our basic Equation 3.1.7 is written in the transient state, all higher moments of θ , c , and v can be evaluated as well. For example, to find the second moments of these quantities we may use Equation 3.2.13 for the gating model and Equation 3.3.16 for the exhaustive model.

In order to examine the basic features of Equations 5.2.1 - 2 let us assume the symmetric nonrandom case where $\bar{w}_k = \bar{w} = 1$ [sec.] for $k \geq 1$.

We study the behaviour of successive cycles and intervisits of one terminal. Define for $l \geq 1$;

$$\bar{c}_{lN+1} = G^{\bar{c}}(l) \qquad \qquad 5.2.3$$

$$\bar{v}_{lN+1} = E^{\bar{v}}(l)$$

Using the computer, we solve Equations 5.2.1 - 2 for various $N \geq 1$, $0 < \rho_0 < 1$, $\bar{w}$, $\underline{\theta}_1$, to find $G^{\bar{c}}(l)$, $E^{\bar{v}}(l)$ for $l \geq 1$.

In Figure 5-1 we sketch $G^{\bar{c}}(l)$, $E^{\bar{v}}(l)$ v.s. l for the symmetric non-random system where $N = 10$ terminal, $\bar{w}_i = \bar{w} = 1$ [sec.], $\rho_0 = 0.8$ for two different $\underline{\theta}_1$. For these values of N, $\bar{w}$, and ρ_0 using Equations 3.2.10, 3.2.11, 3.3.9, 3.3.10, and 3.3.14 we have for both the gating and the exhaustive model

$$\bar{\theta} = 5 \text{ sec.}$$

$$\bar{c} = 50 \text{ sec.} \qquad \qquad 5.2.4$$

$$\bar{v} = 46 \text{ sec.}$$

We choose :

A : $\underline{\theta}_1 = (1, \ldots, 1)$. Since $\bar{w} = 1$, this represents an initially empty system. From this we examine the transient time.

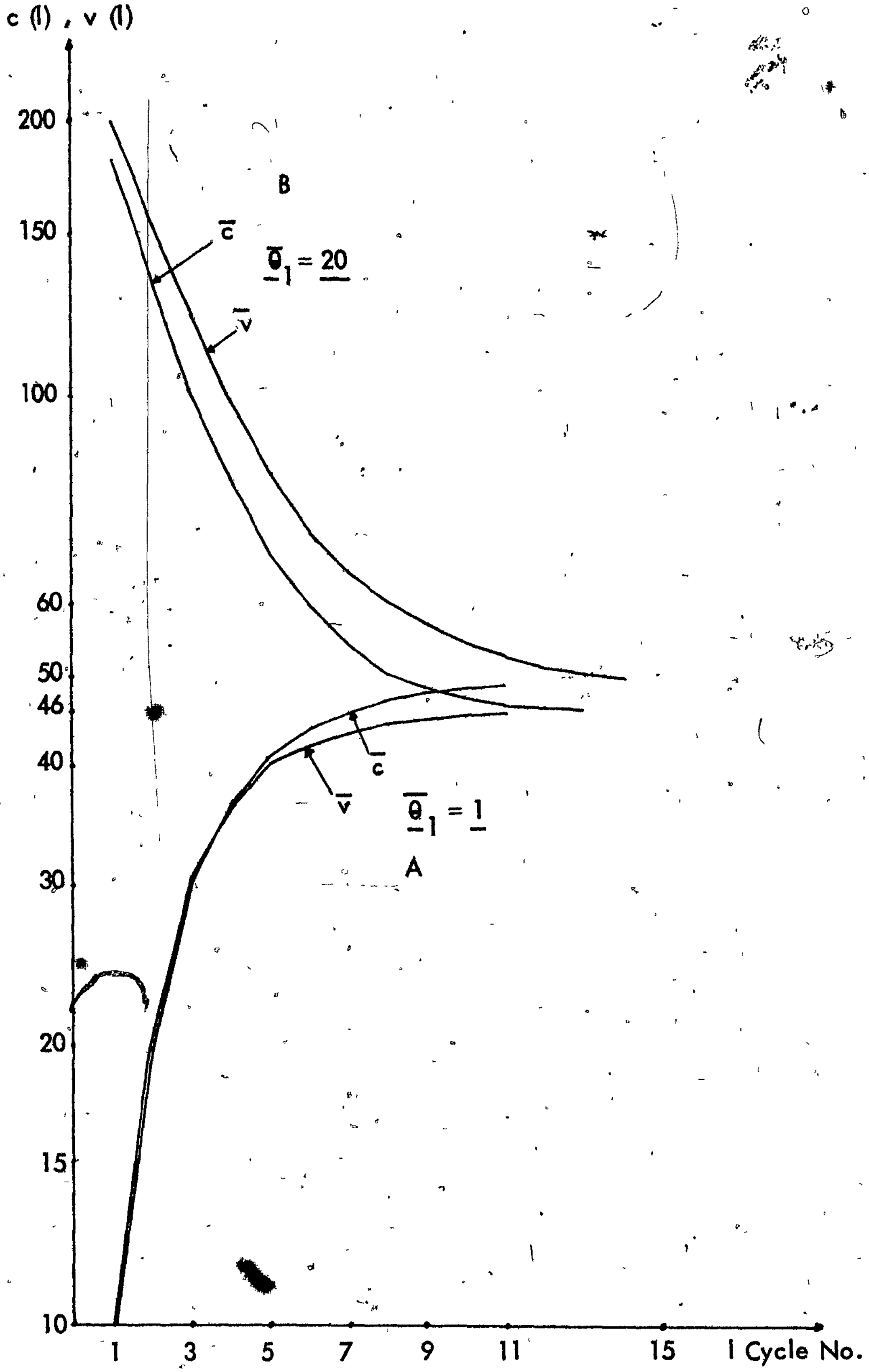

Fig. 5-1. Transient behaviour of the average cycle time, $\bar{c}$, for the gating model, and the average intervisit time, $\bar{v}$, for the exhaustive model. The symmetric nonrandom case with $N = 10$, $\bar{w} = 1$, $\rho = 0.8$, A : $\underline{\bar{\theta}}_1 = (1, \ldots, 1)$ B : $\underline{\bar{\theta}}_1 = (20, \ldots, 20)$.

B : $\underline{\bar{\theta}}_1 = (20, \ldots, 20)$. Since $\underline{\bar{\theta}} = (5, \ldots, 5)$, this represents an initially overloaded system (i.e. suppose all buffers are full). From this we examine the recovery time.

For both cases, 10 - 15 cycles are enough to practically reach the steady state. Different values of $\bar{w}$ yield identical curves (with different scale factors). The general shape of the curves are not very sensitive to the value of N. However, both the transient and recovery time are proportional to ρ_0. Smaller traffic intensity yields faster recovery time. For example, for $\rho_0 = 0.2$ 3 - 4 cycles are sufficient to effectively reach the steady state. For all cases, the recovery time in the exhaustive model is slightly faster than that of the gating model. The numerical results used in Figure 5-1 are in Appendix F.

5.3 Steady State Correlation Between Terminals of Different Cycles

In the steady state for the symmetric case of both the gating and the exhaustive model we have from Equation 3.2.22 for $l = 0, \ldots, N-1$

$$G^R_\theta(l) = {}_E R_\theta(l) = \frac{\overline{(\theta_i - \bar{\theta}_i)(\theta_{i+l} - \bar{\theta}_{i+l})}}{\sigma_\theta^2} = \begin{cases} 1 & l = 0 \\ \dfrac{\dfrac{\rho_0}{N}}{1 - \rho_0 + \dfrac{\rho_0}{N}} & l = 1, \ldots, N-1 \end{cases} \qquad 5.3.1$$

where it should be noted that σ_θ^2 for the gating model (given by Equation 3.2.21) and σ_θ^2 for the exhaustive model (given by Equation 3.3.23) are different.

In Appendix D we derive recursive equations for $G^R_\theta(l)$ and $E^R_\theta(l)$ for $l > N-1$. We show there :

$$G^R_\theta(l) = \frac{\rho_0}{N} \sum_{i=1}^{N} G^R_\theta(l-i) \qquad l \geq N \qquad 5.3.2a$$

$$E^R_\theta(l) = \frac{\frac{\rho_0}{N}}{1-\frac{\rho_0}{N}} \sum_{i=1}^{N-1} E^R_\theta(l-i) \qquad l \geq N-1 \qquad 5.3.2b$$

Lemma 5.3.1. For $N \geq 1$, $0 < \rho_0 < 1$ and $l \geq N$, $G^R_\theta(l)$ is monotonically decreasing and $\lim_{l \to \infty} G^R_\theta(l) = 0$.

Proof : From 5.3.2a we have

$$G^R_\theta(N) = \frac{\frac{\rho_0}{N}}{1-\rho_0+\frac{\rho_0}{N}}$$

$$G^R_\theta(N+1) = \rho_0 \frac{\frac{\rho_0}{N}}{1-\rho_0+\frac{\rho_0}{N}} < G^R_\theta(N) \qquad 5.3.3$$

Using the method of induction, suppose that for some $M \geq N$, $N \leq l \leq M$, $G^R_\theta(l+1) < G^R_\theta(l)$. Using 5.3.2a with $l = M+2$ and $l = M+1$ we obtain

$$G^R_\theta(M+2) = \frac{\rho_0}{N}\sum_{i=1}^{N} G^R_\theta(M+2-i) \qquad 5.3.4a$$

$$G^R_\theta(M+1) = \frac{\rho_0}{N}\sum_{i=1}^{N} G^R_\theta(M+1-i) \qquad 5.3.4b$$

Subtracting 5.3.4b from 5.3.4a, we obtain

$$G^R_\theta(M+2) - G^R_\theta(M+1) = \frac{\rho_0}{N}\left(G^R_\theta(M+1) - G^R_\theta(M+1-N)\right) < 0$$

Hence, for all $l \geq N$

$$G^R_\theta(l+1) < G^R_\theta(l) .$$

Since the infinite series $G^R_\theta(l)$, $l \geq N$, is monotonically decreasing in the compact set $[0, 1]$, the series converges to a single point in this set. Suppose $\lim_{l \to \infty} G^R_\theta(l) = x$, using Equation 5.3.2a we have $x = \rho_0 x$ which implies $x = 0$. Therefore, $\lim_{l \to \infty} G^R_\theta(l) = 0$.

Q.E.D.

In a similar manner, it can be shown that with $N \geq 2$ for all $l \geq N-1$, $E^R_\theta(l)$ is monotonically decreasing and converges to zero. For $N=1$ $E^R_\theta(l) = 0$ for all $l \geq 1$.

At $l = N$

$$ {}_E R_\theta (N) = \frac{(N-1)\,\rho_0}{1 - \frac{\rho_0}{N}} \cdot \frac{\frac{\rho_0}{N}}{1 - \rho_0 + \frac{\rho_0}{N}} \qquad 5.3.5 $$

Comparing Equation 5.3.5 to Equation 5.3.3, it is clear that

$$ {}_G R_\theta (N) > {}_E R_\theta (N) \qquad 5.3.6 $$

From 5.3.2b we have for $l \geq N$

$$ {}_E R_\theta (l) = \frac{\rho_0}{N} \sum_{i=0}^{N-1} {}_E R_\theta (l-i) < \frac{\rho_0}{N} \sum_{i=1}^{N} {}_E R_\theta (l-i) \qquad 5.3.7 $$

Using 5.3.2a, 5.3.5, 5.3.6 and 5.3.7 and a simple induction procedure, we obtain for $l \geq N$

$$ {}_G R_\theta (l) \geq {}_E R_\theta (l) \qquad 5.3.8 $$

Equality in 5.3.8 is obtained for $l \leq N-1$.

<u>The Normalized Cross Correlation Between Cycles</u>

For the gating model, using Equation 5.1.2 in Equation 5.1.3, we have

$$G^R_c(l) = \frac{\overline{\left(\sum_{j=0}^{N-1} (\theta_{kN+j} - \bar{\theta}_{kN+j})\right)\left(\sum_{j=0}^{N-1} \theta_{(k+l)N+j} - \bar{\theta}_{(k+l)N+j}\right)}}{\sigma_c^2} =$$

$$= \frac{\sigma_\theta^2}{\sigma_c^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} \frac{\overline{(\theta_{kN+i} - \bar{\theta}_{kN+i})(\theta_{(k+l)N+j} - \bar{\theta}_{(k+l)N+j})}}{\sigma_\theta^2}$$

$$= \frac{\sigma_\theta^2}{\sigma_c^2} \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} G^R_\theta\,(l\,N+j-i) \qquad 5.3.9$$

Using Equation 3.2.21 in Equation 5.3.9, we obtain for $l \geq 1$

$$G^R_c(l) = \left(1 - \rho_0 + \frac{\rho_0}{N}\right)\left[G^R_\theta(N\,l) + \sum_{i=1}^{N-1} \frac{N-i}{N}\left(G^R_\theta(N\,l+i) + G^R_\theta(N\,l-i)\right)\right] \qquad 5.3.10$$

$$G^R_c(0) = 1\,.$$

Using 5.1.4 in 5.1.3 and following the above derivation, we obtain for $l \geq 1$

$$E^R_c(l) = \left(1 - \rho_0 + \frac{\rho_0}{N}\right)\left[E^R_\theta(N\,l) + \sum_{i=1}^{N-1} \frac{N-i}{N}\left(E^R_\theta(N\,l+i) + E^R_\theta(N\,l-i)\right)\right] \qquad 5.3.11$$

$$E^R_c(0) = 1\,.$$

Using Equation 5.3.8 in order to compare Equation 5.3.10 to Equation 5.3.11, we obtain

$$G^R_c(l) > E^R_c(l) \quad l \geq 1 \qquad 5.3.12$$

For the symmetric nonrandom case, we have from 5.1.8 for both models

$$R_{n_T}(l) = R_c(l)$$

For this case, using Equation 5.1.4 in Equation 5.1.5 we obtain the normalized cross correlation between intervisits

$$ {}_E R_v(l) = \frac{\left(\sum_{i=1}^{N-1} \theta_{kN+i} - \bar{\theta}_{kN+i}\right)\left(\sum_{i=1}^{N-1} \theta_{(k+l)N+i} - \bar{\theta}_{(k+l)N+i}\right)}{\sigma_v^2}$$

$$= \frac{\sigma_\theta^2}{\sigma_v^2} \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} {}_E R_\theta (lN + i - j) \qquad 5.3.13$$

Us[illegible].3.23, we obtain

$$ {}_E R_v(l) = \frac{1 - \rho_0 + \frac{\rho_0}{N}}{1 - \frac{\rho_0}{N}} \Big[{}_E R_\theta (lN)$$

$$+ \sum_{j=1}^{N-2} \frac{N-1-j}{N-1} \left({}_E R_\theta (Nl + j) + {}_E R_\theta (Nl - j)\right) \Big] \qquad 5.3.14$$

Comparing 5.3.14 to 5.3.11, we obtain for $l \geq 1$

$$ {}_G R_c(l) > {}_E R_c(l) > {}_E R_v(l) \qquad 5.3.15$$

To examine the behaviour of ${}_{G}R_{c}(l)$, ${}_{E}R_{c}(l)$, and ${}_{E}R_{v}(l)$ we used the computer to recursively solve 5.3.2 and substitute directly into 5.3.10, 5.3.11 and 5.3.14.

In Figure 5-2, we sketch those ${}_{G}R_{c}(l)$, ${}_{E}R_{c}(l)$, and ${}_{E}R_{v}(l)$ v.s. l for $N = 8$, $\rho_0 = .9$. Figure 5-2 emphasizes the relation of 5.3.15. Note that all quantities are monotonically decreasing. ${}_{G}R_{c}(l) = 0.5$ for $l \simeq 8$ (i.e., 8 cycles) and ${}_{E}R_{c}(l)$ and ${}_{E}R_{v}(l)$ are 0.5 for $l \simeq 6$. For different values of N the curves does not change substantially. As $N \to \infty$, the three curves for ${}_{G}R_{c}(l)$, ${}_{E}R_{c}(l)$, and ${}_{E}R_{v}(l)$ converge to one curve. However, the curves are sensitive to the value of ρ_0; for smaller ρ_0 the curves converge to zero faster.

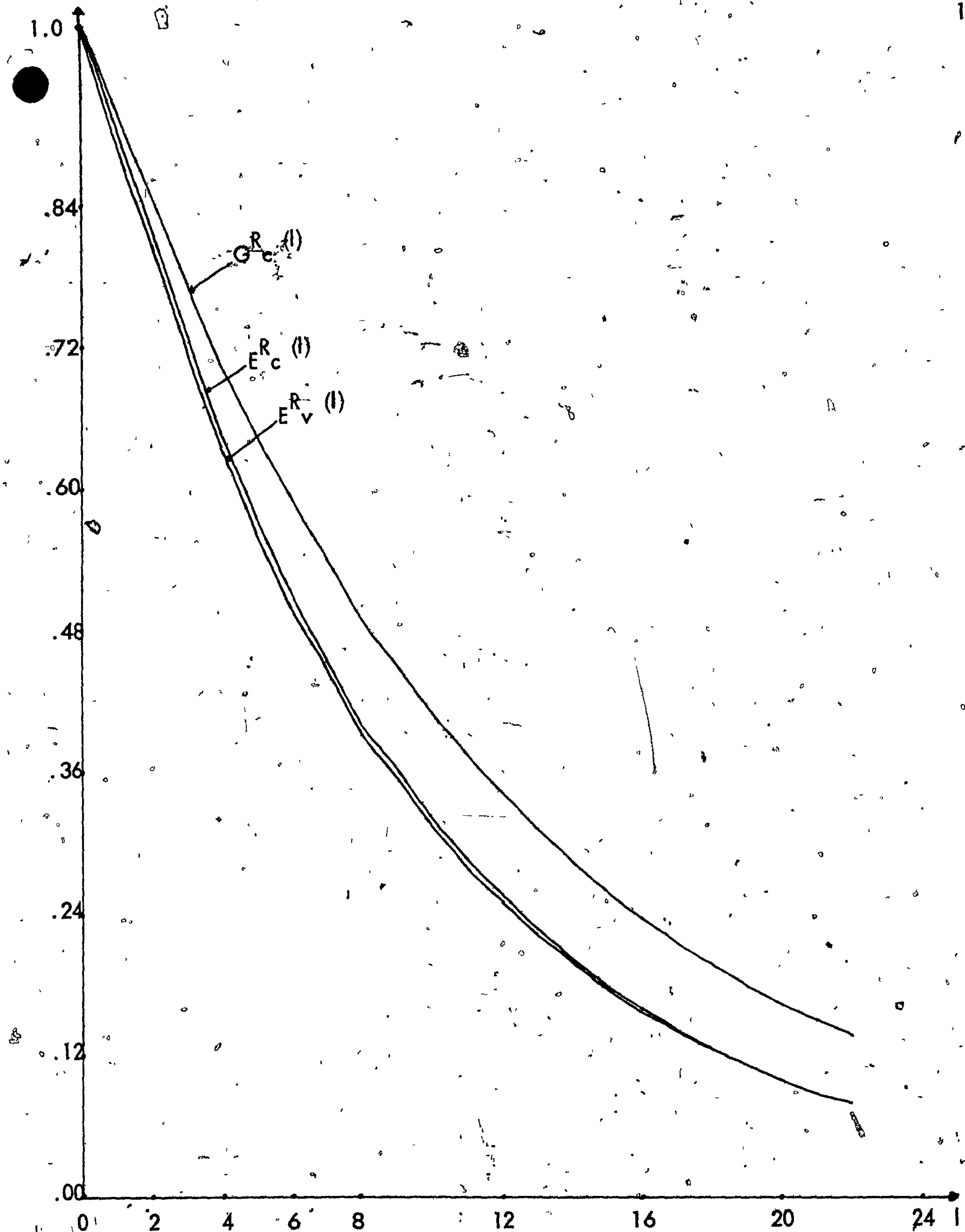

Fig. 5-2. The normalized cross correlation $_G R_c(l)$, $_E R_c(l)$, $_E R_v(l)$ v.s. l for $N = 8$, $\rho_0 = 0.9$.

CHAPTER VI

WAITING TIMES IN BOTH MODELS

6.1 Introduction

In this chapter we study the customers' waiting time characteristics in both the gating and the exhaustive models. The waiting time of a customer is the length of time the customer spends in the buffer of its terminal. In other words, the waiting time of a customer is the time difference between the moment of the customer's arrival to a terminal to the moment his service starts. The waiting time is a non-negative random variable denoted by q. In this chapter we study, for both models, under steady state condition, the first two moments of the waiting time.

For $i = 1, \ldots, N$ we define :

$P_{q_i}(\cdot)$ = the probability density function of a customer's waiting time in T_i ,

$Q_i(\cdot)$ = the Laplace transform of $P_{q_i}(\cdot)$. 6.1.1

In Queueing Theory Problems the average waiting time, $\bar{q}$, primarily because this is the only one that can be found, is considered as an important parameter which indicates the goodness of service in the queue. In Chapter IV, we studied another important parameter of the system, the buffer size requirements in both models. However, the waiting time features are considered to be of greater importance in studying and defining the goodness of a communication network. This is because buffer size indicates hardware requirements for the system to operate properly, while waiting time directly indicates the goodness of service in system after buffer size requirements are met. The major goal of the work done by Eisenberg (1972), Takacs (1968), Avi-Itzhak et al (1965), Cooper (1970), Hayes and Sherman (1972), and Kruskal (1969), was to find the average waiting time in some special models of the polling system. In this chapter we derive exact and explicit expressions concerning the first two moments of q, which are not only new but also play an important role in understanding and designing communication polling systems.

Let c_i and v_i represent the steady state cycle time and intervisit time respectively of T_i. We define :

$$P_{c_i}(\cdot) = \text{the probability density function of } c_i \text{ in } T_i,$$
$$P_{v_i}(\cdot) = \text{the probability density function of } v_i \text{ in } T_i, \qquad 6.1.2$$
$$C_i(\cdot) = \text{the Laplace transform of } P_{c_i}(\cdot),$$
$$V_i(\cdot) = \text{the Laplace transform of } P_{v_i}(\cdot).$$

In order to reduce complexity, we omit the subscript i in Equations 6.1.1 and 6.1.2 but keep in mind that $Q(\cdot)$, $C(\cdot)$, and $V(\cdot)$ refer to some particular T_i. (The same simplification was made in Chapter IV).

In Section 6.2 we derive the relation between $Q(\cdot)$ and $C(\cdot)$ for the gating model. From that we derive general expressions for $\bar{q}$ and σ_q^2. In Section 6.3 we derive the relation between $Q(\cdot)$ and $V(\cdot)$ for the exhaustive model and from that we derive general expressions for $\bar{q}$ and σ_q^2. In Section 6.4 we deal with the symmetric case for both the gating and the exhaustive models. We find explicit expressions for $\bar{q}$ for any $N \geq 1$. For the symmetric nonrandom case we derive explicit expressions for σ_q^2 for $N = 1, 2, \infty$ for the gating model, and $N = 1, 2, 3, \infty$ for exhaustive model. In Section 6.5 we derive, for a general case, explicit expressions for $\bar{q}_i$ for $N = 2$ in the gating model and for $N = 2, 3$ in the exhaustive model. The conclusions are in Section 6.6.

When $P_{q_i}(\cdot)$, as defined in 6.1.1, is the waiting time probability density function of a customer who arrives at T_i, the apriori probability that the customer arrives at T_i given that he arrives at one of the N terminals is λ_i / λ_0. Hence, the waiting time probability density function of a customer who arrives at the system is :

$$P_{q_0}(\cdot) = \frac{1}{\lambda_0} \cdot \sum_{i=1}^{N} \lambda_i P_{q_i}(\cdot) \qquad 6.1.3$$

We refer to q_0 as the waiting time in the system and to q_i as the waiting time in T_i. In Sections 6.4 and 6.5 we derive explicit expressions for $\bar{q}_0$.

Using classical Queueing theory, $Q(\cdot)$ of $M/G/1$ queue (i.e., one terminal and zero walking time) can be found in Takacs (1962) :

$$Q(x) = (1 - \rho) \frac{x}{x - \lambda[1 - S(x)]} \qquad 6.1.4$$

where $N = 1$ and we have : $\rho = \rho_0$, $\lambda = \lambda_0$.

Using L'hôpital technique we may find

$$\bar{q} = \frac{\lambda \bar{s^2}}{2\,(1-\rho)}$$

$$\sigma_q^2 = \frac{\lambda \bar{s^3}}{3\,(1-\rho)} + \frac{(\lambda \bar{s^2})^2}{4\,(1-\rho)^2} \qquad 6.1.5$$

For both the gating and the exhaustive models, under the condition of $N = 1$ and zero walking time, our results are consistent with those of 6.1.5.

Whenever confusion may occur we add the subscript G and E such that q_G and q_E represent waiting times in the gating and the exhaustive models respectively.

6.2 Derivation of Q (·) for the Gating Model

For a customer who arrives at T_i we develop the probability density function, $P_{q_i}(\cdot)$, of his waiting time. Omitting the terminal index i we can write by the law of total probability

$$P_q(t) = \sum_{k=0}^{\infty} \int_0^{\infty} \int_0^{u} \text{Prob}\,(t, k, \tau, u)\, d\tau\, du \,, \qquad 6.2.1$$

where

t = the waiting of "the customer" who arrives at T_i,

k = the number of customers to be served before him in T_i. i.e., the number of customers he finds in the waiting buffer which will be served before him when the service of T_i will start,

τ = the time "the customer" waits until server reaches T_i. i.e., "the customer" arrives τ sec. before server reaches T_i,

u = the time length of the cycle of T_i in which "the customer" arrives. i.e., the time difference between the moment the server reaches T_{i-N} and T_i, during this time "the customer" arrives.

Expanding Equation 6.2.1 we obtain

$$P_q(t) = \sum_{k=0}^{\infty} \int_0^{\infty} \int_0^{u} \text{Prob}\,(^t/k, \tau, u)\ \text{Prob}\,(^k/\tau, u)\ \text{Prob}\,(^\tau/u)\ \text{Prob}\,(u)\ d\tau\, du \qquad 6.2.2$$

Given that "the customer" waits τ sec. until the server reaches his terminal and that there are k customers to be served before him, then, independent of u, the cycle length, the probability that "the customer" waits t sec. before his service starts is equal to the probability that the total service time of the k customers is $t - \tau$. We obtain

$$\text{Prob}\,(^t/k, \tau, u) = \text{Prob}\,(^t/k, \tau) = P_s^{(k)} * (t - \tau) \qquad 6.2.3$$

Given that "the customer" arrives τ seconds before the end of a cycle of length u, the probability that he finds k customers in the waiting buffer, is equal to the probability that k customers arrive during $u - \tau$ (at the beginning of the cycle, the waiting buffer is empty). By the property of a Poisson process, we have :

$$\text{Prob}\,({}^{k}/\tau, u) = \text{Prob}\,({}^{k}/u - \tau) = [(\lambda\,(u - \tau))^{k}/k\,!]\exp\,(-\lambda\,(u - \tau)) \qquad 6.2.4$$

Given that "the customer" arrives during a cycle of length u, the probability that he arrives τ seconds before the end of this cycle is uniformly distributed. This is because aposteriori, the Poisson process is uniformly distributed (Karlin (1969)). We obtain

$$\text{Prob}\,({}^{\tau}/u) = \frac{1}{u} \qquad 6.2.5$$

Given that "the customer" arrives during a cycle, the probability density function of this cycle is :

$$\text{Prob}\,(u) = \frac{u\,P_c\,(u)}{\overline{c}} \qquad 6.2.6$$

A rigorous proof of Equation 6.2.6 was given by Avi-Itzhak et al (1965). In the following we establish relation 6.2.6 from a different point of view.

Let's consider N successive cycles where $N \rightarrow \infty$. From the weak law of large numbers, , the total length of the N cycles is $N\overline{c}$. The number of cycles, from the set of N, whose lengths are equal to or shorter than u

is : $N \int_0^u P_c(z)\, dz$. Given that a cycle is of length equal to or shorter than u, the expected length of such a cycle is : $\dfrac{\int_0^u z P_c(z)\, dz}{\int_0^u P_c(z)\, dz}$.

Hence, the total time length (from the N cycles) that are composed of cycles equal to or shorter than u, is, with probability one,

$$N \cdot \int_0^u P_c(z)\, dz \cdot \frac{\int_0^u z P_c(z)\, dz}{\int_0^u P_c(z)\, dz} = N \int_0^u z P_c(z)\, dz .$$

The ratio : $\lim_{N \to \infty} \dfrac{N \int_0^u z P_c(z)\, dz}{N \bar{c}} = \dfrac{\int_0^u z P_c(z)\, dz}{\bar{c}}$ is the fraction of time, the terminal has cycles of length equal to or shorter than u . Because apriori, "the customer" can arrive at any time with equal probability. The above ratio is the probability that "the customer" arrives during a cycle of length equal to or shorter than u. Hence we have

$$\text{Prob}(u) = \frac{d}{du}\left[\frac{\int_0^u z P_c(z)\, dz}{\bar{c}} \right] = \frac{u P_c(u)}{\bar{c}}$$

Substituting Equations 6.2.3, 6.2.4, 6.2.5, and 6.2.6 into 6.2.2 we obtain

$$P_q(t) = \sum_{k=0}^{\infty} \int_0^{\infty} \int_0^{u} P_s^{*(k)}(t-\tau) \left[(\lambda (u-\tau))^k / k! \right] \exp(-\lambda(u-\tau)) \cdot \frac{1}{u} \cdot \frac{u\, P_c(u)}{\bar{c}}\, d\tau\, du \qquad 6.2.7$$

Applying the Laplace transform, we obtain

$$Q(x) = \frac{1}{\bar{c}} \int_0^{\infty} \sum_{k=0}^{\infty} \int_0^{\infty} \int_0^{u} \exp(-x(t-\tau))\; P_s^{*(k)}(t-\tau) \left[(\lambda (u-\tau))^k / k! \right] \cdot \exp(-\lambda(u-\tau) - x\tau)\, P_c(u)\, d\tau\, du\, dt \qquad 6.2.8$$

Integrating first over t, then summing over k, then integrating over τ we obtain

$$Q(x) = \frac{1}{\bar{c}} \int_0^{\infty} \int_0^{u} \sum_{k=0}^{\infty} \left[(\lambda S(x) (u-\tau))^k / k! \right] \exp(-\lambda(u-\tau) - x\tau)\, P_c(u)\, d\tau\, du$$

$$Q(x) = \frac{1}{\bar{c}} \int_0^{\infty} \int_0^{u} \exp\left[-\lambda(1-S(x))\, u + (-x + \lambda(1-S(x)))\, \tau \right] P_c(u)\, d\tau\, du$$

$$Q(x) = \frac{1}{\bar{c}\,[\lambda(1-S(x)) - x]} \int_0^{\infty} \left[\exp(-xu) - \exp(-\lambda(1-S(x))u) \right] P_c(u)\, du$$

and finally, integrating over u we obtain

$$Q(x) = \frac{1}{\bar{c}} \cdot \frac{C[x] - C[\lambda(1-S(x))]}{\lambda(1-S(x)) - x} \qquad 6.2.9$$

The technique we used to derive $Q(\cdot)$ was hinted at by Hayes and Sherman (1972). Equation (24) there, with minor corrections, is identical to 6.2.9. However, since they had only approximation of the moments of c they were not able to find exact expressions for $\bar{q}$ and σ_q^2.

In Appendix E.1 we use Equation 6.2.9 to derive expressions for $\bar{q}$ and σ_q^2. We obtain from there :

$$\bar{q} = (1+\rho)\frac{\overline{c^2}}{2\bar{c}} = (1+\rho)\left(\frac{\bar{c}}{2} + \frac{\sigma_c^2}{2\bar{c}}\right) \qquad 6.2.10a$$

$$\sigma_q^2 = \frac{\overline{c^3}}{3\bar{c}}(1+\rho+\rho^2) + \bar{q}\left(\frac{\lambda\overline{s^2}}{1+\rho} - \bar{q}\right) \qquad 6.2.10b$$

where $\rho = \lambda\bar{s}$,

6.3. Derivation of Q (·) for the Exhaustive Model

The technique of deriving $Q(\cdot)$ in this case is inherently different from that used in the previous section for the gating model. This is due to the fact that unlike the gating model a customer who arrives at a terminal, which is being served by the server, will be served before the server leaves the terminal. In the gating model this customer is served after the server leaves the terminal and returns to it the next time.

In Appendix A we derived the Laplace transform, $H(\cdot)$, of the probability density function, $P_h(\cdot)$, of the total number of customers that are served in a terminal busy period which is initiated by one customer. By Equation A.17, we have

$$H(x) = \exp(-x)\, S[\lambda(1 - H(x))] \qquad 6.3.1$$

In the third section of Chapter IV we derived the Laplace transform, $N(\cdot)$, of the probability density function, $P_n(\cdot)$, of the number of customers that are served in a terminal. Quoting Equations 4.3.15 and 4.3.17, we have

$$N(x) = V[\lambda(1 - H(x))] \qquad 6.3.2$$

$$\bar{n} = \lambda \bar{c} = \frac{\lambda \bar{v}}{1 - \rho_o} \qquad 6.3.3$$

In order to derive the waiting time probability density function of a customer who arrives at T_i, we define a new integer non negative random variable l with probability density function $P_l(\cdot)$ and its Laplace transform $L(\cdot)$.

Define :

$P_l(k)$ = Prob { Given that "the customer" (the one we compute his waiting time) arrives at T_i, there will be exactly k customers in T_i that will be served after "the customer" before the server walks to T_{i+1} } 6.3.4

By the law of total probability we have

$$P_1(k) = \int_0^{\infty} \sum_{m=0}^{k} \text{Prob}(k, m, t)\, dt \qquad 6.3.5$$

where

k = The number of customers that are served after "the customer", as defined in 6.3.4.

m = The number of customers that exist in T_i at the moment the service of "the customer" is completed.

t = The <u>queueing time</u> of "the customer", the queueing time is the sum of the waiting time and the service time of the customer.

Developing Equation 6.3.5 we obtain

$$P_1(k) = \int_0^{\infty} \sum_{m=0}^{k} \text{Prob}(^k/m, t)\, \text{Prob}(^m/t)\, \text{Prob}(t)\, dt \qquad 6.3.6$$

Given that m customers are in T_i at the moment "the customer" leaves it, independent of t the probability that a total of k customers are served after "the customer" is equal to the probability that the sum of m independent busy periods is k. We have

$$\text{Prob}(^k/m, t) = \text{Prob}(^k/m) = P_h^{(m)} * (k) \qquad 6.3.7$$

Given the queueing time of the customer, the probability that m customers are in the terminal at the moment the service of "the customer" is completed is governed by a Poisson process with parameter λt (since at the moment "the customer" arrives, he is the last one in the terminal) . We have

$$\text{Prob}\,(^m/t) = [(\lambda t)^m/m\,!]\,\exp\,(-\lambda t) \qquad 6.3.8$$

The probability that the queueing time of "the customer" is t is equal to the probability that the sum of the two independent random variables, his waiting time q and his service time s, is equal to t . We have

$$\text{Prob}\,(t) = P_q\,(t) * P_s\,(t) \qquad 6.3.9$$

Substituting Equations 6.3.7, 6.3.8, and 6.3.9 into 6.3.6, we obtain

$$P_l\,(k) = \int_0^{\infty} \sum_{m=0}^{k} P_h^{(m)} * (k)\,[(\lambda t)^m/m\,!]\,\exp\,(-\lambda t)\,P_q\,(t) * P_s\,(t)\,dt \qquad 6.3.10$$

Applying the Laplace transform and using the fact that $P_h^{(m)} * (k)$ is zero for $m > k$, we have

$$L\,(x) = \int_0^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \exp\,(-kx)\,P_h^{(m)} * (k)\,[(\lambda t)^m/m\,!]\,\exp\,(-\lambda t)\,P_q(t) * P_s\,(t)\,dt$$

6.3.11

Summing first over k then summing over m and finally integrating over t, we obtain

$$L(x) = \int_0^{\infty} \sum_{m=0}^{\infty} [(\lambda t H(x))^m / m!] \exp(-\lambda t) P_q(t) * P_s(t) \, dt$$

$$= \int_0^{\infty} \exp[-\lambda(1 - H(x))t] P_q(t) * P_s(t) \, dt$$

$$L(x) = Q[\lambda(1 - H(x))] S[\lambda(1 - H(x))] \qquad 6.3.12$$

At this stage, we have to derive a direct relationship between $L(\cdot)$ and $N(\cdot)$.

Let's consider N successive cycles where $N \to \infty$. From the weak law of large numbers, with probability one, the total number of customers that are served in T_i is $N\bar{n}$, where $\bar{n}$ is given by Equation 6.3.3. The total number of cycles which consist of at least $k+1$ customers' service in T_i is $N \cdot \sum_{i=k+1}^{\infty} P_n(i)$. In every such cycle there is exactly one customer who has exactly k customers behind him. Since "the customer" can be any of the $N \cdot \bar{n}$ customers with equal probability, we have

$$P_l(k) = \lim_{N \to \infty} \frac{N \sum_{i=k+1}^{\infty} P_n(i)}{N \cdot \bar{n}} = \frac{1 - \rho}{\lambda \bar{v}} \cdot \sum_{i=k+1}^{\infty} P_n(i) \qquad 6.3.13$$

Applying the Laplace transform, we have

$$L(x) = \frac{1 - \rho}{\lambda \bar{v}} \sum_{k=0}^{\infty} \sum_{i=k+1}^{\infty} \exp(-kx) \, P_n(i) \qquad 6.3.14$$

Changing the order and the bounds of the summation, we obtain

$$L(x) = \frac{1-\rho}{\lambda \bar{v}} \cdot \frac{N(x)-1}{\exp(-x)-1} \qquad 6.3.15$$

Regrouping the four equations 6.3.1, 6.3.2, 6.3.12, and 6.3.15, we have :

$$H(x) = \exp(-x)\; S[\lambda(1-H(x))] \qquad 6.3.16a$$

$$N(x) = V[\lambda(1-H(x))] \qquad 6.3.16b$$

$$L(x) = Q[\lambda(1-H(x))]\; S[\lambda(1-H(x))] \qquad 6.3.16c$$

$$L(x) = \frac{1-\rho}{\lambda \bar{v}} \cdot \frac{N(x)-1}{\exp(-x)-1} \qquad 6.3.16d$$

Substituting 6.3.16b for $N(x)$ into 6.3.16d and then substituting the result into 6.3.16c, we obtain

$$Q[\lambda(1-H(x))] = \frac{1-\rho}{\bar{v}} \cdot \frac{1-V[\lambda(1-H(x))]}{\lambda S[\lambda(1-H(x))] - \lambda \exp(-x)\, S[\lambda(1-H(x))]} \qquad 6.3.17$$

Define :

$$z = \lambda(1-H(x)), \qquad 6.3.18$$

where z goes from 0 to λ as x goes from 0 to ∞.

Using the above definition in 6.3.16a we have

$$\lambda \exp(-x)\ S[\lambda(1 - H(x))] = \lambda - z \qquad 6.3.19$$

Substituting 6.3.18 and 6.3.19 into 6.3.17 we obtain for $0 \le x \le \lambda$:

$$Q(x) = \frac{1-\rho}{\bar{v}} \cdot \frac{1 - V(x)}{x - \lambda(1 - S(x))} = \frac{1}{\bar{c}} \cdot \frac{1 - V(x)}{x - \lambda(1 - S(x))} \qquad 6.3.20$$

Eisenberg (1972) obtained the same expression (Equation (51) there) by using a different approach. However, since he could not derive the required moments of v (except for the case $N = 2$) he was not able to use 6.3.20 to find $\bar{q}$, σ_q^2 (except $\bar{q}$ for $N \geq 2$).

In Appendix E.2 we use Equation 6.3.20 to derive expressions for $\bar{q}$ and σ_q^2. We obtain from there:

$$\bar{q} = \frac{\overline{v^2}}{2\bar{v}} + \frac{\lambda \overline{s^2}}{2(1-\rho)} = (1-\rho)\frac{\overline{c^2}}{2\bar{c}} = (1-\rho)\left(\frac{\bar{c}}{2} + \frac{\sigma_c^2}{2\bar{c}}\right) \qquad 6.3.21a$$

$$\sigma_q^2 = \frac{\overline{v^3}}{3\bar{v}} + \frac{\lambda \overline{s^3}}{3(1-\rho)} + \bar{q}\left(\frac{\lambda \overline{s^2}}{1-\rho} - \bar{q}\right) \qquad 6.3.21b$$

where $\rho = \lambda \bar{s}$.

6.4 Averages and Variances of the Waiting Time in the Symmetric Case of Both Models

In the symmetric case, defined by 3.1.2, all $P_{q_i}(\cdot)$ for $i = 1, \ldots, N$ are equal. We have for $i = 1, \ldots, N$:

$$P_{q_i}(\cdot) = P_q(\cdot) \qquad 6.4.1$$

Therefore, applying Equation 6.4.1 to 6.1.3, we obtain

$$P_{q_0}(\cdot) = P_q(\cdot) \qquad 6.4.2$$

Hence the waiting time of all terminals and system waiting time have the same probability density function $P_q(\cdot)$.

In order to find the average waiting time, $\bar{q}$, explicitly, in the symmetric case of both models we do the following. For the gating model to find $\bar{q}_G$, we use the explicit expressions for $\bar{c}$ and σ_c^2 obtained in Equations 3.2.10 and 3.2.21 in Equation 6.2.10a. For the exhaustive model, $\bar{q}_E$ is obtained by using Equations 3.3.10 and 3.3.23 in 6.3.21a. We obtain:

$$\bar{q}_G = \frac{\left(1 + \frac{\rho_0}{N}\right)\bar{d} + \lambda_0 \bar{s}^2}{2(1 - \rho_0)} + \frac{\sigma_d^2}{2\bar{d}} \qquad 6.4.3a$$

$$\bar{q}_E = \frac{\left(1 - \frac{\rho_0}{N}\right)\bar{d} + \lambda_0 \bar{s}^2}{2(1 - \rho_0)} + \frac{\sigma_d^2}{2\bar{d}} \qquad 6.4.3b$$

We consider the above expressions to be the most important original contribution of the thesis. From Equation 6.4.3 we see that $\bar{q}_E < \bar{q}_G$. However, as N increases such that $\frac{\rho_0}{N} << 1$, $\bar{q}_E$ and $\bar{q}_G$ converge to the same value.

For the sake of simplicity in the following section we confine our study to the symmetric nonrandom case, defined by 3.1.4.

Using definition 3.1.4 in Equation 6.4.3 we have, for the symmetric nonrandom case :

$$\bar{q}_G = \frac{(1 + \frac{\rho_0}{N})\,\bar{d} + \rho_0\,\bar{s}}{2\,(1 - \rho_0)} \qquad 6.4.4a$$

$$\bar{q}_E = \frac{(1 - \frac{\rho_0}{N})\,\bar{d} + \rho_0\,\bar{s}}{2\,(1 - \rho_0)} \qquad 6.4.4b$$

Because $\sigma_d^2 \geq 0$ and $\overline{s^2} = \bar{s}^2 + \sigma_s^2 \geq \bar{s}^2$, the symmetric nonrandom case has the shortest average waiting time of all possible symmetric cases which have the same ρ_0 and the same averages $\bar{d}$ and $\bar{s}$. This property is satisfied in the classical M / G / 1 queue as well (Karlin (1969)).

Another property of Equation 6.4.4 is revealed when $\bar{d} = 0$, where for all $N \geq 1$, $\bar{q}_G$ and $\bar{q}_E$ are equal. Thus we have :

$$\bar{q}_G = \bar{q}_E = \frac{\rho_0\,\bar{s}}{2\,(1 - \rho_0)} \qquad 6.4.5$$

where $\bar{d} = 0$.

When $\bar{d} \neq 0$ and constant, as N increases $\bar{q}_G$ decreases and $\bar{q}_E$ increases. The smallest $\bar{q}_G$ is obtained with $N = \infty$, the largest with $N = 1$. The smallest $\bar{q}_E$ is obtained with $N = 1$, the largest with $N = \infty$. For any fixed set of $N, \bar{d}, \bar{s}$, both $\bar{q}_G$ and $\bar{q}_E$ are monotonically increasing as ρ_0 increases.

For multiterminal systems where $N >> 1$, both $\bar{q}_G$ and $\bar{q}_E$ are equal, and we have

$$\bar{q}_G = \bar{q}_E = \frac{\bar{d} + \rho_0 \bar{s}}{2(1 - \rho_0)} \qquad 6.4.6$$

where $\frac{\rho_0}{N} << 1$.

In order to visualize the foregoing properties of Equation 6.4.4, we sketch $\bar{q}_{G_N}$ and $\bar{q}_{E_N}$ (N indicates the number of terminals) for $\bar{s} = 1$, for $N = 1, 2, 3, \infty$ and $\bar{d} = 0$. (Figure 6-1), $\bar{d} = 1$. (Figure 6-2), and $\bar{d} = 100$. (Figure 6-3). Figure 6-1 is a numerical example to Equation 6.4.5 where $\bar{d} = 0$. and therefore, for all N, $\bar{q}_E$ is equal to $\bar{q}_G$. Figure 6-2, and Fig. 6-3 are numerical examples to Equation 6.4.4 for two different values of $\bar{d}$. In both we see that as N increases $\bar{q}_{G_N}$ decreases and $\bar{q}_{E_N}$ increases, $\bar{q}_{G_1}$ is always the upper curve, $\bar{q}_{E_1}$ is always the lower curve and $\bar{q}_{E_\infty} = \bar{q}_{G_\infty}$. The numerical values used in these curves are given in Appendix F.

Next we derive explicit expressions for σ_q^2 for both models in the symmetric nonrandom case. For the gating model, we substitute Equation 6.4.4a for $\bar{q}$

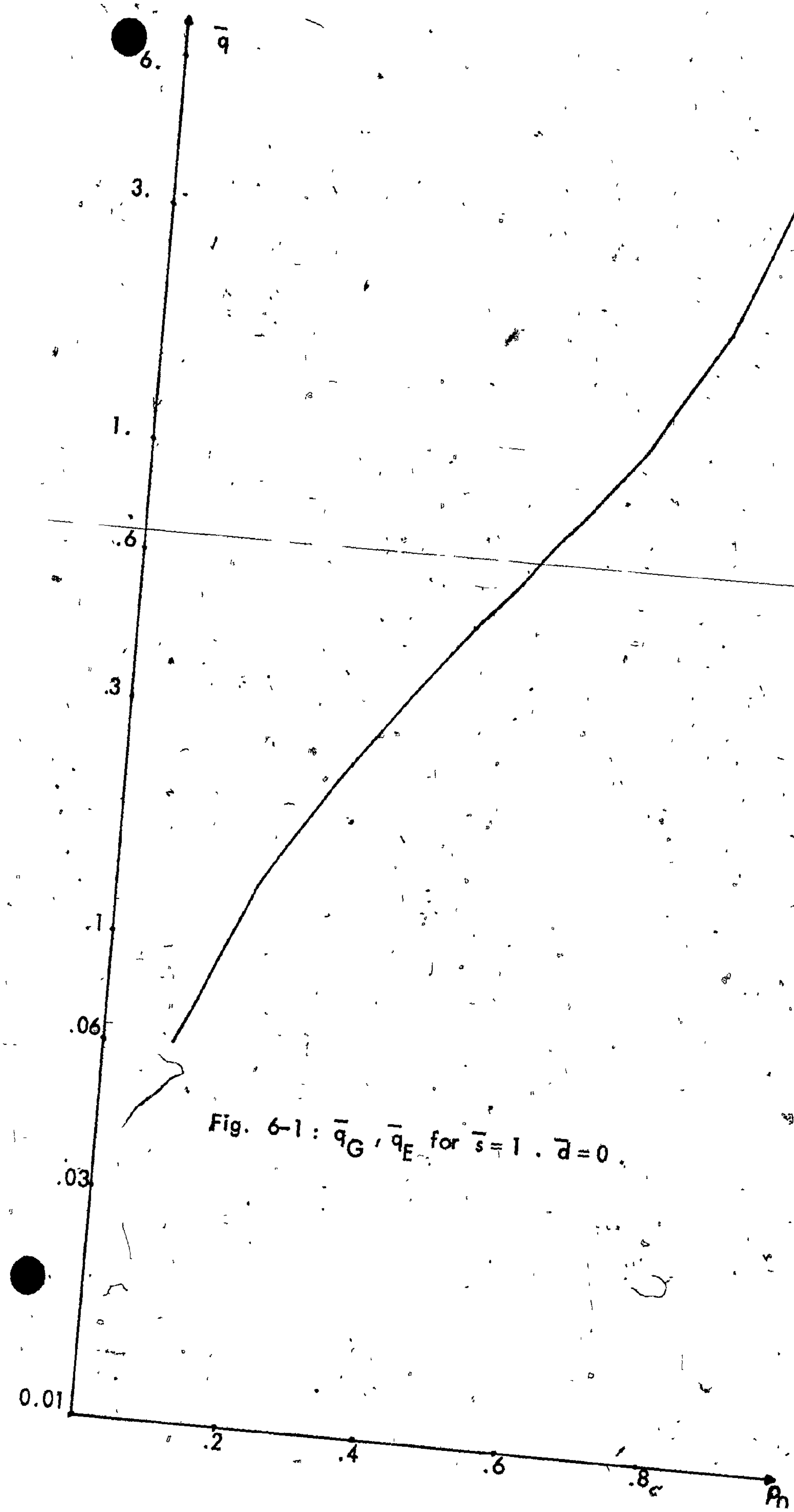

Fig. 6-1 : $\bar{q}_G$, $\bar{q}_E$ for $\bar{s} = 1$. $\bar{d} = 0$.

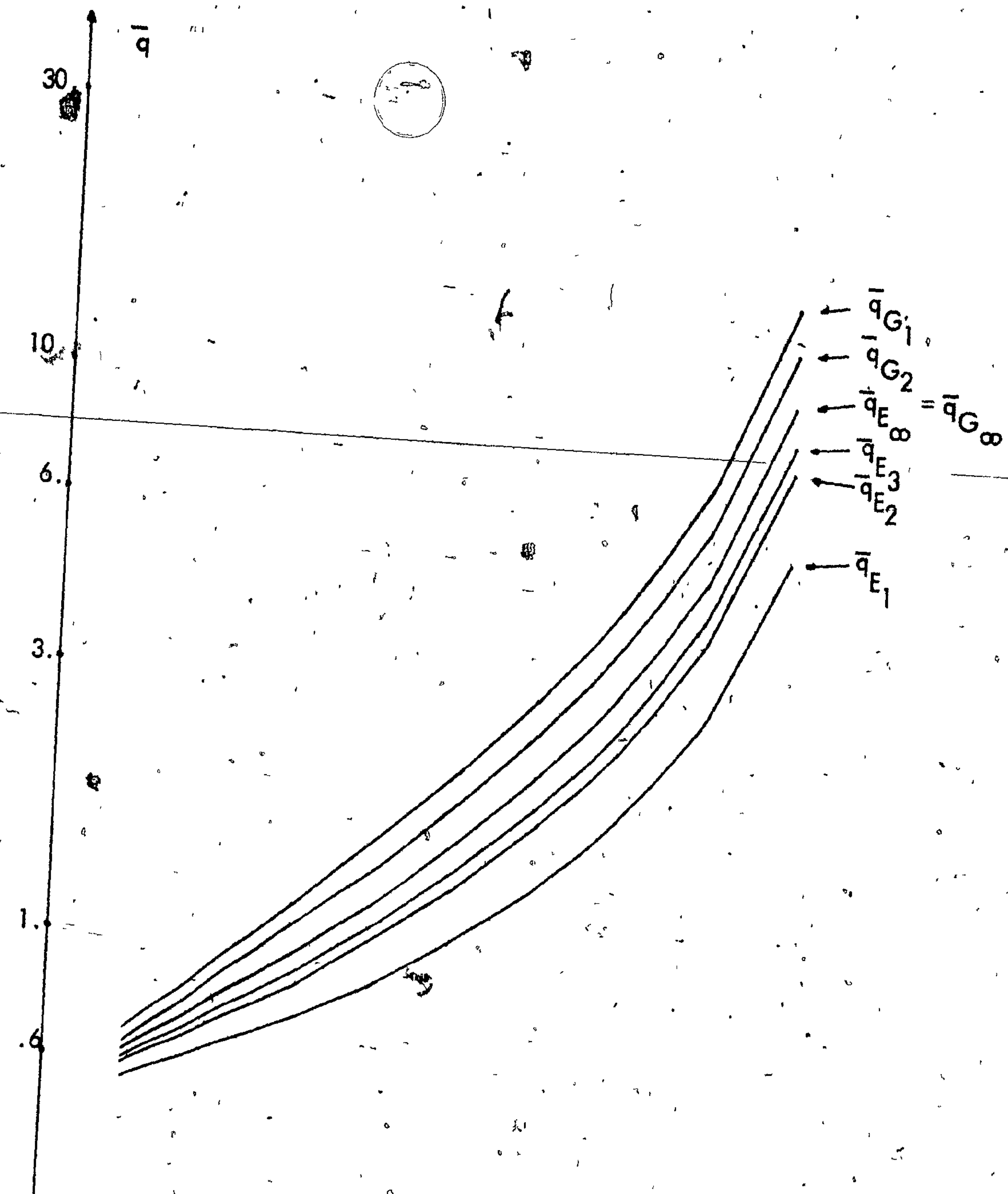

Fig. 6-2 : $\bar{q}_{G_N}$, $\bar{q}_{E_N}$ for $\bar{s} = 1$. $\bar{d} = 1$. $N = 1, 2, 3, \infty$.

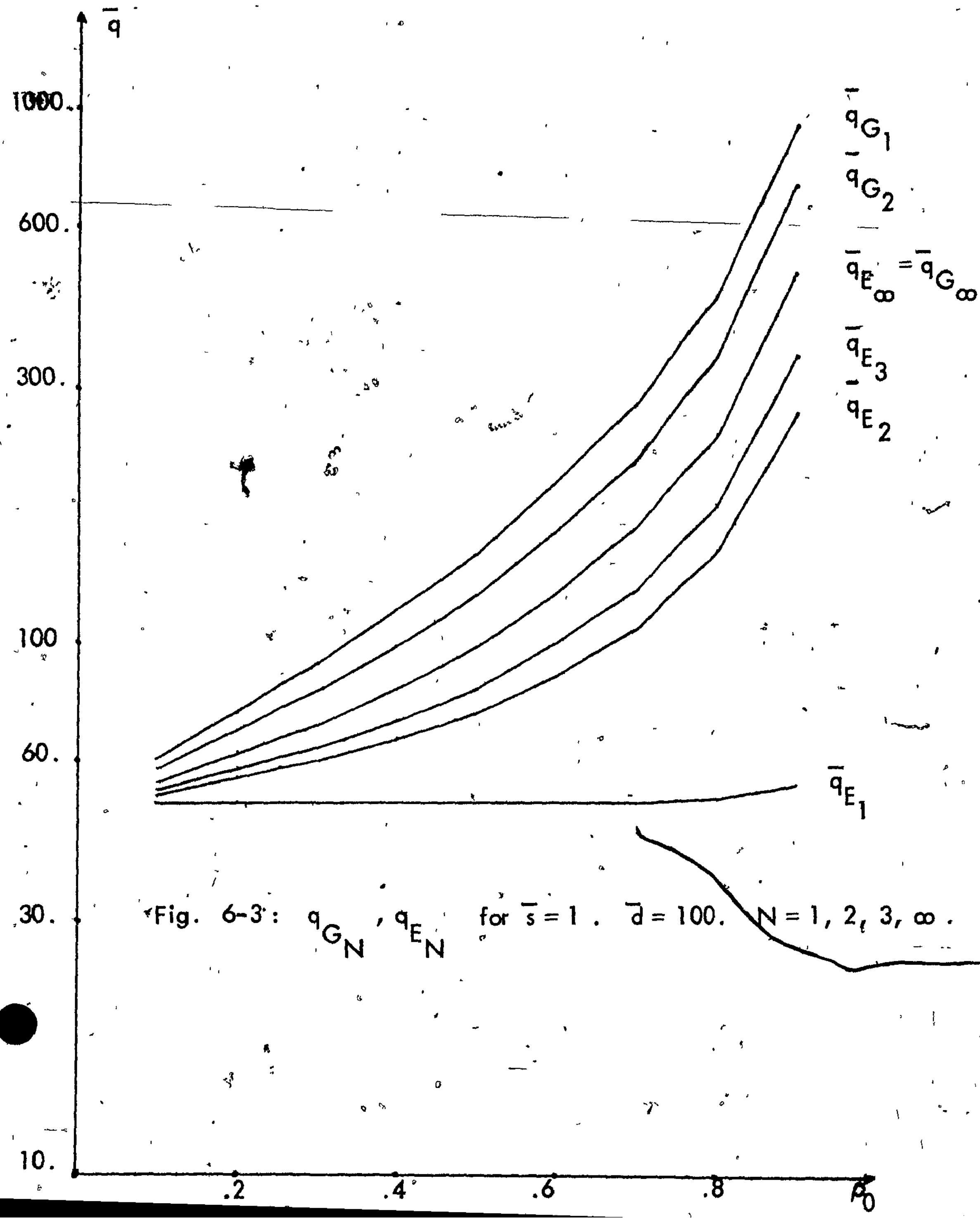

Fig. 6-3: q_{G_N}, q_{E_N} for $\bar{s} = 1$. $\bar{d} = 100$. $N = 1, 2, 3, \infty$.

and Equation 3.2.45 for $\overline{c^3}/\bar{c}$ in Equation 6.2.10b. For the exhaustive model, we substitute Equation 6.4.4b for $\bar{q}$ and Equation 3.3.34 for $\overline{v^3}/\bar{v}$ in Equation 6.3.21b. We obtain :

<u>For N = 1</u>

$$\sigma^2_{q_{G_1}} = \frac{1}{12} \cdot \bar{d}^2 + \frac{\rho_0}{(1+\rho_0)^2(1+\rho_0)} \cdot \bar{d}\,\bar{s} + \frac{\rho_0\left(1-\frac{\rho_0}{4}\right)}{3(1-\rho_0)^2} \cdot \bar{s}^2$$

$$\sigma^2_{q_{E_1}} = \frac{1}{12} \cdot \bar{d}^2 + \frac{\rho_0\left(1-\frac{\rho_0}{4}\right)}{3(1-\rho_0)^2} \cdot \bar{s}^2$$

<u>For N = 2</u>

$$\sigma^2_{q_{G_2}} = \frac{\left(1-\frac{\rho_0}{2}\right)^2}{12(1-\rho_0)^2} \cdot \bar{d}^2 + \frac{3\rho_0\left(1-\frac{1}{6}\rho_0\right)}{4(1-\rho_0)^2\left(1+\frac{\rho_0}{2}\right)} \cdot \bar{d} \cdot \bar{s} + \frac{\rho_0\left(1+\rho_0+\frac{1}{4}\rho_0^3-\frac{3}{16}\rho_0^4-\frac{1}{32}\rho_0^5+\frac{1}{128}\rho_0^6\right)}{3(1-\rho_0)^2\left(1+\frac{\rho_0}{2}\right)^2\left(1+\frac{1}{4}\rho_0^2\left(1-\frac{\rho_0}{2}\right)\right)} \cdot \bar{s}^2$$

$$\sigma^2_{q_{E_2}} = \frac{\left(1-\frac{\rho_0}{2}\right)^2}{12(1-\rho_0)^2} \cdot \bar{d}^2 + \frac{\rho_0}{4(1-\rho_0)^2} \cdot \bar{d} \cdot \bar{s} + \frac{\rho_0\left(1-\frac{1}{2}\rho_0+\frac{1}{2}\rho_0^2-\frac{1}{16}\rho_0^3\right)}{3(1-\rho_0)^2\left(1-\frac{1}{2}\rho_0+\frac{1}{4}\rho_0^2\right)} \cdot \bar{s}^2$$

<u>For N = 3,</u> we have σ_q^2 for the exhaustive model only.

$$\sigma^2_{q_{E_3}} = \frac{(1-\frac{\rho_0}{3})^2}{12(1-\rho_0)^2} \cdot \bar{d}^2 + \frac{\rho_0}{3(1-\rho_0)^2} \cdot \bar{d} \cdot \bar{s} + \frac{\rho_0 (1 - \frac{5}{4}\rho_0 + \frac{8}{9}\rho_0^2 - \frac{53}{108}\rho_0^3 + \frac{5}{36}\rho_0^4 - \frac{1}{108}\rho_0^5)}{3(1-\rho_0)^2 (1-\frac{\rho_0}{3})(1-\rho_0+\frac{4}{9}\rho_0^2 - \frac{1}{9}\rho_0^3)} \cdot \bar{s}^2$$

<u>For N = ∞</u>

$$\sigma^2_{q_{G_\infty}} = \sigma^2_{q_{E_\infty}} = \frac{1}{12(1-\rho_0)^2} \cdot \bar{d}^2 + \frac{\rho_0}{2(1-\rho_0)^2} \cdot \bar{d} \cdot \bar{s} + \frac{\rho_0 (1+\frac{\rho_0}{4})}{3(1-\rho_0)^2} \cdot \bar{s}^2 \qquad 6.4.7$$

We sketch $\sigma^2_{q_{G_N}}$ and $\sigma^2_{q_{E_N}}$ for $\bar{s} = 1$, for N = 1, 2, 3, ∞ and and $\bar{d} = 0$. (Figure 6-4), $\bar{d} = 1$. (Figure 6-5 and Figure 6-6), and $\bar{d} = 100$. (Figure 6-7). From these curves, and the appropriate numerical tables in Appendix F, we see that for those three values of $\bar{d}$ and all ρ_0,

$$\sigma^2_{q_{E_\infty}} > \sigma^2_{q_{E_3}} > \sigma^2_{q_{E_2}} > \sigma^2_{q_{E_1}} \qquad 6.4.8$$

$\sigma^2_{q_{E_\infty}}$ is not only the upper bound of $\sigma^2_{q_{E_N}}$ but might also serve as a close estimate of it for N > 3. For the gating model, relation 6.4.8 does not hold, as can be seen in Figures 6-4, 6-5. However, $\sigma^2_{q_{G_\infty}} = \sigma^2_{q_{E_\infty}}$ may be of use as an estimation of $\sigma^2_{q_{G_N}}$ for N > 2. Yet another feature obtained from

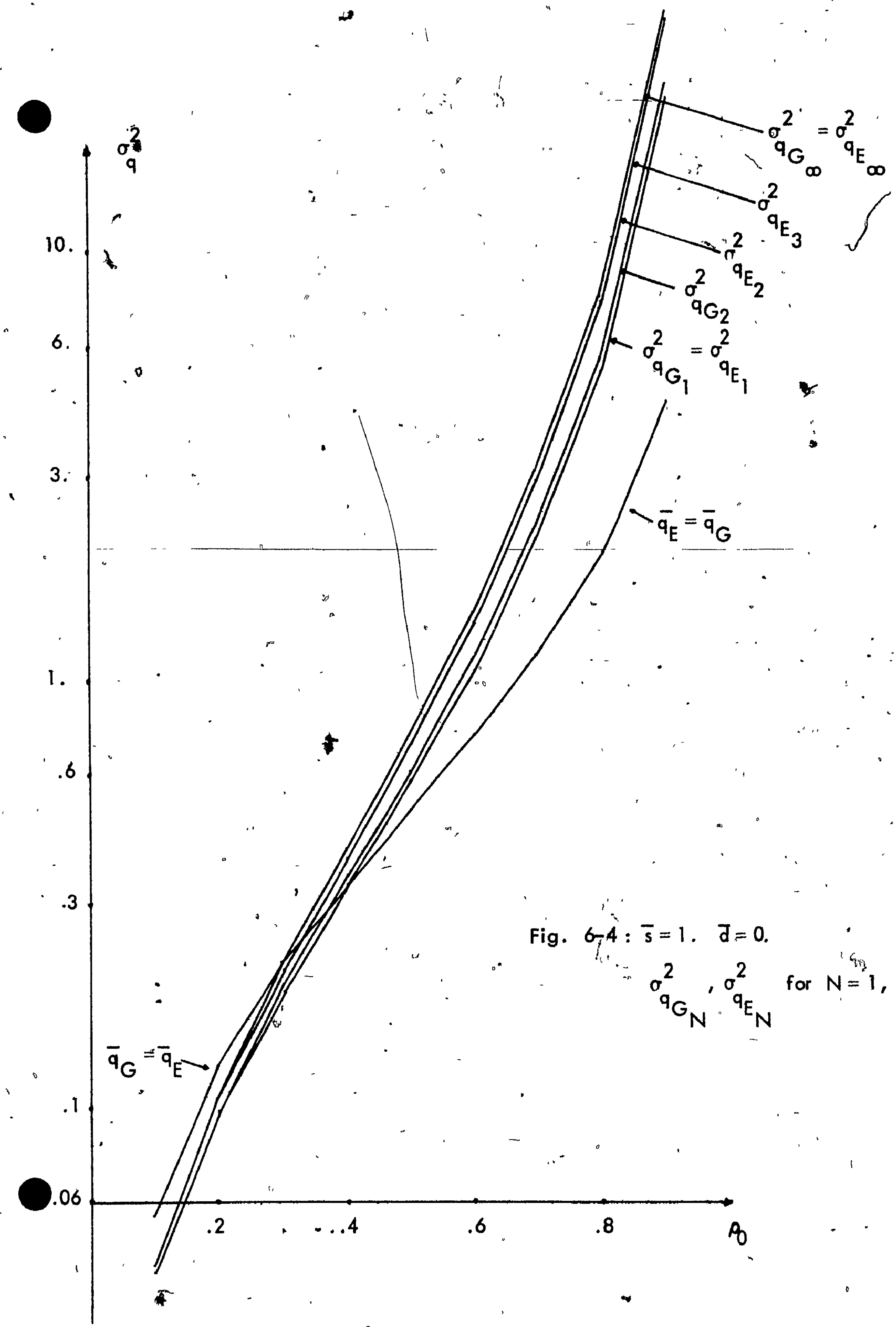

Fig. 6-4 : $\bar{s} = 1.$ $\bar{d} = 0.$

$\sigma^2_{q_{G_N}}$, $\sigma^2_{q_{E_N}}$ for N = 1, 2, 3, ∞ .

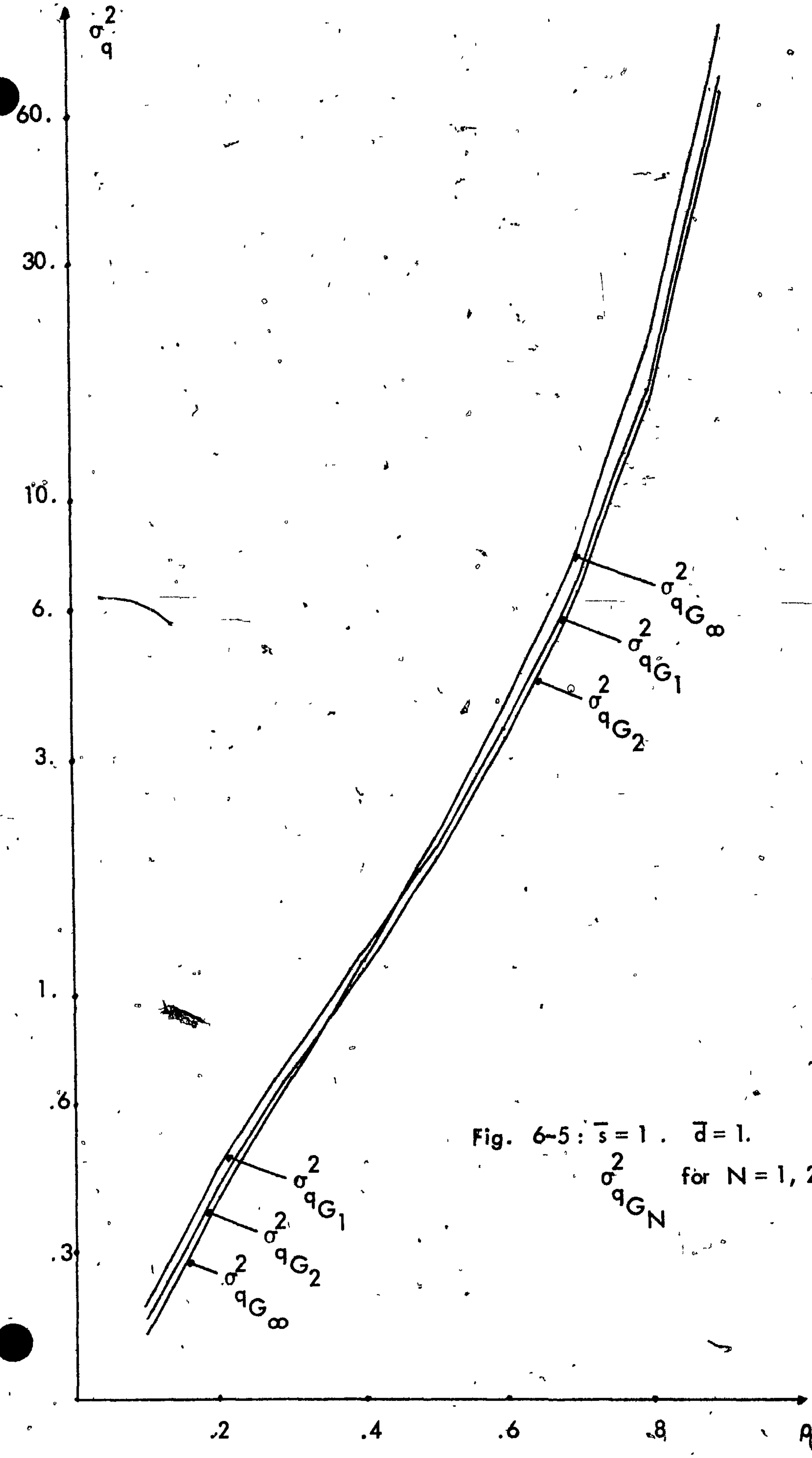

Fig. 6-5 : $\bar{s} = 1$. $\bar{d} = 1$.
$\sigma^2_{qG_N}$ for $N = 1, 2, \infty$.

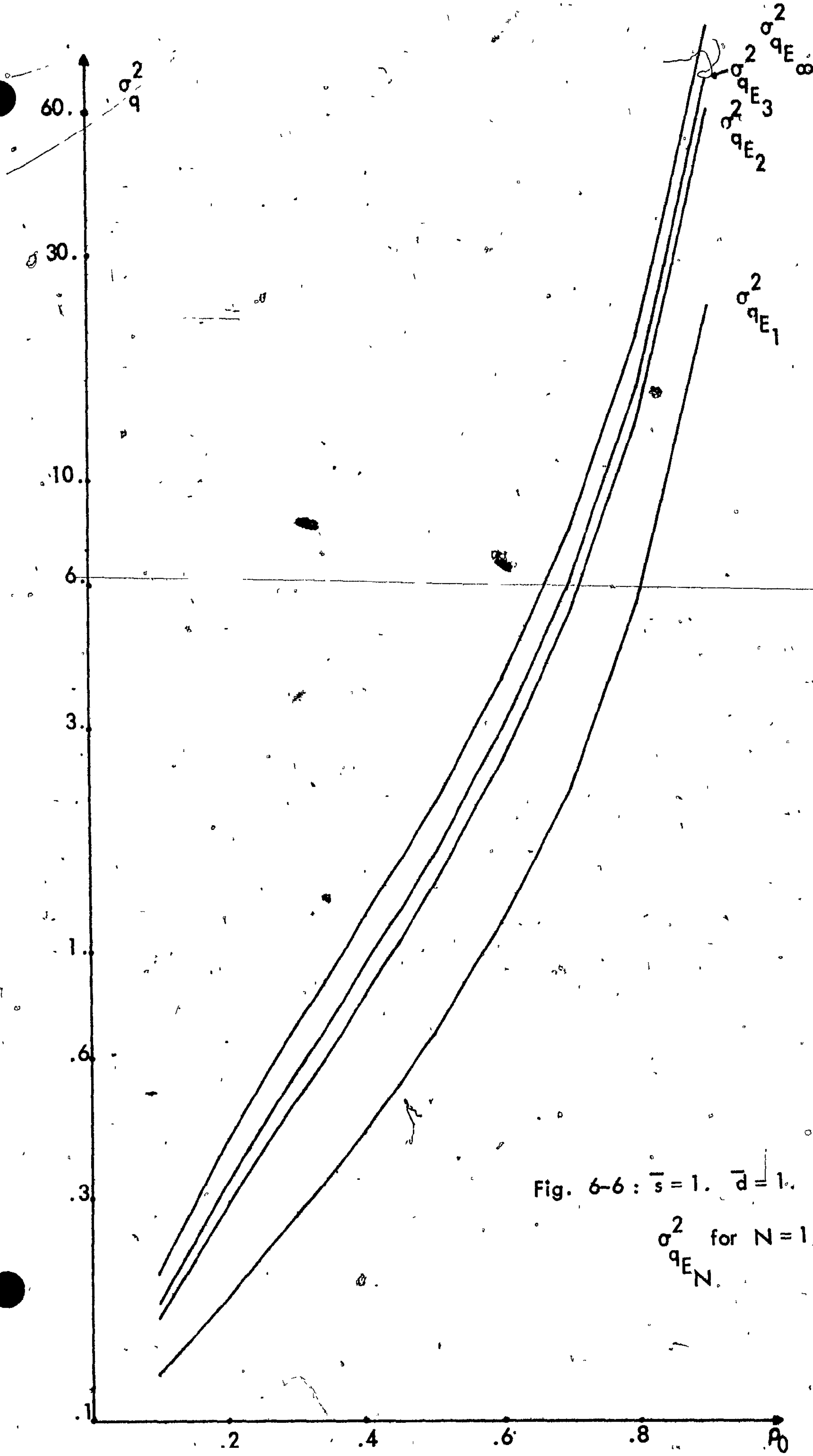

Fig. 6-6 : $\bar{s} = 1.$ $\bar{d} = 1.$

$\sigma^2_{q_{E_N}}$ for $N = 1, 2, 3, \infty$.

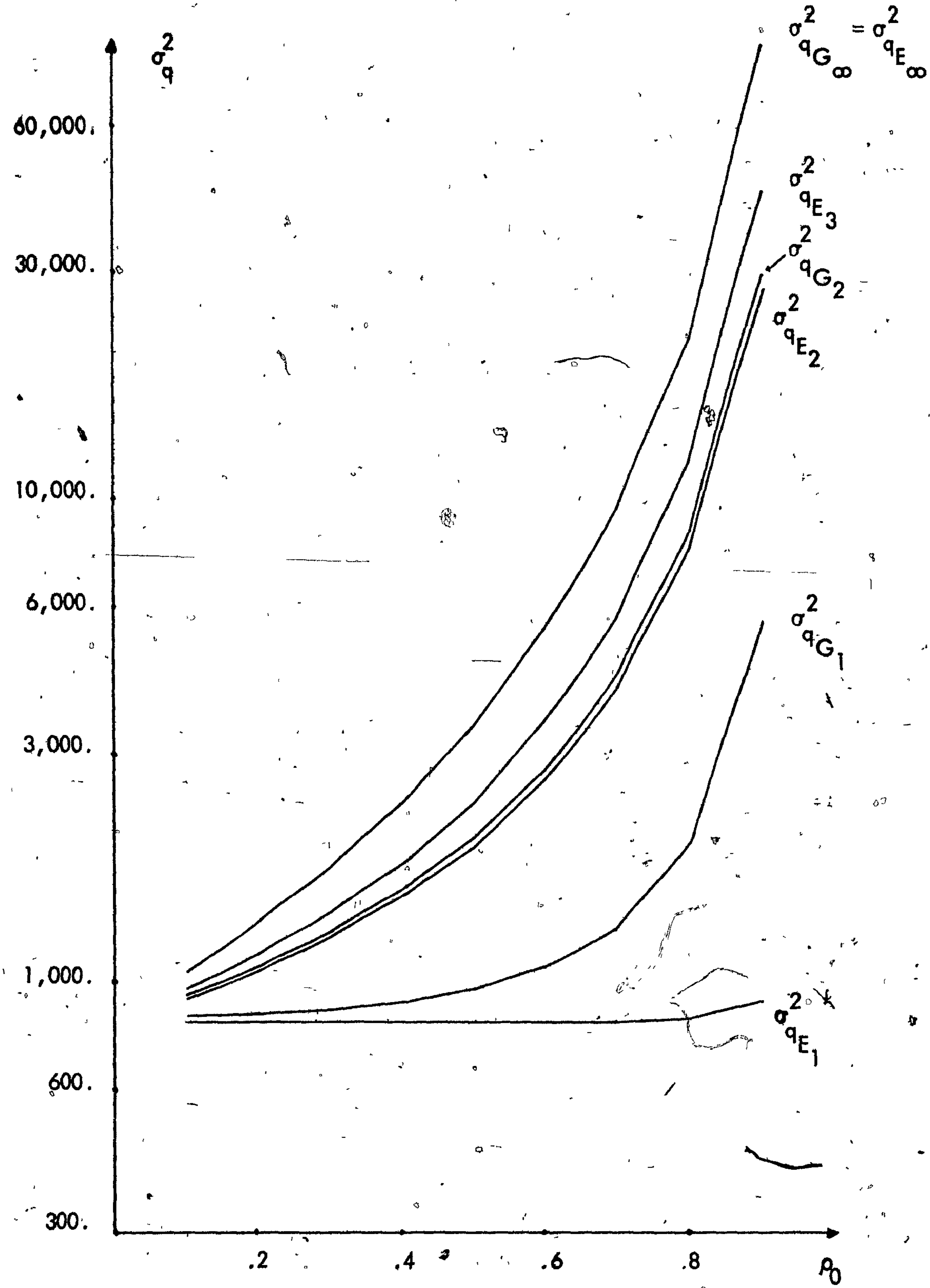

Fig. 6-7 : $\bar{s} = 1$. $\bar{d} = 100$.

$\sigma^2_{q_{G_N}}$, $\sigma^2_{q_{E_N}}$ for N = 1, 2, 3, ∞

Figures 6-5, 6-6, 6-7, is that for $\bar{d} = 1.$ or $\bar{d} = 100.$ (but not $\bar{d} = 0$)

$\sigma^2_{q_{G_N}} > \sigma^2_{q_{E_N}}$ for $N = 1, 2.$

In conclusion we realize that, for the symmetric nonrandom case, the exhaustive model is slightly superior to the gating model (having smaller $\bar{q}$ and σ^2_q). However, as $N >> 1$ we have for the symmetric nonrandom case of both models

$$\bar{q}_G = \bar{q}_E = \frac{\bar{d} + \rho_0 \bar{s}}{2(1 - \rho_0)}$$

$$\sigma^2_{q_G} = \sigma^2_{q_E} = \frac{\bar{d}^2 + 6\rho_0 \bar{d}\bar{s} + \rho_0 (4 + \rho_0) \bar{s}^2}{12(1 - \rho_0)^2}$$

when $N >> 1.$ 6.4.9

Using the above quantities in Chebyshev's inequality, in a similar manner to Equation 4.1.5, we have, for any $a > 0$, for both models

$$\text{Prob}\,(q \geq \bar{q} + a\,\sigma_q) \leq \frac{1}{a^2} \qquad 6.4.10$$

Equations 6.4.9, 6.4.10 enable us to bound the actual waiting time, where such a bound is practically more meaningful than the average waiting time. Thus in the symmetric nonrandom case of multiterminal polling systems (either the gating or the exhaustive model), with probability $(1 - {}^1/a^2)$ the actual waiting time q satisfies

$$q \leq \frac{\bar{d} + \rho_0 \bar{s}}{2(1-\rho_0)} + a \left(\frac{\bar{d}^2 + 6\rho_0 \bar{d}\,\bar{s} + \rho_0 (4+\rho_0) \bar{s}^2}{12(1-\rho_0)^2} \right)^{1/2} \qquad 6.4.11$$

6.5 Average Waiting Time in a General Case of Both Models

In the general case each T_i has a different λ_i, $P_{s_i}(\cdot)$, and $P_{w_i}(t)$. In the following we derive explicit expressions for $\bar{q}_{G_i}$, $\bar{q}_{E_i}$ (average waiting time in T_i) where $N = 2$ and $\bar{q}_{E_i}$ for $N = 3$ as well.

In Equation 3.1.5 we define

$$\rho_0 = \sum_{i=1}^{N} \rho_i$$

$$\lambda_0 = \sum_{i=1}^{N} \lambda_i \qquad 6.5.1$$

For $N = 2$, define for $i = 1, 2$ the set (i, j) equal to $(1, 2)$ and $(2, 1)$, respectively. For the gating model, substituting Equation 3.2.10 for $\bar{c}$ and Equation 3.2.17 for $\sigma^2_{c_i}$ into Equation 6.2.10a yields

$$\bar{q}_{G_i} = \frac{(1+\rho_i)}{2(1-\rho_0)}\bar{d} + (1+\rho_i)\cdot \frac{[(1-\rho_1\rho_2)(1+2\rho_i) - 2\rho_i^3][\lambda_i \bar{s}_i^2 + (1-\rho_0)\frac{\sigma^2_{w_i}}{\bar{d}}] + (1+\rho_1\rho_2)[\lambda_j \bar{s}_j^2 + (1-\rho_0)\frac{\sigma^2_{w_j}}{\bar{d}}]}{2(1-\rho_0)(1+\rho_0+2\rho_1\rho_2)(1-\rho_1\rho_2)}$$

$$i = 1, 2 \qquad 6.5.2$$

For $N = 2$ in the exhaustive model, substituting Equation 3.3.13 for $\bar{v}_i$ and Equation 3.3.19 for $\sigma^2_{v_i}$ into Equation 6.3.21a yields

$$\bar{q}_{E_i} \doteq \frac{(1-\rho_i)}{2(1-\rho_0)}\bar{d} + \frac{1}{2(1-\rho_i)}\left(\lambda_i \overline{s_i^2} + (1-\rho_0)\frac{\sigma^2_{w_i}}{\bar{d}}\right) + \frac{\rho_i^2\left(\lambda_i \overline{s_i^2} + (1-\rho_0)\frac{\sigma^2_{w_i}}{\bar{d}}\right) + (1-\rho_i)^2\left(\lambda_i \overline{s_i^2} + (1-\rho_0)\frac{\sigma^2_{w_i}}{\bar{d}}\right)}{2(1-\rho_i)(1-\rho_0)(1-\rho_0+2\rho_1\rho_2)} \qquad 6.5.3$$

or

$$\bar{q}_{E_i} = \frac{(1-\rho_i)}{2(1-\rho_0)}\bar{d} + \frac{[(1-\rho_i)(1-2\rho_i)+2\rho_i^2]\left[\lambda_i \overline{s_i^2} + (1-\rho_0)\frac{\sigma^2_{w_i}}{\bar{d}}\right] + (1-\rho_i)\left(\lambda_i \overline{s_i^2} + (1-\rho_0)\frac{\sigma^2_{w_i}}{\bar{d}}\right)}{2(1-\rho_0)(1-\rho_0+2\rho_1\rho_2)}$$

$i = 1, 2$ 6.5.4

A two-terminal polling system (mainly with the exhaustive service procedure) was analysed in various papers. Equation 6.5.4 was obtained independently and with different technique by Sykes (1970) and Eisenberg (1972). We are not aware of any study which yields Equation 6.5.2. In the special case where all walking times are zero, we have from Equation 6.5.2 and Equation 6.5.4

for $i = 1, 2$ and $w_1 = w_2 = 0$

$$\bar{q}_{G_i} = (1+\rho_i)\,\frac{[(1-\rho_1\,\rho_2)(1+2\,\rho_i)-2\,\rho_i^3]\,\lambda_i\,\bar{s_i^2}+(1+\rho_1\,\rho_2)\,\lambda_i\,\bar{s}_i^2}{2(1-\rho_0)(1+\rho_0+2\,\rho_1\,\rho_2)(1-\rho_1\,\rho_2^2)} \qquad 6.5.5$$

$$\bar{q}_{E_i} = \frac{[(1-\rho_i)(1-2\,\rho_i)+2\,\rho_i^2]\,\lambda_i\,\bar{s_i^2}+(1-\rho_i)\,\lambda_i\,\bar{s}_i^2}{2\,(1-\rho_0)(1-\rho_0+2\,\rho_1\,\rho_2)} \qquad 6.5.6$$

Equation 6.5.6 is in agreement with the results obtained by Avi-Itzhak et al (1965), Takacs (1968), Jaiswal (1968), and Cooper (1970). Equation 6.5.5 does not exist in the literature.

The general case of the exhaustive model with $N = 3$ is not studied in the literature, because of the mathematical complexity (Eisenberg (1972)). However, a study of three terminal polling system is essential in obtaining basic properties of the general case of multiterminal polling system.

For $N = 3$, we define, for $i = 1, 2, 3$, the set (i, j, l) as $(1, 2, 3)$, $(2, 3, 1)$, and $(3, 1, 2)$ respectively. Substituting Equation 3.3.13 for $\bar{v}_i$ and Equation 3.3.21 for $\sigma^2_{v_i}$ into Equation 6.3.21a, we have :

$$\bar{q}_{E_i} = \frac{(1-\rho_i)}{2(1-\rho_0)}\,\bar{d} + \frac{\lambda_i\,\bar{s_i^2}+(1-\rho_0)\,\frac{\bar{\sigma}^2_{w_l}}{\bar{d}}}{2\,(1-\rho_i)} + \frac{(1-\rho_0)\cdot T\,(i,\,j,\,l)}{2\,(1-\rho_i)\,\bar{d}\,A} \qquad 6.5.7$$

$i = 1, 2, 3$

where A and $T\,(i, j, l)$ are given in determinant form in Equation 3.3.21.

Below we study a special variant of the symmetric case ; we assume for $i = 1, \ldots, N$.

$$P_{s_i}(t) = P_s(t)$$

$$P_{w_i}(t) = P_w(t - \bar{w}_i) \qquad 6.5.8$$

i.e., the symmetric case where all λ_i are different. For this case, we have from Equations 6.5.2, 6.5.4, and 6.5.7 :

<u>For N = 2 :</u>

$$\bar{q}_{G_i} = \frac{(1+\rho_i)}{2(1-\rho_0)}\bar{d} + (1+\rho_i) \cdot \frac{[(1-\rho_1\rho_2)(1+2\rho_i) - 2\rho_i^3][\lambda_i \bar{s}^2 + (1-\rho_0)\frac{\sigma_w^2}{\bar{d}}] + (1+\rho_1\rho_2)(\lambda_i \bar{s}^2 + (1-\rho_0)\frac{\sigma_w^2}{\bar{d}})}{2(1-\rho_0)(1+\rho_0+2\rho_1\rho_2)(1-\rho_1\rho_2)} \qquad 6.5.9$$

$$\bar{q}_{E_i} = \frac{(1-\rho_i)}{2(1-\rho_0)}\bar{d} + \frac{[(1-\rho_i)(1-2\rho_i) + 2\rho_i^2][\lambda_i \bar{s}^2 + (1-\rho_0)\frac{\sigma_w^2}{\bar{d}}] + (1-\rho_i)(\lambda_i \bar{s}^2 + (1-\rho_0)\frac{\sigma_w^2}{\bar{d}})}{2(1-\rho_0)(1-\rho_0+2\rho_1\rho_2)} \qquad 6.5.10$$

$i = 1, 2$

For N = 3 :

$$\bar{q}_{E_i} = \frac{(1 - \rho_i)}{2(1 - \rho_0)} \bar{d} + \frac{\lambda_i \bar{s}^2 + (1 - \rho_0) \frac{\sigma_w^2}{\bar{d}}}{2(1 - \rho_i)} + \frac{(1 - \rho_0) T(i, j, l)}{2(1 - \rho_i) \bar{d} A} \qquad 6.5.11$$

$i = 1, 2, 3$

For this case (the symmetric case with different λ_i, defined by 6.1.8) $\lambda_i / \lambda_0 = \rho_i / \rho_0$. Hence, from Equation 6.1.3, the average system waiting time is

$$\bar{q}_0 = \sum_{i=1}^{N} \frac{\rho_i}{\rho_0} \bar{q}_i \qquad 6.5.12$$

Using Equations 6.5.9, 6.5.10, and 6.5.11 in 6.5.12 we obtain

for N = 2

$$\bar{q}_{G_0} = \frac{\sum_{i=1}^{2} \rho_i (1 + \rho_i)}{2 \rho_0 (1 - \rho_0)} \bar{d} + \frac{\lambda_0 \bar{s}^2}{2(1 - \rho_0)} + \frac{\sigma_d^2}{2\bar{d}}$$

$$\bar{q}_{E_0} = \frac{\sum_{i=1}^{2} \rho_i (1 - \rho_i)}{2 \rho_0 (1 - \rho_0)} \bar{d} + \frac{\lambda_0 \bar{s}^2}{2(1 - \rho_0)} + \frac{\sigma_d^2}{2\bar{d}}$$

and for N = 3

$$\bar{q}_{E_0} = \frac{\sum_{i=1}^{3} \rho_i (1 - \rho_i)}{2 \rho_0 (1 - \rho_0)} \bar{d} + \frac{\lambda_0 \bar{s}^2}{2(1 - \rho_0)} + \frac{\sigma_d^2}{2\bar{d}} \qquad 6.5.13$$

Next, we show that the surprisingly simple expression obtained in Equation 6.5.13 for the average system waiting times are applicable for all $N \geq 1$, for both models, in a general case.

For simplicity, let's confine the study to a <u>discrete case</u> where each T_i has different $\bar{w}_i$, λ_i but the same $\bar{s}$. Define the <u>discrete case</u>

for $i = 1, .., N$

$$P_{s_i}(t) = \delta(t - \bar{s})$$

$$P_{w_i}(t) = \delta(t - \bar{w}_i) \qquad 6.5.14$$

$$\lambda_i = \lambda_i$$

Using Equation 6.5.14 in Equations 6.5.9, 6.5.10, and 6.5.11, we have for the discrete case :

for $N = 2$

$$\bar{q}_{G_i} = \frac{1 + \rho_i}{2(1 - \rho_0)} \cdot \bar{d} + (1 + \rho_i) \frac{[\rho_0 + 2\rho_1 \rho_2 (1 - \rho_1 \rho_2) + \rho_1 \rho_2 (\rho_i (1 - 2\rho_i) - \rho_i)]}{2(1 - \rho_0)(1 + \rho_0 + 2\rho_1 \rho_2)(1 - \rho_1 \rho_2)} \bar{s} \qquad 6.5.15$$

$$\bar{q}_{E_i} = \frac{1 - \rho_i}{2(1 - \rho_0)} \bar{d} + \frac{[\rho_0 (1 - \rho_i) - 2\rho_1 \rho_2 (1 - \rho_0)]}{2(1 - \rho_0)(1 - \rho_0 + 2\rho_1 \rho_2)} \bar{s} \qquad 6.5.16$$

$i = 1, 2$

for $N = 3$,

$$\bar{q}_{E_i} = \frac{1 - \rho_i}{2(1 - \rho_0)} \bar{d} + \frac{\rho_i}{2(1 - \rho_i)} \bar{s} + \frac{(1 - \rho_0)\, T(i, j, 1)}{2(1 - \rho_i)\, \bar{d}\, A} \qquad 6.5.17$$

$i = 1, 2, 3$

where A and $T(i, j, l)$ are given in determinant form in Equation 3.3.21.

From the general structure of Equations 3.2.15 and 3.3.17, we observe that the explicit expressions for the average waiting time in both models, Equations 6.2.10a and 6.3.21a, in the discrete case, have, for all $N \geq 1$, the form :

$$\bar{q}_{G_i} = \frac{(1 + \rho_i)}{2(1 - \rho_0)} \bar{d} + \sum_{j=1}^{N} f_{ij} (\rho_1, \ldots, \rho_N) \rho_j \bar{s}$$

$$\bar{q}_{E_i} = \frac{(1 - \rho_i)}{2(1 - \rho_0)} \bar{d} + \sum_{j=1}^{N} g_{ij} (\rho_1, \ldots, \rho_N) \rho_j \bar{s} \qquad 6.5.18$$

where $f_{ij}(\ldots)$, $g_{ij}(\ldots)$ are N dimensional functions (which we do not have explicitly for all $N \geq 1$).

The average system waiting time, defined by 6.5.12, is

$$\bar{q}_{G_0} = \frac{\sum_{i=1}^{N} \rho_i (1 + \rho_i)}{2 \rho_0 (1 - \rho_0)} \bar{d} + \frac{\bar{s}}{\rho_0} \sum_{i=1}^{N} \sum_{j=1}^{N} \rho_i \rho_j f_{ij} (\rho_1, \ldots, \rho_N)$$

$$\bar{q}_{E_0} = \frac{\sum_{i=1}^{N} \rho_i (1 - \rho_i)}{2 \rho_0 (1 - \rho_0)} \bar{d} + \frac{\bar{s}}{\rho_0} \sum_{i=1}^{N} \sum_{j=1}^{N} \rho_i \rho_j g_{ij} (\rho_1, \ldots, \rho_N) \qquad 6.5.19$$

Temporarily let's assume we have in the discrete case of both models, $\bar{d} = 0$. For this case the system is identical to M/G/1 queue where the single terminal has the parameter λ_0 and all customers require identical service $\bar{s}$. The service procedure, for $N > 1$, is not on a "first come first served" basis; each new customer is directed to T_i with probability $\frac{\lambda_i}{\lambda_0}$ and the service is determined by either the gating or the exhaustive procedure. The average waiting time of a customer who enters the system (and we do not know to which terminal he is going to be directed to) is independent of the service procedure, as long as service is given when customers exist. Hence the average system waiting time, in the discrete case with $\bar{d} = 0$, is given by Equation 6.1.5

$$\bar{q}_{G_0} = \frac{\bar{s}}{\rho_0} \sum_{i=1}^{N} \sum_{j=1}^{N} \rho_i \rho_j f_{ij} (\rho_1, \ldots, \rho_N) = \frac{\rho_0 \bar{s}}{2(1-\rho_0)}$$

$$\bar{q}_{E_0} = \frac{\bar{s}}{\rho_0} \sum_{i=1}^{N} \sum_{j=1}^{N} \rho_i \rho_j g_{ij} (\rho_1, \ldots, \rho_N) = \frac{\rho_0 \bar{s}}{2(1-\rho_0)} \qquad 6.5.20$$

where $\bar{d} = 0$.

Using Equation 6.5.20 in Equation 6.5.19 we finally obtain

$$\bar{q}_{G_0} = \frac{\frac{\bar{d}}{\rho_0} \sum_{i=1}^{N} \rho_i (1+\rho_i) + \rho_0 \bar{s}}{2(1-\rho_0)}$$

$$\bar{q}_{E_0} = \frac{\frac{\bar{d}}{\rho_0} \sum_{i=1}^{N} \rho_i (1-\rho_i) + \rho_0 \bar{s}}{2(1-\rho_0)} \qquad 6.5.21$$

From the above equation we conclude, for the discrete case, that for two identical systems but each with different service procedure (either the exhaustive or the gating) we have

$$\bar{q}_{E_0} \leq \bar{q}_{G_0} \qquad 6.5.22$$

where equality holds only when $\bar{d} = 0$.

When $\bar{d} \neq 0$. and $\rho_0, \bar{s}$ are fixed, $\bar{q}_{G_0}$ is minimum when $\rho_i = \frac{\rho_0}{N}$ $i = 1, \ldots, N$ i.e., the symmetric case. $\bar{q}_{E_0}$ is minimum when all traffic is directed to one terminal, i.e., $\rho_1 = \rho_0$ and $\rho_i = 0$ for $i = 2, \ldots, N$. Equation 6.5.21 expresses the average system waiting time in the discrete case of both models for all $N \geq 1$. This equation is new and surprisingly simple. It emphasises the fact that from system waiting time point of view, which can be regarded as that of Central Data Processor, the order in which the Terminals are served (or located on the loop) is not important and that the exhaustive model is superior to the gating model. In order to study the average terminal waiting in both models we refer back to the special cases of $N = 2, 3$.

For the discrete case with $N = 2$ we have from Equation 6.5.15 and Equation 6.5.16 for $\rho_i < \rho_j$:

$$\bar{q}_{G_i} < \bar{q}_{G_j}$$

$$\bar{q}_{E_i} > \bar{q}_{E_j} \qquad 6.5.23$$

The inequalities in 6.5.23 are explained as follows : for $\bar{d} = 0.$, at the time a new customer arrives, the server is more likely to be at T_{I} than at T_i (since $\rho_{\mathrm{I}} > \rho_i$) . Therefore under the exhaustive service procedure, a customer who arrives at T_{I} is more likely to be served before a customer who arrives at T_i at the same time (since the server may leave T_{I} only when it is empty). But, under the gating service procedure, the customer who arrives at T_{I} (when server is there) is served at the next cycle and therefore after the customer who arrives at T_i . When $\bar{d} \neq 0.$, the contribution of the new terms (in Equations 6.5.15, 6.5.16) strengthen the inequalities of 6.5.23 .

Inequalities 6.5.22 and 6.5.23 yield :

$$\bar{q}_{G_{\mathrm{I}}} > \bar{q}_{E_{\mathrm{I}}}$$

and for $\bar{d} = 0.$: 6.5.24

$$\bar{q}_{G_i} < \bar{q}_{E_i}$$

Hence, from the "big" terminal point of view, the exhaustive procedure is preferable. However, for $\bar{d} = 0.$, from the small terminal point of view the gating model is preferable. As $\bar{d}$ increases, the average waiting time for the gating model increases more rapidly than that of the exhaustive model. Numerical examples of 6.5.15, 6.5.16, and 6.5.21 for the case $\rho_i = \frac{1}{4}\rho_0$, $\rho_{\mathrm{I}} = \frac{3}{4}\rho_0$, and $\bar{s} = 1.$ are given by Figure 6-8 where $\bar{d} = 0.$, Figure 6-9 where $\bar{d} = 1.$, and by Figure 6-10 where $\bar{d} = 100.$. The numerical values used in these curves

Fig. 6-8 : $\bar{d} = 0$. $\bar{s} = 1$. $N = 2$.

$\rho_i = .25\ \rho_0$; $\rho_i = .75\ \rho_0$.

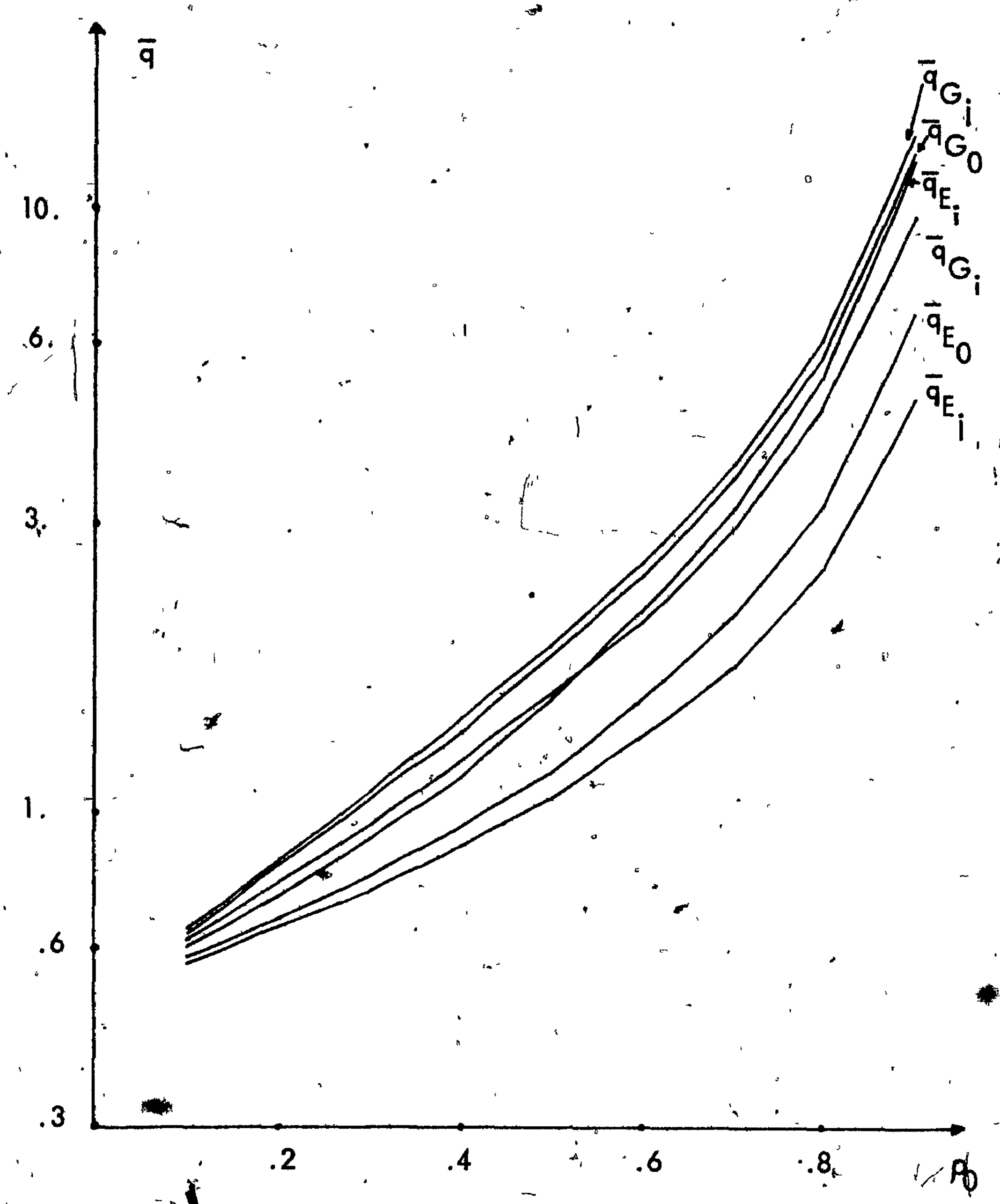

Fig. 6-9 : $\bar{d} = 1.$, $\bar{s} = 1.$ N = 2 .

$\rho_i = .25\,\rho_0$. $\rho_i = .75\,\rho_0$.

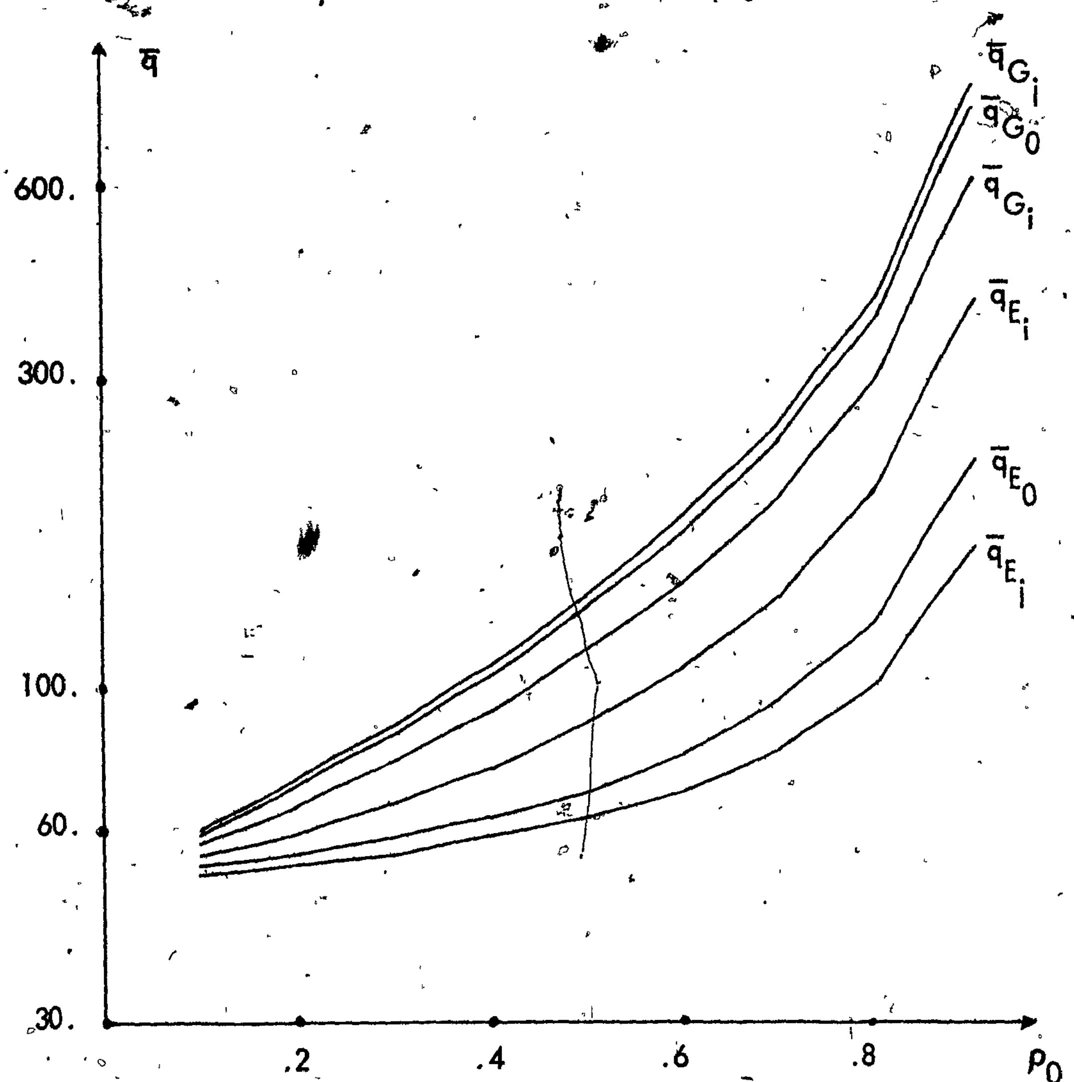

Fig. 6-10 : $\bar{d} = 100$. $\bar{s} = 1$. $N = 2$.

$\rho_i = .25\ \rho_0$. $\rho_j = .75\ \rho_0$.

are given in Appendix F. Inequalities 6.5.22, 6.5.23, and 6.5.24 characterize these curves, as well as many other cases of different sets of $(\rho_i, \rho_j, \bar{s}, \bar{d})$ we studied.

In order to obtain some insight into multiterminal asymmetric polling systems, we conclude this section by performing an exact analysis of three terminal system (N = 3) in the discrete case. We use Equation 6.5.17 to calculate average waiting times in the exhaustive model. From Equation 3.3.21 we see that A is a function of (ρ_1, ρ_2, ρ_3) and $T(i, j, l)$ is $\bar{d} \cdot \bar{s}$ multiplying a function of (ρ_1, ρ_2, ρ_3). Using this fact in Equation 6.5.17, we can write for i = 1, 2, 3

$$\bar{q}_{E_i} = \frac{(1-\rho_i)}{2(1-\rho_0)}\,\bar{d} + f(\rho_i, \rho_j, \rho_l)\cdot\bar{s}\,, \qquad 6.5.25$$

where :

$$f(\rho_i, \rho_j, \rho_l) = \frac{\rho_i}{2(1-\rho_i)} + \frac{(1-\rho_0)\,T(i, j, l)}{2(1-\rho_i)\,\bar{d}\,\bar{s}\,A}$$

and the set (i, j, l) indicates the terminals service order ;

T_j is served after T_i and T_l is served after T_j but before T_i.

The only obvious fact obtained from Equation 6.5.25 is that when $\bar{d}$ increases all $\bar{q}_{E_i}$, i = 1, 2, 3, increase where the contribution to small terminals (with small traffic intensity) is bigger.

Define :

$$\underline{a} = (a_i , a_j , a_l) \qquad 6.5.26$$

where $a_i = \rho_i / \rho_0 \quad a_j = \rho_j / \rho_0 \quad a_l = \rho_l / \rho_0$.

For $\bar{s} = 1$. $\bar{d} = 0.$, $1.$, $100.$ and various sets of $\underline{a}$, we use Equations 3.3.21, 6.5.25, and 6.5.21 to sketch $\bar{q}_{E_i}$, $\bar{q}_{E_j}$, $\bar{q}_{E_l}$, and $\bar{q}_{E_0}$ v.s. ρ_0 . Seven curves are shown : Figures 6-11 to 6-17 and the numerical results used in these curves are given in Appendix F .

From Figure 6-11 where $\bar{d} = 0.$ and $\underline{a} = (.125, .75, .125)$ we see that even so $\rho_i = \rho_l = .125 \rho_0$; $\bar{q}_{E_i} > \bar{q}_{E_l}$. Because of the nature of Equation 6.5.25 this inequality exists also for $\bar{d} \neq 0.$ as shown in Figure 6-12 and Figure 6-13 for $\bar{d} = 1.$ and $\bar{d} = 100.$ respectively. Hence, the order of the terminals in the loop is of importance. It should be mentioned that this feature does not exist for $N = 2$. This "geometric" feature emphasises that $\bar{c}$, $\bar{v}_i$ do not depend on the "geometry" of the system (the order of terminals) but $\sigma^2_{v_i}$ and $\sigma^2_{c_i}$ for $N > 2$ do depend on it. Therefore, the first expression in Equation 6.5.25 $\frac{(1 - \rho_i)}{2(1 - \rho_0)} \bar{d}$ which is $\frac{\bar{v}_i}{2}$ does not contain this geometric feature where the second part $f(\rho_i , \rho_j , \rho_l) \bar{s}$ which is $\frac{(1 - \rho_i) \sigma^2_{c_i}}{2 \bar{c}}$ does depend on the order of the terminals.

An intuitive reason for explaining that for $\underline{a} = (.125, .75, .125)$ $\bar{q}_{E_i} > \bar{q}_{E_l}$ is similar to the one which justifies inequality 6.5.23. Assuming $\bar{d} = 0.$ and because at any given time, the server is at T_i or T_l with equal probability.

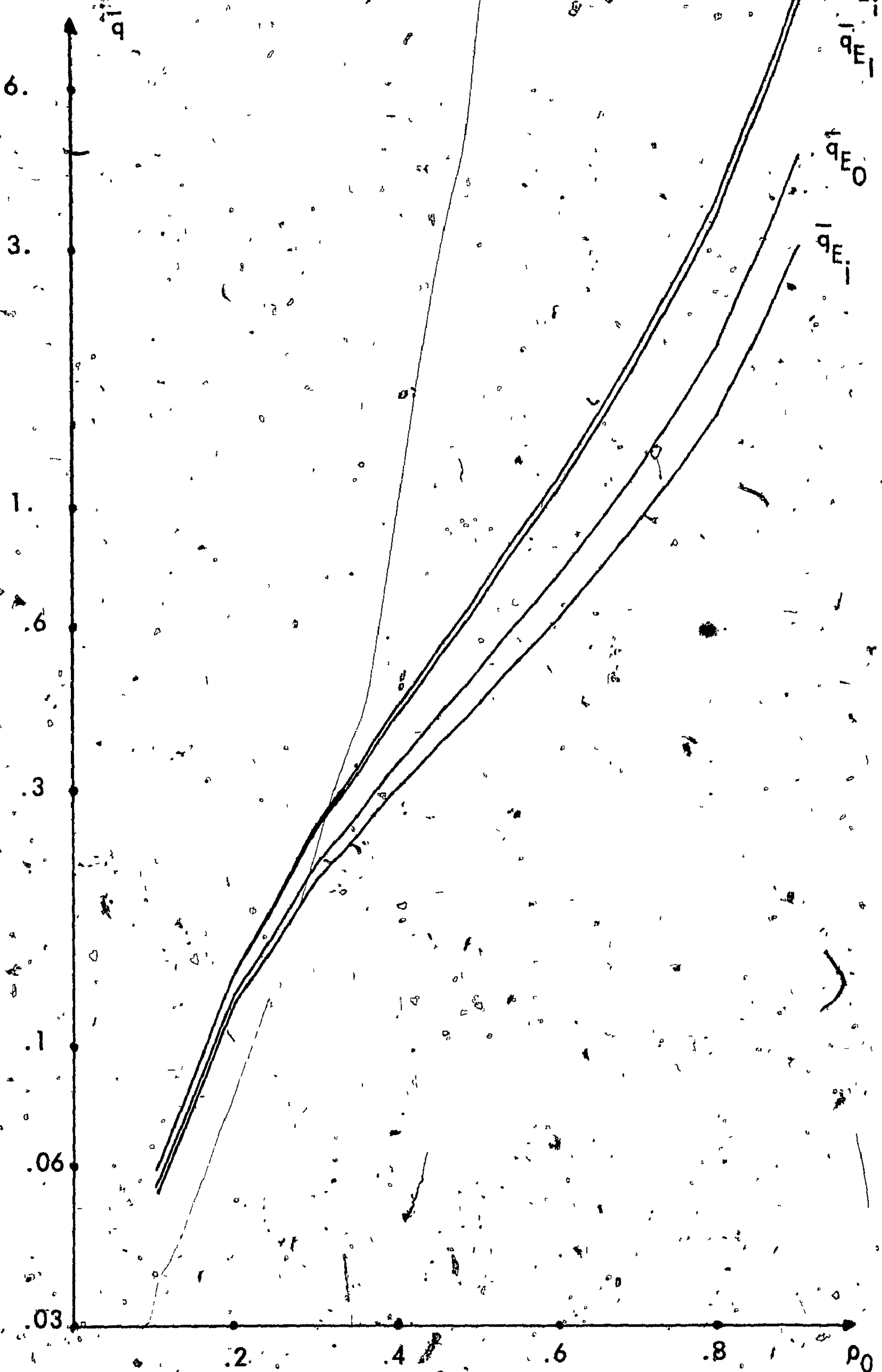

Fig. 6-11 : $\bar{d}_o = 0.$ $\bar{s} = 1$. N = 3 .

$\rho_i = .125\ \rho_0$. $\rho_i = .75\ \rho_0$. $\rho_1 = .125\ \rho_0$.

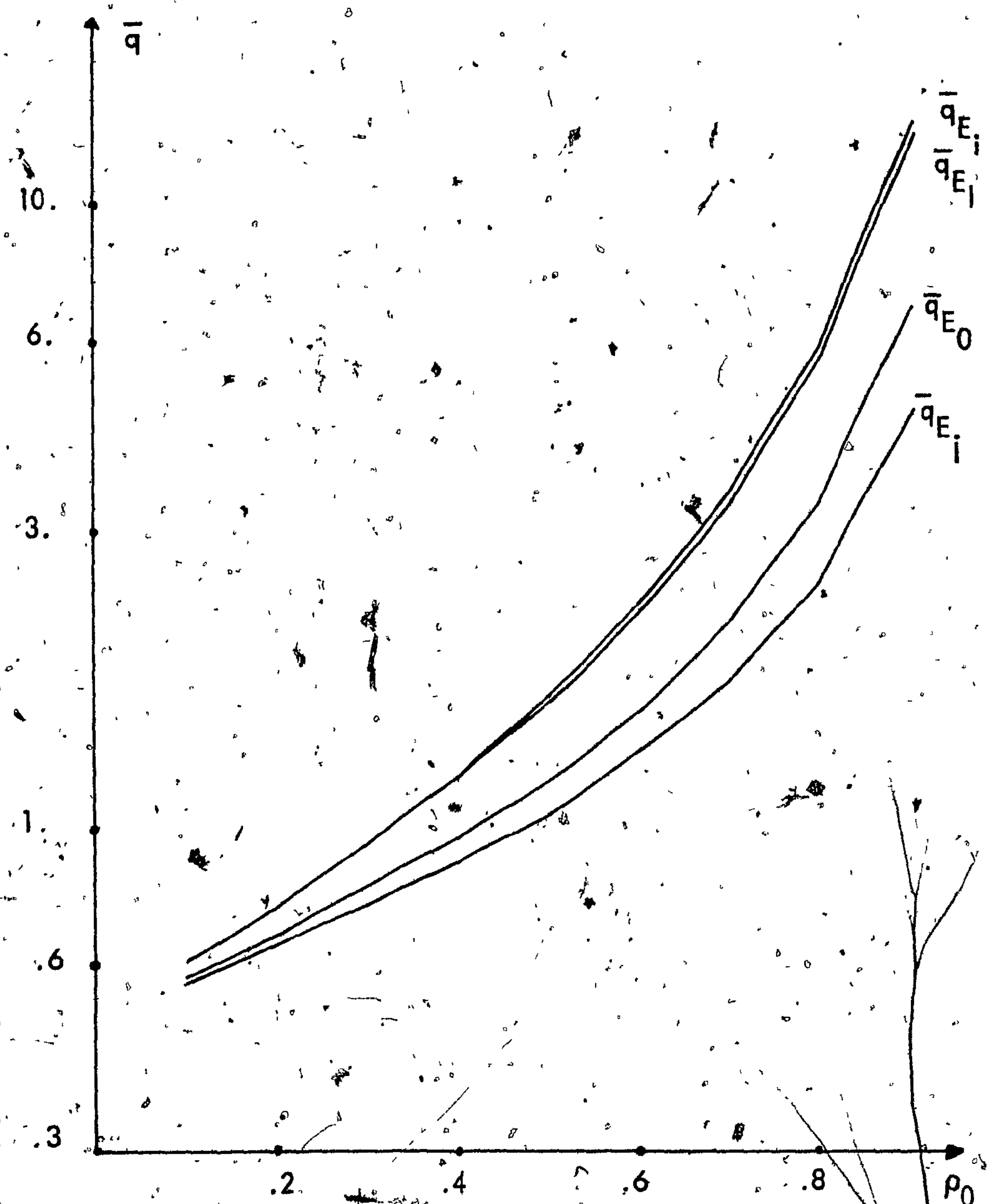

Fig. 6 - 12 : $\bar{d} = 1.$ $\bar{s} = 1.$ $N = 3$.
$\rho_i = .125\ \rho_0$. $\rho_r = .75\ \rho_0$. $\rho_1 = .125\ \rho_0$.

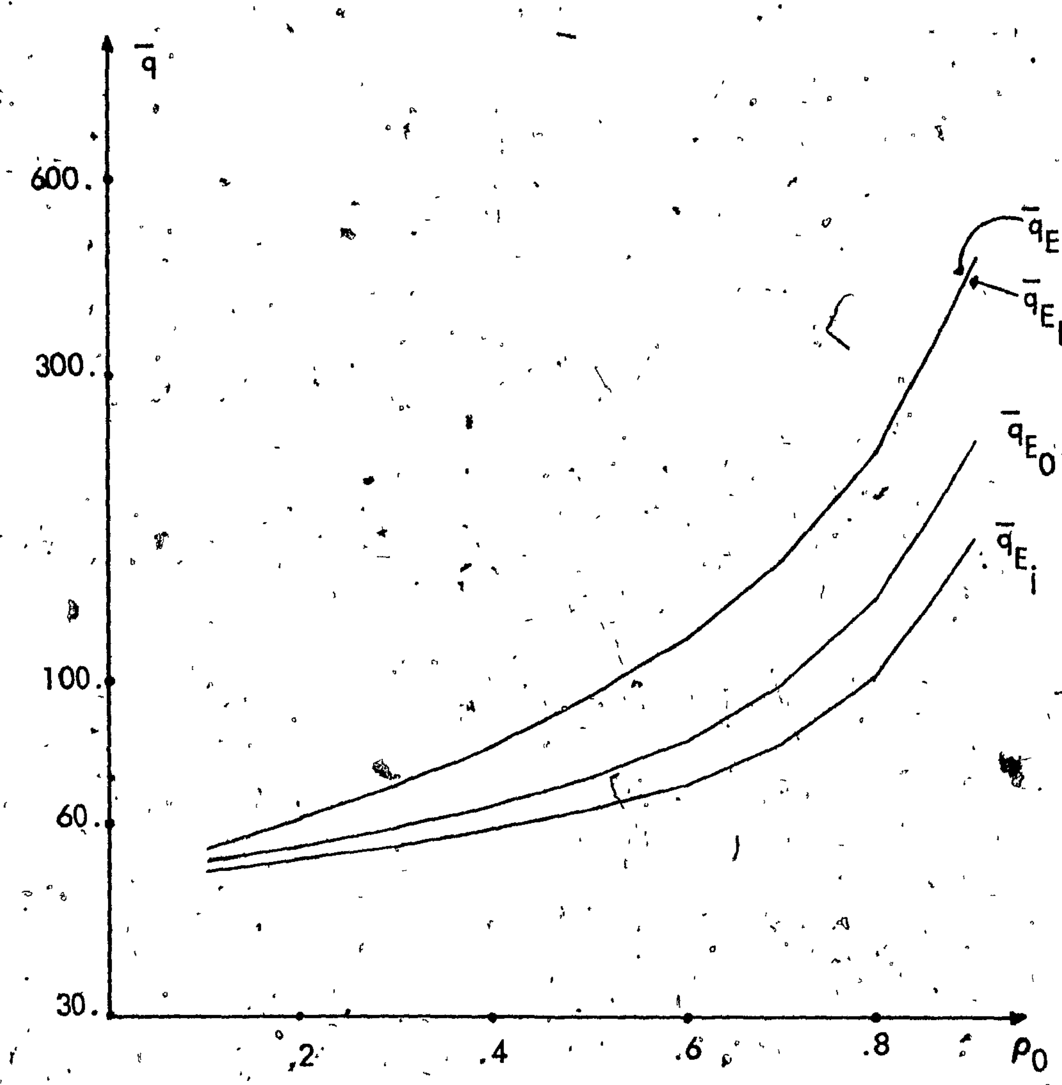

Fig. 6-13 : $\bar{d} = 100$. $\bar{s} = 1$. N = 3 .

$\rho_i = .125\ \rho_0$. $\rho_j = .75\ \rho_0$. $\rho_l = .125\ \rho_0$.

and in T_j with greater probability. Therefore, if at some time instant two customers arrive – one to T_i and one to T_l – the server is more probable to be in T_j or T_l than in T_i and hence the new customer from T_l is served before the new customer from T_i.

Using the above argument for any set of $\underline{a}$ (not necessarily with two equal Terminals), we have for $a_i < a_j$:

$$\bar{q}_{E_l} \{ \text{of } (a_i , a_j , a_l) \} < \bar{q}_{E_l} \{ \text{of } (a_j , a_i , a_l) \} \qquad 6.5.27$$

This inequality can be seen in Figures 6-11 to 6-17 and in Appendix F and was verified for many other sets of $\underline{a}$. Therefore, in order to reduce the average waiting time of a Terminal, we should locate it <u>just after</u> the biggest terminal.

Another important feature which can be seen by comparing the set of Figures 6-8 to 6-10 (where $N = 2$) with the set of Figures 6-11 to 6-17, is that for any $\underline{a}$

$$\bar{q}_{E_l} \{ \text{of } (a_i , a_j , a_l) \} < \bar{q}_{E_l} \{ \text{of } (0. , a_i + a_j , a_l) \} \qquad 6.5.28$$

The intuitive reason for this inequality in three terminal system is that when the server is in T_i, new arrivals to T_l are served <u>before</u> new arrivals to T_j. But in the two terminal system – constructed by combining T_i and T_j to one terminal (T_{ij}), the server leaves T_{ij} (to T_l) only when both T_i and T_j are empty.

Clearly, the same argument holds, in the discrete case of the exhaustive model, for all $N \geq 3$. We have

$$\bar{q}_{E_N} \{ \text{of } (a_1, a_2, \ldots, a_N) \} < \bar{q}_{E_N} \{ \text{of } (\sum_{i=1}^{N-1} a_i , a_N) \} \qquad 6.5.29$$

Hence, to find the upper bound on the average waiting time of T_i in the discrete case of the exhaustive model with any $N \geq 2$, we use Equation 6.5.16 with $\rho_i = \rho_i$ and $\rho_j = \rho_0 - \rho_i$, in inequality 6.5.29, to obtain for all $N \geq 2$

$$\bar{q}_{E_i} \leq \frac{(1-\rho_i)\bar{d}}{2(1-\rho_0)} + \frac{[\rho_0 (1-\rho_i) - 2\rho_i (\rho_0 - \rho_i)(1-\rho_0)] \; \bar{s}}{2(1-\rho_0)(1-\rho_0 + 2\rho_i(\rho_0 - \rho_i))} \qquad 6.5.30$$

For $\bar{s} = 1$ and $\underline{a} = (.125, .75, .125)$ we sketch $\bar{q}_{E_i}$, $\bar{q}_{E_j}$, $\bar{q}_{E_l}$, and $\bar{q}_{E_0}$ for $\bar{d} = 0$. (Figure 6-11), $\bar{d} = 1$. (Figure 6-12), and $\bar{d} = 100$. (Figure 6-13). Since the contribution of $\bar{d} \neq 0$. to the average Terminal waiting time is simple and does not depend on the geometry of the system, we set in all latter Figures $\bar{d} = 0$. For $\bar{s} = 1$. and $\bar{d} = 0$. we sketch $\bar{q}_{E_i}$, $\bar{q}_{E_j}$, $\bar{q}_{E_l}$, and $\bar{q}_{E_0}$ for $\underline{a} = (.25, .5, .25)$ - Figure 6-14, $\underline{a} = (.25, .375, .375)$ - Figure 6-15, $\underline{a} = (.125, .625. .25)$ - Figure 6-16, $\underline{a} = (.125, .25, .625)$ - Figure 6-17.

In these seven curves at least one of the terminals has either $.25\ \rho_0$ or $.75\ \rho_0$ traffic intensity. This enables us to verify inequality 6.5.28, by comparing them to Figure 6-8. This comparison is better performed by using the numerical re-

Fig. 6-14 : $\bar{d} = 0$. $\bar{s} = 1$. N = 3 .

$\rho_i = .25\ \rho_0$. $\rho_l = .5\ \rho_0$; $\rho_l = .25\ \rho_0$.

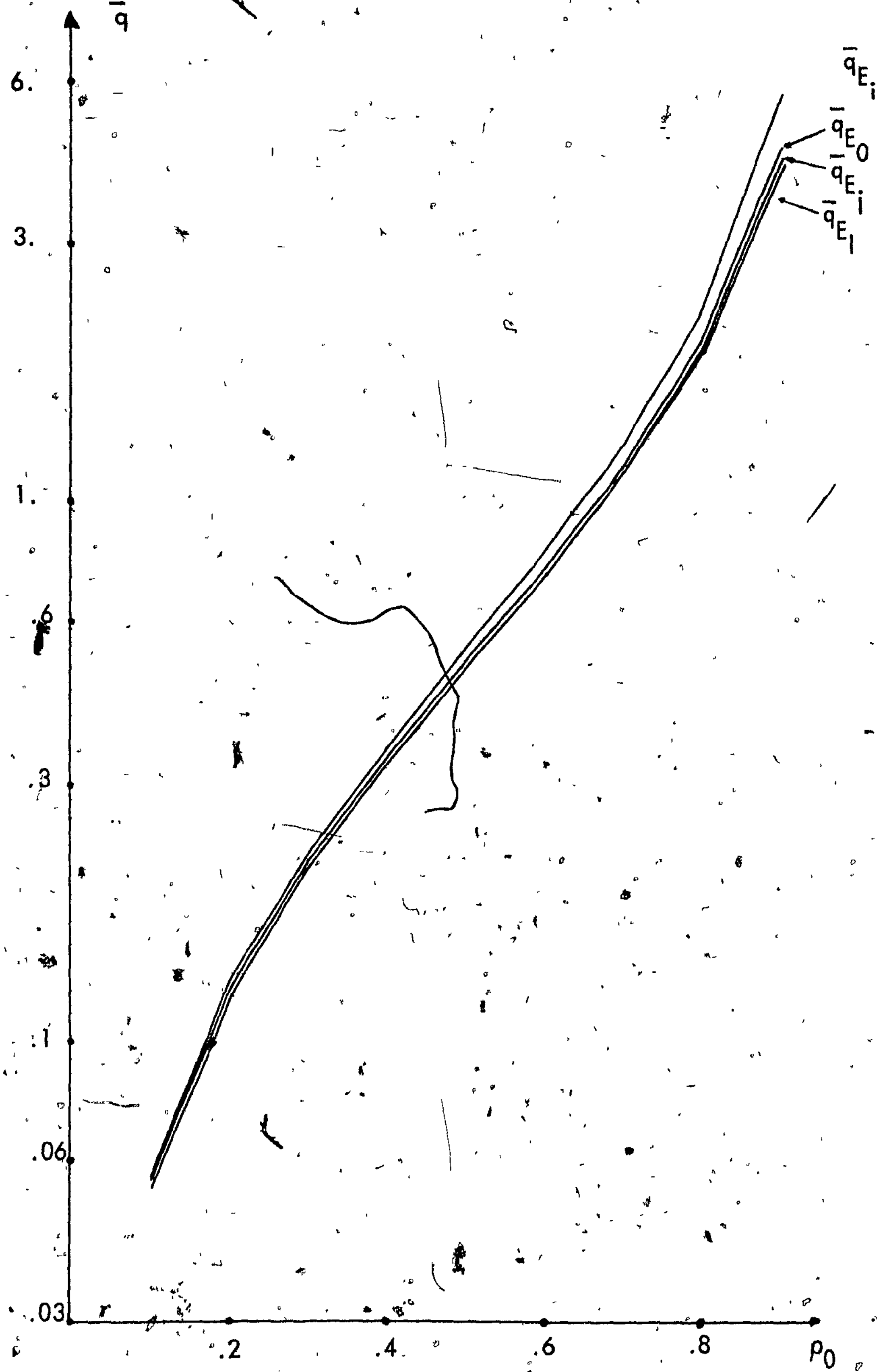

Fig. 6-15 : $\bar{d} = 0$. $\bar{s} = 1$. $N = 3$.

$\rho_i = .25\ \rho_0$. $\rho_j = .375\ \rho_0$. $\rho_1 = .375\ \rho_0$.

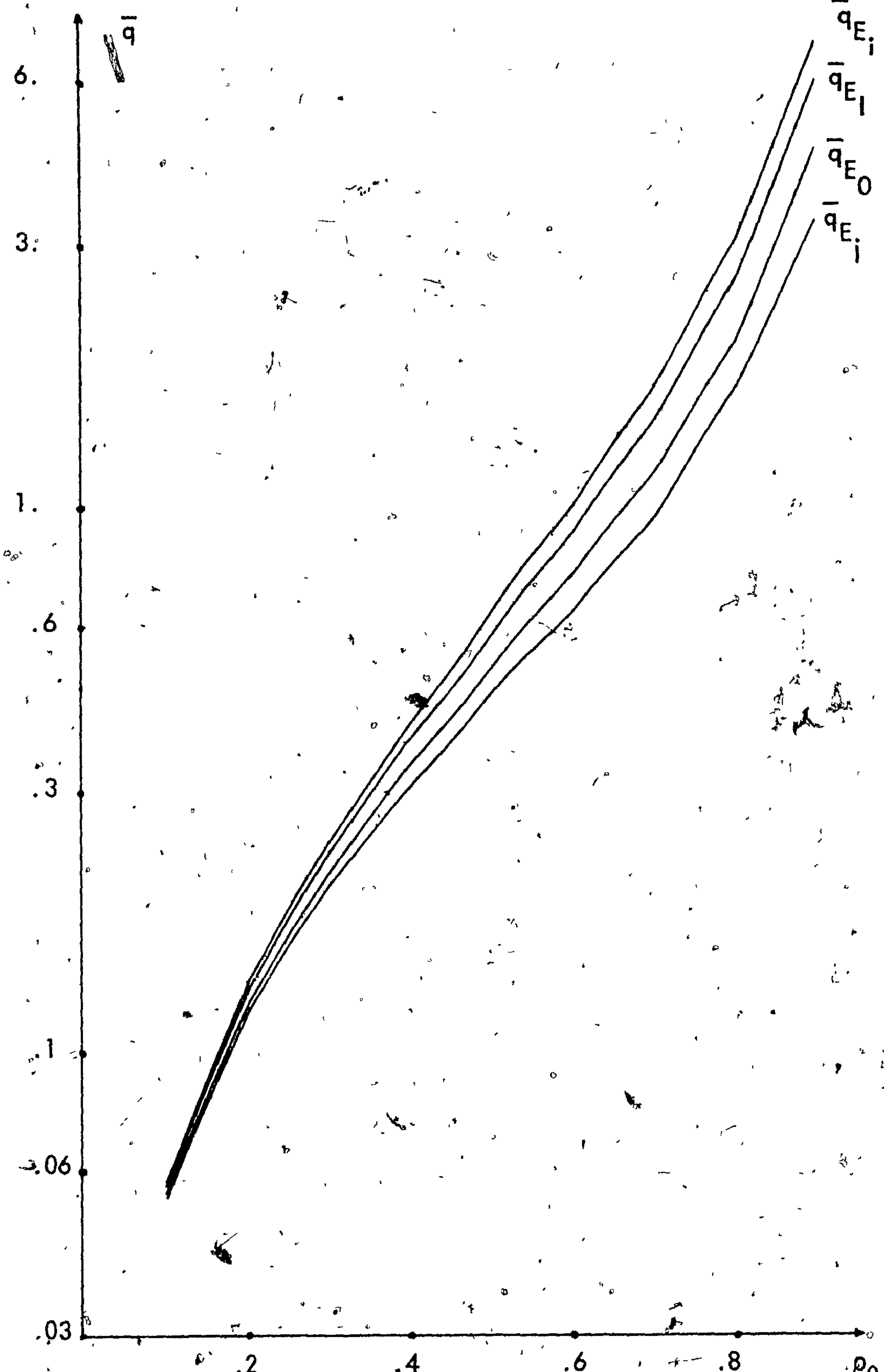

Fig. 6 - 16 : $\bar{d} = 0$. $\bar{s} = 1$. $N = 3$.

$\rho_i = .125\ \rho_0$. $\rho_j = .625\ \rho_0$. $\rho_l = .25\ \rho_0$.

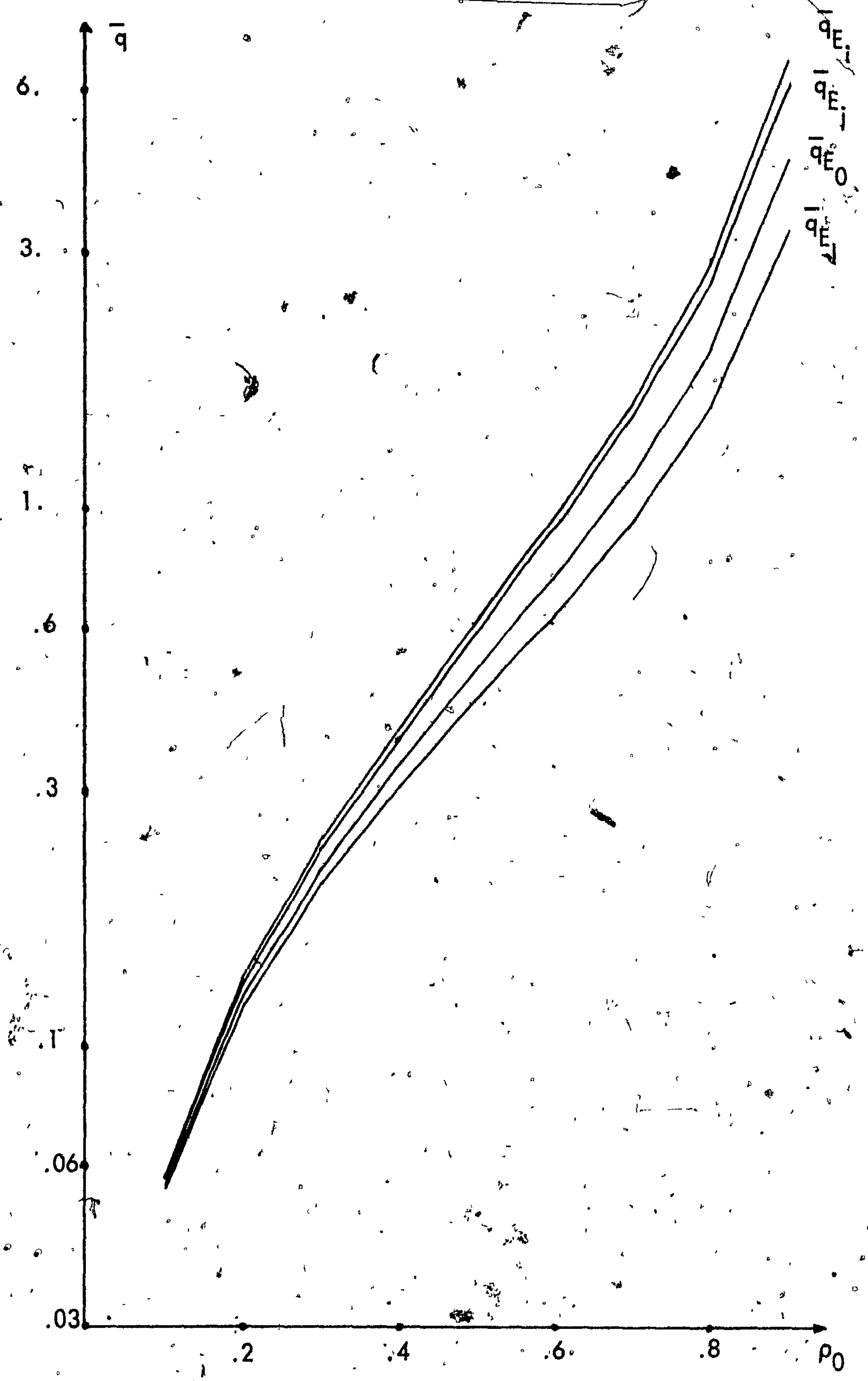

Fig. 6-17 : $\bar{d} = 0$. $\bar{s} = 1$. $N = 3$.

$\rho_i = .125\ \rho_0$. $\rho_j = .25\ \rho_0$. $\rho_l = .625\ \rho_0$.

sults of Appendix F. The inequality 6.5.27 can be verified by those seven curves, the last two curves , Figure 6-16 and Figure 6-17, are supplied to illustrate this inequality for the case $a_i \neq a_j \neq a_l$.

6.6 Conclusion

In this chapter we studied both models from the waiting time point of view. In the symmetric case, we obtained explicit expressions for $\bar{q}_G$ and $\bar{q}_E$ for all $N > 1$ (Equation 6.4.3) . From it we showed that $\bar{q}_E \leq \bar{q}_G$ (equality holds when total average system walking time, $\bar{d}$, is zero). Therefore, from the average waiting time point of view, the exhaustive model is somewhat more attractive. However, for practical symmetric multiterminal system (where $\frac{\rho_0}{N}$ is always much smaller than 1.), the average waiting times in both models are practically equal. For the symmetric nonrandom case of both models we obtained σ_q^2 for $N \to \infty$ to be equal (for small N the exhaustive model, again, looks more attractive). Using $\bar{q}$ and σ_q^2 we obtain a bound on the actual waiting time in the multiterminal case of both models (inequality 6.4.11) .

For the discrete case, defined by 6.5.14, we derived expressions for the average system waiting time (Equation 6.5.21) and showed that $\bar{q}_{E_0} \leq \bar{q}_{G_0}$ (equality holds when $\bar{d} = 0$.) . For N = 2, we showed that the models behave in a reversed manner, i.e., when $\rho_1 < \rho_2$ we have $\bar{q}_{E_1} > \bar{q}_{E_2}$ but $\bar{q}_{G_1} < \bar{q}_{G_2}$. When $\bar{d}$ increases the exhaustive model becomes more and more attractive. For the exhaustive model we gained some insight to the discrete case of multiterminal system by studying the case $N = 3$. From it, we obtained upper bound on $\bar{q}_{E_i}$ for all N (inequality 6.5.30).

CHAPTER VII

CONCLUSION AND SUGGESTIONS FOR FURTHER RESEARCH

In this thesis we performed an exact analysis of a general polling system, where N terminals are attended in a fixed cyclic order by a single server. We analysed two different service procedures. In the first, called the gating model, at each terminal the server attends only to the customers who are present in it at the moment the server arrives. In the second service procedure, called the exhaustive model, the server attends each terminal until it is empty.

In both models, new arrivals at each terminal are governed by an independent Poisson process. The service time of customer and the walking time of the server, between two successive terminals, are independent random variables possessing any probability density function.

The main reason for the success of the study is the choice of a particular N dimensional random vector, whose elements are N successive terminal service times, describing a complete cycle of service.

The basic equations obtained are recursive forms of the Laplace transform of the probability density functions of the random vector. These equations enabled us to derive explicit expressions for some moments of the terminal service time, θ, the cycle time, c, and the intervisit time, v. The moments are obtained in the transient and the steady states.

In deriving those moments we encountered a relatively small number of equations as compared to Leibowitz (1960), Cooper and Murray (1969), Sykes (1970), Cooper (1970), and Eisenberg (1972). This enabled us to derive explicit expressions for some moments of θ, c, v, and customer waiting time, q, that were not derived before. For the symmetric case, the complexity of the equations, obtained by previous researchers, stayed the same. Using our technique, the number of equations needed to be solved, in order to derive the moments, is substantially decreased (by a factor of $^{1}/N$). This feature enabled us to derive explicit expressions for the second moments of θ, c, v, and the first moment of q for the symmetric case. All these expressions are new. At this point, it should be mentioned that the symmetric case is the most practical case of our general model, where each terminal is polled once and only once in a cycle. This polling procedure is the optimal for the symmetric case and it implies no priorities among the terminals. However, for the non-symmetric case, it is better to construct a cycle of service in which a terminal with a high traffic intensity is polled more times than one with low traffic intensity.

Throughout the study a comparison between the gating and the exhaustive model is performed. From the mathematical complexity point of view, the exhaustive model is easier to solve as the number of linear equations that are required to be solved in order to find moments of θ, c, and v, is smaller in this case. This is due to the fact that a basic random vector of dimension N-1 is sufficient for the mathematical description of the exhaustive model, where an N dimensional random vector is necessary for the mathematical description of the gating model.

Comparing the gating and the exhaustive models in the symmetric case, we showed that the exhaustive model is superior to the gating model. The exhaustive model is superior to the gating model in the sense that the transient response and the recovery time of the first moments of θ, c, and v, are shorter in the exhaustive model than in the gating model. The normalized cross correlation of θ or c or v, in both models, converges to zero as the difference between the terminals (or the cycles) increases. However, this convergence is faster in the exhaustive model than in the gating model. A buffer size which guarantees a fixed and arbitrarily small probability of message rejection, in the symmetric case, would be smaller in the exhaustive model than in the gating model. The average customer waiting time (terminal waiting time and system waiting time) is shorter in the exhaustive model than in the gating model. This superiority of the exhaustive model over the gating model, in the symmetric case, is substantial for small N. However, for practical symmetric multiterminal system, where $N >> 1$, we showed that both models are essentially the same. For this multiterminal case, as we derived the variance of the waiting time, using Chebyshev's inequality, we obtained an upper bound on the actual customer waiting time. Hence, in practical multiterminal symmetric polling system, the choice between a gating or an exhaustive service procedure should depend on other criteria such as hardware complexity.

For the discrete case (symmetric nonrandom where each terminal has different new arrival rate) we studied the average waiting time in both models for two terminals system, and in the exhaustive model for three terminals system.

For the two terminals system we showed that the exhaustive and the gating models behave in an opposite way. In the exhaustive model, the terminal with the low traffic intensity has a bigger average waiting time than that with the high traffic intensity, but in the gating model, the opposite holds. For the three terminals system, with the exhaustive service procedure, we showed that the average waiting time of a customer in some terminal also depends on the geometry of system (i.e., the order in which the terminals are polled). For the discrete case in both models and for all N, we derived exact expressions for the average system waiting time. The expressions obtained do not depend on the geometry of the system. The average system waiting time in the exhaustive model is smaller than that in the gating model. However, for $N >> 1$ they are essentially the same.

For the symmetric case results concerning transient behaviour of the first moments of θ, c, and v, steady state second moments of those quantities, and the first two moments of the buffer size and waiting time, are new and of basic importance in understanding and designing a polling system. For the discrete case, all expressions for average waiting times for $N = 2$ in the gating model, $N = 3$ in the exhaustive model, and the average system waiting time for all N, are new as well.

Suggestions for Further Research

1. Derive an explicit solution to the general equations (Asymmetric case), to obtain the second moments of c, and v, in the general case. From it, average waiting times can immediately be derived.

2. Optimize the geometry of a general polling system such that a given terminal has the minimum possible average waiting time.

3. Derive explicit expressions, for second moments of c, v, and first moment of q, for the general polling model in which each terminal may be polled more than once during a cycle of service.

4. Do an exact study of a polling system with finite buffers and / or server with finite service capacity.

5. Do a comparison between the average waiting times of the polling system and the loop system.

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APPENDIX A

Busy Period in M / G / 1 Queue

The method of derivation is due to Takacs (1962). We have one terminal (queue) with a Poisson new-arrivals process having parameter λ, general independent service probability density function $P_s(\cdot)$, and zero walking time.

A busy period starts as a customer enters a previously empty queue and ends as the server becomes idle for the first time.

Define :

$P_b(\cdot) \triangleq$ The probability density function of a busy period.

The basic idea here is that the length of a busy period does not depend on the order in which the customers are served. Therefore if m new customers arrive during the service time of the initiating (first) customer, each of the m customers initiates an independent busy period. Hence, the total busy period is the sum of the initiating customer service time and the m independent "new" busy periods.

By the Law of Total Probability, we have :

$$P_b(t) = \int_o^t \sum_{m=0}^{\infty} \text{Prob}(t, m, \tau)\, d\tau \qquad \text{A.1}$$

where :

t = Length of the busy period ,

τ = service time of the initiating customer,

m = the number of new customers that arrived during the service of the initiating customer.

We have :

$$P_b(t) = \int_0^t \sum_{m=0}^{\infty} \text{Prob}(t/m,\tau)\ \text{Prob}(m/\tau)\ P_s(\tau)\ d\tau \qquad \text{A.2}$$

Prob $(t/m, \tau)$ and Prob (m/τ) can be immediately determined from $P_b(.)$ and the Poisson process respectively, and since $P_b(t)$ is zero for $t < 0$ we obtain

$$P_b(t) = \int_0^{\infty} \sum_{m=0}^{\infty} P_b^{*(m)}(t-\tau)\ \left[(\lambda\tau)^m/m!\right]\ \exp(-\lambda\tau)\, P_s(\tau)\, d\tau \qquad \text{A.3}$$

Define B (.) as the Laplace transform of $P_b(.)$.

We obtain from A.3

$$B(x) = \int_0^{\infty}\int_0^{\infty} \sum_{m=0}^{\infty} \exp\left[-(t-\tau)x\right] P_b^{*(m)}(t-\tau)\ \left[(\lambda\tau)^m/m!\right] \qquad \text{A.4}$$

$$\exp\left[-(\lambda+x)\tau\right]\ P_s(\tau)\, d\tau\, dt$$

Integration over t yields

$$B(x) = \int_0^{\infty} \sum_{m=0}^{\infty} \left[(\lambda \ B(x) \ \tau)^m / m! \right] \ \exp\left[-(\lambda + x)\tau \right] P_s(\tau)\, d\tau \qquad \text{A.5}$$

Summation over m yields

$$B(x) = \int_0^{\infty} \exp\left[-\tau (x + \lambda(1 - B(x))) \right] \ P_s(\tau)\, d\tau \qquad \text{A.6}$$

which implies

$$B(x) = S^{*}(x + \lambda(1 - B(x))) \qquad \text{A.7}$$

at $x = 0$, $B(x) = 0$

To find the first three moments of the busy period $(\bar{b}, \overline{b^2}, \overline{b^3})$ we differentiate A.7 w.r.t. x three times and set $x = 0$.

Notice : $\dot{B}$ is w.r.t. x and $\dot{S}$ is w.r.t. it is all argument, $x + \lambda(1 - B(x))$:

We obtain for all $x \geq 0$

$$\begin{aligned}
\dot{B} &= (1 - \lambda \dot{B}) \dot{S} \\
\ddot{B} &= (1 - \lambda \dot{B})^2 \ddot{S} - \lambda \ddot{B} \dot{S} \\
\dddot{B} &= (1 - \lambda \dot{B})^3 \dddot{S} - 3\lambda \ddot{B} (1 - \lambda \dot{B}) \ddot{S} - \lambda \dddot{B} \dot{S}
\end{aligned} \qquad \text{A.8}$$

At $x = 0$ we have :

$$\dot{S} = -\bar{s} \qquad \ddot{S} = \overline{s^2} \qquad \dddot{S} = -\overline{s^3}$$

$$\dot{B} = -\bar{b} \qquad \ddot{B} = \overline{b^2} \qquad \dddot{B} = -\overline{b^3} \qquad \text{A.9}$$

Substitition of A-9 in A-8 yields

$$\bar{b} = \frac{\bar{s}}{1 - \lambda \bar{s}}$$

$$\bar{b^2} = \frac{\bar{s^2}}{(1 - \lambda \bar{s})^3}$$

$$\bar{b^3} = \frac{\bar{s^3}}{(1 - \lambda \bar{s})^4} + \frac{3 \lambda \bar{s^2}^2}{(1 - \lambda \bar{s})^5} \qquad \text{A.10}$$

Clearly the results hold only for $\lambda \bar{s} < 1$ otherwise we experience a nonzero probability that the length of the busy period is infinitely long.

The Number of Customers in a Busy Period

Define $P_h(\cdot)$ as the probability density function of the number of customers that are served in a busy period. In a manner similar to the derivation of $P_b(t)$ we realize that during the service of the initiating (first) customer, t, $m \geq 0$ of customers might arrive. Each of the new customers independently initiates a busy period. By the law of total probability we have :

$$P_h(n) = \int_o^{\infty} \sum_{m=0}^{n-1} \text{Prob}(n, m, t)\, dt \qquad \text{A.11}$$

where :

n = The total number of customers in the busy period,

m = the number of customers that arrive during the service of the first customer,

t = the service time of the first customer.

Using Bayes chain rule we have :

$$P_h(n) = \int_0^{\infty} \sum_{m=0}^{n-1} \text{Prob}(n/m,t)\,\text{Prob}(m/t)\,P_s(t)\,dt \qquad \text{A.12}$$

$$= \int_0^{\infty} \sum_{m=0}^{\infty} P_h^{(m)*}(n-1)\,[(\lambda t)^m/m!]\,\exp(-\lambda t)\,P_s(t)\,dt \qquad \text{A.13}$$

In deriving A.13 we used the fact that the new m customers create independently a total busy period of n − 1 customer (not including the first one). Therefore :

$$\text{Prob}(n/m,t) = P_h^{(m)*}(n-1), (=0 \text{ for } m > n-1) \qquad \text{A.14}$$

Define the Laplace transform of $P_h(\cdot)$ as $H(\cdot)$ we have :

$$H(x) = \sum_{n=0}^{\infty} \exp(-xn)\,P_h(n) \qquad \text{A.15}$$

Clearly $P_h(0) = 0$,

Applying the Laplace transform to A.13 we obtain :

$$H(x) = \exp(-x) \int_0^\infty \sum_{m=0}^\infty \sum_{n=1}^\infty \exp(-x(n-1)) P_h^{*(m)}(n-1) \; [(\lambda t)^m / m!] \exp(-\lambda t) P_s(t)\, dt \qquad A.16$$

$$= \exp(-x) \int_0^\infty \sum_{m=0}^\infty [(\lambda t H(x))^m / m!] \exp(-\lambda t) P_s(t)\, dt$$

$$= \exp(-x) \int_0^\infty \exp[-\lambda(1 - H(x))\, t]\, P_s(t)\, dt$$

$$H(x) = \exp(-x) \quad S[\lambda(1 - H(x))] \qquad A.17$$

In order to evaluate the first two moments of the number of customers in a busy period $(\bar{h}, \overline{h^2})$ we differentiate Equation A.17 w.r.t. x. Define $H(x) \triangleq H$, $u \triangleq \lambda(1 - H)$, $S(u) \triangleq S$, we have :

$$\dot{u} = -\lambda \dot{H}$$

$$\ddot{u} = -\lambda \ddot{H} \qquad A.18$$

and by differentiation of A.17

$$\dot{H} = \exp(-x)(-S + \dot{u}\dot{S})$$

$$\ddot{H} = \exp(-x)(S - 2\dot{u}\dot{S} + \ddot{u}\dot{S} + (\dot{u})^2 \ddot{S}) \qquad A.19$$

Substituting A.18 into A.19 yields at $x = 0$:

$$\bar{h} = 1 + \lambda \bar{s} \cdot \bar{h}$$

$$\overline{h^2} = 1 + 2\lambda \bar{s}\,\bar{h} + \lambda \bar{s}\,\overline{h^2} + \lambda^2 \overline{s^2}\,\overline{h^2}$$

Solving the above for $\rho = \lambda \bar{s}$

$$\bar{h} = \frac{1}{1-\rho}$$

$$\overline{h^2} = \frac{1+\rho}{(1-\rho)^2} + \frac{\lambda^2 \overline{s^2}}{(1-\rho)^3}$$

$$\overline{h^2} - \bar{h}^2 = \frac{\rho(1-\rho) + \lambda^2 \overline{s^2}}{(1-\rho)^3} \qquad \text{A.20}$$

APPENDIX B

COMPUTATIONAL STEPS IN THE DERIVATION OF THE FIRST THREE MOMENTS OF θ, c, v

B1 Some Properties of $F(\cdot)$

We have, for $N \geq 1$

$\underline{\theta} = (\theta_1, \ldots, \theta_N)$, an N-dimensional random vector with probability density function $P(\underline{\theta})$. The Laplace transform of $P(\underline{\theta})$ is

$$G(\underline{x}) = \int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{i=1}^N x_i \theta_i\right) P(\underline{\theta}) \prod_{i=1}^N d\theta_i \qquad \text{B1.1}$$

Define

$$F(\underline{x}) = \ln G(\underline{x}) \qquad \text{B1.2}$$

It is well known that

differentiating $G(\underline{x})$ w.r.t. the ith, jth, lth arguments where $i, j, l = 1, \ldots N$, at $\underline{x} = \underline{0}$ yields

$$\begin{aligned} G_i(\underline{0}) &= -\bar{\theta}_i \\ G_{i,j}(\underline{0}) &= \overline{\theta_i \theta_j} \\ G_{i,j,l}(\underline{0}) &= -\overline{\theta_i \theta_j \theta_l} \end{aligned} \qquad \text{B1.3}$$

For convenience, let us write G and F instead of $G(\underline{x})$ and $F(\underline{x})$, respectively.

Differentiating B1.2 w.r.t. the ith, jth and lth arguments for all $\underline{x} \geq \underline{0}$, we obtain :

$$F_i = G_i / G$$

$$F_{i,j} = (G\ G_{i,j} - G_i\ G_j) / G^2 \qquad \text{B1.4}$$

$$F_{i,j,l} = [G^2\ G_{i,j,l} - G(G_i\ G_{j,l} + G_j\ G_{i,l} + G_l\ G_{i,j}) + 2\ G_i\ G_j\ G_l\ ] / G^3$$

At $\underline{x} = \underline{0}$, $G(\underline{0}) = \underline{1}$; using B1.3 in B1.4 we obtain

$$F_i(\underline{0}) = -\overline{\theta}_i$$

$$F_{i,j}(\underline{0}) = \overline{\theta_i\theta_j} - \overline{\theta}_i\overline{\theta}_j = \overline{(\theta_i - \overline{\theta}_i)(\theta_j - \overline{\theta}_j)} \qquad \text{B1.5}$$

$$F_{i,j,l}(\underline{0}) = -\overline{\theta_i\theta_j\theta_l} + (\overline{\theta}_i\overline{\theta_j\theta_l} + \overline{\theta}_j\overline{\theta_i\theta_l} + \overline{\theta}_l\overline{\theta_i\theta_j}) - 2\overline{\theta}_i\overline{\theta}_j\overline{\theta}_l$$

$$= -\overline{(\theta_i - \overline{\theta}_i)(\theta_j - \overline{\theta}_j)(\theta_l - \overline{\theta}_l)}$$

The above equation is used to derive Equation 3.1.8.

It should be noted that $F_{i,j,l,m}$ does not have the same structure as $F_{i,j}$ and $F_{i,j,l}$ have .

B2 Development of Equation 3.2.12

We have

$$_{k+1}F_{N,N}(\underline{x}) = \frac{W_k(x_N)\ddot{W}_k(x_N) - (\dot{W}_k(x_N))^2}{(W_k(x_N))^2} - \lambda_k \ddot{S}_k(x_N) \sum_{i=1}^{N} {}_kF_i(\underline{z})$$

$$+ (\lambda_k \dot{S}_k(x_N))^2 \sum_{i=1}^{N} \sum_{l=1}^{N} {}_kF_{i,l}(\underline{z})$$

For $J = 1, \ldots, N-1$

$$_{k+1}F_{N,J}(\underline{x}) = -\lambda_k \dot{S}_k(x_N) \sum_{i=1}^{N} {}_kF_{i,J+1}(\underline{z}) \qquad 3.2.12$$

At $\underline{x} = \underline{0}$ we have from Equations 2.2.16, B1.3 and B1.5

$$\underline{z} = \underline{0}$$

$$W_k(0) = S_k(0) = 1$$

$$\dot{W}_k(0) = -\overline{w}_k \qquad \ddot{W}_k(0) = \overline{w_k^2}$$

$$\dot{S}_k(0) = -\overline{s}_k \qquad \ddot{S}(0) = \overline{s_k^2} \qquad B2.1$$

$$_{k+1}F_i(\underline{0}) = -\overline{\theta}_{k+i}$$

$$_{k+1}F_{i,l}(\underline{0}) = \overline{(\theta_{k+i} - \overline{\theta}_{k+i})(\theta_{k+l} - \overline{\theta}_{k+l})}$$

By defining $\sigma^2_{w_k} = \overline{(w_k - \overline{w}_k)^2} = \overline{w_k^2} - \overline{w}_k^2$, B2.2

and since $c_{k+N} = \sum_{i=0}^{N-1} \theta_{k+i}$, we also have

$$\sigma^2_{c_{k+N}} = \overline{(c_{k+N} - \bar{c}_{k+N})^2} = \overline{(\sum_{i=0}^{N-1} \theta_{k+i} - \sum_{i=0}^{N-1} \bar{\theta}_{k+i})^2}$$

$$= \overline{[\sum_{i=0}^{N-1} (\theta_{k+i} - \bar{\theta}_{k+i})]^2}$$

B2.3

$$= \sum_{i=0}^{N-1} \sum_{l=0}^{N-1} \overline{(\theta_{k+i} - \bar{\theta}_{k+i})(\theta_{k+l} - \bar{\theta}_{k+l})}$$

A direct substitution of B2.1-3, and 3.2.5 into 3.2.12 yields

$$\overline{(\theta_{k+N} - \bar{\theta}_{k+N})^2} = \sigma^2_{w_k} + \lambda_k \bar{s}_k^2 \bar{c}_{k+N} + (\lambda_k \bar{s}_k)^2 \sigma^2_{c_{k+N}}$$

For $J = 1, \ldots, N-1$

$$\overline{(\theta_{k+N} - \bar{\theta}_{k+N})(\theta_{k+J} - \bar{\theta}_{k+J})} = \lambda_k \bar{s}_k \sum_{i=0}^{N-1} \overline{(\theta_{k+J} - \bar{\theta}_{k+J})(\theta_{k+i} - \bar{\theta}_{k+i})}$$

where $\sigma^2_{c_{k+N}} = \sum_{i=0}^{N-1} \sum_{l=0}^{N-1} \overline{(\theta_{k+i} - \bar{\theta}_{k+i})(\theta_{k+l} - \bar{\theta}_{k+l})}$ 3.2.13

B3 Solution of Equation 3.2.15 for N = 2

We have

$$R(1, 1) = \sigma^2_{w_1} + \lambda_1 \bar{s}^2_1 \bar{c} + \rho^2_1 \sigma^2_{c_1} \qquad 3.2.16a$$

$$R(2, 2) = \sigma^2_{w_2} + \lambda_2 \bar{s}^2_2 \bar{c} + \rho^2_2 \sigma^2_{c_2} \qquad 3.2.16b$$

$$R(1, 2) = \rho_1 (R(2, 2) + R(2, 1)) \qquad 3.2.16c$$

$$R(2, 1) = \rho_2 (R(1, 1) + R(1, 2)) \qquad 3.2.16d$$

where

$$\sigma^2_{c_1} = R(1, 1) + R(2, 2) + 2 R(2, 1) \qquad 3.2.16e$$

$$\sigma^2_{c_2} = R(1, 1) + R(2, 2) + 2 R(1, 2) \qquad 3.2.16f$$

Substituting 3.2.16d into 3.2.16c and 3.2.16c into 3.2.16d, we obtain

$$R(1, 2) = \frac{\rho_1 R(2, 2) + \rho_1 \rho_2 R(1, 1)}{1 - \rho_1 \rho_2}$$

$$R(2, 1) = \frac{\rho_2 R(1, 1) + \rho_1 \rho_2 R(2, 2)}{1 - \rho_1 \rho_2} \qquad B3.1$$

Substituting B3.1 into 3.2.16e and 3.2.16f yields

$$\sigma^2_{c_1} = \frac{(1-\rho_1\rho_2+2\rho_2)\,R(1,1)+(1+\rho_1\rho_2)\,R(2,2)}{1-\rho_1\rho_2}$$

$$\sigma^2_{c_2} = \frac{(1+\rho_1\rho_2)\,R(1,1)+(1-\rho_1\rho_2+2\rho_1)\,R(2,2)}{1-\rho_1\rho_2} \qquad \text{B3.2}$$

Substitution of R (1, 1) and R (2, 2) in the above equation and using Equations 3.2.16a and 3.2.16b respectively yields

$$\hat{\sigma}^2_{c_1} = \frac{(1-\rho_1\rho_2+2\rho_2)(\sigma^2_{w_1}+\lambda_1\overline{s^2_1}\,\bar{c}+\rho_1^2\sigma^2_{c_1})+(1+\rho_1\rho_2)(\sigma^2_{w_2}+\lambda_2\overline{s^2_2}\,\bar{c}+\rho_2^2\sigma^2_{c_2})}{1-\rho_1\rho_2}$$

$$\sigma^2_{c_2} = \frac{(1-\rho_1\rho_2+2\rho_1)(\sigma^2_{w_2}+\lambda_2\overline{s^2_2}\,\bar{c}+\rho_2^2\sigma^2_{c_2})+(1+\rho_1\rho_2)(\sigma^2_{w_1}+\lambda_1\overline{s^2_1}\,\bar{c}+\rho_1^2\sigma^2_{c_1})}{1-\rho_1\rho_2} \qquad \text{B3.3}$$

B3.3 is a set of 2 equations with 2 unknowns. Rearranging it, we obtain

$$[1-\rho_1\rho_2-\rho_1^2(1-\rho_1\rho_2+2\rho_2)]\,\sigma^2_{c_1}-\rho_2^2(1+\rho_1\rho_2)\,\sigma^2_{c_2}$$

$$= (1-\rho_1\rho_2+2\rho_2)(\lambda_1\overline{s^2_1}\,\bar{c}+\sigma^2_{w_1})+(1+\rho_1\rho_2)(\lambda_2\overline{s^2_2}\,\bar{c}+\sigma^2_{w_2})$$

$$\text{B3.4}$$

$$-\rho_1^2(1+\rho_1\rho_2)\,\sigma^2_{c_1}+[1-\rho_1\rho_2-\rho_2^2(1-\rho_1\rho_2+2\rho_1)]\,\sigma^2_{c_1}$$

$$= (1+\rho_1\rho_2)(\lambda_1\overline{s^2_1}\,\bar{c}+\sigma^2_{w_1})+(1-\rho_1\rho_2+2\rho_1)(\lambda_2\overline{s^2_2}\,\bar{c}+\sigma^2_{w_2})$$

Applying Cramer's formula for the solution of a set of linear equation, we rearrange the expressions obtained, cancel a common term, $(1 - \rho_1 \rho_2)$, and obtain

$$\sigma_{c_1}^2 = \frac{[(1 - \rho_1 \rho_2)(1 + 2\rho_2) - 2\rho_2^3][\lambda_1 \overline{s_1^2} \bar{c} + \sigma_{w_1}^2] + (1 + \rho_1 \rho_2)(\lambda_2 \overline{s_2^2} \bar{c} + \sigma_{w_2}^2)}{(1 - \rho_1 \rho_2)(1 - \rho_1 - \rho_2)(1 + \rho_1 + \rho_2 + 2\rho_1 \rho_2)} \qquad 3.2.17$$

σ_{c_2} is obtained from 3.2.17 by interchanging the indices 1 and 2.

B4 Solution of Equation 3.2.19

We have

$$R(0) = \sigma_w^2 + \lambda \overline{s^2} \bar{c} + \rho^2 \sigma_c^2 \qquad 3.2.19a$$

for $J = 1, \ldots, N-1$

$$R(N-J) = \rho \left[\sum_{i=0}^{J} R(i) + \sum_{i=1}^{N-J-1} R(i) \right], \qquad 3.2.19b$$

where

$$\sigma_c^2 = N R(0) + 2 \sum_{i=1}^{N-1} (N-i) R(i) \qquad 3.2.19c$$

Using Equation 3.2.19b we show that all $R(i)$, $i = 1, \ldots, N-1$ are equal. This is done in two steps.

<u>Lemma</u> B4.1: For $J = 1, \ldots, N-2$ $R(N-J) = R(J+1)$

Proof : Define $I \triangleq N-(J+1)$. For $J = 1, \ldots, N-2$, I is also in the range $1, \ldots, N-2$. Using 3.2.19b we find

$$R(N-I) = R(J+1) = \rho \left[\sum_{i=0}^{N-(J+1)} R(i) + \sum_{i=1}^{J} R(i) \right]$$

$$= \rho \left[\sum_{i=0}^{J} R(i) + \sum_{i=1}^{N-(J+1)} R(i) \right]$$

Hence, for $J = 1, \ldots, N-2$

$$R(N-J) = R(J+1) \qquad \text{B4.1}$$

Q.E.D.

<u>Lemma</u> B4.2: For $i = 1, \ldots, N-1$ all $R(i)$ are equal

Proof : It is enough to show that for $J = 1, \ldots, N-2$

$R(N-J) = R(N-(J+1))$

Using 3.2.19b to find $R(N-(J+1))$ we obtain

$$R(N-(J+1)) = \rho \left[\sum_{i=0}^{J+1} R(i) + \sum_{i=1}^{N-J-2} R(i) \right] \qquad \text{B4.2}$$

Subtracting B4.2 from 3.2.19b yields

$$R(N-J) - R(N-(J+1)) = -\rho \left[R(J+1) - R(N-(J+1)) \right] \qquad \text{B4.3}$$

Using B4.1 in the above equation we obtain

$$R(N-J) - R(N-(J+1)) = -\rho\,[R(N-J) - R(N-(J+1))]$$

Hence

$$(1+\rho)\,[R(N-J) - R(N-(J+1))] = 0$$

which implies for $J = 1, \ldots, N-2$

$$R(N-J) = R(N-(J+1))$$

Therefore, for $i = 1, \ldots N-1$

$$R(i) = R(1) \qquad \text{B4.4}$$

Q.E.D.

Substituting B4.4. into 3.2.19b yields

$$R(1) = \rho\,(R(0) + (N-1)\,R(1))$$

$$R(1) = \frac{\rho}{1-(N-1)\rho}\,R(0) = \frac{\frac{\rho_0}{N}}{1-\rho_0+\frac{\rho_0}{N}}\,R(0) \qquad \text{B4.5}$$

Substituting B4.4 and B4.5 into 3.2.19c yields

$$\sigma_c^2 = N\,(R(0) + (N-1)\,R(1)) = \frac{N}{\rho}\,R(1) = \frac{N}{1-\rho_0+\frac{\rho_0}{N}}\,R(0) \qquad \text{B4.6}$$

Substituting B4.6 into 3.2.19a yields

$$R(0) = \sigma_w^2 + \lambda \overline{s^2} \bar{c} + \frac{N\rho^2}{1 - \rho_0 + \frac{\rho_0}{N}} R(0) \qquad \text{B4.7}$$

$$R(0) = \frac{(1 - \rho_0 + \frac{\rho_0}{N})(\sigma_w^2 + \lambda \overline{s^2} \bar{c})}{1 - \rho_0 + \frac{\rho_0}{N} - \frac{\rho_0^2}{N}} = \frac{1}{N} \cdot \frac{(1 - \rho_0 + \frac{\rho_0}{N})(\sigma_d^2 + \lambda_0 \overline{s^2} \bar{c})}{(1 - \rho_0)(1 + \frac{\rho_0}{N})}$$

where

$$\sigma_d^2 = N\sigma_w^2, \quad \lambda_0 = N\lambda, \quad \rho_0 = N\rho$$

Substituting B4.7 into B4.6 yields

$$\sigma_c^2 = \frac{\sigma_d^2 + \lambda_0 \overline{s^2} \bar{c}}{(1 - \rho_0)(1 + \frac{\rho_0}{N})} \qquad \text{B4.8}$$

Equations B4.5, B4.7 and B4.8 yield Equation 3.2.21.

B5 The Development of Equation 3.2.25

We have

$$_{k+1}F_{N,N,N}(\underline{x}) = \frac{W^2 \dddot{W} - 3W\dot{W}\ddot{W} + 2(\dot{W})^3}{(W)^3} - \lambda \dddot{S} \sum_{i=1}^{N} {}_kF_i(\underline{z})$$

$$+ 3\lambda^2 \dot{S}\ddot{S} \sum_{i=1}^{N}\sum_{l=1}^{N} {}_kF_{i,l}(\underline{z}) - (\lambda \dot{S})^3 \sum_{i=1}^{N}\sum_{l=1}^{N}\sum_{m=1}^{N} {}_kF_{i,l,m}(\underline{z}) \qquad \text{3.2.25a}$$

for J, P = 1, .., N-1 :

$$ {}_{k+1}F_{N,N,J}(\underline{x}) = -\lambda\ddot{S}\sum_{i=1}^{N} {}_kF_{i,J+1}(\underline{z}) + (\lambda\dot{S})^2\sum_{i=1}^{N}\sum_{l=1}^{N} {}_kF_{i,l,J+1}(\underline{z}) \quad 3.2.25b $$

$$ {}_{k+1}F_{N,J,P}(\underline{x}) = -\lambda\dot{S}\sum_{i=1}^{N} {}_kF_{i,J+1,P+1}(\underline{z}) \quad 3.2.25c $$

At $\underline{x} = \underline{0}$ using 2.1.16 we have for the symmetric nonrandom case

$$ S = 1, \quad \dot{S} = -\bar{s}, \quad \ddot{S} = \overline{s^2} = \bar{s}^2, \quad \dddot{S} = -\overline{s^3} = -\bar{s}^3 $$

$$ W = 1, \quad \dot{W} = -\bar{w}, \quad \ddot{W} = \overline{w^2} = \bar{w}^2, \quad \dddot{W} = -\overline{w^3} = -\bar{w}^3 $$

Therefore

$$ \dddot{W} - 3\dot{W}\ddot{W} + 2(\dot{W})^3 = -\overline{w^3} + 3\bar{w}\,\overline{w^2} - 2\bar{w}^3 = -\overline{(w-\bar{w})^3} = -\delta_w^3 = 0 \quad B5.1 $$

For $k = KN$ where $K \to \infty$ we reach a steady state and are dealing with the steady state joint probability density function of $\underline{\theta} = (\theta_1, .., \theta_i, .. \theta_N)$. For the moment we increase the dimension of the random vector by one to get $(\theta_0, \theta_1, .., \theta_i, \theta_N)$ where θ_0 is the value of TST_N in the previous cycle. At $\underline{x} = 0$, $\underline{z} = 0$ and we have for i,j,l = 1, .., N-1

$$ {}_{k+1}F_{i,j,l}(\underline{0}) = {}_kF_{i+1,j+1,l+1}(\underline{0}) = -\overline{(\theta_i - \bar{\theta}_i)(\theta_j - \bar{\theta}_j)(\theta_l - \bar{\theta}_l)} $$

$$ {}_kF_1(\underline{0}) = -\bar{\theta}_0 \quad B5.2 $$

From Equation 3.2.7 we have

$$\sum_{i=1}^{N} {}_kF_i\,(\underline{0}) = -\bar{c}$$

From Equation 3.2.24 we have

$$\sum_{i=1}^{N}\sum_{l=1}^{N} {}_kF_{i,l}\,(\underline{0}) = \sigma_c^2$$

From Equation B4.6 we have

$$\sum_{i=1}^{N} {}_kF_{i,i+1}\,(\underline{0}) = \frac{1}{N}\,\sigma_c^2 \qquad \text{B5.3}$$

Substituting B5.1 - 3 into 3.2.25 yields

$$\overline{(\theta_N - \bar{\theta}_N)^3} = \rho \bar{s}^2\,\bar{c} + 3\,\rho^2 \bar{s}\,\sigma_c^2 + \rho^3\,\delta_c^3$$

for J, P = 1, .., N-1

$$\overline{(\theta_N - \bar{\theta}_N)^2 (\theta_J - \bar{\theta}_J)} = \rho\,\bar{s}\,\frac{\sigma_c^2}{N} + \rho^2 \sum_{i=0}^{N-1}\sum_{l=0}^{N-1} \overline{(\theta_J - \bar{\theta}_J)\,(\theta_i - \bar{\theta}_i)\,(\theta_l - \bar{\theta}_l)}$$

$$\overline{(\theta_N - \bar{\theta}_N)\,(\theta_J - \bar{\theta}_J)\,(\theta_P - \bar{\theta}_P)} = \rho \sum_{i=0}^{N-1} \overline{(\theta_J - \bar{\theta}_J)\,(\theta_P - \bar{\theta}_P)\,(\theta_i - \bar{\theta}_i)}\,,$$

where

$$\delta_c^3 \triangleq \overline{(c - \bar{c})^3} = \sum_{i=0}^{N-1}\sum_{l=0}^{N-1}\sum_{m=0}^{N-1} \overline{(\theta_i - \bar{\theta}_i)\,(\theta_l - \bar{\theta}_l)\,(\theta_m - \bar{\theta}_m)} \qquad \text{B5.4}$$

$\rho = \lambda \bar{s}$, $N\rho = \rho_0$

From Equation 3.2.24 we have

$$\bar{c} = \frac{(1 - N\rho)(1+\rho)}{N\rho\bar{s}}\,\sigma_c^2 \qquad \text{B5.5}$$

Substituting B5.5 into the first equation of B5.4 yields Equation 3.2.26.

B6 δ_c^3 for $N = 2$

We have

$$R(0,0) = \frac{\rho_0\left(1 - \frac{\rho_0}{2} + \rho_0^2\right)}{2(1-\rho_0)\left(1+\frac{\rho_0}{2}\right)}\,\overline{s^2}\,\bar{c} + \frac{\rho_0^3}{8}\,\delta_c^3 \qquad \text{3.2.30a}$$

$$R(0,1) = \frac{\rho_0^2}{4(1-\rho_0)\left(1+\frac{\rho_0}{2}\right)}\,\overline{s^2}\,\bar{c} + \frac{\rho_0^2}{4}\left(R(0,0) + 2R(0,1) + R(1,0)\right) \qquad \text{3.2.30b}$$

$$R(1,0) = \frac{\rho_0}{2}\left(R(0,1) + R(0,0)\right) \qquad \text{3.2.30c}$$

where

$$\delta_c^3 = 2R(0,0) + 3R(0,1) + 3R(1,0) \qquad \text{3.2.30d}$$

Substitution of 3.2.30c into 3.2.30b yields

$$R(0, 1) = \frac{\rho_0^2}{4(1-\rho_0)(1+\frac{\rho_0}{2})}\,\bar{s}^2\,\bar{c} + \frac{\rho_0^2}{4}\left(R(0,0) + 2R(0,1) + \frac{\rho_0}{2}(R(0,1) + R(0,0))\right)$$

from it we find

$$R(0, 1) = \frac{\rho_0^2\,\bar{s}^2\,\bar{c}}{4(1-\rho_0)(1+\frac{\rho_0}{2})^2(1-\frac{\rho_0}{2}-\frac{\rho_0^2}{4})} + \frac{\rho_0^2}{4(1-\frac{\rho_0}{2}-\frac{\rho_0^2}{4})}\,R(0,0) \qquad \text{B6.1}$$

Substituting 3.2.30c into 3.2.30d yields

$$\delta_c^3 = 2R(0,0) + 3R(0,1) + \frac{3\rho_0}{2}(R(0,1) + R(0,0))$$

$$\delta_c^3 = 2(1+\frac{3}{4}\rho_0)R(0,0) + 3(1+\frac{\rho_0}{2})R(0,1) \qquad \text{B6.2}$$

Substituting B6.1, for R(0, 1), into B6.2 yields

$$\delta_c^3 = 2(1+\frac{3}{4}\rho_0)R(0,0) + \frac{3\rho_0^2(1+\frac{\rho_0}{2})}{4(1-\frac{\rho_0}{2}-\frac{\rho_0^2}{4})}\left(\frac{\bar{s}^2\,\bar{c}}{(1-\rho_0)(1+\frac{\rho_0}{2})^2} + R(0,0)\right) \qquad \text{B6.3}$$

Equations B6.3 and 3.2.30a together yield

$$R(0,0) = \frac{4(1-\frac{\rho_0}{2}-\frac{\rho_0^2}{4})\,\delta_c^3 - \frac{3\rho_0^2\,\bar{s}^2\,\bar{c}}{(1-\rho_0)(1+\frac{\rho_0}{2})}}{8(1+\frac{3}{4}\rho_0)(1-\frac{\rho_0}{2}-\frac{\rho_0^2}{4}) + 3\rho_0^2(1+\frac{\rho_0}{2})} = \frac{\rho_0(1-\frac{\rho_0}{2}+\rho_0^2)\,\bar{s}^2\,\bar{c}}{2(1-\rho_0)(1+\frac{\rho_0}{2})} + \frac{\rho_0^3}{8}\,\delta_c^3 \qquad \text{B6.4}$$

The above equation yields δ_c^3 directly. After rearrangement and cancellation of common terms, we obtain

$$\delta_c^3 = \frac{\rho_0 \left(1 + \frac{\rho_0}{2} + \frac{5}{8}\rho_0^2 + \frac{3}{8}\rho_0^3 - \frac{1}{4}\rho_0^4\right)}{(1-\rho_0)^2 \left(1 + \frac{\rho_0}{2}\right)^2 \left(1 + \frac{\rho_0^2}{4}\left(1 - \frac{\rho_0}{2}\right)\right)} \cdot \bar{s}^2 \bar{c} \qquad 3.2.31$$

B7 The Development of Equation 3.3.15

This part is closely related to B2.

~~We have~~

$$ {}_{k+1}F_{N,N}(\underline{x}) = \frac{W_{k-1}(u_k)\,[\ddot{u}_k \dot{W}_{k-1}(u_k) + (\dot{u}_k)^2 \ddot{W}_{k-1}(u_k)] - (\dot{u}_k)^2 (\dot{W}_{k-1}(u_k))^2}{(W_{k-1}(u_k))^2} $$

$$ - \lambda_k \ddot{B}_k(x_N) \sum_{i=2}^{N} {}_kF_i(\underline{z}) + (\lambda_k \dot{B}_k(x_N))^2 \sum_{i=2}^{N}\sum_{l=2}^{N} {}_kF_{i,l}(\underline{z}) $$

for

$J = 1, \ldots, N-1$

$$ {}_{k+1}F_{N,J}(\underline{x}) = -\lambda_k \dot{B}_k(x_N) \sum_{i=2}^{N} {}_kF_{i,J+1}(\underline{z}) \qquad 3.3.15 $$

At $\underline{x} = \underline{0}$, we have $\underline{z} = \underline{0}$ and $u_k = 0$, and B2.1 is satisfied as well.

$\dot{B}_k(0)$ and $\ddot{B}_k(0)$ are obtained from Equations A-9, A-10. From Equations 2.3.14, 3.3.1, A-9, and A-10 we find

$$\dot{u}_k = 1 - \lambda_k \dot{B}_k(0) = \frac{1}{1-\rho_k}$$

$$\ddot{u}_k = -\lambda_k \ddot{B}_k(0) = \frac{-\lambda_k \overline{s_k^2}}{(1-\rho_k)^3} \qquad \text{B7.1}$$

From these we have

$$\ddot{u}_k \dot{W}_{k-1}(0) + (\dot{u}_k)^2 [\ddot{W}_{k-1}(0) - (\dot{W}_{k-1}(0))^2] = \frac{\lambda_k \overline{s_k^2}}{(1-\rho_k)^3} \bar{w}_{k-1} + \frac{1}{(1-\rho_k)^2} \sigma^2_{w_{k-1}} \qquad \text{B7.2}$$

From Equation 2.1.4 we have

$$\sigma^2_{v_{k+N}} = \overline{(w_{k+N-1} - \bar{w}_{k+N-1} + \sum_{i=1}^{N-1} (\theta_{k+i} - \bar{\theta}_{k+i}))^2} = \sigma^2_{w_{k-1}} + \sum_{i=1}^{N-1}\sum_{l=1}^{N-1} \overline{(\theta_{k+i} - \bar{\theta}_{k+i})(\theta_{k+l} - \bar{\theta}_{k+l})} \qquad \text{B7.3}$$

$$\bar{v}_{k+N} = \bar{w}_{k-1} + \sum_{i=1}^{N-1} \bar{\theta}_{k+i} \qquad 3.3.11$$

Using these relations in 3.3.15 yields

$$\overline{(\theta_{k+N} - \bar{\theta}_{k+N})^2} = \sigma^2_{\theta_{k+N}} = \frac{1+\rho_k}{1-\rho_k} \sigma^2_{w_{k-1}} + \frac{\lambda_k \overline{s_k^2}}{(1-\rho_k)^3} \bar{v}_{k+N} + \frac{\rho_k^2}{(1-\rho_k)^3} \sigma^2_{v_{k+N}} \qquad \text{B7.4}$$

$$\overline{(\theta_{k+N} - \bar{\theta}_{k+N})(\theta_{k+j} - \bar{\theta}_{k+j})} = \frac{\rho_k}{1-\rho_k} \sum_{i=1}^{N-1} \overline{(\theta_{k+j} - \bar{\theta}_{k+j})(\theta_{k+i} - \bar{\theta}_{k+i})}$$

Equations B7.3 and B7.4 are Equation 3.3.16.

B8 Solution of 3.3.17 for N = 2

For $i = 1, 2$, and $j = 3 - i$,

Equation 3.3.18 can be rewritten as

$$R(i, i) = \frac{1}{(1 - \rho_i)^2} \left(\lambda_i \overline{s_i^2}\, \bar{c} + \sigma_{w_j}^2 + \rho_i^2 R(j, j) \right) \qquad \text{B8.1}$$

$i = 1, 2 \qquad j = 3 - i$

two equations with two unknowns, substituting the second equation into the first yields

$$R(1,1) = \frac{1}{(1 - \rho_1)^2} (\lambda_1 \overline{s_1^2}\, \bar{c} + \sigma_{w_2}^2) + \frac{\rho_1^2}{(1-\rho_1)^2 (1-\rho_2)^2} (\lambda_2 \overline{s_2^2}\, \bar{c} + \sigma_{w_1}^2) + \frac{\rho_1^2 \rho_2^2}{(1-\rho_1)^2 (1-\rho_2)^2} R(1, 1)$$

B8.2

Rearrangement of B8.2 yields :

$$R(1, 1) = \frac{\rho_1^2 (\lambda_2 \overline{s_2^2}\, \bar{c} + \sigma_{w_1}^2) + (1 - \rho_2)^2 (\lambda_1 \overline{s_1^2}\, \bar{c} + \sigma_{w_2}^2)}{(1 - \rho_1 - \rho_2)(1 - \rho_1 - \rho_2 + 2 \rho_1 \rho_2)} \qquad \text{B8.3}$$

R (2, 2) is obtained by interchanging the indices 1 and 2 in B8.3.

Using 3.3.18 we find

$$\sigma^2_{v_1} = \sigma^2_{w_2} + R\,(2,\,2) = \sigma^2_{w_2} + \frac{\rho_2^2\,(\lambda_1 \overline{s_1^2}\,\overline{c} + \sigma^2_{w_2}) + (1-\rho_1)^2\,(\lambda_2 \overline{s_2^2}\,\overline{c} + \sigma^2_{w_1})}{(1-\rho_1-\rho_2)\,(1-\rho_1-\rho_2+2\,\rho_1\,\rho_2)} \qquad 3.3.19$$

B9 Solution of Equation 3.3.20

For $i = 1, 2, 3$ the set (i, j, l) is $(1, 2, 3)$, $(2, 3, 1)$ and $(3, 1, 2)$, respectively.

Define

$$\sigma^2_i \triangleq \sigma^2_{v_i} - \sigma^2_{w_l}$$

$$f_i \triangleq \lambda_i\,\overline{s_i^2}\,\overline{c} + \sigma^2_{w_l} \qquad B9.1$$

Equation 3.3.20 can be rewritten as

for $i = 1, 2, 3$

$$R\,(i,\,i) = \frac{1}{(1+\rho_i)^2}\cdot[\,f_i + \rho_i^2\,\sigma_i^2\,] \qquad B9.2a$$

$$R\,(j,\,i) = \frac{\rho_i}{(1-\rho_i)}\cdot[\,R\,(i,\,i) + R\,(i,\,l)\,] \qquad B9.2b$$

where

$$\sigma_i^2 = R(i,i) + R(l,l) + 2R(l,i), \quad \text{B9.2c}$$

6 linear equations with 6 unknowns. From the three equations of B9.2b we find for i = 1, 2, 3 :

$$R(i,i) = \frac{1}{g}\left(\rho_i(1-\rho_l)(1-\rho_i)R(i,i) + \rho_i(1-\rho_l)\rho_i R(l,l) + \rho_i\rho_l\rho_i R(i,i)\right) \quad \text{B9.3}$$

where

$$g = (1-\rho_1)(1-\rho_2)(1-\rho_3) - \rho_1\rho_2\rho_3$$

Employing B9.3 into B9.2c yields for i = 1, 2, 3

$$\sigma_i^2 = \frac{1}{g}[2(1-\rho_i)\rho_i\rho_l R(i,i) + ((1-\rho_i)(1-\rho_i)(1+\rho_l) - \rho_i\rho_i\rho_l)R(i,i)$$
$$+((1-\rho_i)(1-\rho_i)(1-\rho_l) + \rho_i\rho_i\rho_l)] \quad \text{B9.4}$$

Replacing all R (i, i) i = 1, 2, 3 in B9.4 by the expressions of B9.2a, we obtain for i = 1, 2, 3 :

$$\sigma_i^2 = \frac{1}{g}[2(1-\rho_i)\rho_i\rho_l\frac{(f_i+\rho_i^2\sigma_i^2)}{(1-\rho_i)^2} + ((1-\rho_i)(1-\rho_i)(1+\rho_l) - \rho_i\rho_i\rho_l)\frac{(f_i+\rho_i^2\sigma_i^2)}{(1-\rho_i)^2}$$
$$+((1-\rho_i)(1-\rho_i)(1-\rho_l) + \rho_i\rho_i\rho_l)\frac{(f_l+\rho_l^2\sigma_l^2)}{(1-\rho_l)^2}]$$

The above is a set of 3 linear equations with 3 unknowns. A simple rearrangement yields, for i = 1, 2, 3 :

$$A(i)\,\sigma_i^2 - \rho_j^2\, B(i)\,\sigma_j^2 - \rho_l^2\, C(i)\,\sigma_l^2 = F(i) \qquad \text{B9.5}$$

where

$$A(i) = (1-\rho_j)^2 (1-\rho_l)^2 \left[(1-\rho_i)^2 (1-\rho_j)(1-\rho_l) - (1+\rho_i)\,\rho_i\,\rho_j\,\rho_l\right]$$

$$B(i) = (1-\rho_i)(1-\rho_l)^2 \left[(1-\rho_i)(1-\rho_j)(1+\rho_l) - \rho_i\,\rho_j\,\rho_l\right]$$

$$C(i) = (1-\rho_i)(1-\rho_j)^2 \left[(1-\rho_i)(1-\rho_j)(1-\rho_l) + \rho_i\,\rho_j\,\rho_l\right]$$

$$D(i) = 2\,\rho_j\,\rho_l\,(1-\rho_j)^2 (1-\rho_l)^2$$

$$F(i) = D(i)\,f_i + B(i)\,f_j + C(i)\,f_l$$

Using Cramer's formula we find :

$$\sigma_{v_i}^2 = \sigma_{w_l}^2 + \frac{T_{(i,j,l)}}{A} \qquad \text{B9.6}$$

where

$$A = \det \begin{vmatrix} A(1) & -\rho_2^2 B(1) & -\rho_3^2 C(1) \\ -\rho_1^2 C(2) & A(2) & -\rho_3^2 B(2) \\ -\rho_1^2 B(3) & -\rho_2^2 C(3) & A(3) \end{vmatrix}$$

$$T_{(i,j,l)} = \det \begin{vmatrix} F(i) & -\rho_j^2 B(i) & -\rho_l^2 C(i) \\ F(j) & A(j) & -\rho_l^2 B(j) \\ F(l) & -\rho_j^2 C(l) & A(l) \end{vmatrix}$$

To obtain an explicit expression appears to be too lengthy.

Equations B9.1, B9.5 and B9.6 yield Equation 3.3.21.

B10 The Solution of 3.3.22

We have

$$R(0) = \frac{1}{(1-\rho)^2}\left[\lambda \overline{s^2}\,\bar{c} + (1-\rho^2)\,\sigma_w^2 + \rho^2\,\sigma_v^2\right] \qquad \text{B10.1a}$$

and for $J = 1, .., N-1$:

$$R(N-J) = \frac{\rho}{1-\rho}\left(\sum_{i=0}^{J-1} R(i) + \sum_{i=1}^{N-J-1} R(i)\right) \qquad \text{B10.1b}$$

where

$$\sigma_v^2 = \sigma_w^2 + (N-1)\,R(0) + 2\sum_{i=1}^{N-2}(N-1-i)\,R(i) \qquad \text{B10.1c}$$

$$\sigma_c^2 = N\,R(0) + 2\sum_{i=1}^{N-1}(N-i)\,R(i) \qquad \text{B10.1d}$$

In a manner similar to that used in B4, we show that all $R(i)$, $i = 1, .., N-1$ are equal.

<u>Lemma</u> B10.1 : For $J = 1, .., N-1$ $\quad R(N-J) = R(J)$

Proof : Define $I = N-J$, $I = 1, .., N-1$. Using B10.1b, we find

$$R(N-I) = R(J) = \frac{\rho}{1-\rho}\left(\sum_{i=0}^{N-I-1} R(i) + \sum_{i=1}^{I-1} R(i)\right) = \frac{\rho}{1-\rho}\left(\sum_{i=0}^{I-1} R(i) + \sum_{i=1}^{N-I-1} R(i)\right).$$

Hence,

$$R(N-J) = R(J) \qquad J = 1, \ldots, N-1 \qquad \text{B10.2}$$

Q.E.D.

Lemma B10.2: For $i = 1, \ldots, N-1$ all $R(i)$ are equal.

Proof: It is enough to show that for $J = 1, \ldots, N-2$

$$R(N-J) = R(N-(J+1)) \qquad \text{B10.3}$$

Using B10.1b to find $R(N-(J+1))$, we have

$$R(N-(J+1)) = \frac{\rho}{1-\rho}\left[\sum_{i=0}^{J} R(i) + \sum_{i=1}^{N-J-2} R(i)\right] \qquad \text{B10.4}$$

Subtracting B10.1b from B10.4 yields

$$R(N-(J+1)) - R(N-J) = \frac{\rho}{1-\rho}\left(R(J) - R(N-(J+1))\right) \qquad \text{B10.5}$$

Using B10.2 in the above equation, we obtain

$$\frac{1}{1-\rho}\left(R(J) - R(J+1)\right) = 0$$

Hence

for $i = 1, \ldots, N-1$ all $R(i)$ are equal.

Q.E.D.

We have for $i = 1, .., N-1$

$$R(i) = R(1) \tag{B10.6}$$

Substituting B10.6 into B10.1b

$$R(1) = \frac{\rho}{1-\rho} (R(0) + (N-2) R(1))$$

$$R(1) = \frac{\rho}{1-(N-1)\rho} R(0) = \frac{\frac{\rho_0}{N}}{1-\rho_0+\frac{\rho_0}{N}} R(0) \tag{B10.7}$$

Substituting the above into B10.1c yields

$$\sigma_v^2 = \sigma_w^2 + \frac{(N-1)(1-\frac{\rho_0}{N})}{1-\rho_0+\frac{\rho_0}{N}} R(0) \tag{B10.8}$$

Direct substitution of B10.8 into B10.1a yields

$$\sigma_\theta^2 \triangleq R(0) = \frac{1}{N} \cdot \frac{(1-\rho_0+\frac{\rho_0}{N})(\sigma_d^2+\lambda_0 \bar{s}^2 \bar{c})}{(1-\rho_0)(1-\frac{\rho_0}{N})} \tag{B10.9}$$

where

$$\sigma_d^2 = N\sigma_w^2 \qquad \lambda_0 = N\lambda \qquad \rho_0 = N\rho$$

B10.8 is analogous to B4.7 .

Substituting B10.9 into B10.8 yields

$$\sigma_v^2 = \frac{\sigma_d^2}{N} + \frac{N-1}{N} \cdot \frac{\sigma_d^2 + \lambda_0 \overline{s^2}\,\bar{c}}{1-\rho_0} \qquad \text{B10.10}$$

and by B10.1d

$$\sigma_c^2 = \frac{\sigma_d^2 + \lambda_0 \overline{s^2}\,\bar{c}}{(1-\rho_0)(1-\frac{\rho_0}{N})} \qquad \text{B10.11}$$

Equations B10.6, B10.7, B10.9, B10.10 and B10.11 lead to Equation 3.3.23 .

B11 The Development of Equation 3.3.25

This part refers back to Equation 3.3.25 in the main text. At $\underline{x} = \underline{0}$ we have $\underline{z} = 0$, $u = 0$, and B2.1 is satisfied as well. $\dot{u}$ and $\ddot{u}$ are obtained from B7.1. Equations 2.3.14, 3.3.1, A-9 and A-10 together yield for the symmetric nonrandom case :

$$\dddot{u} = -\lambda \dddot{B} = \frac{\rho \overline{s^2}}{(1-\rho)^4} + \frac{3\rho^2 \overline{s^2}^2}{(1-\rho)^5} \qquad \text{B11.1}$$

The expression involving W, u and all their derivatives in 3.3.25 is equal at $\underline{x} = \underline{0}$ to :

$$-\dddot{u}\,\bar{w} + 3\dot{u}\ddot{u}(\overline{w^2} - \bar{w}^2) - (\dot{u})^3(\overline{w^3} - 3\bar{w}\,\overline{w^2} + 2\bar{w}^3)$$

$$= -\dddot{u}\,\bar{w} + 3\dot{u}\ddot{u}\,\sigma_w^2 - (\dot{u})^3 \delta_w^3 = -\dddot{u}\,\bar{w} \qquad \text{B11.2}$$

For $k = KN$ where $K \to \infty$ we reach a steady state and are dealing with the steady state joint probability density function of $\underline{\theta} = (\theta_1, \ldots, \theta_N)'$, and Equation B5.2 is satisfied here as well.

From Equation 3.3.11 we have

$$\bar{w} - \sum_{i=1}^{N-1} {}_kF_i \, (\underline{0}) = \bar{v}$$

From Equation 3.3.22 we have

$$\sum_{i=1}^{N-1} \sum_{l=1}^{N-1} {}_kF_{i,l} \, (\underline{0}) = \sigma_v^2$$

and from Equations 3.3.22 and 3.3.23 we have for $J < N$, $N > 1$ B11.3

$$\sum_{i=1}^{N-1} {}_kF_{i,J+1} \, (\underline{0}) = \frac{1}{N-1} \cdot \sigma_v^2$$

By direct substitution of all the above results into 3.3.25 we obtain for the symmetric nonrandom case, for $N > 1$:

$$\overline{(\theta_N - \bar{\theta}_N)^3} = \left(\frac{1}{(1-\rho)^4} + \frac{3\rho}{(1-\rho)^5}\right) \rho \, \bar{s}^{-2} \, \bar{v} + \frac{3\rho^2}{(1-\rho)^4} \cdot \bar{s} \, \sigma_v^2 + \frac{\rho^3}{(1-\rho)^3} \, \delta_v^3$$

For $J, P = 1, \ldots, N-1$

$$\overline{(\theta_N - \bar{\theta}_N)^2 (\theta_J - \bar{\theta}_J)} = \frac{\rho}{(N-1)(1-\rho)^3} \sigma_v^2 + \frac{\rho^2}{(1-\rho)^2} \sum_{i=1}^{N-1} \sum_{l=1}^{N-1} \overline{(\theta_J - \bar{\theta}_J)(\theta_i - \bar{\theta}_i)(\theta_l - \bar{\theta}_l)}$$

$$\overline{(\theta_N - \bar\theta_N)(\theta_J - \bar\theta_J)(\theta_p - \bar\theta_p)} = \frac{\rho}{1-\rho} \sum_{j=1}^{N-1} \overline{(\theta_J - \bar\theta_J)(\theta_p - \bar\theta_p)(\theta_i - \bar\theta_i)}, \qquad \text{B11.4}$$

where

$$\delta_v^3 = \sum_{i=1}^{N-1} \sum_{l=1}^{N-1} \sum_{m=1}^{N-1} \overline{(\theta_i - \bar\theta_i)(\theta_l - \bar\theta_l)(\theta_m - \bar\theta_m)}$$

Replacing σ_v^2 by the expression given by Equation 3.3.24 yields Equation 3.3.26.

B12 δ_v^3 for $N = 3$

We have

$$R(0, 0) = \frac{\rho_0}{3(1-\rho_0)(1-\frac{\rho_0}{3})^4} \overline{s}^2 \overline{v} + \frac{(\frac{\rho_0}{3})^3}{(1-\frac{\rho_0}{3})^3} \delta_v^3 \qquad 3.3.30a$$

$$R(0, 1) = \frac{(\frac{\rho_0}{3})^2}{(1-\rho_0)(1-\frac{\rho_0}{3})^4} \overline{s}^2 \overline{v} + \frac{(\frac{\rho_0}{3})^2}{(1-\frac{\rho_0}{3})^2} (R(0, 0) + 2R(0, 1) + R(1, 0)) \qquad 3.3.30b$$

$$R(1, 0) = \frac{\frac{\rho_0}{3}}{1-\frac{\rho_0}{3}} (R(0, 0) + R(0, 1)) \qquad 3.3.30c$$

where

$$\delta_v^3 = 2R(0, 0) + 3R(0, 1) + 3R(1, 0) \qquad 3.3.30d$$

The method of the solution is similar to that used in B6 for the solution of the analogous Equation 3.2.30.

Substituting 3.3.30c into 3.3.30b yields

$$R\,(0,\,1) = \frac{\dfrac{\left(\frac{\rho_0}{3}\right)^2}{(1-\rho_0)(1-\frac{\rho_0}{3})}\,\overline{s}^2\,\overline{v} + \left(\frac{\rho_0}{3}\right)^2 R\,(0,\,0)}{\left(1-\frac{\rho_0}{3}\right)^3 - \left(2-\frac{\rho_0}{3}\right)\left(\frac{\rho_0}{3}\right)^2} \qquad \text{B12.1}$$

Substituting 3.3.30c into 3.3.30d yields

$$\delta_v^3 = \frac{2+\frac{\rho_0}{3}}{1-\frac{\rho_0}{3}}\,R\,(0,\,0) + \frac{3}{1-\frac{\rho_0}{3}}\,R\,(0,\,1) \qquad \text{B12.2}$$

Substituting B12.1 into B12.2 yields

$$R\,(0,\,0) = \frac{\left(1-\rho_0+\left(\frac{\rho_0}{3}\right)^2\right)}{\left(2-\rho_0-\left(\frac{\rho_0}{3}\right)^2\right)}\,\delta_v^3 + \frac{3\left(\frac{\rho_0}{3}\right)^2}{(1-\rho_0)\left(1-\frac{\rho_0}{3}\right)^2\left(2-\rho_0-\left(\frac{\rho_0}{3}\right)^2\right)}\,\overline{s}^2\,\overline{v}$$

B12.3

Finally substituting Equation B12.3 into Equation 3.3.30a yields Equation 3.3.31.

APPENDIX C

ON THE RELATION BETWEEN CYCLE TIME AND INTERVISIT TIME IN THE EXHAUSTIVE MODEL

The intervisit time, v, and the cycle time, c, of a terminal in the exhaustive model, are defined by Equation 2.1.4. The intervisit time is the time that separates the server's departure from a terminal from the server's subsequent return to the same terminal. The cycle time is the sum of the intervisit time and the service time of all customers in the terminal until it is empty for the first time. New arrivals to the terminal are according to a Poisson arrival process having parameter λ.

In the following we derive the relation between the Laplace transform of c and v. This relation holds for both transient and steady states.

For any terminal i, $i = 1, \ldots, N$, we define

t = the cycle time of the terminal.

τ = the intervisit time of the terminal.

m = number of customers in the terminal the server fin[illegible]en reaches it, i.e., at the end of the intervisit time (at the [illegible]nning of the intervisit time the terminal is empty).

$P_c(\cdot)$, $P_v(\cdot)$ = the probability density function of c and v respectively.

$C(\cdot)$, $V(\cdot)$ = the Laplace transforms of $P_c(\cdot)$ and $P_v(\cdot)$ respectively.

$P_b(\cdot), B(\cdot)$ = the probability density function and Laplace transform of a busy period.

By the law of total probability, we have :

$$P_c(t) = \int_0^t \sum_{m=0}^{\infty} \text{Prob}(t/m, \tau)\,\text{Prob}(m/\tau)\, P_v(\tau)\, d\tau \qquad \text{C.1}$$

By the same argument used to derive Equations 2.3.6 and 2.3.7, Equation C.1 can be expressed as follows :

$$P_c(t) = \int_0^t \sum_{m=0}^{\infty} P_b^{(m)*}(t-\tau)\,[(\lambda\tau)^m/m!]\exp(-\lambda\tau)\, P_v(\tau)\, d\tau \qquad \text{C.2}$$

Applying the Laplace transform we obtain

$$C(x) = \int_0^{\infty}\int_0^t \sum_{m=0}^{\infty} \exp(-tx)\; P_b^{(m)*}(t-\tau)\,[(\lambda\tau)^m/m!]\exp(-\lambda\tau)\, P_v(\tau)\, d\tau\, dt \qquad \text{C.3}$$

$$C(x) = \int_0^{\infty}\sum_{m=0}^{\infty}\int_{\tau}^{\infty} \exp[-(t-\tau)x]\; P_b^{(m)*}(t-\tau)\,[(\lambda\tau)^m/m!]\exp(-\tau(x+\lambda))P_v(\tau)\,dt\, d\tau \qquad \text{C.4}$$

Integration over t yields

$$C(x) = \int_0^{\infty}\sum_{m=0}^{\infty} [(\lambda\tau B(x))^m/m!]\exp[-\tau(x+\lambda)]\, P_v(\tau)\, d\tau \qquad \text{C.5}$$

summation over m yields

$$C(x) = \int_0^\infty \exp[-\tau(x + \lambda(1 - B(x)))] \, P_v(\tau) \, d\tau \qquad \text{C.6}$$

and we obtain

$$C(x) = V[x + \lambda(1 - B(x))] \qquad \text{C.7}$$

To find the relation between moments of c and v, we differentiate C.7 w.r.t. x.

Omitting all arguments, we obtain

$$\dot{C} = (1 - \lambda \dot{B}) \dot{V} \qquad \text{C.8a}$$

$$\ddot{C} = (1 - \lambda \dot{B})^2 \ddot{V} - \lambda \ddot{B} \dot{V} \qquad \text{C.8b}$$

$$\dddot{C} = (1 - \lambda \dot{B})^3 \dddot{V} - 3 \lambda \ddot{B} (1 - \lambda \dot{B}) \ddot{V} - \lambda \dddot{B} \dot{V} \qquad \text{C.8c}$$

At $x = 0$, $B(0) = 0$, and using Equation A.9 and A.10 in C.8 where $\rho = \lambda \bar{s}$, we obtain :

From C.8a :

$$\bar{c} = \bar{v} / (1 - \rho) \qquad 3.3.14$$

From C.8b :

$$\overline{c^2} = \overline{v^2} / (1 - \rho)^2 + \lambda \overline{s^2} \, \bar{v} / (1 - \rho)^3$$

and hence

$$\sigma_v^2 = (1 - \rho)^2 \sigma_c^2 - \lambda \overline{s^2} \, \bar{c}$$

The above is used in Equation 3.3.23 .

From C.8c :

$$\overline{c^3} = \overline{v^3} / (1 - \rho)^3 + 3 \lambda \overline{s^2}\, \overline{v^2} / (1 - \rho)^4 + [\lambda \overline{s^3} / (1 - \rho)^4 + 3 \lambda^2 \left(\overline{s^2}\right)^2 / (1 - \rho)^5] \, \overline{v} \qquad \text{C.9}$$

Just as we calculate the first three moments of v in Section 3 of Chapter III, Equation C.9 enables us to derive the corresponding moments of c .

Derivation of the relation between c and v in the gating model is very similar ; the only difference is that B (x) is replaced by S (x), and we obtain from Equation C.7 for the gating model

$$C(x) = V [x + \lambda (1 - S(x))] \qquad \text{C.10}$$

APPENDIX D

THE NORMALIZED CROSS CORRELATION BETWEEN TERMINALS IN DIFFERENT CYCLES

The Gating Model

For the steady state in the symmetric case, we have from Equation 3.2.22

$$G^{R}_{\theta}(l) = \begin{cases} 1 & l = 0 \\ \dfrac{\rho_0}{1-\rho_0+\dfrac{\rho_0}{N}} & l = 1, \ldots, N-1 \end{cases} \qquad \text{D.1}$$

For $l \geq N$, we can write :

$$G^{R}_{\theta}(l) = \frac{1}{\sigma^2_{\theta}} \int_0^{\infty}\int_0^{\infty} (\theta_i - \bar{\theta}_i)(\theta_{i+l} - \bar{\theta}_{i+l}) \,\text{Prob}\,(\theta_i, \theta_{i+l})\, d\theta_i\, d\theta_{i+l} \qquad \text{D.2}$$

$$= \frac{1}{\sigma^2_{\theta}} \int_0^{\infty}\cdots\int_0^{\infty} \sum_{n_{i+l}=0}^{\infty} (\theta_i - \bar{\theta}_i)(\theta_{i+l} - \bar{\theta}_{i+l})\,\text{Prob}(\theta_i, \theta_{i+l-N}, \ldots, \theta_{i+l}, n_{i+l})\, d\theta_i \prod_{j=0}^{N} d\theta_{i+l-j}$$

D.3

$$= \frac{1}{\sigma^2_{\theta}} \int_0^{\infty}\cdots\int_0^{\infty} \sum_{n_{i+l}=0}^{\infty} (\theta_i - \bar{\theta}_i)(\theta_{i+l} - \bar{\theta}_{i+l})\,\text{Prob}(\theta_{i+l}/n_{i+l}, \theta_{i+l-N}, \ldots, \theta_{i+l-1})$$

$$\cdot\ \text{Prob}\,(n_{i+l}/\theta_i, \theta_{i+l-N}, \ldots, \theta_{i+l-1}) \cdot \text{Prob}\,(\theta_i, \theta_{i+l-N}, \ldots, \theta_{i+l-1})\, d\theta_i \prod_{j=0}^{N} d\theta_{i+l-j}$$

D.4

where n_{i+1} is the number of customers that are served in T_{i+1}. As in the derivation of Equations 2.2.2 and 2.2.3 we can establish :

$$\text{Prob}\,(\theta_{i+1}/n_{i+1}, \theta_i, \theta_{i+1-N}, \ldots, \theta_{i+1-1}) = \text{Prob}\,(\theta_{i+1}/n_{i+1}) = P_s^{*(n_{i+1})}(\theta_{i+1}) * P_{w_{i+1}}(\theta_{i+1}) \qquad \text{D.5}$$

Paralleling the derivation of Equation 2.2.4, we have

$$\text{Prob}\,(n_{i+1}/\theta_i, \theta_{i+1-N}, \ldots, \theta_{i+1-1}) = \text{Prob}\,(n_{i+1}/\theta_{i+1-N}, \ldots, \theta_{i+1-1})$$

$$= [(\lambda \sum_{j=1}^{N} \theta_{i+1-j})^{n_{i+1}}/n_{i+1}!] \exp(-\lambda \sum_{j=1}^{N} \theta_{i+1-j}) \qquad \text{D.6}$$

Substituting D.5 and D.6 into D.4 yields

$$G^R{}_\theta(1) = \frac{1}{\sigma_\theta^2} \int_0^\infty \cdots \int_0^\infty \sum_{n_{i+1}=0}^{\infty} (\theta_i - \bar{\theta}_i)(\theta_{i+1} - \bar{\theta}_{i+1}) \; P_s^{*(n_{i+1})}(\theta_{i+1}) * P_{w_{i+1}}(\theta_{i+1}) [(\lambda \sum_{j=1}^{N} \theta_{i+1-j})^{n_{i+1}}/n_{i+1}!]$$

$$\cdot \exp(-\lambda \sum_{j=1}^{N} \theta_{i+1-j}) \, \text{Prob}\,(\theta_i, \theta_{i+1-N}, \ldots, \theta_{i+1-1}) \, d\theta_i \prod_{j=0}^{N} d\theta_{i+1-j} \qquad \text{D.7}$$

Integrating over θ_{i+1}, we obtain

$$G^R{}_\theta(1) = \frac{1}{\sigma_\theta^2} \int_0^\infty \cdots \int_0^\infty \sum_{n_{i+1}=0}^{\infty} (\theta_i - \bar{\theta}_i)(n_{i+1}\overline{s+w}_{i+1} - \bar{\theta}_{i+1}) [(\lambda \sum_{j=1}^{N} \theta_{i+1-j})^{n_{i+1}}/n_{i+1}!] \exp(-\lambda \sum_{j=1}^{N} \theta_{i+1-j})$$

$$\cdot \, \text{Prob}\,(\theta_i, \theta_{i+1-N}, \ldots, \theta_{i+1-1}) \, d\theta_i \prod_{j=1}^{N} d\theta_{i+1-j} \qquad \text{D.8}$$

Summing over n_{i+1} we obtain

$$G^R_\theta(l) = \frac{1}{\sigma_\theta^2} \int_0^\infty \cdots \int_0^\infty (\theta_i - \bar{\theta}_i)(\lambda \bar{s} \sum_{j=1}^{N} \theta_{i+l-j} + \bar{w}_{i+l} - \bar{\theta}_{i+l}) \text{Prob}(\theta_i, \theta_{i+l-N}, \ldots, \theta_{i+l-1}) d\theta_i \prod_{j=1}^{N} d\theta_{i+l-j} \qquad \text{D.9}$$

Using Equation 3.2.3 in D.9 we obtain

$$G^R_\theta(l) = \frac{\frac{\rho_0}{N}}{\sigma_\theta^2} \int_0^\infty \cdots \int_0^\infty (\theta_i - \bar{\theta}_i)(\sum_{j=1}^{N} (\theta_{i+l-j} - \bar{\theta}_{i+l-j})) \text{Prob}(\theta_i, \theta_{i+l-N}, \ldots, \theta_{i+l-1}) d\theta_i \prod_{j=1}^{N} d\theta_{i+l-j} \qquad \text{D.10}$$

$$G^R_\theta(l) = \frac{\rho_0}{N} \sum_{j=1}^{N} \int_0^\infty \int_0^\infty \frac{(\theta_i - \bar{\theta}_i)(\theta_{i+l-j} - \bar{\theta}_{i+l-j})}{\sigma_\theta^2} \text{Prob}(\theta_i, \theta_{i+l-j}) \, d\theta_i \, d\theta_{i+l-j} \qquad \text{D.11}$$

Equation D.11 is simply

$$G^R_\theta(l) = \frac{\rho_0}{N} \sum_{j=1}^{N} G^R_\theta(l-i) \qquad \text{D.12}$$

Equation D.12 is Equation 5.3.2a in Chapter V.

<u>The Exhaustive Model</u>

Equation D.1 holds here as well. As in the development for the gating model, for $l \geq N-1$ we can write

$$E^{R_\theta}(l) = \frac{1}{\sigma_\theta^2} \int_0^\infty \int_0^\infty (\theta_i - \bar{\theta}_i)(\theta_{i+l} - \bar{\theta}_{i+l}) \, \text{Prob}(\theta_i, \theta_{i+l}) \, d\theta_i \, d\theta_{i+l} \quad \text{D.13}$$

$$= \frac{1}{\sigma_\theta^2} \int_0^\infty \cdots \int_0^\infty \sum_{m_{i+l}=0}^{\infty} (\theta_i - \bar{\theta}_i)(\theta_{i+l} - \bar{\theta}_{i+l}) \, \text{Prob}(\theta_i, \theta_{i+l-N+1}, \ldots, \theta_{i+l}, w_{i+l-1}, m_{i+l})$$

$$\cdot \, d\theta_i \prod_{j=0}^{N-1} d\theta_{i+l-j} \cdot dw_{i+l-1} \quad \text{D.14}$$

$$= \frac{1}{\sigma_\theta^2} \int_0^\infty \cdots \int_0^\infty \sum_{m_{i+l}=0}^{\infty} (\theta_i - \bar{\theta}_i)(\theta_{i+l} - \bar{\theta}_{i+l}) \, \text{Prob}(\theta_{i+l}/m_{i+l}, w_{i+l-1}, \theta_i, \theta_{i+l-N+1}, \ldots, \theta_{i+l-1})$$

$$\cdot \, \text{Prob}(m_{i+l}/w_{i+l-1}, \theta_i, \theta_{i+l-N+1}, \ldots, \theta_{i+l-1}) \cdot \text{Prob}(w_{i+l-1}/\theta_i, \theta_{i+l-N+1}, \ldots, \theta_{i+l-1})$$

$$\cdot \, \text{Prob}(\theta_i, \theta_{i+l-N+1}, \ldots, \theta_{i+l-1}) \, d\theta_i \prod_{j=0}^{N-1} d\theta_{i+l-j} \, dw_{i+l-1} \quad \text{D.15}$$

where m_{i+l} is the number of customers that exist at T_{i+l} at the moment the server reaches it.

As in Equation 2.3.5 and 2.3.6 we have :

$$\text{Prob}(\theta_{i+l}/m_{i+l}, w_{i+l-1}, \theta_i, \theta_{i+l-N+1}, \ldots, \theta_{i+l-1}) = \text{Prob}(\theta_{i+l}/m_{i+l}, w_{i+l-1})$$

$$= P_b^{(m_{i+l})} * (\theta_{i+l} - w_{i+l-1}) \quad \text{D.16}$$

As in Equation 2.3.7 we have :

$$\text{Prob}\left(^{m_{i+l}}/w_{i+l-1}, \theta_i, \theta_{i+l-N+1}, \ldots, \theta_{i+l-1}\right) = \text{Prob}\left(^{m_{i+l}}/w_{i+l-1}, \sum_{j=1}^{N-1} \theta_{i+l-j}\right)$$

$$= \left[\left(\lambda \left(w_{i+l-1} + \sum_{j=1}^{N-1} \theta_{i+l-j}\right)\right)^{m_{i+l}} / m_{i+l}!\right] \exp\left(-\lambda \left(w_{i+l-1} + \sum_{j=1}^{N-1} \theta_{i+l-j}\right)\right) \qquad \text{D.17}$$

As in 2.3.8 we have :

$$\text{Prob}\left(^{w_{i+l-1}}/\theta_i, \theta_{i+l-N+1}, \ldots, \theta_{i+l-1}\right) = P_{w_{i+l-1}}\left(w_{i+l-1}\right) \qquad \text{D.18}$$

Substituting D.16, D.17, and D.18 into D.15 we obtain :

$$E^{R_\theta}(l) = \frac{1}{\sigma_\theta^2} \int_0^\infty \cdots \int_0^\infty \sum_{m_{i+l}=0}^\infty (\theta_i - \bar{\theta}_i)(\theta_{i+l} - \bar{\theta}_{i+l})\, P_b^{*(m_{i+l})}(\theta_{i+l} - w_{i+l-1}) \left[\left(\lambda\left(w_{i+l-1} + \sum_{j=1}^{N-1} \theta_{i+l-j}\right)\right)^{m_{i+l}} / m_{i+l}!\right]$$

$$\cdot \exp\left[-\lambda\left(w_{i+l-1} + \sum_{j=1}^{N-1} \theta_{i+l-j}\right)\right] \text{Prob}\left(\theta_i, \theta_{i+l-N+1}, \ldots, \theta_{i+l-1}\right)$$

$$\cdot P_{w_{i+l-1}}\left(w_{i+l-1}\right) d\theta_i \prod_{j=0}^{N-1} d\theta_{i+l-j} \cdot dw_{i+l-1} \qquad \text{D.19}$$

Integrating over θ_{i+l}, with the help of Equation A.10, we have :

$$E^{R}_{\theta}(l) = \frac{1}{\sigma_\theta^2} \int_0^\infty \cdots \int_0^\infty \sum_{m_{i+l}=0}^{\infty} (\theta_i - \bar{\theta}_i)\left(m_{i+l}\frac{\bar{s}}{1-\frac{\rho_0}{N}} + w_{i+l-1} - \bar{\theta}_{i+l}\right)\left[\left(\lambda\left(w_{i+l-1} + \sum_{j=1}^{N-1}\theta_{i+l-j}\right)\right)^{m_{i+l}} / m_{i+l}!\right]$$

$$\cdot \exp\left[-\lambda\left(w_{i+l-1} + \sum_{j=1}^{N-1}\theta_{i+l-j}\right)\right] \text{Prob}\,(\theta_i, \theta_{i+l-N+1}, \ldots, \theta_{i+l-1})$$

$$\cdot\, P_{w_{i+l-1}}(w_{i+l-1})\, d\theta_i \prod_{j=1}^{N-1} d\theta_{i+l-j}\, dw_{i+l-1} \qquad \text{D.20}$$

Summing over m_{i+l}, we obtain

$$E^{R}_{\theta}(l) = \frac{1}{\sigma_\theta^2} \int_0^\infty \cdots \int_0^\infty (\theta_i - \bar{\theta}_i)\left(\frac{w_{i+l-1} + \frac{\rho_0}{N}\sum_{j=1}^{N-1}\theta_{i+l-j}}{1-\frac{\rho_0}{N}} - \bar{\theta}_{i+l}\right)$$

$$\cdot\, P_{w_{i+l-1}}(w_{i+l-1})\, \text{Prob}\,(\theta_i, \theta_{i+l-N+1}, \ldots, \theta_{i+l-1})\, dw_{i+l-1}\, d\theta_i \prod_{j=1}^{N-1} d\theta_{i+l-j}$$

D.21

Integrating over w_{i+l-1}, we obtain

$$E^{R}_{\theta}(l) = \frac{1}{\sigma_\theta^2} \int_0^\infty \cdots \int_0^\infty (\theta_i - \bar{\theta}_i)\left(\frac{\bar{w}_{i+l-1} + \frac{\rho_0}{N}\sum_{j=1}^{N-1}\theta_{i+l-j}}{1-\frac{\rho_0}{N}} - \bar{\theta}_{i+l}\right)$$

$$\cdot\, \text{Prob}\,(\theta_i, \theta_{i+l-N+1}, \ldots, \theta_{i+l-1})\, d\theta_i \prod_{j=1}^{N-1} d\theta_{i+l-j} \qquad \text{D.22}$$

Using Equation 3.3.5 in D.22, we obtain

$$E^{R_\theta}(l) = \frac{\frac{\rho_0/N}{1-\rho_0/N}}{\sigma_\theta^2} \int_0^\infty \cdots \int_0^\infty (\theta_i - \bar{\theta}_i)\left(\sum_{j=1}^{N-1} (\theta_{i+l-j} - \bar{\theta}_{i+l-j}\right) \text{Prob}(\theta_i \mid \theta_{i+l-N+1}, \ldots, \theta_{i+l-1}) d\theta_i \prod_{j=1}^{N-1} d\theta_{i+l-j} \tag{D.23}$$

$$E^{R_\theta}(l) = \frac{\rho_0/N}{1-\rho_0/N} \sum_{j=1}^{N-1} \int_0^\infty \int_0^\infty \frac{(\theta_i - \bar{\theta}_i)(\theta_{i+l-j} - \bar{\theta}_{i+l-j})}{\sigma_\theta^2} \text{Prob}(\theta_i, \theta_{i+l-j})\, d\theta_i\, d\theta_{i+l-j} \tag{D.24}$$

The above equation is simply

$$E^{R_\theta}(l) = \frac{\rho_0/N}{1-\rho_0/N} \sum_{j=1}^{N-1} E^{R_\theta}(l-j) \tag{D.25}$$

Equation D.25 is Equation 5.3.2b in Chapter V .

APPENDIX E

DERIVATION OF THE FIRST TWO MOMENTS OF THE WAITING TIME

E.1 The Gating Model

We derived in Chapter VI the Laplace transform of the probability density function of customer waiting time in T_i. From Equation 6.2.9 we have

$$Q(x) = \frac{C(x) - C(z)}{\bar{c}\,[z - x]} \qquad \text{E.1.1}$$

where $z = \lambda(1 - S(x))$

At $x = 0$, we have :

$$z = 0$$

$$\dot{z} = -\lambda \dot{S}(0) = \lambda \bar{s} = \rho$$

$$\ddot{z} = -\lambda \ddot{S}(0) = -\lambda \overline{s^2} \qquad \text{E.1.2}$$

$$\dddot{z} = -\lambda \dddot{S}(0) = \lambda \overline{s^3}$$

At $x = 0$, the right hand side of Equation E.1.1 is $^0/_0$. Therefore we have to use L'hôpital's rule to verify : $Q(0) \stackrel{\Delta}{=} 1$. Differentiating the numerator and the denominator of Equation E.1.1 w.r.t. x we have :

$$Q(0) = \lim_{x \to 0} \frac{\dot{C}(x) - \dot{z}\dot{C}(z)}{\bar{c}(\dot{z} - 1)} = \frac{\bar{c}(1-\rho)}{\bar{c}(1-\rho)} = 1 \qquad \text{E.1.3}$$

In order to find $\bar{q}$, we differentiate Equation E.1.1 w.r.t. x, we obtain :

$$\dot{Q}(x) = \frac{(z-x)\,(\dot{C}(x) - \dot{z}\,\dot{C}(z)) - (\dot{z}-1)(C(x) - C(z))}{\bar{c}\,(z-x)^2} \qquad \text{E.1.4}$$

At $x = 0$ we encounter $^0/_0$. Applying L'hôpital's rule we obtain :

$$\dot{Q}(0) = -\bar{q} = \lim_{x \to 0} \frac{(z-x)\,[\ddot{C}(x) - \ddot{z}\,\dot{C}(z) - (\dot{z})^2\,\ddot{C}(z)] - \ddot{z}\,[C(x) - C(z)]}{\bar{c}\; 2\,(z-x)\,(\dot{z}-1)} \qquad \text{E.1.5}$$

$$\bar{q} = \lim_{x \to 0} \frac{\ddot{z}}{2\,(\dot{z}-1)} \cdot \frac{\dot{C}(x) - C(z)}{\bar{c}\,(z-x)} - \lim_{x \to 0} \frac{\ddot{C}(x) - (\dot{z})^2\,\ddot{C}(z) - \ddot{z}\,\dot{C}(z)}{2\,\bar{c}\,(\dot{z}-1)} \qquad \text{E.1.6}$$

Substituting Equation E.1.1 with $x = 0$ and Equation E.1.2 into E.1.6 we obtain

$$\bar{q} = \frac{\lambda \bar{s}^2}{2(1-\rho)} + \frac{(1-\rho^2)\,\bar{c}^2 - \lambda \bar{s}^2\,\bar{c}}{2\,\bar{c}\,(1-\rho)} = (1+\rho)\,\frac{\bar{c}^2}{2\,\bar{c}} = (1+\rho)\left(\frac{\bar{c}}{2} + \frac{\sigma_c^2}{2\,\bar{c}}\right) \qquad \text{E.1.7}$$

Equation E.1.7 is cited as Equation 6.2.10a in the thesis.

In order to find the variance of q we differentiate Equation E.1.4 w.r.t. x, we obtain

$$\ddot{Q}(x) = \frac{1}{\bar{c}\,(z-x)^3}\Big[(z-x)\,(-\ddot{z}\,(C(x) - \dot{C}(z)) + (z-x)\,(\ddot{C}(x) - (\dot{z})^2\,\ddot{C}(z) - \ddot{z}\,\dot{C}(z)))$$

$$- 2\,(\dot{z}-1)((z-x)\,(\dot{C}(x) - \dot{z}\,\dot{C}(z)) - (\dot{z}-1)\,(C(x) - C(z)))\Big] \qquad \text{E.1.8}$$

At $x = 0$, the right hand side of Equation E.1.8 is $^0/_0$. Using L'hôpital's rule we obtain :

$$\ddot{Q}(0) = \overline{q^2} = \lim_{x \to 0} \frac{1}{3\bar{c}\,(\dot{z}-1)(z-x)^2} \Big[\ddot{z}(z-1)\big(-\ddot{z}\,(C(x) - C(z)) + (z-x)(\ddot{C}(x) - (\dot{z})^2\ddot{C}(z) - \ddot{z}\dot{C}(z))\big)$$

$$- 2\ddot{z}\,\big((z-x)(\dot{C}(x) - \dot{z}\,\dot{C}(z)) - (\dot{z}-1)(C(x) - C(z))\big)$$

$$+ (z-x)\big(-\dddot{z}\,(C(x) - C(z)) - \ddot{z}\,(\dot{C}(x) - \dot{z}\,\dot{C}(z)) + (\dot{z}-1)(\ddot{C}(x) - (\dot{z})^2\ddot{C}(z) - \ddot{z}\,\dot{C}(z))\big)$$

$$+ (z-x)^2\big(\dddot{C}(x) - (\dot{z})^3\,\dddot{C}(z) - 3\,\dot{z}\,\ddot{z}\,\ddot{C}(z) - \dddot{z}\,\dot{C}(z)\big)\Big] \qquad \text{E.1.9}$$

Rearranging the above equation we obtain

$$\overline{q^2} = \lim_{x \to 0} \Big[\frac{\dddot{C}(x) - (\dot{z})^3\,\dddot{C}(z) - 3\,\dot{z}\,\ddot{z}\,\ddot{C}(z) - \dddot{z}\,\dot{C}(z)}{3\bar{c}\,(\dot{z}-1)} - \frac{\dddot{z}}{3(\dot{z}-1)} \cdot \frac{C(x) - C(z)}{\bar{c}\,(z-x)}$$

$$+ \frac{\ddot{z}}{\dot{z}-1} \cdot \frac{(\dot{z}-1)(\dot{C}(x) - \dot{C}(z)) - (z-x)(\ddot{C}(x) - \dot{z}\,\ddot{C}(z))}{\bar{c}\,(z-x)^2} \qquad \text{E.1.10}$$

Using the results of Equations E.1.1, E.1.2, and E.1.4 with $x = 0$ we obtain

$$\overline{q^2} = \frac{(1-\rho^3)\,\overline{C^3} - 3\rho\lambda\overline{s^2}\,\overline{c^2} - \lambda\overline{s^3}\,\bar{c}}{3\bar{c}\,(1-\rho)} + \frac{\lambda\overline{s^3}}{3(1-\rho)} + \frac{\lambda\overline{s^2}}{1-\rho}\,\bar{q}$$

$$\overline{q^2} = \frac{\overline{c^3}\,(1-\rho^3)}{3\bar{c}(1-\rho)} + \bar{q}\,\frac{\lambda\overline{s^2}}{1+\rho} \qquad \text{E.1.11}$$

And

$$\sigma_q^2 = \overline{q^2} - \bar{q}^2 = \frac{\overline{c^3}}{3\bar{c}}\,(1+\rho+\rho^2) + \bar{q}\,\Big(\frac{\lambda\overline{s^2}}{1+\rho} - \bar{q}\Big) \qquad \text{E.1.12}$$

Equation E.1.12 is relabeled by 6.2.10b in the thesis.

E.2 The Exhaustive Model

We deal with Equation 6.3.20

$$Q(x) = \frac{1-\rho}{\bar{v}} \cdot \frac{1-V(x)}{z} \qquad \text{E.2.1}$$

where $z = x - \lambda(1 - S(x))$.

At $x = 0$ we have

$$\begin{aligned} z &= 0 \\ \dot{z} &= 1 + \lambda \dot{S}(0) = 1 - \lambda \bar{s} = 1 - \rho \\ \ddot{z} &= \lambda \ddot{S}(0) = \lambda \overline{s^2} \\ \dddot{z} &= \lambda \dddot{S}(0) = -\lambda \overline{s^3} \end{aligned} \qquad \text{E.2.2}$$

For the sake of simplicity we omit the argument x in Equation E.2.1. and we have

$$Q(x) = \frac{1-\rho}{\bar{v}} \; \frac{1-V}{z} \qquad \text{E.2.3}$$

At $x = 0$ the right hand side of E.2.3 is $^0/0$. Using L'hôpital's rule we have :

$$Q(0) = \lim_{x \to 0} \frac{1-\rho}{\bar{v}} \cdot \frac{-\dot{V}}{\dot{z}} = \frac{1-\rho}{\bar{v}} \cdot \frac{\bar{v}}{1-\rho} = 1 \qquad \text{E.2.4}$$

Differentiating Equation E.2.3 w.r.t. x we have

$$\dot{Q}(x) = \frac{1-\rho}{\bar{v}} \cdot \frac{-\dot{z}V - z(1-\dot{V})}{z^2} \qquad E.2.5$$

At x = 0 we encounter $^0/_0$. Using L'hôpital's rule, we obtain

$$\dot{Q}(0) = -\bar{q} = \lim_{x \to 0} \frac{1-\rho}{\bar{v}} \cdot \frac{-\ddot{z}V - z(1-\ddot{V})}{2 z \dot{z}} = \lim_{x \to 0} \left[\frac{1-\rho}{\bar{v}} \cdot \frac{-\ddot{V}}{2\dot{z}} - \frac{\ddot{z}}{2\dot{z}} \cdot \frac{1-\rho}{\bar{v}} \cdot \frac{1-V}{z} \right]$$

E.2.6

Using Equation E.2.2 and Equation E.2.3 with x = 0, we obtain

$$\bar{q} = \frac{\bar{v}^2}{2\bar{v}} + \frac{\lambda \bar{s}^2}{2(1-\rho)} \qquad E.2.7a$$

Using Equation C.9 in E.2.7 we obtain :

$$\bar{q} = (1-\rho)\frac{\bar{c}^2}{2\bar{c}} = (1-\rho)\left(\frac{\bar{c}}{2} + \frac{\sigma_c^2}{2\bar{c}}\right) \qquad E.2.7b$$

Equation E.2.7 is cited as Equation 6.3.21a in the thesis.

Differentiating Equation E.2.5 w.r.t. x , we obtain

$$\ddot{Q}(x) = \frac{1-\rho}{\bar{v} z^3}\left[-z(z\ddot{V} + \ddot{z}(1-V)) + 2\dot{z}(z\dot{V} + \dot{z}(1-V)) \right] \qquad E.2.8$$

At x = 0 we encounter by $^0/_0$. Using L'hôpital's rule we obtain

$$\ddot{Q}(0) = \overline{q^2} = \lim_{x \to 0} \frac{1-\rho}{3\bar{v}z^2 \dot{z}} \cdot [\dot{z}(z\ddot{V} + \ddot{z}(1-V)) - z(\dot{z}\ddot{V} + z\dddot{V} + \dddot{z}(1-V) - \ddot{z}\dot{V}) + 2\ddot{z}(z\dot{V} + \dot{z}(1-V))] \quad \text{E.2.9}$$

Rearranging the above equation we obtain

$$\overline{q^2} = \lim_{x \to 0} \left[\frac{-(1-\rho)\dddot{V}}{3\bar{v}\dot{z}} - \frac{\dddot{z}}{3\dot{z}} \cdot \frac{1-\rho}{\bar{v}} \cdot \frac{1-V}{z} + \frac{\ddot{z}}{\dot{z}} \cdot \frac{1-\rho}{\bar{v}} \cdot \frac{z\dot{V} + \dot{z}(1-V)}{z^2}\right] \quad \text{E.2.10}$$

Using the results of Equation E.2.1, E.2.2, and E.2.4, we obtain

$$\overline{q^2} = \frac{\overline{v^3}}{3\bar{v}} + \frac{\lambda \overline{s^3}}{3(1-\rho)} + \bar{q}\frac{\lambda \overline{s^2}}{1-\rho} \quad \text{E.2.11}$$

And

$$\sigma_q^2 = \overline{q^2} - \bar{q}^2 = \frac{\overline{v^3}}{3\bar{v}} + \frac{\lambda \overline{s^3}}{3(1-\rho)} + \bar{q}\left(\frac{\lambda \overline{s^2}}{1-\rho} - \bar{q}\right) \quad \text{E.2.12}$$

Equation E.2.12 is quoted as Equation 6.3.21b in the thesis.

APPENDIX F

NUMERICAL RESULTS USED IN THE FIGURES

Fig. 5-1. $\bar{w} = 1$, $\rho_0 = .8$, $N = 10$.

Cycle Number	A : $\bar{\theta}_1 = (1, .., 1)$		B : $\bar{\theta}_1 = (20, .., 20)$	
	$G^{\bar{c}}$	$E^{\bar{v}}$	$C^{\bar{c}}$	$E^{\bar{v}}$
1	10.0	10.0	200	181
2	21.6	22.2	156	135
3	30.6	31.1	123	102
4	36.9	36.7	99.1	80.8
5	41.2	40.2	83.1	67.7
6	44.0	42.4	72.3	59.5
7	46.0	43.8	65.0	54.4
9	48.2	45.1	56.8	49.2
12	49.4	45.8	52.1	46.8
15	49.9	46.0	50.0	46.1
20	50.0	46.0	50.0	46.0

Fig. 5-2. $N = 8$, $\rho_0 = .9$.

Cycle No.	G^R_c	E^R_c	E^R_v
0	1.00	1.00	1.00
1	.850	.822	.813
2	.710	.654	.644
3	.591	.517	.509
4	.491	.408	.402
6	.339	.255	.251
8	.235	.159	.157
10	.162	.099	.098
14	.077	.039	.038
20	.025	.009	.009
30	.004	.001	.001

Fig. 6-1. $\bar{s} = 1.$; $\bar{d} = 0.$, all $N \geq 1$.

ρ_0	$\bar{q}_G = \bar{q}_E$
.1	.0556
.2	.1250
.3	.2143
.4	.3333
.5	.5000
.6	.7500
.7	1.167
.8	2.000
.9	4.500

Fig. 6-2. $\bar{s} = 1.$, $\bar{d} = 1.$, $N = 1, 2, 3, \infty$.

ρ_0	$\bar{q}_{E_1}$	$\bar{q}_{E_2}$	$\bar{q}_{E_3}$	$\bar{q}_{E_\infty} = \bar{q}_{G_\infty}$	$\bar{q}_{G_2}$	$\bar{q}_{G_1}$
.1	.556	.583	.593	.611	.639	.667
.2	.625	.687	.708	.750	.812	.875
.3	.714	.821	.857	.928	1.04	1.14
.4	.833	1.00	1.06	1.16	1.33	1.50
.5	1.00	1.25	1.33	1.50	1.75	2.00
.6	1.25	1.62	1.75	2.00	2.37	2.75
.7	1.67	2.25	2.44	2.83	3.42	4.00
.8	2.50	3.50	3.83	4.50	5.50	6.50
.9	5.00	7.25	8.00	9.50	11.7	14.0

Fig. 6-3. $\bar{s} = 1.$, $\bar{d} = 100.$ $N = 1, 2, 3, \infty$.

ρ_0	$\bar{q}_{E_1}$	$\bar{q}_{E_2}$	$\bar{q}_{E_3}$	$\bar{q}_{E_\infty} = \bar{q}_{G_\infty}$	$\bar{q}_{G_2}$	$\bar{q}_{G_1}$
.1	50.1	52.8	53.8	55.6	58.4	61.2
.2	50.1	56.4	58.5	62.6	68.9	75.1
.3	50.2	60.9	64.5	71.6	82.4	93.1
.4	50.3	67.0	72.6	83.6	100	117
.5	50.5	75.5	83.8	100	125	150
.6	50.7	88.2	101	125	163	201
.7	51.2	109	129	168	226	284
.8	52.0	152	185	252	382	452
.9	54.5	279	354	504	729	954

Fig. 6-4. $\bar{s} = 1.$, $\bar{d} = 0.$, $N = 1, 2, 3, \infty$.

ρ_0	$\sigma^2_{q_{E_1}}$	$\sigma^2_{q_{E_2}}$	$\sigma^2_{q_{E_3}}$	$\sigma^2_{q_{E_\infty}} = \sigma^2_{q_{G_\infty}}$	$\sigma^2_{q_{G_2}}$	$\sigma^2_{q_{G_1}}$
.1	.0401	.0413	.0416	.0422	.0410	.0401
.2	.0990	.105	.107	.109	.103	.0990
.3	.189	.209	.213	.219	.198	.189
.4	.333	.386	.395	.407	.352	.333
.5	.583	.712	.729	.750	.619	.583
.6	1.06	1.37	1.40	1.44	1.18	1.06
.7	2.14	2.93	2.99	3.05	2.29	2.14
.8	5.33	7.79	7.90	8.00	5.72	5.33
.9	23.2	36.3	36.5	36.7	25.0	23.2

Figs. 6-5, 6-6. $\bar{s} = 1.$, $\bar{d} = 1.$ $N = 1, 2, 3, \infty$.

ρ_0	$\sigma^2_{q_{E_1}}$	$\sigma^2_{q_{E_2}}$	$\sigma^2_{q_{E_3}}$	$\sigma^2_{q_{E_\infty}} = \sigma^2_{q_{G_\infty}}$	$\sigma^2_{q_{G_2}}$	$\sigma^2_{q_{G_1}}$
.1	.123	.165	.179	.207	.221	.236
.2	.182	.289	.324	.396	.414	.443
.3	.272	.485	.555	.696	.700	.743
.4	.417	.812	.940	1.19	1.15	1.21
.5	.738	1.40	1.63	2.08	1.91	2.22
.6	1.15	2.56	2.99	3.83	3.33	3.49
.7	2.22	5.27	6.13	7.86	6.50	6.80
.8	5.42	13.5	15.7	20.1	15.8	16.5
.9	23.3	61.3	70.6	90.1	67.1	70.7

Fig. 6-7. $\bar{s} = 1.$, $\bar{d} = 100.$ $N = 1, 2, 3, \infty$.

ρ_0	$\sigma^2_{q_{E_1}}$	$\sigma^2_{q_{E_2}}$	$\sigma^2_{q_{E_3}}$	$\sigma^2_{q_{E_\infty}} = \sigma^2_{q_{G_\infty}}$	$\sigma^2_{q_{G_2}}$	$\sigma^2_{q_{G_1}}$
.1	833	982	966	1040	937	845
.2	833	1060	1140	1320	1080	859
.3	834	1240	1400	1730	1270	881
.4	834	1510	1780	2370	1550	913
.5	834	1930	2380	3430	1990	967
.6	834	2650	3460	5400	2750	1070
.7	835	4110	5700	9650	4300	1290
.8	839	8010	11900	21800	8430	1950
.9	857	27500	43900	37900	29200	5590

Fig. 6-8. $\bar{s} = 1.$, $\bar{d} = 0.$, $N = 2$, $\rho_i = \frac{1}{4}\rho_0$, $\rho_j = \frac{3}{4}\rho_0$

ρ_0	$\bar{q}_{E_i}$	$\bar{q}_{E_j}$	$\bar{q}_{E_0}$ =	$\bar{q}_{G_0}$	$\bar{q}_{G_j}$	$\bar{q}_{G_i}$
.1	.05786	.05479	.05556	.05556	.05619	.05366
.2	.1365	.1212	.1250	.1250	.1276	.1172
.3	.2471	.2033	.2143	.2143	.2205	.1956
.4	.4091	.3081	.3333	.3333	.3455	.2970
.5	.6579	.4474	.5000	.5000	.5215	.4354
.6	1.065	.6449	.7500	.7500	.7871	.6387
.7	1.800	.9556	1.167	1.167	1.232	.9712
.8	3.364	1.545	2.000	2.000	2.125	1.626
.9	8.262	3.246	4.500	4.500	4.812	3.564

Fig. 6-9. $\bar{s} = 1.$, $\bar{d} = 1.$, $N = 2$, $\rho_i = \frac{1}{4}\rho_0$, $\rho_j = \frac{3}{4}\rho_0$

ρ_0	$\bar{q}_{E_i}$	$\bar{q}_{E_j}$	$\bar{q}_{E_0}$	$\bar{q}_{G_0}$	$\bar{q}_{G_j}$	$\bar{q}_{G_i}$
.1	.5995	.5687	.5764	.6458	.6534	.6231
.2	.7303	.6524	.6719	.8281	.8464	.7734
.3	.9079	.7569	.7946	1.062	1.096	.9634
.4	1.159	.8914	.9583	1.375	1.429	1.214
.5	1.533	1.072	1.187	1.812	1.897	1.560
.6	2.128	1.332	1.531	2.469	2.600	2.076
.7	3.175	1.747	2.104	3.562	3.773	2.930
.8	5.364	2.545	3.250	5.750	6.125	4.626
.9	12.14	4.871	6.688	12.31	13.19	9.689

Fig. 6-10. $\bar{s} = 1.$, $\bar{d} = 100.$, $N = 2$, $\rho_i = \frac{1}{4}\rho_0$, $\rho_j = \frac{3}{4}\rho_0$.

ρ_0	$\bar{q}_{E_i}$	$\bar{q}_{E_j}$	$\bar{q}_{E_0}$	$\bar{q}_{G_0}$	$\bar{q}_{G_j}$	$\bar{q}_{G_i}$
.1	54.22	51.44	52.14	59.08	59.78	57.00
.2	59.51	53.25	54.81	70.44	72.00	65.74
.3	66.32	55.56	58.25	85.04	87.72	76.98
.4	75.41	58.64	62.83	104.5	108.7	91.96
.5	88.16	62.95	69.25	131.7	138.0	112.9
.6	107.3	69.39	78.87	172.6	182.0	144.4
.7	139.3	80.12	94.92	240.7	255.4	196.8
.8	203.4	101.5	127.0	377.0	402.1	301.6
.9	395.8	165.7	223.3	785.8	842.3	616.1

Fig. 6-11. $\bar{s} = 1.$, $\bar{d} = 0.$; $N = 3$, $\rho_i = \frac{1}{8}\rho_0$, $\rho_j = \frac{3}{4}\rho_0$, $\rho_l = \frac{1}{8}\rho_0$.

ρ_0	$\bar{q}_{E_i}$	$\bar{q}_{E_j}$	$\bar{q}_{E_l}$	$\bar{q}_{E_0}$
.1	.05844	.05461	.05837	.05556
.2	.1394	.1203	.1387	.1250
.3	.2552	.2011	.2523	.2143
.4	.4275	.3034	.4190	.3333
.5	.6957	.4383	.6747	.5000
.6	1.139	.6279	1.093	.7500
.7	1.943	.9236	1.848	1.167
.8	3.658	1.479	3.470	2.000
.9	8.997	3.061	8.634	4.500

Fig. 6-12. $\bar{s} = 1.$, $\bar{d} = 1.$, $N = 3$, $\rho_i = \frac{1}{8}\rho_0$, $\rho_j = \frac{3}{4}\rho_0$, $\rho_l = \frac{1}{8}\rho_0$.

ρ_0	$\bar{q}_{E_i}$	$\bar{q}_{E_j}$	$\bar{q}_{E_l}$	$\bar{q}_{E_0}$
.1	.6070	.5685	.6070	.5781
.2	.7487	.6516	.7480	.6758
.3	.9427	.7547	.9398	.8013
.4	1.219	.8867	1.211	.9687
.5	1.633	1.063	1.612	1.203
.6	2.295	1.315	2.249	1.555
.7	3.464	1.715	3.369	2.141
.8	5.908	2.479	5.720	3.313
.9	13.43	4.686	13.07	6.828

Fig. 6-13. $\bar{s} = 1.$, $\bar{d} = 100.$, $N = 3$, $\rho_i = \frac{1}{8}\rho_0$, $\rho_j = \frac{3}{4}\rho_0$, $\rho_l = \frac{1}{8}\rho_0$.

ρ_0	$\bar{q}_{E_i}$	$\bar{q}_{E_j}$	$\bar{q}_{E_l}$	$\bar{q}_{E_0}$
.1	54.92	51.44	54.92	52.31
.2	61.08	53.25	61.08	55.20
.3	69.01	55.56	69.00	58.92
.4	79.59	58.64	79.59	63.87
.5	94.45	62.94	94.42	70.81
.6	116.8	69.38	116.7	81.22
.7	154.0	80.09	153.9	98.56
.8	228.7	101.5	228.5	133.3
.9	452.7	165.6	452.4	237.3

Fig. 6-14. $\bar{s} = 1.$, $\bar{d} = 0.$, $N = 3$, $\rho_i = \frac{1}{4}\rho_0$, $\rho_j = \frac{1}{2}\rho_0$, $\rho_l = \frac{1}{4}\rho_0$

ρ_0	$\bar{q}_{E_i}$	$\bar{q}_{E_j}$	$\bar{q}_{E_l}$	$\bar{q}_{E_0}$
.1	.05632	.05481	.05628	.05556
.2	.1288	.1214	.1284	.1250
.3	.2249	.2043	.2235	.2143
.4	.3573	.3114	.3533	.3333
.5	.5481	.4565	.5389	.5000
.6	.8416	.6678	.8229	.7500
.7	1.339	1.011	1.305	1.167
.8	2.343	1.687	2.284	2.000
.9	5.356	3.693	5.259	4.500

Fig. 6-15. $\bar{s} = 1.$, $\bar{d} = 0.$, $N = 3$, $\rho_i = \frac{1}{4}\rho_0$, $\rho_j = \frac{3}{8}\rho_0$, $\rho_l = \frac{3}{8}\rho_0$

ρ_0	$\bar{q}_{E_i}$	$\bar{q}_{E_j}$	$\bar{q}_{E_l}$	$\bar{q}_{E_0}$
.1	.05611	.05538	.05536	.05556
.2	.1276	.1242	.1241	.1250
.3	.2215	.2121	.2116	.2143
.4	.3491	.3288	.3274	.3333
.5	.5310	.4913	.4881	.5000
.6	.8082	.7337	.7275	.7500
.7	1.276	1.136	1.124	1.167
.8	2.222	1.935	1.917	2.000
.9	5.073	4.324	4.294	4.500

Figs. 6-16, 6-17. $\bar{s} = 1.$, $\bar{d} = 0.$, $N = 3$

Fig. 6-16 : $\rho_i = \frac{1}{8}\rho_0$, $\rho_j = \frac{5}{8}\rho_0$, $\rho_l = \frac{1}{4}\rho_0$.

Fig. 6-17 : $\rho_i = \frac{1}{8}\rho_0$, $\rho_j = \frac{1}{4}\rho_0$, $\rho_l = \frac{5}{8}\rho_0$.

← Fig. 6-16 → ← Fig. 6-17 →

ρ_0	$\bar{q}_{E_i}$	$\bar{q}_{E_j}$	$\bar{q}_{E_l}$	$\bar{q}_{E_0}$	$\bar{q}_{E_l}$	$\bar{q}_{E_j}$	$\bar{q}_{E_i}$
.1	.05764	.05462	.05686	.05556	.05461	.05691	.05757
.2	.1353	.1204	.1313	.1250	.1204	.1318	.1346
.3	.2431	.2016	.2317	.2143	.2014	.2335	.2403
.4	.3983	.3049	.3720	.3333	.3044	.3771	.3902
.5	.6320	.4427	.5773	.5000	.4417	.5894	.6125
.6	1.005	.6397	.8979	.7500	.6379	.9231	.9643
.7	1.660	.9542	1.451	1.167	.9509	1.499	1.580
.8	3.014	1.562	2.589	2.000	1.556	2.676	2.868
.9	7.134	3.343	6.075	4.500	3.335	6.223	6.879