FREE FIELD REALIZATION OF EXTENDED CONFORMAL FIELD THEORIES

by

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TO MY MOTHER AND TO THE MEMORY OF MY FATHER

Abstract

I investigate the free field realization (FFR) of various extended conformal field theories (ECFT's). More specifically, I first present a systematic method that allows the construction of the exponential type screening currents in terms of free fields in the case of the ECFT's with Kac-Moody algebras. This method is explicitly illustrated through the $su(n)_k$ and $sp(4)_k$ Kac-Moody algebras. Then, I use the FFR to unravel the embedding structure of the Verma modules of the ECFT with a W_3 algebra. This embedding structure is expressed through a set of intertwining diagrams, which in turn is used to compute the irreducible characters of the W_3 algebra. Next, I construct two FFR's for the ECFT with the $su(n)_k$ parafermion algebra. Finally, I sketch the FFR of the coset model $su(n)_k \times su(n)_\ell / su(n)_{k+\ell}$, which is given in terms of the fields realizing the $su(n)_k$ parafermion model and an extra free field with a background charge.

Résumé

J'examine la représentation par des champs libres (RCL) de plusieurs théories de champs conformes généralisées (TCCG). Plus spécifiquement, je présente une méthode systématique permettant la construction des courants d'écran qui sont exprimés comme des fonctions exponentielles de champs libres, dans le cas des TCCG basées sur les algèbres de Kac-Moody. Cette méthode est explicitement illustrée par les algèbres de Kac-Moody $su(n)_k$ et $sp(4)_k$. Ensuite, j'utilise la RCL afin de déterminer la structure d'inclusion des modules de Verma de la TCCG basée sur l'algèbre W_3. Cette structure d'inclusion est représentée par des diagrammes qui me permettent de calculer les caractères irréductibles de l'algèbre W_3. De plus, je développe deux RCL dans le cas de la TCCG avec l'algèbre $su(n)_k$-parafermionique. Finalement, je considère brièvement la RCL du modèle quotient $su(n)_k \times su(n)_\ell / su(n)_{k+\ell}$. Celle-ci est exprimée en fonction des champs représentant le modèle $su(n)_k$-parafermionique et un champs libre possédant une charge de fond.

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Table of Contents

Chapter 1

Introduction

In the 1960s, the number of known strongly interacting particles (hadrons) increased considerably, with some of them carrying large spins. The squared mass m^2 of the lightest particle with spin J approximately follows a nice pattern, known as a Regge trajectory, $m^2 = J/\alpha'$, where $\alpha' \sim 1(GeV)^{-2}$ is called the Regge slope. The high-energy contribution of these particles was described collectively by this Regge trajectory through Regge-pole theory [1]. At low energies, one expects an *additional* contribution from the s-channel resonances, but this expectation was contradicted by the experiment of Bloom and Gilman [2], who found that the contribution of the s-channel resonances *alone* is on average equal to the t-channel Regge-pole amplitude, and as such it is not an additional contribution. This gave rise to the "duality hypothesis", that is, the equivalence of the s and t channels in describing the hadronic physics. Meanwhile, in 1968, Veneziano postulated an ad hoc formula for the scattering amplitude which is consistent with the stringent requirement of the duality hypothesis [3]. Then Nambu and Goto realized that this formula can be recovered through a classical relativistic bosonic string theory [4]. This turned out to be of far-reaching consequence in that it overthrew an age-old idea of the Greeks that all elementary particles are point-like rather than string-like. Neveu, Schwarz and Ramond [5] quickly generalized the bosonic string theory to include fermions, which led to the superstring theory. After that, Goldstone, Goddard, Rebbi and Thorn worked out the first quantization of the string theory [6]. However, it was soon realized that the theory suffers from two major drawbacks. First, it involves a massless spin-two particle which is not present in the hadronic spectrum. Second, the theory is not consistent in 4 space-time dimensions but rather in the critical dimensions 26 and 10 for the bosonic string and superstring respectively. Furthermore, the great success of QCD in describing the hadronic interactions dashed the hope for the string theory as being the potentially fundamental theory for the hadrons and thus interest in it waned.

Nonetheless, string theory was revived in 1974 when Scherk and Schwarz suggested that the appearance of the spin-two particle in its spectrum is rather a blessing in disguise [7]. Indeed, they identified this spin-two particle with the graviton and thus they proposed string theory as a potential candidate to describe all the interactions including gravity if the energy scale is pushed farther to the Planck energy scale (10^{19} GeV) Interest in string theory increased further in 1984 when Green and Schwarz realized that certain superstring theories are anomaly-free [8] (i.e , self-consistent at the quantum level). However, even the most promising string theory, namely the heterotic string [9], which requires ten-dimensional space-time, failed to fit the most obvious experimental data that our universe exists in four-dimensional space-time. The six remaining dimensions pose a priori a serious problem Nevertheless in the early days of string theory this problem was bypassed by requiring, in the context of the old theory of Kaluza-Klein [10], these extra dimensions to be compactified, that is, by letting them live on a tiny compact manifold with a size of the order the Planck length. This would account for the experimental failure to see these hidden dimensions. A modern approach to this problem is provided in the context of conformal field theory (CFT) [11]. In this case, four-dimensional string theories are directly constructed without any reference whatsoever to the compactified extra dimensions. The previous notion of the compactified dimensions is now replaced by the concept of an internal two-dimensional CFT, from which the particle spectrum and the coupling constants can be fully derived. CFT provides then a powerful framework for making contact with phenomenology in the context of string theory. In fact CFT's describe at the classical level different possible solutions (vacua) of the same second-quantized string theory. However, it is widely believed that dynamical effects such as dynamical symmetry breaking, which are considerably difficult to presently calculate, would single out a unique vacuum for the quantum string theory. A large number of CFT's have been investigated in the literature.

Much interest has also been devoted to the study of CFT's in the context of the critical phenomena of two-dimensional statistical systems at second-order phase transitions [12]. Indeed, it has been realized that some statistical spin systems, consisting of spin variables located at the sites of a two-dimensional lattice and interacting with their near neighbors, can effectively be described by a CFT at the critical point. The reason is that at the critical point (temperature) these systems may undergo a second order phase transition, which is characterized by the divergence of the correlation length ξ compared to the lattice spacing a ($a/\xi \to 0$). Therefore the behavior of these systems is effectively scale-invariant and may be described in this continuum limit by a field theory. Moreover, it was realized by Polyakov [13] that at the critical point this behavior is not only scale-invariant but conformally invariant, that is, invariant under local dilatation, rotation and translation transformations, which preserve the local angles (see Figures 1 and 2 for illustration).

In two dimensions, unlike any other space-time dimension, conformal invariance has far-reaching consequences. In particular, it provides in some cases sufficient information to solve the theory completely. This is mainly because conformal invariance in two dimensions is an infinite-dimensional symmetry (the group of analytic functions on the complex plane) giving rise to an infinite-dimensional algebra called the Virasoro algebra. The representations of the Virasoro algebra, which are well studied in mathematics, turn out to determine the critical exponents. These critical exponents specify the power law behaviors of the thermodynamic functions (free energy, susceptibility, ...) and are experimentally measurable. The fact that these systems are largely related to the representation theory of the Virasoro algebra rather than to the details of the systems underlines the concept of universality classes in condensed matter physics. In fact, each CFT is characterized by a positive c-number called the Virasoro central charge c, which in turn, specifies the universality classes of the systems. For example the following systems are described by CFT's that are specified respectively by the following central charges:

Ising model: $c = \frac{1}{2}$,

Tricritical Ising model: $c = \frac{7}{10}$,

3-state Potts model: $c = \frac{4}{5}$,

Tricritical 3-state Potts model: $c = \frac{6}{7}$.

We also mention that conformal symmetry in two dimensions is so strong that it can almost fix the dynamics in the sense that it allows an exact computation of the correlation functions and the critical exponents for some specific statistical systems. Note that in the above examples the central charge c is always smaller than 1. In fact, the two cases $0 \leq c < 1$ and $c \geq 1$ are significantly different. In the former case, the CFT can be exactly soluble because in this case only a finite number of "fundamental" fields exists. In the latter case however, the CFT may not be exactly soluble because the number of fundamental fields may be infinite. For $c \geq 1$ an intense effort has been devoted to extended CFT's that have extensions of the Virasoro algebra (extended symmetries), which in turn allow the exact solution of the theory by picking only a finite number of fundamental fields. CFT's with Kac-Moody [14], W_n [15], parafermion [16], and superconformal [17] algebras are some examples of the most extensively studied extended CFT's.

In summary, CFT's are a powerful tool in classifying, on the one hand, the universality classes of two-dimensional critical phenomena in statistical mechanics and, on the other hand, classical string vacua. CFT's are also intimately connected to other topics such as non-conformal exactly soluble (integrable) models and combinatorial mathematics.

In this thesis, we investigate various extended CFT's, which are based on some local conserved currents. Most importantly these currents can be represented in terms of free fields. Our main tool in this thesis is this free field realization [18], which turns out to be far more convenient than the abstract algebraic method. In particular, it substantially facilitates the computation of several relevant quantities like the correlation functions and it allows the explicit construction of the spectrum, that is, it makes more apparent the study of the representation theory of the conformal group and its extensions. In the second chapter, we give a concise introduction to two-dimensional CFT. This is necessary in order to make the subsequent chapters more apparent. We review the terminology and the basic concepts of CFT, following the seminal paper of Belavin, Polyakov and Zamolodchikov (BPZ) [19]. We begin with an introduction to the conformal group in any space-time dimension D and then we consider the special case $D = 2$, which is the most interesting. The role of the energy-momentum tensor in generating the conformal transformations and its relation to the infinite-dimensional Virasoro algebra is highlighted. We then discuss the notion of primary fields, fields that have nice transformations under the conformal group, as well as the notion of the operator product expansion (OPE) that unravels the short distance behavior of the local fields and currents. We introduce the highest weight representations, Verma modules, secondary fields, null fields, conformal families, and the completely degenerate representations. Next, we review the minimal models. After that, we discuss the free field realization of the minimal models and the concept of screening currents [18] that will play a crucial role in later chapters. The main virtue of these screening currents is that they allow the derivation of integral representations for the correlation functions, which are often expressed in terms of hypergeometric functions. This method proves to be more convenient than the complicated computation of the correlation functions through the Ward identities, which translate into differential equations that are hard to solve directly. Then in the context of the minimal models, we illustrate how the characters, the fusion rules and the correlation functions can be determined through the free field realization. Finally, we discuss the extended CFT's and rational CFT's (RCFT's) such as CFT's with Kac-Moody, W_n and parafermion algebras, which will be treated in more detail in the subsequent chapters.

In the third chapter, we consider extended CFT's based on Kac-Moody current algebras. These algebras appear in two-dimensional Wess-Zumino-Witten models and play a central role because most other CFT's can be expressed as cosets of them. They are generated from a set of currents generalizing the energy-momentum tensor in the case of the Virasoro algebra. The main topic of this chapter is centered on a full study of the screening currents in CFT's with Kac-Moody current algebras. In particular, we focus on the screening currents of the pure exponential type. We present a systematic method that allows their construction in terms of free scalar fields for any Kac-Moody current algebra

[14]. This chapter is organized as follows. We start with a concise introduction of CFT with Kac-Moody algebras. Then we present a background review of the free field realization of Kac-Moody current algebras, the so-called Wakimoto realization. More specifically, we display the Wakimoto realizations of both the $su(n)_k$ and $sp(4)_k$ Kac-Moody algebras, which are the only ones so far available in the literature. After that, we proceed by working out the technical details of this method. Next, we explicitly apply this procedure to both $su(n)_k$ and $sp(4)_k$. In particular, we solve some discrepancies present in the literature and derive for the first time the correct form of the screening currents, which are expressed as infinite sums of pure exponential terms. The analysis is substantially simplified through the lattice formulation of both the Kac-Moody and screening currents. We conclude this chapter with a few remarks and a note on further possible investigation about the screening currents.

In the fourth chapter, we consider another extended CFT, namely the one based on a W_3 algebra. Here, we mainly present a full description of the embedding structure of the Verma modules of the W_3 algebra. To this end, we use the free field realization of the W_3 RCFT as our main tool. As an application of this embedding structure we compute the irreducible characters of this algebra. We begin with a brief review of the W_3 algebra in the minimal unitary series. In particular, the symmetries and the degeneracy property of the representations of this algebra are highlighted. We then discuss the free field realization of this algebra in terms of a two-dimensional free field. This free field is used to construct the screening currents, which in turn allow the explicit construction of the null states in the completely degenerate Verma modules of W_3. Next, we describe the embedding structure of the Verma submodules generated from these null states in terms of the screening charges. This structure is represented through a set of intertwining diagrams. In particular, we show how these diagrams can be used to derive the irreducible character of a completely degenerate Verma module in terms of the characters of its Verma submodules. The formula for the character thus obtained fully agrees with the one previously conjectured in the literature.

In the fifth chapter, we investigate another extended CFT, namely the $su(n)_k$ parafermion theory. More precisely, we propose a realization of this theory in terms of free fields, that is, the analog of the Wakimoto realization in the case of a Kac-Moody algebra. As a matter of fact, we start from the Wakimoto realization of $su(n)_k$ from which we extract two free field realizations for the associated $su(n)_k$ parafermion theory. Each of these two realizations has its special properties. The first one involves orthonormal fields making it easier to obtain the field realization of the parafermion currents but harder to get that of the primary fields. The second one on the other hand leads to a simple field realization of the primary fields but requires linearly dependent constrained fields, in terms of which the parafermion currents are harder to realize. While working out the details of our two free field realizations we unravel their connections to the related work recently proposed in the literature. Finally, we briefly address the free field realization of the coset model $su(n)_k \times su(n)_\ell / su(n)_{k+\ell}$ in terms of fields realizing the $su(n)_k$ parafermion model. The organization of this chapter is as follows. We begin with a brief introduction to the simplest parafermion theory and that is the Z_k model, which is also called the $su(2)_k$ parafermion model because it is isomorphic to the coset model $su(2)_k/u(1)_k$. Then we present the details of our first free field realization of the $su(n)_k$ parafermion model in terms of orthonormal free fields. Strictly speaking, we express the parafermion currents associated with the negative root and positive simple root $su(n)_k$ currents. For the sake of clarity, all the $su(3)_k$ parafermion currents and the screening currents are explicitly displayed. Then, in order to derive a simple free field realization for the primary fields of the $su(n)_k$ parafermion model, we present our second approach involving constrained non-orthonormal fields. Finally, we consider the coset model $su(n)_k \times su(n)_\ell / su(n)_{k+\ell}$. We show that this model can be represented in terms of the free fields realizing the $su(n)_k$ parafermion model and a free field with a background charge. The free field realization of the screening currents of this coset model is also addressed.

Hereafter, we summarize the original work achieved in this thesis. First, we derive a systematic method that allows the construction of the exponential type screening currents in terms of free scalar fields in the case of extended CFT's with Kac-Moody algebras [20]. We use a lattice approach that considerably simplifies the derivation of these screening currents. Though our procedure is completely general, we apply it to both the $su(n)_k$ and $sp(4)_k$ Kac-Moody algebras whose explicit Wakimoto realizations are worked out in the literature. In particular, our method allows us to derive for the first time the correct expression for those screening currents that are given as infinite sums of terms, and thereby resolve some discrepancies found in the literature. Finally, we make some remarks on the possible applications of these screening currents. Second, we work out the embedding structure of the Verma modules of the extended CFT with a W_3 algebra through the free field realization of this theory [21]. More specifically, we explicitly construct the null states in the completely degenerate Verma modules of this algebra using the screening currents. We then represent the embedding structure of the Verma submodules generated from these null states through a set of intertwining diagrams, which in turn are used to derive the irreducible characters of the W_3 algebra. The result thus obtained confirms the one previously conjectured in the literature. Finally, we investigate the $su(n)_k$

parafermion theory [22]. We basically construct two free field realizations for this model. The first one is suitable for the realization of the parafermion currents in terms of orthonormal fields, whereas the second one accommodates the realization of the parafermion primary fields in terms of linearly dependent constrained fields. The $su(3)_k$ parafermion case is explicitly worked out including the free field realization of its screening currents. Then we address the free field realization of the coset model $su(n)_k \times su(n)_\ell / su(n)_{k+\ell}$. It turns out that it can easily be represented in terms of the fields realizing the $su(n)_k$ parafermion and a free boson with background charge.

Chapter 2

Conformal field theory in two dimensions

This chapter provides a concise technical introduction to conformal field theory (CFT). However, it is by no means complete or exhaustive since the subject of CFT is extremely rich and rapidly developing. Here, we focus mainly on those topics of CFT that will make the next chapters of this thesis more apparent, especially the free field realization. More technical details can be found in the various review articles [12,19,23,24,25,26,27].

2.1 The conformal group

The conformal group is by definition the group of transformations that preserve the local angles. Let $g_{\mu\nu}$ be the metric of the D-dimensional Minkowski space-time R^D with signature (p,q). Let u and v be two vectors in R^D with the scalar product $u \cdot v = g_{\mu\nu}u^\mu v^\nu$. The angle between u and v is preserved under a conformal transformation if the scalar product $u \cdot v/(u^2 v^2)^{1/2}$ is also preserved under this transformation. This means that the metric must be invariant under the conformal group up to a scale change

$$g_{\mu\nu} \to g'_{\mu\nu} = \Omega(x) g_{\mu\nu}. \tag{2.1}$$

The generators of the conformal group can be read off through the infinitesimal transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$. In this case (2.1) becomes

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{D}(\partial \cdot \epsilon) g_{\mu\nu}, \tag{2.2}$$

with $\Omega(x) = 1 + \frac{2}{D}\partial \cdot \epsilon(x)$. For $D > 2$, the solutions of (2.2) correspond to the following infinitesimal conformal transformations:

a) translations: $\epsilon^\mu(x) = a^\mu$,

b) rotations: $\epsilon^\mu(x) = \omega^\mu_\nu x^\nu$, $\quad \omega \in so(p,q)$,

c) scale transformations: $\epsilon^\mu(x) = \lambda x^\mu$,

d) special conformal transformations: $\epsilon^\mu(x) = b^\mu x^2 - 2(b \cdot x)x^\mu$.

Note that the algebra generated by the above transformations is locally isomorphic to the Lie algebra $so(p+1, q+1)$. When $D = 2$, the situation is quite different. In this case, the conformal algebra is infinite-dimensional. Anticipating the conformal invariance and after a Wick rotation to the Euclidean space, one can map the metric $g_{\mu\nu}$ to the flat one $\delta_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, in which case (2.2) translates into

$$\begin{aligned} \partial_1 \epsilon_1 &= \partial_2 \epsilon_2, \\ \partial_1 \epsilon_2 &= -\partial_2 \epsilon_1. \end{aligned} \tag{2.3}$$

The relations (2.3) are nothing but the Cauchy-Riemann equations. It is then natural to introduce the complex coordinates $z = x_1 - ix_2$ and $\bar{z} = x_1 + ix_2$, and define $\epsilon(z, \bar{z}) = \epsilon_1 + i\epsilon_2$, $\bar{\epsilon}(z, \bar{z}) = \epsilon_1 - i\epsilon_2$. Then the relations (2.3) become

$$\partial_z \bar{\epsilon}(z, \bar{z}) = \partial_{\bar{z}} \epsilon(z, \bar{z}) = 0. \tag{2.4}$$

This means that $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$ are respectively analytic and antianalytic functions. Therefore, two-dimensional conformal transformations coincide with the analytic and antianalytic transformations

$$\begin{aligned} z &\rightarrow z' = f(z), \\ \bar{z} &\rightarrow \bar{z}' = \bar{f}(\bar{z}). \end{aligned} \tag{2.5}$$

From now we consider only the analytic sector keeping in mind that the same treatment is valid in the antianalytic sector. The generators of the infinitesimal analytic transformation $\delta z = \epsilon(z)$ can be read off from the Laurent expansion of $\epsilon(z)$

$$\epsilon(z) = -\sum_{n=-\infty}^{\infty} \epsilon_n z^{n+1}. \tag{2.6}$$

It is then clear that the generators of the conformal group coincide with the differential operators

$$\ell_n = -z^{n+1}\partial_z, \quad n = 0, \pm 1, \pm 2, \ldots, \tag{2.7}$$

which satisfy the following commutation relations:

$$[\ell_n, \ell_m] = (n-m)\ell_{n+m}. \tag{2.8}$$

The generators $\bar{\ell}_n$ satisfy the same algebra as (2.8) and commute with the generators ℓ_n. The algebra (2.8) makes it apparent that the conformal algebra in two dimensions is infinite-dimensional. This turns out to be of broad consequence in that in some cases this symmetry is almost sufficient to solve the theory, that is, it allows in principle the exact computation of all the correlation functions. Note that from (2.8) the generators ℓ_{-1}, ℓ_0 and ℓ_1 form a subalgebra (of the conformal algebra) that is isomorphic to $SL(2,C)$. The finite form of these $SL(2,C)$ transformations is

$$z \to z' = \frac{az+b}{cz+d}, \tag{2.9}$$

where $a, b, c, d \in C$ and $ad - bc = 1$. This $SL(2,C)$ group is well known as the projective conformal group.

2.2 The energy-momentum tensor

The energy-momentum tensor (EMT) plays a crucial role in CFT [19]. To see this, let S be the action of the CFT. Then the EMT $T_{\mu\nu}$ is defined as

$$T_{\mu\nu} = -\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}, \tag{2.10}$$

where g is the absolute value of the determinant of $g_{\mu\nu}$. The translational invariance implies that $T_{\mu\nu}$ is conserved, i.e.,

$$\partial_\mu T^{\mu\nu} = 0. \tag{2.11}$$

Moreover, invariance under the scale transformation $\delta g^{\mu\nu} = \delta\lambda g^{\mu\nu}$ requires $T_{\mu\nu}$ to be traceless. Indeed,

$$-\frac{1}{\sqrt{g}}\frac{\delta S}{\delta\lambda} = -\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}}\frac{\delta g^{\mu\nu}}{\delta\lambda} = T_{\mu\nu}g^{\mu\nu} = T^{\mu}_{\mu} = 0. \tag{2.12}$$

In two-dimensional space-time, the equation (2.12) ... ıys a significant role because if we use the complex coordinates z and $\bar{z}$, then it translates into the following form:

$$T_{z\bar{z}} = T_{\bar{z}z} = 0. \tag{2.13}$$

Combining (2.13) with (2.11), which reads in complex coordinates as

$$\partial_{\bar{z}} T_{zz} = \partial_z T_{\bar{z}\bar{z}} = 0, \tag{2.14}$$

we see that the only two nonvanishing components of the EMT are respectively the analytic $T_{zz} \equiv T(z)$ and antianalytic $T_{\bar{z}\bar{z}} \equiv \bar{T}(\bar{z})$ functions on the complex plane. Hereafter, we consider CFT's only in two-dimensional space-time where the above property holds. For an infinitesimal conformal transformation $x^\mu \to x^\mu + \epsilon^\mu(x)$ the conserved current is $T^\mu_\nu \epsilon^\nu$. In complex coordinates the conserved charges associated with $T^\mu_\nu \epsilon^\nu$ are

$$Q_{(\epsilon)} = \frac{1}{2\pi i}\oint dz\epsilon(z)T(z). \tag{2.15}$$

Let $\phi(z)$ be a local field in the CFT. Under the conformal transformation $z \to z + \epsilon(z)$, we expect $Q_{(\epsilon)}$ to generate the transformation of the field $\phi(z)$ as

$$\delta_\epsilon \phi(w) = [Q_\epsilon, \phi(w)]. \tag{2.16}$$

The above commutator is computed as usual at equal time, which in "radial quantization" is equivalent to a commutator computed at equal radius. To see this, let us consider a two-dimensional Minkowski space-time with time and space coordinates τ and σ, and which has the topology of a cylinder (i.e., σ lives on a circle). This cylinder can be parametrized by the complex coordinates $\xi = \tau + i\sigma$ and $\bar{\xi} = \tau - i\sigma$. If we anticipate that the field theory has complete symmetry under the conformal group, we can then conformally map the cylinder to the complex plane by defining the coordinates of the plane to be

$$\begin{aligned} z &= e^{\xi} = e^{\tau + i\sigma}, \\ \bar{z} &= e^{\bar{\xi}} = e^{\tau - i\sigma}. \end{aligned} \tag{2.17}$$

Notice that the origin $z = 0$ corresponds to the infinite past ($\tau = -\infty$) and the point at infinity $z = \infty$ corresponds to the infinite future ($\tau = +\infty$). It is clear from (2.17) that on the plane different times correspond to concentric circles of different radii. Moreover, the time ordering that is understood in (2.16) corresponds to approaching the circle at time τ with circles of slightly bigger and slightly smaller radii respectively; this is referred to as the radial ordering. Therefore in the radial quantization the commutator (2.16) is given by

$$\begin{aligned} \delta_\epsilon \phi(w) = [Q_\epsilon, \phi(w)] &= \frac{1}{2\pi i} \left(\oint_{|z|>|w|} dz - \oint_{|z|<|w|} dz \right) \epsilon(z) T(z) \phi(w) \\ &= \frac{1}{2\pi i} \oint_{C_w} dz \epsilon(z) T(z) \phi(w), \end{aligned} \tag{2.18}$$

where C_w is a small circle around w as shown in Figure 3. Notice that the contour integral in (2.18) will pick up only the contributions from the poles in the singularities, which appear in the "operator product expansion (OPE)" as the short distance singularities when the two operators are considered at nearby points. Thus to compute the commutators of two local operators, it is enough to know their OPE, which is a key ingredient in two-dimensional CFT [19].

2.3 Primary fields and operator product expansions

Among the infinite set of fields in CFT, there are certain fields $\phi(z)$ that transform like tensor fields under the conformal transformation $z \rightarrow f(z)$, namely

$$\phi(z) \rightarrow \phi'(z) = \left(\frac{\partial f(z)}{\partial z}\right)^h \phi\left(f(z)\right). \tag{2.19}$$

These fields $\phi(z)$ are called "primary fields", and h is known as the "conformal dimension" (weight) [19]. As we will see later, unitarity requires h to be a real non-negative number. The remaining fields that do not transform in the nice way defined in (2.19) are called "secondary (or descendant) fields". These secondary fields are usually expressed as combinations of derivatives of the primary fields. For the infinitesimal conformal transformation $z \rightarrow f(z) = z + \epsilon(z)$, (2.19) translates into

$$\delta_{\epsilon(z)}\phi(z) = \epsilon(z)\partial\phi(z) + h\partial\epsilon(z)\phi(z). \tag{2.20}$$

The relations (2.18) and (2.20) imply that the short distance behavior of the product of the EMT $T(z)$ with the primary field $\phi(z)$ must satisfy the following OPE:

$$T(z)\phi(w) = \frac{h\phi(w)}{(z-w)^2} + \frac{\partial_w\phi(w)}{z-w} + \text{regular}. \tag{2.21}$$

Here "regular" stands for the remaining terms of the OPE that are nonsingular as $z \rightarrow w$. This means that they do not contribute in the contour integrals like (2.18). However, they are necessary in defining normal ordered products of local fields. It is worth noting that the above OPE encodes all the information about the conformal transformation of the fields. In fact, the notion of OPE in CFT replaces the customary notion of commutation relations in the usual field theory [19]. The OPE's of secondary fields with the EMT involve higher order singularities than the double pole as in (2.21). The primary fields in CFT are the most important dynamical variables of the theory. For instance, we will see later that the correlation functions of arbitrary fields can be completely determined once only the correlation functions of the primary fields are known.

At the classical level, the EMT (which is a rank 2 tensor) transforms according to (2.20) as

$$\delta_\epsilon T = 2\partial\epsilon(z)T(z) + \epsilon(z)\partial T(z). \tag{2.22}$$

However, at the quantum level a possible Schwinger term may arise due to an anomaly in the transformation properties of the EMT. Consistency with the dimensional analysis implies the following conformal transformation of the EMT at the quantum level:

$$\delta_\epsilon T = 2\partial\epsilon(z)T(z) + \epsilon(z)\partial T(z) + \frac{c}{12}\partial^3\epsilon(z), \tag{2.23}$$

where c is known as the "central charge". Its value depends on the particular CFT under consideration. However as we will see later, unitarity restricts it to be a real positive number. The transformation property (2.23) is equivalent to the following OPE of the EMT with itself:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{regular}. \tag{2.24}$$

This OPE encodes all the information about the quantum conformal algebra, which is commonly known as "the Virasoro algebra" [28].

2.4 The Virasoro algebra

Let us represent the EMT as a Laurent expansion

$$T(z) = \sum_{n=-\infty}^{+\infty} \frac{L_n}{z^{n+2}}, \tag{2.25}$$

where the Laurent modes L_n are given by

$$L_n = \frac{1}{2\pi i} \oint dz z^{n+1} T(z). \tag{2.26}$$

Due to the OPE (2.24) and the definition of the commutation relation in the radial quantization, the modes L_n satisfy the Virasoro algebra [28]

$$\begin{aligned} [L_n, L_m] &= \oint_{C_o} \frac{dw w^{m+1}}{2\pi i} \oint_{C_w} \frac{dz z^{n+1}}{2\pi i} T(z)T(w) \\ &= (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}. \end{aligned} \tag{2.27}$$

Note that the Virasoro algebra (2.27) is the central extension (the quantum version) of the classical conformal algebra given in (2.8). Let us also mention that (2.21) and (2.26) imply that the primary field $\phi(z)$ satisfies

$$[L_n, \phi(z)] = z^n [h(n+1) + z\partial]\phi(z). \tag{2.28}$$

Finally, note that $L_{\pm 1}$, L_0 form a subalgebra with no central charge. Thus, this subalgebra is also isomorphic to $SL(2, C)$.

2.5 Representation theory of the Virasoro algebra

So far we have introduced two real numbers in connection with the Virasoro algebra, namely the conformal dimension h and the central charge c. Therefore it is reasonable to expect these two numbers to label the representations of the Virasoro algebra. Indeed, note from (2.27) that

$$[L_0, L_n] = -nL_n. \tag{2.29}$$

The importance of (2.29) is as follows. Because dilatations on the complex plane correspond to time translations on the cylinder, it is natural to identify the generator L_0 with the Hamiltonian. It is clear then from (2.29) (by analogy with the harmonic oscillator) that L_n $(n > 0)$ and L_n $(n < 0)$ are annihilation and creation operators respectively. This suggests that the irreducible representations of the Virasoro algebra require the existence of a highest weight state $\mid h >$, which is an eigenstate of L_0 and is annihilated by L_n $(n > 0)$. Other states in the representation are obtained by successive applications of L_n $(n < 0)$ on $\mid h >$, that is,

$$\begin{aligned} &L_0 \mid h >= h \mid h >, \\ &L_n \mid h >= 0, \quad n > 0, \\ &L_n \mid h >= \text{new states}, \quad n < 0. \end{aligned} \tag{2.30}$$

The representation satisfying (2.30) is called a "highest weight representation". The set of all the states in this highest weight representation is called a "Verma module". A basis for this Verma module is given by the states

$$L_{-n_1} \dots L_{-n_k} \mid h >, \tag{2.31}$$

with $0 < n_1 \leq n_2 \leq \dots \leq n_k$. It can readily be checked that the secondary state (2.31) has the conformal dimension $h + \sum_{i=1}^{k} n_i$ and therefore it appears at "degree" (level, height) $\sum_{i=1}^{k} n_i$. To summarize, a Verma module $V(c, h)$ of the Virasoro algebra is thus completely specified by the central charge c and the conformal dimension h, which is the L_0 eigenvalue of the highest weight state $\mid h >$. Requiring the CFT to be unitary, amounts to restricting the EMT to be Hermitian and the scalar product of the Hilbert space to be positive definite. The first condition yields [24]

$$L_{-n}^{+} = L_n. \tag{2.32}$$

The second condition is implemented through (2.32), which leads to

$$\| L_{-n} \mid h > \|^2 = < h \mid L_{-n}^{+} L_{-n} \mid h > = < h \mid [L_n, L_{-n}] \mid h > = \left[2nh + \frac{c}{12}(n^3 - n)\right], \tag{2.33}$$

with $n > 0$ and the highest weight state $\mid h >$ normalized as $< h \mid h >= 1$. Unitarity requires the norm in (2.33) to be positive at all levels. In particular, letting $n = 1$ implies that $h \geq 0$, whereas large n leads to $c \geq 0$. A complete analysis of unitary involves the Kac determinant [29]. It has been carried out in reference [30] where it is shown that unitarity requires $V(c, h)$ to either satisfy $c \geq 1$ and h a real positive number, or

$$
\begin{aligned}
c &= 1 - \frac{6}{m(m+1)}, \quad m = 2, 3, \ldots \\
h = h_{r,s} &= \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)},
\end{aligned}
\tag{2.34}
$$

with $1 \leq r \leq m-1$ and $1 \leq s \leq m$. The Verma module $V(c, h)$ is not a priori irreducible. In fact, it is conceivable that it contains a secondary state that itself behaves as a highest weight state. Such a state is called a "null state" [19], which is justified by the fact that it is orthogonal to all the states in $V(c, h)$ and therefore it has, as well as all its descendants, a zero norm. Thus it can effectively be identified with zero $\mid null >\equiv 0$. A Verma module containing a finite number of null states is called a "degenerate Verma module", whereas a Verma module containing an infinite number of null states is called a "completely degenerate Verma module". Therefore, to extract the irreducible highest weight representation from $V(c, h)$, one has to project out from $V(c, h)$ all the null states together with their descendants. It is shown that $V(c, h)$ contains a null state if [29]

$$
h = h_{r,s} = \frac{c-1}{24} + \frac{1}{8}(\alpha_+ r + \alpha_- s)^2,
\tag{2.35}
$$

where

$$
\alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{12}},
\tag{2.36}
$$

and r, s are arbitrary positive integers. Furthermore this state appears at level rs. The null states play an important role in deriving partial differential equations (Ward identities), which are satisfied by the correlation functions. So far the relation between the primary states and the primary fields has not yet been clarified. This will be the subject of the subsequent section.

2.6 Conformal families

Let $\phi(z)$ be a primary field of conformal dimension h. Using (2.28), it is easy to see that the highest weight state $\mid h >$ is created by the action of $\phi(0)$ on the vacuum $\mid 0 >$, that is,

$$\mid h >= \phi(0) \mid 0 > . \tag{2.37}$$

The secondary fields creating the secondary states can be read off from the OPE of the EMT with $\phi(w)$. Indeed, (2.28) implies that

$$T(z)\phi(w) = \sum_{k=0}^{\infty} \frac{\phi^{(-k)}(w)}{(z-w)^{2-k}}. \tag{2.38}$$

Combining (2.21) with (2.38) leads to the identifications $\phi^{(0)}(z) \equiv h\phi(z)$ and $\phi^{(-1)}(z) \equiv \partial\phi(z)$. It can also be checked that the fields $\phi^{(-k)}(z)$ $(k \geq 1)$ are not primary fields, which means that they create from the vacuum $\mid 0 >$ secondary states of $\mid h >$, namely

$$\phi^{(-k)}(0) \mid 0 >= L_{-k} \mid h > . \tag{2.39}$$

The fields $\phi^{(-k)}(z)$ are thus secondary fields of the primary field $\phi(z)$. Similarly, other secondary fields $\phi^{(-k,-k_1)}(z)$ of $\phi(z)$ can be generated from the OPE of $T(z)$ with $\phi^{(-k)}(w)$. Therefore, a general state $L_{-k_1} \ldots L_{-k_n} \mid h >$ in $V(c,h)$ is obtained by the action of the secondary field $\phi^{(-k_1,\ldots,-k_n)}(0)$ on the vacuum, or equivalently

$$\phi^{(-k_1,\ldots,-k_n)}(0)|0 >= L_{-k_1} \ldots L_{-k_n}|h > . \tag{2.40}$$

In this context, note that the EMT $T(z)$ is itself a secondary field of the identity operator. This can be seen from

$$T(z) = \frac{1}{2\pi i}\oint \frac{dwT(w)}{w-z}, \tag{2.41}$$

which underlines the fact that $T(z)$ is not a primary field. The primary field $\phi(z)$ together with its descendants form a "conformal family", which is commonly denoted by $[\phi]$. The usefulness of organizing the fields of a CFT into conformal families stems from the fact that only the primary fields are needed so as to describe the theory completely. If the Verma module $V(c,h)$ contains a null state the corresponding field in the conformal family $[\phi]$ is then called a null field. The null fields are used to derive Ward identities for the correlation functions because any correlation function containing a null field or any of its descendants identically vanishes. As will be discussed later, the null fields are also useful in computing the irreducible characters of the Virasoro algebra.

2.7 Minimal models

Hereafter, we review an interesting class of CFT's that are completely soluble. These theories are characterized by the complete degeneracy of their conformal families. As mentioned earlier, in order to solve the CFT one needs to consider only the primary fields. To corroborate this, let us for example consider the following correlation function:

$$< \phi^{(-k_1,\ldots,-k_m)}(z)\phi_1(z_1)\ldots\phi_n(z_n) >, \tag{2.42}$$

where $\phi_1(z_1)\ldots\phi_n(z_n)$ are primary fields and $\phi^{(-k_1,\ldots,-k_m)}(z)$ is a secondary field belonging to the conformal family $[\phi]$. Using (2.26), (2.38) and (2.40) then $\phi^{(-l_1,\ldots,-k_m)}(z)$ can be rewritten as

$$\phi^{(-k_1,\ldots,-k_m)}(z) = \left[\prod_{j=1}^{m}\oint\frac{dw_j}{2\pi i}(w_j-z)^{1-k_j}T(w_j)\right]\phi(z), \tag{2.43}$$

where $\phi(z)$ is a primary field and the contours encircle all the singularities of the integrand. Using (2.21) together with (2.43) one can easily see that the above correlation function (2.42) can be rewritten in terms of the correlation function $< \phi(z)\phi_1(z_1)\dots\phi_n(z_n) >$, which involves only the primary fields, as

$$< \phi^{(-k_1,\dots,-k_m)}(z)\phi_1(z_1)\dots\phi_n(z_n) >= \prod_{j=1}^{m} \mathcal{D}_{-k_j}(z,\{z_i\}) < \phi(z)\phi_1(z_1)\dots\phi_n(z_n) >, \tag{2.44}$$

where the differential operators $\mathcal{D}_{-k}(z,\{z_i\})$ are given by

$$\mathcal{D}_{-k}(z,\{z_i\}) = \sum_{i=1}^{n}\left[\frac{(1-k)h_i}{(z-z_i)^k} - \frac{1}{(z-z_i)^{k-1}}\partial_i\right]. \tag{2.45}$$

Obviously the above result easily generalizes to express any correlation function with many secondary fields in terms of a correlation function with only primary fields. This is an important result because it allows one to derive the correlation functions of the primary fields as solutions of partial differential equations. To see that, let us assume that the secondary field in the correlation function (2.42) is a null field, that is according to (2.35), it belongs to the conformal family $[\phi] \equiv [\phi_{r,s}]$. However, since this secondary field is a null field then the correlation function (2.42) vanishes. This means that the correlation function $< \phi(z)\phi_1(z_1)\dots\phi_n(z_n) >$ satisfies the following partial differential equation:

$$\prod_{j=1}^{m} \mathcal{D}_{-k_j}(z,\{z_i\}) < \phi(z)\phi_1(z_1)\dots\phi_n(z_n) >= 0. \tag{2.46}$$

Furthermore, these differential equations restrict the OPE among the primary fields. A detailed analysis carried out in reference [19] shows that the case where all the conformal families are completely degenerate is particularly interesting. Indeed, the conformal families form then a closed algebra which can be symbolically represented as

$$\phi_{(m,n)} \times \phi_{(r,s)} = \sum_{\substack{k=|m-r|+1\\ k+m+r=\mathrm{odd}}}^{min(m+r-1,2p-m-r-1)} \sum_{\substack{\ell=|n-s|+1\\ \ell+n+s=\mathrm{odd}}}^{min(n+s-1,2q-n-s-1)} \left[\phi_{(k,\ell)}\right], \tag{2.47}$$

where $[\phi_{(k,\ell)}]$ denotes the entire conformal family of the primary field $\phi_{(k,\ell)}$. The relations (2.47) are well known as the "fusion rules". Moreover, it is shown in reference [19] that if α_+ and α_- are so that

$$\frac{\alpha_-}{\alpha_+} = -\frac{p}{q}, \tag{2.48}$$

that is,

$$c = 1 - \frac{6(p-q)^2}{pq}, \tag{2.49}$$

where q and p are arbitrary positive coprime numbers, then the conformal family $[\phi_{r,s}]$ contains infinitely many null fields, i.e., $[\phi_{r,s}]$ is a completely degenerate conformal family. Thus, any correlation function containing the primary field $[\phi_{r,s}]$ satisfies infinitely many partial differential equations. This suggests that all the correlation functions can be in principle computed as solutions of some of these differential equations. Furthermore, the fusion rules (2.47) close under a finite number of primary fields $\phi_{r,s}$ with $0 < r < p$, $0 < s < q$ and conformal dimensions $h_{r,s}$

$$h_{r,s} = \frac{(qr-ps)^2 - (p-q)^2}{4pq}. \tag{2.50}$$

Such CFT's as specified by (2.48) and (2.49) are called "minimal models". Note that when $c < 1$, the unitary models are also minimal. They are recovered from the set of minimal models by letting $q = p + 1$. As argued in chapter 1, some statistical systems realize these minimal models.

2.8 Free field realization

In this section we propose the explicit realization of CFT's in terms of a free (boson) field [18]. This is also known as the "bosonization" or the "Coulomb gas representation". As mentioned in chapter 1, the free field realization facilitates substantially the resolution of CFT's. In particular, it simplifies the computation of the correlation functions, the fusion rules and the irreducible characters. For the sake of illustration, these quantities will be explicitly worked out in the case of minimal models in the subsequent treatment.

2.8.1 Free field realization of the minimal models

Let us in this section illustrate how most of the properties of the minimal models can be recovered through a single massless free boson $X(\sigma)$. The corresponding action is

$$S = \frac{1}{2}\int d^2\sigma \sqrt{g} g^{\mu\nu}(\sigma)\partial_\mu X(\sigma)\partial_\nu X(\sigma), \tag{2.51}$$

where g is the absolute value of the determinant of $g_{\mu\nu}$. The EMT obtained from the above action is given by

$$T_{\mu\nu} = -\frac{1}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}} = -\frac{1}{2}\partial_\mu X\partial_\nu X + \frac{1}{4}g_{\mu\nu}\partial^\alpha X\partial_\alpha X. \tag{2.52}$$

Note that this EMT is symmetric and traceless, i.e., $T^\mu_\mu = 0$, and since the above action is invariant under conformal transformations, we choose the two-dimensional space to be the complex plane. In this case, because of (2.13) and (2.14) one can easily see that the only two nonvanishing components of the EMT in (2.52) are given by

$$\begin{aligned} T_{zz} &\equiv T(z) = -\frac{1}{2} : \partial_z X\partial_z X :, \\ T_{\bar{z}\bar{z}} &\equiv \bar{T}(\bar{z}) = -\frac{1}{2} : \partial_{\bar{z}} X\partial_{\bar{z}} X :, \end{aligned} \tag{2.53}$$

that is, $T(z)$ and $\bar{T}(\bar{z})$ are respectively holomorphic and antiholomorphic functions, and :: denotes the normal ordering, which is defined as

$$: A(z)B(z) := \frac{1}{2\pi i}\oint_{C_z}\frac{dw}{w-z}A(w)B(z), \tag{2.54}$$

for some local fields $A(z)$ and $B(z)$. Henceforth, unless stated otherwise, fields or products of local fields at the same point are always assumed to be normal ordered, and thus for convenience we will drop the normal ordering symbol :: in the sequel. Again as we have done before, we only consider the holomorphic sector while keeping in mind that the same treatment is also valid for the anti-holomorphic one. The propagator (the Green function) of the analytic free field $X(z)$ reads as follows:

$$< X(z_1)X(z_2) >= -\ln(z_1 - z_2). \tag{2.55}$$

Using (2.55), the Wick rules and Taylor expansions, it can easily be seen that the EMT (2.53) satisfies the OPE

$$T(z)T(w) = \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{regular}. \tag{2.56}$$

Therefore, the above EMT generates the conformal transformations in a CFT of a free massless boson, which is characterized by the central charge $c = 1$. To see that $T(z)$, as represented in (2.53), is indeed the generator of the conformal transformations let us work through a simple example. Consider the field $\partial X(z)$. Under the conformal transformation $z \to z + \epsilon(z)$, this field is expected to transform as

$$\partial X(z) \to \partial X(z) + \partial\epsilon(z)\partial X(z) + \epsilon(z)\partial^2 X(z), \tag{2.57}$$

which is due to

$$X(z) \to X(z+\epsilon) = X(z) + \epsilon\partial X(z). \tag{2.58}$$

On the other hand it can readily be checked that

$$T(z)\partial X(w) = \frac{\partial X(w)}{(z-w)^2} + \frac{\partial^2 X(w)}{z-w} + \text{regular}, \tag{2.59}$$

which asserts the fact that ∂X is a primary field of conformal dimension 1. Moreover, (2.59) implies this transformation for ∂X:

$$\delta_\epsilon \partial X(z) = \left[\oint \frac{dw}{2\pi i} T(w)\epsilon(w), \partial X(z) \right] = \partial\epsilon(z)\partial X(z) + \epsilon(z)\partial^2 X(z), \qquad (2.60)$$

which coincides with (2.57). To appreciate further the usefulness of the free field realization of T as given in (2.53), let us work through another example, namely the vertex field $e^{i\alpha X(z)}$. The OPE of this vertex field with $T(z)$ reads as follows:

$$T(z)e^{i\alpha X(w)} = \frac{(\alpha^2/2)e^{i\alpha X(w)}}{(z-w)^2} + \frac{\partial e^{i\alpha X(w)}}{z-w} + \text{regular}. \qquad (2.61)$$

Therefore, $e^{i\alpha X(z)}$ is a primary field of conformal dimension $h = \alpha^2/2$. This fact can also be deduced from the two-point correlation function

$$< e^{i\alpha X(z)} e^{-i\alpha X(w)} >= e^{\alpha^2 <X(z)X(w)>} = e^{-\alpha^2 \ln(z-w)} = \frac{1}{(z-w)^{\alpha^2}}. \qquad (2.62)$$

Note that the above correlation function can be interpreted in the context of the Coulomb gas representation as being the expectation value of two charged fields $e^{i\alpha X(z)}$ and $e^{-i\alpha X(u)}$ with the "Coulomb charges" α and $-\alpha$ respectively. This just reflects the usual fact that the total Coulomb charge is conserved. The fields $e^{i\alpha X(z)}$ and $e^{-i\alpha X(z)}$ are regarded as conjugates or adjoints of one another. The above correlation function (2.62) can be generalized to any n-point correlation function as follows:

$$< V_{\alpha_1}(z_1)\ldots V_{\alpha_n}(z_n) >= \begin{cases} \prod_{i<j}(z_i - z_j)^{\alpha_i\alpha_j} & \text{if } \sum \alpha_i = 0, \\ 0 & \text{otherwise}, \end{cases} \qquad (2.63)$$

where $V_{\alpha_j}(z) \equiv e^{i\alpha_j X(z)}$.

Since $T(z)$, as represented in (2.53) (with $c = 1$), cannot describe the minimal models that are characterized by $c < 1$, then one is led to deform the above T so that it accommodates this case ($c < 1$) while satisfying the OPE (2.24) should it be the generator of conformal transformations. A way to achieve that is to consider instead

$$\tilde{T}(z) = -\frac{1}{2}(\partial X)^2 + i\alpha_0\partial^2 X, \tag{2.64}$$

where α_0 is a real number. One can easily check that $\tilde{T}(z)$ indeed satisfies (2.24) with the central charge

$$c = 1 - 12\alpha_0^2. \tag{2.65}$$

Thus for special values of α_0, $\tilde{T}(z)$ can be the EMT of the minimal models, i.e., with $c < 1$. The second term in $\tilde{T}$ (2.64) can be interpreted in the context of the Coulomb gas representation as being due to the presence of a background Coulomb charge $-2\alpha_0$ placed at infinity. This is created by the vertex operator $V_{-2\alpha_0} = e^{-2i\alpha_0 X(\infty)}$. Thus, the out-vacuum is charged and defined in terms of the in-vacuum $\mid 0 >$ as [18]

$$< -2\alpha_0 \mid = < 0 \mid V_{-2\alpha_0}(\infty). \tag{2.66}$$

Therefore, with this charged out-vacuum, the nonvanishing two-point correlation function is given by

$$< V_\alpha(z)V_{2\alpha_0-\alpha}(w) > = \frac{1}{(z-w)^{\alpha(\alpha-2\alpha_0)}}. \tag{2.67}$$

The new conjugate of V_α is no longer $V_{-\alpha}$ but rather $V_{\tilde{\alpha}} \equiv V_{2\alpha_0-\alpha}$. Note that both V_α and $V_{\tilde{\alpha}}$ have the same conformal dimension $h(\alpha)$, which reads from (2.67) as follows:

$$h(\alpha) = h(\tilde{\alpha}) = \frac{1}{2}(\alpha^2 - 2\alpha_0\alpha). \tag{2.68}$$

The above two-point correlation function (2.67) generalizes easily as

$$< V_{\alpha_1}(z_1)\ldots V_{\alpha_n}(z_n) > = \begin{cases} \prod_{i<j}(z_i - z_j)^{\alpha_i\alpha_j} & \text{if } \sum \alpha_i = 2\alpha_0, \\ 0 & \text{otherwise,} \end{cases} \tag{2.69}$$

2.8.2 Screening currents

The equation (2.68) implies that there are two vertex fields $S_{\pm}(z)$ of conformal dimension 1

$$S_{\pm}(z) = e^{i\alpha_{\pm} X(z)}, \tag{2.70}$$

with

$$\begin{aligned} \alpha_{+} + \alpha_{-} &= 2\alpha_0, \\ \alpha_{+} \cdot \alpha_{-} &= -2. \end{aligned} \tag{2.71}$$

These vertex fields $S_{\pm}$ are called "screening currents" [18]. The main property of these screening currents is that they satisfy OPE's with the EMT that are total derivatives. This means that the "screening charges" $Q_{\pm}$

$$Q_{\pm} = \oint S_{\pm}(z) dz \tag{2.72}$$

commute with the EMT $\tilde{T}$. Therefore, these screening charges (2.72) can be inserted inside the correlation functions in any number without changing the conformal properties of the original correlation functions [18]. Their only effect is to make the correlation function nonvanishing by screening the background charge placed at infinity. For the sake of illustration let us work through an example, namely the four-point functions of the minimal models.

2.8.3 Four-point correlation functions of the minimal models

First let us require, as is customary in some statistical systems, that the Vertex fields $\{V_{\alpha}\}$ (which are identified with physical operators) are so that the four-point correlation function is nonvanishing for each of them [18], that is*

$$< V_{\alpha}(z_1) V_{\alpha}(z_2) V_{\alpha}(z_3) V_{\alpha}(z_4) > \neq 0. \tag{2.73}$$

---

* This choice is required in order to be able to balance the Coulomb charge placed at infinity by the screening charges.

However, the total Coulomb charge in the above correlation function is $2\alpha + 2\alpha_0 \neq 2\alpha_0$. Therefore, this correlation function vanishes unless we insert $r-1$ and $s-1$ screening charges Q_+ and Q_- respectively. This means that the conformal properties of the correlation function (2.73) can be read off instead from the expectation value

$$< V_\alpha(z_1)V_\alpha(z_2)V_\alpha(z_3)V_{\bar{\alpha}}(z_4)[Q_+]^{r-1}[Q_-]^{s-1} > \neq 0. \tag{2.74}$$

Hence we can build four-point correlation functions with properties natural for some statistical systems (nonzero if the four operators are the same) out of the vertex fields $V_\alpha(z)$ if the Coulomb charge α is restricted to take the discrete set of values $\alpha_{r,s}$ so that

$$\alpha_{r,s} = \frac{1-r}{2}\alpha_+ + \frac{1-s}{2}\alpha_-, \quad r,s \geq 1. \tag{2.75}$$

The corresponding conformal dimensions $h_{r,s}$ are obtained from (2.68), namely

$$h_{r,s} = -\frac{1}{2}\alpha_0^2 + \frac{1}{8}(r\alpha_+ + s\alpha_-)^2. \tag{2.76}$$

The equations (2.75) and (2.76) are precisely the Kac-spectrum for degenerate conformal families. For the minimal models, α_+ and α_- read

$$\begin{aligned} \alpha_+ &= \sqrt{\frac{2q}{p}}, \\ \alpha_- &= -\sqrt{\frac{2p}{q}}, \end{aligned} \tag{2.77}$$

in which case $h_{r,s}$ is given in (2.50). The relations (2.71) and (2.77) imply that

$$\alpha_0 = \frac{q-p}{\sqrt{2pq}}. \tag{2.78}$$

A generic four-point correlation function is then

$$\oint_{C_1} dt_1 \ldots \oint_{C_{r-1}} dt_{r-1} \oint_{C'_1} dt'_1 \ldots \oint_{C'_{s-1}} dt'_{s-1}$$
$$\times < V_\alpha(z_1)V_\alpha(z_2)V_\alpha(z_3)V_{\bar{\alpha}}(z_4)S_+(t_1)\ldots S_+(t_{r-1})S_-(t'_1)\ldots S_-(t'_{s-1}) > . \tag{2.79}$$

The contours in (2.79) wind around the points z_1, z_2, z_3, z_4, so as they do not lead to zero integral. In fact the above expectation value (2.79) stands for the integral representation of the correlation function that satisfies partial differential equations as discussed earlier.

2.8.4 Fusion rules of the minimal models

Here, we rederive the fusion rules of the minimal models (2.47) in the free field realization. The form of the fusion rules in (2.47) suggests that we should only consider the three-point correlation functions. There are three equivalent ways of representing the three-point correlation function in the free field realization, namely

$$< V_{\tilde{\alpha}_{k,\ell}}(z_1)V_{\alpha_{m,n}}(z_2)V_{\alpha_{r,s}}(z_3)Q_+\ldots Q_+Q_-\ldots Q_- >, \tag{2.80}$$

$$< V_{\alpha_{k,\ell}}(z_1)V_{\tilde{\alpha}_{m,n}}(z_2)V_{\alpha_{r,s}}(z_3)Q_+\ldots Q_+Q_-\ldots Q_- >, \tag{2.81}$$

$$< V_{\alpha_{k,\ell}}(z_1)V_{\alpha_{m,n}}(z_2)V_{\tilde{\alpha}_{r,s}}(z_3)Q_+\ldots Q_+Q_-\ldots Q_- >, \tag{2.82}$$

Requiring the conservation of the Coulomb charges in (2.80), (2.81) and (2.82) leads respectively to

$$\begin{aligned}
&k \leq m+r-1, \quad k-m-r+1 \quad \text{even},\\
&\ell \leq n+s-1, \quad \ell-n-s+1 \quad \text{even},
\end{aligned} \tag{2.83}$$

$$\begin{aligned}
&m \leq k+r-1, \quad m-k-r+1 \quad \text{even},\\
&n \leq \ell+s-1, \quad n-\ell-s+1 \quad \text{even},
\end{aligned} \tag{2.84}$$

$$\begin{aligned}
&r \leq k+m-1, \quad r-k-m+1 \quad \text{even},\\
&s \leq \ell+n-1, \quad s-\ell-n+1 \quad \text{even}.
\end{aligned} \tag{2.85}$$

The equations (2.83), (2.84) and (2.85) translate into

$$k \leq m+r-1, \quad \ell \leq n+s-1,$$

$$m \leq r + k - 1, \quad n \leq s + \ell - 1,$$

$$r \leq k + m - 1, \quad s \leq \ell + n - 1,$$

$$k + m + r \quad \text{odd}, \quad \ell + n + s \quad \text{odd}. \tag{2.86}$$

The relation (2.86) implies then the following fusion rules:

$$\phi_{(m,n)} \times \phi_{(r,s)} = \sum_{\substack{k=|m-r|+1 \\ k+m+r=\text{odd}}}^{m+r-1} \sum_{\substack{\ell=|n-s|+1 \\ \ell+n+s=\text{odd}}}^{n+s-1} \left[\phi_{(k,\ell)}\right], \tag{2.87}$$

where $\phi_{(r,s)}$ is a primary field in the minimal model whose free field realization is given by $V_{\alpha_{r,s}}$. From (2.50) it can readily be checked that

$$h_{r,s} = h_{p-r,q-s}. \tag{2.88}$$

Combining (2.87) with (2.88) we get

$$\phi_{(p-m,q-n)} \times \phi_{(p-r,q-s)} = \sum_{\substack{k=|m-r|+1 \\ k+m+r=\text{odd}}}^{2p-m-r-1} \sum_{\substack{\ell=|n-s|+1 \\ \ell+n+s=\text{odd}}}^{2q-n-s-1} \left[\phi_{(k,\ell)}\right]. \tag{2.89}$$

The equations (2.87) and (2.89) are consistent with each other if

$$\phi_{(m,n)} \times \phi_{(r,s)} = \sum_{\substack{k=|m-r|+1 \\ k+m+r=\text{odd}}}^{min(m+r-1,2p-m-r-1)} \sum_{\substack{\ell=|n-s|+1 \\ \ell+n+s=\text{odd}}}^{min(n+s-1,2q-n-s-1)} \left[\phi_{(k,\ell)}\right], \tag{2.90}$$

which are the fusion rules for the minimal models as given earlier in (2.47).

2.8.5 Irreducible characters of the minimal models

Here, we derive the irreducible characters of the minimal models. Let us for example consider the Verma module $V(c, h_{r,s})$, where $h_{r,s}$ is the conformal dimension of a highest weight and is given in (2.50). We know that in the minimal series the Verma module $V(c, h_{r,s})$ contains an infinite number of null states. In the free field realization a highest weight (which can be a null state) is represented by $V_\alpha = e^{i\alpha X}$, where α is to be determined. The screening charges $Q_\pm$ can be used to construct explicitly the null states in the Verma module $[V_\alpha]$. Indeed it can readily be seen that the following two fields $\chi_1(z)$ and $\chi_2(z)$ create from the vacuum two states having the properties of null states in $[V_\alpha]$ and carrying the same Coulomb charges as the states created by $V_{\alpha-\ell\alpha_+}(z)$ and $V_{\alpha-n\alpha_-}(z)$ respectively

$$\begin{aligned}
\chi_1(z) &= \oint \prod_{i=1}^{\ell} du_i S_+(u_i) V_{\alpha-\ell\alpha_+}(z) \equiv Q_+^\ell V_{\alpha-\ell\alpha_+}(z), \\
\chi_2(z) &= \oint \prod_{i=1}^{n} du_i S_-(u_i) V_{\alpha-n\alpha_-}(z) \equiv Q_-^n V_{\alpha-n\alpha_-}(z),
\end{aligned} \tag{2.91}$$

where Q_+^ℓ and Q_-^n are the screening charges and $\alpha_\pm$ are given in (2.77). Working out the OPE's involved in the integrals (2.91), it can be shown that the fields $\chi_1(z)$ and $\chi_2(z)$ have the following general free field realizations:

$$\begin{aligned}
\chi_1 &\sim [\partial X]^{N_1} e^{i\alpha X} \in [V_\alpha], \\
\chi_2 &\sim [\partial X]^{N_2} e^{i\alpha X} \in [V_\alpha],
\end{aligned} \tag{2.92}$$

where $[\partial X]^{N_1}$ (the same definition applies to $[\partial X]^{N_2}$) is a certain linear combination of terms of the type $(\partial X)^{d_1}(\partial^2 X)^{d_2}...(\partial^j X)^{d_j}$ so that

$$\sum_{i=1}^{j} i d_i = N_1. \tag{2.93}$$

N_1 and N_2 are respectively the degrees of $\chi_1(z)$ and $\chi_2(z)$. They measure the level of the null fields χ_1 and χ_2 over the highest weight field V_α. A field with a negative degree is by definition zero. N_1 and N_2 are given in terms of the conformal dimensions as follows:

$$\begin{aligned}
N_1 &= h(\alpha - \ell\alpha_+) - h(\alpha), \\
N_2 &= h(\alpha - n\alpha_-) - h(\alpha),
\end{aligned} \tag{2.94}$$

with the conformal dimension $h(\alpha)$ as defined through (2.68). Requiring the degrees N_1 and N_2 to be positive integers amounts respectively to (see [20] and references therein)

$$\begin{aligned} \alpha &= \frac{1+\ell}{2}\alpha_+ + \frac{1+m}{2}\alpha_-, \qquad m > 0, \\ \alpha &= \frac{1+m'}{2}\alpha_+ + \frac{1+n}{2}\alpha_-, \qquad m' \geq 0, \end{aligned} \tag{2.95}$$

m and m' are arbitrary positive integers. In this case N_1 and N_2 read

$$\begin{aligned} N_1 &= \ell m, \\ N_2 &= nm'. \end{aligned} \tag{2.96}$$

Combining (2.95) with (2.96), clearly the same null state in $[V_\alpha]$ with degree ℓm can be constructed as in (2.91) using either ℓ or m screening currents S_+ or S_- respectively. Thus in the sequel we specify the Coulomb charge α of the highest weight field V_α by a single formula as follows:

$$\alpha = \frac{1+\ell}{2}\alpha_+ + \frac{1+m}{2}\alpha_-, \tag{2.97}$$

where now ℓ and m stand for the numbers of S_+ and S_- respectively. Let us for convenience denote the highest weight field V_α as

$$V_\alpha \equiv (\alpha_{\ell,m}) \equiv (\ell, m). \tag{2.98}$$

Note that ℓ and m are not uniquely defined because of the following translation invariance of α, which can be read off from (2.97):

$$(\ell, m) \equiv (\ell + kp, m + kq), \tag{2.99}$$

where p and q are defined in (2.77) and k is any integer.

Furthermore, we know that there is a Z_2 invariance in the Virasero algebra. Indeed, the conformal dimension $h(\alpha)$ (2.68) is invariant under $\alpha \to \tilde{\alpha} = 2\alpha_0 - \alpha$, which amounts to the identification

$$(\ell, m) \equiv (-\ell, -m). \tag{2.100}$$

That the integers in $(-\ell, -m)$ are negative is not an inconsistency because of the translational invariance (2.99). From the equation (2.91) it is clear that starting from the null field $V_\alpha \equiv (\ell, m)$, one can construct other null fields as $V_{\alpha-\ell\alpha_+} \equiv (-\ell, m)$ and $V_{\alpha-m\alpha_-} \equiv (\ell, -m)$ by means of Q_+^ℓ and Q_-^m respectively

$$
\begin{aligned}
(\ell, m) &\xrightarrow{Q_+^\ell} (-\ell, m), \\
(\ell, m) &\xrightarrow{Q_-^m} (\ell, -m).
\end{aligned}
\tag{2.101}
$$

The screening charges Q_+ and Q_- (the superscript is dropped) map between two null fields so that the Verma submodule generated from the second is embedded in the Verma module generated from the first (as previously remarked null states generate themselves Verma submodules). However, Q_+ and Q_- as given in (2.101) are not uniquely (well) defined because ℓ and m are themselves not uniquely defined. To fix this problem we further require that Q_+ and Q_- map only from a module to its maximal submodule (i.e., so that the degree of the field generating the submodule is the smallest possible positive integer) Moreover, we have to specify the primary field, which creates from the vacuum the highest weight of the primary completely degenerate Verma module. By definition, the latter module is not embedded in any other Verma module except itself. If (r, s) specifies a primary field then it cannot be the image of any other field under Q_+ and Q_-. It can be easily seen that such a primary field exists if

$$
\begin{aligned}
0 < r < p, \\
0 < s < q,
\end{aligned}
\tag{2.102}
$$

which is indeed the result obtained via other algebraic methods. Starting from the primary field (r, s) (2.102) and using the screening charges Q_+ and Q_-, one can construct an infinite number of null states in the primary completely degenerate Verma module $[V_{(r,s)}]$. They fall into two classes A and B that are defined as follows:

$$
\begin{aligned}
A &= \{a(k) \equiv (r + 2kp, s) \equiv (r, s - 2kq), \qquad k \in Z\}, \\
B &= \{b(k) \equiv (-r + 2kp, s) \equiv (r, -s + 2kq), \qquad k \in Z\}.
\end{aligned}
\tag{2.103}
$$

It can be readily checked that the actions of Q_+ and Q_- as described above are now uniquely (well) defined on A and B, namely

$$
\begin{aligned}
&k > 0 : \\
&a(k) \xrightarrow{Q_+^{p-r}} b(k+1), \qquad b(k) \xrightarrow{Q_+^{r}} a(k), \\
&a(k) \xrightarrow{Q_-^{s}} b(-k), \qquad b(k) \xrightarrow{Q_-^{s}} a(-k), \\
&k \leq 0 : \\
&a(k) \xrightarrow{Q_+^{r}} b(k), \qquad b(k) \xrightarrow{Q_+^{p-r}} a(k-1), \\
&a(k) \xrightarrow{Q_-^{q-s}} b(1-k), \qquad b(k) \xrightarrow{Q_-^{q-s}} a(1-k).
\end{aligned}
\tag{2.104}
$$

In (2.104), in order that Q_+ (Q_-) maps only form a module to its maximum submodule, its superscript is then required to be either r or $p-r$ (s or $q-s$). This requirement can always be fulfilled because of the translational invariance (2.99) and the Z_2 invariance (2.100). The embedding relations (2.104) are represented by the embedding diagram of Figure 4. The primary state is identified with $a(0)$ in the notation defined in (2.103).

Let us now show how one can make use of such a diagram to compute the irreducible character of the Verma module $V_{(c,h_{r,s})} \equiv V_{[a(0)]}$ [25,31]. The irreducible character is arduous to compute directly, however the character of the reducible Verma module can be easily computed. Thus, we should first compute the characters of all the Verma (sub)modules involved and then use them to extract the irreducible character through the embedding diagram of Figure 4. The character of the Verma module $[V_{a(0)}]$ is given by

$$
ch_{[V_{r,s}]} = ch_{[V_{a(0)}]} = Tr q^{(L_0 - \frac{c}{24})} = \frac{q^{(h_{r,s} - c/24)}}{\prod_{n=1}^{\infty}(1 - q^n)}, \tag{2.105}
$$

where the trace is taken over all the states in $[V_{a(0)}]$. The character of any Verma submodule generated from $a(k)$ or $b(k)$ is readily obtained from (2.105) by replacing (r, s) with $(r + 2kp, s)$ or $(-r + 2kp, s)$ respectively.

Let us denote the irreducible module by $[I_{r,s}]$. If there were no null states at all in the Verma module $[V_{r,s}]$, the character of the irreducible module $[I_{r,s}]$ would coincide with that of $[V_{r,s}]$ itself. This means that we would have the following relation:

$$ch_{[I_{r,s}]} = ch_{[V_{r,s}]} \equiv ch_{[V_{a(0)}]}. \tag{2.106}$$

According to Figure 4 though, this is not so because $[V_{a(0)}]$ contains two maximal Verma submodules $[V_{b(0)}]$ and $[V_{b(1)}]$, which are generated respectively from the null states $b(0)$ and $b(1)$. Consequently, one has to subtract their contributions from the character of $[V_{r,s}]$. This would then lead to

$$ch_{[I_{r,s}]} = ch_{[V_{a(0)}]} - ch_{[V_{b(0)}]} - ch_{[V_{b(1)}]}. \tag{2.107}$$

This is not the whole story however, because $[V_{b(0)}]$ and $[V_{b(1)}]$ overlap and their intersection contains the maximal Verma submodules $[V_{a(1)}]$ and $[V_{a(-1)}]$. This means that the contributions of the latter submodules to the character of $[I_{r,s}]$ were over subtracted. Therefore, in order to further correct the character of $[I_{r,s}]$ we should add them back, that is

$$ch_{[I_{r,s}]} = ch_{[V_{a(0)}]} - ch_{[V_{b(0)}]} - ch_{[V_{b(1)}]} + ch_{[V_{a(1)}]} + ch_{[V_{a(-1)}]}. \tag{2.108}$$

From the embedding diagram of Figure 4, clearly, one can repeat by induction the same argument of over addition and over subtraction (of the contributions of the null states) infinitely many times to end up finally with the correct irreducible character of $[I_{r,s}]$, which is

$$
\begin{aligned}
ch_{[I_{r,s}]} &= \sum_{k=-\infty}^{+\infty} \left(ch_{[V_{a(k)}]} - ch_{[V_{b(k)}]}\right), \\
&= \frac{q^{-c/24}}{\prod_{n=1}^{\infty}(1-q^n)} \sum_{k=-\infty}^{+\infty} \left\{q^{h_{r+2kp,s}} - q^{h_{-r+2kp,s}}\right\}.
\end{aligned}
\tag{2.109}
$$

Notice how similar the embedding diagram of the CFT with the Virasoro algebra is to that of the CFT with the $su(2)_k$ Kac-Moody algebra [20].

2.9 Extended CFT's and rational CFT's

It has been shown in reference [17] that, for the fractional values of the central charge c as given in (2.49), that is $c < 1$, the CFT is rational (minimal). This means that it has a finite number of primary fields. It is then exactly solvable, i.e., the spectrum, the fusion rules, the irreducible characters and the correlation functions can be exactly computed in principle. However, it has been shown in reference [32] that a CFT which is rational w.r.t. the Virasoro algebra necessarily has central charge $c < 1$ and a CFT with $c \geq 1$ has an infinite number of primary fields w.r.t. the Virasoro algebra and therefore it is not solvable. In order to construct a rational CFT (RCFT) with $c \geq 1$, one is then led to consider a CFT with a larger symmetry algebra than the Virasoro one. In this case, only a finite number of primary fields w.r.t. the extended algebra is allowed. As indicated in chapter 1, the most extensively studied RCFT's with $c \geq 1$ (extended RCFT's) are CFT's with Kac-Moody [14], W_n [15], parafermion [16] and superconformal [17] algebras. The first three of these RCFT's will be respectively the main topics of the next three chapters of this thesis. Therefore, we decide to introduce each of them throughout the corresponding chapter. Finally, let us mention that the classification of all RCFT's is currently an interesting ongoing research subject that is far from being accomplished.

Chapter 3

Free field realization of the screening currents in Kac-Moody conformal field theories

This chapter is devoted to the study of the extended CFT's with Kac-Moody (KM) algebras [14,33,34,35,36,37]. More specifically we present a systematic method that allows the construction of the screening currents [18] that are expressed as pure exponential terms of free fields. Contrary to the screening currents of pure exponential type, the other screening currents are extensively studied in the literature [38] and thus we are not interested in them here. Henceforth, we only consider the screening currents of the pure exponential type. Except for the trivial case of $su(2)_k$, much less is known about the screening currents for general KM algebras in the literature. As we will show later, it turns out that beyond $su(2)_k$ the screening currents are indeed nontrivial in that they are expressed as infinite sums of exponential terms of free fields. We start out this chapter by first reviewing the extended CFT with a KM algebra. We then discuss the free field realization of a KM algebra, which is known as the Wakimoto realization [35,36]. After that, we present the details of the method and finally we apply explicitly this procedure to only those KM algebras whose Wakimoto realizations are presently available in the literature, namely $su(n)_k$ [36] and $sp(4)_k$ [37]. For illustration, the derivation of the screening currents for the $su(2)_k$, $su(3)_k$, $su(4)_k$ and $sp(4)_k$ KM algebras is displayed in detail.

3.1 Introduction to CFT's with Kac-Moody algebras

A Kac-Moody algebra is another infinite-dimensional algebra extending the Virasoro one [14,33]. It plays a central role in string theory and CFT. Indeed, many other RCFT's can be traced back to CFT's with cosets of KM algebras. As indicated in chapter 2, CFT's with KM algebras are rational even with the Virasoro central charge being larger than 1. It also appears as the local symmetry of the Wess-Zumino-Model in two dimensions [14]. A KM algebra is basically the two-dimensional current algebra associated with the Lie algebra. To see that, let J^a, $a = 1, \ldots, d \equiv \dim g$, be the generators of the Lie algebra g, which is defined as

$$[J^a, J^b] = i f^{abc} J^c, \tag{3.1}$$

where f^{abc} are the antisymmetric structure constants of g. Then the KM algebra g_k associated with the algebra (3.1) reads as follows:

$$[J^a_n, J^b_m] = i f^{abc} J^c_{n+m} + kn\delta_{n+m,0}. \tag{3.2}$$

Here, $n, m \in Z$, $2k/\theta^2$ is a c-number called the "level" of the KM algebra and θ is the highest root of g. The level is restricted to be a positive integer by unitarity [33]. Note that in (3.2) the J^a_0's satisfy a subalgebra that is isomorphic to the Lie algebra g given in (3.1). As argued in chapter 2, commutation relations are equivalent to OPE's. As a matter of fact, the KM algebra g_k given in (3.2) is equivalent to the following OPE, which is referred to as the current algebra:

$$J^a(z)J^b(w) = \frac{k\delta^{ab}}{(z-w)^2} + \frac{i f^{abc} J^c(w)}{z-w} + \text{regular}, \tag{3.3}$$

with the KM currents $J^a(z)$ defined on the complex plane as

$$J^a(z) = \sum_{n=-\infty}^{\infty} z^{-n-1} J^a_n. \tag{3.4}$$

The above KM algebra and the Virasoro algebra introduced in chapter 2 are not unrelated. Sugawara has shown that given a KM algebra one can always construct out of it a Virasoro algebra whose EMT $T(z)$ is [33,39]

$$T(z) = \frac{1}{2k + C_2} \sum_a J^a(z)J^a(z), \tag{3.5}$$

where the normal ordering is understood and C_2 stands for the quadratic Casimir in the adjoint representation, namely

$$f^{acd}f^{bcd} = C_2\delta^{ab}. \tag{3.6}$$

It can readily be checked that $T(z)$ satisfies the required OPE (2.24) with the central charge

$$c = \frac{2kd}{2k + C_2}. \tag{3.7}$$

Note that the KM currents $J^a(z)$ are primary field w.r.t. $T(z)$ with conformal dimension 1, i.e.,

$$T(z)J^a(w) \sim \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w}, \tag{3.8}$$

where the symbol $\sim$ means that the equation holds up to regular terms which are being omitted, that is, only the singular terms are displayed. Similarly to the Virasoro algebra case, there are primary fields $\phi_j(z)$ w.r.t. the KM algebra. They are characterized by the OPE

$$J^a(z)\phi_j(w) \sim \frac{t^a_{j\ell}\phi_\ell(w)}{z-w}, \tag{3.9}$$

where $t^a_{j\ell}$ are representation matrices for g. The primary fields $\phi_j(z)$ create from the vacuum a highest weight state $\mid j>$, that is,

$$\mid j > \equiv \phi_j(0) \mid 0 > . \tag{3.10}$$

$\mid j >$ is annihilated by J^a_{+n} (either $n > 0$, $a = 1, \ldots, d$ or $n = 0$ and the index a denotes positive roots), whereas the successive applications of J^a_{-n} (either $n > 0$, $a = 1, \ldots, d$ or $n = 0$ and the index a denotes negative roots) on $\mid j >$ lead to secondary states. The secondary states together with the highest weight state $\mid j >$ constitute a Verma module w.r.t. the KM algebra. Again, this Verma module is a priori reducible, i.e., it contains null states and thus the irreducible module is obtained by modding out the null states together with their descendants from this Verma module. As discussed in chapter 2, a way to locate the null states in the Verma module is provided through the screening charges. For the screening charges to commute with the KM algebra, the corresponding screening currents must then satisfy OPE's with the KM currents that are either regular or total derivative. They must also have conformal dimension 1. These conditions are at the heart of our method presented in the rest of this chapter. As our procedure relies on the free field realization let us first review the Wakimoto realization of KM algebras.

3.2 The Wakimoto realization of KM algebras

In this section we discuss the explicit Wakimoto realization of the KM currents satisfying the KM algebra (3.3) in terms of free massless fields. We choose the Cartan-Weyl basis where the KM currents are denoted by $H_i(z)$, $J_\alpha(z)$ and $J_{-\alpha}(z)$. Here, α labels the positive roots of g, whereas $i = 1, \ldots, r$ (r is the rank of g). These KM currents correspond respectively to the Cartan subalgebra, the raising and lowering step generators. In principle, it is possible to represent any KM algebra for general level $2k/\theta^2$ in terms of free fields, however, the only explicit realizations so far available in the literature are those of $su(n)_k$ [35,36] and $sp(4)_k$ [37]. Therefore, we only give the Wakimoto realizations of these two algebras, though our method to derive the screening currents is completely general and can be applied to any KM algebra provided that its Wakimoto realization is worked out.

3.2.1 The Wakimoto realization of $su(n)_k$

In this case $r = n - 1$ and there are $R = n(n-1)/2$ positive roots $\alpha_{(ij)}$

given by

$$\alpha_{(ij)} = e_i - e_j \equiv (ij), \quad 1 \leq i < j \leq n, \tag{3.11}$$

where $\{e_i,\ i = 1, \dots, n\}$ is the set of orthonormal n-dimensional vectors. In particular, the simple roots α_i of $su(n)$ are in this notation given by

$$\alpha_i = e_i - e_{i+1} \equiv (i, i+1). \tag{3.12}$$

To represent $su(n)_k$ KM currents we need a vector $X = (\varphi_i, u_\alpha; v_\alpha)$ of $d = n^2 - 1$ free massless fields, which are correlated in the following way:

$$X_a(z)X_b(w) \sim -\eta_{ab} \ln(z - w)\,, \quad 1 \leq a, b \leq n^2 - 1 \tag{3.13}$$

where η_{ab} stands for the diagonal metric of a flat Lorentzian space with signature $[(+)^r, (+)^R; (-)^R]$. To write the Wakimoto realization of $su(n)_k$ in a convenient form, namely to avoid carrying square root terms along, let us introduce the $(n^2 - 1)$-dimensional lattice [26]

$$\Lambda_0 = \mathbf{g}^{(k+h)} \oplus Z_R^{(4)} \oplus Z_R^{(4)}, \tag{3.14}$$

where $h = n$ is the dual coxeter number of $su(n)$ and $\mathbf{g}$ is the root lattice of $su(n)$. Its dual lattice, namely the weight lattice of $su(n)$, is denoted by $\mathbf{g}^*$. Z_R is the R-dimensional orthonormal Euclidean lattice. To clarify the significance of the superscript in (3.14), let Γ be a lattice whose dual is Γ^*. Then for any positive integer ℓ, we can construct a scaled lattice $\Gamma^{(\ell)} = \{\sqrt{\ell}x | x \in \Gamma\}$, with $[\Gamma^{(\ell)}]^* = [\Gamma^*]^{(1/\ell)} = \{y/\sqrt{\ell} | y \in \Gamma^*\}$ being its dual. Thus, the dual lattice of Λ_0 is

$$\Lambda_0^* = [\mathbf{g}^*]^{1/(k+h)} \oplus Z_R^{(1/4)} \oplus Z_R^{(1/4)}. \tag{3.15}$$

This lattice Λ_0^* will be useful in avoiding cluttering the notations with square root factors. Indeed, it turns out that all the vectors involved in the Wakimoto realization belong to this lattice. Let us denote a vector in this lattice by

$$(x, y; z) = [x/\sqrt{k+h}, y/\sqrt{4}; z/\sqrt{4}]\,, \tag{3.16}$$

where $x \in \mathbf{g}^*$ and $y, z \in Z_R$. This means that the vectors with round brackets and square bracket span respectively the unscaled lattice and the scaled lattice. Hence, the scalar product of two vectors with round brackets $(x, y; z)$ and $(x', y'; z')$ reads

$$(x, y; z) \cdot (x', y'; z') = \frac{x \cdot x'}{k+h} + \frac{y \cdot y'}{4} - \frac{z \cdot z'}{4}\,, \tag{3.17}$$

The $(n-1)$-dimensional vector $x \in \mathbf{g}^*$ is defined through its components, given in the basis of the fundamental weights $\lambda_i \in \mathbf{g}^*$, as follows:

$$x = (x_1, x_2, \ldots, x_{n-1}), \tag{3.18}$$

which is equivalent to $x = \sum_{i=1}^{n-1} x_i \lambda_i$. With the above notations, the Wakimoto realization of the $su(n)_k$ KM currents is given by† [20]

$$\begin{aligned}
H_i(z) &= i\bar{H}_i \cdot \partial X(z) \exp[iH' \cdot X(z)],\\
J_{-\alpha}(z) &= i\bar{J}_{-\alpha}(0) \cdot \partial X(z) \exp[iJ'_{-\alpha}(0) \cdot X(z)]\\
&\quad + i \sum_{\substack{\sigma > \alpha \\ \sigma \in \Delta_+}} \bar{J}_{-\alpha}(\sigma) \cdot \partial X(z) \exp[iJ'_{-\alpha}(\sigma) \cdot X(z)],\\
J_{+\alpha_i}(z) &= i\bar{J}_{+\alpha_i}(0) \cdot \partial X(z) \exp[iJ'_{+\alpha_i}(0) \cdot X(z)]\\
&\quad + i \sum_{j=i+2}^{n} \bar{J}_{+\alpha_i}(j >) \cdot \partial X(z) \exp[iJ'_{+\alpha_i}(j >) \cdot X(z)]\\
&\quad + i \sum_{j=1}^{i-1} \bar{J}_{+\alpha_i}(j <) \cdot \partial X(z) \exp[iJ'_{+\alpha_i}(j <) \cdot X(z)],
\end{aligned} \tag{3.19}$$

† Note that only the Wakimoto realization of the KM simple root currents $J_{+\alpha_i}$ is given. The Wakimoto realization of the KM compound root currents is very complicated though it can be derived from that of the KM simple root currents through the KM current algebra (3.3).

where

$$
\begin{aligned}
H_i' &= 0, \\
\bar{H}_i &= (-(k+h)\alpha_i, 0; 0) + \sum_{\sigma\in\Delta_+} (\alpha_i \cdot \sigma) q_\sigma, \\
J_{-\alpha}'(0) &= p_\alpha + q_\alpha, \\
\bar{J}_{-\alpha}(0) &= -p_\alpha, \\
J_{-\alpha}'(\sigma) &= p_{\alpha+\sigma} + q_{\alpha+\sigma} - p_\sigma - q_\sigma, \\
\bar{J}_{-\alpha}(\sigma) &= -p_{\alpha+\sigma}, \\
J_{\alpha_i}'(0) &= -p_{\alpha_i} - q_{\alpha_i}, \\
\bar{J}_{\alpha_i}(0) &= ((k+n)\alpha_i, -(k+n-i)t_{\alpha_i}; \xi), \\
\xi &= -(k+n-i+1)t_{\alpha_i} + \sum_{j=1}^{i-1}[t_{(ji)} - t_{(j,i+1)}], \\
J_{\alpha_i}'(j>) &= p_{(i+1,j)} - p_{(ij)} + q_{(i+1,j)} - q_{(ij)}, \\
\bar{J}_{\alpha_i}(j>) &= -p_{(i+1,j)}, \\
J_{\alpha_i}'(j<) &= p_{(ji)} - p_{(j,i+1)} + q_{(ji)} - q_{(j,i+1)}, \\
\bar{J}_{\alpha_i}(j<) &= p_{(ji)}.
\end{aligned}
\tag{3.20}
$$

Here, Δ_+ stands for the set of the positive roots, whereas p_α, q_α and t_α are defined by

$$
\begin{aligned}
p_\alpha &= (0, t_\alpha; 0), \\
q_\alpha &= (0, 0; t_\alpha), \\
(t_\alpha)_\beta &= 2\delta_{\alpha\beta}.
\end{aligned}
\tag{3.21}
$$

In (3.19) $\sigma > \alpha$ means that if $\alpha = (ij) \in \Delta_+$ then $\alpha + \sigma = (jk) \in \Delta_+$ according to the notation defined in (3.11). It can be checked that the energy-momentum tensor $T(z)$, constructed through the Sugawara method described earlier, is given in terms of the free fields as follows:

$$
T(z) = -\frac{1}{2}[\partial X(z)]^2 + i\alpha_0 \cdot \partial^2 X(z),
\tag{3.22}
$$

with

$$\alpha_0 = (-\rho, 1; 1) = [\frac{-\rho}{\sqrt{k+h}}, \frac{1}{2}; \frac{1}{2}], \tag{3.23}$$

where ρ is the sum of the fundamental weights of $su(n)$ and $h = n$ is the dual Coxeter number for $su(n)$. With the above $T(z)$, the conformal dimension of a pure exponential field $V_\beta(z) \equiv \exp[i\beta \cdot \phi(z)]$ is

$$h(\beta) = \frac{1}{2}\beta^2 - \alpha_0 \cdot \beta, \tag{3.24}$$

and the Virasoro central charge is given in (3.7).

3.2.2 The Wakimoto realization of $sp(4)_k$

Here, we will follow closely the treatment of the previous section (with the obvious changes) and therefore many details are omitted while the same notations carry over to this case. For $sp(4)$, $r = 2$, $R = 4$ and the positive roots are given in terms of the two-dimensional orthonormal vectors e_1 and e_2 as

$$\begin{aligned} \alpha_{(12)} &= e_1 - e_2 \equiv (12), \\ \alpha_{(2)} &= 2e_2 \equiv (2), \\ \alpha_{[12]} &= e_1 + e_2 \equiv [12], \\ \alpha_{(1)} &= 2e_1 \equiv (1). \end{aligned} \tag{3.25}$$

The simple roots of $sp(4)$ are (12) and (2). With this normalization, the dual Coxeter number of $sp(4)$ is $h = 6$ and k (which defines the level through $2k/\theta^2$) is a positive even integer. The appropriate lattice is again

$$\Lambda_0^* = [\mathrm{g}^*]^{1/(k+h)} \oplus Z_R^{(1/4)} \oplus Z_R^{(1/4)}, \tag{3.26}$$

where $\mathbf{g}^*$ is the weight lattice of $sp(4)$, which is generated by the fundamental weights

$$\begin{aligned} \lambda_1 &= e_1, \\ \lambda_2 &= e_1 + e_2. \end{aligned} \tag{3.27}$$

The vectors in this Lorentzian lattice, with signature $[(+)^2,(+)^1;(-)^4]$, are again denoted by

$$(x,y;z) = [x/\sqrt{k+h}, y/\sqrt{4}; z/\sqrt{4}]\,, \tag{3.28}$$

with $x \in \mathbf{g}^*$, and $y,z \in Z_R$. The Wakimoto realization of the KM currents of $sp(4)_k$ is also of the form

$$\begin{aligned} H_i(z) &= i\bar{H}_i \cdot \partial X(z), \\ J_{-\alpha}(z) &= i \sum_{\sigma \in I_{-\alpha}} \bar{J}_{-\alpha}(\sigma) \cdot \partial X(z) \exp[iJ'_{-\alpha}(\sigma) \cdot X(z)], \\ J_{+\alpha}(z) &= i \sum_{\sigma \in I_{+\alpha}} \bar{J}_{+\alpha}(\sigma) \cdot \partial X(z) \exp[iJ'_{+\alpha}(\sigma) \cdot X(z)], \end{aligned} \tag{3.29}$$

where the index sets $I_{-\alpha}$ and $I_{+\alpha}$ as well as the vectors $\bar{J}_{-\alpha}$, $J'_{-\alpha}$, $\bar{J}_{+\alpha}$, $J'_{+\alpha}$, which span the lattice Λ_0^*, are given in Table 5 (the reason for introducing this table before Tables 1-4 will become apparent in the section 3.4). As we will see later, this table encodes all the necessary information in order to derive the screening currents. For that matter however, the vectors $\bar{H}_i$ are not needed and thus they are omitted from this table, but for completeness let us write them here

$$\begin{aligned} \bar{H}_1 &= (-2(k+6), k+6, 0,0,0,0; 4,-4,0,4), \\ \bar{H}_2 &= (2(k+6), -2(k+6), 0,0,0,0; -4,8,4,0). \end{aligned} \tag{3.30}$$

The energy-momentum tensor $T(z)$ derived from $sp(4)_k$ through the Sugawara construction is again given in (3.22) and (3.23) with ρ and h being now the sum of the fundamental weights and the dual Coxeter number of $sp(4)$ respectively. The conformal dimension of a pure exponential field and the Virasoro central charge are also given in (3.24) and (3 7) respectively. In fact, these results are completely general and hold for any KM algebra [33].

3.3 Theory of screening currents

This section is the core of this chapter. Here, we propose to develop a systematic method designed for the explicit construction of the screening currents in terms of free fields in a CFT with a KM algebra. The treatment is completely general and can be applied to any KM algebra once its Wakimoto realization is explicitly achieved. We limit the analysis only to the screening currents of the pure exponential type since so far these are the least analyzed in the literature. In fact, the only results available in the literature about this type of screening currents are just inconclusive conjectures based on some guess work, and which are not valid in some cases [36]. On the other hand, our results are derived in a natural way from first principles and most importantly they lead to genuine screening currents which explicitly check out all the necessary criteria [20]. To begin with, let again d be the dimension of the Lie algebra g. The most general form of a screening current in terms of the d-dimensional free field $X(z)$ is

$$S(z) = \sum_{\sigma \in I_S} \exp[\imath S''(\sigma) \cdot X(z)]. \tag{3.31}$$

Clearly, to completely determine the above screening current one needs to specify the index set I_s and the vectors $S''(\sigma)$. As previously discussed, the main property of a screening current is that it must be of conformal dimension 1 and must have an OPE, with any KM current, that is regular or a total derivative. The first condition translates into

$$h[S''(\sigma)] = \frac{1}{2}[S''(\sigma)]^2 - \alpha_0 \cdot S''(\sigma) = 1, \tag{3.32}$$

where α_0 is given in (3.23), which is valid for any KM algebra. The second condition however is subtle. The reason for this subtlety stems from the fact that a KM current may consist of a sum of several terms. Therefore, within the same KM current. $S(z)$ may have an OPE that is regular w.r.t. some of the terms and singular w.r.t. the remaining ones. This makes the second condition mentioned above nontrivial to satisfy. The terms of a given KM current will be then separated into "irreducible units" $J(z)$. An irreducible unit $J(z)$ is defined as the sum of the minimum number of terms so that the OPE $J(z)S(w)$ is either regular or a total derivative in w As we will see shortly, it turns out that $S(z)$ is completely determined from the irreducible unit $J(z)$ whose OPE with $S(w)$ is a total derivative. Therefore the number of the screening currents one can construct will be equal to the number of all the possible irreducible units (leading to total derivative OPE's) available in the Wakimoto realization of the KM currents. However, not all of the screening currents are independent. We will address the issue of the number of the independent screening currents at the end of this chapter. These irreducible units are not arbitrary but rather conform to very stringent conditions. To see that, let $J(z)$ be an irreducible unit consisting of a sum of several terms

$$J(z) = \sum_{j=1}^{\ell} J(j) = i \sum_{j=1}^{\ell} \bar{J}(j) \cdot \partial X(z) \exp[iJ'(j) \cdot X(z)]. \tag{3.33}$$

The vectors $\bar{J}(j)$ and $J'(j)$ lie in the d-dimensional lattice Λ_0^* as indicated in the previous section. For the sake of clarity, let us proceed in steps of increasing order of complexity. Then we first assume that both the irreducible unit $J(z)$ and $S(z)$ consist of a single term, that is,

$$\begin{aligned} J(z) &= i\bar{J} \cdot \partial X(z) \exp[iJ' \cdot X(z)], \\ S(z) &= \exp[iS'' \cdot X(z)]. \end{aligned} \tag{3.34}$$

In this case, the OPE of $J(z)$ with $S(w)$ reads

$$J(z)S(w) \sim 0\,, \qquad J' \cdot S'' \geq 1, \tag{3.35}$$

$$\sim \frac{C}{z-w} \exp[i(J'+S'') \cdot X(w)], \qquad J' \cdot S'' = 0, \tag{3.36}$$

$$\sim [\frac{C}{(z-w)^2} + \frac{i}{z-w}(CJ' + \bar{J}) \cdot \partial X(w)] \exp[i(J' + S'') \cdot X(w)],$$
$$J' \cdot S'' = -1, \tag{3.37}$$

with

$$C \equiv \bar{J} \cdot S''. \tag{3.38}$$

It is clear that S'' can be determined from (3.37) and (3.38). Indeed, requiring (3.37) to be a total derivative implies that

$$\begin{aligned} CS'' &= \bar{J}, \\ J' \cdot \bar{J} &= -C. \end{aligned} \tag{3.39}$$

Thus, S'' is completely determined and so is $S(z)$ through (3.39) because both $\bar{J}$ and J' are a priori known. Combining (3.38) and (3.39), one can easily see that the proportionality constant C satisfies

$$C^2 = \bar{J}^2. \tag{3.40}$$

Moreover, using (3.40) and requiring the conformal dimension of $S(z)$ to be 1 amount to

$$C = -2\alpha_0 \cdot \bar{J}. \tag{3.41}$$

The example just displayed turns out to be trivially simple and well treated in the literature because it corresponds to $su(2)_k$. The next example however, namely when $\ell = 2$, though more complicated, is by far more appealing in that it displays most of the features of the general case and much less is known about it in the literature.

In this case, according to the definition of the irreducible unit, neither the OPE $J(1)S(w)$ nor $J(2)S(w)$ is a total derivative but the sum of the two is. As can be seen from (3.37), this is possible only if $S(z)$ consists of at least 2 terms, in which case a cancellation might occur between the terms whose phases are so that $J'(1)+S''(1)=J'(2)+S''(2)$. However, the OPE of the terms with phases $J'(1)$ and $S''(2)$ requires a priori a third term $S''(3)$ in $S(z)$, for the same reason as above. Continuing this manner, clearly, $S(z)$ must involve a sum of an infinite number of terms whose phases are so that $S''(i)-S''(j)$ is an integral multiple of $\Delta J \equiv J'(2)-J'(1)$. This means that there is a one-to-one correspondence between the set of the integers Z and the set of the phases defining $S(z)$, i.e.,

$$S''(k)=S''(0)-k\Delta J, \quad k\in Z. \tag{3.42}$$

Let us denote by $\Lambda_{\Delta J}$ the one-dimensional lattice generated by ΔJ. The equation (3.42) amounts then to $S''(j)-S''(0)\in\Lambda_{\Delta J}$. In this case, (3.37) generalizes as

$$\begin{aligned} J(z)S(w) \sim \sum_{j=1}^{2}\sum_{k\in Z}[\frac{C(j,k)}{(z-w)^2}+\frac{i}{z-w}(C(j,k)J'(j)+\bar{J}(j))\cdot\partial X(w)] \\ \times \exp[i(J'(j)+S''(k))\cdot X(w)], \end{aligned} \tag{3.43}$$

with

$$\begin{aligned} C(j,k)&=\bar{J}(j)\cdot S''(k), \\ S''(k+1)-S''(k)&=J'(1)-J'(2). \end{aligned} \tag{3.44}$$

The OPE (3.43) is a total derivative provided that the following condition is satisfied:

$$C(1,k)S''(k)+C(2,k+1)S''(k+1)=\bar{J}(1)+\bar{J}(2)\equiv\bar{J}. \tag{3.45}$$

This equation must be true for any $k\in Z$. The quadratic term in k leads to

$$\bar{J}\cdot\Delta J=0. \tag{3.46}$$

The linear term gives

$$\begin{aligned} 2C&=\bar{J}\cdot S''(0), \\ C&\equiv\bar{J}(2)\cdot\Delta J, \end{aligned} \tag{3.47}$$

while the constant term together with (3.47) yield

$$CS''(0) = \bar{J}, \tag{3.48}$$

$$\bar{J}(1) \cdot \bar{J} = \bar{J}(2) \cdot \bar{J} = C^2. \tag{3.49}$$

Let us recall that (3.43) is valid only if

$$J'(j) \cdot S''(k) = -1, \quad j = 1, 2 \quad k \in Z, \tag{3.50}$$

which means that

$$\begin{aligned} J'(j) \cdot \bar{J} &= -C, \\ J'(j) \cdot \Delta J &= 0. \end{aligned} \tag{3.51}$$

The above equations (3.46) through (3.51) fully determine $S''(0)$, that is, $S(z)$ according to (3.42).

We are now in a position to generalize the previous treatment to general ℓ. In this case (3.42) becomes

$$S''(\sigma) = S''(0) + \sigma, \quad \sigma \in \Lambda_{\Delta J}, \tag{3.52}$$

where $\Lambda_{\Delta J}$ is an $(\ell - 1)$-dimensional lattice generated by the vectors $\Delta J(i) = J'(i) - J'(1)$, $i = 2, \ldots, \ell$. This means that the screening current $S(z)$ consists of an $(\ell - 1)$-fold infinite sum of terms, i.e.,

$$S(z) = \sum_{\sigma \in \Lambda_{\Delta J}} S(\sigma, z) = \sum_{\sigma \in \Lambda_{\Delta J}} \exp[iS''(\sigma) \cdot X(z)]. \tag{3.53}$$

The generalization of (3.43) naturally follows as:

$$\begin{aligned} J(z)S(w) \sim \sum_{j=1}^{\ell} \sum_{\sigma \in \Lambda_{\Delta J}} [\frac{C(j, \sigma)}{(z-w)^2} + \frac{i}{z-w}(C(j,\sigma)J'(j) + \bar{J}(j)) \cdot \partial X(w)] \\ \times \exp[i(J'(j) + S''(\sigma)) \cdot X(w)], \end{aligned} \tag{3.54}$$

with

$$C(j,\sigma) = \bar{J}(j)\cdot S''(\sigma).$$

$$S''(\sigma + \Delta J(j)) - S''(\sigma) = -\Delta J(j), \qquad j = 1,\ldots,\ell. \tag{3.55}$$

Again, (3.54) is a total derivative, i.e.,

$$J(z)S(w) \sim \sum_{j=1}^{\ell}\sum_{\sigma\in\Lambda_{\Delta J}} C(j,\sigma)\partial_w\{\frac{1}{(z-w)}\exp[i(J'(j)+S''(\sigma))\cdot X(w)]\}, \tag{3.56}$$

if

$$\sum_{j=1}^{\ell} C(j,\sigma-\Delta(j))S''(\sigma-\Delta(j)) = \sum_{j=1}^{\ell}\bar{J}(j) \equiv \bar{J}. \tag{3.57}$$

Using (3.52) and (3.55), (3.57) becomes

$$\sum_{i=1}^{\ell}[\bar{J}(i)\cdot(S''(0)-\Delta J(i)+\sigma)][S''(0)-\Delta J(i)+\sigma] = J, \tag{3.58}$$

which is true for any $\sigma\in\Lambda_{\Delta J}$. The quadratic term in σ leads to

$$\bar{J}\cdot\Delta J(j) = 0. \tag{3.59}$$

The linear term in σ and (3.59) yield

$$\bar{J}(i)\cdot\Delta J(j) = \mathcal{C}\delta_{ij}\,, \quad (i,j\geq 2), \tag{3.60}$$

$$\mathcal{C} = \bar{J}\cdot S''(0) - \sum_{i=1}^{\ell}\bar{J}(i)\cdot\Delta J(i) = \bar{J}\cdot S''(0) - (\ell-1)\mathcal{C},$$

or

$$\ell\mathcal{C} = \bar{J}\cdot S''(0). \tag{3.61}$$

Furthermore, the constant term together with (3.60) and (3.61) give

$$\mathcal{C}S''(0) = \bar{J}, \tag{3.62}$$

$$\mathcal{C}^2 = \bar{J}(i)\cdot\bar{J}. \tag{3.63}$$

The equation (3.63) gives $\mathcal{C}$, which in turn determines $S''(0)$, i.e., $S(z)$. Again (3.54) is valid only if $J'(i)\cdot S''(0) = -1$, which is equivalent to

$$J'(i)\cdot\bar{J} = -\mathcal{C},$$
$$J'(i)\cdot\Delta J(j) = 0 \qquad 1\leq i,j\leq\ell, \tag{3.64}$$

which is just the natural generalization of (3.51). Moreover, each term $S(\sigma)$ in the sum defining $S(z)$ is required to be of conformal dimension 1. This condition translates into

$$
\begin{aligned}
\Delta J(i) \cdot \Delta J(j) &= 0, \\
\alpha_0 \cdot \Delta J(j) &= 0, \\
2\alpha_0 \cdot \bar{J} &= \mathcal{C}(\ell - 2).
\end{aligned}
\tag{3.65}
$$

In summary, the screening current $S(z) = \sum_{\sigma \in \Lambda_{\Delta J}} \exp[iS''(\sigma) \cdot X(z)]$ has an OPE, with the irreducible unit $J(z) = \sum_{i=1}^{\ell} J(i)$, that is a total derivative if the following conditions are satisfied:

$$\bar{J} \cdot \Delta J(i) = 0, \tag{3.66}$$

$$\bar{J}(i) \cdot J'(j) = K_i + \mathcal{C}\delta_{ij}, \tag{3.67}$$

$$\mathcal{C}S''(0) = \bar{J}, \tag{3.68}$$

$$\mathcal{C}^2 = \bar{J}(i) \cdot \bar{J} \quad \text{or} \quad \ell\mathcal{C}^2 = \bar{J}^2, \tag{3.69}$$

$$\mathcal{C} = -J'(i) \cdot \bar{J}, \tag{3.70}$$

$$J'(i) \cdot \Delta J(j) = 0, \tag{3.71}$$

$$\alpha_0 \cdot \Delta J(j) = 0, \tag{3.72}$$

$$2\alpha_0 \cdot \bar{J} = \mathcal{C}(\ell - 2), \tag{3.73}$$

with $1 \leq i, j \leq \ell$ and K_i is some constant independent of j, whereas α_0 is defined in (3.23).

Let us now pause to make some interesting remarks. First, note from the Wakimoto realizations of $su(n)_k$ and $sp(4)_k$ given in the previous section that the equations (3.71) and (3.72) are trivially satisfied. In fact, the equation (3.71) is a special case of the following more general equation:

$$J'(i) \cdot J'(j) = 0, \qquad 1 \leq i, j \leq \ell. \tag{3.74}$$

Second, we have checked explicitly in the cases of $su(n)_k$ and $sp(4)_k$ that the number of the irreducible units is always larger than the number of the KM currents themselves. More precisely, the total number of the irreducible units from the KM simple root currents is always equal to $2R$ (R being the number of the positive roots). However, not all of the corresponding $2R$ screening currents are linearly independent, in fact, only $R + r$ (r being the number of the simple roots) of them are. Moreover, the screening currents obtained from the irreducible units of the KM compound root currents are always dependent on the $R + r$ independent screening currents obtained from the KM simple root currents. In addition, the KM currents $H(z)$, corresponding to the Cartan subalgebra, do not include any irreducible unit and thus they do not lead to any screening current. The reason is that their phases in the Wakimoto realization are always zero, i.e., $H_i' = 0$, which is imposed by the KM algebra. This can be seen from (3.70). Indeed, $H_i' = 0$ amounts to $C = 0$, which is not consistent with (3.68). To summarize, there are only $r + R$ linearly independent screening currents of the pure exponential type. Third, by explicit computation, the maximum dimensions of the lattices $\Lambda_{\Delta J}$ are $n - 2$ for $su(n)_k$ and 1 for $sp(4)_k$ respectively. This translates into the existence of at least one screening current with an $(n-2)$-fold infinite sum of terms for $su(n)_k$ and a single infinite sum of terms for $sp(4)_k$ respectively. Finally, let us mention that in the case of $su(n)_k$, the screening currents obtained through our procedure coincide with those obtained in reference [36] except those involving infinite sums. Those conjectured in [36] involve only semi-infinite sums rather than lattice sums as is demonstrated through our method. As far as we know the screening currents in the case of $sp(4)_k$ have never been reported on in the literature, so our procedure provides them for the first time.

Although the relations (3.66) through (3.73) convey enough information to construct the screening current $S(z)$ from the irreducible unit $J(z)$, this is not the end of the story however. In fact, $S(z)$ is in addition required to satisfy the condition of having OPE's that are either regular or total a derivative with all the remaining irreducible units (i.e., the ones that are different from $J(z)$). Let then $J_0(z) = \sum_{i=1}^{\ell_0} J_0(i)$ denote any of the remaining irreducible units. If the OPE of $S(z)$ with $J_0(z)$ is a total derivative, then the set of equations (3.66) through (3.73) must still hold with $\bar{J}(i)$, $J'(i)$, ℓ, $\bar{J}$, $\mathcal{C}$, K_i being substituted by $\bar{J}_0(i)$, $J_0'(i)$, ℓ_0, $\bar{J}_0$, $\mathcal{C}_0$, K_{0i} respectively. Here the quantities with the subscript 0 are derived from $J_0(z)$ exactly in the same way as the corresponding quantities without this subscript are obtained from $J(z)$. The OPE $J_0(z)S(w)$ is a total derivative in w only if

$$S''(\lambda) = S''(0) + \lambda, \quad \lambda \in \Lambda_{\Delta J_0}, \tag{3.75}$$

where $S''(\lambda)$ is defined through $S(z) = \sum_{\lambda \in \Lambda_{\Delta J_0}} \exp[iS''(\lambda) \cdot X(z)]$. Combining (3.52) with (3.75), it is clear that $\lambda \in \Lambda_{\Delta J}$, i.e.,

$$\Lambda_{\Delta J_0} \subset \Lambda_{\Delta J}. \tag{3.76}$$

Furthermore, the combination of (3.52), (3.62) and (3.75) implies that

$$\bar{J}_0/\mathcal{C}_0 = \bar{J}/\mathcal{C} + \sigma, \quad \sigma \in \Lambda_{\Delta J}. \tag{3.77}$$

On the other hand, if the OPE $J_0(z)S(w)$ is not a total derivative then $J_0'(i) \cdot S''(\lambda)$ must be either a positive integer or zero. In the former case, this OPE is trivially regular according to (3.35), and thus $S(z)$ survives this test with no more checks. In the latter case however, namely

$$J_0'(i) \cdot S''(\lambda) = 0, \quad (i = 1, \ldots, \ell_0), \tag{3.78}$$

$S(z)$ must also satisfy the following condition should the OPE $J_0(z)S(w)$ be regular:

$$\sum_{i=1}^{\ell_0} \bar{J}_0(i) \cdot [S''(0) + \lambda - \Delta J_0(i)] = 0, \tag{3.79}$$

for every $\lambda \in \Lambda_{\Delta J_0} \subset \Lambda_{\Delta J}$. The equations (3.62) and (3.78) imply that

$$\bar{J} \cdot J_0'(i) = 0, \tag{3.80}$$

$$J_0'(i) \cdot \Delta(j) = 0, \quad 1 \leq i, j \leq \ell_0. \tag{3.81}$$

Moreover, (3.62) and (3.79) lead to

$$\bar{J}_0 \cdot \Delta J(j) = 0, \quad 1 \leq j \leq \ell, \tag{3.82}$$

$$\bar{J} \cdot \bar{J}_0 = C \sum_{i=1}^{\ell_0} \bar{J}_0(i) \cdot \Delta J_0(i). \tag{3.83}$$

Let us conclude this section by the following remark. For $J_0(z) = H(z)$, where $H(z)$ is the Cartan subalgebra current, $H'(i) \cdot S''(\lambda) = 0$ because $H' = 0$. This means that the equations (3.80) through (3.83) must be satisfied for the OPE $H(z)S(w)$ to be regular. $H' = 0$ means that (3.80) and (3.81) are trivially satisfied. For the same reason, the r.h.s. of (3.83) vanishes. Moreover, it can be readily realized that the l.h.s. of both (3.82) and (3.83) vanish also because of the restriction imposed on the OPE $H(z)J(w)$, where $J(z)$ is any KM current, through the KM current algebra (3.3), and thus both these equations are satisfied as well. The fact that $H(z)S(w)$ is always regular, instead of a total derivative, accounts for the remark made earlier and that is the Cartan subalgebra currents $H(z)$ do not lead to any screening current. As our goal is precisely to construct the screening currents, we then omit to consider these currents $H(z)$ in the subsequent treatment. For the same reason, these currents do not appear in the tables we are about to introduce in the next section.

3.4 Examples

The purpose of this section is to illustrate through explicit examples, namely $su(n)_k$, $n = 2, 3, 4$ and $sp(4)_k$, the construction of the screening currents in terms of free fields. To achieve that, our main tool will be the set of equations (3.66) through (3.73) and (3.80) through (3.83). But to make these equations easy to use in a systematic way, let us first introduce some convenient notations and conventions. As argued before, we expect to derive screening currents only from the $2R$ KM root currents and not from the Cartan subalgebra currents. However, each of the KM root currents consists in general of a sum of several terms. Let $J^a(z) = i\bar{J}^a \cdot \partial X(z) \exp[iJ'^a \cdot X(z)]$ $(a = 1, \ldots, N)$ denote a generic term coming from any of these $2R$ KM root currents. N here is the total number of all the terms involved in the $2R$ KM root currents and thus it is expected to be in general larger than $2R$. In addition, we introduce 2 matrices constructed out of the dot products $\bar{J}^a \cdot J'^b$ and $\bar{J}^a \cdot \bar{J}^b$, which are called the P matrix (P for "prime") and the B matrix (B for "bar") respectively. Then we divide these matrices into $2R \times 2R$ submatrices called "blocks", each of them corresponds to the subset of terms coming from a pair of KM currents. As our method relies on irreducible units, we further subdivide these blocks into submatrices of irreducible units. However, this last operation is subtle and nontrivial because the notion of an irreducible unit is screening current dependent and cannot be defined in an intrinsic way only from the KM currents. In fact, we should subdivide the blocks into irreducible units only simultaneously while fulfilling the conditions (3.66) through (3.73). Next, we refer to the submatrices of irreducible units by the name of the corresponding currents, for example, P_{JJ_0} stands for the submatrix of the dot products $\bar{J}(i)J_0'(j)$, where $J = \sum_{i=1}^{\ell} J(i)$ and $J_0 = \sum_{i=1}^{\ell_0} J_0(i)$ define 2 irreducible units. Finally, let us denote the sum of all the entries in any column in the matrices P_{JJ_0} and B_{JJ_0} by CP_{JJ_0} and CB_{JJ_0} respectively. With the above definitions and conventions, the fulfillment of all the necessary conditions becomes more transparent. More specifically, the equation (3.67) means that all the entries in each row in P_{JJ} must be the same except for the diagonal entry which must be larger than them by C; let us refer to this as "the P row rule". The equation (3.69) translates into

$$CB_{JJ} = C^2, \tag{3.84}$$

for all column sums of B_{JJ}. Furthermore, the equation (3.70) reads now as

$$CP_{JJ} = -C, \tag{3.85}$$

for all column sums of P_{JJ} also. In principle, the P row rule together with (3.84) and (3.85) are enough to single out the irreducible units and thereby the construction of the screening currents follows. However, one must make sure that all the screening currents thus constructed are indeed genuine in that they still meet all the criteria w.r.t. all the other irreducible units and not just w.r.t. the irreducible units from which they are derived. For that matter, let $J(z)$ be an irreducible unit giving rise to the screening current $S(z)$ through the P row rule together with the equations (3.84) and (3.85), and let $J_0(z)$ be any other irreducible unit. To check the condition that the OPE $J_0(z)S(w)$ is acceptable, we first compute CP_{JJ_0}. If $CP_{JJ_0} \geq C$, then $S(z)$ survives the test already and no more checks are needed. If $CP_{JJ_0} = -C$, then we must check that the P row rule together with (3.84) and (3.85) are also satisfied with J and C being replaced by J_0 and C_0. In addition, we must make sure that (3.77) is satisfied and $\Lambda_{\Delta J_0} \subset \Lambda_{\Delta J}$. Finally, if $CP_{JJ_0} = 0$, the equations (3.82) and (3.83) must then be satisfied. In particular, (3.82) means that the column sums $CP_{J_0 J}$ must all be identical, let us call this the "P column rule".

Moreover, (3.83) amounts to requiring the sum of all the entries in B_{JJ_0} divided by C to be equal to the sum of the diagonal entries in $P_{J_0J_0}$ minus $CP_{J_0J_0}$; let us refer to this as the "P diagonal rule". Again, one must also check that $\Lambda_{\Delta J_0} \subset \Lambda_{\Delta J}$. Note that if either $J(z)$ or $J_0(z)$ is a one-dimensional irreducible unit, that is the corresponding P and B matrices are 1×1 matrices, then all the above conditions are trivially satisfied except the P diagonal rule, which further requires us to check that the single entry in B_{JJ_0} is zero. We now further proceed by applying the above rules and conditions to explicitly derive the screening currents for $su(n)_k$, $n = 2, 3, 4$ and $sp(4)_k$. To avoid cluttering the notation, let us denote the screening current $S(z)$, which is obtained from the irreducible unit $J(z) = i\sum_{j=1}^{\ell} \bar{J}^{a_j} \cdot \partial X(z) \exp[iJ'^{a_j} \cdot X(z)] \equiv J^{a_1,a_2,\ldots,a_\ell}(z)$, by

$$S^{a_1,a_2,\ldots a_\ell}(z) = \sum_{\sigma \in \Lambda_{\Delta J}} \exp[i(S''(0) + \sigma) \cdot X(z)], \tag{3.86}$$

where $1 \leq a_i \leq N$. What is missing in (3.86) is the lattice $\Lambda_{\Delta J}$, that is the set $\{a_1, \ldots a_\ell\}$, and the vector $S''(0) = \mathcal{C}(\bar{J}^{a_1} + \ldots + \bar{J}^{a_\ell})$. The derivation of these quantities is illustrated below.

Example 1. $su(2)_k$

In this case $R = 1$, the two KM root currents are written in Table 1. Each of them consists of a single term, that is, $N = 2$. The corresponding 2×2 P and B matrices are given in Tables (2a) and (2b) respectively. At this point, we can already deduce that no screening current involving infinite sums of terms exists. This is because the maximum dimension of the lattice $\Lambda_{\Delta J}$ is zero in this case. Moreover, we expect only two screening currents, each of them consisting of a single term. Indeed, Tables (2a) and (2b) show that $\ell = 1$ and $\mathcal{C} = 1$ for both cases, which means according to our notation (3.86) that the screening currents are $S^1(z)$ and $S^2(z)$. Moreover, it can easily be seen that the OPE's $J^1(z)S^2(w)$ and $J^2(z)S^1(w)$ are both trivially regular because $CP_{JJ_0} = C = 1$ for both cases.

Example 2. $su(3)_k$

In this case, $2R = 6$ but 4 KM root currents are expressed as sums of 2 terms, i.e., $N = 10$. These terms are presented in Table 3, whereas their corresponding P and B matrices are respectively given in Tables (4a) and (4b). There, the lines separate the blocks of KM currents. The next step is to look for the irreducible units in each block. For this purpose we resort to the P row rule together with the equations (3.84) and (3.85) to first figure out if the whole block is irreducible. If not, we further proceed by dividing the block to smaller and smaller units until we get irreducible units, in which case we separate them by dotted lines as shown in Tables 3 and 4. It can easily be checked from these tables that the irreducible units are J^1, J^2, J^3, J^4, J^5, J^6, $J^{7,8}$ and $J^{9,10}$ because they satisfy the P row rule and the equations (3.84) and (3.85). All these irreducible units have $C = 1$. The independent screening currents are therefore $S^1(z)$, $S^2(z)$, $S^3(z)$, $S^5(z)$ and $S^{7,8}(z)$. Because there are two indices in $S^{7,8}(z)$, this screening current consists then of a single infinite sum of terms associated with the lattice $\Lambda_{\Delta J}$, which is generated by $J'^8 - J'^7$. The other irreducible units lead to screening currents that are dependent on the above five. In particular, we have the following identifications:

$$
\begin{aligned}
S^4(z) &\equiv S^2(z), \\
S^6(z) &\equiv S^3(z), \qquad\qquad (3.87)\\
S^{9,10}(z) &\equiv S^{7,8}(z).
\end{aligned}
$$

This is not the end of the story though, because we must still check that the above five screening currents are indeed consistent w.r.t. all other KM currents.

Since most of the irreducible units are only of a single term type let us remind the remark made previously and that is all consistency conditions are automatically satisfied except that the single entry of B_{JJ_0} must be zero. To illustrate that, let us content ourselves to work through an example. For that, we present the details of the treatment leading to $S^1(z)$ from $J^1(z)$. We start out by looking at the first row of the P matrix given in Table (4a). If this row contains any nondiagonal entry in $P_{J^1J^a}$, $a \neq 1$, that is larger or equal to 1, then $S^1(z)$ is consistent w.r.t. the irreducible unit J^a. This is the case for $a = 5$ and $a = 9$. If however the entry $P_{J^1J^a} = 0$, then we must make sure from the first row of the B matrix that $B_{J^1J^a} = 0$. It is easy to see from Table (4a) and (4b) that this is so if a is equal to either 2, 3, 4, 6 or 7. Finally, that $P_{J^1J^8} = -1$ simply means that $S^1(z) \equiv S^8(z)$ as expected from $\bar{J}^1 = -\bar{J}^8$. The other 3 screening currents S^2, S^3 and S^5 can similarly be checked to satisfy all the consistency conditions also. Let us now illustrate the process of checking the consistency conditions of $S^{7,8}$, involving an infinite sum of terms, w.r.t. all other KM currents. First we compute the value of CP_{JJ_0}, where now the irreducible unit is $J = J^{7,8}$ according to our notation (3.86). For that, we focus only on the 7th and 8th rows of the P matrix and sum the entries in each column except the 7th and the 8th. If $CP_{JJ_0} \geq 1$, this means that $S^{7,8}$ is consistent w.r.t. J_0 and no more checks are required. This is the case if J_0 is either J^3 or J^4. If $CP_{JJ_0} = 0$, we must check that the P column rule and the P diagonal rule are satisfied and in the process determine the unit J_0 because $CP_{JJ_0} = 0$ alone is not enough to do that. It can readily be checked that this case happens with the above rules satisfied if the unit J_0 is either $J^{1,2}$ or $J^{5,6}$. For the remaining columns however, $CP_{JJ_0} = -1$, that is, we should satisfy the P row rule for J_0 together with $CB_{J_0J_0} = C_0^2$ and $CP_{J_0J_0} = -C_0$. The latter conditions are also satisfied with $J = J^{9,10}$ and $C_0 = 1$. This means that $S^{7,8} \equiv S^{9,10}$ and $\Lambda_{\Delta J} \equiv \Lambda_{\Delta J_0}$, which is confirmed by $\Delta J = J'^8 - J'^7 = J'^{10} - J'^9 = \Delta J_0$.

Example 3. $su(4)_k$

Here, there are $2R = 12$ KM root currents giving rise to $N = 32$ terms. This case, though too long, displays the same features of $su(3)_k$; the only difference is that now a two-dimensional lattice $\Lambda_{\Delta J}$ arises. We have omitted to display the 32×32 P and B matrices for $su(4)_k$ because they are lengthy. However, we carried out the explicit treatment of constructing the screening currents. This is displayed below without much details. The $su(4)_k$ KM root currents together with the terms involved in each of them are

$$
\begin{aligned}
J_{-(12)} &: \quad J^1, J^2, J^3; \\
J_{-(23)} &: \quad J^4, J^5; \\
J_{-(34)} &: \quad J^6; \\
J_{-(13)} &: \quad J^7, J^8; \\
J_{-(24)} &: \quad J^9; \\
J_{-(14)} &: \quad J^{10}; \\
J_{+(12)} &: \quad J^{11}, J^{12}, J^{13}. \\
J_{+(23)} &: \quad J^{14}, J^{15}, J^{16}. \\
J_{+(34)} &: \quad J^{17}, J^{18}, J^{19}; \\
J_{+(13)} &: \quad J^{20}, J^{21}, J^{22}; \\
J_{+(24)} &: \quad J^{23}, J^{24}, J^{25}, J^{26}, J^{27}; \\
J_{+(14)} &: \quad J^{28}, J^{29}, J^{30}, J^{31}, J^{32}.
\end{aligned}
\tag{3.88}
$$

From their respective P and B matrices we consistently construct 9 independent screening currents, which are S^1, S^2, S^3, S^4, S^5, S^6, S^{11}, $S^{14,16}$, $S^{17,18,19}$, and the following dependent ones:

$$
\begin{aligned}
S^7 &\equiv S^2, \\
S^8 &\equiv S^3, \\
S^9 &\equiv S^5, \\
S^{10} &\equiv S^3, \\
S^{12} &\equiv S^1, \\
S^{13} &\equiv S^5, \\
S^{15} &\equiv S^6, \\
S^{20,21} &\equiv S^{11,16}, \\
S^{22} &\equiv S^6, \\
S^{23,24,25} &\equiv S^{17,18,19}, \\
S^{26} &\equiv S^1, \\
S^{27} &\equiv S^1, \\
S^{28} &\equiv S^{11}, \\
S^{29} &\equiv S^{11}, \\
S^{30,31,32} &\equiv S^{17,18,19}.
\end{aligned}
\tag{3.89}
$$

Example 4. $sp(4)_k$

In this case, there are $2R = 8$ KM root currents leading to $N = 20$ terms, which are given in Table 5. The corresponding P and B matrices are displayed in Tables (6a) and (6b). The treatment of constructing the screening currents follows exactly that of $su(3)_k$. More precisely, we find 6 independent screening currents. They are according to our notation (3.86) S^1, S^2, S^3, S^5, $S^{9,10}$ and

$S^{12,13}$. In addition, we find 12 dependent ones, which are the following:

$$\begin{aligned} S^4 &\equiv S^7 \equiv S^8 \equiv S^3, \\ S^6 &\equiv S^{17} \equiv S^2, \\ S^{11} &\equiv S^{20} \equiv S^5, \\ S^{11,16} &\equiv S^{19,20} \equiv S^{12,13}, \\ S^{15} &\equiv S^1, \\ S^{18,19} &\equiv S^{16,17} \equiv S^{9,10}. \end{aligned} \tag{3.90}$$

3.5 Explicit formulas

Here we propose to give the screening currents in a familiar form that is widely used in the literature [20,36,37]. For that, note from the above examples that the number of the independent screening currents is always $r + R$, where r and R are again the numbers of the simple roots and the positive roots respectively. We may then label the independent screening currents as S_{α_i} and η_α, where α_i stands for a simple root while α stands for a positive root. It can easily be checked that the following screening currents are common to $su(n)_k$:

$$\eta_\alpha = \exp(-iu_\alpha), \tag{3.91}$$

and

$$S_{\alpha_1} = \exp\{i[(k+h)v_{\alpha_1} - (k+h-1)u_{\alpha_1} + \sqrt{k+h}\alpha_1 \cdot \varphi]\}, \tag{3.92}$$

with $h = n$. The above formulas with $h = 2$ exhaust all the screening currents for $su(2)_k$. However, for $su(3)_k$ there is one more independent screening current given by

$$\begin{aligned} S_{\alpha_2} = \sum_{n=-\infty}^{\infty} \exp\{i[(k+h-n)v_{\alpha_2} - (k+h-1-n)u_{\alpha_2} - n(v_{\alpha_1} - u_{\alpha_1}) \\ + nv_{\alpha_1+\alpha_2} - (n-1)u_{\alpha_1+\alpha_2} + \sqrt{k+h}\alpha_2 \cdot \varphi]\}, \end{aligned} \tag{3.93}$$

with $h = 3$. For $su(4)_k$, besides all the above screening currents with $h = 4$, there is one more screening current (corresponding to the third simple root α_3) expressed as a double infinite sum of terms. It reads as follows:

$$
\begin{aligned}
S_{\alpha_3} = \sum_{n,m=-\infty}^{\infty} & \exp\{i[(k+h-n-m)v_{\alpha_3} - (k+h-1-n-m)u_{\alpha_3} - n(v_{\alpha_2} - u_{\alpha_2}) \\
& - m(v_{\alpha_1+\alpha_2} - u_{\alpha_1+\alpha_2}) + nv_{\alpha_2+\alpha_3} - (n-1)u_{\alpha_2+\alpha_3} \\
& + mv_{\alpha_1+\alpha_2+\alpha_3} - (m-1)u_{\alpha_1+\alpha_2+\alpha_3} + \sqrt{k+h}\alpha_3 \cdot \varphi]\},
\end{aligned} \tag{3.94}
$$

with $h = 4$. Finally, for $sp(4)_k$ the η_α type screening currents are again given in (3.91), whereas the S_{α_i} type screening currents read as follows:

$$
\begin{aligned}
S_{\alpha_1} &= \sum_{n=-\infty}^{\infty} \exp\{i[(k+6-n)v_{\alpha_1} - (k+5-n)u_{\alpha_1} - n(v_{\alpha_3} - u_{\alpha_3}) \\
&\qquad + nv_{\alpha_4} - (n-1)u_{\alpha_4} + \sqrt{k+6}\alpha_1 \cdot \varphi]\} \\
S_{\alpha_2} &= \sum_{n=-\infty}^{\infty} \exp\{i/2[(k+4+2n)v_{\alpha_2} - (k+2+2n)u_{\alpha_2} + 2(n-1)(v_{\alpha_1} - u_{\alpha_1}) \\
&\qquad - 2(n-1)v_{\alpha_3} + 2nu_{\alpha_3} + \sqrt{k+6}\alpha_2 \cdot \varphi]\},
\end{aligned} \tag{3.95}
$$

where $\alpha_1 \equiv \alpha_{(12)}$, $\alpha_2 \equiv \alpha_{(2)}$, $\alpha_3 \equiv \alpha_{[12]}$ and $\alpha_4 \equiv \alpha_{(1)}$. Note that both of the above screening currents consist of a single infinite sum of terms.

3.6 Conclusions

In this chapter, a systematic method, to construct screening currents in terms of free fields in CFT with a KM algebra, is presented. For the sake of illustration, a detailed treatment is explicitly displayed in the cases of $su(n)_k$, $n = 2, 3, 4$, and $sp(4)_k$. These are the KM algebras whose Wakimoto realizations are presently available in the literature. However our method is completely general and can be applied to any other KM algebra provided that its Wakimoto realization is achieved. Even at this point, building upon the explicit examples we have worked out, we may expect some general results to hold for any KM algebra. In particular, the number of the screening currents is always $r + R$, and the expressions of the screening currents of the η type are still as given in (3.91). Clearly, further investigation is required to clarify this point. Finally, our method gives rise to the lattices $\Lambda_{\Delta J}$ which encode all the information needed to unravel the structure of the screening currents. In fact, these lattices are interesting in their own right in that they may encode enough information to derive the Wakimoto realizations of the KM algebras themselves besides their screening currents. As discussed in chapter 1, the screening currents can be applied to derive the characters, the fusion rules and the correlation functions in the Coulomb gas formalism. Now that the screening currents for KM algebras are available, it is then in principle possible to derive the above quantities in CFT's with KM algebras. In fact, some work in this direction has already been achieved in the case of $su(2)_k$ [20,38,40]. Unfortunately, this case is trivial and does not display most of the features of the $su(n)_k$ or general KM algebras. This is because for $su(2)_k$ $r = R$ and the screening currents involving infinite sums of terms are absent. The use of the latter screening currents is not trivial and investigating it may lead to new insights about KM algebras. Therefore, a full treatment of $su(3)_k$ which embodies all the features of general KM algebras is highly appealing and desirable. Finally, let us mention that though this method is designed for CFT's with KM algebras, we believe that it can also be applied to other CFT's such as the parafermion CFT's for example [41].

Chapter 4

Embedding structure of Verma modules of the W_3 algebra

The purpose of this chapter is to describe the embedding structure of the Verma modules of a W_3 algebra [15]. This is carried out through the free field realization (FFR) of this algebra. This structure is summarized in a set of intertwining diagrams, which in turn allows us to derive the irreducible character of a primary completely degenerate Verma module (PCDVM) of W_3 in terms of the characters of its Verma submodules. The irreducible character thus obtained through our method [21] proves to coincide with the one conjectured in the literature [42]. Our approach adopted here can be regarded as the direct generalization of the Virasoro case presented in chapter 2. The layout of this chapter is as follows. We begin with an introduction to the W_3 algebra in the minimal unitary series and fix the notation. Then we review the free field realization of this algebra. Next, we use the screening charges to explicitly construct, in terms of free fields, the null states in a PCDVM. This allows us to describe the embedding structure of W_3 Verma submodules contained in this PCDVM. After that, we represent this structure through a few intertwining diagrams involving the Verma submodules and the screening charges. Finally, we use these intertwining diagrams to compute the irreducible character of a PCDVM of the W_3 algebra.

4.1 W_3 algebra in the unitary minimal series

A two-dimensional theory with a W_3 algebra is another example of a RCFT based on an infinite-dimensional algebra that extends the Virasoro one. In particular, this means that a CFT with a W_3 algebra is still exactly solvable despite that the central charge c is larger than 1. Here we consider only the unitary W_3 algebra, which is characterized by the central charge

$$c = 2[1 - \frac{12}{p(p+1)}], \tag{4.1}$$

where p is a positive integer larger than 3 Note that W_3 is related to the $su(3)_k$ Kac-Moody algebra. Indeed, the above central charge is the same as that of the coset model $su(3)_p \times su(3)_1/su(3)_{p+1}$. This coset model is obtained through the Goddard-Kent-Olive (GKO) construction [33] and the associated energy-momentum tensor is derived out of the $su(3)_k$ Kac-Moody currents via the Sugawara procedure [39]. In analogy to the Virasoro algebra which is generated only from an energy-momentum tensor, the W_3 algebra is generated from an energy-momentum tensor $T(z)$ and a spin 3 operator $W(z)$. The oscillator modes of $T(z)$ and $W(z)$ satisfy the algebra

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{1}{12}c(n^3 - n)\delta_{n+m,0}, \\
[L_n, W_m] &= (2n-m)W_{n+m}, \\
[W_n, W_m] &= \frac{c}{3.5!}(n^2-4)(n^2-1)n\delta_{n+m,0} + b^2(n-m)\Lambda_{n+m} \\
&\quad + (n-m)[\frac{1}{15}(n+m+2)(n+m+3) - \frac{1}{6}(n+2)(m+2)]L_{n+m},
\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}
b^2 &= \frac{16}{22+5c}, \\
\Lambda_n &= \sum_{k=-\infty}^{+\infty} : L_k L_{n-k} : + \frac{1}{5}x_n L_n, \\
x_{2l} &= (1+l)(1-l), \qquad x_{2l+1} = (2+l)(1-l).
\end{aligned} \tag{4.3}$$

The states in the Verma module $V(h, w)$, with primary state denoted by $| h, w >$, are given by

$$| \phi > = \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} L_{-i}^{n_i} W_{-j}^{n_j} \; | h, w >, \qquad n_i, n_j = 0, ..., \infty \tag{4.4}$$

where $| h, w >$ satisfies

$$
\begin{aligned}
&L_0 \mid h, w > = h \mid h, w >, \\
&W_0 \mid h, w > = w \mid h, w >, \qquad (4.5) \\
&L_n \mid h, w > = W_n \mid h, w > = 0, \qquad n > 0.
\end{aligned}
$$

Note that the above equations (4.5) generalize the relations (2.30), which are required in the case of the Virasoro algebra. The eigenvalues h (conformal dimension) and w are specified by four positive integers n_1, n_2, m_1 and m_2 (or equivalently by the two-dimensional vector $\beta \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ whose components in the orthonormal basis are β_1 and β_2) in the following way†:

$$
\begin{aligned}
&h \begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} = \beta^2 - 2\alpha_0 \cdot \beta = \frac{\xi}{12p(p+1)}, \\
&\xi = 3[(p+1)(n_1+n_2) - p(m_1+m_2)]^2 + [(p+1)(n_1-n_2) - p(m_1-m_2)]^2 - 12, \\
&w \begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} = \frac{2}{3} b[\beta_2^3 - 3\beta_1^2\beta_2 + 6\alpha_0\beta_1\beta_2 - 3\alpha_0^2\beta_2], \qquad (4.6)
\end{aligned}
$$

with the condition that n_1, n_2, m_1 and m_2 are so that

$$
\begin{aligned}
&n_1 + n_2 < p, \\
&m_1 + m_2 < p + 1. \qquad (4.7)
\end{aligned}
$$

In the course of the subsequent treatment and depending on the context we will use the notations

$$
\beta \equiv \beta \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \equiv \beta \begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix}. \qquad (4.8)
$$

However, let us point out that $\beta \begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix}$ does not necessarily specify a primary state unless it is clearly stated that n_1, n_2, m_1 and m_2 satisfy the inequalities (4.7). In the orthonormal basis, the vectors β and α_0 are given by

$$
\begin{aligned}
&\beta \begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} = \begin{pmatrix} (2+n_1+n_2)\alpha_+ + (2+m_1+m_2)\alpha_- \\ \frac{n_1-n_2}{\sqrt{3}}\alpha_+ + \frac{m_1-m_2}{\sqrt{3}}\alpha_- \end{pmatrix}, \\
&\alpha_0 = \begin{pmatrix} 2(\alpha_+ + \alpha_-) \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{p(p+1)}} \\ 0 \end{pmatrix}. \qquad (4.9)
\end{aligned}
$$

† We will elaborate more on the origin of the integers n_1, n_2, m_1 and m_2, and the vector $\beta \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$ in the section 4.2.

b is defined in (4.3) whereas α_+ and α_- read as follows:

$$\begin{aligned} \alpha_+ &= \frac{1}{2}\sqrt{\frac{p+1}{p}}, \\ \alpha_- &= -\frac{1}{2}\sqrt{\frac{p}{p+1}}. \end{aligned} \tag{4.10}$$

4.1.1 Symmetries of the W_3 algebra

Here we summarize the symmetries of W_3 representations, which are pertinent to the subsequent analysis. It can readily be checked through (4.6) that the following transformations are symmetries of the W_3 algebra:

a) Translation symmetry:

$$\begin{aligned} h\begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} &= h\begin{pmatrix} n_1 + k_1 p & m_1 + k_1(p+1) \\ n_2 + k_2 p & m_2 + k_2(p+1) \end{pmatrix}, \\ w\begin{pmatrix} n_1 & m_1 \\ m_1 & m_2 \end{pmatrix} &= w\begin{pmatrix} n_1 + k_1 p & m_1 + k_1(p+1) \\ n_2 + k_2 p & m_2 + k_2(p+1) \end{pmatrix}, \qquad k_1, k_2 \in \mathbf{Z}; \end{aligned} \tag{4.11}$$

b) $\mathbf{Z_2}$ symmetry:

$$\begin{aligned} h\begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} &= h\begin{pmatrix} -n_2 & -m_2 \\ -n_1 & -m_1 \end{pmatrix}, \\ w\begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} &= w\begin{pmatrix} -n_2 & -m_2 \\ -n_1 & -m_1 \end{pmatrix}; \end{aligned} \tag{4.12}$$

c) $\mathbf{Z_3}$ symmetry:

$$\begin{aligned} h\begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} &= h\begin{pmatrix} -n_1 - n_2 & -m_1 - m_2 \\ n_1 & m_1 \end{pmatrix}, \\ w\begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} &= w\begin{pmatrix} -n_1 - n_2 & -m_1 - m_2 \\ n_1 & m_1 \end{pmatrix}. \end{aligned} \tag{4.13}$$

4.1.2 Degenerate representations of the W_3 algebra

As discussed in reference [15], the representations of the minimal series of the W_3 algebra are completely degenerate. This means that the PCDVM $V(h)$ (henceforth, we consider only the eigenvalue h but we should keep in mind that whenever h is known so is w, as is indicated in (4.6)) contains an infinite number of null states $\mid h_N >$, where N stands for the degree of the states, as mentioned in chapter 2. These null states are such that

$$\begin{aligned} L_n \mid h_N > &= W_n \mid h_N >= 0, \qquad n > 0, \\ L_0 \mid h_N > &= h_N \mid h_N >= (h + N) \mid h_N >, \end{aligned} \tag{4.14}$$

where h is given in (4.6). Using the symmetries of the W_3 algebra, we will see in the subsequent sections that h_N can always be written as:

$$h_N \equiv h \begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} \equiv h[\beta \begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix}], \tag{4.15}$$

for some integers n_1, n_2, m_1 and m_2 (not necessarily positive), which will specify the set of null states.

4.2 Free field realization of the W_3 algebra

The RCFT with a W_3 algebra whose central charge is given in (4.1) can be represented in terms of a two-dimensional free field $X \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ [15]. The field components X_1 and X_2 are now correlated like

$$< X_a(z) X_b(w) >= -2\delta_{ab} \log(z - w), \qquad a, b = 1, 2. \tag{4.16}$$

From now on, we only consider the Virasoro part of W_3, though we keep track of all the symmetries induced by the whole W_3 algebra [15]. The energy-momentum tensor $T(z)$ is represented in terms of the vector X as

$$T(z) = -\frac{1}{4}[\partial X(z)]^2 + i\alpha_0 \cdot \partial^2 X(z), \tag{4.17}$$

where the normal ordering is understood. α_0 is the same vector as the one introduced in (4.9). Again, it represents a background charge placed at infinity. The highest weight states are represented as exponential vertex operators, i.e.,

$$V_\beta(z) = \exp[i\beta \cdot X(z)], \tag{4.18}$$

and are specified by singular vectors $\beta \begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix}$. β specifies a primary state if the set of integers n_1, n_2, m_1 and m_2 satisfies the inequalities (4.7); otherwise it specifies the null states for special values of this set of integers. This will be worked out in the subsequent analysis. The conformal dimension of $V_\beta(z)$ is still as given in (4.6). However, the constraints (4.7) are not necessarily satisfied. Furthermore, in this FFR the central charge is given by

$$c = 2(1 - 12\alpha_0^2). \tag{4.19}$$

Substituting α_0 in (4.19) by its value given in (4.9), we recover then the formula (4.1) of the central charge. The other important ingredient in the FFR recipe is the set of screening currents [18,20]. In the present case these screening currents are operators of conformal dimension 1, whose OPE's with both $T(z)$ and $W(z)$ are either regular or a total derivative. Four screening currents meet these criteria. They are expressed as exponential functions of the free field $X(z)$

$$S_a^\pm(z) = \exp[ie_a^\pm \cdot X(z)], \qquad a = 1, 2. \tag{4.20}$$

Here the vectors $e_a^\pm$ are given by

$$e_a^\pm = \alpha_\pm u_a, \qquad a = 1, 2, \tag{4.21}$$

where $\alpha_\pm$ are defined in (4.10), and the vectors u_1 and u_2 are

$$\begin{aligned} u_1 &= (1, \sqrt{3}), \\ u_2 &= (1, -\sqrt{3}). \end{aligned} \tag{4.22}$$

As explained in chapter 2, the screening currents allow the construction of an infinite number of null states in the module V_β. More specifically, these null states are derived by means of the above screening currents as [15,20]

$$\prod_{i=1}^{n_a^\pm} \oint dz_i S_a^\pm(z_i) V_{\beta - n_a^\pm e_a^\pm}(0) \equiv Q V_{\beta - n_a^\pm e_a^\pm}(0), \qquad a = 1,2. \tag{4.23}$$

In $n_a^\pm e_a^\pm$ no sum w.r.t. a is meant and the integers $n_a^\pm$ are obviously non-negative. Let us point out that the states thus constructed in (4.23) do not fully satisfy all the criteria of the null states unless the last integral contour is closed. As discussed in references [18,43,44,45], the closure of the last contour amounts to requiring the difference of the degrees of the states $QV_{\beta-n_a^\pm e_a^\pm}$ and V_β to be a non-negative integer. This difference of degrees can easily be worked out as

$$\begin{aligned} N_a &= h(\beta - n_a^\pm e_a^\pm) - h(\beta) \\ &= -n_a^\pm[1 + 2e_a^\pm \cdot \beta - (n_a^\pm + 1)(e_a^\pm)^2], \qquad a = 1,2, \end{aligned} \tag{4.24}$$

where $h(\beta)$ is defined through (4.6). $QV_{\beta-n_a^\pm e_a^\pm}$, introduced in (4.23), is a null state in the Verma module V_β only if the above N_a is a positive integer. In this case, the equation (4.24) translates into

$$\beta \cdot e_a^\pm = \frac{1 + n_a^\pm}{2}(e_a^\pm)^2 - \frac{1 + m_a^\pm}{2}, \qquad a = 1,2, \tag{4.25}$$

for some positive integer m_a so that $N_a = n_a m_a$. This means that $QV_{\beta-n_a^\pm e_a^\pm}$ is a null state whose degree is $n_a m_a$ above that of V_β. Moreover, it carries the same quantum numbers w.r.t. the W_3 algebra as the one of $V_{\beta-n_a^\pm e_a^\pm}$, and consequently these two modules are identified.

As argued in reference [44], if we require the vacuum state ($\beta = 0$), which corresponds to the identity operator, to be singular, the equation (4.25) then implies that $(e_a^\pm)^2$ is rational. It can be readily seen that this condition is satisfied by the four screening vectors $\epsilon_a^{\pm}$, $a = 1,2$ given in (4.21) and (4.22). Indeed, the latter relations together with (4.10) lead to

$$
\begin{aligned}
(\epsilon_a^+)^2 &= 4\alpha_+^2 = \frac{p+1}{p}, \\
(\epsilon_a^-)^2 &= 4\alpha_-^2 = \frac{p}{p+1}, \qquad a = 1,2
\end{aligned}
\tag{4.26}
$$

According to (4.25), the integers $n_a^{\pm}$ and $m_a^{\pm}$ may be negative as well. This is however not incompatible with (4.23) and (4.24), which suggest that these integers must be positive. The reason is that the dot product $\beta \cdot \epsilon_a^{\pm}$ in (4.25) is invariant under the translations discussed earlier, namely,

$$
\begin{aligned}
&(n_a^+, m_a^+) \to (n_a^+ + k_a^+ p, m_a^+ + k_a^+(p+1)), \\
&(n_a^-, m_a^-) \to (n_a^- + k_a^-(p+1), m_a^- + k_a^- p), \quad k_a^+, k_a^- \in \mathbf{Z}, \quad a = 1,2
\end{aligned}
\tag{4.27}
$$

This means that when these integers are negative it is always possible to use (4.27) to replace them with positive numbers. Consequently, the singular vectors $\beta(n_a^{\pm}, m_a^{\pm})$ may be without ambiguity parametrized by $n_a^{\pm}, m_a^{\pm} \in \mathbf{Z}$, $a = 1,2$. This implies that sums and differences of singular vectors are also singular vectors, that is, the set of all singular vectors $\{\sqrt{2}\beta\}$ spans a lattice, which is denoted by Λ_β. Since the conformal dimension of the screening currents is 1, (4.6) leads to

$$
\alpha_0 \cdot \epsilon_a^{\pm} = \frac{(\epsilon_a^{\pm})^2}{2} - \frac{1}{2}, \qquad a = 1,2. \tag{4.28}
$$

This translates into

$$
\sqrt{2}\alpha_0(n_a^{\pm} = 0, m_a^{\pm} = 0) \in \Lambda_\beta, \qquad a = 1,2. \tag{4.29}
$$

Let us now introduce another lattice, referred to as Λ_ϵ, whose basis is given by the "positive sector" vectors $\{\sqrt{2}\epsilon_a^+, \quad a = 1,2\}$ (we could have chosen the "negative sector" basis instead $\{\sqrt{2}\epsilon_a^-, \quad a = 1,2\}$; the choice of the basis is not relevant). In the orthonormal basis $\{\hat{x}, \hat{y}\}$, the singular vector β is written as $\beta\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}$. Let us define the vector β^* so that

$$\beta^*\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} \equiv \beta\begin{pmatrix}\beta_1\\ -\beta_2\end{pmatrix}. \tag{4.30}$$

This means that β^* is obtained through the Weyl reflection operation $S_{\hat{y}}$ on β w.r.t. the $\hat{x}$ axis, i.e.,

$$\beta^*\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} \equiv S_{\hat{y}}.\beta\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = \beta - 2(\beta\cdot\hat{y})\hat{y} = \beta\begin{pmatrix}\beta_1\\ -\beta_2\end{pmatrix}. \tag{4.31}$$

Note that β^* is also a singular vector, that is, it satisfies (4.25). This is so because of (4.26) and the relations

$$\begin{aligned}\beta^*\cdot e_1^{\pm} &= \beta\cdot e_2^{\pm},\\ \beta^*\cdot e_2^{\pm} &= \beta\cdot e_1^{\pm}.\end{aligned} \tag{4.32}$$

Now taking into account the $\mathbf{Z_2}$ invariance introduced in section 4.1.1 together with the equation (4.32), and replacing in (4.25) β by $2\alpha_0(n_a^{\pm} = 1, m_a^{\pm} = 1)$, it is clear that there exist four null states at level 1 above $V_{2\alpha_0}$, whose quantum numbers are the same as those of

$$V_{2\alpha_0 - e_a^{\pm *}} \equiv V_{e_a^{\pm}}, \qquad a = 1, 2. \tag{4.33}$$

This means that the screening vectors $\{\sqrt{2}e_a^{\pm}, \quad a = 1, 2\}$ are themselves singular vectors and thus belong to Λ_β. Consequently, they must satisfy the equation (4.25), which amounts to the relations

$$\begin{aligned}&2pe_a^{\pm}\cdot e_b^{+} \in \mathbf{Z},\\ &2(p+1)e_a^{\pm}\cdot e_b^{-} \in \mathbf{Z}, \qquad a, b = 1, 2.\end{aligned} \tag{4.34}$$

It is shown in reference [44] that the lattice Λ whose basis is $\{\sqrt{2}pe_a^{+}, \quad a = 1, 2\}$ or $\{\sqrt{2}(p+1)e_a^{-}, \quad a = 1, 2\}$ is an even integral lattice. Furthermore, it satisfies the inclusion relation

$$\Lambda \subset \Lambda_e \subset \Lambda_\beta \equiv \Lambda^*, \tag{4.35}$$

where Λ^* is the dual lattice of Λ. Let $\{\omega_a^{\pm}/\sqrt{2}, \quad a = 1,2\}$ be the basis vectors of the dual lattice of Λ_e, that is,

$$\omega_a^+ \cdot e_b^+ = \omega_a^- \cdot e_b^- = \delta_{ab}, \qquad a, b = 1, 2. \tag{4.36}$$

It can easily be seen that (4.36) is satisfied if

$$\omega_a^{\pm} = -\alpha_{\mp} v_a, \qquad a = 1, 2; \tag{4.37}$$

where $\alpha_{\pm}$ are given in (4.10), and

$$\begin{aligned} v_1 &= 2(1, \frac{1}{\sqrt{3}}), \\ v_2 &= 2(1, -\frac{1}{\sqrt{3}}). \end{aligned} \tag{4.38}$$

The relations (4.36) allow us to solve for the singular vector β from the equation (4.25). We obtain the following solutions, depending on whether we consider the positive sector screening vectors $\{e_a^+, \quad a = 1,2\}$ or the negative sector ones $\{e_a^-, \quad a = 1,2\}$,

$$\beta = \sum_{a=1}^{2} [\frac{1+n_a^+}{2}(e_a^+)^2 - \frac{1+m_a^+}{2}]\omega_a^+, \tag{4.39}$$

$$\beta = \sum_{a=1}^{2} [\frac{1+n_a^-}{2}(e_a^-)^2 - \frac{1+m_a^-}{2}]\omega_a^-. \tag{4.40}$$

Notice that (4.39) and (4.40) are not mutually exclusive; this is because (4.26) and (4.37) lead to

$$\begin{aligned} (e_a^+)^2 &= (e_a^-)^{-2}, \\ \omega_a^{\pm} &= -(e_a^{\mp})^2 \omega_a^{\mp}, \qquad a = 1, 2. \end{aligned} \tag{4.41}$$

These relations, in turn, allow us to rewrite (4.40) as

$$\beta = \sum_{a=1}^{2} [\frac{1+m_a^-}{2}(e_a^+)^2 - \frac{1+n_a^-}{2}]\omega_a^+. \tag{4.42}$$

Thus, (4.39) and (4.40) are consistent with each other provided that

$$\begin{aligned} n_a^- &= m_a^+ \mod p, \\ m_a^- &= n_a^+ \mod (p+1), \qquad a = 1,2. \end{aligned} \tag{4.43}$$

Therefore, in order to specify the set of the singular vectors β, we need only to specify the "Coulomb charges" of the positive sector, namely $\beta(n_a^+, m_a^+)$, $a = 1,2$. Henceforth, we w." consider only the positive sector and omit the superscript $\{+\}$ from the subsequent formulae, but we should keep in mind that the positive sector is equivalent to the negative one through the redefinitions (this remark will considerably simplify the subsequent analysis)

$$\begin{aligned} n_a &\rightarrow m_a \mod p, \\ m_a &\rightarrow n_a \mod (p+1), \qquad a = 1,2. \end{aligned} \tag{4.44}$$

This amounts to the interchange of the first column with the second one in the matrix $\begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix}$, which defines β. A singular vector β may then, without any ambiguity, be written as

$$\beta\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \equiv \beta\begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} = \sum_{a=1}^{2} \left[\frac{1+n_a}{2}(c_a)^2 - \frac{1+m_a}{2} \right] \omega_a. \tag{4.45}$$

This is the notation we will adopt throughout the remaining part of this chapter. Table 7 is a recapitulation of the various lattices that have been introduced in this section. $\Lambda_0 = \{l_1, l_2\}$ is a two-dimensional lattice constructed as a direct product of the one-dimensional lattices $\mathbf{Z}(l_1) = \{\sqrt{l_1}\mathbf{Z}\}$ and $\mathbf{Z}(l_2) = \{\sqrt{l_2}\mathbf{Z}\}$. In Table 7, all the basis vectors (generators) are scaled by $(l_1, l_2) = [2p(p+1), 6p(p+1)]$.

4.3 The Weyl-like group of the W_3 algebra

By analogy with a Kac-Moody algebra [46], where the finite Weyl group and the infinite translation group associated with the root lattice are used [38] to map a highest weight state to its null state descendants, we introduce (as in reference [44]) in the case of the W_3 algebra a finite Weyl-like group called G and an infinite translation group T, associated with the lattice Λ, to fulfill the same role of mapping among null states. In fact, the relation between G and the usual Weyl group is explicitly worked out at the end of section 4.5 of this chapter. From the expression (4.23) of the null states belonging to the Verma module V_β, we read off the following action of G. Let g_a be a generator of G and $\beta(n_a, m_a)$ be a vector in $\Lambda_\beta/\sqrt{2}$, that is it satisfies (4.25), then

$$g_a\beta = \beta - n_a e_a, \qquad a = 1,2, \tag{4.46}$$

where again there is no sum w.r.t. a.

As indicated previously, only the positive sector is considered, and hence the subscript $\{+\}$ is understood on any quantity with the subscript a. Obviously, $g_a\beta$ itself satisfies (4.25). Let us then for convenience rewrite (4.46), which defines the two generators of G, in the "matrix form"

$$\begin{aligned}
g_1\beta\begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} &= \beta\begin{pmatrix} -n_1 & m_1 \\ n_2 - n_1(1+u_{12}) & m_2 - n_1(1+v_{12}) \end{pmatrix} = \beta\begin{pmatrix} -n_1 & m_1 \\ n_1 + n_2 & m_2 \end{pmatrix}, \\
g_2\beta\begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} &= \beta\begin{pmatrix} n_1 - n_2(1+u_{21}) & m_1 - n_2(1+v_{21}) \\ -n_2 & m_2 \end{pmatrix} = \beta\begin{pmatrix} n_1 + n_2 & m_1 \\ -n_2 & m_2 \end{pmatrix}.
\end{aligned} \tag{4.47}$$

where the matrix elements u_{ab} and v_{ab} are given by

$$e_a \cdot e_b = \frac{1+u_{ab}}{2}e_b^2 - \frac{1+v_{ab}}{2}. \tag{4.48}$$

The four remaining group elements of G (including the identity) are obtained through products of g_1 and g_2. Again by analogy with the Kac-Moody algebra, one expects G to be well defined only on Λ_β/Λ, meaning that it is not well defined on Λ_β, which is in fact the lattice of singular vectors. This is because the integers n_a, used in defining G through (4.46), are themselves not uniquely defined. Indeed, according to the translational symmetry described in (4.11), the relation (4.46) translates into

$$g_a\beta = \beta - n_a e_a - k_1 p e_a = \beta - n_a e_a + \beta', \qquad k_1 \in \mathbf{Z}, \tag{4.49}$$

where $\sqrt{2}\beta' \in \Lambda$. As the relation (4.49) suggests, it is the direct product of G with the translation group T, associated with Λ, that is well defined on Λ_β. This, in turn, defines the infinite-dimensional Weyl-like group of the W_3 algebra. Let us denote it by $\hat{G}$

$$\hat{G} = G \times T. \tag{4.50}$$

The generators of $\hat{G}$ are obtained as direct products of the generators of G given in (4.47) and those of T, which are the two fundamental translations with respect to the basis vectors of Λ. As defined in (4.49) and (4.50), $\hat{G}$ can be used to specify the set of null states contained in the Verma module V_β. However, it does not explicitly display the embedding structure of the Verma submodules generated from these null states, that is, which Verma submodule includes which. Nevertheless, one can unravel this embedding structure by appealing to the formula (4.24). To see that, let us note that the transformations (4.47) amount respectively to

$$\begin{aligned} N_1 &= h(g_1\beta) - h(\beta) = h\begin{pmatrix} -n_1 & m_1 \\ n_1 + n_2 & m_2 \end{pmatrix} - h\begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} = n_1 m_1, \\ N_2 &= h(g_2\beta) - h(\beta) = h\begin{pmatrix} n_1 + n_2 & m_1 \\ -n_2 & m_2 \end{pmatrix} - h\begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} = n_2 m_2. \end{aligned} \tag{4.51}$$

Hence, a necessary condition for the Verma module V_β to include the Verma submodule specified by $g_1\beta$ or $g_2\beta$ is respectively

$$n_1 m_1 > 0, \quad \text{or} \quad n_2 m_2 > 0. \tag{4.52}$$

Furthermore, since we are interested in the embedding structure, we should define the operators g_1 and g_2 in such a way that they map any singular state specified by β to its "closest" independent null states (generating maximal submodules in V_β), in the sense that the degree difference they induce must be the smallest possible positive integer for each possible form of β. Recall that there is a $\mathbf{Z_2} \times \mathbf{Z_3}$ symmetry, and therefore β may be written in six different forms before it is mapped by g_1 or g_2. For instance, if $\beta\begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix}$ specifies a primary state, that is, n_1, n_2, m_1 and m_2 satisfy (4.7), the "closest" null states to this primary state are given by

$$
\begin{aligned}
&g_1\beta\begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} = \beta\begin{pmatrix} -n_1 & m_1 \\ n_1+n_2 & m_2 \end{pmatrix}, \quad N_1 = n_1 m_1, \\
&g_2\beta\begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} = \beta\begin{pmatrix} n_1+n_2 & m_1 \\ -n_2 & m_2 \end{pmatrix}, \quad N_2 = n_2 m_2, \\
&g_1\beta\begin{pmatrix} p-n_1 & p+1-m_1 \\ n_1+n_2 & m_1+m_2 \end{pmatrix} = \beta\begin{pmatrix} 2p-n_1 & m_1 \\ n_1+n_2-p & m_2 \end{pmatrix}, \quad N_1 = (p-n_1)(p+1-m_1), \\
&g_2\beta\begin{pmatrix} n_1+n_2 & m_1+m_2 \\ p-n_2 & p+1-m_2 \end{pmatrix} = \beta\begin{pmatrix} n_1+n_2-p & m_1 \\ 2p-n_2 & m_2 \end{pmatrix}, \quad N_2 = (p-n_2)(p+1-m_2), \\
&g_1\beta\begin{pmatrix} n_1+n_2 & m_1+m_2 \\ -n_2 & -m_2 \end{pmatrix} \equiv g_2\beta\begin{pmatrix} -n_1 & -m_1 \\ n_1+n_2 & m_1+m_2 \end{pmatrix} = \beta\begin{pmatrix} -n_2 & m_1 \\ -n_1 & m_2 \end{pmatrix}, \\
&N_1 = (n_1+n_2)(m_1+m_2), \\
&g_1\beta\begin{pmatrix} p-n_1-n_2 & p+1-m_1-m_2 \\ n_1 & m_1 \end{pmatrix} \equiv g_2\beta\begin{pmatrix} n_2 & m_2 \\ p-n_1-n_2 & p+1-m_1-m_2 \end{pmatrix} \\
&= \beta\begin{pmatrix} p-n_2 & m_1 \\ p-n_1 & m_2 \end{pmatrix}, \quad N_2 = (p-n_1-n_2)(p+1-m_1-m_2).
\end{aligned}
\tag{4.53}
$$

Let us now generalize this example to determine the "closest" states included in the Verma module of any singular state, which is specified by $\beta\begin{pmatrix} n_1' & m_1' \\ n_2' & m_2' \end{pmatrix}$. In addition, let us consider the case where this singular state is in turn included in the primary Verma module specified by $\beta\begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix}$. In the sequel, we will reserve the integers n_1, n_2, m_1 and m_2 to label just this primary state, i.e., they satisfy (4.7). As previously discussed, we want to define the generators g_1 and g_2 so that they map the singular vector (written in its six possible forms)

$\beta\begin{pmatrix} n_1' & m_1' \\ n_2' & m_2' \end{pmatrix}$ with the minimum positive degree difference. For that purpose, we resort to the set of screening charges $\{Q_i, \; i = 1,..,6\}$, in such a way that they specify which one among the six forms of β is being considered before the generators g_1 or g_2 is applied on it. Notice that g_1 and g_2, as defined in (4.47), leave the second column invariant. This means that we can uniquely specify the singular vectors β by means of the $\mathbf{Z_2} \times \mathbf{Z_3}$ and the translation symmetries. This is accomplished in such a way that the second entry of the first (second) row is between zero and $p+1$ (zero and p+1), and the first entry of the first (second) row is a positive integer, before g_1 (g_2) is applied on β. Since there are six possible ways for the second entry to be between zero and $p+1$, then we need six operators $Q_i (i = 1,...,6)$ to uniquely define g_1 and g_2 on Λ_β. These Q_i are

$$
\begin{aligned}
Q_1\beta\begin{pmatrix} n_1' & m_1' \\ n_2' & m_2' \end{pmatrix} &\equiv g_1\beta\begin{pmatrix} n_1'' & m_1 \\ n_2'' & m_2'' \end{pmatrix}, \quad n_1'' > 0, \\
Q_2\beta\begin{pmatrix} n_1' & m_1' \\ n_2' & m_2' \end{pmatrix} &\equiv g_2\beta\begin{pmatrix} n_1'' & m_1'' \\ n_2'' & m_2 \end{pmatrix}, \quad n_2'' > 0, \\
Q_3\beta\begin{pmatrix} n_1' & m_1' \\ n_2' & m_2' \end{pmatrix} &\equiv g_1\beta\begin{pmatrix} n_1'' & p+1-m_1 \\ n_2'' & m_2'' \end{pmatrix}, \quad n_1'' > 0, \\
Q_4\beta\begin{pmatrix} n_1' & m_1' \\ n_2' & m_2' \end{pmatrix} &\equiv g_2\beta\begin{pmatrix} n_1'' & m_1'' \\ n_2'' & p+1-m_2 \end{pmatrix}, \quad n_2'' > 0, \\
Q_5\beta\begin{pmatrix} n_1' & m_1' \\ n_2' & m_2' \end{pmatrix} &\equiv g_1\beta\begin{pmatrix} n_1'' & m_1+m_2 \\ n_2'' & m_2'' \end{pmatrix} \equiv g_2\beta\begin{pmatrix} n_1''' & m_1''' \\ n_1'' & m_1+m_2 \end{pmatrix}, \quad n_1'' > 0, \\
Q_6\beta\begin{pmatrix} n_1' & m_1' \\ n_2' & m_2' \end{pmatrix} &\equiv g_1\beta\begin{pmatrix} n_1'' & p+1-m_1-m_2 \\ n_2'' & m_2'' \end{pmatrix} \\
&\equiv g_2\beta\begin{pmatrix} n_1''' & m_1''' \\ n_1'' & p+1-m_1-m_2 \end{pmatrix}, \quad n_1'' > 0.
\end{aligned}
\tag{4.54}
$$

In the next section we will use these screening charges to classify the Verma submodules of the W_3 algebra and to describe their embedding structure.

4.4 The embedding structure of the W_3 Verma submodules

Let again V_β be a primary module specified by the primary vector $\beta\begin{pmatrix} n_1 & m_2 \\ n_2 & m_2 \end{pmatrix}$, which means that the integers n_1, n_2, m_1 and m_2 satisfy the inequalities (4.7) According to (4.53) and (4.54), if we apply a few times the screening charges Q_i on β, then clearly the submodules of V_β fall into six different classes of modules, whose highest weight states are specified by the following singular vectors:

$$
\begin{aligned}
A &= \left\{\beta\begin{pmatrix} k_1p+n_1 & m_1 \\ k_2p+n_2 & m_2 \end{pmatrix} \equiv A(k_1,k_2), \quad k_1,k_2\in\mathbf{Z}, \quad k_1-k_2\in 3\mathbf{Z}\right\}, \\
B &= \left\{\beta\begin{pmatrix} k_1p+n_2 & m_1 \\ k_2p-n_1-n_2 & m_2 \end{pmatrix} \equiv B(k_1,k_2), \quad k_1,k_2\in\mathbf{Z}, \quad k_1-k_2\in 3\mathbf{Z}\right\}, \\
C &= \left\{\beta\begin{pmatrix} k_1p-n_1-n_2 & m_1 \\ k_2p+n_1 & m_2 \end{pmatrix} \equiv C(k_1,k_2), \quad k_1,k_2\in\mathbf{Z}, \quad k_1-k_2\in 3\mathbf{Z}\right\}, \\
D &= \left\{\beta\begin{pmatrix} k_1p-n_1 & m_1 \\ k_2p+n_1+n_2 & m_2 \end{pmatrix} \equiv D(k_1,k_2), \quad k_1,k_2\in\mathbf{Z}, \quad k_1-k_2\in 3\mathbf{Z}\right\}, \\
E &= \left\{\beta\begin{pmatrix} k_1p+n_1+n_2 & m_1 \\ k_2p-n_2 & m_2 \end{pmatrix} \equiv E(k_1,k_2), \quad k_1,k_2\in\mathbf{Z}, \quad k_1-k_2\in 3\mathbf{Z}\right\}, \\
F &= \left\{\beta\begin{pmatrix} k_1p-n_2 & m_1 \\ k_2p-n_1 & m_2 \end{pmatrix} \equiv F(k_1,k_2), \quad k_1,k_2\in\mathbf{Z}, \quad k_1-k_2\in 3\mathbf{Z}\right\}.
\end{aligned} \tag{4.55}
$$

With this notation, the primary state reads

$$
\beta\begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} \equiv A(0,0). \tag{4.56}
$$

It is noteworthy that this classification of null states of the W_3 algebra may be thought of as a generalization of the Virasoro algebra case [25]. In order to find the embedding structure of the Verma submodules generated from these null states, we need to work out through (4.54) the mappings of the screening charges Q_i between the various classes of the submodules (4.55) in V_β. Table 8 summarizes these mappings (the conditions set on k_1 and/or k_2 fix the domain of definition of Q_i). Let us recall that the superscript $\{+\}$ has been understood in the previous treatment. It is now explicitly restored in Table 8. This is to emphasize that, in order to determine the complete embedding structure of these Verma submodules, we need to consider the negative sector as well. As explained below the equation (4.43), the negative sector screening charges Q_i^- are equivalent to the positive sector ones Q_i^+ by simply interchanging the "roles" of the $n's$ and the $m's$ in (4.55). Thus, using the same arguments and notations as those leading to Q_i^+, we define Q_i^- in the following way:

$$
\begin{aligned}
&Q_1^-\beta\begin{pmatrix} n_1' & m_1' \\ n_2' & m_2' \end{pmatrix} \equiv g_1^-\beta\begin{pmatrix} n_1 & m_1'' \\ n_2'' & m_2'' \end{pmatrix}, \quad m_1''>0,\\
&Q_2^-\beta\begin{pmatrix} n_1' & m_1' \\ n_2' & m_2' \end{pmatrix} \equiv g_2^-\beta\begin{pmatrix} n_1'' & m_1'' \\ n_2 & m_2'' \end{pmatrix}, \quad m_2''>0,\\
&Q_3^-\beta\begin{pmatrix} n_1' & m_1' \\ n_2' & m_2' \end{pmatrix} \equiv g_1^-\beta\begin{pmatrix} p-n_1 & m_1'' \\ n_2'' & m_2'' \end{pmatrix}, \quad m_1''>0,\\
&Q_4^-\beta\begin{pmatrix} n_1' & m_1' \\ n_2' & m_2' \end{pmatrix} \equiv g_2^-\beta\begin{pmatrix} n_1'' & m_1'' \\ p-n_2 & m_2'' \end{pmatrix}, \quad m_2''>0,\\
&Q_5^-\beta\begin{pmatrix} n_1' & m_1' \\ n_2' & m_2' \end{pmatrix} \equiv g_1^-\beta\begin{pmatrix} n_1+n_2 & m_1'' \\ n_2'' & m_2'' \end{pmatrix} \equiv g_2^-\beta\begin{pmatrix} n_1''' & m_1''' \\ n_1+n_2 & m_1'' \end{pmatrix}, \quad m_1''>0,\\
&Q_6^-\beta\begin{pmatrix} n_1' & m_1' \\ n_2' & m_2' \end{pmatrix} \equiv g_1^-\beta\begin{pmatrix} p-n_1-n_2 & m_1'' \\ n_2'' & m_2'' \end{pmatrix} \equiv g_2^-\beta\begin{pmatrix} n_1''' & m_1''' \\ p-n_1-n_2 & m_1'' \end{pmatrix}, m_1''>0.
\end{aligned}
\tag{4.57}
$$

Again, we need to work out the Q_i^- mappings between the six classes of submodules (4.55) in order to unravel their intertwining structure. These mappings are summarized in Table 9. It can readily be seen from Tables 8 and 9 that the screening charges Q_i^+ satisfy the independent relations

$$
\begin{aligned}
Q_1^+Q_2^+ &= Q_5^+Q_1^+,\\
Q_2^+Q_1^+ &= Q_5^+Q_2^+,\\
Q_1^+Q_2^+Q_1^+ &= Q_2^+Q_1^+Q_2^+,\\
Q_1^+Q_6^+ &= Q_4^+Q_1^+,
\end{aligned}
\tag{4.58}
$$

$$Q_6^+ Q_1^+ = Q_1^+ Q_6^+,$$

$$Q_6^+ Q_1^+ Q_6^+ = Q_1^+ Q_6^+ Q_1^+, \tag{4.59}$$

$$Q_6^+ Q_2^+ = Q_3^+ Q_6^+,$$

$$Q_2^+ Q_6^+ = Q_3^+ Q_2^+,$$

$$Q_6^+ Q_2^+ Q_6^+ = Q_2^+ Q_6^+ Q_2^+, \tag{4.60}$$

together with a similar set of relations with the superscript $\{+\}$ being replaced by $\{-\}$, and the equations

$$Q_i^+ Q_j^+ = Q_j^- Q_i^-, \quad i \neq j, \quad i, j = 1, \ldots, 6; \tag{4.61}$$

$$Q_i^+ Q_j^- = Q_j^- Q_i^+, \quad i, j = 1, \ldots, 6. \tag{4.62}$$

The relations (4.58), (4.59) and (4.60) are schematically represented through the basic hexagons (a), (b) and (c) of Figure 5, whereas the relations (4.61) and (4.62) are schematically represented through the basic squares (d) and (e) of Figure 6. Two plane projections of the three-dimensional intertwining diagram are respectively drawn in Figures 7 and 8. Let us point out that Figure 8 displays the embedding of the Virasoro representations in those of the W_3 algebra. Indeed, if we single out any straight line of Figure 7 and folds it at its unique point (submodule), from which two arrows emerge in opposite directions, then we obtain exactly the same embedding structure as in the Virasoro case, which is illustrated in Figure 4. The examples displayed in Figure 8 are about the straight lines of Figure 7 that cross the primary state $A(0,0)$.

4.5 Irreducible characters of the W_3 algebra

In this section we will make use of the embedding structure, which is displayed through Figures 7 and 8, to compute the irreducible character of the

PCDVM $V_{A(0,0)}$, which is specified by the primary vector $A(0,0)$, in terms of the characters of its Verma submodules $V_\beta \subset V_{A(0,0)}$, where β is any singular vector. Let us denote the irreducible module contained in $V_{A(0,0)}$ by $I_{A(0,0)}$. Our approach to derive the irreducible character is the extension to the W_3 algebra of the method of Rocha-Caridi [31] and Feigin-Fuchs [47], which has been applied to the Virasoro algebra. The "Virasoro contribution" to the character of the Verma module V_β, specified by the singular vector β, is

$$\chi_{V_\beta}(q) = q^{-\frac{c}{24}} Tr\, q^{L_0}. \tag{4.63}$$

The states in V_β are given in (4.4). They lead to this more explicit formula for χ_{V_β}:

$$\chi_{V_\beta} = \frac{1}{\eta^2(q)} q^{\frac{2-c}{24}+h(\beta)}, \tag{4.64}$$

where $\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is the usual Dedekind eta function, and c and $h(\beta)$ are respectively given in (4.1) and (4.6); but the relations (4.7) are not meant unless $\beta = A(0,0)$. The character shown in (4.64) can be simplified even more to read

$$\chi_{V_\beta} = \frac{1}{\eta^2(q)} q^{\frac{1}{p(p+1)}+h(\beta)} = \frac{1}{\eta^2(q)} q^{(\beta-\alpha_0)^2}. \tag{4.65}$$

Let us now compute more specifically the irreducible character of $I_{A(0,0)}$. If there were no null states in $V_{A(0,0)}$, we would have according to (4.65)

$$\chi_{I_{A(0,0)}} = \chi_{V_{A(0,0)}}. \tag{4.66}$$

Figures 7 and 8 indicate, though, that $V_{A(0,0)}$ contains the maximal submodules $V_{D(0,0)}$, $V_{E(0,0)}$ and $V_{F(1,1)}$, which are not contained in any other submodule of $V_{A(0,0)}$. Consequently, the character given in (4.66) would be corrected as

$$\chi_{I_{A(0,0)}} = \chi_{V_{A(0,0)}} - (\chi_{V_{D(0,0)}} + \chi_{V_{E(0,0)}} + \chi_{V_{F(1,1)}}). \tag{4.67}$$

However, the equation (4.67) is not quite true because Figure 7 and diagrams analogous to those in Figure 8 show that

$$
\begin{aligned}
V_{D(0,0)} \bigcap V_{E(0,0)} &= \{V_{B(0,0)} + V_{C(0,0)}\}, \\
V_{D(0,0)} \bigcap V_{F(1,1)} &= \{V_{C(1,1)} + V_{B(-1,2)}\}, \\
V_{E(0,0)} \bigcap V_{F(1,1)} &= \{V_{B(1,1)} + V_{C(2,-1)}\}.
\end{aligned} \tag{4.68}
$$

This means that the characters of the submodules in traces were over subtracted in computing the character through (4.67), and hence they should be added back, that is,

$$
\begin{aligned}
\chi_{I_{A(0,0)}} = \chi_{V_{A(0,0)}} &- (\chi_{V_{D(0,0)}} + \chi_{V_{E(0,0)}} + \chi_{V_{F(1,1)}}) \\
&+ (\chi_{V_{B(0,0)}} + \chi_{V_{C(0,0)}} + \chi_{V_{C(1,1)}} + \chi_{V_{B(-1,2)}} + \chi_{V_{B(1,1)}} + \chi_{V_{C(2,-1)}}).
\end{aligned} \tag{4.69}
$$

The same argument as before still holds and we must then further correct the character given in (4.69). In fact, it can be shown by induction that the correct final answer for $\chi_{I_{A(0,0)}}$ is given by

$$
\chi_{I_{A(0,0)}} = \sum_{X=A}^{F} \epsilon(X) \chi_X, \tag{4.70}
$$

where $A, .., F$ are as defined in (4.55), and the sign function $\epsilon(X)$ is defined as

$$
\begin{aligned}
\epsilon(X) &= +1, \qquad X = A, B, C; \\
\epsilon(X) &= -1, \qquad X = D, E, F;
\end{aligned} \tag{4.71}
$$

and

$$
\chi_X = \sum_{\substack{k_1, k_2 \in \mathbf{Z} \\ k_1 - k_2 \in 3\mathbf{Z}}} \chi_{V_{X(k_1,k_2)}} = \frac{1}{\eta^2} \sum_{\substack{k_1, k_2 \in \mathbf{Z} \\ k_1 - k_2 \in 3\mathbf{Z}}} q^{\frac{1}{p(p+1)} + h(X(k_1,k_2))}. \tag{4.72}
$$

Due to the relations (4.6), (4.55), (4.70) and (4.72), it is possible to explicitly write $\chi_{I_{A(0,0)}}$ in the following form, which agrees with the conjectured result [42]:

$$
\chi_{I_{A(0,0)}} = \frac{1}{\eta^2} \sum_{X=A}^{F} \sum_{k,j \in \mathbf{Z}} \epsilon(X) q^{\xi(X)/12p(p+1)},
$$

$$\xi(X) = 3[(n_1(X) + n_2(X))(p+1) - (m_1 + m_2)p + p(p+1)(2k+j)]^2$$
$$+ [(n_1(X) - n_2(X))(p+1) - m_1 - m_2)p + 3p(p+1)j]^2, \tag{4.73}$$

where m_1 and m_2 satisfy (4.7), and $n_1(X)$ and $n_2(X)$ are defined through (4.55) as

$$\beta \begin{pmatrix} n_1(X) & m_1 \\ n_2(X) & m_2 \end{pmatrix} \equiv X(0,0). \qquad X = A, ..., F. \tag{4.74}$$

For instance

$$\begin{aligned} n_1(C) &= -n_1 - n_2, \\ n_2(C) &= n_1. \end{aligned} \tag{4.75}$$

The character given in (4.73) can also be written as a lattice sum over Λ, which is defined in (4.35). This can be readily achieved if we use the third member of the relation (4.65) and the following formula for β (being replaced by $X(k_1, k_2)$), which is obtained from (4.45) and (4.55):

$$\begin{aligned} X(k_1, j) =& (1 + \frac{2n_1(X) + n_2(X)}{3})e_1^+ + (1 + \frac{2n_2(X) + n_1(X)}{3})e_2^+ \\ & - \frac{p}{p+1}\left[(1 + \frac{2m_1 + m_2}{3})e_1^+ + (1 + \frac{2m_2 + m_1}{3})e_2^+\right] \\ & + (k_1 - j)pe_1^+ + (k_1 - 2j)pe_2^+, \qquad X = A, ..., F; \end{aligned} \tag{4.76}$$

where $n_1(X)$ and $n_2(X)$ are defined through (4.74), and $j = (k_1 - k_2)/3$. Let us note that $X(k_1, j)$ may in general be written as

$$X(k_1, j) = X(0,0) + \delta, \qquad X = A, ..., F; \tag{4.77}$$

δ being a vector in $\Lambda/\sqrt{2}$. Substituting (4.77) in (4.65) and using (4.70) together with (4.72), we derive the character in terms of Λ

$$\chi_{I_{A(0,0)}} = \frac{1}{\eta^2} \sum_{X=A}^{F} \sum_{\delta \in \Lambda/\sqrt{2}} \epsilon(X) q^{(X(0,0) - \alpha_0 + \delta)^2}, \tag{4.78}$$

where α_0 is the background charge, which in this notation reads

$$\alpha_0 = -\frac{1}{p+1}(e_1^+ + e_2^+). \tag{4.79}$$

Furthermore, we can readily see from the Weyl-like group G as defined in (4.47) that the vectors $X(0,0)$, $X = A, \ldots, F$, are all related to the primary vector $A(0,0)$ as

$$X(0,0) = g_i A(0,0), \qquad g_i \in G. \tag{4.80}$$

Consequently, substituting (4.80) in (4.78) yields the following formula for the character:

$$\chi_{\Gamma_{A(0,0)}} = \frac{1}{\eta^2} \sum_{g \in G} \sum_{\Lambda/\sqrt{2}} \epsilon(g) q^{(gA(0,0)-\alpha_0+\delta)^2}. \tag{4.81}$$

Let us further express the character given in (4.81) as a lattice sum over the root lattice Λ_R of the finite $su(3)$ Lie algebra and the associated finite Weyl group W. For that purpose, let us first unravel the relation between G and W. We choose as a basis for Λ_R the vectors

$$\begin{aligned} \alpha_1 &= (\sqrt{2}, 0), \\ \alpha_2 &= \left(-\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\right). \end{aligned} \tag{4.82}$$

Moreover, $A(0,0)$, as defined through (4.76) and (4.77), can always be rewritten in the form

$$A(0,0) = A_n(0,0) + A_m(0,0), \tag{4.83}$$

where

$$\begin{aligned} A_n(0,0) &= \frac{2n_1+n_2}{3\sqrt{2}}\sqrt{\frac{p+1}{p}}\alpha_1 + \frac{n_1-n_2}{3\sqrt{2}}\sqrt{\frac{p+1}{p}}\alpha_2 \\ &= \frac{p}{2(p+1)}(n_1\omega_1^+ + n_2\omega_2^+), \\ A_m(0,0) &= -\frac{1}{\sqrt{2}}\left(1+\frac{2m_1+m_2}{3}\right)\sqrt{\frac{p}{p+1}}\alpha_1 - \frac{m_1-m_2}{3\sqrt{2}}\sqrt{\frac{p}{p+1}}\alpha_2 + \sqrt{\frac{p+1}{2p}}\alpha_1 \\ &= -\frac{m_1+1}{2}\omega_1^+ - \frac{m_2+1}{2}\omega_2^+ + \frac{2p}{p+1}(\omega_1^+ + \omega_2^+). \end{aligned} \tag{4.84}$$

Using this notation, we can easily see that

$$g_i A(0,0) = A_m(0,0) + w_i A_n(0,0), \qquad i = 1, ..., 6; \tag{4.85}$$

where $g_i \in G$ and $w_i \in W$. Furthermore, after making a change of basis from Λ to Λ_R and substituting (4.85) in (4.81), we realize that the character formula then becomes

$$\chi_{I_{A(0,0)}} = \frac{1}{\eta^2} \sum_{w \in W} \epsilon(w) \Theta^k_{\sqrt{2k}(A_m(0,0)+wA_n(0,0))-\alpha_1}(0, q, 0), \tag{4.86}$$

where the "level" of the W_3 algebra is given by

$$k = p(p+1), \tag{4.87}$$

and Θ is the usual theta function, which is defined here on the root lattice of $su(3)$ as

$$\Theta^k_\lambda(u, q, z) = e^{-2\pi i k u} \sum_{p \in \Lambda_R + \frac{\lambda}{k}} q^{p^2/2} e^{-2\pi i p z}. \tag{4.88}$$

4.6 Conclusions

In this chapter, we have fully described the embedding structure of the Verma submodules of the W_3 algebra through the free field realization. This embedding structure is schematically represented through Figures 5, 6, 7 and 8, which in turn allow the computation of the irreducible character of the PCDVM of the W_3 algebra. The form of the character thus obtained proves to coincide with the one conjectured in [42]. The analogy between the Kac-Moody and the W_3 algebras has been emphasized. This method can be in principle applied to any other RCFT. In particular, it will be interesting to carry the present analysis to the RCFT with a Kac-Moody algebra, where some screening currents are expressed as infinite sums of terms.

Chapter 5

Free field realization of the $su(n)_k$ parafermion theory

This chapter is devoted to the study of the $su(n)_k$ parafermion theory, which is another rational CFT. More specifically, we proceed from the Wakimoto realization of $su(n)_k$ presented in chapter 3 to derive two free field realizations for the associated $su(n)_k$ parafermion theory. In particular, we express the parafermion currents, associated with the negative root and the simple root $su(n)_k$ Kac-Moody currents, in terms of free fields. For the sake of illustration, the full free field realization of $su(3)_k$ parafermion currents and screening currents is explicitly displayed. Each of our two free field realizations has its advantages and drawbacks. The first one involves a set of orthonormal free fields so that the field realization of the parafermion currents is easily obtained from the Wakimoto realization of $su(n)_k$. However, the field realization of the parafermion primary fields turns out to be messy. On the contrary, the second one leads to simple field realization of the parafermion primary fields but involves some linearly dependent constrained fields. The relations between our free field realizations and those recently proposed in the literature are outlined. Finally, as the $su(n)_k$ parafermion theory is closely related to the coset model $su(n)_k \times su(n)_\ell / su(n)_{k+\ell}$, we then use the free field realization of the former to sketch that of the latter. This chapter is organized as follows. We begin with a background review of the simplest parafermion theory, namely, the $su(2)_k$ parafermion, which is commonly known as the Z_k parafermion CFT. Then we present our first free field realization of $su(n)_k$ parafermion theory in terms of orthonormal free fields. As an example, the $su(3)_k$ parafermion currents and screening currents are explicitly written in terms of these free fields. After that, we display our second free field realization, which is based on some constrained non-orthonormal fields. Finally, we sketch the free field realization of the coset model $su(n)_k \times su(n)_\ell / su(n)_{k+\ell}$ together with its screening currents.

5.1 Introduction to the Z_k parafermion theory

This is yet another extended RCFT based on an infinite-dimensional algebra [16,41]. It is called the Z_k parafermion because it describes self-dual critical points in Z_k symmetric statistical systems in terms of a set of fields with fractional spins, i.e., parafermion fields generalizing the usual fermion field of the Ising model, which is nothing but the Z_2 model. Unless otherwise stated, we consider in the sequel only the holomorphic sector bearing in mind that the same treatment is valid for the antiholomorphic one. The Z_k parafermion theory involves the parafermion currents $\psi_\ell(z)$, $\ell = 0, 1, \ldots, k-1$; where ψ_0 is identified with the identity operator and k is a positive integer. These parafermion currents are the analog of the Kac-Moody currents in that they, too, give rise to a current algebra (i.e., infinite-dimensional algebra). Moreover, they satisfy the hermiticity condition $\psi_\ell^\dagger = \psi_{k-\ell}$. The conformal dimension (which is also the spin) d_ℓ of ψ_ℓ is given by

$$d_\ell = \frac{\ell(k-\ell)}{k}, \tag{5.1}$$

for $\ell = 0, 1, \ldots, k-1$. The parafermion (current) algebra is defined through the following OPE's (only the most singular terms are shown):

$$\begin{aligned}
\psi_\ell(z)\psi_m(w) &= c_{\ell,m}(z-w)^{\frac{-2\ell m}{k}}\left[\psi_{\ell+m}(w)+\ldots\right], \quad \ell+m<k, \\
\psi_\ell(z)\psi_m^\dagger(w) &= c_{\ell,k-m}(z-w)^{\frac{-2\ell(k-m)}{k}}\left[\psi_{\ell-m}(w)+\ldots\right], \quad m<\ell, \\
\psi_\ell(z)\psi_\ell^\dagger(w) &= (z-w)^{\frac{-2\ell(k-\ell)}{k}}\left[1+\frac{2d_\ell}{c}(z-w)^2 T(w)+\ldots\right],
\end{aligned} \tag{5.2}$$

where the structure constants $c_{\ell,m}$ are so that

$$c_{k,m}^2 = \frac{\Gamma(\ell+m+1)\Gamma(k-\ell+1)\Gamma(k-m+1)}{\Gamma(\ell+1)\Gamma(m+1)\Gamma(k-\ell-m+1)\Gamma(k+1)}, \tag{5.3}$$

with $\Gamma(n)$ being the usual gamma function. In (5.2), $T(z)$ stands for the energy-momentum tensor of the parafermion algebra and satisfies the usual OPE

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}, \tag{5.4}$$

with the central charge

$$c = \frac{2(k-1)}{k+2}. \tag{5.5}$$

The parafermion currents ψ_ℓ are primary w.r.t. $T(z)$, that is,

$$T(z)\psi(w) \sim \frac{d_\ell \psi_\ell(w)}{(z-w)^2} + \frac{\partial\psi_\ell(w)}{z-w}. \tag{5.6}$$

Note from the equations (5.2) that all the parafermion currents can be generated from $\psi_1(z) \equiv \psi(z)$ and $\psi_1^\dagger(z) \equiv \psi^\dagger(z)$. Therefore the representation theory of the Z_k parafermion model can be described in terms of these two parafermion currents only. For that purpose, let us define, in analogy with the Virasoro case as shown in (2.30), the primary states $|\ell>$ of the parafermion algebra to be

$$A_{\ell/k+n}|\ell>= A^\dagger_{-\ell/k+n+1}|\ell>=0, \tag{5.7}$$

where $n \geq 0$, $\ell = 0,1,\ldots,k-1$, and the parafermion operators A and $A^\dagger$ are defined by

$$\begin{aligned} \psi(z)|\ell> &= \sum_{n=-\infty}^{\infty} z^{-\ell/k+n-1} A_{\ell/k-n}|\ell>, \\ \psi^\dagger(z)|\ell> &= \sum_{n=-\infty}^{\infty} z^{-\ell/k+n-1} A^\dagger_{-\ell/k-n}|\ell>. \end{aligned} \tag{5.8}$$

The equations (5.2) lead to the following conformal dimensions of the parafermion primary states $|\ell>$:

$$h_\ell = \frac{\ell(k-\ell)}{2k(k+2)}. \tag{5.9}$$

The other states in the parafermion Verma module V^ℓ are obtained through successive applications of the operators A and $A^\dagger$ on $|\ell>$. Moreover, as discussed in the references [16,41,48], V^ℓ splits into a sum of submodules V_m^ℓ as

$$V_\ell = \frac{1}{2} \bigoplus_{m=-\ell}^{2k-\ell} V_m^\ell. \qquad (5.10)$$

The submodules V_m^ℓ are generated from the Virasoro primary states, which are defined as

$$\begin{aligned} |\ell, m> &= A_{(m-2)/k-1} A_{(m-4)/k-1} \cdots A_{\ell/k-1} |\ell>, \quad \ell \leq m \leq 2k-\ell, \\ |\ell, m> &= A^\dagger_{-(m+2)/k} A^\dagger_{(m+4)/k} \cdots A^\dagger_{\ell/k} |\ell>, \quad -\ell \leq m \leq \ell. \end{aligned} \qquad (5.11)$$

These states $|\ell, m>$ have conformal dimensions

$$\begin{aligned} h_m^\ell &= \frac{(k-\ell)(k-\ell+2)}{4(k+2)} - \frac{(k-m)^2}{4k}, \quad \ell \leq m \leq 2k-\ell, \\ h_m^\ell &= \frac{\ell(\ell+2)}{4(k+2)} - \frac{m^2}{4k}, \quad -\ell \leq m \leq \ell. \end{aligned} \qquad (5.12)$$

To conclude this section let us mention that the Z_k parafermion model is isomorphic to the coset model $su(2)_k/u(1)_k$, that is, to the $su(2)_k$ Kac-Moody algebra with the $u(1)_k$ (Cartan) subalgebra factored out. This can be seen from the central charges. Indeed, the central charge of $su(2)_k$ which is $\frac{3k}{k+2}$ can be rewritten as

$$\frac{3k}{k+2} = 1 + \frac{2(k-1)}{k+2}. \qquad (5.13)$$

In (5.13), we recognize $\frac{2(k-1)}{k+2}$ as being the central charge of the Z_k parafermion given in (5.5), whereas 1 is the central charge of the $u(1)_k$ factor. Therefore, the Z_k parafermion theory is sometimes referred to as the $su(2)_k$ parafermion model. In the remaining part of this chapter we will concentrate on the free field realization of the $su(n)_k$ parafermion theory, namely, $su(n)_k/u(1)_k^{n-1}$ [49].

5.2 Free field realizations of the $su(n)_k$ parafermion model

As our approach relies on the Wakimoto realization of the Kac-Moody algebra $su(n)_k$ [20], let us then briefly remind some of its characteristics relevant for the subsequent investigation; more details can be found in chapter 3. This Wakimoto realization requires an r-component free scalar field φ and two R-component free fields u and v, whose components u_α and v_α are associated with the positive roots α of $su(n)$. Here $r = n-1$ is the rank and $R = n(n-1)/2$ is the number of positive roots of $su(n)$. Furthermore, as already indicated through (3.13), all the components of the field $X = (\varphi, u; v)$ are orthogonal, with the Lorentzian signature $(+,+;-)$. The Wakimoto realization of the $su(n)_k$ currents $H(z)$ and $J_{\pm\alpha}$ is given in (3.19), whereas that of the energy-momentum tensor is displayed through (3.22), which can be rewritten explicitly as

$$\begin{aligned} T &= -\frac{1}{2}(\partial X)^2 + i\alpha_0 \cdot \partial^2 X \\ &= -\frac{1}{2}(\partial\varphi)^2 + \frac{i}{\alpha_+}\rho \cdot \partial^2\varphi - \frac{1}{2}\sum_{\alpha\in\Delta_+}[(\partial u_\alpha)^2 - i\partial^2 u_\alpha - (\partial v_\alpha)^2 + i\partial^2 v_\alpha], \end{aligned} \tag{5.14}$$

where Δ_+ is the set of the positive roots and α_0 is the background charge, which reads

$$\alpha_0(\varphi, u; v) = \left[\frac{\rho}{\alpha_+}, (\frac{1}{2})^R; (\frac{1}{2})^R\right]. \tag{5.15}$$

Here, ρ is the sum of the fundamental weights of $su(n)$ and $\alpha_+ = \sqrt{k+n}$; k being the level of the Kac-Moody algebra $su(n)_k$.

5.2.1 Realization in terms of orthonormal fields

In analogy with the $su(2)_k$ (Z_k) parafermion model, which is equivalent to the coset model $su(2)_k/u(1)_k$, we construct the $su(n)_k$ parafermion theory through the coset model $su(n)_k/u(1)_k^{n-1}$, that is, we must factor out the fields that give rise to the Cartan subalgebra $u(1)_k^{n-1}$ associated with the Kac-Moody (KM) current $H(z)$. To this end, we must rewrite the KM currents, which were expressed in terms of $(\varphi, u; v)$ in (3.19), in such a way that this factorization becomes apparent. This translates into imposing the following field realization on the $su(n)_k$ currents:

$$H = i\sqrt{k}\partial B, \tag{5.16}$$

$$J_{\pm\alpha} = \sqrt{k}\psi_{\pm\alpha}\exp(\pm i\alpha B/\sqrt{k}). \tag{5.17}$$

This defines the $su(n)_k$ parafermion currents $\psi_{\pm\alpha}$ and the r-component field B. Note that $\psi_{+\alpha}$ and $\psi_{-\alpha}$ are respectively the analogue of ψ_1 and $\psi_1^\dagger$, which generate the Z_k parafermion algebra (5.2). Combining the equation (5.16) with the expression of H as given in (3.19), we can easily read off the definition of B in terms of $(\varphi, u; v)$, namely,

$$B = \frac{1}{\sqrt{k}}(\alpha_+\varphi - \sum_{\alpha\in\Delta_+} \alpha v_\alpha). \tag{5.18}$$

As the number of the components of the old free fields φ, u and v is $r+2R$, clearly besides the r-component new free field B, we need to introduce two new R-component free fields f and g, whose components f_α and g_α are associated with the positive roots of $su(n)$. The components of the new set of fields $(B, f; g)$ are mutually orthonormal, with the Lorentzian signature $(+,+;-)$. As can readily be seen from (3.19) and (3.20), the J'_α's always depend on the combinations $u-v$. This is consistent with (5.17) only if

$$u_\alpha - v_\alpha = -\alpha B/\sqrt{k} + \cdots. \tag{5.19}$$

The equation (5.19) alone does not determine the fields f and g in terms of $(\varphi, u; v)$. However, for simplicity we propose the following ansatz which is consistent with (5.19) (we are proceeding in a similar way to that of reference [50]):

$$u_\alpha = f_\alpha, \tag{5.20}$$

$$v_\alpha = \frac{1}{\sqrt{k}}(\alpha B + \sum_{\gamma\in\Delta_+} A_{\alpha\gamma} g_\gamma). \tag{5.21}$$

The equation (5.21) together with (5.18) yield

$$\varphi = \frac{1}{\sqrt{k}}(\alpha_+ B + \frac{1}{\alpha_+}\sum_{\gamma\in\Delta_+}\sum_{\beta\in\Delta_+} \beta A_{\beta\gamma} g_\gamma). \tag{5.22}$$

The matrix elements $A_{\alpha\beta}$ introduced in (5.21) are determined through the orthonormality of the fields $(\varphi, u; v)$ as

$$A_{\alpha\beta} = \sqrt{k}(\delta_{\alpha\beta} + \frac{\alpha\cdot\beta}{\sqrt{k}(\alpha_+ + \sqrt{k})}). \tag{5.23}$$

Therefore, the relations (5.20) through (5.23) define the transformation from the new set of fields $(B, g; f)$ to the old one $(\varphi, u; v)$. This transformation is equivalent to that found in reference [50] (in fact, it is the inverse transformation that is given there). As our aim is to factorize the $u(1)_k$ fields in the KM currents given in (3.19), it is convenient to define the transformation in the above way.

The energy-momentum tensor in this realization $(B, f; g)$ reads

$$\begin{aligned} T &= T_B + T_P, \\ T_B &= -\frac{1}{2}(\partial B)^2, \\ T_P &= \sum_{\alpha\in\Delta_+}\left[-\frac{1}{2}(\partial f_\alpha)^2 + \frac{i}{2}\partial^2 f_\alpha + \frac{1}{2}(\partial g_\alpha)^2 - i\alpha_g^\alpha \partial^2 g_\alpha\right]. \end{aligned} \tag{5.24}$$

This means that the background charge α_0 takes on the following form in the basis $(B, f; g)$:

$$\alpha_0(B, f; g) = \left[0^r, (\frac{1}{2})^R; \alpha_g^\alpha\right], \tag{5.25}$$

with

$$\alpha_g^\alpha = \frac{1}{2} - \frac{\rho\cdot\alpha}{\alpha_+(\alpha_+ + \sqrt{k})}. \tag{5.26}$$

That the background charge of g is more complicated is expected according to the equation (5.21).

Let us now elaborate on the free field realization of the $su(n)_k$ parafermion theory. The substitution of $(\varphi, u; v)$ by $(B, f; g)$ in (3.19) (according to (5.20) through (5.23)) and the relation (5.17) enable us to read off the free field realization of the $su(n)_k$ parafermion currents. Indeed, the parafermion currents corresponding to the negative roots and positive simple roots of $su(n)$ are thus represented in terms of $(B, f; g)$ as

$$\psi_{-\alpha} = \frac{-i}{\sqrt{k}}\partial f_\alpha \exp(iF_\alpha) + \sum_{\substack{\beta\in\Delta_+\\ \beta>\alpha}} \partial f_{\alpha+\beta}\exp(i(F_{\alpha+\beta} - F_\beta)), \tag{5.27}$$

$$\psi_{\alpha_j} = \psi_{\alpha_j}(0) + \sum_{l=1}^{j-1}\psi_{\alpha_j}(l<) + \sum_{l=j+2}^{n}\psi_{\alpha_j}(l>), \tag{5.28}$$

where α stands for positive roots, α_j $(j = 1,\ldots,n-1)$ denotes the simple positive roots, while the relation $\beta > \alpha$ has already been explained below the equation (3.21) in chapter 3, and

$$\begin{aligned}\psi_{\alpha_j}(0) = &\frac{-i}{\sqrt{k}}\{(\alpha_+^2 - j)\partial f_j + \frac{1}{\sqrt{k}}\sum_{\beta\in\Delta_+}\partial g_\beta[-(\alpha_+^2 + 1 - j)A_{\alpha_j\beta} \\ &+ \sum_{\alpha\in\Delta_+}\alpha\cdot\alpha_j A_{\alpha\beta} + \sum_{l=1}^{j-1}(A_{(lj)\beta} - A_{(l,j+1)\beta})]\}\exp(-iF_j)\end{aligned} \tag{5.29}$$

$$\psi_{\alpha_j}(l<) = \frac{i}{\sqrt{k}}\partial f_{(lj)}\exp i[F_{(lj)} - F_{(l,j+1)}] \tag{5.30}$$

$$\psi_{\alpha_j}(l>) = \frac{-i}{\sqrt{k}}\partial f_{(j+1,l)}\exp i[F_{(j+1,l)} - F_{(j,l)}] \tag{5.31}$$

The notation (ij), sometimes used for $su(n)$ roots, is explained in (3.11), and the fields F_α are defined by

$$F_\alpha = f_\alpha - g_\alpha - \sum_{\beta\in\Delta_+}\frac{\alpha\cdot\beta}{\sqrt{k}(\alpha_+ + \sqrt{k})}g_\beta. \tag{5.32}$$

In summary, the equations (5.27) through (5.32) constitute the free field realization of the $su(n)_k$ parafermion currents. This is the analog of the Wakimoto realization of $su(n)_k$ given in (3.19) through (3.21). To conclude this section let us mention that a full field realization of the $su(n)_k$ parafermion model must include, in addition to the parafermion currents, the screening currents. In this regard, we note that provided that the screening currents of $su(n)_k$ are derived in terms of $(\varphi, u; v)$, the screening currents of the $su(n)_k$ parafermion model can easily be obtained in terms of $(B, g; f)$ through the relations (5.20) to (5.23) and the modding out of the B fields.

5.2.2 Example: the $su(3)_k$ parafermion model

For the sake of illustration, let us present the explicit field realization of the $su(3)_k$ parafermion currents and screening currents in terms of $(B, f; g)$. To avoid encumbering the notation, the two simple roots of $su(3)$ are denoted by 1 and 2, whereas the compound one by 3. The Wakimoto realization of $su(3)_k$ given in chapter 3 and the relation (5.17) imply the following free field realization of the $su(3)_k$ parafermion currents:

$$
\begin{aligned}
\psi_{-1} &= -\frac{i}{\sqrt{k}}[\partial f_1 \exp(iF_1) + \partial f_3 \exp(i(F_3 - F_2))], \\
\psi_{-2} &= -\frac{i}{\sqrt{k}}\partial f_2 \exp(iF_2), \\
\psi_{-3} &= -\frac{i}{\sqrt{k}}\partial f_3 \exp(iF_3), \\
\psi_1 &= -\frac{i}{\sqrt{k}}[(k+2)\partial f_1 + \frac{1}{\sqrt{k}}\sum_{il}(\alpha_1 \cdot \alpha_i - (k+3)\delta_{i1})A_{il}\partial g_l]\exp(-iF_1) \\
&\quad - \frac{i}{\sqrt{k}}\partial f_2 \exp(i(F_2 - F_3)),
\end{aligned}
$$

$$
\begin{aligned}
\psi_2 =& -\frac{i}{\sqrt{k}}[(k+1)\partial f_2 - \sqrt{k}\sum_l A_{2l}\partial g_l]\exp(-iF_2) \\
&+ \frac{i}{\sqrt{k}}\partial f_1 \exp(i(F_1 - F_3)), \\
\psi_3 =& \frac{i}{\sqrt{k}}[(k+2)\partial f_1 + \frac{1}{\sqrt{k}}\sum_{il}(\alpha_1\cdot\alpha_i - (k+3)\delta_{i1})A_{il}\partial g_l]\exp(-i(F_1+F_2)) \\
&+ \frac{i}{\sqrt{k}}[-(k+1)\partial f_3 + \sqrt{k}\sum_l A_{3l}\partial g_l]\exp(-iF_3).
\end{aligned} \tag{5.33}
$$

The $su(3)_k$ parafermion screening currents split into two classes. The first class screening currents are not expressed as pure exponential terms and are associated with the two simple roots; they read

$$
\begin{aligned}
S_1^{(-)} &= -i\partial f_1 \exp(iG_1), \\
S_2^{(-)} &= -i\partial f_2 \exp(iG_2) - i\partial f_3 \exp[i(G_3 - G_1)].
\end{aligned} \tag{5.34}
$$

On the contrary, the second class screening currents are expressed as pure exponential terms and are associated with the negative and the simple positive roots. They are given by

$$
\begin{aligned}
S_{-1}^{(+)} &= \exp(-if_1), \\
S_{-2}^{(+)} &= \exp(-if_2), \\
S_{-3}^{(+)} &= \exp(-if_3), \\
S_{+1}^{(+)} &= \exp i(f_1 - \alpha_+^2 G_1), \\
S_{+2}^{(+)} &= \sum_{n\in Z}\exp i\{f_2 + f_3 - \alpha_+^2 G_2 \\
&\qquad + n[f_1 + f_2 - f_3 - (g_1 + g_2 - g_3)]\}.
\end{aligned} \tag{5.35}
$$

The G_α fields introduced in (5.34) and (5.35) are defined by

$$
G_\alpha = f_\alpha - g_\alpha + \sum_{\beta\in\Delta_+}\frac{\alpha\cdot\beta}{\alpha_+(\alpha_+ + \sqrt{k})}g_\beta. \tag{5.36}
$$

5.2.3 Realization in terms of constrained fields

As we have seen, $su(n)_k$ KM currents written in terms of the orthonormal fields $(B, f; g)$, allow the $u(1)_k^{n-1}$ factorization in a natural way, and thereby the free field realization of the $su(n)_k$ parafermion model can be readily derived. However, as far as the primary fields, that is, the representations of the parafermion algebra, are concerned, this set of fields $(B, f; g)$ proves to be inconvenient in that it makes harder the derivation of the free field realization for the $su(n)_k$ parafermion primary fields from that of the $su(n)_k$ primary fields, through the modding out of B. To see that, let us consider the Wakimoto realization of the primary fields of $su(n)_k$. It reads in terms of the fields $(\varphi, u; v)$ as [20]

$$\Phi_{\Lambda\lambda} = \exp\left[-\frac{i}{\alpha_+}\Lambda\cdot\varphi - \frac{i}{n}\sum_{\alpha\in\Delta_+}\alpha\cdot(\Lambda-\lambda)(u_\alpha - v_\alpha)\right], \tag{5.37}$$

where the r-component vector Λ is a highest weight of $su(n)$ and λ is a weight belonging to this highest weight representation. Again to factor out the B field let us substitute the fields $(\varphi, u; v)$ in (5.37) with the fields $(B, f; g)$ by means of the relations (5.20) through (5.23). This leads to

$$\Phi_{\Lambda\lambda} = \phi_{\Lambda\lambda}\exp\left(-\frac{i}{\sqrt{k}}\lambda\cdot B\right). \tag{5.38}$$

After factorization, the parafermion primary field $\phi_{\Lambda\lambda}$ reads

$$\phi_{\Lambda\lambda} = \exp\left[-\frac{i}{\alpha_+\sqrt{k}}\sum_{\alpha\in\Delta_+}\Lambda\cdot\alpha g_\alpha - \frac{i}{\bar{h}}\sum_{\alpha\in\Delta_+}(\Lambda-\lambda)\cdot\alpha F_\alpha\right]. \tag{5.39}$$

As a check, let us verify that the conformal dimensions of these fields are consistent with each other. They are evaluated independently using their respective background charges. Thus, they are derived respectively for $\Phi_{\Lambda\lambda}$, $\phi_{\Lambda\lambda}$ and

$\exp[-\frac{i}{\sqrt{k}}\lambda \cdot B]$ as

$$\begin{aligned} h_1 &= \frac{\Lambda \cdot (\Lambda + 2\rho)}{2\alpha_+^2}, \\ h_2 &= \frac{\Lambda \cdot (\Lambda + 2\rho)}{2\alpha_+^2} - \frac{\lambda^2}{2k}, \\ h_3 &= \frac{\lambda^2}{2k}. \end{aligned} \tag{5.40}$$

Clearly, $h_1 = h_2 + h_3$ as expected. Taking into account the equation (5.32), the above expression (5.39) is indeed cumbersome and not suitable for the computation of the parafermion correlation functions for example. To make it more tractable let us define the following r-component fields:

$$\begin{aligned} \phi_1 &= \frac{\alpha_+}{n} \sum_{\alpha \in \Delta_+} \alpha E_\alpha, \\ \phi_2 &= -\frac{\sqrt{k}}{n} \sum_{\alpha \in \Delta_+} \alpha F_\alpha, \end{aligned} \tag{5.41}$$

where the fields E_α are defined as

$$\begin{aligned} E_\alpha &= f_\alpha - \sum_{\beta \in \Delta_+} B_{\alpha\beta} g_\beta, \\ B_{\alpha\beta} &= \left(1 - \frac{n}{\alpha_+ \sqrt{k}}\right) \delta_{\alpha\beta} + \frac{1}{\sqrt{k}(\alpha_+ + \sqrt{k})} \alpha \cdot \beta. \end{aligned} \tag{5.42}$$

It can easily be checked through (5.42) and (5.32) that the fields ϕ_1 and ϕ_2 are orthonormal with a Lorentzian signature $(+;-)$. In terms of these fields, the cumbersome expression (5.39) of the $su(n)_k$ parafermion primary fields reduces to the following simple form:

$$\phi_{\Lambda,\lambda} = \exp\left[-i\left(\frac{1}{\alpha_+}\Lambda \cdot \phi_1 + \frac{1}{\sqrt{k}}\lambda \cdot \phi_2\right)\right]. \tag{5.43}$$

Consequently, the $2r$ fields ϕ_1 and ϕ_2 prove to be convenient to represent the parafermion primary fields. However, as the old basis $(f; g)$ involves $2R$ fields, we must then complete the new basis with $2(R-r)$ new fields. As ϕ_1 and ϕ_2

are expressed as sums over positive roots of f_α and g_α, we propose then the following form for the latter fields:

$$
\begin{aligned}
f_\alpha &= \frac{\alpha}{n}\cdot(\alpha_+\phi_1+\sqrt{k}\phi_2)+\tilde{f}_\alpha, \\
g_\alpha &= \frac{\alpha}{n}\cdot(\sqrt{k}\phi_1+\alpha_+\phi_2)+\tilde{g}_\alpha,
\end{aligned}
\tag{5.44}
$$

where $\tilde{f}_\alpha$ and $\tilde{g}_\alpha$ are new fields assumed to be independent of ϕ_1 and ϕ_2. But then their number $2R$ is $2r$ larger than what is needed. In fact, this problem is naturally solved because these fields are not linearly independent but satisfy $2r$ constraints. This can be read off from the orthonormality of f_α and g_α together with that of ϕ_1 and ϕ_2, which yield

$$
\sum_{\alpha\in\Delta_+}\alpha\tilde{f}_\alpha=\sum_{\alpha\in\Delta_+}\alpha\tilde{g}_\alpha=0, \tag{5.45}
$$

$$
\tilde{f}_\alpha(z)\tilde{f}_\beta(0)\sim-\ln z\left(\delta_{\alpha\beta}-\frac{1}{n}\alpha\cdot\beta\right), \tag{5.46}
$$

$$
\tilde{g}_\alpha(z)\tilde{g}_\beta(0)\sim\ln z\left(\delta_{\alpha\beta}-\frac{1}{n}\alpha\cdot\beta\right). \tag{5.47}
$$

Thus, the new basis, which is suitable for the field realization of the parafermion primary fields, involves the fields $(\phi_1,\tilde{f}_\alpha;\phi_2,\tilde{g}_\alpha)$, with the constraints (5.45). In this new basis, the energy-momentum tensor of the parafermion theory T_P reads

$$
T_P=\frac{1}{2}\left[-\phi_1'^2+\phi_2'^2+\frac{2i}{\alpha_+}\rho\cdot\phi_1''\right]+\frac{1}{2}\sum_{\alpha\in\Delta_+}\left[-\tilde{f}_\alpha'^2+\tilde{g}_\alpha'^2+i(\tilde{f}_\alpha''-\tilde{g}_\alpha'')\right], \tag{5.48}
$$

where the primes denote derivatives. Our energy-momentum tensor, containing the constrained fields $\tilde{f}_\alpha$ and $\tilde{g}_\alpha$, coincides with that obtained in reference [36]. It also proves that the set of orthonormal fields $(f;g)$ is equivalent to the set of constrained fields $(\phi_1,\tilde{f};\phi_2,\tilde{g})$ in representing the $su(n)_k$ parafermion model. To conclude this section, let us note that the expression T_P in (5.48) can easily be obtained from the energy-momentum tensor in (5.14) through the following redefinitions of the fields:

$$\begin{aligned}
\varphi &= \frac{\alpha_+ B}{\sqrt{k}} + \frac{1}{\sqrt{k}}(\sqrt{k}\phi_1 + \alpha_+\phi_2), \\
u_\alpha &= \frac{1}{n}\alpha\cdot(\alpha_+\phi_1 + \sqrt{k}\phi_2) + \tilde{f}_\alpha, \\
v_\alpha &= \frac{\alpha\cdot B}{\sqrt{k}} + \frac{\alpha_+}{n\sqrt{k}}\alpha\cdot(\sqrt{k}\phi_1 + \alpha_+\phi_2) + \tilde{g}_\alpha.
\end{aligned} \tag{5.49}$$

5.3 Free field realization of the coset model $su(n)_k \times su(n)_\ell/su(n)_{k+\ell}$

In this section, we address the free field realization of the coset model $C \equiv su(n)_k \times su(n)_\ell/su(n)_{k+\ell}$. As argued in the previous sections the $su(n)_k$ parafermion model is isomorphic to the coset model $su(n)_k/u(1)^r_k$, without any background charge associated with the $u(1)^r$ B field. Similarly, we may expect the $su(n)_k$ parafermion model to be schematically expressed as $C/u(1)^r_{\bar{\alpha}}$. $u(1)^r_{\bar{\alpha}}$ means that now the B field does have a background charge denoted by $\bar{\alpha}$. This can be seen from the central charges. Indeed, since the central charge of $su(n)_k$ is $kd/(k+n)$ ($d = n^2-1$), the central charges c, c_p and c_B of C, $su(n)_k$ parafermion model and $u(1)^r_{\bar{\alpha}}$ are respectively given by

$$c = c_P + c_B, \tag{5.50}$$

$$c_P = \frac{k\,d}{k+n} - r = \frac{2R(k-1)}{k+n}, \tag{5.51}$$

$$c_B = r - \frac{knd}{k'(k+k')}, \tag{5.52}$$

where $r = n-1$, $R = n(n-1)/2$ and $k' = \ell + n$. The equation (5.52) clearly reveals that the background charge associated with the B field is now nonvanishing.

As indicated in references [50,51], the free field realization of the coset C is derived through the Goddard-Kent-Olive procedure [33] in a nontrivial asymmetric way. The currents of $su(n)_{k+\ell}$ are not, as usual, direct sums of those of

$su(n)_k$ and $su(n)_\ell$, except the Cartan subalgebra currents, which are still constructed in the standard way. In fact, $su(n)_k$ is expressed in terms of $(B, f; g)$ with the energy-momentum tensor as given in (5.24), whereas both $su(n)_\ell$ and $su(n)_{k+\ell}$ are realized in terms of $(\varphi, u_\alpha; v_\alpha)$, whose energy-momentum tensors read as in (5.14). Furthermore, the fields $(u_\alpha; v_\alpha)$ corresponding to both $su(n)_\ell$ and $su(n)_{k+\ell}$ are identified. These conditions translate into the following relations (the subscripts of the fields indicate which algebras the fields are associated with):

$$\sqrt{k+\ell+n}\,\varphi_{k+\ell} = \sqrt{k}B_k + \sqrt{\ell+n}\,\varphi_\ell, \tag{5.53}$$

$$\begin{aligned} T &= T_k + T_\ell - T_{k+\ell} \\ &= T_P - \frac{1}{2}(\partial B_k)^2 - \frac{1}{2}(\partial\varphi_\ell)^2 + \frac{1}{2}(\partial\varphi_{k+\ell})^2 \\ &\quad + i\rho\cdot\partial^2\left(\frac{1}{\sqrt{\ell+n}}\varphi_\ell - \sqrt{k+\ell+n}\,\varphi_{k+\ell}\right), \end{aligned} \tag{5.54}$$

where T, T_k and T_P stand respectively for the energy-momentum tensors of C', $su(n)_k$ and the $su(n)_k$ parafermion model. In (5.54), $T - T_P$ is then the energy-momentum tensor of $u(1)^r_{\bar\alpha}$. It can be further simplified if we express it in terms of the field B, which is defined to be orthogonal to $\varphi_{k+\ell}$. It can readily be checked through (5.53) that B reads as

$$B = \frac{1}{\sqrt{k+\ell+n}}(-\sqrt{\ell+n}B_k + \sqrt{k}\varphi_\ell), \tag{5.55}$$

In terms of B, the energy-momentum tensor $T_B \equiv T - T_P$ of $u(1)^r_{\bar\alpha}$ reduces to

$$T_B = -\frac{1}{2}(\partial B)^2 + i\bar\alpha_0\rho\cdot\partial^2 B. \tag{5.56}$$

This T_B yields the following central charge c_B:

$$c_B = r - 12(\bar\alpha_0\rho)^2 = r - \bar\alpha_0^2 n(n^2-1). \tag{5.57}$$

The identification of this c_B with that given in (5.52) determines the background charge associated with B as

$$\bar{\alpha}_0 = k/\sqrt{kk'(k+k')}. \tag{5.58}$$

Let us now sketch the free field realization of the screening currents of the coset model C. These screening currents can be expressed as the products of the parafermion currents and a vertex operator $\exp(i\gamma\alpha \cdot B)$ in such a way that their conformal dimension is 1, that is,

$$S_{\pm}^{(i)} = \psi_{\pm i} \exp(i\gamma_{\pm}\alpha_i \cdot B), \tag{5.59}$$

$$\frac{\alpha_i^2}{2}\gamma_{\pm}^2 - \gamma_{\pm}\bar{\alpha}_0\alpha_i \cdot \rho + \Delta_i = 1, \tag{5.60}$$

where $\Delta_i = 1 - \alpha_i^2/2k$ and the screening currents $S_{\pm}^{(i)}$ are associated with the positive and negative simple roots α_i. Solving the quadratic equation (5.60) in $\gamma_{\pm}$, we obtain

$$\begin{aligned} \gamma_+ &= (k+k')/\sqrt{kk'(k+k')}, \\ \gamma_- &= -k'/\sqrt{kk'(k+k')}. \end{aligned} \tag{5.61}$$

To conclude this section let us briefly sketch the free field realization of the primary fields Φ of the coset model C. Again, they are obtained as products of the parafermion primary field $\phi_{\Lambda,\Lambda}$, as given in (5.43), and the vertex operators $\exp(i\beta \cdot B)$, for some β,

$$\Phi = \phi_{\Lambda,\Lambda} \exp(i\beta \cdot B). \tag{5.62}$$

The equation (5.62) means that the conformal dimension h of Φ is given by

$$h = \Delta(\Lambda,\Lambda) + h_\beta, \tag{5.63}$$

where

$$\begin{aligned} \Delta(\Lambda,\Lambda) &= \frac{\Lambda \cdot (\Lambda + 2\rho)}{2\alpha_+^2} - \frac{\Lambda^2}{2k}, \\ h_\beta &= \frac{1}{2}\beta \cdot (\beta - \bar{\alpha}_0\rho). \end{aligned} \tag{5.64}$$

Now let $p \in P_+^\ell$ and $q \in P_+^{k+\ell}$, with the symbol P_+^k being the set of dominant weights of level k, i.e.,

$$P_+^k = \{p | p \cdot \lambda_i \geq 0; \quad p \cdot \rho \leq k\}, \tag{5.65}$$

where λ_i denotes the fundamental weights of $su(n)$. Then if we choose β as

$$\beta = -\frac{(k+k')p - k'q}{\sqrt{kk'(k+k')}}, \tag{5.66}$$

we recover the spectrum of the conformal dimensions of the primary fields Φ as given in reference [42].

5.4 Conclusions

In this chapter, we have constructed two free field realizations of the $su(n)_k$ parafermion model. The first one is appropriate for the parafermion currents whereas the second one is suitable for the parafermion primary fields. We have also derived the free field realization of the screening currents of the $su(3)_k$ parafermion model. Finally, we have sketched the free field realization of the coset model $C \equiv su(n)_k \times su(n)_\ell / su(n)_{k+\ell}$. Its energy-momentum tensor, screening currents and primary fields are expressed in terms of the fields realizing the $su(n)_k$ parafermion model and $u(1)^r_{\tilde{\alpha}}$. A case of particular interest is $\ell = 1$, which means that C is nothing but the W_n algebra [15]. Therefore, the analysis presented in the section 5.3 is also valid for the W_n algebra. Moreover, the free field realization of the $su(3)_k$ parafermion model suggests that the theory of the screening currents presented in chapter 3 in the case of Kac-Moody algebras can also be applied to the parafermion algebras. Finally, let us mention that the results obtained in this chapter can be straightforwardly generalized to a parafermion model associated with any other Kac-Moody algebra.

Conclusions and outlook

In this thesis we have shed some light on two-dimensional conformal field theories (CFT's), and we hope that new insights into this rapidly developing subject have thereby been gained. We have highlighted the fact that the free field realization provides a comprehensive and solid framework to describe several extended CFT's. Although most of the analysis presented in this thesis can be easily generalized to more complicated cases, further investigation of the free field realization is certainly needed, in particular, the use of the screening currents, which are expressed as infinite sums of terms, in order to explicitly compute the characters, the fusion rules and the correlation functions of CFT's with Kac-Moody algebras. A detailed study of the $su(3)_k$ Kac-Moody algebra which features all of the characteristics of general Kac-Moody algebras may serve as a useful orientation. So far this free field realization has been applied only to CFT's that are valid just at some critical points (dimensions), it would be interesting to explore its possible applications to non-conformal two-dimensional theories such as perturbed CFT's and integrable models. Finally, let us note that the classification of all rational CFT's is still an open problem, which is far from being solved.

Table captions

Table 1. : The vectors $J'_{\mp\alpha}$ and $\bar{J}_{\mp\alpha}$ for the KM root currents of $su(2)_k$. The first column denotes the KM current, the second column the enumeration of the individual terms, the third, fourth, and fifth columns are the components of these vectors, in the round-bracket notation (see equations (3.16) and (3.18)) along the directions φ, $u_{(12)}$, and $v_{(12)}$ respectively. The first row in each case is J', and the second row is $\bar{J}$. k is the level of the KM algebra.

Table 2. : (a): The dot product $\bar{J}^a \cdot J'^b$; (b) the dot product $\bar{J}^a \cdot \bar{J}^b$, calculated from Table 1 and formula (3.17).

Table 3. : The vectors $J'_{\mp\alpha}$ and $\bar{J}_{\mp\alpha}$ for the KM root currents of $su(3)_k$. The first column denotes the KM current, the second column the enumeration of the individual terms, the third, fourth, and fifth columns are the components of these vectors, in the round-bracket notation (see equations (3.16) and (3.18)), along the directions φ, u_α, and v_α respectively. The first row in each case is J', and the second row is $\bar{J}$. In order to fit everything into the table, we have used the abbreviation $k_i \equiv k + i$, where k is the level of the KM algebra.

Table 4. : (a): The dot product $\bar{J}^a \cdot J'^b$; (b) the dot product $\bar{J}^a \cdot \bar{J}^b$, calculated from Table 3 and formula (3.17). The solid horizontal lines separate the different KM currents; the dotted horizontal lines delineate the different irreducible units within a KM current.

Table 5. : The vectors $J'_{\mp\alpha}$ and $\bar{J}_{\mp\alpha}$ for the KM root currents of $sp(4)_k$. The

first column denotes the KM current, the second column the enumeration of the individual terms, the third, fourth, and fifth columns are the components of these vectors, in the round-bracket notation (see equations (3.16) and (3.18)), along the directions φ, u_α, and v_α respectively. See the text for notations of the roots. The first row in each case is J', and the second row is $\bar{J}$. In order to fit everything into the table, we have used the abbreviation $k_i \equiv k + i$, and $k'_i \equiv (k/2) + i$, where k is the level of the KM algebra, which in this case must be a positive even integer.

Table 6. : (a): The dot product $\bar{J}^a \cdot J'^b$; (b) the dot product $\bar{J}^a \cdot \bar{J}^b$, calculated from Table 5 and formula (3.17). The solid horizontal lines separate the different KM currents; the dotted horizontal lines delineate the different irreducible units within a KM current.

Table 7. : A recapitulation of the various lattices involved in the free field realization of the CFT with a W_3 algebra.

Table 8. : The mappings of the positive sector screening charges $Q_i^+, i = 1, \ldots, 6$, in (4.54) among the six classes of the W_3 submodules, which are defined in (4.55).

Table 9. : The mappings of the negative screening charges $Q_i^-, i = 1, \ldots, 6$, in (4.57) among the six classes of the W_3 submodules, which are defined in (4.55).

Figure Captions

Figure 1. : An example of a conformal transformation that preserves the local

Table 1

		1	(12)	(12)
$J_{-(12)}$	1	0 0	2 -2	2 0
$J_{+(12)}$	2	0 $2(k+2)$	-2 $-2(k+1)$	-2 $-2(k+2)$

Table 2a

	1	2
1	-1	1
2	1	-1

Table 2b

	1	2
1	1	$k+1$
2	$k+1$	1

Table 3

		1	2	(12)	(23)	(13)	(12)	(23)	(13)
$J_{-(12)}$	1	0	0	2	0	0	2	0	0
		0	0	-2	0	0	0	0	0
$J_{-(12)}$	2	0	0	0	-2	2	0	-2	2
		0	0	0	0	-2	0	0	0
$J_{-(23)}$	3	0	0	0	2	0	0	2	0
		0	0	0	-2	0	0	0	0
$J_{-(13)}$	4	0	0	0	0	2	0	0	2
		0	0	0	0	-2	0	0	0
$J_{+(12)}$	5	0	0	-2	0	0	-2	0	0
		$2k_3$	$-k_3$	$-2k_2$	0	0	$-2k_3$	0	0
$J_{+(12)}$	6	0	0	0	2	-2	0	2	-2
		0	0	0	-2	0	0	0	0
$J_{+(23)}$	7	0	0	0	-2	0	0	-2	0
		$-k_3$	$2k_3$	0	$-2k_1$	0	2	$-2k_2$	-2
$J_{+(23)}$	8	0	0	2	0	-2	2	0	-2
		0	0	2	0	0	0	0	0
$J_{+(13)}$	9	0	0	-2	-2	0	-2	-2	0
		$-2k_3$	k_3	$2k_2$	0	0	$2k_3$	0	0
$J_{+(13)}$	10	0	0	0	0	-2	0	0	-2
		k_3	k_3	0	0	$-2k_1$	-2	-2	$-2k_2$

Table 4a

	1	2	3	4	5	6	7	8	9	10
1	−1	0	0	0	1	0	0	−1	1	0
2	0	−1	0	−1	0	1	0	1	0	1
3	0	1	−1	0	0	−1	1	0	1	0
4	0	−1	0	−1	0	1	0	1	0	1
5	1	0	0	0	−1	0	0	1	−1	0
6	0	1	−1	0	0	−1	1	0	1	0
7	−1	0	1	1	1	0	−1	−2	0	−1
8	1	0	0	0	−1	0	0	1	−1	0
9	−1	0	0	0	1	0	0	−1	1	0
10	1	0	1	1	−1	0	−1	0	−2	−1

Table 4b

	1	2	3	4	5	6	7	8	9	10
1	1	0	0	0	k_2	0	0	−1	$-k_2$	0
2	0	1	0	1	0	0	0	0	0	k_1
3	0	0	1	0	0	1	k_1	0	0	0
4	0	1	0	1	0	0	0	0	0	k_1
5	k_2	0	0	0	1	0	0	$-k_2$	−1	0
6	0	0	1	0	0	1	k_1	0	0	0
7	0	0	k_1	0	0	k_1	1	0	0	1
8	−1	0	0	0	$-k_2$	0	0	1	k_2	0
9	$-k_2$	0	0	0	−1	0	0	k_2	1	0
10	0	k_1	0	k_1	0	0	$-k$	0	0	-1

Table 5

		1	2	(12)	(2)	[12]	(1)	(12)	(2)	[12]	(1)
$J_{-(12)}$	1	0	0	2	0	0	0	2	0	0	0
		0	0	−2	0	0	0	0	0	0	0
$J_{-(12)}$	2	0	0	0	−2	2	0	0	−2	2	0
		0	0	0	0	−2	0	0	0	0	0
$J_{-(12)}$	3	0	0	0	0	−2	2	0	0	−2	2
		0	0	0	0	0	2	0	0	0	0
$J_{-(12)}$	4	0	0	−2	−2	0	2	−2	−2	0	2
		0	0	0	0	0	−2	0	0	0	0
$J_{-(2)}$	5	0	0	0	2	0	0	0	2	0	0
		0	0	0	−2	0	0	0	0	0	0
$J_{-[12]}$	6	0	0	0	0	2	0	0	0	2	0
		0	0	0	0	−2	0	0	0	0	0
$J_{-[12]}$	7	0	0	−2	0	0	2	−2	0	0	2
		0	0	0	0	0	−2	0	0	0	0
$J_{-(1)}$	8	0	0	0	0	0	2	0	0	0	2
		0	0	0	0	0	−2	0	0	0	0
$J_{+(12)}$	9	0	0	−2	0	0	0	−2	0	0	0
		$2k_6$	$-k_6$	$-2k_4$	0	0	0	$-2k_5$	0	2	−2
$J_{+(12)}$	10	0	0	0	0	2	−2	0	0	2	−2
		0	0	0	0	2	0	0	0	0	0
$J_{+(12)}$	11	0	0	0	2	−2	0	0	2	−2	0
		0	0	0	−4	0	0	0	0	0	0
$J_{+(2)}$	12	0	0	0	−2	0	0	0	−2	0	0
		$-k_6$	k_6	0	$-k_2$	0	0	2	$-k_4$	−2	0
$J_{+(2)}$	13	0	0	2	0	−2	0	2	0	−2	0
		0	0	2	0	0	0	0	0	0	0
$J_{+[12]}$	14	0	0	0	0	−2	0	0	0	−2	0
		0	k_6	0	0	$-2k_2$	0	−2	−4	$-2k_3$	−2
$J_{+[12]}$	15	0	0	2	0	0	−2	2	0	0	−2
		0	0	−2	0	0	0	0	0	0	0
$J_{+[12]}$	16	0	0	−2	−2	0	0	−2	−2	0	0
		$-2k_6$	k_6	$2k_4$	0	0	0	$2k_5$	0	−2	2
$J_{+[12]}$	17	0	0	0	−2	2	−2	0	−2	2	−2
		0	0	0	0	−2	0	0	0	0	0
$J_{+(1)}$	18	0	0	0	0	0	−2	0	0	0	−2
		k_6	0	0	0	0	$-k_2$	−2	0	−2	$-k_4$
$J_{+(1)}$	19	0	0	−2	0	−2	0	−2	0	−2	0
		k_6	$-k_6$	$-k_6$	0	k_4	0	$-k_6$	0	k_6	0
$J_{+(1)}$	20	0	0	0	2	−4	0	0	2	−4	0
		0	0	0	−2	0	0	0	0	0	0

Table 6a

	1	2	3	4	5	6	7	8	9	10
1	−1	0	0	1	0	0	1	0	1	0
2	0	−1	1	0	0	−1	0	0	0	−1
3	0	0	1	1	0	0	1	1	0	−1
4	0	0	−1	−1	0	0	−1	−1	0	1
5	0	1	0	1	−1	0	0	0	0	0
6	0	−1	1	0	0	−1	0	0	0	−1
7	0	0	−1	−1	0	0	−1	−1	0	1
8	0	0	−1	−1	0	0	−1	−1	0	1
9	1	−1	2	0	0	−1	0	1	−1	−2
10	0	1	−1	0	0	1	0	0	0	1
11	0	2	0	2	−2	0	0	0	0	0
12	−1	0	−1	0	1	1	1	0	1	1
13	1	0	0	−1	0	0	−1	0	−1	0
14	1	−1	0	−2	2	1	0	1	−1	0
15	−1	0	0	1	0	0	1	0	1	0
16	−1	1	−2	0	0	1	0	−1	1	2
17	0	−1	1	0	0	−1	0	0	0	−1
18	1	1	0	0	0	1	0	1	−1	0
19	0	−1	1	0	0	−1	0	0	0	−1
20	0	1	0	1	−1	0	0	0	0	0

	11	12	13	14	15	16	17	18	19	20
1	0	0	−1	0	−1	1	0	0	1	0
2	1	0	1	1	0	0	−1	0	1	2
3	0	0	0	0	−1	0	−1	−1	0	0
4	0	0	0	0	1	0	1	1	0	0
5	−1	1	0	0	0	1	1	0	0	−1
6	1	0	1	1	0	0	−1	0	1	2
7	0	0	0	0	1	0	1	1	0	0
8	0	0	0	0	1	0	1	1	0	0
9	1	0	2	1	0	−1	−2	−1	0	2
10	−1	0	−1	−1	0	0	1	0	−1	−2
11	−2	2	0	0	0	2	2	0	0	−2
12	0	−1	−2	−1	−1	0	0	0	0	−1
13	0	0	1	0	1	−1	0	0	−1	0
14	1	−2	0	−1	0	−3	−2	−1	−2	0
15	0	0	−1	0	−1	1	0	0	1	0
16	−1	0	−2	−1	0	1	2	1	0	−2
17	1	0	1	1	0	0	−1	0	1	2
18	−1	0	0	−1	0	−1	0	−1	−2	−2
19	1	0	1	1	0	0	−1	0	1	2
20	−1	1	0	0	0	1	1	0	0	−1

Table 6b

	1	2	3	4	5	6	7	8	9	10
1	1	0	0	0	0	0	0	0	k_4	0
2	0	1	0	0	0	1	0	0	0	-1
3	0	0	1	-1	0	0	-1	-1	0	0
4	0	0	-1	1	0	0	1	1	0	0
5	0	0	0	0	1	0	0	0	0	0
6	0	1	0	0	0	1	0	0	0	-1
7	0	0	-1	1	0	0	1	1	0	0
8	0	0	-1	1	0	0	1	1	0	0
9	k_4	0	0	0	0	0	0	0	1	0
10	0	-1	0	0	0	-1	0	0	0	1
11	0	0	0	0	2	0	0	0	0	0
12	0	0	0	0	k_1'	0	0	0	0	0
13	-1	0	0	0	0	0	0	0	$-k_4$	0
14	0	k_2	0	0	0	k_2	0	0	-3	$-k_2$
15	1	0	0	0	0	0	0	0	k_4	0
16	$-k_4$	0	0	0	0	0	0	0	-1	0
17	0	1	0	0	0	1	0	0	0	-1
18	0	0	$-k_1'$	k_1'	0	0	k_1'	k_1'	$-k_0'$	0
19	k_6	$-k_2'$	0	0	0	$-k_2'$	0	0	0	k_2'
20	0	0	0	0	1	0	0	0	0	0

	11	12	13	14	15	16	17	18	19	20
1	0	0	-1	0	1	$-k_4$	0	0	k_6	0
2	0	0	0	k_2	0	0	1	0	$-k_2'$	0
3	0	0	0	0	0	0	0	$-k_1'$	0	0
4	0	0	0	0	0	0	0	k_1'	0	0
5	2	k_1'	0	0	0	0	0	0	0	1
6	0	0	0	k_2	0	0	1	0	$-k_2'$	0
7	0	0	0	0	0	0	0	k_1	0	0
8	0	0	0	0	0	0	0	k_1'	0	0
9	0	0	$-k_4$	-3	k_4	-1	0	$-k_0'$	0	0
10	0	0	0	$-k_2$	0	0	-1	0	k_2'	0
11	4	k_2	0	0	0	0	0	0	0	2
12	k_2	1	0	$-k$	0	0	0	0	0	k_1'
13	0	0	1	0	-1	k_4	0	0	$-k_3'$	0
14	0	$-k$	0	1	0	3	k_2	$-k_0'$	-4	0
15	0	0	-1	0	1	$-k_4$	0	0	k_6	0
16	0	0	k_4	3	$-k_4$	1	0	k_0'	0	0
17	0	0	0	k_2	0	0	1	0	$-k_2'$	0
18	0	0	0	$-k_0'$	0	k_0'	0	1	0	0
19	0	0	$-k_3$	-4	k_6	0	$-k_2'$	0	1	0
20	2	k_1'	0	0	0	0	0	0	0	1

Table 7

Λ_0	basis vectors	$l_1 = 2p(p+1)$	$l_2 = 6p(p+1)$
Λ	$\sqrt{2}pe_1^+$	$p(p+1)$	$3p(p+1)$
	$\sqrt{2}pe_2^+$	$p(p+1)$	$-3p(p+1)$
	$\sqrt{2}(p+1)e_1^-$	$-p(p+1)$	$-3p(p+1)$
	$\sqrt{2}(p+1)e_2^-$	$-p(p+1)$	$3p(p+1)$
$\sqrt{2}\alpha_0$	$\sqrt{2}\alpha_0$	2	0
Λ_e	$\sqrt{2}e_1^+$	$p+1$	$3(p+1)$
	$\sqrt{2}e_2^+$	$p+1$	$-3(p+1)$
	$\sqrt{2}e_1^-$	$-p$	$-3p$
	$\sqrt{2}e_2^-$	$-p$	$3p$
$\Lambda_\beta \equiv \Lambda^*$	$\sqrt{2}\beta_1$	0	2
	$\sqrt{2}\beta_2$	1	1

Table 8

	$A(k_1,k_2)$	$B(k_1,k_2)$	$C(k_1,k_2)$
Q_1^+	$D(-k_1,k_1+k_2)$ $k_1 \geq 0$	$F(-k_1,k_1+k_2)$ $k_1 \geq 0$	$E(-k_1,k_1+k_2)$ $k_1 \geq 1$
Q_2^+	$E(k_1+k_2,-k_2)$ $k_2 \geq 0$	$D(k_1+k_2,-k_2)$ $k_2 \geq 1$	$F(k_1+k_2,-k_2)$ $k_2 \geq 0$
Q_3^+	$D(2-k_1,k_1+k_2-1)$ $k_1 \leq 0$	$F(2-k_1,k_1+k_2-1)$ $k_1 \leq 0$	$E(2-k_1,k_1+k_2-1)$ $k_1 \leq 1$
Q_4^+	$E(k_1+k_2-1,2-k_2)$ $k_2 \leq 0$	$D(k_1+k_2-1,2-k_2)$ $k_2 \leq 1$	$F(k_1+k_2-1,2-k_2)$ $k_2 \leq 0$
Q_5^+	$F(-k_2,-k_1)$ $k_1+k_2 \geq 0$	$E(-k_2,-k_1)$ $k_1+k_2 \geq 1$	$D(-k_2,-k_1)$ $k_1+k_2 \geq 1$
Q_6^+	$F(1-k_2,1-k_1)$ $k_1+k_2 \leq 0$	$E(1-k_2,1-k_1)$ $k_1+k_2 \leq 1$	$D(1-k_2,1-k_1)$ $k_1+k_2 \leq 1$

	$D(k_1,k_2)$	$E(k_1,k_2)$	$F(k_1,k_2)$
Q_1^+	$A(-k_1,k_1+k_2)$ $k_1 \geq 1$	$C(-k_1,k_1+k_2)$ $k_1 \geq 0$	$B(-k_1,k_1+k_2)$ $k_1 \geq 1$
Q_2^+	$B(k_1+k_2,-k_2)$ $k_2 \geq 0$	$A(k_1+k_2,-k_2)$ $k_2 \geq 1$	$C(k_1+k_2,-k_2)$ $k_2 \geq 1$
Q_3^+	$A(2-k_1,k_1+k_2-1)$ $k_1 \leq 1$	$C(2-k_1,k_1+k_2-1)$ $k_1 \leq 0$	$B(2-k_1,k_1+k_2-1)$ $k_1 \leq 1$
Q_4^+	$B(k_1+k_2-1,2-k_2)$ $k_2 \leq 0$	$A(k_1+k_2-1,2-k_2)$ $k_2 \leq 1$	$C(k_1+k_2-1,2-k_2)$ $k_2 \leq 1$
Q_5^+	$C(-k_2,-k_1)$ $k_1+k_2 \geq 0$	$B(-k_2,-k_1)$ $k_1+k_2 \geq 0$	$A(-k_2,-k_1)$ $k_1+k_2 \geq 1$
Q_6^+	$C(1-k_2,1-k_1)$ $k_1+k_2 \leq 0$	$B(1-k_2,1-k_1)$ $k_1+k_2 \leq 0$	$A(1-k_2,1-k_1)$ $k_1+k_2 \leq 1$

Table 9

	$A(k_1,k_2)$	$B(k_1,k_2)$	$C(k_1,k_2)$
Q_1^-	$D(k_1,k_2)$ $k_1 \le 0$	$E(k_1,k_2)$ $k_1+k_2 \ge 1$	$F(k_1,k_2)$ $k_2 \le 0$
Q_2^-	$E(k_1,k_2)$ $k_2 \le 0$	$F(k_1,k_2)$ $k_1 \le 0$	$D(k_1,k_2)$ $k_1+k_2 \ge 1$
Q_3^-	$D(k_1+2,k_2-1)$ $k_1 \ge 0$	$E(k_1-1,k_2-1)$ $k_1+k_2 \le 1$	$F(k_1-1,k_2+2)$ $k_2 \ge 0$
Q_4^-	$E(k_1-1,k_2+2)$ $k_2 \ge 0$	$F(k_1+2,k_2-1)$ $k_1 \ge 0$	$D(k_1-1,k_2-1)$ $k_1+k_2 \le 1$
Q_5^-	$F(k_1,k_2)$ $k_1+k_2 \le 0$	$D(k_1,k_2)$ $k_2 \ge 1$	$E(k_1,k_2)$ $k_1 \ge 1$
Q_6^-	$F(k_1+1,k_2+1)$ $k_1+k_2 \ge 0$	$D(k_1+1,k_2-2)$ $k_2 \le 1$	$E(k_1-2,k_2+1)$ $k_1 \le 1$

	$D(k_1,k_2)$	$E(k_1,k_2)$	$F(k_1,k_2)$
Q_1^-	$A(k_1,k_2)$ $k_1 \ge 1$	$B(k_1,k_2)$ $k_1+k_2 \le 0$	$C(k_1,k_2)$ $k_2 \ge 1$
Q_2^-	$C(k_1,k_2)$ $k_1+k_2 \le 0$	$A(k_1,k_2)$ $k_2 \ge 1$	$B(k_1,k_2)$ $k_1 \ge 1$
Q_3^-	$A(k_1-2,k_2+1)$ $k_1 \le 1$	$B(k_1+1,k_2+1)$ $k_1+k_2 \ge 0$	$C(k_1+1,k_2-2)$ $k_2 \le 1$
Q_4^-	$C(k_1+1,k_2+1)$ $k_1+k_2 \ge 0$	$A(k_1+1,k_2-2)$ $k_2 \le 1$	$B(k_1-2,k_2+1)$ $k_1 \le 1$
Q_5^-	$B(k_1,k_2)$ $k_2 \le 0$	$C(k_1,k_2)$ $k_1 \le 0$	$A(k_1,k_2)$ $k_1+k_2 \ge 1$
Q_6^-	$B(k_1-1,k_2+2)$ $k_2 \ge 0$	$C(k_1+2,k_2-1)$ $k_1 \ge 0$	$A(k_1-1,k_2-1)$ $k_1+k_2 \le 1$

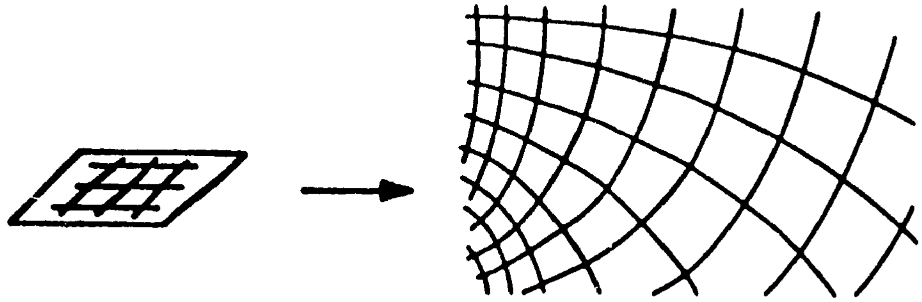


Figure 1.

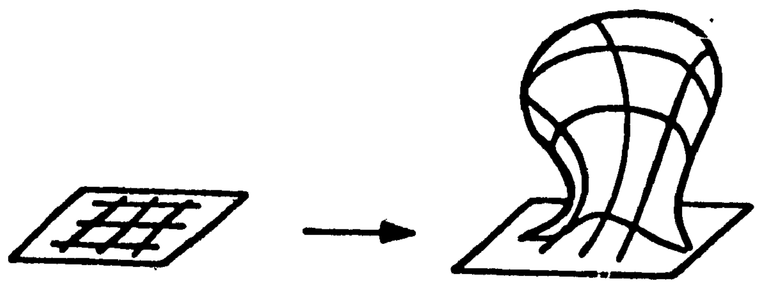
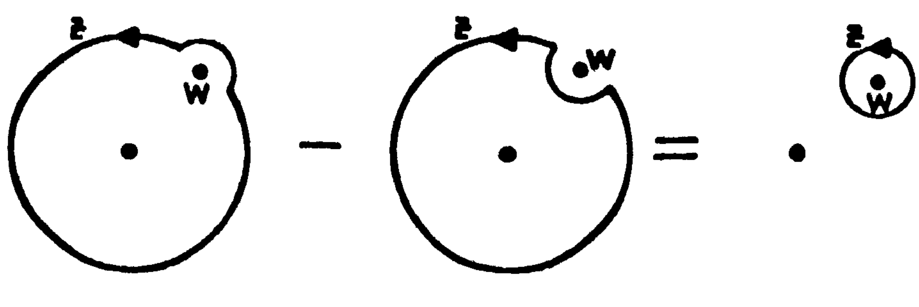


Figure 2.



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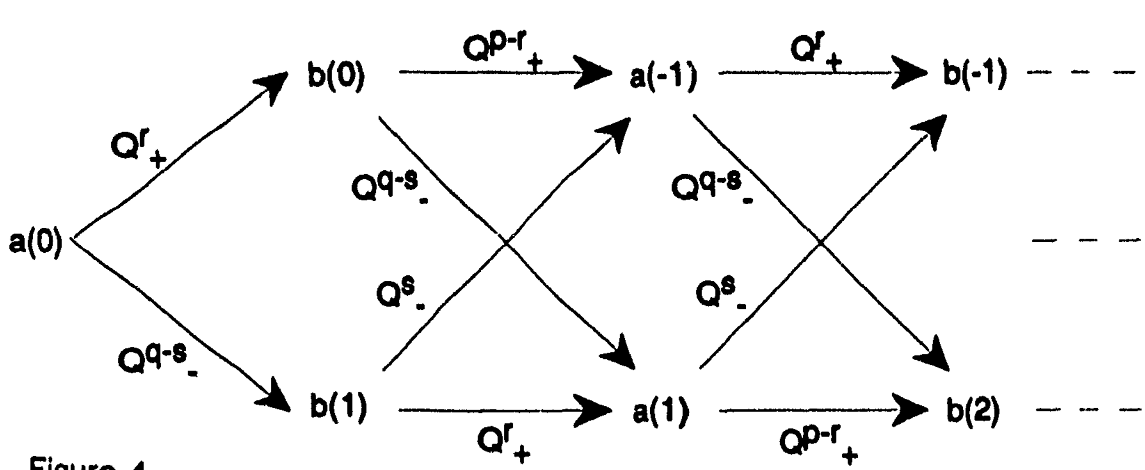

Figure 4.

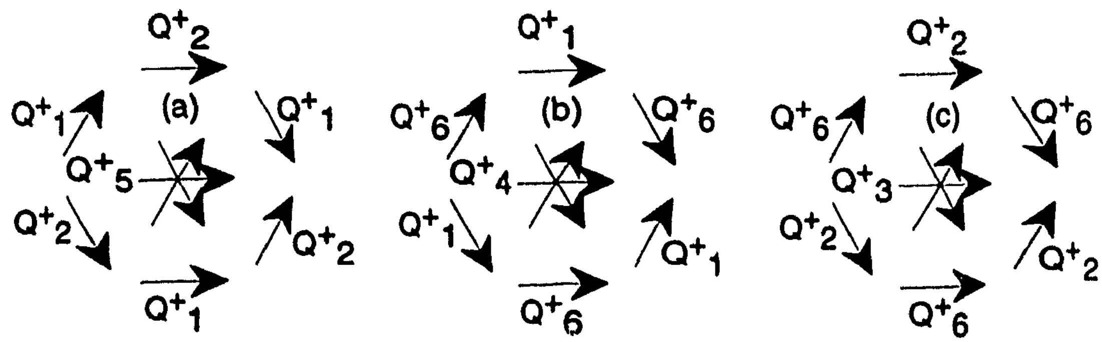

Figure 5.

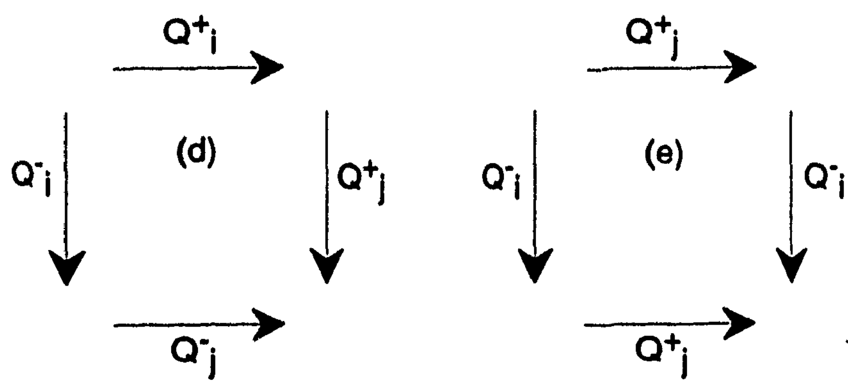

Figure 6.

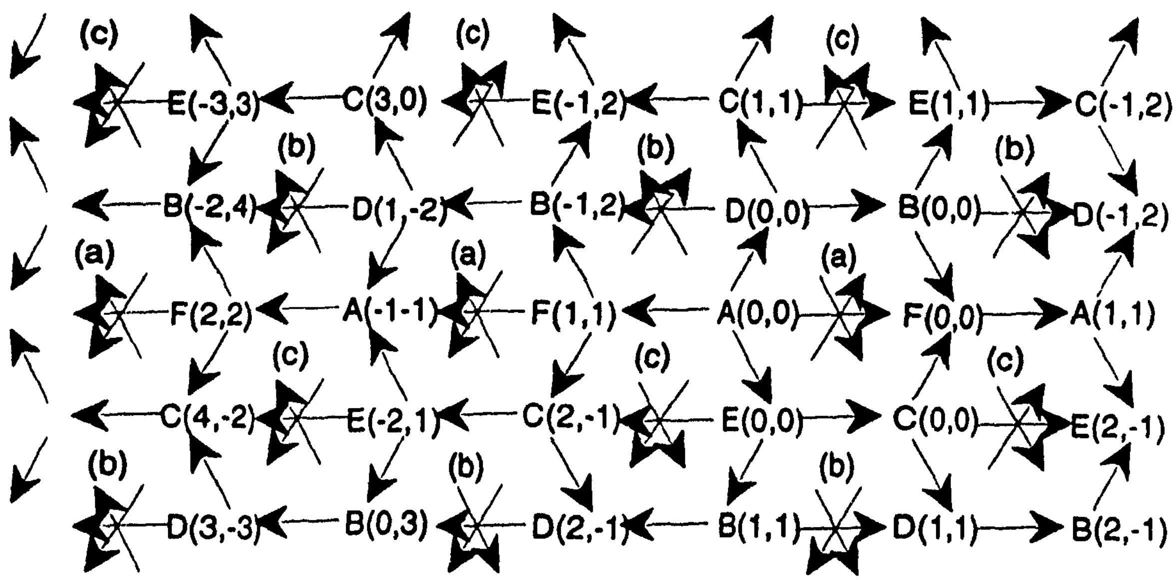

Figure 7.

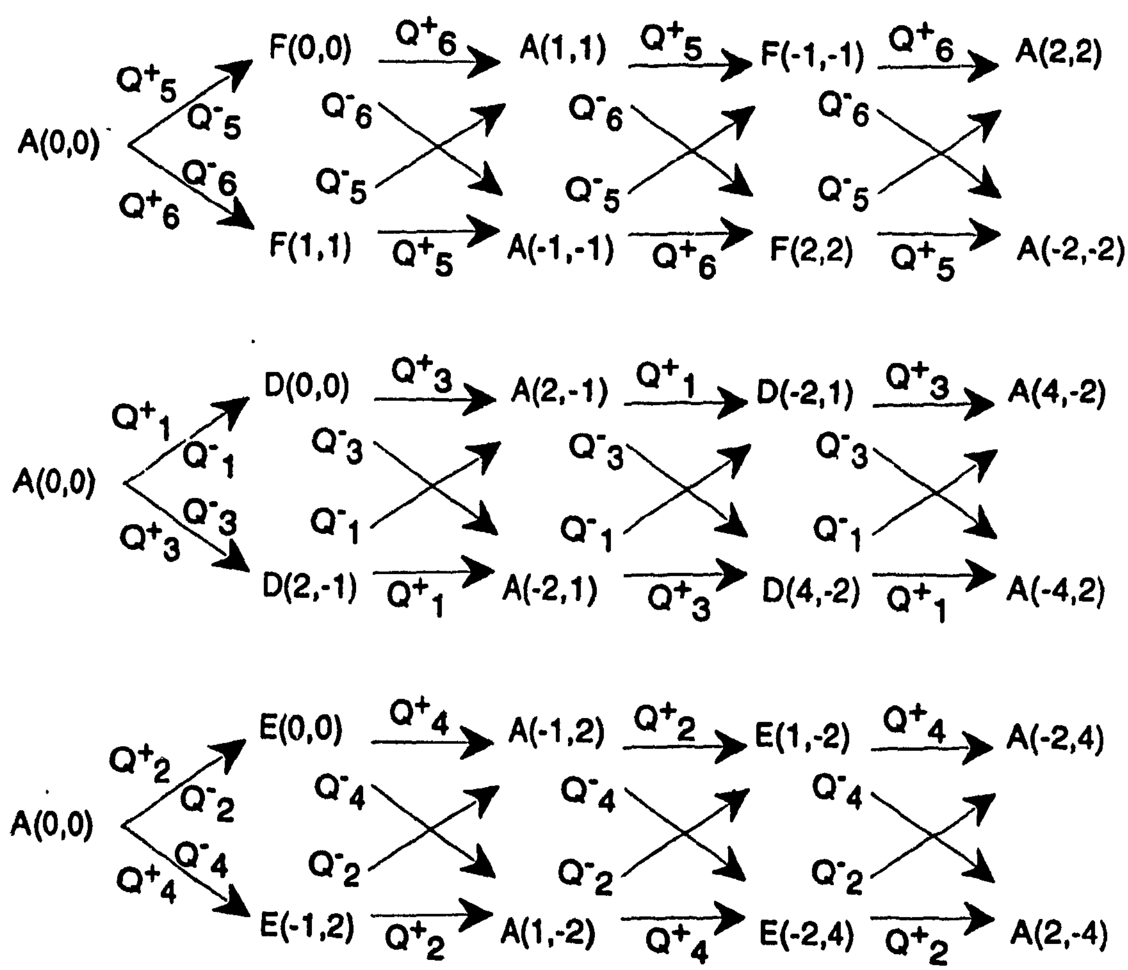

Figure 8.

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