COTORSION THEORIES, TORSION THEORIES OVER PERFECT RINGS

COTORSION THEORIES AND TORSION THEORIES OVER PERFECT RINGS

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Abstract

Let R be an associative ring with unit element, and let P be a projective right R-module with $E = \mathrm{End}_R(P)$. A cotorsion theory in Mod-R associated with P is defined, and for any M in Mod-R a colocalization of M at P is then defined which is unique up to isomorphism. It is shown that this colocalization of M at P is $\mathrm{Hom}_R(P, M) \otimes_E P$. Equivalent conditions are given for the colocalization functor to be exact. The colocalization of R is an associative ring, in general without unit element, and equivalent conditions are given for it to have a two-sided (left, right) unit element. It is shown that this colocalization coincides with the localization in $(\text{Mod-R})^{op}$ given by Lambek and Rattray. The localization in Mod-R associated with the trace ideal T of P is investigated, and this leads to a characterization of left perfect rings.

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Résumé

Soit R un anneau associatif unitaire, et P un R-module projectif à droite. On établit une théorie de cotorsion de Mod-R associée à P. Ainsi, pour tout M de Mod-R, on peut définir un colocalisé de M à P, qui est unique sauf par isomorphisme. On démontre que ce colocalisé de M à P est $\mathrm{Hom}_R(P, M) \otimes_E P$. Ensuite, on obtient des conditions équivalentes pour que le foncteur colocalisation soit exact. Le colocalisé de R étant un anneau associatif, en général sans élément unitaire, on donne des conditions équivalentes pour qu'il possède un élément unitaire bilatéral (à gauche, à droite). De plus, on démontre que cette colocalisation correspond à la localisation de $(\text{Mod-R})^{op}$ utilisée par Lambek et Rattray. Finalement, on étudie la localisation de Mod-R associée à T, l'idéal de trace de P; on est alors amené à charactériser les anneaux parfaits à gauche.

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INTRODUCTION

The primary aim of this thesis is to develop and investigate a theory of colocalization in the category Mod-R of right R-modules, where R is an associative ring with unit element. If R is left perfect, there is a natural relationship between the usual theory of localization with respect to a torsion theory in Mod-R, and this colocalization.

Chapter I includes some fundamental results on torsion theories in Mod-R, and establishes the terminology used throughout. Cotorsion theories in Mod-R are introduced by defining cotorsion modules, cotorsionfree modules, codivisible modules, and codivisible covers of modules. The colocalization of a given module is then defined uniquely up to isomorphism, and an explicit form is given for this colocalization.

In Chapter II the category of cotorsionfree codivisible modules is studied, and in Chapter III equivalent conditions are given for the colocalization functor, regarded as an endofunctor of Mod-R, to be exact. In Chapter IV a multiplication is defined on the colocalization of R which gives it the structure of an associative ring, but in general without unit element. Equivalent conditions are given for it to have a two-sided (left, right) unit element. The colocalization of a given module is then compared with that obtained by Lambek and Rattray in their work on localization at injectives in complete categories [25].

Given a colocalization functor in Mod-R, there is an obvious localization functor associated with it, and in Chapter V it is shown that with a certain condition (which is satisfied by R left perfect) on the colocalization functor, the localization is given by a dual construction. This leads to an investigation of rings for which this condition is always satisfied, and a characterization of left perfect rings is obtained. A brief exposition on torsion theories in Mod-R for R semiperfect and left perfect, respectively, which is relevant to this discussion, has been included in the form of an appendix since the results contained in it are not original, even though the proofs, except as noted, are due to the author. In Chapter VI, chain conditions on the localization of R are investigated under certain conditions on the localization functor, generalizing results obtained by Storrer [37] for the maximal (Utumi, Lambek) ring of quotients.

To mention all previous relevant work would be a formidable task, considering the vast literature on torsion theories in Mod-R. Here, as in the bibliography, reference is made only to research directly connected to this investigation.

Courter [10] dualized rational extensions of modules, and defined a maximal co-rational extension over a module. He showed that if a module M has a projective cover $\varphi: P(M) \longrightarrow M$, then up to isomorphism $P(M)/X$ is the maximal co-rational extension of M, where $X = \sum_{f \in Hom_R(P(M), K)} f(P(M))$ and $K = Ker\ \varphi$. We shall see that this is a certain codivisible cover of M.

Sandomiersky [35] has defined a module M to be T-accessible if $MT = M$, where T is the trace ideal of a projective right R-module P, and has showed that there exists a one-to-one inclusion preserving correspondence between the T-accessible submodules of a module M, and the submodules of $Hom_R({}_EP_R, M_R)_E$ where $E = End_R(P)$. A T-accessible module is simply a cotorsionfree module in the cotorsion theory determined by P.

Miller [28,29] has also studied T-accessible modules, and defined a module M to be strongly T-accessible if every submodule of M is T-accessible, and defined P to be a quasi-generator if every T-accessible module is strongly T-accessible, i.e. if the class of cotorsionfree modules is closed under submodules.

Beachy [4] has called a subfunctor ρ of the identity functor on Mod-R a cotorsion radical if the dual of the functor $1/\rho$: $M \longrightarrow M/\rho(M)$ is a torsion radical for $(\text{Mod-R})^{op}$. He showed that this is equivalent to ρ being an idempotent radical and every factor module of a ρ-torsionfree module being ρ-torsionfree. Hence what has been called a TTF class of modules is the torsionfree class of a cotorsion radical, and conversely the torsionfree class of a cotorsion radical is a TTF class. With our definition of cotorsion theories there is not in general a one-to-one correspondence between cotorsion theories and TTF classes, unless for the given ring R every idempotent ideal of R is the trace ideal of a projective module.

Ulmer [39] has considered a set M of objects in a Grothendieck category $\underline{A}$. If $J: \underline{\tilde{M}} \longrightarrow \underline{A}$ denotes the inclusion of the full subcategory of $\underline{A}$ consisting of all finite coproducts of objects of M, then the functor : $\underline{A} \longrightarrow [\underline{\tilde{M}}^{op}, \underline{Ab}.\underline{Gr}.]$, $A \longmapsto [J(\_), A]$ where $[\underline{\tilde{M}}^{op}, \underline{Ab}.\underline{Gr}.]$ is the category of contravariant additive functors on $\underline{\tilde{M}}$ with values in the category $\underline{Ab}.\underline{Gr}.$ of abelian groups, has a left adjoint denoted by $\otimes M$. The full subcategory of $\underline{A}$ consisting of all fixpoints of of the composite $\underline{A} \longrightarrow [\underline{\tilde{M}}^{op}, \underline{Ab}.\underline{Gr}.] \longrightarrow \underline{A}$, $A \longmapsto [J(\_), A] \otimes M$, i.e. all objects $X \in \underline{A}$ such that the evaluation morphism $\epsilon(X)$: $[J(\_), X] \otimes M \longrightarrow X$ is an isomorphism, corresponds to a subcategory of cotorsionfree codivisible modules if we let $\underline{A}$ = Mod-R and $M_0 = \{P\}$ where P is a projective right R-module.

The author has just recently seen a paper by Bland [6] in which he defines codivisible modules and codivisible covers of modules, but in the context of a pre-torsion theory rather than a cotorsion theory.

Lambek and Rattray [25] have studied localizations at injectives in complete categories. Their work has been the inspiration for investigating colocalizations in Mod-R, i.e. localizations in $(\text{Mod-R})^{op}$.

Finally, the term "cotorsion" is not a new one. If R is an integral domain with quotient field Q then an R-module C has been called cotorsion if $\text{Hom}_R(Q,C) = 0$ and $\text{Ext}^1(Q,C) = 0$ (see e.g. Matlis [26]).

With ~~the~~ exception of the Appendix and the background material at the beginning of Chapter I, all results and proofs are due to the author unless indicated otherwise, and to the best of his knowledge are original.

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CHAPTER I

Cotorsion Theories and Colocalization

Let R be an associative ring with unit element. Mod-R and R-Mod will denote the categories of unitary right and left R-modules, respectively, and all modules are assumed to be in Mod-R unless otherwise specified. $\forall M, N \in$ Mod-R, $\mathrm{Hom}_R(M, N)$ will usually be abbreviated as $[M, N]$. For an exposition on torsion theories in Mod-R the reader is referred to Lambek [22]. However, the definitions of the basic terms are repeated here to avoid any confusion arising from the existence in the literature of a somewhat different way of defining the same terms.

If $\mathcal{B}$ and $\mathcal{C}$ are classes of modules, let $\mathcal{B}^{r} = \{X \in \text{Mod-R} \mid [B, X] = 0 \ \forall B \in \mathcal{B}\}$, and $\mathcal{C}^{\ell} = \{X \in \text{Mod-R} \mid [X, C] = 0 \ \forall C \in \mathcal{C}\}$. A class $\mathcal{B}$ of modules is called a <u>pre-torsion class</u> if it is closed under homomorphic images, direct sums, group extensions (i.e. whenever N is a submodule of M, and N and M/N are in $\mathcal{B}$, then so is M), and isomorphic images. If $\mathcal{C} = \mathcal{B}^{r}$, then $\mathcal{C}$ is closed under submodules, direct products, group extensions, and isomorphic images, i.e. it is what is called a <u>pre-torsionfree class</u>, and the pair $(\mathcal{B}, \mathcal{C})$ is called a <u>pre-torsion theory</u>. If $(\mathcal{B}, \mathcal{C})$ is a pre-torsion theory then $\mathcal{C} = \mathcal{B}^{r}$ and $\mathcal{B} = \mathcal{C}^{\ell}$, and in fact this also defines a pre-torsion theory. A pre-torsion class is closed under submodules if and only if the corresponding pre-torsionfree class is closed under injective hulls. A pre-torsion class $\mathfrak{T}$ which is closed under submodules is called a *torsion class*, a pre-torsionfree class $\mathfrak{F}$ which is closed under injective hulls is called a *torsionfree class*, and a pre-torsion theory $(\mathfrak{T}, \mathfrak{F})$ in which $\mathfrak{T}$ is a torsion class is called a *torsion theory*.

Some authors, notably Dickson [15] and Jans [20], have called a pre-torsion theory a torsion theory, and a torsion theory a hereditary torsion theory. Jans [20] has also called a class of modules which is closed under submodules, direct products, homomorphic images, group extensions, and isomorphic images a TTF (torsion-torsionfree) class. Since such a class $\mathfrak{T}$ is not closed under injective hulls, we find this terminology misleading, and shall instead (following a suggestion by J. Golan) call $\mathfrak{T}$ a *Jansian class* from now on. A torsion class $\mathfrak{T}$ which is closed under injective hulls is called *stable*, and hence a stable Jansian class is a true torsion-torsionfree class.

If $(\mathfrak{T}, \mathfrak{F})$ is a torsion theory then modules in $\mathfrak{T}$ are called *torsion*, and modules in $\mathfrak{F}$ are called *torsionfree*. Each $M \in$ Mod-R has a unique maximal torsion submodule, denoted by $\mathfrak{T}(M)$. (It is the unique submodule $X \subseteq M$ such that X is torsion and M/X is torsionfree.) A submodule D of M is called *dense* if M/D is torsion. Let $\mathfrak{D}_{\mathfrak{T}}$ denote the set of all dense right ideals of R. $\mathfrak{D}_{\mathfrak{T}}$ forms an *idempotent* (or *Gabriel*) *filter*, i.e. it satisfies the following conditions:

(0) $R \in \mathfrak{D}_{\mathfrak{T}}$

(1) $D \in \mathfrak{D}_{\mathfrak{T}}$ and $D \subseteq K \Rightarrow K \in \mathfrak{D}_{\mathfrak{T}}$

(2) $D \in \mathfrak{D}_{\mathfrak{T}}$ and $r \in R \Rightarrow (r:D) \in \mathfrak{D}_{\mathfrak{T}}$, where $(r:D) = \{x \in R \mid rx \in D\}$

(3) $D \in \mathfrak{D}_{\mathfrak{T}}$ and $(d:K) \in \mathfrak{D}_{\mathfrak{T}}$ $\forall d \in D \Rightarrow D \cap K \in \mathfrak{D}_{\mathfrak{T}}$

Gabriel [17] has showed that there is a one-to-one correspondence between torsion classes in Mod-R and idempotent filters of right ideals of R: to a torsion class $\mathfrak{T}$ associate the idempotent filter $\mathfrak{D}_{\mathfrak{T}}$, and to an idempotent filter $\mathfrak{D}$ associate the torsion class $\mathfrak{T}_{\mathfrak{D}} = \{M \in \text{Mod-}R \mid (m:0) \in \mathfrak{D} \ \forall m \in M\}$.

Jans [20] showed that a torsion class $\mathfrak{T}$ is a Jansian class if and only if $\mathfrak{D}_{\mathfrak{T}}$ contains a unique minimal right ideal T, in which case T is an idempotent two-sided ideal, and $T = C(R)$ where $(C, \mathfrak{T})$ is the pre-torsion theory with $\mathfrak{T}$ as the pre-torsionfree class. Thus there is a one-to-one correspondence between Jansian classes and idempotent ideals of R, with the inverse correspondence given by $T \longrightarrow \{M \in \text{Mod-}R \mid MT = 0\}$.

Given an injective module I_R, one can form the largest torsion theory for which I is torsionfree (where $(\mathfrak{T}, \mathfrak{F}) \subseteq (\mathfrak{T}', \mathfrak{F}')$ if $\mathfrak{T} \subseteq \mathfrak{T}'$), and in fact every torsion theory is of this form for some injective I. For a given torsion theory $(\mathfrak{T}, \mathfrak{F})$, a module M is called <u>divisible</u> (or <u>$\mathfrak{T}$-injective</u>) if $I(M)/M \in \mathfrak{F}$, where I(M) denotes the injective hull of M. Every module M has a <u>divisible hull</u> D(M) defined by $D(M)/M = \mathfrak{T}(I(M)/M)$. One also defines the <u>quotient module</u> Q(M) of M by $Q(M) = D(M/\mathfrak{T}(M))$. This is also called the <u>localization</u> of M at I, where I is an injective module such that $(\mathfrak{T}, \mathfrak{F})$ is the largest torsion theory for which I is torsionfree. Q(R) is a ring, and the canonical mapping $R \longrightarrow Q(R)$ is a ring homomorphism. It is also well known that every torsionfree divisible module is a right $Q(R)$-module, and every R-homomorphism between torsionfree divisible modules is a $Q(R)$-homomorphism (see [22, Sec. 1]).

Let P_R be a projective module, let $E = [P, P]$, and let $P^* = [P, R]$. As mentioned above, every torsion theory can be thought of as the largest torsion theory for which some injective module I_R is torsionfree, where a module M is torsion if and only if $[M, I] = 0$. We duallize this in the following definitions:

DEFINITION 1.1. (a) A module M is <u>cotorsion</u> if $[P, M] = 0$.

(b) A module M is <u>cotorsionfree</u> if $[M, X] = 0 \ \forall X$ cotorsion.

(c) If $\mathfrak{J}^*$ denotes the class of cotorsion modules, and $\mathfrak{F}^*$ the class of cotorsionfree modules, then $(\mathfrak{F}^*, \mathfrak{J}^*)$ is a <u>cotorsion theory</u>.

(d) $\epsilon(M)$ is the evaluation mapping $[P, M] \otimes_E P \longrightarrow M$, i.e. $\epsilon(M)(\Sigma g_i \otimes p_i) = \Sigma g_i(p_i)$.

(e) $T = \epsilon(R)(P^* \otimes_E P)$, the trace ideal of P.

LEMMA 1.2. $M \in$ Mod-R is cotorsion if and only if $MT = 0$.

<u>Proof</u>: Suppose $[P, M] = 0$, then $\forall p \in P$, $\forall m \in M$, $\forall f \in P^*$, $mf(p) = 0$ since $mf \in [P, M]$, and hence $MT = 0$. Conversely, if $MT = 0$ then $\forall g \in$

$[P, M]$, $g(P) = g(PT) = g(P)T \subseteq MT = 0$, and therefore M is cotorsion.

PROPOSITION 1.3. $\forall M \in$ Mod-R, the following conditions are equivalent:

(1) M is cotorsionfree

(2) $MT = M$

(3) $M \otimes_R R/T = 0$

(4) $\epsilon(M)$ is an epimorphism

(5) M is an epimorphic image of a direct sum of copies of P

Proof: (1) ⇒ (2) M/MT is cotorsion since $(M/MT)T = 0$, hence the projection mapping $M \longrightarrow M/MT = 0$.

(2) ⇔ (3) $M/MT \cong M \otimes_R R/T$

(2) ⇒ (4) $\forall m \in M$, $\forall t \in T$, $mt = m\Sigma f_i(p_i)$ for some $f_i \in P^*$, $p_i \in P$, $i = 1, \ldots, n$

$$= \Sigma mf_i(p_i) \in \operatorname{Im} \epsilon(M), \text{ since } mf_i \in [P, M],$$

and hence $M = MT \subseteq \operatorname{Im} \epsilon(M)$, i.e. $\epsilon(M)$ is an epimorphism.

(4) ⇔ (5) clear

(4) ⇒ (2) $\epsilon(M)([P, M] \otimes_E P) = \epsilon(M)([P, M] \otimes_E PT)$

$$= \epsilon(M)([P, M] \otimes_E P)T \subseteq MT$$

Therefore $\epsilon(M)$ an epimorphism ⇒ $MT = M$.

(2) ⇒ (1) $\forall X$ cotorsion, $\forall \varphi \in [M, X]$, $\varphi(M) = \varphi(MT) = \varphi(M)T \subseteq XT = 0$.

Therefore M is cotorsionfree.

It is clear from the proof that $\operatorname{Im} \epsilon(M) = MT$, and of course $[P, M] \otimes_E P$

is cotorsionfree. The class $\mathfrak{J}^*$ of cotorsion modules is closed under submodules, direct products, homomorphic images, group extensions, and isomorphic images, i.e. it is a Jansian class. The class $\mathfrak{F}^*$ of cotorsionfree modules is closed under homomorphic images, direct sums, group extensions, isomorphic images, and by [34, Prop. 1] minimal extensions (and hence projective covers if they exist).

DEFINITION 1.4. A module M is <u>codivisible</u> if for any homomorphism φ: $B \longrightarrow A$ such that Ker φ is cotorsion, any homomorphism $M \longrightarrow A$ can be extended to a homomorphism $M \longrightarrow B$, i.e.

$$\begin{tikzcd} & M \arrow[dl, dashed, "\exists\psi'"'] \arrow[dr, "\forall\psi"] & \\ B \arrow[rr, "\varphi"] & & A \end{tikzcd}$$

PROPOSITION 1.5. $\forall M \in$ Mod-R, $[P, M] \otimes_E P$ is codivisible.

<u>Proof</u>: Let φ: $B \longrightarrow A$ be any epimorphism such that Ker φ is cotorsion. Let ψ be any homomorphism: $[P, M] \otimes_E P \longrightarrow A$. Define ψ_f: $P \longrightarrow A$ by $\psi_f(p) = \psi(f \otimes p)$ $\forall f \in [P, M]$, $\forall p \in P$. Then since P is projective there exists ψ_f': $P \longrightarrow B$ such that $\varphi\psi_f' = \psi_f$. Define α: $[P, M] \times_E P \longrightarrow B$ by $\alpha((f, p)) = \psi_f'(p)$. Since P is projective and $[P, \text{Ker}\,\varphi] = 0$, $[P, B] \cong [P, A]$, and it is now easily shown that α is bilinear. Therefore there exists ψ': $[P, M] \otimes_E P \longrightarrow B$ such that $\varphi\psi'(\Sigma f_i \otimes p_i) =$

$$\varphi(\Sigma\,\psi_{f_i}'(p_i)) = \Sigma\,\psi_{f_i}(p_i) = \Sigma\,\psi(f_i \otimes p_i) = \psi(\Sigma\, f_i \otimes p_i) \quad \forall \Sigma\, f_i \otimes p_i \in [P, M] \otimes_E P.$$

Thus $\varphi\psi' \overset{\wedge}{=} \psi$, and hence $[P, M] \otimes_E P$ is codivisible.

PROPOSITION 1.6. $\forall M \in$ Mod-R, Ker $\epsilon(M)$ is cotorsion.

<u>Proof</u>: Let $\Sigma\, f_i \otimes p_i \in [P, M] \otimes_E P$ such that $\epsilon(M)(\Sigma\, f_i \otimes p_i) = \Sigma\, f_i(p_i) = 0$. Then $\forall f \in P^*$, $\forall p \in P$, $(\Sigma\, f_i \otimes p_i)f(p) = \Sigma\, f_i \otimes p_i f(p) = \Sigma\, f_i p_i f \otimes p = 0$, since $\forall x \in P$, $(\Sigma\, f_i p_i f)(x) = \Sigma\, f_i(p_i f(x)) = \Sigma\,(f_i(p_i))f(x) = (\Sigma\, f_i(p_i))f(x) = 0$. Therefore $(\Sigma\, f_i \otimes p_i)T = 0$, and Ker $\epsilon(M)$ is cotorsion.

COROLLARY 1.7. P is a generator $\Leftrightarrow$ $\epsilon(M)$ is an isomorphism $\forall M \in$ Mod-R.

<u>Proof</u>: P is a generator $\Leftrightarrow T = R$, i.e. $\epsilon(R)$ is an epimorphism

$\Leftrightarrow$ Ker $\epsilon(M) = 0$ and $MT = M$ $\forall M \in$ Mod-R

$\Leftrightarrow$ $\epsilon(M)$ is an isomorphism $\forall M \in$ Mod-R

The next theorem is due mainly to Miller [29, Thm. 2.1], in particular the equivalence of statements (2) to (7). (2) $\Leftrightarrow$ (5) was also proved by Azumaya [2, Thm. 6], along with several more equivalent statements. (2) $\Leftrightarrow$ (8) was proved independently of any knowledge of the dual result given by Teply [38, Thm. 3.1] for torsion theories, but (2) $\Leftrightarrow$ (9) is the result of duallizing another equivalent statement from Teply's theorem. First we need a lemma.

LEMMA 1.8. Let $\mathfrak{H} = \{X \in \text{Mod-}R \mid X'T = X' \ \forall X' \subseteq X\}$. Then $X \in \mathfrak{H}$ if and only if $x \in xT \ \forall x \in X$. Also, $\mathfrak{H}$ is a torsion class.

Proof: Let $X \in \mathfrak{H}$, then $\forall x \in X$, $xR = xRT = xT$, and therefore $x \in xT$. Conversely, let $X' \subseteq X$, then $\forall x \in X'$, $x \in xT$ and hence $X' = X'T$. Thus $X \in \mathfrak{H}$. The only non-trivial step in proving that $\mathfrak{H}$ is a torsion class is to show that it is closed under direct sums, and this is done by an argument given by Chase [9, Prop. 2.2]. Let $X = \bigoplus_{i \in I} X_i$, where $X_i \in \mathfrak{H} \ \forall i \in I$. Let $X' \subseteq X$, and let $x_{i_1} + \ldots + x_{i_n} \in X'$. We will show by induction on n that $\exists t \in T$ such that $x_{i_j} = x_{i_j} t \ \forall j = 1, \ldots, n$. It is true for $n = 1$ since each $X_i \in \mathfrak{H}$. Assume it is true for $n = k-1$, and let $t_k \in T$ such that $x_{i_k} = x_{i_k} t_k$. Then $\exists t' \in T$ such that $x_{i_j} - x_{i_j} t_k = (x_{i_j} - x_{i_j} t_k)t' \ \forall j = 1, \ldots, k-1$. Let $t = t' - t_k t' + t_k$, then $x_{i_j} t = x_{i_j} t' - x_{i_j} t_k t' + x_{i_j} t_k = x_{i_j} \ \forall j = 1, \ldots, k-1$, and $x_{i_k} t = x_{i_k} t' - x_{i_k} t_k t' + x_{i_k} t_k = x_{i_k}$. Hence it is true for all n, and therefore $X' \in \mathfrak{H}$, since $x_{i_1} + \ldots + x_{i_n} \in (x_{i_1} + \ldots + x_{i_n})T$.

THEOREM 1.9. The following statements are equivalent:

(1) $\mathfrak{T}^*$, the class of cotorsion modules, is closed under injective hulls.

(2) $\mathfrak{F}^*$, the class of cotorsionfree modules, is closed under submodules, i.e. $\mathfrak{F}^* = \mathfrak{H}$.

(3) $P \in \mathfrak{H}$

(4) $T \in \mathfrak{H}$

(5) R/T is flat as a left R-module.

(6) $(p:0) + T = R$

(7) $(t:0) + T = R$

(8) Every cotorsionfree module is codivisible.

(9) $F: M \longrightarrow M/MT \ \forall M \in \text{Mod-}R$ is an exact functor.

Proof: (1) ⇔ (2) well known

(2) ⇒ (7) Since $\mathfrak{J}^* = \mathfrak{H}$, $\mathfrak{J}^*$ is a torsion class by Lemma 1.8, and thus has a corresponding idempotent filter $\mathfrak{D}_{\mathfrak{J}}*$. Since $T \in \mathfrak{J}^*$, $(t:0) \in \mathfrak{D}_{\mathfrak{J}}* \ \forall t \in T$, i.e. $R/(t:0) \in \mathfrak{J}^*$ and hence $(t:0) + T = R$.

(7) ⇒ (5) $R = (t:0) + T \ \forall t \in T$, and therefore $1 = x + t'$ for some $x \in (t:0)$ and $t' \in T$, $\forall t \in T$. Hence $t = tx + tt' = tt' \in tT$, $\forall t \in T$, and ${}_R(R/T)$ is flat by [9, Prop. 2.2].

(5) ⇒ (2) Let $X \in \mathfrak{J}^*$, then $\forall X' \subseteq X$, $0 \longrightarrow X' \otimes_R R/T \longrightarrow X \otimes_R R/T$ is exact since ${}_R(R/T)$ is flat. But then $X' \otimes_R R/T = 0$ since $X \otimes_R R/T = 0$ by Proposition 1.3, and $X' \in \mathfrak{J}^*$. Therefore $\mathfrak{J}^* = \mathfrak{H}$.

(3) ⇔ (6) By Lemma 1.8, $\forall p \in P \ \exists t \in T$ such that $p = pt$. Therefore $p(1 - t) = 0$, i.e. $(1 - t) \in (p:0)$, and $R = (p:0) + T \ \forall p \in P$. Conversely, if $(p:0) + T = R \ \forall p \in P$, then $1 = x + t$ for some $x \in (p:0)$ and $t \in T$, $\forall p \in P$. Hence $p = px + pt = pt \in pT \ \forall p \in P$, and $P \in \mathfrak{H}$ by Lemma 1.8.

(4) ⇔ (7) This is proved in the same way as (3) ⇔ (6).

(2) ⇒ (3) clear

(3) ⇒ (2) Let $X \in \mathfrak{J}^*$, then by Proposition 1.3 X is an epimorphic image of a direct sum of copies of P. But $P \in \aleph$ and $\aleph$ is a torsion class, hence $X \in \aleph$ and $\mathfrak{J}^* = \aleph$.

(2) ⇒ (9) Let $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ be an exact sequence in Mod-R. Then $A/AT \xrightarrow{f'} B/BT \xrightarrow{g'} C/CT \longrightarrow 0$ is always exact. Suppose $f'(a + AT) = 0$, i.e. $f(a) \in BT$, for some $a \in A$. Then since $\mathfrak{J}^*$ is closed under submodules, $f(a)R = f(a)RT = f(a)T$, and therefore $\exists t \in T$ such that $f(a) = f(a)t = f(at)$. But f is a monomorphism, and hence $a = at$, i.e. $a + AT = 0$, and f' is a monomorphism.

(9) ⇒ (8) $0 \longrightarrow \text{Ker } \epsilon(M) \longrightarrow [P, M] \otimes_E P \xrightarrow{\epsilon(M)} MT \longrightarrow 0$ is an exact sequence $\forall M \in$ Mod-R, and therefore, in particular,

$0 \longrightarrow \text{Ker } \epsilon(M)/(\text{Ker } \epsilon(M))T \longrightarrow [P, M] \otimes_E P/([P, M] \otimes_E P)T$ is exact.

But $[P, M] \otimes_E P$ is cotorsionfree, and hence so is Ker $\epsilon(M)$. By Proposition 1.6 Ker $\epsilon(M)$ is also cotorsion, and thus it is zero. Therefore $MT \cong [P, M] \otimes_E P$, and hence is codivisible by Proposition 1.5.

(8) ⇒ (1) Let M be a cotorsion module, i.e. $MT = 0$. Let I(M) denote the injective hull of M. We show that $I(M)T = 0$ also. Let π be the projection map: $I(M)T \longrightarrow I(M)T/I(M)T \cap M$.

$$\begin{tikzcd}
 & I(M)T \arrow[d, "\pi"] \arrow[dl, dashed, "\exists g"'] \\
 & I(M)T/I(M)T \cap M \arrow[d, equal, "1_{I(M)T/I(M)T \cap M}"] \arrow[dl, dashed, "\exists f"'] \\
I(M)T \arrow[r, "\pi"] & I(M)T/I(M)T \cap M
\end{tikzcd}$$

$I(M)T$ and $I(M)T/I(M)T \cap M$ are both cotorsionfree, and hence codivisible, and $I(M)T \cap M = \mathrm{Ker}\ \pi$ is cotorsion since M is cotorsion. Therefore $\exists!f$ such that $\pi f = 1_{I(M)T/I(M)T \cap M}$ and $\exists!g$ such that $\pi g = 1_{I(M)T/I(M)T\cap M}\pi$, hence $g = f\pi = 1_{I(M)T}$, and π is an isomorphism. (The uniqueness of f and g follows from the fact that any mapping from a cotorsionfree module to a cotorsion module is zero.) Thus $I(M)T \cap M = 0$, but M is essential in $I(M)$ and therefore $I(M)T = 0$.

The next result is the dual of a well-known characterization of the localization of M at I, $\forall M \in$ Mod-R and $\forall$ injective $I \in$ Mod-R [24, Prop. 1.1].

PROPOSITION 1.10. $\forall M \in$ Mod-R, let $\varphi: X \longrightarrow M$ and $\psi: Y \longrightarrow M$ be homomorphisms such that X and Y are cotorsionfree and codivisible modules, and φ and ψ have cotorsion kernels and cokernels. Then $X \cong Y$.

Proof: Since X is cotorsionfree, $X = XT$ and therefore $\varphi(X) \subseteq MT$. But $\mathrm{Cok}\ \varphi = M/\varphi(X)$ is cotorsion, and therefore $MT \subseteq \varphi(X)$. Hence $\varphi(X) = MT$, and similarly $\psi(Y) = MT$. We may regard φ as an epimorphism from X to MT, and ψ as an epimorphism from Y to MT. Since $\mathrm{Ker}\ \varphi$ is cotorsion and Y is codivisible, $\exists f: Y \longrightarrow X$ such that $\varphi f = \psi$. Similarly $\exists g: X \longrightarrow Y$ such that $\psi g = \varphi$. Then $\varphi(1_X - fg) = \varphi - \varphi fg = \varphi - \psi g = \varphi - \varphi = 0$, and therefore $(1_X - fg): X \longrightarrow \mathrm{Ker}\ \varphi$. Hence $1_X = fg$ since X is cotorsionfree and $\mathrm{Ker}\ \varphi$ is cotorsion. Similarly $1_Y = gf$, and $X \cong Y$.

We are now able to make the following definition.

DEFINITION 1.11. $\forall M \in$ Mod-R, $\varphi: X \longrightarrow M$ is (up to isomorphism) the colocalization of M at P if X is cotorsionfree and codivisible, and Ker φ and Cok φ are cotorsion.

Lambek and Rattray [25] have formed a colocalization at P in Mod-R by taking the cotriple (S, ϵ', δ') on Mod-R: S: Mod-R $\longrightarrow$ Mod-R is defined by $S(M) = \sum_{f:P \longrightarrow M} P$ where $\sum_f (f, p_f)$ denotes an element of S(M), $\forall M \in$ Mod-R, and $1_{\text{Mod-R}} \xleftarrow{\epsilon'} S \xrightarrow{\delta'} S^2$ where $\epsilon'(M): S(M) \longrightarrow M$ is given by the evaluation mapping, i.e. $\epsilon'(M)(\sum_f (f, p_f)) = \sum_f f(p_f)$. Then their colocalization of M is given by the coequalizer of the pair of mappings $S^2(M) \underset{S\epsilon'(M)}{\overset{\epsilon' S(M)}{\rightrightarrows}} S(M)$. For P a finitely generated projective module, they showed that this colocalization of M at P is $[P, M] \otimes_E P$. The next theorem states that this is our colocalization of M at P for any projective P. We will later verify that the two colocalizations are the same for any projective P.

THEOREM 1.12. $\forall M \in$ Mod-R, $[P, M] \otimes_E P$ is the colocalization of M at P.

Proof: Since clearly $[P, M] \otimes_E P$ is cotorsionfree and Cok $\epsilon(M) = M/MT$ is cotorsion, the result follows from Propositions 1.5 and 1.6.

If we let $F = \_ \otimes_E P$: Mod-E $\longrightarrow$ Mod-R and $U = [P, \_]\cdot$ Mod-R $\longrightarrow$ Mod-E, then F is the left adjoint of U, i.e. there exist natural transformations $\eta: 1_{\text{Mod-E}} \longrightarrow UF$, given by $\eta(B)(b)(p) = b \otimes p$ $\forall B \in$ Mod-E, $\forall b \in B$, $\forall p \in P$, and $\epsilon: FU \longrightarrow 1_{\text{Mod-R}}$, given by $\epsilon(A)(\Sigma g_i \otimes p_i) = \Sigma g_i(p_i)$ $\forall A \in$ Mod-R, $\forall \Sigma g_i \otimes p_i \in [P,A] \otimes_E P$, such that $U\epsilon \circ \eta U = 1_U$ and $\epsilon F \circ F\eta = 1_F$.

We can then form the cotriple ($S^* = FU, \epsilon, \delta$) on Mod-R. $S^*(M)$ is by Theorem 1.12 the colocalization of M at P, $\forall M \in$ Mod-R. The coequalizer of the mappings $\epsilon S^*(M)$, $S^*\epsilon(M) : S^{*2}(M) \longrightarrow S^*(M)$ is just the identity on $S^*(M)$, since $\epsilon S^*(M)$ is an isomorphism and therefore $\epsilon S^*(M) = S^*\epsilon(M)$ (since $\epsilon S^*(M)\delta = 1_{S^*(M)} = S^*\epsilon(M)\delta$).

The dual situation (see [24, Sec. 3]) is more complicated. If I is an injective module and $H = [I, I]$, then $[\_, I]$: Mod-R $\longrightarrow$ $(\text{H-Mod})^{op}$ has a right adjoint $\text{Hom}_H(\_, {}_H I)$. If we form the triple ($S = \text{Hom}_H([\_, I], {}_H I), \eta, \mu$) arising from this pair of adjoint functors, then Q(M), the localization of M at I, $\forall M \in$ Mod-R, is given by the equalizer of the pair of mappings $\eta S(M)$, $S\eta(M): S(M) \longrightarrow S^2(M)$. S(M) is torsionfree and divisible, and Ker $\eta(M)$ is torsion, but in general $S(M) \neq Q(M)$. (They are equal if [M, I] is a finitely generated left H-module.) In general, then, Cok $\eta(M)$ is not torsion.

For example, if $R = \mathbb{Z}$, and we take the largest torsion theory in Mod-$\mathbb{Z}$ for which $\mathbb{Z}/p\mathbb{Z}$ is torsionfree, where p is a prime number. A $\mathbb{Z}$-module M is torsion if and only if $\forall m \in M$, $(m:0) \not\subset p\mathbb{Z}$, and $Q(\mathbb{Z})$ is the usual localization of the commutative ring $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$, i.e. $Q(\mathbb{Z})$ consists of all rational numbers whose denominators are prime to p. Every torsionfree factor module of $Q(\mathbb{Z})$ is divisible (in fact, if D is any dense ideal $DQ(\mathbb{Z}) = Q(\mathbb{Z})$ and hence the localization functor Q preserves all colimits), and therefore $S(\mathbb{Z})$ is the $I(\mathbb{Z}/p\mathbb{Z})$-adic completion of $Q(\mathbb{Z})$ [24, Thm. 4.2]. But the $I(\mathbb{Z}/p\mathbb{Z})$- adic topology on $Q(\mathbb{Z})$ coincides with the p-adic topology [23, Prop. 4], and thus $S(\mathbb{Z})$ is the ring of p-adic integers. But $S(\mathbb{Z})/\mathbb{Z} = \mathrm{Cok}\,(\eta(\mathbb{Z}):\mathbb{Z} \longrightarrow S(\mathbb{Z})\,)$ is not torsion, since $\forall\ z + z_1p + z_2p^2 + \ldots \in S(\mathbb{Z})$, if $\exists\ n, m \in \mathbb{Z}$ such that $n \notin p\mathbb{Z}$ and $n(z + z_1p + z_2p^2 + \ldots) = m$, then $z + z_1p + z_2p^2 + \ldots = \frac{m}{n} \in Q(\mathbb{Z})$.

DEFINITION 1.13. $\varphi: X \longrightarrow M$ is a <u>codivisible cover</u> of $M \in$ Mod-R if
(1) φ is a minimal epimorphism
(2) Ker φ is cotorsion
(3) X is codivisible

PROPOSITION 1.14. $\forall M \in$ Mod-R, if M has a codivisible cover, then it is unique up to isomorphism.

<u>Proof</u>: Let $\varphi: X \longrightarrow M$ and $\psi: Y \longrightarrow M$ be codivisible covers of M. Then $\exists\ f: X \longrightarrow Y$ such that $\psi f = \varphi$ since X is codivisible and Ker ψ is cotorsion. φ an epimorphism and Ker ψ small in Y implies that f is an epimorphism, and Ker f is cotorsion and small in X since Ker f $\subseteq$

Ker φ. Therefore $\exists\ g: Y \longrightarrow X$ such that $fg = 1_Y$, hence $X = g(Y) \oplus \text{Ker } f$. But then Ker $f = 0$ since Ker f is small in X, and hence f is an isomorphism.

We will show that if $M \in$ Mod-R has a projective cover, then it has a codivisible cover.

LEMMA 1.15. If $M \in$ Mod-R is codivisible and $M' \subseteq M$ is a cotorsionfree submodule of M, then M/M' is codivisible.

Proof: Let $\pi: M \longrightarrow M/M'$ be the projection map, and let $\varphi: B \longrightarrow A$ be any homomorphism with Ker φ cotorsion, and $\psi: M/M' \longrightarrow A$. Since M is codivisible $\exists\ \psi': M \longrightarrow B$ such that $\varphi\psi' = \psi\pi$. $\varphi\psi'(M') = \psi\pi(M') = 0$, and therefore $0 = \psi'|_{M'}: M' \longrightarrow \text{Ker } \varphi$ since M' is cotorsionfree and Ker φ is cotorsion. Therefore ψ' induces a homomorphism $\psi'': M/M' \longrightarrow B$ such that $\varphi\psi'' = \psi$, and hence M/M' is codivisible.

PROPOSITION 1.16. If $\varphi: P(M) \longrightarrow M$ is the projective cover of $M \in$ Mod-R, then $\bar{\varphi}: P(M)/(\text{Ker } \varphi)T \longrightarrow M$ is the codivisible cover of M, where $\bar{\varphi}$ is the homomorphism induced by φ.

Proof: Clearly $\bar{\varphi}: P(M)/(\text{Ker } \varphi)T$ is a minimal epimorphism, and Ker $\bar{\varphi} = \text{Ker } \varphi/(\text{Ker } \varphi)T$ is cotorsion. It remains to show that $P(M)/(\text{Ker } \varphi)T$ is codivisible, but this follows from the preceding lemma.

COROLLARY 1.17. If φ: $P(M) \longrightarrow M$ is the projective cover of $M \in$ Mod-R, then the codivisible cover of M in the cotorsion theory determined by P(M) is the maximal co-rational extension over M.

Proof: Courter [10, Thhm. 2.12] showed that $P(M)/X$ is the maximal co-rational extension over M, where $X = \sum_{f \in [P(M), \mathrm{Ker}\ \varphi]} f(P(M))$. But if $T_{P(M)}$ denotes the trace ideal of P(M), then it is clear from the proof of Proposition 1.3 that $X = (\mathrm{Ker}\ \varphi)T_{P(M)}$.

COROLLARY 1.18. If φ: $P(M) \longrightarrow M$ is the projective cover of $M \in$ Mod-R, then M is codivisible if and only if Ker φ is cotorsionfree.

Proof: Ker φ cotorsionfree implies that Ker $\bar{\varphi} = 0$, and hence $M \cong P(M)/(\mathrm{Ker}\ \varphi)T$ which is codivisible. Conversely, if M is codivisible then $\exists\ \psi: M \longrightarrow P(M)/(\mathrm{Ker}\ \varphi)T$ such that $\bar{\varphi}\psi = 1_M$. Therefore $P(M)/(\mathrm{Ker}\ \varphi)T = \psi(M) \oplus \mathrm{Ker}\ \bar{\varphi}$, but Ker $\bar{\varphi}$ is small in $P(M)/(\mathrm{Ker}\ \varphi)T$ and hence zero, and therefore Ker $\varphi = (\mathrm{Ker}\ \varphi)T$.

THEOREM 1.19. $[P, M] \otimes_E P = [P, MT] \otimes_E P$ is the codivisible cover of MT.

Proof: We have already shown that $\epsilon(M)$: $[P, M] \otimes_E P \longrightarrow MT$ is an epimorphism (Proposition 1.3) with cotorsion kernel (Proposition 1.6), and that $[P, M] \otimes_E P$ is codivisible (Proposition 1.5). Ker $\epsilon(M)$ is small in $[P,M] \otimes_E P$, since if $\mathrm{Ker}\ \epsilon(M) + U = [P,M] \otimes_E P$ for some submodule $U \subseteq [P,M] \otimes_E P$, then $U \supseteq UT = (\mathrm{Ker}\ \epsilon(M))T + UT = ([P,M] \otimes_E P)T = [P,M] \otimes_E P$. Hence $[P,M] \otimes_E P$ is the codivisible cover of MT.

$\forall\ M \in$ Mod-R, the torsion submodule $\mathcal{T}(M)$ with respect to a torsion theory $(\mathcal{T},\mathcal{F})$ is uniquely defined by $\mathcal{T}(M)$ torsion and $M/\mathcal{T}(M)$ torsionfree. Dually, M/MT is unique in that M/MT is cotorsion and MT is cotorsionfree. We call M/MT the <u>cotorsion factor module</u> of M. And, we can colocalize in two steps, namely

$$\begin{tikzcd}
{[P,M] \otimes_E P} \arrow[r] & MT \arrow[r] & M \\
\text{codivisible cover of } MT & &
\end{tikzcd}$$

duallizing

$$\begin{tikzcd}
M \arrow[r] & M/\mathcal{T}(M) \arrow[r] & Q(M) . \\
& & \text{divisible hull of } M/\mathcal{T}(M)
\end{tikzcd}$$

CHAPTER II

The Category of Cotorsionfree Codivisible Modules

PROPOSITION 2.1. The cotorsionfree codivisible modules form a coreflective subcategory of Mod-R. Let $\underline{A}$ denote this subcategory.

<u>Proof</u>: $\forall N \in \underline{A}$, , $\forall M \in$ Mod-R, $\forall$ homomorphism $f: N \longrightarrow M$, since N is cotorsionfree $\exists ! f': N \longrightarrow MT$ such that $if' = f$, where i denotes the inclusion mapping: $MT \longrightarrow M$, defined by $f'(n) = f(n)$ $\forall n \in N$. And, since N is cotorsionfree and codivisible $\exists ! f'': N \longrightarrow [P, M] \otimes_E P$ such that $\epsilon(M)f'' = f'$. Since i is a monomorphism, f'' is the unique homomorphism such that $i\epsilon(M)f'' = f$.

$$\begin{tikzcd}
{[P,M] \otimes_E P} \arrow[d, "\epsilon(M)"'] & \\
MT \arrow[d, "i"'] & \\
M & N \arrow[l, "\forall f"'] \arrow[ul, dashed, "\exists ! f'"'] \arrow[uul, dashed, "\exists ! f''"'] \\
& \text{cotorsionfree} \\
& \text{codivisible}
\end{tikzcd}$$

Let Q^* denote the coreflector, i.e. let $Q^*(M)$ denote the colocalization of M. Note that M cotorsionfree and codivisible implies that $M \cong Q^*(M)$, and in particular $Q^*(M) \cong Q^*(Q^*(M))$. $\forall$ N, M$\in$ Mod-R, $\forall$ homomorphism $f: N \longrightarrow M$, the unique homomorphism $Q^*(f)$ which makes the diagram below commute is given by $Q^*(f)(\Sigma f_i \otimes p_i) = \Sigma ff_i \otimes p_i$ $\forall \Sigma f_i \otimes p_i \in$

$[P,N] \otimes_E P$.

$$\begin{tikzcd}
Q^*(N) \arrow[r, "Q^*(f)"] \arrow[d, "\epsilon(N)"'] & Q^*(M) \arrow[d, "\epsilon(M)"] \\
N \arrow[r] & M
\end{tikzcd}$$

LEMMA 2.2. For any homomorphism $f\colon M \longrightarrow N$ in Mod-R, $Q^*(f)$ is a monomorphism in $\underline{A}$ if and only if Ker f (in Mod-R) is cotorsion. In particular, a homomorphism $f\colon M \longrightarrow N$ in $\underline{A}$ is a monomorphism in $\underline{A}$ if and only Ker f (in Mod-R) is cotorsion.

Proof: Suppose $Q^*(f)$ is a monomorphism in $\underline{A}$, then if $k = \ker f$ (in Mod-R), $fk = 0 \Rightarrow Q^*(fk) = 0 \Rightarrow Q^*(f)Q^*(k) = 0 \Rightarrow Q^*(k) = 0 \Rightarrow \epsilon(\text{Ker } f) = 0 \Rightarrow (\text{Ker } f)T = 0$.

$$\begin{tikzcd}
\text{Ker } f \arrow[r, "k"] & M \arrow[r, "f"] & N \\
Q^*(\text{Ker } f) \arrow[u, "\epsilon(\text{Ker } f)"] \arrow[r, "Q^*(k)"] & Q^*(M) \arrow[u, "\epsilon(M)"] \arrow[r, "Q^*(f)"] & Q^*(N) \arrow[u, "\epsilon(N)"]
\end{tikzcd}$$

Conversely, Q^* is the right adjoint of the inclusion functor $U\colon \underline{A} \longrightarrow$ Mod-R, hence preserves kernels. Therefore $Q^*(f)$ has (in $\underline{A}$) the kernel $Q^*(k)\colon Q^*(\text{Ker } f) \longrightarrow Q^*(M)$. But since Ker f is cotorsion, $Q^*(\text{Ker } f) = 0$, therefore the kernel of $Q^*(f)$ in $\underline{A}$ is zero, and $Q^*(f)$ is a monomorphism in $\underline{A}$.

THEOREM 2.3. $\underline{A}$, the category of cotorsionfree codivisible modules, is a cocomplete abelian category.

Proof: $\underline{A}$ is additive, contains the module 0, and is closed under direct sums. It remains to show that $\underline{A}$ has cokernels and every epimorphism is a cokernel, and that $\underline{A}$ has kernels and every monomorphism is a kernel.

$\underline{A}$ has cokernels (and in fact is closed under cokernels), since $\forall$ f: X $\longrightarrow$ Y in $\underline{A}$, Y/f(X) is cotorsionfree since Y is cotorsionfree, and codivisible by Lemma 1.15. To show that in $\underline{A}$ every epimorphism is a cokernel, let f: X $\longrightarrow$ Y be an epimorphism in $\underline{A}$. U: $\underline{A}$ $\longrightarrow$ Mod-R is the left adjoint of Q^*, hence preserves epimorphisms. Therefore f is an epimorphism in Mod-R, and hence a cokernel in Mod-R. Since both Y $\cong$ X/Ker f and X/(Ker f)T are cotorsionfree and codivisible, it is easily shown that Ker f = (Ker f)T, and therefore ϵ(Ker f) is an epimorphism. Thus f = cok $(Q^*(\text{Ker } f) \xrightarrow{\epsilon(\text{Ker } f)} \text{Ker } f \longrightarrow X)$ in $\underline{A}$, since f = cok(Ker f $\longrightarrow$ X) in Mod-R.

To show that $\underline{A}$ has kernels, let f: X $\longrightarrow$ Y in $\underline{A}$. Ker f (in Mod-R) is in general not in $\underline{A}$, but since Q^* is a right adjoint it preserves kernels, and therefore in $\underline{A}$ f has the kernel

$$\begin{tikzcd}
Q^*(\text{Ker } f) \arrow[rr, "Q^*(\ker f)"] \arrow[dr, "\epsilon(\text{Ker } f)"'] & \arrow[d, phantom, "="] & X. \\
& \text{Ker } f \arrow[ur, "\ker f"'] &
\end{tikzcd}$$

To show that in $\underline{A}$ every monomorphism is a kernel, let $f\colon X \longrightarrow Y$ be a monomorphism in $\underline{A}$, and let $X \xrightarrow{e} f(X) \xrightarrow{m} Y$ be the factoring of f through its image f(X). We show that $f = \ker(\operatorname{cok} f)$ in $\underline{A}$. Suppose $z\colon Z \longrightarrow Y$ in $\underline{A}$ such that $(\operatorname{cok} f)z = 0$. Since $m = \ker(\operatorname{cok} f)$ in Mod-R $\exists! y\colon Z \longrightarrow f(X)$ such that $my = z$. Ker e = Ker f and is cotorsion by Lemma 2.2, hence since Z is codivisible and cotorsionfree $\exists! x\colon Z \longrightarrow X$ such that $ex = y$. Therefore $fx = mex = my = z$. If for some $g\colon Z \longrightarrow X$, $fg = z$, then $meg = fg = z$ and hence $eg = y$, hence $g = x$. Therefore $f = \ker(\operatorname{cok} f)$ in $\underline{A}$.

It is a well-known theorem in category theory that a cocomplete abelian category $\underline{C}$ is equivalent to a module category Mod-R for some ring R if and only if $\underline{C}$ has a small projective generator (see e.g. [30, Thm. IV.4.1]). Recall that $C \in \underline{C}$ is small if and only if for any morphism from C to a coproduct $\bigoplus_{i \in I} C_i$ there is a factorization

$$\begin{tikzcd} C \arrow[rr] \arrow[dr] & & \bigoplus_{i \in I} C_i \\ & \bigoplus_{j \in J} C_j \arrow[ur] & \end{tikzcd}$$

for some finite set $J \subseteq I$.

PROPOSITION 2.4. P finitely generated $\Rightarrow$ $\underline{A}$ is a module category.

Proof: Since P is projective in Mod-R and P generates all cotorsionfree modules, $\underline{A}$ is a cocomplete abelian category with a projective generator. It is easy to see that P is small in Mod-R if and only if P is finitely generated, and therefore P finitely generated implies P is small in $\underline{A}$.

PROPOSITION 2.5. If T is codivisible, then P small in $\underline{A} \Rightarrow$ P is finitely generated.

Proof: Let $f\colon \oplus R \longrightarrow P$ be an epimorphism from a direct sum of copies of R to P. Since P is projective $\exists\, g\colon P \longrightarrow \oplus R$ such that $fg = 1_P$. $g(P) = g(PT) \subseteq (\oplus R)T = \oplus RT = \oplus T$. T codivisible implies $\oplus T$ is codivisible and hence is in $\underline{A}$. Therefore $g\colon P \longrightarrow \oplus T$ has a finite factorization $g'\colon P \longrightarrow \underset{\text{finite}}{\oplus} T \longrightarrow \oplus T$, and hence there is an epimorphism $f'\colon \underset{\text{finite}}{\oplus} R \longrightarrow P$. Thus P is finitely generated.

The question arises as to whether $(\text{Mod-}R)^{\text{op}}$ is a module category. Although it is tempting to think that with certain conditions on R, for example R right perfect, this will be true, it is in fact never true. Duallizing the definition of a small object, we call an object C in a category $\underline{C}$ cosmall if and only if for any morphism from a product $\prod_{i\in I} C_i$ to C there is a finite factorization

$$\begin{tikzcd} f\colon \prod_{i\in I} C_i \arrow[rr] \arrow[dr] & & C \\ & \prod_{j\in J} C_j \arrow[ur] & \end{tikzcd}$$

for some finite set $J \subseteq I$, (i.e., assuming that $\underline{C}$ is additive, $f = \sum_{j\in J} fu_j k_j$ where u_j and k_j denote the canonical injections and projections, respectively.)

PROPOSITION 2.6. There does not exist a cosmall injective cogenerator in Mod-R, and hence $(\text{Mod-}R)^{\text{op}}$ is not a module category.

Proof: Let A be an injective cogenerator in Mod-R, and let $\{A_i | i \in I\}$ be a non-finite family of non-zero modules. Since A is a cogenerator in Mod-R, $[A_i, A] \neq 0 \ \forall i \in I$. Let $0 \neq a_i \in [A_i, A] \ \forall i \in I$, and let (a_i) be the canonical map : $\bigoplus_{i\in I} A_i \longrightarrow A$ such that $(a_i)u_i = a_i \ \forall i \in I$. Since A is injective $\exists\, a: \prod_{i\in I} A_i \longrightarrow A$ such that $(a_i) = au$. But a does not have a finite factorization, since if it factors through $\prod_{j\in J} A_j$ where $J \subseteq I$ is finite, then $\forall I \notin J$ we have $a_i = (a_i)u_i = auu_i = \sum_{j\in J} au_j k_j uu_i = 0$.. Therefore A is not cosmall, and hence $(\text{Mod-R})^{op}$ is not a module category.

$$\begin{tikzcd}
A_i \arrow[r, "u_i"] \arrow[dr, "a_i"'] & \bigoplus_{i\in I} A_i \arrow[r, "u"] \arrow[d, "(a_i)"] & \prod_{i\in I} A_i \arrow[dl, "a"] \\
& A &
\end{tikzcd}$$

If P is a generator, then we have seen (Corollary 1.7) that $\epsilon(M)$ is an isomorphism $\forall M \in$ Mod-R, and therefore $\underline{A}$ = Mod-R. However, even though the cotorsion theory is trivial, it is still of interest to see what it means for P to be finitely generated in this situation.

PROPOSITION 2.7. If P is a generator then the following statements are equivalent:

(1) P is finitely generated

(2) $[P, \_]: \text{Mod-R} \longrightarrow \text{Mod-E}$ is an equivalence

(3) $B_E \cong [P, B \otimes_E P] \; \forall B \in \text{Mod-E}$

Proof: (1) $\Leftrightarrow$ (2) See, e.g. [30, Thm. IV.4.1].

(2) $\Leftrightarrow$ (3) A functor $U: \underline{B} \longrightarrow \underline{C}$ is an equivalence between categories $\underline{B}$ and $\underline{C}$ if and only if there is a functor $F: \underline{C} \longrightarrow \underline{B}$ together with natural equivalences $\varphi: 1_{\underline{C}} \approx UF$ and $\psi: FU \approx 1_{\underline{B}}$. If such is the case, we can always choose ψ such that $U\psi = (\varphi U)^{-1}$ and $F\varphi = (\psi F)^{-1}$. Therefore if $U = [P, \_]: \text{Mod-R} \longrightarrow \text{Mod-E}$ is an equivalence there exists such a functor $F: \text{Mod-E} \longrightarrow \text{Mod-R}$. F is then, in particular, a left adjoint of $[P, \_]$, and since $\_ \otimes_E P$ is also a left adjoint of $[P, \_]$, $F \approx \_ \otimes_E P$ and $B_E \cong [P, B \otimes_E P] \; \forall B \in \text{Mod-E}$. The converse holds since P is a generator, and therefore by Corollary 1.7 $[P, \_] \otimes_E P \approx 1_{\text{Mod-R}}$.

CHAPTER III

The Colocalization Functor Q^*

From now on, we regard Q^* as an endofunctor of Mod-R, i.e. it is the right adjoint of the inclusion functor $U: \underline{A} \longrightarrow$ Mod-R, where $\underline{A}$ is the subcategory of cotorsionfree codivisible modules, followed by U.

PROPOSITION 3.1. Q^* preserves epimorphisms, and hence is right exact since $\underline{A}$ is abelian.

Proof: Let $f: A \longrightarrow B$ be an epimorphism in Mod-R, then $f' = f_{|AT}$ is an epimorphism since $f'(AT) = f(AT) = f(A)T = BT$. Therefore $f'\epsilon(A)$ is an epimorphism, and since Ker $\epsilon(B)$ is small in $Q^*(B)$, $Q^*(f)$ is an epimorphism.

$$\begin{tikzcd}
A \arrow[r, "f"] & B \\
AT \arrow[u] \arrow[r, "f'"] & BT \arrow[u] \\
Q^*(A) \arrow[u, "\epsilon(A)"] \arrow[r, "Q^*(f)"] & Q^*(B) \arrow[u, "\epsilon(B)"']
\end{tikzcd}$$

In the next theorem, the equivalence of (3), (4), and (5) is due to Ulmer [39, Thm. 2], who proved these results in a more general setting. The proof of (5) ⇒ (4) is a straight "translation" of his proof. (1) ⇔ (2) is the dual of Goldman's criterion for the right exactness of the localisation functor Q [19, Thm. 4.5].

THEOREM 3.2. The following statements are equivalent:

(1) Q^* preserves monomorphisms, and hence is left exact (hence exact).

(2) Every cotorsionfree submodule of a cotorsionfree codivisible module is codivisible.

(3) $\underline{A}$ is closed under kernels.

(4) P is flat as a left E-module.

(5) P generates the kernel of every homomorphism $f\colon \bigoplus_{\text{finite}} P \longrightarrow P$.

Proof: (1) ⇒ (2) Let N be a cotorsionfree submodule of a cotorsionfree codivisible module M. Let $i\colon N \hookrightarrow M$ denote the inclusion mapping, then $Q^*(i)$ is a monomorphism and thus so is $\epsilon(N)$. But $\epsilon(N)$ is an epimorphism since N is cotorsionfree, and therefore $N \cong Q^*(N)$.

$$\begin{tikzcd}
N \arrow[r, "i"] & M \\
Q^*(N) \arrow[u, "\epsilon(N)"] \arrow[ur, "Q^*(i)"'] &
\end{tikzcd}$$

(2) ⇒ (1) Let $f\colon N \longrightarrow M$ be a monomorphism in Mod-R. Then $Q^*(f)(Q^*(N))$ is cotorsionfree and contained in $Q^*(M)$, hence is codivisible. $\operatorname{Ker} Q^*(f) \subseteq \operatorname{Ker} f\epsilon(N) = \operatorname{Ker} \epsilon(N)$ since f is a monomorphism, and is hence cotorsion and small in $Q^*(N)$. Therefore $\exists\, g\colon Q^*(f)(Q^*(N)) \cong Q^*(N)/\operatorname{Ker} Q^*(f) \longrightarrow Q^*(N)$ such that $Q^*(f)g = 1_{Q^*(N)/\operatorname{Ker} Q^*(f)}$, and thus $Q^*(N) =$

Im g $\oplus$ Ker $Q^*(f)$. But Ker $Q^*(f)$ is small in $Q^*(N)$, hence zero, and $Q^*(f)$ is a monomorphism.

$$\begin{tikzcd}
 & \begin{array}{c} Q^*(N)/\mathrm{Ker}\,Q^*(f) \\ \cong Q^*(f)(Q^*(N)) \end{array} \arrow[dl, dashed, "\exists g"'] \arrow[d, equal, "1_{Q^*(N)/\mathrm{Ker}\,Q^*(f)}"] \\
Q^*(N) \arrow[r, "Q^*(f)"] & Q^*(N)/\mathrm{Ker}\,Q^*(f)
\end{tikzcd}$$

(1) $\Rightarrow$ (3) Let $f: A \longrightarrow A'$ in $\underline{A}$, and let $K = \mathrm{Ker}\, f$ in Mod-R, then $0 \longrightarrow K \longrightarrow A \xrightarrow{f} A' \longrightarrow 0$ is an exact sequence in Mod-R. Q^* left exact implies $0 \longrightarrow Q^*(K) \longrightarrow A \xrightarrow{f} A' \longrightarrow 0$ is also exact. Therefore $K \cong Q^*(K)$, and $\underline{A}$ is closed under kernels.

(3) $\Rightarrow$ (5) P generates every cotorsionfree module, and hence every module in $\underline{A}$. Since $\underline{A}$ is closed under kernels, the kernel of every homomorphism $f: \bigoplus_{\text{finite}} P \longrightarrow P$ is in $\underline{A}$.

(5) $\Rightarrow$ (4) ${}_E P$ is flat if and only if for every finitely generated right ideal H of E the inclusion $j: H \longrightarrow E$ yields a monomorphism $j \otimes_E P$: $H \otimes_E P \longrightarrow E \otimes_E P$. (See e.g. [21, Prop. 5.4.1].) Let $H = \sum_{i \in I} f_i E$ where each $f_i \in E$ $\forall i \in I$ finite. H is the image of the canonical mapping $([P, f_i]): \bigoplus_{i \in I} [P, P] \longrightarrow [P, P]$. Let $k: K \longrightarrow \bigoplus_{i \in I} P$ be the kernel of of the canonical mapping $(f_i): \bigoplus_{i \in I} P \longrightarrow P$, and let (g) be the canonical mapping: $\bigoplus_{g:P \to K} P \longrightarrow K$.

Then $0 \longrightarrow [P,K] \xrightarrow{[P,k]} [P, \bigoplus_{i\in I} P] \xrightarrow{[P,(f_i)]} [P,P]$ is exact, and hence the top row of the diagram below is exact at $\bigoplus_{i\in I}[P,P]$.

$$\begin{tikzcd}
\bigoplus_{g:P\to K}[P,P] \arrow[rr, "can^{-1}\cdot([P,k\cdot g])"] \arrow[dd, "{([P,g])}"'] \arrow[ddrr, "{([P,k\cdot g])}"] & & \bigoplus_{i\in I}[P,P] \arrow[rr, "{([P,f_i])}"] \arrow[dr] \arrow[dd, Rightarrow, "can"] & & {[P,P]} \\
 & & & H \arrow[ur, "j"] & \\
{[P,K]} \arrow[rr, "{[P,k]}"] & & {[P, \bigoplus_{i\in I} P]} \arrow[uurr, "{[P,(f_i)]}"'] & &
\end{tikzcd}$$

Therefore, in particular, $\bigoplus_{i\in I}[P,P] \longrightarrow H$ is the cokernel of $can^{-1}\circ([P,k\circ g])$. Since $\_\otimes_E P$ preserves coproducts and cokernels, there are canonical isomorphisms $(\bigoplus_{g:P\to K}[P,P])\otimes_E P \xrightarrow{\cong} \bigoplus_{g:P\to K} P$, $(\bigoplus_{i\in I}[P,P])\otimes_E P \xrightarrow{\cong} \bigoplus_{i\in I} P$, $[P,P]\otimes_E P \xrightarrow{\cong} P$, and $H\otimes_E P \xrightarrow{\cong} Cok\,(k\cdot(g))$ which make the following diagram commute.

$$\begin{tikzcd}
\bigoplus_{g:P\to K} P \arrow[rr, "k\cdot(g)"] \arrow[dr, "(g)"'] \arrow[dd, "\cong"] & & \bigoplus_{i\in I} P \arrow[rr, "(f_i)"] \arrow[dr, "can"] \arrow[dd, "\cong"] & & P \arrow[dd, "\cong"] \\
 & K \arrow[ur, "k"'] & & Cok\,(k\cdot(g)) \arrow[ur, "can"'] \arrow[d, "\cong"] & \\
(\bigoplus_{g:P\to K}[P,P])\otimes_E P \arrow[rr] & & (\bigoplus_{i\in I}[P,P])\otimes_E P \arrow[r] & H\otimes_E P \arrow[r, "j\otimes P"] & {[P,P]\otimes_E P}
\end{tikzcd}$$

By assumption $(g)\colon \bigoplus_{g:P\to K} P \longrightarrow K$ is an epimorphism, and hence

$\text{Cok}(k\circ(g)) \longrightarrow P$ and therefore $H \otimes_E P \longrightarrow E \otimes_E P$ are monomorphisms. Hence ${}_E P$ is flat.

(4) ⇒ (1) This is clear since $Q^*(M) = [P, M] \otimes_E P \ \forall M \in \text{Mod-}R$.

Since $\underline{A}$ is always closed under cokernels and direct sums, the statements of the above theorem are also equivalent to those of the following theorem by Ulmer [39, Thm.3], which we state without proof.

THEOREM 3.3. The following statements are equivalent:

(1) P generates the kernel of every homomorphism $f\colon \bigoplus_{\text{finite}} P \longrightarrow X$ where X is a cokernel of some homomorphism $\bigoplus_{i\in I} P \longrightarrow \bigoplus_{j\in J} P$.

(2) $\underline{A}$ is closed under kernels, cokernels, and direct sums.

(3) ${}_E P$ is flat, and the composite

$$\underline{A} \subseteq \text{Mod-}R \xrightarrow{[P,\_]} \text{Mod-}E \xrightarrow{\text{can.proj.}} \text{Mod-}E/\text{Ker}\ \_ \otimes_E P$$

is an equivalence, where $\text{Mod-}E/\text{Ker}\ \_ \otimes_E P$ is the quotient category corresponding to the torsion class $\text{Ker}\ \_ \otimes_E P$ (see [17]).

Lambek has showed in [24] that the localization functor Q corresponding to an injective $I \in \text{Mod-}R$ is exact if I has zero singular submodule, i.e. $[R/E, I] = 0$ for any essential right ideal $E \subseteq R$. We are able, with the restriction that every cotorsionfree codivisible module have a projective cover, to dualize this result.

PROPOSITION 3.4. Assume that every cotorsionfree codivisible module $M \in$ Mod-R has a projective cover $\varphi_M: P(M) \longrightarrow M$. Then if M is a cotorsionfree codivisible module, Ker φ_M is codivisible if and only if for any epimorphism $f: N \longrightarrow M$ where N is a cotorsionfree codivisible module, Ker f is codivisible.

Proof: Assume that Ker φ_M is codivisible, and let $f: N \longrightarrow M$ be an epimorphism for a cotorsionfree codivisible module N. Then Ker f = Ker $f\varphi_N$/Ker φ_N, and therefore Ker f will be codivisible if Ker $f\varphi_N$ is codivisible and Ker φ_N is cotorsionfree (Lemma 1.15). Ker φ_N is cotorsionfree since N is codivisible (Corollary 1.18). Since P(N) is projective and φ_M is an epimorphism, $\exists g: P(N) \longrightarrow P(M)$ such that $\varphi_M g = f\varphi_N$. g is an epimorphism since $f\varphi_N$ is an epimorphism and Ker φ_M is small in P(M), and therefore since P(M) is projective $\exists h: P(M) \longrightarrow P(N)$ such that $gh = 1_{P(M)}$. Then $P(N) = h(P(M)) \oplus$ Ker g, and Ker $f\varphi_N$ = Ker $\varphi_M g$ = Ker $\varphi_M \oplus$ Ker g. But Ker φ_M is codivisible by assumption, and Ker g is codivisible since it is projective, hence Ker $f\varphi_N$ and thus Ker f is codivisible.

$$\begin{tikzcd}
 & P(M) \arrow[dl, dashed, "\exists h"'] \arrow[d, equal, "1_{P(M)}"] \\
P(N) \arrow[r, dashed, "\exists g"] \arrow[d, "\varphi_N"'] & P(M) \arrow[d, "\varphi_M"] \\
N \arrow[r, "f"] & M
\end{tikzcd}$$

Conversely, P(M) is cotorsionfree since $\mathfrak{F}^*$ is closed under minimal epimorphisms [34, Prop. 1], and codivisible since it is projective, hence Ker φ_M is codivisible.

COROLLARY 3.5. Assume that every cotorsionfree codivisible module $M \in$ Mod-R has a projective cover $\varphi_M: P(M) \longrightarrow M$. Then Q^* is exact if and only if Ker φ_M is codivisible for any cotorsionfree codivisible module M.

Proof: If Q^* is exact, then for any cotorsionfree codivisible module M, P(M) is cotorsionfree and codivisible, and Ker φ_M is cotorsionfree (Corollary 1.18), and hence by Theorem 3.2 Ker φ_M is codivisible. Conversely, suppose A is a cotorsionfree submodule of a cotorsionfree codivisible module B. Then B/A is cotorsionfree and (by Lemma 1.15) codivisible, and therefore Ker $\varphi_{B/A}$ is codivisible. But A = Ker (B $\longrightarrow$ B/A), and hence by the above proposition A is codivisible. Q^* is thus exact by Theorem 3.2.

PROPOSITION 3.6. Assume that every codivisible module has a projective cover, then the following statements are equivalent:

(1) Every codivisible module is projective

(2) $[P, D] = 0$ for any small right ideal $D \subseteq R$.

(3) For any module M with a projective cover $\varphi_M: P(M) \longrightarrow M$, Ker φ_M is cotorsion.

Proof: (1) ⇒ (2) Let D be a small right ideal of R, then R/DT is codivisible by Lemma 1.15 and hence by assumption is projective. Therefore DT = 0 since it is small in R, i.e. [P, D] = 0.

(2) ⇒ (3) Let $M \in$ Mod-R have a projective cover $\varphi_M: P(M) \longrightarrow M$ and suppose Ker φ_M is not cotorsion, i.e. [P, Ker φ_M] $\neq 0$. Let $0 \neq f: P \longrightarrow$ Ker φ_M, and let $p \in P$ such that $f(p) = x \neq 0$. Let k be a projection mapping from Ker φ_M to R (every projective module is a direct summand of a free module) such that $k(x) \neq 0$, then k(Ker φ_M) = D is a small right ideal of R and $0 \neq kf \in [P, D]$.

(3) ⇒ (1) For any codivisible module M, M has a projective cover $\varphi_M: P(M) \longrightarrow M$, and Ker φ_M is then both cotorsionfree (Corollary 1.18) and cotorsion, hence zero. Therefore $M \cong P(M)$.

Note that (2) ⇔ (3) is true in general, and we therefore have the following result.

COROLLARY 3.7. Assume that every cotorsionfree codivisible module has a projective cover, then Q^* is exact if for any small right ideal D of R, [P, D] = 0.

Proof: Let M be a cotorsionfree and codivisible module, then M has by assumption a projective cover $\varphi_M: P(M) \longrightarrow M$. Ker φ_M is then cotorsion and cotorsionfree, hence zero, and the result now follows from Corollary 3.5.

CHAPTER IV

$Q^*(R)$, and an Alternate Construction of Q^*

PROPOSITION 4.1. $Q^*(R)$ is an associative ring, in general without unit element, and $\epsilon(R)\colon Q^*(R) \longrightarrow R$ is a ring homomorphism.

<u>Proof</u>: $\forall f, f' \in [P, R]$, $\forall p, p' \in P$, $f \otimes_E p(f'(p')) = f \otimes_E (pf')(p') = f(pf') \otimes_E p' = (f(p))f' \otimes_E p'$. Therefore $\forall\ a = \sum_{i \in I} f_i \otimes p_i$, $b = \sum_{j \in J} f_j \otimes p_j$, $c = \sum_{k \in K} f_k \otimes p_k \in Q^*(R)$, define $ab = \sum_{i \in I} (f_i \otimes p_i(\sum_{j \in J} f_j(p_j))) = \sum_{j \in J} ((\sum_{i \in I} f_i(p_i))f_j \otimes p_j)$. This is clearly well-defined, since $\epsilon(R)$ is well-defined, and $\sum_{i \in I} f_i(p_i) = \epsilon(R)(\sum_{i \in I} f_i \otimes p_i)$. $Q^*(R)$ is associative since $(ab)c =$

$$(\sum_{j \in J} (\sum_{i \in I} f_i(p_i))f_j \otimes p_j)(\sum_{k \in K} f_k \otimes p_k)$$

$$= \sum_{j \in J} (\sum_{i \in I} f_i(p_i))f_j \otimes p_j(\sum_{k \in K} f_k(p_k))$$

$$= (\sum_{i \in I} f_i \otimes p_i)(\sum_{j \in J} f_j \otimes p_j(\sum_{k \in K} f_k(p_k))) = a(bc).$$

$Q^*(R)$ is clearly an abelian (additive) group, and the distributive laws hold, therefore it is an associative ring (without unit element). $\epsilon(R)(ab) = \sum_{j \in J} (\sum_{i \in I} f_i(p_i))f_j(p_j) = (\sum_{i \in I} f_i(p_i))(\sum_{j \in J} f_j(p_j)) = \epsilon(R)(a)\epsilon(R)(b)$, and therefore $\epsilon(R)$ is a ring homomorphism.

COROLLARY 4.2. Every right R-module is a right $Q^*(R)$-module.

Proof: Since $\epsilon(R): Q^*(R) \longrightarrow R$ is a ring homomorphism, it induces the functor Mod-$\epsilon(R)$: Mod-R $\longrightarrow$ Mod-$Q^*(R)$. $\forall M \in$ Mod-R, Mod-$\epsilon(R)(M)$ is then a $Q^*(R)$-module as follows: $\forall s = \sum_{i\in I} f_i \otimes p_i \in Q^*(R)$, $\forall m \in M$, $ms = m\epsilon(R)(s) = m(\sum_{i\in I} f_i(p_i))$.

PROPOSITION 4.3. $Q^*(R)$ has a unit element if and only if T is a ring direct summand of R, i.e. $T = eR$ where e is a central idempotent of R.

Proof: Suppose $\sum_{i\in I} f_i \otimes p_i$ is the unit element of $Q^*(R)$, then $\sum_{i\in I} f_i(p_i) =$ e is an idempotent in R such that $Re = Te = T = eT = eR$. Conversely, if $T = Re = eR$ for some central idempotent e, then there exists an element $\sum_{i\in I} f_i \otimes p_i$ of $Q^*(R)$ such that $\sum_{i\in I} f_i(p_i) = e$, and for any $\sum_{j\in J} f_j \otimes p_j \in Q^*(R)$, $(\sum_{i\in I} f_i \otimes p_i)(\sum_{j\in J} f_j \otimes p_j) = \sum_{j\in J} (\sum_{i\in I} f_i(p_i)) f_j \otimes p_j = \sum_{j\in J} f_j \otimes p_j$ since $\forall f \in [P, R]$, $(\sum_{i\in I} f_i(p_i))f = f$. By the dual basis lemma (see e.g. [8, Prop. VII.3.1]), a module P is projective if and only if $\exists \{f_k \in [P,R] | k \in K\}$, $\{p_k \in P | k \in K\}$ such that $\forall p \in P$, $p = \sum_{k\in K} p_k f_k(p)$ where $f_k(p) \neq 0$ for only finitely many $k \in K$, and therefore $p(\sum_{i\in I} f_i(p_i)) = p$ $\forall p \in P$. Thus we also have $(\sum_{j\in J} f_j \otimes p_j)(\sum_{i\in I} f_i \otimes p_i) = \sum_{j\in J} f_j \otimes p_j(\sum_{i\in I} f_i(p_i)) = \sum_{j\in J} f_j \otimes p_j$, and $\sum_{i\in I} f_i \otimes p_i$ is the unit element of $Q^*(R)$.

COROLLARY 4.4. The following statements are equivalent.

(1) $Q^*(R)$ has a left (right) unit element.

(2) $T = eR$ $(T = Re)$ for some idempotent $e \in R$.

(3) $(\mathfrak{J}^*)^r$ $((\mathfrak{J}^*)^\ell = \mathfrak{J}^*)$ is a Jansian class.

Proof: (1) $\Leftrightarrow$ (2) This is clear by the above proposition.

(2) $\Leftrightarrow$ (3) This proof is due to Azumaya [2]. First of all, it is clear that if $\mathcal{C}$ is a Jansian class with corresponding idempotent two-sided ideal C, then $\mathcal{C} = \{X \in \text{Mod-}R \mid XC = 0\}$, $\mathcal{C}^\ell = \{X \in \text{Mod-}R \mid XC = X\}$, and $\mathcal{C}^r = \{X \in \text{Mod-}R \mid \forall x \in X,\ xC = 0 \Rightarrow x = 0\}$. Suppose $T = eR$ for some idempotent $e \in R$. Let $D = R(1 - e)$, then D is an idempotent two-sided ideal of R. Also $D + T = R$ since $Re \subseteq T$ and $D + Re = R$, and $DT = 0$. Let $X \in \text{Mod-}R$ be a member of the Jansian class corresponding to D, i.e. $XD = 0$. Then $\forall x \in X$, $x = x(1 - e) + xe = xe$, and therefore $xT = 0 \Rightarrow x = 0$. Hence $X \in (\mathfrak{J}^*)^r$. On the other hand, let $X \in (\mathfrak{J}^*)^r$ and let $x \in XD$. Then since $DT = 0$, $xT = 0$, hence $x = 0$ and X is a member of the Jansian class corresponding to D. $(\mathfrak{J}^*)^r$ is thus the Jansian class corresponding to D.

Conversely, suppose $(\mathfrak{J}^*)^r$ is a Jansian class. Let D be the corresponding idempotent ideal of R. Then since $R/T \in \mathfrak{J}^*$, $(R/T)D = R/T$ and hence $D + T = R$. Also $D \in \mathfrak{J}^*$ since $DD = D$, and therefore $DT = 0$. Therefore $\exists f \in D$ and $\exists e \in T$ such that $f + e = 1$. Then $\forall t \in T$, $t = ft + et = et$, and in particular $e = e^2$. Thus $T = eR$ where $e \in R$ is an idempotent.

A similar proof holds for $(\mathfrak{J}^*)^\ell = \mathfrak{J}^*$.

We now return to the colocalization at P obtained by Lambek and Rattray [25], and we will show that it is the same as our colocalization at P. Recall that they started with a cotriple (S, ϵ', δ') on Mod-R, where S: Mod-R $\longrightarrow$ Mod-R is defined by $S(M) = \sum_{f: P \rightarrow M} P \quad \forall M \in$ Mod-R, and an element of $S(M)$ is written as $\sum_f (f, p_f)$. $S(M)$ is a right R-module in view of the definitions $\sum_f (f, p_f) + \sum_f (f, q_f) = \sum_f (f, p_f + q_f)$, and $(\sum_f (f, p_f))r = \sum_f (f, p_f r) \ \forall r \in R$. $\epsilon'(M): S(M) \longrightarrow M$ is given by $\epsilon'(M)(\sum_f (f, p_f)) = \sum_f f(p_f)$. If $k_f: P \longrightarrow \sum_f P$ is the canonical injection then $\epsilon'(M)k_f = f$. For any $g: M \longrightarrow N$ in Mod-R, $S(M) \longrightarrow S(N)$ is given by $S(g)(\sum_f (f, p_f)) = \sum_f (gf, p_f)$, i.e. for the canonical injection k_f, $S(g)k_f = k_{gf}$. Their colocalization $Q'(M)$ of M at P is given by the coequalizer $\varkappa(M)$: $S(M) \longrightarrow Q'(M)$ of the pair of mappings $\epsilon' S(M)$, $S\epsilon'(M): S^2(M) \longrightarrow S(M)$. The following lemma is the dual of [25, Lemma 1].

LEMMA 4.5. $\forall M \in$ Mod-R, $\varkappa(M)$ is the joint coequalizer of all pairs of mappings $u, v: P \longrightarrow S(M)$ which equalize $\epsilon'(M): S(M) \longrightarrow M$.

Proof: Let $u: P \longrightarrow S(M)$, then $\epsilon' S(M)k_u = u$ and $S\epsilon'(M)k_u = k_{\epsilon'(M)u}$. Therefore $\varkappa(M)$ coequalizes all mappings $(u, k_{\epsilon'(M)u})$. Now let $v: P \longrightarrow S(M)$ be such that $\epsilon'(M)u = \epsilon'(M)v$. Then $\varkappa(M)$ coequalizes (u, v) since $\varkappa(M)u = \varkappa(M)k_{\epsilon'(M)u} = \varkappa(M)k_{\epsilon'(M)v} = \varkappa(M)v$. Conversely, any mapping which coequalizes all (u, v) such that $\epsilon'(M)u = \epsilon'(M)v$ coequalizes $(u, k_{\epsilon'(M)u})$ in particular since $\epsilon'(M)k_{\epsilon'(M)u} = \epsilon'(M)u$ by definition of $\epsilon'(M)$, and hence coequalizes $(\epsilon' S(M), S\epsilon'(M))$. It follows that $\epsilon'(M)$ is the joint coequalizer.

LEMMA 4.6. Let $f: B \longrightarrow A$ be an epimorphism where B is a cotorsionfree module and A is a codivisible module. Then Ker f is cotorsionfree.

Proof: Let $\bar{f}: B/(\operatorname{Ker} f)T \longrightarrow A$ be the homomorphism induced by f. Then since A is codivisible and $\bar{f}$ is an epimorphism with cotorsion kernel, $\exists\ g: A \longrightarrow B/(\operatorname{Ker} f)T$ such that $\bar{f}g = 1_A$. Therefore $(B/(\operatorname{Ker} f)T)T = B/(\operatorname{Ker} f)T = \operatorname{Im} g \oplus \operatorname{Ker} \bar{f} = (\operatorname{Im} g)T \oplus (\operatorname{Ker} \bar{f})T = (\operatorname{Im} g)T$ and hence $\operatorname{Ker} \bar{f} = 0$, i.e. $\operatorname{Ker} f = (\operatorname{Ker} f)T$.

LEMMA 4.7. $\forall M \in$ Mod-R, MT is the smallest submodule $M' \subseteq M$ such that $\forall f: P \longrightarrow M$, $0 = (P \xrightarrow{f} M \longrightarrow M/M')$.

Proof: $\forall f: P \longrightarrow M$, $(P \xrightarrow{f} M \longrightarrow M/MT) = 0$ since $f(P) \subseteq MT$. Suppose $M' \subseteq M$ is such that $\forall f: P \longrightarrow M$, $(P \xrightarrow{f} M \longrightarrow M/M') = 0$, then $\forall g \in [P, M/M']$ since P is projective $\exists f: P \longrightarrow M$ such that the diagram below commutes, and hence $g = 0$. M/M' is therefore cotorsion, and $MT \subseteq M'$.

$$\begin{tikzcd}
 & M \arrow[d] \\
P \arrow[ur, dashed, "\exists f"] \arrow[r, "\forall g"] & M/M'
\end{tikzcd}$$

THEOREM 4.8. $\forall M \in$ Mod-R, $Q^*(M)$ is the coequalizer of the pair of mappings $\epsilon' S(M)$, $S\epsilon'(M)$: $S^2(M) \longrightarrow S(M)$.

Proof:

$$\begin{tikzcd}
 & S(M) \arrow[dl, "e"'] \arrow[d, "\epsilon'(M)"] \\
Q^*(M) \arrow[r, "\epsilon(M)"] & MT
\end{tikzcd}$$

$\epsilon(M)$ and $\epsilon'(M)$ both have the same image, namely MT, and we consider then as mappings from $Q^*(M)$ to MT and from S(M) to MT, respectively. Then since S(M) is projective (since it is a coproduct of copies of P) and Ker $\epsilon(M)$ is small in $Q^*(M)$, there exists an epimorphism e: S(M) $\longrightarrow$ $Q^*(M)$, such that $\epsilon(M)e = \epsilon'(M)$. By Lemma 4.6 Ker e is cotorsionfree since S(M) is cotorsionfree and $Q^*(M)$ is codivisible. But since Ker e is cotorsionfree and Ker $\epsilon(M)$ is cotorsion, Ker $\epsilon(M)$ = Ker $\epsilon'(M)$/Ker e is the cotorsion factor module of Ker $\epsilon'(M)$, i.e. Ker e = (Ker $\epsilon'(M)$)T. Hence by Lemma 4.7 Ker e is the smallest submodule X of Ker $\epsilon'(M)$ such that $\forall f$: P $\longrightarrow$ Ker $\epsilon'(M)$, $0 = (P \xrightarrow{f} \text{Ker } \epsilon'(M) \longrightarrow \text{Ker } \epsilon'(M)/X)$. Therefore Ker e is the smallest submodule X of S(M) such that $\forall f$: P $\longrightarrow$ S(M) such that $\epsilon'(M)f = 0$, $0 = (P \xrightarrow{f}$

$S(M) \longrightarrow S(M)/X$). Hence Ker e is the smallest submodule X of $S(M)$ such that $\forall f, f' \colon P \longrightarrow S(M)$ such that $\epsilon'(M)f = \epsilon'(M)f'$, $(P \xrightarrow{f} S(M) \longrightarrow S(M)/X) = (P \xrightarrow{f'} S(M) \longrightarrow S(M)/X)$, i.e. $S(M) \xrightarrow{e} S(M)/\text{Ker } e \cong Q^{*}(M)$ is the joint coequalizer of all pairs of mappings $f, f' \colon P \longrightarrow S(M)$ which equalize $\epsilon'(M)$. Thus by Lemma 4.5 $Q^{*}(M) \cong Q'(M)$.

CHAPTER V

Localizations in Mod-R for R Left Perfect

If T' is the trace ideal of a finitely generated projective left R-module P', then Cunningham, Rutter, and Turnidge have showed [12, Prop. 1.6] that the localization Q in Mod-R corresponding to the idempotent filter of all right ideals of R containing T' is given by $Q(M) = \mathrm{Hom}_{E'}(P'_{E'}, (M \otimes_R P')_{E'})$, where $E' = \mathrm{Hom}_R({}_RP', {}_RP')$, $\forall M \in$ Mod-R.

THEOREM 5.1. The colocalization functor Q^* is given by the composite of $[P, \_]$: Mod-R $\longrightarrow$ Mod-E and its left adjoint $\_ \otimes_E P$: Mod-E $\longrightarrow$ Mod-R. If P is finitely generated then Q is given by the composite of $[P, \_]$ and its right adjoint $\mathrm{Hom}_E(P^*_E, \_)$.

Proof: It is routine to check (see remarks following Theorem 1.12) that $\_ \otimes_E P$ is the left adjoint of $[P, \_]$, and of course $Q^*(M) = [P, M] \otimes_E P$ $\forall M \in$ Mod-R. Assume now that P is finitely generated. Define $\alpha: 1_{\text{Mod-R}} \longrightarrow \mathrm{Hom}_E(P^*_E, [P, \_])$ by $((\alpha(A)(a))(p^*))(p) = a(p^*(p))$ $\forall A \in$ Mod-R, $\forall a \in A$, $\forall p^* \in P^*$, $\forall p \in P$. Define $\beta: [P, \mathrm{Hom}_E(P^*_E, \_)] \longrightarrow 1_{\text{Mod-E}}$ by $\beta(B)(g) = \sum_{i=1}^{n} (g(p_i))(f_i)$ $\forall B \in$ Mod-E, $\forall g \in [P, \mathrm{Hom}_E(P^*_E, B_E)]$, where $\{p_i \mid i = 1, \ldots, n\}$ is a set of generators for P and $\{f_i \mid i = 1, \ldots, n\}$ is the "dual basis" such that $\forall p \in P$ $p = \sum_{i=1}^{n} p_i f_i(p)$ (see e.g. [8, Prop. VII.3.1]).

Then α and β are natural transformations, and it is easy to check that $U\beta \circ \alpha U = 1_U$ and $\beta F \circ F\alpha = 1_F$ where $U = \mathrm{Hom}_E(P^*_E, \_)$ and $F = [P, \_]$. Therefore $\mathrm{Hom}_E(P^*_E, \_)$ is the right adjoint of $[P, \_]$.

The localization Q in Mod-R corresponding to the idempotent filter of all right ideals of R containing T is, by [12, Prop. 1.6], given by $Q(M) = \mathrm{Hom}_{E^*}(P^*_{E^*}, (M \otimes_R P^*)_{E^*})$ since P^* is a finitely generated left R-module with trace ideal T. ($E^* = \mathrm{End}_R({}_R P^*)$.) But by [35, Lemma 1] E and E^* are ring isomorphic, and therefore Q(M) is given by $\mathrm{Hom}_E(P^*_E, (M \otimes_R P^*)_E)$, which is in turn isomorphic to $\mathrm{Hom}_E(P^*_E, [P, M])$ (see e.g. [8, p.120]).

The remainder of this chapter investigates the property that every idempotent ideal of R is the trace ideal of a finitely generated projective module. It is known that this is the case if R is left or right perfect. The Appendix contains a new proof of this and other relevant results, and introduces notation and theory which it might be helpful to read at this point.

Let J(R), or J if there is no ambiguity, denote the Jacobson radical of R.

PROPOSITION 5.2. **If R is semiperfect, then the following statements are equivalent:**

(1) Every idempotent ideal of R is of the form ReR, where $e \in R$ is an idempotent central module J.

(2) Every idempotent ideal of R is the trace ideal of a finitely generated projective right R-module.

(3) There are exactly 2^n different idempotent ideals of R, where n is the number of isomorphism classes of simple right R-modules.

(4) The simple right R-modules determine the Jansian classes in Mod-R, i.e. if $\mathfrak{J}_T$ and $\mathfrak{J}_{T'}$ are Jansian classes in Mod-R such that for any simple right R-module S, $S \in \mathfrak{J}_T \Leftrightarrow S \in \mathfrak{J}_{T'}$, then $T = T'$.

Proof: Since R is semiperfect, $1 = e_1 + \ldots + e_m$, a sum of orthogonal local idempotents, and $\{e_1R/e_1J, \ldots, e_nR/e_nJ\}$ denotes a representative set of simple right R-modules for some $n \leq m$.

(1) ⇒ (2) $ReR = tr(eR)$ (where tr(M) denotes the trace ideal of M, $\forall M \in$ Mod-R).

(2) ⇒ (3) P finitely generated projective module ⇒ $P \cong \bigoplus_{j=1}^{r} e_jR$ (see [32] or [33]). Therefore $T = tr(P) = R(e_{i_1} + \ldots + e_{i_k})R$ for some $k \leq n$, where $\bigoplus_{j=1}^{k} e_{i_j}R$ is the basic submodule of $\bigoplus_{j=1}^{r} e_jR$, i.e. each distinct isomorphism class of indecomposable direct summands appearing in the (external) direct sum is represented by a single summand.

(3) ⇒ (4) Let $\mathfrak{J}_T$ and $\mathfrak{J}_{T'}$ be two Jansian classes in Mod-R such that for any simple right R-module S, $S \in \mathfrak{J}_T \Leftrightarrow S \in \mathfrak{J}_{T'}$, or equivalently, $S \in \mathfrak{J}_T$

$\Leftrightarrow S \in \mathfrak{I}_{T'}$. Then for some i, $1 \le i \le n$, $S \cong e_iR/e_iJ$, and $e_i \in T \Leftrightarrow e_iT = e_iR$ $\Leftrightarrow ST = S \Leftrightarrow S \in \mathfrak{I}_T \Leftrightarrow S \in \mathfrak{I}_{T'} \Leftrightarrow e_i \in T'$, and hence $T = T' = R\Sigma e_iR$ where the sum is over all e_i such that $e_iR/e_iJ \in \mathfrak{I}_{T'}$, $1 \le i \le n$.

(4) ⇒ (1) Let T be a nonzero idempotent ideal of R, then there exists a simple right R-module $S \in \mathfrak{I}_T$, $S \cong e_hR/e_hJ$ for some h, $1 \le h \le n$, or equivalently there exists an $e_h \in T$, since otherwise every simple module is in $\mathfrak{I}_T$ and hence by assumption $T = 0$. Therefore T/TJ is a nonzero R/J-module, hence $T/TJ \cong \bigoplus_{i \in I} e_iR/e_iJ$ where $\forall i \in I$, $e_i \in \{e_1, \ldots, e_n\}$. $T^2 = T \Rightarrow \forall i \in I$, $e_iR = e_iT$ and thus $e_i \in T$. Therefore $R(e_{i_1} + \ldots + e_{i_k})R \subseteq T$ where $\bigoplus_{j=1}^{k} e_{i_j}R/e_{i_j}J$ is the basic submodule of $\bigoplus_{i \in I} e_iR/e_iJ$. Also, $e_h \in T$ $\Rightarrow e_hR/e_hJ = e_hR/e_hR \cap TJ \cong (e_hR + TJ)/TJ \subseteq T/TJ \cong \bigoplus_{i \in I} e_iR/e_iJ$, hence $e_hR/e_hJ \cong e_iR/e_iJ$ for some $i \in I$, or equivalently $e_hR \cong e_iR$ for some $i \in I$, and therefore $e_h \in R(e_{i_1} + \ldots + e_{i_k})R$. Therefore $S \cong e_hR/e_hJ \in \mathfrak{I}_T \Leftrightarrow$ $e_h \in T \Leftrightarrow e_h \in R(e_{i_1} + \ldots + e_{i_k})R \Leftrightarrow S \in \mathfrak{I}_{R(e_{i_1} + \ldots + e_{i_k})R}$ and thus by assumption $T = R(e_{i_1} + \ldots + e_{i_k})R$. As in Corollary A.6 of the Appendix, T then has the form ReR, where e is an idempotent of R central modulo J.

COROLLARY 5.3. R is left perfect if and only if idempotents lift modulo J and every torsion class in Mod-R is of the form $\mathfrak{I}_{ReR}$, where e is an idempotent of R central modulo J.

Proof: By Theorem A.4 of the Appendix, if every torsion class in Mod-R of simple type is Jansian, then R/J is semiperfect, and R is left perfect if and only if all torsion classes in Mod-R are Jansian classes of simple type. Therefore if idempotents lift modulo J and every torsion class in Mod-R is of the form $\mathfrak{J}_{ReR}$ for an idempotent e of R central modulo J, then R/J is semiperfect, and by the above proposition the Jansian classes in Mod-R are of simple type.

It does not seem to be possible to eliminate the condition that idempotents lift modulo J, although one can show that if every torsion class in Mod-R is of the form $\mathfrak{J}_{ReR}$, where e is an idempotent of R central modulo J, then central idempotents lift modulo J. Let $\bar{R}$ denote R/J and $\bar{r}$ denote $r + J$, $\forall r \in R$, and let x be a central idempotent modulo J, i.e. $\bar{x}\bar{x} = \bar{x}$, and $\bar{x}\bar{r} = \bar{r}\bar{x}$ $\forall \bar{r} \in \bar{R}$. Then $\bar{x}\bar{R}$ is a direct sum of simple R-modules, and the smallest torsion class in Mod-R containing $\bar{x}\bar{R}$ is of the form $\mathfrak{J}_{ReR}$ where e is an idempotent of R central modulo J. Then $\bar{x}\bar{R} \subseteq \overline{(1 - e)}\bar{R}$. But the smallest torsion class in Mod-R containing $\overline{(1 - e)}\bar{R}$ is also $\mathfrak{J}_{ReR}$, and therefore the isomorphism class of every simple summand of $\overline{(1 - e)}\bar{R}$ is also represented by a simple summand of $\bar{x}\bar{R}$. Hence $\bar{R}\bar{x}\bar{R} = \bar{R}\overline{(1 - e)}\bar{R}$, and $\bar{x} = \overline{1 - e}$ since both are central idempotents.

We do not have an example of a ring R which is not left perfect, but for which every torsion class in Mod-R is of the form $\mathfrak{J}_{ReR}$ where e is an idempotent of R central modulo J. Dlab has provided an example [13, Ex. 2] which shows that the two conditions, idempotents lift modulo J and every torsion class in Mod-R is Jansian, do not imply that R is left perfect. Also, the two conditions, R semiperfect and every idempotent ideal of R has the form ReR, where e is an idempotent of R central modulo J, do not imply that R is left perfect, since there are rings which are right perfect but not left perfect.

CHAPTER VI

Chain Conditions

Of special interest is the case where the localization functor Q_T: Mod-R $\longrightarrow$ Mod-R associated with a Jansian class $\mathcal{J}_T$ preserves all colimits, i.e. it is right exact and commutes with direct sums. (Some authors, using Stenstrom's terminology [36], have called a torsion class perfect if the associated localization functor preserves all colimits. Others have used Goldman's terminology [19] and say such a torsion class has property T.) We will show that, in this situation, if R is respectively left, right, or two-sided Artinian or Noetherian, a right cogenerator, or quasi-Frobenius (QF), then so is $Q_T(R)$.

LEMMA 6.1. $Q_T(M) = [T, M/\mathcal{J}_T(M)] = [T/\mathcal{J}_T(T), M/\mathcal{J}_T(M)] = \mathrm{Hom}_K(T/\mathcal{J}_T(T), M/\mathcal{J}_T(M))$ where $K = R/\mathcal{J}_T(R)$, $\forall M \in$ Mod-R.

Proof: The first equality follows from the well-known construction of the quotient module of a module M, for a localization functor Q associated with a torsion class $\mathcal{J}$, given by the direct limit

$$Q(M) = \varinjlim_{D \in \mathcal{D}_{\mathcal{J}}} [D, M/\mathcal{J}(M)] \qquad \text{(see e.g. [22, Sec. 0])}$$

and the fact that T is the minimal element of $\mathcal{D}_{\mathcal{J}}$. Since $\mathcal{J}_T(T)$ is torsion and $M/\mathfrak{T}_T(M)$ is torsionfree we have the second equality, and the third follows since a torsionfree R-module is a $R/\mathfrak{T}_T(R)$-module, i.e. a K-module.

Let $\bar{T}$ denote $T/\mathfrak{T}_T(T)$. Note that $\bar{T} = T/\mathfrak{T}_T(R) \cap T \cong (T + \mathfrak{T}_T(R))/\mathfrak{T}_T(R) \subseteq R/\mathfrak{T}_T(R) = K$. Also, by the above lemma, $Q_T(R)\bar{T} \subseteq \bar{T}$ (and, in fact, $Q_T(R)\bar{T} = \bar{T}$).

THEOREM 6.2. The following statements are equivalent:

(1) Every right $Q_T(R)$-module is torsionfree (as an R-module).

(2) $\bar{T}Q_T(R) = Q_T(R)$ (or, equivalently, $TQ_T(R) = Q_T(R)$).

(3) Every right $Q_T(R)$-module is torsionfree and divisible (as an R-module).

(4) $Q_T(M) \cong M \otimes_R Q_T(R)$ $\forall M \in$ Mod-R.

(5) Q_T is right exact and commutes with direct sums.

(6) $\bar{T}$ is a finitely generated projective right K-module.

(7) T is a finitely generated right R-module and T is projective with respect to the class of epimorphisms $\mathcal{E} = \{ A \longrightarrow B \longrightarrow 0 \mid A, B \in \mathfrak{T}_T \}$.

(8) There exists a ring F and an epimorphism of rings: $R \longrightarrow F$ such that ${}_RF$ is flat and $\mathfrak{T}_T = \{M \in \text{Mod-R} \mid M \otimes_R F = 0\}$.

<u>Proof</u>: The equivalence of (1) through (5) is proved in [19, Thm. 3].

(2) $\Leftrightarrow$ (6) This is a generalization of an equivalence in [37, Thm. 5.6].

Assume $\overline{T}Q_T(R) = Q_T(R)$, then $\exists \bar{t}_i \in \overline{T}$, $q_i \in Q_T(R)$, $i = 1, \ldots, n$, such that $\sum_{i=1}^{n} \bar{t}_i q_i = 1_{Q_T(R)}$. Define $f_i : \overline{T} \longrightarrow K$ by $f_i(\bar{t}) = q_i \bar{t}$ $\forall \bar{t} \in \overline{T}$, $i = 1, \ldots, n$, then $\bar{t} = \sum_{i=1}^{n} \bar{t}_i f_i(\bar{t})$. Therefore $\overline{T}$ is a finitely generated projective K-module by the dual basis lemma [8, Prop. VII.3.1].

Conversely, let $f_i : \overline{T} \longrightarrow K$, $\bar{t}_i \in \overline{T}$, $i = 1, \ldots, n$, be such that $\forall \bar{t} \in \overline{T}$, $\bar{t} = \sum_{i=1}^{n} \bar{t}_i f_i(\bar{t})$. But $f_i(\bar{t}) = q_i \bar{t}$ $\forall \bar{t} \in \overline{T}$ for some unique $q_i \in Q_T(R)$, $1 = 1, \ldots, n$, since $K/\overline{T}$ is torsion and $Q_T(R)$ is torsionfree and divisible, hence $\forall \bar{t} \in \overline{T}$, $(1_{Q_T(R)} - \sum_{i=1}^{n} \bar{t}_i q_i)\bar{t} = 0$. This implies that $1_{Q_T(R)} = \sum_{i=1}^{n} \bar{t}_i q_i$, and therefore $\overline{T}Q_T(R) = Q_T(R)$, since $\bar{t} \longrightarrow (1_{Q_T(R)} - \sum_{i=1}^{n} \bar{t}_i q_i)\bar{t}$ $\forall \bar{t} \in \overline{T}$ defines an R-homomorphism: $\overline{T} \longrightarrow Q_T(R)$ which has a unique extension to an R-homomorphism: $K \longrightarrow Q_T(R)$ (i.e. $q\overline{T} = 0 \Rightarrow q = 0$ $\forall q \in Q_T(R)$).

(5) $\Leftrightarrow$ (8) See e.g. [24, Prop. 2.2].

The following proposition is a generalization of [37, Prop. 3.1].

PROPOSITION 6.3. Let $\mathcal{J}_T$ be a Jansian class in Mod-R such that $\overline{T}$ is a finitely generated right K-module. Then K left Artinian (Noetherian) $\Rightarrow$ $Q_T(R)$ left Artinian (Noetherian).

Proof: For some n, there exists an epimorphism: $K^n \longrightarrow \overline{T}$ as right K-modules, hence $0 \longrightarrow \mathrm{Hom}_K(\overline{T}, K) \longrightarrow \mathrm{Hom}_K(K^n, K)$ is exact as left K-modules. But $\mathrm{Hom}_K(K^n, K) \cong \overset{n}{\Sigma}\mathrm{Hom}_K(K, K) \cong \overset{n}{\Sigma} K$ is left Artinian (Noetherian), hence $Q_T(R) \cong \mathrm{Hom}_K(\overline{T}, K)$ is left Artinian (Noetherian).

COROLLARY 6.4. Let $\mathfrak{I}_T$ be a Jansian class in Mod-R such that $\overline{T}$ is a finitely generated right K-module. Then R left Artinian (Noetherian) $\Rightarrow$ $Q_T(R)$ left Artinian (Noetherian).

Proof: R left Artinian (Noetherian) $\Rightarrow$ K left Artinian (Noetherian) since K is a factor ring of R.

PROPOSITION 6.5. Let $\mathfrak{I}_T$ be a Jansian class in Mod-R such that $\overline{T}Q_T(R) = Q_T(R)$. Then R a cogenerator for Mod-R $\Rightarrow$ $Q_T(R)$ a cogenerator for Mod-$Q_T(R)$.

Proof: $\forall M \in$ Mod-$Q_T(R)$, $\exists$ a monomorphism $f: M \longrightarrow R^I$ for some index set I. M is torsionfree by Theorem 6.2, therefore if $g: R^I \longrightarrow K^I$ is the canonical mapping induced by the projection mapping: $R \longrightarrow R/\mathfrak{I}_T(R) = K$, and if $h: K^I \longrightarrow Q_T(R)^I$ is the inclusion mapping, then $\forall m \in M$, $hgf(m) = 0 \Rightarrow gf(m) = 0 \Rightarrow f(m)T = 0 \Rightarrow f(mT) = 0 \Rightarrow mT = 0 \Rightarrow m = 0$. Hence hgf is a monomorphism. $Q_T(R)^I$ is a torsionfree divisible module since $Q_T(R)$ is a torsionfree divisible module, and therefore hgf is a $Q_T(R)$-homomorphism (see e.g. [22, Sec. 1]). Thus $Q_T(R)$ is a cogenerator for Mod-$Q_T(R)$.

COROLLARY 6.5. Let $\mathfrak{J}_T$ be a Jansian class in Mod-R such that $\overline{T}Q_T(R) = Q_T(R)$. Then R QF (quasi-Frobenius) $\Rightarrow$ $Q_T(R)$ QF.

Proof: A ring R is QF $\Leftrightarrow$ R is left Noetherian and a cogenerator for Mod-R. The result now follows from Corollary 6.4 and the above proposition.

The following proposition is a generalization of [37, Prop. 5.9].

PROPOSITION 6.7. Let $\mathfrak{J}_T$ be a Jansian class in Mod-R such that $\overline{T}Q_T(R) = Q_T(R)$. Then there is a one-to-one correspondence between the right ideals of $Q_T(R)$ and the right ideals of K of the form $I\overline{T}$ for some right ideal $I \subseteq K$.

Proof: $\overline{T} = T/\mathfrak{J}_T(T)$ is an idempotent ideal of $K = R/\mathfrak{J}_T(R)$, since T is an idempotent ideal of R, and $\mathfrak{J}_T(T) = T \cap \mathfrak{J}_T(R)$ is an ideal of R. Therefore for any right ideal X of $Q_T(R)$, the correspondence $X \longrightarrow X\overline{T}$ is of the desired form since $X\overline{T} = X\overline{T}\,\overline{T}$ and $X\overline{T} \subseteq \overline{T} \subseteq K$. For any right ideal I of K, the inverse correspondence is given by $I\overline{T} \longrightarrow I\overline{T}Q_T(R)$. Then $X\overline{T}Q_T(R) = XQ_T(R) = X$, and $I\overline{T}Q_T(T)\overline{T} = I\overline{T}\,\overline{T} = I\overline{T}$.

COROLLARY 6.8. Let $\mathfrak{J}_T$ be a Jansian class in Mod-R such that $\overline{T}Q_T(R) = Q_T(R)$. Then R right Artinian (Noetherian) $\Rightarrow$ $Q_T(R)$ right Artinian (Noetherian).

Proof: R right Artinian (Noetherian) $\Rightarrow$ K right Artinian (Noetherian). The result now follows from the proposition, or from the observation that for any right ideal $X \subseteq Q_T(R)$, $X = XQ_T(R) \supseteq (X \cap K)Q_T(R) \supseteq (X \cap \bar{T})Q_T(R) \supseteq X\bar{T}Q_T(R) = XQ_T(R) = X$.

It is clear that the result holds for two-sided ideals as well as for right ideals.

APPENDIX

A ring R is called <u>semiperfect</u> if every finitely generated right (or left) R-module has a projective cover. Equivalent conditions are: R/J is semisimple Artinian and idempotents can be lifted modulo J, where J is the Jacobson radical of R (see [3]); every simple right (left) R-module is of the form eR/eJ (Re/Je), $e^2 = e \in R$ (and in particular, if $e \in R$ is a primitive idempotent, a simple right (left) R-module M $\cong$ eR/eJ (Re/Je) if and only if $Me \neq 0$ $(eM \neq 0)$); there are mutually orthogonal idempotents $e_1, \ldots, e_m$ such that $\sum_{i=1}^{m} e_i = 1$ and each e_i is local, i.e. each e_iRe_i is a local (or completely primary, or sum-irreducible) ring (see [31] or [32]).

If R is semiperfect, $R = e_1R \oplus \ldots \oplus e_mR$ where $\{e_1, \ldots, e_m\}$ is a set of orthogonal local idempotents. Let $\{e_1R, \ldots, e_nR\}$ be a complete set of representatives for the isomorphism classes of the e_iR's, for some $n \leq m$. Let $\bar{R}$ denote R/J and $\bar{r}$ denote $r + J$ $\forall r \in R$. We can then form 2^n different idempotent ideals T of the form $T = R(e_{i_1} + \ldots + e_{i_k})R = Re_{i_1}R + \ldots + Re_{i_k}R$, since $\bar{R} = \bar{R}\bar{e}_1\bar{R} \oplus \ldots \oplus \bar{R}\bar{e}_n\bar{R}$. Now $\mathfrak{J}_{\mathfrak{D}}$ denotes the torsion class in Mod-R which corresponds to the idempotent filter of right ideals $\mathfrak{D}$. If $\mathfrak{D}$ has a minimal element T, we will also denote the corresponding torsion class by $\mathfrak{J}_T$ (and the corresponding torsion theory by $(\mathfrak{J}_T, \mathfrak{F}_T)$).

Let M be a simple right R-module, then in the torsion theory $(\mathfrak{J}_T, \mathfrak{F}_T)$ where $T = R(e_{i_1} + \ldots + e_{i_k})R$, M is torsion $\Leftrightarrow$ $MT = 0 \Leftrightarrow M \cong e_jR/e_jJ$ where $e_j \notin \{e_{i_1}, \ldots, e_{i_k}\}$, and M is torsionfree $\Leftrightarrow$ $MT = M$ $\Leftrightarrow$ $M \cong e_{i_j}R/e_{i_j}J$ for some j, $1 \leq j \leq k$.

A ring R is called <u>left perfect</u> if every left R-module has a projective cover. Equivalent conditions include: R/J is semisimple Artinian and every nonzero right R-module has a nonzero simple submodule (see [3]); R satisfies the descending chain condition on finitely generated right ideals (see [5]).

Therefore if R is left perfect, R is torsionfree with respect to the torsion theory $(\mathfrak{J}_T, \mathfrak{F}_T)$ $\Leftrightarrow$ every simple right ideal is torsionfree $\Leftrightarrow$ $T \supseteq T_0 = R(e_{j_1} + \ldots + e_{j_h})R$ where $\{e_{j_1}R/e_{j_1}J, \ldots, e_{j_h}R/e_{j_h}J\}$ is a representative set of nonisomorphic simple right ideals of R.

DEFINITION A.1. Let C be any class of right R-modules. Then define $C^R = \{X \in \text{Mod-}R \mid [C, I(X)] = 0 \ \forall C \in \mathcal{C}\}$ and $C^L = \{X \in \text{Mod-}R \mid [X, I(C)] = 0 \ \forall C \in \mathcal{C}\}$ where recall that $\forall M \in \text{Mod-}R$, I(M) denotes the injective hull of M. The largest torsion theory in which C is torsionfree is (C^{L}, C^{Lr}), and the smallest torsion theory in which C is torsion is $(C^{R\ell}, C^{R})$ (see [22]).

Let $\mathcal{S} = \{S_1, \ldots, S_n\}$ be a representative set of nonisomorphic simple right R-modules. For any $C \subseteq \mathcal{S}$, the class of all right R-modules M such that every nonzero homomorphic image of M has a nonzero submodule isomorphic to a member of C is a torsion class, and was called a <u>torsion</u> <u>class</u> <u>of</u> <u>simple</u> <u>type</u> by Alin and Armendariz [1]. We denote this torsion class by ${}_{C}\mathfrak{T}$, and if $C = \{S\}$, by ${}_{S}\mathfrak{T}$.

PROPOSITION A.2. $\forall C \subseteq \mathcal{S}$, ${}_{C}\mathfrak{T} = C^{R\ell}$, the smallest torsion class containing C.

<u>Proof</u>: Suppose $M \in {}_{C}\mathfrak{T}$, then $[M, I(X)] \neq 0 \Rightarrow [S, I(X)] \neq 0$ for some $S \in C \Rightarrow X \notin C^{R}$. Therefore $X \in C^{R} \Rightarrow [M, I(X)] = 0$ and hence $M \in C^{R\ell}$. Conversely, let $M \in C^{R\ell}$, and suppose $\exists\, N \subseteq M$ such that M/N does not have a nonzero submodule isomorphic to a member of C. Then $[S, I(M/N)] = 0$ $\forall S \in C$, and therefore $M/N \in C^{R} \cap C^{R\ell} = 0$. Thus $M \in {}_{C}\mathfrak{T}$.

Dlab [13] defined a prime (fundamental) torsion class to be a torsion class associated with the smallest idempotent filter of right ideals containing a class (union of classes) of equivalent maximal right ideals, where M_1 and M_2 are equivalent maximal right ideals if $R/M_1 \cong$

R/M_2. The fundamental torsion classes are thus the same as torsion classes of simple type, with the prime torsion classes being the torsion classes ${}_S\mathfrak{T}$. Alin and Armendariz [1] and Dlab [13] both showed that every torsion class in Mod-R is of simple type if and only if every nonzero R-module has a nonzero simple submodule, and they also showed that if R is left perfect then every torsion class in Mod-R is Jansian (i.e. closed under direct products). Thus every torsion theory in Mod-R, for R left perfect, is of the form $(\mathfrak{T}_T, \mathfrak{F}_T)$ and there are exactly 2^n torsion theories, where n is the number of isomorphism classes of simple right R-modules. Rutter [34] has also given a proof of this result. A simple proof, presented by Vamos in a seminar at McGill University (1971-72), makes use of the one-to-one correspondence between Jansian classes and idempotent ideals of R. His proof is as follows:

As we have seen, if R is semiperfect, there are at least 2^n idempotent ideals T of R, and hence Jansian classes in Mod-R, where n is the number of isomorphism classes of simple right R-modules. But every torsion theory can be described as the largest torsion theory for which a certain injective module I_R is torsionfree, and naturally two injectives I and I' give rise to the same torsion theory if and only if $I \subseteq (I')^{\alpha}$ and $I' \subseteq I^{\beta}$ for some powers α and β. If R is left perfect, the socle S of I is essential in I, and hence $I = I(S)$. $S = \bigoplus_{\alpha \in A} S_\alpha = B_1 \oplus \ldots \oplus B_k$ where B_i is the sum of all the S_α's isomorphic to S_i, $i = 1, \ldots, k \leq n$ (where recall that $\mathcal{S} = \{S_1, \dots, S_n\}$ denotes a representative set of nonisomorphic simple right R-modules), and A is some index set. Therefore $I = I(S) = I(B_1) \oplus \dots \oplus I(B_k)$. Let $I' = I(S_1) \oplus \dots \oplus I(S_k)$, then clearly I and I' induce the same torsion theory, and hence there are no more than 2^n torsion theories in Mod-R. Thus there are exactly 2^n torsion theories in Mod-R and each one is of the form $(\mathcal{T}_T, \mathcal{F}_T)$ for an idempotent ideal $T = R(e_{i_1} + \dots + e_{i_k})R$.

As a consequence of this and previous remarks, we have the following theorem, which was proved by Storrer [37, Thm. 2.5].

THEOREM (Storrer) A.3. If R is left perfect, then the largest torsion theory in which R is torsionfree has the minimal dense ideal $T_o = R(e_{j_1} + \dots + e_{j_h})R$ where $\{e_{j_1}R/e_{j_1}J, \dots, e_{j_h}R/e_{j_h}J\}$ is a representative set of nonisomorphic simple right ideals of R.

Dlab [13] also proved the following result:

THEOREM (Dlab) A.4. If every torsion class of simple type in Mod-R is Jansian, then R/J is semisimple Artinian. Hence a ring R is left perfect if and only if all torsion classes in Mod-R are Jansian classes of simple type.

Dlab [14, Thm. 1] also examined rings R for which 0 and Mod-R are the only torsion classes in Mod-R, and gave several equivalent properties. He did not explicitly give the following one, although it is perhaps clear from his proof that it is equivalent. Gardner [18] has also studied these rings.

COROLLARY A.5. R is left perfect and R/J is simple if and only if 0 and Mod-R are the only torsion classes in Mod-R.

Proof: If R is left perfect, then clearly R/J is simple if and only if R has no nontrivial idempotent ideals if and only if 0 and Mod-R are the only torsion classes in Mod-R. If 0 and Mod-R are the only torsion classes in Mod-R they are of course both Jansian classes of simple type, hence R is left perfect.

We also have the following result, which was proved by Michler [27, Prop. 2.1].

COROLLARY A.6. If R is left or right perfect, then every idempotent ideal in R is of the form ReR, where e is an idempotent of R central modulo J.

Proof: There are 2^n idempotent ideals in R, each one of the form $R(e_{i_1}+\ldots+e_{i_k})R$. If we let $\{e_{i_{k+1}}, \ldots, e_{i_j}\}$ be those idempotents from the decomposition $1 = e_1 + \ldots + e_m$ which generate a right ideal isomorphic to $e_{i_h}R$ for some h, $1 \leq h \leq k$, then $R(e_{i_1} + \ldots + e_{i_k})R = R(e_{i_1} + \ldots + e_{i_j})R$ and $e_{i_1} + \ldots + e_{i_j}$ is central modulo J, since $(\bar{e}_{i_1} + \ldots + \bar{e}_{i_j})\bar{R}$ is a sum of simple (or homogeneous) components of $\bar{R}$.

Dlab [14, Thm. 2] gave several properties equivalent to a ring R being a finite direct sum of rings R_i for which Mod-R_i has only the two trivial torsion classes, 0 and Mod-R_i. Vamos, again in a seminar at McGill University (1971-72), gave another equivalent property, and since it does not seem to have appeared in the literature, we also give a proof, basically as Vamos presented it except for the use of the primary decomposition of R.

THEOREM (Vamos) A.7. A ring R is a finite direct sum of rings R_i, where each R_i is left perfect and R_i/J_i is simple, if and only if every torsion class in Mod-R is Jansian and stable.

<u>Proof</u>: Suppose $R = R_1 \oplus \ldots \oplus R_n$, where R_i is left perfect and R_i/J_i is simple, $i = 1, \ldots, n$. Then for any right R-module M, $M = MR_1 \oplus \ldots \oplus MR_n$, and for any torsion theory $(\mathfrak{T}, \mathfrak{F})$ in Mod-R, the torsion submodule $\mathfrak{T}(M) = \mathfrak{T}(MR_1) \oplus \ldots \oplus \mathfrak{T}(MR_n)$. Let $\mathfrak{T}_i(MR_i) = \mathfrak{T}(MR_i)$, then $\{X \in \text{Mod-}R_i \mid \mathfrak{T}_i(X) = X\}$ is a torsion class in Mod-R_i and hence by Corollary A.5 is

either 0 or Mod-R_i, $i = 1, \ldots, n$. Suppose $M = \mathfrak{T}(M)$, then $\mathfrak{T}_i(MR_i) = MR_i$, $i = 1, \ldots, n$. If $MR_i = 0$, then $I(MR_i) = 0$ hence $I(MR_i) = \mathfrak{T}_i(I(MR_i))$; if $MR_i \neq 0$, then $\mathfrak{T}_i = \text{Mod-}R_i$ and hence $I(MR_i) = \mathfrak{T}_i(I(MR_i))$. Therefore $I(M) = I(MR_i) \oplus \ldots \oplus I(MR_n) = \mathfrak{T}_1(I(MR_1)) \oplus \ldots \oplus \mathfrak{T}_n(I(MR_n)) = \mathfrak{T}(I(M))$, and thus $\mathfrak{T}$ is stable. R is of course left perfect, and hence by Theorem A.4 $\mathfrak{T}$ is Jansian.

Conversely, suppose every torsion class in Mod-R is Jansian and stable. By Theorem A.4, to show that R is left perfect it suffices to show that every torsion class in Mod-R is of simple type, or equivalently, that every nonzero right R-module contains a nonzero simple submodule. Suppose $\exists M \in$ Mod-R such that M has no nonzero simple submodule, then every simple right R-module is in the torsion class $\{I(M)\}^L$. But then every right R-module is in $\{I(M)\}^L$, since $\prod_{S \text{ simple}} I(S)$ is a cogenerator for Mod-R and $\{I(M)\}^L$ is stable and Jansian. Therefore $I(M) = 0$, hence $M = 0$, and therefore R is left perfect. R is then an essential extension of its right socle, hence $R \in {}_{\mathcal{S}}\mathfrak{T}$ (the smallest torsion class in Mod-R containing every simple right R-module). By a result of Dickson [15, Thm. 2.2], since every torsion class in Mod-R is stable, R has <u>primary decomposition</u>, i.e. $\forall M \in {}_{\mathcal{S}}\mathfrak{T}$, $M = \bigoplus_{i=1}^{n} {}_{S_i}\mathfrak{T}(M)$ where recall that $\mathcal{S} = \{S_1, \ldots, S_n\}$ is a representative set of nonisomorphic simple right R-modules (see e.g. [15], [7]). Thus $R = \bigoplus_{i=1}^{n} {}_{S_i}\mathfrak{T}(R)$, and ${}_{S_i}\mathfrak{T}(R)$ is an essential extension of a sum of simple modules each isomorphic to S_i, $i = 1, \ldots, n$. Then $\forall\, i \neq j$, $S_j \in {}_{S_i}\mathfrak{T}(R)$, and $[{}_{S_i}\mathfrak{T}(R), {}_{S_j}\mathfrak{T}(R)] = 0$, $1 \le i, j \le n$. Each ${}_{S_i}\mathfrak{T}(R)$ may be written in the form $e_{S_i}R$ where $e_{S_i}^2 = e_{S_i} \in R$, and $0 = [e_{S_i}R, e_{S_j}R] \cong e_{S_j}Re_{S_i}$ implies $re_{S_i} = e_{S_i}r$ $\forall r \in R$, $1 \le i \le n$. Let $R_i = e_{S_i}R$, $i = 1, \ldots, n$, then $R = R_1 \oplus \ldots \oplus R_n$ is a finite direct sum of of left perfect rings. $R_i/J_i \in {}_{S_i}\mathfrak{T}$ since $R_i \in {}_{S_i}\mathfrak{T}$, hence R_i/J_i is a sum of simple right ideals each isomorphic to S_i and thus is simple, $i = 1, \ldots, n$.

The above theorem, as well as the results by Dlab referred to in the remarks preceding the theorem, thus characterize those rings R such that every torsion class in Mod-R is a torsion-torsionfree class, i.e. Jansian and stable. Bronowitz and Teply [7, Thm. 3] have characterized those rings R such that every pre-torsion class is a pre-torsion - pre-torsionfree class, or what has been called TTF, i.e. Jansian. No one, it seems, has been able to characterize those rings R such that every torsion class is Jansian.

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