Rigorous parameterization of stable and unstable manifolds of fixed points from Ordinary differential equations with polynomial vector field depending on parameters

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April 20, 2020

A thesis submitted in partial fulfillment of the requirements of the degree of M.Sc. Mathematics and Statistics

Abstract

Back in 1675, Ordinary differentials equations were already under the scope. Throughout the years, their use has risen. Scientists began using them with parameters to model real-life problems. Along the way, it gave birth to stable and unstable manifolds of fixed points with a dependency on parameters. Nowadays, these objects are really useful, amongst others, to spacecraft missions. Yet, their computations are not simple and often carry errors of approximation. To handle those matters, scientists need to develop general rigorous methods. This thesis introduces a reliable method that, under not too restrictive assumptions, is used to rigorously compute a local approximation of these objects. The method is based on a parameterization via power series whose coefficients are computed exactly from a conjugacy relation between the vector field of the studied system and its linearization. The method provides control of the errors of approximation depending on the number of coefficients computed and the size of the domain of the parameterization.

Résumé

L'étude des Équations différentielles ordinaires remonte jusqu'en 1675. À travers les années, elles ont été utilisées de plus en plus. Les scientifiques ont commencé à y ajouter des paramètres afin de modéliser des problèmes dans la vie courante. Ce faisant, cela a donné naissance aux variétés stables et instables de points fixes avec une dépendance aux paramètres. De nos jours, ces objets sont très utiles, entre autres, pour les missions spaciales. Pourtant, leur calcul n'est pas simple et il est souvent accompagné d'erreurs d'approximation. Afin de traiter ces inconvénients, les scientifiques ont besoin de développer des méthodes générales rigoureuses. Ce mémoire introduit une méthode fiable qui, sous des hypothèses non trop contraignantes, est utilisée pour calculer rigoureusement une approximation locale de ces objets. Cette méthode est basée sur une paramétrisation en séries de puissance dont les coefficients sont calculés exactement à l'aide d'une relation de conjugaison entre le champ de vecteurs du système étudié et sa linéarisation. Cette méthode fournit du contrôle sur les erreurs d'approximation qui dépend du nombre de coefficients calculés et de la taille du domaine de paramétrisation.

Acknowledgments

I would like to thank my supervisor, Jean-Philippe Lessard, for all the advice he gave me throughout my thesis, I would not have been able to do it without him.

I also want to give a special thanks to my family for the support and encouragements they gave me to finish the thesis, it helped me go through.

Moreover, I would like to thank my girlfriend and her family who always encouraged me to advance and look for the best.

Finally, to all who supported me throughout my thesis, I thank you with the bottom of my heart.

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1 Introduction

The study of Ordinary differential equations (ODEs) dates back from 1675 (see [13]). Hence, there is a lot we know on ODEs. There are natural mathematical objects that arise from ODEs. Some of them are fixed points, periodic orbits, stable and unstable manifolds of fixed points, stable and unstable manifolds of periodic orbits, etc. In particular, stable and unstable manifolds of fixed points are quite helpful to characterize the set of initial conditions that give convergent solutions as time goes to infinity.

Let us focus on stable and unstable manifolds of fixed points (see Section 2). These objects are found, among others, in biology and physics. For instance, in predator-prey models like in [15], the study of stable manifolds enables looking at different scenarios for the convergence of solutions depending on where are located the initial conditions and the value of the parameters. Furthermore, those models often depends on parameters. Therefore, studying the stable manifolds with a dependency on those parameters provides answers regarding the parameters values to input in order to get desired results. Moreover, in physics, especially in space-craft missions like in [16], the study of stable and unstable manifolds of fixed points and periodic orbits is very useful since it optimizes the use of fuel in order for satellites to go farther in space with less fuel. Once again, the use of parameters is useful, so it is natural to have parameter-dependent stable and unstable manifolds.

Now, when it comes to the computation of these objects, it is often hard nay impossible to do it by hand. Nonetheless, since the arrival of computers, the computation of these objects has become easier, although it still is hard, and has received more attention from scientists. However, although there exists a theorem for their existence and uniqueness, there exists no general constructive approach to compute them. The same is true for solution of ODEs : There exists a theorem of existence and uniqueness for the solutions (see [4]) but there exists no general constructive approach to compute them rigorously. Nonetheless, researches have been done and specific-cases methods have been developped to compute stable and unstable manifolds of fixed points.

In this thesis, we consider the class of ODEs given as polynomial vector fields. Although they are easy to write and apply to a broad variety of real-life problems, for instance predator-prey models (see [15]) and space-craft missions (see [16]) as mentioned above – one can recover polynomial vector fields from the analytic ones of

[16] through automatic differentiation (see [14]) –, they are way harder to solve than linear systems. Several methods have been developpped to compute specific solutions of them. Some of these methods include looking for homoclinic and heteroclinic orbits (see [15]) and low-energy transfers (see [16]). Both of these two methods require the computation of stable and unstable manifolds of fixed points. [15] uses a qualitative approach that does not compute them but tell them the behavior of the solutions – they left the computation of the stable and unstable manifolds of their fixed points as an open question. [16] computes them by means of Newton's method. Note that our method applies to most of the stable and unstable manifolds of fixed points of [15]. Moreover, note that our method applies to the stable and unstable manifolds of fixed points of [16] through automatic differentiation (see [14]) – the later is mandatory to convert their vector fields to polynomial ones.

The goal of this thesis is to develop a rigorous computer-assisted method to compute efficiently parameterized families of the local unstable and stable manifolds of fixed points with respect to some parameters and subject to some assumptions that are not too restraining. For that purpose, here are the assumptions that will be considered :

A0. We consider an ODE that depends on some variables $x(t) \in \mathbb{R}^n$ and parameters $\omega \in \mathbb{R}^p$, say

$$\dot{x} = f(x, \omega) \quad ; \tag{1.1}$$

A1. (1.1) has at least one hyperbolic fixed point – a constant solution whose Jacobian Matrix has no eigenvalue lying on the subspace $\{z \in \mathbb{C} : \Re(z) = 0\}$ – for some value $\tilde{\omega}$ of the parameters ω ;

A2. The vector field $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is polynomial in both x and ω ;

A3. The Jacobian Matrix is diagonalizable at this hyperbolic fixed point for the value $\tilde{\omega}$ of the parameters ω. Moreover, all its eigenvalues are distinct.

Remark.

— $\tilde{\omega}$ will be chosen later on.

— There is one further assumption not mentioned above (no resonance). It is going to be stated later on when its needs occur.

Let us have a closer look at these assumptions.

1. Hyperbolicity of fixed points is often found in real-life problems. However, it is a restriction since our results do not hold otherwise. Nevertheless, variations of the Stable Manifold Theorem, which we are going to cover later on, have been

studied in recent years, so one could go over our study again with a variation of the previous theorem.

2. Many ODEs are given by polynomials. Moreover, polynomials are dense in the set of continuous functions. Furthermore, consider that many ODEs are piecewise continuous – $\dot{x} = \frac{1}{x}$ is continuous everywhere but $x = 0$. As seen previously, the addition of parameter-dependency for the vector field actually allows to consider many practical applications that need control to be optimized – the control is done through parameter variation (again, see [15] and [16]). Moreover, it gives rise to chaos and bifurcations in dynamical systems, two subjects that received a lot of attention in the past 30 years.

3. The diagonalizability of the Jacobian Matrix is not too restrictive. Indeed, diagonalization is a generic property – the set of diagonalizable matrices is dense in the set of all matrices over the field $\mathbb{C}$ –, whence not restrictive. Unless one chooses or makes an example with this property not being verified, the Jacobian Matrix will be diagonalizable with probability one. Moreover, the set of matrices with distinct eigenvalues is dense in the set of all matrices over the field $\mathbb{C}$. Hence, even this assumption, stronger than the diagonalizability, is not too restrictive.

As mentioned previously, the method developped in this thesis, with respect to the above framework, allows one to compute both the stable and unstable manifolds of fixed points of [15] and [16]. Our method uses Taylor series to parameterize the stable and unstable manifolds of fixed points (see Section 3). Hence, our approximations are given by polynomials. Nonetheless, it is done without Newton's method. As opposed to Newton's method, our method does not require to invert a matrix that depends on the size of the polynomial approximation in order to get a bound on the latter. The method developped in this thesis has already been introduced in [2]. Nevertheless, the proofs here differ from the ones of [2] to focus more on the polynomial form of the vector fields and the explicit computations of both the approximations and the bounds. Indeed, in Section 4, we focus on the easiness of the computations of the polynomial approximations of the stable and unstable manifolds of fixed points, as well as on giving a proper and detailed explanation of the computation of the bounds in Section 5. Moreover, this thesis covers a 4-dimensional example in Section 6, thus giving more insights on the strength of the method developped.

Let us briefly mention that the work done in this thesis can be applied to other problems that do not satisfy assumptions **A1**, **A2** and **A3**. They can be studied by going over our approach with slightly different theorems that can be deduced from ours by using other settings and going over our proofs with these in mind.

That being said, we are first going to talk about stable and unstable manifolds of fixed points (Section 2) as the subject of this thesis is to compute them. Then, we are going to talk about the method we will be using to compute them (Section 3). Finally, we will go over the practice (Sections 4 and 5) to show two examples (Section 6) before wrapping up this thesis with a conclusion (Section 7).

2 Stable and unstable manifolds

2.1 Topological manifolds

To present a formal definition of the stable and unstable manifolds, we must first go through some others. We borrow some statements of [5].

Definition 1 (Coordinate system, chart, parameterization)**.** Let $\mathcal{M}$ be a topological space and $\mathcal{U} \subseteq \mathcal{M}$ an open set. Let $\mathcal{V} \subseteq \mathbb{R}^n$ be open. A homeomorphism $\phi : \mathcal{U} \to \mathcal{V}, \phi(u) = (x_1(u), \ldots, x_n(u))$ is called a *coordinate system* on $\mathcal{U}$, and the functions $x_1, \ldots, x_n$ the *coordinate functions*. The pair $(\mathcal{U}, \phi)$ is called a *chart* on $\mathcal{M}$. The inverse map ϕ^{-1} is a *parameterization* of $\mathcal{U}$.

Definition 2 (Cover, atlas, transition maps)**.** A *cover* of $\mathcal{M}$ is a collection $\{\mathcal{U}_\alpha\}_{\alpha\in\mathcal{I}}$ of open subsets of $\mathcal{M}$ such that this collection covers $\mathcal{M}$, i.e. $\mathcal{M} = \bigcup_{\alpha\in\mathcal{I}} \mathcal{U}_\alpha$. An *atlas* on $\mathcal{M}$ is a collection of charts $\{\mathcal{U}_\alpha, \phi_\alpha\}_{\alpha\in\mathcal{I}}$ such that $\mathcal{U}_\alpha$ covers $\mathcal{M}$. The homeomorphisms $\phi_\beta\phi_\alpha^{-1} : \phi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \to \phi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ are the *transition maps* or *coordinate transformations*.

Recall that a topological space is *second countable* if the topology has a countable base, and *Hausdorff* if distinct points can be separated by neighbourhoods.

Definition 3 (Topological manifold, C^r-differentiable manifold, smooth manifold, analytic manifold)**.** A second countable, Hausdorff topological space $\mathcal{M}$ is an *n-dimensional topological manifold* if it admits an atlas $\{\mathcal{U}_\alpha, \phi_\alpha\}, \phi_\alpha : \mathcal{U}_\alpha \to \mathbb{R}^n, n \in \mathbb{N}$. It is a *$C^r$-differentiable manifold* if all transition maps are of class C^r. It is a *smooth manifold* if all transition maps are C^∞ diffeomorphisms, that is, all partial derivatives exist and are continuous. It is an *analytic manifold* if all transition maps are analytic.

Notice that an atlas could contain only one chart, say $\{\mathcal{U}, \phi\}$. Thus, we would have $\mathcal{U} = \mathcal{M}$, ϕ^{-1} would be a parameterization of the whole topological space $\mathcal{M}$ and $\mathcal{M}$ would be a smooth manifold. Nonetheless, if $\mathcal{M} \subseteq \mathbb{R}^m, m \in \mathbb{N}$, ϕ^{-1} would not have to be differentiable at all. Furthermore, differentiability of transition maps does not imply differentiability of coordinate systems or parameterizations.

Let us go over two simple examples to highlight the fact that all the coordinate systems of an n-dimensional manifold map to $\mathbb{R}^n$. Figure 1 is a 1-dimensional topological manifold. Indeed, if an atom were to move on the object, it would move locally on a 1-dimensional curve, regardless of where it is. Therefore, for any point on the object, one can always find a homeomorphism from some neighbourhood of that point to the 1-dimensional open unit ball. Nevertheless, Figure 2 is not a

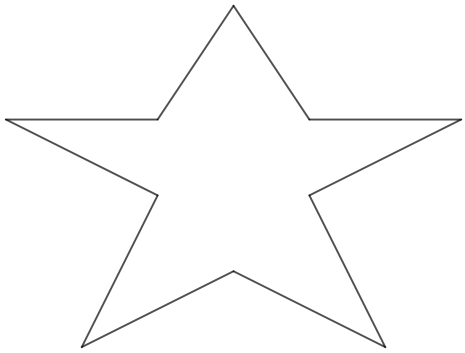

Figure 1 – Example of a 1-dimensional topological manifold

topological manifold. Indeed, consider any point on the object besides A and B. An atom can only move in 1-dimension from that point. Now, consider A or B. An atom could move in two dimensions at those points. Therefore, there is no homeomorphism from some neighbourhood of either of them to the 1-dimensional open unit ball. Since all the coordinate systems must map to $\mathbb{R}^n$ for the same n, it cannot be a topological manifold.

2.2 Stable and unstable manifolds

To state the definition of a stable/unstable manifold, we must first talk about flows.

Definition 4 (Flow)**.** A *flow* is a continuous function $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ that satisfies the following conditions :

1. $\phi(0, x) = x \quad (\forall x \in \mathbb{R}^n)$;
2. $\phi(t_1 + t_2, x) = \phi(t_2, \phi(t_1, x)) \quad (\forall t_1, t_2 \in \mathbb{R}, x \in \mathbb{R}^n)$.

For the sake of simplicity, we are sometimes going to write $\phi_t(x)$ instead of $\phi(t, x)$.

Flows are very important in ODEs since their solutions naturally give rise to flows. Indeed, let $\phi(t, x_0)$ be the solution of some ODE with x_0 as the initial condition

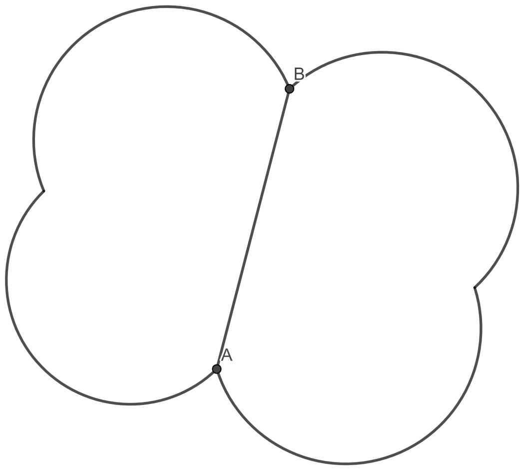

Figure 2 – Example of a non-topological manifold

at time $t = 0$. Then, $\phi(\cdot\,, x_0)$ is a flow. Furthermore, the solution of an ODE always exists given obvious assumptions.

Theorem 2.2.1 (The Fundamental Existence-Uniqueness Theorem)**.** *Let E be an open subset of $\mathbb{R}^n$ containing x_0 and assume that $f \in C^1(E)$. Then, there exists an $a > 0$ such that the initial value problem*

$$\begin{aligned} \dot{x} &= f(x) \\ x(0) &= x_0 \end{aligned}$$

has a unique solution $x(t)$ on the interval $[-a, a]$.

Proof. See [4].

□

The domain of the unique solution $\phi(t, x_0) \overset{\text{def}}{=} x(t)$ of Theorem 2.2.1 can always be extended to a *maximal interval of existence and uniqueness* that is open and contains $[-a, a]$ (see [4]). Moreover, the unique solution $\phi(t, x_0)$ is always *topologically conjugated* –two functions f and g are topologically conjugated if there exists a homeomorphism h such that $h \circ f = g \circ h$– to a flow (see [4]). Henceforth, when we refer to a flow, we are going to refer to the solution of an ODE.

Recall that a *fixed point* $\tilde{x}$ of an ODE is a point such that the flow of the ODE satisfies $\phi_t(\tilde{x}) = \tilde{x} \quad (\forall t \in \mathbb{R})$. We are now ready to state the definition of a stable/unstable manifold.

Definition 5 (Stable/unstable manifold)**.** Let $\tilde{x} \in \mathbb{R}^n$ be a fixed point of some ODE with flow $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. The *unstable manifold* of the ODE at $\tilde{x}$ is

$$W^u(\tilde{x}) \stackrel{\text{def}}{=} \left\{ x_0 \in \mathbb{R}^n : \lim_{t \to -\infty} \phi_t(x_0) = \tilde{x} \right\}.$$

In the same manner, the *stable manifold* of the ODE at $\tilde{x}$ is

$$W^s(\tilde{x}) \stackrel{\text{def}}{=} \left\{ x_0 \in \mathbb{R}^n : \lim_{t \to +\infty} \phi_t(x_0) = \tilde{x} \right\}.$$

Let us go over one general example to illustrate the computation of stable and unstable manifolds.

Example 1 (Linear ODE)**.** Consider the ODE

$$\dot{x} = Ax \quad , \tag{2.1}$$

where $A \in M_n(\mathbb{R})$. Let $x(0) = x_0$. The solution is given by $x(t) = e^{At}x_0$ and 0 is the only fixed point. Assume A is diagonalizable with real nonzero eigenvalues. Let $\lambda_1, \ldots, \lambda_k$ be the negative eigenvalues and $\lambda_{k+1}, \ldots, \lambda_n$ be the positive eigenvalues. Let $v_1, \ldots, v_n$ be their associated eigenvector respectively. It is well-known that they form a basis for $\mathbb{R}^n$. Let $(a_1, \ldots, a_n) \in \mathbb{R}^n$ be the unique n-tuple such that $x_0 = a_1 v_1 + \cdots + a_n v_n$. Then,

$$x(t) = \underbrace{a_1 e^{\lambda_1 t} v_1 + \cdots + a_k e^{\lambda_k t} v_k}_{x_s(t)} + \underbrace{a_{k+1} e^{\lambda_{k+1} t} v_{k+1} + \cdots + a_n e^{\lambda_n t} v_n}_{x_u(t)} \quad .$$

As $t \to \infty$, $x_s(t) \to 0$. However, the limit of $x_u(t)$ as $t \to \infty$ does not exist but for $a_{k+1} = \cdots = a_n = 0$. Therefore, the only way to have convergence at infinite time is to pick an initial condition given by a linear combination of eigenvectors associated to negative eigenvalues and the limit will always be 0. Let

$$\mathbb{E}^s \stackrel{\text{def}}{=} \langle v_1, \ldots, v_k \rangle \quad .$$

We have just shown that $W^s(0) = \mathbb{E}^s$. A similar argument shows that $W^u(0)$ is equal to $\mathbb{E}^u$ defined as

$$\mathbb{E}^u \stackrel{\text{def}}{=} \langle v_{k+1}, \ldots, v_n \rangle \quad .$$

The notations $\mathbb{E}^s$ and $\mathbb{E}^u$ from Example 1 generalize as follows :

Definition 6 (Linearized ODE, stable and unstable subspaces)**.** Let $\dot{x} = f(x)$, where $f : E \subset \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable over E, be an ODE such that $x_0 \in \mathbb{R}^n$ is a fixed point. Assume $Df(x_0)$ is diagonalizable with real nonzero eigenvalues. Let $v_1, \ldots, v_k$ and $v_{k+1}, \ldots, v_n$ be the eigenvectors associated to the negative and positive eigenvalues respectively. The *linearized ODE* at x_0 is $\dot{y} = Df(x_0)y$. We define the *stable subspace* $\mathbb{E}^s$ and the *unstable subspace* $\mathbb{E}^u$ of the linearized ODE as

$$\begin{aligned} \mathbb{E}^s &\overset{\text{def}}{=} y_0 + \langle v_1, \ldots, v_k \rangle \\ \mathbb{E}^u &\overset{\text{def}}{=} y_0 + \langle v_{k+1}, \ldots, v_n \rangle \end{aligned} .$$

One can verify the stable and unstable manifolds of the linearized ODE at the fixed point 0 are always going to be $W^s(0) = \langle v_1, \ldots, v_k \rangle$ and $W^u(0) = \langle v_{k+1}, \ldots, v_n \rangle$. Definition 6 will be useful later on.

Notice the stable and unstable manifolds of Example 1 are k-dimensional smooth manifold and $(n-k)$-dimensional smooth manifold respectively. Moreover, both of them are global manifolds – they are not restricted to an open neighbourhood of the fixed point. Let us go through a quick concrete example to illustrate the dynamics around the fixed point of a linear ODE.

Example 2. Consider the ODE

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}.$$

One can check 0 is the only fixed point and

$$W^s(0) = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \quad \& \quad W^u(0) = \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle \quad .$$

Figure 3 shows both the stable and unstable manifolds at 0 of Example 2. They are both 1-dimensional smooth manifolds. Let us now go over the same example but with non linear terms added to the ODE this time.

Example 3. Consider the ODE

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x - y^2 \\ x^2 + y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix} + \underbrace{\begin{pmatrix} -y^2 \\ x^2 \end{pmatrix}}_{G(x,y)}.$$

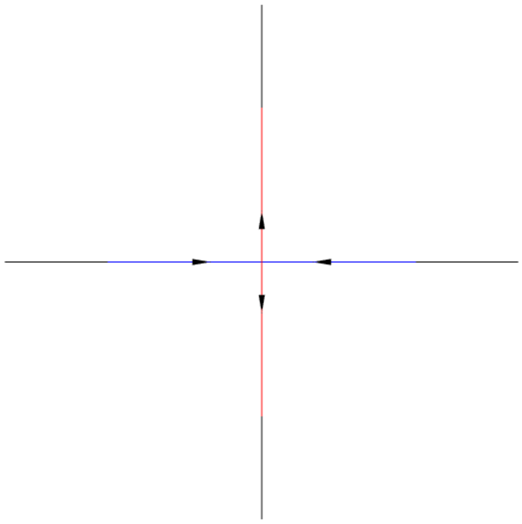

Figure 3 – $W^s(0)$ in blue and $W^u(0)$ in red of Example 2

The origin is a fixed point. The addition of the non linear terms $G(x, y)$ renders the computation of the stable and unstable manifold really hard by hand. Therefore, the computation of the stable and unstable manifolds have been done numerically. Figure 4a and Figure 4b show respectively the stable and unstable manifolds at the

(a) Stable manifold $W^s(0)$

(b) Unstable manifold $W^u(0)$

Figure 4 – Stable and unstable manifolds at 0 – $W^s(0)$ and $W^u(0)$ respectively

origin. They are locally homeomorphic to the 1-dimensional open unit ball at every point but the origin. The same argument as for Figure 2 applies to prove they are not topological manifolds.

Although we cannot guarantee that stable and unstable manifolds are globally topological manifold, the Stable and Unstable Manifold Theorem, which we will cover later on, provides us with the existence of local stable and unstable manifold at a fixed point that are at least C^1-differentiable manifolds under some non-restrictive conditions. Nevertheless, we need to make a definition before stating the Stable and Unstable Manifold Theorem.

Definition 7 (Smooth function, tangent vector, tangent space)**.** Let $\mathcal{M}$ be a smooth manifold. A *smooth function* on $\mathcal{M}$ is a real valued function from $\mathcal{M}$ such that its precomposition with a parameterization of $\mathcal{M}$ is smooth wherever it is defined. The set of all smooth functions on $\mathcal{M}$ is denoted $C^\infty(\mathcal{M})$. Let $x \in \mathcal{M}$. A *tangent vector* of $\mathcal{M}$ at x is a function $v : C^\infty(\mathcal{M}) \to \mathbb{R}$ such that, for every $f, g \in C^\infty(\mathcal{M})$ and $a \in \mathbb{R}$,

1. $v(f+g) = v(f) + v(g)$;
2. $v(af) = av(f)$;
3. $v(fg) = v(f)g(x) + f(x)v(g)$.

The *tangent space* of $\mathcal{M}$ at x is the set of all tangent vectors at x and is denoted $T_x\mathcal{M}$.

Though this definition is formal, since our stable and unstable manifolds are going to be real manifolds, one can think of tangent vectors in the same way as in $\mathbb{R}^n$, i.e. as directional derivatives. With this in mind, we are now ready to state the Stable and Unstable Manifold Theorem.

Theorem 2.2.2 (Stable and Unstable Manifold Theorem)**.** *Let E be an open subset of $\mathbb{R}^n$ containing the origin. Consider the ODE $\dot{x} = f(x)$, where $f : E \to \mathbb{R}^n$. Let $f \in C^1(E)$, and let ϕ_t be the flow of the ODE. Suppose that $f(0) = 0$ and that $Df(0)$ has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. Then there exists a k-dimensional differentiable manifold S tangent to the stable subspace $\mathbb{E}^s$ of the linearized ODE at 0 such that for all $t \geq 0$, $\phi_t(S) \subset S$ and for all $x_0 \in S$,*

$$\lim_{t\to\infty} \phi_t(x_0) = 0$$

and there exists an $n-k$ differentiable manifold U tangent to the unstable subspace $\mathbb{E}^u$ of the linearized ODE at 0 such that for all $t \leq 0$, $\phi_t(U) \subset U$ and for all $x_0 \in U$,

$$\lim_{t\to-\infty} \phi_t(x_0) = 0 \ .$$

Proof. See [4].

□

Remark. There exists a more general version of this theorem : If $f \in C^r(E)$, then the stable and unstable manifold are C^r-differentiable manifolds. Moreover, if f is analytic over E, then the stable and unstable manifolds are analytic manifolds. This can easily be derived directly from the proof of Perko [4].

We call *local stable manifold* at $\tilde{x}$, denoted $W^s_{\text{loc}}(\tilde{x})$, the k-dimensional differentiable manifold S whose existence is guaranteed by Theorem 2.2.2. In the same way, we call *local unstable manifold* at $\tilde{x}$, denoted $W^u_{\text{loc}}(\tilde{x})$, the $n-k$-dimensional differentiable manifold U whose existence is guaranteed by Theorem 2.2.2.

Let us quickly come back on Example 3. Theorem 2.2.2 applied to this example states there exists local stable and unstable manifolds that are 1-dimensional differentiable manifolds tangent respectively to the stable subspace and unstable subspace. Figure 5 shows the statement. One may notice the tangency of the manifolds at the fixed point 0. Moreover, one may note that the local stable manifold and the stable subspace share the same dimension. The same holds for the local unstable manifold and the unstable subspace. One may verify this holds under the same conditions of Theorem 2.2.2.

Since our vector field is analytic by Assumption **A2** and our manifolds are real manifolds, we would like to have analyticity of the parameterizations of our manifolds. However, this is not the same as having an analytic manifold since the latter means the transition maps are analytic. Although these two matters seem unrelated, we do have the result we strive for with the settings of Theorem 2.2.2.

Corollary 2.2.1 (Differentiability of Stable and Unstable Manifold Theorem)**.** *Let us work with the same settings as Theorem 2.2.2. Both the stable and unstable manifolds derived from Theorem 2.2.2 admit a parameterization that inherits the same order of differentiability as the vector field.*

Proof. As one goes through the proof of Theorem 2.2.2 given in [4], one can notice the manifold considered is defined by only one parameterization which possess the same order of differentiability as the vector field. We may talk about differentiability of parameterizations here since the manifold considered is real. □

Corollary 2.2.1 tells us that, given an hyperbolic fixed point of an ODE, there

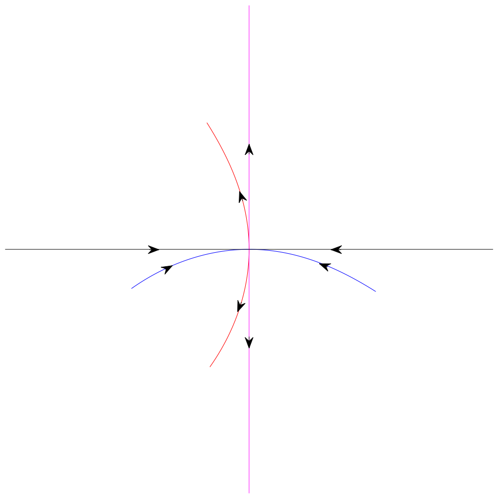

Figure 5 – Local stable manifold $W^s_{\text{loc}}(0)$ (blue), local unstable manifold $W^u_{\text{loc}}(0)$ (red), stable subspace $\mathbb{E}^s$ (black) and unstable subspace $\mathbb{E}^u$ (magenta)

will always exist a parameterization of the stable manifold at the fixed point and that it will be as differentiable as the vector field is. The same goes for the unstable manifold at the fixed point.

Such a powerful theorem as Theorem 2.2.2 deserves a bit of history. It was first introduced by Hadamard in 1901 (see [6]), though it was in 2 dimensions and the formulation was not as concise as today. A couple of years later, in 1907, Liapunov came up with three theorems (Théorème I, Théorème II and Théorème III, see [7]) that better described the mathematics behind Theorem 2.2.2 as he introduced Spectral Theory in the statements of his theorems. Many years later, in 1928, Perron came up with a theorem of its own (Satz 11, see [8]) that introduced integral equations as part of the proof of his theorem as well as a close formulation of his theorem to Theorem 2.2.2. Decades later, in 1991, Perko wrote a book (see [4]) in which Theorem 2.2.2 is proven using techniques introduced by Liapunov and Perron. As of today, proofs of Theorem 2.2.2 refer to Perko's book. Nowadays, research has been developped around this theorem. One may look at [1], [2] and [3] for further readings. This thesis is a continuum of the work done in [2] as it has the same settings as we do, including the dependency on parameters. We will come back on that later.

Consider the stable manifold of a fixed point $\tilde{x}$ of some ODE $\dot{x} = f(x)$ with $f : E \subset \mathbb{R}^n \to \mathbb{R}^n$, $f \in C^1(E, \mathbb{R}^n)$ and flow ϕ_t. Assume $\tilde{x}$ is a *hyperbolic fixed point*, i.e. $Df(\tilde{x})$ has no eigenvalue with zero real part. Assume all the eigenvalues are real – we will cover the complex case later on. Assume $Df(\tilde{x})$ is diagonalizable. Let $k \in \mathbb{N}$ be the number of negative eigenvalues. Let $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ be the negative eigenvalues of $Df(\tilde{x})$, $\lambda_{k+1}, \ldots, \lambda_n \in \mathbb{R}$ the positive ones and $v_1, \ldots, v_n \in \mathbb{R}^n$ their associated eigenvector.

Let

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} \quad , \quad A = \left(\begin{array}{c|c|c} v_1 & \cdots & v_n \end{array} \right) \quad .$$

The linearized ODE of $\dot{x} = f(x)$ at $\tilde{x}$ is

$$\begin{aligned} \dot{y} &= Df(\tilde{x})y \\ &= A\Lambda A^{-1}y. \end{aligned}$$

By performing the change of variable $z = A^{-1}y$, we get the new ODE

$$\dot{z} = \Lambda z. \tag{2.2}$$

Note that (2.2) is isomorphic to the linearized ODE because the change of variable performed is an isomorphism. Therefore, we can work with either one of them. Let us work with (2.2). The stable and unstable manifold of (2.2) at 0 are also called the *stable subspace*, denoted $\mathbb{E}^s$, and *unstable subspace*, denoted $\mathbb{E}^u$, respectively. It will be clear whether we use the definition for the stable and unstable manifolds at 0 of (2.2) or the one of Example 2. Let

$$\Lambda_s = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_k \end{pmatrix} \quad , \quad \Lambda_u = \begin{pmatrix} \lambda_{k+1} & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

and

$$A_s = \left(\begin{array}{c|c|c} v_1 & \cdots & v_k \end{array} \right) \quad , \quad A_u = \left(\begin{array}{c|c|c} v_{k+1} & \cdots & v_n \end{array} \right) \quad .$$

Notice that

$$\Lambda = \begin{pmatrix} \Lambda_s & 0 \\ 0 & \Lambda_u \end{pmatrix} \quad , \quad A = \left(\begin{array}{c|c} A_s & A_u \end{array} \right) \quad .$$

Moreover, let

$$\mathbb{E}^s = \{z_s \in \mathbb{R}^n : (z_s)_i = 0 \quad \forall k+1 \leq i \leq n\}$$

and

$$\mathbb{E}^u = \{z_u \in \mathbb{R}^n : (z_u)_i = 0 \quad \forall 1 \leq i \leq k\}$$

be the stable and unstable subspace respectively of (2.2) at 0. Hence, we have

$$\forall z \in \mathbb{R}^n, \exists!(z_s \in \mathbb{E}^s, z_u \in \mathbb{E}^u), z = z_s + z_u,$$

i.e. $\mathbb{R}^n = \mathbb{E}^s \oplus \mathbb{E}^u$. Thus, (2.2) is equivalent to

$$\dot{z}_s + \dot{z}_u = \begin{pmatrix} \Lambda_s & 0 \\ 0 & 0 \end{pmatrix} z_s + \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_u \end{pmatrix} z_u \ .$$

The latter gives rise to two new ODEs that are solved separately to solve (2.2) :

$$\dot{z} = \Lambda_s z \quad (z \in \mathbb{R}^k)$$

and

$$\dot{z} = \Lambda_u z \quad (z \in \mathbb{R}^{n-k}) \quad .$$

Note that the 0 components have been dropped in both of these ODEs. On the same note, since the 0 components may be ignored, without loss of generality, we set

$$\mathbb{E}^s = \mathbb{R}^k, \quad \mathbb{E}^u = \mathbb{R}^{n-k} \quad .$$

Definition 8 (Open ball)**.** We denote by $B^k(a, r) \subset \mathbb{R}^k$ the real *open ball* of center $a \in \mathbb{R}^k$, radius $r > 0$ and dimension k, i.e.

$$B^k(a, r) \stackrel{\text{def}}{=} \left\{x \in \mathbb{R}^k : \|x - a\| < r\right\} \quad .$$

The norm $\|\cdot\|$ need not be specified because all the norms are equivalent in spaces of finite dimension (see [17]). Moreover, for a general normed space $(X, \|\cdot\|_X)$, we denote by $B(a, r) \subset X$ the *open ball* of center $a \in X$ and radius $r > 0$, i.e.

$$B(a, r) \stackrel{\text{def}}{=} \{x \in X : \|x - a\|_X < r\} \quad .$$

The notation $B^k(a, r)$ will prime over $B(a, r)$ when the normed space considered is $\mathbb{R}^k$.

By Theorem 2.2.2, the stable manifold at $\tilde{x}$ is a local C^1-differentiable manifold of dimension k. Let $P : B^k(0, \tilde{r}) \to W^s_{\text{loc}}(\tilde{x})$ be a parameterization of the local stable manifold at $\tilde{x}$. The existence of $\tilde{r} > 0$ is given by Theorem 2.2.2 because P is a parameterization of a k-dimensional manifold. Furthermore, given $\theta \in B^k(0, \tilde{r})$ and $t > 0$, notice $P(\theta) \subset W^s_{\text{loc}}(\tilde{x}) \implies \phi_t(P(\theta)) \subset W^s_{\text{loc}}(\tilde{x})$ by Theorem 2.2.2 and $e^{\Lambda_s t}\theta \subset B^k(0, \tilde{r})$ since Λ_s is the diagonal matrix of the negative eigenvalues. We would like to derive an equation with P and its derivatives as the only unknowns. We are going to set

$$\begin{aligned} P(0) &= \tilde{x} \\ DP(0) &= A_s \\ \phi_t(P(\theta)) &= P(e^{\Lambda_s t}\theta) \end{aligned} \quad . \tag{2.3}$$

The third equation of (2.3) is known as the *conjugacy relation*. Basically, we assume that P maps orbits of the stable subspace $\mathbb{E}^s$ of (2.2) at 0 to orbits of the local stable manifold $W^s_{\text{loc}}(\tilde{x})$ of $\dot{x} = f(x)$ at $\tilde{x}$. It can be seen as a commutative diagram as illustrated in Figure 6. Figure 7 provides some insights on what the conjugacy does. Moreover, one may notice the resemblance with the Hartman-Grobmann Theorem (see [4]). The big difference there is that the Hartman-Grobmann Theorem is a theorem on the full domain $E \subset \mathbb{R}^n$ of the vector field f, whereas the conjugacy only holds on the local stable manifold $W^s_{\text{loc}}(\tilde{x})$ at $\tilde{x}$. This assumption gives us the

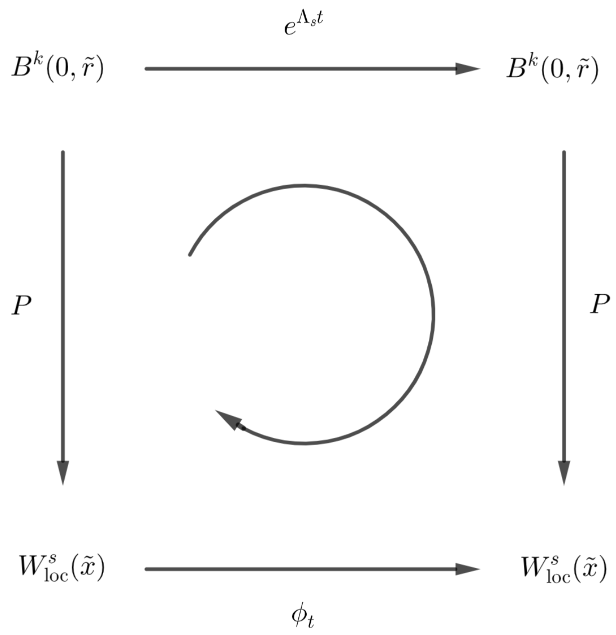

Figure 6 – Conjugacy as a commutative diagram

following theorem which is the heart of our method to solve for P as we will see later on.

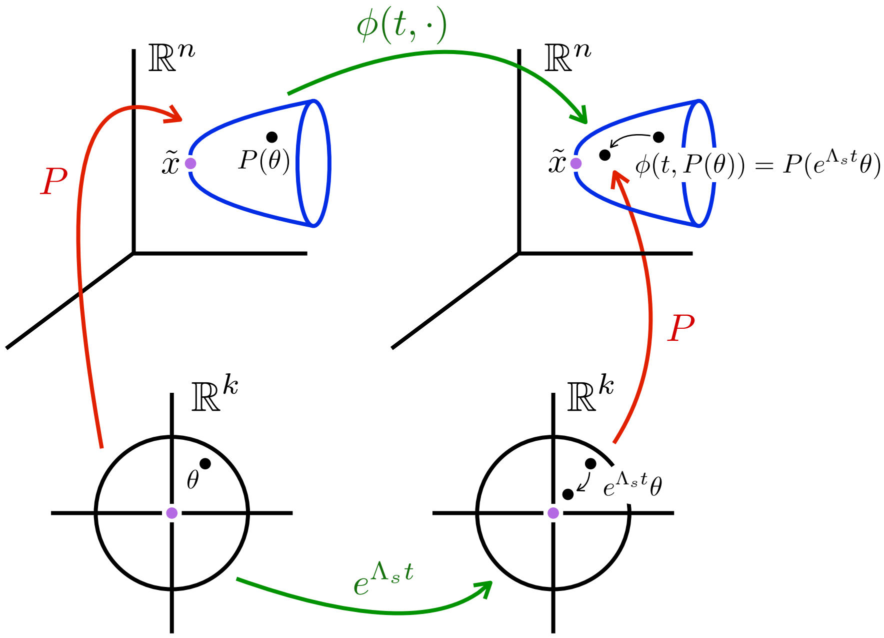

Figure 7 – Work done by the conjugacy

Theorem 2.2.3. (2.3) *is equivalent to*

$$\begin{aligned} P(0) &= \tilde{x} \\ DP(0) &= A_s \\ f(P(\theta)) &= DP(\theta)\Lambda_s\theta \end{aligned} \quad . \tag{2.4}$$

Proof. The first two conditions of (2.3) and (2.4) are shared, so only the equivalence of the third ones needs to be proven.

$\Longrightarrow$ We derive both sides of the third equation of (2.3) with respect to the time to get

$$f(\phi_t(P(\theta))) = DP(e^{\Lambda_s t}\theta)\Lambda_s e^{\Lambda_s t}\theta \quad .$$

Just let $t \downarrow 0$ to get that

$$f(P(\theta)) = DP(\theta)\Lambda_s\theta \quad ,$$

i.e. (2.4) holds.

$\Longleftarrow$ Fix θ. Let

$$\gamma(t) = P(e^{\Lambda_s t}\theta) \quad .$$

Then, using the third equation of (2.4), we get

$$\begin{aligned} \dot{\gamma}(t) &= DP(e^{\Lambda_s t}\theta)\Lambda_s e^{\Lambda_s t}\theta \\ &= f(P(e^{\Lambda_s t}\theta)) \end{aligned}$$

$$= f(\gamma(t)) \quad .$$

Therefore, by uniqueness of solutions, we have

$$\gamma(t) = \phi_t(\gamma(0)) \quad ,$$

which is equivalent to

$$P(e^{\Lambda_s t}\theta) = \phi_t(P(\theta)) \quad .$$

Since the argument is valid for every θ, we recover (2.3).

□

It is worth noticing that Theorem 2.2.3 has only P and its derivative as unknowns. Furthermore, Theorem 2.2.3 does not depend on the parameterization P of the stable manifold. Indeed, it holds for the parameterization P of the stable manifold at $\tilde{x}$ but it also holds for any other parameterization of the stable manifold at $\tilde{x}$, might they exist. Moreover, Theorem 2.2.3 also holds for complex eigenvalues and eigenvectors. We are going to tackle this matter later. Finally, P is a parameterization of the local stable manifold at $\tilde{x}$. Hence, it is a chart, which means it defined in a neighbourhood of $\tilde{x}$. Thus, Theorem 2.2.3 is a local theorem. Indeed, it does not hold globally a priori because P is not a parameterization of the global stable manifold at $\tilde{x}$ in general.

Henceforth, we are going to refer to the third equation of (2.4) as the *homological equation*. Also, Theorem 2.2.3 may be applied to Q a parameterization of the local unstable manifold. One just needs to substitute P for Q, Λ_s for Λ_u and A_s for A_u. Overall, the proof remains the same, one just needs to take limits as $t \to -\infty$ instead of limits as $t \to \infty$.

We can now carry over these results to the case with ODEs depending on parameters as well.

2.3 Stable and unstable manifolds with parameters

Nowadays, among others, with the enthusiasm around space missions, stable and unstable manifolds often get to depend on parameters. One can see [12] for further reading. Many situations have already been tackled, like when the parameters come as Lyapunov functions (see [10]). Moreover, the study of chaos led to the study of stable and unstable manifolds with dependency on parameters. The interested reader may read [9] for deeper understanding. Furthermore, like the goal of this thesis, methods for proving existence of stable and unstable manifolds depending on parameters have been researched as it has been done in [11] for instance.

Fortunately, the work done in the previous subsection can be carried over to the case where stable and unstable manifolds depend on parameters. Indeed, recall the discussion that follows Corollary 2.2.1. We may as well add a dependency on some parameters $\omega \in \mathbb{R}^p$ for the ODE $\dot{x} = f(x)$. This would result in the ODE

$$\frac{\partial}{\partial t} x(t,\omega) = f(x,\omega) \quad . \tag{2.5}$$

Then, the subsequent objects of the discussion are all going to inherit the dependency on ω. One can go over the details by himself, but we resume at the settings for the conjugacy, which now become

$$\begin{aligned} P(0,\omega) &= \tilde{x}(\omega) \\ DP(0,\omega) &= A_s(\omega) \\ \phi_t(P(\theta,\omega),\omega) &= P(e^{\Lambda_s(\omega)t}\theta,\omega) \end{aligned} \quad . \tag{2.6}$$

Fix $\omega \in \mathbb{R}^p$. Note that, by Theorem 2.2.2, the stable manifold at $\tilde{x}(\omega)$ is a local C^1-differentiable manifold of dimension k and $P : B^k(0,\tilde{r}(\omega))\times\mathbb{R}^p \to W^s_{\text{loc}}(\tilde{x}(\omega))$ is a parameterization of the local stable manifold at $\tilde{x}(\omega)$. The existence of $\tilde{r}(\omega) > 0$ is given by Theorem 2.2.2 because $P(\cdot\,,\omega)$ is a parameterization of a k-dimensional manifold. Furthermore, notice $P(\theta,\omega) \subset W^s_{\text{loc}}(\tilde{x}(\omega)) \implies \phi_t(P(\theta,\omega),\omega) \subset W^s_{\text{loc}}(\tilde{x}(\omega))$ by Theorem 2.2.2 and $e^{\Lambda_s(\omega)t}\theta \subset B^k(0,\tilde{r}(\omega))$ since $\Lambda_s(\omega)$ is the diagonal matrix of the negative eigenvalues at ω and $t > 0$. Figure 8 illustrates the conjugacy for a fixed value of $\omega \in \mathbb{R}^p$. The important matter is that the commutative diagram shown in Figure 8 holds for fixed values of the parameters ω as it is easier to understand what is going on if we study what occurs for a fixed value of ω and then what occurs when we change this value. With that in mind, let us rewrite Theorem 2.2.3 to add the parameters ω :

Theorem 2.3.1. (2.6) *is equivalent to*

$$\begin{aligned} P(0,\omega) &= \tilde{x}(\omega) \\ D_\theta P(0,\omega) &= A_s(\omega) \\ f(P(\theta,\omega),\omega) &= D_\theta P(\theta,\omega)\Lambda_s(\omega)\theta \end{aligned} \tag{2.7}$$

for each $\omega \in \mathbb{R}^p$.

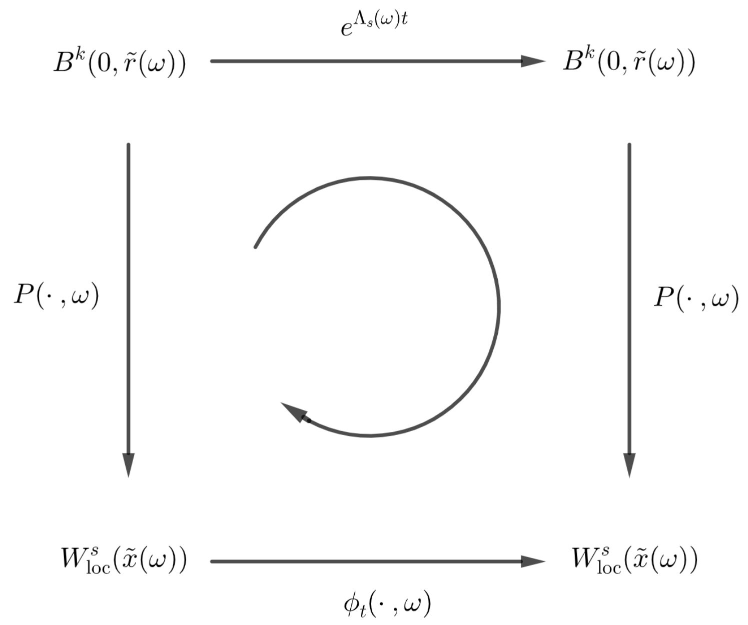

Figure 8 – Conjugacy with parameters as a commutative diagram

Proof. The proof mimics the one of Theorem 2.2.3. One just needs to fix $\omega \in \mathbb{R}^p$ to recover the settings of Theorem 2.2.3. Therefore, Theorem 2.2.3 is valid for every $\omega \in \mathbb{R}^p$, which is exactly the statement of Theorem 2.3.1.

□

Let us be more formal here. Fix $\omega = \tilde{\omega}$. $\tilde{x}(\tilde{\omega})$ is an hyperbolic fix point of (2.5) at $\tilde{\omega}$. $\Lambda_s(\tilde{\omega})$ is the diagonal matrix of the eigenvalues of $Df(\tilde{x}(\tilde{\omega}), \tilde{\omega})$ with negative real part and $A_s(\tilde{\omega})$ is the matrix of their associated eigenvector.

The discussion that follows Theorem 2.2.3 still applies for Theorem 2.3.1 for any $\omega \in \mathbb{R}^p$. The only difference is that the *homological equation* is henceforth going to refer to the third equation of (2.7). The only thing that we highlight again, even though it is mentioned in the discussion that follows Theorem 2.2.3, is that P and its derivative are the only unknowns in the homological equation. Therefore, if we come up with a parameterization of P, then (2.7) allows us to solve for this parameterization.

Remark. Parameterization has another meaning in the above paragraph. The parameterization P is the one defined in Definition 1. The *parameterization* of P is the way we write P with respect to some basis of $C^1(\mathbb{R}^k \times \mathbb{R}^p, \mathbb{R}^n)$. Basically, we choose a basis and rewrite P with respect to it. This is why we say we solve for parameterized families of the local stable manifold : We parameterize P to ease our computations of the local stable manifold. Henceforth, when we talk about a parameterization of P, the parameterization is going to refer to the definition given in this remark, whereas P will be the parameterization defined in Definition 1.

We can now move on to the matter of parameterizing P.

3 Parameterization Method via power series

3.1 Power series

For the purpose of this thesis, we are going to study power series because we are going to parameterize the chart P of the local stable manifold using power series. We will see later on that doing so leads to a natural computation of local stable manifold.

Definition 9 (Multi-index)**.** Let $\alpha \in \mathbb{N}^d$. Then, we define $|\alpha|$, the *order* of α, by

$$|\alpha| \overset{\text{def}}{=} \alpha_1 + \cdots + \alpha_d \quad .$$

Therefore, the set $\{|\alpha| = n\}$ for $n \in \mathbb{N}$ is the set

$$\{|\alpha| = n\} \overset{\text{def}}{=} \{\alpha \in \mathbb{N}^d : |\alpha| = n\} \quad .$$

Henceforth, for the sake of simplicity, we are just going to write $|\alpha| = n$ instead of $\{|\alpha| = n\}$. Furthermore, for $\alpha, \beta \in \mathbb{N}^d$, we define the relation of order $\leq$ by

$$\alpha \leq \beta \iff \alpha_i \leq \beta_i \quad (\forall 1 \leq i \leq d)$$

and, similarly, the relation of strict order $<$ by

$$\alpha < \beta \iff \big(\alpha_i \leq \beta_i \quad (\forall 1 \leq i \leq d) \quad \& \quad \exists \tilde{i} \in \{1, \ldots, d\}, \alpha_{\tilde{i}} < \beta_{\tilde{i}}\big) \quad .$$

The relations $\geq$ and $>$ are defined in a similar manner, just substitute $\leq$ for $\geq$ and $<$ for $>$ in this definition. Moreover, notice that

$$|\alpha_1 + \alpha_2| = |\alpha_1| + |\alpha_2| \quad .$$

Finally, for $x \in \mathbb{R}^d$, we define

$$x^\alpha \overset{\text{def}}{=} x_1^{\alpha_1} \cdots x_d^{\alpha_d} \quad .$$

It will be clear whether we use $|\cdot|$ to denote the absolute values or the multi-index notation. Indeed, given that we pick $\alpha \in \mathbb{N}^d$, it would not make any sense to speak of the absolute value of α since α is already nonnegative in every component. Therefore, the reader can assume $|\cdot|$ is always going to denote the multi-index notation whenever its argument is in $\mathbb{N}^d$ for some $d > 0$ and is always going to denote the absolute values whenever its argument is in $\mathbb{R}^n$ for some $n > 0$.

We can now talk about power series.

Definition 10 (Power series)**.** A *power series* for $g \in C^\omega(\mathbb{R}^n, \mathbb{R}^n)$ – $g \in C^\omega(\mathbb{R}^n, \mathbb{R}^n)$ means $g : \mathbb{R}^n \to \mathbb{R}^n$ is analytic in every one of its components – at $\tilde{x} \in \mathbb{R}^n$ is a series such that

$$g(x) = \sum_{|\alpha| \geq 0} a_\alpha (x - \tilde{x})^\alpha \quad ,$$

where $\alpha \in \mathbb{N}^n$ and $a_\alpha \in \mathbb{R}^n \stackrel{\text{def}}{=} (a_{\alpha_1}, \ldots, a_{\alpha_n}) \in \mathbb{R}^n$ $(\forall \alpha)$.

Remark. The coefficient of g at $\alpha = (0, \ldots, 0) \in \mathbb{N}^n$ is henceforth going to be denoted by a_0. We are going to refer to the coefficient a_α as the *α-th coefficient.*

Basically, a power series for $g \in C^\omega(\mathbb{R}^n, \mathbb{R}^n)$ at $\tilde{x} \in \mathbb{R}^n$ is a Taylor series for g at $\tilde{x}$ whose existence is guaranteed by the fact that g is analytic. The *radius of convergence* R of the power series for g at $\tilde{x}$ is a nonnegative real number such that the power series converges for every $x \in B(\tilde{x}, R)$.

Having defined power series, it is natural to talk about recurrence relations.

Definition 11 (Recurrence relation)**.** Let $a = (a_\alpha)_{|\alpha| \geq 0}$ be a sequence of real numbers with $\alpha \in \mathbb{N}^d$ $(d > 0)$. A *recurrence relation* onto a is a relation R such that

$$a_\alpha = \mathrm{R}(a_{\alpha^-}) \quad (\forall |\alpha| \geq 1) \quad ,$$

where $a_{\alpha^-} = (a_{\alpha^*})_{\alpha^* < \alpha}$. In other words, a recurrence relation onto a is a relation R that gives every member of the sequence, besides the first one, as a computation depending on members of lesser order only.

In the case of power series, if one has an equation $F(x, \omega) = 0$, where $F \in C^\omega(\mathbb{R}^m \times \mathbb{R}^p, \mathbb{R}^n)$ and $\omega \in \mathbb{R}^p$ are parameters, then one can look for a solution $x(\omega)$ given as power series, say $x(\omega) = \sum_{|\beta| \geq 0} b_\beta \omega^\beta$. With these settings and b_0 given, the coefficients $b = (b_\beta)_{|\beta| \geq 0}$ of the power series of $x(\omega)$ are going to be given by a recurrence relation. This remains true when F also depends on derivatives of $x(\omega)$.

Before going over an example, let us introduce further notation for the sake of simplicity. Let $P : \mathbb{R}^m \to \mathbb{R}^n$ and $Q : \mathbb{R}^m \to \mathbb{R}^n$ be power series given by

$$P(x) = \sum_{|\alpha| \geq 0} a_\alpha x^\alpha \quad , \quad Q(x) = \sum_{|\alpha| \geq 0} b_\alpha x^\alpha \quad ,$$

where $\alpha \in \mathbb{R}^m$ and $a_\alpha, b_\alpha \in \mathbb{R}^n$ $(\forall \alpha)$. Recall that the sum of two power series is

$$P(x) + Q(x) = \sum_{|\alpha| \geq 0} (a_\alpha + b_\alpha) x^\alpha \quad .$$

Before recalling the product of two power series, let us make a useful definition :

Definition 12 (Cauchy product)**.** Let $a = (a_\alpha)_{\alpha\in\mathbb{N}^d}$ and $b = (b_\alpha)_{\alpha\in\mathbb{N}^d}$ be sequences of real numbers. The *Cauchy product* of a and b, denoted $a*b$, is defined componentwise by

$$(a * b)_\alpha \stackrel{\text{def}}{=} \sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \alpha_1,\,\alpha_2\in\mathbb{N}^d}} a_{\alpha_1} b_{\alpha_2} \quad .$$

Moreover, for $\beta \in \mathbb{N}^l$ and $a_1 = ((a_1)_\alpha)_{\alpha\in\mathbb{N}^d}, \dots, a_l = ((a_l)_\alpha)_{\alpha\in\mathbb{N}^d}$ sequences of real numbers, we define

$$a_1^{\beta_1} * \cdots * a_l^{\beta_l} \stackrel{\text{def}}{=} \underbrace{a_1 * \cdots * a_1}_{\beta_1 \text{ times}} * \cdots * \underbrace{a_l * \cdots * a_l}_{\beta_l \text{ times}} \quad .$$

Notice that, for $\eta \in \mathbb{R}$ and $a = (a_\alpha)_{\alpha\in\mathbb{N}^d}, b = (b_\alpha)_{\alpha\in\mathbb{N}^d}, c = (c_\alpha)_{\alpha\in\mathbb{N}^d}$ sequences of real numbers, we have

$$a*b = b*a \quad \& \quad \eta\cdot(a*b) = (\eta\cdot a)*b = a*(\eta\cdot b) \quad \& \quad a*b*c \stackrel{\text{def}}{=} (a*b)*c = a*(b*c) \quad ,$$

so there is no ambiguity in Definition 12. Moreover, one can check that

$$(a+b)*c = (a*c)+(b*c) \quad .$$

With Definition 12 in mind, notice the product of two power series is given by

$$P(x)\cdot Q(x) = \sum_{|\alpha|\geq 0} (a*b)_\alpha x^\alpha \quad .$$

We can now make an example to illustrate how one can retrieve recurrence relations.

Example 4. Consider the Cauchy problem

$$\begin{cases} \dot{x} &= x(x-1) \\ x(0) &= x_0 \end{cases} \quad ,$$

where x depends on the time t and $x_0 \in \mathbb{R}$. Suppose x is given by $x(t) = \sum_{\beta\geq 0} b_\beta t^\beta$, where $\beta \in \mathbb{R}$ and $b_\beta \in \mathbb{R}$ $(\forall\beta)$. Then, by substituting the power series of x into the equation, gathering the coefficients of the same power of t and rearranging them, one can check we get

$$\beta\cdot b_\beta = (b*b)_{\beta-1} - b_{\beta-1} \quad (\forall\beta\geq 1) \quad .$$

Hence, we get the relation

$$b_\beta = R(b_{\beta^-}) \stackrel{\text{def}}{=} \frac{1}{\beta}\left((b*b)_{\beta-1} - b_{\beta-1}\right) \quad (\forall\beta\geq 1) \quad .$$

According to Definition 11, $R(b_{\beta^-})$ is a recurrence relation.

Let us do another example with some parameters in it this time to see how one can retrieve recurrence relations when parameters are involved.

Example 5. Consider the equation

$$\begin{pmatrix} -x_1^2 + x_2 + \omega^2 \\ x_2^2 + x_1\omega \end{pmatrix} = 0 \quad \text{subject to} \quad x(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad ,$$

where x depends on ω. Suppose x is given by the power series $x(\omega) = \sum_{|\beta|\geq 0} b_\beta \omega^\beta$, where $\beta \in \mathbb{N}$ and $b_\beta \in \mathbb{R}^2$ $(\forall \beta)$. Then, by substituting the power series of x into the equation, gathering the coefficients of the same power of ω and rearranging them, one can check we get

$$\underbrace{\begin{pmatrix} -2(b_1)_0 & 1 \\ 0 & 2(b_2)_0 \end{pmatrix}}_{B} \cdot \begin{pmatrix} (b_1)_\beta \\ (b_2)_\beta \end{pmatrix} = \begin{pmatrix} (b_1 \hat{*} b_1)_\beta - \delta_{\beta,2} \\ -(b_2 \hat{*} b_2)_\beta - (b_1)_{\beta-1} \end{pmatrix} \quad (\forall \beta \geq 1) \quad ,$$

where $\delta_{i,j} \stackrel{\text{def}}{=} \begin{cases} 1 & , \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$ $(i, j \in \mathbb{N})$ is the *Kronecker delta*, $(b_1 \hat{*} b_1)_\beta \stackrel{\text{def}}{=} (b_1 * b_1)_\beta - 2(b_1)_0(b_1)_\beta$ and $(b_2 \hat{*} b_2)_\beta \stackrel{\text{def}}{=} (b_2 * b_2)_\beta - 2(b_2)_0(b_2)_\beta$. Notice both $(b_1 \hat{*} b_1)_\beta$ and $(b_2 \hat{*} b_2)_\beta$ do not depend on b_β. Moreover, notice $b_0 = (3, 4)$. Hence, B is invertible and we get the relation

$$\begin{pmatrix} (b_1)_\beta \\ (b_2)_\beta \end{pmatrix} = R(b_{\beta^-}) \stackrel{\text{def}}{=} B^{-1} \begin{pmatrix} (b_1 \hat{*} b_1)_\beta - \delta_{\beta,2} \\ -(b_2 \hat{*} b_2)_\beta - (b_1)_{\beta-1} \end{pmatrix} \quad (\forall \beta \geq 1) \quad .$$

According to Definition 11, $R(b_{\beta^-})$ is a recurrence relation.

As we have seen in Examples 4 and 5, when we have an ODE with a polynomial vector field, we can solve for analytic solutions. The thing to notice here is that retrieving the recurrence relation is by far not a hard task and not computational-heavy. Thus, it is an efficient method to find solutions of ODE. However, we need the invertibility of the matrix B. This matter is going to be discussed thoroughly in Section 4 when we talk explicitly about our computations.

Even though we are not going to compute solutions of ODEs, our computations are going to resemble the ones for an analytic solution of an ODE with a polynomial vector field. We are going to use power series for all of them. We are going to discuss the existence and uniqueness of our computations as well. Nevertheless, without properly speaking of existence, one could argue that it is natural to look for power series solutions of polynomial equations – we mean by that every member of these equations is a polynomial. Nonetheless, we are going to develop a rigorous computer-assisted proof for our computations that will ensure their existence and uniqueness afterward (see Subsection 3.4).

Let us now move to the next subsection and talk about the kind of operator that will be considered in Section 4 for our computations.

3.2 Operators

Recall Equations 2.7. Recall our goal is to compute a parameterization of the local stable manifold $W^s_{\text{loc}}(\tilde{x}(\omega))$ at the fixed point $\tilde{x}(\omega)$ depending on the parameters ω – without loss of generality, we consider the local stable manifold instead of the local unstable manifold because the computations are the same, only the inputs differ slightly. We are going to parameterize P using power series. The first two equations of (2.7) are the initial conditions for the coefficients of the power series of P while the third is the homological equation.

Suppose the fixed point, eigenvalues, eigenvectors and P are given by power series. Equations for computing fixed points, eigenvalues and eigenvectors are well-known. Moreover, the homological equation gives us a way to compute P given a parameterization of the latter. Hence, as seen in Example 5 and discussed thoroughly in Section 4, we will have recurrence relations onto the coefficients of those power series using the aforementioned equations – they will be covered explicitly in Section 4. Nevertheless, since we can only compute finitely many coefficients, we want to compute enough so we have "good" approximations – we will see later what we mean by a good approximation – of these power series but not too much in order to keep control on the error associated to the computer itself.

Consider any of the power series mentioned above. Let $z = (z_\delta)_{|\delta|\geq 0}$ be its coefficients. Assume we have already computed all its coefficients up to order $N-1 > 0$, i.e. all its coefficients z_δ such that $|\delta| \leq N-1$ – we will see in the next section there is a way to choose N. Recall Definition 11. Assume we have the recurrence relation

$$z_\delta = \mathrm{R}(z_{\delta^-})$$

onto the coefficients. Let $z^N \overset{\text{def}}{=} \{z_\delta : |\delta| \leq N-1\}$ be the set of the coefficients of order less than N, i.e. those that have already been computed. Let $\bar{z}$ be defined component-wise by

$$\bar{z}_\delta \overset{\text{def}}{=} \begin{cases} z_\delta & , \text{if } |\delta| \leq N-1 \\ 0 & , \text{otherwise} \end{cases} .$$

$\bar{z}$ is the coefficients of the approximation of the power series, i.e. those of z^N and 0s for all the coefficients at indices $|\delta| \geq N$. The operator T we solve to get existence and uniqueness of the "true" power series – the power series with all the coefficients, not only finitely many computed – is defined component-wise by

$$(T(u))_\delta \overset{\text{def}}{=} \begin{cases} 0 & , \text{if } \bar{z}_\delta \in z^N \\ \mathrm{R}\big((\bar{z}+u)_{\delta^-}\big) & , \text{otherwise} \end{cases} . \tag{3.1}$$

The spaces in which the domain and the image of T lie are going to be explicited in Section 4. Basically, we want to check that the approximation of the power series we computed is close enough to the true power series. Since the coefficients z^N we have already computed are the exact coefficients, we just need to have a bound on the tail of the power series for which we can guarantee the smallness. Hence, for now, try to think of the operator T as the distance coefficient-wise between the approximation of the power series and the true power series. Since the distance coefficient-wise between the computed coefficients of the approximation of the power series – the coefficients $\bar{z}$ – and the corresponding coefficients of the true power series – the coefficients z – is 0, T takes the value of 0 at the indices referring to them – the indices δ such that $|\delta| \leq N-1$. For the rest of the indices, the distance component-wise is just the value of the coefficient of the true power series for each index. Thus, the value of T at those indices is the value of the recurrence relation at these. Since we cannot evaluate the recurrence relation at the coefficients of the true power series, we feed T the coefficients of the approximation of the power series plus a small variation u. The purpose of the operator T and the number N is to prove $\bar{z}$ is a good enough approximation of the coefficients of the true power series.

Now is the time to define the spaces on which the operator T is going to act so we can speak afterward of what a good enough approximation is to us.

3.3 Weighted spaces

Whenever one works with power series, there is always the lurking question of convergence, namely how big the radius of convergence is. Weighted spaces are natural spaces to work with power series because the radius of convergence can be seen as a weight added on a functional space. We are going to consider certain weighted spaces that fit the use of operators defined as in (3.1). Though they can be defined for complex sequences, for the sake of simplicity, they are defined for real sequences – we will discuss in Section 4 the case of complex sequences when it occurs.

Definition 13 (ℓ^1_ν spaces)**.** Let $d, p, n, N \in \mathbb{N}$. Let $\nu \in \mathbb{R}^d_+$ and $\mu \in \mathbb{R}^p_+$ – $\mathbb{R}^d_+$ is the subset of $\mathbb{R}^d$ whose elements have positive components. We define the ℓ^1_ν *spaces* as

$$\begin{aligned}
\ell^1_\nu &\stackrel{\text{def}}{=} \left\{ a = (a_\alpha)_{\alpha \in \mathbb{N}^d} \subset \mathbb{R}^n : \sum_{|\alpha| \geq 0} |a_\alpha| \nu^\alpha < \infty \right\} \\
\ell^1_{\nu,\mu} &\stackrel{\text{def}}{=} \left\{ a = (a_{\alpha,\beta})_{(\alpha,\beta) \in \mathbb{N}^{d+p}} \subset \mathbb{R}^n : \sum_{|\beta| \geq 0} \sum_{|\alpha| \geq 0} |a_{\alpha,\beta}| \nu^\alpha \mu^\beta < \infty \right\} \\
\ell^{1,N}_\nu &\stackrel{\text{def}}{=} \left\{ a \in \ell^1_\nu : a_\alpha = 0 \ (\forall |\alpha| < N) \right\} \\
\ell^{1,N}_{\nu,\mu} &\stackrel{\text{def}}{=} \left\{ a \in \ell^1_{\nu,\mu} : a_{\alpha,\beta} = 0 \ (\forall |\alpha| < N) \right\}
\end{aligned} .$$

For $M \in \mathbb{N}$, we can consider the product of each of these ℓ^1_ν spaces with itself M times, i.e. $(\ell^1_\nu)^M$, $\left(\ell^1_{\nu,\mu}\right)^M$, $\left(\ell^{1,N}_\nu\right)^M$ and $\left(\ell^{1,N}_{\nu,\mu}\right)^M$ respectively.

Remark. Let X be any of the spaces of Definition 13. Then, for any $a = (a_\delta)_{\delta \in \mathbb{N}^l} \in X$ $(l > 0)$, we have $|a| \in X$, where $|a|$ is defined component-wise by $\big(|a|\big)_\delta \stackrel{\text{def}}{=} |a_\delta|$.

If one sees the sequences involved in the definition of those spaces as coefficients of some power series, then the condition for these sequences to belong to those spaces is just the convergence of their power series. The next definition sets the condition to be a norm.

Definition 14 (Norms on ℓ^1_ν spaces)**.** Recall Definition 13. Then,

$$\begin{aligned}
\|a\|_{1,\nu} &\stackrel{\text{def}}{=} \sum_{|\alpha| \geq 0} |a_\alpha| \nu^\alpha && (a \in \ell^1_\nu) \\
\|a\|_{1,(\nu,\mu)} &\stackrel{\text{def}}{=} \sum_{|\beta| \geq 0} \sum_{|\alpha| \geq 0} |a_\alpha| \nu^\alpha \mu^\beta && (a \in \ell^1_{\nu,\mu}) \\
\|a\|_{1,\nu,N} &\stackrel{\text{def}}{=} \sum_{|\alpha| \geq N} |a_\alpha| \nu^\alpha && (a \in \ell^{1,N}_\nu) \\
\|a\|_{1,(\nu,\mu),N} &\stackrel{\text{def}}{=} \sum_{|\beta| \geq 0} \sum_{|\alpha| \geq N} |a_\alpha| \nu^\alpha \mu^\beta && (a \in \ell^{1,N}_{\nu,\mu})
\end{aligned} .$$

Furthermore,

$$\begin{aligned}
\|a\|_{1,\nu}^{(M)} &\stackrel{\text{def}}{=} \max\{\|a_1\|_{1,\nu},\dots,\|a_M\|_{1,\nu}\} && \left(a\in(\ell_\nu^1)^M\right)\\
\|a\|_{1,(\nu,\mu)}^{(M)} &\stackrel{\text{def}}{=} \max\left\{\|a_1\|_{1,(\nu,\mu)},\dots,\|a_M\|_{1,(\nu,\mu)}\right\} && \left(a\in\left(\ell_{\nu,\mu}^1\right)^M\right)\\
\|a\|_{1,\nu,N}^{(M)} &\stackrel{\text{def}}{=} \max\{\|a_1\|_{1,\nu,N},\dots,\|a_M\|_{1,\nu,N}\} && \left(a\in\left(\ell_\nu^{1,N}\right)^M\right)\\
\|a\|_{1,(\nu,\mu),N}^{(M)} &\stackrel{\text{def}}{=} \max\left\{\|a_1\|_{1,(\nu,\mu),N},\dots,\|a_M\|_{1,(\nu,\mu),N}\right\} && \left(a\in\left(\ell_{\nu,\mu}^{1,N}\right)^M\right)
\end{aligned}.$$

One can verify that $\|\cdot\|_{1,\nu}:\ell_\nu^1\to\mathbb{R}$, $\|\cdot\|_{1,(\nu,\mu)}:\ell_{\nu,\mu}^1\to\mathbb{R}$, $\|\cdot\|_{1,\nu,N}:\ell_\nu^{1,N}\to\mathbb{R}$, $\|\cdot\|_{1,(\nu,\mu),N}:\ell_{\nu,\mu}^{1,N}\to\mathbb{R}$, $\|\cdot\|_{1,\nu}^{(M)}:(\ell_\nu^1)^M\to\mathbb{R}$, $\|\cdot\|_{1,(\nu,\mu)}^{(M)}:\left(\ell_{\nu,\mu}^1\right)^M\to\mathbb{R}$, $\|\cdot\|_{1,\nu,N}^{(M)}:\left(\ell_\nu^{1,N}\right)^M\to\mathbb{R}$ and $\|\cdot\|_{1,(\nu,\mu),N}^{(M)}:\left(\ell_{\nu,\mu}^{1,N}\right)^M\to\mathbb{R}$ are norms on their respective space of Definition 13.

The purpose of the norms of Definition 14 is they define a *Banach space* – a complete metric space with a norm – on their respective space. One can prove this using the fact $\mathbb{R}$ is a complete space. As we will see in the next subsection, our method to ensure that our computations are good enough approximations requires Banach spaces to work on.

Recall a *Banach algebra* is a Banach space $(B,\|\cdot\|_B)$ together with an algebra $*:B\times B\to B$ such that $\|a*b\|_B\le\|a\|_B\|b\|_B$ for all $a,b\in B$.

Theorem 3.3.1. *Recall Definition 12. Let $\nu\in\mathbb{R}_+^d$, $\mu\in\mathbb{R}_+^p$ and N be a positive integer. Then, $(X,\|\cdot\|_X)$ together with the Cauchy product $*$ is a Banach algebra – $(X,\|\cdot\|_X)$ is any Banach space of Definition 13 coupled with its norm of Definition 14.*

Proof. Let $a,b\in\ell_\nu^1$. We already know the Cauchy product is an algebra by Subsection 3.1. Thus, we only need to verify that $\|a*b\|_{1,\nu}\le\|a\|_{1,\nu}\|b\|_{1,\nu}$. Indeed, it will also prove $a*b\in\ell_\nu^1$ since $\|a\|_{1,\nu},\|b\|_{1,\nu}<\infty$ by assumption. Observe that

$$\begin{aligned}
\sum_{|\alpha|=0}^{m}|(a*b)_\alpha|\nu^\alpha &= \sum_{|\alpha|=0}^{m}\left|\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \alpha_1,\alpha_2\in\mathbb{N}^d}} a_{\alpha_1}b_{\alpha_2}\right|\nu^\alpha\\
&\le \sum_{|\alpha|=0}^{m}\left(\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \alpha_1,\alpha_2\in\mathbb{N}^d}} |a_{\alpha_1}b_{\alpha_2}|\right)\nu^\alpha\\
&\le \left(\sum_{|\alpha|=0}^{m}|a_\alpha|\nu^\alpha\right)\left(\sum_{|\alpha|=0}^{m}|b_\alpha|\nu^\alpha\right)
\end{aligned}.$$

Hence,

$$\begin{aligned}
\|a * b\|_{1,\nu} &= \lim_{m\to\infty} \sum_{|\alpha|=0}^{m} |(a*b)_\alpha|\nu^\alpha \\
&\leq \lim_{m\to\infty} \left(\sum_{|\alpha|=0}^{m} |a_\alpha|\nu^\alpha\right)\left(\sum_{|\alpha|=0}^{m} |b_\alpha|\nu^\alpha\right) \\
&= \|a\|_{1,\nu}\|b\|_{1,\nu} \\
&< \infty
\end{aligned}$$

where the strict inequality comes from the fact both a and b lie in ℓ^1_ν.

The same argument can be applied to show $*$ is also a Banach algebra over the other Banach spaces. Indeed, notice $\ell^{1,N}_\nu$ is a subspace of ℓ^1_ν, $\ell^1_{\nu,\mu}$ is exactly $\ell^1_{\tilde{\nu}}$ with $\tilde{\nu} = (\nu, \mu)$ and $\ell^{1,N}_{\nu,\mu}$ is a subspace of $\ell^1_{\nu,\mu}$, so the argument works for each of them. Finally, knowing $*$ is a Banach algebra over each of the previous four spaces, one can easily prove $*$ is also a Banach algebra over the finite product of any of them with itself.

□

Theorem 3.3.1 is really handy because the Cauchy product arises naturally in product of power series and it gives us a bound on the coefficients of those products. Notice Theorem 3.3.1 also applies to the hat Cauchy product of Definition 15. Nonetheless, for the later, we would like to get a better bound on the coefficients of the product of power series than the bound given by the Banach algebra. To this end, we must first make a proposition.

Proposition 3.3.1. *Recall Definition 13. Let* $a \in \ell^1_\nu$ *and* $b \in \ell^{1,N}_\nu$. *Then,* $a*b \in \ell^{1,N}_\nu$ *and*

$$\|a * b\|_{1,\nu,N} \leq \|a\|_{1,\nu}\|b\|_{1,\nu,N} \quad .$$

Proof. First of all, notice that

$$(a*b)_\alpha = \sum_{\alpha_1+\alpha_2=\alpha} a_{\alpha_1} b_{\alpha_2} = 0 \quad \forall |\alpha| < N$$

since $b_\alpha = 0$ for all $|\alpha| < N$. Now, showing the estimate on the $\ell^{1,N}_\nu$-norm of $a*b$ will also prove that the latter belongs to $\ell^{1,N}_\nu$. Since $\ell^{1,N}_\nu \subset \ell^1_\nu$, both b and $a*b$ belong to ℓ^1_ν. By Theorem 3.3.1, we have

$$\|a * b\|_{1,\nu} \leq \|a\|_{1,\nu}\|b\|_{1,\nu} \quad .$$

Therefore, since $\|b\|_{1,\nu} = \|b\|_{1,\nu,N}$ and $\|a * b\|_{1,\nu} = \|a * b\|_{1,\nu,N}$, we have

$$\|a * b\|_{1,\nu,N} \leq \|a\|_{1,\nu} \|b\|_{1,\nu,N} \quad .$$

□

Proposition 3.3.1 also works for $a \in \ell^1_{\nu,\mu}$ and $b \in \ell^{1,N}_{\nu,\mu}$. As mentioned above, this proposition is a tool to prove the next theorem. Nonetheless, we also need a lemma to prove it, lemma that requires to make a definition first.

Definition 15 (Hat Cauchy product)**.** Recall Definition 12. Let $a = (a_\alpha)_{\alpha \in \mathbb{N}^d}$ and $b = (b_\alpha)_{\alpha \in \mathbb{N}^d}$ be sequences of real numbers. The *hat Cauchy product* of a and b, denoted $\widehat{a * b}$, is defined component-wise by

$$(\widehat{a * b})_\alpha \overset{\text{def}}{=} \begin{cases} 0 & \text{, if } \alpha = 0 \\ (a * b)_\alpha - a_\alpha b_0 - a_0 b_\alpha & \text{, otherwise} \end{cases} \quad .$$

Moreover, for $\beta \in \mathbb{N}^l$ and $a_1 = ((a_1)_\alpha)_{\alpha \in \mathbb{N}^d}, \dots, a_l = ((a_l)_\alpha)_{\alpha \in \mathbb{N}^d}$ sequences of real numbers, we define $\widehat{a_1^{\beta_1} * \cdots * a_l^{\beta_l}}$ component-wise by

$$(\widehat{a_1^{\beta_1} * \cdots * a_l^{\beta_l}})_\alpha \overset{\text{def}}{=} \begin{cases} 0 & \text{, if } \alpha = 0 \\ (a_1^{\beta_1} * \cdots * a_l^{\beta_l})_\alpha - \beta_1 (a_1)_\alpha (a_1)_0^{\beta_1 - 1} (a_2)_0^{\beta_2} \dots (a_{l-1})_0^{\beta_{l-1}} (a_l)_0^{\beta_l} & \\ \qquad - \cdots - \beta_l (a_1)_0^{\beta_1} (a_2)_0^{\beta_2} \dots (a_{l-1})_0^{\beta_{l-1}} (a_l)_0^{\beta_l - 1} (a_l)_\alpha & \text{, otherwise} \end{cases} \quad .$$

For the sake of simplicity, we are henceforth going to write $a \hat{*} b$ instead of $\widehat{a * b}$ whenever the hat Cauchy product only involves two sequences of real numbers.

Remark.

— One may notice the hat Cauchy product is not an associative operation in the sense that, in general, for a, b, c sequences of real numbers,

$$\widehat{a * b * c} \neq (a \hat{*} b) \hat{*} c \neq a \hat{*} (b \hat{*} c) \quad .$$

— Definition 15 naturally arises with the use of recurrence relation. For instance, recall Example 5. The hat Cauchy product had already been introduced there.

With Definition 15 in hand, we can state the lemma needed to prove the next theorem.

Lemma 3.3.1. *Recall Definition 15. Let $a_1, \dots, a_q \in \ell^1_\nu$ and $M > 0$. Then,*

$$\sum_{|\alpha|=0}^{M} |(\widehat{a_1 * \cdots * a_q})_\alpha| \nu^\alpha \leq \sum_{|\alpha|=0}^{M} (\widehat{|a_1| * \cdots * |a_q|})_\alpha \nu^\alpha \quad .$$

Proof. Notice

$$\begin{aligned}
\sum_{|\alpha|=0}^{M} |(\widehat{a_1 * \cdots * a_q})_\alpha \nu^\alpha| &= \sum_{|\alpha|=0}^{M} \left| \sum_{\substack{\alpha_1+\cdots+\alpha_q=\alpha \\ |\alpha_1|,\dots,|\alpha_q|\geq 0 \\ \alpha_1,\dots,\alpha_q\neq\alpha}} a_{\alpha_1} \cdot \ldots \cdot a_{\alpha_q} \right| \nu^\alpha \\
&\leq \sum_{|\alpha|=0}^{M} \left(\sum_{\substack{\alpha_1+\cdots+\alpha_q=\alpha \\ |\alpha_1|,\dots,|\alpha_q|\geq 0 \\ \alpha_1,\dots,\alpha_q\neq\alpha}} |a_{\alpha_1}| \cdot \ldots \cdot |a_{\alpha_q}| \right) \nu^\alpha \\
&= \sum_{|\alpha|=0}^{M} (\widehat{|a_1| * \cdots * |a_q|})_\alpha \nu^\alpha \quad .
\end{aligned}$$

□

With Proposition 3.3.1 and Lemma 3.3.1 in hand, we can now state a theorem that strengthens the bound of the Banach algebra when the hat Cauchy product is used.

Theorem 3.3.2. *Recall Definition 15. Let $r > 0$. If $a_1, a_2, \dots, a_{q-1} \in \ell^1_\nu$ and $a_q \in \ell^{1,N}_\nu$ with $\|a_q\|_{1,\nu,N} = r$, then $\widehat{a_1 * a_2 * \cdots * a_q} \in \ell^{1,N}_\nu$ and*

$$\|\widehat{a_1 * a_2 * \cdots * a_q}\|_{1,\nu,N} \leq \|a_1\|_{1,\nu}\|a_2\|_{1,\nu} \cdots \|a_{q-1}\|_{1,\nu} \cdot r - |(a_1)_0| \cdot |(a_2)_0| \cdots |(a_{q-1})_0| \cdot r \quad .$$

Proof. First of all, note that

$$(\widehat{a_1 * a_2 * \cdots * a_q})_\alpha = \sum_{\substack{\alpha_1+\alpha_2+\cdots+\alpha_q=\alpha \\ \alpha_1,\alpha_2,\dots,\alpha_q\neq\alpha}} (a_1)_{\alpha_1}(a_2)_{\alpha_2} \cdots (a_q)_{\alpha_q} = 0 \quad (\forall |\alpha| < N)$$

since $|\alpha_q| \leq |\alpha| < N$ and $(a_q)_{\alpha_q} = 0$ for all $|\alpha_q| < N$. Now, showing the estimate on the $\ell^{1,N}_\nu$-norm of $\widehat{a_1 * a_2 * \cdots * a_q}$ will also prove that the latter belongs to $\ell^{1,N}_\nu$. For $M > N$, by lemma 3.3.1, notice we have

$$\sum_{|\alpha|=N}^{M} |(\widehat{a_1 * a_2 * \cdots * a_q})_\alpha| \nu^\alpha \leq \sum_{|\alpha|=N}^{M} (\widehat{|a_1| * |a_2| * \cdots * |a_q|})_\alpha \nu^\alpha$$

$$= \sum_{|\alpha|=N}^{M} \left((|a_1| * |a_2| * \cdots * |a_q|)_\alpha - |(a_1)_0| \cdot |(a_2)_0| \cdots |(a_{q-1})_0| \cdot |(a_q)_\alpha| \right) \nu^\alpha \quad .$$

Hence, taking the limit as $M \to \infty$ of the above, we get

$$\begin{aligned}
\|\widehat{a_1 * a_2 * \cdots * a_q}\|_{1,\nu,N} &= \lim_{M\to\infty} \sum_{|\alpha|=N}^{M} |(\widehat{a_1 * a_2 * \cdots * a_q})_\alpha| \nu^\alpha \\
&\le \lim_{M\to\infty} \sum_{|\alpha|=N}^{M} \left((|a_1| * |a_2| * \cdots * |a_q|)_\alpha - |(a_1)_0| \cdot |(a_2)_0| \cdots |(a_{q-1})_0| \cdot |(a_q)_\alpha| \right) \nu^\alpha \\
&= \||a_1| * |a_2| * \cdots * |a_q|\|_{1,\nu,N} - |(a_1)_0| \cdot |(a_2)_0| \cdots |(a_{q-1})_0| \cdot \||a_q|\|_{1,\nu,N} \quad .
\end{aligned}$$

By proposition 3.3.1, we get

$$\begin{aligned}
\|\widehat{a_1 * a_2 * \cdots * a_q}\|_{1,\nu,N} &\le \||a_1|\|_{1,\nu} \||a_2|\|_{1,\nu} \cdots \||a_{q-1}|\|_{1,\nu} \||a_q|\|_{1,\nu,N} - |(a_1)_0| \cdot |(a_2)_0| \cdots |(a_{q-1})_0| \cdot \||a_q|\|_{1,\nu,N} \\
&= \|a_1\|_{1,\nu} \|a_2\|_{1,\nu} \cdots \|a_{q-1}\|_{1,\nu} \|a_q\|_{1,\nu,N} - |(a_1)_0| \cdot |(a_2)_0| \cdots |(a_{q-1})_0| \cdot \|a_q\|_{1,\nu,N} \\
&= \|a_1\|_{1,\nu} \|a_2\|_{1,\nu} \cdots \|a_{q-1}\|_{1,\nu} \cdot r - |(a_1)_0| \cdot |(a_2)_0| \cdots |(a_{q-1})_0| \cdot r \quad .
\end{aligned}$$

The last equality uses the assumption $\|a_q\|_{1,\nu,N} = r$.

□

One can go over the same argument to show Theorem 3.3.2 also holds for $a_1, a_2, \ldots, a_{q-1} \in \ell^1_{\nu,\mu}$ and $a_q \in \ell^{1,N}_{\nu,\mu}$ with $\|a_q\|_{1,(\nu,\mu),N} = r$. This theorem is going to be handy later on when we go over some examples.

Definition 16. Recall Theorem 3.3.2. For the sake of simplicity, we define

$$\widehat{\|a_1\|_{1,\nu} \|a_2\|_{1,\nu} \cdots \|a_{q-1}\|_{1,\nu}} \stackrel{\text{def}}{=} \|a_1\|_{1,\nu} \|a_2\|_{1,\nu} \cdots \|a_{q-1}\|_{1,\nu} - |(a_1)_0 (a_2)_0 \cdots (a_{q-1})_0|$$

for $a_1, a_2, \ldots, a_{q-1} \in \ell^1_\nu$ and $a_q \in \ell^{1,N}_\nu$ with $\|a_q\|_{1,\nu,N} = 1$. In the same way, we also define

$$\widehat{\|a_1\|_{1,(\nu,\mu)} \|a_2\|_{1,(\nu,\mu)} \cdots \|a_{q-1}\|_{1,(\nu,\mu)}} \stackrel{\text{def}}{=} \|a_1\|_{1,(\nu,\mu)} \|a_2\|_{1,(\nu,\mu)} \cdots \|a_{q-1}\|_{1,(\nu,\mu)} - |(a_1)_{0,0} (a_2)_{0,0} \cdots (a_{q-1})_{0,0}|$$

for $a_1, a_2, \ldots, a_{q-1} \in \ell^1_{\nu,\mu}$ and $a_q \in \ell^{1,N}_{\nu,\mu}$ with $\|a_q\|_{1,(\nu,\mu),N} = 1$.

Definition 16 is going to simplify a lot the notation when we go on to the next section.

We can finally talk about the method that proves our computations are good enough approximations.

3.4 Radii polynomials

We are going to state a theorem that is fundamental in applied mathematics. Recall that a *contraction* is an operator $T : X \to X$ from a Banach space to itself such that T is Lipschitz continuous with constant $\kappa < 1$.

Theorem 3.4.1 (Contraction Mapping Theorem)**.** *Let (X, d) be a non-empty complete metric space. Let $T : X \to X$ be a contraction. Then, T admits a unique fixed point in X.*

Proof. See [17].

□

Note that a *fixed point* for an operator is an element $\tilde{x}$ of its domain such that $T(\tilde{x}) = \tilde{x}$.

This theorem is omnipresent in applied analysis because it is a useful tool to prove existence and uniqueness of solutions of Partial Differential Equations (PDEs). In our case, it will be a big part of the proof of our main theorem to ensure existence and uniqueness of the true power series in a neighbourhood of our approximated power series. Nonetheless, we need to make another Definition before stating this theorem.

Definition 17 (Fréchet derivative)**.** Let X and Y be normed linear spaces. The *Fréchet derivative* of an operator $T : X \to Y$ at $a \in X$ is the bounded linear operator $DF(a) : X \to Y$ which satisfies

$$\lim_{h \to 0} \frac{\|F(a+h) - F(a) - DF(a)h\|}{\|h\|} = 0 \quad .$$

We say the operator T is *Fréchet differentiable* if it has a Fréchet derivative at every point in its domain.

As explained in [17], the Fréchet derivative is a generalization of the derivative we are used to in the real numbers. It allows one to compute "derivatives" of infinite-dimensional operators – one gets to define the notion of derivative in infinite-dimensional spaces, e.g. Fréchet derivative. We are now ready to state the main theorem to valid our computations, theorem that is a tool to study the fixed point of T as introduced in Equation 3.1.

Theorem 3.4.2 (Radii polynomials theorem)**.** *Suppose we have a Fréchet differentiable operator $T : X \to X$, where X is a Banach space. Suppose that, for some $\bar{x} \in X$, we have*

$$\|T(\bar{x}) - \bar{x}\|_X \leq Y_0, \qquad \sup_{x \in B(\bar{x},r)} \|DT(x)\|_{B(X)} \leq Z(r) \quad , \tag{3.2}$$

where $\|\cdot\|_{B(X)}$ is the norm on bounded linear operators from X to X induced by the norm $\|\cdot\|_X$ on X. Moreover, suppose that $Y_0 < \infty$, $Z(r) < \infty$ ($\forall r \in B(\bar{x}, r)$) and there exists $r_0 > 0$ such that

$$Z(r_0)r_0 + Y_0 < r_0 \quad .$$

Then, there exists a unique fixed point $\tilde{x}$ of T in $B(\bar{x}, r_0)$.

Proof. This proof is an application of Theorem 3.4.1. Let $x \in B(\bar{x}, r_0)$. Then,

$$\begin{aligned}
\|T(x) - \bar{x}\|_X &\leq \|T(x) - T(\bar{x})\|_X + \|T(\bar{x}) - \bar{x}\|_X \\
&\overset{(*)}{\leq} \sup_{\xi \in [x,\bar{x}]} \|DT(\xi)\|_{B(X)} \|x - \bar{x}\|_X + \|T(\bar{x}) - \bar{x}\|_X \\
&\leq Z(r_0)r_0 + Y_0 \\
&< r_0 \quad ,
\end{aligned}$$

where $[x, \bar{x}]$ is the line segment joining x and $\bar{x}$. Hence, T maps $B(\bar{x}, r_0)$ onto itself. Let $y_1, y_2 \in B(\bar{x}, r_0)$. Then,

$$\begin{aligned}
\|T(y_1) - T(y_2)\|_X &\overset{(*)}{\leq} \sup_{\xi \in [y_1,y_2]} \|DT(\xi)\|_{B(X)} \|y_1 - y_2\|_X \\
&\leq Z(r_0)\|y_1 - y_2\|_X \\
&< \left(1 - \frac{Y_0}{r_0}\right) \|y_1 - y_2\|_X \\
&= \kappa \|y_1 - y_2\|_X \quad ,
\end{aligned}$$

where $\kappa = \left(1 - \frac{Y_0}{r_0}\right)$. Notice $\kappa \in (0,1)$. Indeed, since $Z(r_0)r_0 + Y_0 < r_0$ and all of these 3 terms are nonnegative, it follows that $Y_0 < r_0$. Assuming $Y_0 > 0$ – we will discuss that right after the proof but this is no restriction whatsoever –, this proves $\kappa \in (0,1)$. Therefore, $T : B(\bar{x}, r_0) \to B(\bar{x}, r_0)$ is a contraction. Hence, by Theorem 3.4.1, T has a unique fixed point in $B(\bar{x}, r_0)$, say $\tilde{x}$.

□

Remark. We need to clarify a couple of things regarding Theorem 3.4.2.

1. The theorem does not work if $r_0 = 0$. Nevertheless, $r_0 = 0$ means that $Y_0 = 0$ in the theorem, so $\bar{x}$ would already be the fixed point, which is never the case in practice. If it was, then there would be no need to prove anything because we would already have the fixed point as wanted.
2. In the same regard, we cannot have $Y_0 = 0$. Once again, this would mean we already have the fixed point as wanted.
3. The two inequalities $\overset{(*)}{\leq}$ presented in the proof of the theorem use a result we have yet to proof. The result is exactly the statement of the second inequality for Fréchet differentiable operators, i.e. a variation for normed spaces of the well-known Mean Value Theorem. One needs to use one of the derivations of the Hahn-Banach Theorem to prove this result.

Given the bounds (3.2), Theorem 3.4.2 is usable if and only if one can find $r_0 > 0$ such that

$$Z(r_0)r_0 + Y_0 < r_0 \quad .$$

Therefore, we would like to have some control over those bounds to be able to get the existence of such an r_0. Recall Definition 13. Let $X = \ell^1_\nu$. This choice of Banach space allows us to get control over the bounds (3.2). Indeed, we choose the weight ν to lessen the value of the norms involved in the computation of the bounds 3.2. Even though it may have several dimensions, we can still choose each one of its components to lessen the bounds 3.2. Hence, a good choice of ν can ensure the existence of an $r_0 > 0$ because we will have control over the bounds of the Radii Polynomials Theorem.

Let us make a corollary to Theorem 3.4.2 in order to show its use when one works with power series.

Corollary 3.4.1. *Suppose we have a Fréchet differentiable operator $T : X \to X$, where X is a Banach space. Suppose that, for some $\bar{x} \in X$, we have*

$$\|T(\bar{x}) - \bar{x}\|_X \leq Y_0, \qquad \sup_{x \in B(\bar{x},r)} \|DT(x)\|_{B(X)} \leq Z(r) = Z_1 + Z_2(r) \quad ,$$

where $\|\cdot\|_{B(X)}$ is the norm on bounded linear operators from X to X induced by the norm $\|\cdot\|_X$ on X. Moreover, suppose that $Y_0, Z_1 < \infty$, $Z_2(r) < \infty$ $(\forall r \in B(\bar{x}, r))$ and there exists $r_0 > 0$ such that

$$Z_2(r_0)r_0^2 + (Z_1 - 1)r_0 + Y_0 < 0 \quad . \tag{3.3}$$

Then, there exists a unique fixed point $\tilde{x}$ of T in $B(\bar{x}, r_0)$.

Proof. Notice

$$Z(r_0) + Y_0 < r_0 \iff Z_2(r_0)r_0^2 + (Z_1 - 1)r_0 + Y_0 < 0 \quad .$$

Therefore, applying Theorem 3.4.2, we get the result.

□

Remark. Since $Z_2(r_0), Z_1, Y_0 > 0$ and $r_0 > 0$, we must have $Z_1 < 1$, otherwise r_0 cannot be positive. In practice, we do not care too much about $Z_2(r_0)$ and put a lot of focus on Z_1 to make sure it is less than 1. One can do so by diminishing components of the weight ν and increasing the number of computed coefficients.

Recall Operator 3.1. Assume $X = \ell_\nu^{1,N}$. Since we are working on solving for coefficients of power series and **A2** holds, the operator T will be given componentwise by a finite combination of Cauchy products – this fact will become clear in Section 4. Hence, the same goes for its derivative – this fact will become clear in Section 5. Recall Corollary 3.4.1. Let Z_1 be a bound for the constant terms of the derivative $DT(x)$ – it does not depend on x. Let $Z_2(r)$ be a polynomial bound in r for the higher order terms of $DT(x)$ for all $x \in B(\bar{x}, r)$ – this fact will become clear in Section 5. As stated in the above remark, Corollary 3.4.1 will be applicable if there exists an $r_0 > 0$ such that Equation 3.3 is verified, i.e. if $Z_1 < 1$. By the current discussion, the left hand side of Equation 3.3 is going to be a polynomial in r – substitute r_0 for r. Hence, solving for r_0 boils down to solving for the roots of the latter polynomial. Therefore, given that the bound Y_0 will be very small compared to Z_1 and $Z_2(r)$ – provided $|\nu| < 1$ and N big enough –, we will have the existence of a small $r_0 > 0$ satisfying Equation 3.3. Thus, Theorem 3.4.2 is a powerful tool for validating our computations because a fixed point of the operator T is a solution for the coefficients of the power series we are looking at.

Theorem 3.4.2 being stated, we are now ready to cover exactly which operators we have to solve in practice (Section 4) and how we manage to use the theory on Radii Polynomials to ensure we have good enough approximations of the power series we are looking at (Section 5).

4 Practical operators

Theorem 3.4.2 provides us with existence and uniqueness of our power series a posteriori. The *Strong Implicit Function Theorem* (see the Appendix) provides us with existence and uniqueness of some of our power series – fixed point, eigenvalues and eigenvectors (Subsections 4.1 and 4.2) – a priori. The Strong Implicit Function Theorem is really important : It will provide us with analytic dependency of our computations with respect to the parameters.

Recall (2.6). Recall our goal is to compute a parameterization, given by a power series, of P. We want to compute the coefficients of the power series of P. To do so, we first need to compute parameterizations, given by power series, of the fixed point $\tilde{x}(\omega)$ and the eigenvalues and eigenvectors of the Jacobian Matrix $D_x f(\tilde{x}(\omega), \omega)$. Then, we will be able to compute the parameterization of P. Therefore, we split our computations into three parts, namely Subsections 4.1, 4.2 and 4.3.

Remark. Recall Definitions 9, 12 and 13. Let $X = (\ell_\nu^1)^l$ $(l > 0)$ with $\nu \in \mathbb{R}^d$ $(d > 0)$. Let $\alpha \in \mathbb{Z}^d$, $\gamma \in \mathbb{Z}^l$ and $z \in X$. Henceforth, for the sake of simplicity, we are going to consider that

$$z_\alpha = 0 \qquad \& \qquad z_i^{\gamma_i} = 1 \quad (i \in \{1, \dots, l\})$$

whenever $\alpha_j < 0$ for any $j \in \{1, \dots, d\}$ and $\gamma_i < 0$ for any $i \in \{1, \dots, l\}$. This remark is very useful when working with power series because it allows to disregard negative indices as well as negative exponents of Cauchy products (see Definition 12).

4.1 Fixed point operator

The first part of the method consists of the computation of a parameterization of the fixed point $\tilde{x}(\omega)$ of the vector field of ODE (1.1) depending on the parameters ω. By Assumption **A2**, suppose the vector field is parameterized by

$$f(x, \omega) = \sum_{|\gamma_2|=0}^{M_2} \sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,\gamma_2} x^{\gamma_1} (\omega - \tilde{\omega})^{\gamma_2} \quad . \tag{4.1}$$

By Assumption **A1**, it has an hyperbolic fixed point $\tilde{x} \stackrel{\text{def}}{=} \tilde{x}(\tilde{\omega})$ for the value $\tilde{\omega}$ of the parameters ω. Fixed points of the ODE (1.1) are zeros of the vector field. Therefore, consider the equation

$$f(x, \omega) = 0 \quad . \tag{4.2}$$

$\tilde{x}$ satisfies Equation (4.2) for $\omega = \tilde{\omega}$. By Assumption **A3**, $D_x f(\tilde{x}, \tilde{\omega})$ is invertible since it has no eigenvalue with zero real part. Therefore, the Strong Implicit Function Theorem implies that the fixed point can be written as a power series in ω in some neighbourhood of $\tilde{\omega}$. Let

$$\tilde{x}(\omega) = \sum_{|\beta| \geq 0} b_\beta (\omega - \tilde{\omega})^\beta \tag{4.3}$$

where $b_\beta = (b_1, \ldots, b_n)_\beta = ((b_1)_\beta, \ldots, (b_n)_\beta)$. Basically, b is a vector of sequences. Plugging $\tilde{\omega}$ into Equation 4.3, we get $b_0 = \tilde{x}$. Recall Definitions 12 and 15. Since $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is analytic at $\tilde{x}$ with respect to both variables and parameters, we can substitute x by the series of $\tilde{x}(\omega)$ in (4.2) and rewrite everything as a power series with respect to $\omega - \tilde{\omega}$. This gives

$$\sum_{|\beta| \geq 0} \sum_{|\gamma_2|=0}^{M_2} \sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1, \gamma_2} \left(b_1^{(\gamma_1)_1} * \cdots * b_n^{(\gamma_1)_n} \right)_{\beta - \gamma_2} (\omega - \tilde{\omega})^\beta = 0 \quad .$$

Since a power series is zero everywhere if and only if its coefficients are all 0, we get

$$\sum_{|\gamma_2|=0}^{M_2} \sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1, \gamma_2} \left(b_1^{(\gamma_1)_1} * \cdots * b_n^{(\gamma_1)_n} \right)_{\beta - \gamma_2} = 0 \quad (\forall |\beta| \geq 0) \quad .$$

Hence, that gives us a relation onto the coefficients of $\tilde{x}(\omega)$, namely

$$B \begin{pmatrix} (b_1)_\beta \\ \vdots \\ (b_n)_\beta \end{pmatrix} = (g(b,c))_\beta \quad (\forall |\beta| \geq 1) \quad ,$$

where $g(b,c)$ is defined component-wise by

$$(g(b,c))_\beta \overset{\text{def}}{=} - \sum_{|\gamma_2|=1}^{M_2} \sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1, \gamma_2} \left(b_1^{(\gamma_1)_1} * \cdots * b_n^{(\gamma_1)_n} \right)_{\beta - \gamma_2} - \sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1, 0} \left(\widehat{b_1^{(\gamma_1)_1} * \cdots * b_n^{(\gamma_1)_n}} \right)_\beta$$

and

$$B_{1,1} \overset{\text{def}}{=} \sum_{|\gamma_1|=0}^{M_1} (\gamma_1)_1 (c_1)_{\gamma_1, 0} (b_1)_0^{(\gamma_1)_1 - 1} \cdot (b_2)_0^{(\gamma_1)_2} \cdot \ldots \cdot (b_n)_0^{(\gamma_1)_n}$$

$$B_{1,n} \overset{\text{def}}{=} \sum_{|\gamma_1|=0}^{M_1} (\gamma_1)_n (c_1)_{\gamma_1, 0} (b_1)_0^{(\gamma_1)_1} \cdot \ldots \cdot (b_{n-1})_0^{(\gamma_1)_{n-1}} \cdot (b_n)_0^{(\gamma_1)_n - 1}$$

$$B_{n,1} \overset{\text{def}}{=} \sum_{|\gamma_1|=0}^{M_1} (\gamma_1)_1 (c_n)_{\gamma_1, 0} (b_1)_0^{(\gamma_1)_1 - 1} \cdot (b_2)_0^{(\gamma_1)_2} \cdot \ldots \cdot (b_n)_0^{(\gamma_1)_n}$$

$$B_{n,n} \overset{\text{def}}{=} \sum_{|\gamma_1|=0}^{M_1} (\gamma_1)_n (c_n)_{\gamma_1, 0} (b_1)_0^{(\gamma_1)_1} \cdot \ldots \cdot (b_{n-1})_0^{(\gamma_1)_{n-1}} \cdot (b_n)_0^{(\gamma_1)_n - 1}$$

$$B \stackrel{\text{def}}{=} \begin{pmatrix} B_{1,1} & \cdots & B_{1,n} \\ \vdots & \ddots & \vdots \\ B_{n,1} & \cdots & B_{n,n} \end{pmatrix} \quad .$$

Thus, we can use this relation to get the recurrence relation

$$b_\beta = B^{-1} \left(g(b,c)\right)_\beta \quad (\forall |\beta| \geq 1) \quad . \tag{4.4}$$

Let us justify the invertibility of B. Notice $B = D_x f(x(\tilde{\omega}), \tilde{\omega})$. By Assumption **A3**, B has no eigenvalues with zero real part, so it is invertible.

The recurrence relation (4.4) requires an initial condition, which is b_0. Notice that

$$\tilde{x}(\tilde{\omega}) = b_0 \quad .$$

Since the computation of $\tilde{x}(\tilde{\omega})$ is the classic computation of fixed points of a vector field, we know how to compute b_0. Therefore, we have a way to compute exactly the coefficients of the power series of the fixed point $\tilde{x}(\omega)$. Recall the discussion in Subsection 3.2, especially the definition of the general operator (3.1). Recall Definition 13. Assume we have computed all the coefficients b_β up to order $N-1$. We are going to set $T : \left(\ell_\mu^{1,N}\right)^n \to \left(\ell_\mu^{1,N}\right)^n$ as

$$(T(u))_\beta \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } 0 \leq |k| \leq N-1 \\ B^{-1} \left(g(\bar{b}+u, c)\right)_\beta & \text{if } |\beta| \geq N \end{cases} \quad , \tag{4.5}$$

where

$$\bar{b}_\beta \stackrel{\text{def}}{=} \begin{cases} b_\beta & \text{if } 0 \leq |\beta| \leq N-1 \\ 0 & \text{if } |\beta| \geq N \end{cases} \quad .$$

We want to prove T has a unique fixed point and that $u = 0$ is a good approximation of it. We have proven the existence of a series in ω for the fixed point $\tilde{x}(\omega)$, which means that it has a radius of convergence $R > 0$ with $R \in \mathbb{R}^p$. We know by basic analysis that this implies

$$\lim_{|\beta| \to \infty} |b_\beta \delta^\beta| = 0 \quad (-R < \delta < R) \quad ,$$

where $\delta \in \mathbb{R}^p$. Thus, we can get $|b_\beta \delta^\beta|$ as small as we want if we take $|\beta|$ big enough. Notice that the terms $|b_\beta \delta^\beta|$ are exactly the terms involved in the $\left(\ell_\mu^{1,N}\right)^n$-norm, just substitute δ for μ. Hence, the $\left(\ell_\mu^{1,N}\right)^n$-norm of $T(0)$ is going to be very small provided N big enough. The latter is a blueprint to choose N : Choose N so that each $|b_\beta \mu^\beta|$

is below some tolerance. This justifies that $u = 0$ is a good approximation of the fixed point of T in $\left(\ell_\mu^{1,N}\right)^n$. We say "the" because T is by definition the recurrence relation, so the coefficients of the series of $\tilde{x}(\omega)$ are a fixed point by definition of the recurrence relation. Furthermore, they are the only fixed point because a solution to the recurrence relation is totally determined by the initial condition.

Remark.

- For the sake of the argument, we assume here that the fixed point $\tilde{x}(\tilde{\omega})$ is isolated, i.e. not a cluster point with respect to the set of fixed points.
- Note that we do not prove the coefficients b_β for $|\beta| \leq N-1$. Indeed, we need not to because they have been computed exactly using the recurrence relation, so there is no need to prove them.

Now that we know $u = 0$ is a good approximation of the unique fixed point of T for some given tolerance, provided N big enough, it remains to rigorously prove it. This is where we use Theorem 3.4.2, the Radii Polynomial Theorem. We just have to compute the bounds (3.2). To ensure that the theorem works, we must ensure the polynomial (3.3) has a positive root. To that end, we can take $|\mu|$ small enough and N big enough. There is no proper ratio for the latters, their choice will be governed by the needs for the proof and the software limitations.

4.2 Eigenvalues and eigenvectors operator

The second part consists of the computation of a parameterization of the eigenvalues and eigenvectors of the vector field of the ODE (1.1) at the fixed point $\tilde{x}(\omega)$. Assume we have already computed its parameterization with respect to ω, i.e. we have already computed the coefficients (4.4). Eigenvalues and eigenvectors for the fixed point $\tilde{x}(\omega)$ around $\tilde{\omega}$ are zeros of the equation

$$D_x F(\tilde{x}(\omega), \omega) v(\omega) - \lambda(\omega) v(\omega) = 0 \quad , \tag{4.6}$$

where $\lambda(\omega)$ is an eigenvalue and $v(\omega)$ one of its eigenvectors for each value of the parameters ω. Assume $\lambda(\omega) \in \mathbb{R}$ – the complex case will be covered later on. Let $(\lambda, v) \overset{\text{def}}{=} (\lambda(\tilde{\omega}), v(\tilde{\omega}))$ satisfy (4.6) at $\omega = \tilde{\omega}$. (λ, v) is an eigencouple – an eigenvalue and one of its eigenvectors – by definition. The variables are λ and $v_1, \ldots, v_n$, the components of v. Since a scalar multiple of an eigenvector is again an eigenvector, there are infinitely many solutions to (4.6) at $\omega = \tilde{\omega}$, so there is no way to get uniqueness of the eigenvector at $\omega = \tilde{\omega}$ without at least another assumption.

Remark. An eigenvector has at least one nonzero component. Without loss of generality, suppose $v_j \neq 0$ ($j \in \{1, \dots, n\}$). Therefore, $\frac{v}{v_j}$ is also an eigenvector of λ because it is a scalar multiple of the eigenvector v and $v_j \neq 0$.

Using the remark, without loss of generality, suppose that $v_j = 1$. Thus, the couple (λ, v) is the unique one to satisfy (4.6) at $\omega = \tilde{\omega}$. Indeed, the dimension of the eigenspace of λ is one – it is a consequence of Assumption **A3**, namely the eigenvalues are distinct.

Given $v_j = 1$, since (4.6) has a unique solution at $\omega = \tilde{\omega}$, the Jacobian matrix is invertible at $\omega = \tilde{\omega}$, so the Strong Implicit Function Theorem applies and we get that $\lambda(\omega)$ and $v(\omega)$ can both be written as a series in ω in some neighbourhood of $\tilde{\omega}$. The fact that the Jacobian matrix is invertible can be derived from a straight computation of the Jacobian matrix coupled with the assumption that the eigenspace is of dimension one. We will come back shortly to that. Hence, let us write $\lambda(\omega)$ and $v(\omega)$ as

$$\lambda(\omega) = \sum_{|\beta| \geq 0} \lambda_\beta (\omega - \tilde{\omega})^\beta, \quad v(\omega) = \sum_{|\beta| \geq 0} v_\beta (\omega - \tilde{\omega})^\beta \quad ,$$

where $v_\beta = (v_1, \dots, v_n)_\beta = ((v_1)_\beta, \dots, (v_n)_\beta)$. Since $v_j(\tilde{\omega}) = 1$ and $v_j(\omega)$ is analytic, in particular continuous, there exists a neighbourhood of $\tilde{\omega}$ such that $v_j(\omega) \neq 0$ for all ω in this neighbourhood. Hence, without loss of generality, consider the eigenvector $V(\omega) = \frac{v(\omega)}{v_j(\omega)}$. It is a power series depending on ω around $\tilde{\omega}$ because it is holomorphic in the neighbourhood of $\tilde{\omega}$ (see [18]). Therefore, we now have

$$\lambda(\omega) = \sum_{|\beta| \geq 0} \lambda_\beta (\omega - \tilde{\omega})^\beta, \quad V(\omega) = \sum_{|\beta| \geq 0} V_\beta (\omega - \tilde{\omega})^\beta \quad , \tag{4.7}$$

where $V_\beta = (V_1, \dots, V_n)_\beta = ((V_1)_\beta, \dots, (V_n)_\beta)$. Noticing that $V_j(\omega) = 1$ for all ω in a proper neighbourhood of $\tilde{\omega}$, we deduce that

$$(V_j)_\beta = \begin{cases} 1 & \text{, if } |\beta| = 0 \\ 0 & \text{, otherwise} \end{cases} \quad .$$

In the end, we have isolated the couple $(\lambda(\omega), V(\omega))$ with respect to the eigenspace depending on ω. As for the computation of the coefficients (4.4), by substituting (4.7) into (4.6), we get

$$\sum_{|\beta|\geq 0}\sum_{|\gamma_2|=0}^{M_2}\sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,\gamma_2}\Bigg((\gamma_1)_1\left(\bar{b}_1^{(\gamma_1)_1-1}*\bar{b}_2^{(\gamma_1)_2}*\cdots*\bar{b}_n^{(\gamma_1)_n}*V_1\right)_{\beta-\gamma_2}+\cdots+$$

$$(\gamma_1)_n\left(\bar{b}_1^{(\gamma_1)_1}*\cdots*\bar{b}_{n-1}^{(\gamma_1)_{n-1}}*\bar{b}_n^{(\gamma_1)_n-1}*V_n\right)_{\beta-\gamma_2}\Bigg)(\omega-\tilde{\omega})^\beta-\sum_{|\beta|\geq 0}(\lambda*V)_\beta(\omega-\tilde{\omega})^\beta=0 \quad ,$$

where $\bar{b}$ is defined as in Subsection 4.1 and c is defined as in Equation 4.1. For the sake of simplicity, let $V_\lambda^{*j}=(V_1,\ldots,V_{j-1},\lambda,V_{j+1},\ldots,V_n)$. Since a power series is zero everywhere if and only if its coefficients are all 0, we get the relation

$$B(V_\lambda^{*j})_\beta = B\begin{pmatrix}(V_1)_\beta\\ \vdots\\ (V_{j-1})_\beta\\ \lambda_\beta\\ (V_{j+1})_\beta\\ \vdots\\ (V_n)_\beta\end{pmatrix}=\left(g(V_\lambda^{*j},V_j,\bar{b},c)\right)_\beta \quad (\forall|\beta|\geq 1) \quad ,$$

where $g(V_\lambda^{*j},V_j,b,c)$ is defined component-wise by

$$\left(g^1(V_\lambda^{*j},V_j,\bar{b},c)\right)_\beta \stackrel{\text{def}}{=} -\sum_{|\gamma_2|=1}^{M_2}\sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,\gamma_2}\sum_{1\leq i\leq n}(\gamma_1)_i\left(\bar{b}_1^{(\gamma_1)_1}*\cdots*\bar{b}_{i-1}^{(\gamma_1)_{i-1}}*\bar{b}_i^{(\gamma_1)_i-1}*\bar{b}_{i+1}^{(\gamma_1)_{i+1}}*\cdots*\bar{b}_n^{(\gamma_1)_n}*V_i\right)_{\beta-\gamma_2}$$

$$\left(g^2(V_\lambda^{*j},V_j,\bar{b},c)\right)_\beta \stackrel{\text{def}}{=} -\sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,0}\Bigg((\gamma_1)_j\left(\bar{b}_1^{(\gamma_1)_1}*\cdots*\bar{b}_{j-1}^{(\gamma_1)_{j-1}}*\bar{b}_j^{(\gamma_1)_j-1}*\bar{b}_{j+1}^{(\gamma_1)_{j+1}}*\cdots*\bar{b}_n^{(\gamma_1)_n}*V_j\right)_\beta$$

$$+\sum_{\substack{1\leq i\leq n\\ i\neq j}}(\gamma_1)_i\left(\overline{\bar{b}_1^{(\gamma_1)_1}*\cdots*\bar{b}_{i-1}^{(\gamma_1)_{i-1}}*\bar{b}_i^{(\gamma_1)_i-1}*\bar{b}_{i+1}^{(\gamma_1)_{i+1}}*\cdots*\bar{b}_n^{(\gamma_1)_n}*V_i}\right)_\beta\Bigg)$$

$$\left(g^3(V_\lambda^{*j},V_j,\bar{b},c)\right)_\beta \stackrel{\text{def}}{=} -\sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,0}\sum_{\substack{1\leq i\leq n\\ i\neq j}}(\gamma_1)_i\Big((\gamma_1)_1(\bar{b}_1)_0^{(\gamma_1)_1-1}(\bar{b}_2)_0^{(\gamma_1)_2}\cdots(\bar{b}_{i-1})_0^{(\gamma_1)_{i-1}}(\bar{b}_i)_0^{(\gamma_1)_i-1}(\bar{b}_{i+1})_0^{(\gamma_1)_{i+1}}\cdots(\bar{b}_n)_0^{(\gamma_1)_n}(V_i)_0(\bar{b}_1)_\beta$$

$$+\cdots+((\gamma_1)_i-1)(\bar{b}_1)_0^{(\gamma_1)_1}\cdots(\bar{b}_{i-1})_0^{(\gamma_1)_{i-1}}(\bar{b}_i)_0^{(\gamma_1)_i-2}(\bar{b}_{i+1})_0^{(\gamma_1)_{i+1}}\cdots(\bar{b}_n)_0^{(\gamma_1)_n}(V_i)_0(\bar{b}_i)_\beta$$

$$+\cdots+(\gamma_1)_n(\bar{b}_1)_0^{(\gamma_1)_1}\cdots(\bar{b}_{i-1})_0^{(\gamma_1)_{i-1}}(\bar{b}_i)_0^{(\gamma_1)_i-1}(\bar{b}_{i+1})_0^{(\gamma_1)_{i+1}}\cdots(\bar{b}_{n-1})_0^{(\gamma_1)_{n-1}}(\bar{b}_n)_0^{(\gamma_1)_n-1}(V_i)_0(\bar{b}_n)_\beta\Big)$$

$$\left(g^4(V_\lambda^{*j},V_j,\bar{b},c)\right)_\beta \stackrel{\text{def}}{=} (\lambda\hat{*}V)_\beta+\lambda_0(V_j)_\beta\cdot e_j$$

$$\left(g(V_\lambda^{*j},V_j,\bar{b},c)\right)_\beta \stackrel{\text{def}}{=} \left(g^1(V_\lambda^{*j},V_j,\bar{b},c)\right)_\beta+\left(g^2(V_\lambda^{*j},V_j,\bar{b},c)\right)_\beta+\left(g^3(V_\lambda^{*j},V_j,\bar{b},c)\right)_\beta+\left(g^4(V_\lambda^{*j},V_j,\bar{b},c)\right)_\beta \quad ,$$

e_j is the j-th canonical vector from $\mathbb{R}^n$ and B is given column-wise by

$$B \overset{\text{def}}{=} \left(\begin{array}{c|c|c|c|c|c|c} K_1 & \cdots & K_{j-1} & K_j & K_{j+1} & \cdots & K_n \end{array} \right) - \begin{pmatrix} \lambda_0 & & & & & & \\ & \ddots & & & & & \\ & & \lambda_0 & & & & \\ & & & 0 & & & \\ & & & & \lambda_0 & & \\ & & & & & \ddots & \\ & & & & & & \lambda_0 \end{pmatrix}$$

$$\underbrace{\qquad}_{i\text{-th column}}$$

with

$$B_i \overset{\text{def}}{=} \begin{cases} \displaystyle\sum_{|\gamma_1|=0}^{M_1} (\gamma_1)_i \cdot c_{\gamma_1,0} \cdot (\bar{b}_1)_0^{(\gamma_1)_1} \cdots (\bar{b}_{i-1})_0^{(\gamma_1)_{i-1}} \cdot (\bar{b}_i)_0^{(\gamma_1)_i - 1} \cdot (\bar{b}_{i+1})_0^{(\gamma_1)_{i+1}} \cdots (\bar{b}_n)_0^{(\gamma_1)_n} & , \text{if } i \neq j \\ -V_0 & , \text{if } i = j \end{cases} .$$

Thus, we can use this relation to get the recurrence relation

$$(V_\lambda^{*j})_\beta = B^{-1} \left(g(V_\lambda^{*j}, V_j, \bar{b}, c) \right)_\beta \quad (\forall |\beta| \geq 1) \quad . \tag{4.8}$$

Let us justify the invertibility of B. Let $K = D_x F(\tilde{x}(\tilde{\omega}), \tilde{\omega}) - \lambda(\tilde{\omega}) \cdot I_n$. Recall $(\lambda, v) = (\lambda(\tilde{\omega}), v(\tilde{\omega}))$ satisfies Equation 4.6. Let $V \overset{\text{def}}{=} V_0$. It follows (λ, V) also satisfies Equation 4.6. Notice the columns of B and K are the same but the i-th ones. Since the kernel of K is of dimension one by Assumption **A3**, we know all its columns but the i-th one are linearly independent. Therefore, if B is not invertible, then its i-th column is a linear combination of the others. Denote by $B_1, \ldots, B_n$ its columns from first to last respectively. Since $B_j = -V$, there exits $k_1, \ldots, k_{j-1}, k_{j+1}, \ldots, k_n \in \mathbb{R}$ such that $V = k_1 B_1 + \cdots + k_{j-1} B_{j-1} + k_{j+1} B_{j+1} + \cdots + k_n B_n$. Thus, V belongs to the image of K. Let $y \in \mathbb{R}^n$ be such that $Ky = V$. We have

$$K^2 y = K(Ky) = KV = 0 \quad .$$

Hence, y is a generalized vector of $D_x F(\tilde{x}(\tilde{\omega}), \tilde{\omega})$ for the eigenvalue λ. However, the dimension of the eigenspace of λ is one because the kernel of K is of dimension one. Therefore, y is a multiple of V, say $y = \kappa V$ $(\kappa \in \mathbb{R})$, and we have

$$V = Ky = K(\kappa V) = \kappa K V = \kappa \cdot 0 = 0 \quad .$$

This is a contradiction because an eigenvector is not zero by definition. Thereby, B is invertible.

The recurrence relation (4.8) requires an initial condition, which is $(V_\lambda^{*j})_0$. Notice

$$\lambda(\tilde{\omega}) = \lambda_0, \quad V(\tilde{\omega}) = V_0 \quad .$$

We can use classic computations to get $(\lambda_0, V_0) = (\lambda, V)$. Indeed, V_0 is the unique eigenvector associated to the eigenvalue λ_0 such that $(V_j)_0 = 1$ because the dimension of the eigenspace of λ_0 is one. Therefore, we have a way to compute exactly the coefficients of both the power series of the eigenvalue $\lambda(\omega)$ and its associated eigenvector $V(\omega)$. Recall the discussion in Subsection 3.2, especially the definition of the general operator (3.1). Recall Definition 13. Assume we have computed all the coefficients $(V_\lambda^{*j})_\beta$ up to order $N-1$. We are going to set $T : \left(\ell_\mu^{1,N}\right)^n \to \left(\ell_\mu^{1,N}\right)^n$ as

$$(T(u))_\beta \stackrel{\text{def}}{=} \begin{cases} 0 & , \text{if } 0 \leq |\beta| \leq N-1 \\ B^{-1}\left(g(\bar{V}_\lambda^{*j} + u, \bar{V}_j, \bar{b}, c)\right)_\beta & , \text{if } |\beta| \geq N \end{cases} , \tag{4.9}$$

where

$$(\bar{V}_\lambda^{*j})_\beta = (\bar{V}_1, \dots, \bar{V}_{j-1}, \bar{\lambda}, \bar{V}_{j+1}, \dots, \bar{V}_n)_\beta \stackrel{\text{def}}{=} \begin{cases} (V_\lambda^{*j})_\beta & , \text{if } 0 \leq |\beta| \leq N-1 \\ 0 & , \text{if } |\beta| \geq N \end{cases}$$

and

$$(\bar{V}_j)_\beta \stackrel{\text{def}}{=} \begin{cases} 1 & , \text{if } |\beta| = 0 \\ 0 & , \text{otherwise} \end{cases} .$$

The goal will be to prove that T has a unique fixed point and that $u = 0$ is a good approximation of it. We have two power series that have a radius of convergence $R > 0$ – without loss of generality, we can assume they both have the same radius of convergence – and T is by definition the recurrence relation. Hence, using the same argument as for the fixed point in Subsection 4.1, we can get the bounds (3.2) as small as we want by take $|\mu|$ small enough and N big enough. Therefore, Theorem 3.4.2 will apply and we will have existence and uniqueness of both the power series for the eigenvalue and its associated eigenvector.

Remark. Since it is going to become handy for the next computations, we suppose the eigenvalue does not change sign within its neighbourhood of definition around $\tilde{\omega}$. It is always going to be the case if we shrink this neighbourhood enough.

Note we have to repeat this procedure for every eigenvalue and their associated eigenvector before moving on to the computation of the parameterizations of the stable and unstable manifolds.

4.3 Stable and unstable manifolds coefficients operator

The third part consists of the computation of a parameterization of P (see Equation 2.6) around the value $\tilde{\omega}$ of the parameters. Assume we have already computed the parameterization with respect to ω of the fixed point and the eigenvalues and eigenvectors, i.e. we have already computed the coefficients (4.4) and (4.8). Recall Theorem 2.3.1 and Equation (2.7). Assume $P : \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}^n$ is given by a power series, say

$$P(\theta, \omega) = \sum_{|\beta|\geq 0} \sum_{|\alpha|\geq 0} a_{\alpha,\beta} \theta^\alpha (\omega - \tilde{\omega})^\beta \quad . \tag{4.10}$$

Let us motivate this assumption. Recall Assumptions **A1**, **A2** and **A3**. We have an ODE with a polynomial vector field. By Theorem 2.2.2, for any fixed value of the parameters ω, there exists a parameterization of the local stable manifold $W^s_{\text{loc}}(\tilde{x}(\omega))$ at $\tilde{x}(\omega)$ that is analytic and whose domain is a subset of the stable subspace $\mathbb{E}^s$ containing the origin. This motivates the power series for P.

We assume Equations (2.6) stand. The first two equations are consequences of Theorem 2.2.2. The third one is an assumption that leads to Equations (2.7) of Theorem 2.3.1. These equations allow us to retrieve a recurrence relation over the coefficients of the power series of P. Among them, the only restrictive assumption is the third equation of Equations (2.6). However, recall that it is similar to the Hartman-Grobmann Theorem (see [4]), so it is sort of natural to make it.

Recall we assume that P is given by (4.10) and the vector field (1.1) is given by $f(x, \omega) = \sum_{|\gamma_2|=0}^{M_2} \sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,\gamma_2} (x - \tilde{x})^{\gamma_1} (\omega - \tilde{\omega})^{\gamma_2}$. Substituting the power series of P into the homological equation in (2.7), we get

$$\sum_{|\gamma_2|=0}^{M_2} \sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,\gamma_2} \left(a_1^{(\gamma_1)_1} * \cdots * a_n^{(\gamma_1)_n} \right)_{\alpha,\beta-\gamma_2} = \left((\alpha \cdot \bar{\lambda}) * a_\alpha \right)_\beta \tag{4.11}$$

wherever it is defined. Note that c is defined as in Equation 4.1 and $\bar{\lambda}$ as in Subsection 4.2. Hence, that gives us a relation onto the coefficients of $P(\theta, \omega)$, namely

$$\left(A - (\alpha \cdot \lambda)_0 \cdot I_n \right) \begin{pmatrix} (a_1)_{\alpha,\beta} \\ \vdots \\ (a_n)_{\alpha,\beta} \end{pmatrix} = \left(g(a, \bar{\lambda}, c) \right)_{\alpha,\beta} \quad (\forall |\alpha| \geq 2, |\beta| \geq 0) \quad , \tag{4.12}$$

where $g(a,\bar{\lambda},c)$ is defined component-wise by

$$\left(g(a,\bar{\lambda},c)\right)_{\alpha,\beta} = -\sum_{|\gamma_2|=1}^{M_2}\sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,\gamma_2}\left(a_1^{(\gamma_1)_1} * \cdots * a_n^{(\gamma_1)_n}\right)_{\alpha,\beta-\gamma_2} - \sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,0}\left(\widehat{a_1^{(\gamma_1)_1} * \cdots * a_n^{(\gamma_1)_n}}\right)_{\alpha,\beta}$$
$$+\left((\alpha\cdot\bar{\lambda})\hat{*}a_\alpha\right)_\beta + (1-\delta_{\beta,0})\cdot(\alpha\cdot\bar{\lambda})_\beta\cdot a_{\alpha,0} \quad ,$$

with

$$\delta_{i,j} \overset{\text{def}}{=} \begin{cases} 1 & \text{, if } i=j \\ 0 & \text{, if } i\neq j \end{cases} \qquad (i,j\in\mathbb{N}^p)$$

being the *Kronecker Delta*, and where

$$A_{1,1} = \sum_{|\gamma_1|=0}^{M_1}(c_1)_{\gamma_1,0}(\gamma_1)_1(a_1)_{0,0}^{(\gamma_1)_1-1}(a_2)_{0,0}^{(\gamma_1)_2}\dots(a_{n-1})_{0,0}^{(\gamma_1)_{n-1}}(a_n)_{0,0}^{(\gamma_1)_n}$$
$$A_{1,n} = \sum_{|\gamma_1|=0}^{M_1}(c_1)_{\gamma_1,0}(\gamma_1)_n(a_1)_{0,0}^{(\gamma_1)_1}(a_2)_{0,0}^{(\gamma_1)_2}\dots(a_{n-1})_{0,0}^{(\gamma_1)_{n-1}}(a_n)_{0,0}^{(\gamma_1)_n-1}$$
$$A_{n,1} = \sum_{|\gamma_1|=0}^{M_1}(c_n)_{\gamma_1,0}(\gamma_1)_1(a_1)_{0,0}^{(\gamma_1)_1-1}(a_2)_{0,0}^{(\gamma_1)_2}\dots(a_{n-1})_{0,0}^{(\gamma_1)_{n-1}}(a_n)_{0,0}^{(\gamma_1)_n}$$
$$A_{n,n} = \sum_{|\gamma_1|=0}^{M_1}(c_n)_{\gamma_1,0}(\gamma_1)_n(a_1)_{0,0}^{(\gamma_1)_1}(a_2)_{0,0}^{(\gamma_1)_2}\dots(a_{n-1})_{0,0}^{(\gamma_1)_{n-1}}(a_n)_{0,0}^{(\gamma_1)_n-1}$$
$$A = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{pmatrix} \quad .$$

Remark. We used the following definitions in Equations (4.11) and (4.12) :

$$\begin{aligned} a_\alpha &\overset{\text{def}}{=} (a_{\alpha,\beta})_{|\beta|\geq 0} \\ \alpha\cdot\bar{\lambda} &\overset{\text{def}}{=} \alpha_1\bar{\lambda}_1 + \cdots + \alpha_k\bar{\lambda}_k \\ (\alpha\cdot\bar{\lambda})_\beta &\overset{\text{def}}{=} \alpha_1(\bar{\lambda}_1)_\beta + \cdots + \alpha_k(\bar{\lambda}_k)_\beta \end{aligned} \quad .$$

Notice that $A = D_x f(P(0,\tilde{\omega}),\tilde{\omega})$. Hence, Equation (4.12) leads to a recurrence relation onto the coefficients of the power series of P if and only if $A - (\alpha\cdot\bar{\lambda})_0\cdot I_n$ is invertible for all $|\alpha|\geq 0$. Namely, Equation (4.12) leads to a recurrence relation onto the coefficients of the power series of P if and only if $(\alpha\cdot\bar{\lambda})_0$ is never an eigenvalue of $D_x f(P(0,\tilde{\omega}),\tilde{\omega})$ for all $|\alpha|\geq 0$. This is what we call a *resonance condition.*

Definition 18 (Resonance). Let $A\in M_n(\mathbb{R})$ be a square matrix of size $n\times n$ over the field of real numbers. Let $\alpha\in\mathbb{N}^d$ for $d>0$. Let $\Lambda=\{\lambda_1,\dots,\lambda_d\}$ be a subset of the eigenvalues of A. We say the matrix A has a *resonance* on Λ if there is an eigenvalue of A that is a nonnegative integer linear combination – an integer linear combination such that all the integers are nonnegative – of the eigenvalues of A belonging to Λ.

We mention here that the resonance condition of Equation 4.12 is formulated only with the negative eigenvalues of the derivative $D_x f(\tilde{x}(\tilde{\omega}), \tilde{\omega})$ of the vector field (1.1) at the fixed point $(\tilde{x}(\tilde{\omega}), \tilde{\omega})$. Like in dynamical systems, we see that the study of the eigenvalues of the linearized system at a fixed point is a primordial asset. Moreover, it is a phenomena that often arises when one is looking at conjugacy relations with parameterizations given by power series (see [2] and [1]).

As mentioned in Section 1, we were missing one assumption :

A4. $D_x f(\tilde{x}(\tilde{\omega}), \tilde{\omega})$ has no resonance on $\Lambda = \{\lambda_1, \ldots, \lambda_k\}$.

Thus, we can use Equation 4.12 to get the recurrence relation

$$a_{\alpha,\beta} = (A - (\alpha \cdot \bar{\lambda})_0 \cdot I_n)^{-1} \left(g(a, \bar{\lambda}, c) \right)_{\alpha,\beta} \quad (\forall |\alpha| \geq 2, |\beta| \geq 0) \quad . \tag{4.13}$$

Let us justify the invertibility of $A - (\alpha \cdot \bar{\lambda})_0 \cdot I_n$ for all $|\alpha| \geq 2$. Assumption **A4** states A has no resonance on the set of negative eigenvalues. Therefore, the matrix $A - (\alpha \cdot \bar{\lambda})_0 \cdot I_n$ is invertible for all $|\alpha| \geq 0$. In particular, it is invertible for all $|\alpha| \geq 2$.

The recurrence relation (4.13) has the initial conditions

$$a_{0,\beta} = \bar{b}_\beta, \quad a_{e_i,\beta} = \bar{V}^i_\beta \quad (\forall 1 \leq i \leq j) \quad ,$$

where $e_i \in \mathbb{N}^k$ is the vector with all components equal to 0 except for the i-th one that is equal to 1, $\bar{b}_\beta$ are defined as in Subsection 4.1 and $\bar{V}^i_\beta$ are the coefficients of the power series of the i-th eigenvector at the fixed point $\tilde{x}(\omega)$ – the eigenvector associated to the i-th negative eigenvalue –, computed as in Subsection 4.2. Those initial conditions are derived from the first two equations of (2.4). Therefore, we have a way to compute exactly the coefficients of the series of P. However, as opposed to Subsections 4.1 and 4.2, the invertible matrix does depend on the index of the coefficients. As we will see in Section 5, this causes an issue concerning the computation of the bounds (3.2). To make up for it, we need to modify slightly $g(a, c)$ to introduce an intermediate operator.

Definition 19. Let $a_1, \ldots, a_n \in \ell^1_{\nu,\mu}$. We define $a_1 \star \cdots \star a_n$ component-wise by

$$\begin{aligned}(a_1 \star \cdots \star a_n)_{\alpha,\beta} \overset{\text{def}}{=} \; & (a_1 * \cdots * a_n)_{\alpha,\beta} - ((a_1)_\alpha * (a_2)_0 * \cdots * (a_n)_0)_\beta \\ & - \cdots - ((a_1)_0 * \cdots * (a_{n-1})_0 * (a_n)_\alpha)_\beta \end{aligned} \quad .$$

Definition 19 allows us to rewrite $g(a,c)$ as $g_\alpha(a_\alpha, a_{\alpha^-}, c)$ defined component-wise by

$$\left(g_\alpha(a_\alpha, a_{\alpha^-}, \bar{\lambda}, c)\right)_\beta \overset{\text{def}}{=} \left(g^1_\alpha(a_\alpha, a_{\alpha^-}, c)\right)_\beta + \left(g^2_\alpha(a_\alpha, a_{\alpha^-}, c)\right)_\beta + \left(g^3_\alpha(a_\alpha, a_{\alpha^-}, c)\right)_\beta + \left(g^4_\alpha(a_\alpha, \bar{\lambda})\right)_\beta \quad ,$$

where

$$\begin{aligned}
\left(g^1_\alpha(a_\alpha, a_{\alpha^-}, c)\right)_\beta \overset{\text{def}}{=} & - \sum_{|\gamma_2|=1}^{M_2} \sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,\gamma_2} \Big((a_1^{(\gamma_1)_1} \star \cdots \star a_n^{(\gamma_1)_n})_{\alpha,\beta-\gamma_2} + \Big[(\gamma_1)_1 \left((a_1)_0^{(\gamma_1)_1 - 1} * (a_2)_0^{(\gamma_1)_2} * \cdots * (a_n)_0^{(\gamma_1)_n} * (a_1)_\alpha \right) \\
& + \cdots + (\gamma_1)_n \left((a_1)_0^{(\gamma_1)_1} * \cdots * (a_{n-1})_0^{(\gamma_1)_{n-1}} * (a_n)_0^{(\gamma_1)_n - 1} * (a_n)_\alpha \right) \Big]_{\beta-\gamma_2} \Big) \\
\left(g^2_\alpha(a_\alpha, a_{\alpha^-}, c)\right)_\beta \overset{\text{def}}{=} & - \sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,0} \Big((a_1^{(\gamma_1)_1} \star \cdots \star a_n^{(\gamma_1)_n})_{\alpha,\beta} + \Big[(\gamma_1)_1 \left(\overbrace{(a_1)_0^{(\gamma_1)_1 - 1} * (a_2)_0^{(\gamma_1)_2} * \cdots * (a_n)_0^{(\gamma_1)_n} * (a_1)_\alpha} \right) \\
& + \cdots + (\gamma_1)_n \left(\overbrace{(a_1)_0^{(\gamma_1)_1} * \cdots * (a_{n-1})_0^{(\gamma_1)_{n-1}} * (a_n)_0^{(\gamma_1)_n - 1} * (a_n)_\alpha} \right) \Big]_\beta \Big) \\
\left(g^3_\alpha(a_\alpha, a_{\alpha^-}, c)\right)_\beta \overset{\text{def}}{=} & \sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,0} \sum_{1 \le i \le n} (\gamma_1)_i \Big((\gamma_1)_1 (a_1)_{0,0}^{(\gamma_1)_1 - 1} (a_2)_{0,0}^{(\gamma_1)_2} \cdots (a_{i-1})_{0,0}^{(\gamma_1)_{i-1}} (a_i)_{0,0}^{(\gamma_1)_i - 1} (a_{i+1})_{0,0}^{(\gamma_1)_{i+1}} \cdots (a_n)_{0,0}^{(\gamma_1)_n} (a_i)_{\alpha,0} (a_1)_{0,\beta} \\
& + \cdots + ((\gamma_1)_i - 1) (a_1)_{0,0}^{(\gamma_1)_1} \cdots (a_{i-1})_{0,0}^{(\gamma_1)_{i-1}} (a_i)_{0,0}^{(\gamma_1)_i - 2} (a_{i+1})_{0,0}^{(\gamma_1)_{i+1}} \cdots (a_n)_{0,0}^{(\gamma_1)_n} (a_i)_{\alpha,0} (a_i)_{0,\beta} \\
& + \cdots + (\gamma_1)_n (a_1)_{0,0}^{(\gamma_1)_1} \cdots (a_{i-1})_{0,0}^{(\gamma_1)_{i-1}} (a_i)_{0,0}^{(\gamma_1)_i - 1} (a_{i+1})_{0,0}^{(\gamma_1)_{i+1}} \cdots (a_{n-1})_{0,0}^{(\gamma_1)_{n-1}} (a_n)_{0,0}^{(\gamma_1)_n - 1} (a_i)_{\alpha,0} (a_n)_{0,\beta} \Big) \\
\left(g^4_\alpha(a_\alpha, \bar{\lambda})\right)_\beta \overset{\text{def}}{=} & \left((\alpha \cdot \bar{\lambda}) \hat{*} a_\alpha \right)_\beta + (\alpha \cdot \bar{\lambda})_\beta \cdot a_{\alpha,0}
\end{aligned}$$

and $a_{\alpha^-} \overset{\text{def}}{=} (a_{\alpha^*})_{\alpha^* < \alpha}$. Notice $\left(g_\alpha(a_\alpha, a_{\alpha^-}, \bar{\lambda}, c)\right)_\beta = \left(g(a, \bar{\lambda}, c)\right)_{\alpha,\beta}$ $(\forall |\alpha| \ge 2, |\beta| \ge 1)$. Basically, $g_\alpha(a_\alpha, a_{\alpha^-}, \bar{\lambda}, c)$ is a rewriting of $g(a, \bar{\lambda}, c)$ for a fixed α and all $|\beta| \ge 1$. This allows us to simplify the computation of the bounds (3.2) by splitting the computation of the coefficients of the series of P in two parts : Firstly, we will compute them for fixed α and, secondly, we will compute the rest of them. The reason for this split will be covered in Section 5.

Firstly, fix α and suppose we have computed the coefficients a_α up to order $|\beta| < N_1$ for some positive integer N_1 that we choose. Recall Definition 13. We are going to set $T_1 : \left(\ell^{1,N_1}_\mu\right)^n \to \left(\ell^{1,N_1}_\mu\right)^n$ as

$$(T_1(u))_\beta \overset{\text{def}}{=} \begin{cases} 0 & , \text{ if } 0 \le |\beta| \le N_1 - 1 \\ \left(A - (\alpha \cdot \bar{\lambda})_0 \cdot I_n\right)^{-1} \left(g_\alpha(\bar{a}_\alpha + u, \bar{a}_{\alpha^-}, \bar{\lambda}, c)\right)_\beta & , \text{ if } |\beta| \ge N_1 \end{cases} \quad , \tag{4.14}$$

where

$$(\bar{a}_\alpha)_\beta = \bar{a}_{\alpha,\beta} \overset{\text{def}}{=} \begin{cases} a_{\alpha,\beta} & , \text{ if } 0 \le |\beta| \le N_1 - 1 \\ 0 & , \text{ if } |\beta| \ge N_1 \end{cases}$$

and

$$\bar{a}_{\alpha^-} \overset{\text{def}}{=} (\bar{a}_{\alpha^*})_{\alpha^*<\alpha} \quad \& \quad (\bar{a}_{\alpha^*})_\beta = \bar{a}_{\alpha^*,\beta} \overset{\text{def}}{=} \begin{cases} a_{\alpha^*,\beta} & \text{, if } 0 \le |\beta| \le N_1 - 1 \\ 0 & \text{, if } |\beta| \ge N_1 \end{cases} \quad (\alpha^* < \alpha) \quad .$$

Remark. $\bar{a}_\alpha + u$ means that $u \in \left(\ell^{1,N_1}_\mu\right)^n$ is added only to $\bar{a}_\alpha$ and not to $\bar{a}_{\alpha^*}$ for $\alpha^* < \alpha$.

The goal will be to prove that T_1 has a unique fixed point and that $u = 0$ is a good approximation of it. We know that the series has two radius of convergence, say, $R_1, R_2 > 0$, the first one being for the convergence in θ and the second one for the convergence in ω. Moreover, T_1 is by definition the recurrence relation (4.13) for α fixed so, using the same argument as for the fixed point and the eigenvalues and eigenvectors, we can get the bounds (3.2) as small as we want by taking $|\mu|$ small enough and N_1 big enough, whence Theorem 3.4.2 will apply and we will have existence and uniqueness of the series for the fixed α. Notice we have to repeat the computations for every α. Since we can only do a finite number of computations, we will have to set a new operator to prove the existence and uniqueness of the remaining coefficients of P.

Remark. Even if α is fixed, the recurrence relation 4.13 still depends on every coefficients a_{α^*} such that $\alpha^* < \alpha$. Moreover, at some point, each component of $\bar{a}_\alpha$ is 0 for every α that we fix ; we take it to be past order $N_1 - 1$ without loss of generality.

Secondly, suppose that we have computed a rigorous approximation of the coefficients $a_{\alpha,\beta}$ for every $|\alpha| < N_2$ where N_2 is a positive integer that we choose, i.e. we successfully applied Theorem 3.4.2 to every a_α such that $|\alpha| < N_2$. We are now going to set $T_2 : \left(\ell^{1,N_2}_{\nu,\mu}\right)^n \to \left(\ell^{1,N_2}_{\nu,\mu}\right)^n$ as

$$(T_2(u))_{\alpha,\beta} \overset{\text{def}}{=} \begin{cases} 0 & \text{, if } 0 \le |\alpha| \le N_2 - 1 \\ \left(A - (\alpha\cdot\bar{\lambda})_0 \cdot I_n\right)^{-1} \left(g(\bar{a} + u, \bar{\lambda}, c)\right)_{\alpha,\beta} & \text{, if } |\alpha| \ge N_2 \end{cases} , \tag{4.15}$$

where

$$\bar{a}_{\alpha,\beta} = \begin{cases} a_{\alpha,\beta} & \text{, if } 0 \le |\alpha| \le N_2 - 1 \;\&\; 0 \le |\beta| \le N_1 - 1 \\ 0 & \text{, otherwise} \end{cases} .$$

We again want to prove that $u = 0$ is a good approximation of the fixed point of T_2. Since T_2 is the recurrence relation 4.13, by the same argument as for the fixed point and the eigenvalues and eigenvectors, we will have control over the bounds (3.2) by taking N_2 big enough and both $|\nu|$ and $|\mu|$ small enough. Hence, Theorem 3.4.2 will apply and we will get existence and uniqueness of the series of P.

Remark.

— In practice, we want to take $|\nu|$ as big as possible because we will get a bigger radius of convergence for the power series of P.

— Recall Subsection 2.3. For the unstable manifold, the same argument applies with $\Lambda_u(\omega)$ instead of $\Lambda_s(\omega)$ and it will yield the same kind of operator. Furthermore, the computations of the bounds (3.2) is going to be almost the same as for the stable manifold, so there is no need to go over it in details.

Let us come back on the complex case. Suppose the eigenvalues of Subsection 4.2 are complex – they can be a mix of real and complex ones. The argument of Subsections 4.2 and 4.3 still applies by taking the absolute values $|\cdot|$ as the complex absolute values. Nonetheless, we will end up with a complex parameterization P of the local stable manifold. Fortunately, there is a way to retrieve a real parameterization of P out of the complex one. Sort the eigenvalues such that $\lambda_1, \ldots, \lambda_j$ are the complex ones and $\lambda_{j+1}, \ldots, \lambda_k$ the real ones. Since $D_x f(\tilde{x}(\tilde{\omega}), \tilde{\omega})$ is a real matrix, we know the complex eigenvalues are complex conjugate, i.e. they come in pair. Sort again the complex eigenvalues such that $\lambda_1 = \overline{\lambda_2}, \ldots, \lambda_{j-1} = \overline{\lambda_j}$ – here, $\overline{z}$ denotes the conjugate of $z \in \mathbb{C}$. Define $Q : \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}^n$ as

$$Q(\varphi, \omega) \stackrel{\text{def}}{=} P\left((\varphi_1 + i\cdot \varphi_2), \ldots, (\varphi_{j-1} + i\cdot \varphi_j), \varphi_{j+1}, \ldots, \varphi_k, \omega\right) \quad .$$

Now, notice

$$a_{\alpha_1,\alpha_2,\ldots,\alpha_{j-1},\alpha_j,\alpha_{j+1},\ldots,\alpha_k,\beta} = \overline{a_{\alpha_2,\alpha_1,\ldots,\alpha_j,\alpha_{j-1},\alpha_{j+1},\ldots,\alpha_k,\beta}} \quad (\forall |\alpha| \geq 2, |\beta| \geq 0) \quad .$$

Indeed, recall $\bar{V}^i$ is the eigenvector associated to the i-th eigenvalue and e_i is the i-th canonical unit vector of $\mathbb{R}^k$. Notice $\bar{V}^1 = \overline{\bar{V}^2}, \ldots, \bar{V}^{j-1} = \overline{\bar{V}^j}$, $a_{e_i} = \bar{V}^i$ ($\forall 1 \leq i \leq k$) and $a_{\alpha,\beta}$ is given by a recurrence relation depending only on coefficients of lesser order. One can verify that those facts together imply the above equation. Furthermore, the above equation implies that

$$Q(\varphi, \omega) = \overline{Q(\varphi, \omega)} \quad ,$$

i.e. Q is a real parameterization of the local stable manifold. Thus, our method also works for the complex case, it just requires the extra step of defining Q as above.

The next section will cover how one can estimate the bounds (3.2) for the operators of Section 4. Then, two examples will follow in Section 6 to apply our theory in practice.

5 Bounds

As mentioned in Subsection 3.4, when using power series as parameterizations of the objects we are computing – fixed point, eigenvalues and eigenvectors and parameterization of the local stable manifold –, in order to apply Theorem 3.4.2, we need to make sure Equation 3.3 holds. In practice, this means we need sharp estimates for some of the bounds (3.2) and coarse estimates for the others. As mentioned briefly in Subsection 3.2, the bound Y_0 is already going to be small by definition of T, the operator we are working with. Indeed, regardless of which one of the operators from Section 4 we work with, it has its components equal to 0 up to a certain order. Therefore, since our norms from Definition 14 all include a weight, taking each component of the latter less than 1 and small enough, we have the norm of $T(0) - 0$ as small as we want. This is precisely a justification to the bound Y_0 being as small as desired.

Y_0 being a positive number as close to 0 as needed, the polynomials from Equation 3.3 will have a positive root close to 0 if $Z_1 - 1$ is negative, i.e. $Z_1 < 1$. In practice, since the bound Z_1 is the bound for the linear terms of our operator T, we need to be sharper on it so it does not exceed 1. However, regarding the bound $Z_2(r)$, in general, one can just take a coarse bound of it without any consequences.

For the sake of simplicity, let us make a definition that limits the notation used regarding matrices.

Definition 20 (Matrix operator)**.** Let X be any of the spaces from Definition 13. Let $h \in X$ and A be a matrix of size $m \times m$ over the field of real numbers. The *matrix operator* $A : X \to X$ is a linear functional whose action on $h \in X$ is given by

$$\left(Ah\right)_\delta = A \cdot h_\delta \quad .$$

If $\delta = (\alpha, \beta)$ and A depends α, i.e. $A = A(\alpha)$, then the *matrix operator* $A(\alpha) : X \to X$ is a linear functional whose action on $h \in X$ is given by

$$\left(A(\alpha)h\right)_{\alpha,\beta} \stackrel{\text{def}}{=} A(\alpha) \cdot h_{\alpha,\beta} \quad .$$

Remark. We are going to use the same symbol for a matrix and its matrix operator. It is always going to be clear whether it is used for the matrix or the matrix operator.

For a norm $\|\cdot\|_X$ on a Banach space X, we are going to denote the norm induced by $\|\cdot\|_X$ of a bounded linear operator from X to X by $\|\cdot\|_{B(X)}$. Since it is going to be useful for the computation of the bounds (3.2), let us make a theorem.

Theorem 5.0.1. *Recall Definition 20. Let X be any of the spaces from Definition 13 and $\|\cdot\|_X$ be its norm from Definition 14. Let A be a matrix operator over X. Then, A is a bounded linear operator and its norm induced by $\|\cdot\|_X$ is bounded by*

$$\|A\|_{B(X)} \leq \|A\|_\infty \quad ,$$

where $\|\cdot\|_\infty$ is the well-know ∞-norm of a linear operator, i.e. the maximum absolute row sum norm (see [19]) for a finite dimensional matrix.

Proof. This is a straightforward computation.

□

Remark.

— Theorem 5.0.1 also holds for $|A|$ defined component-wise as $|A|_{i,j} \stackrel{\text{def}}{=} |A_{i,j}|$ $(1 \leq i, j \leq n)$. Then, notice that $\|\,|A|\,\|_\infty = \|A\|_\infty$.

— Theorem 5.0.1 does apply to the matrix operator $A(\alpha)$, but the result is then $\|A(\alpha)\|_{B(X)} \leq \max_{\alpha} \|A(\alpha)\|_\infty$.

Moreover, for the sake of simplicity, let us make two definitions.

Definition 21. Let $d > 0$. For $\gamma \in \mathbb{N}^d$, we define $\gamma^{*_i} \in \mathbb{N}^d$ component-wise by

$$(\gamma^{*_i})_k \stackrel{\text{def}}{=} \begin{cases} \gamma_k & , \text{if } k \neq i \\ \gamma_k - 1 & , \text{if } k = i \end{cases} \quad ,$$

where $i \in \{1, \ldots, d\}$. Since $\gamma^{*_i} \in \mathbb{N}^d$, we set $(\gamma^{*_i})_i = 0$ if $\gamma_i = 0$.

Definition 22. Let $a, b \in \mathbb{N}$ with $a \geq b$. Then, we define the binomial coefficient $\binom{a}{b}$ by

$$\binom{a}{b} \stackrel{\text{def}}{=} \frac{a!}{b!(a-b)!} \quad .$$

We are now ready to discuss the computation of the bounds (3.2) for each subsection of Section 4. Henceforth, unless stated otherwise, the absolute values $|\cdot|$ of a number $z \in \mathbb{R}^d$ $(d > 0)$ are defined as $|z| \stackrel{\text{def}}{=} (|z_1|, \ldots, |z_d|)$, where $|z_i|$ are the usual absolute values of z_i $(1 \leq i \leq d)$. We resume after the work done in Section 4 and start off with the fixed point (Subsection 4.1).

5.1 Fixed Point

Recall the operator $T : \left(\ell_\mu^{1,N}\right)^n \to \left(\ell_\mu^{1,N}\right)^n$ given by (4.5). Its derivative $DT(u) : \left(\ell_\mu^{1,N}\right)^n \to \left(\ell_\mu^{1,N}\right)^n$ evaluated at an element $u \in \left(\ell_\mu^{1,N}\right)^n$ is a linear operator. One can verify its action on an element $h \in \left(\ell_\mu^{1,N}\right)^n$ is given by

$$(DT(u)h)_\beta = B^{-1}\left(D_u g(\bar{b}+u,c)h\right)_\beta \quad , \tag{5.1}$$

where

$$\begin{aligned}\left(D_u g(\bar{b}+u,c)h\right)_\beta = &-\sum_{|\gamma_2|=1}^{M_2}\sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,\gamma_2}\cdot\Big[(\gamma_1)_1\cdot\left((\bar{b}_1+u_1)^{(\gamma_1)_1-1} * (\bar{b}_2+u_2)^{(\gamma_1)_2} * \cdots * (\bar{b}_n+u_n)^{(\gamma_1)_n} * h_1\right)\\ &+\cdots+(\gamma_1)_n\cdot\left((\bar{b}_1+u_1)^{(\gamma_1)_1} * \cdots * (\bar{b}_{n-1}+u_{n-1})^{(\gamma_1)_{n-1}} * (\bar{b}_n+u_n)^{(\gamma_1)_n-1} * h_n\right)\Big]_{\beta-\gamma_2}\\ &-\sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,0}\cdot\Big[(\gamma_1)_1\cdot\left(\overline{(\bar{b}_1+u_1)^{(\gamma_1)_1-1} * (\bar{b}_2+u_2)^{(\gamma_1)_2} * \cdots * (\bar{b}_n+u_n)^{(\gamma_1)_n} * h_1}\right)\\ &+\cdots+(\gamma_1)_n\cdot\left(\overline{(\bar{b}_1+u_1)^{(\gamma_1)_1} * \cdots * (\bar{b}_{n-1}+u_{n-1})^{(\gamma_1)_{n-1}} * (\bar{b}_n+u_n)^{(\gamma_1)_n-1} * h_n}\right)\Big]_\beta \quad .\end{aligned}$$

Recall the bounds (3.2). Recall Definition 14. Y_0 is given by

$$Y_0 \stackrel{\text{def}}{=} \|T(0)-0\|_{1,\mu,N}^{(n)} = \|T(0)\|_{1,\mu,N}^{(n)} \quad . \tag{5.2}$$

Notice the computation of Y_0 involves a finite number of terms because $\bar{b}$ is a sequence with finitely many nonzero elements and T is evaluated at $u=0$.

Let us move on to the computation of $Z(r)$. Notice

$$\|DT(u)h\|_{1,\mu,N}^{(n)} \le \|B^{-1}\|_{B\left(\left(\ell_\mu^{1,N}\right)^n\right)}\cdot\|D_u g(\bar{b}+u,c)h\|_{1,\mu,N}^{(n)} \quad .$$

By Theorem 5.0.1, we have

$$\|DT(u)h\|_{1,\mu,N}^{(n)} \le \|B^{-1}\|_\infty\cdot\|D_u g(\bar{b}+u,c)h\|_{1,\mu,N}^{(n)} \quad .$$

Therefore, what remains to bound is $\|D_u g(\bar{b}+u,c)h\|_{1,\mu,N}^{(n)}$.

Lemma 5.1.1. *Recall* (5.1) *and Definitions 16, 21 and 22. Let* $r>0$. *Suppose* $\|u\|_{1,\mu,N}^{(n)} \le r$ *and* $\|h\|_{1,\mu,N}^{(n)} = 1$. *We have*

$$\|D_u g(\bar{b}+u,c)h\|_{1,\mu,N}^{(n)} \le Z^g(r) \quad ,$$

where

$$Z^g(r) \stackrel{\rm def}{=} \max\Bigg\{ \sum_{|\gamma_2|=1}^{M_2} \mu^{\gamma_2} \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,\gamma_2}| \Bigg(|(\gamma_1)_1| \sum_{\eta\le\gamma_1^{*1}} \binom{(\gamma_1^{*1})_1}{\eta_1} \cdots \binom{(\gamma_1^{*1})_n}{\eta_n} \|\bar b_1\|_{1,\mu}^{(\gamma_1^{*1})_1-\eta_1} \cdots \|\bar b_n\|_{1,\mu}^{(\gamma_1^{*1})_n-\eta_n} r^{|\eta|}$$
$$+\cdots+|(\gamma_1)_n| \sum_{\eta\le\gamma_1^{*n}} \binom{(\gamma_1^{*n})_1}{\eta_1} \cdots \binom{(\gamma_1^{*n})_n}{\eta_n} \|\bar b_1\|_{1,\mu}^{(\gamma_1^{*n})_1-\eta_1} \cdots \|\bar b_n\|_{1,\mu}^{(\gamma_1^{*n})_n-\eta_n} r^{|\eta|} \Bigg)$$
$$+ \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,0}| \Bigg(|(\gamma_1)_1| \sum_{\eta\le\gamma_1^{*1}} \binom{(\gamma_1^{*1})_1}{\eta_1} \cdots \binom{(\gamma_1^{*1})_n}{\eta_n} \overline{\|\bar b_1\|_{1,\mu}^{(\gamma_1^{*1})_1-\eta_1} \cdots \|\bar b_n\|_{1,\mu}^{(\gamma_1^{*1})_n-\eta_n} r^{|\eta|}}$$
$$+\cdots+|(\gamma_1)_n| \sum_{\eta\le\gamma_1^{*n}} \binom{(\gamma_1^{*n})_1}{\eta_1} \cdots \binom{(\gamma_1^{*n})_n}{\eta_n} \overline{\|\bar b_1\|_{1,\mu}^{(\gamma_1^{*n})_1-\eta_1} \cdots \|\bar b_n\|_{1,\mu}^{(\gamma_1^{*n})_n-\eta_n} r^{|\eta|}} \Bigg) \Bigg\}. \tag{5.3}$$

Proof. Let $\big(D_u g(\bar b+u,c)h\big)_\beta = (D_1(u)h)_\beta + (D_2(u)h)_\beta$, where $D_1(u)h$ and $D_2(u)h$ are defined component-wise by

$$(D_1(u)h)_\beta \stackrel{\rm def}{=} -\sum_{|\gamma_2|=1}^{M_2}\sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,\gamma_2}\cdot\Big[(\gamma_1)_1\cdot\Big((\bar b_1+u_1)^{(\gamma_1)_1-1} * (\bar b_2+u_2)^{(\gamma_1)_2} * \cdots * (\bar b_n+u_n)^{(\gamma_1)_n} * h_1\Big)$$
$$+\cdots+(\gamma_1)_n\cdot\big((\bar b_1+u_1)^{(\gamma_1)_1} * \cdots * (\bar b_{n-1}+u_{n-1})^{(\gamma_1)_{n-1}} * (\bar b_n+u_n)^{(\gamma_1)_n-1} * h_n\big)\Big]_{\beta-\gamma_2}$$

and

$$(D_2(u)h)_\beta \stackrel{\rm def}{=} -\sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,0}\cdot\Big[(\gamma_1)_1\cdot\Big(\overline{(\bar b_1+u_1)^{(\gamma_1)_1-1} * (\bar b_2+u_2)^{(\gamma_1)_2} * \cdots * (\bar b_n+u_n)^{(\gamma_1)_n} * h_1}\Big)$$
$$+\cdots+(\gamma_1)_n\cdot\Big(\overline{(\bar b_1+u_1)^{(\gamma_1)_1} * \cdots * (\bar b_{n-1}+u_{n-1})^{(\gamma_1)_{n-1}} * (\bar b_n+u_n)^{(\gamma_1)_n-1} * h_n}\Big)\Big]_\beta .$$

Let $\Upsilon_1(u,h) = \sum_{|\beta|\ge0} \Big|(D_1(u)h)_\beta\Big| \mu^\beta$. We have

$$\Upsilon_1(u,h) = \sum_{|\beta|\ge0}\Bigg| \sum_{|\gamma_2|=1}^{M_2}\sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,\gamma_2}\cdot\Big[(\gamma_1)_1\cdot\Big((\bar b_1+u_1)^{(\gamma_1)_1-1} * (\bar b_2+u_2)^{(\gamma_1)_2} * \cdots * (\bar b_n+u_n)^{(\gamma_1)_n} * h_1\Big)$$
$$+\cdots+(\gamma_1)_n\cdot\big((\bar b_1+u_1)^{(\gamma_1)_1} * \cdots * (\bar b_{n-1}+u_{n-1})^{(\gamma_1)_{n-1}} * (\bar b_n+u_n)^{(\gamma_1)_n-1} * h_n\big)\Big]_{\beta-\gamma_2}\Bigg|\mu^\beta$$
$$\le \sum_{|\gamma_2|=1}^{M_2} \mu^{\gamma_2} \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,\gamma_2}| \sum_{|\beta|\ge0} \Big[|(\gamma_1)_1|\cdot\Big|(\bar b_1+u_1)^{(\gamma_1)_1-1} * (\bar b_2+u_2)^{(\gamma_1)_2} * \cdots * (\bar b_n+u_n)^{(\gamma_1)_n} * h_1\Big|$$
$$+\cdots+|(\gamma_1)_n|\cdot\big|(\bar b_1+u_1)^{(\gamma_1)_1} * \cdots * (\bar b_{n-1}+u_{n-1})^{(\gamma_1)_{n-1}} * (\bar b_n+u_n)^{(\gamma_1)_n-1} * h_n\big|\Big]_{\beta-\gamma_2} \mu^{\beta-\gamma_2} .$$

The change of order of summation was possible because the triple series converges absolutely. Let $i \in \{1,\ldots,n\}$. Notice

$$(\bar{b}_1+u_1)^{(\gamma_1)_1-1} * \cdots * (\bar{b}_{i-1}+u_{i-1})^{(\gamma_1)_{i-1}} * (\bar{b}_i+u_i)^{(\gamma_1)_i-1} * (\bar{b}_{i+1}+u_{i+1})^{(\gamma_1)_{i+1}} * \cdots * (\bar{b}_n+u_n)^{(\gamma_1)_n} * h_i$$
$$= \sum_{\eta \le {\gamma_1^*}^i} \binom{({\gamma_1^*}^i)_1}{\eta_1} \cdots \binom{({\gamma_1^*}^i)_n}{\eta_n} \cdot \bar{b}_1^{({\gamma_1^*}^i)_1-\eta_1} * u_1^{\eta_1} * \cdots * \bar{b}_n^{({\gamma_1^*}^i)_n-\eta_n} * u_n^{\eta_n} * h_i \quad .$$

Hence, using Proposition 3.3.1 along with $\|u\|_{1,\mu,N}^{(n)} \le r$ and $\|h\|_{1,\mu,N}^{(n)} = 1$, we get

$$\begin{aligned}\Upsilon_1(u,h) \ \le\ & \sum_{|\gamma_2|=1}^{M_2} \mu^{\gamma_2} \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,\gamma_2}| \Bigg(|(\gamma_1)_1| \sum_{\eta \le {\gamma_1^*}^1} \binom{({\gamma_1^*}^1)_1}{\eta_1} \cdots \binom{({\gamma_1^*}^1)_n}{\eta_n} \|\bar{b}_1\|_{1,\mu}^{({\gamma_1^*}^1)_1-\eta_1} \cdots \|\bar{b}_n\|_{1,\mu}^{({\gamma_1^*}^1)_n-\eta_n} r^{|\eta|} \\ & + \cdots + |(\gamma_1)_n| \sum_{\eta \le {\gamma_1^*}^n} \binom{({\gamma_1^*}^n)_1}{\eta_1} \cdots \binom{({\gamma_1^*}^n)_n}{\eta_n} \|\bar{b}_1\|_{1,\mu}^{({\gamma_1^*}^n)_1-\eta_1} \cdots \|\bar{b}_n\|_{1,\mu}^{({\gamma_1^*}^n)_n-\eta_n} r^{|\eta|} \Bigg) \quad .\end{aligned}$$

Moreover, let $\Upsilon_2(u,h) = \sum_{|\beta|\ge 0} \left| (D_2(u)h)_\beta \right| \mu^\beta$. We have

$$\begin{aligned}\Upsilon_2(u,h) \ =\ & \sum_{|\beta|\ge 0} \Bigg| \sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,0} \cdot \Big[(\gamma_1)_1 \cdot \Big(\overline{(\bar{b}_1+u_1)^{(\gamma_1)_1-1} * (\bar{b}_2+u_2)^{(\gamma_1)_2} * \cdots * (\bar{b}_n+u_n)^{(\gamma_1)_n} * h_1} \Big) \\ & + \cdots + (\gamma_1)_n \cdot \Big(\overline{(\bar{b}_1+u_1)^{(\gamma_1)_1} * \cdots * (\bar{b}_{n-1}+u_{n-1})^{(\gamma_1)_{n-1}} * (\bar{b}_n+u_n)^{(\gamma_1)_n-1} * h_n} \Big) \Big]_\beta \Bigg| \mu^\beta \\ \le\ & \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,0}| \sum_{|\beta|\ge 0} \Big[|(\gamma_1)_1| \cdot \Big| \overline{(\bar{b}_1+u_1)^{(\gamma_1)_1-1} * (\bar{b}_2+u_2)^{(\gamma_1)_2} * \cdots * (\bar{b}_n+u_n)^{(\gamma_1)_n} * h_1} \Big| \\ & + \cdots + |(\gamma_1)_n| \cdot \Big| \overline{(\bar{b}_1+u_1)^{(\gamma_1)_1} * \cdots * (\bar{b}_{n-1}+u_{n-1})^{(\gamma_1)_{n-1}} * (\bar{b}_n+u_n)^{(\gamma_1)_n-1} * h_n} \Big| \Big]_\beta \mu^\beta \quad .\end{aligned}$$

The change of order of summation was possible because the double series converges absolutely. Let $i \in \{1, \ldots, n\}$. Notice

$$\overline{(\bar{b}_1+u_1)^{(\gamma_1)_1-1} * \cdots * (\bar{b}_{i-1}+u_{i-1})^{(\gamma_1)_{i-1}} * (\bar{b}_i+u_i)^{(\gamma_1)_i-1} * (\bar{b}_{i+1}+u_{i+1})^{(\gamma_1)_{i+1}} * \cdots * (\bar{b}_n+u_n)^{(\gamma_1)_n} * h_i}$$
$$= \sum_{\eta \le {\gamma_1^*}^i} \binom{({\gamma_1^*}^i)_1}{\eta_1} \cdots \binom{({\gamma_1^*}^i)_n}{\eta_n} \cdot \overline{\bar{b}_1^{({\gamma_1^*}^i)_1-\eta_1} * u_1^{\eta_1} * \cdots * \bar{b}_n^{({\gamma_1^*}^i)_n-\eta_n} * u_n^{\eta_n} * h_i} \quad .$$

Hence, using Theorem 3.3.2 along with $\|u\|_{1,\mu,N}^{(n)} \le r$ and $\|h\|_{1,\mu,N}^{(n)} = 1$, we get

$$\begin{aligned}\Upsilon_2(u,h) \ \le\ & \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,0}| \Bigg(|(\gamma_1)_1| \sum_{\eta \le {\gamma_1^*}^1} \binom{({\gamma_1^*}^1)_1}{\eta_1} \cdots \binom{({\gamma_1^*}^1)_n}{\eta_n} \overline{\|\bar{b}_1\|_{1,\mu}^{({\gamma_1^*}^1)_1-\eta_1} \cdots \|\bar{b}_n\|_{1,\mu}^{({\gamma_1^*}^1)_n-\eta_n}} r^{|\eta|} \\ & + \cdots + |(\gamma_1)_n| \sum_{\eta \le {\gamma_1^*}^n} \binom{({\gamma_1^*}^n)_1}{\eta_1} \cdots \binom{({\gamma_1^*}^n)_n}{\eta_n} \overline{\|\bar{b}_1\|_{1,\mu}^{({\gamma_1^*}^n)_1-\eta_1} \cdots \|\bar{b}_n\|_{1,\mu}^{({\gamma_1^*}^n)_n-\eta_n}} r^{|\eta|} \Bigg) \quad .\end{aligned}$$

Finally, the result follows from

$$\|D_u g(\bar{b}+u,c)h\|^{(n)}_{1,\mu,N} \leq \max\Big\{\Upsilon_1(u,h)+\Upsilon_2(u,h)\Big\} \leq Z^g(r) \quad .$$

□

We can now apply the Radii Polynomials Approach (see Subsection 3.4) to see what are the conditions needed for the operator $T : \left(\ell^{1,N}_\mu\right)^n \to \left(\ell^{1,N}_\mu\right)^n$ given by (4.5) to have a fixed point close to $u = 0$.

Theorem 5.1.1. *Recall* $T : \left(\ell^{1,N}_\mu\right)^n \to \left(\ell^{1,N}_\mu\right)^n$*, the operator given by* (4.5)*. Let* $K \stackrel{\text{def}}{=} \|B^{-1}\|_\infty$*. Recall the bounds* Y_0 *and* $Z^g(r)$ *defined in* (5.2) *and* (5.3) *respectively. Let*

$$\begin{aligned} Z_1 \stackrel{\text{def}}{=} K\cdot\max\Bigg\{ &\sum_{|\gamma_2|=1}^{M_2} \mu^{\gamma_2} \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,\gamma_2}| \left(|(\gamma_1)_1|\cdot \|\bar{b}_1\|_{1,\mu}^{(\gamma_1^{*1})_1} \cdots \|\bar{b}_n\|_{1,\mu}^{(\gamma_1^{*1})_n} + \cdots + |(\gamma_1)_n|\cdot \|\bar{b}_1\|_{1,\mu}^{(\gamma_1^{*n})_1} \cdots \|\bar{b}_n\|_{1,\mu}^{(\gamma_1^{*n})_n} \right) \\ &+ \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,0}| \left(|(\gamma_1)_1|\cdot \widehat{\|\bar{b}_1\|_{1,\mu}^{(\gamma_1^{*1})_1} \cdots \|\bar{b}_n\|_{1,\mu}^{(\gamma_1^{*1})_n}} + \cdots + |(\gamma_1)_n|\cdot \widehat{\|\bar{b}_1\|_{1,\mu}^{(\gamma_1^{*n})_1} \cdots \|\bar{b}_n\|_{1,\mu}^{(\gamma_1^{*n})_n}} \right) \Bigg\} \end{aligned} \tag{5.4}$$

and

$$Z_2(r) \stackrel{\text{def}}{=} K\cdot Z^g(r) - Z_1 \quad . \tag{5.5}$$

If there exists $r_0 > 0$ *such that*

$$Z_2(r_0)r_0^2 + (Z_1 - 1)r_0 + Y_0 < 0 \quad , \tag{5.6}$$

then T *has a unique fixed point in* $B(0, r_0)$*.*

Proof. This is an application of Corollary 3.4.1 using Lemma 5.1.1 and the bounds Y_0, Z_1 and $Z_2(r)$ defined in (5.2), (5.4) and (5.5) respectively.

□

Remark. The reason for the use of the bounds Z_1 (Equation 5.4) and $Z_2(r)$ (Equation 5.5) instead of the bound $Z(r) \stackrel{\text{def}}{=} K\cdot Z^g(r)$ (Equation 5.3) is going to be explained in Subsection 5.4.

Assuming the existence of an $r_0 > 0$ satisfying Theorem 5.1.1 – this matter is going to be addressed in Subsection 5.4 –, we can now move on to the computation of the bounds (3.2) for the eigenvalues and eigenvectors (Subsection 4.2).

5.2 Eigenvalues and eigenvectors

Recall the operator $T : \left(\ell_\mu^{1,N}\right)^n \to \left(\ell_\mu^{1,N}\right)^n$ given by (4.9). Its derivative $DT(u) : \left(\ell_\mu^{1,N}\right)^n \to \left(\ell_\mu^{1,N}\right)^n$ evaluated at an element $u \in \left(\ell_\mu^{1,N}\right)^n$ is a linear operator. Its action on an element $h \in \left(\ell_\mu^{1,N}\right)^n$ is given by

$$(DT(u)h)_\beta = B^{-1}\left(D_u g(\bar{V}_\lambda^{*j} + u, \bar{V}_j, \bar{b}, c)h\right)_\beta \quad , \tag{5.7}$$

where

$$\begin{aligned}\left(D_u g(\bar{V}_\lambda^{*j}+u, \bar{V}_j, \bar{b}, c)h\right)_\beta \overset{\text{def}}{=} & -\sum_{|\gamma_2|=1}^{M_2}\sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,\gamma_2}\sum_{\substack{1\le i\le n\\ i\neq j}}(\gamma_1)_i\left(\bar{b}_1^{(\gamma_1)_1} * \cdots * \bar{b}_{i-1}^{(\gamma_1)_{i-1}} * \bar{b}_i^{(\gamma_1)_i-1} * \bar{b}_{i+1}^{(\gamma_1)_{i+1}} * \cdots * \bar{b}_n^{(\gamma_1)_n} * h_i\right)_{\beta-\gamma_2} \\ & -\sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,0}\sum_{\substack{1\le i\le n\\ i\neq j}}(\gamma_1)_i\left(\overline{\bar{b}_1^{(\gamma_1)_1} * \cdots * \bar{b}_{i-1}^{(\gamma_1)_{i-1}} * \bar{b}_i^{(\gamma_1)_i-1} * \bar{b}_{i+1}^{(\gamma_1)_{i+1}} * \cdots * \bar{b}_n^{(\gamma_1)_n} * h_i}\right)_\beta \\ & +(\bar{V}_j \hat{*} h_j)_\beta \cdot e_j + \sum_{\substack{1\le i\le n\\ i\neq j}}\left(\left((\bar{V}_i + u_i)\hat{*}h_j\right)_\beta + \left((\bar{\lambda} + u_j)\hat{*}h_i\right)_\beta\right)\cdot e_i \quad ,\end{aligned}$$

with e_i ($i \in \{1, \ldots, n\}$) being the i-th canonical vector from $\mathbb{R}^n$. Recall the bounds (3.2). Recall Definition 14. Y_0 is given by

$$Y_0 \overset{\text{def}}{=} \|T(0) - 0\|_{1,\mu,N}^{(n)} = \|T(0)\|_{1,\mu,N}^{(n)} \quad . \tag{5.8}$$

Notice the computation of Y_0 involves a finite number of terms because $\bar{b}$, $\bar{V}_j$ and $\bar{V}_\lambda^{*j}$ are sequences with finitely many nonzero elements and T is evaluated at $u = 0$.

Let us move on to the computation of $Z(r)$. Notice

$$\|DT(u)h\|_{1,\mu,N}^{(n)} \le \|B^{-1}\|_{B\left(\left(\ell_\mu^{1,N}\right)^n\right)} \cdot \|D_u g(\bar{V}_\lambda^{*j} + u, \bar{V}_j, \bar{b}, c)h\|_{1,\mu,N}^{(n)} \quad .$$

By Theorem 5.0.1, we have

$$\|DT(u)h\|_{1,\mu,N}^{(n)} \le \|B^{-1}\|_\infty \cdot \|D_u g(\bar{V}_\lambda^{*j} + u, \bar{V}_j, \bar{b}, c)h\|_{1,\mu,N}^{(n)} \quad .$$

Therefore, what remains to bound is $\|D_u g(\bar{V}_\lambda^{*j} + u, \bar{V}_j, \bar{b}, c)h\|_{1,\mu,N}^{(n)}$.

Lemma 5.2.1. *Recall* (5.7) *and Definitions 16 and 21. Let $r > 0$. Suppose $\|u\|_{1,\mu,N}^{(n)} \le r$ and $\|h\|_{1,\mu,N}^{(n)} = 1$. We have*

$$\|D_u g(\bar{V}_\lambda^{*j} + u, \bar{V}_j, \bar{b}, c)h\|_{1,\mu,N}^{(n)} \le Z^g(r) \quad ,$$

where

$$Z^g(r) \stackrel{\text{def}}{=} \max\left\{ \sum_{|\gamma_2|=1}^{M_2} \mu^{\gamma_2} \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,\gamma_2}| \sum_{\substack{1\le i\le n\\ i\ne j}} |(\gamma_1)_i| \cdot \|\bar{b}_1\|_{1,\mu}^{(\gamma_1^{*i})_1} \cdots \|\bar{b}_n\|_{1,\mu}^{(\gamma_1^{*i})_n} \right.$$
$$+ \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,0}| \sum_{\substack{1\le i\le n\\ i\ne j}} |(\gamma_1)_i| \cdot \widehat{\|\bar{b}_1\|_{1,\mu}^{(\gamma_1^{*i})_1} \cdots \|\bar{b}_n\|_{1,\mu}^{(\gamma_1^{*i})_n}} \qquad (5.9)$$
$$\left. + \sum_{\substack{1\le i\le n\\ i\ne j}} \left(\widehat{\|\bar{V}_i\|_{1,\mu}} + \widehat{\|\bar{\lambda}\|_{1,\mu}} + 2r \right) \cdot e_i \right\} \quad .$$

Proof. Let $\left(D_u g(\bar{V}_\lambda^{*j} + u, \bar{V}_j, \bar{b}, c)h\right)_\beta = (D_1(u)h)_\beta + (D_2(u)h)_\beta$, where $D_1(u)h$ and $D_2(u)h$ are defined component-wise by

$$(D_1(u)h)_\beta \stackrel{\text{def}}{=} -\sum_{|\gamma_2|=1}^{M_2} \sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,\gamma_2} \sum_{\substack{1\le i\le n\\ i\ne j}} (\gamma_1)_i \left(\bar{b}_1^{(\gamma_1)_1} * \cdots * \bar{b}_{i-1}^{(\gamma_1)_{i-1}} * \bar{b}_i^{(\gamma_1)_i - 1} * \bar{b}_{i+1}^{(\gamma_1)_{i+1}} * \cdots * \bar{b}_n^{(\gamma_1)_n} * h_i \right)_{\beta-\gamma_2}$$

and

$$(D_2(u)h)_\beta \stackrel{\text{def}}{=} -\sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,0} \sum_{\substack{1\le i\le n\\ i\ne j}} (\gamma_1)_i \left(\widehat{\bar{b}_1^{(\gamma_1)_1} * \cdots * \bar{b}_{i-1}^{(\gamma_1)_{i-1}} * \bar{b}_i^{(\gamma_1)_i - 1} * \bar{b}_{i+1}^{(\gamma_1)_{i+1}} * \cdots * \bar{b}_n^{(\gamma_1)_n} * h_i} \right)_\beta$$
$$+ (\bar{V}_j \hat{*} h_j)_\beta \cdot e_j + \sum_{\substack{1\le i\le n\\ i\ne j}} \left(\left((\bar{V}_i + u_i) \hat{*} h_j \right)_\beta + \left((\bar{\lambda} + u_j) \hat{*} h_i \right)_\beta \right) \cdot e_i \quad .$$

Let $\Upsilon_1(u,h) = \sum_{|\beta|\ge 0} \left| (D_1(u)h)_\beta \right| \mu^\beta$. We have

$$\Upsilon_1(u,h) = \sum_{|\beta|\ge 0} \left| \sum_{|\gamma_2|=1}^{M_2} \sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,\gamma_2} \sum_{\substack{1\le i\le n\\ i\ne j}} (\gamma_1)_i \left(\bar{b}_1^{(\gamma_1)_1} * \cdots * \bar{b}_{i-1}^{(\gamma_1)_{i-1}} * \bar{b}_i^{(\gamma_1)_i - 1} * \bar{b}_{i+1}^{(\gamma_1)_{i+1}} * \cdots * \bar{b}_n^{(\gamma_1)_n} * h_i \right)_{\beta-\gamma_2} \right| \mu^\beta$$
$$\le \sum_{|\gamma_2|=1}^{M_2} \mu^{\gamma_2} \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,\gamma_2}| \sum_{\substack{1\le i\le n\\ i\ne j}} |(\gamma_1)_i| \sum_{|\beta|\ge 0} \left| \left(\bar{b}_1^{(\gamma_1)_1} * \cdots * \bar{b}_{i-1}^{(\gamma_1)_{i-1}} * \bar{b}_i^{(\gamma_1)_i - 1} * \bar{b}_{i+1}^{(\gamma_1)_{i+1}} * \cdots * \bar{b}_n^{(\gamma_1)_n} * h_i \right)_{\beta-\gamma_2} \right| \mu^{\beta-\gamma_2} \quad .$$

The change of order of summation was possible because the quadruple series converges absolutely. Let $i \in \{1, \ldots, n\}$. Notice

$$\bar{b}_1^{(\gamma_1)_1} * \cdots * \bar{b}_{i-1}^{(\gamma_1)_{i-1}} * \bar{b}_i^{(\gamma_1)_i - 1} * \bar{b}_{i+1}^{(\gamma_1)_{i+1}} * \cdots * \bar{b}_n^{(\gamma_1)_n} * h_i = \bar{b}_1^{(\gamma_1^{*i})_1} * \cdots * \bar{b}_n^{(\gamma_1^{*i})_n} * h_i \quad .$$

Hence, using Proposition 3.3.1 along with $\|h\|_{1,\mu,N}^{(n)} = 1$, we get

$$\Upsilon_1(u,h) \le \sum_{|\gamma_2|=1}^{M_2} \mu^{\gamma_2} \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,\gamma_2}| \sum_{\substack{1\le i\le n\\ i\neq j}} |(\gamma_1)_i| \cdot \|\bar{b}_1\|_{1,\mu}^{(\gamma_1^{*_i})_1} \cdots \|\bar{b}_n\|_{1,\mu}^{(\gamma_1^{*_i})_n} \quad .$$

Moreover, let $\Upsilon_2(u,h) = \sum_{|\beta|\ge 0} \left|(D_2(u)h)_\beta\right| \mu^\beta$. We have

$$\begin{aligned}
\Upsilon_2(u,h) &\stackrel{\text{def}}{=} \sum_{|\beta|\ge 0} \Bigg| \sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,0} \sum_{\substack{1\le i\le n\\ i\neq j}} (\gamma_1)_i \left(\overline{\bar{b}_1^{(\gamma_1)_1} * \cdots * \bar{b}_{i-1}^{(\gamma_1)_{i-1}} * \bar{b}_i^{(\gamma_1)_i - 1} * \bar{b}_{i+1}^{(\gamma_1)_{i+1}} * \cdots * \bar{b}_n^{(\gamma_1)_n} * h_i} \right)_\beta \\
&\quad + (\bar{V}_j \hat{*} h_j)_\beta \cdot e_j + \sum_{\substack{1\le i\le n\\ i\neq j}} \Big(\big((\bar{V}_i + u_i)\hat{*} h_j\big)_\beta + \big((\bar{\lambda} + u_j)\hat{*} h_i\big)_\beta \Big) \cdot e_i \Bigg| \mu^\beta \\
&\le \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,0}| \sum_{\substack{1\le i\le n\\ i\neq j}} |(\gamma_1)_i| \sum_{|\beta|\ge 0} \left| \left(\overline{\bar{b}_1^{(\gamma_1)_1} * \cdots * \bar{b}_{i-1}^{(\gamma_1)_{i-1}} * \bar{b}_i^{(\gamma_1)_i - 1} * \bar{b}_{i+1}^{(\gamma_1)_{i+1}} * \cdots * \bar{b}_n^{(\gamma_1)_n} * h_i} \right)_\beta \right| \mu^\beta \\
&\quad + \sum_{\substack{1\le i\le n\\ i\neq j}} \sum_{|\beta|\ge 0} \left| \Big(\big((\bar{V}_i + u_i)\hat{*} h_j\big)_\beta + \big((\bar{\lambda} + u_j)\hat{*} h_i\big)_\beta \Big) \cdot e_i \right| \mu^\beta \quad .
\end{aligned}$$

The change of order of summation was possible because the triple and double series both converge absolutely. Else, we used the fact that $(\bar{V}_j \hat{*} h_j)_\beta = 0$ $(|\beta| \ge 0)$ by definition of $\bar{V}_j$ (see Subsection 4.2). Let $i \in \{1, \dots, n\}$. Notice

$$\overline{\bar{b}_1^{(\gamma_1)_1} * \cdots * \bar{b}_{i-1}^{(\gamma_1)_{i-1}} * \bar{b}_i^{(\gamma_1)_i - 1} * \bar{b}_{i+1}^{(\gamma_1)_{i+1}} * \cdots * \bar{b}_n^{(\gamma_1)_n} * h_i} = \overline{\bar{b}_1^{(\gamma_1^{*_i})_1} * \cdots * \bar{b}_n^{(\gamma_1^{*_i})_n} * h_i}$$

and

$$(\bar{V}_i + u_i)\hat{*} h_j = \bar{V}_i \hat{*} h_j + u_i \hat{*} h_j \quad \& \quad (\bar{\lambda} + u_j)\hat{*} h_i = \bar{\lambda} \hat{*} h_i + u_j \hat{*} h_i \quad .$$

Hence, using Theorem 3.3.2 along with $\|u\|_{1,\mu,N}^{(n)} \le r$ and $\|h\|_{1,\mu,N}^{(n)} = 1$, we get

$$\begin{aligned}
\Upsilon_2(u,h) &\le \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,0}| \sum_{\substack{1\le i\le n\\ i\neq j}} |(\gamma_1)_i| \cdot \overline{\|\bar{b}_1\|_{1,\mu}^{(\gamma_1^{*_i})_1} \cdots \|\bar{b}_n\|_{1,\mu}^{(\gamma_1^{*_i})_n}} \\
&\quad + \sum_{\substack{1\le i\le n\\ i\neq j}} \left(\widehat{\|\bar{V}_i\|_{1,\mu}} + \widehat{\|\bar{\lambda}\|_{1,\mu}} + 2r \right) \cdot e_i
\end{aligned} \quad .$$

Finally, the result follows from

$$\|D_u g(\bar{V}_\lambda^{*_j} + u, \bar{V}_j, \bar{b}, c)h\|_{1,\mu,N}^{(n)} \le \max\left\{ \Upsilon_1(u,h) + \Upsilon_2(u,h) \right\} \le Z^g(r) \quad .$$

□

We can now apply the Radii Polynomials Approach (see Subsection 3.4) to see what are the conditions needed for the operator $T : \left(\ell_{\mu}^{1,N}\right)^n \to \left(\ell_{\mu}^{1,N}\right)^n$ given by (4.9) to have a fixed point close to $u = 0$.

Theorem 5.2.1. *Recall $T : \left(\ell_{\mu}^{1,N}\right)^n \to \left(\ell_{\mu}^{1,N}\right)^n$, the operator given by (4.9). Let $K \stackrel{\text{def}}{=} \|B^{-1}\|_{\infty}$. Recall the bounds Y_0 and $Z^g(r)$ defined in (5.8) and (5.9) respectively. Let*

$$
\begin{aligned}
Z_1 \quad \stackrel{\text{def}}{=} \quad K \cdot \max \Bigg\{ & \sum_{|\gamma_2|=1}^{M_2} \mu^{\gamma_2} \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,\gamma_2}| \sum_{\substack{1 \le i \le n \\ i \ne j}} |(\gamma_1)_i| \cdot \|\bar{b}_1\|_{1,\mu}^{(\gamma_1^{*i})_1} \cdots \|\bar{b}_n\|_{1,\mu}^{(\gamma_1^{*i})_n} \\
& + \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,0}| \sum_{\substack{1 \le i \le n \\ i \ne j}} \overline{|(\gamma_1)_i| \cdot \|\bar{b}_1\|_{1,\mu}^{(\gamma_1^{*i})_1} \cdots \|\bar{b}_n\|_{1,\mu}^{(\gamma_1^{*i})_n}} \\
& + \sum_{\substack{1 \le i \le n \\ i \ne j}} \left(\widehat{\|\bar{V}_i\|_{1,\mu}} + \widehat{\|\bar{\lambda}\|_{1,\mu}} \right) \cdot e_i \Bigg\}
\end{aligned}
\tag{5.10}
$$

and

$$Z_2(r) \stackrel{\text{def}}{=} K \cdot Z(r) - Z_1 \quad . \tag{5.11}$$

If there exists $r_0 > 0$ such that

$$Z_2(r_0) r_0^2 + (Z_1 - 1) r_0 + Y_0 < 0 \quad , \tag{5.12}$$

then T has a unique fixed point in $B(0, r_0)$.

Proof. This is an application of Corollary 3.4.1 using Lemma 5.2.1 and the bounds Y_0, Z_1 and $Z_2(r)$ defined in (5.8), (5.10) and (5.11) respectively.

□

Remark. The reason for the use of the bounds Z_1 (Equation 5.10) and $Z_2(r)$ (Equation 5.11) instead of the bound $Z(r) \stackrel{\text{def}}{=} K \cdot Z^g(r)$ (Equation 5.9) is going to be explained in Subsection 5.4.

Assuming the existence of an $r_0 > 0$ satisfying Theorem 5.2.1 – this matter is going to be addressed in Subsection 5.4 –, we can now move on to the computation of the bounds (3.2) for the coefficients of the stable and unstable manifolds parameterizations (Subsection 4.3).

5.3 Stable and unstable manifolds coefficients

As opposed to Subsections 5.1 and 5.2, this subsection is going to address the computation of the bounds (3.2) for two operators instead of one, namely the two operators of Subsection 4.3.

Recall the operator $T_1 : \left(\ell_\mu^{1,N_1}\right)^n \to \left(\ell_\mu^{1,N_1}\right)^n$ given by (4.14). Its derivative $DT_1(u) : \left(\ell_\mu^{1,N_1}\right)^n \to \left(\ell_\mu^{1,N_1}\right)^n$ evaluated at an element $u \in \left(\ell_\mu^{1,N_1}\right)^n$ is a linear operator. Its action on an element $h \in \left(\ell_\mu^{1,N_1}\right)^n$ is given by

$$\left(DT_1(u)h\right)_\beta = \left(A - (\alpha\cdot\bar{\lambda})_0 \cdot I_n\right)^{-1} \left(D_u g_\alpha(\bar{a}_\alpha + u, \bar{a}_{\alpha^-}, \bar{\lambda}, c)h\right)_\beta \quad , \tag{5.13}$$

where

$$\begin{aligned}\left(D_u g_\alpha(\bar{a}_\alpha + u, \bar{a}_{\alpha^-}, \bar{\lambda}, c)h\right)_\beta \;=\; & -\sum_{|\gamma_2|=1}^{M_2}\sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,\gamma_2}\Big[(\gamma_1)_1\left((\bar{a}_1)_0^{(\gamma_1)_1-1} * (\bar{a}_2)_0^{(\gamma_1)_2} * \cdots * (\bar{a}_n)_0^{(\gamma_1)_n} * h_1\right) \\ & \qquad + \cdots + (\gamma_1)_n\left((\bar{a}_1)_0^{(\gamma_1)_1} * \cdots * (\bar{a}_{n-1})_0^{(\gamma_1)_{n-1}} * (\bar{a}_n)_0^{(\gamma_1)_n-1} * h_n\right)\Big]_{\beta-\gamma_2} \\ & -\sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,0}\Big[(\gamma_1)_1\left(\overline{(\bar{a}_1)_0^{(\gamma_1)_1-1} * (\bar{a}_2)_0^{(\gamma_1)_2} * \cdots * (\bar{a}_n)_0^{(\gamma_1)_n} * h_1}\right) \\ & \qquad + \cdots + (\gamma_1)_n\left(\overline{(\bar{a}_1)_0^{(\gamma_1)_1} * \cdots * (\bar{a}_{n-1})_0^{(\gamma_1)_{n-1}} * (\bar{a}_n)_0^{(\gamma_1)_n-1} * h_n}\right)\Big]_\beta \\ & + \left((\alpha\cdot\bar{\lambda})\hat{*}h\right)_\beta \quad .\end{aligned}$$

Recall the bounds (3.2). Let $(Y_1)_0 = Y_0$ and $Z^1(r) = Z(r)$. Recall Definition 14. $(Y_1)_0$ is given by

$$(Y_1)_0 \stackrel{\text{def}}{=} \|T_1(0) - 0\|_{1,\mu,N_1}^{(n)} = \|T_1(0)\|_{1,\mu,N_1}^{(n)} \quad . \tag{5.14}$$

Notice the computation of $(Y_1)_0$ involves a finite number of terms because $\bar{\lambda}$, $\bar{a}_{\alpha^-}$ and $\bar{a}_\alpha$ are sequences with finitely many nonzero elements and T_1 is evaluated at $u = 0$.

Let us move on to the computation of $Z^1(r)$. Notice

$$\|DT_1(u)h\|_{1,\mu,N_1}^{(n)} \le \left\| \left(A - (\alpha\cdot\bar{\lambda})_0\cdot I_n\right)^{-1}\right\|_{B\left(\left(\ell_\mu^{1,N_1}\right)^n\right)} \cdot \|D_u g_\alpha(\bar{a}_\alpha + u, \bar{a}_{\alpha^-}, \bar{\lambda}, c)h\|_{1,\mu,N_1}^{(n)} \quad .$$

By Theorem 5.0.1, we have

$$\|DT_1(u)h\|_{1,\mu,N_1}^{(n)} \le \left\| \left(A - (\alpha\cdot\bar{\lambda})_0\cdot I_n\right)^{-1}\right\|_\infty \cdot \|D_u g_\alpha(\bar{a}_\alpha + u, \bar{a}_{\alpha^-}, \bar{\lambda}, c)h\|_{1,\mu,N_1}^{(n)} \quad .$$

Notice $\left\| \left(A - (\alpha\cdot\bar{\lambda})_0\cdot I_n\right)^{-1}\right\|_\infty$ is a constant because α is fixed for the operator T_1. Therefore, what remains to bound is $\|D_u g_\alpha(\bar{a}_\alpha + u, \bar{a}_{\alpha^-}, \bar{\lambda}, c)h\|_{1,\mu,N_1}^{(n)}$.

Lemma 5.3.1. *Recall* (5.13) *and Definitions 16 and 21. Let* $r > 0$. *Suppose* $\|u\|^{(n)}_{1,\mu,N_1} \le r$ *and* $\|h\|^{(n)}_{1,\mu,N_1} = 1$. *We have*

$$\|D_u g_\alpha(\bar a_\alpha + u, \bar a_{\alpha^-}, \bar\lambda, c)h\|^{(n)}_{1,\mu,N_1} \le Z^g_1(r) \quad ,$$

where

$$\begin{aligned} Z^g_1(r) \overset{\text{def}}{=} \max\Bigg\{ & \sum_{|\gamma_2|=1}^{M_2} \mu^{\gamma_2} \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,\gamma_2}| \sum_{1\le i\le n} |(\gamma_1)_i| \cdot \|(\bar a_1)_0\|_{1,\mu}^{({\gamma_1^*}^i)_1} \cdots \|(\bar a_n)_0\|_{1,\mu}^{({\gamma_1^*}^i)_n} \\ & + \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,0}| \sum_{1\le i\le n} |(\gamma_1)_i| \cdot \widehat{\|(\bar a_1)_0\|_{1,\mu}^{({\gamma_1^*}^i)_1} \cdots \|(\bar a_n)_0\|_{1,\mu}^{({\gamma_1^*}^i)_n}} \\ & + \alpha \cdot \widehat{\|\bar\lambda\|_{1,\mu}} \Bigg\} \end{aligned} \tag{5.15}$$

with

$$\alpha \cdot \widehat{\|\bar\lambda\|_{1,\mu}} \overset{\text{def}}{=} \alpha_1 \cdot \widehat{\|\bar\lambda_1\|_{1,\mu}} + \cdots + \alpha_k \cdot \widehat{\|\bar\lambda_k\|_{1,\mu}} \quad .$$

Proof. Let $\left(D_u g_\alpha(\bar a_\alpha + u, \bar a_{\alpha^-}, \bar\lambda, c)h\right)_\beta = (D_1(u)h)_\beta + (D_2(u)h)_\beta$, where $D_1(u)h$ and $D_2(u)h$ are defined component-wise by

$$\begin{aligned} (D_1(u)h)_\beta \overset{\text{def}}{=} & -\sum_{|\gamma_2|=1}^{M_2} \sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,\gamma_2} \Big[(\gamma_1)_1 \left((\bar a_1)_0^{(\gamma_1)_1-1} * (\bar a_2)_0^{(\gamma_1)_2} * \cdots * (\bar a_n)_0^{(\gamma_1)_n} * h_1 \right) \\ & + \cdots + (\gamma_1)_n \left((\bar a_1)_0^{(\gamma_1)_1} * \cdots * (\bar a_{n-1})_0^{(\gamma_1)_{n-1}} * (\bar a_n)_0^{(\gamma_1)_n-1} * h_n \right) \Big]_{\beta-\gamma_2} \end{aligned}$$

and

$$\begin{aligned} (D_2(u)h)_\beta \overset{\text{def}}{=} & -\sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,0} \Big[(\gamma_1)_1 \left(\widehat{(\bar a_1)_0^{(\gamma_1)_1-1} * (\bar a_2)_0^{(\gamma_1)_2} * \cdots * (\bar a_n)_0^{(\gamma_1)_n} * h_1} \right) \\ & + \cdots + (\gamma_1)_n \left(\widehat{(\bar a_1)_0^{(\gamma_1)_1} * \cdots * (\bar a_{n-1})_0^{(\gamma_1)_{n-1}} * (\bar a_n)_0^{(\gamma_1)_n-1} * h_n} \right) \Big]_\beta \\ & + \left((\alpha \cdot \bar\lambda) \hat{*} h \right)_\beta \end{aligned} .$$

Let $\Upsilon_1(u,h) = \sum_{|\beta|\ge 0} \left| (D_1(u)h)_\beta \right| \mu^\beta$. We have

$$\begin{aligned}\Upsilon_1(u,h) &= \sum_{|\beta|\geq 0}\left|\sum_{|\gamma_2|=1}^{M_2}\sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,\gamma_2}\Big[(\gamma_1)_1\left((\bar a_1)_0^{(\gamma_1)_1-1}*(\bar a_2)_0^{(\gamma_1)_2}*\cdots*(\bar a_n)_0^{(\gamma_1)_n}*h_1\right)\right.\\
&\qquad\left.+\cdots+(\gamma_1)_n\left((\bar a_1)_0^{(\gamma_1)_1}*\cdots*(\bar a_{n-1})_0^{(\gamma_1)_{n-1}}*(\bar a_n)_0^{(\gamma_1)_n-1}*h_n\right)\Big]_{\beta-\gamma_2}\right|\mu^\beta\\
&\leq \sum_{|\gamma_2|=1}^{M_2}\sum_{|\gamma_1|=0}^{M_1}|c_{\gamma_1,\gamma_2}|\left(|(\gamma_1)_1|\sum_{|\beta|\geq 0}\left|\left((\bar a_1)_0^{(\gamma_1)_1-1}*(\bar a_2)_0^{(\gamma_1)_2}*\cdots*(\bar a_n)_0^{(\gamma_1)_n}*h_1\right)_{\beta-\gamma_2}\right|\mu^\beta\right.\\
&\qquad\left.+\cdots+|(\gamma_1)_n|\sum_{|\beta|\geq 0}\left|\left((\bar a_1)_0^{(\gamma_1)_1}*\cdots*(\bar a_{n-1})_0^{(\gamma_1)_{n-1}}*(\bar a_n)_0^{(\gamma_1)_n-1}*h_n\right)_{\beta-\gamma_2}\right|\mu^\beta\right)\quad .\end{aligned}$$

The change of order of summation was possible because the two triple series both converge absolutely. Let $i\in\{1,\dots,n\}$. Notice

$$(\bar a_1)_0^{(\gamma_1)_1}*\cdots*(\bar a_{i-1})_0^{(\gamma_1)_{i-1}}*(\bar a_i)_0^{(\gamma_1)_i-1}*(\bar a_{i+1})_0^{(\gamma_1)_{i+1}}*\cdots*(\bar a_n)_0^{(\gamma_1)_n}*h_i=(\bar a_1)_0^{(\gamma_1^{*i})_1}*\cdots*(\bar a_n)_0^{(\gamma_1^{*i})_n}*h_i\quad .$$

Hence, using Proposition 3.3.1 along with $\|h\|_{1,\mu,N_1}^{(n)}=1$, we get

$$\Upsilon_1(u,h)\leq\sum_{|\gamma_2|=1}^{M_2}\mu^{\gamma_2}\sum_{|\gamma_1|=0}^{M_1}|c_{\gamma_1,\gamma_2}|\sum_{1\leq i\leq n}|(\gamma_1)_i|\cdot\|(\bar a_1)_0\|_{1,\mu}^{(\gamma_1^{*i})_1}\cdots\|(\bar a_n)_0\|_{1,\mu}^{(\gamma_1^{*i})_n}\quad .$$

Moreover, let $\Upsilon_2(u,h)=\sum_{|\beta|\geq 0}\left|(D_2(u)h)_\beta\right|\mu^\beta$. We have

$$\begin{aligned}\Upsilon_2(u,h) &= \sum_{|\beta|\geq 0}\left|\sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,0}\Big[(\gamma_1)_1\left(\overline{(\bar a_1)_0^{(\gamma_1)_1-1}*(\bar a_2)_0^{(\gamma_1)_2}*\cdots*(\bar a_n)_0^{(\gamma_1)_n}*h_1}\right)\right.\\
&\qquad+\cdots+(\gamma_1)_n\left(\overline{(\bar a_1)_0^{(\gamma_1)_1}*\cdots*(\bar a_{n-1})_0^{(\gamma_1)_{n-1}}*(\bar a_n)_0^{(\gamma_1)_n-1}*h_n}\right)\Big]_\beta\\
&\qquad\left.+\left((\alpha\cdot\bar\lambda)\hat{*}h\right)_\beta\right|\mu^\beta\\
&\leq \sum_{|\gamma_1|=0}^{M_1}|c_{\gamma_1,0}|\left(|(\gamma_1)_1|\sum_{|\beta|\geq 0}\left|\left(\overline{(\bar a_1)_0^{(\gamma_1)_1-1}*(\bar a_2)_0^{(\gamma_1)_2}*\cdots*(\bar a_n)_0^{(\gamma_1)_n}*h_1}\right)_\beta\right|\mu^\beta\right.\\
&\qquad\left.+\cdots+|(\gamma_1)_n|\sum_{|\beta|\geq 0}\left|\left(\overline{(\bar a_1)_0^{(\gamma_1)_1}*\cdots*(\bar a_{n-1})_0^{(\gamma_1)_{n-1}}*(\bar a_n)_0^{(\gamma_1)_n-1}*h_n}\right)_\beta\right|\mu^\beta\right)\\
&\quad+\sum_{|\beta|\geq 0}\left|\left((\alpha\cdot\bar\lambda)\hat{*}h\right)_\beta\right|\mu^\beta\quad .\end{aligned}$$

The change of order of summation was possible because the two double series both converge absolutely, as well as the single series. Let $i \in \{1, \dots, n\}$. Notice

$$\overbrace{(\bar{a}_1)_0^{(\gamma_1)_1} * \cdots * (\bar{a}_{i-1})_0^{(\gamma_1)_{i-1}} * (\bar{a}_i)_0^{(\gamma_1)_i - 1} * (\bar{a}_{i+1})_0^{(\gamma_1)_{i+1}} * \cdots * (\bar{a}_n)_0^{(\gamma_1)_n} * h_i} = \overbrace{(\bar{a}_1)_0^{(\gamma_1^{*i})_1} * \cdots * (\bar{a}_n)_0^{(\gamma_1^{*i})_n} * h_i} \quad .$$

Let $\mathbf{1} \overset{\text{def}}{=} (1, \dots, 1) \in \mathbb{R}^n$. Hence, using Theorem 3.3.2 along with $\|h\|_{1,\mu,N_1}^{(n)} = 1$, we get

$$\begin{aligned} \Upsilon_2(u,h) \ \leq \ & \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,0}| \sum_{1 \leq i \leq n} |(\gamma_1)_i| \cdot \overbrace{\|(\bar{a}_1)_0\|_{1,\mu}^{(\gamma_1^{*i})_1} \cdots \|(\bar{a}_n)_0\|_{1,\mu}^{(\gamma_1^{*i})_n}} \\ & + \mathbf{1} \cdot \left(\alpha \cdot \widehat{\|\bar{\lambda}\|_{1,\mu}} \right) \end{aligned} \quad .$$

Finally, the result follows from

$$\|D_u g_\alpha(\bar{a}_\alpha + u, \bar{a}_{\alpha^-}, \bar{\lambda}, c)h\|_{1,\mu,N_1}^{(n)} \leq \max\left\{\Upsilon_1(u,h) + \Upsilon_2(u,h)\right\} \leq Z_1^g(r) \quad .$$

□

We can now apply the Radii Polynomials Approach (see Subsection 3.4) to see what are the conditions needed for the operator $T_1 : \left(\ell_\mu^{1,N_1}\right)^n \to \left(\ell_\mu^{1,N_1}\right)^n$ given by (4.14) to have a fixed point close to $u = 0$.

Theorem 5.3.1. *Recall* $T_1 : \left(\ell_\mu^{1,N_1}\right)^n \to \left(\ell_\mu^{1,N_1}\right)^n$, *the operator given by* (4.14). *Let* $K \overset{\text{def}}{=} \left\| \left(A - (\alpha \cdot \bar{\lambda})_0 \cdot I_n\right)^{-1} \right\|_\infty$. *Recall the bounds* $(Y_1)_0$ *and* $Z_1^g(r)$ *defined in* (5.14) *and* (5.15) *respectively. Let*

$$Z^1(r) \overset{\text{def}}{=} K \cdot Z_1^g(r) \quad . \tag{5.16}$$

If there exists $r_0 > 0$ *such that*

$$(Z^1(r) - 1)r_0 + (Y_1)_0 < 0 \quad , \tag{5.17}$$

then T_1 *has a unique fixed point in* $B(0, r_0)$.

Proof. This is an application of Theorem 3.4.2 using Lemma 5.3.1 and the bounds $(Y_1)_0$ and $Z^1(r)$ defined in (5.14) and (5.16) respectively.

□

Remark. Notice the bound $Z_1(r)$ is constant, i.e. it does not depend on r. Therefore, the left hand side of Equation 5.17 is a polynomial of degree 1.

Assuming the existence of an $r_0 > 0$ satisfying Theorem 5.3.1 – this matter is going to be addressed in Subsection 5.4 –, we can now address the computation of the bounds (3.2) for the second operator of Subsection 4.3.

Recall the operator $T_2 : \left(\ell^{1,N_2}_{\nu,\mu}\right)^n \to \left(\ell^{1,N_2}_{\nu,\mu}\right)^n$ given by (4.15). Its derivative $DT_2(u) : \left(\ell^{1,N_2}_{\nu,\mu}\right)^n \to \left(\ell^{1,N_2}_{\nu,\mu}\right)^n$ evaluated at an element $u \in \left(\ell^{1,N_2}_{\nu,\mu}\right)^n$ is a linear operator. Its action on an element $h \in \left(\ell^{1,N_2}_{\nu,\mu}\right)^n$ is given by

$$\left(DT_2(u)h\right)_{\alpha,\beta} = \left(A - (\alpha\cdot\bar{\lambda})_0\cdot I_n\right)^{-1}\left(D_u g(\bar{a}+u,\bar{\lambda},c)h\right)_{\alpha,\beta} \quad , \tag{5.18}$$

where

$$\begin{aligned}
\left(D^1_u g(\bar{a}+u,\bar{\lambda},c)h\right)_{\alpha,\beta} &\overset{\text{def}}{=} -\sum_{|\gamma_2|=1}^{M_2}\sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,\gamma_2}\cdot\Big[(\gamma_1)_1\cdot\left((\bar{a}_1+u_1)^{(\gamma_1)_1-1}*(\bar{a}_2+u_2)^{(\gamma_1)_2}*\cdots*(\bar{a}_n+u_n)^{(\gamma_1)_n}*h_1\right) \\
&\qquad +\cdots+(\gamma_1)_n\cdot\left((\bar{a}_1+u_1)^{(\gamma_1)_1}*\cdots*(\bar{a}_{n-1}+u_{n-1})^{(\gamma_1)_{n-1}}*(\bar{a}_n+u_n)^{(\gamma_1)_n-1}*h_n\right)\Big]_{\alpha,\beta-\gamma_2} \\
&\quad -\sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,0}\cdot\Big[(\gamma_1)_1\cdot\left(\widehat{(\bar{a}_1+u_1)^{(\gamma_1)_1-1}*(\bar{a}_2+u_2)^{(\gamma_1)_2}*\cdots*(\bar{a}_n+u_n)^{(\gamma_1)_n}*h_1}\right) \\
&\qquad +\cdots+(\gamma_1)_n\cdot\left(\widehat{(\bar{a}_1+u_1)^{(\gamma_1)_1}*\cdots*(\bar{a}_{n-1}+u_{n-1})^{(\gamma_1)_{n-1}}*(\bar{a}_n+u_n)^{(\gamma_1)_n-1}*h_n}\right)\Big]_{\alpha,\beta} \\
\left(D^2_u g(\bar{a}+u,\bar{\lambda},c)h\right)_{\alpha,\beta} &\overset{\text{def}}{=} \left((\alpha\cdot\bar{\lambda})\hat{*}h_\alpha\right)_\beta + (1-\delta_{\beta,0})\cdot(\alpha\cdot\bar{\lambda})_\beta\cdot h_{\alpha,0} \\
\left(D_u g(\bar{a}+u,\bar{\lambda},c)h\right)_{\alpha,\beta} &\overset{\text{def}}{=} \left(D^1_u g(\bar{a}+u,\bar{\lambda},c)h\right)_{\alpha,\beta} + \left(D^2_u g(\bar{a}+u,\bar{\lambda},c)h\right)_{\alpha,\beta}
\end{aligned}$$

with

$$\delta_{i,j} \overset{\text{def}}{=} \begin{cases} 1 & , \text{if } i = j \\ 0 & , \text{if } i \neq j \end{cases} \qquad (i,j \in \mathbb{N}^p)$$

being the *Kronecker Delta*. Recall the bounds (3.2). Let $(Y_2)_0 = Y_0$ and $Z^2(r) = Z(r)$. Recall Definition 14. $(Y_2)_0$ is given by

$$(Y_2)_0 \overset{\text{def}}{=} \|T_2(0) - 0\|^{(n)}_{1,(\nu,\mu),N_2} = \|T_2(0)\|^{(n)}_{1,(\nu,\mu),N_2} \quad . \tag{5.19}$$

Notice the computation of $(Y_2)_0$ involves a finite number of terms because $\bar{\lambda}$ and $\bar{a}$ are sequences with finitely many nonzero elements and T_2 is evaluated at $u = 0$.

Let us move on to the computation of $Z^2(r)$. Since $\left\|\left(A - (\alpha\cdot\bar{\lambda})_0\cdot I_n\right)^{-1}\right\|_\infty$ depends on α and the latter is not constant this time, we cannot use the same trick to bound the norm of the matrix involved in the recurrence relation considered as we did for the derivatives (5.1), (5.7) and (5.13). Nevertheless, we can retrieve a bound of this norm for any fixed α. Recall Assumption **A3**. Let $\boldsymbol{\lambda} \stackrel{\text{def}}{=} (\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_n)$ be the eigenvalues of A and $\boldsymbol{V} \stackrel{\text{def}}{=} (\boldsymbol{V}^1, \ldots, \boldsymbol{V}^n)$ be their associated eigenvectors. Notice

$$(\bar{\lambda})_0 = \big((\bar{\lambda}_1)_0, \ldots, (\bar{\lambda}_k)_0\big) = (\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k) \quad \& \quad (\bar{V})_0 = \big((\bar{V}^1)_0, \ldots, (\bar{V}^k)_0\big) = (\boldsymbol{V}^1, \ldots, \boldsymbol{V}^k) \quad .$$

Since $A = D_x f(\tilde{x}(\tilde{\omega}), \tilde{\omega})$, A is diagonalizable, we have

$$A = PDP^{-1} \quad ,$$

where

$$P \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{V}^1 & \Big| \cdots \Big| & \boldsymbol{V}^n \end{pmatrix} \quad \& \quad D \stackrel{\text{def}}{=} \begin{pmatrix} \boldsymbol{\lambda}_1 & & \\ & \ddots & \\ & & \boldsymbol{\lambda}_n \end{pmatrix} \quad .$$

Thus, we can rewrite $A - (\alpha \cdot \bar{\lambda})_0 \cdot I_n$ as

$$A - (\alpha \cdot \bar{\lambda})_0 \cdot I_n = PDP^{-1} - (\alpha \cdot \bar{\lambda})_0 \cdot I_n = P\left(D - (\alpha \cdot \bar{\lambda})_0 \cdot I_n\right) P^{-1} \quad .$$

The latter implies that

$$\left(A - (\alpha \cdot \bar{\lambda})_0 \cdot I_n\right)^{-1} = P^{-1}\left(D - (\alpha \cdot \bar{\lambda})_0 \cdot I_n\right)^{-1} P \quad .$$

Therefore, for a fixed α, we have

$$\left\|\left(A - (\alpha \cdot \bar{\lambda})_0 \cdot I_n\right)^{-1}\right\|_\infty \leq \|P^{-1}\|_\infty \cdot \left\|\left(D - (\alpha \cdot \bar{\lambda})_0 \cdot I_n\right)^{-1}\right\|_\infty \cdot \|P\|_\infty \quad .$$

Since $\|P^{-1}\|_\infty$ and $\|P\|_\infty$ are both constant, what remains to bound for any fixed α is $\left\|\left(D - (\alpha \cdot \bar{\lambda})_0 \cdot I_n\right)^{-1}\right\|_\infty$.

Lemma 5.3.2. *Let the eigenvalues of A be complex. Fix α. For N_2 big enough, we have*

$$\left\|\left(D - (\alpha \cdot \bar{\lambda})_0 \cdot I_n\right)^{-1}\right\|_\infty \leq \frac{10}{9|\alpha| \cdot \min\{|\mathrm{Re}(\bar{\lambda}_0)|\}} \quad .$$

Proof. Let $\alpha \cdot |\mathrm{Re}(\bar{\lambda}_0)| \stackrel{\text{def}}{=} \alpha_1 \cdot |\mathrm{Re}\left((\bar{\lambda}_1)_0\right)| + \cdots + \alpha_k \cdot |\mathrm{Re}\left((\bar{\lambda}_k)_0\right)|$. We have

$$\begin{aligned}
\left\|\left(D - (\alpha \cdot \bar{\lambda})_0 \cdot I_n\right)^{-1}\right\|_\infty &\stackrel{(1)}{=} \max_{1 \leq i \leq n} \left\{ \frac{1}{\left|\boldsymbol{\lambda}_i - (\alpha \cdot \bar{\lambda})_0\right|} \right\} \\
&\leq \max_{1 \leq i \leq n} \left\{ \frac{1}{\left|\mathrm{Re}\left(\boldsymbol{\lambda}_i - (\alpha \cdot \bar{\lambda})_0\right)\right|} \right\} \\
&\stackrel{(2)}{\leq} \max_{1 \leq i \leq n} \left\{ \frac{1}{\alpha \cdot \left|\mathrm{Re}(\bar{\lambda}_0)\right| - |\mathrm{Re}\left(\boldsymbol{\lambda}_i\right)|} \right\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \max_{1\leq i\leq n}\left\{\frac{1}{|\alpha|\cdot\min\{\left|\mathrm{Re}(\bar{\lambda}_0)\right|\}-|\mathrm{Re}\,(\boldsymbol{\lambda}_i)|}\right\}\\
&\overset{(3)}{\leq} \max_{1\leq i\leq n}\left\{\frac{1}{|\alpha|\cdot\min\{\left|\mathrm{Re}(\bar{\lambda}_0)\right|\}-\dfrac{0.1|\alpha|\cdot|\,\mathrm{Re}(\boldsymbol{\lambda}_i)|\cdot\min\{\left|\mathrm{Re}(\bar{\lambda}_0)\right|\}}{|\,\mathrm{Re}(\boldsymbol{\lambda}_1)|+\cdots+|\,\mathrm{Re}(\boldsymbol{\lambda}_n)|}}\right\}\\
&\leq \max_{1\leq i\leq n}\left\{\frac{1}{|\alpha|\cdot\min\{\left|\mathrm{Re}(\bar{\lambda}_0)\right|\}-0.1|\alpha|\min\{\left|\mathrm{Re}(\bar{\lambda}_0)\right|\}}\right\}\\
&=\frac{10}{9|\alpha|\cdot\min\{\left|\mathrm{Re}(\bar{\lambda}_0)\right|\}} \quad .
\end{aligned}
$$

□

Remark.

(1) The absolute values of a complex number $z \in \mathbb{C}^d$ $(d > 0)$ are defined as $|z| \overset{\text{def}}{=} (|z_1|, \ldots, |z_d|)$, where $|z_i|$ is the usual norm of the complex number z_i $(1 \leq i \leq d)$.

(2) Without mentioning it, we used $\alpha\cdot\left|\mathrm{Re}(\bar{\lambda}_0)\right| \overset{\text{def}}{=} \alpha\cdot\left|\mathrm{Re}\left((\bar{\lambda}_1)_0\right)\right|+\cdots+\alpha\cdot\left|\mathrm{Re}\left((\bar{\lambda}_k)_0\right)\right|$. Notice $\mathrm{Re}\left((\bar{\lambda}_1)_0\right), \ldots, \mathrm{Re}\left((\bar{\lambda}_k)_0\right)$ all have the sign and $|\alpha| \geq N_2$, so the inequality stands provided N_2 big enough.

(3) Since $|\alpha| \geq N_2$, the inequality stands provided N_2 big enough.

Lemma 5.3.2 is valid for complex eigenvalues. For now, we consider the real case because we have always done so far. The complex case is going to be discussed later on. The reason for allowing complex eigenvalue for this specific theorem is that it is not obvious it should stand for complex eigenvalues. Thus, we incorporate them so it will be clear later on, when we speak of the complex case, that this lemma still holds.

Lemma 5.3.2 allows us to get the following bound :

Theorem 5.3.2. *Recall Definition 20. For N_2 big enough, we have*

$$
\left\|\left(A-(\alpha\cdot\bar{\lambda})_0\cdot I_n\right)^{-1}\right\|_{B\left(\left(\ell^{1,N_2}_{\nu,\mu}\right)^n\right)} \leq \frac{10}{9\cdot N_2\cdot\min\{\left|\bar{\lambda}_0\right|\}}\cdot\|P^{-1}\|_\infty\cdot\|P\|_\infty \quad .
$$

Proof. Notice

$$
\left\|\left(A-(\alpha\cdot\bar{\lambda})_0\cdot I_n\right)^{-1}\right\|_{B\left(\left(\ell^{1,N_2}_{\nu,\mu}\right)^n\right)} = \left\|P^{-1}\left(D-(\alpha\cdot\bar{\lambda})_0\cdot I_n\right)^{-1}P\right\|_{B\left(\left(\ell^{1,N_2}_{\nu,\mu}\right)^n\right)}
$$

$$\leq \|P^{-1}\|_{B\left(\left(\ell^{1,N_2}_{\nu,\mu}\right)^n\right)} \cdot \left\| \left(D - (\alpha\cdot \bar{\lambda})_0 \cdot I_n\right)^{-1} \right\|_{B\left(\left(\ell^{1,N_2}_{\nu,\mu}\right)^n\right)} \cdot \|P\|_{B\left(\left(\ell^{1,N_2}_{\nu,\mu}\right)^n\right)} \quad .$$

Since we take N_2 big enough, by Theorem 5.0.1, we have

$$\left\| \left(A - (\alpha\cdot \bar{\lambda})_0 \cdot I_n\right)^{-1} \right\|_{B\left(\left(\ell^{1,N_2}_{\nu,\mu}\right)^n\right)} \leq \|P^{-1}\|_\infty \cdot \max_\alpha \left\{ \left\| \left(D - (\alpha\cdot \bar{\lambda})_0 \cdot I_n\right)^{-1} \right\|_\infty \right\} \cdot \|P\|_\infty \quad .$$

By Lemma 5.3.2, we have

$$\left\| \left(A - (\alpha\cdot \bar{\lambda})_0 \cdot I_n\right)^{-1} \right\|_{B\left(\left(\ell^{1,N_2}_{\nu,\mu}\right)^n\right)} \leq \max_\alpha \left\{ \frac{10}{9|\alpha| \cdot \min\{|\bar{\lambda}_0|\}} \right\} \cdot \|P^{-1}\|_\infty \cdot \|P\|_\infty \quad .$$

The result follows from the fact $|\alpha| \geq N_2$ for all α.

□

We indirectly proved a very useful bound while proving Theorem 5.3.2, so let us state it as a corollary.

Corollary 5.3.1. *Recall Definition 20. Let $i \in \{1, \ldots, k\}$. For N_2 big enough, we have*

$$\left\| \alpha_i \cdot \left(A - (\alpha\cdot \bar{\lambda})_0 \cdot I_n\right)^{-1} \right\|_{B\left(\left(\ell^{1,N_2}_{\nu,\mu}\right)^n\right)} \leq \frac{10}{9 \cdot \min\{|\bar{\lambda}_0|\}} \cdot \|P^{-1}\|_\infty \cdot \|P\|_\infty \quad .$$

Proof. This is basically the same proof as to Theorem 5.3.2, just notice that

$$\frac{\alpha_i}{|\alpha|} \leq 1 \quad (\forall |\alpha| \geq N_2) \quad .$$

□

We are now set to go over the computation of the bound $Z^2(r)$ from (3.2) for the operator $T_2 : \left(\ell^{1,N_2}_{\nu,\mu}\right)^n \to \left(\ell^{1,N_2}_{\nu,\mu}\right)^n$ given by (4.15). Notice

$$\|DT_2(u)h\|^{(n)}_{1,(\nu,\mu),N_2} \leq \left\| \left(A - (\alpha\cdot \bar{\lambda})_0 \cdot I_n\right)^{-1} \right\|_{B\left(\left(\ell^{1,N_2}_{\nu,\mu}\right)^n\right)} \cdot \|D_u g(\bar{a}+u, \bar{\lambda}, c)h\|^{(n)}_{1,(\nu,\mu),N_2} \quad .$$

Theorem 5.3.2 provides us with a bound on $\left\| \left(A - (\alpha\cdot \bar{\lambda})_0 \cdot I_n\right)^{-1} \right\|_{B\left(\left(\ell^{1,N_2}_{\nu,\mu}\right)^n\right)}$. Therefore, what remains to bound is $\|D_u g(\bar{a}+u, \bar{\lambda}, c)h\|^{(n)}_{1,(\nu,\mu),N_2}$. Unfortunately, it is not bounded due to the term $\alpha\cdot \bar{\lambda}$ of $\|D^2_u g(\bar{a}+u, \bar{\lambda}, c)h\|^{(n)}_{1,(\nu,\mu),N_2}$. However, the action of $\left(A - (\alpha\cdot \bar{\lambda})_0 \cdot I_n\right)^{-1}$ on the latter is bounded, and that is where Corollary 5.3.1 comes in handy.

That being said, let us state two lemmas that provide us with the bound $Z_2(r)$ from (3.2).

Lemma 5.3.3. *Recall* (5.18) *and Definitions 16, 21 and 22. Let* $r > 0$. *Suppose* $\|u\|^{(n)}_{1,(\nu,\mu),N_2} \leq r$ *and* $\|h\|^{(n)}_{1,(\nu,\mu),N_2} = 1$. *We have*

$$\|D^1_u g(\bar{a}+u,\bar{\lambda},c)h\|^{(n)}_{1,(\nu,\mu),N_2} \leq Z^{g^1}_2(r) \quad ,$$

where

$$\begin{aligned} Z^{g^1}_2(r) \overset{\text{def}}{=} \max\Bigg\{ & \sum_{|\gamma_2|=1}^{M_2} \mu^{\gamma_2} \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,\gamma_2}| \Bigg(|(\gamma_1)_1| \sum_{\eta \leq \gamma_1^{*1}} \binom{(\gamma_1^{*1})_1}{\eta_1} \cdots \binom{(\gamma_1^{*1})_n}{\eta_n} \|\bar{a}_1\|_{1,(\nu,\mu)}^{(\gamma_1^{*1})_1-\eta_1} \cdots \|\bar{a}_n\|_{1,(\nu,\mu)}^{(\gamma_1^{*1})_n-\eta_n} r^{|\eta|} \\ & + \cdots + |(\gamma_1)_n| \sum_{\eta \leq \gamma_1^{*n}} \binom{(\gamma_1^{*n})_1}{\eta_1} \cdots \binom{(\gamma_1^{*n})_n}{\eta_n} \|\bar{a}_1\|_{1,(\nu,\mu)}^{(\gamma_1^{*n})_1-\eta_1} \cdots \|\bar{a}_n\|_{1,(\nu,\mu)}^{(\gamma_1^{*n})_n-\eta_n} r^{|\eta|} \Bigg) \\ & + \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,0}| \Bigg(|(\gamma_1)_1| \sum_{\eta \leq \gamma_1^{*1}} \binom{(\gamma_1^{*1})_1}{\eta_1} \cdots \binom{(\gamma_1^{*1})_n}{\eta_n} \overline{\|\bar{a}_1\|_{1,(\nu,\mu)}^{(\gamma_1^{*1})_1-\eta_1} \cdots \|\bar{a}_n\|_{1,(\nu,\mu)}^{(\gamma_1^{*1})_n-\eta_n}} r^{|\eta|} \\ & + \cdots + |(\gamma_1)_n| \sum_{\eta \leq \gamma_1^{*n}} \binom{(\gamma_1^{*n})_1}{\eta_1} \cdots \binom{(\gamma_1^{*n})_n}{\eta_n} \overline{\|\bar{a}_1\|_{1,(\nu,\mu)}^{(\gamma_1^{*n})_1-\eta_1} \cdots \|\bar{a}_n\|_{1,(\nu,\mu)}^{(\gamma_1^{*n})_n-\eta_n}} r^{|\eta|} \Bigg) \Bigg\} . \end{aligned} \tag{5.20}$$

Proof. Let $\left(D^1_u g(\bar{a}+u,\bar{\lambda},c)h\right)_{\alpha,\beta} = (D_1(u)h)_{\alpha,\beta} + (D_2(u)h)_{\alpha,\beta}$, where $D_1(u)h$ and $D_2(u)h$ are defined component-wise by

$$\begin{aligned} (D_1(u)h)_{\alpha,\beta} \overset{\text{def}}{=} & -\sum_{|\gamma_2|=1}^{M_2} \sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,\gamma_2} \cdot \Big[(\gamma_1)_1 \cdot \Big((\bar{a}_1+u_1)^{(\gamma_1)_1-1} * (\bar{a}_2+u_2)^{(\gamma_1)_2} * \cdots * (\bar{a}_n+u_n)^{(\gamma_1)_n} * h_1 \Big) \\ & + \cdots + (\gamma_1)_n \cdot \Big((\bar{a}_1+u_1)^{(\gamma_1)_1} * \cdots * (\bar{a}_{n-1}+u_{n-1})^{(\gamma_1)_{n-1}} * (\bar{a}_n+u_n)^{(\gamma_1)_n-1} * h_n \Big) \Big]_{\alpha,\beta-\gamma_2} \end{aligned}$$

and

$$\begin{aligned} (D_2(u)h)_{\alpha,\beta} \overset{\text{def}}{=} & -\sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,0} \cdot \Big[(\gamma_1)_1 \cdot \Big(\overline{(\bar{a}_1+u_1)^{(\gamma_1)_1-1} * (\bar{a}_2+u_2)^{(\gamma_1)_2} * \cdots * (\bar{a}_n+u_n)^{(\gamma_1)_n} * h_1} \Big) \\ & + \cdots + (\gamma_1)_n \cdot \Big(\overline{(\bar{a}_1+u_1)^{(\gamma_1)_1} * \cdots * (\bar{a}_{n-1}+u_{n-1})^{(\gamma_1)_{n-1}} * (\bar{a}_n+u_n)^{(\gamma_1)_n-1} * h_n} \Big) \Big]_{\alpha,\beta} . \end{aligned}$$

Let $\Upsilon_1(u,h) = \sum_{|\beta|\geq 0} \sum_{|\alpha|\geq N_2} \left| (D_1(u)h)_{\alpha,\beta} \right| \nu^\alpha \mu^\beta$. We have

$$\begin{aligned} \Upsilon_1(u,h) = & \sum_{|\beta|\geq 0} \sum_{|\alpha|\geq N_2} \Bigg| \sum_{|\gamma_2|=1}^{M_2} \sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,\gamma_2} \cdot \Big[(\gamma_1)_1 \cdot \Big((\bar{a}_1+u_1)^{(\gamma_1)_1-1} * (\bar{a}_2+u_2)^{(\gamma_1)_2} * \cdots * (\bar{a}_n+u_n)^{(\gamma_1)_n} * h_1 \Big) \\ & + \cdots + (\gamma_1)_n \cdot \Big((\bar{a}_1+u_1)^{(\gamma_1)_1} * \cdots * (\bar{a}_{n-1}+u_{n-1})^{(\gamma_1)_{n-1}} * (\bar{a}_n+u_n)^{(\gamma_1)_n-1} * h_n \Big) \Big]_{\alpha,\beta-\gamma_2} \Bigg| \nu^\alpha \mu^\beta \\ \leq & \sum_{|\gamma_2|=1}^{M_2} \mu^{\gamma_2} \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,\gamma_2}| \sum_{|\beta|\geq 0} \sum_{|\alpha|\geq N_2} \Big[|(\gamma_1)_1| \cdot \Big| (\bar{a}_1+u_1)^{(\gamma_1)_1-1} * (\bar{a}_2+u_2)^{(\gamma_1)_2} * \cdots * (\bar{a}_n+u_n)^{(\gamma_1)_n} * h_1 \Big| \\ & + \cdots + |(\gamma_1)_n| \cdot \Big| (\bar{a}_1+u_1)^{(\gamma_1)_1} * \cdots * (\bar{a}_{n-1}+u_{n-1})^{(\gamma_1)_{n-1}} * (\bar{a}_n+u_n)^{(\gamma_1)_n-1} * h_n \Big| \Big]_{\alpha,\beta-\gamma_2} \nu^\alpha \mu^{\beta-\gamma_2} . \end{aligned}$$

The change of order of summation was possible because the triple series converges absolutely. Let $i \in \{1, \dots, n\}$. Notice

$$(\bar{a}_1 + u_1)^{(\gamma_1)_1 - 1} * \cdots * (\bar{a}_{i-1} + u_{i-1})^{(\gamma_1)_{i-1}} * (\bar{a}_i + u_i)^{(\gamma_1)_i - 1} * (\bar{a}_{i+1} + u_{i+1})^{(\gamma_1)_{i+1}} * \cdots * (\bar{a}_n + u_n)^{(\gamma_1)_n} * h_i$$
$$= \sum_{\eta \le \gamma_1^{*i}} \binom{(\gamma_1^{*i})_1}{\eta_1} \cdots \binom{(\gamma_1^{*i})_n}{\eta_n} \cdot \bar{a}_1^{(\gamma_1^{*i})_1 - \eta_1} * u_1^{\eta_1} * \cdots * \bar{a}_n^{(\gamma_1^{*i})_n - \eta_n} * u_n^{\eta_n} * h_i \quad .$$

Hence, using Proposition 3.3.1 along with $\|u\|^{(n)}_{1,(\nu,\mu),N_2} \le r$ and $\|h\|^{(n)}_{1,(\nu,\mu),N_2} = 1$, we get

$$\Upsilon_1(u,h) \le \sum_{|\gamma_2|=1}^{M_2} \mu^{\gamma_2} \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,\gamma_2}| \left(|(\gamma_1)_1| \sum_{\eta \le \gamma_1^{*1}} \binom{(\gamma_1^{*1})_1}{\eta_1} \cdots \binom{(\gamma_1^{*1})_n}{\eta_n} \|\bar{a}_1\|_{1,(\nu,\mu)}^{(\gamma_1^{*1})_1 - \eta_1} \cdots \|\bar{a}_n\|_{1,(\nu,\mu)}^{(\gamma_1^{*1})_n - \eta_n} r^{|\eta|} \right.$$
$$\left. + \cdots + |(\gamma_1)_n| \sum_{\eta \le \gamma_1^{*n}} \binom{(\gamma_1^{*n})_1}{\eta_1} \cdots \binom{(\gamma_1^{*n})_n}{\eta_n} \|\bar{a}_1\|_{1,(\nu,\mu)}^{(\gamma_1^{*n})_1 - \eta_1} \cdots \|\bar{a}_n\|_{1,(\nu,\mu)}^{(\gamma_1^{*n})_n - \eta_n} r^{|\eta|} \right) \quad .$$

Moreover, let $\Upsilon_2(u,h) = \sum_{|\beta| \ge 0} \sum_{|\alpha| \ge N_2} \left| (D_2(u)h)_{\alpha,\beta} \right| \nu^\alpha \mu^\beta$. We have

$$\Upsilon_2(u,h) = \sum_{|\beta| \ge 0} \sum_{|\alpha| \ge N_2} \left| \sum_{|\gamma_1|=0}^{M_1} c_{\gamma_1,0} \cdot \left[(\gamma_1)_1 \cdot \left(\overline{(\bar{a}_1 + u_1)^{(\gamma_1)_1 - 1} * (\bar{a}_2 + u_2)^{(\gamma_1)_2} * \cdots * (\bar{a}_n + u_n)^{(\gamma_1)_n} * h_1} \right) \right. \right.$$
$$\left. \left. + \cdots + (\gamma_1)_n \cdot \left(\overline{(\bar{a}_1 + u_1)^{(\gamma_1)_1} * \cdots * (\bar{a}_{n-1} + u_{n-1})^{(\gamma_1)_{n-1}} * (\bar{a}_n + u_n)^{(\gamma_1)_n - 1} * h_n} \right) \right]_{\alpha,\beta} \right| \nu^\alpha \mu^\beta$$
$$\le \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,0}| \sum_{|\beta| \ge 0} \sum_{|\alpha| \ge N_2} \left[|(\gamma_1)_1| \cdot \left| \overline{(\bar{a}_1 + u_1)^{(\gamma_1)_1 - 1} * (\bar{a}_2 + u_2)^{(\gamma_1)_2} * \cdots * (\bar{a}_n + u_n)^{(\gamma_1)_n} * h_1} \right| \right.$$
$$\left. + \cdots + |(\gamma_1)_n| \cdot \left| \overline{(\bar{a}_1 + u_1)^{(\gamma_1)_1} * \cdots * (\bar{a}_{n-1} + u_{n-1})^{(\gamma_1)_{n-1}} * (\bar{a}_n + u_n)^{(\gamma_1)_n - 1} * h_n} \right| \right]_{\alpha,\beta} \nu^\alpha \mu^\beta \quad .$$

The change of order of summation was possible because the double series converges absolutely. Let $i \in \{1, \dots, n\}$. Notice

$$\overline{(\bar{a}_1 + u_1)^{(\gamma_1)_1 - 1} * \cdots * (\bar{a}_{i-1} + u_{i-1})^{(\gamma_1)_{i-1}} * (\bar{a}_i + u_i)^{(\gamma_1)_i - 1} * (\bar{a}_{i+1} + u_{i+1})^{(\gamma_1)_{i+1}} * \cdots * (\bar{a}_n + u_n)^{(\gamma_1)_n} * h_i}$$
$$= \sum_{\eta \le \gamma_1^{*i}} \binom{(\gamma_1^{*i})_1}{\eta_1} \cdots \binom{(\gamma_1^{*i})_n}{\eta_n} \cdot \overline{\bar{a}_1^{(\gamma_1^{*i})_1 - \eta_1} * u_1^{\eta_1} * \cdots * \bar{a}_n^{(\gamma_1^{*i})_n - \eta_n} * u_n^{\eta_n} * h_i} \quad .$$

Hence, using Theorem 3.3.2 along with $\|u\|^{(n)}_{1,(\nu,\mu),N_2} \le r$ and $\|h\|^{(n)}_{1,(\nu,\mu),N_2} = 1$, we get

$$\Upsilon_2(u,h) \le \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,0}| \Bigg(|(\gamma_1)_1| \sum_{\eta \le \gamma_1^{*1}} \binom{(\gamma_1^{*1})_1}{\eta_1} \cdots \binom{(\gamma_1^{*1})_n}{\eta_n} \overbrace{\|\bar a_1\|_{1,(\nu,\mu)}^{(\gamma_1^{*1})_1-\eta_1} \cdots \|\bar a_n\|_{1,(\nu,\mu)}^{(\gamma_1^{*1})_n-\eta_n} r^{|\eta|}}$$
$$+ \cdots + |(\gamma_1)_n| \sum_{\eta \le \gamma_1^{*n}} \binom{(\gamma_1^{*n})_1}{\eta_1} \cdots \binom{(\gamma_1^{*n})_n}{\eta_n} \overbrace{\|\bar a_1\|_{1,(\nu,\mu)}^{(\gamma_1^{*n})_1-\eta_1} \cdots \|\bar a_n\|_{1,(\nu,\mu)}^{(\gamma_1^{*n})_n-\eta_n} r^{|\eta|}} \Bigg) \quad .$$

Finally, the result follows from

$$\|D_u^1 g(\bar a + u, c)h\|_{1,(\nu,\mu),N_2}^{(n)} \le \max\Big\{\Upsilon_1(u,h) + \Upsilon_2(u,h)\Big\} = Z_2^{g^1}(r) \quad .$$

□

The first lemma being stated and proved, let us move on to the second one.

Lemma 5.3.4. *Recall* (5.18) *and Definitions 16, 21 and 22. Let* $r > 0$*. Suppose* $\|u\|_{1,(\nu,\mu),N_2}^{(n)} \le r$ *and* $\|h\|_{1,(\nu,\mu),N_2}^{(n)} = 1$*. Let* $h_* \in \left(\ell_{\nu,\mu}^{1,N_2}\right)^n$ *be defined component-wise by*

$$(h_*)_{\alpha,\beta} \overset{\text{def}}{=} \begin{cases} 0 & , \text{ if } |\alpha| \le N_2 - 1 \\ \frac{1}{|\alpha|}\cdot h_{\alpha,\beta} & , \text{ if } |\alpha| \ge N_2 \end{cases} \quad .$$

Then, we have

$$\|D_u^2 g(\bar a + u, \bar\lambda, c)h_*\|_{1,(\nu,\mu),N_2}^{(n)} \le Z_2^{g^2}(r) \quad ,$$

where

$$Z_2^{g^2}(r) \overset{\text{def}}{=} 2\cdot \left(\widehat{\|\bar\lambda_1\|_{1,\mu}} + \cdots + \widehat{\|\bar\lambda_k\|_{1,\mu}} \right) \quad . \tag{5.21}$$

Proof. Let $\Upsilon(u,h) = \sum_{|\beta|\ge 0} \sum_{|\alpha|\ge N_2} \left| \left(D_u^2 g(\bar a + u, \bar\lambda, c)h_*\right)_{\alpha,\beta} \right| \nu^\alpha \mu^\beta$. We have

$$\begin{aligned}
\Upsilon(u,h) &= \sum_{|\beta|\ge 0} \sum_{|\alpha|\ge N_2} \left| \Big((\alpha\cdot\bar\lambda) \hat{*} (h_*)_\alpha \Big)_\beta + (1-\delta_{\beta,0})\cdot(\alpha\cdot\bar\lambda)_\beta\cdot(h_*)_{\alpha,0} \right| \nu^\alpha\mu^\beta \\
&= \sum_{|\beta|\ge 0} \sum_{|\alpha|\ge N_2} \left| \left((\alpha\cdot\bar\lambda) \hat{*} \left(\frac{1}{|\alpha|} h_\alpha\right) \right)_\beta + (1-\delta_{\beta,0})\cdot(\alpha\cdot\bar\lambda)_\beta\cdot\left(\frac{1}{|\alpha|} h_{\alpha,0}\right) \right| \nu^\alpha\mu^\beta \\
&\le \sum_{|\beta|\ge 0} \sum_{|\alpha|\ge N_2} \left[\left((\alpha\cdot|\bar\lambda|) \hat{*} \left(\frac{1}{|\alpha|} |h_\alpha|\right) \right)_\beta + (1-\delta_{\beta,0})\cdot(\alpha\cdot|\bar\lambda|)_\beta\cdot\left(\frac{1}{|\alpha|} |h_{\alpha,0}|\right) \right] \nu^\alpha\mu^\beta \\
&\le \sum_{|\beta|\ge 0} \sum_{|\alpha|\ge N_2} \sum_{i=1}^{k} \left[\Big(|\bar\lambda_i| \hat{*} |h_\alpha| \Big)_\beta + (1-\delta_{\beta,0})\cdot(|\bar\lambda_i|)_\beta\cdot|h_{\alpha,0}| \right] \nu^\alpha\mu^\beta
\end{aligned}$$

Let $\bar{\lambda}_i^{(\nu,\mu)}$ be defined component-wise by

$$\left(\bar{\lambda}_i^{(\nu,\mu)}\right)_{\alpha,\beta} \stackrel{\text{def}}{=} \begin{cases} \left(\bar{\lambda}_i\right)_\beta & , \text{ if } |\alpha| = 0 \\ 0 & , \text{ if } |\alpha| \neq 0 \end{cases} \qquad (i \in \{1, \ldots, k\}) \quad .$$

Notice $\bar{\lambda}_i^{(\nu,\mu)} \in \left(\ell^{1,N_2}_{\nu,\mu}\right)^n$ for all $i \in \{1, \ldots, k\}$. Hence,

$$\Upsilon(u,h) \leq \sum_{|\beta|\geq 0} \sum_{|\alpha|\geq N_2} \sum_{i=1}^{k} \left[\left(|\bar{\lambda}_i^{(\nu,\mu)}| \widehat{*} |h| \right)_{\alpha,\beta} + (1 - \delta_{\beta,0}) \cdot \left(|\bar{\lambda}_i| \right)_\beta \cdot \left(|h| \right)_{\alpha,0} \right] \nu^\alpha \mu^\beta \quad .$$

Let $\mathbf{1} \stackrel{\text{def}}{=} (1, \ldots, 1) \in \mathbb{R}^n$. Therefore, since $\|h\|^{(n)}_{1,(\nu,\mu),N_2} = 1$ and $\|\bar{\lambda}_i^{(\nu,\mu)}\|_{1,(\nu,\mu)} = \|\bar{\lambda}_i\|_{1,\mu}$ ($\forall i \in \{1, \ldots, k\}$) – one can verify the latter is indeed true –, we get

$$\begin{aligned} \Upsilon(u,h) &\leq \sum_{i=1}^{k} \mathbf{1} \cdot \left[\widehat{\|\bar{\lambda}_i\|_{1,\mu}} + \widehat{\|\bar{\lambda}_i\|_{1,\mu}} \right] \\ &= 2 \sum_{i=1}^{k} \mathbf{1} \cdot \widehat{\|\bar{\lambda}_i\|_{1,\mu}} \quad . \end{aligned}$$

Finally, the result follows from

$$\|D_u^2 g(\bar{a} + u, \bar{\lambda}, c) h_*\|^{(n)}_{1,(\nu,\mu),N_2} \leq \max\left\{ \Upsilon(u,h) \right\} \leq Z_2^{g^2}(r) \quad .$$

□

We can now apply the Radii Polynomials Approach (see Subsection 3.4) one last time to see what are the conditions needed for the operator $T_2 : \left(\ell^{1,N_2}_{\nu,\mu}\right)^n \to \left(\ell^{1,N_2}_{\nu,\mu}\right)^n$ given by (4.14) to have a fixed point close to $u = 0$.

Theorem 5.3.3. *Recall $T_2 : \left(\ell^{1,N_2}_{\nu,\mu}\right)^n \to \left(\ell^{1,N_2}_{\nu,\mu}\right)^n$, the operator given by* (4.15). *Let*

$$K_1 \stackrel{\text{def}}{=} \frac{10}{9 \cdot N_2 \cdot \min\{|\bar{\lambda}_0|\}} \cdot \|P^{-1}\|_\infty \cdot \|P\|_\infty \quad \& \quad K_2 \stackrel{\text{def}}{=} \frac{10}{9 \cdot \min\{|\bar{\lambda}_0|\}} \cdot \|P^{-1}\|_\infty \cdot \|P\|_\infty \quad .$$

Recall the bounds $(Y_2)_0$, $Z_2^{g^1}(r)$ and $Z_2^{g^2}(r)$ defined in (5.19), (5.20) *and* (5.21) *respectively. Let*

$$\begin{aligned} Z_1 \stackrel{\text{def}}{=} K_1 \cdot \max \Bigg\{ & \sum_{|\gamma_2|=1}^{M_2} \mu^{\gamma_2} \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,\gamma_2}| \left(|(\gamma_1)_1| \cdot \|\bar{a}_1\|_{1,(\nu,\mu)}^{(\gamma_1^{*1})_1} \cdots \|\bar{a}_n\|_{1,(\nu,\mu)}^{(\gamma_1^{*1})_n} + \cdots + |(\gamma_1)_n| \cdot \|\bar{a}_1\|_{1,(\nu,\mu)}^{(\gamma_1^{*n})_1} \cdots \|\bar{a}_n\|_{1,(\nu,\mu)}^{(\gamma_1^{*n})_n} \right) \\ & + \sum_{|\gamma_1|=0}^{M_1} |c_{\gamma_1,0}| \left(|(\gamma_1)_1| \cdot \widehat{\|\bar{a}_1\|_{1,(\nu,\mu)}^{(\gamma_1^{*1})_1} \cdots \|\bar{a}_n\|_{1,(\nu,\mu)}^{(\gamma_1^{*1})_n}} + \cdots + |(\gamma_1)_n| \cdot \widehat{\|\bar{a}_1\|_{1,(\nu,\mu)}^{(\gamma_1^{*n})_1} \cdots \|\bar{a}_n\|_{1,(\nu,\mu)}^{(\gamma_1^{*n})_n}} \right) \Bigg\} \\ & + K_2 \cdot Z_2^{g^2}(r) \end{aligned} \tag{5.22}$$

and

$$Z_2(r) \overset{\text{def}}{=} K_1 \cdot Z_2^{g^1}(r) - Z_1 \quad . \tag{5.23}$$

If there exists $r_0 > 0$ such that

$$Z_2(r_0)r_0^2 + (Z_1 - 1)r_0 + (Y_1)_0 < 0 \quad , \tag{5.24}$$

then T_2 has a unique fixed point in $B(0, r_0)$.

Proof. This is an application of Corollary 3.4.1 using Lemmas 5.3.3 and 5.3.4 and the bounds $(Y_2)_0$, Z_1 and $Z_2(r)$ defined in (5.19), (5.22) and (5.23) respectively.

□

Remark. The reason for the use of the bounds Z_1 (Equation 5.22) and $Z_2(r)$ (Equation 5.23) instead of the bound $Z(r) \overset{\text{def}}{=} K_1 \cdot Z_2^{g^1}(r) + K_2 \cdot Z_2^{g^2}(r)$ (Equations 5.20 and 5.21) is going to be explained in Subsection 5.4.

Assuming the existence of an $r_0 > 0$ satisfying Theorem 5.3.3 – this matter is going to be addressed in Subsection 5.4 –, we have now derived all the tools to rigorously prove an approximation of a parameterization of the stable manifold. Again, the same method applies to the unstable manifold, one just needs to consider the positive eigenvalues instead of the negative ones. Now, let us talk briefly about the complex case. Consider the general case, i.e. the eigenvalues may be complex. Notice all our theorems also holds for complex numbers – this includes our lemmas, corollaries and propositions too. As discussed in Subsection 4.3, we have a way to retrieve a real parameterization of the (un)stable manifold out of the (complex) parameterization we consider. The only difference is that the absolute values $|z|$ of a complex number $z = a + b \cdot i \in \mathbb{C}$ $(a, b \in \mathbb{R})$ are taken to be its usual norm, i.e. $|z| = (a^2 + b^2)^{\frac{1}{2}}$.

The reason for the definition of the operators T_1 and T_2 in Subsection 4.3 is that it allows us to get the bounds (3.2) much easily. Indeed, as one may have already noticed, the fact that $|\alpha| \geq N_2$ for the operator T_2 was the key to prove Lemma 5.3.2. If we had done the work in Subsection 4.3 using only one operator, α would have been such that $|\alpha| \geq 2$ and we would not have been able to prove Lemma 5.3.2. Hence, getting the bounds (3.2) would have been much harder, whence the definition of the operators T_1 and T_2.

We can now move on to Subsection 5.4 where we will discuss the existence of the r_0 from Theorem 3.4.2 for each one of Subsections 5.1, 5.2 and 5.3. This will justify the usefulness of the Parameterization method (see Subsection 2.3) we introduced in this thesis.

5.4 Control of the error via Radii Polynomials

Recall Theorems 5.1.1, 5.2.1, 5.3.1 and 5.3.3. Consider Equations 5.6, 5.12, 5.17 and 5.24. Notice each of their left hand side member is a polynomial in r. Hence, trying to get the existence of an $r_0 > 0$ such that those polynomials are negative – this r_0 need not be the same for each of those polynomials – boils down to solving for the roots of those polynomials. Indeed, a positive root for one of them implies the existence of a desired $r_0 > 0$ for it by continuity – we disregard the case of multiple roots because one only needs to diminish the weight(s) to get a lower Y_0 bound while keeping the previous values for the other bounds. Since all the bounds derived in Subsections 5.1, 5.2 and 5.3 are nonnegative, the only way for those polynomials to have a positive root is to have a negative coefficient for their first order term. This coefficient has the same form for all four polynomials. Without loss of generality, say it is $Z - 1$. Notice that, for all four polynomials, Z does not depend on r. Moreover, each member of Z depends on the weight(s) of the Banach space considered (see Definition 13). Hence, we can make Z as small as we want by taking the weight(s) as small as needed. Therefore, we can ensure Z to be less than 1, i.e. $Z - 1$ to be negative. Finally, we can always ensure the existence of a positive root for each four polynomials by taking the bounds as small as needed – we can always do so because each member of the bounds depends on the weight(s).

Let us talk about our approximations being "good enough". Let S be any of the series we considered. Let $\bar{S}$ be the approximation of S we considered. The weight of the Banach space considered for its proof is less than the radius of convergence of S. Moreover, the distance between S and $\bar{S}$ – the distance is taken using the norm of the Banach space considered – is at most $r_0 > 0$ – it exists by the above discussion. Since Y_0 can be taken very small by diminishing the weight of the Banach space considered, $r_0 > 0$ can be taken as small as we want. A dilemma arises here : A better approximation – this means a lesser r_0 – is taken at the cost of a lesser radius of convergence of S. Nevertheless, we will see in Section 7 that, depending on the purpose of the parameterization of the (un)stable manifold, one can choose the better approximation over the radius of convergence.

Let us now talk about the propagation of the error throughout the computations.

As mentioned above, the distance between S and $\bar{S}$ is r_0. Let s_δ and $\bar{s}_\delta$ be the coefficients at index δ of the power series S and $\bar{S}$ respectively. Let υ be the weight considered for the validation of $\bar{S}$. We have

$$\max_{i\in\{1,\ldots,n\}} |(s_i)_\delta - (\bar{s}_i)_\delta| < \frac{r_0}{\upsilon^\delta} \quad (\forall\delta) \quad .$$

Hence, we know that

$$(s_i)_\delta \in \left((\bar{s}_i)_\delta - \frac{r_0}{\upsilon^\delta}, (\bar{s}_i)_\delta + \frac{r_0}{\upsilon^\delta}\right) \quad (\forall i \in \{1,\ldots,n\}, \forall\delta) \quad .$$

Therefore, we can use this interval in our computations to keep track of the error. The computations can then be done using the Interval arithmetic (see [21]) for the sake of completeness. This ensures the results include the errors on all the approximations. A direct consequence of the Interval arithmetic is that the r_0 from Theorem 3.4.2 may be bigger than intended – this is still acceptable – or negative – this would require a better job at getting the bounds (3.2), namely diminishing the weight(s). Nonetheless, as shown in this section, we can guarantee the existence of r_0 for Theorems 5.1.1, 5.2.1, 5.3.1 and 5.3.3, so it is not mandatory to use the Interval arithmetic.

As discussed above, given Assumptions **A1**, **A2**, **A3** and **A4**, our method allows to retrieve rigorously – this is because we can always guarantee the existence of r_0 – and efficiently – this is because the recurrence relations only involve basic operations – a parameterization of the (un)stable manifold. We are now ready to go over two examples to see what the method looks like in practice.

6 Applications

We are going to go over two examples (Subsections 6.1 and 6.2) to see how the computations look like in practice. In sections 4 and 5, we considered Assumption **A2**, i.e. the vector field is polynomial. This implied considering all the terms of the polynomial vector field up to its order – the order was M_1 in space and M_2 in parameters in our settings (see Equation 4.1). Nonetheless, in practice, one often has only a couple of terms. This makes the computations much easier in the sense that there are fewer of them.

For Subsections 6.1 and 6.2, we are only going to state the operators and their derivative, as well as the bounds (3.2) used for the theorems we derived in Section 5 to validate our computations. We refer the reader to Sections 4 and 5 for the proofs.

6.1 Lorenz system

Let us start with a famous example : The Lorenz system (see [20]). In the Lorenz system (LS), the vector field $f : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$f(x, y, z) \stackrel{\text{def}}{=} \begin{pmatrix} \sigma(y - x) \\ (\rho - z)x - y \\ xy - \beta z \end{pmatrix}$$

where $\sigma, \rho, \beta \in \mathbb{R}$ are parameters that will be fixed. Let us introduce another parameter in it, say $\omega \in \mathbb{R}$. We will consider the new vector field $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ given by

$$f(x, \omega) \stackrel{\text{def}}{=} \begin{pmatrix} \sigma_p(x_2 - x_1) \\ (\rho_p + \omega - x_3)x_1 - x_2 \\ x_1 x_2 - \beta_p x_3 \end{pmatrix} \tag{6.1}$$

where $(\sigma_p, \rho_p, \beta_p) = (10, 28, 8/3)$ – we use the notation $(\sigma_p, \rho_p, \beta_p)$ instead of (σ, ρ, β) to avoid confusion between the parameter β and the index β. One can verify that $x_0 \stackrel{\text{def}}{=} (0, 0, 0)$ is a fixed point of (6.1) for $\omega = 0$. Recall Section 4. We need to compute the fixed point along with its associated eigenvalues and eigenvectors with respect to the parameter ω in order to get to the computation of the parameterization of the (un)stable manifold at each of them. We are going to consider the value $\tilde{\omega} = 0$ of ω. Assumptions **A1** and **A3** will be verified for $\tilde{\omega} = 0$ and the aforementioned fixed point, as well as Assumption **A4**. Notice that Assumption **A2** is verified because the vector field defined in (6.1) is polynomial.

Let us move on to the computation of a parameterization of the (un)stable manifold in a neighbourhood of the fixed point x_0 and $\omega = \tilde{\omega}$.

6.1.1 Fixed point (LS)

We start with the computation of the fixed point. Recall the recurrence relation onto the coefficients of $\tilde{x}(\omega)$ is given by Equation 4.4. Thereby, for $\beta \geq 1$, we have

$$\underbrace{\begin{pmatrix} -\sigma_p & \sigma_p & 0 \\ \rho_p - (b_3)_0 & -1 & -(b_1)_0 \\ (b_2)_0 & (b_1)_0 & -\beta_p \end{pmatrix}}_{B} b_\beta = \begin{pmatrix} 0 \\ -(b_1)_{\beta-1} + (b_1 \hat{*} b_3)_\beta \\ -(b_1 \hat{*} b_2)_\beta \end{pmatrix} .$$

Since $D_x f(\tilde{x}(\tilde{\omega}), \tilde{\omega}) = B$ has no purely imaginary eigenvalues for $\tilde{x}(\tilde{\omega}) = x_0$ – one can verify the latter is indeed true –, $\tilde{x}(\tilde{\omega})$ is an hyperbolic fixed point, so Assumption **A1** is verified.

Recall Theorem 5.1.1. Using the bounds Y_0, Z_1 and $Z_2(r)$, respectively defined in (5.2), (5.4) and (5.5), we numerically get the results of Table 1. Hence, using the

	x_0
	$\mu = 0.9$ & $N = 12$
Y_0	0
Z_1	0.3375
$Z_2(r)$	$0.75r$
r_0	$[0, 0.88\bar{3}]$

Table 1 – Lorenz Fixed Point Results

data from Table 1, Theorem 5.1.1 validates our approximation $\bar{b}$ of the coefficients of the power series of $\tilde{x}(\omega)$.

Let us now move on to the computation of the eigenvalues and their associated eigenvector.

6.1.2 Eigenvalues and eigenvectors (LS)

We follow the computation of the fixed point by the computations of the eigenvalues and their associated eigenvector. One can verify that the eigenvalues $\lambda_0^1, \lambda_0^2, \lambda_0^3$ of $D_x f(\tilde{x}(\tilde{\omega}), \tilde{\omega})$ are all distinct for $\tilde{x}(\tilde{\omega}) = x_0$, whence Assumption **A3** is verified.

Recall the recurrence relation onto the coefficients of $\lambda(\omega)$ and $v(\omega)$ is given by Equation 4.6 – Equation 4.8 is the recurrence relation used once we have fixed one of the components of $v(\omega)$ for all ω in a neighbourhood of $\tilde{\omega}$. Thereby, for $\beta \geq 1$, we have

$$\begin{pmatrix} -\sigma_p - \lambda_0 & \sigma_p & 0 \\ -\rho_p - (\bar{b}_3)_0 & -1 - \lambda_0 & -(\bar{b}_1)_0 \\ -(\bar{b}_2)_0 & (\bar{b}_1)_0 & -\beta_p - \lambda_0 \end{pmatrix} \begin{pmatrix} (v_1)_\beta \\ (v_2)_\beta \\ (v_3)_\beta \end{pmatrix} = \begin{pmatrix} (v_1 \hat{*} \lambda)_\beta + \lambda_\beta (v_1)_0 \\ -(v_1)_{\beta-1} + (\bar{b}_3 \hat{*} v_1)_\beta + (\bar{b}_3)_\beta (v_1)_0 + (\bar{b}_1 \hat{*} v_3)_\beta \\ +(\bar{b}_1)_\beta (v_3)_0 + (\lambda \hat{*} v_2)_\beta + \lambda_\beta (v_2)_0 \\ -(\bar{b}_2 \hat{*} v_1)_\beta - (\bar{b}_2)_\beta (v_1)_0 - (\bar{b}_1 \hat{*} v_2)_\beta - (\bar{b}_1)_\beta (v_2)_0 \\ +(\lambda \hat{*} v_3)_\beta + \lambda_\beta (v_3)_0 \end{pmatrix}.$$

As mentioned in Section 4, we have to fix one of the components of $v(\omega)$ to use Equation 4.8. We need to do so for every eigenvalue of the fixed point considered.

Recall Theorem 5.2.1. Using the bounds Y_0, Z_1 and $Z_2(r)$, respectively defined in (5.8), (5.10) and (5.11), we numerically get the results of Table 2. Hence, using

	λ_0^1		λ_0^2		λ_0^3	
x_0		$\mu = 0.1$ & $N = 12$		$\mu = 0.1$ & $N = 12$		$\mu = 0.1$ & $N = 12$
	Y_0	$1.471241 \cdot 10^{-29}$	Y_0	$7.519219 \cdot 10^{-31}$	Y_0	0
	Z_1	0.1493786	Z_1	0.1581509	Z_1	0.1
	$Z_2(r)$	$2.987572r$	$Z_2(r)$	$3.163019r$	$Z_2(r)$	$2r$
	r_0	$[1.729608 \cdot 10^{-29}, 0.28472]$	r_0	$[8.931790 \cdot 10^{-31}, 0.2661537]$	r_0	$[0, 0.45]$

Table 2 – Lorenz Eigenvalues And Eigenvectors Results

the data from Table 2, Theorem 5.2.1 validates our approximations $\bar{\lambda}$ and $\bar{v}$ of the coefficients of the power series of $\lambda(\omega)$ and $v(\omega)$ respectively – it does not matter which eigenvalues of $D_x f(\tilde{x}(\tilde{\omega}), \tilde{\omega})$ are λ_0^1, λ_0^2 and λ_0^3 because we can pick any r_0 that verifies Equation 5.12 for all of them. Note that we need to repeat this procedure for every eigenvalue of $D_x f(\tilde{x}(\omega), \omega)$ and their associated eigenvector.

Let us now move on to the computation of the parameterization of the (un)stable manifold.

6.1.3 Stable and unstable manifolds coefficients (LS)

We follow the computations of the eigenvalues and their associated eigenvector by the computation of a parameterization of the (un)stable manifold. Recall Definition 18. One can verify that $D_x f(\tilde{x}(\tilde{\omega}), \tilde{\omega})$ has no resonance for $\tilde{x}(\tilde{\omega}) = x_0$, whence Assumption **A4** is verified.

Recall the recurrence relation onto the coefficients of $P(\theta, \omega)$ is given by Equation 4.13. Thereby, for $|\alpha| \geq 2$, we have

$$a_{\alpha,\beta} = B_\alpha^{-1} \left(\begin{pmatrix} 0 \\ -(a_1)_{\alpha,\beta-1} + (a_1 \hat{*} a_3)_{\alpha,\beta} \\ -(a_1 \hat{*} a_2)_{\alpha,\beta} \end{pmatrix} + \left((\alpha \cdot \lambda) \hat{*} a_\alpha \right)_\beta + (1 - \delta_{\beta,0}) \cdot (\alpha \cdot \lambda)_\beta a_{\alpha,0} \right)$$

where

$$B_\alpha = \begin{pmatrix} -\sigma_p - (\alpha \cdot \bar{\lambda})_0 & \sigma_p & 0 \\ \rho_p - (a_3)_{0,0} & -1 - (\alpha \cdot \bar{\lambda})_0 & -(a_1)_{0,0} \\ (a_2)_{0,0} & (a_1)_{0,0} & -\beta_p - (\alpha \cdot \bar{\lambda})_0 \end{pmatrix} .$$

Recall Theorem 5.3.1. Let $P^s = P^s(\theta, \omega)$ and $P^u = P^u(\theta, \omega)$ respectively be the parameterizations of the stable and unstable manifolds at $\tilde{x}(\omega)$. Using the bounds $(Y_1)_0$ and $Z^1(r)$, respectively defined in (5.14) and (5.16), we numerically get the results of Table 3. For the sake of simplicity, Table 3 only displays the results of the α for which the interval containing r_0 has the highest lower bound. Notice this interval is contained in the ones from the other values of α up to order $N_2 - 1$ (see Table 4). Hence, using the data from Table 3, Theorem 5.3.1 validates our approximation $\bar{a}_\alpha$ of the coefficients of the power series of $P(\theta, \omega)$ for α fixed up to order $N_2 - 1$ – the notation $P(\theta, \omega)$ is used for both P^s and P^u.

	P^s		P^u	
x_0		$\mu = 0.01$ & $N_1 = 12$		$\mu 0.1 =$ & $N_1 = 12$
	α	[1 1]	α	2
	$(Y_1)_0$	$3.162076 \cdot 10^{-43}$	$(Y_1)_0$	$2.220395 \cdot 10^{-33}$
	Z^1	0.005632	Z^1	0.017691
	r_0	$[3.179983 \cdot 10^{-43}, \infty)$	r_0	$[2.260385 \cdot 10^{-33}, \infty)$

Table 3 – Lorenz Stable And Unstable Manifolds Results for α fixed

Finally, recall Theorem 5.3.3. Using the bounds $(Y_2)_0$, Z_1 and $Z_2(r)$, respectively defined in (5.19), (5.22) and (5.23), we numerically get the results of Table 4. Hence,

	P^s		P^u	
x_0		$\nu = [0.1\ 0.1]$ & $\mu = 0.01$ & $N_2 = 8$		$\nu = 1$ & $\mu = 0.1$ & $N_2 = 11$
	$(Y_2)_0$	$7.315016 \cdot 10^{-16}$	$(Y_2)_0$	$3.048495 \cdot 10^{-17}$
	Z_1	0.353093	Z_1	0.040743
	$Z_2(r)$	$0.319654r$	$Z_2(r)$	$0.050449r$
	r_0	$[1.130768 \cdot 10^{-15}, 2.023774]$	r_0	$[3.177974 \cdot 10^{-17}, 19.0147]$

Table 4 – Lorenz Stable And Unstable Manifolds Results

using the data from Table 4, Theorem 5.3.3 validates our approximation $\bar{a}$ of the coefficients of the power series of $P(\theta, \omega)$ – the notation $P(\theta, \omega)$ is again used for both P^s and P^u.

Figures 9 and 11 respectively show our approximations of the stable and unstable manifolds at $\tilde{x}(\omega) = x_0$. Figures 10 and 12 respectively show a globalization of the stable and unstable manifolds at $\tilde{x}(\omega) = x_0$ we computed. The globalizations are done using the *ode45* function in MATLAB. The red dot is the origin. The Lorenz attractor (see [20]) was highlighted in green in Figure 10.

Let us now move on to an example in higher dimensions.

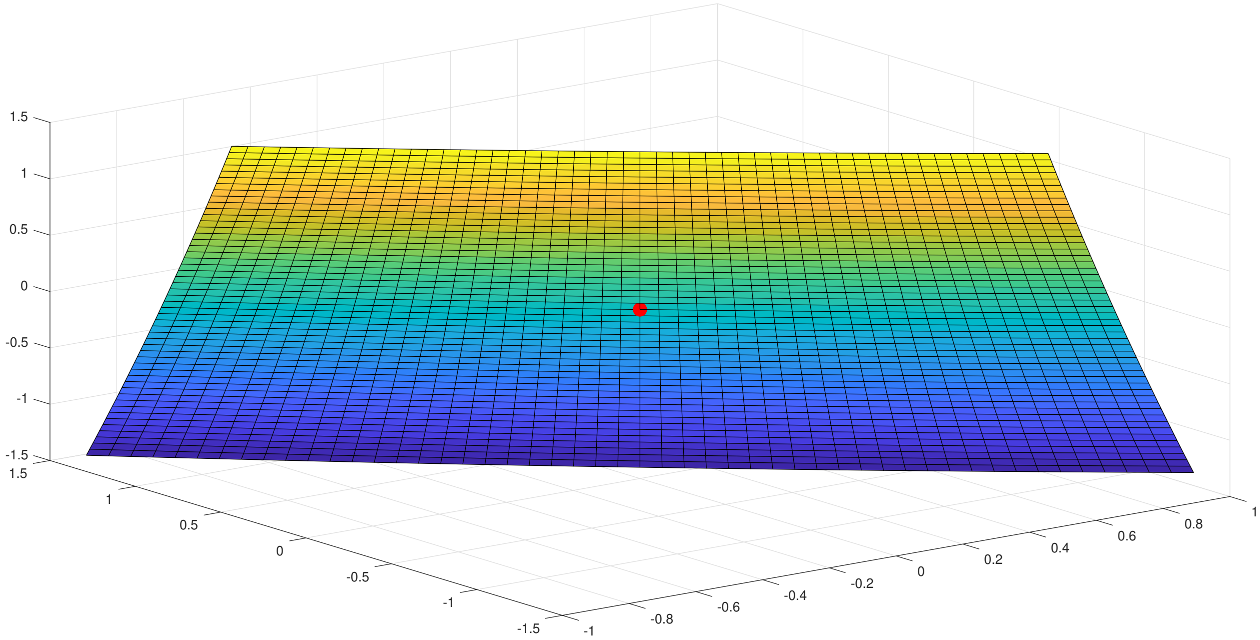

Figure 9 – Lorenz local stable manifold at 0

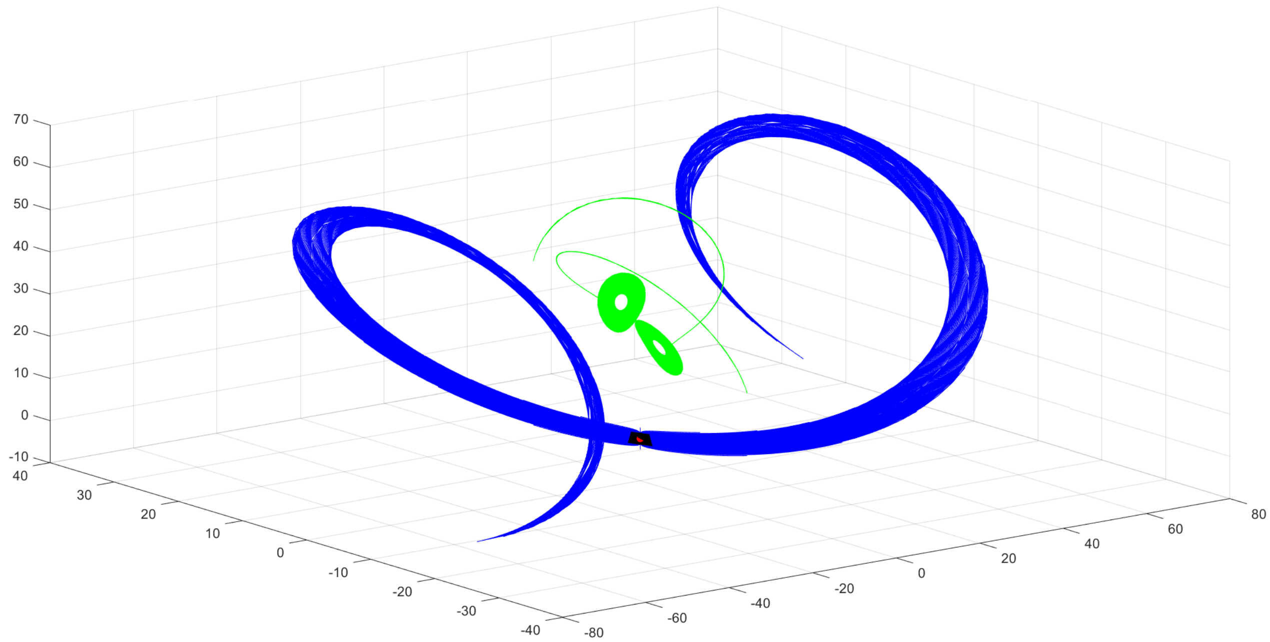

Figure 10 – Lorenz global stable manifold at 0

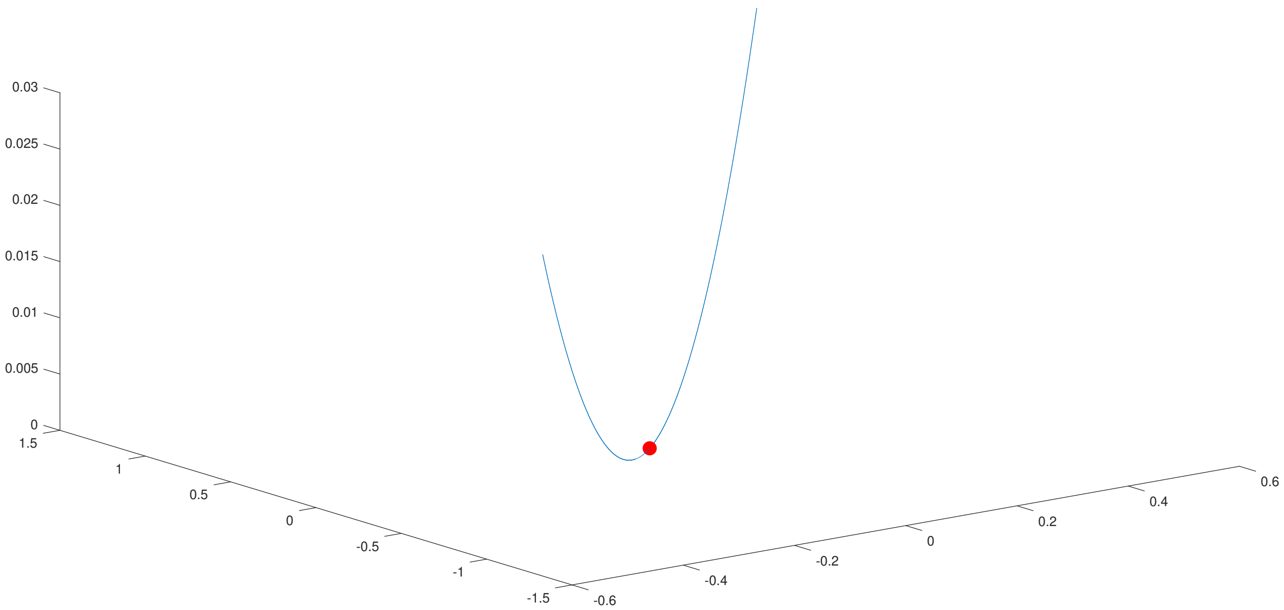

Figure 11 – Lorenz local unstable manifold at 0

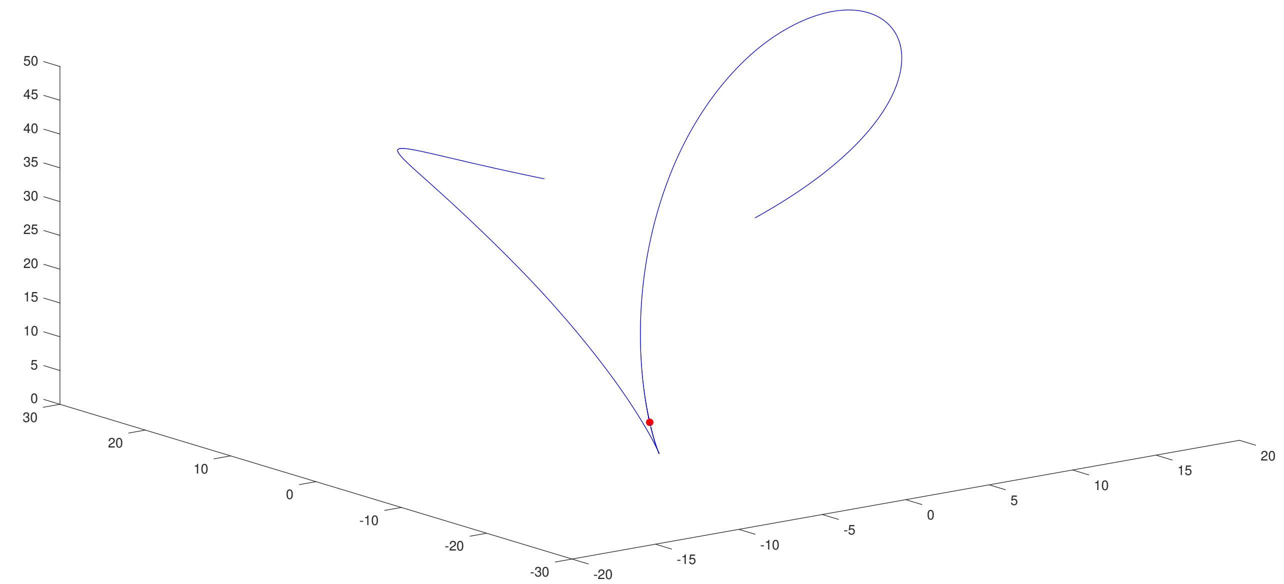

Figure 12 – Lorenz global unstable manifold at 0

6.2 Rolls and Hexagons system

Let us continue with another example : The Rolls and Hexagons system (RH) from [3]. In this system, the vector field $\Psi : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$ is given by

$$\Psi(U, \tilde{c}) = \begin{pmatrix} U_2 \\ -\frac{1}{4}\frac{\tilde{c}}{\tilde{\beta}}U_2 - \frac{\gamma}{4}U_1 - \frac{\sqrt{2}}{4}U_3^2 + \frac{3}{8}U_1^3 + 3U_1U_3^2 \\ U_4 \\ -\frac{\tilde{c}}{\tilde{\beta}}U_4 - \gamma U_3 - \frac{\sqrt{2}}{2}U_1U_3 + 9U_3^3 + 3U_1^2U_3 \end{pmatrix}$$

where $\gamma, \tilde{\beta} \in \mathbb{R}$ are parameters that will be fixed. Let us rewrite it using our notation, i.e. consider the vector field $f : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$ given by

$$f(x, \omega) \stackrel{\text{def}}{=} \begin{pmatrix} x_2 \\ -\frac{1}{4}\frac{\omega}{\beta_p}x_2 - \frac{\gamma_p}{4}x_1 - \frac{\sqrt{2}}{4}x_3^2 + \frac{3}{8}x_1^3 + 3x_1x_3^2 \\ x_4 \\ -\frac{\omega}{\beta_p}x_4 - \gamma_p x_3 - \frac{\sqrt{2}}{2}x_1x_3 + 9x_3^3 + 3x_1^2x_3 \end{pmatrix} \tag{6.2}$$

where $(\gamma_p, \beta_p) = \big((7 + 3\sqrt{6})/30, 1\big)$ – we use the notation (γ_p, β_p) instead of $(\gamma, \tilde{\beta})$ to avoid confusion between the parameter $\tilde{\beta}$ and the index β as well as between the parameter γ and the index γ. One can verify that $x_0 \stackrel{\text{def}}{=} (-\sqrt{2\gamma_p/3}, 0, 0, 0)$ is a fixed point of (6.2) for $\omega = 0$. Recall Section 4. We need to compute this fixed point along with its associated eigenvalues and eigenvectors with respect to the parameter ω in order to get to the computation of the parameterization of the (un)stable manifold at it. We are going to consider the value $\tilde{\omega} = 0$ of ω. Assumptions **A1** and **A3** will be verified for $\tilde{\omega} = 0$ and the aforementioned fixed point, as well as Assumption **A4**. Notice that Assumption **A2** is verified because the vector field defined in (6.2) is polynomial.

Let us move on to the computation of a parameterization of the (un)stable manifold in a neighbourhood of x_0 and $\omega = \tilde{\omega}$.

6.2.1 Fixed point (RH)

We start with the computation of the fixed point. Recall the recurrence relation onto the coefficients of $\tilde{x}(\omega)$ is given by Equation 4.4. Thereby, for $\beta \geq 1$, we have

$$b_\beta = B^{-1}\begin{pmatrix} 0 \\ \frac{1}{4\beta_p}(b_2)_{\beta-1} + \frac{\sqrt{2}}{4}(b_3 \widehat{*} b_3)_\beta - \frac{3}{8}(\widehat{b_1 * b_1 * b_1})_\beta - 3(\widehat{b_1 * b_3 * b_3})_\beta \\ 0 \\ \frac{1}{\beta_p}(b_4)_{\beta-1} + \frac{\sqrt{2}}{2}(b_1 \widehat{*} b_3)_\beta - 9(\widehat{b_3 * b_3 * b_3})_\beta - 3(\widehat{b_1 * b_1 * b_3})_\beta \end{pmatrix},$$

where

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{\gamma_p}{4} + \frac{9}{8}(b_1 * b_1)_0 + 3(b_3 * b_3)_0 & 0 & -\frac{\sqrt{2}}{2}(b_3)_0 + 6(b_1 * b_3)_0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{\sqrt{2}}{2}(b_3)_0 + 6(b_1 * b_3)_0 & 0 & -\gamma_p - \frac{\sqrt{2}}{2}(b_1)_0 + 27(b_3 * b_3)_0 + 3(b_1 * b_1)_0 & 0 \end{pmatrix}.$$

Since $D_x f(\tilde{x}(\tilde{\omega}), \tilde{\omega}) = B$ has no purely imaginary eigenvalues for $\tilde{x}(\tilde{\omega}) = x_0$ – one can verify the latter is indeed true –, $\tilde{x}(\tilde{\omega})$ is an hyperbolic fixed point, so Assumption **A1** is verified.

Recall Theorem 5.1.1. Using the bounds Y_0, Z_1 and $Z_2(r)$, respectively defined in (5.2), (5.4) and (5.5), we numerically get the results of Table 5. Hence, using the data from Table 5, Theorem 5.1.1 validates our approximation $\bar{b}$ of the coefficients of the power series of $\tilde{x}(\omega)$.

Let us now move on to the computation of the eigenvalues and their associated eigenvector.

6.2.2 Eigenvalues and eigenvectors (RH)

We follow the computation of the fixed point by the computations of the eigenvalues and their associated eigenvector. One can verify that the eigenvalues

x_0	
	$\mu = 0.05$ & $N = 12$
Y_0	0
Z_1	0.209082
$Z_2(r)$	$5.913719r + 150.538707r^2$
r_0	$[0, 0.055456]$

Table 5 – Rolls And Hexagons Fixed Point Results

$\lambda_0^1, \lambda_0^2, \lambda_0^3, \lambda_0^4$ of $D_x f(\tilde{x}(\tilde{\omega}), \tilde{\omega})$ are all distinct for $\tilde{x}(\tilde{\omega}) = x_0$, whence Assumption **A3** is verified.

Recall the recurrence relation onto the coefficients of $\lambda(\omega)$ and $v(\omega)$ is given by Equation 4.6 – Equation 4.8 is the recurrence relation used once we have fixed one of the components of $v(\omega)$ for all ω in a neighbourhood of $\tilde{\omega}$. Thereby, for $\beta \geq 1$, we have

$$\begin{pmatrix} (v_1)_\beta \\ (v_2)_\beta \\ (v_3)_\beta \\ (v_4)_\beta \end{pmatrix} = B^{-1} \begin{pmatrix} (\lambda \hat{*} v_1)_\beta + \lambda_\beta (v_1)_0 \\ \\ -\frac{9}{8}\widehat{(b_1 * b_1 * v_1)}_\beta - \frac{9}{4}(b_1)_\beta (b_1)_0 (v_1)_0 - 3\widehat{(b_3 * b_3 * v_1)}_\beta - 6(b_3)_\beta (b_3)_0 (v_3)_0 + \frac{1}{4\beta_p}(v_2)_{\beta-1} \\ +\frac{\sqrt{2}}{2}(b_3 \hat{*} v_3)_\beta + \frac{\sqrt{2}}{2}(b_3)_\beta (v_3)_0 - 6\widehat{(b_1 * b_3 * v_3)}_\beta - 6(b_1)_\beta (b_3)_0 (v_3)_0 - 6(b_1)_0 (b_3)_\beta (v_3)_0 \\ +(\lambda \hat{*} v_2)_\beta + \lambda_\beta (v_2)_0 \\ \\ (\lambda \hat{*} v_3)_\beta + \lambda_\beta (v_3)_0 \\ \\ \frac{\sqrt{2}}{2}(b_3 \hat{*} v_1)_\beta + \frac{\sqrt{2}}{2}(b_3)_\beta (v_1)_0 - 6\widehat{(b_1 * b_3 * v_1)}_\beta - 6(b_1)_\beta (b_3)_0 (v_1)_0 - 6(b_1)_0 (b_3)_\beta (v_1)_0 \\ +\frac{\sqrt{2}}{2}(b_1 \hat{*} v_3)_\beta + \frac{\sqrt{2}}{2}(b_1)_\beta (v_3)_0 - 27\widehat{(b_3 * b_3 * v_3)}_\beta - 54(b_3)_\beta (b_3)_0 (v_3)_0 - 3\widehat{(b_1 * b_1 * v_3)}_\beta \\ -6(b_1)_\beta (b_1)_0 (v_3)_0 + \frac{1}{\beta_p}(v_4)_{\beta-1} + (\lambda \hat{*} v_4)_\beta + \lambda_\beta (v_4)_0 \end{pmatrix}$$

where

$$
B = \begin{pmatrix} -\lambda_0 & 1 & 0 & 0 \\ -\frac{\gamma_p}{4} + \frac{9}{8}(b_1 * b_1)_0 + 3(b_3 * b_3)_0 & -\lambda_0 & -\frac{\sqrt{2}}{2}(b_3)_0 + 6(b_1 * b_3)_0 & 0 \\ 0 & 0 & -\lambda_0 & 1 \\ -\frac{\sqrt{2}}{2}(b_3)_0 + 6(b_1 * b_3)_0 & 0 & -\gamma_p - \frac{\sqrt{2}}{2}(b_1)_0 + 27(b_3 * b_3)_0 + 3(b_1 * b_1)_0 & -\lambda_0 \end{pmatrix}.
$$

As mentioned in Section 4, we have to fix one of the components of $v(\omega)$ to use Equation 4.8. We need to do so for every eigenvalue of the fixed point considered.

Recall Theorem 5.2.1. Using the bounds Y_0, Z_1 and $Z_2(r)$, respectively defined in (5.8), (5.10) and (5.11), we numerically get the results of Table 6. Hence, using

x_0					
λ_0^1		$\mu = 0.1$ & $N = 12$	λ_0^2		$\mu = 0.1$ & $N = 12$
	Y_0	$1.027719 \cdot 10^{-17}$		Y_0	$1.027719 \cdot 10^{-17}$
	Z_1	0.572760		Z_1	0.572760
	$Z_2(r)$	6.067386		$Z_2(r)$	6.067386
	r_0	$[2.405486 \cdot 10^{-17}, 0.070416]$		r_0	$[2.405486 \cdot 10^{-17}, 0.070416]$
λ_0^3		$\mu = 0.1$ & $N = 12$	λ_0^4		$\mu = 0.1$ & $N = 12$
	Y_0	$7.804082 \cdot 10^{-22}$		Y_0	$7.804082 \cdot 10^{-22}$
	Z_1	0.26276		Z_1	0.26276
	$Z_2(r)$	4.66467		$Z_2(r)$	4.66467
	r_0	$[1.058554 \cdot 10^{-21}, 0.158048]$		r_0	$[1.058554 \cdot 10^{-21}, 0.158048]$

Table 6 – Lorenz Eigenvalues And Eigenvectors Results

the data from Table 6, Theorem 5.2.1 validates our approximations $\bar{\lambda}$ and $\bar{v}$ of the coefficients of the power series of $\lambda(\omega)$ and $v(\omega)$ respectively – it does not matter which eigenvalues of $D_x f(\tilde{x}(\tilde{\omega}), \tilde{\omega})$ are λ_0^1, λ_0^2, λ_0^3 and λ_0^4 because we can pick any r_0 that verifies Equation 5.12 for all of them. Note that we need to repeat this procedure for every eigenvalue of $D_x f(\tilde{x}(\omega), \omega)$ and their associated eigenvector.

Let us now move on to the computation of the parameterization of the (un)stable manifold.

6.2.3 Stable and unstable manifolds coefficients (RH)

We follow the computations of the eigenvalues and their associated eigenvector by the computation of a parameterization of the (un)stable manifold. Recall Definition 18. One can verify that $D_x f(\tilde{x}(\tilde{\omega}),\tilde{\omega})$ has no resonance for $\tilde{x}(\tilde{\omega}) = x_0$, whence Assumption **A4** is verified.

Recall the recurrence relation onto the coefficients of $P(\theta,\omega)$ is given by Equation 4.13. Thereby, for $|\alpha| \geq 2$, we have

$$a_{\alpha,\beta} = B_\alpha^{-1} \begin{pmatrix} \left((\alpha\cdot\lambda)\hat{*}(a_1)_\alpha\right)_\beta + (1-\delta_{\beta,0})\cdot(\alpha\cdot\lambda)_\beta (a_1)_{\alpha,0} \\ \frac{1}{4\beta_p}(a_2)_{\alpha,\beta-1} + \frac{\sqrt{2}}{4}(a_3\hat{*}a_3)_{\alpha,\beta} - \frac{3}{8}(\widehat{a_1*a_1*a_1})_{\alpha,\beta} - 3(\widehat{a_1*a_3*a_3})_{\alpha,\beta} \\ + \left((\alpha\cdot\lambda)\hat{*}(a_2)_\alpha\right)_\beta + (1-\delta_{\beta,0})\cdot(\alpha\cdot\lambda)_\beta (a_2)_{\alpha,0} \\ \left((\alpha\cdot\lambda)\hat{*}(a_3)_\alpha\right)_\beta + (1-\delta_{\beta,0})\cdot(\alpha\cdot\lambda)_\beta (a_3)_{\alpha,0} \\ \frac{1}{\beta_p}(a_4)_{\alpha,\beta-1} + \frac{\sqrt{2}}{2}(a_1\hat{*}a_3)_{\alpha,\beta} - 9(\widehat{a_3*a_3*a_3})_{\alpha,\beta} - 3(\widehat{a_1*a_1*a_3})_{\alpha,\beta} \\ + \left((\alpha\cdot\lambda)\hat{*}(a_4)_\alpha\right)_\beta + (1-\delta_{\beta,0})\cdot(\alpha\cdot\lambda)_\beta (a_4)_{\alpha,0} \end{pmatrix}$$

where

$$B_\alpha = \begin{pmatrix} -(\alpha\cdot\lambda)_0 & 1 & 0 & 0 \\ -\frac{\gamma_p}{4} + \frac{9}{8}(a_1)_{0,0}^2 + 3(a_3)_{0,0}^2 & -(\alpha\cdot\lambda)_0 & -\frac{\sqrt{2}}{2}(a_3)_{0,0} + 6(a_1)_{0,0}(a_3)_{0,0} & 0 \\ 0 & 0 & -(\alpha\cdot\lambda)_0 & 1 \\ -\frac{\sqrt{2}}{2}(a_3)_{0,0} + 6(a_1)_{0,0}(a_3)_{0,0} & 0 & -\gamma_p - \frac{\sqrt{2}}{2}(a_1)_{0,0} + 27(a_3)_{0,0}^2 + 3(a_1)_{0,0}^2 & -(\alpha\cdot\lambda)_0 \end{pmatrix}.$$

Recall Theorem 5.3.1. Let $P^s = P^s(\theta,\omega)$ and $P^u = P^u(\theta,\omega)$ respectively be the parameterizations of the stable and unstable manifolds at $\tilde{x}(\omega)$. Using the bounds $(Y_1)_0$ and $Z^1(r)$, respectively defined in (5.14) and (5.16), we numerically get the results of Table 7. For the sake of simplicity, Table 7 only displays the results of the α for which the interval containing r_0 has the highest lower bound. Notice this interval is contained in the ones from the other values of α up to order $N_2 - 1$ (see Table 8). Hence, using the data from Table 7, Theorem 5.3.1 validates our approximation $\bar{a}_\alpha$

x_0					
P^s		$\mu = 10^{-6}$ & $N_1 = 12$	P^u		$\mu = 10^{-6}$ & $N_1 = 12$
	α	[1 1]		α	[1 1]
	$(Y_1)_0$	0		$(Y_1)_0$	0
	Z^1	$3.41188 \cdot 10^{-6}$		Z^1	$3.41188 \cdot 10^{-6}$
	r_0	$[0, \infty)$		r_0	$[0, \infty)$

Table 7 – Rolls And Hexagons Stable And Unstable Manifolds Results for α fixed

of the coefficients of the power series of $P(\theta, \omega)$ for α fixed up to order $N_2 - 1$ – the notation $P(\theta, \omega)$ is used for both P^s and P^u.

Finally, recall Theorem 5.3.3. Using the bounds $(Y_2)_0$, Z_1 and $Z_2(r)$, respectively defined in (5.19), (5.22) and (5.23), we numerically get the results of Table 8. Hence,

x_0					
P^s		$\nu = (10^{-3}, 10^{-3})$ & $\mu = 10^{-6}$ & $N_2 = 8$	P^u		$\nu = (10^{-3}, 10^{-3})$ & $\mu = 10^{-6}$ & $N_2 = 8$
	$(Y_2)_0$	$3.990597 \cdot 10^{-18}$		$(Y_2)_0$	$3.964935 \cdot 10^{-18}$
	Z_1	$3.928334 \cdot 10^{-3}$		Z_1	$3.928334 \cdot 10^{-3}$
	$Z_2(r)$	$0.702889r + 16.604092r^2$		$Z_2(r)$	$0.702889r + 16.604092r^2$
	r_0	$[4.006335 \cdot 10^{-18}, 0.224674]$		r_0	$[3.980572 \cdot 10^{-18}, 0.224674]$

Table 8 – Rolls And Hexagons Stable And Unstable Manifolds Results

using the data from Table 8, Theorem 5.3.3 validates our approximation $\bar{a}$ of the coefficients of the power series of $P(\theta, \omega)$ – the notation $P(\theta, \omega)$ is again used for both P^s and P^u.

We show no figures for this example because one has to make sense of a 3D or 2D projection of a 4-dimensional problem.

7 Conclusion

Recall the purpose of this thesis was to develop a rigorous method to compute stable and unstable manifolds of ODEs satisfying Assumptions **A1**, **A2**, **A3** and **A4**. Section 2 introduced the notion of stable and unstable manifolds of fixed points depending on parameters. Section 3 went over the necessary tools required to develop our method. Sections 4 and 5 were the core of the method as it was developed mainly in these two sections. Finally, Section 6 covered two examples to illustrate our method. Note we saw in Subsection 5.4 that one can always guarantee the accuracy of our computations using the Radii Polynomials approach (see Subsection 3.4), thus ensuring the viability of this method.

Even though the work in this thesis has its limits, as it only allows to compute local parameterizations of stable and unstable manifolds of fixed points, it has several applications. For instance, one might be interested in the intersection of stable and unstable manifolds of fixed points. More often than not, the presence of parameters is needed in order to get such an intersection (see [22]), which is part of the settings of our method. These intersections are called homoclinic orbits when the fixed points for both manifolds are the same and heteroclinic orbits otherwise. For example, in biology, predator-prey models make use of these orbits and have a parameter dependency (see [15]). Indeed, it is used to predict the evolution of a species as time goes by. Moreover, in physics, chaos control theory also makes use of these orbits and requires parameters too (see [9]). Indeed, unusual behaviours, like homoclinic and heteroclinic orbits, occur for certain values of the parameters.

As mentioned in Section 1, stable and unstable manifolds of fixed points with a parameter dependency receive a lot of attention nowadays. A follow-up to this thesis could be the globalization of the local parameterizations of the stable and unstable manifolds of fixed points with respect to parameters we obtain through our method. For instance, spacecraft missions make use of these globalizations (see [16]). Indeed, it allows scientists to move satellites in space using as low fuel as possible for as long as possible. Therefore, the method developed in this thesis could be adapted to this end. Moreover, it would provide computations as well as accuracy using the same ideas as we developed in this thesis.

Finally, the method developed in this thesis could be applied to the computation of local parameterizations of stable and unstable manifolds of periodic orbits. Those are more than useful in spacecraft missions and have been used for 20 years (see [12]).

Indeed, periodic solutions always show up in spacecraft missions and, as opposed to fixed point, are lengthy. Therefore, the idea is to send satellites onto those orbits, using as low fuel as possible, via their stable and unstable manifolds. The same tools as the ones used in this thesis could be applied to compute those new objects with the accuracy of our method.

Appendix

Theorem A.1 (Implicit Function Theorem)**.** *Let W be an open subset of $\mathbb{R}^m \times \mathbb{R}^n$. Let $f = f(x, y) : W \to \mathbb{R}^n$ be continuously differentiable over W. Let $(a, b) \in W$ satisfy*

1. $f(a, b) = 0$ *;*
2. $f'(a, b) = (A|B)$*,*

where $A = \partial f/\partial x \in M_{n\times m}(\mathbb{R})$ and $B = \partial f/\partial y \in M_{n\times n}(\mathbb{R})$ with B invertible. Then, there exists neighbourhoods U and V of a and b respectively such that, for all $x \in U$, there exists a unique $y \in V$ satisfying $f(x, y) = 0$. Moreover, writing $y = g(x)$, the map $g : U \to V$ is continuously differentiable over U and satisfies $g(a) = b$ and $g'(a) = -B^{-1}A$.

Proof. See [17].

□

Theorem A.2 (Strong Implicit Function Theorem)**.** *Recall Theorem A.1. If the function f is l times differentiable, then so is the function g. Moreover, if the function f is analytic, then so is the function g.*

Proof. See [17].

□

References

[1] Xavier Cabré, Ernest Fontich, Raphael de la Llave. *The parameterization method for invariant manifolds III: Overview and applications.* Journal of Differential Equations, 218(2):444-515, 2005.

[2] J. D. Mireles-James. *Polynomial approximation of one parameter families of (un)stable manifolds with rigorous computed assisted error bounds* Indagationes Mathematicae, 26(1), 2014.

[3] Jan Bouwe van den Berg, Andréa Deschênes, Jean-Philippe Lessard, Jason D. Mireles James. *Stationary Coexistence of Hexagons and Rolls via Rigorous Computations.* Siam J. Applied Dynamical Systems, 2015.

[4] Lawrence Perko. *Differential Equations and Dynamical Systems, Third Edition.* Springer, 2001.

[5] Jenny Wilson. *Manifolds.*
`http://www.math.lsa.umich.edu/ jchw/WOMPtalk-Manifolds.pdf`

[6] J. Hadamard. *Sur l'itération et les solutions asymptotiques des équations différentielles.* Bulletin Société Mathématique de France, **29** (1901), 224-228.
`https://gallica.bnf.fr/ark:/12148/bpt6k4862034/f256.image.texteImage`

[7] A. Liapunov. *Problème général de la stabilité du mouvement.* Annales de la Faculté des Sciences de Toulouse, **9** (1907), 203-475.
`http://www.numdam.org/article/AFST_1907_2_9__203_0.pdf`

[8] O. Perron. *Über Stabilität und Asymptotisches Verhalten der Integrale von Differentialgleiehnngssystemen.* Math. Z., **29** (1928), 129-160.
`https://gdz.sub.uni-goettingen.de/id/PPN266833020_0029`

[9] S. Chen, Q. Zhang, J. Xie, C. Wang. *A stable-manifold-based method for chaos control and synchronization.* Chaos, Solitons and Fractals, Volume 20, Issue 5, 947-954. `https://doi.org/10.1016/j.chaos.2003.09.021`

[10] H. Nakamura, Y. Fukui, N. Nakamura, H. Nishitani. *Multilayer minimum projection method for nonsmooth strict control Lyapunov function design.* Systems & and Control Letters 59 (2010), 563-570.
`https://www.journals.elsevier.com/systems-and-control-letters`

[11] P. Zgliczyński. *Covering relations, cone conditions and the stable manifold theorem.*
`https://ww2.ii.uj.edu.pl/z̃gliczyn/papers/invman/cncv.pdf`

[12] R. Serban, W. S. Koon, M. W. Lo, J. E. Marsden, L. R. Petzold, S. D. Ross, R. S. Wilson. *Halo orbit mission correction maneuvers using optimal control.* Automatica 38 (2002), 571-583. https://www.sciencedirect.com/science/article/pii/S0005109801002795

[13] J. E. Sasser. *History of Ordinary Differential Equations, the first hundred years*, 1992.
http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.112.3646&rep=rep1&type=pdf

[14] R. D. Neidinger. *Introduction to Automatic Differentiation and MATLAB Object-Oriented Programming.* SIAM Review, Vol. 52, No. 3 (2010), 545-563. https://epubs.siam.org/doi/pdf/10.1137/080743627

[15] S.-B. Hsu, T.-W. Hwang, Y. Kuang. *Global analysis of the Michaelis-Menten-type ratio-dependent predator-prey system.* Springer-Verlag, 2001.

[16] G. Gómez, E. Barrabés. *Space Manifold Dynamics.* Scholarpedia, 6(2):10597 (2011). http://www.scholarpedia.org/article/Space_Manifold_dynamics

[17] J. Mashreghi. *Analyse abstraite.* Loze-Dion éditeur, 2006.

[18] W. Rudin. *Real And Complex Analysis, Third Edition.* 1987.

[19] E. Weisstein. *Maximum Absolute Row Sum Norm.*
http://mathworld.wolfram.com/MaximumAbsoluteRowSumNorm.html

[20] P. Bourke. *The Lorenz Attractor in 3D.* 1997.
paulbourke.net/fractals/lorenz/

[21] N. M. Patrikalakis, T. Maekawa, W. Cho. *Interval arithmetic.* 2009.
http://web.mit.edu/hyperbook/Patrikalakis-Maekawa-Cho/node45.html

[22] Beyn, Wolf-Jürgen. *The numerical computation of connecting orbits in dynamical systems.* BiPrints Repository, Bielefeld University, 2009.
http://repositories.ub.uni-bielefeld.de/biprints/volltexte/2009/3783