The variational bicomplex for hyperbolic equations

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Abstract

This thesis presents the geometric investigation of hyperbolic partial differential equations in the plane as carried out by Niky Kamran, Ian Anderson, and Martin Juraš. In particular, the relation between the Darboux integrability of an arbitrary hyperbolic equation and the Laplace invariants of the linearization of this equation is established. This extends to non-linear hyperbolic equations in the plane a classical result of Goursat for linear hyperbolic equations. The formal setting for this geometric investigation is afforded by the constrained variational bicomplex, which allows the solution to a partial differential equation to be viewed as a manifold on which standard differential geometric operations such as exterior differentiation and Lie differentiation can be performed. The key element in this investigation is the judicious construction and use of appropriate moving coframes which will reflect the properties of the equations under investigation.

Résumé

Cette thèse décrit les grandes lignes de l'étude géométrique des équations aux dérivées partielles hyperboliques telle que réalisée par Niky Kamran, Ian Anderson et Martin Juraš. En particulier, la relation liant l'intégrabilité de Darboux d'une équation hyperbolique arbitraire et les invariants de Laplace de la linéarisation de cette même équation est étudiée. Cette relation étend au cas des équations aux dérivées partielles hyperboliques non-linéaires un résultat classique de Goursat sur les équations hyperboliques linéaires. Le cadre formel de cette étude géométrique est fourni par le complexe bi-variationnel contraint, exprimant la solution d'une équation aux dérivées partielles en tant que variété sur laquelle les manipulations ordinaires de géometrie differentielle, telles la differentiation extèrieure ou la dérivée de Lie, peuvent avoir lieu. L'élément clé de cette étude est la construction et l'utilisation judicieuses d'un co-repère mobile approprié reflètant les propriétés de l'équation étudiée.

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Contents

Chapter 1

Introduction

In this thesis, we present aspects of the geometric theory of hyperbolic partial differential equations in the plane, following [3] and [9]. The cornerstone of this theory is to view a partial differential equation as the locus, assumed to be a manifold, of a particular equation in some higher-dimensional space; functional-theoretic properties of the partial differential equation then manifest themselves as geometric properties of this manifold. In particular, various differential operators can be defined on this manifold, and integrability conditions can be expressed in terms of these differential operators. Furthermore, vector fields and differential forms can also be defined, and solutions to the original partial differential equation can be defined as integral manifolds of differential ideals generated by certain one-forms. The choice of an appropriate family, called a moving coframe, of one-forms adapted to the partial differential equation leads to the expression of properties of the partial differential equation in purely differential-geometric terms as structure equations of the moving coframe.

The specific result we shall study in this thesis is the equivalence established in [3] and [9] between the Darboux integrability of an arbitrary hyperbolic equation and the vanishing of the Laplace invariants of a linear hyperbolic equation derived from the original partial differential equation. Such an equivalence had already been established by Goursat [8] for linear hyperbolic equations. It is a very instructive exercise to compare the treatment of Goursat for linear hyperbolic equations to that of [3] and [9] for arbitrary hyperbolic equations: Whereas Goursat's approach is constructive and yields explicit solutions in the process of proving the equivalence, the geometric treatment is far more implicit and hence far-reaching. Also, whereas Goursat's approach hinges on solving the hyperbolic equation itself, the geometric treatment hinges on computing structure equations of particular moving coframes. There is no reason a priori why moving coframes should embody any information of interest about the partial differential equation, and the whole difficulty and challenge of the geometric approach is the definition and use of appropriate moving coframes.

In the second chapter of this thesis, we present the classical theory of hyperbolic equations in the plane as far as Darboux integrability and the Laplace transform are concerned. We shall, in particular, state and partially establish Goursat's result to the effect that the termination of both sequences of Laplace invariants implies the Darboux integrability of the hyperbolic equation. In the third chapter, we present the formal geometric setting of jet bundles and the variational bicomplex, together with the main technical results allowing us to fully exploit this geometric formalization. In the fourth chapter, we shall establish the relation between Darboux integrability and Laplace invariants using the method of moving coframes, and we conclude with the last chapter.

Chapter 2

Aspects of the classical theory of hyperbolic partial differential equations in the plane

In this chapter, we present the classical theory of Laplace transformations and Darboux integrability for hyperbolic equations in the plane, as well as their relations for linear hyperbolic equation, following Goursat's classical treatise [7] and [8], and also the short treatise of Gosse [6].

2.1 Characteristics and the Cauchy problem

Consider the partial differential equation

$$F(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}) = 0 \tag{2.1}$$

defined for (x, y) in an open subset of $\mathbb{R}^2$. We shall use the classical Monge notation and denote the partial derivatives $z_x, z_y, z_{xx}, z_{xy}, z_{yy}$ by p, q, r, s, t, respectively. The Cauchy problem for equation (2.1) consists in determining the graph $(x, y) \mapsto (x, y, z(x, y))$ of a solution to this equation such that it contains a given curve Γ along which the tangent plane at every point is known. The Cauchy data is thus given by a curve

$$\lambda \mapsto (x(\lambda), y(\lambda), z(\lambda), p(\lambda), q(\lambda))$$

subject only to the condition $dz = pdx + qdy$. By the implicit function theorem, the three conditions

$$\begin{cases} dp = rdx + sdy, \\ dq = sdx + tdy, \\ F(x, y, z, p, q, r, s, t) = 0 \end{cases}$$

will allow us to determine r, s, and t as functions of λ if and only if the functional determinant

$$\Delta \equiv \begin{vmatrix} R & S & T \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = Rdy^2 - Sdxdy + Tdx^2$$

is non-zero, where $R = \frac{\partial F}{\partial r}, S = \frac{\partial F}{\partial s}$, and $T = \frac{\partial F}{\partial t}$. Δ is a bilinear form in (dx, dy), and is called the characteristic form of the partial differential equation (2.1). We can attempt to determine partial derivatives of z of order larger than two in terms of the Cauchy data by differentiating both sides of equation (2.1) with respect to x and y. Denoting $\frac{\partial^{i+k} z}{\partial x^i \partial y^k}$ by p_{ik}, with $z = p_{00}$, the values of p_{ik} with $i + k = 3$ are provided by the following system of equations:

$$\begin{cases} dp_{20} = p_{30}dx + p_{21}dy, \\ dp_{11} = p_{21}dx + p_{12}dy, \\ dp_{02} = p_{12}dx + p_{03}dy, \\ \frac{dF}{dx} = 0, \\ \frac{dF}{dy} = 0. \end{cases}$$

We can write $\frac{dF}{dy} = Rp_{21} + Sp_{12} + Tp_{03} + (\frac{dF}{dy}) = 0$, where $(\frac{dF}{dy})$ contains no term in p_{ik} with $i + k = 3$. Since $dF = \frac{dF}{dx}dx + \frac{dF}{dy}dy$, the equation $\frac{dF}{dx} = 0$ is a consequence of $F = 0$ and $\frac{dF}{dy} = 0$. The partial derivatives p_{ik} with $i + k = 3$ are thus to be determined from the system of seven equations in seven unknowns given by

$$\begin{cases} F = 0, \\ \frac{dF}{dy} = 0, \\ dp_{ik} = p_{i+1,k}dx + p_{i,k+1}dy, \quad i+k \leq 2. \end{cases}$$

The p_{ik} with $i + k = 3$ can be determined from this system if and only if the functional determinant

$$\begin{vmatrix} dx & dy & 0 & 0 \\ 0 & dx & dy & 0 \\ 0 & 0 & dx & dy \\ 0 & R & S & T \end{vmatrix} = \Delta dx$$

is non-zero, where Δ is the characteristic form. We can continue this procedure so as to determine all the p_{ik} with $i + k > 3$. Suppose then that we have determined all the p_{ik} with $i + k = n$ as a function of λ, through the system

$$(S) \begin{cases} F = 0, \\ \frac{dF}{dy} = 0, \\ \dots \\ \frac{d^{n-2}F}{dy^{n-2}} = 0, \\ dp_{ik} = p_{i+1,k}dx + p_{i,k+1}dy, \quad i+k \leq n-1. \end{cases}$$

To determine the p_{ik} with $i + k = n + 1$, we shall, as before, augment this system by the equations $\frac{d^{n-1}F}{dx^j dy^{n-1-j}} = 0$, and the equations $dp_{ik} = p_{i+1,k}dx + p_{i,k+1}dy$, with $i + k = n$. It is a simple matter to show that, just as previously, the $n + 2$ partial derivatives p_{ik} with $i + k = n + 1$ can be determined from this augmented system if and only if the functional determinant

$$\begin{vmatrix} dx & dy & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & dx & dy & 0 & \dots & 0 & 0 & 0 \\ & & & & \dots & & & \\ 0 & 0 & 0 & 0 & \dots & 0 & dx & dy \\ 0 & 0 & 0 & 0 & 0 & R & S & T \end{vmatrix} = \Delta dx^{n-1}$$

is non-zero. If there does exist a value λ_0 of λ for which $\Delta(\lambda_0) \neq 0$, the solution z of (2.1) can then be given locally by a formal power series in $x - x(\lambda_0), y - y(\lambda_0)$, the convergence of which can be established whenever the Cauchy data (i.e. the function F and the curve Γ) are analytic. This is the essence of Cauchy-Kovalewskaya's theorem. The characteristic form $\Delta = Rdy^2 - Sdxdy + Tdx^2$ is a fundamental attribute of the partial differential equation (2.1) and is ubiquitous in the study of partial differential equations. If the characteristic form has no real roots at a certain point, then the equation $F = 0$ is said to be elliptic there; if it has exactly one real root, the equation is said to be parabolic; the case of interest to us is the one where the characteristic polynomial has two real distinct roots at every point, and we shall assume this throughout the thesis. In this case, the equation $F = 0$ is called a hyperbolic equation. For equation (2.1) to be hyperbolic, therefore, we need to have $4RT - S^2 < 0$ everywhere.

The notion of characteristic can be easily generalized. Suppose, for simplicity and without loss of generality, that in the equation $F = 0$ we have solved for r in terms of x, y, z, p, q, s, t, and that $F = 0$ can be rewritten as $r + f(x, y, z, p, q, s, t) = 0$, or equivalently, $p_{20} + f(x, y, p_{00}, p_{10}, p_{01}, p_{11}, p_{02}) = 0$. Differentiating both sides of this equation with respect to y allows us to compute all of the $p_{2,k}$ in terms of the p_{ik}, with $i = 0, 1$. Differentiating with respect to x, on the other hand, will allow us to compute all the p_{ik} with $i \geq 3$ in terms of the p_{ik} with $i \leq 2$, and hence in terms of the p_{ik} with $i = 0, 1$. Thus, all of the p_{ik} with $i + k = n - 1$ will be determined once $p_{0,n-1}, p_{1,n-2}$ and the p_{ik} with $i + k < n - 1$ are known. The system (S), which allowed the determination of the p_{ik} with $i + k = n$, is then equivalent to the system

$$(S') \begin{cases} dz = p_{10}dx + p_{01}dy, \\ dp_{10} = p_{20}dx + p_{11}dy, \\ \dots \\ dp_{1,n-2} = p_{2,n-2}dx + p_{1,n-1}dy, \\ dp_{01} = p_{11}dx + p_{02}dy, \\ \dots \\ dp_{0,n-1} = p_{1,n-1}dx + p_{0,n}dy, \end{cases}$$

where the $p_{2,i}$ are determined from the $p_{0,i}$ and $p_{1,i}$ by differentiating both sides of $p_{2,0} + f = 0$ with respect to y. Doing so i times, we obtain

$$p_{2,i} + (\frac{d^i f}{dy^i}) + \frac{\partial f}{\partial s} p_{1,i+1} + \frac{\partial f}{\partial t} p_{0,i+2} = 0, \quad i = 0, ..., n-2,$$

where $(\frac{d^i f}{dy^i})$ contains no terms in p_{jk} with $j + k = i + 2$. The p_{ik} with $i + k = n$ are obtained from the following three relations:

$$\begin{cases} dp_{1,n-2} = p_{2,n-2}dx + p_{1,n-1}dy, \\ dp_{0,n-1} = p_{1,n-1}dx + p_{0,n}dy, \\ p_{2,n-2} + (\frac{d^{n-2}f}{dy^{n-2}}) + \frac{\partial f}{\partial s} p_{1,n-1} + \frac{\partial f}{\partial t} p_{0,n} = 0, \end{cases}$$

where $(\frac{d^{n-2}f}{dy^{n-2}})$ contains no terms in p_{ik} with $i + k = n$. Solving for $p_{0,n}$, we obtain the equation

$$p_{0n}[(\frac{dy}{dx})^2 - \frac{\partial f}{\partial s}\frac{dy}{dx} + \frac{\partial f}{\partial t}] + \frac{dp_{1,n-2}}{dx} + \frac{dp_{0,n-1}}{dx}[\frac{\partial f}{\partial s} - \frac{dy}{dx}] + (\frac{d^{n-2}f}{dy^{n-2}}) = 0.$$

The characteristic equation $(\frac{dy}{dx})^2 - \frac{\partial f}{\partial s}\frac{dy}{dx} + \frac{\partial f}{\partial t}$ has, by assumption, two real distinct roots m_1 and m_2. If $\frac{dy}{dx} = m_1$, then $p_{0,n}$ can be chosen arbitrarily, provided

$$\frac{dp_{1,n-2}}{dx} + \frac{dp_{0,n-1}}{dx}[\frac{\partial f}{\partial s} - m_1] + (\frac{d^{n-2}f}{dy^{n-2}}) = 0,$$

which is equivalent to

$$\frac{dp_{1,n-2}}{dx} + m_2\frac{dp_{0,n-1}}{dx} + (\frac{d^{n-2}f}{dy^{n-2}}) = 0.$$

We thus define the first system of characteristics of order n as the system

$$(I) \begin{cases} \frac{dy}{dx} = m_1, \\ dz = p_{10}dx + p_{01}dy, \\ dp_{10} = p_{20}dx + p_{11}dy, \cdots, dp_{1,n-2} = p_{2,n-2}dx + p_{1,n-1}dy, \\ dp_{01} = p_{11}dx + p_{02}dy, \cdots, dp_{0,n-2} = p_{1,n-2}dx + p_{0,n-1}dy, \\ dp_{1,n-2} + m_2 dp_{0,n-1} + (\frac{d^{n-2}f}{dy^{n-2}})dx = 0. \end{cases}$$

The second system of characteristics of order n is defined by interchanging m_1 and m_2 in the above system.

2.2 Involution and invariants

Consider a hyperbolic equation given in the form

$$r + f = 0,$$

and consider an arbitrary function ϕ of order n, that is, ϕ is a function of x, y, and the p_{ik}, with $i + k \leq n$. As we saw before, each p_{ik} can be expressed in terms of the p_{0j} and the p_{1j} with $j \leq i + k$. We can thus assume that ϕ is a function of $x, y, z, p_{1,0}, \cdots, p_{1,n-1}, p_{0,1}, \cdots, p_{0,n}$. For ϕ to be of order n, we need to have at least one of $\frac{\partial \phi}{\partial p_{1,n-1}}$ and $\frac{\partial \phi}{\partial p_{0,n}}$ non-zero. We assume, without loss of generality, that $\frac{\partial \phi}{\partial p_{1,n-1}} \neq 0$. Consider now the system of equations:

$$\begin{cases} r + f = 0, \\ \phi = 0. \end{cases}$$

We wish to determine whether or not this system has a solution. The p_{ik} with $i + k = n + 1$ can be obtained from the system of equations

$$\begin{cases} \frac{d\phi}{dx} = \left(\frac{d\phi}{dx}\right) + \frac{\partial \phi}{\partial p_{1,n-1}} p_{2,n-1} + \frac{\partial \phi}{\partial p_{0,n}} p_{1,n} = 0, \\ \frac{d\phi}{dy} = \left(\frac{d\phi}{dy}\right) + \frac{\partial \phi}{\partial p_{1,n-1}} p_{1,n} + \frac{\partial \phi}{\partial p_{0,n}} p_{0,n+1} = 0, \\ \frac{d^{n-1}(r+f)}{dy^{n-1}} = p_{2,n-1} + \frac{\partial f}{\partial s} p_{1,n} + \frac{\partial f}{\partial t} p_{0,n+1} + \left(\frac{d^{n-1}(r+f)}{dy^{n-1}}\right) = 0, \end{cases}$$

where none of $\left(\frac{d\phi}{dx}\right), \left(\frac{d\phi}{dy}\right), \left(\frac{d^{n-1}(r+f)}{dy^{n-1}}\right)$ contain terms in p_{ik} with $i + k = n + 1$. We say that the hyperbolic equation $r + f = 0$ is in involution with the equation $\phi = 0$ if these three equations reduce to two. It is easy to see that this will be the case if and only if

$$\left(\frac{\partial \phi}{\partial p_{0,n}}\right)^2 - \frac{\partial f}{\partial s} \frac{\partial \phi}{\partial p_{1,n-1}} \frac{\partial \phi}{\partial p_{0,n}} + \frac{\partial f}{\partial t} \left(\frac{\partial \phi}{\partial p_{1,n-1}}\right)^2 = 0 \tag{2.2}$$

and

$$\frac{\partial \phi}{\partial p_{0,n}} \left(\frac{d\phi}{dx}\right) + \frac{\partial f}{\partial s} \frac{\partial \phi}{\partial p_{1,n-1}} \frac{d\phi}{dy} - \frac{\partial \phi}{\partial p_{1,n-1}} \frac{\partial \phi}{\partial p_{0,n}} \left(\frac{d^{n-1} f}{dy^{n-1}}\right) = 0.$$

Equation (2.2) is readily identified as the characteristic equation of the hyperbolic equation $r + f = 0$. There will thus be two distinct types of equations in involution with $r+f=0$, depending on whether the ratio $\frac{\partial\phi}{\partial p_{0,n}}/\frac{\partial\phi}{\partial p_{1,n-1}}$ is equal to m_1 or to m_2. We will say that the first system of characteristics is in involution with a function ϕ of order n if the following holds:

$$(A)\left\{\begin{array}{l}\frac{\partial\phi}{\partial p_{0,n}}-m_1\frac{\partial\phi}{\partial p_{1,n-1}}=0,\\ (\frac{d\phi}{dx})+m_2(\frac{d\phi}{dy})-\frac{\partial\phi}{\partial p_{1,n-1}}(\frac{d^{n-1}f}{dy^{n-1}})=0.\end{array}\right.$$

Similarly, the second system of characteristics will be said to be in involution with a function ϕ of order n if we have

$$(B)\left\{\begin{array}{l}\frac{\partial\phi}{\partial p_{0,n}}-m_2\frac{\partial\phi}{\partial p_{1,n-1}}=0,\\ (\frac{d\phi}{dx})+m_1(\frac{d\phi}{dy})-\frac{\partial\phi}{\partial p_{1,n-1}}(\frac{d^{n-1}f}{dy^{n-1}})=0.\end{array}\right.$$

The system (A) could be verified as a consequence of the equation $\phi=0$; in this case, $r+f=0$ and $\phi=0$ are simply said to be in involution. It could also happen, however, that the system (A) is verified identically, independently of the condition $\phi=0$. Since ϕ intervenes in (A) only through its derivatives, it is easy to see that whenever (A) holds identically irrespective of $\phi=0$, then it holds as well for $\phi=c$, for any real constant c. In this case, the equation $r+f=0$ is in involution with all the equations $\phi=c$, with c any real constant. We say in this case that ϕ is an invariant of order n of the first system of characteristics. The reason for calling ϕ an invariant is that ϕ does indeed stay constant along the characteristics of order n of the hyperbolic equation $r+f=0$. To see this, note that

$$\begin{aligned} d\phi &= \frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz+\frac{\partial\phi}{\partial p_{1,0}}dp_{1,0}+\cdots\frac{\partial\phi}{\partial p_{1,n-1}}dp_{1,n-1}+\\ &\quad \frac{\partial\phi}{\partial o_{0,1}}dp_{0,1}+\cdots\frac{\partial\phi}{\partial p_{0,n}}dp_{0,n},\end{aligned}$$

and that replacing the expressions dp_{ik} by the ones obtained from the system of characteristics of order n, we are left with

$$d\phi=[(\frac{d\phi}{dx})+m_1(\frac{d\phi}{dy})]dx+\frac{\partial\phi}{\partial p_{0,n}}dp_{0,n}-\frac{\partial\phi}{\partial p_{1,n-1}}[(\frac{d^{n-1}f}{dy^{n-1}})dx+m_2dp_{0,n}],$$

which is zero, since ϕ is assumed to satisfy the system (A) identically. We thus see that if ϕ is an invariant of order n of the first system of characteristics of $r+f=0$, then $d\phi$ can be expressed as a linear combination of the forms:

$$\begin{cases} dz - p_{1,0}dx - p_{0,1}dy, \\ dy - m_1 dx, \\ dp_{1,0} - p_{2,0}dx - p_{1,1}dy, \cdots, dp_{1,n-2} - p_{2,n-2}dx - p_{1,n-1}dy, \\ dp_{0,1} - p_{1,1}dx - p_{0,2}dy, \cdots, dp_{0,n-1} - p_{1,n-1}dx - p_{0,n}dy, \\ dp_{1,n-1} - p_{2,n-1}dx - p_{1,n}dy, \end{cases}$$

which define the first system of characteristics of order n of $r+f=0$. Conversely, if $d\phi$ is in the linear span of the one-forms defining one of the systems of characteristics of $r+f=0$, then ϕ remains constant along any characteristic of order n of that system, and ϕ is an invariant of order n of $r+f=0$.

Consider now the system of equations

$$(S)\begin{cases} r+f=0, \\ \phi=0, \\ \psi=0 \end{cases}$$

where ϕ and ψ are of arbitrary order. Assume that both ϕ and ψ are in involution with $r+f=0$, and that ϕ is in involution according to the system (A) and ψ according to the system (B) of characteristics; that is, we have:

$$\begin{cases} \frac{\partial\phi}{\partial p_{0,n}} - m_1\frac{\partial\phi}{\partial p_{1,n-1}} = 0, \\ (\frac{d\phi}{dx}) + m_2(\frac{d\phi}{dy}) - \frac{\partial\phi}{\partial p_{1,n-1}}(\frac{d^{n-1}f}{dy^{n-1}}) = 0, \\ \frac{\partial\psi}{\partial p_{0,n}} - m_2\frac{\partial\psi}{\partial p_{1,n-1}} = 0, \\ (\frac{d\psi}{dx}) + m_1(\frac{d\psi}{dy}) - \frac{\partial\psi}{\partial p_{1,n-1}}(\frac{d^{n-1}f}{dy^{n-1}}) = 0. \end{cases}$$

The functional determinant $\frac{D(\phi,\psi)}{D(p_{0,n},p_{1,n-1})} = (m_1-m_2)(\frac{\partial\phi}{\partial p_{1,n-1}}\frac{\partial\psi}{\partial p_{1,n-1}})^2$ is non-zero by virtue of the assumption of hyperbolicity of the equation $F=0$ (or equivalently $r+f=0$). Hence, by the implicit function theorem, $p_{0,n}$ and $p_{1,n-1}$ can be obtained from the equations $\phi=0$ and $\psi=0$. If $z \mapsto z(x,y)$ is a solution of the system (S), all the partial derivatives of order n of z can thus be expressed in terms of $x, y, z, p_{1,0}, p_{1,1}, \cdots, p_{1,n-2}, p_{0,1}, p_{0,2}, \cdots, p_{0,n-1}$. Thus, z and its partial derivatives satisfy the differential system:

$$(E)\begin{cases} dz = p_{1,0}dx + p_{0,1}dy, \\ dp_{1,0} = p_{2,0}dx + p_{1,1}dy, \\ \cdots \\ dp_{1,n-2} = p_{2,n-2}dx + p_{1,n-1}dy, \\ dp_{0,n-1} = p_{1,n-1}dx + p_{0,n}dy, \end{cases}$$

where $p_{1,n-1}$ and $p_{0,n}$ are obtained from the equations $\phi = 0$ and $\psi = 0$, whereas the $p_{2,i}$ are expressed in terms of the $p_{0,j}$ and $p_{1,k}$. The existence of a solution z to the system (S) is thus equivalent to the complete integrability of the system (E). The system (E), on the other hand, will be completely integrable if and only if the three equations $r + f = 0, \phi = 0, \psi = 0$ yield a unique set of values for the $(n+1)^{th}$ order derivatives $p_{0,n+1}, p_{1,n}$, and $p_{2,n-1}$. The equations that $p_{0,n+1}, p_{1,n}$, and $p_{2,n-1}$ should satisfy are obtained by differentiating both sides of the equations in the system (S), yielding:

$$(E')\begin{cases} (\frac{d^{n-1}f}{dy^{n-1}}) + p_{2,n-1} + \frac{\partial f}{\partial s}p_{1,n} + \frac{\partial f}{\partial t}p_{0,n+1} = 0, \\ (\frac{d\phi}{dx}) + \frac{\partial \phi}{\partial p_{0,n}}p_{1,n} + \frac{\partial \phi}{\partial p_{1,n-1}}p_{2,n-1} = 0, \\ (\frac{d\phi}{dy}) + \frac{\partial \phi}{\partial p_{0,n}}p_{0,n+1} + \frac{\partial \phi}{\partial p_{1,n-1}}p_{1,n} = 0, \\ (\frac{d\psi}{dx}) + \frac{\partial \psi}{\partial p_{0,n}}p_{1,n} + \frac{\partial \psi}{\partial p_{1,n-1}}p_{2,n-1} = 0, \\ (\frac{d\psi}{dy}) + \frac{\partial \psi}{\partial p_{0,n}}p_{0,n+1} + \frac{\partial \psi}{\partial p_{1,n-1}}p_{1,n} = 0, \end{cases}$$

where none of the terms $(\frac{d^{n-1}f}{dy^{n-1}}), (\frac{d\phi}{dx}), (\frac{d\phi}{dy}), (\frac{d\psi}{dx}), (\frac{d\psi}{dy})$ contain terms in p_{ik} with $i + k = n + 1$. Since ϕ is in involution with $r + f = 0$ according to the system (A) and ψ is in involution with $r + f = 0$ according to the system (B), the five equations in system (E') reduce to three independent equations; the system (E) is thus completely integrable.

2.3 Darboux integrability

Consider the equation $r + f = 0$, which we assume to be hyperbolic. Assume u and v are both independent invariants (of whatever order) of the same system of characteristics, say (A). Assume that on a solution surface $(x,y)\mapsto(x,y,z(x,y))$ of $r+f=0$, we move along characteristic curves of the first system, given by $\frac{dy}{dx}=m_1$. Along one such curve, which we can locally parametrize by either x or y (say x), y, z, and the partial derivatives p_{ik} are all functions of x; so too then are u and v, and since they remain constant along these characteristics, they satisfy a functional relationship of the form $\phi(u,v)=0$, which we can assume, without loss of generality, can be written $u=\phi(v)$. Every solution of $r+f=0$ thus satisfies $u=\phi(v)$ for a particular choice of ϕ. Conversely, u and v being invariants of the system (A) of characteristics of $r+f=0$, du and dv are in the linear span of the system of 1-forms (I) of characteristics, and so is then $du-\phi'(v)dv$, for any choice of function ϕ. The equation $u-\phi(v)=0$ is thus in involution (according to the system (A)) with the hyperbolic equation $r+f=0$. Assume now that u_1 and v_1 are both independent invariants of the second system (B) of characteristics. Just as before, the equation $u_1-\psi(v_1)=0$ is always in involution (according to the system (B)) with $r+f=0$, for any choice of function ψ. Putting these two observations together, we arrive at the cornerstone of Darboux's method of integration, namely that if u,v are both independent invariants of the same system of characteristics, and u_1,v_1 independent invariants of the other system, then the system

$$\begin{cases} r+f=0,\\ u=\phi(v),\\ u_1=\psi(v_1),\end{cases}$$

is completely integrable, for any choice of functions ϕ and ψ. The essential step in Darboux's method of integration is therefore the computation of the invariants u,v and u_1,v_1, if they do exist. An attempt to classify all Darboux-integrable equations has been made by Vessiot ([11] and [12]).

The simplest, almost trivial, example of the Darboux method is given by the wave equation $s = 0$. The first system of characteristics is given by

$$(I) \begin{cases} dx = 0, \\ dz - q dy = 0, \\ dp = 0, \\ dq - t dy = 0, \end{cases}$$

and the second system of characteristics is given by

$$(II) \begin{cases} dy = 0, \\ dz - p dx = 0, \\ dp - r dx = 0, \\ dq = 0. \end{cases}$$

The first system of characteristics thus has independent invariants x, p, and the second system has independent invariants y, q. For any choice of functions ϕ and ψ, the system

$$\begin{cases} s = 0, \\ p = \phi(x), \\ q = \psi(y), \end{cases}$$

is thus completely integrable, and its general solution is given by $(x, y) \mapsto z(x, y) = \Phi(x) + \Psi(y)$, where $\Phi'(x) = \phi(x)$ and $\Psi'(y) = \psi(y)$.

The simplest non-trivial example of the Darboux method of integration is provided by Liouville's equation $s = e^z$. The first and second systems of characteristics of second order are given by

$$(I) \begin{cases} dx = 0, \\ dz - q dy = 0, \\ dp - e^z dy = 0, \\ dq - t dy = 0, \\ dr - p e^z dy = 0, \end{cases}$$

and

$$(II)\begin{cases} dy = 0, \\ dz - pdx = 0, \\ dp - rdx = 0, \\ dq - e^z dx = 0, \\ dt - qe^z dx = 0, \end{cases}$$

respectively. From the first system, we obtain the independent invariants x and $r - p^2/2$, and from the second system, we obtain the independent invariants y and $t - q^2/2$.Therefore, for any choice of functions ϕ and ψ, the system

$$\begin{cases} dz = pdx + qdy, \\ dp = (p^2/2 + \phi(x))dx + e^z dy, \\ dq = e^z dx + (q^2/2 + \psi(y))dy, \end{cases}$$

is completely integrable. It is easy to see that the general solution of this system is given by $z = Log\frac{2X'Y'}{(X+Y)^2}$, where X is a function of x only, and Y a function only of y. Conversely, any function z of this form is a solution of Liouville's equation.

2.4 The Laplace transform

Consider the linear hyperbolic equation

$$\frac{\partial^2 z}{\partial x \partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz = M, \tag{2.3}$$

where the coefficients a, b, c, and M are given functions of the independent variables x, y. We can rewrite this equation as

$$\frac{\partial}{\partial x}(\frac{\partial z}{\partial y} + az) + b(\frac{\partial z}{\partial y} + az) - (\frac{\partial a}{\partial x} + ab - c)z = M.$$

If $(\frac{\partial a}{\partial x} + ab - c) = 0$, then, writing $z_1 = \frac{\partial z}{\partial y} + az$, we obtain $\frac{\partial z_1}{\partial x} + bz_1 = M$, the general solution of which is given by $z_1 = e^{-\int ady}\{X + \int\{Y + \int Me^{\int bdx}dx\}$, where Y is an arbitrary function of y. z is in turn obtained from z_1 by solving the differential equation $\frac{\partial z}{\partial y} + az = z_1$, yielding $z = e^{-\int ady}\{X + \int\{Y + \int Me^{\int bdx}dx\}e^{\int ady - bdx}dy\}$, where X is an arbitrary function of x. Equivalently, the original equation can be rewritten as

$$\frac{\partial}{\partial y}(\frac{\partial z}{\partial x}+bz)+a(\frac{\partial z}{\partial x}+bz)-(\frac{\partial b}{\partial y}+ab-c)z=M.$$

If now $\frac{\partial b}{\partial y}+ab-c=0$, then, writing $z_{-1}=\frac{\partial z}{\partial x}+bz$, we obtain the ordinary differential equation $\frac{\partial z_{-1}}{\partial y}+bz_{-1}=M$, the general solution of which is given by $z_{-1}=e^{-\int ady}\{X+\int Me^{\int ady}dy\}$, where X is an arbitrary function of x. Just as before, z can be obtained from z_{-1} by solving the ordinary differential equation $\frac{\partial z}{\partial x}+bz=z_{-1}$, yielding $z=e^{-\int bdx}\{Y+\{X+\int Me^{\int ady}dy\}e^{\int bdx-ady}dx\}$, where Y is an arbitrary function of y. In conclusion, if either of $\frac{\partial a}{\partial x}+ab-c$ and $\frac{\partial b}{\partial y}+ab-c$ vanishes, then the linear hyperbolic equation (2.3) can be integrated out. We call $\frac{\partial a}{\partial x}+ab-c$ and $\frac{\partial b}{\partial y}+ab-c$ the Laplace invariants of the linear hyperbolic equation (2.3), and denote them by h and k, respectively.

Assume now that $h\neq 0$. We can rewrite equation (2.3) as $\frac{\partial z_1}{\partial x}+bz_1-hz=M$, with $z_1=\frac{\partial z}{\partial y}+az$. Differentiating both sides of this equation with respect to y, and expressing z in terms of z_1 in the resulting equation yields the equation

$$\frac{\partial^2 z_1}{\partial x\partial y}+a_1\frac{\partial z_1}{\partial x}+b_1\frac{\partial z_1}{\partial y}+c_1z_1=M_1, \tag{2.4}$$

which is of the same form as (2.3), and where the coefficients a_1,b_1,c_1,M_1 are given by

$$\begin{cases} a_1=a-\frac{\partial logh}{\partial y}, \\ b_1=b, \\ c_1=c-\frac{\partial a}{\partial x}+\frac{\partial b}{\partial y}-b\frac{\partial logh}{\partial y}, \\ M_1=M(a-\frac{\partial logh}{\partial y})+\frac{\partial M}{\partial y}. \end{cases}$$

If a solution z to (2.3) is known, then a solution z_1 to (2.4) is immediately obtained by $z_1=\frac{\partial z}{\partial y}+az$. Conversely, if a solution to (2.4) is known, then a solution z to (2.3) is immediately given by $z=\frac{\frac{\partial z_1}{\partial x}+bz_1-M}{h}$. Thus, the integration of equation (2.3) is equivalent to the integration of equation (2.4). Since equation (2.4) is of the same form as equation (2.3), we can compute its Laplace invariants, respectively given by $h_1=\frac{\partial a_1}{\partial x}+a_1b_1-c_1$, and $k_1=\frac{\partial b_1}{\partial y}+a_1b_1-c_1$, or equivalently, $h_1=2h-k-\frac{\partial^2 logh}{\partial x\partial y}$, and $k_1 = h$. By assumption h is non-zero, but we could very well have $h_1 = 0$, leading to a solution of (2.4) and thereby to a solution of (2.3).

We call z_1 the $Y-$Laplace transform of z, and equation (2.4) the $Y-$ Laplace transform of equation (2.3). It is important to note that the $Y-$ Laplace transform of z is always defined, whereas the $Y-$Laplace transform of equation (2.3) is defined only when its $Y-$Laplace invariant h is non-zero.

Similarly, the $X-$Laplace transform of z is defined as $z_{-1} = \frac{\partial z}{\partial x} + bz$, and whenever $k \neq 0$, the $X-$Laplace transform of equation (2.3) is defined as

$$\frac{\partial^2 z_{-1}}{\partial x \partial y} + a_{-1}\frac{\partial z_{-1}}{\partial x} + b_{-1}\frac{\partial z_{-1}}{\partial y} + c_{-1}z_{-1} = M_{-1}, \tag{2.5}$$

where the coefficients are given by

$$\begin{cases} a_{-1} = a, \\ b_{-1} = b - \frac{\partial logk}{\partial x}, \\ c_{-1} = c - \frac{\partial b}{\partial y} + \frac{\partial a}{\partial x} - a\frac{\partial logk}{\partial x}, \\ M_{-1} = M(b - \frac{\partial logk}{\partial x}) + \frac{\partial M}{\partial x}. \end{cases}$$

The laplace invariants of the $X-$Laplace transform of equation (2.3) are given by $h_{-1} = \frac{\partial a_{-1}}{\partial x} + a_{-1}b_{-1} - c_{-1} = k$, and $k_{-1} = \frac{\partial b_{-1}}{\partial y} + a_{-1}b_{-1} - c_{-1} = 2k - h - \frac{\partial^2 logk}{\partial x \partial y}$. Just as with (2.4), the integration of (2.5) is equivalent to the integration of (2.3).

Since (2.4) is of the same form as (2.3), we can apply the $X-$ and $Y-$Laplace transforms to it in case both h_1 and k_1 are non-zero; in case either of h_1 and k_1 vanishes, we can solve for z_1, and subsequently for z, as was seen earlier. Similarly, we can apply the $X-$ and $Y-$Laplace transforms to equation (2.5) in case both h_{-1} and k_{-1} are non-zero. in case either of h_{-1} and k_{-1} vanishes, we can solve for z_{-1}, and subsequently for z. A simple calculation shows that with $h \neq 0$, applying the $X-$Laplace transform to the $Y-$Laplace transform of equation (2.3) yields an equation which can be deduced from equation (2.3) by replacing the dependent variable z by hz. Similarly, with $k \neq 0$, applying the $Y-$Laplace transform to the $X-$Laplace transform of equation (2.3) yields an equation which can be deduced from equation (2.3) by replacing the dependent variable z by kz. Thus the repeated application, whenever the corresponding Laplace invariants do not vanish, of the $X-$

and $Y-$Laplace transforms to equation (2.3) yields only a singly-indexed sequence $\cdots, (E_{-2}), (E_{-1}), (E), (E_1), (E_2), \cdots$ of linear hyperbolic equations, where (E) denotes the original equation (2.3), and where (E_{i+1}) is the $Y-$Laplace transform of (E_i) and (E_{i-1}) its $X-$Laplace transform. It follows from our preceding discussion that all of the equations in this sequence, and in particular the original equation (E), can be integrated as soon as any of them can be integrated. The Laplace method of integration consists in generating successive elements of this sequence, starting from (E), until the $X-$ or $Y-$Laplace invariant of the last generated equation vanishes.

Consider now the homogeneous linear hyperbolic equation

$$\frac{\partial^2 z}{\partial x \partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz = 0,$$

which we denote by (E). Assume that the $Y-$Laplace invariant h of (E) is non-zero. Taking the $Y-$Laplace transform of (E) yields a new equation (E_1), given by

$$\frac{\partial^2 z_1}{\partial x \partial y} + a_1\frac{\partial z_1}{\partial x} + b_1\frac{\partial z_1}{\partial y} + c_1 z_1 = 0,$$

which is homogeneous as well. Assume that $h_1 \neq 0, h_2 \neq 0, \cdots, h_{i-1} \neq 0$, so that we can take i successive $Y-$Laplace transforms, and obtain equation (E_i):

$$\frac{\partial^2 z_i}{\partial x \partial y} + a_i\frac{\partial z_i}{\partial x} + b_i\frac{\partial z_i}{\partial y} + c_i z_i = 0.$$

Assume now that $h_i = 0$. Then, the general solution of (E_i) is of the form $z_i = \alpha(X + \int \beta Y dy)$, where α, β are given functions of x and y, X is an arbitrary function of x, and Y and arbitrary function of y. Working our way backwards, to z, we obtain

$$z = A(X + \int \beta Y dy) + A_1(X' + \int Y\frac{\partial \beta}{\partial x}dy) + \cdots + A_i(X^{(i)} + \int Y\frac{\partial^i \beta}{\partial x^i}dy)$$

where $A, A_1, \cdots, A_i$ are given functions of x, y. Since Y is an arbitrary function of y, we may choose it as being identically zero. We then obtain:

$$z = AX + A_1X' + \cdots + A_iX^{(i)}. \tag{2.6}$$

We conclude that whenever the sequence (h_l) of $Y-$Laplace invariants of (E) terminates at $l = i$, i.e. $h_l \neq 0$ for $l < i$ and $h_i = 0$, then (E) has a solution of the form (2.6) depending on an arbitrary function of x. It is also a simple matter to show that conversely, whenever (E) admits a solution of the form (2.6), then its sequence (h_l) of $Y-$Laplace invariants terminates at most for $l = i$. We arrive at a similar conclusion concerning the $X-$Laplace invariants by exchanging x and y. In particular, if the sequence (k_l) of $X-$Laplace invariants is such that $k_l \neq 0$ for $l < j$ and $k_j = 0$, then (E) has a general solution of the form

$$z = BY + B_1 Y' + \cdots + B_j Y^{(j)} \tag{2.7}$$

where $B, B_1, \cdots, B_j$ are given functions of x, y, and Y is an arbitrary function of y; conversely, whenever (E) has a solution of the form (2.7), then the sequence (k_l) of $X-$Laplace invariants of (E) terminates at index j at the latest. If now both sequences (h_l) and (k_l) of Laplace invariants of (E) terminate, say (h_l) at index i and (k_l) at index j, then, by virtue of its linearity, (E) admits a general solution of the form

$$z = AX + A_1 X' + \cdots + A_i X^{(i)} + BY + B_1 Y' + \cdots + B_j Y^{(j)} \tag{2.8}$$

where $A, A_1, \cdots, A_i, B, B_1, \cdots, B_j$ are given functions of x, y, X is an arbitrary function of x, and Y and arbitrary function of y; and conversely, if (E) has a general solution of the form (2.8), then both of its sequences of Laplace invariants vanish.

2.5 Darboux integrability and Laplace invariants

Consider the linear hyperbolic equation given by:

$$\Phi(z) \equiv \frac{\partial^2 z}{\partial x \partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz = 0. \tag{2.9}$$

Assume that the sequence of $X-$Laplace invariants of (2.9) terminates at index p. Then, it follows that (2.9) admits a solution of the form

$$z = BY + B_1 Y' + \cdots + B_{p-1} Y^{(p-1)} \tag{2.10}$$

where $B, B_1, \cdots, B_{p-1}$ are given functions of x, y, and Y an arbitrary function of y. Differentiating both sides of (2.10) with respect to x yields

$$\frac{dz}{dx} = \frac{dB}{dx}Y + \frac{dB_1}{dx}Y' + \cdots + \frac{dB_{p-1}}{dx}Y^{(p-1)}$$

and p successive differentiations of both sides of (2.10) with respect to x yield p equations of the form

$$\frac{d^i z}{dx^i} = \frac{d^i B}{dx^i}Y + \frac{d^i B_1}{dx^i}Y' + \cdots + \frac{d^i B_{p-1}}{dx^i}Y^{(p-1)}, \quad i = 1, \cdots, p,$$

to which we can append equation (2.10). We thus obtain a system of $p+1$ linear equations in $Y, Y', \cdots, Y^{(p-1)}$, from which it follows that z must satisfy an ordinary differential equation of the form

$$F_1(z) \equiv \frac{\partial^p z}{\partial x^p} + T_1 \frac{\partial^{p-1} z}{\partial x^{p-1}} + \cdots + T_p = 0, \tag{2.11}$$

where the T_i are functions of x and y. Differentiating both sides of (2.11) with respect to y, differentiating both sides of (2.9) $(p-1)$ times with respect to x, and combining the results, we obtain an equation of the form

$$\frac{d^{p-1}\Phi}{dx^{p-1}} + \alpha_1 \frac{d^{p-2}\Phi}{dx^{p-2}} + \cdots + \alpha_{p-1}\Phi(z) - \frac{dF_1(z)}{dy} - \beta F_1(z) = 0, \tag{2.12}$$

where $\alpha_1, \alpha_2, \cdots, \beta$ are functions of x and y. The coefficient of $\frac{\partial^p z}{\partial x^p}$ in this equation is $a - \beta$, and equating it to zero, we can rewrite (2.12) as

$$e^{\int a dy}\{\frac{d^{p-1}\Phi}{dx^{p-1}} + \alpha_1 \frac{d^{p-2}\Phi}{dx^{p-2}} + \cdots + \alpha_{p-1}\Phi(z)\} = \frac{d}{dy}\{e^{\int a dy}F_1(z)\}.$$

Thus, any solution z of (2.9) automatically satisfies $\frac{d}{dy}\{e^{\int a dy}F_1(z)\} = 0$ as well. Noting that the first system of characteristics of (2.9) is given by

$$(I)\begin{cases} dx = 0, \\ dz = q_1 dy, \\ dq_1 = q_2 dy, \\ \cdots, \\ dq_{n-1} = q_n dy, \\ dp_1 = p_2 dy, \\ \cdots, \\ dp_n = \frac{\partial p_n}{\partial y} dy, \end{cases}$$

we have that

$$d(e^{\int ady}F_1(z)) = \frac{d(e^{\int ady}F_1(z))}{dx}dx + \frac{d(e^{\int ady}F_1(z))}{dy}dy = \frac{d(e^{\int ady}F_1(z))}{dx}dx,$$

and hence $e^{\int ady}F_1(z)$ is a function of order p such that its differential lies in the linear span of the first system of characteristics of (2.9); this yields two independent invariant functions for the first system of characteristics, namely x and $e^{\int ady}F_1(z)$. Similarly, if the sequence of $X-$Laplace invariants of (2.9) terminates at index q, then we can construct a function of order q of the form $e^{\int bdx}G_1(z)$ such that its differential $d(e^{\int bdx}G_1(z))$ lies in the linear span of the second system of characteristics of (2.9); this yields two independent invariant functions for the second system of characteristics, namely y and $e^{\int bdx}G_1(z)$. We thus arrive at the important conclusion that if both sequences of Laplace invariants of the linear hyperbolic equation (2.9) terminate, then (2.9) is Darboux-integrable. It is also shown in [8] that conversely, whenever (2.9) is Darboux-integrable, then both of its sequences of Laplace invariants do terminate.

As a simple example, consider the linear hyperbolic equation

$$s = z. \tag{2.13}$$

The $Y-$Laplace invariant of (2.13) is $h = 1$, and its $Y-$Laplace transform yields the same equation. By symmetry, the same is true for the $X-$ Laplace invariant and transform. In particular, the sequences of $X-$ and $Y-$ Laplace invariants of (2.13) never terminate. Thus, (2.13) is not Darboux integrable.

Chapter 3

The jet bundle and the variational bicomplex

In this chapter, we present the formal framework in which the geometric study of differential equations is performed. Although this study has its roots in the work of Élie Cartan [4], we follow here the modern treatment described in [1] and based on the variational bicomplex originally introduced by Tulczyjew [10].

3.1 Pfaffian systems

Consider the partial differential equation

$$F(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}) = 0. \tag{3.1}$$

As we saw earlier, we can replace the partial derivatives $z_x, z_y, z_{xx}, z_{xy}, z_{yy}$ with the variables p, q, r, s, t, respectively, so as to treat them as variables in their own right. Let now $U \subset \mathbb{R}^2$ be an open set and assume that the equation

$$F(x, y, z, p, q, r, s, t) = 0 \tag{3.2}$$

defines a seven-dimensional submanifold Σ_7 of $U \times \mathbb{R}^6$ which is parametrized by (x, y). Let $i : \Sigma_7 \hookrightarrow U \times \mathbb{R}^6$ be the inclusion mapping. Consider the 1-forms on

$U \times \mathbb{R}^6$ given by

$$\begin{cases} \omega^1 = dz - pdx - qdy, \\ \omega^2 = dp - rdx - sdy, \\ \omega^3 = dq - sdx - tdy. \end{cases}$$

Let $z : (x, y) \mapsto z(x, y)$ be any $\mathbb{R}$-valued function on U. The graph of z is defined as the image of the map $(x, y) \mapsto (x, y, z(x, y))$; we define the 1-jet of z as the map

$$j^1 z : (x, y) \mapsto (x, y, z(x, y), z_x(x, y), z_y(x, y)),$$

and its two-jet as the map

$$j^2 z : (x, y) \mapsto (x, y, z(x, y), z_x(x, y), z_y(x, y), z_{xx}(x, y), z_{xy}(x, y), z_{yy}(x, y)).$$

If z is a solution to (3.1), then $j^2 z(x, y) \in \Sigma_7, \forall (x, y) \in U$, and $(j^2)^\star \omega^1 = (j^2)^\star \omega^2 = (j^2)^\star \omega^3 = 0$. In other words, the map $j^2 z$ defines an integral manifold for the system of 1-forms $\{\omega^1, \omega^2, \omega^3\}$ which is a submanifold of Σ_7. Now since $i^\star d\omega = d(i^\star \omega)$, if z is a solution to (3.1), then $j^2 z$ defines an integral submanifold of Σ_7 for the system $\{\omega^1, \omega^2, \omega^3, d\omega^1, d\omega^2, d\omega^3\}$ as well, and since $i^\star(\omega \wedge \eta) = i^\star(\omega) \wedge i^\star(\eta)$, $j^2 z$ defines an integral submanifold of Σ_7 for the ideal generated by the forms $\{\omega^1, \omega^2, \omega^3, d\omega^1, d\omega^2, d\omega^3\}$. Such an ideal is called a differential ideal. If, as is the case here, this ideal is generated by a family of 1-forms and their exterior derivatives, then such an ideal is called a Pfaffian system. Thus, two-jets of solutions to (3.1) are submanifolds of Σ_7 which are integral manifolds of the Pfaffian system generated by the forms $\{\omega^1, \omega^2, \omega^3\}$. Conversely, if $i : (x, y) \mapsto (x, y, z, p, q, r, s, t)$ is a two-dimensional submanifold of Σ_7 such that $i^\star \omega^1 = i^\star \omega^2 = i^\star \omega^3 = 0$, then $z : (x, y) \mapsto z(x, y)$ is a solution to (3.1). Thus the problem of finding solutions to (3.1) is equivalent to the problem of finding two-dimensional integral manifolds of the Pfaffian system generated by $\{\omega^1, \omega^2, \omega^3\}$ on the seven-dimensional submanifold Σ_7 of $U \times \mathbb{R}^6$ defined by (3.2).

Equation (3.1) gives one set of relations between the various partial derivatives of order 2 of a solution. Relations between higher-order partial derivatives can also be established by differentiating both sides of (3.1) arbitrarily many times with respect to x and y. For example, differentiating both sides of (3.1) once with respect to x and once with respect to y yields the two additional relations:

$$\begin{aligned}
\frac{dF}{dx} &= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} z_x + \frac{\partial F}{\partial z_x} z_{xx} + \frac{\partial F}{\partial z_y} z_{xy} + \\
&\quad \frac{\partial F}{\partial z_{xx}} z_{xxx} + \frac{\partial F}{\partial z_{xy}} z_{xxy} + \frac{\partial F}{\partial z_{yy}} z_{xyy} = 0, \\
\frac{dF}{dy} &= \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} z_y + \frac{\partial F}{\partial z_x} z_{xy} + \frac{\partial F}{\partial z_y} z_{yy} + \\
&\quad \frac{\partial F}{\partial z_{xx}} z_{xxy} + \frac{\partial F}{\partial z_{xy}} z_{xyy} + \frac{\partial F}{\partial z_{yy}} z_{yyy} = 0.
\end{aligned}$$

Here again, solving (3.1) is equivalent to finding two-dimensional integral manifolds of a particular Pfaffian system (generated by six one-forms) on a nine-dimensional submanifold of $U \times \mathbb{R}^{10}$ (more precisely, the one defined by $F = 0$, $\frac{dF}{dx} = 0$, $\frac{dF}{dy} = 0$). Note that working on this nine-dimensional submanifold, we can consider functions which involve partial derivatives of the third order of the solution to (3.1), something which could not be done in the context of the former, seven-dimensional submanifolds. There is a real advantage gained in not restraining oneself a priori by the derivative order; this suggests that one must work not on a finite dimensional submanifold of some $U \times \mathbb{R}^n$, no matter how large n, but rather on an infinite dimensional submanifold of another infinite dimensional manifold. This is what we shall make precise next.

3.2 Jet bundles

Let $\pi : E \to M$ be a fiber bundle over a connected base manifold M. Since all our considerations will be local, we can think of M as an open connected subset of $\mathbb{R}^n$, for some $n \in \mathbb{N}^\star$. Similarly, we can think of E as the product $M \times \mathbb{R}^m$ for some $m \in \mathbb{R}^\star$, and of π as the trivial bundle. Let $s : M \to E$ be a section of π. The image $s(M)$ of s is the graph of a particular $\mathbb{R}^m$-valued function on M. Conversely, if $f : M \to \mathbb{R}^m$ is a given function, then the map $x \mapsto (x, f(x))$ is a section of π. We extend this basic formalism as follows: if $s : U \to E$ and $s' : U \to E$ are two local sections of E, and U, U' open subsets of M, then we say that s and s' are equivalent at $p \in U \cap U'$ to order k if their partial derivatives to order k agree at p. Note that this definition is independent of any trivialization used. The equivalence class of s at p under this equivalence relation is denoted by $j^k(s)(p)$, and is called the k–jet of s at p. We thus define the bundle $\pi^k_M : J^k(E) \to M$ of k-jets of local sections of E. For any point of M, the fiber $(\pi^k_M)^{-1}(p)$ consists of equivalence classes of local sections of π at p. If the local coordinates on M are given by $(x^i)_{i=1,\cdots,n}$, and the fiber coordinates on E by $(z^\alpha)_{\alpha=1,\cdots,m}$, then the induced local coordinates on $J^k(E)$ are $(x^i, z^\alpha, z^\alpha_{i_1}, z^\alpha_{i_1 i_2}, \cdots, z^\alpha_{i_1 i_2 \cdots i_k})$, where $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n$, and where $z^\alpha(j^k(s)(p)) = s^\alpha(p), z^\alpha_i(j^k(s)(p)) = \frac{\partial s^\alpha}{\partial x^i}(p), z^\alpha_{ij}(j^k(s)(p)) = \frac{\partial^2 s^\alpha}{\partial x^i \partial x^j}(p)$, and so on. We define $J^0(E)$ as being E itself. For any $k, l \in \mathbb{N}$ with $k \geq l$, we have the obvious projection $\pi^k_l : J^k(E) \to J^l(E)$. This defines an inverse system $\{J^k(E), \pi^k_l\}$ of topological spaces. The inverse limit of this system is denoted by $J^\infty(E)$ and yields the bundle $\pi^\infty_M : J^\infty(E) \to M$, together with the projection maps $\pi^\infty_k : J^\infty(E) \to J^k(E)$, and $\pi^\infty_E : J^\infty(E) \to E$. $\pi^\infty_M : J^\infty(E) \to M$ is called the infinite jet bundle of the bundle $\pi : E \to M$. The fiber at p of the infinite jet bundle is defined as equivalence classes, denoted $j^\infty(s)(p)$ of local sections of p, where two local sections s, s' are said to be equivalent at p if they have the same partial derivatives to all orders at p. The equivalence class of s at p is called the infinite jet of s at p, and is denoted by $j^\infty(s)(p)$. Clearly, any section $s : U \to E$ of the bundle π lifts to a unique section $j^\infty(s) : U \to J^\infty(E)$, called the infinite jet of s. Conceptually, if E is the space coordinatized by the independent variables and the dependent variables, the latter representing function values, $J^\infty(E)$ is the space coordinatized by the independent variables, the dependent variables, and all partial derivatives of the dependent variables with respect to the independent variables.

Let $f : J^k(E) \to P$, where P is some arbitrary smooth manifold, be a smooth function. For any $l \geq k$, we can pull back f to a smooth P–valued function on $J^l(E)$ through the map $f \mapsto f \circ \pi^l_k = (\pi^l_k)^\star(f)$. This defines the direct sequence $\{C^\infty(J^k(E), P), (\pi^l_k)^\star\}$, and we define the set $C^\infty(J^\infty(E), P)$ of smooth functions from $J^\infty(E)$ to P as the direct limit of this sequence. By definition of the direct limit, a function $f : J^\infty(E) \to P$ is smooth if and only if it factors through a smooth function on $J^k(E)$ for some $k \in \mathbb{N}$, i.e. there exists $f_k : J^k(E) \to P$ such that $f = f_k \circ \pi_k^\infty$. We call k the order of f. Viewing functions as zero-forms, we can extend this construction to differential forms as well. Let $\bigwedge^p(J^k(E))$ be the p^{th} exterior product bundle. The sequence $\{\bigwedge^p(J^k(E)), (\pi_k^l)^\star\}$ is a direct sequence with direct limit $\bigwedge^p(J^\infty(E))$, the p^{th} exterior product bundle of $J^\infty(E)$. A section of $\bigwedge^p(J^\infty(E))$ is called a differential form on E. Every smooth differential $p-$form ω on $J^\infty(E)$ is represented by a $p-$form $\hat{\omega}$ on $J^k(E)$ for some $k \in \mathbb{N}$, i.e. $\omega = (\pi_k^\infty)^\star(\hat{\omega})$. In local coordinates, a $p-$form ω on $J^\infty(E)$ is therefore a finite sum of terms of the type

$$A[x,z]dz_{I_1}^{\alpha_1} \wedge \cdots \wedge dz_{I_s}^{\alpha_s} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r}, \tag{3.3}$$

where $r + s = p$, and where the coefficient A is a smooth function on $J^\infty(E)$. The order of the form (3.3) is the maximum of the orders of the coefficient function A and the differentials dz_I^α.

A vector field X on $J^\infty(E)$ is defined as a derivation on the ring $C^\infty(J^\infty(E))$ of smooth functions on $J^\infty(E)$. In local coordinates, we have

$$X = A^i \frac{\partial}{\partial x^i} + \sum_{k=0,\alpha}^{\infty} B^\alpha_{i_1 i_2 \cdots i_k} \frac{\partial}{\partial z^\alpha_{i_1 i_2 \cdots i_k}}, \tag{3.4}$$

where the coefficients A^i and $B^\alpha_{i_1 \cdots i_k}$ are smooth functions on $J^\infty(E)$. Since smooth functions f and forms ω on $J^\infty(E)$ are of finite order, the expressions $< df, X >$ and $< \omega, X >$ always reduce to finite sums and are therefore well-defined.

Definition 3.1 *A differential form ω on $J^\infty(E)$ is called a contact form if, for any local section s of E, we have $(j^\infty(s))^\star(\omega) = 0$.*

Contact forms are of fundamental importance in all that follows since they characterize submanifolds of $J^\infty(E)$ which are graphs of infinite jets of functions on M. The distributivity of the pullback operation with respect to the cup product on forms implies that the set of contact forms on $J^\infty(E)$ is actually an ideal in $\Omega^\star(J^\infty(E))$, denoted by $\mathcal{C}(J^\infty(E))$, and called the contact ideal of $J^\infty(E)$. $\mathcal{C}(J^\infty(E))$ is generated by the contact one-forms

$$\theta^\alpha_{i_1\cdots i_k} = dz^\alpha_{i_1\cdots i_k} - z^\alpha_{i_1\cdots i_k j}dx^j,$$

for all $k \in \mathbb{N}$. We denote by $\mathcal{C}^s(J^\infty(E))$ the s^{th} exterior product of the contact ideal $\mathcal{C}(J^\infty(E))$.

Definition 3.2 *Let $\pi' : E' \to M'$ be another fiber bundle. A smooth map*

$$\Phi : J^\infty(E) \to J^\infty(E')$$

is called a generalized contact transformation if it preserves the contact ideal, i.e.

$$\Phi^\star(\mathcal{C}(J^\infty(E')) \subset \mathcal{C}(J^\infty(E)).$$

If, in addition, Φ covers a smooth map $\Phi_1 : J^1(E) \to J^1(E')$, i.e. $\Phi_1 \circ \pi^\infty_1 = (\pi')^\infty_1 \circ \Phi$, then Φ is called a classical contact transformation.

Definition 3.3 *A vector field X on $J^\infty(E)$ such that $X \neg \omega = 0$ for every contact one-form ω is called a total vector field.*

A total vector field on $J^\infty(E)$ can be seen as a vector tangent to the graph of the infinite jet of a section of E. If the vector field in (3.4) is a total vector field, then we should have $X \neg \theta^\alpha_{i_1\cdots i_k} = 0$, $\forall \alpha \in \{1, \cdots, m\}, \forall k \in \mathbb{N}$, yielding $X = A^j D_j$, with D_j the total vector field

$$D_j = \frac{\partial}{\partial x^j} + \frac{\partial}{\partial z^\alpha} z^\alpha_j + \frac{\partial}{\partial z^\alpha_{i_1}} z^\alpha_{i_1 j} + \cdots .$$

The Lie bracket $[X, Y]$ of the two total vector fields $X = A^i D_i$ and $Y = B^i D_i$ is the total vector field defined as $[X, Y] = (A^j D_j B^i - B^j D_j A^i) D_i$.

3.3 The variational bicomplex

We say a form ω on $J^\infty(E)$ is of type (r, s), with $r + s = p$, if $\omega(X_1, \cdots, X_p) = 0$ whenever either more than s of the vector fields X_i are π^∞_M−vertical, or more than

r of them are total vector fields. Let $\Omega^{r,s}(J^\infty(E))$ be the space of type (r,s) forms on $J^\infty(E)$. In local coordinates, a type (r,s) form is a finite sum of terms

$$a(x, z^{(k)})dx^{i_1} \wedge \cdots \wedge dx^{i_r} \wedge \theta^{\alpha_1}_{j_1\cdots j_{p_1}} \wedge \cdots \theta^{\alpha_1}_{k_1\cdots k_{p_s}}.$$

By virtue of working on the infinite jet bundle $J^\infty(E)$, we have the direct sum decomposition

$$\Omega^p(J^\infty(E)) = \bigoplus_{r+s=p} \Omega^{r,s}(J^\infty(E)),$$

and we define the corresponding projection map $\pi^{r,s} : \Omega^p(J^\infty(E)) \to \Omega^{r,s}(J^\infty(E))$. Using local coordinates, it can be seen that the exterior derivative is a map

$$d : \Omega^{r,s}(J^\infty(E)) \to \Omega^{r+1,s}(J^\infty(E)) \oplus \Omega^{r,s+1}(J^\infty(E)).$$

We can thus write $d = d_H + d_V$, where $d_H : \Omega^{r,s}(J^\infty(E)) \to \Omega^{r+1,s}(J^\infty(E))$ and $d_V : \Omega^{r,s}(J^\infty(E)) \to \Omega^{r,s+1}(J^\infty(E))$ are defined as $d_H = \pi^{r+1,s} \circ d$ and $d_V = \pi^{r,s+1} \circ d$, respectively. d_H is called the horizontal, and d_V the vertical, exterior derivative. Since $0 = d^2 = (d_H + d_V)^2 = d_H^2 + d_H d_V + d_V d_H + d_V^2$, grouping these terms by degree, we obtain $d_H^2 = d_V^2 = 0$ and $d_H d_V = -d_V d_H$. We define the free variational bicomplex for the fiber bundle $\pi : E \to M$ as the double complex $(\Omega^{\star,\star}(J^\infty(E)), d_H, d_V)$ of differential forms on the infinite jet bundle $J^\infty(E)$ of E.

$$\begin{tikzcd}
 & & \uparrow d_V & \uparrow d_V & & \uparrow d_V \\
 & 0 \arrow[r] & \Omega^{0,2}(\mathcal{C}^\infty) \arrow[r, "d_H"] \arrow[u, "d_V"] & \Omega^{1,2}(\mathcal{C}^\infty) \arrow[r, "d_H"] \arrow[u, "d_V"] & \cdots \arrow[r, "d_H"] & \Omega^{n,2}(\mathcal{C}^\infty) \arrow[u, "d_V"] \\
 & 0 \arrow[r] & \Omega^{0,1}(\mathcal{C}^\infty) \arrow[r, "d_H"] \arrow[u, "d_V"] & \Omega^{1,1}(\mathcal{C}^\infty) \arrow[r, "d_H"] \arrow[u, "d_V"] & \cdots \arrow[r, "d_H"] & \Omega^{n,1}(\mathcal{C}^\infty) \arrow[u, "d_V"] \\
0 \arrow[r] & \mathbb{R} \arrow[r] & \Omega^{0,0}(\mathcal{C}^\infty) \arrow[r, "d_H"] \arrow[u, "d_V"] & \Omega^{1,0}(\mathcal{C}^\infty) \arrow[r, "d_H"] \arrow[u, "d_V"] & \cdots \arrow[r, "d_H"] & \Omega^{n,0}(\mathcal{C}^\infty) \arrow[u, "d_V"]
\end{tikzcd}$$

As an example, if $f \in C^\infty(J^\infty(E))$ is a smooth function on $J^\infty(E)$,

$$\begin{aligned}
df &= \frac{\partial f}{\partial x^i}dx^i + \frac{\partial f}{\partial z^\alpha}dz^\alpha + \frac{\partial f}{\partial z^\alpha_i}dz^\alpha_i + \cdots \\
&= [\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial z^\alpha}z^\alpha_i + \frac{\partial f}{\partial z^\alpha_j}z^\alpha_{ij} + \cdots]dx^i + \\
&\quad [\frac{\partial f}{\partial z^\alpha}\theta^\alpha + \frac{\partial f}{\partial z^\alpha_i}\theta^\alpha_i + \frac{\partial f}{\partial z^\alpha_{ij}}\theta^\alpha_{ij} + \cdots].
\end{aligned}$$

Thus, $d_H f = \pi^{1,0}(df) = [\frac{\partial f}{\partial x^i} + \frac{\partial f}{\partial z^\alpha} z_i^\alpha + \frac{\partial f}{\partial z_j^\alpha} z_{ij}^\alpha + \cdots:]dx^i = D_i f dx^i$, and $d_V f = \pi^{0,1}(df) = [\frac{\partial f}{\partial z^\alpha}\theta^\alpha + \frac{\partial f}{\partial z_i^\alpha}\theta_i^\alpha + \frac{\partial f}{\partial z_{ij}^\alpha}\theta_{ij}^\alpha + \cdots]$.

Now let $\pi' : E' \to M'$ be another fiber bundle, and let

$$\Phi : J^\infty(E) \to J^\infty(E')$$

be a smooth map. In general, the pullback

$$\Phi^\star : \Omega^p(J^\infty(E')) \to \Omega^p(J^\infty(E'))$$

will not preserve the horizontal and vertical bigrading on forms. This leads us to the following:

Definition 3.4 *For $\omega \in \Omega^{r,s}(J^\infty(E))$, we define the projected pullback map $\Phi^\sharp$ of Φ by*

$$\Phi^\sharp(\omega) = \pi^{r,s}(\Phi^\star(\omega)).$$

We have the following important result:

Proposition 3.1 *If $\Phi : J^\infty(E) \to J^\infty(E')$ is a generalized contact transformation, then the projected pull-back map $\Phi^\sharp$ commutes with d_H:*

$$\Phi^\sharp \circ d_H = d_H \circ \Phi^\sharp.$$

Proof. Since $\Phi^\star$ preserves the contact ideal, we have

$$\begin{aligned}\Phi^\star(J^\infty(E')) &\subset \mathcal{C}(J^\infty(E)) \\ &\subset \Omega^{r,s}(J^\infty(E)) \oplus \Omega^{r-1,s+1}(J^\infty(E)) \oplus \Omega^{r-2,s+2}(J^\infty(E)) \oplus \cdots .\end{aligned}$$

This shows that

$$\pi^{r+1,s}(d\Phi^\star\omega) = d_H(\Phi^\sharp\omega) \quad \text{and} \quad d_V(\Phi^\sharp\omega) = 0,$$

and the conclusion follows. $\square$

We have a similar definition for total vector fields:

Definition 3.5 *If X is a total vector field on $J^\infty(E)$ and $\omega \in \Omega^{r,s}(J^\infty(E))$, we define $X(\omega) \in \Omega^{r,s}(J^\infty(E))$ to be the projected Lie derivative $\pi^{r,s}(\mathcal{L}_X\omega)$ of ω with respect to X.*

We have the following important Cartan-type formula for the projected Lie derivative:

$$X(\omega) = X\neg d_H(\omega) + d_H(X\neg\omega).$$

Note that we have in particular $D_j\theta^\alpha_{i_1\cdots i_k} = \theta^\alpha_{i_1\cdots i_k j}$. Note also that $X(d_H\omega) = X\neg d_H^2(\omega) + d_H(X\neg d_H\omega) = d_H(X\neg d_H\omega) + d_H^2(X\neg\omega) = d_H(X(w))$. We shall make constant use of these relations in all that follows. We shall also make constant use of the following commutation relations, which are easily proved:

Proposition 3.2 *Let $\omega \in \Omega^{r,s}(J^\infty(E))$. If X and Y are total vector fields on $J^\infty(E)$ and Z is any π^∞_M-vertical vector field on $J^\infty(E)$, then*

$$\begin{aligned} X(Y(\omega)) - Y(X(\omega)) &= [X,Y](\omega), \\ Z\neg X(w) &= [Z,X](\omega) + X(Z\neg\omega). \end{aligned}$$

We will also make crucial use of the following operators:

Definition 3.6 *We define the interior Euler operator J in local coordinates by*

$$\begin{aligned} J \ &: \ \Omega^{n,s}(J^\infty(E)) \to \Omega^{n,s}(J^\infty(E)) \\ &\omega \mapsto J(\omega) = \frac{1}{s}\sum_{|I|=0}^{\infty}(-D)_I[\partial^I_\alpha\neg\omega] \end{aligned}$$

where I is a multi-index and $|I|$ its length.

We have the following important proposition:

Proposition 3.3 *The interior Euler operator satisfies $J \circ d_H = 0$ and $J^2 = J$.*

For the proof, we refer to [1].

Definition 3.7 *For $s \geq 1$, we define the horizontal homotopy operator*

$$\begin{aligned} h_H^{r,s} &: \Omega^{r,s}(J^\infty(E)) \to \Omega^{r-1,s}(J^\infty(E)) \\ &\omega \mapsto h_H^{r,s}(\omega) = \frac{1}{s}\sum_{|I|=0}^{k-1} \frac{|I|+1}{n-r+|I|+1} D_I[\theta^\alpha \wedge F_\alpha^{Ij}(\omega_j)], \end{aligned}$$

where $w_j = D_j \neg \omega$ and

$$F_\alpha^I = \sum_{|J|=0}^{k-|I|} \begin{pmatrix} |I|+|J| \\ |I| \end{pmatrix} (-D)_J(\partial_\alpha^{IJ} \neg \omega).$$

As its name indicates, the horizontal homotopy operator $h_H^{r,s}$ satisfies the following relation, which is of key importance in what follows:

Proposition 3.4 *For $s \geq 1$, the horizontal homotopy operator $h_H^{r,s}$ satisfies the relation*

$$h_H^{r+1,s}(d_H\omega) + d_H(h_H^{r,s}\omega) = \omega, \quad \forall \omega \in \Omega^{r,s}(J^\infty(E)).$$

For the proof, we refer again to [1].

3.4 The constrained variational bicomplex and conservation laws

Consider a second-order partial differential equation defined in an open connected subset U of $\mathbb{R}^2$ and given by

$$F(x, y, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}) = 0, \tag{3.5}$$

or, equivalently

$$F(x, y, z, p, q, r, s, t) = 0, \tag{3.6}$$

with the classical Monge notation. Equation (3.6) defines a submanifold of E, where E is the total space of the bundle $U \times \mathbb{R}^6 \to U$. Let $\mathcal{R}$ denote an open connected and contractible subset of this submanifold. Let $i : \mathcal{R}^2 \hookrightarrow J^2(E)$ be the inclusion map. The successive prolongations of $\mathcal{R}^2$ are defined recursively by the total derivatives of (3.5), e.g. $\mathcal{R}^3 = \{j^3(s)(p) | j^2(s)(p) \in \mathcal{R}^2 | (D_xF)(j^3(s)(p)) = (D_yF)(j^3(s)(p)) = 0\}$. Each prolongation $i : \mathcal{R}^k \hookrightarrow J^k(E)$ fibers over $\mathcal{R}^{k-1}$ and this yields and inverse system with inverse limit $i : \mathcal{R}^\infty \hookrightarrow J^\infty(E)$. $\mathcal{R}^\infty$ is called the infinite prolongation of $\mathcal{R}^2$. We have the obvious projections $\pi^\infty_k : \mathcal{R}^\infty \to \mathcal{R}^k$ and $\pi^\infty_U : \mathcal{R}^\infty \to U$. We denote by $\mathcal{C}(\mathcal{R}^\infty)$ the pullback of the contact ideal on $J^\infty(E)$ to $\mathcal{R}^\infty$, i.e., $\mathcal{C}(\mathcal{R}^\infty) = i^\star(\mathcal{C}(J^\infty(E)))$, and we let $\mathcal{R}$ be the triple $\mathcal{R} = (\mathcal{R}^\infty, \pi^\infty_U, \mathcal{C}(\mathcal{R}^\infty))$. It can be easily shown ([5]) that local solutions to (3.5) are in one-to-one correspondence with sections σ of $\pi^\infty_U : \mathcal{R}^\infty \to U$ which satisfy $\sigma^\star(\mathcal{C}(\mathcal{R}^\infty)) = 0$. A total vector field on $\mathcal{R}^\infty$ is a vector field X such that $X \neg \omega = 0$ for any 1-form $\omega \in \mathcal{C}(\mathcal{R}^\infty)$. We can thus bigrade the differential forms on $\mathcal{R}^\infty$ by horizontal and vertical degree, just as on $J^\infty(E)$. We then define the variational bicomplex for $\mathcal{R} = \{\mathcal{R}^\infty, \pi^\infty_U, \mathcal{C}(\mathcal{R}^\infty)\}$ to be the pullback of the free variational bicomplex $(\Omega^{\star,\star}(J^\infty(E)), d_H, d_V)$ to $\mathcal{R}^\infty$.

$$
\begin{array}{ccccccccc}
 & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} \\
 & & 0 & \longrightarrow & \Omega^{0,2}(\mathcal{R}^\infty) & \xrightarrow{d_H} & \Omega^{1,2}(\mathcal{R}^\infty) & \xrightarrow{d_H} & \Omega^{2,2}(\mathcal{R}^\infty) \\
 & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} \\
 & & 0 & \longrightarrow & \Omega^{0,1}(\mathcal{R}^\infty) & \xrightarrow{d_H} & \Omega^{1,1}(\mathcal{R}^\infty) & \xrightarrow{d_H} & \Omega^{2,1}(\mathcal{R}^\infty) \\
 & & & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} & & \uparrow{\scriptstyle d_V} \\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \Omega^{0,0}(\mathcal{R}^\infty) & \xrightarrow{d_H} & \Omega^{1,0}(\mathcal{R}^\infty) & \xrightarrow{d_H} & \Omega^{2,0}(\mathcal{R}^\infty)
\end{array}
$$

Consider the one-form

$$\omega = M \wedge dx + N \wedge dy,$$

where M and N are smooth functions on $\mathcal{R}^\infty$. If ω is d_H−closed, that is, $d_H\omega = 0$, we call ω a classical conservation law of the hyperbolic equation $\mathcal{R}$. If M and N are type $(0, s)$ contact forms $(s \geq 1)$ and $d_H\omega = 0$, we call ω a type $(0, s)$ contact form-valued conservation law of $\mathcal{R}$. If in addition ω is d_H−exact, i.e. there exists a type $(s-1, 1)$ form γ such that $\omega = d_H\gamma$, then ω is called a trivial conservation law. In general, a type (r, s) form which is d_H closed is called a type (r, s) conservation law of $\mathcal{R}$. We shall be interested in non-trivial conservation laws of $\mathcal{R}$, i.e. the cohomology groups $H^{r,s}(\mathcal{R}^\infty, d_H)$. Since type $(2, s)$ forms are trivially d_H—closed, and since, by a theorem of Vinogradov [13], $H^{0,s}(\mathcal{R}^\infty) = 0$ for $s \geq 1$, we shall focus solely on type $(1, s)$ contact form-valued conservation laws, that is, on elements of

$$H^{1,s}(\mathcal{R}^\infty, d_H) = \frac{Ker(d_H : \Omega^{1,s}(\mathcal{R}^\infty) \to \Omega^{2,s}(\mathcal{R}^\infty))}{Im(d_H : \Omega^{0,s}(\mathcal{R}^\infty) \to \Omega^{1,s}(\mathcal{R}^\infty))}, \quad s \geq 1.$$

Consider now equation (3.5). With no loss of generality, we can assume that we can rewrite this equation as

$$z_{xx} + f(x, y, z, z_x, z_y, z_{xy}, z_{yy}) = 0.$$

The natural coordinates for $\mathcal{R}^\infty$ are then

$$(x, y, z, z_x, z_y, z_{xy}, z_{yy}, \cdots, z_{xy^{k-1}}, z_{xy^k}, \cdots),$$

and a basis for the contact ideal on $\mathcal{R}^\infty$ is

$$\{\theta, \theta_x, \theta_y, \theta_{xy}, \theta_{yy}, \cdots, \theta_{xy^{k-1}}, \theta_{xy^k}, \cdots\},$$

where

$$\begin{cases} \theta = dz - z_x dx - z_y dy, \\ \theta_{xy^{k-1}} = dz_{xy^{k-1}} + (D_y^{k-1} f)dx - z_{xy^k} dy, \\ \theta_{y^k} = dz_{y^k} - z_{xy^k} dx - z_{y^{k+1}} dy. \end{cases}$$

We conclude this section with an example adapted from [2]. Let then E and E' be two copies of the trivial bundle $\pi : \mathbb{R}^3 \to \mathbb{R}^2$, with respective jet coordinates

$$(x, y, u, u_x, u_y, \cdots) \quad \text{and} \quad (v, w, z, z_v, z_w, \cdots).$$

Consider the wave equation

$$z_{vv} - z_{ww} = 0 \tag{3.7}$$

and the hyperbolic Monge-Ampère equation

$$u_{xx} u_{yy} - u_{xy}^2 = -1. \tag{3.8}$$

Consider the unique contact transformation $\Phi : J^\infty(E) \to J^\infty(E')$ determined by the relations

$$v \circ \Phi = u_x, \quad w \circ \Phi = y, \quad z \circ \Phi = u - x u_x,$$
$$z_v \circ \Phi = -x, \quad z_w \circ \Phi = u_y,$$

It immediately follows from these relations that $\Phi^\star(\phi) = \theta$, where ϕ is the contact form $\phi = dz - z_v dv - z_w dw$ on $J^\infty(E')$, and θ is the contact form $\theta = du - u_x dx - u_y dy$ on $J^\infty(E)$. We have

$$\begin{aligned} \Phi^\star(dz_v - z_{vv} dv - z_{vw} dw) &= -dx - (z_{vv} \circ \Phi) du_x - (z_{vw} \circ \Phi) dy \\ &= -(z_{vv} \circ \Phi)[du_x + \frac{1}{z_{vv} \circ \Phi} dx + \frac{z_{vw} \circ \Phi}{z_{vv} \circ \Phi} dy], \end{aligned}$$

and hence, the contact condition

$$\Phi^\star(\phi_v) \equiv 0 \quad \bmod \{\theta, \theta_x, \theta_y\}$$

implies that

$$z_{vv} \circ \Phi = -\frac{1}{u_{xx}} \quad \text{and} \quad z_{vw} \circ \Phi = \frac{u_{xy}}{u_{xx}}$$

In addition, the contact condition

$$\Phi^\star(\phi_w) \equiv 0 \quad \bmod \{\theta, \theta_x, \theta_y\}$$

implies that

$$z_{ww} \circ \Phi = u_{yy} - \frac{u_{xy}^2}{u_{xx}}$$

and consequently, we obtain

$$(z_{vv} - z_{ww}) \circ \Phi = -\frac{1}{u_{xx}}(u_{xx} u_{yy} - u_{xy}^2 + 1).$$

Thus, equations (3.7) and (3.8) are equivalent under the contact transformation Φ. Furthermore, it is easy to verify that Φ preserves characteristic directions. Indeed, the characteristic equation of (3.7) is

$$dw^2 - dv^2 = 0,$$

with characteristic directions

$$D_v \pm D_w,$$

and, $(dw^2 - dv^2)(D_v \pm D_w) = 0$. Similarly, the characteristic equation of (3.8) is

$$u_{yy}dy^2 + 2u_{xy}dxdy + u_{xx}dx^2 = 0,$$

with characteristic directions

$$u_{yy}D_x + (-u_{xy} \pm 1)D_y,$$

and, $(u_{yy}dy^2+2u_{xy}dxdy+u_{xx}dx^2)(u_{yy}D_x+(-u_{xy}\pm1)D_y) = u_{yy}(u_{xx}u_{yy}-u_{xy}^2+1) = 0$ as a consequence of (3.8). Now,

$$\begin{aligned} < dv, \Phi_\star(D_x) > &= < \Phi^\star(dv), D_x > \\ &= < du_x, D_x > \\ &= u_{xx}, \end{aligned}$$

and hence,

$$\Phi_\star^{-1}(D_v) = \frac{1}{u_{xx}}D_x.$$

Similarly, we have that

$$\Phi_\star^{-1}(D_w) = -\frac{u_{xy}}{u_{xx}}D_x + D_y,$$

and we obtain therefore

$$\begin{aligned} \Phi_\star^{-1}(D_v \pm D_w) &= \frac{1}{u_{xx}} \pm (-\frac{u_{xy}}{u_{xx}}D_x + D_y) \\ &= \frac{1}{u_{xx}}[(1 \mp u_{xy})D_x \pm u_{xx}D_y] \\ &= \frac{1}{u_{xx}(1 \pm u_{xy})}[(1 - u_{xy}^2)D_x + u_{xx}(u_{xy} \pm 1)D_y] \\ &= -\frac{1}{1 \pm u_{xy}}[u_{yy}D_x + (-u_{xy} \mp 1)D_y]. \end{aligned}$$

Thus, Φ preserves the characteristic directions.

Finally, consider the one-form

$$\omega = \frac{1}{2}(z_v^2 + z_w^2)dv + z_v z_w dw.$$

We have $d_H\omega = 0$ as a consequence of (3.7) and hence, ω is a classical conservation law for (3.7). The pullback of ω under Φ is given by

$$\Phi^\star(\omega) = \frac{1}{2}(x^2 + u_y^2)du_x - xu_y dy,$$

and hence, the projected pullback of ω is

$$\Phi^\sharp(\omega) = \frac{1}{2}(x^2 + u_y^2)u_{xx}dx + (-xu_y + \frac{1}{2}(x^2 + u_y^2)u_{xy})dy$$

which is then a conservation law for the hyperbolic Monge-Ampère equation (3.8) by virtue of proposition 3.1.

Chapter 4

The variational bicomplex for hyperbolic PDEs

In this chapter, we prove the equivalence between the Darboux integrability of a hyperbolic equation in the plane and the vanishing of Laplace invariants of the linearization of this hyperbolic equation. This equivalence is established through the construction of suitable moving coframes and their structure equations. The presentation in this chapter follows [3], and to a lesser extent, [9].

4.1 Characteristics of hyperbolic PDEs and the characteristic coframe

The characteristic equation for the second-order partial differential equation

$$F(x, y, z, p, q, r, s, t) = 0 \tag{4.1}$$

is the quadratic equation

$$F_r\lambda^2 - F_s\lambda\mu + F_t\mu^2 = 0.$$

Let $\sigma = j^2(S)(x, y)$ be a point in $J^2(E)$ satisfying (4.1). Since (4.1) is assumed to be of hyperbolic type, there exist two distinct characteristic directions at σ, given by the linearly independent vectors (m_x, m_y) and (n_x, n_y), such that

$$(m_x\lambda - m_y\mu)(n_x\lambda - n_y\mu) = \kappa(F_r\lambda^2 - F_s\lambda\mu + F_t\mu^2), \tag{4.2}$$

where κ is a non-vanishing smooth function on $\mathcal{R}^\infty$, and m_x, m_y, n_x, n_y smooth functions on $\mathcal{R}^\infty$. We define the following basis for the total vector fields on $\mathcal{R}^\infty$:

$$\begin{cases} X = m_x D_x + m_y D_y, \\ Y = n_x D_x + n_y D_y, \end{cases}$$

and we call $\{X, Y\}$ the characteristic frame of (4.1). We express the Lie bracket of X and Y in the $\{X, Y\}$ basis as

$$[X, Y] = PX + QY. \tag{4.3}$$

The coframe on $\mathcal{R}^\infty$ dual to $\{X, Y\}$ is denoted by $\{\sigma, \tau\}$ and is defined by the relations

$$\begin{cases} \sigma(X) = 1, \sigma(Y) = 0, \\ \tau(X) = 0, \tau(Y) = 1. \end{cases}$$

From the Lie bracket expression (4.3), we easily deduce the following d_H structure equations:

$$\begin{cases} d_H\sigma = -P\sigma \wedge \tau, \\ d_H\tau = -Q\sigma \wedge \tau. \end{cases}$$

Furthermore, for any type-(r, s) form $\omega \in \Omega^{r,s}(\mathcal{R}^\infty)$, we have

$$d_H\omega = \sigma \wedge X(\omega) + \tau \wedge Y(\omega).$$

On the equation manifold $\mathcal{R}^\infty$, we have $F = 0$, and hence $dF = 0$ as well. Since $dF = d_H F + d_V F$, and since $d_H F$ and $d_V F$ have different degrees, we have $d_H F = 0$ and $d_V F = 0$ on $\mathcal{R}^\infty$ as well. In particular, the relation $d_V F = 0$ can be written as

$$F_r\theta_{xx} + F_s\theta_{xy} + F_t\theta_{yy} + F_p\theta_x + F_q\theta_y + F_u\theta = 0, \tag{4.4}$$

or equivalently as

$$(F_r D_x^2 + F_s D_x D_y + F_t D_y^2 + F_p D_x + F_q D_y + F_u)\theta = 0,$$

where θ is the contact form

$$\theta = dz - pdx - qdy.$$

Equation (4.4) is called the universal linearization of the original partial differential equation (4.1) and is verified on the equation manifold $\mathcal{R}^\infty$ of (4.1). The universal linearization (4.4) is of fundamental importance in all that follows for the following two reasons: On the one hand, it is a linear hyperbolic equation, and is amenable to Laplace transformation following a suitable change of coordinates; on the other hand, it has the same characteristic directions as the original equation (4.1). We now recall the classical result of chapter **2** establishing the equivalence between Darboux integrability and termination of Laplace invariant sequences for linear hyperbolic equations, and we are led to believe that such a result may hold for the universal linearization (4.4) as well. Now since (4.4) and (4.1) have the same characteristic directions, one could expect a relation between the Laplace invariants of (4.4) and the Darboux integrability of (4.1). The remainder of this chapter is devoted to proving precisely such a result.

As was hinted above, the universal linearization can be expressed in a simpler form by a suitable choice of total differential operators:

Lemma 4.1 *Let $\{X, Y\}$ be the characteristic frame of (4.1). Then, in the basis $\{X, Y\}$, the universal linearization (4.4) of (4.1) can be written as*

$$(XY + AX + BY + C)\theta = 0 \tag{4.5}$$

where

$$\begin{aligned} A &= [(\kappa F_p - X(n_x))n_y - (\kappa F_q - X(n_y))n_x], \\ B &= [-(\kappa F_p - X(m_x))m_y + (\kappa F_q - X(m_y))m_x], \\ C &= \kappa F_u, \end{aligned}$$

where κ is given by the relation (4.2).

The proof follows from a simple calculation.

Clearly, in the characteristic frame $\{X, Y\}$, the universal linearization (4.5) is in a form which is reminiscent of the equations to which the Laplace transform can be applied. This is precisely the route we shall take, and it is precisely the universal linearization expressed in the characteristic frame basis that will provide the link between Laplace invariants and Darboux integrability. The universal linearization is of such fundamental importance as to warrant the following extension and definition:

Definition/Proposition 4.1 *The universal linearization operator associated to the scalar second-order hyperbolic partial differential equation (4.1) is the total differential operator $\mathcal{L} : \Omega^{0,s}(\mathcal{R}^\infty) \to \Omega^{0,s+1}(\mathcal{R}^\infty)$ defined for $\omega \in \Omega^{0,s}(\mathcal{R}^\infty)$ by*

$$\mathcal{L}(\omega) = XY(\omega) + AX(\omega) + BY(\omega) + C\omega,$$

where $\{X, Y\}$ is the characteristic frame associated to (4.1), and the coefficients $A, B,$ and C are defined by

$$\begin{cases} A = A_0 - \frac{Y(\rho)}{\rho}, \\ B = B_0 - \frac{X(\rho)}{\rho}, \\ C = C_0 - \frac{XY(\rho)}{\rho} - A_0\frac{X(\rho)}{\rho} - B_0\frac{Y(\rho)}{\rho} + 2\frac{X(\rho)Y(\rho)}{\rho^2}, \end{cases}$$

where

$$\begin{cases} A_0 = \frac{1}{\delta}[(\kappa F_p - X(n_x))n_y - (\kappa F_q - X(n_y))n_x], \\ B_0 = \frac{1}{\delta}[-(\kappa F_p - X(m_x))m_y - (\kappa F_q - X(m_y))m_x], \\ C_0 = \kappa F_u, \end{cases}$$

where ρ is a non-vanishing function on $\mathcal{R}^\infty$, κ is given by (4.2), and $\delta = m_x n_y - m_y n_x$. With $\Theta = \rho\theta$, the identity $d_V F = 0$ on $\mathcal{R}^\infty$ becomes

$$\mathcal{L}(\Theta) = XY(\Theta) + AX(\Theta) + BY(\Theta) + C\Theta = 0.$$

This proposition is proved by a simple but tedious calculation.

Starting from the contact one-form $\Theta = \rho\theta$, where ρ is an arbitrary smooth function on $\mathcal{R}^\infty$, we can define higher order contact forms by repeated application of the total differential operators X and Y. In particular, we define the contact forms $\xi_k = X^k(\theta), \eta_k = Y^k(\Theta)$, and we say that a form $\omega \in \Omega^p(\mathcal{R}^\infty)$ is of adapted order k if it lies in the exterior algebra generated over $\mathcal{C}^\infty(\mathcal{R}^\infty)$ by the one-forms $\{\sigma, \tau, \Theta, \xi_1, \eta_1, \cdots, \xi_k, \eta_k\}$ with k the minimal such integer. It is important to note that the adapted order of a form may be different from its order as a form on $\mathcal{R}^\infty$ and that a form of adapted order k does not necessarily factor through $\mathcal{R}^{k+1}$. It is proved in [5] that the adapted order of a form is invariant under contact transformations in $\mathcal{R}^\infty$. Note that we have

$$\begin{aligned} Y(\xi_k) &= YX(\xi_{k-1}) = XY(\xi_{k-1}) - [X, Y](\xi_{k-1}) \\ &= XY(\xi_{k-1}) - P\xi_k - QY(\xi_{k-1}), \\ X(\eta_k) &= XY(\eta_{k-1}) = YX(\eta_{k-1}) - [X, Y](\eta_{k-1}) \\ &= YX(\eta_{k-1}) - PX(\eta_{k-1}) - Q\eta_k, \end{aligned}$$

and hence, by induction on k, we obtain that $\forall k \geq 1$, $Y(\xi_k)$ and $X(\eta_k)$ are of adapted order less than or equal to k. Since the total differential operators $D_x^i D_y^j$ can be expressed in terms of the total differential operators $X^k Y^l$, and since for $k \geq 1$ the contact forms $Y(\xi_k)$ and $X(\eta_k)$ have adapted order $\leq k$ on $\mathcal{R}^\infty$, we obtain the following theorem:

Theorem 4.1 *Let $\mathcal{R}$ be a hyperbolic partial differential equation. A coframe on the equation manifold $\mathcal{R}^\infty$ is given by the one-forms*

$$\{\sigma, \tau, \Theta, \xi_1, \eta_1, \xi_2, \eta_2, \cdots, \xi_k, \eta_k, \cdots\}.$$

We call this coframe the characteristic coframe of $\mathcal{R}$. The d_H structure equations for the characteristic coframe are easily obtained. Indeed, as seen above, $d_H\sigma = -P\sigma \wedge \tau$ and $d_H\tau = -Q\sigma \wedge \tau$, $d_H\Theta = \sigma \wedge X(\Theta) + \tau \wedge Y(\Theta) = \sigma \wedge \xi_1 + \tau \wedge \eta_1$, $d_H\xi_k = \sigma \wedge X(\xi_k) + \tau \wedge Y(\xi_k) = \sigma \wedge \xi_{k+1} + \tau \wedge \mu_k$, where $\mu_k = Y(\xi_k)$ is a contact form of adapted order $\leq k$. Similarly, $d_H\eta_k = \sigma \wedge X(\eta_k) + \tau \wedge Y(\eta_k) = \sigma \wedge \nu_k + \tau \wedge \eta_{k+1}$, where $\nu_k = X(\eta_k)$ is a contact form of adapted order $\leq k$.

We shall later need to consider the behaviour of the universal linearization operator under rescaling. For this purpose, let $X' = mX, Y' = nY$, and $\Theta' = l\Theta$, where m, n, l are non-vanishing functions on $\mathcal{R}^\infty$. We have the following lemma, which is proved by simple but tedious calculations:

Lemma 4.2 *The universal linearization of $\mathcal{R}$ in terms of X', Y', and Θ' is given by*

$$X'Y'(\Theta') + A'X'(\Theta') + B'Y'(\Theta') + C'\Theta' = 0,$$

where

$$\begin{cases} A = \frac{A'}{n} + \frac{Y(l)}{l}, \\ B = \frac{B'}{m} + \frac{X(n)}{n} + \frac{X(l)}{l}, \\ C = \frac{C'}{mn} + \frac{X(l)}{nl}A' + \frac{Y(l)}{ml}B' + \frac{X(n)Y(l)}{nl} + \frac{XY(l)}{l} \end{cases}$$

With the characteristic coframe $\{\sigma, \tau, \Theta, \xi_1, \eta_1, \xi_2, \eta_2, \cdots, \xi_k, \eta_k, \cdots\}$ in hand, we can define the $X-$ and $Y-$ characteristic Pfaffian systems of order k by

$$C_k(X) = \Omega^1(\tau, \Theta, \xi_1, \eta_1, \xi_2, \eta_2, \cdots, \xi_k, \eta_k), \tag{4.6}$$

$$C_k(Y) = \Omega^1(\sigma, \Theta, \xi_1, \eta_1, \xi_2, \eta_2, \cdots, \xi_k, \eta_k), \tag{4.7}$$

respectively. The characteristic Pfaffian systems of infinite order are similarly defined by

$$C_\infty(X) = \Omega^1(\tau, \Theta, \xi_1, \eta_1, \xi_2, \eta_2, \cdots),$$

$$C_\infty(Y) = \Omega^1(\sigma, \Theta, \xi_1, \eta_1, \xi_2, \eta_2, \cdots).$$

As was seen earlier, the hyperbolic equation (4.1) is Darboux-integrable if there exist two independent functions $I, \hat{I}$ (that is $dI \wedge d\hat{I} \neq 0$ at all points) and two independent functions $J, \hat{J}$ such that $dI, d\hat{I}$ are in the span of the linear span of the first system of characteristics of(4.1), and $dJ, d\hat{J}$ in the linear span of the second system; or, in other words, $dI, d\hat{I} \in C_k(X)$, and $dJ, d\hat{J} \in C_k(Y)$. Since $dI = d_H I + d_V I = \sigma \wedge X(I) + \tau \wedge Y(I) + d_V(I)$, we have that $dI \in C_k(X)$ if and only if $X(I) = 0$. A function I on $\mathcal{R}^\infty$ for which $X(I) = 0$ is called an $X-$invariant function. Similarly, we have that $dJ \in C_k(Y)$ if and only if $Y(J) = 0$, in which case we call J a $Y-$invariant function on $\mathcal{R}^\infty$. Thus, Darboux integrability of (4.1) ultimately hinges on the existence of $X-$ and $Y-$ invariant functions on $\mathcal{R}^\infty$. The notion of $X-$ and

$Y-$ invariance of functions can be readily extended to arbitrary forms on $\mathcal{R}^\infty$, with $X(\omega)$ and $Y(\omega)$ denoting the projected $X-$ and $Y-$ Lie derivatives of $\omega \in \Omega^{r,s}(\mathcal{R}^\infty)$, respectively. $\omega \in \Omega^{r,s}(\mathcal{R}^\infty)$ is then said to be $X-$invariant if $X(\omega) = 0$, and $Y-$invariant if $Y(\omega) = 0$. We shall see later how Darboux integrability of (4.1) yields suitably many $X-$ and $Y-$invariant functions from which $X-$ and $Y-$invariant contact forms may be constructed. This notion of invariance can also be extended by considering relative invariants: A form $\omega \in \Omega^{r,s}(\mathcal{R}^\infty)$ is a relative $X-$invariant if $X(\omega) = \lambda\omega$ for some $\lambda \in \mathcal{C}^\infty(\mathcal{R}^\infty)$, and a relative $Y-$invariant if $Y(\omega) = \lambda\omega$. Clearly, any invariant form is also a relative invariant.

Consider now the universal linearization given by

$$\mathcal{L}(\Theta) = XY(\Theta) + AX(\Theta) + BY(\Theta) + C\Theta = 0$$

in terms of the characteristic frame $\{X, Y\}$. As alluded to earlier, this equation is in a form similar to an equation of the type

$$\frac{\partial^2 z}{\partial x \partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz = 0 \tag{4.8}$$

to which the Laplace transform can be applied. The $Y-$Laplace transform of a solution z to (4.8) being $z_1 = \frac{\partial z}{\partial y} + az$, equation (4.8) becomes

$$\frac{\partial}{\partial x} z_1 + bz_1 - hz = 0, \tag{4.9}$$

where h is the Laplace invariant of (4.9), given by $h = \frac{\partial a}{\partial x} + ab - c$. Mimicking these steps, the $Y-$Laplace transform of a contact form ω would be

$$\mu = Y(\omega) + A\omega,$$

in terms of which the universal linearization would become

$$X(\mu) + B\mu - H\omega = 0, \tag{4.10}$$

with $H = X(A) + AB - C$. As can be seen from (4.10), the relative $X-$ invariance of μ is directly tied to the vanishing of H, and H can thus be seen as an obstruction to μ being a relative $X-$invariant. Since, as mentionned above, the existence of such relative invariants is guaranteed by Darboux-integrability of (4.1), this suggests a link between Darboux-integrability of (4.1) and properties of the Laplace transform of the universal linearization (4.5). It is precisely this link we shall make precise in the remainder of the thesis. First, however, we shall suitably extend the theory of the classical Laplace transform.

4.2 The generalized Laplace transform and the Laplace-adapted coframe

Consider the hyperbolic total differential operator $\mathcal{F} : \Omega^{0,s}(\mathcal{R}^\infty) \to \Omega^{0,s}(\mathcal{R}^\infty)$ defined by

$$\mathcal{F}(\omega) = XY(\omega) + AX(\omega) + BY(\omega) + C\omega, \tag{4.11}$$

on the equation manifold $\mathcal{R}^\infty$ of the second-order hyperbolic equation (4.1) with characteristic frame $\{X, Y\}$. The commutator of X and Y being given by (4.3), $\mathcal{F}$ can also be expressed in the equivalent form

$$\mathcal{F}(\omega) = YX(\omega) + DX(\omega) + EY(\omega) + G\omega,$$

where

$$\begin{aligned} D &= A + P, \\ E &= B + Q, \\ G &= C. \end{aligned}$$

There are two generalized Laplace transforms associated to the operator $\mathcal{F}$, one for each of the characteristic vector fields X and Y. To define the $Y-$Laplace transform, we define the first-order total differential operator $\mathcal{Y}_{\mathcal{F}} : \Omega^{0,s}(\mathcal{R}^\infty) \to \Omega^{0,s}(\mathcal{R}^\infty)$ by $\mathcal{Y}_{\mathcal{F}}(\omega) = Y(\omega) + A\omega$. The form $\eta = \mathcal{Y}_{\mathcal{F}}(\omega)$ is called the $Y-$Laplace transform of the contact form ω associated to $\mathcal{F}$. Note the similarity with the classical Laplace transform $z_1 = \frac{\partial z}{\partial y} + az$. In terms of η, the contact form $\mathcal{F}(\omega)$ can be expressed as

$$\mathcal{F}(\omega) = X(\eta) + B\eta - H\omega,$$

where $H = H(\mathcal{F}) \equiv X(A) + AB - C$. Again, we note the similarity between H and the classical Laplace invariant $h = \frac{\partial a}{\partial x} + ab - c$. We call $H(\mathcal{F})$ the generalized $Y-$Laplace invariant for the total differential operator $\mathcal{F}$. Just as with the classical Laplace transform and classical Laplace invariants, we may construct a new total differential operator of the same form as (4.11) if $H(\mathcal{F}) \neq 0$. Indeed, $\mathcal{F}(\omega) = 0 \Rightarrow X(\eta) + B\eta - H\omega = 0 \Rightarrow YX(\eta) + Y(B\eta) - Y(H\omega) = 0$, and this last equation can be rewritten as

$$XY(\eta) - BX(\eta) + (B - Q)Y(\eta) + (Y(B) - H)\eta + (AH - Y(H))\omega = 0. \quad (4.12)$$

Since $H \neq 0$ by assumption, we can solve for ω in $X(\eta) + B\eta - H\omega = 0$ and substitute the resulting expression in (4.12), obtaining:

$$[\mathcal{Y}(\mathcal{F})](\eta) \equiv XY(\eta) + \mathcal{Y}(A)X(\eta) + \mathcal{Y}(B)Y(\eta) + \mathcal{Y}(C)\eta = 0,$$

with

$$\begin{cases} \mathcal{Y}(A) = A - \frac{Y(H)}{H} - P, \\ \mathcal{Y}(B) = B - Q, \\ \mathcal{Y}(C) = C - X(A) - B\frac{Y(H)}{H} + Y(B). \end{cases}$$

We call $\mathcal{Y}(\mathcal{F})$ the $Y-$Laplace transform of $\mathcal{F}$. Note that for commuting characteristic vector fields ($P = Q = 0$), the coefficients of $\mathcal{Y}(\mathcal{F})$ are analogous to those of the classical Laplace transform. Just as with the classical Laplace transform, the $Y-$Laplace transform $\mathcal{Y}_{\mathcal{F}}(\omega) = Y(\omega) + A\omega$ of the form ω is defined for any total differential operator $\mathcal{F}$ of the form (4.11), but the $Y-$Laplace transform $\mathcal{Y}(\mathcal{F})$ of $\mathcal{F}$ is defined only when the generalized Laplace invariant $H(\mathcal{F})$ does not vanish. We can similarly define the $X-$Laplace transform of the form ω by $\mathcal{X}_{\mathcal{F}}(\omega) = X(\omega) + E\omega$, where $\mathcal{X}_{\mathcal{F}} : \Omega^{0,s}(\mathcal{R}^{\infty}) \to \Omega^{0,s}(\mathcal{R}^{\infty})$. In terms of $\xi = \mathcal{X}_{\mathcal{F}}(\omega)$, the contact form $\mathcal{F}(\omega)$ can be expressed as

$$\mathcal{F}(\omega) = Y(\xi) + D\xi - K\omega,$$

where $K = K(\mathcal{F}) \equiv Y(E) + ED - G$. We call $K(\mathcal{F})$ the generalized $X-$Laplace invariant of the total differential operator $\mathcal{F}$. Here again, $\mathcal{F}(\omega) = 0 \Rightarrow Y(\xi) +$

$D\xi - K\omega = 0 \Rightarrow XY(\xi) + X(D\xi) - X(K\omega) = 0$; if now $K(\mathcal{F}) \neq 0$, then we can express ω in terms of ξ by virtue of $Y(\xi) + D\xi - K\omega = 0$, and substituting in $XY(\xi) + X(D\xi) - X(K\omega) = 0$, we obtain

$$[\mathcal{X}(\mathcal{F})](\xi) \equiv YX(\xi) + \mathcal{X}(D)X(\xi) + \mathcal{X}(E)Y(\xi) + \mathcal{X}(G)\xi = 0,$$

where

$$\begin{cases} \mathcal{X}(D) = D + P, \\ \mathcal{X}(E) = E - \frac{X(K)}{K} + Q, \\ \mathcal{X}(G) = G - Y(E) - D\frac{X(K)}{K} + X(D). \end{cases}$$

We call $\mathcal{X}(\mathcal{F})$ the generalized $X-$Laplace transform of the hyperbolic total differential operator $\mathcal{F}$. Just as with the $Y-$Laplace transform, the $X-$Laplace transform $\mathcal{X}_{\mathcal{F}}(\omega) = X(\omega) + E\omega$ of ω is always defined, whereas the $X-$Laplace transform $\mathcal{X}(\mathcal{F})$ of $\mathcal{F}$ is defined only when $K(\mathcal{F}) \neq 0$. Just as with the classical Laplace transform, solutions to $\mathcal{F} = 0$ yield solutions to $\mathcal{Y}(\mathcal{F}) = 0$ and $\mathcal{X}(\mathcal{F}) = 0$ whenever these exist, and vice-versa. In other words, the solutions to $\mathcal{F}(\omega) = 0$ and $[\mathcal{Y}(\mathcal{F})](\omega) = 0$ (respectively, $[\mathcal{X}(\mathcal{F})](\omega) = 0$) are in bijective correspondence. This is the subject of our next proposition, which is easily proved:

Proposition 4.1 *For any hyperbolic second-order total differential operator $\mathcal{F}$ and for any forms $\omega, \eta \in \Omega^{0,s}(\mathcal{R}^\infty)$, $[\mathcal{X}_{\mathcal{Y}(\mathcal{F})} \circ \mathcal{Y}_{\mathcal{F}}](\omega) = H(\mathcal{F})\omega + \mathcal{F}(\omega)$, and, provided $H(\mathcal{F}) \neq 0$, $\mathcal{Y}_{\mathcal{F}}(\frac{1}{H(\mathcal{F})}\mathcal{X}_{\mathcal{Y}(\mathcal{F})}(\eta)) = \eta + \frac{1}{H(\mathcal{F})}[\mathcal{Y}(\mathcal{F})](\eta)$. If $H(\mathcal{F}) \neq 0$ and $\eta = \mathcal{Y}_{\mathcal{F}}(\omega)$, then $\mathcal{F}(\omega) = 0$ implies $[\mathcal{Y}(\mathcal{F})](\eta) = 0$; conversely, if $\hat{\omega} = \frac{1}{H(\mathcal{F})}\mathcal{X}_{\mathcal{Y}(\mathcal{F})}(\eta)$, then $[\mathcal{Y}(\mathcal{F})](\eta) = 0$ implies $\mathcal{F}(\hat{\omega}) = 0$. Similar identities hold for $K(\mathcal{F})$ and $\mathcal{X}(\mathcal{F})$.*

As discussed earlier, if the $Y-$Laplace invariant $H(\mathcal{F})$ of $\mathcal{F}$ vanishes, then the $Y-$Laplace transform $\mathcal{Y}_{\mathcal{F}}(\omega) = Y(\omega) + A\omega$ is a relative $X-$invariant form. If however $H(\mathcal{F}) \neq 0$, we can draw no conclusion as to the $X-$invariance of the form ω, but instead we can consider the $Y-$Laplace transform $\mathcal{Y}(\mathcal{F})$ of $\mathcal{F}$. If now $H(\mathcal{Y}(\mathcal{F})) = 0$, then we have a relative $X-$invariant form, given by $\mathcal{Y}_{\mathcal{Y}(\mathcal{F})}(\omega)$. If however $H(\mathcal{Y}(\mathcal{F})) \neq 0$, then we can take another $Y-$Laplace transform $\mathcal{Y}(\mathcal{Y}(\mathcal{F}))$

and continue this procedure. The same conclusion holds for the $X-$Laplace transform, with $Y-$invariance instead of $X-$invariance, and the $X-$Laplace invariant K instead of the $Y-$Laplace invariant H. This suggests that we can consider successive applications of the $Y-$ and $X-$Laplace transforms of the total differential operator $\mathcal{F}$. Let then $H_0 = H(\mathcal{F})$, $H_1 = H(\mathcal{Y}(\mathcal{F}))$ provided $H_0 \neq 0$. Provided $H_0 \neq 0, H_1 \neq 0, \cdots, H_{i-1} \equiv H(\mathcal{Y}^{i-1}(\mathcal{F})) \neq 0$, we define the i^{th} generalized $Y-$Laplace invariant of $\mathcal{F}$ to be $H_i = H(Y^i(\mathcal{F}))$. The first integer p for which $H_p = 0$ is called the $Y-$Laplace index of $\mathcal{F}$ and is denoted by $ind_{\mathcal{Y}}(\mathcal{F})$. If $H_i \neq 0$ for all integers $i \geq 0$, $ind_{\mathcal{Y}}(\mathcal{F})$ is defined as ∞. In a similar fashion, we define $K_0 = K(\mathcal{F})$ and $K_j = K(\mathcal{X}^j(\mathcal{F}))$ provided $K_0 \neq 0, K_1 \neq 0, \cdots K_{j-1} \neq 0$, and we define the $X-$Laplace index $ind_{\mathcal{X}}(\mathcal{F})$ of $\mathcal{F}$ to be the first integer q for which $K_q = 0$. Note that it follows from the above proposition that $H(\mathcal{X}_{\mathcal{Y}(\mathcal{F})} \circ \mathcal{Y}(\mathcal{F})) = H(\mathcal{F})$ and $K(\mathcal{Y}_{\mathcal{X}(\mathcal{F})} \circ \mathcal{X}(\mathcal{F})) = K(\mathcal{F})$. Hence, there is no advantage in combining the $X-$ and $Y-$ generalized Laplace transforms.

Let $\mathcal{L} : \Omega^{0,s}(\mathcal{R}^\infty) \to \Omega^{0,s}(\mathcal{R}^\infty)$ be the second order total differential operator $\mathcal{L} = XY + AX + BY + C$ defining the universal linearization of the hyperbolic equation $\mathcal{R}$, with the coefficients A, B, C given in definition/proposition (4.1). If Θ is the contact form $\rho\theta$, then $\mathcal{L}(\Theta) = 0$. If $ind_{\mathcal{Y}}(\mathcal{L}) = p$, then $H_p = H(\mathcal{Y}^p(\mathcal{L})) = 0$ and $H(\mathcal{Y}^{p-1}(\mathcal{L})) \neq 0$, and hence the contact form $\mathcal{Y}_{\mathcal{Y}^{p-1}(\mathcal{L})}(\Theta)$ is a relative $X-$invariant form; similarly, if $ind_{\mathcal{X}}(\mathcal{L}) = q$, then $K_q = K(\mathcal{X}^q(\mathcal{L})) = 0$ and $K(\mathcal{X}^{q-1}(\mathcal{L})) \neq 0$, and the form $X_{X^{q-1}(\mathcal{L})}(\Theta)$ is a relative $Y-$invariant contact form. This suggests that there may be some advantage in suitably modifying the characteristic coframe via applications of the $X-$ and $Y-$ Laplace transforms, whenever possible, so as to include these invariant contact forms. For simplicity of notation, we denote the Laplace indices of $\mathcal{L}$ by $ind(\mathcal{Y}) \equiv ind_{\mathcal{Y}}(\mathcal{L})$ and $ind(\mathcal{X}) \equiv ind_{\mathcal{X}}(\mathcal{L})$, respectively, and the generalized Laplace invariants of $\mathcal{L}$ by $H_i \equiv H(\mathcal{Y}^i(\mathcal{L})), K_j \equiv K(\mathcal{X}^j(\mathcal{L}))$, respectively. We modify the characteristic coframe constructed in the previous section in order to obtain a new coframe on $\mathcal{R}^\infty$, as follows: If $ind(\mathcal{Y}) = p$, we define $\eta_1 = \mathcal{Y}_{\mathcal{L}}(\Theta) = Y(\Theta) + A\Theta, \eta_i = \mathcal{Y}_{\mathcal{Y}^{i-1}(\mathcal{L})}(\eta_{i-1}) = Y(\eta_{i-1}) + A_{i-1}\eta_{i-1}, i = 2, \cdots, p+1$, and $\eta_{p+i} = Y(\eta_{p+i-1})$ for $i \geq 2$, where A_{i-1} is the coefficient of X in the operator $\mathcal{Y}^{i-1}(\mathcal{L})$. If on the other hand $ind(\mathcal{Y}) = \infty$, then the contact one-forms η_i are defined by $\eta_i = Y(\eta_{i-1}) + A_{i-1}\eta_{i-1}$ for all $i \geq 2$. It immediately follows from the above that $X(\eta_1) + B\eta_1 - H_0\Theta = 0$ and that if $H_0 \neq 0$, $\mathcal{Y}(\mathcal{L}(\eta_1)) = XY(\eta_1) + A_1X(\eta_1) + B_1Y(\eta_1) + C_1\eta_1 = 0$. A simple calculation also shows that $X(\eta_i) + B_{i-1}\eta_i - H_{i-1}\eta_{i-1} = 0$ for $i = 2, 3, \cdots, p+1$, and that for $i = 1, \cdots, p$, $[\mathcal{Y}^i(\mathcal{L})](\eta_i) = XY(\eta_i) + A_iX(\eta_i) + B_iY(\eta_i) + C_i\eta_i = 0$. We define the forms ξ_j similarly with respect to the generalized $X-$Laplace transform. If $ind(\mathcal{X}) = q$, we define $\xi_1 = \mathcal{X}_{\mathcal{L}}(\Theta) = X(\Theta) + E\Theta, \xi_j = \mathcal{X}_{\mathcal{X}^{j-1}(\mathcal{L})}(\xi_{j-1}) = X(\xi_{j-1}) + E_{j-1}\xi_{j-1}$, for $j = 2, \cdots, q+1$, and $\xi_{q+j} = X(\xi_{q+j-1}$ for $j \geq 2$, where E_{j-1} is the coefficient of Y in the operator $\mathcal{X}^{j-1}(\mathcal{L})$. If $ind(\mathcal{X}) = \infty$, then the contact one-forms ξ_j are defined by $\xi_j = X(\xi_{j-1}) + E_{j-1}\xi_{j-1}$, for all $j \geq 2$. The resulting coframe $\{\sigma, \tau, \Theta, \eta_1, \xi_1, \eta_2, \xi_2, \cdots, \}$ is called the Laplace-adapted coframe on $\mathcal{R}$ and will play a fundamental role in all that follows. Before it can be used, however, its structure equations have to be defined.

4.3 Structure equations for the Laplace-adapted coframe

We compute the d_H and d_V structure equations separately. To compute the d_H structure equations, we use the basic equation $d_H\omega = \sigma \wedge X(\omega) + \tau \wedge Y(\omega), \forall \omega \in \Omega^{r,s}(\mathcal{R}^\infty)$. For example, for the first three contact forms of the Laplace-adapted coframe, we obtain:

$$\begin{aligned}
d_H(\Theta) &= \sigma \wedge X(\Theta) + \tau \wedge Y(\Theta) \\
&= \sigma \wedge (\xi_1 - E\Theta) + \tau \wedge (\eta_1 - A\Theta), \\
d_H\eta_1 &= \sigma \wedge X(\eta_1) + \tau \wedge Y(\eta_1) \\
&= \sigma \wedge (-B\eta_1 + H_0\Theta) + \tau \wedge (\eta_2 - A_1\eta_1), \\
d_H\xi_1 &= \sigma \wedge X(\xi_1) + \tau \wedge Y(\xi_1) \\
&= \sigma \wedge (\xi_2 - E_1\xi_1) + \tau \wedge (-D\xi_1 + K_0\Theta).
\end{aligned}$$

Gathering all of these, we have the following important proposition:

Proposition 4.2 *Suppose $0 \leq p = ind(\mathcal{Y}) < \infty$ and $0 \leq q = ind(\mathcal{X}) < \infty$. The d_H structure equations for the Laplace-adapted coframe for the hyperbolic equation $\mathcal{R}$ are given by*

$$\begin{aligned}
d_H\sigma &= -P\sigma \wedge \tau, \\
d_H\tau &= -Q\sigma \wedge \tau, \\
d_H(\Theta) &= \sigma \wedge (\xi_1 - E\Theta) + \tau \wedge (\eta_1 - A\Theta), \\
d_H\eta_1 &= \sigma \wedge (-B\eta_1 + H_0\Theta) + \tau \wedge (\eta_2 - A_1\eta_1), \\
d_H\eta_i &= \sigma \wedge (-B_{i-1}\eta_i + H_{i-1}\eta_{i-1}) + \tau \wedge (\eta_{i+1} - A_i\eta_i), \quad 2 \leq i \leq p, \\
d_H\eta_{p+1} &= \sigma \wedge (-B_p\eta_{p+1}) + \tau \wedge \eta_{p+1}, \\
d_H\eta_{p+i} &= \sigma \wedge \nu_{p+i} + \tau \wedge \eta_{p+i+1}, \quad i \geq 2, \\
d_H\xi_1 &= \sigma \wedge (\xi_2 - E_1\xi_1) + \tau \wedge (-D\xi_1 + K_0\Theta), \\
d_H\xi_i &= \sigma \wedge (\xi_{i+1} - E_i\xi_i) + \tau \wedge (-D_{i-1}\xi_i + K_{i-1}\xi_{i-1}, \quad 2 \leq i \leq q, \\
d_H\xi_{q+1} &= \sigma \wedge \xi_{q+2} + \tau \wedge (-D_q\xi_{q+1}), \\
d_H\xi_{q+i} &= \sigma \wedge \xi_{q+i+1} + \tau \wedge \mu_{q+i}, \quad i \geq 2,
\end{aligned}$$

where ν_{p+i} is a contact one-form such that

$$\nu_{p+i} \equiv [(i-1)Q - B_p]\eta_{p+i} \quad \mod \{\eta_{p+1}, \cdots, \eta_{p+i-1}\},$$

and μ_{q+i} is a contact form such that

$$\mu_{q+i} \equiv [(i-1)P - D_q]\xi_{q+i} \quad \mod \{\xi_{q+1}, \cdots, \xi_{q+i-1}\}$$

If ind$(\mathcal{Y}) = \infty$, then the structure equations

$$d_H \eta_i = \sigma \wedge (-B_{i-1}\eta_i + H_{i-1}\eta_{i-1}) + \tau \wedge (\eta_{i+1} - A_i \eta_i)$$

remain valid for all $i \geq 2$. Similarly, if ind$(\mathcal{X}) = \infty$, then the structure equations

$$d_H \xi_i = \sigma \wedge (\xi_{i+1} - E_i \xi_i) + \tau \wedge (-D_{i-1}\xi_i + K_{i-1}\xi_{i-1})$$

remain valid for all $i \geq 2$.

We shall need to know the behaviour of the generalized Laplace invariants and the Laplace-adapted coframe under contact transformations. Let then $\mathcal{R}'$ be another second-order hyperbolic system, and let $\Phi : \mathcal{R} \to \mathcal{R}'$ be a contact transformation. Let X', Y' be the characteristic vector fields for $\mathcal{R}'$, and let $X'Y'(\Theta') + A'X'(\Theta') + B'Y'(\Theta') + C'\Theta' = 0$ be the universal linearization on $\mathcal{R}^{\infty\prime}$. Φ being a contact transformation, the characteristic directions are preserved, and we can write $X' = mX, Y' = nY$, and since contact forms are also preserved, we have $\Theta' = l\Theta$, for non-vanishing functions l, m, n on $\mathcal{R}^\infty$. We have the following theorem, which is established by simple but tedious computations:

Theorem 4.2 *Suppose $p = ind(\mathcal{Y}) < \infty$ and $q = ind(\mathcal{X}) < \infty$. Then the Laplace indices of $\mathcal{R}$ and $\mathcal{R}'$ coincide. The generalized Laplace invariants H, K of $\mathcal{R}$ and H', K' of $\mathcal{R}'$ are related by $H'_i = mnH_i, K'_i = mnK_i$. Furthermore, the Laplace-adapted coframes are related by $\sigma' = \frac{1}{m}\sigma, \tau' = \frac{1}{n}\tau, \eta_i{}' = n^i l \eta_i$ for $i \leq i \leq p+1$, $\eta'_{p+i} \equiv n^{p+i} l \eta_{p+i} \mod \{\eta_{p+1}, \cdots, \eta_{p+i-1}\}$ for $2 \leq i < \infty$; similarly, $\xi_j{}' = m^j l \xi_j$ for $1 \leq j \leq q+1$, $\xi'_{q+j} \equiv m^{q+j} l \xi_{q+j} \mod \{\xi_{q+1}, \cdots, \xi_{q+j-1}\}$ for $2 \leq j < \infty$; when $p = \infty$, we have $\eta_i{}' = n^i l \eta_i$ for all $i \geq 1$, and when $q = \infty$, we have $\xi_j{}' = m^j l \xi_j$ for all $j \geq 1$.*

In view of further computations, we also need to express the structure equations of the Laplace-adapted coframe as structure equations of the corresponding dual frame on $\mathcal{R}^\infty$, which we define as follows:

Definition 4.1 *The vertical vector fields $U, V^1, W^1, V^2, W^2, \cdots$ dual to the contact forms $\Theta, \eta_1, \xi_1, \eta_2, \xi_2, \cdots$ of the Laplace-adapted coframe are defined by the relations*

$$\begin{cases} \Theta(U) = 1, \eta_i(U) = 0, \xi_i(U) = 0, \\ \Theta(V^h) = 0, \eta_i(V^h) = \delta_i^h, \xi_j(V^h) = 0, \\ \Theta(W^k) = 0, \eta_i(W^k) = 0, \xi_j(W^k) = \delta_j^k, \end{cases}$$

We call $\{X, Y, U, V^1, W^1, V^2, W^2, \cdots\}$ the Laplace-adapted frame on $\mathcal{R}^\infty$.

Let Z be any total vector field on $\mathcal{R}^\infty$, and ω any contact one-form. Then $d_V\omega$ is a contact two-form and $Z\neg d_V\omega = 0$, and hence, for any vector field V on $\mathcal{R}^\infty$ we have

$$\begin{aligned} d\omega(Z, V) &= d_H\omega(Z, V) = Z(\omega(V)) - V(\omega(Z)) - \omega([Z, V]) \\ &= Z(\omega(V)) - \omega([Z, V]). \end{aligned}$$

In particular, if $\omega(V)$ is constant, then $\omega([Z, V]) = -d_H\omega(Z, V)$. This formula relates the d_H structure equations of the Laplace-adapted frame to the Lie bracket structure equations of the dual Laplace-adapted frame. We thus obtain the following simple, but useful, proposition:

Proposition 4.3 *Let $\mathcal{R}$ be a second-order hyperbolic equation, and suppose that $0 \leq p = ind(\mathcal{Y}) \leq \infty$ and $0 \leq q = ind(\mathcal{X}) \leq \infty$. Then the following congruences hold for the Lie brackets of the total vector field X with the vertical vector fields U, V^i, W^j:*

$$\begin{aligned} [X, U] &\equiv EU - H_0V^1 \quad \bmod \{X, Y\}, \\ [X, V^1] &\equiv BV^1 - H_1V^2 \quad \bmod \{X, Y\}, \\ [X, V^i] &\equiv B_{i-1}V^i - H_iV^{i+1} \quad \bmod \{X, Y\}, 2 \leq i \leq p, \\ [X, V^{p+i}] &\equiv (B_p - (i-1)Q)V^{p+i} \quad \bmod \{X, Y, v^{p+i+1}, \cdots\}, 1 \leq i < \infty, \\ [X, W^1] &\equiv -U + E_1W^1 \quad \bmod \{X, Y\}, \\ [X, W^j] &\equiv -W^{j-1} + E_jW^j \quad \bmod \{X, Y\}, \\ [X, W^{q+j}] &\equiv -W^{q+j-1} + E_jW^j \quad \bmod \{X, Y\}, 1 \leq j < \infty. \end{aligned}$$

Similar congruences are obtained for the Lie brackets of the characteristic vector field Y with the vertical vector fields U, V^i, W^j. We shall also need the d_V structure equations for the Laplace-adapted coframe. We begin with a simple proposition.

Proposition 4.4 *The d_V structure equations for the horizontal forms σ and τ are*

$$\begin{aligned} d_V\sigma &= \sigma \wedge \mu_1 + \tau \wedge \alpha, \\ d_V\tau &= \sigma \wedge \beta + \tau \wedge \mu_2, \end{aligned}$$

where $\alpha, \beta, \mu_1, \mu_2$ are contact one-forms, and α, β are of adapted order 2. Moreover,

$$\begin{aligned} d_V P &= X(\alpha) - Y(\mu_1) + P\mu_2 - Q\alpha, \\ d_V Q &= X(\mu_2) - Y(\beta) + Q\mu_1 - P\beta, \\ d_V\beta &= \beta \wedge (\mu_2 - \mu_1), \\ d_V\mu_2 &= \alpha \wedge \beta = -d_V\mu_1, \\ d_V\alpha &= \alpha \wedge (\mu_1 - \mu_2). \end{aligned}$$

Proof. Since $\sigma, \tau \in \Omega^{1,0}(\mathcal{R}^\infty)$, $d_V\sigma, d_V\tau \in \Omega^{1,1}(\mathcal{R}^\infty)$, and hence can be written as $d_V\sigma = \sigma\wedge\mu_1 + \tau\wedge\alpha$ and $d_V\tau = \sigma\wedge\beta + \tau\wedge\mu_2$, for $\alpha, \beta, \mu_1, \mu_2 \in \Omega^{0,1}(\mathcal{R}^\infty)$. Now write the horizontal form σ in terms of the natural coordinate coframe $\{dx, dy\}$ of the plane as $\sigma = a dx + b dy$. From the relations $X = m_x D_x + m_y D_y$ and $Y = n_x D_x + n_y D_y$ and the fact that $\{\sigma, \tau\}$ is dual to $\{X, Y\}$, we obtain

$$\begin{aligned} a &= \frac{n_y}{m_x n_y - m_y n_x}, \\ b &= \frac{-n_x}{m_x n_y - m_y n_x}. \end{aligned}$$

Since the functions m_x, m_y, n_x, and n_y are smooth functions on $\mathcal{R}^\infty$ that can be factored through $\mathcal{R}^2$, the same holds for a and b, and hence for σ as well. Hence the adapted order of σ is 2, and as a result that of α as well. It is shown similarly that the adapted order of β is 2. Now the three relations $d_H\sigma = -P\sigma\wedge\tau$, $d_V\sigma = \sigma\wedge\mu_1 + \tau\wedge\alpha$, and $d_H d_V \sigma + d_V d_H \sigma = 0$ yield

$$\begin{aligned} d_H(\sigma \wedge \mu_1 + \tau \wedge \alpha) &= d_V(P\sigma \wedge \tau) \\ &= d_V P \wedge \sigma \wedge \tau + P \wedge d_V\sigma \wedge \tau - P \wedge \sigma \wedge d_V\tau, \end{aligned}$$

that is,

$$\begin{aligned} d_H\sigma \wedge \mu_1 &- \sigma \wedge d_H\mu_1 + d_H\tau \wedge \alpha - \tau \wedge d_H\alpha \\ &= d_V P \wedge \sigma \wedge \tau + P\sigma \wedge \mu_1 \wedge \tau - P\sigma \wedge \tau \wedge \mu_2, \end{aligned}$$

which simplifies to

$$\begin{aligned} -P\sigma \wedge \tau \wedge \mu_1 &- \sigma \wedge \tau \wedge Y(\mu_1) - Q\sigma \wedge \tau \wedge \alpha - \tau \wedge \sigma \wedge X(\alpha) \\ &= d_V P \wedge \sigma \wedge \tau + P\sigma \wedge \mu_1 \wedge \tau - P\sigma \wedge \tau \wedge \mu_2. \end{aligned}$$

Grouping terms, we obtain the desired result. Similarly, the integrability condition $d_V^2\sigma = 0$ and $d_V^2\tau = 0$ yield the last two relations. □

We have $d_H(d_V\omega) = \sigma \wedge X(d_V\omega) + \tau \wedge Y(d_V\omega)$. But we also have $d_H(d_V\omega) = -d_v(d_H\omega) = -d_V(\sigma \wedge X(\omega) + \tau \wedge Y(\omega) = d_V(X(\omega)) \wedge \sigma - X(\omega) \wedge d_V\sigma + d_V(Y(\omega)) \wedge \tau - Y(\omega) \wedge d_V\tau$. Using the expressions for $d_V\sigma$ and $d_V\tau$ computed in the previous proposition, we obtain the following result:

Proposition 4.5 $\forall \omega \in \Omega^\star(\Theta, \eta_1, \xi_1, \cdots) : d_V[X(\omega)] - X(d_V\omega) = \mu_1 \wedge X(\omega) + \beta \wedge Y(\omega)$, *and* $d_V[Y(\omega)] - Y(d_V\omega) = \sigma \wedge X(\omega) + \mu_2 \wedge Y(\omega)$, *where* $\alpha, \beta, \mu_1, \mu_2$ *are as in proposition (4.4).*

The commutation rules established in the previous proposition allow us to establish the following d_V structure equations for the Laplace-adapted coframe on $\mathcal{R}^\infty$:

Proposition 4.6 *The Laplace-adapted coframe satisfies the following congruences:*

$$\begin{aligned} d_V\Theta &\equiv 0 \quad \mathrm{mod}\ \{\Theta\}, \\ d_V\eta_i &\equiv 0 \quad \mathrm{mod}\ \{\xi_1, \Theta, \eta_1, \cdots, \eta_i\}, \forall i \geq 1, \\ d_V\xi_i &\equiv 0 \quad \mathrm{mod}\ \{\eta_1, \Theta, \xi_1, \cdots, \xi_i\}, \forall i \geq 1. \end{aligned}$$

Proof. We have

$$\begin{aligned} d_H(d_V\Theta) &= -d_V(d_H\Theta) \iff \\ \sigma\wedge X(d_V\Theta)+\tau\wedge Y(d_V\Theta) &= -d_V(\sigma\wedge X(\Theta)+\tau\wedge Y(\Theta)) = \\ d_V(X(\Theta))\wedge\sigma - X(\Theta)\wedge d_V\sigma &+ d_V(Y(\Theta))\wedge\tau - Y(\Theta)\wedge d_V\tau. \end{aligned}$$

The d_V structure equations for the horizontal forms σ, τ, together with the commutation rules of proposition (4.5) yield $d_V\Theta \equiv 0 \mod \{\Theta\}$. The congruences for $d_V\eta_i$ and $d_V\xi_i$ for $i = 1$ similarly follow from the integrability condition $d_H d_V + d_V d_H = 0$, and are established for $i > 1$ by a simple induction. □

It is important to note that the congruences in these structure equations are not as sharp as they could be; for one thing, they are totally independent of the Laplace indices on which the Laplace-adapted coframe itself is based. Armed with the d_H and d_V structure equations, we are now able to characterize $X-$ and $Y-$ invariant forms.

4.4 Characterization of relative invariants

Consider a contact one-form $\omega \in \Omega^1(\xi_1, \xi_2, \cdots, \xi_l)$, for some $l \geq 1$; ω can be written as a linear combination $\omega = \sum_{i=1}^{l} a_i\xi_i$, where the a_i are smooth functions on $\mathcal{R}^\infty$. Assume now that ω is $X-$invariant, that is, there exists a smooth function λ on $\mathcal{R}^\infty$ such that $X(\omega) = \lambda\omega$. $X(\omega) = \sum_{i=1}^{l}(X(a_i)\xi_i + a_i X(\xi_i))$, and by construction of the Laplace-adapted coframe itself, we have $X(\omega) \equiv a_l\xi_{l+1} \mod \Omega^1(\xi_1, \cdots, \xi_l)$. On the other hand, $\lambda\omega \equiv 0 \mod \Omega^1(\xi_1, \cdots, \xi_l)$, and this implies that $a_l = 0$, that is, $\omega \in \Omega^1(\xi_1, \cdots, \xi_{l-1})$. Repeating this procedure l times, we deduce that $\omega = 0$. Thus, for any positive integer l, no contact one-form in $\Omega^1(\xi_1, \cdots, \xi_l)$ can be a relative $X-$invariant. Similarly, no contact one-form in $\Omega^1(\eta_1, \cdots, \eta_l)$ can be a relative $Y-$invariant form. The following key theorem makes this basic observation more precise:

Theorem 4.3 *Let $\mathcal{R}$ be a hyperbolic equation with characteristic vector fields X and Y, Laplace indices ind($\mathcal{Y}$) $= p$ and ind($\mathcal{X}$) $= q$, and Laplace-adapted coframe*

$\{\sigma, \tau, \Theta, \xi_1, \eta_1, \xi_2, \eta_2, \cdots\}$. *Let* $s \geq 1$*:*

1. *If* $\omega \in \Omega^{0,s}(\mathcal{R}^\infty)$ *is a relative* $X-$*invariant form, then*

$$\omega \in \Omega^s(\eta_{p+1}, \eta_{p+2}, \cdots).$$

   If ind$(\mathcal{Y}) = \infty$*, then there are no non-zero relative* $X-$*invariant type* $(0, s)$ *forms;*

2. *If* $\omega \in \Omega^{0,s}(\mathcal{R}^\infty)$ *is a relative* $Y-$*invariant form, then*

$$\omega \in \Omega^s(\xi_{q+1}, \xi_{q+2}, \cdots).$$

   If ind$(\mathcal{X}) = \infty$*, then there are no non-zero relative* $Y-$*invariant type* $(0, s)$ *forms.*

Proof. Suppose ω is a relative $X-$invariant type$-(0, s)$ form of adapted order k; that is, $\omega \in \Omega^s(\Theta, \xi_1, \eta_1, \cdots, \xi_k, \eta_k)$ and $X(\omega) = \lambda\omega$, for some function $\lambda \in \mathcal{C}^\infty(\mathcal{R}^\infty)$. Let U, V^i, W^j be the vertical vector fields defined in (4.1) dual to the Laplace-adapted coframe. Since ω has no dependence on ξ_{k+1} by assumption of it being of order k, we have that $W^{k+1} \neg \omega = 0$. Thus $W^{k+1} \neg X(\omega) = \lambda W^{k+1} \neg \omega = 0$ as well, and since $W^{k+1} \neg X(\omega) = X(W^{k+1} \neg \omega) - [X, W^{k+1}] \neg \omega$, we obtain $[X, W^{k+1}] \neg \omega = 0$. From the Lie bracket structure equations of proposition (4.3) it follows that $W^k \neg \omega = 0$. Repeating this procedure, we obtain $W^{k-1} \neg \omega = W^{k-2} \neg \omega = \cdots = W^1 \neg \omega = U \neg \omega = 0$. This shows that $\omega \in \Omega^s(\eta_1, \eta_2, \cdots, \eta_k)$. Since $[X, U] \equiv EU - H_0 V^1 \mod \{X, Y\}$, $U \neg X = 0$ and $X(\omega) = \lambda\omega$, we obtain $H_0 V^1 \neg \omega = 0$, and since $H_0 \neq 0$, we have that $V^1 \neg \omega = 0$. From the Lie bracket structure equation $[X, V^1] \equiv BV^1 - H_1 V^2 \mod \{X, Y\}$, we obtain $V^2 \neg \omega = 0$. Repeating this procedure, we find $V^1 \neg \omega = V^2 \neg \omega = \cdots = V^p \neg \omega = 0$. $V^p \neg \omega = 0$ yields $H_p V^{p+1} \neg \omega = 0$, and since $H_p = 0$ by assumption, we cannot deduce anything about $V^{p+1} \neg \omega$. $\square$

We refine the previous theorem through the following proposition:

Proposition 4.7 *Suppose* $H_p = 0$*. Let* l *be a non-negative integer, let*

$$\omega \in \Omega^\star(\Theta, \eta_1, \xi_1, \cdots),$$

and suppose

$$X(\omega) \equiv \lambda\omega \bmod \{\eta_{p+1}, \eta_{p+2}, \cdots, \eta_{p+l}\}.$$

Then ω decomposes uniquely into a sum $\omega = \omega_1 + \omega_2$, where

$$\begin{aligned} \omega_1 &\equiv 0 \bmod \{\eta_{p+1}, \cdots, \eta_{p+l}\}, \\ \omega_2 &\in \Omega^*(\eta_{p+l+1}, \eta_{p+l+2}, \cdots). \end{aligned}$$

Proof. We can decompose ω uniquely as $\omega = \omega_1 + \omega_2$, where

$$\omega \equiv 0 \bmod \{\eta_{p+1}, \eta_{p+2}, \cdots, \eta_{p+l}\}$$

and

$$\omega_2 \in \Omega^s(\xi_k, \xi_{k-1}, \cdots, \xi_1, \Theta, \eta_1, \cdots, \eta_{p+l}, \eta_{p+l+1}, \cdots).$$

We can further explicit the dependence of ω_2 on ξ_k by writing

$$\omega_2 = \xi_k \wedge \gamma + \epsilon,$$

where

$$\gamma, \epsilon \in \Omega^s(\xi_{k-1}, \cdots, \xi_1, \Theta, \eta_1, \cdots, \eta_{p+l}, \eta_{p+l+1}, \cdots).$$

Now, $X(\omega) = \lambda\omega \Rightarrow X(\omega_1) + X(\omega_2) = \lambda\omega_1 + \lambda\omega_2$. But $\omega_1 \equiv 0 \bmod \{\eta_{p+1}, \cdots, \eta_{p+l}\}$ by assumption, and hence $X(\omega_1) \equiv 0 \bmod \{\eta_{p+1}, \cdots, \eta_{p+l}\}$ as well, by construction of the Laplace-adapted coframe. Thus, we have

$$X(\omega_2) \equiv \lambda\omega_2 \bmod \{\eta_{p+1}, \cdots, \eta_{p+l}\}.$$

Now,

$$X(\omega_2) = X(\xi_k) \wedge \gamma + \xi_k \wedge X(\gamma) + X(\epsilon),$$

and hence, by construction of the Laplace-adapted coframe,

$$X(\omega_2) \equiv \xi_{k+1} \wedge \gamma + \delta \bmod \{\eta_{p+1}, \eta_{p+2}, \cdots, \eta_{p+l}\},$$

where

$$\delta \in \Omega^s(\xi_k, \cdots, \xi_1, \Theta, \eta_1, \cdots, \eta_p, \eta_{p+l+1}, \eta_{p+l+2}, \cdots).$$

Now since

$$X(\omega_2) \equiv \lambda\omega_2 \bmod \{\eta_{p+1}, \cdots, \eta_{p+l}\}$$

and

$$\omega_2 \in \Omega^s(\xi_k, \xi_{k-1}, \cdots, \Theta, \eta_1, \cdots, \eta_p, \eta_{p+l+1}, \cdots),$$

we obtain that $\gamma = 0$. As a result,

$$\omega_2 \in \Omega^s(\xi_{k-1}, \cdots, \Theta, \eta_1, \cdots, \eta_p, \eta_{p+l+1}, \cdots).$$

Repeating this argument yields

$$\omega_2 \in \Omega^s(\eta_1, \cdots, \eta_p, \eta_{p+l+1}, \eta_{p+l+2}, \cdots).$$

This proves the theorem for $p = 0$. For $p \geq 1$, we can write $\omega_2 = \eta_1 \wedge \gamma + \epsilon$, where $\gamma, \epsilon \in \Omega^s(\eta_2, \cdots, \eta_p, \eta_{p+l+1}, \eta_{p+l+2}, \cdots)$. This yields the congruence

$$X(\omega_2) \equiv H_0\Theta \wedge \gamma + \delta \bmod \{\eta_{p+1}, \eta_{p+2}, \cdots, \eta_{p+l}\},$$

where

$$\delta \in \Omega^s(\eta_1, \cdots, \eta_p, \eta_{p+l+1}, \eta_{p+l+2}, \cdots).$$

Since $H_0 \neq 0$, we conclude that $\gamma = 0$, and hence

$$\omega_2 \in \Omega^s(\eta_2, \cdots, \eta_p, \eta_{p+l+1}, \eta_{p+l+2}, \cdots).$$

Repeating this argument p times, until $H_p = 0$, yields the proof of the theorem. □

Relative invariants are closely tied to conservation laws: Assume $ind(\mathcal{Y}) = p$, consider the form $\alpha \in \Omega^{0,s-1}(\mathcal{R}^\infty)$, and construct the type$-(1,s)$ form $\omega_1 = \tau \wedge \eta_{p+1} \wedge \alpha$; then

$$d_H\omega_1 = \sigma \wedge \tau \wedge \eta_{p+1} \wedge [X(\alpha) - (B - (p-1)Q)\alpha].$$

Thus, if α is a relative $X-$invariant contact form such that $X(\alpha) = (B-(p-1)Q)\alpha$, then the form ω_1 is a conservation law of the hyperbolic equation $\mathcal{R}$, i.e. $d_H\omega_1 = 0$. This construction can be further generalized; indeed, consider the form $\omega = \tau \wedge \eta_{p+1} \wedge \eta_{p+2} \wedge \alpha$. It follows from the d_H structure equations for the Laplace-adapted coframe that

$$d_H\omega = \sigma \wedge \tau \wedge \eta_{p+1} \wedge \eta_{p+2} \wedge [X(\alpha) - 2(B - pQ)].$$

Thus, if α is a relative invariant form such that $X(\alpha) = 2(B-pQ)\alpha$, then the form ω is d_H-closed. We can repeat this construction with $\omega = \tau\wedge\eta_{p+1}\wedge\eta_{p+2}\wedge\cdots\wedge\eta_{p+l}\wedge\alpha$, for all $l \in \mathbb{N}^{\star}$. Similarly, if $ind(\mathcal{X}) = q$ and the form β satisfies $Y(\beta) = [A + qP]\beta$, then the form $\sigma \wedge \xi_{q+1} \wedge \beta$ is a conservation law of the hyperbolic equation. In the following section, we shall give a detailed characterization of type$-(1, s)$ conservation laws for hyperbolic equations.

4.5 Structure theorem for type$-(1, s)$ conservation laws

Consider functions $f_i : \mathbb{R}^n \to \mathbb{R}, i = 1, \cdots, m, m < n$, and assume $M = \bigcap_{i=1}^m f_i^{-1}(0)$ is a submanifold of $\mathbb{R}^n$ of dimension $n-m$, with $i : M \hookrightarrow \mathbb{R}^n$ the inclusion map. Let $\{\theta_1, \cdots, \theta_{n-m}\}$ be a coframe on M. We can complete this coframe to the coframe $\{\theta_1, \cdots, \theta_{n-m}, df_1, \cdots, df_m\}$ on $\mathbb{R}^n$. Let now ω be a one-form on $\mathbb{R}^n$ which vanishes on M, i.e. $i^{\star}\omega = 0$. We can express ω in the coframe as $\omega = \sum_{j=1}^{n-m} \alpha^j\theta_j + \sum_{j=1}^m \beta^j df_j$. Since $i^{\star}(df_j) = d(i^{\star}f_j) = d(f_j \circ i) = 0$, we obtain $i^{\star}\omega = \sum_{j=1}^n (\alpha^j \circ i) i^{\star}\theta_j = 0$, and this implies that all the α^j vanish, and hence, $\omega = \sum_{j=1}^m \beta^j df_j$. A similar result holds for forms of arbitrary degree. Furthermore, this basic result can be generalized to forms on jet bundles. This generalization is provided by the following lemma, and is proved in a manner similar to the result above:

Lemma 4.3 *Let $\omega \in \Omega^p(J^\infty(E))$, and let $i : \mathcal{R}^\infty \hookrightarrow J^\infty(E)$ be the inclusion map of* *the infinitely prolonged equation manifold. If $i^\star\omega = 0$, then*

$$\omega = \sum_{i,j=0}^{k} \alpha_{ij}(D_x^i D_y^j F) + \sum_{i,j=0}^{k} \beta_{ij} \wedge d_V(D_x^i D_y^j F),$$

where $\alpha_{ij} \in \Omega^p(J^\infty(E))$ and $\beta_{ij} \in \Omega^{p-1}(J^\infty(E))$.

The main application of this lemma is to provide us with a first characterization of conservation laws of the second-order hyperbolic equation $\mathcal{R}$. Indeed, let $\omega \in \Omega^{1,s}(\mathcal{R}^\infty)$ be a d_H-closed form. By extending the natural coordinates

$$(x, y, z, z_x, z_y, z_{xy}, z_{yy}, \cdots),$$

and the natural coframe on $\mathcal{R}^\infty$ to the natural coordinate system and the natural coframe on $J^\infty(E)$, there exists a type $(1, s)-$form ω_0 on $J^\infty(E)$ such that $i^\star\omega_0 = \omega$. Since $i^\star d_H(\omega_0) = d_H(i^\star\omega_0) = d_H\omega = 0$, we can write

$$d_H\omega_0 = dx \wedge dy \wedge [\sum_{i,j=0}^{k} (D_x^i D_y^j F)\alpha_{ij} + \sum_{i,j=0}^{k} d_V(D_x^i D_y^j F) \wedge \beta_{ij}],$$

where $\alpha_{ij} \in \Omega^{0,s}(J^\infty(E))$ and $\beta_{ij} \in \Omega^{s-1}(J^\infty(E))$. Now note that

$$\begin{aligned} dx \wedge dy \wedge (D_xF)\alpha &= (D_xF)dx \wedge dy \wedge \alpha = d_HF \wedge dy \wedge \alpha \\ &= d_H(Fdy \wedge \alpha) - Fd_H(dy) \wedge \alpha + Fdy \wedge d_H\alpha \\ &= d_H(Fdy \wedge \alpha) - dx \wedge dy \wedge F(D_x\alpha). \end{aligned}$$

Thus, $dx \wedge dy \wedge (D_xF)\alpha$ differs from $dx \wedge dy \wedge F(D_x\alpha)$ by the term $d_H(Fdy \wedge \alpha)$, which vanishes when pulled back to $\mathcal{R}^\infty$. Repeating this basic integration by parts operation starting from the highest derivative terms $D_x^k D_y^k F$ and $d_V(D_x^k D_y^k F)$, we can rewrite $d_H\omega_0$ as

$$d_H\omega_0 = dx \wedge dy \wedge [F\hat{\zeta} + (d_VF) \wedge \hat{\rho}] + \{\star\}$$

where $\{\star\}$ denotes terms which vanish when pulled back to $\mathcal{R}^\infty$. We thus have:

Proposition 4.8 *Let $\omega \in \Omega^{1,s}(\mathcal{R}^\infty)$ be a d_H-closed form on $\mathcal{R}^\infty$. Then $\exists\hat{\omega} \in \Omega^{1,s}(J^\infty(E)), \hat{\zeta} \in \Omega^{0,s}(J^\infty(E)), \hat{\rho} \in \Omega^{0,s-1}(J^\infty(E))$ such that $i^\star(\hat{\omega}) = \omega$ and $d_H\hat{\omega} = dx \wedge dy \wedge [F\hat{\zeta} + (d_VF) \wedge \hat{\rho}]$.*

This characterization of d_H−closed forms on $\mathcal{R}^\infty$ in terms of forms on $J^\infty(E)$ is of fundamental importance, as it allows us to make use of the horizontal homotopy operators and thereby complete our characterization of conservation laws. Recall that the horizontal homotopy operators $h_H^{r,s}$ satisfy the identity $\hat{\omega} = h_H^{2,s}(d_H\hat{\omega}) + d_H h_H^{1,s}(\hat{\omega})$, for every type−$(1,s)$ form $\hat{\omega}$ on $J^\infty(E)$. With $d_H\hat{\omega} = dx \wedge dy \wedge [F\hat{\zeta} + (d_V F) \wedge \hat{\rho}]$, we obtain

$$h_H^{2,s}(d_H\hat{\omega}) = \frac{1}{s}\nu_i \wedge \theta \wedge [\frac{\partial F}{\partial z_i}\hat{\rho} - D_j(\frac{\partial F}{\partial z_{ij}}\hat{\rho})] + \frac{1}{s}\nu_i \wedge \theta_j \wedge \frac{\partial F}{\partial z_{ij}}\hat{\rho} + \{\star\},$$

where $\{\star\}$ denotes terms which vanish when pulled back to $\mathcal{R}^\infty$, and $\nu_i = \frac{\partial}{\partial x^j} \neg dx \wedge dy$. From $\omega = i^\star\hat{\omega} = i^\star(h_H^{2,s}(d_H\hat{\omega})) + d_H(i^\star h_H 1, s(\hat{\omega}))$, we obtain $\omega = \Psi_c(\rho_c) + d_H\gamma$, where $\rho_c = \frac{1}{s}i^\star(\hat{\rho})$ and $\Psi_c(\rho_c) = \nu_j \wedge \theta \wedge [\frac{\partial F}{\partial z_i}\rho_c - D_i(\frac{\partial F}{\partial z_{ij}}\rho_c)] + \nu_j \wedge \theta_i \wedge [\frac{\partial F}{\partial z_{ij}}\rho_c)]$.

To complete our characterization of the d_H−closed form ω in the natural coordinate coframe, we need to characterize the contact form ρ_c as well. To do so, we recall that the interior Euler-Lagrange operator $J : \Omega^{2,s}(J^\infty(E)) \to \Omega^{2,s-1}(J^\infty(E))$ has the fundamental property that $J(d_H\hat{\alpha}) = 0, \forall \hat{\alpha} \in \Omega^{1,s}(J^\infty(E))$. Furthermore, it is explicitely given by

$$J(\hat{\alpha}) = \frac{\partial}{\partial z}\neg\hat{\alpha} - D_i(\frac{\partial}{\partial z_i}\neg\hat{\alpha}) + D_{ij}(\frac{\partial}{\partial z_{ij}}\neg\hat{\alpha}) + \cdots.$$

We can thus rewrite $J[dx \wedge dy \wedge (F\hat{\zeta} + (d_V F) \wedge \hat{\rho})] = 0$ as

$$\frac{\partial F}{\partial z}\hat{\rho} - D_i(\frac{\partial F}{\partial z_i}\hat{\rho}) + D_{ij}(\frac{\partial F}{\partial z_{ij}}\hat{\rho}) + \{\star\} = 0 \tag{4.13}$$

on $J^\infty(E)$, where again $\{\star\}$ denotes terms which vanish when pulled back to $\mathcal{R}^\infty$. Pulling (4.13) back to $\mathcal{R}^\infty$, we thus obtain

$$\frac{\partial F}{\partial z}\rho_c - D_i(\frac{\partial F}{\partial z_i}\rho_c) + D_{ij}(\frac{\partial F}{\partial z_{ij}}\rho_c) = 0.$$

This completes our characterization of d_H−closed forms in the natural coordinate coframe $\{dx, dy, \theta, \theta_x, \theta_y, \cdots\}$. We can thus state:

Theorem 4.4 *Let $\omega \in \Omega^{1,s}(\mathcal{R}^\infty)$ be a d_H−closed form on $\mathcal{R}^\infty$. Then there exists a form $\rho_c \in \Omega^{0,s-1}(\mathcal{R}^\infty)$, and a form $\gamma \in \Omega^{0,s}(\mathcal{R}^\infty)$ such that ω is given by*

$$\omega = \Psi_c(\rho_c) + d_H\gamma$$

where $\Psi_c : \Omega^{0,s-1}(\mathcal{R}^\infty) \to \Omega^{1,s}(\mathcal{R}^\infty)$ *is defined by*

$$\Psi_c(\rho_c) = \nu_j \wedge \theta \wedge [\frac{\partial F}{\partial z_i}\rho_c - D_i(\frac{\partial F}{\partial z_{ij}}\rho_c)] + \nu_j \wedge \theta_i \wedge [\frac{\partial F}{\partial z_{ij}}\rho_c)]$$

and where ρ_c *satisfies the equation*

$$\frac{\partial F}{\partial u}\rho_c - D_i(\frac{\partial F}{\partial z_i}\rho_c) + D_{ij}(\frac{\partial F}{\partial z_{ij}}\rho_c) = 0.$$

Let now $\omega \in \Omega^{0,s'}(\mathcal{R}^\infty)$ be a contact form. Note that we can write

$$\begin{aligned}
[\frac{\partial F}{\partial u}\rho_c \quad &- \quad D_i(\frac{\partial F}{\partial z_i}\rho_c) + D_{ij}(\frac{\partial F}{\partial z_{ij}}\rho_c)] \wedge \omega \wedge dx \wedge dy \\
&= \quad \rho_c \wedge [\frac{\partial F}{\partial u}\omega + \frac{\partial F}{\partial z_i}D_i(\omega) + \frac{\partial F}{\partial z_{ij}}D_{ij}(\omega)] \wedge dx \wedge dy + d_H\alpha \\
&= \quad \rho_c \wedge \mathcal{L}_c(\omega) \wedge dx \wedge dy + d_H\alpha
\end{aligned}$$

where $\alpha \in \Omega^{1,s+s'}(\mathcal{R}^\infty)$, and $\mathcal{L}_c$ is the universal linearization of the hyperbolic equation $\mathcal{R}$ expressed in the natural coordinate coframe. Thus, at the level of cohomology, the operator $\mathcal{L}_c^\star : \Omega^{0,s}(\mathcal{R}^\infty) \to \Omega^{0,s}(\mathcal{R}^\infty)$ defined by

$$\mathcal{L}_c^\star(\rho_c) = \frac{\partial F}{\partial u}\rho_c - D_i(\frac{\partial F}{\partial z_i}\rho_c) + D_{ij}(\frac{\partial F}{\partial z_{ij}}\rho_c)$$

is the adjoint of the universal linearization operator $\mathcal{L}_c : \Omega^{0,s}(\mathcal{R}^\infty) \to \Omega^{0,s}(\mathcal{R}^\infty)$ for the pairing

$$(\rho, \omega) \mapsto < \rho, \omega > = \rho \wedge \omega \wedge dx \wedge dy.$$

To characterize d_H−closed forms in the Laplace-adapted coframe, we need to re-express all these definitions.

Consider then the universal linearization $\mathcal{L} : \Omega^{0,s}(\mathcal{R}^\infty) \to \Omega^{0,s}(\mathcal{R}^\infty)$ of the second order hyperbolic operator equation $\mathcal{R}$, given by the total differential operator $\mathcal{L} = XY + AX + BY + C$. We define the adjoint $\mathcal{L}^\star$ of $\mathcal{L}$ as being the unique total differential operator $\mathcal{L}^\star : \Omega^{0,s}(\mathcal{R}^\infty) \to \Omega^{0,s}(\mathcal{R}^\infty)$ such that

$$\forall \rho \in \Omega^{0,s}(\mathcal{R}^\infty), \forall \omega \in \Omega^{0,s'}(\mathcal{R}^\infty), [\rho \wedge \mathcal{L}(\omega) - \mathcal{L}^\star(\rho) \wedge \omega] \wedge \sigma \wedge \tau = d_H\gamma,$$

for some $\gamma \in \Omega^{1,s+s'}(\mathcal{R}^\infty)$. The formula defining the adjoint operator is nothing other than an integration by parts formula in the Laplace-adapted coframe. Since

$$\begin{aligned}
d_H(\rho \wedge \omega \wedge \tau) \quad &= \quad \sigma \wedge X(\rho \wedge \omega \wedge \tau) \\
&= \quad \sigma \wedge X(\rho) \wedge \omega \wedge \tau + \sigma \wedge \rho \wedge X(\omega) \wedge \tau - Q\sigma \wedge \rho \wedge \omega \wedge \tau,
\end{aligned}$$

we have that the adjoint $X^\star$ of X is $X^\star = -X + Q$. Similarly, the adjoint $Y^\star$ of Y is given by $Y^\star = -Y - P$. It follows easily that the adjoint $\mathcal{L}^\star$ of $\mathcal{L}$ is given by

$$\mathcal{L}^\star(\rho) = XY(\rho) + A^\star X(\rho) + B^\star Y(\rho) + C^\star \rho,$$

where

$$\begin{cases} A^\star = -A, \\ B^\star = -B - 2Q, \\ C^\star = -X(A) - Y(B+Q) + C - AB + (A-P)(B+Q). \end{cases}$$

Let $\{\sigma, \tau, \Theta, \eta_1, \xi_1, \eta_2, \xi_2, \cdots\}$ be the Laplace-adapted coframe. With respect to this coframe, we define, for each $s \geq 1$, a map $\Psi : \Omega^{0,s-1} \to \Omega^{0,s-1}$ by

$$\Psi(\rho) = \frac{1}{2}\sigma \wedge [\Theta \wedge \psi_1 - \xi_1 \wedge \rho] + \frac{1}{2}\tau \wedge [\Theta \wedge \psi_2 + \eta_1 \wedge \rho],$$

with

$$\begin{aligned} \psi_1 &= X(\rho) - (B+Q)\rho, \\ \psi_2 &= -Y(\rho) + a\rho. \end{aligned}$$

A simple calculation shows that $d_H\Psi(\rho) = -\sigma \wedge \tau \wedge \Theta \wedge \mathcal{L}^\star(\rho)$ and hence, if the contact form ρ satisfies the adjoint equation $\mathcal{L}^\star(\rho) = 0$, then the form $\Psi(\rho)$ is d_H-closed. A simple but tedious calculation shows also that $\Psi_c(\rho_c) = \Psi(\rho)$, with $\rho = \frac{\delta}{\kappa}\rho_c$, where δ, κ are given in definition/proposition (4.1). Combining this result with the previous theorem, we obtain the following characterization of d_H-closed forms in the Laplace-adapted coframe:

Theorem 4.5 *Let $s \geq 1$ and let $\omega \in \Omega^{1,s}(\mathcal{R}^\infty)$ be a d_H-closed form. Then there exist contact forms $\rho \in \Omega^{0,s-1}$ and $\gamma \in \Omega^{0,s}(\mathcal{R}^\infty)$ such that ω is given by*

$$\omega = \Psi(\rho) + d_H\gamma,$$

and where ρ satisfies the adjoint equation

$$XY(\rho) + A^\star X(\rho) + B^\star Y(\rho) + C^\star \rho = 0.$$

Using this characterization of conservation laws of $\mathcal{R}$, we deduce the following theorem:

Theorem 4.6 *Let $\mathcal{R}$ be a second-order hyperbolic equation and suppose $ind(\mathcal{Y}) = \infty$ and $ind(\mathcal{X}) = \infty$. Then, for all $s \geq 3$, all type$-(1,s)$ conservation laws are trivial, i.e. $H^{1,s}(\mathcal{R}^\infty) = 0$.*

Proof. According to theorem (4.5), we need only prove that there are no non-zero type$-(0, s-1)$ solutions ρ to the adjoint equation $XY(\rho)+A^\star X(\rho)+B^\star Y(\rho)+C^\star\rho = 0$. We rewrite this second order total differential equation as a system of first-order total differential equations

$$\begin{aligned} X(\rho) &= (Q+B)\rho + \psi_1, \\ Y(\psi_1) &= H_0\rho + (A-P)\psi_1. \end{aligned}$$

Suppose ρ is a non-zero solution of adapted order k of this first-order system. We can therefore assume, with no loss of generality, $V^k\neg\rho \neq 0, V^{k+i}\neg\rho = 0, \forall i \geq 1$. Then

$$\begin{aligned} V^{k+1}\neg X(\rho) &= X(V^{k+1}\neg\rho) - [X, V^{k+1}]\neg\rho \\ &= -[X, V^{k+1}]\neg\rho \\ &= -B_k(V^{k+1}\neg\rho) + H_{k+1}(V^{k+2}\neg\rho) \\ &= 0, \end{aligned}$$

and hence $V^{k+1}\neg\psi_1 = 0$. From

$$V^{k+1}\neg Y(\psi_1) = H_0 V^{k+1}\neg\rho + (A-P)V^{k+1}\neg\psi_1 = 0,$$

we obtain $V^k\neg\psi_1 = 0$. Now

$$\begin{aligned} V^k\neg X(\rho) &= (Q+B)V^k\neg\rho + V^k\neg\psi_1 \\ &= (Q+B)V^k\neg\rho \\ &= X(V^k\neg\rho) - [X, V^k]\neg\rho \\ &= X(V^k\neg\rho) - B_{k-1}V^k\neg\rho. \end{aligned}$$

Thus, $V^k \neg \rho$ is a non-zero relative $X-$invariant contact form. But $ind(\mathcal{Y}) = \infty$ and this contradicts theorem (4.3). Thus ρ cannot satisfy the adjoint equation, and hence $H^{1,s}(\mathcal{R}^\infty) = 0$. $\square$

We conclude this section with a structure theorem for conservation laws in $H^{1,s}(\mathcal{R}^\infty)$, $s \geq 3$, for the proof of which we refer to [3].

Theorem 4.7 *Let $\mathcal{R}$ be a second-order hyperbolic equation. Suppose $ind(\mathcal{Y}) = p$ and $ind(\mathcal{X}) = q$. Then, for $s \geq 3$, every d_H-closed form $\omega \in \Omega^{1,s}(\mathcal{R}^\infty)$ may be written as*

$$\omega = \sigma \wedge \xi_{q+1} \wedge \beta + \tau \wedge \eta_{p+1} \wedge \alpha + d_H \gamma,$$

where $\gamma \in \Omega^{0,s}(\mathcal{R}^\infty)$ and $\alpha \in \Omega^1(\eta_{p+1}, \eta_{p+2}, \cdots)$ and $\beta \in \Omega^{s-1}(\xi_{q+1}, \xi_{q+2}, \cdots)$ satisfy

$$\begin{aligned} X(\alpha) + [-B + (p-1)Q]\alpha &= 0, \\ Y(\beta) - [-A + qP]\beta &= 0. \end{aligned} \tag{4.14}$$

If $ind(\mathcal{X}) = \infty$, then (4.14) remains valid with $\beta = 0$.

4.6 Darboux integrability and Laplace invariants

We recall from chapter **2** that a second-order hyperbolic equation $\mathcal{R}$ is Darboux-integrable if there exist two functionally independent real-valued $X-$ invariant functions $I, \hat{I}$ on $\mathcal{R}^\infty$, and two functionally independent real-valued $Y-$ invariant functions $J, \hat{J}$ on $\mathcal{R}^\infty$, that is, $X(I) = X(\hat{I}) = 0, dI \wedge d\hat{I} \neq 0$, and $X(J) = X(\hat{J}) = 0, dJ \wedge d\hat{J} \neq 0$. With the $X-$ and $Y-$ characteristic Pfaffian systems of order k as defined in equations (4.6) and (4.7), respectively, it was shown in section (3.1) that $I \in \mathcal{C}^\infty(\mathcal{R}^\infty)$ of adapted order k is $X-$invariant if and only if $dI \in C_k(X)$, and that $J \in \mathcal{C}^\infty(\mathcal{R}^\infty)$ of adapted order k is $Y-$invariant if and only if $dJ \in C_k(Y)$. Consider now a Pfaffian system $\mathcal{I}$ on a manifold M. The i^{th} derived Pfaffian system $\mathcal{I}^{(i)}$ is defined inductively by the short exact sequence $0 \to \mathcal{I}^{(i+1)} \overset{j}{\hookrightarrow} \mathcal{I}^{(i)} \overset{\delta_i}{\to} d\mathcal{I}^{(i)} \bmod \mathcal{I}(i) \to 0$ where j is the inclusion map, and δ_i

is the composition of the exterior derivative d with the quotient modulo the ideal generated by $\mathcal{I}^{(i)}$. The derived Pfaffian systems form a sequence

$$\cdots \subset \mathcal{I}^{(i)} \subset \cdots \subset \mathcal{I}^{(2)} \subset \mathcal{I}^{(1)} \subset \mathcal{I}$$

which stabilizes at the maximal completely integrable subsystem of $\mathcal{I}$, which we denote by $\mathcal{I}^{(\infty)}$. Applying this construction to the characteristic systems $C_k(X)$ and $C_k(Y)$ yields the maximal completely integrable subsystems $C_k^{(\infty)}(X)$ and $C_k^{(\infty)}(Y)$, respectively. The structure equations of the Laplace-adapted coframe allow us to give the following characterization of $C_k^{(\infty)}(X)$ and $C_k^{(\infty)}(Y)$; note that this characterization is valid for the characteristic coframe as well.

Lemma 4.4 *Let $\mathcal{R}$ be a second-order hyperbolic equation with characteristic vector fields X and Y, and let $\{\sigma, \tau, \Theta, \xi_1, \eta_1, \cdots\}$ be the Laplace-adapted coframe on $\mathcal{R}^\infty$. Then, $\forall k \in \mathbb{N}^\star$, $C_k^{(\infty)}(X)$ and $C_k^{(\infty)}(Y)$ satisfy*

$$C_k^{(\infty)}(X) \subset \Omega^1(\tau, \Theta, \xi_1, \eta_1, \eta_2, \cdots, \eta_k),$$
$$C_k^{(\infty)}(Y) \subset \Omega^1(\sigma, \Theta, \eta_1, \xi_1, \xi_2, \cdots, \xi_k).$$

Proof. Assume $\xi_k \in C_k^{(1)}(X)$. Then, exactness of the sequence $0 \to C_k^{(1)}(X) \overset{j}{\hookrightarrow} C_k(X) \xrightarrow{\delta_i} dC_k(X) \bmod C_k(X) \to 0$ implies that $d\xi_k \equiv 0 \mod C_k(X)$. But $d\xi_k = d_H\xi_k + d_V\xi_k \equiv \sigma \wedge \xi_{k+1} + d_V\xi_k \mod C_k(X)$, and hence $d\xi_k \neq 0$, $\bmod C_k(X)$. Thus $\xi_k \notin C_k^{(1)}(X)$. Since $d\xi_{k-1} \equiv \sigma \wedge \xi_k + d_V\xi_{k-1} \bmod C_k^{(1)}(X)$, and $\xi_k \notin C_k^{(1)}(X)$, we deduce that $\xi_{k-1} \notin C_k^{(2)}(X)$. Continuing in this way, we eliminate the ξ_l up to and including ξ_2. But we cannot eliminate ξ_1, since $d_V\tau = \sigma \wedge \beta + \tau \wedge \mu_2$ and β, being of adapted order 2, may contain ξ_2. $\square$

If I is an $X-$invariant function on $\mathcal{R}^\infty$, then $dI \in C_k(X)$. Since $d^2I = 0$, $dI \in C_k^{(\infty)}(X)$ as well. Thus, given a Darboux-integrable equation, $\dim C_k^{(\infty)}(X)$ will be at least 2 for some $k \geq 1$, and $\dim C_l^{(\infty)}(Y)$ will be at least 2 for some $l \geq 1$. The characterization of $\dim C_k^{(\infty)}(X)$ and $\dim C_l^{(\infty)}(Y)$ has been performed by Goursat [8], and in what follows, we mention this characterization for $C_k^{(\infty)}(X)$ only, that for $C_k^{(\infty)}(Y)$ being identical.

Theorem 4.8 *Let m be the minimum integer for which $\dim C_k^{(\infty)}(X) \neq 0$. Then $\dim C_k^{(\infty)}(X)$ is either 1, 2, or 3, and we have the following cases:*

1. $\dim C_m^{(\infty)}(X) = 1$: *Then there exists an integer $n > m$ such that* $\dim C_m^{(\infty)}(X) = 1, \cdots, \dim C_{n-1}^{(\infty)}(X) = 1, \dim C_n^{(\infty)}(X) = 2, \cdots,$ $\dim C_{n+i}^{(\infty)}(X) = 2 + i, \cdots$

2. $\dim C_m^{(\infty)}(X) = 2$: *Then m is either 1 or 2, and we have the following subcases:*

   (a) $m = 1$: *Then* $\dim C_2^{(\infty)}(X) = 3, \cdots, \dim C_{i+1}^{(\infty)}(X) = 2 + i, \cdots$.

   (b) $m = 2$: *Then* $\dim C_3^{(\infty)}(X) = 3, \cdots, \dim C_{i+2}^{(\infty)}(X) = 2 + i, \cdots$.

3. $\dim C_m^{(\infty)}(X) = 3$: *Then $m = 3$ and* $\dim C_{i+3}^{(\infty)}(X) = 3 + i, \cdots$

The important thing to note here is that Darboux-integrability of the hyperbolic equation $\mathcal{R}$ immediately yields a plethora of $X-$ and $Y-$ invariant functions of increasing order. In particular, in case (1), there exists a sequence of functionally independent $X-$invariant functions $I_m, I_n, I_{n+1}, I_{n+2}, \cdots$ of order $m, n, n+1, n+2, \cdots$ such that any $X-$invariant function may be expressed as a function of these; in case (2a), there is a similar sequence of $X-$ invariant functions $I_1, I_1', I_2, I_3, \cdots$, where I_1, I_1' are of order 1 and I_j of order j; in case (2b), there is a sequence $I_2, I_2', I_3, \cdots$ of $X-$invariant functions, where I_2, I_2' are of order 2, and I_j of order j; finally, in case (3), there is a sequence $I_2'', I_2', I_2, I_3, \cdots$ of functionally independent $X-$invariant functions, where I_2'', I_2', I_2 are of order 2, and I_j of order j.

This abundance of $X-$ and $Y-$ invariant functions allows us to rescale the characteristic total vector fields X and Y so that they commute:

Lemma 4.5 *Let $\mathcal{R}$ be a hyperbolic equation with characteristic total vector fields X and Y. If I and J are non-trivial $X-$ and $Y-$ invariant functions, respectively, then the characteristic vector fields $\hat{X} = \frac{1}{X(J)} X and \hat{Y} = \frac{1}{Y(I)} Y$ commute.*

Proof. We have

$$\begin{aligned} [\hat{X}, \hat{Y}] &= [\frac{1}{X(J)} X, \frac{1}{Y(I)} Y] \\ &= \frac{1}{X(J)Y(I)} (-\frac{X(Y(I))}{Y(I)} Y + \frac{Y(X(J))}{X(J)} X + [X, Y]). \end{aligned}$$

Now

$$\begin{aligned} X(Y(I)) &= Y(X(I)) + [X,Y](I) \\ &= [X,Y](I) = PX(I) + QY(I) = QY(I). \end{aligned}$$

We obtain similarly that $Y(X(J)) = -PX(J)$. These two equalities yield $[\hat{X}, \hat{Y}] = 0$. $\square$

If $\mathcal{R}$ is Darboux-integrable and we assume X and Y are rescaled so as to commute, then as an immediate consequence we have that if I is an $X-$invariant function, then so is $Y(I)$. It is proved in Goursat [8] that the sequences of invariants in all cases (1), (2a), (2b), and (3), are all generated in this manner, that is, by choosing $X-$ and $Y-$ invariant functions of minimal order and successively applying the $Y-$ and $X-$ operators, respectively. We can thus find $X-$invariant functions I, J, K such that $K = Y(J)$ and, rescaling Y if necessary, $Y(I) = 1$. Consider now the contact form $\omega = d_V J - K d_V I$. We have $d_H\omega = d_H d_V J - d_H K \wedge d_V I + K d_V d_H I = \tau \wedge [d_V K - Y(K) d_V I]$, which implies that $X(\omega) = X \neg d_H \omega = 0$; in other words, the contact form ω is an $X-$invariant form. $Y-$invariant forms are similarly constructed. Together with theorem (4.3), this leads to the following result [3]:

Theorem 4.9 *Let $\mathcal{R}$ be a second-order hyperbolic equation. If $\mathcal{R}$ is Darboux-integrable, then the Laplace indices ind($\mathcal{X}$) and ind($\mathcal{Y}$) are finite.*

Such a result was to be expected in light of the classical theory of linear hyperbolic equations. The classical theory goes even further, establishing Darboux-integrability of linear hyperbolic equations for which the Laplace indices are finite. We now prove that the converse of theorem (4.9) holds as well. First, however, we shall prove the following proposition:

Proposition 4.9 *If $H_p = 0$, then there is a unique form*

$$\Upsilon \in \Omega^1(\xi_1, \Theta, \eta_1, \cdots, \eta_p)$$

such that for some contact form $\hat{\eta}$,

$$d_V \eta_{p+1} = \eta_{p+2} \wedge \Upsilon + \eta_{p+1} \wedge \hat{\eta}.$$

The form Υ satisfies

$$\begin{aligned} X(\Upsilon) &\equiv -Q\Upsilon + \beta \bmod \{\eta_{p+1}, \eta_{p+2}\},, \\ d_V\Upsilon &\equiv \Upsilon \wedge [\mu_2 - Y(\Upsilon)] \bmod \{\eta_{p+1}, \eta_{p+2}\}. \end{aligned}$$

The forms $\eta_{p+i}, i \geq 1$, satisfy the d_V structure equations $d_V\eta_{p+i} \equiv \eta_{p+i+1} \wedge \Upsilon \bmod \{\eta_{p+1}, \cdots, \eta_{p+i}\}$.

Proof. We have

$$\begin{aligned} X(d_V\eta_{p+1}) &\equiv d_V(X(\eta_{p+1})) - \mu_1 \wedge X(\eta_{p+1}) - \beta \wedge Y(\eta_{p+1}) \bmod \{\eta_{p+1}, \eta_{p+2}\} \\ &\equiv -B_p d_V(\eta_{p+1}) \bmod \{\eta_{p+1}, \eta_{p+2}\} \end{aligned}$$

and hence $d_V\eta_{p+1}$ is a relative $X-$invariant $\bmod \{\eta_{p+1}, \eta_{p+2}\}$. We apply the characterization of relative invariants in proposition (4.7) and we write

$$d_V\eta_{p+1} = \eta_{p+2} \wedge \Upsilon + \eta_{p+1} \wedge \hat{\eta} + \omega,$$

where $\omega \in \Omega^2(\eta_{p+3}, \eta_{p+4}, \cdots)$. The d_V structure equation

$$d_V\eta_i \equiv 0 \bmod \{\xi_1, \Theta, \eta_1, \cdots, \eta_i\}, \forall i \geq 1,$$

implies $\omega = 0$ and $\Upsilon \in \Omega^1(\xi_1, \Theta, \eta_1, \cdots, \eta_{p+1})$. Now

$$\begin{aligned} d_H(d_V\eta_{p+1}) &\equiv (d_H\eta_{p+2}) \wedge \Upsilon - \eta_{p+2} \wedge d_H\Upsilon + d_H\eta_{p+1} \wedge \hat{\eta} \bmod \{\eta_{p+1}\} \\ &\equiv \sigma \wedge [(Q - B_p)\eta_{p+2} \wedge \Upsilon + \eta_{p+2} \wedge X(\Upsilon)] \\ &\quad + \tau \wedge [\eta_{p+3} \wedge \Upsilon + \eta_{p+2} \wedge Y(\Upsilon) + \eta_{p+2} \wedge \hat{\eta}] \bmod \{\eta_{p+1}\}; \end{aligned}$$

on the other hand,

$$\begin{aligned} d_H(d_V\eta_{p+1}) &= d_V(-B_p\sigma \wedge \eta_{p+1} + \tau \wedge \eta_{p+2}) \\ &\equiv \sigma \wedge [Q\Upsilon + X(\Upsilon) - \beta] \bmod \{\eta_{p+1}\} \\ &\equiv 0 \bmod \{\eta_{p+1}\}, \end{aligned}$$

that is, $X(\Upsilon) \equiv -Q\Upsilon + \beta \bmod \{\eta_{p+1}, \eta_{p+2}\}$. Now,

$$\begin{aligned} d_V\eta_{p+2} &= d_V(Y(\eta_{p+1})) \\ &= Y(d_V(\eta_{p+1})) + \alpha \wedge X(\eta_{p+1}) + \mu_2 \wedge Y(\eta_{p+1}) \\ &\equiv \eta_{p+3} \wedge \Upsilon + \eta_{p+2} \wedge Y(\Upsilon) + \eta_{p+2} \wedge \hat{\eta} + \mu_2 \wedge \eta_{p+2} \bmod \{\eta_{p+1}\}, \end{aligned}$$

and from $d_V^2\eta_{p+1} = d_V(\eta_{p+2} \wedge \Upsilon + \eta_{p+1} \wedge \hat{\eta}) = 0$, we obtain $d_V\eta_{p+2} \equiv \eta_{p+3} \wedge \Upsilon \bmod \{\eta_{p+1}, \eta_{p+2}\}$. Repeating this procedure, we obtain $d_V\eta_{p+i} \equiv \eta_{p+i+1} \wedge \Upsilon \bmod \{\eta_{p+1}, \eta_{p+2}, \cdots, \eta_{p+i}\}, \forall i \geq 1$. $\square$

The significance of the contact form Υ defined in the previous structure theorem is given by the following theorem [9]:

Theorem 4.10 *The Pfaffian system $\mathcal{D}_{p+i}(X) = \Omega^1(\tau - \Upsilon, \eta_{p+1}, \cdots, \eta_{p+i})$ is completely integrable for $i \geq 2$ if $H_p = 0$.*

Proof.

$$\begin{aligned} d(\tau - \Upsilon) &= d_H\tau + d_V\tau - d_H\Upsilon - d_V\Upsilon \\ &\equiv -\sigma \wedge [X(\Upsilon) + Q\Upsilon - \beta] + \Upsilon \wedge [\mu_2 - Y(\Upsilon)] - d_V\Upsilon \bmod \mathcal{D}_{p+2}(X) \\ &\equiv 0 \bmod \mathcal{D}_{p+2}(X); \end{aligned}$$

furthermore,

$$\begin{aligned} d\eta_{p+1} &= d_H\eta_{p+1} + d_V\eta_{p+1} \\ &= \sigma \wedge X(\eta_{p+1}) + \tau \wedge \eta_{p+2} + \eta_{p+2} \wedge \Upsilon + \eta_{p+1} \wedge \hat{\eta} \\ &\equiv 0 \bmod \mathcal{D}_{p+2}(X); \end{aligned}$$

finally,

$$\begin{aligned} d\eta_{p+2} &= d_H\eta_{p+2} + d_V\eta_{p+2} \\ &\equiv \sigma \wedge X(\eta_{p+2}) + \tau \wedge \eta_{p+3} + \eta_{p+3} \wedge \Upsilon \bmod \mathcal{D}_{p+2}(X) \\ &\equiv 0 \bmod \mathcal{D}_{p+2}(X). \end{aligned}$$

This proves the theorem for $i = 2$. Using the same d_H and d_V structure equations, the theorem is proved by induction on i. $\square$

Recalling now that the hyperbolic equation $\mathcal{R}$ is Darboux-integrable if for sufficiently large order k, the derived characteristic Pfaffian systems $C_k^{(\infty)}(X)$ and $C_k^{(\infty)}(Y)$ each stabilize at a completely integrable subsystem of dimension at least 2, we deduce the following converse to theorem (4.9):

Theorem 4.11 *Let $\mathcal{R}$ be a second-order hyperbolic equation. If the Laplace indices $ind(\mathcal{X})$ and $ind(\mathcal{Y})$ are finite, then $\mathcal{R}$ is Darboux-integrable.*

Chapter 5

Conclusion

In this thesis, we have illustrated two classical techniques for integrating partial differential equations, namely the Darboux method, and the Laplace method for linear hyperbolic equations, as well as the relations between the two techniques for this latter class of equations. Following the work in [3] and [9], we have shown how the variational bicomplex can provide a formal differential geometric setting in which such relations can be studied for non-linear hyperbolic equations as well, and we have demonstrated the key role that moving coframes and their structure equations play in this study.

Such a study can be extended in many directions, and it is reasonable to expect that numerous properties of hyperbolic and other partial differential equations would manifest themselves geometrically; in particular, the questions of existence, unicity, and regularity of solutions. In the classical setting, these questions are usually settled through estimates obtained via some form or other of integration by parts and classical conservation laws. In this geometric setting, the analogous technique would be based on a Stokes formula for the d_H operator and contact-form valued conservation laws, and an important problem is to determine under what conditions such a Stokes formula does indeed hold, and what information these contact-form valued conservation laws do provide. The moving coframe technique is also a potentially rich source of developments: Being non-constructive in its application, one could expect that it would be much further-reaching than the techniques based on explicit solutions . The question remains how to build a moving coframe on a solution manifold in a systematic manner, so that its structure equations embed as much information as possible about the original partial differential equation.

Bibliography

[1] I. M. ANDERSON, *The Variational Bicomplex,* forthcoming.

[2] I. M. ANDERSON and NIKY KAMRAN, *The variational bicomplex for second order scalar partial differential equations in the plane,* Technical Report, Department of Mathematics, Utah State University, September 1994.

[3] I. M. ANDERSON and NIKY KAMRAN, *The variational bicomplex for hyperbolic second-order scalar partial differential equations in the plane,* Duke Mathematical Journal **87** (1997), 265-319.

[4] É. CARTAN, *Les systèmes différentiels extérieurs et leurs applications géométriques,* Hermann, 1971.

[5] R. B. GARDNER and N. KAMRAN, *Characteristics and the geometry of hyperbolic equations in the plane,* J. Differential Equations **104** (1993), 60-116.

[6] M. R. GOSSE *La méthode de Darboux et les équations $s = f(x, y, z, p, q)$*, Mémorial des Sciences Mathématiques, Paris, 1926.

[7] E. GOURSAT, *Leçons sur l'intégration des équations aux dérivées partielles du second ordre à deux variables indépendantes, Tome I,* Hermann, Paris, 1896.

[8] E. GOURSAT, *Leçons sur l'intégration des équations aux dérivées partielles du second ordre à deux variables indépendantes, Tome II,* Hermann, Paris, 1896.

[9] M. JURAŠ and I. M. ANDERSON, *Generalized Laplace invariants and the method of Darboux,* Duke Mathematical Journal **89** (1997), 351-375.

[10] W. M. TULCZYJEW, "The Euler-Lagrange resolution," in *Differential Geometrical Methods in Mathematical Physics,* Lecture Notes in Math. **836**, Springer-Verlag, New-York, 1980, 22-48.

[11] E. VESSIOT, *Sur les équations aux dérivées partielles du second ordre,* $F(x, y, z, p, q, r, s, t) = 0$, *intégrables par la méthode de Darboux,* J. Math. Pures Appl. (9) **18** (1939), 1-61.

[12] E. VESSIOT, *Sur les équations aux dérivées partielles du second ordre,* $F(x, y, z, p, q, r, s, t) = 0$, *intégrables par la méthode de Darboux (suite),* J. Math. Pures Appl. (9) **21** (1942), 1-66.

[13] A. M. VINOGRADOV, *The C-spectral sequence, Lagrangian formalism, and conservation laws I, II,* J. Math. Anal. Appl. **100** (1984), 1-40, 41-129.