ON ILIEFF'S CONJECTURE AND RELATED PROBLEMS

FRANK GACS

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The following is a conjecture due to Goodman, Rahman, and Ratti: if P, a polynomial of degree n, has all its zero in $|z| \leq 1$, and $P(a) = 0$, $0 \leq a \leq 1$, then P' must have a zero in $|z-a/2| \leq 1-a/2$. This is proved for $2 \leq n \leq 5$. It is further conjectured that the radius $1-a/2$ can be replaced by $r_n(a)$, where $1/2 \leq r_n(a) \leq 1-a/2$ and $r_n(a)$ is the zero of $(1/a)\ [(x + a/2)^n - (x-a/2)^n] -1$. This is proved for $n = 3$. It is also proved that if, under similar conditions, $P(1) = 0$, $P'(1) \neq 0$, $n \geq 4$, and P' does not have a zero in $|z-1/2| < 1/2$, then $P(z) = c(z^n-1)$.

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by

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The following is a conjecture due to Goodman, Rahman, and Ratti: if P, a polynomial of degree n, has all its zero in $|z| \leq 1$, and $P(a) = 0$, $0 \leq a \leq 1$, then P' must have a zero in $|z-a/2| \leq 1-a/2$. This is proved for $2 \leq n \leq 5$. It is further conjectured that the radius $1-a/2$ can be replaced by $r_n(a)$, where $1/2 \leq r_n(a) \leq 1-a/2$ and $r_n(a)$ is the zero of $(1/a)\ [(x + a/2)^n - (x-a/2)^n] -1$. This is proved for $n = 3$. It is also proved that if, under similar conditions, $P(1) = 0$, $P'(1) \neq 0$, $n \geq 4$, and P' does not have a zero in $|z-1/2| < 1/2$, then $P(z) = c(z^n-1)$.

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INTRODUCTION

The fundamental result in the study of geometric relationships between the zeros of a polynomial over the complex field and the zeros of its derivative is the Gauss-Lucas Theorem. It asserts that if all the zeros of a polynomial are inside a circle, then so are the zeros of its derivative. A closely related problem which is far from being solved is as follows: given that all the zeros of a polynomial P are in a circle and one of the zeros is specified, describe the minimal region that will always contain a zero of P'.

By a suitable translation and rotation followed by an expansion or contraction this problem can be reduced to the following: given that $P(a,n)$, $0 \leq a \leq 1$, $n \geq 2$, is the set of monic polynomials of degree n that have all their zeros in $|z| \leq 1$ and have at least one zero at a, and that $\mathcal{D}(a,n)$ is the class of all regions $D(a,n)$ that have the property that if $P \in P(a,n)$ then P' has a zero in $D(a,n)$, describe $D^*(a,n)$ which is the intersection of all elements of $\mathcal{D}(a,n)$.

As a start, the Gauss-Lucas Theorem assures us that the region $|z| \leq 1$ is an element of $\mathcal{D}(a,n)$, $0 \leq a \leq 1$, $n \geq 2$. The next step in the solution was made by a conjecture due to Ilieff which asserts that $|z-a| \leq 1$ is a member of $\mathcal{D}(a,n)$. In [1], Goodman, Rahman and Ratti further conjectured that the region $|z-a/2| \leq 1-a/2$ is also a member of $\mathcal{D}(a,n)$, $0 \leq a \leq 1$, $n \geq 2$. This is a much better result because

$$\{|z-a/2| \leq 1-a/2\} \subset \{|z-a| \leq 1\} \cap \{|z| \leq 1\} .$$

The boundary case of the above conjecture, i.e. when $a = 1$, $n \geq 2$, has been proved in [1], [4], [5]. For $2 \leq n \leq 4$, the conjecture has been proved in [5], and implicitly in [2].

In this thesis we will extend this result to $n = 5$. Also, we will describe a sequence of regions contained in $|z-a/2| \leq 1-a/2$ which we will conjecture are elements of $\mathcal{D}(a,n)$. This will be proved in the cases $n = 2$ and 3, and moreover, in these cases they will describe $D^*(a,2)$ and $D^*(a,3)$.

1. Extension to Quintics.

Let $\mathcal{P}$ denote the set of monic polynomials over the complex numbers. Let D_∞ denote the operation of differentiation with normalization, so that if

$$f(z) = z^n + a_1 z^{n-1} + \dots + a_n ,$$

$$D_\infty f(z) = z^{n-1} + \frac{(n-1)}{n} a_1 z^{n-2} + \dots + \frac{a_n}{n} .$$

Let D_λ denote polar differentiation with respect to λ with normalization. If

$$T(z) = \frac{az+b}{cz+d} , \qquad ad \neq bc ,$$

is a linear transformation, and $f \in \mathcal{P}$ has zeros $z_1, z_2, \dots, z_n$ such that $T(z_i) \neq \infty$, $1 \leq i \leq n$, then Tf is defined as the monic polynomial whose zeros are $T(z_i)$, i.e.,

$$Tf(z) = t(cz-a)^n f(T^{-1}z),$$

where

$$\frac{1}{t} = \begin{cases} c^n f(-d/c) & \text{if} \quad c \neq 0 , \\ (ad)^n & \text{if} \quad c = 0 . \end{cases}$$

The following lemmas call attention to some well known facts which we will be using. The first lemma describes the fundamental property of the polar derivative.

LEMMA 1. Let μ, λ belong to the extended complex plane, and $f \in \mathcal{P}$. If $\mu = T(\lambda)$, then $T(D_\lambda f) = D_\mu(Tf)$.

The next lemma contains some elementary facts from complex analysis.

LEMMA 2. If

$$T(z) = \frac{az + b}{cz + b}, \quad ad \neq bc, \quad c \neq 0,$$

is a linear transformation and C is a circle such that $-d/c$ is outside C, then $T(C) = \{T(z): z \in C\}$ is another circle, with the interior of C being mapped into the interior of T(C). In particular, if a,b,c,d are real and C has a diameter on the real axis with ends at p and q, the T(C) will have a diameter with ends at T(p) and T(q).

The next lemma is the Coincidence Theorem of Walsh [3; p.46].

LEMMA 3. Let $g(z_1,\ldots,z_n)$ be a linear, symmetric function of $z_1,\ldots,z_n$, i.e.,

$$g(z_1,\ldots,z_n) = a_o + a_1\sigma_1 + \ldots + a_n\sigma_n,$$

where the σ_i $(i = 1,\ldots,n)$ are the elementary symmetric functions of the z_i. If $w_1,\ldots,w_n$ are on or outside a circle C, then there exists a w on or outside C such that

$$g(w_1,\ldots,w_n) = g(w,\ldots,w)$$

provided that $a_n \neq 0$.

COROLLARY. If

$$g(w_1,\dots,w_n) = a_o +\dots+ a_n\sigma_n ,$$

$$h(w_1,\dots,w_n) = b_o +\dots + b_n\sigma_n \neq 0,$$

we can extend Walsh's Theorem to apply to

$$q = \frac{g(w_1,\dots,w_n)}{h(w_1,\dots,w_n)}$$

provided $q \neq a_n/b_n$.

Proof: By Walsh's Theorem applied to

$$qh(w_1,\dots,w_n) - g(w_1,\dots,w_n) = 0$$

there exist a w on or outside C such that

$$qh(w,\dots,w) - g(w,\dots,w) = 0.$$

Hence

$$q = \frac{g(w,\dots,w)}{h(w,\dots,w)} .$$

LEMMA 4. Let $f,g \in \mathcal{P}$ be both of degree n. If $D_\lambda f = D_\lambda g$, then

$$f(z) = (1-t)(z-\lambda)^n - tg(z), \qquad t \neq 0 .$$

Proof: We have

$$nf(z) - (z-\lambda)f'(z) = tng(z) - t(z-\lambda)g'(z), \qquad t \neq 0 .$$

Let $h(z) = f(z) - tg(z)$. Thus, $nh(z) = (z-\lambda)h'(z)$.

Assume that $h(z) \not\equiv 0$. Then there exists a domain D, in the upper half plane, where $h'(z)/h(z)$ is analytic. But $h'(z)/h(z)=n/(z-\lambda)$, $z \in D$, and hence

$$f(z) = C(z-\lambda)^n , \qquad z \in D.$$

Because h is a polynomial, this equation is valid in the entire plane. Thus

$$h(z) = C(z-\lambda)^n - tg(z).$$

Because f and g are monic, $C = 1-t$. Note that this equation is correct even when $h(z) \equiv 0$.

COROLLARY. If $g \in \mathcal{P}$ and $\lambda \neq x$, then $f \in \mathcal{P}$ is uniquely determined by the following conditions:

1) $f(x) = 0$; 2) $D_\lambda f = g$; 3) $\deg(f) = \deg(g) + 1$.

Thus we can introduce the notation $f = I_\lambda^x g$. It should be noted that if the zeros of g are $w_1,\ldots,w_{n-1}$, then

$$I_\lambda^x g(z) = Q(z,w_1,\ldots,w_{n-1}),$$

where Q is the ratio of two linear symmetric functions in $w_1,\ldots,w_{n-1}$. Now, direct verification yields .

LEMMA 5. Let $g(z) = (z-w)^{n-1}$, $I_\lambda^x g = f$. Then

$$f(z) = \frac{(z-w)^n(\lambda-x)^n - (z-\lambda)^n(w-x)^n}{(\lambda-x)^n - (w-x)^n} .$$

If $w = \lambda$, define f by continuity, i.e.

$$f(z) = (z-\lambda)^{n-1}(z-x) .$$

If

$$x = \frac{\lambda - \exp(2\pi ki/n)w}{1-\exp(2\pi ki/n)} , \qquad k = 1,\ldots,n-1,$$

$I_\lambda^x g$ does not exist.

We are now ready to prove

THEOREM 1. Let $P \in \mathcal{P}$ be of degree n. Let $P(a) = 0$, $0 < a < 1$. If $2 \leq n \leq 5$ and all the zeros of P are in $|z| \leq 1$, then $D_\infty P$ must have a zero in $|z-a/2| < 1-a/2$.

Proof. Consider the linear transformation

$$T_x(z) = \frac{(x+a) - (1+ax)z}{1 + ax - (x + a)z} , \qquad 0 < x < 1 .$$

Clearly,

$$T_x(1) = -1, \quad T_x(-1) = 1, \quad T_x(a) = x,$$

$$T_x(\infty) = \frac{1+ax}{x + a} = \lambda(x), \quad T_x(a-1) = \frac{1+x+ax-a^2x}{1 + x + a-a^2} = t(x).$$

Hence by Lemma 2, under the action of T_x the unit circle is mapped into itself, while $|z-a/2| = 1-a/2$ is mapped into the circle with diameter ends at -1 and t.

Assume all the zeros of $D_\infty P$ are in $|z-a/2| \geq 1-a/2$. Then, all zeros of $T_x(D_\infty P)$ are in $|z| \geq t$.

However, all the zeros of T_xP are in $|z| \leq 1$. We will thus arrive at a contradiction by showing that $|T_xP(0)|/x > 1$ for $x \in (1-\epsilon,1)$.

Let $f = T_x(D_\infty P)$. Hence by Lemma 1, $f = D_\lambda(T_xP)$, and thus $T_xP = I_\lambda^x f$. If the zeros of f are $w_1,\ldots,w_{n-1}$, then $|w_i| \geq t$, $i = 1,\ldots,n-1$, and

$$T_xP(z) = Q(z,w_1,\ldots,w_{n-1})$$

where Q is the ratio of two linear symmetric functions in $w_1,\ldots,w_{n-1}$. Applying Lemma 3,

$$T_xP(0) = Q(0,w,\ldots,w), \qquad |w| \geq t.$$

Hence by Lemma 5,

$$|T_xP(0)| = \left|\frac{w^n(\lambda-x)^n - \lambda^n(w-x)^n}{(\lambda-x)^n - (w-x)^n}\right|.$$

Lemma 3 cannot be applied if

$$|T_xP(0)| = \lambda^n - (\lambda-x)^n = \frac{(1+ax)^n - (1-x^2)^n}{(x+a)^n} = s(x) .$$

Because

$$s(1) = 1, \quad s'(1) = \frac{n(a-1)}{a+1} < 0,$$

$|T_xP(0)| < 1$ for $x \in (1-\epsilon_1,1)$.

Let

$$|h(w)| = 1/|T_xP(0)| = \left|\frac{(\lambda-x)^n - (w-x)^n}{w^n(\lambda-x)^n - \lambda^n(w-x)^n}\right| .$$

We will show that h(w) is analytic in $|w| \geq t$ for $x \in (1-\epsilon_2,1)$; i.e., if w_o is a zero of $p(w) = w^n(\lambda-x)^n - \lambda^n(w-x)^n$, then $|w_o| < t$. Now, $w_o = \lambda$, or

$$w_o = -\frac{\exp(2\pi ki/n)\lambda x}{\lambda-x-\exp(2\pi ki/n)\lambda}, \qquad k = 1,\ldots,n-1$$

If $w_o = \lambda$, h can be defined by continuity. Thus, we need to have

$$|w_o|^2 = \frac{(1+ax)^2x^2}{x^2(x+a)^2 + 2(1+ax)(1-x^2)(1-\cos 2\pi k/n)} < t^2, \quad k=1,\ldots,n-1.$$

Cross multiplying, factoring out $(1-x^2)$, and letting $x \to 1$, we need to have

$$(1-a)/(2-a) < 1-\cos(2\pi k/n), \qquad k = 1,\ldots,n-1$$

which is valid for $2 \leq n \leq 6$. Thus $|w_\theta| < t$ for $x \in (1-\epsilon_2,1)$.

Now, $h(\infty)$ can also be defined by continuity. Hence, for $x \in (1-\epsilon_2,1)$, by the Maximum Modulus Theorem we have

$$|h(w)| \leq |h(te^{i\phi})|$$

where $\phi = \phi(x)$, $0 \leq \phi < 2\pi$. Let

$$s_\phi(x) = |h(te^{i\phi})|^2, \qquad 0 < \phi < 2\pi.$$

$$s_\phi(x) = \left| \frac{(1-x^2)^n(1+x+a-a^2)^n - (x+a)^nC^n}{e^{in\phi}(1-x^2)^n(1+x+ax-a^2x)^n-(1+ax)^nC^n} \right|^2$$

where

$$C = (1+x+ax-a^2x)e^{i\phi} - (x+x^2+ax-a^2x).$$

Because

$$s_\phi(1) = 1, \quad s'_\phi(1) = 2n(1-a)/(1+a) > 0,$$

$s_\phi(x) < 1$ for $x \in (1 - \epsilon_3, 1)$, $0 < \phi < 2\pi$.

Let

$$s_o(x) = x|h(t)| = \frac{x(1+x+a-a^2)^n - x(x+a)^n}{(1+x+ax-a^2x)^n - (1+ax)^n} .$$

$$s_o(1) = 1, \quad s'_o(1) = \frac{(2-a)^{n-1}(2+a-a^2-na+na^2)-(n-na+1+a)}{(1+a)[(2-a)^n - 1]} .$$

Hence for $2 \leq n \leq 5$, $s'_o(1) > 0$. Thus in our case $s_o(x) < 1$, for $x \in (1-\epsilon_4, 1)$.

Let $\epsilon = \min(\epsilon_1, \ldots \epsilon_4)$. Consequently $|T_xP(0)|/x \gtrless 1$ for $x \in (1-\epsilon, 1)$. But this is a contradiction, because all the zeros of T_xP should be in $|z| \leq 1$.

2. A More General Conjecture.

Under the conditions of Theorem 1, can the circle $|z-a/2| = 1-a/2$ be made smaller? When $a = 1$, the circle $|z-1/2| = 1/2$ is the best as shown by the polynomials $|z-1/2| = 1/2$ is the best as shown by the polynomials $(z-1)^n$ and z^n-1. When $a = 0$, the region $|z| \leq (1/n)^{1/(n-1)}$ satisfies the requirements of Theorem 1, as can be seen from the inequality $|P'(0)| \leq 1$. This is the best result as shown by $P(z) = z(z^{n-1} - e^{i\phi})$.

These observations suggest the following conjecture.

If $P \in \mathcal{P}$ is of degree n, $P(a) = 0$, $0 \leq a \leq 1$, and P has all its zeros in $|z| \leq 1$, then P' must have a zero in $|z-a/2| \leq p_n(a)$, where $1/2 \leq p_n(a) \leq 1-a/2$ and $p_n(a)$ is a zero of

$$g_a(x) = \frac{(x+a/2)^n - (x-a/2)^n}{a} - 1 .$$

Remarks.

1) When $n = 2$, $g_a(x) = 2x-1$, and obviously the conjecture is valid. In the following notes it is assumed that $n \geq 3$.

2) When $0 \leq a < 1$ we have $g_a(1/2) < 0$, $g_a(1-a/2) > 0$, and $g_a'(x) > 0$ for $1/2 \leq x \leq 1-a/2$. Thus, $p_n(a)$ is uniquely defined.

3) $g_o(x) = nx^{n-1}-1$; thus $p_n(0) = (1/n)^{1/(n-1)}$.

$g_1(1/2) = 0$; thus $p_n(1) = 1/2$.

4) $$\frac{(x+a/2)^n - (x-a/2)^n}{a} - 1 > \frac{(x+a/2)^{n+1} - (x-a/2)^{n+1}}{a} - 1$$

for $1/2 \leq x \leq 1-a/2$, $0 \leq a < 1$. Thus $\{p_n(a)\}$ is an increasing sequence.

5) When $a, a', x > 0$,

$$a' > a \implies g_{a'}(x) > g_a(x) .$$

Thus $p_n(a)$ is strictly decreasing in a.

6) By implicit differentiation, $p_n'(1) = (1/n) - (1/2)$. Hence $\lim_{n \to \infty} p_n'(1) = - 1/2$.

Thus, by the results of 4) and 5) we have

$$\lim_{n \to \infty} p_n(a) = 1-a/2.$$

7) For n = 3 this is the best result as shown by

$$P(z) = (z-a)(z-e^{i\phi})(z-e^{-i\phi}).$$

The zeros of P' are

$$[a+2\cos\phi \pm (a^2-3-2a\cos\phi+4\cos^2\phi)^{1/2}]/3 .$$

As

$$[a-(12-3a^2)^{1/2}]/4 \le \cos\phi \le [a+(12-3a^2)^{1/2}]/4$$

and $0 \le \phi \le \pi$, $|z-a/2| = [(4-a^2)/12]^{1/2}$ is traced out by the two roots.

3. The Case of Cubics.

THEOREM 2. Let $P \in \mathcal{P}$ be of degree 3. If P has all its zeros in $|z| \le 1$, and $P(a) = 0$, $0 \le a \le 1$, then P' must have a zero in $|z-a/2| \le [(4-a^2)/12]^{1/2}$.

Proof. Let $P'(z+a/2)$ have zeros w_1, w_2. Thus,

$$P(z)/(z-a) = z^2 + (1/2)[-a-3(w_1+w_2)]z+(1/4)[a^2+12w_1w_2] .$$

Let us take the polar derivative with respect to 1.

$$2P_1(z) = [-a-3(w_1+w_2)+4]z - [a+3(w_1+w_2) -a^2-12w_1w_2] .$$

P_1 is linear unless 1 is the center of mass of the zeros of $P(z)/(z-a)$; i.e., $P(z) = (z-1)^2(z-a)$.

Assume $|w_i| > [(4-a^2)/12]^{1/2} = t$, $i = 1,2$. Thus if z_o is the zero of P_1, by Lemma 3

$$z_o = \frac{a+6w-a^2-12w^2}{-a-6w+4} , \qquad |w| > t.$$

Let $y = w/t$. Then $|y| > 1$, and

$$z_o = \frac{y^2(4-a^2)-y(12-3a^2)^{1/2} + a^2-a}{y(12-3a^2)^{1/2} + a-4} = \frac{p(y)}{q(y)} .$$

Now, it is easily checked that $q(y)/p(y)$ is analytic for $|y| \geq 1$. If $a = 1$, $q(1)/p(1)$ is defined by continuity. $q(\infty)/p(\infty)$ can also be defined by continuity. Thus, by the Maximum Modulus Theorem,

$$\frac{1}{|z_o|} < \left|\frac{q(e^{i\phi})}{p(e^{i\phi})}\right| .$$

Let

$$r(\phi) = |p(e^{i\phi})|^2 - |q(e^{i\phi})|^2 .$$

$$r'(\phi) = 8a(1-a)(4-a^2) \sin\phi \ \cos\phi .$$

Thus

$$|z_o| > \left|\frac{p(1)}{q(1)}\right| = 1 .$$

Therefore by Laguerre's Theorem, $P(z)/(z-a)$ must have a zero outside the unit circle. But this is a contradiction, and thus the theorem is proved.

4. A Refinement of the Boundary Case.

In what follows we shall assume that f has the following properties:

1) f is in $\mathcal{P}$ and is of degree n. 2.) $f(1) = 0$, and $f'(1) \neq 0$.

3) All the zeros of f are in $|z| \leq 1$.

THEOREM 3. If $n \geq 4$ and f' does not have a zero in $|z-1/a| < 1/2$, then

$$f(z) = z^n - 1.$$

LEMMA 6. If f' does not have a zero in $|z-1/2| < 1/2$, then f has all its zeros on $|z| = 1$, and f has all its zeros on $|z-1/2| = 1/2$.

Proof. The proof can be found in [4; Theorem 1].

LEMMA 7. Let f_1 represent the polar derivative of f with respect to 1. If f_1' has a zero in $|z-1/2| < 1/2$, then so does f'.

Proof. The proof follows from Laguerre's Theorem. See [5; Lemma 1].

LEMMA 8. Let $n = 3$. If f' does not have a zero in $|z-1/2| < 1/2$, then

$$f(z) = z^3 - 3cz^2 + 3cz-1 , \quad 0 \leq c < 1.$$

Proof. By Lemma 6,

$$f(z) = (z-1)(z-e^{i\theta})(z-e^{i\phi}),$$

and $|f'(1/2)|/ 3 = 1/4$. Consequently we have

$$|4\exp(i\theta - i\phi)- 1|/12 = 1/4 .$$

Thus, $\phi = - \theta$. Let $c = (1+2 \cos \phi)/3$, and result follows.

LEMMA 9. Let

$$g(z) = (n-t)(z-1)^{n-1} + tz^{n-1} , \qquad t \neq 0.$$

If $n \geq 4$ and g has all its zeros on $|z-1/2| = 1/2$, then $t = n$.

Proof.

$$h(z) = g(\tfrac{z+1}{2}) = (n-t) (\tfrac{z-1}{2})^{n-1} + t(\tfrac{z+1}{2})^{n-1}$$

has all its zeros on $|z| = 1$.

$$p(w) = (w+i)^{n-1}h[(w-i)/(w+i)] = (-i)^{n-1}(n-t)+tw^{n-1}$$

has only real zeros. But this is only possible if $n \leq 3$, or $t = n$.

LEMMA 10. The theorem is true for $n = 4$.

Proof. Suppose f' does not have a zero in $|z-1/2| < 1/2$. Let

$$f(z) = z^4 + a_3z^3 + a_2z^2 + a_1z + a_o .$$

By Lemmas 7 and 8,

$$f_1(z) = tz^3 - 3ctz^2 + 3ctz - t, \quad 0 \leq c < 1, \quad t \neq 0.$$

Thus,

$$a_3 = t-4, \qquad a_2 = (1/2)(12-3t-3ct),$$

$$a_1 = t+2ct-4, \quad a_o = (1/2)(-t-ct+2).$$

By Lemma 6, $|a_o| = 1$; i.e.,

$$t = (2 + 2e^{i\phi})(1 + c), \quad -\pi < \phi < \pi,$$

and

$$|f'(1/2)|/4 = 1/8.$$

Hence,

$$c(2c + 1)(\cos\phi + 1) = 0.$$

Therefore we must have $c = 0$. Thus, f' satisfies the conditions of Lemma 9. Consequently we have $t = 4$, and

$$f(z) = z^4 - 1.$$

Proof of Theorem 3. Proof is by induction on n, the degree of f.

Because $f(z) \neq (z-1)^n$, f_1 is of degree n-1. Also since $f(1) = 0$, $f_1(1) = 0$. By Laguerre's Theorem, f_1 has all its zeros in $|z| \leq 1$, and $f_1'(1) \neq 0$.

Thus by Lemma 7 and inductional hypothesis,

$$f_1(z) = tz^{n-1} - t, \qquad t \neq 0.$$

$$f(z) = (1-t/n)(z-1)^n + (t/n)(z^n-1).$$

$$f'(z) = (n-t)(z-1)^{n-1} + tz^{n-1}.$$

Hence by Lemma 9, $t = n$, and

$$f(z) = z^n - 1.$$

5. A Special Case of Ilieff's Conjecture.

LEMMA 11. The right bisector of the line segment joining two zeros of a polynomial either separates the zeros of its derivative or passes through at least one of them.

Proof. This follows from the proof of the Grace-Heawood Theorem [3; p.84].

The following theorem was proved in [5]. Here we will use a geometric point of view.

THEOREM 4. Let P be a polynomial with all its zeros in $|z| \leq 1$. If $P(0) = 0$, and $P(a) = 0$, then P' must have a zero in $|z-a| \leq 1$.

Proof. Without loss of generality we may assume $0 < a \leq 1$. Thus by Lemma 11 and Gauss-Lucas Theorem, P' must have a zero in

$$S = \{Re(z) \geq a/2\} \cap \{|z| \leq 1\}.$$

But $S \subset \{|z-a| \leq 1\}$. This completes the proof.

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