A review of six-dimensional braneworld solutions

by

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Contents

Abstract

In the last years, brane world scenarios have been studied extensively, but most of these studies have been done in the case of five-dimensional spacetime. It is therefore of interest to investigate which of the particular features observed are proper to one extra dimension and which are generic to any number of dimensions. In this thesis, I present an overview of models and solutions to Einstein's equations for six-dimensional brane world scenarios. Solutions for a simple setup with cylindrically symmetric bulk centered about a three-brane are derived and classified. There are two main kinds of topology: either solutions are compactified in a spherical topology, closed up by another three-brane, or they have a disc topology, which must be terminated by a four-brane. One of the particular features of codimension-two branes is demonstrated, namely that their tension, or vacuum energy, induces a deficit angle in the bulk. Solutions for different arrangements of codimension-one and codimension-two branes are also reviewed. Although the review focuses on topological and cosmological properties of the solutions, models using a field theoretical approach to the brane-world scenario, i.e. considering the brane as a topological defect arising from higher dimensional fields, are also considered.

Résumé

Au cours des dernières années, les modèles d'univers branes ont fait l'objet de nombreuses études, la plupart d'entre elles ayant cependant été réalisées dans le contexte d'un espace-temps à cinq dimensions. Il est donc pertinent de se demander lesquelles des caractéristiques observées sont propres à ce nombre de dimensions et lesquelles s'appliquent de manière plus générale. Je présenterai dans ce mémoire un éventail de modèles et de solutions aux équations d'Einstein pour des modèles d'univers branes dans un espace-temps à six dimensions. En particulier, les solutions pour un modèle simple consistant en un espace-temps à symétrie cylindrique centrée sur une "brane" à trois dimensions spatiales seront dérivées puis classifiées. Les solutions se divisent en deux catégories principales selon leur topologie: soit elles sont compactifiées de telle sorte qu'on obtienne une topologie sphérique terminée par une autre "brane" à trois dimensions, soit elles affichent la topologie d'un disque, bordée cette fois par une "brane" à quatre dimensions spatiales. Une propriété importante des "branes" à deux codimensions sera démontrée, soit le fait que leur tension introduise un angle déficitaire dans l'espace-temps. D'autres solutions pour des configurations différentes de "branes" seront aussi présentées. Malgré le fait que cette revue soit davantage centrée sur les propriétés topologiques et cosmologiques des différentes solutions, certains modèles abordant la question des champs à l'origine du défaut topologique qu'est la "brane" seront aussi examinés.

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Statement of Originality

Most of the results presented in chapter 3 can be found in [11]. I have contributed to the study of these solutions, for example by verifying some of the calculations and making observations. I particular, the observation that equation (3.66) does not automatically eliminate the possibility of a pure brane tension 4-brane in the case of generic Λ_4 is one of mine. Of course, since this thesis is mainly a review of six-dimensional braneworld solutions, most of the material is not original but has been taken from the cited sources.

Chapter 1

Introduction

The standard model has accumulated successes, but there are still some unresolved problems. Two of the major ones are the hierarchy problem and the cosmological constant problem. The hierarchy problem refers to the huge difference between the electroweak scale and the Planck scale. It is usually assumed that the fundamental dynamic scale underlying gravity is the Planck scale,

$$M_{Pl} \sim 10^{18} \text{ GeV}, \tag{1.1}$$

set by the observed value of Newton's constant. If so, one is faced with the problem of understanding the mechanism which stabilizes the very large hierarchy between this scale and the electroweak scale, $m_{EW} = 246\ GeV$. The cosmological constant problem is that of explaining why the observed vacuum energy is so small in comparison to the theoretical value. The net cosmological constant can be seen as the sum of apparently disparate contributions, including potential energies from scalar fields (that may change with time as the universe passes through different phases) and zero-point fluctuations of each field theory degree of freedom, as well as a bare cosmological constant Λ_0. Unlike the last of these, in the first two cases we can make educated guesses at the magnitudes, which leads us to expect a contribution of order

$$\rho_\Lambda^{Pl} \sim (M_{Pl})^4 \sim (10^{18}\ \mathrm{GeV})^4 \sim 10^{122}\ \mathrm{eV/cm^3}. \tag{1.2}$$

On the other hand, cosmological observations imply

$$\rho_\Lambda^{obs} \sim 10^2\ \mathrm{eV/cm^3}. \tag{1.3}$$

There is no obstacle to imagine that all of the large and apparently unrelated contributions mentioned add together, with different signs, to produce a net cosmological constant as small as (1.3), other than the fact that it seems very unnatural. The ratio of (1.2) to (1.3) is the origin of the famous discrepancy of 120 orders of magnitude between the theoretical and observational values of the cosmological constant. But since energy density can be expressed as a mass scale to the fourth power, a more fair characterization of the problem would be

$$\frac{M_{vac}^{(theory)}}{M_{vac}^{(obs)}} = \frac{10^{18}\ \mathrm{GeV}}{10^{-3}\ \mathrm{eV}} \sim 10^{30}. \tag{1.4}$$

Of course, this difference of thirty orders of magnitude still constitutes a tough problem.

A lot of attention has lately been devoted to the use of extra dimensions in trying to resolve these two problems, mostly because the solutions such models point towards offer simplicity. Originally, models considering space as multidimensional had been motivated by theories which incorporate gravity in a reliable manner, namely string theory and its derivatives. In fact, most of these theories need to be formulated in space-time of more than four dimensions to be consistent. The recent interest towards multi-dimensional scenarios has itself been brought up mainly by phenomenological studies, which by using simplified field theoretical models allow the consideration of various models, revealing a wide range of possibilities for how those long-standing problems of particle theory mentioned above might be solved.

The framework of these studies is the brane world scenario, which consists in the idea that we live on a hypersurface, a three-dimensional "brane", embedded in a larger dimensional warped spacetime bulk. A lot of effort has been invested in exploring such scenario in five-dimensional spacetime, giving rise to interesting possibilities for solving the hierarchy problem, but also to all sorts of unexpected features, like the modification of the Friedmann equation, or the requirement of negative tension branes. We already know that the latter does not arise in six dimensions, and so it is natural to wonder if the other observed features are artifacts of five-dimensional models, or if they are generic to any number of dimensions. That alone is a good reason for studying six-dimensional models, but there is more. Rubakov and Shaposhnikov [40] first proposed the idea of solving the cosmological constant problem using a six-dimensional model in 1983, and this hope is still one of the rationales for studying six-dimensional braneworlds.

In this thesis will be presented a short review of possible solutions to Einstein's equation in a six dimensional bulk, with the standard model confined to a four-dimensional submanifold. This review could not have the pretension of being exhaustive. I rather intended to describe and compare, with more or less details, the variety of models and solutions that I have encountered throughout my own search for six-dimensional cosmological solutions, either in the literature or working on it myself. Rather than remaining an eclectic ensemble of fruitless attempts to finding new solutions, the research that I have done could then be very helpful to someone eager to search in that direction or simply wanting to know more about it. The reader should keep in mind, however, that it was of course impossible to cover everything that has been done in six-dimension, and therefore a choice had to be made.

Following the lines of a more phenomenological approach, the accent will not be put here on the field theoretical properties of the solutions reviewed, but rather on the more general cosmological and topological properties of the models. This choice have been made in order to simplify the discussion and therefore allow the consideration of many different models and make easier the comparison between those models. For details regarding a certain model or family of solutions, the reader may refer to the cited source.

The organization of the thesis is as follows: The theoretical background will first be sketched in Chapter 2, mostly building up on the work that has been done in 5 dimensions. I will then present, in Chapter 3, a general model of six-dimensional braneworld scenario, with a cylindrically symmetric bulk, centered on the codimension-two brane, or string, where the standard model particles are confined. Solutions to Einstein's equation for this configuration will then be classified according to their topology and other characteristics. Most of the results presented in this chapter can be found in [11]. In Chapter 4, other six-dimensional models and solutions from the literature will be described. I will finally conclude with a summary of the commonalities and differences amongst different solutions.

Chapter 2

Theoretical Background: The Brane World Scenario

2.1 The Kaluza-Klein Idea

The investigation of higher dimensional theories can be traced back to the idea of Kaluza, later elaborated by Klein, that the appearance of electromagnetism may be viewed as an artifact produced by the existence of a five-dimensional spacetime that contains a version of general relativity. Long before the brane world scenario made its appearance, Kaluza considered in 1921 the case of a $(4+1)$-dimensional spacetime. The symmetric metric tensor in 5 dimensions can be decomposed in terms of arbitrary 4×4 matrix $g_{\mu\nu}$, vector

A^μ and scalar ϕ as

$$g^{(5)}_{\mu\nu} = \phi^{-1/3} \begin{pmatrix} g_{\mu\nu} + \phi A_\mu A_\nu & \phi A_\mu \\ \phi A_\nu & \phi \end{pmatrix}. \tag{2.1}$$

If we substitute this metric into the five-dimensional action of general relativity,

$$S = \frac{1}{16\pi G^{(5)}} \int d^5x \sqrt{g^{(5)}} R^{(5)} \tag{2.2}$$

assuming that there is no dependence in the fifth dimension, the action can be rearranged into

$$S = \frac{1}{16\pi G} \int d^4x^\mu \sqrt{g} \left(R + \frac{1}{4}\phi F^{\mu\nu} F_{\mu\nu} + \frac{1}{6\phi^2} \partial^\mu \phi \partial_\mu \phi \right) \tag{2.3}$$

where $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. In the last equation, g and R are now the metric determinant and Ricci scalar calculated from the four-dimensional metric $g_{\mu\nu}$, and G corresponds to the usual four-dimensional Newton's constant, in contrast with the quantities of equation (2.2): $G^{(5)}$ is the five-dimensional Newton's constant while $g^{(5)}$ and $R^{(5)}$ are derived from the full five-dimensional metric $g^{(5)}_{\mu\nu}$. The field A_μ have to be correctly normalized by removing the factor $\phi/16\pi G$ from the $F^{\mu\nu}F_{\mu\nu}$ term in the rearranged action.[1] The passage from equation (2.2) to (2.3) is merely a rearrangement, but the interpretation is that from a five-dimensional spacetime with general relativity, we can get ordinary four-dimensional general relativity, plus electromagnetism, plus a scalar field known as the dilaton.

The physical significance of such extra dimensions is an important question in all multi-dimensional theories. If the extra dimensions are real, there has to be a mechanism by which they are hidden, so that the spacetime looks four-dimensional to us. In theories of

[1]More details on the calculations involved can be found in Chapter 13 of ref. [15].

Kaluza-Klein type, the extra dimensions are assumed to be compactified: the spacetime is so highly curved that typical extent in the extra directions is of order the Planck length. That would explain why chunks of ordinary matter do not disappear entirely by a displacement in one of the new dimensions, and why these extra dimensions do not manifest themselves in everyday physics. In five dimensions, we would have a cylinder of a certain compactification radius (say R) on which the ordinary spacial dimensions are infinite, and the extra dimension runs from 0 to $2\pi R$. If we consider the five-dimensional Klein-Gordon equation on this cylinder, it will give rise to a set of wave functions of a free massless particle, with extra-dimensional angular momentum eigenvalue n. The $n = 0$ mode will correspond to an ordinary particle obeying the 4-dimensional Klein-Gordon equation. The other modes will carry energy of order $1/R$, and cannot be excited in low-energy processes. Thus, below the energy scale $1/R$, only $n = 0$ modes are relevant, and physics is effectively four-dimensional. But at energy $E \sim 1/R$, the extra dimension will start to show up. From a four-dimensional point of view, each Kaluza-Klein (KK) mode can be interpreted as a separate particle of mass $m_n = n/R$, and in fact, every multidimensional field will correspond to a Kaluza-Klein tower of particles with increasing masses. Since the KK partners of ordinary particles have not been observed yet, the energy scale must be at least in the few hundred GeV range, so according to this scenario the size of the extra dimensions must be microscopic.

2.2 Large Extra Dimensions and the ADD Proposal

More recently, another kind of proposal with extra dimensions aroused enthusiasm, because it opened up a new way to address the hierarchy problem: in this scenario, the fundamental dynamical scale of gravity, M, is not much larger than the electroweak scale, $m_{EW} \sim 1\ TeV$. The way this is realized is by only allowing gravity to propagate into the extra dimensions, the SM particles being trapped on a four-dimensional submanifold. Localization of matter on a four-dimensional brane explains why low energy physics is effectively four-dimensional insofar as all interactions except gravity are concerned, while the fact that gravity is diluted into the extra dimensions resolves the problem of why it is so weak compared to the other interactions. The fundamental parameter of gravity is not the four-dimensional Planck scale in this model; rather it is the multi-dimensional mass scale M, that enters the full multi-dimensional gravitational action,

$$S = \frac{1}{16\pi G_{(D)}} \int d^D X \sqrt{-g^{(D)}} R^{(D)} \tag{2.4}$$

where

$$G_{(D)} = \frac{1}{M^{D-2}} \equiv \frac{1}{M^{d+2}} \tag{2.5}$$

is the fundamental D-dimensional Newton's constant and $d = D-4$ is the number of extra dimensions. Here again the metric is independent of the extra-dimensional coordinates, hence the effective four-dimensional gravitational action can be obtained by performing the trivial integration over the d extra dimensions. We obtain

$$S_{eff} = \frac{V_d}{16\pi G_{(D)}} \int d^4x \sqrt{-g^{(4)}} R^{(4)}$$

where $V_d \sim R^d$ is the volume of extra dimensions (we assume that they have the same compactification radius R, for simplicity). Comparing this with the ordinary four-dimensional gravitational action,

$$S_4 = \frac{1}{16\pi G} \int d^4x \sqrt{-g^{(4)}} R^{(4)}, \tag{2.6}$$

we obtain that the scale governing effective four-dimensional gravity (the Planck mass) is given in terms of the fundamental scale by

$$M_{Pl}^2 = M^2 (MR)^d \tag{2.7}$$

up to a constant of order one. Hence it is possible to get a Planck mass which is much higher than the fundamental gravity scale M, if the size of extra dimensions is large compared to the fundamental length M^{-1}. In the original proposal [1], N. Arkani-Hamed, S. Dimopoulos and G. Dvali (ADD) assume that $M = m_{EW}$, taking the philosophy that m_{EW} is the only fundamental short distance scale in nature, thus eliminating the hierarchy problem. Doing so, for the case of one extra dimension, $d = 1$, demanding that R be chosen to reproduce the observed M_{Pl} yields $R \sim 10^{13}$ cm. This is experimentally excluded, since for distances smaller than R, the gravitational potential should be, in this theory, that dictated by Gauss's law in D dimensions

$$V(r) \sim \frac{m_1 m_2}{M^{d+2}} \frac{1}{r^{d+1}}. \tag{2.8}$$

For such a large R, it would imply deviations from Newtonian gravity over solar system distances, so the case $d = 1$ is excluded. Now for distances much larger than R, the flux lines between two test masses will not penetrate in the small extra dimensions, so the gravitational potential felt will be the usual $1/r$ potential,

$$V(r) \sim \frac{m_1 m_2}{M^{d+2} R^d} \frac{1}{r}. \tag{2.9}$$

For the case $d = 2$, we obtain $R \sim 0.1 - 1$ mm, so that at scales where gravity has been tested, the large distance limit applies and we recover the usual four-dimensional gravity potential. However, the regime where full multi-dimensional potential applies is close enough that deviations from gravity could be observed by new experiments.

The major difference between this proposal and Kaluza-Klein models is that the extra-dimensions need not be microscopic. This is because the Standard Model physics is not averaged over the extra dimensions anymore, but localized on a brane. Only gravity can propagate into the extra dimensions in this model, and gravity has not been probed at distances smaller than a millimeter, so the extra dimensions can be that large. The Standard Model gauge forces, on the other hand, have been accurately measured at weak scale distances, and so the "slice" of spacetime to which they are confined must be very thin in the extra dimension, so that no deviation from their four-dimensional form could have been observed. Following the philosophy that m_{EW} is the only short-distance scale in the theory, this thickness should be $\sim m_{EW}^{-1}$ in the extra dimensions. Of course, a non-trivial task is the explicit realization of this confinement. In [1], the authors show that trapping zero modes on a topological defect is one way to do it, but they stress that other mechanisms could be used, either in the context of field theory or of string theory, without affecting the key ideas and consequences of their proposal. We defer the discussion about topological defects and trapping mechanisms to a subsequent section. We will simply say here that for the localization of the Standard Model on the defect, there are different mechanisms for different spins, and generally a trapped mode arises from the coupling of the corresponding higher dimensional field with the vortex field (the field responsible for creating the defect). This issue requires a long and complicated discussion not particularly relevant for the present purpose, and so we refer the reader to more detailed treatments of this matter, for example in [1, 39].

To conclude this section, the ADD proposal has brought new ways of thinking about the hierarchy problem. ADD showed that it was possible for the hierarchy between M_{Pl} and M_{EW} to be entirely due to the geometry of the space. In their model, it is the large size of the extra dimension which creates this difference of scales. Hence the new hierarchy that has to be explained is that between the fundamental short-distance scale and the largeness of the extra dimensions. In the six-dimensional case, this corresponds to the hierarchy between the weak and the millimeter scales. This hierarchy is stable in the sense that small changes of parameters have small effects on the physics (so there is no fine tuning problem).

2.3 Warped Extra Dimension and the RS Mechanism

2.3.1 The Model

In view of this new hierarchy problem, Randall and Sundrum (RS) proposed an alternative solution [37, 38]. In their model, an exponential hierarchy of mass scales arises without the need for the extra dimensions to be large. This is because the space is not anymore the simple product of a four-dimensional spacetime with a d-dimensional compact space. Rather, the four-dimensional metric is multiplied by a "warp" factor which is a rapidly changing function of an additional dimension. In their model, Randall and Sundrum consider the case of one extra dimension. The full five-dimensional metric is

$$ds^2 = e^{-2kr_c\phi}\eta_{\mu\nu}dx^\mu dx^\nu + r_c^2 d\phi^2, \tag{2.10}$$

where k is a scale of order the Planck scale, x^μ are coordinates for the familiar four dimensions, while $0 \leq \phi \leq \pi$ is the coordinate for an extra dimension, which is a finite interval whose size is set by r_c. It is the exponential factor $e^{-2kr_c\phi}$ that causes the exponential hierarchy of scale, by a mechanism which depends on the particular setup and will be explained shortly.

This particular metric comes about because, contrary to the ADD scenario, the effect of the brane(s) on the bulk gravitational metric is taken into account. In [1], the energy density of the brane itself, i.e. the gravitational field produced by the brane, was ignored. Randall and Sundrum showed that a gravitating brane in five-dimensional space can induce the interesting geometry characterized by the above metric. When considering distance scales much larger that the brane thickness, the gravitating brane can be seen as a delta-function source of gravity. Moreover, if we are interested in determining the metric in the ground state (in the absence of particle excitations), we can characterize the brane by only one parameter, the energy density per unit three-volume, or brane tension. We will denote this quantity by τ. The five-dimensional action including the effect of a brane is

$$S = \int d^4x \int d\phi \sqrt{-g^{(5)}} \left[\frac{R^{(5)}}{16\pi G_{(5)}} + \Lambda_5 \right] + \int d^4x \sqrt{-g^{(4)}}\tau \tag{2.11}$$

where Λ_5 is the cosmological constant in the bulk, $G_{(5)}$ is the fundamental five-dimensional Newton's constant and $g^{(5)}$ and $R^{(5)}$ are the metric determinant and Ricci scalar calculated from the full five-dimensional metric while $g^{(4)}$ is the determinant of the effective four-dimensional metric. In [37], Randall and Sundrum consider a setup that has two branes, and work on the space S^1/Z_2. That is, the fifth coordinate ϕ is taken to be periodic and each point (x, ϕ) is identified with $(x, -\phi)$. In principle, the range of ϕ is $[-\pi, \pi]$, so we only need to solve for $[0, \pi]$ for the metric to be completely specified. The orbifold fixed points (the two boundaries of the space) at $\phi = 0$ and $\phi = \pi$ are taken as the locations of the two 3-branes. Note that the presence of a second (gravitating) brane adds a term to the action, identical to the last term in equation (2.11), with the tension being that of the second brane now. The Einstein's equations for this action are given by

$$R^{(5)}_{MN} - \frac{1}{2} g^{(5)}_{MN} R^{(5)} = -8\pi G_{(5)} \left[\Lambda_5 g^{(5)}_{MN} + \tau_\pi \frac{\sqrt{-g^{(4)}_\pi}}{\sqrt{-g^{(5)}}} \, g^{(4)}_{\mu\nu}\big|_\pi \, \delta^\mu_M \delta^\nu_N \delta\left(\phi - \pi\right) + \tau_0 \frac{\sqrt{-g^{(4)}_0}}{\sqrt{-g^{(5)}}} \, g^{(4)}_{\mu\nu}\big|_0 \, \delta^\mu_M \delta^\nu_N \delta\left(\phi\right) \right]. \tag{2.12}$$

where we use the subscripts 0 and π to mean that these quantities are those measured on the branes located at $\phi = 0$ and $\phi = \pi$, respectively.

A solution to these equations that respects four-dimensional Poincaré invariance in the x^μ directions will take the form

$$ds^2 = e^{-2\sigma(\phi)} \eta_{\mu\nu} dx^\mu dx^\nu + r_c^2 d\phi^2. \tag{2.13}$$

With this ansatz, the Einstein's equations reduce to

$$\frac{6\sigma'^2}{r_c^2} = -8\pi G_{(5)}\Lambda_5, \tag{2.14a}$$

$$\frac{3\sigma''}{r_c^2} = \frac{8\pi G_{(5)}}{r_c}\left[\tau_0\delta\left(\phi\right) + \tau_\pi\delta\left(\phi - \pi\right)\right] \tag{2.14b}$$

Equation (2.14a) together with the requirement of symmetry $\phi \longrightarrow -\phi$ yield the solution

$$\sigma = r_c\left|\phi\right|\sqrt{-2\pi G_{(5)}\frac{\Lambda_5}{3}} + C. \tag{2.15}$$

C is simply an additive integration constant that sets the overall constant scaling of the x^μ coordinates. We keep it explicit because we will wish to do such a rescaling shortly. Now plugging this solution into the second equation (2.14b) and matching the delta functions will lead to the following condition for the existence of a solution:

$$\tau_0 = -\tau_\pi = \sqrt{-3\frac{\Lambda_5}{2\pi G_{(5)}}}. \tag{2.16}$$

The fine-tuning of each tension with the bulk cosmological constant remains even in the case of a single brane. Also, it can be shown that for this condition not to hold, there must be a non-zero four-dimensional cosmological constant on the brane. This fine-tuning can thus be associated with the cosmological constant problem.

2.3.2 Solving Hierarchy

Now to understand how the above solution can solve the hierarchy problem, it is convenient to take the viewpoint of a four-dimensional observer residing on our brane, i.e. the brane where the SM is located. In this setup, this is at $\phi = \pi$. Intuitively, we can see why it is a good choice for solving the hierarchy problem by looking at the warp factor multiplying the Minkowski metric, $e^{-2kr_c\phi}$ ($k = \sqrt{-2\pi G_{(5)}\frac{\Lambda_5}{3}}$), noting that gravity will be more weakly coupled away from $\phi = 0$. Now to show this properly, we will make a change of coordinates, $x^\mu \longrightarrow e^{kr_c\pi}x^\mu$, that is, we set $C = \sqrt{-2\pi G_{(5)}\frac{\Lambda_5}{3}}$. With this choice, the solution is now

$$ds^2 = e^{-2kr_c(|\phi|-\pi)}\eta_{\mu\nu}dx^\mu dx^\nu + r_c^2 d\phi^2 \tag{2.17}$$

so that we recover the ordinary four-dimensional metric at $\phi = \pi$ (the warp factor is unity there).

We first replace the Minkowski metric by a more general four-dimensional metric $g^{(4)}_{\mu\nu}(x) = \eta_{\mu\nu} + h^{(4)}_{\mu\nu}$, where $h^{(4)}_{\mu\nu}$ represents the tensor fluctuations about Minkowski space and is the physical graviton of the four-dimensional effective theory.[2] We then substitute into the gravitational action and evaluate the curvature at $\phi = \pi$ to get the four-dimensional effective action,

$$S = \frac{1}{16\pi G_{(5)}}\int d^4x \int_{-\pi}^{\pi} d\phi\ r_c\ e^{-2kr_c(|\phi|-\pi)}\sqrt{-g^{(4)}_\pi}R^{(4)}, \tag{2.18}$$

where, again, $R^{(4)}$ is the four-dimensional Ricci scalar constructed from $g^{(4)}_{\mu\nu}\Big|_\pi$. We can compare this effective action with the four-dimensional gravitational action to get the relation between the five-dimensional and the effective four-dimensional Newton's con-

[2]The physical graviton also corresponds to the massless mode, or zero-mode, in the Kaluza-Klein decomposition of the metric.

stants:

$$\frac{1}{G_{(4)}} = \frac{2}{G_{(5)}} \int_0^{\pi} d\phi r_c e^{-2kr_c(|\phi|-\pi)} \tag{2.19}$$

$$= \frac{1}{G_{(5)}} \frac{e^{2kr_c\pi} - 1}{k} \tag{2.20}$$

We now express this result in terms of the fundamental five-dimensional mass scale M and the four-dimensional effective Planck scale M_{Pl} , related to the Newton's constants through equation (2.5), and get

$$M_{Pl}^2 = \frac{M^3}{k} \left(e^{2kr_c\pi} - 1\right). \tag{2.21}$$

If we take the philosophy that the only fundamental scale is the TeV scale, then the Planck scale is a derived scale, that can be exponentially larger than the fundamental scale because of the exponential factor in equation (2.21). Because of the warped geometry of this space, we don't need large hierarchies among the fundamental parameters; in fact, we only require $kr_c \sim 10$ to get $e^{kr_c\pi} \sim 10^{16}$ and generate the Planck scale from the TeV scale. Also, since we have normalized the warp factor so that it is unity at the location of our brane, the matter Lagrangian is unaffected by the warp factor, because the matter action

$$S_{\phi=\pi} = \int d^4x \sqrt{-g^{(5)}\,(\phi = \pi)}\mathcal{L}_{\phi=\pi} = \int d^4x \sqrt{-g_\pi^{(4)}}\mathcal{L}_{\phi=\pi} \tag{2.22}$$

gives us the right coupling between matter and gravity. Hence the mass scale for other interactions remains the fundamental TeV scale. In comparison, if we were to put matter on the brane located at $\phi = 0$, the 3-brane action thus obtained,

$$S_{\phi=0} = \int d^4x \sqrt{-g^{(5)}\,(\phi = 0)}\mathcal{L}_{\phi=0} = \int d^4x \sqrt{-g_0^{(4)}} e^{2kr_c\pi}\mathcal{L}_{\phi=0}, \tag{2.23}$$

shows us that we would need to renormalize the Lagrangian in order to get the observed coupling between matter and gravity. Basically, on that brane any mass parameter of the fundamental higher dimensional theory (of order TeV if we follow our philosophy) would correspond to a physical mass $e^{kr_c\pi}$ bigger. So on that brane, everything is of order M_{Pl} and there is no hierarchy. That is why resolving the hierarchy problem by the Randall-Sundrum mechanism implies that the Standard Model is trapped on the negative tension brane.

Of course, we could also take the alternative point of view of taking the Planck scale to be the fundamental scale, by not making any change of coordinates. But that wouldn't change anything for the brane located at $\phi = 0$ if we tried to put matter on it, since although we would automatically obtain the right coupling between gravity and the matter Lagrangian, since the fundamental scale would be the Planck scale everything would still be of order the Planck scale on that brane. Indeed, the strength of gravity on the branes from this point of view is of the same order as the fundamental scale:

$$M_{Pl}^2 = \frac{M^3}{k}\left(1 - e^{-2kr_c\pi}\right). \tag{2.24}$$

It is now the TeV scale that is a derived scale, since it is on the Standard Model (SM) brane, at $\phi = \pi$, that the Lagrangian must be renormalized. We would have in these units

$$\sqrt{-g^{(5)}\left(\phi = \pi\right)} = \sqrt{-g_\pi^{(4)}}e^{-2kr_c\pi}, \tag{2.25}$$

and so any mass parameter m_0 in the fundamental theory (of order M_{Pl} in this alternative philosophy) would correspond to a four-dimensional physical mass

$$m = e^{-kr_c\pi} m_0. \tag{2.26}$$

So from this point of view, the one originally taken by Randall and Sundrum, it is the TeV scale that is generated from the warp factor, and the hierarchy is still resolved on the negative tension brane.

2.3.3 Infinite Extra Dimension

Randall and Sundrum also showed [38] that it is possible to have infinite extra-dimensional volume and still get localized gravity.[3] For that, they consider another setup where there is only one 3-brane, with positive tension. The second brane with negative tension is sent to infinity by letting $r_c \longrightarrow \infty$. We will work from the point of view of a four-dimensional observer on the positive brane, with the warp factor being unity there. Recall the metric in this case is given by equations (2.13) and (2.15):

$$ds^2 = e^{-2kr_c|\phi|}\eta_{\mu\nu}dx^\mu dx^\nu + r_c^2 d\phi^2, \quad k = \sqrt{-2\pi G_{(5)}\frac{\Lambda_5}{3}} \tag{2.27}$$

In the $r_{c\longrightarrow\infty}$ limit, we see from equation (2.24) that the Planck scale is still well defined :

$$M_{Pl}^2 = \frac{M^3}{k}. \tag{2.28}$$

This a good indication that we can get a sensible effective four-dimensional theory with the usual Newtonian force law. This is radically different from what we could expect if

[3]For any higher dimensional Universe model, it is essential to confine gravity, in order not to modify the three-dimensional laws of gravity, that have been experimentally tested on many different scales, ranging from the millimeter to intergalactic distances (a few Mega-parsec).

the space was a product space like in the ADD proposal, $M_{Pl}^2 = M^3 r_c \pi$ (see equation 2.7).

More formally, one can prove (although we will not do it here) that the curved background of the model supports a bound state of the higher-dimensional graviton. Although space is infinite in extent, the graviton is confined to a small region within the space. Gravitational fluctuations can be written as a superposition of modes that are eigenmodes of a wave equation depending only on the extra dimensional coordinate. This corresponds to the Kaluza-Klein decomposition of the higher-dimensional gravitational fluctuations in terms of four-dimensional KK states. Every eigenmode has a given mass, and the zero-mode is the wave function associated with the four-dimensional graviton. One can verify that this state is indeed a bound state and falls off rapidly away from the brane. To calculate how "strongly" it is bound to the brane amounts to calculating the relation between M and M_{Pl}, like we have done a few times already. But apart from this zero mode, there is a tower of KK modes that we have to take into account. In factorizable geometries like the ADD model, there is a gap after the zero mode, and four-dimensional gravity is reproduced up to the scale determined by the gap. Here things are different; because there is no gap, we have a continuous KK spectrum. Fortunately, the authors show that the coupling of other KK modes with matter is greatly suppressed relative to the zero mode. Hence four-dimensional physics is very well approximated: the bound state mode reproduces conventional four-dimensional gravity, and the other KK modes give only a small correction.[4] But this setup doesn't have the virtue of solving the hierarchy problem as the previous one.

2.3.4 Phenomenology of the Brane-World Scenario

For a viable theory, the "brane world" must reproduce correct gravity and cosmology of our Universe. In the RS picture, the negative cosmological constant of the bulk is used to cancel the cosmological constant or tension on the brane. On a brane with positive tension, like in the single brane scenario, gravity is effectively confined to the brane by the steep warp factor generated by the tension dominating the brane. In realistic cosmologies, of course, the energy density of the Universe must be dominated by matter instead. Taking that into account, an important observation about the cosmology of the (five-dimensional) brane world scenario was made, for example in [3]. It was shown that the equation governing the expansion of the scale factor on the brane was not the usual Friedmann equation. The dependence of the Hubble parameter H on the energy density on the brane was $H^2 \sim \rho^2$, instead of the expected $H^2 \sim \rho$. In another series of papers (see for example [12, 16, 27]), it was shown that for the Randall-Sundrum scenario, the full energy density ρ is a sum of a vacuum energy density, i.e. brane tension, and a matter energy density ρ_m, and thus the correct expression for H^2 can be obtained by cancelling the Λ_{brane}^2 term of $(\Lambda_{brane} + \rho_m)^2$ with the term Λ_5 coming from the negative

[4]This correction is still experimentally testable, depending on the AdS curvature scale.

bulk cosmological constant, so that we get

$$H^2 \sim \Lambda_{brane}\rho_m \tag{2.29}$$

plus a correction which is quadratic in the density. From this equation we see that we cannot live on a negative tension brane, since this would imply either negative energy density, or imaginary Hubble constant, and neither of these are observed.

In [32], general rules for arbitrary number of dimensions have been derived. These rules, the consistency conditions, are a family of one-parameter conditions relating the geometry of the brane-world to its stress-energy content. They were shown to imply that negative tension branes are required in five-dimensional scenarios, but that requirement is evaded for more than five dimensions. As an example, in the same paper Leblond *et al.* construct a six-dimensional braneworld model with only positive tension branes. From this, and from the question of whether or not the Friedmann equation also gets modified in a higher number of dimensions, the search for six-dimensional solutions gets its motivation.

2.4 The Topological Defect Approach

2.4.1 The Trapping Mechanism

Up to now, we have avoided the question of how it is possible for ordinary matter to be trapped on a brane, and what this brane is exactly. The term "brane" actually has quite different meaning in different contexts, but here we are using it for any 3-dimensional submanifold to which ordinary matter could be trapped, irrespective of the trapping mechanism. There are many ways of obtaining this trapping, and in this thesis we will most of the time assume that it is done by one means or another, without mentioning any mechanism in particular, or worrying about this aspect explicitly. But I will nevertheless mention one example of such mechanisms, that will hopefully help clarify what a brane could be.[5] In a five-dimensional theory, a simple way to do it is to introduce in the bulk a scalar field ϕ that obeys the following action:

$$S_\phi = \int d^4x dz \left(\frac{1}{2}\partial_A\phi\partial^A\phi - V(\phi)\right) \tag{2.30}$$

where z is the extra-dimensional coordinate, and the subscript A denotes all five coordinates. The key to obtaining a domain wall (the brane) where particles will be trapped is that the potential $V(\phi)$ has a double-well shape, with two degenerate minima at $\phi = \pm\eta$. The potential $V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \eta^2\right)^2$, for example, has this property. The resulting field equation will have, in addition to the two ground state solutions $\phi = \pm\eta$, another solution, the "kink", which interpolates between the ground states. This solution has asymptotics[6]

$$\phi(z \to +\infty) = +\eta \tag{2.31}$$

$$\phi(z \to -\infty) = -\eta \tag{2.32}$$

[5]For other trapping mechanisms, see refs. [20, 29, 30]. Note that in string theory the trapping is automatic, and you don't need to resort to field theory mechanisms.

[6]There is another solution with opposite signs.

and the interpolating region of rapid change of the scalar field corresponds to the domain wall.

Now if one introduces, say, fermions in that model, it turns out that there is a zero mode, a solution to the equation of motion with mass $m = 0$, which is localized near $z = 0$ (the location of the domain wall) and decaying exponentially at large $|z|$. These massless four-dimensional fermions localized on the domain wall are meant to play the rôle of our matter. Interestingly, the number of zero modes can be larger than one, so that from one family of multi-dimensional fermions it is possible to obtain several four-dimensional families. This could help explain the origin of the number of Standard Model generations. It is more difficult to localize gauge fields, but various mechanisms have been proposed through which it can be done.

2.4.2 Topological Defects

The domain wall of the above example is one of a larger group of topological defects, that all have in common that they are boundaries of phases of the theory. In the example just given, the domain wall was separating the two regions $\phi = +\eta$ and $\phi = -\eta$. In a one-dimensional (sub)space, a domain wall is a point defect, and that is why in the case of one extra dimension it can be associated with our universe, that has no extension in the extra dimension. Strings and monopoles are other kinds of topological defects. In three dimensions, domain walls still separate regions of definite discrete states of the field, strings are linear defects, where the phase of the field ϕ (which is no longer a scalar, but a vector) changes by 2π in making a loop about the string, and monopoles are point defects, where the field points radially away from the defect. In two dimensions, which is the situation of interest in this thesis, point defects (vortices) are the equivalent of the strings in three dimensions, the field making one internal rotation as we travel on a loop about the defect.

Topological defects are related to some form of symmetry breaking, which gives rise to a set of degenerate ground states, just like in the example given above. To understand how topological defects can form, suppose that as temperature drops, for example, the direction of ϕ within its internal space (its phase, or in the case of a scalar, its sign) is selected at random independently at each point in space. This is not a minimum-energy state, because the field derivatives are non-zero, and so the fields in different places will attempt to align themselves. This process may not be able to go to completion if there are gradients that cannot be removed by any small local changes of the field. This situation is familiar from paramagnetism: below the Curie temperature, magnetization occurs spontaneously, but in different directions in different places, leading to magnetic domains. Now in the case we will be interested in, the point defect with two codimensions, it would arise if there was a loop such that the field ϕ makes one or more rotations in its internal space as one moves around the loop. This is a topological configuration that cannot be erased by a continuous change in the phase of ϕ. As we shrink the loop to zero size, the field derivatives increase, and so some energy is associated with the field configuration. We have a topological defect.

2.4.3 Global vs. Local Defects

Defects can arise from the breaking of a global or a gauge symmetry. There is an important distinction to make between these two cases. In both cases, there is energy associated with fields derivatives in the core of the defect, but in the case of a gauge symmetry, both scalar and gauge fields can adjust in order to minimize the energy. The result is that defect energy is more localized in the case of the gauge (or local) defect than for the global defect. For global strings, for example, the field energy falls as $\rho \propto 1/r^2$ at large distances from the string, so that the mass per unit length $M = \int \rho \, 2\pi r \, dr$ diverges logarithmically. On the contrary, the energy per unit length for a gauge string is perfectly well defined. Since its energy is really localized in its core, a codimension-two gauge string, or gauge vortex, is a good model for a codimension-2 brane. The metric induced by static cosmic strings (in an ordinary three-dimensional space) has been known for a while. In 1981, Vilenkin [42] found it to be flat with a conical defect at the core and a deficit angle around the string.[7] Cohen and Kaplan [14] did the equivalent for global strings in 1988, finding an exact solution to Einstein's equation, and they noted that there was a singularity at finite distance from the core. Global strings have been used in the framework of the braneworld scenario, and it was shown that this singularity can be removed by having a negative bulk cosmological constant [23], or by adding a time dependence in the metric [24], but as mentioned before, the global string has a strong effect far from the core and so it is not a realistic model for our three-brane universe. We will discuss in section 4.1.3 how

[7] We will come back on the meaning of these in the next chapter.

the gauge, or local vortex can be used in a six-dimensional braneworld model. I will only mention that the model that is generally used when introducing strings in braneworld scenarios, the Abelian-Higgs model, comes back to Nielsen and Olesen [35] who showed in 1973 that the Higgs Lagrangian

$$\mathcal{L}_H = \frac{1}{2}\partial_A\phi\partial^A\phi - V(\phi)\,, \quad V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \eta^2\right)^2 \tag{2.33}$$

allowed for vortex-line solutions. [8]

2.4.4 Topological Defect vs. Phenomenological Approach

Many people have been working on the topological defect aspect of brane world scenarios, trying to show for various models that the fields responsible for creating the brane could be consistently incorporated in the action without affecting the beneficial characteristics of the models. But there is another approach which favours the exploration of the physical consequences of a large number of models in order to be able to associate them with the different observations that could be done with the next generation of accelerators. In this approach, the models are simplified so that a larger number of them can be examined. We will be working along these lines in the present thesis, since our goal is to present many different solutions. In fact, for practical purposes, one can just assume that the Standard Model particles are trapped on a three-dimensional brane and work from there, keeping in mind that effective trapping mechanisms exist, and that it should be possible to find such a mechanism for models which will be proven to agree with future observations.

[8] We used the Higgs Lagrangian in the example at the beginning of this section.

To simplify the model, one can consider the brane as a delta-function source for the gravitational field. In the simplest case, the gravitating brane is characterized by just one parameter: the energy density per unit three-volume (or **brane tension**) τ. This is the kind of procedure that we will use in the next chapter where we examine a simple generic six-dimensional model.

Chapter 3

General Model with Codimension-two Brane

We now look at a general model of six-dimensional braneworld scenario. The objective is to get a classification of the solutions to Einstein's equation for a six-dimensional braneworld scenario that is as general as possible, while the model is simple enough so that we can actually study the solutions. The model is allowed to have arbitrary bulk and physical (effective four-dimensional) cosmological constants, solutions with vanishing physical cosmological constants having already been studied extensively. We will be of course interested in introducing a three-brane at some point of the space, where the Standard Model particles will be confined. However we leave open the possibility for more than one 3-brane, and for 4-branes. We demand that the metric be cylindrically symmetric, and that it depends only on the extra coordinates and thus do not break general covariance along the four physical dimensions. One of the extra dimensions will be assumed to be compact. The other extra coordinate is not required to be either compact or non-compact a priori.

We begin by looking at the different cylindrically symmetric 6D geometries without source terms, to have a first classification of the possible scenarios. We then introduce a brane tension at the origin, see how that changes the solutions, and finally look at those solutions which will contain another brane: either those for which we need to cut out part of the space, or those in which the space is naturally compactified, and that can consistently closed up with another 3-brane.

3.1 Setup

Consider the total action

$$S = S_{grav} + S_{brane} \tag{3.1}$$

with the bulk action

$$S_{grav} = \int d^6x \sqrt{-\widetilde{g}} \left[\frac{\widetilde{R}}{2\kappa_6^2} + \Lambda_6 \right], \tag{3.2}$$

where Λ is a bulk cosmological constant, $\kappa_6^2 = 8\pi G_6 = 8\pi / M_6^4$ where M_6 denotes the six-dimensional Planck mass, $\widetilde{g}$ denotes the determinant of the full 6-dimensional metric and $\widetilde{R}$ is the Ricci scalar constructed from this metric. The value of the cosmological constant is arbitrary at this point, so it includes the case of vanishing cosmological constant.

Our metric ansatz for this action is

$$ds^2 = \widetilde{g}_{AB}dx^A dx^B = M(\rho)^2 g_{\mu\nu}dx^\mu dx^\nu + d\rho^2 + L(\rho)^2 d\theta^2 \tag{3.3}$$

where ρ and θ are, respectively, the bulk radius and the bulk angle, $g_{\mu\nu}$ is the four-dimensional metric (with mostly + signature) and $M(\rho)$, $L(\rho)$ are the warp factors.

We compute the Einstein equations $\widetilde{R}_{AB} - \frac{1}{2}\widetilde{g}_{AB}\widetilde{R} = \kappa_6^2(T_{AB} + \Lambda_6)$ for this metric:

$$\frac{3H^2}{M^2} - 3\left(\frac{M'}{M}\right)^2 - 3\frac{M''}{M} - 3\frac{M'}{M}\frac{L'}{L} - \frac{L''}{L} = \kappa_6^2(T_0^0 + \Lambda_6) \tag{3.4}$$

$$\frac{6H^2}{M^2} - 6\left(\frac{M'}{M}\right)^2 - 4\frac{M'}{M}\frac{L'}{L} = \kappa_6^2(T_\rho^\rho + \Lambda_6) \tag{3.5}$$

$$\frac{6H^2}{M^2} - 6\left(\frac{M'}{M}\right)^2 - 4\frac{M''}{M} = \kappa_6^2(T_\theta^\theta + \Lambda_6) \tag{3.6}$$

We have assumed the effective four-dimensional metric to be $g_{\mu\nu} = -dt^2 + a(t)^2 d\vec{x}^2$, with $a(t) = e^{Ht}$ Alternatively, the above equations could be expressed in terms of a physical cosmological constant, Λ_4, related to H through $\kappa_4^2\Lambda_4 = 3H^2$.[1] Similarly to its six-dimensional equivalent, κ_4^2 is defined by $\kappa_4^2 = 8\pi G_4$, with G_4 the ordinary four-dimensional Newton's constant.

3.2 Bulk Solutions

We wish to classify the possible solutions to the 6-dimensional gravitational action with arbitrary bulk and physical cosmological constants, which exhibit cylindrical symmetry

[1] In the case $H^2 < 0$, corresponding to an anti de Sitter brane, the four-dimensional line element will take a different form $(a(t) = \sin(iHt))$ than the one given, which is appropriate in the de Sitter case.

and preserve general covariance along the four physical dimensions. This classification of 6-dimensional geometries could then be the starting point for a further classification of solutions, once source terms from the brane part of the action are introduced. Actually, since we will find out that the effect of introducing a brane tension on the solution is to introduce a deficit angle, and that it does not otherwise affect the geometry of the space, the classification of bulk solutions that we make hereafter stays relevant even in the case where source terms have been added, and so it is clearly a useful step at this point.

3.2.1 Boundary Conditions

Because of cylindrical symmetry, $M(\rho)$ and $M'(\rho)$ are constant for all angles. Now if we take the directional derivative of M along a straight line passing through the origin, we can determine the jump in M' at the origin:

$$\Delta M' = \vec{\nabla} M(\epsilon) \cdot \hat{\rho} - \vec{\nabla} M(\epsilon) \cdot (-\hat{\rho}) \tag{3.7}$$

$$= \left.\frac{dM}{d\rho}\right|_{\epsilon} \hat{\rho} \cdot \hat{\rho} - \left.\frac{dM}{d\rho}\right|_{\epsilon} \hat{\rho} \cdot (-\hat{\rho}) \tag{3.8}$$

$$= 2M'(\epsilon) \tag{3.9}$$

So in the absence of singular source terms, the continuity of the slope requires that $M'(0) = 0$. That is the boundary condition that we will use in deriving the bulk solutions. It should be noted, however, that this condition will remain true in the case where we want to add a 3-brane at the origin, as we will see in a later section, in such a way that the solutions derived below are still relevant in that case.

3.2.2 General Bulk Solutions

We wish to study the properties of the six-dimensional cylindrically symmetric geometry by considering only the gravitational part of the action, not yet including the effects of the brane. In this case, all source terms vanish in equations 3.4-3.6 and only the cosmological constant term remains on the right hand side. Equating equations (3.5) and (3.6), we obtain

$$M'' = M'\frac{L'}{L} \Longrightarrow \textit{either } M' = 0 \textit{ or } M' = RL \tag{3.10}$$

where R is an integration constant.

We first examine the unwarped solution, that is, the case $M' = 0$. The complete solution for that case is

$$M(\rho) = M_0,\ L(\rho) = \begin{cases} L_1 \sinh\left(\kappa_6\sqrt{\frac{\Lambda_6}{2}}\rho\right) + L_2 \cosh\left(\kappa_6\sqrt{\frac{\Lambda_6}{2}}\rho\right) & if \quad \Lambda_6 < 0 \\ L_1 \sin\left(\kappa_6\sqrt{\frac{\Lambda_6}{2}}\rho\right) + L_2 \cos\left(\kappa_6\sqrt{\frac{\Lambda_6}{2}}\rho\right) & if \quad \Lambda_6 > 0 \\ L_0\rho & if \quad \Lambda_6 = 0 \end{cases} \tag{3.11}$$

and

$$\kappa_4^2\Lambda_4 = \frac{\kappa_6^2\Lambda_6}{2}M_0^2. \tag{3.12}$$

Of course, since we will eventually be interested in introducing a three-brane at the origin, meaning that $L(\rho)$ vanishes at that point, L_2 will be set to 0 (see section 3.3.1). Hence we get one solution, the case $\Lambda_6 > 0$, where the extra-dimensional coordinate warp factor $L(\rho)$ vanishes at some other radius ($\kappa_6\sqrt{\frac{\Lambda_6}{2}}\rho = \pi$), so that there is only extension in the four ordinary dimensions there. We can then consistently introduce another three-brane at that point. The two other unwarped solutions have $L(\rho)$ increasing with ρ, in a way that prevents localization of gravity. The condition for gravity to be localized, with this metric, amounts to:

$$M_P^2 = \frac{4\pi M_6^4}{m_H^2} \int d\rho M^2(\rho)\mathcal{L}(\rho) < \infty. \tag{3.13}$$

Hence for the present case of unwarped solutions, with M constant, this condition becomes $\int d\rho\mathcal{L}(\rho) < \infty$. In both cases above, $L \sim \sinh\left(\kappa_6\sqrt{\frac{\Lambda_6}{2}}\rho\right)$ and $L \sim \rho$, this integral blows up for an infinite extra dimension. Thus we will need to cut the space with a four-brane at some radius if we are interested in using these solutions.

Now for the case $M' = RL$, we closely follow the elegant analysis of Rubakov and Shaposhnikov [40]. They showed that solving the 6D Einstein equations for this situation is equivalent to solving for the classical motion of a particle of unit mass in a potential U, i.e. solving the equation

$$\ddot{z} = -\frac{dU(z)}{dz} \tag{3.14}$$

where ρ plays the role of the time variable, i.e. $\dot{z} = \frac{dz}{d(\rho)}$, the variable z is related to the warp factor M through $z(\rho) = M(\rho)^{\frac{5}{2}}$ and the potential depends on the dimensionless 4D and 6D cosmological constants $\hat{\Lambda}_4 = \kappa_4^2\Lambda_4$ and $\hat{\Lambda}_6 = \kappa_6^2\Lambda_6$ via

$$U(z) = az^2 - bz^{6/5},\ a = \frac{5}{16}\hat{\Lambda}_6,\ b = \frac{25}{24}\hat{\Lambda}_4. \tag{3.15}$$

To show this, we start with bulk equation (3.6)

$$2\left(\frac{\hat{\Lambda}_4}{M^2} - 3\left(\frac{M'}{M}\right)^2 - 2\frac{M''}{M}\right) = \hat{\Lambda}_6. \tag{3.16}$$

In order to solve this equation, let $\nu = \frac{dM}{d\rho}$, so that $M'' = v\frac{dv}{dM}$. We get

$$-\left(3\frac{\nu^2}{M^2} + 2\frac{v}{M}\frac{dv}{dM}\right) = \frac{\hat{\Lambda}_6}{2} - \frac{\hat{\Lambda}_4}{M^2}. \tag{3.17}$$

We multiply by M^4 on both sides:

$$\left(3\nu^2 M^2 + 2v\frac{dv}{dM}M^3\right) = -M^4\left[\frac{\hat{\Lambda}_6}{2} - \frac{\hat{\Lambda}_4}{M^2}\right], \tag{3.18}$$

integrate with respect to M to get

$$\nu^2 M^3 = \frac{\hat{\Lambda}_4}{3}M^3 - \frac{\hat{\Lambda}_6}{10}M^5 + C_1. \tag{3.19}$$

We now have

$$\frac{dM}{d\rho} = \pm\sqrt{\frac{\hat{\Lambda}_4}{3} - \frac{\hat{\Lambda}_6}{10}M^2 + \frac{C_1}{M^3}} \tag{3.20}$$

and we obtain

$$\rho = \left|\int_{M(0)}^{M(\rho)} \frac{M^{3/2}dM}{\sqrt{\frac{\hat{\Lambda}_4}{3}M^3 - \frac{\hat{\Lambda}_6}{10}M^5 + C_1}}\right|. \tag{3.21}$$

Finally, we can express the result in term of $z = M^{5/2}$, yielding

$$\rho = \left|\int_{z(0)}^{z(\rho)} \frac{dz}{\frac{5}{2}\sqrt{\frac{\hat{\Lambda}_4}{3}z^{6/5} - \frac{\hat{\Lambda}_6}{10}z^2 + C_1}}\right|, \tag{3.22}$$

which is precisely

$$\rho = \left|\int_{z_0}^{z} \frac{dz}{\sqrt{2\left(E - U(z)\right)}}\right|, \tag{3.23}$$

the integral form of the equation of energy conservation for a classical particle motion, $E = \frac{1}{2}\dot{z}^2 + U(z)$, with the time variable ρ and the potential $U(z) = \frac{5}{16}\hat{\Lambda}z^2 - \frac{25}{24}\hat{\Lambda}_4 z^{6/5} =$

$az^2 - bz^{6/5}$, $a = \frac{5}{16}\hat{\Lambda}_6$, $b = \frac{25}{24}\hat{\Lambda}_4$. E simply appeared in the equation as a constant of integration.[2]

The shape of the potential $U(z)$ will depend on the signs of $\hat{\Lambda}_4$ and $\hat{\Lambda}_6$. To find solutions to the Einstein equations then amounts to finding the possible trajectories for a classical particle in this potential, starting with an initial "velocity" $\dot{z}(0) = 0$. This corresponds to $M'(0) = 0$, which we have seen has to be so when no source term is present, but will also remain true if we add a three-brane at $\rho = 0$, as we will see later. The variable z corresponds to the position of the particle, and as we have said, ρ plays the role of time. Once the trajectory $z(\rho)$ is known (and so $M(\rho)$), equation (3.10) gives us the corresponding $L(\rho)$.

3.2.3 Classification of Bulk Solutions

Different shapes of the potential, and different initial conditions for the classical particle will yield different types of trajectories, which for us correspond to different types of solutions. It is interesting to classify those different types of solutions, especially because their general properties will not be affected with the introduction of a brane tension, which only yields a deficit angle.

[2]Note that this equation is derived solely from the $\theta\theta$ component of Einstein bulk equations (eq.3.6), which will be valid everywhere but at the location of a 4-brane, because both in the bulk and at a 3-brane we have $T^\theta_\theta = 0$.

At first sight when looking at equation (3.23), it seems that there are two different initial conditions to specify: the total energy E and the initial position of the particle z_0. In fact, with the additional condition $\dot{z}(0) = 0$, knowing z_0 will fix the integration constant E through $E = \frac{1}{2}\dot{z}(0)^2 + U(z_0)$. We are always free to rescale the units so that $z_0 = 1$ (unless it is zero). That is the approach that we will take here, so that time and energy are normalized in the usual way. (We are basically taking the viewpoint of a four-dimensional observer on our brane, just like we did on page 21 when interpreting the Randall-Sundrum model.) The constant E can then be expressed in terms of the cosmological constants, $E = a - b$, and specifying a particular value for E will select a particular solution. Basically, we are using the following boundary conditions:

$$M'\big|_{\rho=0} = 0 \quad \text{and} \quad M\big|_{\rho=0} = 1 \tag{3.24}$$

to fix the constants of integration in our equation. For example equation (3.20) becomes

$$M'^2 = \frac{\hat{\Lambda}_4}{3} - \frac{\hat{\Lambda}_6}{10}M^2 + \frac{1}{M^3}\left(\frac{\hat{\Lambda}_6}{10} - \frac{\hat{\Lambda}_4}{3}\right). \tag{3.25}$$

This equation, or its equivalent in integral form, equation (3.23), will remain unchanged in the presence of a 3-brane, and actually stays valid as long as one doesn't encounter a 4-brane. This is because this equation is derived solely from the $(\theta\theta)$ component of the bulk Einstein's equation (i.e. with vanishing T^θ_θ), but in fact T^θ_θ will only be non-zero at the location of a four-brane. So these equations, and the solutions that we will extract from these equations, will remain valid everywhere, except at the location of a four-brane.

Before classifying the types of solution, we examine the shape of the potential for each case to see what happens:(Recall that $a = \frac{5}{16}\hat{\Lambda}_6,\ b = \frac{25}{24}\hat{\Lambda}_4$)

$$\hat{\Lambda} > 0,\ \hat{\Lambda}_{ph} > 0: \quad U(z) \geq 0 \iff z \geq \left(\tfrac{b}{a}\right)^{5/4}$$

$$U'(z) \geq 0 \iff z \geq \left(\tfrac{3b}{5a}\right)^{5/4}$$

$$U''(z) \geq 0 \iff z \geq \left(\tfrac{3b}{25a}\right)^{5/4}$$

$$\hat{\Lambda} \geq 0,\ \hat{\Lambda}_{ph} \leq 0: \quad U(z) \geq 0 \quad \forall \quad z$$

$$U'(z) \geq 0 \quad \forall \quad z$$

$$U''(z) \geq 0 \quad \forall \quad z$$

$$\hat{\Lambda} \leq 0,\ \hat{\Lambda}_{ph} \geq 0: \quad U(z) \leq 0 \quad \forall \quad z$$

$$U'(z) \leq 0 \quad \forall \quad z$$

$$U''(z) \leq 0 \quad \forall \quad z$$

$$\hat{\Lambda} < 0,\ \hat{\Lambda}_{ph} < 0: \quad U(z) \leq 0 \iff z \geq \left(\tfrac{b}{a}\right)^{5/4}$$

$$U'(z) \leq 0 \iff z \geq \left(\tfrac{3b}{5a}\right)^{5/4}$$

$$U''(z) \leq 0 \iff z \geq \left(\tfrac{3b}{25a}\right)^{5/4}$$

These four distinctive shapes of the potential are summarized in figure 3.1. Depending on the particular potential behavior and on the initial conditions, different types of trajectories/solutions are possible once the particle starts rolling in the potential. We will distinguish between bulk solutions with $z'(0) = M'(0) = L(0) = 0$ and the others, concentrating on the former because we are mostly interested in solutions which allow for

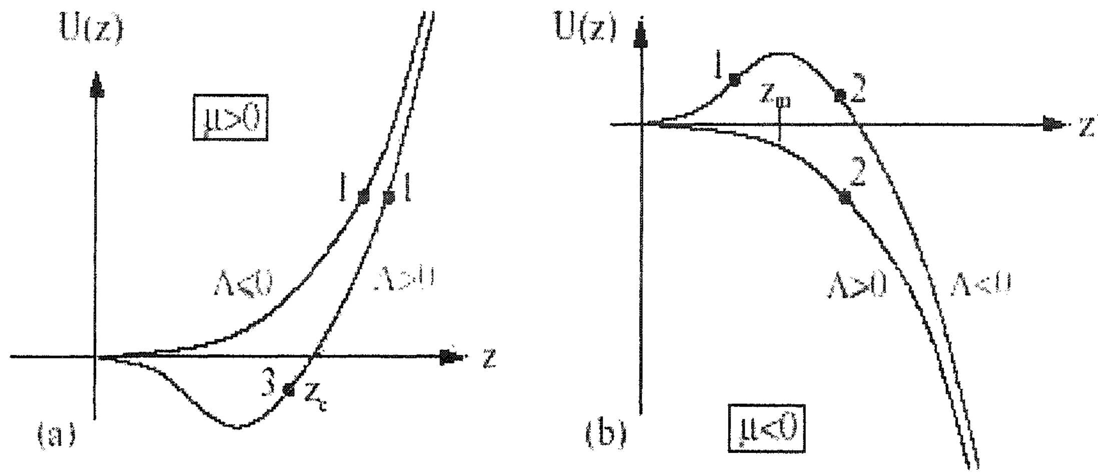

Figure 3.1: Shapes of the potential $U(z)$ depending on the sign of Λ_4 for (a) $\Lambda_6 > 0$ and (b) $\Lambda_6 < 0$. Numbered dots show initial conditions leading to the three kinds of solutions discussed below.

consistently introducing a 3-brane at $\rho = 0$. (see section 3.3.1). We describe below these different types of trajectories/solutions:

1. In solutions of type 1, the particle reach $z = 0$ in a finite amount of "time" ρ. That means $M \to 0$ at some ρ. Some curvature invariants diverge at that point so the subspace ends at a singularity. To use this solution, we need to insert a 4-brane at $\rho < \rho_s$ to cut off the space before reaching the point where it becomes singular. This happens when:

- $\Lambda_6 \geq 0,\ \Lambda_4 \geq 0$ *and* $E > 0$ (*i.e.* $\hat{\Lambda}_4 > \frac{3}{10}\hat{\Lambda}_6$)

- $\Lambda_6 > 0,\ \Lambda_4 < 0$

- $\Lambda_6 \leq 0,\ \Lambda_4 \leq 0,\ and\ z_0 < z_m = \left(\frac{3b}{5a}\right)^{5/4}\ (i.e.\ \hat{\Lambda}_6 > 2\hat{\Lambda}_4)$.

2. In type 2 solutions, the particle eventually reaches $z \longrightarrow \infty$. We therefore have $M \rightarrow \infty$ as $\rho \rightarrow \infty$, which prevents gravity from being localized,.so we have to put a 4-brane at large ρ again. This happens when:

- $\Lambda_6 \leq 0,\ \Lambda_4 \leq 0,\ z_0 > z_m\ (i.e.\ \hat{\Lambda}_6 < 2\hat{\Lambda}_4)$

- $\Lambda_6 < 0,\ \Lambda_4 > 0$.

3. In the third type of solutions, the particle comes to rest again, $z' = 0$ for some ρ. The corresponding solution is characterized by the internal space closing off again at some point ρ_m ($L = 0$ there) so that another 3-brane can be consistently introduced. The extra dimensions look like S_2 with a wedge cut out. (the 3-branes induce a deficit angle). This solution is valid for

- $\Lambda_6 \geq 0,\ \Lambda_4 \geq 0,\ E < 0\ (i.e.\ \hat{\Lambda}_4 < \frac{3}{10}\hat{\Lambda}_6)$.

As we have said, all of the above types of solutions assume the particle starts from rest, i.e. at $\rho = 0$, $M' = L = 0$. But there also are solutions with $z'(0) \neq 0$, trajectories where the particle starts with a non-zero initial velocity. Those solutions cannot have a three-brane placed at the origin since $\rho = 0$ doesn't correspond to a four-dimensional (3 spatial dimensions) point anymore. At this location, we still have an extension in the other coordinate θ, since L doesn't vanish there in these solutions. We will not study these solutions in detail here, since we are interested in a model that allows for a 3-brane that could correspond with the four-dimensional universe.

We now examine in more detail each one of these three types of solutions, and give examples of particular analytic solutions where they exist.

Type 1 Solutions

To show that in type 1 bulk solutions, the warp factor vanishes at some radius ρ, or in the analogy that the classical particle reaches $z = 0$ in a finite amount of time, one has to show that the integral

$$\rho = \left| \int_{z_0}^{0} \frac{dz}{\sqrt{2\,(E - U(z))}} \right| \tag{3.26}$$

is finite for the three cases:

a. $a \geq 0,\ b \geq 0,\ E = (a - b) > 0$

b. $a > 0,\ b < 0$

c. $a \leq 0,\ b \leq 0,\ z_0 < z_m$ *i.e.* $\frac{3b}{5a} > 1$.

If $\dot{z}(0) = L(0) = 0$ (for a three-brane at the origin) and in units where $z(0) = M(0) = 1$, this integral becomes

$$\rho = \left| \int_{1}^{0} \frac{dz}{\sqrt{2\,(a - b - az^2 - bz^{6/5})}} \right|. \tag{3.27}$$

Now the integrand only diverges at $z = 1$, so we can split up the integral into two parts

$$\rho = \left| \int_{1}^{1-\epsilon} \frac{dz}{\sqrt{2\,(a - b - az^2 - bz^{6/5})}} \right| + \left| \int_{1-\epsilon}^{0} \frac{dz}{\sqrt{2\,(a - b - az^2 - bz^{6/5})}} \right|. \tag{3.28}$$

(both parts of the trajectory take a positive amount of time) The second integral is obviously finite, so we only need to examine the first one more closely. In the vincinity of $z = 1 - \epsilon$, we can Taylor-expand the function in the square root,

$$a - b - az^2 + bz^{6/5} = 2(a - \frac{3}{5}b)\epsilon + \mathcal{O}(\epsilon^2) \tag{3.29}$$

So we have

$$\left| \int_1^{1-\epsilon} \frac{dz}{\sqrt{2\left(a - b - az^2 - bz^{6/5}\right)}} \right| = \int_0^{\epsilon} \frac{d\epsilon}{\sqrt{4(a - \frac{3}{5}b)\epsilon + \mathcal{O}(\epsilon^2)}}. \tag{3.30}$$

In all three cases above, we have that $a - \frac{3}{5}b > 0$. Since the integral $\int_0^{\epsilon} \frac{d\epsilon}{\sqrt{\epsilon}}$ converges, the original integral will converge too. That means the classical particle in the analogy take a finite amount of time ρ to go from $z(0) = 1$ to $z(\rho) = 0$. The translation of that in terms of the warp factor $M = z^{2/5}$ is that given $M(0) = 1$, $M(\rho)$ will vanish at some finite radius ρ. Now at that point, some curvature invariants diverge. For example, if we look at the Ricci scalar:

$$R = -12\frac{H^2}{M(\rho)^2} + \frac{12}{M(\rho)^2}\left(\frac{dM}{d\rho}\right)^2 + \frac{8}{M(\rho)}\left(\frac{d^2M}{d\rho^2}\right) + \frac{8}{M(\rho)L(\rho)}\frac{dM}{d\rho}\frac{dL}{d\rho} + \frac{2}{L(\rho)}\left(\frac{d^2L}{d\rho^2}\right) \tag{3.31}$$

we see that it generically diverges when $M \to 0$. In the special case $H = \Lambda_4 = b = 0$ where this is not as evident, we find the solution ($\Lambda_6 > 0$)

$$M(\rho) = \cos^{2/5}(k\rho);\ L(\rho) = Rk\sin^{2/5}(k\rho)\,;\ k = \sqrt{\frac{5}{8}\hat{\Lambda}_6}. \tag{3.32}$$

For this solution, we see that when $M \to 0$ the derivative of M diverges, which automatically gives us a divergence in the Ricci scalar. Hence we have shown that for the three cases above, the warp factor vanishes at some radius, at which point some curvature invariant generically diverges. We will thus need to cut the space with a four-brane before that radius is reached.

Type 2 Solutions

We use the classical mechanical analogy and look at the shape of the potential for the cases:

- $\Lambda_6 \leq 0,\ \Lambda_4 \leq 0,\ z_0 > z_m$
- $\Lambda_6 < 0,\ \Lambda_4 > 0.$

We easily see that in these situations, the particle escapes toward infinity. That means as ρ increases, $z(\rho)$ keeps on increasing, and so does $M(\rho)$. These solutions are therefore reminiscent of the Randall-Sundrum solution to the hierarchy problem, with a warp factor increasing away from our brane.[37] That makes them very interesting solutions. In fact, one particular solution of type 2 is the AdS soliton, (for $\Lambda_4 = 0$) which has been studied before: [4][32]

$$M(\rho) = \cosh^{\frac{2}{5}}(k\rho);\ L(\rho) = \frac{2}{5}Rk\frac{\sinh^{\frac{2}{5}}(k\rho)}{\cosh^{\frac{3}{5}}(k\rho)};\ k = \sqrt{-\frac{5}{8}\hat{\Lambda}_6}. \tag{3.33}$$

It is not possible to generalize this solution to the non-static case $\Lambda_4 > 0$ analytically. It is nevertheless possible to obtain approximate analytic solutions by considering Λ_4 as a perturbation, yielding:[11]

$$M(\rho) = [z_0 \cosh(k(\rho + d\rho))]^{2/5};\ L(\rho) = RM'(\rho)\,, \tag{3.34}$$

where $d\rho$ increases with ρ and approaches a constant at large ρ, so at leading order the only difference between the perturbed and unperturbed solutions at large ρ is a shift in the radial size of the extra dimension.

Now for the case $\Lambda_6 \leq 0,\ \Lambda_4 \leq 0,\ z_0 > z_m$, we find a solution with $E = 0$,

$$M(\rho) = \cosh(k\rho);\ L(\rho) = Rk \sinh(k\rho)\,;\ k = \sqrt{\frac{\left|\hat{\Lambda}_6\right|}{10}} = iH, \tag{3.35}$$

Note that H is imaginary here, and the 4D metric is AdS space. Again, this particular solution does not generalize easily to the case $E \neq 0$.

Type 3 Solutions

Examining the case $\Lambda_6 > 0,\ \Lambda_4 > 0,\ E \leq 0$ through the analogy, we can use our classical mechanical intuition to see that the particle will come to rest again at some time. Looking for example at the equation $E = \frac{1}{2}\dot{z}^2 + U(z)$, which actually follows directly from the $(\theta\theta)$ component of Einstein equation, but corresponds in the analogy to the conservation of energy equation, we can deduce some general features of the solution. Rescaling $z(0) = 1$ will fix $E = a - b$ as before. Now depending on whether the particle starts at the left or the right of the stable equilibrium point, integrating the above equation will tell us the sign of the acceleration, and therefore the direction the particle will initially take:

$$\ddot{z} = -2az + \frac{6}{5}bz^{1/5} \begin{cases} \geq 0 & whenever \quad z \leq \left(\frac{3b}{5a}\right)^{5/4} \\ \leq 0 & whenever \quad z \geq \left(\frac{3b}{5a}\right)^{5/4} \end{cases} . \tag{3.36}$$

Looking for example at a case where $z_0 \leq \left(\frac{3b}{5a}\right)^{5/4}$, $\dot{z}$ will initially increase and so it will be positive. It will only start decreasing once z reaches $\left(\frac{3b}{5a}\right)^{5/4}$, the location of the equilibrium point. Now we use our classical mechanical intuition, which as we just saw is perfectly applicable. Looking at the shape of the potential, shown in figure 3.1, it is easy to see that the particle will reach the point where $U(z) = E$ again, on the other side of the equilibrium point, whatever side it started from. The only case which maybe cannot be determined that simply is the one with $E = 0$. This solution is actually at the boundary between types 1 and 3:($\Lambda_6 > 0,\ \Lambda_4 > 0,\ E = 0$):

$$M(\rho) = \cos(k\rho);\ L(\rho) = Rk\sin(k\rho);\ k = \sqrt{\frac{\hat{\Lambda}_6}{10}} = \sqrt{\frac{\hat{\Lambda}_4}{3}} = H. \tag{3.37}$$

On one side, we see that for this solution the space will close up again at $k\rho = \pi$,.like for the other solutions of type 3 On the other hand, at the point $k\rho = \frac{\pi}{2}$, we have $M \to 0$. But unlike the solutions of type 1, no curvature invariant diverge at that point. In fact, they are all constant for this solution. The Ricci Scalar, for example, is $6H^2$. The 4D part of the spacetime seems to disappear at that point leaving only a 2D Euclidean space there. In fact, that strange point is a horizon (light cannot propagate across that radius[3]), and the space can be continued normally for values of rho on the other side. Unfortunately,

[3]For a radial null trajectory, $d\rho/dt = \cos(k\rho) = \sin(k\rho_m - k\rho)$ and so it takes infinite time for a photon to reach ρ_m from either smaller or larger radii.

general solutions cannot be found exactly, but in the case where the particle stays close to the bottom of the well, we can approximate the potential there as a harmonic oscillator. The result is

$$M(\rho) \cong (z_0 + \epsilon \cos(k\rho))^{2/5};\ k^2 = \frac{1}{2}\Lambda_6. \tag{3.38}$$

z_0 is the position of the minimum of the potential, $z_0 = \left(\frac{2\Lambda_4}{\Lambda_6}\right)^{5/4}$.

For all solutions of this type, we found that there is a point where the particle in the analogy comes to rest ($z' = 0$) again, i.e. a point where $L = 0$ and the extra-dimensional space closes off. At that point we will be able to consistently introduce another three-brane, in addition to the three-brane where the standard model is supposed to lie, that we will introduce at the origin.

3.3 Brane Tension

So far we have a classification for bulk solutions. Now we would like to get a classification of all solutions, including the effect of the branes themselves, i.e. the brane tension. This is harder because there are many possible combinations of branes, but also, since we don't have an explicit general solution, we must add the brane tension by perturbing specific analytic solutions. Nevertheless there are still some overall properties that we can compute without going through that.

3.3.1 3-brane at $\rho = 0$

To include the effects of the brane tension, we examine the metric behavior in the $\rho \to 0$ limit. In order to introduce a 3-brane at $\rho = 0$, we demand that $L = 0$ at this point so that there is only extension in three spatial dimensions. Therefore in the vicinity of the 3-brane equation eq.(3.10) (which remains unchanged in the case of a 3-brane because $T^\rho_\rho = T^\theta_\theta = 0$) gives us

$$L \to 0 \Longrightarrow M \to M_0 \tag{3.39}$$

where M_0 is a constant. The (00) Einstein equation in this limit becomes

$$\frac{3H^2}{M_0^2} - \frac{L''}{L} = \kappa_6^2(T^0_0 + \Lambda). \tag{3.40}$$

So in the small ρ limit:

$$L = \begin{cases} L_0 \sinh\left(\sqrt{3\frac{H^2}{M_0^2} - \kappa_6^2(T_0^0 + \Lambda)}\rho\right) & if \quad \kappa_6^2(T_0^0 + \Lambda) < 3\frac{H^2}{M_0^2} \\ L_0 \sin\left(\sqrt{\kappa_6^2(T_0^0 + \Lambda) - 3\frac{H^2}{M_0^2}}\rho\right) & if \quad \kappa_6^2(T_0^0 + \Lambda) > 3\frac{H^2}{M_0^2} \\ L_0\rho & if \quad \kappa_6^2(T_0^0 + \Lambda) = 3\frac{H^2}{M_0^2} \end{cases} \tag{3.41}$$

In any case, L can be expanded in a Taylor series around $\rho = 0$, which gives

$$L \approx L'(0) \cdot \rho \tag{3.42}$$

for small ρ. It is clear from the form of the metric that for general values of $L'(0)$ there will be a conical singularity associated with a deficit angle δ. Computing the ratio of the circumference over the radius of a small circle around the singularity, C/R, we get the value of the deficit angle

$$\delta = 2\pi - C/R = 2\pi(1 - L'(0)). \tag{3.43}$$

The existence of this conical singularity is connected to the presence of a 3-brane at $\rho = 0$. To determine this connection one has to carefully define the brane tension and examine the Einstein tensor in the vincinity of the brane. For doing this, it is easier to write the metric in a conformally flat form for the extra-dimensional space (in order to obtain the simple form of equation 3.46):

$$ds^2 = M_0^2 g_{\mu\nu} dx^\mu dx^\nu + f(r)(dr^2 + r^2 d\theta^2) \tag{3.44}$$

where $f = f_o r^{2(L'(0)-1)}$ and $\rho = \frac{r^{L'(0)}}{L'(0)}$. In these coordinates, the Einstein (00) equation reads

$$\frac{1}{2f}\left\{\frac{6H^2 f}{M_0^2} - \left(\frac{f'}{f}\right)^2 - \frac{f'}{rf} - \frac{f''}{f}\right\} = \kappa_6^2(T_0^0 + \Lambda) \tag{3.45}$$

or

$$-\frac{\nabla^2 \ln f}{2f} = \kappa_6^2(T_0^0 + \Lambda) - \frac{3H^2}{M_0^2}. \tag{3.46}$$

Now we look at the right-hand side of the equation. We first separate the stress-energy tensor into a brane tension part and an additional energy source δT_0^0 :[4]

$$T_0^0 = \tau_3 \frac{\sqrt{-g}}{\sqrt{-\tilde{g}}} \delta(\vec{r}) + \delta T_0^0, \tag{3.47}$$

where g is the determinant of the 4-dimensional effective metric. Now, one must be very careful with the choice of the delta function used in this definition, which depends on

[4]Strictly speaking, since the delta function is not a proper function but a linear operator, the non-linear equation (3.45) is mathematically ill defined if the energy density contains a delta function. However, our analysis can be made well defined by considering a highly peaked source and taking the appropriate limit at the end of the calculation. For simplicity, we keep the delta function notation throughout.

the coordinates $\widetilde{g}$ refers to. By definition, the tension τ_3 corresponds to what a four-dimensional observer would calculate as the vacuum energy. Hence, writing the part of the six-dimensional action due to the brane tension:

$$S_{3-brane} = \int d^4x \sqrt{-g}\tau_3 = \int d^4x \int d^2y \sqrt{-\widetilde{g}} \left(\frac{\sqrt{-g}}{\sqrt{-\widetilde{g}}} \tau_3 \right) \delta(\vec{r}) \tag{3.48}$$

we see that the appropriate combination of coordinates and delta function must satisfy:

$$\int d^2y \ \delta(\vec{r}) = 1. \tag{3.49}$$

For example, when using Cartesian coordinates y_1, y_2, the 2-D Cartesian coordinates delta function $\delta^2(\overrightarrow{y})$ is the correct delta function to use. In the present case, since we are using cylindrical coordinates, we use a properly weighted 1-D radial cylindrical coordinate delta function:[5]

$$\int dr d\theta \left(\frac{\delta(r)}{2\pi} \right) = 1 \Longrightarrow \delta(\vec{r}) = \frac{\delta(r)}{2\pi}. \tag{3.50}$$

We can also express it in terms of the 2-D Cartesian delta function, and the correct definition of the brane tension for this metric is:

$$T_0^0 |_{brane} = \tau_3 \frac{1}{fr} \frac{\delta(r)}{2\pi} = \tau_3 \frac{1}{f} \delta^2(\vec{r}) \tag{3.51}$$

Noting further that $\nabla^2 \ln r = 2\pi\delta^2(\vec{r})$ and matching the delta functions, equation 3.46 gives us:

$$-\left(L'(0) - 1\right) \cdot 2\pi = \kappa_6^2 \tau_3, \tag{3.52}$$

[5] Alternatively, we could keep $\delta^2(y_1, y_2)$ and refer to $\sqrt{-\widetilde{g}}$ as the determinant of the metric with its extra-dimensional part in cartesian coordinates.

so that the deficit angle is

$$\delta = 2\pi \left(1 - L'(0)\right) = \kappa_6^2 \tau_3 \tag{3.53}$$

It is important to note that staying in the region of small radius ρ, the region we would expect to be perturbed by the introduction of the brane, we find that the 3-brane tension is not associated with anything but the deficit angle, so that the solutions are not affected in any other way than the conical singularity. This important result is typical of codimension-two branes in six-dimensions (see for example [5], [9], [33]) It is what makes our classification for the bulk solutions still relevant.

Now once we have determined the behavior of the geometry close to the origin, we also need to look at the boundary conditions elsewhere. More specifically, as we have seen in section 3.2.3, all geometries with $\dot{z}(0) = 0$ (those with a 3-brane at the origin) will either close off at some radius ρ_m or will have to be cut off at some radius ρ_m to avoid some undesirable behavior (divergence of curvature invariants, or divergence of the four-dimensional Planck mass). So here we will include the effect of those boundary conditions at ρ_m, distinguishing between the case where a 3-brane is introduced at that point and that where there is a 4-brane.

3.3.2 3-brane at $\rho = \rho_m$

We already found (eq. 3.52) that the requirement that there be a 3-brane at the point $\rho = 0$, and so that $L(0) = 0$, leads to $\kappa_6^2 \tau_3 = 2\pi \left(1 - L'(0)\right)$. The exact same reasoning will lead to the equivalent condition for the point ρ_m :

$$\kappa_6^2 \tau_3' = 2\pi \left(1 + L'(\rho_m)\right). \tag{3.54}$$

The sign difference comes about because if the radius of a small circle around the point ρ_m is ϵ, the Taylor expansion around that point (on that circle) is

$$L \approx L'(\rho_m) \cdot (\rho - \rho_m) = -L'(\rho_m) \cdot \epsilon. \tag{3.55}$$

Since the circumference of the circle is $\int_0^{2\pi} L d\theta = -2\pi\epsilon L'(\rho_m)$, the deficit angle is

$$\delta = 2\pi - C/R = 2\pi(1 + L'(\rho_m)). \tag{3.56}$$

Condition 3.54 will apply to all solutions that were classified as type 3 in section 3.2.3 because for these solutions, there exist a radius $\rho_m > 0$ where $\dot{z} = 0$ (i.e. $L = 0$). In general, the value of L' need not be the same at $\rho = 0$ and $\rho = \rho_m$. If we take the particular case of the unwarped ($M' = 0$) solution with two 3-branes ($\Lambda_6 > 0$ case), an interesting characteristic is that in this particular geometry the deficit angle at the two 3-branes are the same, and so the tensions must also be the same. One could say that a fine-tuning of the stress-tensor is required. This happens because for this solution $L'(0) = L'(\rho_m)$. In general, for solutions with two 3-branes (type 3), we have a spherical topology with a deficit angle induced at each end. Hence the topology here is very close to that of the "football-shape solution" that we will mention in the next chapter [5, 11, 33], except not necessarily symmetric with respect to a certain radius ρ and not necessarily having equal deficit angles at both ends. The static unwarped solution (football-shaped) necessitates a magnetic field, that we have not included in the present model.

3.3.3 4-brane at $\rho = \rho_m$

Now if we want to cut the space at some radius ρ_m by introducing a 4-brane at that point, (as we have seen is generally necessary for type 1 and type 2 solutions) things are different than for a 3-brane. We do not have $L(\rho_m) = 0$, so $M'(\rho_m)$ can be anything, and the Einstein equations for this situation are, in the vincinity of ρ_m :

$$\left(\frac{M'}{M}\right)^2 + \frac{M''}{M} + \frac{M'}{M}\frac{L'}{L} + \frac{L''}{3L} - \frac{H^2}{M^2} = -\frac{\kappa_6^2}{3}\left(\delta\left(\rho - \rho_m\right) T_0^0\big|_{4-brane} + \Lambda\right) \tag{3.57}$$

$$\left(\frac{M'}{M}\right)^2 + \frac{2}{3}\frac{M'}{M}\frac{L'}{L} - \frac{H^2}{M^2} = -\frac{\kappa_6^2 \Lambda}{6} \tag{3.58}$$

$$\left(\frac{M'}{M}\right)^2 + \frac{2}{3}\frac{M''}{M} - \frac{H^2}{M^2} = -\frac{\kappa_6^2}{6}\left(\delta\left(\rho - \rho_m\right) T_\theta^\theta\big|_{4-brane} + \Lambda\right) \tag{3.59}$$

We now impose the Z_2 symmetry at the 4-brane, that is,

$$M(\rho_m - \epsilon) = M(\rho_m + \epsilon) \tag{3.60}$$

Taking the directional derivative along the direction of the jump in a similar way as we did before, we calculate the jump to be:

$$\Delta M' = \vec{\nabla} M\Big|_{\rho_m+\epsilon} \cdot \hat{\rho} - \vec{\nabla} M\Big|_{\rho_m-\epsilon} \cdot \hat{\rho} \tag{3.61}$$

$$= \frac{dM}{d\rho}\Big|_{\rho_m+\epsilon} (-\hat{\rho}) \cdot \hat{\rho} - \frac{dM}{d\rho}\Big|_{\rho_m-\epsilon} \hat{\rho} \cdot \hat{\rho} \tag{3.62}$$

$$= 2M'(\rho_m). \tag{3.63}$$

So the delta functions on the right of eq..3.57-3.59 can be matched with the second derivative terms on the left to give:

$$\kappa_6^2 \, T^0_0 \big|_{4-brane} = 6 \, \frac{M'}{M}\bigg|_{\rho_m^-} + 2 \, \frac{L'}{L}\bigg|_{\rho_m^-} \tag{3.64}$$

$$\kappa_6^2 \, T^\theta_\theta \big|_{4-brane} = 8 \, \frac{M'}{M}\bigg|_{\rho_m^-} \tag{3.65}$$

Now if the stress-energy tensor of the four-brane was pure tension, this would mean that $T^0_0|_{4-brane} = T^\theta_\theta|_{4-brane}$, which implies

$$\frac{L'}{L}\bigg|_{\rho_m^-} = \frac{M'}{M}\bigg|_{\rho_m^-} \tag{3.66}$$

This will impose a condition on the location of the four-brane. To see that, first express the difference between equations 3.5 and 3.4 in the form

$$\frac{M''}{M} + 3\left(\frac{M'}{M}\right)^2 - 3\frac{M'}{M}\frac{L'}{L} - \frac{L''}{L} = \frac{\Lambda_4}{M^2} - \kappa_6^2\left(T^\theta_\theta - T^0_0\right)$$

which is the same as

$$\left(M^4 L\left(\frac{M'}{M} - \frac{L'}{L}\right)\right)' = M^2 L b^2 \hat{\Lambda}_4. \tag{3.67}$$

Now this can be easily integrated between $\rho = 0$ and ρ_m, giving

$$M^4 L\left(\frac{M'}{M} - \frac{L'}{L}\right)\bigg|_{\rho_m} - M^4 L\left(\frac{M'}{M} - \frac{L'}{L}\right)\bigg|_{\rho=0} = \int_0^{\rho_m} M^2 L \Lambda_{ph} \tag{3.68}$$

So far, this equation is completely general since it only relies on Einstein's equation. Now if there is a three-brane at $\rho = 0$, the boundary conditions at this point, $L(0) = M'(0) = 0$ reduces it to

$$M^4 L\left(\frac{M'}{M} - \frac{L'}{L}\right)\bigg|_{\rho_m} + M^4 L'\big|_{\rho=0} = \int_0^{\rho_m} M^2 L \Lambda_{ph} \tag{3.69}$$

Furthermore, if a pure tension 4-brane is introduced at ρ_m, we get

$$M^4 L'\big|_{\rho=0} = \int_0^{\rho_m} M^2 L \Lambda_{ph}, \tag{3.70}$$

which is the promised condition on the location of that (pure tension) 4-brane.

We can apply this condition to the few analytic solutions that we have mentioned in the last section. For solutions of type 2, equations (3.33) and (3.35), condition (3.70) gives that the pure tension brane would have to be placed at $\rho \to \infty$, which is obviously not what is desired. For type 1 solution (3.32), the conditions force Λ_6 to be 0, Hence in all those examples it is not possible to insert a pure tension four-brane at a finite radius. One must add some additional energy on it, to force $T^0_0|_{4-brane} \neq T^\theta_\theta\big|_{4-brane}$. This will generally be the case, unless one finds a solution for which condition (3.70) can be satisfied. This additional source of energy must be anisotropic, and one way to obtain that is to "smear" a 3-brane along the compact dimension of the 4-brane[32]. This will give an extra contribution only to the T^μ_μ components, because the 3-brane will have $T^\theta_\theta = 0$. Another way to get a deviation from $T^0_0|_{4-brane} = T^\theta_\theta\big|_{4-brane}$ is through the Casimir energy (vacuum quantum effect) of a massless field confined to the 4-brane [8].

Summarizing, for solutions of type 1 and 2 we have a disk topology with a three-brane at the center, whose tension induces a conical singularity and deficit angle; at the boundary of the disk is a four-brane, generally provided with an anisotropic source of energy, like the tension of a smeared three-brane or the Casimir energy of a massless field. For some solutions, it might be possible to have a pure tension four-brane, but in such a case the location of that brane is fixed by the boundary conditions.

To conclude this chapter, solutions to a general model of six-dimensional gravity with a 3-brane and cylindrically symmetric bulk can be classified, depending on the signs of the full six-dimensional and effective four-dimensional cosmological constants, under two different topologies. Solutions of type 3, with $0 < \frac{3}{10}\hat{\Lambda}_6 \leq \hat{\Lambda}_4$, have a spherical topology, with two three-branes at the poles, one of which is our four-dimensional universe. The tensions of those three-branes induce conical singularities with deficit angles. For other values of the cosmological constants, we have the disk topology as explained above, with a warp factor $M(\rho)$ *increasing* away from our brane for type 2 solutions ($\hat{\Lambda}_6 \leq 0$, except when $2\hat{\Lambda}_4 \leq \hat{\Lambda}_6 \leq 0$), and *decreasing* for type 1 ($0 \leq \hat{\Lambda}_4 < \frac{3}{10}\hat{\Lambda}_6$, $\hat{\Lambda}_4 \leq 0 \leq \hat{\Lambda}_6$, or $2\hat{\Lambda}_4 < \hat{\Lambda}_6 < 0$). In our general model, unwarped solutions exist when $\hat{\Lambda}_6 = 2\hat{\Lambda}_4$, for any sign of $\hat{\Lambda}_6$. Maybe the most important feature of models with codimension-two branes remains the fact, already known for some time [41, 9, 30], that if one introduces a 3-brane at the center of a six-dimensional bulk with cylindrically symmetric background solution, the non-zero brane tension induces a conical singularity (or a deficit angle) in the transverse space. This particular way in which the 2D manifold's internal geometry responds to the tension of a codimension-two brane will lead to unexpected results. The four-dimensional effective cosmological constant will in some cases be independent of this tension; in other instances it may depend on the tension in an unusual way. In general one needs to examine the stability of a solution to confirm the dependence of the four-dimensional cosmological constant on the brane tension. This and other extensions of the general model presented above will be looked at in the next chapter.

Chapter 4

Other Six-Dimensional Solutions

In the previous chapter, I classified and described in some detail solutions to a general model with a codimension-two brane. These were the solutions I have been most involved with in the course of my research. Nevertheless it would be beneficial to the reader to be aware that other types of solutions exist, with different setups. I will therefore mention in this chapter alternatives and additions to the solutions presented above. It should be noted that the goal here is rather one of covering a wide range of possibilities rather than exploring any one very deeply. Therefore, I do not intent to fill out all the details, but rather to direct the reader towards appropriate sources by describing what has been done.

4.1 Other solutions with codimension-two branes

We first wish to complete the classification of solutions that was done in the last section by mentioning additional work that has been done on those solutions, and existing solutions that are very close to the ones just described.

4.1.1 Stabilization of previous solutions

Stabilization of some solutions of the last chapter (namely the AdS soliton and the unwarped case) was considered in [11], by adding either a scalar field or a magnetic flux. Stable solutions are those for which the size of the extra dimensions is determined by the dynamics of the model, so to make it stable one generally adds a field whose potential will have a minimum when the extra dimension has a certain size.

In this paper, we first studied the stabilization of the warped solution corresponding to the AdS soliton model (3.33), since it was known that the radial size of the extra dimension is unstable in the AdS soliton model, at least for the static case [4]. A scalar field was used for that stabilization, first concentrating on the static solution $\Lambda_4 = 0$. Λ_4 is subsequently treated as a perturbation. This is reasonable because the present universe has an energy density (corresponding to Λ_4 here) which is much less than that corresponding to the Planck scale. So once the solution for the scalar field is found using the static solution for the metric, the back reaction of the scalar on the metric is calculated, next we obtain the stabilized size of the radial extra dimension ρ_m, and only then is the Λ_4 perturbation considered. That led to a surprising result: the dependence of the rate of expansion on the tension of the standard model 3-brane (the relation between Λ_4 and $\delta\tau_3$) is not what we would have expected. A *decrease* in the energy density (or tension) on the 3-brane leads to cosmological *expansion* in this model:

$$\Lambda_4 = -\frac{3k^2}{10\pi}\frac{\delta\tau_3}{1-\frac{\tau_3}{2\pi}}\cosh^{-6/5}(k\rho_m). \tag{4.1}$$

Hence the Friedmann equation $H^2 \sim \rho$ of conventional four-dimensional cosmology cannot hold in this case and must somehow be modified. This result comes from the difference of the two jump conditions at the four-brane,

$$\frac{L'}{L} - \frac{M'}{M} = \frac{\alpha\tau'}{2L^\alpha}, \tag{4.2}$$

where α determines the equation of state of the energy density of the 4-brane source τ'. Substituting into this the result of the Λ_4 perturbation to bulk solution $\Delta\left(\frac{L'}{L} - \frac{M'}{M}\right)$ combined with the zeroth order relation for $\frac{L'}{L} - \frac{M'}{M}$, we can deduce equation (4.1) and even a more general form of it, for arbitrary values of α (see ref.[11]).

In the same paper, stabilization of unwarped solutions is also analyzed. Stable unwarped solutions were also recently studied by Carroll and Guica [5] and Navarro [33]. They are similar in topology to solutions of type 3 described in the previous chapter, except that in order for the static case to be stable, it requires a nonvanishing magnetic flux $F_{\rho\theta}$. The magnetic field strength satisfies the Maxwell equation $\partial_A\left(\sqrt{|G|}F^{AB}\right) = 0$, and adds a term to the Einstein equations that now take the form:

$$\frac{3H^2}{M^2} - 3\left(\frac{M'}{M}\right)^2 - 3\frac{M''}{M} - 3\frac{M'}{M}\frac{L'}{L} - \frac{L''}{L} = \kappa_6^2(T_0^0 + \Lambda) + \frac{\beta^2}{2M^8} \tag{4.3}$$

$$\frac{6H^2}{M^2} - 6\left(\frac{M'}{M}\right)^2 - 4\frac{M'}{M}\frac{L'}{L} = \kappa_6^2(T_\rho^\rho + \Lambda) - \frac{\beta^2}{2M^8} \tag{4.4}$$

$$\frac{6H^2}{M^2} - 6\left(\frac{M'}{M}\right)^2 - 4\frac{M''}{M} = \kappa_6^2(T_\theta^\theta + \Lambda) - \frac{\beta^2}{2M^8} \tag{4.5}$$

where β is related to the magnetic field strength by $F_{\rho\theta} = \beta L/M^4$. In our usual units where $M_0 = 1$, the static solution takes the form:

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu + d\rho^2 + R\sin(k\rho)d\theta^2;\ k^2 = \frac{1}{2}\hat{\Lambda}_6 + \frac{1}{4}\beta^2 \tag{4.6}$$

[5] and [33] use sightly different coordinates but have similar solutions. The relation between the different parameters for the general unwarped solution with magnetic flux is

$$\Lambda_4 = \frac{1}{2}\Lambda_6 - \frac{1}{4}\beta^2 \tag{4.7}$$

so that we get $\frac{1}{2}\Lambda_6 = \frac{1}{4}\beta^2$ for the static case. In the last chapter we had $\beta = 0$, so that this relation (same as equation 3.12) forces the bulk cosmological constant to vanish if Λ_4 does so. Another characteristic of the static solution is that the two branes have equal tensions, or equivalently induce the same deficit angle.

As we have already outlined in the last chapter, a remarkable feature of these solutions with spherical topology (two 3-branes) is that the brane geometry (e.g. the rate of expansion of the universe) is insensitive to the value of the brane tension, whose only effect is to induce a deficit angle in the bulk. It was primarily to verify that this effect was not associated with a massless radion (typical of unstable solutions) that a thorough analysis of the stability of these solutions was performed in [11]. The fact that they are indeed stable makes this feature even more surprising. And since the tension comes from the energy of the brane fields, but does not induce cosmological expansion on the brane, the conventional Friedmann equation that links the field energy density to the cosmological expansion cannot hold in its original form. Also, the fact that this brane tension represents what an observer on the brane would calculate as the vacuum energy moves the cosmological constant problem completely into the extra dimensions, since it does not affect the rate of expansion. Of course the problem is not completely resolved, as there is still the need to tune the bulk magnetic field against the bulk cosmological constant, but the cosmological constant problem is sufficiently difficult that transforming it into a different problem is a worthy endeavor, since the new formulation might suggest new solutions. One hope would be to use bulk supersymmetry to ensure $\Lambda_4 = 0$. If Λ_4 is independent of the brane tension, it will be insensitive to the quantum corrections to the vacuum energy that usually give the large, irreconcilable with observations, contribution to the theoretical value of the cosmological constant.

4.1.2 Other Models with 3-Branes

There is another solution studied in [33], and it is again very close to the ones we have been discussed in chapter 3. It has a disc topology, like the solutions of type 1 and 2, but the author simply obtains it by taking only half of the sphere, i.e. identifying points in the northern and southern hemispheres that are symmetric under reflection through the equatorial plane., so that ρ (he uses θ for that coordinate) ranges from 0 to $\pi/2$. For his particular solution, the first derivative of the metric warp factor are zero at the orbifold fixed points ($\rho = \pi/2$), and the matching conditions are satisfied trivially, so that there is no need for adding a 4-brane carrying any energy at this position.

Disc topology solutions are discussed in many other papers. The original AdS soliton spacetime has been first studied by Horowitz and Myers [26]. It is a double analytic continuation of a planar AdS black hole metric, and as we can expect it involves two compact dimensions having the topology of a disc with a conical singularity at its center. It has been applied to the braneworld idea by Leblond et al. [32] and Burgess et al.[4]. The metric that they use is

$$ds^2 = a(r)\left[\eta_{\mu\nu}dx^\mu dx^\nu\right] + a(r)f(r)d\theta^2 + a^{-1}(r)f^{-1}(r)dr^2. \tag{4.8}$$

The setup and particularities of the model described in those paper is pretty much identical to those of the AdS soliton solution of chapter 3 (equation 3.33). As for the solution we have seen, the boundary of the disc occurs at a four-brane placed at a certain radius $r = R$ and a 3-brane is placed at the conical defect. The geometry is everywhere smooth and nonsingular including at the location of the three brane, where the circle parametrized by θ smoothly shrinks to a point and the internal space ends. As usual, the 3-brane induces a conical defect of size $\delta = \kappa_6^2\tau_3$ at this point. In agreement with the discussion of section 3.3.3, the stress-energy of the 4-brane requires an anisotropic form which could arise from the smearing of 3-branes around the 4-brane or from Casimir energy of light particles confined to the four-brane. The way the hierarchy problem is resolved in this model is through a combination of warping and having a large extra dimension. The 3-brane has a TeV energy scale, while the four-brane is the Planck brane. [4] provides a complete accounting of the metric modes. This model is also closely related to that of [9]. In [11], stabilization of the soliton is studied, an expanding soliton model is also considered and it is found that the model leads to a nonstandard Friedmann equation. As we have seen in the last section, a decrease in the energy density on the 3-brane leads to cosmological expansion.

We have already discussed in the last section solutions with two three-branes found in the literature, like those in [5] and [33]. Another family of solutions, encountered in [10], is closely related to those classified in the last chapter. It has two 3-branes with different tensions embedded in a de Sitter bulk. The topology is again that of a sphere with a wedge cut out, but the metric,

$$ds^2 = e^{2\lambda(x^i)}[g_{\mu\nu}dx^\mu dx^\nu + e^{2K(x^i)}dx^i dx^j],$$

does not allow for solutions with flat 4D spacetime. This can be understood in terms of the limitations of this particular metric: there is no room for having different warp factors for radial and angular extra coordinates, contrary to the metric discussed in section 4.1.1 which has flat 4D spacetime. Despite this fact, Chodos and Poppitz show that it is always possible to tune the parameters of the model so that the solution agrees with observation, a fine-tuning that corresponds to the cosmological constant problem. In the language of chapter 3, Chodos and Poppitz's solution belongs to the category of type 3 solutions, with both the bulk and the metric induced on the brane being De Sitter. Their model encompasses a particular solution with constant 4D metric warp factor (unwarped solution), but otherwise, that warp factor generally depends on the extra dimensional coordinates, and no other specific examples of solutions are given, the author concentrating on parameter space analysis. The deficit angles characteristic of point-like sources in 2 dimensions are also present in their model. Many of the characteristics just mentioned demonstrate a similarity of the model with our, except for the extra limitations imposed on the metric, as discussed before. Also, this solution is just one six-dimensional example of their more general constructions in d+2 dimensions with d-1 branes.

Finally, I will mention one last model which uses three-branes but in a particular setup. In [41], Sundrum considers a scenario where several parallel 3-branes are present. The Standard Model particles are confined to live on one of the 3-branes while different four-dimensional field theories may inhabit the others. These parallel, four-dimensional sub-universes interact weakly with each other via the bulk six-dimensional gravity, so that they can be considered as hidden sectors relative to each other. As in the other models we have seen thus far, each 3-brane induces a conical geometry in the two dimensions transverse to it. Collectively, they act as sources for the curvature needed to compactify the extra dimensions into a space of spherical topology. Basically once the author has shown that a single three-brane induces a conical geometry in the two transverse dimensions, he proceeds by patching together the cones from several three-branes. The bulk spacetime is the product of the compact two-dimensional manifold $\mathcal{M}_2$ with four-dimensional Minkowski space. The conical singularity are the only source of curvature for $\mathcal{M}_2$, and the deficit angles must add up to 4π in order to yield a surface of spherical topology. A simple example for visualizing that is the tetrahedron, with the four vertices corresponding to the positions of four 3-branes. The effective field theory is then constructed, introducing an abelian gauge field with a non-trivial magnetic flux through the compact space for stabilization. In fact, the size of the compact space is determined by the balance between two effects: the potential due to a small bulk cosmological constant, and that due to the flux density. The $U(1)$ gauge field has non-zero magnetic flux through the closed two-dimensional surface $\mathcal{M}_2$. This flux is fixed, and so as the area of the surface ($\mathcal{A}$) increases, the flux density decreases, leading to a *lower* magnetic energy density. On the other hand, the potential due to the bulk cosmological constant Λ_6 *increases* as the compactified area $\mathcal{A}$ increases:

$$\delta S_\Lambda = -\int d^4x\sqrt{-g}\mathcal{A}\Lambda_6. \tag{4.9}$$

It is the competition between those two opposing forces which will stabilize the model. The resulting theory is consistent with all experimental Standard Model and gravitational tests. In this model, it is possible to solve the hierarchy problem as in ADD, by taking the 6D fundamental scale to be close to the weak scale, and the size of the compact space not much smaller than a millimeter. About the cosmological constant problem, he notes the now familiar result, that before compactification the problem disappears into the extra dimensions, in the sense that Λ_4 can vanish without any need to fine-tune the 3-brane tension, since the only effect of the latter is to induce a conical singularity in the extra dimension.

4.1.3 Topological Defect Approach

So far we have taken a phenomenological approach to the braneworld scenario, in the sense that we have considered the brane as a fundamental object, modelled as a delta function source, without worrying about the origin of this brane. As we said in section

2.4, one can also use a field theoretical approach to the problem, by not only assuming some distribution for the energy-momentum tensor of the brane, but explicitly including in the total action of the system the contribution of the field(s) responsible for creating the topological defect (the vortex, or string, in our case). In the Abelian-Higgs model, for example, a string defect arises naturally. Giovannini, Meyer and Shaposhnikov [21] considered essentially the same general setup with the same ansatz as we did in the last chapter (but only in the static case $\Lambda_4 = 0$), except that they added those field theoretical source terms causing the defect.

Gherghetta and Shaposhnikov[19] had already added arbitrary source terms to the setup, similar to ours, of [40], but not in the context of a proper field theoretical model justifying those terms. They had shown that a thin (local) string could lead to localization of gravity if certain relations between the tension components were satisfied. While [18] generalizes these results to an arbitrary number of dimensions, in [21] a field theoretical realization of this idea is proposed. Both [19] and [21] examine the case of static four-dimensional spacetime ($\Lambda_4 = 0$) in a bulk with negative cosmological constant $\Lambda_6 < 0$. We will examine more closely the model described in these two papers, first because it is very close to that discussed in Chapter 3, and in order to present the other approach to brane-world scenario, that which regards our universe submanifold as a topological defect created by some scalar fields.

Concretely, Giovannini, Meyer and Shaposhnikov use the action of a gravitating Abelian Higgs model in six dimensions: added to the six-dimensional gravity bulk action S_{grav} (equation 3.2) is a gauge-Higgs action describing the brane,

$$S_{brane} = \int d^6x\sqrt{-g}\mathcal{L}_{brane}, \; \mathcal{L}_{brane} = \frac{1}{2}(\mathcal{D}_A\phi)^*\mathcal{D}^A\phi - \frac{1}{4}F_{AB}F^{AB} - \frac{\lambda}{4}(\phi^*\phi - v^2)^2 \quad (4.10)$$

In this equation, $\mathcal{D}_A = \nabla_A - ieA_A$ is the gauge covariant derivative while ∇_A is the generally covariant derivative, and v is the vacuum expectation value of the Higgs field determining the masses of the Higgs and of the gauge boson

$$m_H = \sqrt{2\lambda}v, \; m_V = ev. \quad (4.11)$$

Because of the gauge field, the vortex (or string) arising in this particular model is a gauged (or localized) defect. In fact the solution to this action is a version of the Nielsen-Olesen vortex, or string [35]. As we said in an earlier section, a gauged string is a better candidate for getting localized gravity than a global string because the energy density of the latter falls off too slowly away (like $\frac{1}{r}$) to be considered localized.

The metric ansatz is as before,

$$ds^2 = g_{AB}dx^A dx^B = M(\rho)^2 g_{\mu\nu}dx^\mu dx^\nu + d\rho^2 + L(\rho)^2 d\theta^2, \quad (4.12)$$

while the Nielsen-Olesen ansatz for the gauge-Higgs system reads:

$$\begin{aligned} \phi(\rho, \theta) &= vf(\rho)e^{in\theta}, \\ A_\theta(\rho, \theta) &= \frac{1}{e}[n - P(\rho)], \end{aligned} \quad (4.13)$$

where n is the winding number. In [21], they concentrate mostly on the case $n = 1$. One can then compute the equations of motion with the ansatz:

$$f'' + (4m + l)\, f' + \frac{m_H^2}{2}(1 - f^2)f - \frac{P^2}{L^2}f = 0 \tag{4.14a}$$

$$P'' + (4m - l)\, P' - m_V^2 f^2 P = 0 \tag{4.14b}$$

$$l' + 3m' + l^2 + 6m^2 + 3lm = -\kappa^2\Lambda - \frac{m_H^4}{4\lambda}\kappa^2 T_0^0 + \kappa^2\frac{M_6^4}{M_P^2}\frac{\Lambda_4}{M^2} \tag{4.14c}$$

$$4m' + 10m^2 = -\kappa^2\Lambda - \frac{m_H^4}{4\lambda}\kappa^2 T_\theta^\theta + 2\cdot\kappa^2\frac{M_6^4}{M_P^2}\frac{\Lambda_4}{M^2} \tag{4.14d}$$

$$2lm + 3m^2 = -\frac{\kappa^2\Lambda}{2} - \frac{m_H^4}{4\lambda}\frac{\kappa^2}{2}T_\rho^\rho + \kappa^2\frac{M_6^4}{M_P^2}\frac{\Lambda_4}{M^2} \tag{4.14e}$$

where the prime denotes the derivative with respect to the rescaled variable $x = \frac{m_H\rho}{\sqrt{2}}$, and the functions $m(x)$ and $l(x)$ are simply

$$m(x) = \frac{M'(x)}{M(x)}, \quad l(x) = \frac{L'(x)}{L(x)}. \tag{4.15}$$

The difference with [19] resides in the two first equations, (4.14a) and (4.14b), which constraint the source fields. Both papers thus find the same bulk solution (away from the defect):

$$ds^2 = e^{-c\rho}g_{\mu\nu}dx^\mu dx^\nu + d\rho^2 + L_0^2 e^{-c\rho}d\theta^2, \quad c = \sqrt{\frac{2}{5}\frac{-\Lambda_6}{M_6^4}} \tag{4.16}$$

Giovannini *et al.* actually present a more general solution in the form

$$M(x) = M_0 e^{-kx}\left|1 + \epsilon e^{5kx}\right|^{\frac{2}{5}}, \tag{4.17a}$$

$$L(x) = L_0 e^{-kx}\frac{\left|\epsilon e^{5kx} - 1\right|}{\left|1 + \epsilon e^{5kx}\right|^{\frac{3}{5}}}, \; k = \sqrt{-\frac{\hat{\Lambda}_6}{5m_H}}, \tag{4.17b}$$

but only the case $\epsilon = 0$ will lead to localization of gravity and everywhere regular geometry. In the cases $\epsilon > 0$ or $\epsilon \leq -1$, gravity is not localized, while for $-1 < \epsilon < 0$, the geometry is singular at some point. This discussion on the properties of six-dimensional cylindrically symmetric warped geometry in the absence of defects corresponds to what we did in section 3.2, restricted to the case $\Lambda_4 = 0$, $\Lambda_6 \leq 0$.[1]

In [19], applying the boundary conditions to the solution (4.16) directly results in a condition on string tension per unit length components, μ_0 and μ_θ:

$$\mu_0 = \mu_\theta + A^2 M_6^2 \tag{4.18}$$

So with this solution, there is no tuning between the cosmological constant and the brane tension, but there is some tuning between components of the brane tension. Giovannini *et al.* actually find the same condition (with A now related to the parameters λ and m_H), which is for them a particular case (for $\epsilon = 0$) of a more general condition for generic ϵ. In their case, the tension components are directly related to the scalar and vector fields. Also, when the defect is introduced at the origin, ϵ will now relate to the parameters of the model, m_H, m_V, λ and Λ_6, that will be constrained by the requirement $\epsilon = 0$. The authors use these conditions and others coming from the boundary conditions at the origin and at infinity[2] to get numerical solutions and scan the parameter space for sets of parameters that satisfy all the requirements (boundary conditions obeyed, regular

[1] For example, note that for $\epsilon = 1$, this is exactly the AdS soliton solution of equation (3.33). On the other hand, the metric solution they focus on ($\epsilon = 0$) does not fit into the classification of bulk solutions of Section 3.2, simply because their bulk solution does not obey the condition that $L(0) = 0$, condition that we imposed to all the bulk solutions entering the classification.

[2] In addition to the same boundary conditions for $M(\rho)$ and $L(\rho)$ as we had in the last chapter, we now have boundary conditions for the scalar and gauge fields, that insure that they describe a string-like defect. We demand that the scalar field reaches, for large ρ, its vacuum expectation value, i.e. $|\phi(\rho)| \to v$

geometry and localization of gravity).

One important requirement is the localization of gravity, which is equivalent to demanding that the four-dimensional Planck mass is finite:

$$M_P^2 = \frac{4\pi M_6^4}{\sqrt{2}m_H} \int dx M^2(x) L(x) < \infty. \tag{4.21}$$

This is necessary for the world to look 4-dimensional to an observer on the brane. In [19], Gherghetta and Shaposhnikov show Newton's law is not affected significantly by considering the equations of motion for the linearized metric fluctuations, i.e. for $h_{\mu\nu}$ defined by:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) . \tag{4.22}$$

The zero mode tensor fluctuation (corresponding to the massless graviton) is localized near the origin $\rho = 0$ and is normalizable. Moreover, although the contribution from the non-zero modes modify Newton's law on the 3-brane, this correction actually grows like $1/r^3$. It is more suppressed than in five dimensions because now the gravitational field of the bulk continuum modes spreads out in one extra dimension and so their effect is weaker.

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for $\rho \to \infty$. In the same limit,the magnetic field should go to zero. Also, close to the core both fields should be regular. That gives:

$$f(0) = 0, \qquad P(0) = n, \tag{4.19}$$

$$\lim_{\rho\to\infty} f(\rho) = 1, \qquad \lim_{\rho\to\infty} P(\rho) = 0. \tag{4.20}$$

Finally, in this model it is not possible to solve hierarchy in the manner of RS1 [37], using the sole effect of the warping, because the extra-dimensional space is infinite. However, it is possible to obtain $M_6 \ll M_{Pl}$ by adjusting the brane tension or the bulk cosmological constant, solving the hierarchy more along the lines of ADD [1].

4.2 Other Types of Setup

We now review and discuss different possibilities that have been left out in the previous analysis and that are found in the literature, like models with four-branes only, models with intersections of branes and models with our three-brane embedded in a four-brane. Because the five-dimensional case had been so well studied, the easiest models to work with seemed to be ones with codimension-one branes, that are in close parallel with the five-dimensional case. I will first describe a few of these models. A possible path in working toward a codimension-two brane is to use the well known codimension-one branes and study the intersection of two of these. That is the type of model that will be reviewed next. I will then discuss an interesting type of setup, which uses both features of ADD-like and Randall-Sundrum-like scenarios, trying to take the best of both worlds by embedding a 3-brane in a four-brane. The natural continuation of that whole series of models is the use of genuine 3-branes, which is the case we have been discussing up to now.

4.2.1 Codimension-one Brane

In [28], P. Kanti, R. Madden and K. A. Olive derive both static and non-static solutions for a six-dimensional bulk with a cosmological constant (the gravitational action is that of equation 3.2), starting from the ansatz:

$$ds^2 = a^2(t,\theta,\phi)\eta_{\mu\nu}dx^\mu dx^\nu + b^2[d\theta^2 + f^2(\theta)d\phi]. \tag{4.23}$$

Both extra dimensions are initially non-compact; one of the two coordinates (ϕ) is quite similar to the fifth dimension in the Randall-Sundrum scenario. The size of this dimension is made finite by introducing 2 branes (4-branes here, 3-branes in RS1) at two different points along its coordinate. Periodic boundary conditions are imposed at those two points ($\phi = 0$ and $\phi = L$), and the inter-brane distance defines the size of this extra dimension. This setup already has many similarities with the Randall-Sundrum scenario [37], and if we look at the static solution,

$$f(\theta) = \frac{1}{4\lambda}\left[e^{\pm\lambda(\theta-\theta_0)} + 4\omega^2 e^{\mp\lambda(\theta-\theta_0)}\right];\ \lambda^2 = -\frac{\kappa_6^2 b_0^2}{10}\Lambda_6, \tag{4.24}$$

$$a(\theta,\phi) = a_0 e^{\pm\omega\phi} f(\theta), \tag{4.25}$$

we see that the dependence on the Randall-Sundrum-like coordinate, ϕ, is a purely exponential one, like the warp factor of the Randall-Sundrum model. This solution also has the same fine-tuning requirement between the two brane tensions as in the RS model: they have to be equal and opposite in order to be consistent with the boundary conditions. However, the second RS fine-tuning between each brane tension and the bulk cosmological constant is absent. Instead, there is a new fundamental parameter, the size of the longitudinal (θ) dimension, which is fixed in terms of the brane tensions. The size of the transverse ϕ-dimension, on the other hand, remains a free parameter. Note that in the case of a single-brane configuration, we simply send the second brane to infinite distance from the first, just like Randall and Sundrum did [38].

The most promising difference with respect to the five-dimensional case occurs in the non-static case. In this case, the fine-tuning between the two brane tensions is relaxed, rendering the model free of any fine-tuning. In return, the locations of the two branes are then fixed: they are determined by the jump conditions in terms of the two brane tensions. The solution for the non-static case is still given by equation 4.24 for the metric function $f(\theta)$ since the bulk equation of motion for the function remains unchanged. As for the warp factor, however, its time-dependence radically changes the solution for the ϕ-dependent part. It is now:

$$a(\theta, \phi, t) = \frac{1}{H(t-t_0)}\Phi(\phi)f(\theta), \tag{4.26}$$

$$\text{with } \Phi(\phi) = \begin{cases} \frac{bH}{\omega}\sinh\left[\omega\left|\phi-\phi_0\right|\right] & \text{if} \quad H \text{ is real} \\ \frac{b\,\mathrm{Im}(H)}{\omega}\cosh\left[\omega\left|\phi-\phi_0\right|\right] & \text{if} \quad H \text{ is imaginary} \end{cases} \tag{4.27}$$

The first case, H real, corresponds to a de Sitter four-dimensional submanifold, while the second, H imaginary, describes an anti-de Sitter submanifold. The time coordinate t above is the conformal time. Since the tuning between the two brane tensions is now relaxed, the non-static case can accommodate pairs of positive-positive tension branes. This is another advantageous difference relative to the 5D case. But in fact, the solution that corresponds to two branes with positive tension, the first case of equation (4.27), proves to be unstable under small time-dependent perturbations. The second case of equation (4.27), for its part, is stable but corresponds to two branes with negative tensions.

In another paper, Z. Chacko and A. E. Nelson build from a setup with two four-branes a slightly different model. Again they assume that the only source in the bulk is a cosmological constant, so the action is the same as usual. The metric ansatz is also quite similar to the one we used in Chapter 3:

$$ds^2 = f(r)\eta_{\mu\nu}dx^\mu dx^\nu + dr^2 + s(r)d\theta^2, \tag{4.28}$$

θ being the compact coordinate, that runs from 0 to 2π, and r the radial coordinate. The only difference with the metric we used before, apart from the choice of labels for the warp factors, is the restriction to the static case.[3] However, the geometric setup, and thus the boundary conditions, will be different. Here the geometry of the model consists of two four-branes localized at different radii, $r = a$ and $r = b$. The branes can be viewed as the surfaces of two infinitely long concentric cylinders in the higher dimensional space, with the regular four dimensions parallel to the common axis of the cylinders, the fifth dimension going around the surface but perpendicular to the axis, and the sixth dimension being the radius. A particularity of this model is that they do not assume that the bulk cosmological constant has to be the same for every region of the space-time. The rationale for doing this is that the branes, being topological defects, might be separating different phases of the theory. Hence a priori the cosmological constant takes three values: α_1

[3]It should be noted that the general bulk solutions which are the starting point for the model elaborated in this paper are coordinate transformations of the bulk solutions found in ref. [10].

when $0 < r < a$, α_2 when $a < r < b$ and α_3 when $r > b$. Note that our universe does not lie at the origin in this model, but on the four-brane located at $r = a$. The singularity at the origin is simply smoothed out. Now the form of the solutions in the three regions are, explicitly,

$$\text{For } \; r < a \; : \; \begin{aligned} f &= f_1 e^{\alpha_1 r}\left[1 + e^{-\frac{5}{2}\alpha_1 r}\right]^{\frac{4}{5}}, \\ s &= s_1 e^{\alpha_1 r}\left[1 + e^{-\frac{5}{2}\alpha_1 r}\right]^{-\frac{6}{5}}\left[1 - e^{-\frac{5}{2}\alpha_1 r}\right]^{2}; \end{aligned} \tag{4.29}$$

$$\text{For } \; a < r < b \; : \; \begin{aligned} f &= f_2 e^{\alpha_2 r}\left[1 - c_2 e^{-\frac{5}{2}\alpha_2 r}\right]^{\frac{4}{5}}, \\ s &= s_2 e^{\alpha_2 r}\left[1 - c_2 e^{-\frac{5}{2}\alpha_2 r}\right]^{-\frac{6}{5}}\left[1 + c_2 e^{-\frac{5}{2}\alpha_2 r}\right]^{2}; \end{aligned} \tag{4.30}$$

$$\text{For } \; r > b \; : \; \begin{aligned} f &= f_3 e^{-\alpha_3 r}, \\ s &= s_3 e^{-\alpha_3 r}. \end{aligned} \tag{4.31}$$

The constants in these formulae need to be determined through boundary conditions, i.e. by matching the solutions across the branes.

In this model the hierarchy can be resolved by the enormous difference in the warp factors at the positions of the two branes. Gravity is mostly localized on the outer brane while the standard model lives on the inner brane, explaining the apparent weakness of gravity in our world. Now the resolution of the hierarchy problem depends on the positions of the branes, and it is important that these positions be fixed. This is done by a combination of two effects comparable in size: the anisotropic contribution of quantum fields to the stress tensor of each brane, and a scalar bulk field, using the Goldberger-Wise mechanism [22]. Together those two effects yield a large enough value of brane spacing to solve hierarchy without much fine-tuning.

In a little more detail, the first effect, the anisotropic contribution coming from quantum effects of fields, arises when we consider the stress-energy tensor for the four-branes in order to match the solutions at the branes. Because of four-dimensional Lorentz invariance, the stress tensor is of the form:[4]

$$\kappa^2 T_{AB} = - \begin{pmatrix} \beta^2 f \eta_{\mu\nu} & \\ & \gamma^2 s \end{pmatrix} . \tag{4.32}$$

Now this tensor comprises a contribution from the cosmological constant on the brane, and the rest comes from the matter Lagrangian, more precisely the expectation value of the stress tensor of the matter fields living on the brane. If only the cosmological constant was contributing, we would have $\beta^2 = \gamma^2$. Now the deviation from that is the Casimir effect arising from the vacuum energy of quantum fields, which we have mentioned before. They calculate that the difference between T^n_m and T^5_5 is of order $\left(\frac{1}{a}\right)^5$, where a is the size of the compact dimension. It vanishes in the limit where the compact dimension becomes infinite.[5] Now the size of the compact dimension is in general an r-dependent function, determined by the Einstein equations, but we just saw that the brane tension depends on this size; as a result, the Einstein's equations fix the brane location. After this process of matching the solutions across the four-branes, with anisotropic contributions to their

[4]We are assuming the matter on the brane is in its ground state.

[5]The calculation is lenghty and will not be presented here, but the result can be understood intuitively this way: since the only counterterm allowed by general covariance is the cosmological constant which contributes equally to both T^n_m and T^5_5, the difference between these two must be finite and regulator independent in the limit that the cutoff is taken to infinity. For a massless field, a is the only available dimensionful parameter.

stress-energy tensor, the brane spacing is fixed, however to obtain a large hierarchy an exponentially precise fine-tuning of the parameter is still necessary. This is in addition to the fine-tuning necessary to set the four-dimensional cosmological constant to zero, a problem which is not addressed in this paper.

Now comes the other effect, the introduction of a Goldberger-Wise scalar field in the bulk, which will lessen the fine-tuning necessary for getting a large hierarchy. The metric and brane locations will then be completely determined. Also, we can see from the equation that determines the location of the outer brane b in terms of the mass of the scalar field m and the cosmological constant,

$$b = \mathcal{O}\left(\frac{\alpha}{m^2}\right), \ \alpha = \frac{\Lambda_6}{10M^4}, \tag{4.33}$$

that m need not be much smaller than α to get sizeable hierarchy. Exponentially precise fine-tuning is not necessary anymore. As before there is one fine-tuning left, which is that of the cosmological constant problem. Also, it should be noted that both branes in the theory have positive tensions. Finally, they examine physical implications of the model, showing that deviations from Newtonian gravity are highly suppressed at long distances.

I will mention one last model with two four-branes, that of I. I. Kogan et al. [30]. This paper is somewhat different than the two preceding ones, and we will not describe it in detail here, because its main interest is the realization of multigravity in six dimensions, which has to do mainly with field theoretical phenomenology of the model, and that is not the focus of the present thesis. In multigravity models, gravity at some scale is mediated by both the massless graviton and other state(s). This possibility was introduced in a five-dimensional model in [31].The motivation for the model was not different than others, (to find a brane world model in which we would live on a positive brane, thus avoiding the cosmological problems associated with the usual negative branes of five-dimensional model) but their particular model gave rise to an unusual possibility. Gravity (in all experimentally analyzed regions) may result from the exchange of the ordinary graviton plus an ultralight KK state (hence the name of "bi-gravity") and modifications of gravity may occur at both small and extremely large scales. This ultralight first KK state can be so light that the corresponding wavelength can be of order of 1% of the observable size of the Universe, while the second KK mode is in the submillimeter region. Only at scales larger than $10^{26}cm$ will the first KK mode decouple leading to a much smaller gravitational coupling beyond this length scale.

Coming back to the six-dimensional generalization, the setup of [30] consists of a double disc topology, bounded by two four-branes and with or without the presence of a conical singularity associated with a 3-brane in between (see figure 4.1). Gravity is localized on the four-branes, where the standard model lies. One of the four-branes' dimension is compact, unwarped and of Planck length, so in the low energy limit the spacetime on the branes appears three dimensional. The rationale for going to six-dimensional spacetime was that in five dimensions, all multigravity models had a ghost field (the radion associated with moving negative tension branes) The hope was thus to find a multigravity model of flat branes without ghost fields by adding another extra dimension, since the requirement of negative tension branes is relaxed in six dimensions. Now the construction

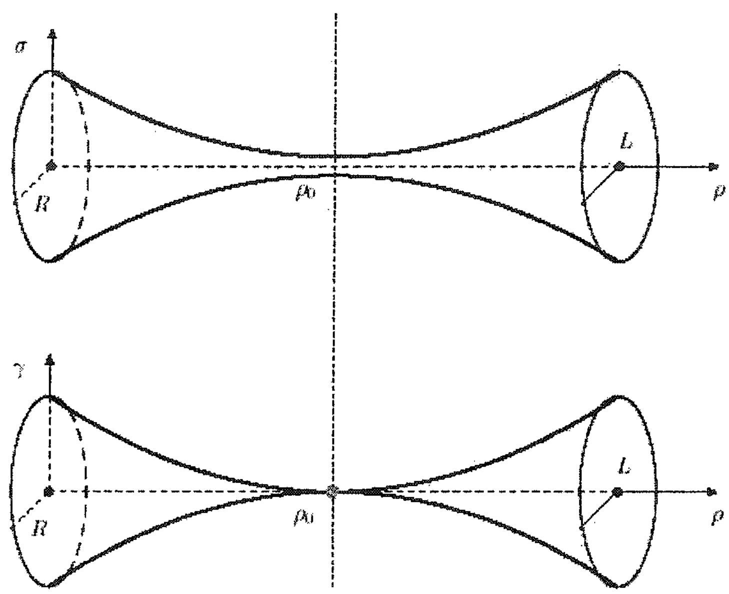

Figure 4.1: Warp factors $\sigma(\rho)$ and $\gamma(\rho)$ for Kogan *et al.* model of bi-gravity. The point ρ_0 corresponds to a conical singularity.

of this model is done by pasting two single four-brane solutions. The simple single four-brane solution that is their starting point is similar to that in [21] (see equations 4.17 for $\epsilon = 0$):

$$ds^2 = e^{-k\rho}\eta_{\mu\nu}dx^\mu dx^\nu + d\rho^2 + R^2 e^{-k\rho}d\theta^2, \; k^2 = -\frac{2}{5}\kappa_6^2\Lambda_6, \tag{4.34}$$

R is the radius of the four-brane. They then consider solutions that allow for a brane which is not necessary flat, with generic tension components. Ultimately, they construct two different models where two single four-brane solutions are pasted together, one model with a three-brane at the center, and one which is singularity-free. Those models are six-dimensional generalizations of the quasi-localized and crystalline models, both bi-gravity models.

4.2.2 Intersection of Codimension-one Branes

When Randall and Sundrum proposed their mechanism for localizing gravity on a three-brane in a locally AdS five-dimensional bulk, it seemed quite difficult to extend the idea to any number of dimensions, since it appeared to rely on the peculiar properties of co-dimension one objects in gravity. Arkani-Hamed *et al.* thus suggested that this obstacle can be overcome by considering n intersecting $(2+n)$ branes separating sections of the $4+n$ dimensional AdS space [2]. In this paper, the branes (two four-branes in six dimensions) intersect othogonally in a single 3-brane where gravity is localized. Hence in contrast with the models of last section, where Standard Model particles lived on a four-brane which appear to us as four-dimensional because the fifth dimension was small, in this model ordinary particles are truly confined to a four-dimensional "object", the intersection of the two four-branes. This solution maintains a fine-tuning relationship between the brane tensions and the bulk cosmological constants, like in the Randall-Sundrum model. In addition, it requires that there is no additional contribution to the tension of the intersection from brane-brane interactions. In this paper it is assumed for simplicity that both intersecting branes have the same tension.

In two papers that appeared at about the same time, Csaki and Shirman [17], and Nelson [34] consider more general constructions of intersecting branes. In particular, they study "brane junctions", where (many) semi-infinite branes intersect in a single 3-brane. Static solutions to Einstein's equations for branes intersecting at various angles are obtained, by gluing patches of AdS space together, with the branes as the boundaries. Unlike the Arkani-Hamed *et al.* model, in these more general models, different regions may have different bulk cosmological constants, and the branes may have different tensions. The branes all meet at the 3+1 dimensional Minkowski junction where gravity is localized, as pictured in figure 4.2. It is found that the existence of a static solution determines the angle between the branes uniquely, and moreover, there are other constraints that the parameters of the theory (the brane tensions, angles between the branes and cosmological constants) have to satisfy. There is one fine-tuning condition involving the brane tensions and the cosmological constants, the same condition that was obtained in [2]. Since it is independent of the brane angles, it cannot have a dynamical origin. Both papers end with a discussion on possible resolution of the cosmological constant problem following

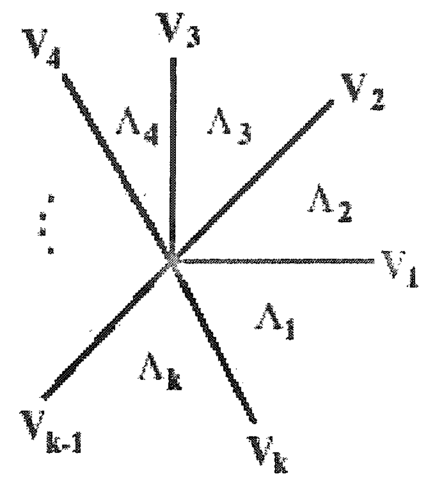

Figure 4.2: Semi-infinite 4-branes intersecting in a single 3-brane in 5+1 dimensions. The brane tensions are denoted by V_i, while the bulk cosmological constants are given by Λ_i.

this avenue. Csaki and Shirman suggest that brane configurations might exist where the cosmological constant might be set to zero by adjusting only the orientations of the branes, unlike the one just described. Then one could translate the cosmological constant problem to a completely dynamical problem in the given brane setup.

In [13], Cline *et al.* generalize these solutions to the nonstatic case. They find expanding solutions for a setup of intersecting codimension-one branes in arbitrary number of dimensions. Again, the junction of these branes plays the role of our four-dimensional universe. Standard cosmology Friedmann equation is recovered if the brane tensions and bulk cosmological constant are tuned. It is also possible to create a hierarchy between branes, but as usual this brings in negative tension branes. Finally, one problem with this model is that inflation of the bulk (it is inflating at the same rate as the branes) causes the strength of gravity to decrease on the TeV brane (where the hierarchy problem would be solved).

Finally, although the present thesis does not examine the vast range of new possibilities allowed if one considers supersymmetry, I will just mention here as an example one last paper on the subject of intersecting branes: Carroll *et al.*, in [6], study $(3+1)$-dimensional junctions of domain walls in higher dimensional supersymmetric theories.

In general, those solutions for codimension-one brane junctions are mathematically quite different than the other solutions reviewed in this thesis, and that is why their details have not been discussed here. Qualitatively, however, certain features are constant whatever the setup, like the fine-tuning required in models with two codimension-one branes, involving the brane tensions. We have only seen it completely disappears in [8], when the Goldberger-Wise mechanism is used to stabilize the size of one of the extra dimensions, and in [30], for the non-static case. Also, this fine tuning was only involving the two tensions in the case of two parallel branes, while for intersecting brane it concerns the relation between branes and cosmological constants.

4.2.3 3-brane embedded in a four-brane

A last kind of model involving codimension-one branes in six-dimensional spacetime uses features of both ADD-like and RS-like scenarios to resolve the hierarchy problem by embedding a three-brane in a four-brane. In [7], an ADD-like scenario is generated from RS-like warped geometry. More precisely, the large dimensions required for solving the hierarchy in the manner of the ADD proposal arise naturally from a higher dimensional warped geometry. This model with two compact extra dimensions is very close to that of ref. [8], except for the important difference in the boundary conditions: instead of two concentric four-branes, in [7] there is only one four-brane in which a three-brane is embedded, the three-brane where the Standard Model particles are confined. This difference will impact on the way hierarchy is obtained. Otherwise, the setup is very close to that of [8]: apart from the four-brane, the sources of gravity are a bulk cosmological constant, a bulk scalar field which stabilizes the compact dimensions and another form of matter which is localized to the brane and gives an anisotropic contribution to the brane tension, such as the Casimir energy of massless fields, a flux or a complex scalar field with a non-trivial winding number in the compact direction. In [8] this anisotropic contribution was given by the Casimir effect. The metric ansatz is also the same:

$$ds^2 = f(r)\eta_{\mu\nu}dx^\mu dx^\nu + dr^2 + s(r)d\theta^2, \tag{4.35}$$

and the bulk solutions are expressed as

$$f(r) = \frac{\cosh^{\frac{4}{5}} \alpha r}{\cosh^{\frac{4}{5}} \alpha b}, \tag{4.36}$$

$$s(r) = \frac{\sinh^2 \alpha r}{\alpha^2 \cosh^{\frac{6}{5}} \alpha r}, \tag{4.37}$$

b is the radius at which the four-brane is situated, f is normalized so that $f(b) = 1$.and $\alpha = -\frac{5}{8}\kappa_6^2\Lambda_6$ (The bulk cosmological constant is negative). θ runs from 0 to 2π as usual, and the radial dimension is orbifolded about the four-brane, i.e. there exists a symmetry that identifies points on one side of the brane to points on the other side. Since we will place the standard model on the same four-brane which localizes gravity, there is no need for a second brane here. This bulk solution is almost exactly identical to one of the solutions of chapter 3, the AdS soliton one (equation 3.33), although once again the boundary conditions are different. Here there is no requirement for a three-brane at the origin since the three-brane will be embedded in the four-brane.

Now the important particularity of this model is that unlike in [8], where the SM was occupying all of the four-brane, the fifth dimension being invisible to us because it was compactified on a circle of small radius, here ordinary particles are trapped on a three-brane inside the four-brane. Hierarchy does not arise because of the warping as in [8], rather it is due to the large compact angular dimension ϕ:

$$M_{Pl}^2 = \frac{M^4}{\alpha} V, \ \alpha = -\frac{5}{8}\kappa_6^2 \Lambda_6, \tag{4.38}$$

where V corresponds here to the size of the ϕ dimension. Conceptually, gravity is localized on the four-brane, and it is weak due to the presence of the large compact dimension, and the fact that gravity spreads evenly over the entire four-brane. We still have warped space in this model, along the radial coordinate r (the RS coordinate), but here its rôle is to lead, together with the scalar field, to a large ADD-like ϕ dimension. This is achieved by stabilizing the four-brane at a radius where the warping is important. If the mass of the scalar is lighter than the fundamental scale by a factor of a few, the setup will be stabilized at finite but exponentially large volume for the ADD dimension. Thus we can say that the large dimension necessary to solve the hierarchy problem in the ADD fashion is generated naturally by the presence of an other RS-like warped dimension. Actually, the six-dimensional model is only an example of their more general D-dimensional model. In the generalized version, there is one warped RS dimension and the other dimensions form a large ADD-like volume.

Chapter 5

Conclusion

I have presented in this thesis an overview of six-dimensional braneworld models and solutions. A certain emphasis has been put on a general model with a cylindrically symmetric bulk, centered on the codimension-two brane, where the standard model particles are confined. Arbitrary bulk and effective four-dimensional cosmological constants were considered. This model is the same as in ref.[11]. Working on such a model, which was intended to be as general as possible but simple enough so that its solutions could be studied, provided a structure for classifying six-dimensional solutions. Solutions to Einstein's equation for this configuration can be classified, depending on the signs of the full six-dimensional and effective four-dimensional cosmological constants, under two different topologies. Solutions with $0 < \frac{3}{10}\hat{\Lambda}_6 \leq \hat{\Lambda}_4$ have a spherical topology, with two three-branes at the poles, one of which is our four-dimensional universe. For other values of the cosmological constants, we have a disk topology with a three-brane at the center and a four-brane at the boundary of the disk (generally provided with an anisotropic source of energy). For solutions with this topology the warp factor $M(\rho)$ can either be increasing away from our brane, or decreasing. Unwarped solutions (with constant $M(\rho)$) arise when $\hat{\Lambda}_6 = 2\hat{\Lambda}_4$, for any sign of $\hat{\Lambda}_6$.

We also looked at some extensions of this general model; we mentioned for example the effect of stabilizing of the model with a scalar field or a magnetic flux. The stability analysis in fact reveals the surprising result that the Friedmann equation is modified in some cases: the four-dimensional cosmological constant could be independent of the brane tension, or increase as the latter decreases. Extensions of the model also include models from the literature that closely resemble ours. For example, one of Navarro's solutions, which has a double disc topology, with symmetry with respect to ρ_m instead of a four-brane there; Chodos' model, with solution encompassed in our general model, but provided with a parameter space analysis and extension of the model to more dimensions; the spherical topology model of Carroll and Navarro, which correspond to our unwarped solution stabilized with magnetic flux; details on the much studied AdS soliton model, which is also related to one of our general model solutions; and finally the Sundrum model with many parallel three-branes. These are the different extensions of the general model that were considered.

We finally looked at other possibilities, broadening our horizons. We saw that there are six-dimensional models with other combinations of branes: models with four-branes only, with intersection of four-branes as our universe, the Chacko *et al.* model with our three-brane embedded in a four-brane are examples we discussed. We looked at another approach to braneworld models, which considers the fields responsible for creating the vortex where the Standard Model particles are trapped. We concentrated on the work of Giovannini *et al.*, which builds on that of Gherghetta and Shaposhnikov and is close enough to the model we studied in Chapter 3, in terms of the general setup.

Most of all models studied try to find a solution to either the cosmological constant problem or the hierarchy problem (or both). The first of these gets hope of solution in the fact that the four-dimensional cosmological constant is in some cases observed to be independent of the three-brane tension. Hence, in many of the papers discussed in the thesis, it is mentioned that the problem could be transferred into extra dimensions in a case where Λ_4 can vanish without any need to fine-tune the three-brane tension. On the other side many authors admit that their six-dimensional model still requires a fine-tuning which corresponds to the cosmological constant problem. The hierarchy problem is in general more successfully addressed in the models we considered. It is generally either solved using the warping of the extra dimensions, like in the Chacko and Nelson model, where gravity is localized on a different brane than the Standard Model (explaining the apparent weakness of gravity in our world), or the hierarchy is rather due to the largeness of the extra dimensions, like in the Sundrum model, in which the hierarchy can be obtained by taking the 6D fundamental scale to be close to the weak scale and the size of the compact space not much smaller than a millimeter. Sometimes it is through a combination of both that the problem can be solved. For example, in the

Chacko *et al.* model with a three-brane embedded in a four-brane, hierarchy is obtained from the large ϕ dimension, but the largeness of this dimension is in turn due to the warping along the radial coordinate.

We have already noticed that an important feature of codimension-two branes, characteristic of six-dimensional models, is that their tension, or vacuum energy, induces a deficit angle in the transverse space. One result of this particularity of six-dimensional geometry is the peculiar way in which the four-dimensional effective cosmological constant will depend on the three-brane tension. In some cases it will be independent of this tension; in other instances it may depend on the tension in an unusual way, even for stable solutions. This is an important result because the Friedmann equation will thus get modified [11]. Hence, we may conclude that although turning to six-dimensional models resolves some of the problems intrinsic to the five-dimensional case, like the need for a negative tension brane for example, new problematic features appear. On the other hand, insensitivity to the three-brane tension seems to suggest a solution to the cosmological constant problem, although the standard Friedmann equation would have to be recovered for this to be applicable. The next step could be to link the models to fundamental theory, which would probably shed new light on the problem. Supersymmetric versions of six-dimensional models, for example, appear to fix certain problems. The last word on braneworld models has certainly not be said.

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