REFLECTIVE SUBCATEGORIES

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Submitted in partial fulfilment of the requirements of the Ph.D. degree at McGill University.

by S. BARON

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Dedication

To Dr B. Rattray for his patience, encouragement, guidance and constructive criticism,

to Dr J. Lambek who introduced me to category theory,

to Dr E. Rosenthall who provided employment and took a thoughtful interest in my progress,

to my parents Mr. and Mrs Melvin Baron who aided materially and spiritually, through many trying years,

and to my wife, Arlette, and our three children who sometimes let me work in peace,

this thesis is respectfully dedicated.

Reflective Subcategories

Reflective subcategories correspond to solutions of some universal mapping problems. In the category of topological spaces, the most famous example is the reflective subcategory of compact Hausdorff spaces which corresponds to the Stone-Cech compactification [1]. Some work has been done on characterizing reflective and coreflective subcategories. Freyd [2] gives a necessary condition for a subcategory of a well-powered, complete category to be reflective. Isbell [5] generalizes this using bicategorical structures. Kennison [3] gives necessary and sufficient conditions for coreflective subcategories and three types of reflective subcategory of the category of topological spaces. In this thesis, necessary and sufficient conditions are given for a full subcategory of a cowell powered category with products to be reflective with epi-reflection map. It is then shown that every reflective subcategory of such a category is an epi-reflective subcategory of an epi-reflective subcategory, epi being with respect to the respective containing category. Putting these two theorems together yields necessary and sufficient conditions for a full, cowell powered subcategory to be reflective. A corollary involving the concept of extremal subobject is a generalization of Kennison's theorem and a special case of the Freyd-Isbell theorem. These results are applied to the problems of generation and intersection of reflective subcategories.

The author claims originality for theorems 3, 4, 5, 6.1-2, 6.4, 6.7, 7.1-3, 8, 9.1-2, and the concepts of qer subcategory and intermediate subcategory.

It was discovered after the typing of this thesis that a remark of Isbell [5, p.9] implies 6.6, and that all well powered, left complete categories are factorable. Thus in 6.7, 6.8, 7.1, 8, 9.1 and 9.2, it is not necessary to assume factorability.

1. - Fundamental definitions

For definitions of category, subcategory, full, mono, epi, product, coproduct, object, subobject, quotient object, functor, and natural transformation, see Freyd [2]. We shall follow the convention of prefixing these terms by the name of the relevant category, wherever there is a chance of confusion.

1.1 - Let $\mathcal{A}$ be a full, replete subcategory of a category, $\mathcal{B}$; B, a $\mathcal{B}$-object ; R (B), an $\mathcal{A}$-object ; and $r_B : B \dashrightarrow R(B)$. R (B) is called a reflection of B and r_B a reflection map if for every map $f : B \dashrightarrow A$ where A is an $\mathcal{A}$-object, there is a unique $R(f) : R(B) \dashrightarrow A$ such that $f = R(f) \circ r$. If every $\mathcal{B}$ - object has a reflection in $\mathcal{A}$, than $\mathcal{A}$ is said to be a reflective subcategory.

Remarks : (1) In the work of some authors, a reflective subcategory is not necessarily full or replete.

(2) All reflections of a given object in a given subcategory are isomorphic, and all reflection maps from a given object differ only by an isomorphism.

1.2 - A category is said to be <u>(co)well powered</u> if every object has a representative set of sub(quotient) objects. Let $\mathcal{A} \subseteq \mathcal{B}$. $\mathcal{B}$ is said to be <u>$\mathcal{A}$(co)well-powered</u> if for every object of $\mathcal{B}$, there is a representative set of the class of those sub(quotient) objects that are $\mathcal{A}$-objects.

1.3 - A full subcategory $\mathcal{A}$ of a category $\mathcal{B}$ is said to be <u>quasi-epi-reflective</u> (qer) if for any map $f : B \longrightarrow A$, with B, an object of $\mathcal{B}$, A, an object of $\mathcal{A}$, there is an object A' of $\mathcal{A}$ and maps $f_1 : B \longrightarrow A'$, $f_2 : A' \longrightarrow A$ such that f_1 is epi, f_2 is mono and $f = f_2 \circ f_1$.

1.4 - A category is <u>factorable</u>, if every map may be factored into an epi followed by a mono.

1.5 - A reflective subcategory is said to be <u>epi(mono) reflective</u> if every object of the containing category has an epi (mono) reflection map.

1.6 - f is said to distinguish a pair of maps $g_1 \neq g_2$ whose domain is the range of f, if $g_1 \circ f \neq g_2 \circ f$.

1.7 – A <u>diagram</u> in a category, $\mathcal{C}$, is a functor from a small category to $\mathcal{C}$.

1.8 – A functor $F : \mathcal{A} \to \mathcal{B}$ is said to be a <u>constant functor</u> if F maps all $\mathcal{A}$-objects to the same $\mathcal{B}$-object and all $\mathcal{A}$-maps to the identity map of that $\mathcal{B}$-object. We will sometimes identify a constant functor with its image.

1.9 – Let $F : \mathcal{D} \to \mathcal{C}$ be a diagram and $L : \mathcal{D} \to \mathcal{C}$ ($R : \mathcal{D} \to \mathcal{C}$) be a constant functor with a natural transformation $\mu : L \to F$ ($\mu' : F \to R$). If for any natural transformation, $\alpha : M \to F$ ($\alpha' : F \to M'$), where M (M') is any constant functor, there is a unique map $m : M \to L$ ($m' : R \to M'$) such that $\alpha = \mu \circ m$ ($\alpha' = m' \circ \mu'$) then (L, μ) [(R, μ')] is said to be a <u>left-root</u> (<u>right root</u>) of the diagram, F.

1.10 – If every diagram in a category, $\mathcal{C}$, has a left (right) root, then $\mathcal{C}$ is said to be <u>left</u> (<u>right</u>) <u>complete</u>.

1.11 – The left root of $\genfrac{}{}{0pt}{}{\xrightarrow{f}}{\xrightarrow[f]{}}$ is said to be the <u>kernel pair</u> of f ; the right root of $\genfrac{}{}{0pt}{}{\xrightarrow{f}}{\xrightarrow[f]{}}$ is said to be the <u>cokernel pair</u> of f.

2. Preliminary results

We shall present those results that will be needed in the sequel. Most of these are proved elsewhere, but they are collected here for convenient reference.

2.1 – If $\mathcal{A} \subseteq \mathcal{B}$ and the $\mathcal{B}$-left root of some $\mathcal{A}$-diagram has a reflection in $\mathcal{A}$, then its reflection map is an isomorphism.

<u>Proof</u> : Let (L, μ) be the $\mathcal{B}$-left root of a diagram, $F : \mathcal{D} \longrightarrow \mathcal{A}$. Let $d : D_i \longrightarrow D_j$ be any $\mathcal{D}$-map.
Then $\mu(D_j) = F(d) \circ \mu(D_i)$. Let $R(L)$ and r_L be the reflection and reflection map, respectively, of L.
Then there are unique maps $R[\mu(D_i)] : R(L) \longrightarrow F(D_i)$ and $R[\mu(D_j)] : R(L) \longrightarrow F(D_j)$. By this uniqueness, $R[\mu(D_j)] = F(d) \circ R[\mu(D_i)]$. Thus we have a natural transformation, $\alpha : R(L) \longrightarrow F$, such that $\alpha(D_i) = R[\mu(D_i)]$.
Since (L, μ) is the $\mathcal{B}$-left root of F, there is a map $m : R(L) \longrightarrow L$ such that $\alpha = \mu \circ m$.

$$\begin{tikzcd}
R(L) \arrow[dr, "R(\mu(D_i))"] \arrow[drr, bend left, "R(\mu(D_j))"] \arrow[dd, bend right, "m"'] & & \\
 & F(D_i) \arrow[r, "F(d)"] & F(D_j) \\
L \arrow[uu, "r_L"'] \arrow[ur, "\mu(D_i)"] \arrow[urr, bend right, "\mu(D_j)"'] & &
\end{tikzcd}$$

Now $\mu \circ m \circ r_L = \alpha \circ r_L = \mu = \mu \circ 1_L$. Since there is unique factorization through a left root, $m \circ r_L = 1_L$.

Left-multiplying by r_L, we obtain $r_L \circ m \circ r_L = r_L$ which gives us the following commutative diagram :

$$\begin{tikzcd}
R(L) \arrow[r, bend left, "r_L \circ m"] \arrow[r, bend right, "1_{R(L)}"'] & R(L) \\
L \arrow[u, "r_L"] \arrow[ur, "r_L"'] &
\end{tikzcd}$$

Since $R(r_L)$ must be unique $r_L \circ m = 1_{R(L)}$, $m = r_L^{-1}$ and r_L is an isomorphism. ▯

<u>Corollary 1</u> - A reflective subcategory contains all existing left roots of its diagrams.

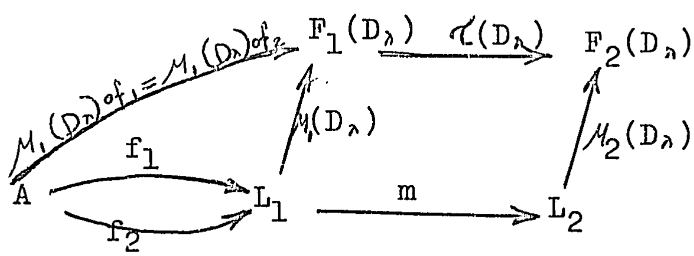
<u>Corollary 2</u> - A reflective subcategory of a left complete category is left complete.

2.2 - If $\mathcal{A}$ is a full subcategory of $\mathcal{B}$, then any $\mathcal{A}$-object that is a $\mathcal{B}$-left (right) root of an $\mathcal{A}$-diagram is an $\mathcal{A}$-left (right) root.

<u>Proof</u> : Let (L, μ) be a $\mathcal{B}$-left root of $F : \mathcal{D} \longrightarrow \mathcal{A} \subseteq \mathcal{B}$, L an $\mathcal{A}$-object. Since $\mathcal{A}$ is full, $\mu : L \longrightarrow F|\mathcal{A}$, where $F|\mathcal{A}$ is F with range restricted to $\mathcal{A}$. If $M : \mathcal{D} \longrightarrow \mathcal{A}$ is any constant functor with $\tau : M \longrightarrow F|\mathcal{A}$, we have a unique $\mathcal{B}$-map, $m : M \longrightarrow L$ such that $\tau = \mu \circ m$. However, since $\mathcal{A}$ is full, it follows that m is an $\mathcal{A}$-map and (L, μ) is an $\mathcal{A}$-left root of $F|\mathcal{A}$. Similarly with right roots. ▯

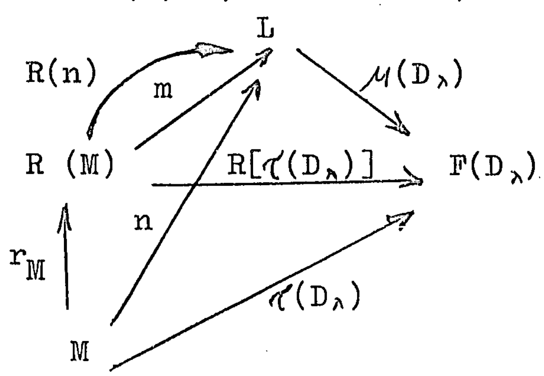
2.3 - If $F_1, F_2 : \mathcal{D} \longrightarrow \mathcal{A}$ are diagrams, $\tau : F_1 \longrightarrow F_2$, a natural transformation, and for any $\mathcal{D}$-object, D, $F_1(D)$ is a subobject of $F_2(D)$ with subobject map $\tau(D)$, then the left root of F_1 is a subobject of the left root of F_2.

<u>Proof</u> : Suppose that (L_1, μ_1) and (L_2, μ_2) are the left roots of F_1 and F_2, respectively. Then there is a unique map $m : L_1 \longrightarrow L_2$ such that $\tau \circ \mu_1 = \mu_2 \circ m$. Suppose $f_1, f_2 : A \longrightarrow L_1$, such that $m \circ f_1 = m \circ f_2$. Then $\mu_2 (D_\lambda) \circ m \circ f_1 = \mu_2 (D_\lambda) \circ m \circ f_2$ and $\tau(D_\lambda) \circ \mu_1(D_\lambda) \circ f_1 = \tau(D_\lambda) \circ \mu_1(D_\lambda) \circ f_2$. Since $\tau(D_\lambda)$ is mono, $\mu_1 (D_\lambda) \circ f_1 = \mu_1 (D_\lambda) \circ f_2$. These maps define a natural transformation from the constant functor with image A to the diagram F and therefore the map from $A \longrightarrow L_1$ must be unique. Hence $f_1 = f_2$; m is mono, and L_1 is a subobject of L_2. ▯



2.4 - If $\mathcal{A}$ is a reflective subcategory of $\mathcal{B}$, then the $\mathcal{A}$-left root of an $\mathcal{A}$-diagram is a $\mathcal{B}$-left root.

<u>Proof</u> : Let $F : \mathcal{D} \to \mathcal{A}$ be a diagram and (L, μ), its $\mathcal{A}$-left root. Suppose $M : \mathcal{D} \to \mathcal{B}$ is a constant functor and $\tau : M \longrightarrow F$ a natural transformation. Let R (M) and r_M be respectively the reflection and reflection map of M. Then $R(\tau) : R(M) \longrightarrow F$ is an $\mathcal{A}$-natural transformation and there is a unique $\mathcal{A}$-map, $m : R(M) \longrightarrow L$ such that $\mu \circ m = R(\tau)$. By reflection property $\tau = R(\tau) \circ r_M$ and $\mu \circ m \circ r_M = \tau$



If $\mu \circ n = \tau$, for some map $n : M \longrightarrow L$, then $\mu \circ R(n) = R(\tau)$. Then $R(n) = m$ and $n = m \circ r_M$. Thus $m \circ r_M$ is unique. Thus L is a $\mathcal{B}$-left root of F. ▯

2.5 – A morphism is a mono iff it has a kernel pair which is the identity of its domain. A morphism is epi iff it has a cokernel pair which is the identity of its range.

<u>Proof</u> $:\Longrightarrow$ Let $m : A \longrightarrow B$ be a mono. Certainly, $m \circ 1_A = m \circ 1_A$. If $m \circ f = m \circ g$ where $f, g : C \longrightarrow A$, then $f = g$. Thus there is a unique map $f : C \longrightarrow A$ such that $1_A \circ f = f$ and $1_A \circ f = g$. $\square$

$$\begin{tikzcd} & & A \arrow[dr, "m"] & \\ C \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] \arrow[r, dashed, "f"] & A \arrow[ur, "1_A"] \arrow[dr, "1_A"'] & & B \\ & & A \arrow[ur, "m"'] & \end{tikzcd}$$

$\Longleftarrow$ Let $m : A \longrightarrow B$ be a morphism whose kernel pair is $(1_A, 1_A)$. Suppose $m \circ f = m \circ g$ where $f, g : C \longrightarrow A$. Then there must be a unique map $h : C \longrightarrow A$ such that $1_A \circ h = f$ and $1_A \circ h = g$ Thus $f = g$ and m is mono. $\square$

$$\begin{tikzcd} & & A \arrow[dr, "m"] & \\ C \arrow[urr, bend left, "f"] \arrow[drr, bend right, "g"'] \arrow[r, "h"] & A \arrow[ur, "1_A"] \arrow[dr, "1_A"'] & & B \\ & & A \arrow[ur, "m"'] & \end{tikzcd}$$

The proof of the second part is similar.

<u>Corollary 1</u> – If $\mathcal{A}$ is a reflective subcategory of $\mathcal{B}$, then any $\mathcal{A}$-mono is a $\mathcal{B}$-mono.

<u>Corollary 2</u> – A reflective subcategory of a well powered category is well powered.

<u>Corollary 3</u> – If $\mathcal{A}$ is a full subcategory of $\mathcal{B}$, then any $\mathcal{B}$-mono (epi) between $\mathcal{A}$-objects is an $\mathcal{A}$-mono (epi).

2.6 – Let $\mathcal{A}$ be a full subcategory of $\mathcal{B}$ such that every $\mathcal{B}$-object is a $\mathcal{B}$-subobject of an object of $\mathcal{A}$. Then any $\mathcal{B}$-map, $f : B \longrightarrow A$ (where A is an $\mathcal{A}$-object) which distinguishes all distinct pairs of $\mathcal{A}$-maps with domain A, is $\mathcal{B}$-epi.

Proof : Let $h_1, h_2 : A \longrightarrow B'$, where B' is a $\mathcal{B}$-object, satisfy $h_1 \circ f = h_2 \circ f$. But B' is a subobject of some $\mathcal{A}$-object, A' ; let m be the subobject map. Thus $m \circ h_1 \circ f = m \circ h_2 \circ f$. But by hypothesis, since $m \circ h_1$ and $m \circ h_2$ are $\mathcal{A}$-maps, this implies $m \circ h_1 = m \circ h_2$. Also m is mono and $h_1 = h_2$. Thus f is $\mathcal{B}$-epi. ▯

Corollary – All $\mathcal{A}$-epis are $\mathcal{B}$-epis.

2.7 – $f \circ g$ epi implies f epi ; $f \circ g$ mono implies g mono [2].

2.8 – f,g epi implies $f \circ g$ epi ; f,g mono implies $f \circ g$ mono [2].

2.9 – If $\mathcal{B}$ is a full subcategory of a factorable category $\mathcal{C}$ and if $\mathcal{B}$ contains all $\mathcal{C}$-subobjects of its objects, then (1) $\mathcal{B}$ is factorable and (2) $\mathcal{B}$ is qer subcategory of $\mathcal{C}$.

Proof : Consider any $\mathcal{C}$-map, $f : C \longrightarrow B$ where B is a $\mathcal{B}$-object. Then since $\mathcal{C}$ is factorable $f = m \circ e$ where $e : C \longrightarrow B'$ is epi and $m : B' \longrightarrow B$ is mono. By hypothesis, B' is a $\mathcal{B}$-object. Thus we have (2). (1) follows when we let C be a $\mathcal{B}$-object, and f any $\mathcal{B}$-map in view of Corollary 3 of 2.5.

3. Fundamental Lemma

Let $\mathcal{A}$ be a full, replete subcategory of $\mathcal{B}$, and $\mathcal{B}$ be an $\mathcal{A}$-cowell powered category with products. $\mathcal{A}$ is a $\mathcal{B}$-epi-reflective subcategory of $\mathcal{B}$ iff :

(1) $\mathcal{A}$ contains all $\mathcal{B}$-products of $\mathcal{A}$-objects

(2) $\mathcal{A}$ is qer.

Proof $\Rightarrow$ (1) This follows from Corollary 1 of 2.1.

(2) Since $\mathcal{A}$ is epi-reflective, a map from any $\mathcal{B}$-object, B, to an $\mathcal{A}$-object, A, may be factored through R (B) via the epi reflection map. ‖

$\Leftarrow$ Consider an object, B, of $\mathcal{B}$. Because $\mathcal{B}$ is $\mathcal{A}$ cowell-powered, the class of quotient objects of B that are also objects of $\mathcal{A}$ has a representative set, $\{B_\lambda\}$. $\prod B_\lambda$ is an object of $\mathcal{A}$. We have an evaluation map, $e : B \rightarrow \prod B_\lambda$. Because $\mathcal{A}$ is qer, $e = f \circ r_B$ where $r_B : B \rightarrow R(B)$ is epi, and R (B) is an object of $\mathcal{A}$.

We maintain that R (B) is a reflection of B with reflection map r_B. Let $f : B \rightarrow A$ where A is an object of $\mathcal{A}$. Since $\mathcal{A}$ is qer, f may be factored through a B_λ ; thus through $\prod B_\lambda$ by the evaluation map ; thus through R (B) by r_B.

$$\begin{tikzcd}
R(B) \arrow[r] \arrow[rrr, bend left, "R(f)"] & \prod B_\lambda \arrow[r] & B_\lambda \arrow[r] & A \\
B \arrow[u, "r_B"] \arrow[ur, "e"] \arrow[urr] \arrow[urrr, "f"'] & & &
\end{tikzcd}$$

The map from R (B) to A will be called R (f). It is unique because r_B is epi. Thus every object of $\mathcal{B}$ has a reflection in $\mathcal{A}$ and $\mathcal{A}$ is a $\mathcal{B}$ -epi reflective subcategory of $\mathcal{B}$. ▯

4. <u>Theorem</u>

If $\mathcal{A}$ is a reflective subcategory of $\mathcal{C}$, and $\mathcal{C}$ is a factorable, cowell-powered category with products, then the full subcategory, $\mathcal{B}$, of subobjects of objects of $\mathcal{A}$ is a $\mathcal{C}$-epi-reflective subcategory of $\mathcal{C}$ and $\mathcal{A}$ is a $\mathcal{B}$-epi-reflective subcategory of $\mathcal{B}$.

<u>Proof</u> : $\mathcal{A}$ contains all $\mathcal{C}$-products of its objects (Corollary 1 of 2.1) ; hence, since the product of subobjects is a subobject of the product (2.3), $\mathcal{B}$ contains all $\mathcal{C}$-products of its objects.

Suppose $f : C \longrightarrow B$ where B is an object of $\mathcal{B}$. Since $\mathcal{C}$ is factorable, f may be factored through a subobject of B which is a quotient object of C. However, $\mathcal{B}$ is closed under subobjects and f is thus factored through an object of $\mathcal{B}$ that is a quotient object of C ; therefore, $\mathcal{B}$ is a qer subcategory of $\mathcal{C}$. By the Lemma, $\mathcal{B}$ is an epi-reflective subcategory of $\mathcal{C}$.

Since every object of $\mathcal{C}$ has a reflection in $\mathcal{A}$, certainly every object of $\mathcal{B}$ has and $\mathcal{A}$ is a reflective subcategory of $\mathcal{B}$. We would like to show that for each $\mathcal{B}$-object B, the reflection map r_B is $\mathcal{B}$-epi. Since for any map $f : B \longrightarrow A$, A, an $\mathcal{A}$-object, R (f) is unique, it follows that, if $g_1, g_2 : R(B) \longrightarrow A$ are any $\mathcal{A}$-maps such that $g_1 \circ r_B = g_2 \circ r_B$, then $g_1 = g_2$. Thus by 2.6, r_B is $\mathcal{B}$-epi and $\mathcal{A}$ is a $\mathcal{B}$-epi-reflective subcategory of $\mathcal{B}$. ∎

Note that B is the subobject of some $\mathcal{A}$-object, A', and the subobject map may be factored through r_B, by the properties of reflection. By 2.7, r_B is $\mathcal{C}$-mono (by 2.5 Corollary 3, $\mathcal{B}$-mono as well) and $\mathcal{A}$ is a mono-$\mathcal{B}$-epi-reflective subcategory of $\mathcal{B}$.

5. Principal theorem

If $\mathcal{A}$ is a full, replete subcategory of $\mathcal{C}$, a factorable, cowell-powered category with products, then : $\mathcal{A}$ is a cowell-powered, reflective subcategory of $\mathcal{C}$ iff :

(1) the category, $\mathcal{B}$, of $\mathcal{C}$-subobjects of objects of $\mathcal{A}$ is $\mathcal{A}$-cowell powered,

(2) $\mathcal{A}$ contains each $\mathcal{C}$-product of $\mathcal{A}$-objects, and

(3) $\mathcal{A}$ is a qer subcategory of $\mathcal{B}$.

Proof : $\Longrightarrow$ By Corollary 1 of 2.1, we have (2). By 4., we know that $\mathcal{A}$ is a $\mathcal{B}$-epi-reflective subcategory of $\mathcal{B}$. Hence by 3., we have (3). Next, if an $\mathcal{A}$-object, A, is a $\mathcal{B}$-quotient object of a $\mathcal{B}$-object, B, with quotient map f, then $R(f) : R(B) \longrightarrow A$ is also $\mathcal{B}$-epi by 2.7, since $f = R(f) \circ r_B$. By 2.5, Corollary 3, R(f) is $\mathcal{A}$-epi and A is an $\mathcal{A}$-quotient object of R(B). Thus every $\mathcal{B}$-quotient object of B that is an $\mathcal{A}$-object is an $\mathcal{A}$-quotient object of R(B). Since $\mathcal{A}$ is cowell powered, there can be only a representative set of the class of $\mathcal{B}$-quotient objects of B that are also $\mathcal{A}$-objects, i.e. $\mathcal{B}$ is $\mathcal{A}$-cowell powered which is (1).

$\Longleftarrow$ By (2), $\mathcal{B}$ contains every $\mathcal{C}$-product of $\mathcal{A}$-objects. Thus by 2.2, any $\mathcal{B}$-product of $\mathcal{A}$-objects is the $\mathcal{C}$-product. Thus $\mathcal{A}$ contains each $\mathcal{B}$-product of $\mathcal{A}$-objects. This, together with (1), (3) and 3., shows that $\mathcal{A}$ is an epi-reflective subcategory of $\mathcal{B}$.

By 2.3 and (2), $\mathcal{B}$ contains each $\mathcal{C}$-product of $\mathcal{B}$-objects. Since $\mathcal{C}$ is factorable, any $f : C \longrightarrow B$ with C, a $\mathcal{C}$-object and B, a $\mathcal{B}$-object, can be factored as $m \circ e$, where m is mono and e is epi. The intermediate object is a subobject of B, hence a $\mathcal{B}$-object. Thus $\mathcal{B}$ is a qer subcategory of $\mathcal{C}$. Also $\mathcal{C}$ is cowell powered, hence $\mathcal{B}$-cowell powered. Thus by 3., $\mathcal{B}$ is a reflective subcategory of $\mathcal{C}$. Since the composition of two reflections is a reflection, $\mathcal{A}$ is a reflective subcategory of $\mathcal{C}$.

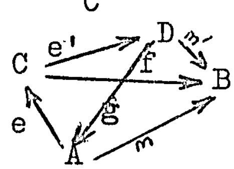
By Corollary of 2.6, every $\mathcal{A}$-epi is a $\mathcal{B}$-epi and thus every $\mathcal{A}$-quotient object is a $\mathcal{B}$-quotient object. Therefore $\mathcal{A}$ is cowell powered. $\Box$

6. Extremal and regular monos

<u>Definition</u> – A mono $m : A \rightarrowtail B$ is said to be <u>extremal</u> if in any factorization $m = m_1 \circ e$, where m_1 is mono and e epi, e must be iso. (A,m) (and loosely A) is said to be an <u>extremal subobject</u> of B.

6.1 – In a factorable category, if an extremal mono, m, is factored as $f \circ e$, where e is epi, then e is iso.

<u>Proof</u> – Suppose $m : A \rightarrowtail B$ is factored as $f \circ e$ where $e : A \twoheadrightarrow C$ and $f : C \rightarrowtail B$, e epi. Factor f as $m' \circ e'$ where $m' : D \rightarrowtail B$ is mono and $e' : C \twoheadrightarrow D$ is epi. Then $m = m' \circ (e' \circ e)$ which is an epi (2.8) followed by a mono. Thus since m is extremal, $e' \circ e$ is iso. Let $g = (e' \circ e)^{-1}$. Thus $(g \circ e') \circ e = 1_A$. Also, $e \circ (g \circ e') \circ e = e$ and since e is epi, $e \circ (g \circ e') = 1_C$.



thus $g \circ e' = e^{-1}$ and e is iso. ▯

6.2 – A qer subcategory, $\mathcal{A}$, of a factorable category, $\mathcal{B}$, contains all $\mathcal{B}$–extremal subobjects of its objects.

<u>Proof</u> – Let (B,m) be a $\mathcal{B}$–extremal subobject of an $\mathcal{A}$–object, A. Since $\mathcal{A}$ is qer, m may be factored as $f \circ e$ where $e : B \twoheadrightarrow A'$ is epi; A', an $\mathcal{A}$–object. By 6.1, e is iso and A' is an isomorphic copy of B. ▯

<u>Definition</u> : A mono $f : A \longrightarrow B$ is said to be regular if for some C and morphisms $g_1, g_2 : B \longrightarrow C$, f is the difference kernel of the pair(g_1, g_2).

6.3 - All regular subobjects are extremal.

<u>Proof</u> : Let (A,m) be a regular subobject of B and m, the difference kernel of $f,g : B \longrightarrow C$. Suppose m is factored as $m_1 \circ e_1$ where $m_1 : D \longrightarrow B$ is mono and $e_1 : A \longrightarrow D$, epi. Thus m_1 may be factored through m ; say $m_1 = m \circ h$ where $h : D \longrightarrow A = \ker(f,g)$. Thus $m_1 = m_1 \circ e_1 \circ h$ and since m_1 is mono, $1_D = e_1 \circ h$; also $m = m \circ h \circ e_1$ and $1_A = h \circ e$ since m is mono. Thus e is iso and (A,m) is extremal.$\square$

$$\begin{tikzcd}
A \arrow[r, "m"] \arrow[d, bend left, "e_1"] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & C \\
D \arrow[u, bend left, "h"] \arrow[ur, "m_1"'] & &
\end{tikzcd}$$

6.4 - If two functors $F_1, F_2 : \mathcal{D} \longrightarrow \mathcal{A}$, where $\mathcal{D}$ is small and $\mathcal{A}$ is well powered and left complete, have a natural transformation $\mu : F_1 \longrightarrow F_2$ such that for every $\mathcal{D}$-object, $D, \mu(D)$ is an extremal mono, then the induced morphism, f, mapping the left root of F_1 to the left root of F_2 is an extremal mono.

<u>Proof</u> : By 2.3, f is mono. Let (L_1, φ_1) and (L_2, φ_2) be the left roots of F_1 and F_2, respectively. Now suppose $f = m \circ e$, where $e : L_1 \longrightarrow C$ is epi and $m : C \longrightarrow L_2$ is mono. For each λ, let (C_λ, m_λ) be the intersection of all those subobjects of $F_2(D_\lambda)$ through which both $\mu(D_\lambda)$ and $\phi_2(D_\lambda)\circ m$ factor.

$$\begin{tikzcd}
& F_1(D_\lambda) \arrow[rrr, "\mathcal{M}(D_\lambda)"] \arrow[dr, "e_\lambda"] & & & F_2(D_\lambda) \\
& & C_\lambda \arrow[urr, "m_\lambda"] \arrow[r, shift left, "g_\lambda"] \arrow[r, shift right, "g'_\lambda"'] & E_\lambda & \\
L_1 \arrow[uur, "\varphi_1(D_\lambda)"] \arrow[rr, "e"'] \arrow[rrrr, bend right, "f"'] & & C \arrow[ll, bend right, "h"'] \arrow[u, "m^*_\lambda"] \arrow[rr, "m"] & & L_2 \arrow[uu, "\varphi_2(D_\lambda)"']
\end{tikzcd}$$

Suppose $g_\lambda, g'_\lambda : C_\lambda \longrightarrow E_\lambda$ such that $g_\lambda \circ e_\lambda = g'_\lambda \circ e_\lambda$. Then e_λ and hence $\mathcal{M}(D_\lambda)$ factor through the difference kernel (K_λ, k_λ) of (g_λ, g'_λ). Also $g_\lambda \circ e_\lambda \circ \varphi_1(D_\lambda) = g'_\lambda \circ e_\lambda \circ \varphi_1(D_\lambda)$ and $g_\lambda \circ m^*_\lambda \circ e = g'_\lambda \circ m^*_\lambda \circ e$. Since e is epi, it follows that $g_\lambda \circ m^*_\lambda = g'_\lambda \circ m^*_\lambda$. Thus m^*_λ and hence $\phi_2(D_\lambda) \circ m$ factor through (K_λ, k_λ). Thus m_λ may be factored through $m_\lambda \circ k_\lambda$, say $m_\lambda = m_\lambda \circ k_\lambda \circ k'_\lambda$. Since m_λ is mono $k_\lambda \circ k'_\lambda = 1_{C_\lambda}$; also $k_\lambda \circ k'_\lambda \circ k_\lambda = k_\lambda$ and $k'_\lambda \circ k_\lambda = 1_{K_\lambda}$, since k_λ is mono. Thus k_λ is an isomorphism, $g_\lambda = g'_\lambda$ and e_λ is epi. Since $\mathcal{M}(D_\lambda)$ is extremal and m_λ is mono, it follows that e_λ is iso. Thus $m^*_\lambda : C \longrightarrow F_1(D_\lambda)$. Thus there is a unique $h : C \longrightarrow L_1$, such that $\varphi_1(D_\lambda) \circ h = m^*_\lambda$. Also $\mathcal{M}(D_\lambda) \circ m^*_\lambda \circ e = \mathcal{M}(D_\lambda) \circ \varphi_1(D_\lambda)$ and $m^*_\lambda \circ e = \varphi_1(D_\lambda)$. Thus $\varphi_1(D_\lambda) \circ h \circ e = m^*_\lambda \circ e = \varphi_1(D_\lambda)$ and $h \circ e = 1_{L_1}$. Also $e \circ h \circ e = e$ and $e \circ h = 1_C$. Thus f is extremal. ▯

<u>Corollary 1</u> - The product of extremal subobjects of a set of objects is an extremal subobject of the product of the set.

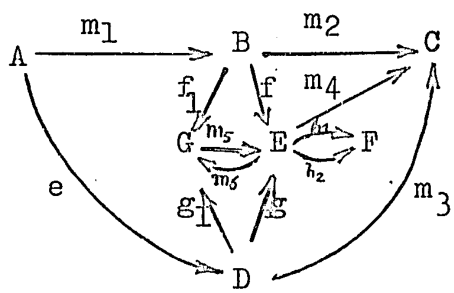
<u>Corollary 2</u> - The intersection of a family of extremal subobjects of a given object is again an extremal subobject. [5, p.8].

Proof - The subobject maps define a natural transformation from the diagram determined by the family of extremal sub-objects to the constant diagram consisting of the object. Since the left root of a constant diagram is itself and the left root of the first diagram is the intersection of the extremal subobjects, there must be an extremal mono from the intersection to the containing object. ▯

6.5 - In a well-powered, left complete category, if $m_1 : A \longrightarrow B$ and $m_2 : B \longrightarrow C$ are extremal monos, then $m_2 \circ m_1 : A \longrightarrow C$ is an extremal mono. [5, p.8].

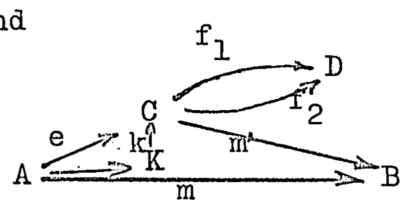
Proof. Suppose $m_2 \circ m_1 = m_3 \circ e$ where $e : A \longrightarrow D$ is epi and $m_3 : D \longrightarrow C$ is mono. Let (E, m_4) be the intersection of all those subobjects of C through which both m_2 and m_3 factor. By the definition of intersection, both factor through m_4 say $m_2 = m_4 \circ f$ and $m_3 = m_4 \circ g$; thus $m_4 \circ f \circ m_1 = m_4 \circ g \circ e$ and since m_4 is mono, $f \circ m_1 = g \circ e$. Now suppose $h_1, h_2 : E \longrightarrow F$ such that $h_1 \circ f = h_2 \circ f$. Let (G, m_5) be the difference kernel of (h_1, h_2). $h_1 \circ f \circ m_1 = h_2 \circ f \circ m_1$ and $h_1 \circ g \circ e = h_2 \circ g \circ e$. Since e is epi, $h_1 \circ g = h_2 \circ g$. By the definition of a difference kernel, f and g factor through m_5, say $f = m_5 \circ f_1$, $g = m_5 \circ g_1$. Then $m_2 = m_4 \circ m_5 \circ f_1$ and $m_3 = m_4 \circ m_5 \circ g_1$ (i.e. m_2 and m_3 factor through $m_4 \circ m_5$) and by the definition of intersection, m_4 must factor through $m_4 \circ m_5$, say $m_4 = m_4 \circ m_5 \circ m_6$.

Since m_4 is mono, $1_E = m_5 \circ m_6$. Also $m_5 = m_5 \circ m_6 \circ m_5$ and since m_5 is mono, $1_G = m_6 \circ m_5$ and m_5 is iso. Thus $h_1 = h_2$. Therefore f is epi. Since m_2 is extremal, f is iso. Thus m_3 factors through m_2. Since m_1 is extremal, e is iso. Thus $m_2 \circ m_1$ is extremal.▯



6.6 – In a well-powered, left complete category, any mono may be factored as an epi followed by an extremal mono.

Proof : Let $m : A \longrightarrow B$ be mono. Let (C, m') be the intersection of those extremal subobjects of B containing A. By corollary 2 to 6.4, m' is extremal. Suppose $m = m' \circ e$; we will show that e is epi. Suppose $f_1, f_2 : C \longrightarrow D$ such that $f_1 \circ e = f_2 \circ e$. Let (K,k) be the difference kernel of (f_1, f_2). e may be factored through k because it equalizes f_1 and f_2. However, k is extremal by 6.3 and $m' \circ k$ is extremal by 6.5. Thus m' may be factored through $m' \circ k$ uniquely, from which we may show that k is iso. Thus $f_1 = f_2$ and e is epi.



6.7 - In a factorable, left complete, well powered category $\mathcal{B}$, if $\mathcal{A}$ is a full subcategory that contains the extremal subobjects of each of its objects then $\mathcal{A}$ is a qer subcategory of $\mathcal{B}$.

<u>Proof</u> - Let $f : B \longrightarrow A$, where B is a $\mathcal{B}$-object and A is an $\mathcal{A}$-object. Since $\mathcal{B}$ is factorable, $f = m \circ e$ where $e : B \longrightarrow B'$ is epi and $m : B' \longrightarrow A$ is mono. Now by 6.6, $m = m' \circ e'$ where $e' : B' \longrightarrow A'$ is epi and $m' : A' \longrightarrow A$ is extremal mono. Thus A' is an $\mathcal{A}$-object. Also, $f = m \circ e = m' \circ e' \circ e$ where $e' \circ e : B \longrightarrow A'$ is epi, and $m' : A' \longrightarrow A$ is mono. Thus $\mathcal{A}$ is a qer subcategory of $\mathcal{B}$.

<u>Corollary 1</u> - A full, replete subcategory of a factorable, left complete, well-powered category is a qer subcategory iff it contains the extremal subobjects of each of its objects.

<u>Corollary 2</u> - A full, replete subcategory, $\mathcal{A}$, of a factorable, left complete, well powered, $\mathcal{A}$-cowell powered category, $\mathcal{B}$, is an epi-reflective subcategory iff it contains all $\mathcal{B}$-products and $\mathcal{B}$-extremal subobjects of its objects.

<u>Proof</u> : This follows from 3. and Corollary 1. ∎

Remark : This is a special case of the Freyd-Isbell theorem [6, p.1276].

6.8 - A qer subcategory of a well-powered, left complete, factorable category is factorable.

<u>Proof</u> : Let $\mathcal{A}$ be a qer subcategory of $\mathcal{B}$, a well powered, left complete, factorable category, and $f : A \longrightarrow A'$ an $\mathcal{A}$-map. f may be factored in $\mathcal{B}$ as $m \circ e$ where $m : B \longrightarrow A'$ is mono and $e : A \longrightarrow B$ is epi. B is a subobject of A' and $m = m' \circ e'$ where $e' : B \longrightarrow C$ is epi and $m' : C \longrightarrow A'$ is extremal mono (6.6). Thus $f = m' \circ (e' \circ e)$. C is in $\mathcal{A}$ by 6.2. Thus f may be factored in $\mathcal{A}$ as an epi followed by a mono and $\mathcal{A}$ is factorable.$\|$

7. Intermediate categories

Definition - If $\mathcal{A}$ is a reflective subcategory of $\mathcal{C}$; $\mathcal{B}$, a $\mathcal{C}$-epi-reflective subcategory of $\mathcal{C}$; and $\mathcal{A}$, a $\mathcal{B}$-epi-reflective subcategory of $\mathcal{B}$, then $\mathcal{B}$ is said to be an intermediate category of the pair $(\mathcal{A}, \mathcal{C})$.

Thus 4. states that if $\mathcal{C}$ is a factorable, cowell powered category with products and $\mathcal{A}$ is any reflective subcategory of $\mathcal{C}$, then the category of subobjects of $\mathcal{A}$ is an intermediate category of $(\mathcal{A}, \mathcal{C})$. The following theorem shows that under certain conditions on $\mathcal{C}$, there is a minimum intermediate category.

7.1 - If $\mathcal{A}$ is a reflective subcategory of a factorable, left complete, well powered, cowell powered category with products then the full subcategory $\mathcal{B}'$ of all $\mathcal{C}$-extremal subobjects of objects of $\mathcal{A}$ is a minimum intermediate category of $(\mathcal{A}, \mathcal{C})$.

Proof : By the same argument as in 4., $\mathcal{A}$ is a $\mathcal{B}'$-epi-reflective subcategory of $\mathcal{B}'$. (Note that, as in 4., $\mathcal{A}$ is a mono -$\mathcal{B}'$-epi-reflective subcategory of $\mathcal{B}'$).

Also, $\mathcal{A}$ contains all $\mathcal{C}$-products of its objects by corollary 1 to 2.1. Since by Corollary 1 to 6.4, the product of extremal subobjects of a set of objects is an extremal subobject of the product of the set, $\mathcal{B}'$ contains all $\mathcal{C}$-products of its objects.

By 6.5, a $\mathcal{C}$-extremal subobject of a $\mathcal{C}$-extremal subobject of an $\mathcal{A}$-object A, is again a $\mathcal{C}$-extremal subobject of A. Thus $\mathcal{B}'$ contains all of the $\mathcal{C}$-extremal subobjects of its objects. Thus by 6.7, Corollary 2, $\mathcal{B}'$ is a $\mathcal{C}$-epi-reflective subcategory of $\mathcal{C}$. Thus $\mathcal{B}'$ is an intermediate category. Since any intermediate category must contain $\mathcal{A}$ and by Corollary 2 to 6.7, must contain all $\mathcal{C}$-extremal subobjects of its objects, $\mathcal{B}'$ is minimal.▯

7.2 - Let $\mathcal{B}$ be a full subcategory of $\mathcal{C}$, a factorable, cowell powered category with products and let $\mathcal{X}$ be a subclass of the class of $\mathcal{B}$-maps. We define a full subcategory, $\mathcal{B}''$ of $\mathcal{C}$ as follows : B'' is a $\mathcal{B}''$-object iff any pair of maps from a $\mathcal{B}$-object, B, to B'' is distinguished by every $\mathcal{X}$-map into B. $\mathcal{B}''$ is a $\mathcal{C}$-epi-reflective subcategory of $\mathcal{C}$.

<u>Proof</u> : We show that $\mathcal{B}''$ is closed under formation of $\mathcal{C}$-subobjects. Let B be a $\mathcal{B}''$-object and C a $\mathcal{C}$-subobject of B. Suppose there are $\mathcal{B}$-objects, A, A', an $\mathcal{X}$-map $x : A \longrightarrow A'$ two maps $y_1, y_2 : A' \longrightarrow C$ such that $y_1 \circ x = y_2 \circ x$. Then if m is the subobject map of C, we have two maps $m \circ y_1, m \circ y_2 : A' \longrightarrow B$ such that $m \circ y_1 \circ x = m \circ y_2 \circ x$. But B is a $\mathcal{B}''$-object ; thus $m \circ y_1 = m \circ y_2$. Since m is mono, $y_1 = y_2$ and thus C is a $\mathcal{B}''$-object. By 2.9, $\mathcal{B}''$ is a qer subcategory of $\mathcal{C}$. Similarly, we show that $\mathcal{B}''$ is closed under $\mathcal{C}$-products.

Let $\{B_\lambda\}$ be a set of $\mathcal{B}$"-objects. Suppose there are $\mathcal{B}$-objects, A, A', an $\mathcal{X}$-map $x : A \longrightarrow A'$ and two maps $y_1, y_2 : A' \longrightarrow \prod B_\lambda$ such that $y_1 \circ x = y_2 \circ x$. For any λ, $p_\lambda \circ y_1 \circ x = p_\lambda \circ y_2 \circ x$ where p_λ is the λ-projection map. Since B_λ is $\mathcal{B}$"-object, $p_\lambda \circ y_1 = p_\lambda \circ y_2$. Since this is true for any λ and the evaluation map is unique, it follows that $y_1 = y_2$ and $\prod B_\lambda$ is a $\mathcal{B}$"-object.

By 3., $\mathcal{B}$" is a $\mathcal{C}$-epi-reflective subcategory of $\mathcal{C}$. ‖

7.3 - Let $\mathcal{A}$ be a reflective subcategory of $\mathcal{C}$, a factorable, cowell-powered category with products, and $\mathcal{B}$ be the full category of subobjects of objects of $\mathcal{A}$. Let $\mathcal{B}$" be defined as in 7.2 with $\mathcal{X}$ the class of $\mathcal{B}$-epis. Then $\mathcal{B}$" is an intermediate category.

Proof : By 7.2, $\mathcal{B}$" is a $\mathcal{C}$-epi-reflective subcategory of $\mathcal{C}$. Every $\mathcal{B}$"-object, B", has a reflection in $\mathcal{A}$, say R(B"). We must show that the reflection map r_B is $\mathcal{B}$"-epi. Since $\mathcal{C}$ is factorable, $r_{B''} = m \circ e$ where $e : B'' \longrightarrow B$ is $\mathcal{C}$-epi (thus $\mathcal{B}$"-epi) and $m : B \longrightarrow R(B'')$ is $\mathcal{C}$-mono. Thus B is a $\mathcal{C}$-subobject of R(B") and therefore a $\mathcal{B}$-object. Suppose $g_1, g_2 : R(B) \longrightarrow A$, A any $\mathcal{A}$-object, such that $g_1 \circ m = g_2 \circ m$. Then $g_1 \circ r_{B''} = g_2 \circ r_{B''}$. But R $(g_1 \circ r_{B''})$ is unique ; hence $g_1 = g_2$. Thus m distinguishes all distinct pairs of $\mathcal{A}$-maps whose domain is R(B") and by 2.6 is $\mathcal{B}$-epi.

$$\begin{tikzcd}
R(B'') \arrow[r, bend left, "g_1"] \arrow[r, bend right, "g_2"'] & A \\
B \arrow[u, "m"] & \\
B'' \arrow[u, "e"] \arrow[uu, bend left, "r_{B''}"] \arrow[uur] &
\end{tikzcd}
\qquad g_1 \circ r_{B''} = g_2 \circ r_{B''}$$

Since $m \in \mathcal{Z}$, it distinguishes all $\mathcal{B}''$-maps whose domain is R (B) and is thus $\mathcal{B}''$-epi. $r_{B''} = m \circ e$ is $\mathcal{B}''$-epi by 2.8. Thus $\mathcal{A}$ is a $\mathcal{B}''$-epi subcategory of $\mathcal{B}''$ and $\mathcal{B}''$ is an intermediate category. ‖

<u>Definition</u> - If $\mathcal{A}$ is a full subcategory of $\mathcal{C}$, we define a <u>potential intermediate category</u> (pic) of ($\mathcal{A}$, $\mathcal{C}$) to be one of the following full subcategories of $\mathcal{C}$:

(1) $\mathcal{B}$, whose objects are the $\mathcal{C}$-subobjects of $\mathcal{A}$

(2) $\mathcal{B}'$, whose objects are the $\mathcal{C}$-extremal subobjects of $\mathcal{A}$

(3) $\mathcal{B}''$, whose objects are those $\mathcal{C}$-objects, B", such that each $\mathcal{B}$-epi distinguishes every pair of distinct maps to B".

8. - Theorem

If $\mathcal{C}$ is a factorable, left complete, well powered, cowell powered category ; $\mathcal{A}$, a full, replete subcategory of $\mathcal{C}$ and $\mathcal{B}$, a pic of ($\mathcal{A}$, $\mathcal{C}$) then : $\mathcal{A}$ is a cowell powered reflective subcategory of $\mathcal{C}$ iff (1) $\mathcal{A}$ contains all $\mathcal{C}$ -products and $\mathcal{B}$ -extremal subobjects of its objects and (2) $\mathcal{B}$ is $\mathcal{A}$ -cowell powered.

Proof. $\Longleftarrow$ Since $\mathcal{A}$ contains all $\mathcal{C}$-products of its object so does $\mathcal{B}$ by 2.3, 6.4 Corollary 1, and 7.2. $\mathcal{B}$ contains all $\mathcal{C}$ -extremal subobjects of its objects by 2.8, 6.5, and 7.2. Thus by Corollary 2 of 6.7, $\mathcal{B}$ is $\mathcal{C}$ epi-reflective. By 6.8, $\mathcal{B}$ is factorable. By corollary 2 of 2.1, $\mathcal{B}$ is left complete. By corollary 2 of 2.5, $\mathcal{B}$ is well powered. Thus by (1) and Corollary 2 of 6.7, since all $\mathcal{B}$ -products are $\mathcal{C}$ -products, $\mathcal{A}$ is a $\mathcal{B}$ -epi-reflective subcategory of $\mathcal{B}$. By Corollary of 2.6 or definition of third type of pic, all $\mathcal{A}$ -epis are $\mathcal{B}$ -epis ; thus, $\mathcal{A}$ is cowell powered. By composition of reflections, $\mathcal{A}$ is a cowell powered reflective subcategory of $\mathcal{C}$. ▯

$\Longrightarrow$ By Corollary 1 of 2.1, $\mathcal{A}$ contains all $\mathcal{C}$ -products of $\mathcal{A}$ -objects. By 4, 7.1 or 7.3, $\mathcal{B}$ is an intermediate category. By 3. and 6.8, $\mathcal{B}$ is factorable. We can now apply 3. and 6.2 to see that $\mathcal{A}$ contains a copy of all $\mathcal{B}$ -extremal subobjects of its objects. $\mathcal{B}$ is $\mathcal{A}$ -cowell powered as in the proof of 5. ▯

9. Generation and Intersection of reflective subcategories

This chapter partially solves a problem posed by Isbell [4, p.33, problem 8].

9.1 - Let $\mathcal{C}$ be a factorable, left complete, well powered, cowell powered category and S, a class of $\mathcal{C}$-objects.

(1) - The full subcategory, $\mathcal{B}'$, of $\mathcal{C}$-extremal subobjects of products of elements of S is the smallest epi-reflective subcategory of $\mathcal{C}$ that contains S.

(2) - If $\mathcal{B}'$ is cowell powered, the full subcategory, $\mathcal{A}$, of $\mathcal{B}'$-extremal subobjects of products of elements of S is the smallest reflective subcategory of $\mathcal{C}$ that contains S.

Proof :

(1). By 6.5 and Corollary 1 of 6.4, $\mathcal{B}'$ contains all $\mathcal{C}$-extremal subobjects and $\mathcal{C}$-products of its objects. Thus by Corollary 2 of 6.7, $\mathcal{B}'$ is a $\mathcal{C}$-epi-reflective subcategory of $\mathcal{C}$. By 6.2, any qer subcategory and thus any epi-reflective subcategory containing S must contain $\mathcal{B}'$. ▯

(2). $\mathcal{B}'$ is factorable by 6.8, left complete by 2.1, Corollary 2, well powered by 2.5, Corollary 2 and cowell powered by hypothesis. Thus by the same reasoning as in (1), $\mathcal{A}$ is a $\mathcal{B}'$-epi-reflective subcategory of $\mathcal{B}'$, and a reflective subcategory of $\mathcal{C}'$.

Now suppose $\mathcal{A}'$ is a reflective subcategory of $\mathcal{C}$ and $S \subseteq \mathcal{A}'_{Ob}$. Let $\mathcal{B}^*$ be the full subcategory of $\mathcal{C}$ whose objects are the $\mathcal{C}$-extremal subobjects of $\mathcal{A}'$. It follows that $\mathcal{B}' \leq \mathcal{B}^*$. By 2.5, Corollary 3, every $\mathcal{B}^*$-epi between $\mathcal{B}'$-objects is a $\mathcal{B}'$-epi ; every $\mathcal{B}^*$-mono between $\mathcal{B}'$-objects is a $\mathcal{B}'$-mono. Let T be an $\mathcal{A}$-object, say (T,f) is a $\mathcal{B}'$-extremal subobject of $U \in S$. Factor f into $m \circ e$, where e is $\mathcal{B}^*$-epi, m is $\mathcal{B}^*$-mono. But by above, e is $\mathcal{B}'$-epi, m is $\mathcal{B}'$-mono. Thus e is iso and f is $\mathcal{B}^*$-extremal mono. (2.5, Corollary 1 is used to show that f is mono ; since $\mathcal{B}'$ is a reflective subcategory of $\mathcal{C}$ it is also a reflective subcategory of $\mathcal{B}^*$.). Thus by 7.1 and 6.2, T is an $\mathcal{A}'$-object. Thus $\mathcal{A}$ is the smallest reflective subcategory of $\mathcal{C}$ that contains S. ▯

9.2 - Let $\mathcal{C}$ satisfy the conditions of 9.1, and let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two reflective subcategories of $\mathcal{C}$.

(1). If $\mathcal{A}_1$ and $\mathcal{A}_2$ are epi-reflective subcategories, then so is $\mathcal{A}_1 \cap \mathcal{A}_2$.

(2). If $(\mathcal{A}_1, \mathcal{C})$ and $(\mathcal{A}_2, \mathcal{C})$ have a common intermediate category that is cowell powered, then $\mathcal{A}_1 \cap \mathcal{A}_2$ is reflective.

(3). If $\mathcal{B}_i$ is an intermediate category of $(\mathcal{A}_i, \mathcal{C})$ $i = 1, 2$, and if $\mathcal{B}_1 \cap \mathcal{B}_2$ is cowell powered, then $\mathcal{A}_1 \cap \mathcal{A}_2$ is a reflective subcategory with $\mathcal{B}_1 \cap \mathcal{B}_2$, an intermediate category of $(\mathcal{A}_1 \cap \mathcal{A}_2, \mathcal{C})$.

<u>Proof</u> :

(1) - By 6.7, Corollary 2, $\mathcal{A}_1$, and $\mathcal{A}_2$ contain all $\mathcal{C}$-products and $\mathcal{C}$-extremal subobjects of their objects. Therefore $\mathcal{C}$-products and $\mathcal{C}$-extremal subobjects of $\mathcal{A}_1 \cap \mathcal{A}_2$ -objects are in both $\mathcal{A}_1$, and $\mathcal{A}_2$, thus in $\mathcal{A}_1 \cap \mathcal{A}_2$. Again by 6.7, Corollary 2, $\mathcal{A}_1 \cap \mathcal{A}_2$ is an epi-reflective subcategory. $\square$

(2) - If $\mathcal{B}$ is the common intermediate category, then by (1) and the same reasoning as in 9.1 (2), $\mathcal{A}_1 \cap \mathcal{A}_2$ is a $\mathcal{B}$-epi-reflective subcategory of $\mathcal{B}$ and a reflective subcategory of $\mathcal{C}$. $\square$

(3) - By (1), $\mathcal{B}_1 \cap \mathcal{B}_2$ is an epi-reflective subcategory of $\mathcal{C}$. It satisfies the appropriate conditions as in 9.1 (2). $\mathcal{A}_1 \cap \mathcal{A}_2$ contains all of its $\mathcal{B}_1$ and $\mathcal{B}_2$ -products and -extremal subobjects, thus certainly all of its $\mathcal{B}_1 \cap \mathcal{B}_2$ -products and -extremal subobjects. Therefore by 6.7, Corollary 2, $\mathcal{A}_1 \cap \mathcal{A}_2$ is an epi-reflective subcategory of $\mathcal{B}_1 \cap \mathcal{B}_2$ and a reflective category of $\mathcal{C}$. $\mathcal{B}_1 \cap \mathcal{B}_2$ is an intermediate category of $(\mathcal{A}_1 \cap \mathcal{A}_2, \mathcal{C})$. $\square$

<u>Remark</u> : This theorem may be generalized to intersections of arbitrary sets of reflective subcategories.

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