A class of nonlinear partial differential equations and their applications in differential geometry

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2022-03-22

Requirements Statement

ACKNOWLEDGEMENTS

First and foremost, I would like to express my deep appreciation to my supervisor Prof. Pengfei Guan for his guidance and constant support. I'm really lucky to have him as my PhD supervisor. His insight into mathematics has influenced me significantly and has taught me how to be a good researcher. Many results in this thesis are enlightened by the conversations with him. This thesis can't be finished without his help.

I would convey my sincere gratitude to my committee member Prof. Jérôme Vétois, who has helped me a lot during my years at McGill. I would also thank Prof. Stancu Alina at Concordia University from whom I learned much about convex geometry. I would thank Prof. Valentino Tosatti and Prof. Xi-Nan Ma as well for their help in the past.

Parts of the thesis are the results in collaborations with Dr. Jiawei Liu and Min Chen during their visit at McGill University. In particular, Chapter 2 is taken from the paper [40] joint with Jiawei Liu and Chapter 3 is taken from the paper [17] joint with Min Chen. I would like to give my special thanks to their contributions in the collaborations.

I would also thank my friends Fengrui Yang, Yuzhen Cao, Zhao Wang and Dongliang Zhang, who has accompanied me during my time at McGill. I would thank specially Claire Heffernan who has helped translate the abstract of the thesis into French.

At last, I would like to thank my parents for their selfless love. They have given me much freedom so that I can pursue my own career and interests. I couldn't have gone so far without their support.

ABSTRACT

The thesis is about a class of nonlinear partial differential equations and their applications in differential geometry. The main threads consist of three parts.

The first part is about the Ricci flow starting from an embedded closed convex surface in $\mathbb{R}^3$. This considers the Ricci flow with initial value an embedded closed convex surface in $\mathbb{R}^3$ without regularity assumption. We prove the convergence in metric and uniqueness of the flow. We hope this can be used as the first step to give a PDE proof of Pogorelov's rigidity theorem about convex surfaces.

The second part is about the flows by powers of the Gauss curvature in space forms. We prove that the Gauss curvature type flow $X_t = -K^\alpha \nu$ in an $n+1$ dimensional simply connected space form $\mathbb{N}^{n+1}(\kappa)$ of constant curvature ($\kappa = \pm 1$) converges to a point in finite time $T^* > 0$ for any initial strictly convex smooth hypersurface and $\alpha > 0$. Moreover, we prove the convergence to a geodesic sphere after rescaling for $\alpha > \frac{1}{n+2}$. This is the complete analogue of the corresponding results in Euclidean space.

The last part consists of some partial results about the Weyl's embedding problem. It includes a new proof of the closedness of Weyl's embedding problem and some discussions of the variational problem.

ABRÉGÉ

La thèse porte sur une classe d'équations aux dérivées partielles non linéaires et leurs applications en géométrie différentielle. Les fils principaux se composent de trois parties.

Le première partie concerne le flux de Ricci à partir d'une surface convexe fermée plongée dans $\mathbb{R}^3$. Ceci considère le flux de Ricci avec pour valeur initiale une surface convexe fermée plongée dans $\mathbb{R}^3$ sans hypothèse de régularité. Nous prouvons la convergence en métrique et l'unicité du flux. Nous espérons que cela pourra être utilisé comme première étape pour fournir une preuve du théorème de rigidité de Pogorelov sur les surfaces convexes en utilisant des EDP.

La deuxième partie concerne les écoulements des puissances de la courbure de Gauss dans les formes spatiales. Nous montrons que la courbure de Gauss de type de flux $X_t = -K^\alpha \nu$ dans une forme spatiale $\mathbb{N}^{n+1}(\kappa)$ simplement connexe de dimension $n+1$ de courbure constante ($\kappa = \pm 1$) converge vers un point en temps fini $T^* > 0$ pour une hypersurface lisse strictement convexe initiale et $\alpha > 0$. De plus, nous prouvons la convergence vers une sphère géodésique après redimensionnement pour $\alpha > \frac{1}{n+2}$. C'est l'analogue complet des résultats correspondants dans l'espace euclidien.

La dernière partie consiste de quelques résultats partiels sur le problème de plongement de Weyl. Cela inclut une nouvelle preuve du problème de la fermeture de plongement de Weyl et quelques discussions sur le problème variationnel.

TABLE OF CONTENTS

CHAPTER 1
Introduction

The thesis is about a class of nonlinear PDEs and their applications in differential geometry. This is a topic of long history. Basically speaking, there are two kinds of PDEs that are most often used in differential geometry: the elliptic PDEs and the parabolic ones. The early applications are mainly about elliptic PDEs. For instance, by solving Monge-Ampère type PDEs, Nirenberg [48], Cheng-Yau [18] solved the Minkowski problem, and Yau [69] solved the Calabi's conjecture. In the last four decades, the parabolic PDEs has began to play more important roles. A typical example is the crucial use of Ricci flow in the solution of the Poincaré conjecture [36, 50, 51] and the differentiable sphere theorem [13].

In this thesis, I will use both the elliptic and parabolic PDEs to study the problems in differential geometry. This consists of three parts. The first part is based on a joint work with Jiawei Liu [40], and the second part is based on a joint work with Min Chen [17]. These two parts are about parabolic PDEs. Finally, the third part is related to an elliptic PDE [39] . I will introduce all these three parts in each of the following sections and introduce the notations in Riemannian geometry which will be used in the thesis in the last section of this chapter.

1.1 Ricci flow starting from an embedded closed convex surface in $\mathbb{R}^3$

The Ricci flow is the parabolic system

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij}, \tag{1.1.1}$$

where $g_{ij}(t)$ is a family of Riemannian metrics on a fixed smooth manifold M, and R_{ij} is the Ricci curvature of the metric g_{ij}. It was first introduced by Hamilton [36] in 1982 and has many applications in differential geometry and topology since its birth. For example, Perelman [50, 51] gave the solution of Poincaré conjecture, and Brendle-Schoen [13] proved the differentiable sphere theorem via Ricci flow.

The reason that Ricci flow is so powerful for studying geometric objects is that the parabolic PDE has many good properties. One of them is its smoothing effect which will evolve the geometric objects into more regular ones as time evolves. It looks that this effect is far from being fully utilized. Recently, Simon [64, 65] studied Ricci flow admitting a class of irregular metric spaces of dimension two or three as metric initial condition. This can be seen as an approximation of irregular spaces by Ricci flow since the Ricci flow will converge to the irregular initial condition as $t \to 0^+$. In particular Simon got the following result

Theorem 1.1.1 ([65] Theorem 7.1)**.** *Let (M, g_0) be a complete smooth surface without boundary such that*

$$\begin{aligned} &(a)\ Ricci(g_0) \geqslant k; \\ &(b)\ vol({}^{g_0}B_1(x)) \geqslant v_0 > 0 \text{ for all } x \in M; \\ &(c)\ \sup_M |Riem(g_0)| < \infty. \end{aligned} \tag{1.1.2}$$

Then there are constants $c_1 = c_1(v_0, k) > 0$*,* $c_2 = c_2(v_0, k) > 0$*,* $S = S(v_0, k) > 0$ *and* $K = K(v_0, k)$ *and a solution* $(M, g(t))_{t\in[0,T)}$ *to Ricci flow which satisfies* $T \geqslant S$*, and*

$$
\begin{aligned}
&(a_t)\ Ricci(g(t)) \geqslant -K^2;\\
&(b_t)\ vol({}^{g_t}B_1(x)) \geqslant \frac{v_0}{2} > 0 \text{ for all } x \in M \text{ and } t \in (0,T);\\
&(c_t)\ \sup_M |Riem(g(t))| \leqslant \frac{K^2}{t} \text{ for all } t \in (0,T);\\
&(d_t)\ d(p,q,s) - c_2(\sqrt{t} - \sqrt{s}) \leqslant d(p,q,t) \leqslant e^{c_1(t-s)} d(p,q,s),\\
&\qquad \text{for all } 0 < s \leqslant t < T \text{ and } p,\ q \in M.
\end{aligned}
\tag{1.1.3}
$$

(Note that the estimates are trivial for $t = 0$*.)*

Based on Simon's work, Richard [58] studied the existence and uniqueness of Ricci flow whose metric initial condition is a closed Alexandrov surface with curvature bounded from below. This provides a canonical approximation for an Alexandrov surface with curvature bounded from below.

Theorem 1.1.2 ([65] Theorem 1.11 and [58] Theorem 0.5)**.** *Let* (M_i, g_i) *be a sequence of smooth surfaces satisfying (a), (b) and (c) in* (1.1.2) *with uniform constants* k *and* v_0 *for all* i *and let* (X, d) *be a Gromov-Hausdorff limit of this sequence. Let* $(M_i, g_i(t))_{t\in[0,T)}$ *be the solutions to Ricci flows starting from* (M_i, g_i) *in Theorem 1.1.1. Then (after taking a subsequence if necessary) there exists smooth Ricci flow* $(M, g(t))_{t\in(0,T)} := \lim_{i\to\infty} (M_i, g_i(t))_{t\in(0,T)}$ *satisfying* (a_t)*,* (b_t)*,* (c_t)*,* (d_t) *in* (1.1.3) *and* $(M, d_{g(t)})$ *converge to* (X, d) *in Gromov-Hausdorff sense as* $t \to 0$*.*

If $(N, \hat{g}(t))_{t\in(0,T)}$ *is another smooth Ricci flow starting from* (X, d) *in the same sense as* $(M, g(t))_{t\in(0,T)}$ *above, then there is a diffeomorphism* $\varphi : M \to N$ *such that* $g(t) = \varphi^* \hat{g}(t)$*.*

We note that the Ricci flow $(M, g(t))_{t\in(0,T)}$ converge uniformly to (X, d) up to an isometry in Theorem 1.1.2. A natural question is when such a Ricci flow will converge to the initial metric in classical sense (without isometry), and what kind of uniqueness one can claim. In chapter 2, when the metric initial condition (X, d) is an embedded closed convex surface in $\mathbb{R}^3$, by using smooth approximation, we prove that there is a Ricci flow on X such that the induced distance along this flow converge uniformly to d as $t \to 0$ and that such flows keep the isometries between their metric initial conditions.

We remark that the convex surface in chapter 2 is in the sense of Alexandrov (see section 2.1) and that both the C^∞-topology and the induced metric d of X as an embedded surface in $\mathbb{R}^3$ in our situation are induced by the radial graph parameterization from the unit sphere $(\mathbb{S}^2, \delta)$ there, where δ is the standard metric on $\mathbb{S}^2$. Also, throughout the construction of the smooth approximation of (X, d), we fix this radial graph parameterization (see the proof of Lemma 2.2.1 for details).

Our first result about the Ricci flow starting from an embedded closed convex surface in $\mathbb{R}^3$ is the following existence theorem.

Theorem 1.1.3 ([40] Theorem 1.3)**.** *If (X, d) is an embedded closed convex surface in $\mathbb{R}^3$, then there exists a $T > 0$ and a smooth Ricci flow $(X, g(t))_{t\in(0,T)}$ such that the distance functions $d_{g(t)}$ induced by $g(t)$ converge uniformly to d as $t \to 0$, that is,*

$$\lim_{t\to 0} \max_{p,q\in X} |d_{g(t)}(p, q) - d(p, q)| = 0. \tag{1.1.4}$$

The difference between Theorem 1.1.3 and Theorem 1.1.2 is that we remove the isometry coming from the Gromov-Hausdorff convergence in Theorem 1.1.2 when the metric initial condition (X, d) is an embedded closed convex surface in $\mathbb{R}^3$. This is due to that we can construct a sequence of smooth convex surfaces that approximate (X, d) in Hausdorff sense (Lemma 2.2.1) instead of the Gromov-Hausdorff approximation in Theorem 1.1.2. Since the Hausdorff convergence is stronger than Gromov-Hausdorff convergence, we can prove some stronger results. In fact, removing the isometry is crucial for proving the uniqueness of such Ricci flow. Moreover, we hope this improved smooth approximation to the metric initial condition can be used to study the rigidity problem of closed convex surfaces in $\mathbb{R}^3$ (see section 2.3).

In the following, unless otherwise specified, by saying that $(X, g(t))_{t\in(0,T)}$ is a Ricci flow admitting an embedded closed convex surface (X, d) in $\mathbb{R}^3$ as metric initial condition, we mean that it is a Ricci flow in the sense of Theorem 1.1.3.

The second result of chapter 2 is the following uniqueness theorem.

Theorem 1.1.4 ([40] Theorem 1.5)**.** *Assume that (X_1, d_1) and (X_2, d_2) are two non-degenerate embedded closed convex surfaces in $\mathbb{R}^3$ and $f : (X_1, d_1) \to (X_2, d_2)$ is an isometry. Let $(X_1, g_1(t))_{t\in(0,T)}$ and $(X_2, g_2(t))_{t\in(0,T)}$ be Ricci flows admitting (X_1, d_1) and (X_2, d_2) as metric initial conditions respectively. Then $g_1(t) = f^*g_2(t)$.*

In Theorem 1.1.2, if we assume that $(M, d_{g(t)})$ and $(N, d_{\hat{g}(t)})$ converge uniformly as $t \to 0$ to metric spaces $(M, \tilde{d})$ and $(N, \hat{d})$ respectively. Then from the existence part of Theorem 1.1.2, both $(M, \tilde{d})$ and $(N, \hat{d})$ are isometric to (X, d), and so there is an isometry ϕ between $(M, \tilde{d})$ and $(N, \hat{d})$. But ϕ may not be differentiable, so it may not be the diffeomorphism φ between M and N in Theorem 1.1.2. Here, when the metric initial conditions are embedded closed convex surfaces in $\mathbb{R}^3$, Theorem 1.1.4 means that the Ricci flows obtained in Theorem 1.1.3 keep the isometries between their metric initial conditions, which implies that $\varphi = \phi$ in this case. The main step is to prove that the isometry between the two metric initial conditions in this case is differentiable (Lemma 2.2.8). Therefore, the pull back metrics under this isometry still satisfy Ricci flow equation. Then Theorem 1.1.4 follows from a uniqueness of Ricci flow.

The motivation we consider the Ricci flow starting from embedded closed convex surfaces is to use the Ricci flow to study rigidity of the convex surfaces in the sense of Alexandrov. Pogorelov's famous rigidity theorem [52] says that any two isometric convex surfaces embedded in $\mathbb{R}^3$ are congruent. This is the generalization of Cohn-Vesson's classical rigidity result [23] about smooth convex surfaces to Alexandrov sense. Since there is no regularity assumption on such surfaces, Pogorelov's theorem is highly non-trivial. Pogorelov [55] gave a proof of the theorem by using his theory of convex surfaces, but it is still inaccessible to many geometers. We hope to use the results here to give a new proof of Pogorelov's rigidity theorem from the PDE point in the future.

1.2 Flow by powers of the Gauss curvature in space forms

Ricci flow is a kind of intrinsic flow since its description only needs the intrinsic geometric quantities of the manifold. Another kind of parabolic flows in geometric analysis is the extrinsic flows which describes the evolutions of an embedding of a manifold into an ambient space. When the codimension of the submanifold in the ambient space is one, the extrinsic flow is called a hypersurface flow. One important hypersurface flow is the flow by Gauss curvature.

Gauss curvature flow was first introduced by Firey [27] to model the erosion of strictly convex stones as they tumble on a beach. To obtain the model, Firey assumed that the stones were of uniform density, that their wear was isotropic, and that the number of collisions in a region was proportional to the set of normal directions of the region. In this case the rate of wear is proportional to the density per unit surface area of contact directions, which is the Gauss curvature. If we denote X the position vector of the stones, K the Gauss curvature of the boundary surface, and ν the unit outer normal of the boundary surface. Then the mathematical formulation of the problem is

$$X_\tau(x,\tau) = -K(x,\tau)\nu(x,\tau)$$

This is the original Gauss curvature flow.

More generally, we consider the flow of convex hypersurfaces $X(\cdot,\tau) : M \to \mathbb{R}^{n+1}$ by powers of Gauss curvature:

$$X_\tau(x,\tau) = -K^\alpha(x,\tau)\nu(x,\tau), \quad \alpha > 0, \tag{1.2.1}$$

with a strictly convex initial hypersurface, where X is the position vector of the hypersurface, $\nu(x,\tau)$ is the unit outer normal at $M_\tau = X(x,\tau)$ and $K(x,\tau)$ is the Gauss curvature at $X(x,\tau)$.

It was proved in [66] for $\alpha = 1$, and in [21] for any $\alpha > 0$ that the flow shrinks to a point in finite time $T^* > 0$ for any smooth strictly convex initial hypersurfaces M. A Harnack type inequality for Gauss curvature flow of compact convex hypersurfaces for all $\alpha > 0$, and an entropy estimate for $\alpha = 1$ were proved in [22]. Hamilton [37] used these results to get the sharp upper bound of Gauss curvature and the diameter.

The main interest is to understand the asymptotic behavior of the flows (1.2.1) as the time τ approaches the singular time T^*.

When $n = 1, \alpha = 1$, (1.2.1) is the curve shortening flow, convergence to circle was proved by Gage-Hamilton [28] for initial convex curve, and Grayson [31] for general initial curve. Convergence to a circle was proved for $n = 1$ and $\alpha > 1$ in [4], for $n = 1$ and $\frac{1}{3} < \alpha < 1$ in [6] with convex initial curve. For general $n > 1$, Chow [21] analyzed the case $\alpha = \frac{1}{n}$ and proved that solutions of the normalized flow converge to the unit sphere as $t \to \infty$. The convergence to sphere when $n = 2, \alpha = 1$ was established by Andrews in [2], see also [7] for the case $n = 2$ and $\frac{1}{2} < \alpha < 1$. The exponent $\alpha = \frac{1}{n+2}$ is critical as it's the affine curvature flow. In this case, the convergence to ellipsoids was established by Andrews in [5] (see also [59] for $n = 1$). Convergence to solitons was established for $\alpha \in (\frac{1}{n+2}, \frac{1}{n})$ in [3] for a family of anisotropic Gauss curvature flows (more general situation). For the normalized flow of (1.2.1) with strictly convex initial hypersurfaces in $\mathbb{R}^{n+1}, \forall n \geq 1$, the convergence to solitons (self-similar solutions) was established for the case $\alpha = 1$ by Guan-Ni [32], and by Andrews-Guan-Ni [9] for $\forall \alpha > \frac{1}{n+2}$. In [9], the uniqueness of soliton (round sphere) was proved when it is centrally symmetric. The final resolution of the uniqueness of solitons of the normalized flow of (1.2.1) was obtained by Choi-Daskalopoulos in [19] ($\frac{1}{n} < \alpha < 1 + \frac{1}{n}$) and by Brendle-Choi-Daskalopoulos [12] for all $\alpha > \frac{1}{n+2}$.

Parabolic flows for hypersurfaces in general Remannian manifolds were also considered by many authors. Generalization of flows by mean curvature in Euclidean space to general Riemannian manifold [41] was a fundamental contribution by Huisken. More recently, a new type of mean curvature flow in space forms was introduced by Guan and Li [33]. Gerhardt [30] demonstrated a correspondence between contracting and expanding flows of hypersurfaces in the sphere. Andrews, Han, Li and Wei [10] generalized Andrew's noncollapsing estimates for curvature flows in Euclidean space to fully nonlinear curvature flows in space forms.

It is natural to consider flows by powers of Gauss curvature in more general ambient spaces. Very little is known except for the case $\alpha = 1$, $n = 2$ or $\alpha = 1$, $n \geq 3$ and initial hypersurfaces are axially symmetric [47].

In the second part of the thesis (chapter 3), we establish complete analogous results of the flow by power of Gauss curvature in space forms:

$$\begin{cases} \tilde{X}_\tau(x,\tau) = -\tilde{K}^\alpha(x,\tau)\nu(x,\tau), \\ \quad \tilde{X}(0) = \tilde{X}_0, \end{cases} \tag{1.2.2}$$

where $\nu(x,\tau)$ is the unit outer normal at $\tilde{X}(x,\tau)$ and $\tilde{K}(x,\tau)$ is the Gauss curvature of $\tilde{M}_\tau$, $\mathbb{N}^{n+1}(\kappa)$ is the $(n+1)$ dimensional simply connected space form of constant sectional curvature $\kappa = \pm 1$ (the tildes distinguish these from the normalized counterparts introduced later). Below is our main theorem.

Theorem 1.2.1 ([17] Theorem 1.1)**.** *If $\tilde{X}_0$ represents a strictly convex smooth hypersurface in $\mathbb{N}^{n+1}(\kappa)$, then for any $\alpha > 0$, the initial value problem (1.2.2) has a unique solution on a maximum finite time interval $[0, T^*)$ such that the $\tilde{M}_\tau$ converges to a point as $\tau \to T^*$. Moreover, for $\alpha > \frac{1}{n+2}$, $\tilde{M}_\tau$ converges to a geodesic sphere in $\mathbb{N}^{n+1}(\kappa)$ in the C^∞-topology after re-scaling.*

The theorem generalizes the known results in Euclidean space to space forms. The first statement is a generalization of [66, 21]. The second statement extends results in [32, 9, 12].

Our approach to flow (1.2.2) is to reduce it to a flow in the Euclidean space by proper projections. Choosing the projection π_p (see details in section 3.1), the Gauss curvature of the image satisfies (3.1.6). It suffices to consider the following type of flow (the image of projection) in Euclidean space:

$$\hat{\tilde{X}}_\tau(x,\tau) = -(1+\kappa|\hat{\tilde{X}}|^2)^{\frac{n+2}{2}\alpha+\frac{1}{2}}(1+\kappa\langle\hat{\tilde{X}},\hat{\nu}\rangle^2)^{-\frac{n+2}{2}\alpha+\frac{1}{2}}\hat{\tilde{K}}^\alpha, \tag{1.2.3}$$

where $\kappa = 1$ when $\tilde{X}(x,\tau)$ is the flow of convex hypersurfaces in $\mathbb{S}^{n+1}$ and where $\kappa = -1$ when $\tilde{X}(x,\tau)$ is the flow of convex hypersurfaces in $\mathbb{H}^{n+1}$ (the hat distinguish these from the counterparts before the projection). It is well known that any strictly convex hypersurfaces $\hat{\tilde{M}}$ in $\mathbb{R}^{n+1}$ can be recovered completely from the support function u by $\hat{\tilde{M}} = \{\nabla\hat{\tilde{u}} + \hat{\tilde{u}}x, x \in \mathbb{S}^n\}$, see e.g. [60]. Then the support function satisfies equation

$$\hat{\tilde{u}}_\tau(x,\tau) = -\left(1+\kappa(\hat{\tilde{u}}^2+|\nabla\hat{\tilde{u}}|^2)\right)^{\frac{n+2}{2}\alpha+\frac{1}{2}}(1+\kappa\hat{\tilde{u}}^2)^{-\frac{n+2}{2}\alpha+\frac{1}{2}}\det{}^{-\alpha}(\nabla^2\hat{\tilde{u}}+\hat{\tilde{u}}I). \tag{1.2.4}$$

For flow (1.2.4), we obtain the estimates of the lower bound of principal curvatures and the upper bound of the Gauss curvature and pinching estimate of the inner and outer radii.

The key in the proof is an almost monotonicity formula for associated entropies considered in [32, 9]. In this respect, the normalized flow (3.3.6) of (1.2.4) will be used in section 3.3. A crucial observation is the decay estimate (3.3.11) in section 3.3.

It allows us to obtain a monotone quantity $\mathcal{E}_\alpha(\hat{\Omega}_t) + C(n, \alpha, \tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}t}$ along the normalized flow (3.3.6) by modifying the monotone quantity used in [32, 9]. From this, we can use the methods in [32, 9] to obtain a uniformly lower and upper bound of support function. This in turn implies a uniform C^2-estimate, and to conclude that the normalized flow for any smooth initial convex body converges smoothly as $t \to \infty$ to a uniformly convex soliton. By the soliton classification result in [12], we obtain that the limit is a round sphere. This implies the convergence of the normalized flow in $\mathbb{N}^{n+1}(\kappa)$ for $\alpha > \frac{1}{n+2}$.

1.3 A warped product metric and the Weyl problem

The Weyl problem is an isometric embedding problem in differential geometry. It was introduced by Weyl [68] in 1916. The problem concerns the realization of a closed surface $(\mathbb{S}^2, g)$ with positive Gauss curvature as a convex surface in $\mathbb{R}^3$. More precisely, suppose we have a closed surface $M = (\mathbb{S}^2, g)$ whose metric g has positive Gauss curvature $K_g > 0$, can we embed M into $\mathbb{R}^3$ with induced metric exactly g ? The problem was solved by Nirenberg [48] and Pogorelov [53] independently for smooth g. Before that, Levi [45] solved the problem when the metric g is analytic. The problem also makes sense for more general ambient spaces. The problem in hyperbolic space was considered by Pogorelov [54] for $K_g > -1$, he also considered the embedding into general 3−dimensional Riemannian manifolds [55]. The problem has attracted renewed attention recently due to its relation to the definition of quasi-local mass in general relativity [67].

In the third part of the thesis (chapter 4), we first give a new proof of the closedness of Weyl problem by using the warped product metric introduced in [42].

Theorem 1.3.1 ([39] Theorem 1)**.** *Suppose $M = (\mathbb{S}^2, g)$ is a smooth closed surface with positive Gauss curvature K, then $\mathcal{I}$ is closed, where $\mathcal{I}$ is defined in (4.2.15).*

In [42], Izmestiev also showed that the Euclidean embedding of $(\mathbb{S}^2, g)$ is the critical point of the Hilbert-Einstein (HE) functional (see (4.0.2) for the definition of HE) in the warped product metric class related to the metric g. We will discuss the variation and the stability of the HE functional at the critical point there. In particular, we get the following result

Theorem 1.3.2 ([39] Theorem 2)**.** *Let δ be the standard metric on $\mathbb{S}^2$, $\sqrt{2\rho(0)} \equiv 1$ (be the radial function of the embedding of $(\mathbb{S}^2, \delta)$ into $\mathbb{R}^3$ with origin at the center of the embedding). Let $\eta(x)$ be a smooth function on $(\mathbb{S}^2, \delta)$ s.t. $\int_{\mathbb{S}^2} \eta(x) d\delta = 0$, $\varepsilon > 0$ small, $\mathcal{A}_\eta = \{\rho(t, x) = \rho(0) + t\eta(x) \text{ admissible} | t \in (-\varepsilon, \varepsilon),\ x \in (\mathbb{S}^2, \delta)\}$, then $\rho(0)$ is a local maximum of HE in $\mathcal{A}_\eta$.*

The motivation for this part is that I hope the study of the warped product metric there can develop some new insights into the quantitive rigidity and openness of the Weyl problem. More precisely, the quantitive rigidity problem (or the stability problem) concerns how much will two embedded convex surfaces differ in $\mathbb{R}^3$ if their intrinsic metric differ by ε ? This is a natural question in Weyl's embedding problem. On the other hand, Li-Wang [46] recently established the openness of the isometric embedding into a warped product space. This combined with Guan-Lu's result [34] of closedness provided the isometric embedding of $(\mathbb{S}^2, g)$ into $(\mathbb{N}^3, \bar{g})$ which contains no horizon. It is not clear if Li-Wang's proof can be applied to the case when the embedding contains a horizon, since their proof needs the path of the embedding to keep the system elliptic when using method of continuity, while such a path is not known yet in this case. This forms a barrier for its wide applications in general relativity. I hope the research here can shed some lights for the openness of the embedding when the embedding contains a horizon.

1.4 Notations and conventions

We conclude the introduction by introducing our conventions for notations in Riemannian geometry which will be used throughout the thesis. The notations and preliminaries used in each chapter will be introduced at the beginning of that chapter.

Let (M, g) be a smooth Riemannian manifold with metric g, ∇ be the unique Levi-Civita connection associated to g. For any smooth vector fields X, Y, Z on M, we use the definition

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{1.4.1}$$

for the Riemannian curvature tensor.

Let $(e_1, \dots, e_n)$ be a local frame for the tangent bundle TM, we use the convention

$$R(e_i, e_j)e_k = R^l_{ijk} e_l \tag{1.4.2}$$

and

$$R_{ijkl} = \langle R(e_i, e_j)e_k, e_l \rangle_g = R^m_{ijk} g_{ml} \tag{1.4.3}$$

for the index of curvature tensor.

We define the Ricci curvature and scalar curvature by

$$\begin{aligned} R_{ij} &= g^{kl} R_{iklj} \\ R &= g^{ij} R_{ij} \end{aligned} \tag{1.4.4}$$

Moreover, let f be a smooth function on M, we use the notation

$$f_{ij} = \nabla_{e_j}\nabla_{e_i} f = e_j(e_i f), \quad f_{ijk} = \nabla_{e_k}\nabla_{e_j}\nabla_{e_i} f = e_k(e_j(e_i f)), \quad etc. \tag{1.4.5}$$

for the higher covariant derivatives of f. In particular, we have the following Ricci identity

$$\begin{aligned} f_{ijk} - f_{ikl} &= R^l_{jki} f_l \\ f_{ijkl} - f_{ijlk} &= R^m_{kli} f_{mj} + R^m_{klj} f_{im} \end{aligned} \tag{1.4.6}$$

The organization of this thesis is as follows. In Chapter 2, we prove the existence and uniqueness of the Ricci flow starting from an embedded closed convex surface in $\mathbb{R}^3$. We give a short discussion of our plan to apply the result there to give a PDE proof of Pogorelov's rigidity theorem. In Chapter 3, we give the convergence of the flow by $\alpha - th$ power of Gauss curvature to a point for $\alpha > 0$ and convergence to a geodesic sphere after rescaling for $\alpha > \frac{1}{n+2}$ with initial smooth strictly convex hypersurface in the sphere and hyperbolic space. In Chapter 4, we give a new proof of the closedness of Weyl problem and discuss the corresponding variational problem.

CHAPTER 2
Ricci flow starting from an embedded closed convex surface in $\mathbb{R}^3$

In this chapter, we prove the results on Ricci flow starting from an embedded closed convex surface in $\mathbb{R}^3$. In section 2.1, we recall some basic facts about convex surfaces in the sense of Alexandrov. Then, in the second section, we first prove the existence result by constructing a sequence of smooth convex surfaces that approximate the metric initial condition in Hausdorff distance (Lemma 2.2.1) and using Theorem 1.1.1. Then we prove the uniqueness of such Ricci flow by giving an exact expression of the initial metric (Theorem 2.2.7). Finally, in section 2.3, we introduce our plan which aims to study the rigidity of convex surfaces by using Ricci flow.

2.1 Preliminaries

In this section, we recall some basic results about convex surfaces in the sense of Alexandrov. These are mainly taken from [1, 55], see also the Appendix of [58].

Let (X, d) be a metric space, it is called a geodesic metric space if any two points a and b in X can be connected by a continuous path of shortest length on X. Suppose (X_1, d_1) and (X_2, d_2) are two metric spaces, an isometry between X_1 and X_2 is a bijection $f : X_1 \to X_2$ such that

$$d_2(f(a), f(b)) = d_1(a, b), \quad for\ all\ a,\ b \in X_1.$$

Let a, b and c be three different points in a geodesic metric space (X, d), we define the comparison angle $\tilde{\angle} a_b^c$ as the angle at $\tilde{a}$ of the triangle $\tilde{a}\tilde{b}\tilde{c}$ in S_0 whose sides have length $d_0(\tilde{a}, \tilde{b}) = d(a, b)$, $d_0(\tilde{a}, \tilde{c}) = d(a, c)$ and $d_0(\tilde{b}, \tilde{c}) = d(b, c)$, where S_0 is the Euclidean space, and d_0 is the standard distance in S_0.

Definition 2.1.1. Let (X, d) be a geodesic metric space, it is said to satisfy the convexity condition if for any point $a \in X$, and any two shortest paths $(\gamma_1(s))_{s\in[0,T]}$ and $(\gamma_2(s))_{s\in[0,T]}$ in X parameterized by arc length issuing from a, the comparison angle $\tilde{\angle} a_{\gamma_1(s)}^{\gamma_2(t)}$ is an non-increasing function of s and t.

Definition 2.1.2. Let (X, d) be a geodesic metric space, it is called a closed convex surface in the sense of Alexandrov, if it is at the same time a compact topological surface without boundary, and satisfies the convexity condition.

We also have the following equivalent definition.

Definition 2.1.3. A closed convex surface in the sense of Alexandrov is a geodesic metric space (X, d) which is at the same time a compact topological surface without boundary and a metric space with non-negative curvature in the sense of Alexandrov.

A geodesic metric space has non-negative curvature in the sense of Alexandrov if its geodesic triangles are bigger than the geodesic triangles in S_0. To be more precise, a geodesic metric space (X, d) has non-negative curvature in the sense of Alexandrov if and only if the following condition is satisfied:

Let a, b and c be any three points in (X, d), and m be any point on a shortest path from b to c. Let $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ be points in S_0 such that $d_0(\tilde{a}, \tilde{b}) = d(a, b)$, $d_0(\tilde{a}, \tilde{c}) = d(a, c)$ and $d_0(\tilde{b}, \tilde{c}) = d(b, c)$. If $\tilde{m}$ is a point on $\tilde{b}\tilde{c}$ such that $d_0(\tilde{b}, \tilde{m}) = d(b, m)$. Then $d(a, m) \geqslant d_0(\tilde{a}, \tilde{m})$.

In the following of this chapter, we call a closed convex surface in the sense of Alexandrov a closed convex surface if there is no confusion. By Toponogov's theorem, every closed smooth surface with non-negative Gauss curvature is a closed convex surface. The boundary of a convex set with the induced metric in $\mathbb{R}^3$ is also a closed convex surface (Theorem 10.2.6 in [14]). Alexandrov proved the following isometric embeding theorem.

Theorem 2.1.4. *(Page 269 in [1]) Any closed convex surface (X, d) can be isometrically embedded into $\mathbb{R}^3$ as the boundary of a (possibly degenerate) convex body.*

The following lemma will be used in the proof of Theorem 1.1.3. Its geometric meaning is that the Hausdorff distance of two convex surfaces in $\mathbb{R}^3$ controls their intrinsic distance functions.

Lemma 2.1.5. *(Theorem 2 in Chapter 3 of [1]) For every closed convex surface F and for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever the deviation of a closed convex surface S from F is less than δ and the distance of some points X and Y on F from some points A and B on S are also less than δ, we have*

$$|d_F(X, Y) - d_S(A, B)| < \varepsilon,$$

where d_F and d_S are the distance functions on F and S respectively.

2.2 Existence and Uniqueness of Ricci flow

In this section, assuming that (X, d) is an embedded closed convex surface in $\mathbb{R}^3$, we prove the existence and uniqueness of the Ricci flow that admits (X, d) as a metric initial condition. As an embedded closed convex surface in $\mathbb{R}^3$, the smooth topology of X and metric d are induced by the radial graph parameterization from the unit sphere $(\mathbb{S}^2, \delta)$, and the smooth approximation we construct are also from this fixed parameterization.

2.2.1 Existence of the Ricci flow

In this subsection, we prove the existence of Ricci flow $(X, g(t))_{t\in(0,T)}$ that admits (X, d) as metric initial condition. First, we have the following lemma.

Lemma 2.2.1. *Let (X, d) be an embedded closed convex surface in $\mathbb{R}^3$. Then there exists a sequence of closed smooth convex surfaces $\{(X_i, \tilde{g}_i)\}_{i=1}^{\infty}$ and bijections $f_i : X \to X_i$ such that $d_i(x, y)$ converges to $d(x, y)$ uniformly for $x,\ y \in X$ as $i \to \infty$, where $d_i(x, y) = \tilde{d}_i(f_i(x), f_i(y))$, and $\tilde{d}_i$ is the distance on X_i induced by $\tilde{g}_i$.*

The proof depends on Lemma 2.1.5 and the following lemma, which is about approximating a convex body by smooth convex bodies in Hausdorff sense in $\mathbb{R}^3$.

Lemma 2.2.2. *(Theorem 3.4.1 in [60]) Let $\varepsilon > 0$ and let $\varphi : [0, \infty) \to [0, \infty)$ be a function of class C^∞ with support in $[\frac{\varepsilon}{2}, \varepsilon]$ and with*

$$\int_{\mathbb{R}^3} \varphi(|z|)dz = 1.$$

If $f :\ \mathbb{R}^3 \to \mathbb{R}$ is a support function, then the function defined by

$$\tilde{f}(x) := \int_{\mathbb{R}^3} f(x + |x|z)\varphi(|z|)dz \text{ for } x \in \mathbb{R}^3$$

is a support function of class C^∞ on $\mathbb{R}^3 \setminus \{0\}$ and satisfies

$$d_H(K, \tilde{K}) \leqslant R\varepsilon,$$

where K and $\tilde{K}$ are convex bodies corresponding to support functions f and $\tilde{f}$ respectively, $R > 0$ is a constant satisfying that $K \subset B(0, R) \subset \mathbb{R}^3$, and d_H is the Hausdorff distance in $\mathbb{R}^3$.

We now prove Lemma 2.2.1.

Proof of Lemma 2.2.1. Since (X, d) is closed, it is contained in a ball $B(O, R) \subseteq \mathbb{R}^3$ for some $R > 0$.

Case 1. (X, d) is non-degenerate. Let K be the convex body enclosed by X, then K is non-degenerate, there is a ball B of radius $\frac{1}{R}$ contained inside K (enlarge R if necessary). Take the origin to be the center of B and consider the dual body K^* of K. Then K^* will stay inside the ball $B(O, R)$. Let $\rho : (\mathbb{S}^2, \delta) \to X, x \mapsto \tilde{x}$ be the radial function of X. With an abuse of nation, we also use ρ to denote $\rho(x) = |x|$. Then $\frac{1}{\rho}$ will be the support function of K^*. By applying Lemma 2.2.2 to $f = \frac{1}{\rho}$ with $\varepsilon_i = \frac{1}{2iR}$ $(i = 1, 2, \ldots,)$, we obtaining a sequence of convex bodies $\tilde{K}_i^*$, such that $d_H(\tilde{K}_i^*, K^*) \leq \frac{1}{2i}$. Let $K_i^* = \tilde{K}_i^* + B(O, \frac{1}{2i})$ be the Minkowski sum of $\tilde{K}_i^*$ and $B(O, \frac{1}{2i})$. We see that K_i^* is strictly convex and smooth, and $d_H(K_i^*, K^*) \leq \frac{1}{i}$. Denote the support function of K_i^* by $\frac{1}{{}^i\rho}$, then $\frac{1}{{}^i\rho}$ is smooth,

$$\frac{1}{2R} \leq \frac{1}{{}^i\rho} \leq 2R \tag{2.2.1}$$

for i large, and

$$|\frac{1}{{}^i\rho(x)} - \frac{1}{\rho(x)}| \leq d_H(K_i^*, K^*) \leq \frac{1}{i}, \quad \forall x \in (\mathbb{S}^2, \delta). \tag{2.2.2}$$

Denote $K_i := (K_i^*)^*$ be the dual body of K_i^*, then $\rho_i(x)$ will be the radial function of K_i, and by (2.2.2) and (2.2.1), we have

$$|{}^i\rho(x) - \rho(x)| \leq \frac{2R^2}{i}, \quad \forall x \in (\mathbb{S}^2, \delta). \tag{2.2.3}$$

We denote the boundary of K_i with induced metric in $\mathbb{R}^3$ by $(X_i, \tilde{g}_i)$. Then this implies that $(X_i, \tilde{g}_i)$ are smooth convex surfaces and that

$$d_H(X, X_i) \leqslant \frac{2R^2}{i}. \tag{2.2.4}$$

Moreover, let $f_i := \rho_i \circ \rho^{-1} : X \to X_i$, $\tilde{x} \mapsto \tilde{x}_i$, then (2.2.3) implies that

$$\|\overline{\tilde{x}\tilde{x}_i}\| \leq \frac{2R^2}{i}, \quad for\ all\ \tilde{x} \in X, \tag{2.2.5}$$

where $\|\cdot\|$ denotes the distance in $\mathbb{R}^3$.

Let $\tilde{d}_i$ be the distance function of X_i defined by $\tilde{g}_i$. From Lemma 2.1.5, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|\tilde{d}_i(\tilde{x}_i, \tilde{y}_i) - d(\tilde{x}, \tilde{y})| < \varepsilon. \tag{2.2.6}$$

for all $\tilde{x},\ \tilde{y} \in X$ and $\tilde{x}_i,\ \tilde{y}_i \in X_i$ s.t. $\|\overline{\tilde{x}\tilde{x}_i}\| < \delta$, $\|\overline{\tilde{y}\tilde{y}_i}\| < \delta$ and $d_H(X_i, X) < \delta$.

For such δ, there exists N such that for $i > N$, $\frac{2R^2}{i} < \delta$, and (2.2.6) holds by (2.2.4) and (2.2.5). Since $d_i(\tilde{x}, \tilde{y}) = \tilde{d}_i(f_i(\tilde{x}), f_i(\tilde{y})) = \tilde{d}_i(\tilde{x}_i, \tilde{y}_i)$,

$$|d_i(\tilde{x}, \tilde{y}) - d(\tilde{x}, \tilde{y})| < \varepsilon \ \ for\ all\ \tilde{x},\ \tilde{y} \in X,\ i > N. \tag{2.2.7}$$

Case 2. (X, d) is degenerate, that is, it is a doubly-covered convex domain in a plane. Suppose X lies in the xOy plane and the origin O is in the interior of X. Then there is $r > 0$ such that the disk $B(O, r) \subset \mathbb{R}^2$ is contained in X. Connecting the point $N_i = (0, 0, \frac{1}{i})$, $S_i = (0, 0, -\frac{1}{i})$ with the boundary points of X, we get a sequence of two-sided cones V_i with height $\frac{2}{i}$. Since V_i are non-degenerate. By the arguments in *Case 1*, there are smooth convex surfaces $(X_i, \tilde{d}_i)$ and bijections $h_i : V_i \to X_i$

satisfying

$$|\tilde{d}_i(h_i(\tilde{x}), h_i(\tilde{y})) - d_{V_i}(\tilde{x}, \tilde{y})| < \frac{1}{i} \quad for\ all\ \tilde{x}, \tilde{y} \in V_i. \tag{2.2.8}$$

Let $P : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection onto the xOy plane. Then

$$d(\tilde{x}, \tilde{y}) \leqslant d_{V_i}(P^{-1}(\tilde{x}), P^{-1}(\tilde{y})) \leqslant \frac{d(\tilde{x}, \tilde{y})}{\cos\theta_i} \quad for\ all\ x,\ y \in X, \tag{2.2.9}$$

where θ_i is the largest angle between xOy plane and the segments connecting $N_i,\ S_i$ and the boundary points of X. Since $B(O, r)$ is contained in X, $\tan\theta_i \leqslant \frac{1}{ir}$ and then $\frac{1}{\cos\theta_i} \leqslant \frac{\sqrt{1+i^2r^2}}{ir}$. Thus for $\tilde{x},\ \tilde{y} \in X$ and i large enough,

$$|d_{V_i}((P^{-1}(\tilde{x}), P^{-1}(\tilde{y}))) - d(\tilde{x}, \tilde{y})| \leqslant (\frac{\sqrt{1+i^2r^2}}{ir} - 1) d(\tilde{x}, \tilde{y}) \leqslant (\frac{\sqrt{1+i^2r^2}}{ir} - 1) diam(X).$$

Let $f_i = h_i \circ P^{-1} : X \to X_i$, then for all $\tilde{x},\ \tilde{y} \in X$,

$$\begin{aligned} &|\tilde{d}_i(f_i(\tilde{x}), f_i(\tilde{y})) - d(\tilde{x}, \tilde{y})| \\ \leq &|\tilde{d}_i(h_i(P^{-1}(\tilde{x})), h_i(P^{-1}(\tilde{y}))) - d_{V_i}(P^{-1}(\tilde{x}), P^{-1}(\tilde{y}))| \\ &+ |d_{V_i}(P^{-1}(\tilde{x}), P^{-1}(\tilde{y})) - d(\tilde{x}, \tilde{y})| \\ \leq &\frac{1}{i} + (\frac{\sqrt{1+i^2r^2}}{ir} - 1) diam(X). \end{aligned}$$

Since the last terms go to 0, $\tilde{d}_i(f_i(\tilde{x}), f_i(\tilde{y}))$ converge to $d(\tilde{x}, \tilde{y})$ uniformly for all $\tilde{x}, \tilde{y} \in X$ as $i \to \infty$. We complete the proof of Lemma 2.2.1. □

Now we prove Theorem 1.1.3 by using Lemma 2.2.1.

Proof of Theorem 1.1.3. First, for a sequence of subsets in the same metric space, Gromov-Hausdorff distance by definition is not greater than Hausdorff distance. Thus the Hausdorff convergence of $(X_i, \tilde{g}_i)$ to (X, d) in Lemma 2.2.1 implies the Gromov-Hausdorff convergence of $(X_i, \tilde{g}_i)$ to (X, d). Suppose

$$diam(X, d) \leqslant D \quad and \quad vol(X) \geqslant \tilde{\nu}_0 > 0 \tag{2.2.10}$$

for some positive constants D and $\tilde{\nu}_0$. Since the diameter and volume are continuous with respect to Gromov-Hausdorff convergence with curvature bounded from below (see Exercise 7.3.14 and Theorem 10.10.10 of [14]), and all the surfaces $(X_i, \tilde{g}_i)$ and (X, d) are convex, we have

$$diam(X_i, \tilde{g}_i) \leqslant 2D \quad and \quad {}^{\tilde{g}_i}vol(X_i) \geqslant \frac{\tilde{\nu}_0}{2} > 0 \tag{2.2.11}$$

for $(X_i, \tilde{g}_i)$ with large i. By Bishop-Gromov comparison theorem, we get

$$\frac{vol({}^{\tilde{g}_i}B_1(x))}{vol(B_1(x))} \geqslant \frac{vol({}^{\tilde{g}_i}B_{2D}(x))}{vol(B_{2D}(x))} = \frac{{}^{\tilde{g}_i}vol(X_i)}{vol(B_{2D}(x))} \geqslant \frac{\tilde{\nu}_0}{2vol(B_{2D}(x))}, \tag{2.2.12}$$

which implies that

$$vol({}^{\tilde{g}_i}B_1(x)) \geqslant \tilde{\nu}_0 \frac{vol(B_1(x))}{2vol(B_{2D}(x))} := \nu_0 > 0. \tag{2.2.13}$$

Due to Theorem 1.1.1, we know that there are smooth Ricci flows $(X_i, g_i(t))_{t\in[0,T)}$ with $g_i(0) = \tilde{g}_i$ satisfying

(a'_t) $Ricci(g_i(t)) \geqslant 0$ *for all* $t \in [0, T)$;

(b'_t) $vol({}^{g_i(t)}B_1(x)) \geqslant \frac{\nu_0}{2} > 0$, *for all* $x \in X$ *and* $t \in [0, T)$;

(c'_t) $\sup_{X_i} |Riem(g_i(t))| \leqslant \frac{K}{t}$ *for all* $t \in [0, T)$;

(d'_t) $d_{g_i(s)}(p, q) - c_2(\sqrt{t} - \sqrt{s}) \leqslant d_{g_i(t)}(p, q) \leqslant e^{c_1(t-s)} d_{g_i(s)}(p, q)$,

for all $0 \leqslant s \leqslant t \in [0, T)$ *and* $p,\ q \in X_i$, where $K = K(\nu_0)$, $c_1 = c_1(\nu_0)$, $c_2 = c_2(\nu_0)$ and $T = T(\nu_0)$ are constants independent of i. Combining (c'_t) and Shi's higher derivative estimates [61, 62] with Arzelà-Ascoli theorem, there exists a subsequence (which we also denote by $g_i(t)$) converge to a metric $g(t)$ on X, and $(X, g(t))_{t\in(0,T)}$ is a smooth Ricci flow. Let $s \to 0$ in (d'_t), for all $t \in (0, T)$ and $p,\ q \in X$, we have

$$\tilde{d}_i(f_i(p), f_i(q)) - c_2\sqrt{t} \leqslant d_{g_i(t)}(f_i(p), f_i(q)) \leqslant e^{c_1 t}\tilde{d}_i(f_i(p), f_i(q)). \tag{2.2.14}$$

Since X_i converge to X in Hausdorff distance and $g_i(t)$ converge to $g(t)$ in local smooth sense of $(0, T)$, by letting $i \to \infty$ and using Lemma 2.2.1, we have

$$d(p, q) - c_2\sqrt{t} \leqslant d_{g(t)}(p, q) \leqslant e^{c_1 t}d(p, q). \tag{2.2.15}$$

The uniform convergence of $d_{g(t)}$ to d follows by letting $t \to 0$ in (2.2.15). □

2.2.2 Uniqueness

In this subsection, we prove the uniqueness of Ricci flow that admits a non-degenerate embedded closed convex surface (X, d) in $\mathbb{R}^3$ as a metric initial condition. In this case, up to a translation, we can always assume that the origin is in the interior of the region enclosed by the surface. Hence the radial function of (X, d) is bounded from below and above by positive constants. The main step is to give an exact expression of the initial metric d (Theorem 2.2.7), which is used to improve the regularity of the isometry between the initial conditions by Lemma 2.2.8.

Let $(X, g(t))_{t\in(0,T)}$ be the Ricci flow obtained in subsection 2.2.1. It is easy to see that its Gauss curvature $K_{g(t)}$ is positive. In fact, the non-negativity of the Gauss curvature $K_{\tilde{g}_i}$ of the smooth convex surfaces $(X_i, \tilde{g}_i)$ implies that the Gauss curvature $K_{g_i(t)}$ along Ricci flow $(X_i, g_i(t))_{t\in[0,T)}$ is also non-negative by applying maximum principle to the evolution equation of $K_{g_i(t)}$,

$$\frac{\partial}{\partial t}K_{g_i(t)} = \Delta_{g_i(t)}K_{g_i(t)} + |Ric_{g_i(t)}|^2_{g_i(t)}. \tag{2.2.16}$$

From Gauss-Bonnet theorem for $g_i(t)$ and the fact that $g_i(t)$ converges to $g(t)$ smoothly, for $t \in (0,T)$, $K_{g(t)}$ is non-negative and satisfies

$$\int_X K_{g(t)} dV_{g(t)} = 4\pi. \tag{2.2.17}$$

Hence there must be a point x_0 such that $K_{g(t)}$ is positive at x_0. Then the strong maximum principle implies that $K_{g(t)}$ is positive everywhere.

Fix $t_0 \in (0,T)$ and denote $(X, g(t_0)) = (X, g_{t_0})$. By the Uniformization theorem, there is a conformal equivalence (holomorphic isomorphism) $\Phi : (\mathbb{S}^2, \tilde{h}) \to (X, g_{t_0})$, where $\tilde{h}$ is a smooth metric of positive constant curvature, i.e. $\Phi^*(g_{t_0}) = e^{\tilde{u}(t_0,x)}\tilde{h}(x)$ for some smooth function $\tilde{u}(t_0,x)$ on $\mathbb{S}^2$. On the other hand, the 2-dimensional Ricci flow can be written as

$$\frac{\partial}{\partial t}g(t) = -R_{g(t)}g(t). \tag{2.2.18}$$

Hence we can write $g(t) = e^{-\int_{t_0}^t R_{g(s)}ds} g_{t_0} := \tilde{\omega}(t,x)g_{t_0}$, which implies that

$$g(t) = \tilde{\omega}(t,x)(\Phi^{-1})^*(e^{\tilde{u}(t_0,x)}\tilde{h}(x)) = \tilde{\omega}(t,x)e^{\tilde{u}(t_0,\Phi^{-1}(x))}\tilde{h}(\Phi^{-1}(x)) := e^{2u(t,x)}h(x).$$

where $h(x) := \tilde{h}(\Phi^{-1}(x))$ and $u(t,x) := \frac{\ln(\tilde{\omega}(t,x)) + \tilde{u}(t_0,\Phi^{-1}(x))}{2}$. We call $u(t) := u(t,x)$ the conformal potential along Ricci flow $(X, g(t))_{t\in(0,T)}$. Since $g(t) = e^{2u(t,x)}h(x)$ and metric $h(x)$ is fixed, Ricci flow $(X, g(t))_{t\in(0,T)}$ keeps the conformal class. Here we obtain $u(t)$ and $h(x)$ from the limiting Ricci flow $(X, g(t))_{t\in(0,T)}$ directly. They are not the limits of $u_i(t)$ and $h_i(x)$ along the smooth approximation flows $(X_i, g_i(t))_{t\in[0,T)}$ as $i \to \infty$.

From (2.2.18), the evolution equation of $u(t)$ reads

$$\frac{\partial u(t)}{\partial t} = e^{-2u(t)}(\Delta_h u(t) - K_h) = -K_{g(t)}. \tag{2.2.19}$$

Since $K_{g(t)}$ is positive, $u(t)$ increases as t decreases to 0, and then $u(t) \geqslant u(T)$ for $t \in (0, T]$. It is proved in Lemma 2.2 of [58] that $u(t)$ is bounded in L^1-sense and converges to an integrable function $u_0(x)$ in L^1-sense. Here, we prove that $u(t)$ is bounded for $t \in (0, T]$ in the classical sense, so $u_0(x)$ is also bounded in the classical sense.

Before starting the proof, we remark that due to Proposition 0.6 in [58], every Ricci flow $(X, g(t))_{t\in(0,T)}$ admitting (X, d) as metric initial condition can be obtained through the process in subsection 2.2.1. Hence we only need to consider the uniqueness for the Ricci flow obtained in subsection 2.2.1.

Lemma 2.2.3. *Assume that (X, d) is a non-degenerate embedded closed convex surface in $\mathbb{R}^3$. Let $(X, g(t))_{t\in(0,T]}$ be a Ricci flow admitting (X, d) as metric initial condition. Then the conformal potential $u(t)$ along $(X, g(t))_{t\in(0,T]}$ is uniformly bounded. Thereby $u_0(x)$ is bounded on X.*

To prove this lemma, we first show the uniform equivalence of the smooth approximating metrics $\tilde{g}_i$ obtained in Lemma 2.2.1.

Let (X, g) be a smooth embedded closed convex surface in $\mathbb{R}^3$ and ρ be the radial function of (X, g). Then the induced metric g and the second fundamental form II with respect to the radial graph parameterization from the unit sphere $(\mathbb{S}^2, \delta)$ can be written as

$$\begin{aligned} g_{ij} &= \rho^2\delta_{ij} + \rho_i\rho_j, \\ II_{ij} &= \frac{1}{\sqrt{\rho^2+|\nabla\rho|^2}}(\rho^2\delta_{ij} + 2\rho_i\rho_j - \rho\rho_{ij}) \\ &= \frac{\rho^3}{\sqrt{\rho^2+|\nabla\rho|^2}}(v_{ij} + v\delta_{ij}), \end{aligned} \tag{2.2.20}$$

where $v = \frac{1}{\rho}$, the derivatives are taken with respect to the connection of $(\mathbb{S}^2, \delta)$. Since $(X_i, \tilde{g}_i)$ have the same radial parameterizations from $(\mathbb{S}^2, \delta)$ as (X, d), $\tilde{g}_i$ can be written as the form in (2.2.20). Now we prove that $\tilde{g}_i$ are uniformly equivalent to δ in Lemma 3.3.

Lemma 2.2.4. *There is a uniform positive constant C such that*

$$\frac{1}{C}\delta \leqslant \tilde{g}_i \leqslant C\delta \quad for\ large\ i. \tag{2.2.21}$$

Proof. Let ${}^i\rho$ be the radial function of $(X_i, \tilde{g}_i)$ and ${}^iv = \frac{1}{{}^i\rho}$. We claim that

$$\max_{\mathbb{S}^2}(|\nabla\ {}^iv|^2 + \ {}^iv^2) \leqslant \max_{\mathbb{S}^2}\ {}^iv^2. \tag{2.2.22}$$

Define $f = |\nabla\ {}^iv|^2 + (1+\eta)\ {}^iv^2$ for $\eta > 0$. At the maximum point of f,

$$0 = \nabla_l f = \nabla_l(|\nabla\ {}^iv|^2 + (1+\eta)\ {}^iv^2) = 2\ {}^iv_j(\ {}^iv_{lj} + (1+\eta)\ {}^iv\delta_{lj}). \tag{2.2.23}$$

Since $(X_i, \tilde{g}_i)$ is convex, $(\ {}^iII_{lj}) \geqslant 0$ and thus $({}^iv_{lj} + \ {}^iv\delta_{lj}) \geqslant 0$. Then we have $({}^iv_{lj} + (1+\eta)\ {}^iv\delta_{lj}) > 0$. Hence $\nabla v = 0$ at the maximum point of f, which implies

$$\max_{\mathbb{S}^2}(|\nabla\ {}^iv|^2 + (1+\eta)\ {}^iv^2) \leqslant (1+\eta)\max_{\mathbb{S}^2}\ {}^iv^2. \tag{2.2.24}$$

Let $\eta \to 0$, we complete the proof of the claim.

Since the origin is in the interior of the region enclosed by (X, d) by assumption and $(X_i, \tilde{g}_i)$ converges to (X, d) in Hausdorff sense, there exists a constant C such that for large i,

$$\frac{1}{C} \leqslant {}^i\rho \leqslant C. \tag{2.2.25}$$

From the claim, $|\nabla\ {}^i\rho|^2$ are also uniformly bounded for large i. Taking trace with respect to δ on both sides of $(\tilde{g}_i)_{lj} = {}^i\rho^2 \delta_{lj} + {}^i\rho_l\ {}^i\rho_j$, we conclude that there exists a uniform constant C such that

$$tr_\delta \tilde{g}_i \leqslant C \quad and \quad tr_\delta \tilde{g}_i \geqslant \frac{1}{C} > 0 \ \ for\ large\ i. \tag{2.2.26}$$

Therefore, there is a uniform constant C such that

$$\frac{1}{C}\delta \leqslant \tilde{g}_i \leqslant C\delta \tag{2.2.27}$$

for large i. □

Next, we extend this equivalence to Ricci flow.

Lemma 2.2.5. *Let $(X, g(t))_{t\in[0,T]}$ be a 2-dimensional smooth Ricci flow with initial metric g_0, then we have*

$$g(t) \leqslant e^{L_1 T} g_0 \quad and \quad tr_{g(t)} g_0 \leqslant L_2 (\frac{dV_{g_0}}{dV_{g(T)}})^2, \tag{2.2.28}$$

where $-L_1$ is the lower bound of R_{g_0}, and L_2 is a positive constant which depends on L_1, T.

Proof. Applying maximum principle to the evolution equation of the scalar curvature $R_{g(t)}$

$$\frac{\partial}{\partial t}R_{g(t)} = \Delta_{g(t)}R_{g(t)} + 2|Ric_{g(t)}|^2_{g(t)}, \tag{2.2.29}$$

we get

$$R_{g(t)} \geqslant R_{g_0} \geqslant -L_1. \tag{2.2.30}$$

In dimension 2, Ricci flow can be written as

$$\frac{\partial}{\partial t}g(t) = -R_{g(t)}g(t) \leqslant L_1 g(t). \tag{2.2.31}$$

Hence we have $g(t) \leqslant e^{L_1 t} g_0$ and get the first estimate in (2.2.28).

For the second estimates, we need the following inequality.

$$n\Big(\frac{detg_1}{detg_2}\Big)^{\frac{1}{n}} \leqslant tr_{g_2}g_1 \leqslant n\Big(\frac{detg_1}{detg_2}\Big)(tr_{g_1}g_2)^{n-1}, \tag{2.2.32}$$

where g_1 and g_2 are any two smooth n-dimensional metrics. In our case,

$$tr_{g(t)}g_0 \leqslant 2\Big(\frac{detg_0}{detg(t)}\Big)(tr_{g_0}g(t)) = 2\Big(\frac{dV_{g_0}}{dV_{g(t)}}\Big)^2(tr_{g_0}g(t)). \tag{2.2.33}$$

Hence we only need to prove that $dV_{g(t)}$ is bounded from below uniformly. The volume form evolves as

$$\frac{\partial}{\partial t}dV_{g(t)} = -RdV_{g(t)} \leqslant L_1 dV_{g(t)}, \tag{2.2.34}$$

which implies that $e^{-L_1 t}dV_{g(t)}$ decrease and then $dV_{g(t)} \geqslant e^{-L_1(T-t)}dV_{g(T)} \geqslant e^{-L_1 T}dV_{g(T)}$. So we have

$$tr_{g(t)}g_0 \leqslant 4\Big(\frac{dV_{g_0}}{e^{-L_1 T}dV_{g(T)}}\Big)^2 e^{L_1 T}. \tag{2.2.35}$$

Let $L_2 = 4e^{3L_1T}$, we complete this Lemma. $\square$

For the sequence of smooth Ricci flow $(X_i, g_i(t))_{t\in[0,T]}$ with $\tilde{g}_i$ as initial condition, we have by Lemma 2.2.4 and Lemma 2.2.5, that

$$g_i(t) \leqslant C\delta \quad and \quad tr_{g_i(t)}\delta \leqslant 4C^3(\frac{dV_\delta}{dV_{g_i(T)}})^2, \tag{2.2.36}$$

where C is the constant in Lemma 2.2.4. Letting $i \to \infty$ gives

$$g(t) \leqslant C\delta \quad and \quad tr_{g(t)}\delta \leqslant 4C^3(\frac{dV_\delta}{dV_{g(T)}})^2 \quad for\ t \in (0,T], \tag{2.2.37}$$

which is equivalent to

$$\frac{1}{A}\delta \leqslant g(t) \leqslant C\delta \quad for\ t \in (0,T], \tag{2.2.38}$$

where A depends on C and T. In fact, we proved the following Lemma.

Lemma 2.2.6. *Assume that (X, d) is a non-degenerate embedded closed convex surface in $\mathbb{R}^3$. Let $(X, g(t))_{t\in(0,T]}$ be a Ricci flow admitting (X, d) as a metric initial condition. Then there exists a positive constant C such that*

$$\frac{1}{C}\delta \leqslant g(t) \leqslant C\delta \quad for\ t \in (0,T]. \tag{2.2.39}$$

Now Lemma 2.2.3 follows immediately.

Proof of Lemma 2.2.3. Since $g(t) = e^{2u(t)}h(x)$ is uniform equivalent to δ for $t \in (0,T]$, $u(t)$ is uniformly bounded. Since $u(t)$ increases to u_0 as t decreases to 0, u_0 is also bounded. $\square$

Next, for an L^1-function u and a smooth Riemannian metric h on M, there is a metric $d_{h,u}$ defined as

$$d_{h,u}(x,y) = \inf_{\gamma\in\Gamma(x,y)} \int_0^1 e^{u(\gamma(\tau))}|\dot{\gamma}(\tau)|_h d\tau, \tag{2.2.40}$$

where $\Gamma(x,y)$ is the space of C^1 paths γ from $[0,1]$ to M with $\gamma(0) = x$ and $\gamma(1) = y$. This metric was studied by Reshetnyak [56]. For more details, please see the appendix of [58].

In [58], when (X,d) is a compact Alexandrov surface with curvature bounded from below, the metric $d_{g(t)}$ induced by $g(t)$ (here, $g(t)$ is a Ricci flow on M with metric initial condition (X,d) in the sense of Theorem 1.1.2) converges to d_{h,u_0} uniformly, where u_0 is the L^1-limit of the conformal potential $u(t)$ along Ricci flow $(M,g(t))_{t\in(0,T]}$ as $t\to 0$ in [58]. But d may not be d_{h,u_0} there. In fact, we can only conclude that (M,d_{h,u_0}) is isometric to (X,d) from the Lemma 2.4 in [58]. In our case when (X,d) is a non-degenerate embedded closed convex surface in $\mathbb{R}^3$, we can prove that indeed $d = d_{h,u_0}$.

Theorem 2.2.7. *Assume that (X,d) is a non-degenerate embedded closed convex surface in $\mathbb{R}^3$. Let $(X,g(t))_{t\in(0,T]}$ be the Ricci flow admitting (X,d) as a metric initial condition and $u(t)$ be the conformal potential along $(X,g(t))_{t\in(0,T]}$. Then*

$$d = d_{h,u_0}, \tag{2.2.41}$$

where $u_0(x)$ is the pointwise limit of $u(t,x)$ as $t\to 0$.

Proof. By definition (2.2.40), and the definition of conformal potential $g(t) = e^{2u(t,x)}h(x)$, we have

$$\begin{aligned} d_{h,u(t)}(x,y) &= \inf_{\gamma\in\Gamma(x,y)} \int_0^1 e^{u(t,\gamma(\tau))}|\dot\gamma(\tau)|_h d\tau \\ &= \inf_{\gamma\in\Gamma(x,y)} \int_0^1 |\dot\gamma(\tau)|_{g_t(\tau)} d\tau = d_{g(t)}(x,y). \end{aligned} \tag{2.2.42}$$

From the proof of Lemma 2.4 in [58], $d_{h,u(t)}$ converges to d_{h,u_0} uniformly as $t \to 0$. Since $d_{g(t)}$ converges to d uniformly as $t \to 0$ on X, then (2.2.41) follows by letting $t \to 0$ on both sides of (2.2.42). □

We now prove a regularity lemma for the isometry between the metric spaces $(X_1, e^{2u_1}h_1)$ and $(X_2, e^{2u_2}h_2)$, where u_1 and u_2 are two bounded functions, and h_1 and h_2 are pull back metrics of two metrics on $\mathbb{S}^2$ with constant Gauss curvature.

Lemma 2.2.8. *Assume that $F : (X_1, e^{2u_1}h_1) \to (X_2, e^{2u_2}h_2)$ is an isometry. Then F is differentiable, where u_1 and u_2 are two bounded functions, and h_1 and h_2 are two pull back metrics of the metrics on $\mathbb{S}^2$ with constant Gauss curvature.*

Proof. Since F is an isometry and u_1, u_2 are bounded, F is bi-Lipschitz with respect to h_1, h_2. Then F is differentiable almost everywhere. Write locally $h_1 = \lambda_1(du^2 + dv^2)$ and $h_2 = \lambda_2(dx^2 + dy^2)$ for some positive functions λ_1 and λ_2. At differentiable points of F, we have

$$\begin{aligned} e^{2u_1}\lambda_1(du^2+dv^2) = e^{2u_1}h_1 = F^*(e^{2u_2}h_2) &= e^{2u_2\circ F}(\lambda_2\circ F)F^*h_2 \\ &= e^{2u_2\circ F}(\lambda_2\circ F)\big((x_u du + x_v dv)^2 + (y_u du + y_v dv)^2\big) \\ &= e^{2u_2\circ F}(\lambda_2\circ F)\big((x_u^2+y_u^2)du^2 + (x_v^2+y_v^2)dv^2 \\ &\quad + 2(x_u x_v + y_u y_v)du dv\big) \end{aligned}$$

Since u_1 and u_2 are bounded functions, we have

$$x_u x_v + y_u y_v = 0 \quad and \quad x_u^2 + y_u^2 = x_v^2 + y_v^2, \tag{2.2.43}$$

which is equivalent to the fact that

$$x_u = -y_v \; and \; x_v = y_u \quad \text{or} \quad x_u = y_v \; and \; x_v = -y_u. \tag{2.2.44}$$

In the former case $\det \begin{pmatrix} x_u, x_v \\ y_u, y_v \end{pmatrix} = -x_u^2 - x_v^2 \leq 0$ at differentiable points and F is orientation-reversing, while in the later case $\det \begin{pmatrix} x_u, x_v \\ y_u, y_v \end{pmatrix} = x_u^2 + x_v^2 \geq 0$ at differentiable points and F is orientation-preserving. Since X_1 and X_2 are both orientable (since they are both topologically $\mathbb{S}^2$), thus only one case can happen in a small open coordinate chart U. Without loss of generality, we may assume F is orientation-preserving, so that $x_u = y_v \, and \, x_v = -y_u$ almost everywhere in U. Then we know that F is differentiable everywhere in U by the analytic extension theorem in [11]. Since U is arbitrary, we know F is differentiable everywhere. □

Since $u(t)$ increases to u_0 and both of them are bounded, by using (2.2.42) and Lebesgue's monotone convergence theorem, we conclude that d equals to d_{h,u_0} which is the metric induced by $e^{2u_0}h$. Now we can apply Lemma 3.7 to the isometry between the initial conditions.

Proof of Theorem 1.1.4. By Lemma 2.2.7, $d_i = d_{h_i,u_i}$ $(i = 1, 2)$ is the metric induced by $e^{2u_i}h_i$, where $u_i = \lim_{t \to 0} u_i(t)$, $u_i(t)$ is the conformal potential along

$(X_i, g_i(t))_{t\in(0,T]}$ and h_i is the pull back metric of a metric on $\mathbb{S}^2$ with constant Gaussian curvature.

From Lemma 2.2.8, the isometry f is differentiable. Then $(X_1, f^*g_2(t))_{t\in(0,T)}$ is also a Ricci flow admitting (X_1, d_1) as metric initial condition in the sense that the distance function induced by $f^*g_2(t)$ converges uniformly to d_1 as $t \to 0$. By using Proposition 0.6 in [58], we have $g_1(t) = f^*g_2(t)$. $\square$

2.3 Further discussions

In this section, we introduce our plan which aims to study Pogorelov's uniqueness theorem by Ricci flow.

Pogorelov's uniqueness theorem [52] states that any two closed isometric convex surfaces in $\mathbb{R}^3$ are congruent. It is a generalization of the classical Cohn-Vesson's rigidity theorem [23].

Our basic idea to study this theorem is to use Ricci flow to construct two families of smooth convex surfaces approximating the two isometric closed convex surfaces in Pogorelov's theorem. We then apply Cohn-Vesson's rigidity theorem to the smooth surfaces and take a limit as $t \to 0$ to see if "the limit of Cohn-Vesson's rigidity theorem will imply Pogorelov's rigid theorem". More precisely, given two isometric embedded closed convex surfaces (X_1, d_1) and (X_2, d_2) in $\mathbb{R}^3$, Theorem 1.1.3 implies that there are two Ricci flows $(X_1, g_1(t))_{t\in(0,T)}$ and $(X_2, g_2(t))_{t\in(0,T)}$ admitting (X_1, d_1) and (X_2, d_2) as metric initial conditions. Then for every positive time t, we embed the Ricci flow $(X_1, g_1(t))$ and $(X_2, g_2(t))$ smoothly and isometrically into $\mathbb{R}^3$ as $(X_1^t, G_1(t))$ and $(X_2^t, G_2(t))$ respectively. The validity for these embeddings is due to the fact that $(X_1, g_1(t))$ and $(X_2, g_2(t))$ are smooth strictly convex surfaces and the solvability of Weyl's problem proved by Nirenberg [48]. By Theorem 1.1.4, $(X_1^t, G_1(t))$ and $(X_2^t, G_2(t))$ are isometric. Then Cohn-Vesson's rigidity theorem implies that there is a congruence $F(t) \in O(3)$ between them. We hope to investigate the limits of $(X_1^t, G_1(t))$ and $(X_2^t, G_2(t))$ as $t \to 0$. If $(X_1^t, G_1(t))$ and $(X_2^t, G_2(t))$ converge to (X_1, d_1) and (X_2, d_2) in Hausdorff distance up to an isometry in $O(3)$ respectively, the compactness of $O(3)$ will imply the congruence between (X_1, d_1) and (X_2, d_2).

CHAPTER 3
Flow by powers of the Gauss curvature in space forms

In this chapter, we show our result about the flow by powers of the Gauss curvature in space forms. In section 3.1, we will recall some basic facts which will be used later in this chapter. In section 3.2, we prove the flow (3.2.1) converges to a point in finite time $T^* > 0$ for $\alpha > 0$. As a corollary, we prove that the flow (1.2.2) converges to a point at finite time T^* for $\alpha > 0$. In section 3.3, we obtain the modified monotone quantity and the a priori estimates of the normalized flow. In section 3.4, we prove the normalized flow in $\mathbb{N}^{n+1}(\kappa)$ converges to a geodesic sphere centered at the extinct point q_0 for $\alpha > \frac{1}{n+2}$.

3.1 Preliminaries

In this section, we present some basic facts about space forms and the stereographic type projections which will be used later.

Recall that $\mathbb{N}^{n+1}(\kappa)$ is the $n+1$ dimensional simply connected space form of constant sectional curvature $\kappa = \pm 1$. In the geodesic polar coordinates, the metric of $\mathbb{N}^{n+1}(\kappa)$ can be denoted as

$$\bar{g} = d\rho^2 + \phi^2(\rho)dz^2, \tag{3.1.1}$$

where $\phi(\rho) = \sin(\rho)$, $\rho \in [0, \pi)$ when $\kappa = 1$; and $\phi(\rho) = \sinh(\rho)$ when $\kappa = -1$; and dz^2 is the standard induced metric on $\mathbb{S}^n$ in Euclidean space.

Let Ω be a convex body in $\mathbb{N}^{n+1}(\kappa)$. Suppose $\mathcal{M} = \partial\Omega$ is smooth and strictly convex, denote the metric and the unit outer normal of $\mathcal{M}$ by g_{ij}, and ν respectively. Let h_{ij} be the second fundamental form of $\mathcal{M}$ with respect to ν and $u = \langle \phi \frac{\partial}{\partial\rho}, \nu \rangle$ the support function of $\mathcal{M}$. Suppose $q \in \mathcal{M}$ and, there is an open subset $\mathcal{N}$ of $\mathcal{M}$ containing q such that $\langle \frac{\partial}{\partial\rho}, \nu \rangle$ is strictly positive or negative (doesn't change sign) in $\mathcal{N}$, then $\mathcal{N}$ can be represented as a radial graph locally. As a local radial graph, it is well-known (see e.g. [33]) that in $\mathcal{N}$

$$\begin{aligned} g_{ij} &= \rho_i \rho_j + \phi^2 \delta_{ij}, \\ \nu &= \frac{\sigma}{\omega}\left(\frac{\partial}{\partial\rho} - \frac{\nabla\rho}{\phi^2}\right), \\ h_{ij} &= \sigma\left(\sqrt{\phi^2 + |\nabla\rho|^2}\right)^{-1}(-\phi\rho_{ij} + 2\phi'\rho_i\rho_j + \phi^2\phi'\delta_{ij}), \end{aligned} \tag{3.1.2}$$

where $\rho_i = \nabla_i \rho$ and ∇ is the covariant derivative on $\mathbb{S}^n$ with respect to an orthonormal frame, $\omega = \frac{\sqrt{\phi^2+|\nabla\rho|^2}}{\phi}$, $\sigma = 1$ when $\langle \frac{\partial}{\partial\rho}, \nu \rangle > 0$, and $\sigma = -1$ when $\langle \frac{\partial}{\partial\rho}, \nu \rangle < 0$. Therefore, the Gauss curvature of $\mathcal{N}$ is given by

$$K = \frac{\det h_{ij}}{\det g_{ij}} = \frac{\sigma^n \det(-\phi\rho_{ij} + 2\phi'\rho_i\rho_j + \phi^2\phi'\delta_{ij})}{(\phi^2 + |\nabla\rho|^2)^{\frac{n+2}{2}} \phi^{2n-2}}. \tag{3.1.3}$$

To investigate the flow (1.2.2) in $\mathbb{N}^{n+1}(\kappa)$, we project it to the tangent plane of $\mathbb{N}^{n+1}(\kappa)$ at a certain point. The sterographic projection can be found in many references, see e.g. [29, 35]. Here we describe it briefly for completeness.

When $\kappa = 1$, $\mathbb{N}^{n+1}(\kappa) = \mathbb{S}^{n+1}$. For any $p \in \mathbb{S}^{n+1}$, denote $\mathcal{H}(p) = \{z \in \mathbb{S}^{n+1} : d_{\mathbb{S}^{n+1}}(p, z) < \frac{\pi}{2}\}$ the open hemisphere centered at p. We consider the projection π_p

of $\mathcal{H}(p)$ onto the tangent plane L_p of $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ at p, defined by

$$\pi_p : z \in \mathcal{H}(p) \mapsto \frac{z}{\langle z, p \rangle} \in L_p, \tag{3.1.4}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^{n+2}$. For a strictly convex hypersurface $\mathcal{M} \subset \mathbb{S}^{n+1}$, it must enclose a convex body and is contained in a hemisphere $\mathcal{H}$, see, for example [26]. Suppose $\mathcal{H} = \mathcal{H}(p)$ is centered at p, we can use the above projection π_p to project $\mathcal{M}$ onto L_p.

When $\kappa = -1$, $\mathbb{N}^{n+1}(\kappa) = \mathbb{H}^{n+1}$ is the hyperbolic space. For any point $p \in \mathbb{H}^{n+1}$, we can consider $\mathbb{H}^{n+1}$ as a submanifold of $\mathbb{R}^{n+2}$ with vertex p,

$$\mathbb{H}^{n+1} = \{(x_1, \dots, x_{n+1}, x_{n+2}) \in \mathbb{R}^{n+2} | x_{n+2}^2 - \sum_{i=1}^{n+1} x_i^2 = 1, x_{n+2} > 0\},$$

and

$$p = (0, \dots, 0, 1).$$

Let L_p be the tangent plane of $\mathbb{H}^{n+1}$ at p, we define π_p as

$$\pi_p : z \in \mathbb{H}^{n+1} \mapsto \frac{z}{\langle z, p \rangle} \in L_p. \tag{3.1.5}$$

Note that if $z = (x_1, \dots, x_{n+1}, x_{n+2}) \in \mathbb{H}^{n+1}$, then $\pi_p(z) = (\frac{x_1}{x_{n+2}}, \dots, \frac{x_{n+1}}{x_{n+2}}, 1)$. Thus, $\pi_p(\mathbb{H}^{n+1})$ is contained in the unit ball $B^{n+1}(p, 1)$ of L_p centered at p.

Lemma 3.1.1. *Let $\mathcal{M}$ be a closed smooth strictly convex hypersurface in $\mathcal{N}^{n+1}(\kappa)$, Ω be the set enclosed by $\mathcal{M}$. Let $\pi := \pi_p$ be defined as above and $\hat{\Omega} = \pi(\Omega) \subset L_p$ be the image of Ω under the projection, $\hat{u} : \mathbb{S}^n \to \mathbb{R}$, $x \mapsto \sup\{\langle y, x \rangle : y \in \hat{\Omega}\}$ be the* *support function of* $\hat{\Omega}$*, and* $\hat{K}$ *be the Gauss curvature of* $\hat{\mathcal{M}} := \partial\hat{\Omega}$*. Then*

$$K(q) = \Big(\frac{1+\kappa(\hat{u}^2 + |\nabla \hat{u}|^2)}{1+\kappa\hat{u}^2}\Big)^{\frac{n+2}{2}} \hat{K}(\pi(q)), \qquad \forall q \in \mathcal{M}. \tag{3.1.6}$$

where $x \in \mathbb{S}^n$ *is the unique point such that* $\pi(q) = \nabla u + ux$*.*

Proof. For the case $\kappa = 1$, [35] gives a detailed proof. We give a proof for the general case of space forms.

We identify the tangent plane L_p with $\mathbb{R}^{n+1}$ and choose p as the origin of $\mathbb{R}^{n+1}$. Recall that $\rho(q)$ is the geodesic distance from q to the origin, let r be the Euclidean distance from the origin. It is easy to see that

$$r = \frac{\phi}{\phi'} = \begin{cases} \tan(\rho), & \kappa = 1; \\ \tanh(\rho), & \kappa = -1. \end{cases} \tag{3.1.7}$$

Since $\mathcal{M}$ is strictly convex, we claim that the co-dimension of the set $\mathcal{Z} = \{q' \in \mathcal{M} | u(q') = 0\}$ is at least one. In fact, let ∇^g denote the gradient of $\mathcal{M}$, $\Phi(\rho) = \int_0^\rho \phi(s)ds$, then $\nabla^g_i u = h_i^j \nabla^g_j \Phi$ (see e.g. [33]). If $\nabla^g u = 0$, then $\nabla^g \Phi = 0$. On the other hand, $u^2 + |\nabla^g \Phi|^2 = \phi^2$, so $\{q' \in \mathcal{Z} | \nabla^g u(q') = 0\} = \{q' \in \mathcal{M} | \phi(q') = 0\}$, i.e. a single point. If $\nabla^g u \neq 0$, then the set $\{q' \in \mathcal{Z} | \nabla^g u(q') \neq 0\}$ is co-dimension one by the implicit function theorem. Thus the claim is true. For $q \in \mathcal{M} \setminus \mathcal{Z}$, there is a neighbourhood $\mathcal{N}$ of q in $\mathcal{M}$ such that $u > 0\,(< 0)$ in $\mathcal{N}$, and $\mathcal{N}$ can be represented as local radial graph. Moreover, $\hat{u} > 0\,(< 0)$ in $\hat{\mathcal{N}} = \pi(\mathcal{N})$ if and only if $u > 0\,(< 0)$ in $\mathcal{N}$, and $\hat{\mathcal{N}}$ can be represented as a radial graph in the polar coordinates of $\mathbb{R}^{n+1}$ with origin p.

Similar to (3.1.2) and (3.1.3), we have in $\hat{\mathcal{N}}$ that

$$\begin{aligned} &\hat{g}_{ij} = r_i r_j + r^2\delta_{ij}, \\ &\hat{\nu} = \frac{\sigma r}{\sqrt{r^2+|\nabla r|^2}}\left(\frac{\partial}{\partial r} - \frac{\nabla r}{r^2}\right), \\ &\hat{h}_{ij} = \sigma(\sqrt{r^2+|\nabla r|^2})^{-1}(-r r_{ij} + 2 r_i r_j + r^2\delta_{ij}), \end{aligned} \tag{3.1.8}$$

$$\hat{K} = \frac{\det \hat{h}_{ij}}{\det \hat{g}_{ij}} = \frac{\sigma^n \det(-r r_{ij} + 2 r_i r_j + r^2\delta_{ij})}{(r^2+|\nabla r|^2)^{\frac{n+2}{2}} r^{2n-2}}, \tag{3.1.9}$$

where $\hat{g}_{ij}, \hat{\nu}, \hat{h}_{ij}$ and $\hat{K}$ are the metric, the unit outer normal, the second fundamental form, and the Gauss curvature of $\hat{\mathcal{M}}$ respectively.

On the other hand, by (3.1.7) and the fact that $\phi'^2 - \phi\phi'' = 1$, we get

$$K = \frac{\sigma^n \det(-r r_{ij} + 2 r_i r_j + r^2\delta_{ij})}{r^{2n-2}(r^2 + \phi'^2|\nabla r|^2)^{\frac{n+2}{2}}}. \tag{3.1.10}$$

Moreover, since $\phi(\rho) = \sinh(\rho)$ for $\kappa = -1$, $\phi(\rho) = \sin(\rho)$ for $\kappa = 1$, thus $r^2 = \frac{\phi^2}{\phi'^2} = \kappa(\frac{1}{\phi'^2} - 1)$, i.e.

$$\phi'^2 = \frac{1}{1+\kappa r^2}. \tag{3.1.11}$$

Comparing this to (3.1.9), we get

$$\frac{K}{\hat{K}} = \left(\frac{r^2+|\nabla r|^2}{r^2 + \frac{|\nabla r|^2}{1+\kappa r^2}}\right)^{\frac{n+2}{2}}. \tag{3.1.12}$$

It is well known that for $x \in \mathbb{S}^n$, x is an unit outer normal of hypersurface defined by $\nabla\hat{u} + \hat{u}x \in \hat{\mathcal{N}}$, thus

$$r^2 = \hat{u}^2 + |\nabla\hat{u}|^2, \tag{3.1.13}$$

$$\hat{u}(x) = \langle r\frac{\partial}{\partial r}, \hat{\nu}\rangle = \frac{\sigma r^2}{\sqrt{r^2+|\nabla r|^2}}. \tag{3.1.14}$$

Plugging the above two equations into (3.1.12), we get

$$\frac{K}{\hat{K}} = \Big(\frac{1+\kappa(\hat{u}^2+|\nabla\hat{u}|^2)}{1+\kappa\hat{u}^2}\Big)^{\frac{n+2}{2}}. \tag{3.1.15}$$

This proves (3.1.6) for $q \in \mathcal{M} \setminus \mathcal{Z}$. Since $\mathcal{M} \setminus \mathcal{Z}$ is dense in $\mathcal{M}$, (3.1.6) holds for $q \in \mathcal{M}$. □

Let $\tilde{X}(\tau)$ be a family of hypersurfaces evolving by the flow (1.2.2). Suppose we can represent $\tilde{X}(\tau)$ as $\{(\tilde{\rho}(z,t)z, z)\}$ as a radial graph locally over $\mathbb{S}^n$ in the polar coordinates with center p, then we obtain the scalar curvature flow equation (locally)

$$\tilde{\rho}_t = -\tilde{K}^\alpha \tilde{\omega}, \tag{3.1.16}$$

where $\tilde{\omega} = \frac{\sqrt{\phi(\tilde{\rho})^2+|\nabla\tilde{\rho}|^2}}{\phi(\tilde{\rho})}$.

Suppose that (1.2.2) exists on $[0,T^*)$. Then we can project $\tilde{X}(\tau)$ into L_{p_0} through the projection π_{p_0}, where p_0 is the outer center of $\tilde{X}(0)$.

Lemma 3.1.2. *The image* $\hat{\tilde{X}}(\tau) := \pi_{p_0}(\tilde{X}(\tau))$ *evolves by*

$$\hat{\tilde{X}}_\tau = -(1+\kappa|\hat{\tilde{X}}|^2)^{\frac{n+2}{2}\alpha+\frac{1}{2}}(1+\kappa\langle\hat{\tilde{X}},\hat{\nu}\rangle^2)^{-\frac{n+2}{2}\alpha+\frac{1}{2}}\hat{\tilde{K}}^\alpha(x,\tau)\hat{\nu}, \qquad \tau\in[0,T^*), \tag{3.1.17}$$

where $\hat{\tilde{K}}$ *and* $\hat{\nu}$ *are the Gauss curvature and the unit outer normal of* $\hat{\tilde{X}}(\tau)$ *respectively. The support function satisfies*

$$\hat{\tilde{u}}_\tau(x,\tau) = -\,(1+\kappa(\hat{\tilde{u}}^2+|\nabla\hat{\tilde{u}}|^2))^{\frac{n+2}{2}\alpha+\frac{1}{2}}(1+\kappa\hat{\tilde{u}}^2)^{-\frac{n+2}{2}\alpha+\frac{1}{2}}\hat{\tilde{K}}^\alpha(x,\tau), \qquad \tau\in[0,T^*). \tag{3.1.18}$$

Proof. Since $\tilde{X}_0$ is strictly convex, $\tilde{X}(\tau)$ will stay strictly convex on a short time interval $[0,\delta)$. Thus, there is a set $\mathcal{Z}(\tau) \subset \tilde{X}(\tau)$ of measure zero, such that $\tilde{X}(\tau)$

can be represented as local radial graph and (3.1.16) holds on $\tilde{X}(\tau) \setminus \mathcal{Z}(\tau)$. Plugging (3.1.7) into (3.1.16), we get the evolution equation of $\tilde{r}$

$$\tilde{r}_\tau = -\sqrt{(1+\kappa\tilde{r}^2)^2 + \frac{|\nabla\tilde{r}|^2}{\tilde{r}^2}(1+\kappa\tilde{r}^2)}\tilde{K}^\alpha \tag{3.1.19}$$

holds in $\tilde{X}(\tau) \setminus \mathcal{Z}(\tau)$ for $\tau \in [0,\delta)$.

Note that $\frac{\hat{\tilde{u}}(x,\tau)_\tau}{\hat{\tilde{u}}} = \frac{\tilde{r}(z,\tau)_\tau}{\tilde{r}}$, by Lemma 3.1.1, we obtain

$$\hat{\tilde{u}}_\tau(x,\tau) = -\left(1+\kappa(\hat{\tilde{u}}^2 + |\nabla\hat{\tilde{u}}|^2)\right)^{\frac{n+2}{2}\alpha+\frac{1}{2}}(1+\kappa\hat{\tilde{u}}^2)^{-\frac{n+2}{2}\alpha+\frac{1}{2}}\hat{\tilde{K}}^\alpha(x,\tau)$$

holds in $\tilde{X}(\tau) \setminus \mathcal{Z}(\tau)$ for $\tau \in [0,\delta)$. This is the evolution equation of the support funtion when the hypersurfaces evolve by (3.1.17). By applying Lemma 3.2.3 with $T = \delta$, we obtain that the principal curvatures of $\hat{\tilde{X}}$ have a uniform positive lower bound ε_0 depending only on $n, \alpha, \tilde{X}_0$. This implies that $\tilde{X}(\tau)$ is uniformly convex on $[0,\delta]$. Then repeating this process, (3.1.17) and (3.1.18) hold on $\tilde{X}(\tau) \setminus \mathcal{Z}(\tau)$ for any $\tau \in [0, T^*)$. Since $\mathcal{Z}(\tau)$ is of measure zero for any fixed $\tau \in [0, T^*)$, this finishes the proof. □

3.2 Convergence to a point

In this section, we prove that the flow (1.2.2) converges to a point in finite time $T^* > 0$. This is proved by proving the image flow of its projection in $\mathbb{R}^{n+1}$ converges to a point at T^*. More generally, we prove the following theorem.

Theorem 3.2.1. *Suppose $\{\hat{\tilde{X}}(\tau)\} \subset \mathbb{R}^{n+1}$ is a family of hypersurfaces in $\mathbb{R}^{n+1}$ evolving by*

$$\hat{\tilde{X}}_\tau = -\psi(\langle\hat{\tilde{X}},\hat{\nu}\rangle, \hat{\nu}, \nabla\langle\hat{\tilde{X}},\hat{\nu}\rangle)\hat{\tilde{K}}^\alpha(\hat{\tilde{X}},\tau)\hat{\nu}, \tag{3.2.1}$$

with $\hat{\hat{X}}(0) = \hat{\hat{X}}_0$ *strictly convex, where* $\hat{\hat{K}}, \hat{\nu}$ *are the Gauss curvature, and unit outer normal of* $\hat{\hat{X}}$ *respectively,* $\alpha > 0$ *is a positive constant,* $\psi : (\mathbb{R} \times T\mathbb{S}^n) \to \mathbb{R}$ *is a smooth function satisfying*

$$\begin{aligned} \frac{1}{A} \leq \psi \leq A, \\ \|\psi\|_{C^2} \leq A, \end{aligned} \tag{3.2.2}$$

for some positive constant $A > 0$ *as long as the flow (3.2.1) exists. Then the flow (3.2.1) converges to a point in finite time* $T^* > 0$ *with* T^* *depending only on* $n, \alpha, \hat{\hat{X}}_0$ *and* A.

When we consider the flow (3.1.17), $\psi = (1+\kappa|\hat{\hat{X}}|^2)^{\frac{n+2}{2}\alpha+\frac{1}{2}}(1+\kappa\langle\hat{\hat{X}}, \hat{\nu}\rangle^2)^{-\frac{n+2}{2}\alpha+\frac{1}{2}}$ is uniformly bounded from below, since the flow is contracting.

Corollary 3.2.2. *The flow (1.2.2) converges to a point in finite time* T^* *with* T^* *depending only on* $\tilde{X}_0, n, \alpha$.

Proof. Let p_0 be the outer center of $\tilde{X}_0$, we consider the projection π_{p_0} of $\tilde{X}(\tau)$ into L_{p_0} on $[0, T^*)$. By Lemma 3.1.2, the image $\hat{\hat{X}}(\tau) = \pi_{p_0}(\tilde{X}(\tau))$ will evolve by (3.1.17), which is a special case of (3.2.1) with $\psi = (1 + \kappa|\hat{\hat{X}}|^2)^{\frac{n+2}{2}\alpha+\frac{1}{2}}(1 + \kappa\langle\hat{\hat{X}}, \hat{\nu}\rangle^2)^{-\frac{n+2}{2}\alpha+\frac{1}{2}}$. We can check that $\hat{\hat{X}}_0$ is strictly convex and ψ satisfies (3.2.2). By Theorem 3.2.1, $\hat{\hat{X}}(\tau)$ converges to a point $\hat{q}$ in finite time $T^* > 0$. Thus, $\pi_{p_0}^{-1}(\hat{\hat{X}}(\tau))$ converges to the point $\pi_{p_0}^{-1}(q)$. □

Next, we prove Theorem 3.2.1. Note that under the flow (3.2.1), the support function $\hat{\hat{u}}$ evolves by

$$\hat{\hat{u}}_\tau = -\psi(\hat{\hat{u}}, x, \nabla\hat{\hat{u}})\hat{\hat{K}}^\alpha(x, \tau). \tag{3.2.3}$$

Lemma 3.2.3. *Suppose ψ satisfies (3.2.2), then under the flow (3.2.1), there exists a constant $\varepsilon_0 = \varepsilon_0(n, \alpha, A, \hat{\tilde{X}}_0) > 0$ such that the principal curvatures $\hat{\tilde{\kappa}}_i$ of $\hat{\tilde{X}}$ satisfies*

$$\hat{\tilde{\kappa}}_i \geq \varepsilon_0, \qquad \tau \in [0, T], \tag{3.2.4}$$

for any $T < T^*$ fixed, where T^* is the maximal existence time of (3.2.1).

Proof. In the following of the proof, we omit ˆ and ˜ and use t instead of τ for simplicity.

Let $(W_{ij}) = (u_{ij} + u\delta_{ij})$ be the inverse of the Weingarten matrix of X, whose eigenvalues $(\lambda_1, \ldots, \lambda_n)$ are the principal radii of curvature of X. To prove the lower bound of κ_i, it suffices to prove the upper bound of $\lambda(x,t) := \max_{i=1,\ldots,n} \lambda_i(x,t)$. We consider the function $\bar{G}(x,t) := \log \lambda + \frac{L}{2} r^2$, where $L > 0$ is a large constant to be determined. Suppose the maximum of $\bar{G}$ on $\mathbb{S}^n \times [0, T]$ is attained at (x_0, t_0), we choose a local orthonormal frame $e_1, \ldots, e_n$ around x_0 such that $\{W_{ij}(x,t)\}$ is diagonal at (x_0, t_0) and $W_{11}(x_0, t_0) = \lambda_1(x_0, t_0) = \lambda(x_0, t_0)$. Then the function

$$G(x,t) := \log W_{11}(x,t) + \frac{L}{2} r^2 \tag{3.2.5}$$

also attains its maximum at (x_0, t_0). Let $(W^{ij}) = (W_{ij})^{-1}$ be the inverse of (W_{ij}), $F^{pq} := \alpha \psi K^\alpha W^{pq}$, then at (x_0, t_0)

$$\begin{aligned} W_{11,t} =& (u_{11} + u)_t \\ =& -K^\alpha [\psi + \psi_u u_{11} + \psi_{u_i} u_{i11} + \psi_{x_1 x_1} + \psi_{uu} u_1^2 + \psi_{u_1 u_1} u_{11}^2 + 2\psi_{x_1 u} u_1 \\ & + 2\psi_{x_1 u_1} u_{11} + 2\psi_{u u_1} u_{11} u_1 - 2\alpha W^{ii} W_{ii1} (\psi_{x_1} + \psi_u u_1 + \psi_{u_1} u_{11}) \\ & + \alpha^2 \psi (W^{ii} W_{ii1})^2 + \alpha \psi W^{ii} W^{jj} W_{ij1}^2 - \alpha \psi W^{ii} W_{ii,11}]. \end{aligned}$$

Using the formula for commutating covariant derivatives on $\mathbb{S}^n$, we have at (x_0, t_0),

$$W_{i11} = W_{11i},$$
$$W_{ii,11} = W_{11,ii} + W_{ii} - W_{11},$$

Thus

$$\begin{aligned}
&W_{11,t} - F^{pp}W_{11,pp} \\
=& - K^{\alpha}[\psi + \psi_u W_{11} - \psi_u u + \psi_{u_i} W_{11i} - \psi_{u_1} u_1 + \psi_{x_1x_1} + \psi_{uu} u_1^2 + \psi_{u_1u_1} W_{11}^2 \\
&- 2\psi_{u_1u_1} W_{11} u + \psi_{u_1u_1} u^2 + 2\psi_{x_1u} u_1 + 2\psi_{x_1u_1} W_{11} - 2\psi_{x_1u_1} u \\
&+ 2\psi_{uu_1} W_{11} u_1 - 2\psi_{uu_1} u u_1 - 2\alpha W^{ii} W_{ii1}(\psi_{x_1} + \psi_u u_1 + \psi_{u_1} W_{11} \\
&- \psi_{u_1} u) + \alpha^2 \psi (W^{ii} W_{ii1})^2 + \alpha\psi W^{ii} W^{jj} W_{ij1}^2 + \alpha\psi W^{ii}(W_{11} - W_{ii})].
\end{aligned} \tag{3.2.6}$$

On the other hand, we have at (x_0, t_0)

$$r_t^2 = 2uu_t + 2u_i u_{it} = -2K^{\alpha}[u\psi + \psi_{x_i} u_i + \psi_u |\nabla u|^2 + \psi_{u_i} u_{ii} u_i - \alpha\psi W^{pp} W_{ppi} u_i],$$
$$(r^2)_{pp} = 2(W_{pi} u_i)_p = 2W_{ppi} u_i + 2W_{pp}^2 - 2uW_{pp},$$

which implies

$$\begin{aligned}
&r_t^2 - F^{pp}(r^2)_{pp} \\
=& - 2K^{\alpha}[u\psi + \psi_{x_i} u_i + \psi_u |\nabla u|^2 + \psi_{u_i} u_{ii} u_i + \alpha\psi W^{pp}(W_{pp}^2 - uW_{pp})] \\
=& - 2K^{\alpha}[(-n\alpha + 1)u\psi + \psi_{x_i} u_i + \psi_u |\nabla u|^2 + \psi_{u_i} u_{ii} u_i + \alpha\psi W_{pp}].
\end{aligned} \tag{3.2.7}$$

By maximum principle, at (x_0, t_0), we have

$$W_{11i} = -\frac{L}{2} W_{11}(r^2)_i = -LW_{11} W_{ii} u_i, \tag{3.2.8}$$

and

$$
\begin{aligned}
0 \leq & G_t - F^{pp}G_{pp} \\
= & \frac{W_{11t} - F^{pp}W_{11,pp}}{W_{11}} + F^{pp}\frac{W_{11p}^2}{W_{11}^2} + \frac{L}{2}(r_t^2 - F^{pp}(r^2)_{pp}) \\
= & - K^{\alpha}[\psi_{u_1u_1}W_{11} + \psi_{u_i}\frac{W_{11i}}{W_{11}} + \psi_u - 2\psi_{u_1u_1}u + 2\psi_{x_1u_1} + 2\psi_{uu_1}u_1 \\
& + \frac{1}{W_{11}}(\psi - \psi_u u - \psi_{u_1}u_1 + \psi_{x_1x_1} + \psi_{uu}u_1^2 + \psi_{u_1u_1}u^2 + 2\psi_{x_1u}u_1 - 2\psi_{x_1u_1}u - 2\psi_{uu_1}uu_1) \\
& - 2\alpha W^{ii}W_{ii1}(\psi_{u_1} + \frac{\psi_{x_1} + \psi_u u_1 - \psi_{u_1}u}{W_{11}}) + \alpha^2\frac{\psi(W^{ii}W_{ii1})^2}{W_{11}} + \alpha\frac{\psi W^{ii}W^{jj}W_{ij1}^2}{W_{11}} \\
& + \alpha\psi W^{ii}(1 - \frac{W_{ii}}{W_{11}}) - \alpha\psi W^{pp}\frac{W_{11p}^2}{W_{11}^2} + (1 - n\alpha)Lu\psi + L\psi_{x_i}u_i + L\psi_u|\nabla u|^2 \\
& + L\psi_{u_i}W_{ii}u_i - L\psi_{u_i}uu_i + \alpha L\psi W_{pp}] \\
\leq & - K^{\alpha}[-C(n, \alpha, A, L, X_0) + \psi_{u_1u_1}W_{11} - L\psi_{u_i}W_{ii}u_i - 2\alpha W^{ii}W_{ii1}(\psi_{u_1} + \frac{\psi_{x_1} + \psi_u u_1 - \psi_{u_1}u}{W_{11}}) \\
& + \alpha^2\frac{\psi(W^{ii}W_{ii1})^2}{W_{11}} + \alpha\frac{\psi W^{ii}W^{jj}W_{ij1}^2}{W_{11}} + \alpha\psi W^{ii}(1 - \frac{W_{ii}}{W_{11}}) - \alpha\psi W^{pp}\frac{W_{11p}^2}{W_{11}^2} \\
& + L\psi_{u_i}W_{ii}u_i + \alpha L\psi W_{pp}]
\end{aligned}
$$

where we used (3.2.8), (3.2.2) and the fact that $u^2 \le u^2 + |\nabla u|^2 \le C(X_0)$ in the last step since the flow is contracting. By Cauchy-Schwarz inequality,

$$
\begin{aligned}
&G_t - F^{pp} G_{pp} \\
\le& - K^\alpha [-C(n,\alpha,A,L,X_0) + (\psi_{u_1u_1} - C(n,\alpha,A,X_0))W_{11} \\
&+ \alpha \frac{\psi W^{ii} W^{jj} W_{ij1}^2}{W_{11}} - \alpha \psi W^{pp} \frac{W_{11p}^2}{W_{11}^2} + \alpha L \psi W_{pp}] \\
\le& - K^\alpha [-C(n,\alpha,A,L,X_0) + (\psi_{u_1u_1} - C(n,\alpha,A,X_0))W_{11} \\
&+ \alpha \frac{\psi W^{ii} W^{11} W_{11i}^2}{W_{11}} - \alpha \psi W^{pp} \frac{W_{11p}^2}{W_{11}^2} + \alpha L \psi W_{pp}] \\
\le& - K^\alpha [-C(n,\alpha,A,L,X_0) - C(n,\alpha,A,X_0) W_{11} + \alpha L \psi W_{11}].
\end{aligned}
\tag{3.2.9}
$$

Since $\psi \ge \frac{1}{A} > 0$ is bounded from below by (3.2.2), we can take $L \ge C_1(n,\alpha,A,X_0)$ sufficiently large such that $W_{11} \le C_2(n,\alpha,A,X_0)$. This implies that

$$
\lambda(x,t) \le C_2(n,\alpha,A,X_0). \tag{3.2.10}
$$

□

Lemma 3.2.4. *Suppose ψ satisfies the condition (3.2.2), and the inner radius $r_-(\hat{\tilde{\Omega}}(\tau))$ and outer radius $r_+(\hat{\tilde{\Omega}}(\tau))$ satisfies $0 < r_0 \le r_-(\hat{\tilde{\Omega}}(\tau)) \le r_+(\hat{\tilde{\Omega}}(\tau)) \le r_1$ for $\tau \in [0,T]$, then there is a constant $C(n,\alpha,\hat{\tilde{X}}_0,A)$ s.t.*

$$
\psi \hat{\tilde{K}}^\alpha(x,\tau) \le C(n,\alpha,\hat{\tilde{X}}_0,A) \frac{r_1}{r_0} \max\{\frac{1}{r_0^{n\alpha}}, \max_{\tau=0} \psi K^\alpha\} \tag{3.2.11}
$$

for $(x,\tau) \in \mathbb{S}^n \times [0,T]$.

Proof. In the following of the proof, we omit ˜ and ˆ, and use t instead of τ for simplicity.

First, by the definition of inner radius, there is a point $z_0 \in \Omega(T)$ s.t. $u_{z_0}(x,T) := u(x,T) - \langle z_0, x\rangle \geq r_0$, $\forall x \in \mathbb{S}^n$. Since X is a contracting flow, we also have

$$u_{z_0}(x,t) = u(x,t) - \langle z_0, x\rangle \geq r_0, \quad (x,t) \in \mathbb{S}^n \times [0,T]. \tag{3.2.12}$$

Consider the function $Q = \frac{-u_t}{u_{z_0} - \frac{r_0}{2}}$ on $(x,t) \in \mathbb{S}^n \times [0,T]$. Suppose that Q attains its maximum on $\mathbb{S}^n \times [0,T]$ at (x_0,t_0), then $\log Q$ will also attains its maximum at (x_0,t_0). If $t_0 = 0$, then we are done. Suppose $t_0 > 0$. A direct computation shows

$$\begin{aligned}
(\log Q)_t =& \frac{(-u_t)_t}{-u_t} - \frac{u_t}{u_{z_0} - \frac{r_0}{2}},\\
(\log Q)_i =& \frac{(-u_t)_i}{-u_t} - \frac{u_{z_0,i}}{u_{z_0} - \frac{r_0}{2}},\\
(\log Q)_{ij} =& \frac{(-u_t)_{ij}}{-u_t} - \frac{(-u_t)_i(-u_t)_j}{(-u_t)^2} - \frac{u_{z_0,ij}}{u_{z_0} - \frac{r_0}{2}} + \frac{u_{z_0,i}u_{z_0,j}}{(u_{z_0} - \frac{r_0}{2})^2}.
\end{aligned}$$

Let $F^{ij} := \alpha\psi K^\alpha W^{ij}$, where $W_{ij} = u_{ij} + u\delta_{ij}$ and $(W^{ij}) = (W_{ij})^{-1}$ as before. Then we have by maximum principle that at (x_0,t_0)

$$u_{ti} = \frac{u_t u_{z_0,i}}{u_{z_0} - \frac{r_0}{2}} \tag{3.2.13}$$

and

$$0 \leq (\log Q)_t - F^{ij}(\log Q)_{ij} = \frac{(-u_t)_t - F^{ij}(-u_t)_{ij}}{-u_t} - \frac{u_t - F^{ij}u_{z_0,ij}}{u_{z_0} - \frac{r_0}{2}}. \tag{3.2.14}$$

Let $\sigma_k (1 \leq k \leq n)$ be the $k-th$ elementary symmetric function, $\sigma_n^{ij} := \frac{\partial \sigma_n(W_{ij})}{\partial W_{ij}}$, then $\sigma_n(W_{ij}) = K^{-1}$, $F^{ij} = \alpha\psi\sigma_n^{-\alpha-1}\sigma_n^{ij}$, and

$$\begin{aligned}
-u_{tt} =& (\psi\sigma_n^{-\alpha})_t = \psi_t\sigma_n^{-\alpha} - \alpha\psi\sigma_n^{-\alpha-1}\sigma_n^{ij}(u_{tij} + u_t\delta_{ij})\\
=& \sigma_n^{-\alpha}(\psi_u u_t + \psi_{u_i}u_{it} - \alpha\psi W^{ij}u_{tij} - \alpha\psi\frac{\sigma_{n-1}}{\sigma_n}u_t).
\end{aligned} \tag{3.2.15}$$

Moreover, we have

$$\begin{aligned}F^{ij}u_{z_0,ij} =&F^{ij}(u_{ij} + \langle z_0, x\rangle\delta_{ij}) = F^{ij}(u_{ij} + u\delta_{ij} - u_{z_0}\delta_{ij})\\ =&n\alpha\psi\sigma_n^{-\alpha} - \alpha\psi\sigma_n^{-\alpha}\frac{\sigma_{n-1}}{\sigma_n}u_{z_0}\\ =&-\alpha u_t(n - \frac{\sigma_{n-1}}{\sigma_n}u_{z_0}).\end{aligned}\tag{3.2.16}$$

Plugging the above two equations into (3.2.14), and using (3.2.13), we get at (x_0, t_0)

$$\begin{aligned}0 \leq&\sigma_n^{-\alpha}(-\psi_u - \frac{\psi_{u_i}u_{z_0,i}}{u_{z_0} - \frac{r_0}{2}} + \alpha\psi\frac{\sigma_{n-1}}{\sigma_n}) - \frac{(n\alpha + 1) - \alpha u_{z_0}\frac{\sigma_{n-1}}{\sigma_n}}{u_{z_0} - \frac{r_0}{2}}u_t\\ =&Q[-\frac{\psi_u}{\psi}(u_{z_0} - \frac{r_0}{2}) - \frac{\psi_{u_i}u_{z_0,i}}{\psi} + \alpha\frac{\sigma_{n-1}}{\sigma_n}(u_{z_0} - \frac{r_0}{2}) + (n\alpha + 1) - \alpha u_{z_0}\frac{\sigma_{n-1}}{\sigma_n}]\\ \leq&Q[C(n, \alpha, X_0, A) - \alpha\frac{r_0}{2}\frac{\sigma_{n-1}}{\sigma_n}],\end{aligned}\tag{3.2.17}$$

where we used (3.2.2) and the fact that X is contracting and strictly convex, which implies that $|\nabla u_{z_0}|_{C^0} \leq |u_{z_0}|_{C^0} \leq C(X_0)$ in the last inequality. Note that

$$\frac{\sigma_{n-1}}{\sigma_n} \geq C(n)\sigma_n^{-\frac{1}{n}} = C(n)\Big(\frac{Q(u_{z_0} - \frac{r_0}{2})}{\psi}\Big)^{\frac{1}{n\alpha}} \geq C(n, \alpha, A)r_0^{\frac{1}{n\alpha}}Q^{\frac{1}{n\alpha}}. \tag{3.2.18}$$

Plugging this into (3.2.17), we get

$$Q(x_0, t_0) \leq \frac{C(n, \alpha, X_0, A)}{r_0^{n\alpha+1}}, \tag{3.2.19}$$

and

$$\psi K^\alpha(x, t) \leq (u_{z_0}(x, t) - \frac{r_0}{2})\max_{(x,t)\in\mathbb{S}^n\times[0,T]}Q(x, t) \leq C(n, \alpha, X_0, A)\frac{r_1}{r_0}\max\{\frac{1}{r_0^{n\alpha}}, \max_{t=0}\psi K^\alpha\}$$

for $(x, t) \in \mathbb{S}^n \times [0, T]$. □

Proof of Theorem 3.2.1. Since $\hat{\tilde{X}}(\tau)$ is a contracting flow, $|\hat{\tilde{u}}|_{C^0} \leq C(\hat{\tilde{X}}_0)$. Since $\hat{\tilde{X}}$ is strictly convex, the C^0 estimate implies the C^1 estimate of $\hat{\tilde{u}}$. By the definition of T^*, $r_+ \geq \varepsilon > 0$ on $[0, T]$ for any $T < T^*$. On the other hand, due to Lemma 2.2 in [20] and Lemma 3.2.3, we have

$$r_+^2 \leq C(n, \alpha, A, \hat{\tilde{X}}_0) r_-, \tag{3.2.20}$$

as long as the flow exists. By (3.2.20), Lemma 3.2.3 and Lemma 3.2.4

$$\|\hat{\tilde{u}}(x, \tau)\|_{C^2(\mathbb{S}^n \times [0,T])} \leq C(n, \alpha, A, \hat{\tilde{X}}_0, \varepsilon). \tag{3.2.21}$$

Since equation (3.2.3) is a concave parabolic equation, by Krylov-Safanov's theorem and the standard theory on parabolic equations, this implies that $\hat{\tilde{X}}(\tau)$ is smooth on $[0, T]$. Thus, $\lim_{\tau \to T^*} r_- = \lim_{\tau \to T^*} r_+ = 0$. That is, $\hat{\tilde{X}}(\tau)$ converges to a point $\hat{q}$ as $\tau \to T^*$. Set the initial value of the flow (3.2.1) to be the boundary of the outer ball of $\hat{\tilde{X}}_0$, then the solution will be a family of geodesic spheres with radii $\{\tilde{r}(\tau)\}$ satisfying the ODE

$$\frac{\partial \tilde{r}}{\partial \tau} = -\frac{\psi}{\tilde{r}^{n\alpha}} \leq -\frac{1}{A\tilde{r}^{n\alpha}}, \tag{3.2.22}$$

which converges to zero in finite time. By comparison principle, $\hat{\tilde{X}}(\tau)$ will converge to a point in finite time as well. □

3.3 The rescaled flow

The un-normalized flow (1.2.2) converges to a point $q_0 \in \mathbb{N}^{n+1}(\kappa)$ as $\tau \to T^*$ by Corollary 3.2.2. For $\kappa = 1$ and sufficiently small δ_1, $\tilde{X}(\tau)$ is contained in the open hemi-sphere $\mathcal{H}(q_0)$ when $\tau \in [T^* - \delta_1, T^*]$. We consider a new geodesic polar coordinate centered at q_0 and use the new projection π_{q_0} of $\tilde{X}(\tau)$ for $\tau \in [T^* - \delta_1, T^*]$

onto the hyperplane of $\mathbb{R}^{n+2}$ which is tangent to $\mathbb{N}^{n+1}(\kappa)$ at q_0. We re-scale $\hat{\hat{X}}(\tau)$ to keep the enclosed volume fixed. Let $\hat{X}(x,t) = e^t \hat{\hat{X}}(x,\tau)$, we have

$$\begin{aligned}\hat{X}_t(x,t) =& \hat{X}(x,t) + e^t \hat{\hat{X}}_\tau(x,\tau)\tau'(t)\\ =& \hat{X}(x,t) - e^{(n\alpha+1)t}\tau'(t)(1+\kappa|\hat{\hat{X}}|^2)^{\frac{n+2}{2}\alpha+\frac{1}{2}}(1+\kappa|\langle\hat{\hat{X}},\hat{\nu}\rangle|^2)^{-\frac{n+2}{2}\alpha+\frac{1}{2}}\hat{K}^\alpha\\ =& \hat{X}(x,t) - e^{(n\alpha+1)t}\tau'(t)(1+\kappa\frac{|\hat{X}|^2}{e^{2t}})^{\frac{n+2}{2}\alpha+\frac{1}{2}}(1+\kappa\frac{|\langle\hat{X},\hat{\nu}\rangle|^2}{e^{2t}})^{-\frac{n+2}{2}\alpha+\frac{1}{2}}\hat{K}^\alpha,\end{aligned} \tag{3.3.1}$$

and the corresponding support function evolves by

$$\hat{u}_t = \hat{u} - e^{(n\alpha+1)t}\tau'(t)(1+\kappa\frac{\hat{r}^2}{e^{2t}})^{\frac{n+2}{2}\alpha+\frac{1}{2}}(1+\kappa\frac{\hat{u}^2}{e^{2t}})^{-\frac{n+2}{2}\alpha+\frac{1}{2}}\hat{K}^\alpha. \tag{3.3.2}$$

Since

$$\begin{aligned}0 = \frac{d\mathrm{Vol}(\hat{\Omega})}{dt} &= \int_{\mathbb{S}^n}\hat{u}_t\sigma_n d\theta\\ &= \int_{\mathbb{S}^n}\hat{u}\sigma_n d\theta - \tau'(t)e^{(n\alpha+1)t}\int_{\mathbb{S}^n}(1+\kappa\frac{\hat{r}^2}{e^{2t}})^{\frac{n+2}{2}\alpha+\frac{1}{2}}(1+\kappa\frac{\hat{u}^2}{e^{2t}})^{-\frac{n+2}{2}\alpha+\frac{1}{2}}\hat{K}^{\alpha-1}d\theta.\end{aligned}$$

We have

$$\tau'(t) = \frac{\int_{\mathbb{S}^n}\hat{u}\sigma_n d\theta}{e^{(n\alpha+1)t}\int_{\mathbb{S}^n}(1+\kappa\frac{\hat{r}^2}{e^{2t}})^{\frac{n+2}{2}\alpha+\frac{1}{2}}(1+\kappa\frac{\hat{u}^2}{e^{2t}})^{-\frac{n+2}{2}\alpha+\frac{1}{2}}\hat{K}^{\alpha-1}d\theta} = \frac{(n+1)|\hat{\hat{\Omega}}(\tau)|}{\frac{d|\hat{\hat{\Omega}}(\tau)|}{d\tau}}, \tag{3.3.3}$$

that is

$$t = \frac{1}{n+1}\log\Big(\frac{|B(1)|}{|\hat{\hat{\Omega}}(\tau)|}\Big), \tag{3.3.4}$$

where $B(1)$ denotes the unit ball in $\mathbb{R}^{n+1}$. Note that $|\hat{\hat{\Omega}}(\tau)|$ approaches zero as τ approaches T^*, and consequently t approaches infinity as τ approaches T^* and the solution $\hat{X}(x,t)$ exists for all positive time. We get

$$\hat{X}_t(x,t) = \hat{X}(x,t) - \frac{(1+\kappa\frac{|\hat{X}|^2}{e^{2t}})^{\frac{n+2}{2}\alpha+\frac{1}{2}}(1+\kappa\frac{\langle \hat{X},\hat{\nu}\rangle^2}{e^{2t}})^{-\frac{n+2}{2}\alpha+\frac{1}{2}}\hat{K}^{\alpha}}{\fint_{\mathbb{S}^n}(1+\kappa\frac{|\hat{X}|^2}{e^{2t}})^{\frac{n+2}{2}\alpha+\frac{1}{2}}(1+\kappa\frac{\langle \hat{X},\hat{\nu}\rangle^2}{e^{2t}})^{-\frac{n+2}{2}\alpha+\frac{1}{2}}\hat{K}^{\alpha-1}d\theta}\hat{\nu}, \tag{3.3.5}$$

with initial value $\hat{X}_0 = \left(\frac{B(1)}{|\hat{\tilde{\Omega}}(T^*-\delta_1)|}\right)^{\frac{1}{n+1}}\pi_{q_0}(\tilde{X}_{T^*-\delta_1})$. The support function satisfies

$$\begin{aligned}\hat{u}_t &= \hat{u} - \frac{(1+\kappa\frac{\hat{r}^2}{e^{2t}})^{\frac{n+2}{2}\alpha+\frac{1}{2}}(1+\kappa\frac{\hat{u}^2}{e^{2t}})^{-\frac{n+2}{2}\alpha+\frac{1}{2}}\hat{K}^{\alpha}}{\fint_{\mathbb{S}^n}(1+\kappa\frac{\hat{r}^2}{e^{2t}})^{\frac{n+2}{2}\alpha+\frac{1}{2}}(1+\kappa\frac{\hat{u}^2}{e^{2t}})^{-\frac{n+2}{2}\alpha+\frac{1}{2}}\hat{K}^{\alpha-1}d\theta} \\ &= \hat{u} - \frac{\psi\hat{K}^{\alpha}}{\fint_{\mathbb{S}^n}\psi\hat{K}^{\alpha-1}d\theta},\end{aligned} \tag{3.3.6}$$

with initial value $\hat{u}_0 = \langle \hat{X}_0, \hat{\nu}\rangle$, $\psi = (1+\kappa\frac{\hat{r}^2}{e^{2t}})^{\frac{n+2}{2}\alpha+\frac{1}{2}}(1+\kappa\frac{\hat{u}^2}{e^{2t}})^{-\frac{n+2}{2}\alpha+\frac{1}{2}}$.

3.3.1 Entropy and monotonicity

Recall for a convex body $\Omega \subset \mathbb{R}^{n+1}$, $z_0 \in \Omega$ [9] defined the entropy

$$\mathcal{E}_\alpha(\Omega, z_0) = \begin{cases} \frac{\alpha}{\alpha-1}(\log\fint_{\mathbb{S}^n} u_{z_0}^{1-\frac{1}{\alpha}}d\theta), & \alpha \neq 1; \\ \fint_{\mathbb{S}^n}\log u_{z_0}d\theta, & \alpha = 1; \end{cases} \tag{3.3.7}$$

and the entropy

$$\mathcal{E}_\alpha(\Omega) := \sup_{z_0\in\Omega}\mathcal{E}_\alpha(\Omega, z_0), \tag{3.3.8}$$

where $u_{z_0}(x) := \sup_{z\in\Omega}\langle z - z_0, x\rangle = u(x) - \langle z_0, x\rangle$. We know from Lemma 2.5 of [9] that there is a unique point $z_e \in \text{Int}(\Omega)$ ($\text{Int}(\Omega)$ denotes the interior of Ω) for the convex body Ω with non-empty interior such that $\mathcal{E}_\alpha(\Omega) = \mathcal{E}_\alpha(\Omega, z_e)$, z_e is called the 'entropy point' of Ω. Moreover, $\mathcal{E}_\alpha(\Omega) \geq 0$ for Ω with $|\Omega| = |B(1)|$ and $\alpha > \frac{1}{n+2}$ by Corollary 2.2 of [9].

In the following, we denote $\hat{\Omega}(t)$ and $\hat{\hat{\Omega}}(\tau)$ by $\hat{\Omega}_t$ and $\hat{\hat{\Omega}}_\tau$ respectively. We remark that for any $z_0 \in \hat{\hat{\Omega}}_\tau$, the entropy of $\hat{\Omega}$ and $\hat{\hat{\Omega}}$ is related by

$$\mathcal{E}_\alpha(\hat{\Omega}_t, e^t z_0) = \mathcal{E}_\alpha(\hat{\hat{\Omega}}_{\tau(t)}, z_0) - \frac{1}{n+1}\log\Big(\frac{|\hat{\hat{\Omega}}_{\tau(t)}|}{|B(1)|}\Big). \tag{3.3.9}$$

Therefore, $z_e \in \hat{\hat{\Omega}}_\tau$ is the entropy point of $\hat{\hat{\Omega}}_\tau$ if and only if $e^t z_0 \in \hat{\Omega}_{t(\tau)}$ is the entropy point of $\hat{\Omega}_{t(\tau)}$. We have the following key monotonicity property for $\mathcal{E}_\alpha(\hat{\Omega}_t) + C(n,\alpha,\tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}t}$, where $C(n,\alpha,\tilde{X}_0)$ is a positive constant depending only on $n,\alpha,\tilde{X}_0$.

Theorem 3.3.1. *There exists a positive constant $C(n,\alpha,\tilde{X}_0) > 0$ depending only on $n,\alpha,\tilde{X}_0$ such that $\mathcal{E}_\alpha(\hat{\Omega}_t) + C(n,\alpha,\tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}t}$ is none-increasing along the normalized flow (3.3.6) for $\alpha \geq \frac{1}{n+2}$ when $t \geq T_0(n,\alpha,\tilde{X}_0) \geq t(T^* - \delta_1)$. In particular, the entropy $\mathcal{E}_\alpha(\hat{\Omega}_t)$ is bounded from above for $\alpha \geq \frac{1}{n+2}$.*

Proof. Let $\tau_0 \geq T^* - \delta_1$ to be determined and $\tau_1 \geq \tau_0$. Let $t_i = t(\tau_i) \quad (i = 0,1)$, $z_1 \in \mathrm{Int}(\hat{\hat{\Omega}}_{\tau_1})$ be the unique entropy point of $\hat{\hat{\Omega}}_{\tau_1}$, then $e^{t_1}z_1 \in \mathrm{Int}(\hat{\Omega}_{t_1})$ will be the entropy point of $\hat{\Omega}_{t_1}$. Since $\hat{\hat{X}}$ is contracting, $\hat{\hat{\Omega}}_\tau \supset \hat{\hat{\Omega}}_{\tau_1} \ni z_1$ for any $\tau \leq \tau_1$. Therefore, $\hat{\Omega}_t \supset \hat{\Omega}_{t_1} \ni e^t z_1$, $\hat{u}_{e^t z_1}(x,t) > 0$ for every $t \leq t_1$. Denote $r_+ = r_+(\hat{\hat{\Omega}}_t)$ and $r_- = r_-(\hat{\hat{\Omega}}_t)$ the outer and inner radius of $\hat{\hat{\Omega}}_t$ respectively, we have by (3.2.20) that

$$C(n,\alpha,\tilde{X}_0)r_+^{2n+1} \leq C(n)r_-^n r_+ \leq \mathrm{Vol}(\hat{\hat{\Omega}}) = e^{-(n+1)t} \leq C(n)r_- r_+^n \leq C(n,\alpha,\tilde{X}_0)r_-^{1+\frac{n}{2}}. \tag{3.3.10}$$

Thus

$$C(n,\alpha,\tilde{X}_0)e^{-\frac{2(n+1)}{n+2}t} \leq r_- \leq r_+ \leq C(n,\alpha,\tilde{X}_0)e^{-\frac{n+1}{2n+1}t}. \tag{3.3.11}$$

Therefore,

$$
\begin{aligned}
&1 \geq \frac{1}{\psi} \geq (1+\hat{\tilde{r}}_+^2)^{-\frac{n+2}{2}\alpha-\frac{1}{2}} \geq 1-(\frac{n+2}{2}\alpha+\frac{1}{2})C(n,\alpha,\tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}t}, \text{ if } \kappa=1;\\
&1 \geq \psi \geq (1-\hat{\tilde{r}}^2)^{\frac{n+2}{2}\alpha+\frac{1}{2}} \geq 1-(\frac{n+2}{2}\alpha+\frac{1}{2})C(n,\alpha,\tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}t}, \text{ if } \kappa=-1.
\end{aligned}
\tag{3.3.12}
$$

This implies the following inequalities for $\mathcal{E}_\alpha(\hat{\Omega}_t)$.

Case 1: $\alpha \neq 1$.

$$
\begin{aligned}
\mathcal{E}_\alpha(\hat{\Omega}_{t_1}) =&\mathcal{E}_\alpha(\hat{\Omega}_{t_1}, e^{t_1}z_1)\\
=&\mathcal{E}_\alpha(\hat{\Omega}_{t_0}, e^{t_0}z_1) + \int_{t_0}^{t_1} \frac{\frac{\alpha}{\alpha-1} d\log\left(⨍_{\mathbb{S}^n} \hat{u}_{e^tz_1}^{1-\frac{1}{\alpha}} d\theta\right)}{dt} dt\\
=&\mathcal{E}_\alpha(\hat{\Omega}_{t_0}, e^{t_0}z_1) + \int_{t_0}^{t_1} \frac{⨍_{\mathbb{S}^n} \hat{u}_{e^tz_1}^{-\frac{1}{\alpha}}(\hat{u}_{e^tz_1} - \frac{\psi\hat{K}^\alpha}{⨍_{\mathbb{S}^n}\psi\hat{K}^{\alpha-1}d\theta})d\theta}{⨍_{\mathbb{S}^n} \hat{u}_{e^tz_1}^{1-\frac{1}{\alpha}} d\theta} dt\\
=&\mathcal{E}_\alpha(\hat{\Omega}_{t_0}, e^{t_0}z_1) + \int_{t_0}^{t_1} 1 - \frac{⨍_{\mathbb{S}^n} \psi\hat{u}_{e^tz_1}^{-\frac{1}{\alpha}}\hat{K}^\alpha d\theta}{⨍_{\mathbb{S}^n} \hat{u}_{e^tz_1}^{1-\frac{1}{\alpha}} d\theta ⨍_{\mathbb{S}^n}\psi\hat{K}^{\alpha-1}d\theta} dt\\
\leq&\mathcal{E}_\alpha(\hat{\Omega}_{t_0}, e^{t_0}z_1) + \int_{t_0}^{t_1} 1 - [1-C(n,\alpha,\tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}t}]\frac{⨍_{\mathbb{S}^n} \hat{u}_{e^tz_1}^{-\frac{1}{\alpha}}\hat{K}^\alpha d\theta}{⨍_{\mathbb{S}^n} \hat{u}_{e^tz_1}^{1-\frac{1}{\alpha}} d\theta ⨍_{\mathbb{S}^n}\hat{K}^{\alpha-1}d\theta} dt\\
\leq&\mathcal{E}_\alpha(\hat{\Omega}_{t_0}, e^{t_0}z_1) + \int_{t_0}^{t_1} \left(1 - \frac{⨍_{\mathbb{S}^n} \hat{u}_{e^tz_1}^{-\frac{1}{\alpha}}\hat{K}^\alpha d\theta}{⨍_{\mathbb{S}^n} \hat{u}_{e^tz_1}^{1-\frac{1}{\alpha}} d\theta ⨍_{\mathbb{S}^n}\hat{K}^{\alpha-1}d\theta}\right)[1-C(n,\alpha,\tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}t}] dt\\
&+ C(n,\alpha,\tilde{X}_0)\int_{t_0}^{t_1} e^{-\frac{2(n+1)}{2n+1}t}dt\\
=&\mathcal{E}_\alpha(\hat{\Omega}_{t_0}, e^{t_0}z_1) + \int_{t_0}^{t_1} \left(1 - \frac{⨍_{\mathbb{S}^n} \hat{u}_{e^tz_1}^{-\frac{1}{\alpha}-1}\hat{K}^{\alpha+1} d\tilde{\theta} ⨍_{\mathbb{S}^n} d\tilde{\theta}}{⨍_{\mathbb{S}^n} \hat{u}_{e^tz_1}^{-\frac{1}{\alpha}}\hat{K} d\tilde{\theta} ⨍_{\mathbb{S}^n} \hat{u}_{e^tz_1}^{-1}\hat{K}^\alpha d\tilde{\theta}}\right)[1-C(n,\alpha,\tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}t}] dt\\
&+ C(n,\alpha,\tilde{X}_0)\int_{t_0}^{t_1} e^{-\frac{2(n+1)}{2n+1}t}dt
\end{aligned}
$$

$$
\begin{aligned}
&\leq \mathcal{E}_\alpha(\hat{\Omega}_{t_0}, e^{t_0}z_1) + C(n,\alpha,\tilde{X}_0)\int_{t_0}^{t_1} e^{-\frac{2(n+1)}{2n+1}t}dt \\
&\leq \mathcal{E}_\alpha(\hat{\Omega}_{t_0}) + C(n,\alpha,\tilde{X}_0)(e^{-\frac{2(n+1)}{2n+1}t_0} - e^{-\frac{2(n+1)}{2n+1}t_1})
\end{aligned}
$$

for $t_0 \geq T_0(n,\alpha,\tilde{X}_0) \geq t(T^* - \delta_1)$ large such that $1 - C(n,\alpha,\tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}T_0} \geq 0$, where $d\tilde{\theta} := \frac{\hat{u}_{e^t z_1}}{\hat{K}}d\theta$ and we used (3.3.12) in the first inequality and the fact $⨍_{\mathbb{S}^n} \frac{\hat{u}_{e^t z_1}}{\hat{K}}d\theta = \mathrm{Vol}(\hat{\Omega}_t) \equiv 1$ with Hölder inequality in the second to the last inequality. That is,

$$
\mathcal{E}_\alpha(\hat{\Omega}_{t_1}) + C(n,\alpha,\tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}t_1} \leq \mathcal{E}_\alpha(\hat{\Omega}_{t_0}) + C(n,\alpha,\tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}t_0}. \tag{3.3.13}
$$

Case 2: $\alpha = 1$.

$$
\begin{aligned}
\mathcal{E}_1(\hat{\Omega}_{t_1}) =& \mathcal{E}_1(\hat{\Omega}_{t_1}, e^{t_1}z_1) \\
=& \mathcal{E}_1(\hat{\Omega}_{t_0}, e^{t_0}z_1) + \int_{t_0}^{t_1} \frac{d\Big(⨍_{\mathbb{S}^n} \log \hat{u}_{e^t z_1} d\theta\Big)}{dt}dt \\
=& \mathcal{E}_1(\hat{\Omega}_{t_0}, e^{t_0}z_1) + \int_{t_0}^{t_1} ⨍_{\mathbb{S}^n} \frac{\hat{u}_{e^t z_1} - \frac{\psi\hat{K}}{⨍_{\mathbb{S}^n}\psi d\theta}}{\hat{u}_{e^t z_1}}dt \\
=& \mathcal{E}_1(\hat{\Omega}_{t_0}, e^{t_0}z_1) + \int_{t_0}^{t_1} 1 - \frac{⨍_{\mathbb{S}^n}\psi \hat{u}^{-1}_{e^t z_1}\hat{K}d\theta}{⨍_{\mathbb{S}^n}\psi d\theta}dt \\
\leq& \mathcal{E}_1(\hat{\Omega}_{t_0}, e^{t_0}z_1) + \int_{t_0}^{t_1} 1 - [1 - C(n,\tilde{X}_0)e^{-2\frac{n+1}{2n+1}t}]⨍_{\mathbb{S}^n} \hat{u}^{-1}_{e^t z_1}\hat{K}d\theta dt \\
\leq& \mathcal{E}_1(\hat{\Omega}_{t_0}, e^{t_0}z_1) - \int_{t_0}^{t_1}[1 - C(n,\tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}t}]⨍_{\mathbb{S}^n} \Big(\sqrt{\frac{\hat{u}_{e^t z_1}}{\hat{K}}} - \sqrt{\frac{\hat{K}}{\hat{u}_{e^t z_1}}}\Big)^2 d\theta dt \\
&+ C(n,\tilde{X}_0)\int_{t_0}^{t_1} e^{-\frac{2(n+1)}{2n+1}t}dt \\
\leq& \mathcal{E}_1(\hat{\Omega}_{t_0}, e^{t_0}z_1) + C(n,\tilde{X}_0)\int_{t_0}^{t_1} e^{-\frac{2(n+1)}{2n+1}t}dt \\
\leq& \mathcal{E}_1(\hat{\Omega}_{t_0}) + C(n,\tilde{X}_0)(e^{-\frac{2(n+1)}{2n+1}t_0} - e^{-\frac{2(n+1)}{2n+1}t_1})
\end{aligned}
$$

for $t_0 \geq T_0(n, \alpha, \tilde{X}_0) \geq t(T^* - \delta_1)$ large such that $1 - C(n, \tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}T_0} \geq 0$, where we used (3.3.12) in the first inequality and the fact that $\fint_{\mathbb{S}^n} \frac{\hat{u}_{e^t z_1}}{\hat{K}} d\theta = \mathrm{Vol}(\hat{\Omega}_t) \equiv 1$ in the second inequality. That is,

$$\mathcal{E}_1(\hat{\Omega}_{t_1}) + C(n, \tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}t_1} \leq \mathcal{E}_1(\hat{\Omega}_{t_0}) + C(n, \tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}t_0}. \tag{3.3.14}$$

This completes the proof of the Theorem since $t_1 \geq t_0 \geq T_0(n, \alpha, \tilde{X}_0)$ are arbitrary.

□

Remark 3.3.2. The method to modify the "standard entropy" by adding a correction term here to obtain a proper monotone entropy follows the argument in [16], where in-homogeneous Gauss curvature type flows was treated.

Corollary 3.3.3. *Let $\hat{M}_t = \partial\hat{\Omega}_t$ be a solution to the normalized flow (3.3.6) with $|\hat{\Omega}_t| = |B(1)|$ and $\alpha > \frac{1}{n+2}$. Then there exists $C(n, \alpha, \tilde{X}_0)$ such that*

$$\max\{w_+(\hat{\Omega}_t), r_+(\hat{\Omega}_t)\} \leq C, \quad \min\{w_-(\hat{\Omega}_t), r_-(\hat{\Omega}_t)\} \geq \frac{1}{C} \tag{3.3.15}$$

for all $t \geq T_0(n, \alpha, \tilde{X}_0)$, where $w_-(\hat{\Omega}_t)$, $w_+(\hat{\Omega}_t)$ are the minimum and maximum of the width function $w(\hat{\Omega}_t)(x) := \hat{u}(x, t) + \hat{u}(-x, t), x \in \mathbb{S}^n$ respectively.

Proof. Since $|\hat{\Omega}_t| = |B(1)|$, by Proposition 2.7 of [9], there exist positive constants β and C depending only on n, α such that

$$\max\{r_+(\hat{\Omega}_t), w_+(\hat{\Omega}_t)\} \leq Ce^{n\beta\mathcal{E}_\alpha(\hat{\Omega}_t)} \tag{3.3.16}$$

and

$$\min\{r_-(\hat{\Omega}_t), w_-(\hat{\Omega}_t)\} \geq C^{-1}e^{-\beta\mathcal{E}_\alpha(\hat{\Omega}_t)}. \tag{3.3.17}$$

By Theorem 3.3.1, $\mathcal{E}_\alpha(\hat{\Omega}_t) + C(n,\alpha,\tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}t}$ is non-increasing and $\mathcal{E}_\alpha(\hat{\Omega}_t)$ is bounded from above for $t \geq T_0(n,\alpha,\tilde{X}_0)$. The corollary follows directly from (3.3.16) and (3.3.17). □

3.3.2 C^0 estimates

Next, we derive the uniform C^0 bound for $\hat{u}(x,t)$ along the normalized flow (3.3.6) with $|\hat{\Omega}_t| = |B(1)|$. The main effort is to derive the uniform lower bound for $\hat{u}(x,t)$ since the upper bound of $\hat{u}(x,t)$ follows directly from Corollary 3.3.3.

Define $\mathcal{E}_{\alpha,\infty} := \lim_{t\to\infty} \mathcal{E}_\alpha(\hat{\Omega}_t) = \lim_{t\to\infty} \mathcal{E}_\alpha(\hat{\Omega}_t) + C(n,\alpha,\tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}t}$, where $C(n,\alpha,\tilde{X}_0)$ is the constant in Theorem 3.3.1. From the monotonicity property in Theorem 3.3.1 and the fact that $\mathcal{E}_\alpha(\hat{\Omega}_t) \geq 0$ since $|\hat{\Omega}_t| = |B(1)|$ (see Corollary 2.2 of [9]), we know the limit $\mathcal{E}_{\alpha,\infty}$ exists and is finite. Now we derive a relation of $\mathcal{E}_{\alpha,\infty}$ and $\mathcal{E}_\alpha(\hat{\Omega}_t, 0)$.

Lemma 3.3.4. *Let $\hat{u}(x,t)$ be the unique positive solution of (3.3.6) with $|\hat{\Omega}_0| = |B(1)|$. Let $T_0(n,\alpha,\tilde{X}_0)$ and $C(n,\alpha,\tilde{X}_0)$ be the constants in Theorem 3.3.1, then for any $t \geq T_0$, we have*

(1) For $\alpha \neq 1$, $\alpha \geq \frac{1}{n+2}$.

$$\begin{aligned}
&\mathcal{E}_\alpha(\hat{\Omega}_t, 0) - \mathcal{E}_{\alpha,\infty} \\
&\geq \int_t^\infty \Big(\frac{\fint_{\mathbb{S}^n} \hat{u}^{-\frac{1}{\alpha}}(x,s)\hat{K}^\alpha(x,s)d\theta}{\fint_{\mathbb{S}^n} \hat{u}^{1-\frac{1}{\alpha}}(x,s)d\theta \fint_{\mathbb{S}^n} \hat{K}^{\alpha-1}(x,s)d\theta} - 1\Big)[1 - C(n,\alpha,\tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}s}]ds - C(n,\alpha,\tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}t}.
\end{aligned} \tag{3.3.18}$$

(2) For $\alpha = 1$.

$$\mathcal{E}_1(\hat{\Omega}_t, 0) - \mathcal{E}_{1,\infty}$$

$$\geq \int_t^{\infty} [1 - C(n, \tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}s}] \fint_{\mathbb{S}^n} \Big(\sqrt{\frac{\hat{u}(x,s)}{\hat{K}(x,s)}} - \sqrt{\frac{\hat{K}(x,s)}{\hat{u}(x,s)}}\Big)^2 d\theta ds \quad -C(n, \tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}t}. \tag{3.3.19}$$

Proof. We adopt the arguments in [32]. Let T_0 be the positive constant in Theorem 3.3.1, for each $\bar{t} \geq t_0 \geq T_0$ fixed, pick $t_1 > \bar{t}$. Let $e^{t_1} z_1 = a^{t_1} = (a_1^{t_1}, \dots, a_{n+1}^{t_1})$ be the entropy point of $\hat{\Omega}_{t_1}$, where $z_1 = e^{-t_1} a^{t_1} \in \hat{\tilde{\Omega}}_{t_1}$. Note that since both the origin and the entropy point a^{t_1} are in $\mathrm{Int}(\hat{\Omega}_{t_1})$,

$$|a^{t_1}| \leq 2r_+(\hat{\Omega}_{t_1}) \leq C(n, \alpha, \tilde{X}_0) \tag{3.3.20}$$

by corollary 3.3.3.

Set $\hat{u}_{e^t z_1}(x,t) = \hat{u}(x,t) - e^t\langle z_1, x\rangle$, then from of the proof of Theorem 3.3.1, we have

For $\alpha \neq 1$.

$$\begin{aligned}
\mathcal{E}_\alpha(\hat{\Omega}_{t_1}) \leq & \mathcal{E}_\alpha(\hat{\Omega}_{t_0}, e^{t_0} z_1) + \int_{t_0}^{t_1} \Big(1 - \frac{\fint_{\mathbb{S}^n} \hat{u}_{e^t z_1}^{-\frac{1}{\alpha}} \hat{K}^\alpha d\theta}{\fint_{\mathbb{S}^n} \hat{u}_{e^t z_1}^{1-\frac{1}{\alpha}} d\theta \fint_{\mathbb{S}^n} \hat{K}^{\alpha-1} d\theta}\Big)[1 - C(n, \alpha, \tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}t}]dt \\
& + C(n, \alpha, \tilde{X}_0) \int_{t_0}^{t_1} e^{-\frac{2(n+1)}{2n+1}t} dt \\
= & \frac{\alpha}{\alpha - 1} \log \fint_{\mathbb{S}^n} \hat{u}_{e^{t_0} z_1}^{1-\frac{1}{\alpha}} d\theta + \int_{t_0}^{t_1} \Big(1 - \frac{\fint_{\mathbb{S}^n} \hat{u}_{e^t z_1}^{-\frac{1}{\alpha}} \hat{K}^\alpha d\theta}{\fint_{\mathbb{S}^n} \hat{u}_{e^t z_1}^{1-\frac{1}{\alpha}} d\theta \fint_{\mathbb{S}^n} \hat{K}^{\alpha-1} d\theta}\Big)[1 - C(n, \alpha, \tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}t}]dt \\
& + C(n, \alpha, \tilde{X}_0)(e^{-\frac{2(n+1)}{2n+1}t_0} - e^{-\frac{2(n+1)}{2n+1}t_1}) \\
\leq & \frac{\alpha}{\alpha - 1} \log \fint_{\mathbb{S}^n} \hat{u}_{e^{t_0} z_1}^{1-\frac{1}{\alpha}} d\theta + \int_{t_0}^{\bar{t}} \Big(1 - \frac{\fint_{\mathbb{S}^n} \hat{u}_{e^t z_1}^{-\frac{1}{\alpha}} \hat{K}^\alpha d\theta}{\fint_{\mathbb{S}^n} \hat{u}_{e^t z_1}^{1-\frac{1}{\alpha}} d\theta \fint_{\mathbb{S}^n} \hat{K}^{\alpha-1} d\theta}\Big)[1 - C(n, \alpha, \tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}t}]dt
\end{aligned}$$

$$+ C(n, \alpha, \tilde{X}_0)(e^{-\frac{2(n+1)}{2n+1}t_0} - e^{-\frac{2(n+1)}{2n+1}t_1}).$$

For $\alpha = 1$.

$$\begin{aligned}
\mathcal{E}_1(\hat{\Omega}_{t_1}) \leq & \mathcal{E}_1(\hat{\Omega}_{t_0}, e^{t_0} z_1) - \int_{t_0}^{t_1} [1 - C(n, \tilde{X}_0) e^{-\frac{2(n+1)}{2n+1}t}] \fint_{\mathbb{S}^n} \Big(\sqrt{\frac{\hat{u}_{e^t z_1}}{\hat{K}}} - \sqrt{\frac{\hat{K}}{\hat{u}_{e^t z_1}}}\Big)^2 d\theta dt \\
& + C(n, \tilde{X}_0) \int_{t_0}^{t_1} e^{-\frac{2(n+1)}{2n+1}t} dt \\
= & \fint_{\mathbb{S}^n} \log \hat{u}_{e^{t_0} z_1} d\theta + \int_{t_0}^{t_1} -[1 - C(n, \tilde{X}_0) e^{-\frac{2(n+1)}{2n+1}t}] \fint_{\mathbb{S}^n} \Big(\sqrt{\frac{\hat{u}_{e^t z_1}}{\hat{K}}} - \sqrt{\frac{\hat{K}}{\hat{u}_{e^t z_1}}}\Big)^2 d\theta dt \\
& + C(n, \tilde{X}_0)(e^{-\frac{2(n+1)}{2n+1}t_0} - e^{-\frac{2(n+1)}{2n+1}t_1}) \\
\leq & \fint_{\mathbb{S}^n} \log \hat{u}_{e^{t_0} z_1} d\theta + \int_{t_0}^{\bar{t}} -[1 - C(n, \tilde{X}_0) e^{-\frac{2(n+1)}{2n+1}t}] \fint_{\mathbb{S}^n} \Big(\sqrt{\frac{\hat{u}_{e^t z_1}}{\hat{K}}} - \sqrt{\frac{\hat{K}}{\hat{u}_{e^t z_1}}}\Big)^2 d\theta dt \\
& + C(n, \tilde{X}_0)(e^{-\frac{2(n+1)}{2n+1}t_0} - e^{-\frac{2(n+1)}{2n+1}t_1})
\end{aligned}$$

since $\bar{t} \leq t_1$ and the integrand is non-positive (same reasoning as the proof of Theorem 3.3.1).

On the other hand

$$\hat{u}_{e^t z_1}(x, t) = \hat{u}(x, t) - e^t \langle z_1, x \rangle = \hat{u}(x, t) - e^{t - t_1} \langle a^{t_1}, x \rangle \tag{3.3.21}$$

converges to $\hat{u}(x, t)$ uniformly for $t_0 \leq t \leq \bar{t}$, $x \in \mathbb{S}^n$ as $t_1 \to \infty$.

Letting $t_1 \to \infty$, we get from the two inequalities above that

For $\alpha \neq 1$.

$$\mathcal{E}_{\alpha,\infty} \leq \frac{\alpha}{\alpha-1}\log \fint_{\mathbb{S}^n} \hat{u}(x,t_0)^{1-\frac{1}{\alpha}}d\theta + \int_{t_0}^{\bar{t}} \Big(1 - \frac{\fint_{\mathbb{S}^n} \hat{u}^{-\frac{1}{\alpha}}(x,t)\hat{K}^{\alpha}(x,t)d\theta}{\fint_{\mathbb{S}^n} \hat{u}^{1-\frac{1}{\alpha}}(x,t)d\theta \fint_{\mathbb{S}^n} \hat{K}^{\alpha-1}(x,t)d\theta}\Big)$$
$$[1 - C(n,\alpha,\tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}t}]dt + C(n,\alpha,\tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}t_0}, \tag{3.3.22}$$

and

For $\alpha = 1$.

$$\mathcal{E}_{1,\infty} \leq \fint_{\mathbb{S}^n} \log \hat{u}(x,t_0)d\theta - \int_{t_0}^{\bar{t}} [1 - C(n,\tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}t}] \fint_{\mathbb{S}^n} \Big(\sqrt{\frac{\hat{u}(x,t)}{\hat{K}(x,t)}} - \sqrt{\frac{\hat{K}(x,t)}{\hat{u}(x,t)}}\Big)^2 d\theta dt$$
$$+ C(n,\tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}t_0}. \tag{3.3.23}$$

This proves (3.3.18) and (3.3.19) for $t = t_0$ since $\bar{t} \geq t_0$ is arbitrary. Since $t_0 \geq T_0(n,\alpha,\tilde{X}_0)$ is also arbitray, we obtain (3.3.18) and (3.3.19) for all $t \geq T_0(n,\alpha,\tilde{X}_0)$.

□

Define for each $\rho \in (0,1)$ the following collection of convex bodies:

$$\Gamma_\rho := \{\Omega \subset \mathbb{R}^{n+1} \text{ compact, convex } | \{r_+(\Omega), r_-(\Omega)\} \subset [\rho, \frac{1}{\rho}]\} \tag{3.3.24}$$

Theorem 3.3.5. *Suppose $\alpha > \frac{1}{n+2}$ and $\hat{u}(x,t) > 0$ is the solution of (3.3.6) with initial data $\hat{u}_0$, where $\hat{u}_0$ is the support function of a convex body $\hat{\Omega}_0$ with $|\hat{\Omega}_0| = |B(1)|$, then there exists $\varepsilon(n,\alpha,\tilde{X}_0) > 0$ and $T_1(n,\alpha,\tilde{X}_0) \geq T_0$, such that for $t \geq T_1$ and $x \in \mathbb{S}^n$,*

$$\hat{u}(x,t) \geq \varepsilon. \tag{3.3.25}$$

Proof. Note that under (3.3.6), we have for all $t \geq T_0$ that $|\hat{\Omega}_t| = |B(1)|$, and by Corollary 3.3.3, there exists $\rho > 0$ such that $\hat{\Omega}_t \in \Gamma_\rho$ for every $t \geq T_0$.

By Lemma 3.3.4

$$\mathcal{E}_{\alpha,\infty} - \mathcal{E}_\alpha(\hat{\Omega}_t) \leq \mathcal{E}_{\alpha,\infty} - \mathcal{E}_\alpha(\hat{\Omega}_t, 0) \leq C(n, \alpha, \tilde{X}_0)e^{-\frac{2(n+1)}{2n+1}t}, \quad t \geq T_0. \tag{3.3.26}$$

This implies that $\lim_{t\to\infty} \mathcal{E}_\alpha(\hat{\Omega}_t, 0) = \mathcal{E}_{\alpha,\infty}$ and $\lim_{t\to\infty}(\mathcal{E}_\alpha(\hat{\Omega}_t, 0) - \mathcal{E}_\alpha(\hat{\Omega}_t)) = 0$. Let $z_e(\hat{\Omega}_t)$ be the entropy point of $\hat{\Omega}_t$. By Lemma 4.2 of [9], there is a positive constant $D > 0$ such that when $\mathcal{E}_\alpha(\hat{\Omega}_t, 0) > \mathcal{E}_\alpha(\hat{\Omega}_t) - 1$,

$$|z_e(\hat{\Omega}_t) - 0|^2 \leq \frac{1}{D}|\mathcal{E}_\alpha(\hat{\Omega}_t, 0) - \mathcal{E}_\alpha(\hat{\Omega}_t)| \tag{3.3.27}$$

which approaches to zero as $t \to \infty$. The claimed result then follows from Lemma 4.4 of [9]. $\square$

Corollary 3.3.6. *Let $\alpha > \frac{1}{n+2}$ and $\hat{u}(x,t)$ be as in Theorem 3.3.5. Then there exists $\Lambda(n, \alpha, \tilde{X}_0)$ such that*

$$\frac{1}{\Lambda} \leq \hat{u}(x,t) \leq \Lambda \tag{3.3.28}$$

for all $(x,t) \in \mathbb{S}^n \times [T_0, \infty)$, where $T_0 = T_0(n, \alpha, \tilde{X}_0) \geq t(T^* - \delta_1)$ is the T_0 in Theorem 3.3.1.

Proof. The upper bound is immediate since the diameter of $\hat{\Omega}_t$ is bounded by Corollary 3.3.3 and the fact that origin is in $\hat{\Omega}_t$ for $t \geq T_0$. The lower bound for $t \geq T_1$ is proved by Theorem 3.3.5. For the time $T_0 \leq t < T_1$, we use the fact that $\hat{\hat{u}}(x,\tau) = \hat{u}(x,t)e^{-t}$ is non-increasing in τ hence in t, so we have

$$\hat{u}(x,t) \geq e^{t-T_1}\hat{u}(x,T_1) \geq \varepsilon e^{-T_1}.$$

$\square$

3.3.3 C^2-estimates

Theorem 3.3.7. *Suppose $\alpha > \frac{1}{n+2}$ and $\hat{u}(x,t) > 0$ is the solution of (3.3.6) with initial data $\hat{u}_0$, where $\hat{u}_0$ is the support function of the convex body $\hat{\Omega}_0$ with $|\hat{\Omega}_0| = |B(1)|$. Then there exists a constant $\bar{K} = \bar{K}(n, \alpha, \tilde{X}_0) > 0$ such that*

$$\hat{K}(x,t) \leq \bar{K} \tag{3.3.29}$$

for $t \geq T_0$.

Proof. This is immediate by re-scaling the upper bound of $\hat{\tilde{K}}$ obtained in Lemma 3.2.4 and Corollary 3.3.3. $\square$

Theorem 3.3.8. *Suppose $\alpha > \frac{1}{n+2}$ and $\hat{u}(x,t) > 0$ is the solution of (3.3.6) with initial data $\hat{u}_0$, where $\hat{u}_0$ is the support function of the convex body $\hat{\Omega}_0$ with $|\hat{\Omega}_0| = |B(1)|$. Then there exists a constant $\underline{K} = \underline{K}(n, \alpha, \tilde{X}_0) > 0$ such that*

$$\hat{K}(x,t) \geq \underline{K} \tag{3.3.30}$$

for $t \geq t_1(n, \alpha, \tilde{X}_0) \geq T_0$.

Proof. It follows the same lines of the alternate proof of Theorem 5.2 in [9]. Let

$$f = \psi \hat{K}^{\alpha} = (\hat{u} - \hat{u}_t)\eta,$$

where $\eta = \fint_{\mathbb{S}^n} \psi \hat{K}^{\alpha-1} d\theta$. Let $\hat{W} = (\hat{u}_{ij} + \hat{u}\delta_{ij})$, and $\mathcal{L} := \partial_t - \frac{\alpha\psi\sigma_n^{-\alpha-1}}{\eta}\sigma_n^{ij}\nabla_i\nabla_j$. Then

$$
\begin{aligned}
f_t =&(\psi \hat{K}^{\alpha})_t \\
=&\psi_t \hat{K}^{\alpha} - \alpha\psi\sigma_n^{-\alpha-1}\sigma_n^{ij}(\hat{u}_{ij} - \frac{f_{ij}}{\eta} + \hat{u}\delta_{ij} - \frac{f}{\eta}\delta_{ij}) \\
=&\psi_t \hat{K}^{\alpha} - n\alpha f + \frac{\alpha\psi\sigma_n^{-\alpha-1}}{\eta}\sigma_n^{ij} f_{ij} + \frac{\alpha f \sigma_n^{-1}\sigma_{n-1}}{\eta} f.
\end{aligned}
\tag{3.3.31}
$$

Since

$$
\begin{aligned}
\mathcal{L}\hat{u} =&\hat{u} - \frac{f}{\eta} - \frac{\alpha\psi\sigma_n^{-\alpha-1}}{\eta}\sigma_n^{ij}(\hat{u}_{ij} + \hat{u}\delta_{ij}) + \hat{u}\frac{\alpha\psi\sigma_n^{-\alpha-1}}{\eta}\sigma_n^{ij}\delta_{ij} \\
=&\hat{u} - \frac{(n\alpha+1)f}{\eta} + \frac{\alpha f \sigma_n^{-1}\sigma_{n-1}}{\eta}\hat{u},
\end{aligned}
\tag{3.3.32}
$$

thus

$$
\begin{aligned}
\mathcal{L}(\log(f\hat{u}^l)) \geq& \frac{\mathcal{L}f}{f} + l\frac{\mathcal{L}\hat{u}}{\hat{u}} \\
=&(\frac{\psi_t}{\psi} - n\alpha + l) - \frac{l(n\alpha+1)f}{\hat{u}\eta} + (l+1)\frac{\alpha f \sigma_n^{-1}\sigma_{n-1}}{\eta}.
\end{aligned}
\tag{3.3.33}
$$

Since at the minimum point of $\log(fu^l)$, we have

$$
\frac{f_i}{f} + l\frac{\hat{u}_i}{\hat{u}} = 0.
\tag{3.3.34}
$$

Thus

$$
\hat{u}_{ti} = (1 + \frac{lf}{\hat{u}\eta})\hat{u}_i,
\tag{3.3.35}
$$

and

$$
\begin{aligned}
\psi_t =&\frac{1}{e^t}[\psi_{\hat{\hat{u}}}(\hat{u}_t - \hat{u}) + \psi_{\hat{\hat{u}}_i}(\hat{u}_{it} - \hat{u}_i)] \\
=&\frac{1}{e^t}[\psi_{\hat{\hat{u}}}(\hat{u} - \frac{f}{\eta} - \hat{u}) + \psi_{\hat{\hat{u}}_i}((1 + \frac{lf}{\hat{u}\eta})\hat{u}_i - \hat{u}_i)] \\
=&\frac{1}{e^t}[-\hat{u}\psi_{\hat{\hat{u}}} + l\psi_{\hat{\hat{u}}_i}\hat{u}_i]\frac{f}{\hat{u}\eta}.
\end{aligned}
\tag{3.3.36}
$$

Let $0 < \varepsilon < 1$ small, note that

$$\begin{aligned}|\psi_{\hat{\hat{u}}}| =&2\kappa\psi|[(\frac{n+2}{2}\alpha+\frac{1}{2})(1+\kappa\frac{\hat{r}^2}{e^{2t}})^{-1}+(-\frac{n+2}{2}\alpha+\frac{1}{2})(1+\kappa\frac{\hat{u}^2}{e^{2t}})^{-1}]|\frac{\hat{u}}{e^t}\leq\varepsilon\psi,\\ |\psi_{\hat{\hat{u}}_i}| =&|2\kappa\psi(\frac{n+2}{2}\alpha+\frac{1}{2})(1+\kappa\frac{\hat{r}^2}{e^{2t}})^{-1}\frac{\hat{u}_i}{e^t}|\leq\varepsilon\psi\end{aligned} \tag{3.3.37}$$

for $t > \bar{T}(n, \alpha, \tilde{X}_0) \geq T_0$ large enough since $\frac{\hat{u}}{e^t} = \hat{\hat{u}}$, $\frac{|\hat{u}_i|}{e^t} \leq \frac{\hat{r}}{e^t} = \hat{\hat{r}} \to 0$ as $t \to \infty$. Thus

$$\psi_t \geq -\frac{\psi f}{\hat{u}\eta} \tag{3.3.38}$$

for $t > T_2(n, \alpha, \tilde{X}_0) \geq \bar{T}$ large enough, which implies that

$$\mathcal{L}(\log(f\hat{u}^l)) \geq (l - n\alpha) - \frac{l(n\alpha+2)f}{\hat{u}\eta}. \tag{3.3.39}$$

Since $\eta = ⨍_{\mathbb{S}^n} \psi\hat{K}^{\alpha-1}d\theta \geq \frac{1}{2}⨍_{\mathbb{S}^n} \hat{K}^{\alpha-1}d\theta$ for t large and $(⨍_{\mathbb{S}^n} \hat{K}^{\alpha-1})^{\frac{1}{\alpha-1}} \geq e^{\mathcal{E}_\alpha(\hat{\Omega}_t)}$ by Lemma 5.4 of [9]. Thus $\eta \geq \frac{1}{2}e^{(\alpha-1)\mathcal{E}_\alpha(\hat{\Omega}_t)} \geq \frac{1}{2}$ for $\alpha \geq 1$ and $\eta \geq \frac{1}{2}\Lambda^{\alpha-1}$ for $\alpha < 1$. Let $\lambda = \frac{100}{(n\alpha+2)^2\Lambda^{(n-1)\alpha+4}}$, take Λ large enough such that $\min_{t=0}(f\hat{u}^l) \geq 2\lambda$. Then we claim that $\min(f\hat{u}^l) \geq \lambda > 0$ for all $t > 0$. In fact, suppose t' is the first time when $\min(f\hat{u}^l) = f\hat{u}^l(x', t')$ touch λ, take $l = n\alpha + 2$, then at (x', t'), we have

$$0 \geq 2-(n\alpha+2)^2\frac{f}{\hat{u}\eta} = 2-(n\alpha+2)^2\frac{f\hat{u}^l}{\eta\hat{u}^{l+1}} \geq 2-2(n\alpha+2)^2\frac{f\hat{u}^l}{\Lambda^{\alpha-l-2}}, \tag{3.3.40}$$

where we used the fact that $\hat{u} \geq \frac{1}{\Lambda}$ in the last inequality. Thus $f\hat{u}^l(x', t') \leq \frac{\Lambda^{\alpha-2-l}}{(n\alpha+2)^2} = \frac{1}{(n\alpha+2)^2\Lambda^{(n-1)\alpha+4}} = \frac{1}{100}\lambda$ which is a contradiction since $f\hat{u}^l(x', t') = \lambda$. Thus the claim is true and $\hat{K} \geq (\frac{\lambda}{\hat{u}^l})^{\frac{1}{\alpha}} \geq \underline{K} > 0$. □

Theorem 3.3.9. *Suppose $\alpha > \frac{1}{n+2}$ and $\hat{u}(x,t) > 0$ is the solution of (3.3.6) with initial data $\hat{u}_0$, where $\hat{u}_0$ is the support function of the convex body $\hat{\Omega}_0$ with $|\hat{\Omega}_0| =$*

$|B(1)|$. *Then there exist constants* C_1, C_2 *depending only on* $n, \alpha, \tilde{X}_0 > 0$ *such that*

$$C_1 I \leq \hat{W}_{ij} = \hat{u}_{ij} + \hat{u}\delta_{ij} \leq C_2 I. \tag{3.3.41}$$

for $t \geq t_1(n, \alpha, \tilde{X}_0) \geq T_0$.

Proof. Since we already have the upper and lower bound of $\hat{K}$, it suffices to prove an upper bound of the eigenvalues of $(\hat{W}_{ij})$. Similar to the proof of Lemma 3.2.3, let $(\lambda_1, \dots, \lambda_n)$ be the eigenvalues of $(\hat{W}_{ij})$, $\lambda(x,t) := \max_{i=1,\dots,n} \lambda_i(x,t)$. For any $T > 0$, suppose λ attains its maximum on $\mathbb{S}^n \times [0,T]$ at (x', t'). We take a local orthonormal frame $\{e_1, \dots, e_n\}$ on $\mathbb{S}^n$ around x', such that $(\hat{W}_{ij}(x', t'))$ is diagonal and $\lambda(x', t') = \lambda_1(x', t') = \hat{W}_{11}(x', t')$. Then $\hat{W}_{11}(x, t)$ also attains its maximum at (x', t'). If $t' = 0$, $\lambda(x, t) \leq \lambda(x', 0)$, we are done. Assume that $t' > 0$, by (3.2.6), we have

$$\begin{aligned}
\hat{W}_{11,t}(x,t) =& (e^t \hat{\tilde{W}}_{11}(x,\tau))_t = \hat{W}_{11}(x,t) + e^t \hat{\tilde{W}}_{11,\tau}(x,\tau)\tau'(t) \\
=& \hat{W}_{11}(x,t) + \frac{e^{-n\alpha t}}{\eta} \hat{\tilde{W}}_{11,\tau}(x,\tau) \\
=& \hat{W}_{11} - \frac{1}{\eta}\hat{K}^{\alpha}[\psi + \psi_{\hat{\tilde{u}}} \hat{\tilde{W}}_{11} - \psi_{\hat{\tilde{u}}} \hat{\tilde{u}} + \psi_{\hat{\tilde{u}}_i} \hat{\tilde{W}}_{11i} - \psi_{\hat{\tilde{u}}_1} \hat{\tilde{u}}_1 \\
& + \psi_{\hat{\tilde{u}}\hat{\tilde{u}}} \hat{\tilde{u}}_1^2 + \psi_{\hat{\tilde{u}}_1\hat{\tilde{u}}_1} \hat{\tilde{W}}_{11}^2 - 2\psi_{\hat{\tilde{u}}_1\hat{\tilde{u}}_1} \hat{\tilde{W}}_{11} \hat{\tilde{u}} + \psi_{\hat{\tilde{u}}_1\hat{\tilde{u}}_1} \hat{\tilde{u}}^2 + 2\psi_{\hat{\tilde{u}}\hat{\tilde{u}}_1} \hat{\tilde{W}}_{11} \hat{\tilde{u}}_1 \\
& - 2\psi_{\hat{\tilde{u}}\hat{\tilde{u}}_1} \hat{\tilde{u}}\hat{\tilde{u}}_1 - 2\alpha \hat{\tilde{W}}^{ii} \hat{\tilde{W}}_{ii1}(\psi_{\hat{\tilde{u}}} \hat{\tilde{u}}_1 + \psi_{\hat{\tilde{u}}_1} \hat{\tilde{W}}_{11} - \psi_{\hat{\tilde{u}}_1} \hat{\tilde{u}}) + \alpha^2 \psi (\hat{\tilde{W}}^{ii} \hat{\tilde{W}}_{ii1})^2 \\
& + \alpha\psi \hat{\tilde{W}}^{ii} \hat{\tilde{W}}^{jj} \hat{\tilde{W}}_{ij1}^2 + \alpha\psi \hat{W}^{ii}(\hat{W}_{11} - \hat{W}_{ii}) - \alpha\psi \hat{W}^{ii} \hat{W}_{11,ii}],
\end{aligned}$$

since $\hat{\tilde{\hat{W}}}^{ii}(\hat{\tilde{\hat{W}}}_{11} - \hat{\tilde{\hat{W}}}_{ii})$ and $\hat{\tilde{\hat{W}}}^{ii}\hat{\tilde{\hat{W}}}_{11,ii}$ are scaling invariant. Since we already proved that $\hat{\tilde{\hat{W}}}_{11} \leq C(n, \alpha, \tilde{X}_0)$ in Lemma 3.2.3, and by maximum principle at (x', t')

$$\hat{\tilde{\hat{W}}}_{11i} = 0, \tag{3.3.42}$$

thus,

$$\begin{aligned}
\mathcal{L}\hat{W}_{11} &\leq \hat{W}_{11} - \frac{1}{\eta}\hat{K}^{\alpha}[-2\alpha\hat{\tilde{\hat{W}}}^{ii}\hat{\tilde{\hat{W}}}_{ii1}(\psi_{\hat{\tilde{u}}}\hat{\tilde{u}}_1 + \psi_{\hat{\tilde{u}}_1}\hat{\tilde{\hat{W}}}_{11} - \psi_{\hat{\tilde{u}}_1}\hat{\tilde{u}}) + \alpha^2\psi(\hat{\tilde{\hat{W}}}^{ii}\hat{\tilde{\hat{W}}}_{ii1})^2 \\
&+ \alpha\psi\hat{\tilde{\hat{W}}}^{ii}\hat{\tilde{\hat{W}}}^{jj}\hat{\tilde{\hat{W}}}_{ij1}^2 + \alpha\psi\hat{W}^{ii}(\hat{W}_{11} - \hat{W}_{ii}) - C(n, \alpha, \tilde{X}_0)] \\
&\leq \hat{W}_{11} - \frac{1}{\eta}\hat{K}^{\alpha}[-C(n, \alpha, \tilde{X}_0)|\hat{\tilde{\hat{W}}}^{ii}\hat{\tilde{\hat{W}}}_{ii1}| + \alpha^2\psi(\hat{\tilde{\hat{W}}}^{ii}\hat{\tilde{\hat{W}}}_{ii1})^2 + \alpha\psi\hat{W}_{11}\hat{W}^{ii} \\
&- C(n, \alpha, \tilde{X}_0)] \\
&\leq \hat{W}_{11} - \frac{1}{\eta}\hat{K}^{\alpha}[\alpha\psi\hat{W}_{11}\sum_i \hat{W}^{ii} - C(n, \alpha, \tilde{X}_0)] \\
&\leq \hat{W}_{11} - C(n, \alpha, \tilde{X}_0, \bar{K}, \underline{K})\hat{W}_{11}\sum_i \hat{W}^{ii} + C(n, \alpha, \tilde{X}_0, \bar{K}, \underline{K}) \\
&= \hat{W}_{11} - C(n, \alpha, \tilde{X}_0, \bar{K}, \underline{K})\hat{W}_{11}\frac{\sigma_{n-1}}{\sigma_n} + C(n, \alpha, \tilde{X}_0, \bar{K}, \underline{K}),
\end{aligned}$$

where we used the Cauchy-Schwartz inequality in the third step, and $\eta = ⨍_{\mathbb{S}^n} \psi\hat{K}^{\alpha-1}d\theta \leq C(n, \alpha, \tilde{X}_0, \bar{K}, \underline{K})$. By Newton 's inequality

$$\frac{\sigma_{n-1}}{n} \geq (\frac{\sigma_1}{n})^{\frac{1}{n-1}}\sigma_n^{\frac{n-2}{n-1}}, \tag{3.3.43}$$

we get

$$\mathcal{L}\hat{W}_{11} \leq \hat{W}_{11} - C(n, \alpha, \tilde{X}_0, \bar{K}, \underline{K})\hat{W}_{11}^{\frac{n}{n-1}} + C(n, \alpha, \tilde{X}_0, \bar{K}, \underline{K}). \tag{3.3.44}$$

This implies that

$$\hat{W}_{11}(x_0, t_0) \le C(n, \alpha, \tilde{X}_0), \tag{3.3.45}$$

since $\bar{K}$ and $\underline{K}$ only depend on $n, \alpha, \tilde{X}_0$. □

Combining Corollary 3.3.6, Theorem 3.3.9, we conclude that there exists a positive constant C depending only on $n, \alpha, \tilde{X}_0$ such that for the unique solution to (3.3.6)

$$\|\hat{u}(\cdot, t)\|_{C^2} \le C. \tag{3.3.46}$$

3.4 Convergence to a sphere

Since (3.3.6) is a concave parabolic equation, by Krylov's theorem and the standard theory of parabolic equations, the estimates (3.3.46) and (3.3.41) imply bounds on all derivatives of $\hat{u}(x,t)$. More precisely, for any $k \ge 3$, there exists $C_k \ge 0$, depending only on $n, \alpha, \tilde{X}_0$ such that for $t \ge t_1(n, \alpha, \tilde{X}_0)$,

$$\|\hat{u}(\cdot, t)\|_{C^k(\mathbb{S}^n)} \le C_k. \tag{3.4.1}$$

Proposition 3.4.1. *Let $\hat{X}(t)$ be the solution of (3.3.5) with $\alpha > \frac{1}{n+2}$, then $\hat{X}(x,t)$ converges in C^∞-topology to a round sphere as $t \to \infty$.*

Proof. First, given a sequence $t_j \to \infty$ and $T > 0$, define $\hat{u}_j(x,t) = \hat{u}(x, t + t_j)$. Since by (3.4.1), $\hat{u}_j$ are uniformly bounded in $C^k(\mathbb{S}^n \times [0,T])$, for every $k \in \mathbb{N}$. By Arzelà-Ascoli theorem, $\hat{u}_j$ has a subsequence converging in C^∞-topology to a limit $\hat{u}_\infty$ on $\mathbb{S}^n \times [0,T]$ and $\hat{u}_\infty$ is a solution of

$$\hat{u}(x,t)_{\infty,t} = \hat{u}_\infty(x,t) - \frac{\hat{K}^\alpha_\infty(x,t)}{\fint_{\mathbb{S}^n} \hat{u}_\infty \hat{K}^{\alpha-1}_\infty d\theta} \tag{3.4.2}$$

on $\mathbb{S}^n \times [0, T]$.

We claim that $\hat{u}_\infty(x,t)$ is a soliton, i.e.

$$\lambda(t)\hat{u}_\infty(x,t) = \hat{K}_\infty^\alpha(x,t), \tag{3.4.3}$$

for some $\lambda(t) > 0$. In fact, otherwise, there is $(x', t') \in \mathbb{S}^n \times [0, T]$, a sequence $j_k \to \infty$ and positive constants $\varepsilon, \delta > 0$ independent of j_k such that

$$\begin{aligned} &\Big(\sqrt{\frac{\hat{u}(x,t)}{\hat{K}(x,t)}} - \sqrt{\frac{\hat{K}(x,t)}{\hat{u}(x,t)}}\Big)^2 > \delta, \quad (\alpha = 1); \\ &\text{or} \\ &1 - \frac{⨍_{\mathbb{S}^n} (\hat{u})^{-\frac{1}{\alpha}} \hat{K}^\alpha d\theta}{⨍_{\mathbb{S}^n} (\hat{u})^{1-\frac{1}{\alpha}} \hat{K}^{\alpha-1} d\theta \cdot ⨍_{\mathbb{S}^n} (\hat{u})^{1-\frac{1}{\alpha}} d\theta} > \delta, \quad (\alpha \neq 1) \end{aligned} \tag{3.4.4}$$

on $B(x', \varepsilon) \times [t' + t_{j_k} - \varepsilon, t' + t_{j_k} + \varepsilon]$ by (3.4.1). This implies that

$$\int_{T_0}^\infty ⨍_{\mathbb{S}^n} \Big(\sqrt{\frac{\hat{u}(x,t)}{\hat{K}(x,t)}} - \sqrt{\frac{\hat{K}(x,t)}{\hat{u}(x,t)}}\Big)^2 \Big(1 - C(n, \tilde{X}_0) e^{-\frac{2(n+1)}{2n+1}t}\Big) d\theta dt = \infty, \quad (\alpha = 1);$$

or

$$\int_{T_0}^\infty (1 - \frac{⨍_{\mathbb{S}^n} (\hat{u})^{-\frac{1}{\alpha}} \hat{K}^\alpha d\theta}{⨍_{\mathbb{S}^n} (\hat{u})^{1-\frac{1}{\alpha}} \hat{K}^{\alpha-1} d\theta \cdot ⨍_{\mathbb{S}^n} (\hat{u})^{1-\frac{1}{\alpha}} d\theta})(1 - C(n, \alpha, \tilde{X}_0) e^{-\frac{2(n+1)}{2n+1}t}) dt = \infty, \quad (\alpha \neq 1),$$

which is a contradiction to (3.3.19) and (3.3.18). Thus (3.4.3) is true. Multiplying $\hat{K}^{-1}(x,t)$ on both side of (3.4.3) and integrating on $\mathbb{S}^n$, we get $\lambda(t) = ⨍_{\mathbb{S}^n} \hat{K}_\infty^{\alpha-1} d\theta$. Plugging this into (3.4.2), we get $\hat{u}_{\infty,t} = 0$, $\lambda(t) \equiv ⨍_{\mathbb{S}^n} \hat{K}_\infty^{\alpha-1} d\theta$ is a constant. By Theorem 1 of [12], the solution of (3.4.3) is a round sphere.

Next, we claim that $\{\hat{u}(x,t)\}$ converges in C^∞-topology to $\hat{u}_\infty$ itself as $t \to \infty$. In fact, otherwise, there is $k \in \mathbb{N}$, a positive constant $\gamma > 0$ and a sequence $\{t_l\} \to \infty$

such that

$$\sup_{x\in\mathbb{S}^n} |\hat{u}^{(k)}(x,t_l) - \hat{u}_\infty^{(k)}(x,t_l)| \geq \gamma, \quad \forall\, l \geq 1. \tag{3.4.5}$$

On the other hand, applying the above argument to $\{\hat{u}_l(x,t) := \hat{u}(x,t_l+t)\}$, we can find a subsequence $\{\hat{u}(x,t_{l_j}+t)\}$ of $\{\hat{u}(x,t_l+t)\}$ converging to $\hat{u}_\infty(x,t)$ in C^∞-topology on $\mathbb{S}^n \times \{0\}$ as $j \to \infty$, i.e. $\hat{u}_{t_j}(x,0)$ converges in C^∞-topology to $\hat{u}_\infty(x)$ on $\mathbb{S}^n$, which is a contradiction to (3.4.5). □

Recall that q_0 is the extinct point of $\tilde{M}_\tau$ as $\tau \to T^*$.

Theorem 3.4.2. *$\tilde{M}_\tau$ converges to a geodesic sphere centered at q_0 in $\mathbb{N}^{n+1}(\kappa)$ as $\tau \to T^*$ after the rescaling.*

By Corollary 3.2.2 and Theorem 3.4.2, we finish the proof of Theorem 1.2.1.

CHAPTER 4
A warped product metric and the Weyl problem

In this chapter, we present the result about a warped product metric and the Weyl problem.

Suppose $M := (\mathbb{S}^2, g)$ is a compact smooth surface with Riemannian metric g, let r be a positive function defined on M, satisfying $1 - |\nabla r|^2 > 0$. In [42], Izmestiev considered the manifold $\mathbb{R}^+ \times \mathbb{S}^2 = \{(l, x) | l \in \mathbb{R}^+, x \in \mathbb{S}^2\}$ with the warped product metric

$$\tilde{g} = dl^2 + l^2 \frac{g - dr \otimes dr}{r^2} = dl^2 + l^2 \hat{g}, \tag{4.0.1}$$

where $\hat{g} = \frac{g - dr \otimes dr}{r^2}$. Since $1 - |\nabla r|^2 > 0$, $\tilde{g}$ is indeed a warped product Riemannian metric on $\mathbb{R}^+ \times \mathbb{S}^2$. In [42], Izmestiev used the metric $\tilde{g}$ and the Hilbert-Einstein (HE) functional

$$HE(\tilde{g}) = \frac{1}{2} \int_P R_{\tilde{g}} dvol + \int_M H darea \tag{4.0.2}$$

to reprove the infinitesimal rigidity of the Weyl's isometric embedding problem. Here $P = \{(\rho, x) \in \mathbb{R}^+ \times \mathbb{S}^2 | 0 < \rho \leq r(x)\}$, $R_{\tilde{g}}$ is the scalar curvature of $\tilde{g}$, H is the mean curvature of boundary ∂P (the trace of the second fundamental form of ∂P). In this chapter, we will use the warped product metric $\tilde{g}$ to give a new proof of the closedness of Weyl's embedding problem and study the stability of HE near the critical point. First, we recall some known results from [42].

4.1 Reduction of Weyl's embedding problem to a single equation

In this section, we recall some known results in [42].

In the following, we will denote the Riemannian manifold $(\mathbb{R}^+ \times \mathbb{S}^2, \tilde{g})$ by N. It is easy to see that the map $f : M = (\mathbb{S}^2, g) \to N = (\mathbb{R}^+ \times \mathbb{S}^2, \tilde{g}), x \mapsto (r(x), x)$ is an isometric embedding. This was mentioned by Izmestiev in [42], we give a short proof here for completeness.

Lemma 4.1.1. *Given a metric g on $\mathbb{S}^2$, a positive function r on $\mathbb{S}^2$ s.t. $1-|\nabla r|^2 > 0$, then for the metric $\tilde{g} = dl^2 + l^2 \frac{g - dr \otimes dr}{r^2}$ on $\mathbb{R}^+ \times \mathbb{S}^2$, the map*

$$f : (\mathbb{S}^2, g) \to (\mathbb{R}^+ \times \mathbb{S}^2, \tilde{g}), \quad x \mapsto (r(x), x) \tag{4.1.1}$$

is an isometric embedding.

Proof. In general, for a smooth manifold M^m, a Riemannian manifold $(N^n, \tilde{g})$, and a map $f : M \to N$, the pull-back metric on M in local coordinates $(x^1, \ldots, x^m)$ of M and $(y^1, \ldots, y^n)$ of N can be expressed as

$$f^*(\tilde{g})_{ij} = \sum_{l,m=1}^{n} \frac{\partial f^l}{\partial x^i} \frac{\partial f^m}{\partial x^j} \tilde{g}_{lm}.$$

Here, take local coordinates (x^1, x^2) on $\mathbb{S}^2$, then (l, x^1, x^2) will be local coordinates on $\mathbb{R}^+ \times \mathbb{S}^2$, and under these coordinates

$$\tilde{g}_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{l^2(g_{11} - r_1^2)}{r^2} & \frac{l^2(g_{12} - r_1 r_2)}{r^2} \\ 0 & \frac{l^2(g_{21} - r_2 r_1)}{r^2} & \frac{l^2(g_{22} - r_2^2)}{r^2} \end{pmatrix},$$

and f takes the form

$$f : (x^1, x^2) \mapsto (r(x^1, x^2), x^1, x^2).$$

Thus the pullback metric is

$$f^*(\tilde{g})_{ij} = \sum_{l,m=1}^{3} \frac{\partial f^l}{\partial x^i} \frac{\partial f^m}{\partial x^j} \tilde{g}_{ij} = r_i r_j + \sum_{l,m=2}^{3} \delta_{li} \delta_{mj} \frac{r^2(g_{lm} - r_l r_m)}{r^2} = g_{ij}. \tag{4.1.2}$$

That is $f^*(\tilde{g}) = g$, f is an isometric embedding. □

One key feature of the warped product metric $\tilde{g}$ is that there is a simple relation between the sectional curvatures of different sectional 2-planes of $(N, \tilde{g})$. In [42], Izmestiev showed that for any two sectional plane $\pi \subset T_{(l,x)}N$, the sectional curvature of N along π is

$$\sec_{(l,x)}(\pi) = \cos^2 \varphi \sec_{(l,x)}(\partial l^\perp) = \cos^2 \varphi \frac{r(x)^2}{l^2} \sec_{(r(x),x)}(\partial l^\perp) = \frac{\cos^2 \varphi}{\cos^2 \alpha} \frac{r(x)^2}{l^2} \sec_{(r(x),x)}, \tag{4.1.3}$$

where $\partial l^\perp \in T_{(l,x)}N$ is the 2-plane perpendicular to ∂l, φ is the angle between π and $\partial l^\perp$, $\sec_{(x,r(x))}$ is the sectional curvature of the tangent plane Σ of $f(M) = \partial P$ at $((x, r(x))$, and α is the angle between Σ and $\partial l_{(r(x),x)}$ (see B.1.3 and B.1.4 in [42]). This means that the sectional curvature of the tangent planes of ∂P determines the sectional curvature of $(N, \tilde{g})$ in all directions.

On the other hand, by Gauss equation, we have

$$\sec_{(r(x),x)} = K - \det(B), \tag{4.1.4}$$

where K is the Gauss curvature of M, and B is the Weingarten tensor of ∂P with respect to the unit outer normal ν at $(r(x), x)$ (the eigenvalues of B are the principal curvatures of ∂P).

Let $\rho(x) = \frac{1}{2}r^2(x)$, then it was shown in Lemma 4.2.3 of [42] that

$$B = \frac{Id - Hess(\rho)}{(2\rho - |\nabla\rho|^2)^{\frac{1}{2}}}. \tag{4.1.5}$$

Combining (4.1.3), (4.1.4), and (4.1.5), we have

$$\sec_{(l,x)}(\pi) = \frac{\cos^2\varphi}{\cos^2\alpha}\frac{r(x)^2}{l^2}(K - \frac{\det(Id - Hess(\rho)}{2\rho - |\nabla\rho|^2}) \tag{4.1.6}$$

for any $(l, x) \in N$ and any $\pi \in T_{(l,x)}N$. In particular, if we can show that there is a function r on $(\mathbb{S}^2, g)$ s.t.

$$K \equiv \frac{\det(Id - Hess(\rho)}{2\rho - |\nabla\rho|^2}. \tag{4.1.7}$$

Then, N will have constant sectional curvature zero, and thus will be locally isometric to $\mathbb{R}^3$, since N is simply connected and $\tilde{g}$ is complete. Suppose $\Phi : N \to \mathbb{R}^3$ is this isometry, then $\tilde{f} := \Phi \circ f$ will be an isometric embedding of M into $\mathbb{R}^3$. Thus, in order to give an isometric embedding of M into $\mathbb{R}^3$, we only need to find a positive solution r of (4.1.7) s.t. $1 - |\nabla r|^2 > 0$.

4.2 C^2 estimate and closedness

In the following, we only consider the metric g on $\mathbb{S}^2$ with positive Gauss curvature $K > 0$. Taking a local frame on $M = (\mathbb{S}^2, g)$, we can write equation (4.1.7) as

$$K(2\rho - |\nabla\rho|^2)\det(g) = \det(\rho_{ij} - g_{ij}) \tag{4.2.1}$$

for a function $\rho > 0$ defined on M.

Definition 4.2.1. A function $\rho > 0$ on M is called an admissible solution of (4.2.1) if $2\rho - |\nabla\rho|^2 > 0$ and $(g_{ij} - \rho_{ij}) > 0$ is strictly positive definite on M.

Theorem 4.2.2. *(C^2 estimate) Suppose that ρ is an admissible solution of (4.2.1) satisfying*

$$c_1 \leq \rho \leq c_2 \tag{4.2.2}$$

for some positive constants c_1, c_2. Then there exists positive constants $c, C > 0$ depending only on g, c_1, c_2 such that

$$0 < c \leq 2\rho - |\nabla\rho|^2 \tag{4.2.3}$$

and

$$\|\rho\|_{C^2(M)} \leq C \tag{4.2.4}$$

Proof. C^1 estimate: First, since ρ is an admissible solution, we have $2\rho - |\nabla\rho|^2 > 0$, thus

$$|\nabla\rho| \leq \sqrt{2\rho} \leq \sqrt{2c_2} \tag{4.2.5}$$

by (4.2.2). This gives the upper bound of $|\nabla\rho|$.

Moreover, suppose $2\rho - |\nabla\rho|^2$ attains its minimum at $x_0 \in M$. Take a local orthonormal frame on M near x_0, by maximum principle, we have at x_0

$$0 = (2\rho - |\nabla\rho|^2)_i = 2(\delta_{ki} - \rho_{ki})\rho_k, \qquad i = 1, 2. \tag{4.2.6}$$

Since the matrix $(\delta_{ij} - \rho_{ij}) > 0$ is strictly positive definite, we have $\rho_k = 0$ for $k = 1, 2$. That is

$$(2\rho - |\nabla\rho|^2)(x) \geq (2\rho - |\nabla\rho|^2)(x_0) = 2\rho(x_0) \geq 2\rho_{min} \geq 2c_1 > 0 \tag{4.2.7}$$

by (4.2.2) for any $x \in M$. This gives the lower bound of $2\rho - |\nabla\rho|^2$.

C^2 estimate: let

$$F(\rho, \nabla\rho, \nabla^2\rho) = K(2\rho - |\nabla\rho|^2)\det(g) - \det(\rho_{ij} - g_{ij}). \tag{4.2.8}$$

To estimate $\|\rho_{ij}\|_{C^0}$, we only need to estimate $\rho_{11} - g_{11}, \rho_{22} - g_{22}$ since $(\rho_{ij} - g_{ij})$ is negative definite. Without loss of generality, we may assume $\rho_{11} - g_{11} \leq \rho_{22} - g_{22}$. Consider the function

$$G = (\rho_{11} - 1)f(\rho), \tag{4.2.9}$$

where $f > 0$ is a function to be determined.

Suppose G attains its minimum at x_0. We take a local orthonormal frame of $(\mathbb{S}^2, g)$ around x_0, such that (ρ_{ij}) is diagonal at x_0. Then $g_{ij} = \delta_{ij}$ around x_0, the equation (4.2.1) can be written as

$$K(2\rho - |\nabla\rho|^2) = \det(\rho_{ij} - \delta_{ij}) \tag{4.2.10}$$

around x_0. Moreover, $(\rho_{ij} - g_{ij})$ will also be diagonal at x_0 and we have

$$\begin{aligned} G_i &= \rho_{11i}f + (\rho_{11} - 1)f'\rho_i, \\ G_{ij} &= \rho_{11ij}f + \rho_{11i}f'\rho_j + \rho_{11j}f'\rho_i + (\rho_{11} - 1)f''\rho_i\rho_j + (\rho_{11} - 1)f'\rho_{ij}. \end{aligned} \tag{4.2.11}$$

Let $F^{ij} = \frac{\partial F}{\partial \rho_{ij}} = -A_{ij} > 0$ and $A_{ij,kl} = \frac{\partial \det(\rho_{ij} - \delta_{ij})}{\partial \rho_{kl}\partial \rho_{ij}}$, where $(A_{ij}) = (\frac{\partial \det(\rho_{ij} - \delta_{ij})}{\partial \rho_{ij}}) = \begin{pmatrix} \rho_{22} - 1 & -\rho_{21} \\ -\rho_{12} & \rho_{11} - 1 \end{pmatrix}$. By maximum principle, we have at x_0

$$\rho_{11i} = -\frac{f'}{f}(\rho_{11} - 1)\rho_i \tag{4.2.12}$$

and

$$
\begin{aligned}
0 \leq & F^{ij}G_{ij} = -A_{ij}(\rho_{11ij} f + 2\rho_{11i} f'\rho_j + (\rho_{11}-1)f''\rho_i\rho_j + (\rho_{11}-1)f'\rho_{ij}) \\
=& - f A_{ii}\rho_{11ii} - A_{ij}(-\frac{2f'^2}{f}(\rho_{11}-1)\rho_i\rho_j + f''(\rho_{11}-1)\rho_i\rho_j + (\rho_{11}-1)f'\rho_{ij}) \\
=& - f[A_{ii}\rho_{ii11} + A_{22}(R_{1221,1}\rho_1 + 2R_{1221}\rho_{11} + 2R_{1212}\rho_{22} + R_{1212,2}\rho_2)] \\
& - A_{ij}(\rho_{11}-1)((-\frac{2f'^2}{f} + f'')\rho_i\rho_j + f'(\rho_{ij} - \delta_{ij} + \delta_{ij})) \\
=& f[A_{ij,kl}\rho_{ij1}\rho_{kl1} - K_{11}(2\rho - |\nabla\rho|^2) - 2K_1(2\rho_1 - 2\rho_k\rho_{k1}) - K(2\rho_{11} - 2\rho_{k1}\rho_{k1} - 2\rho_k\rho_{k11})] \\
& + f(1-\rho_{11})(K_1\rho_1 - K_2\rho_2 + 2K(\rho_{11}-\rho_{22})) - A_{ij}(\rho_{11}-1)(-\frac{2f'^2}{f} + f'')\rho_i\rho_j \\
& + f'(1-\rho_{11})(2\det + \sum_i A_{ii}),
\end{aligned}
\tag{4.2.13}
$$

where we used the equation (4.2.10) and the fact that $K = R_{1221}$ in the last equality.

On the other hand, by differentiating (4.2.10) once with respect to the first variable and using (4.2.12), we have at x_0

$$
\begin{aligned}
\rho_{111}\rho_{221} =& -\frac{f'}{f}\rho_1(\rho_{11}-1)\rho_{221} = \frac{f'}{f}\rho_1(-K_1(2\rho - |\nabla\rho|^2) - 2K\rho_1(1-\rho_{11}) + (\rho_{22}-1)\rho_{111}) \\
=& \frac{f'}{f}\rho_1(-K_1(2\rho - |\nabla\rho|^2) - 2K\rho_1(1-\rho_{11}) - \frac{f'}{f}\rho_1(\rho_{22}-1)(\rho_{11}-1)), \\
\rho_{121}^2 =& (\rho_{112} + R_{2112}\rho_2)^2 = \rho_2^2(-\frac{f'}{f}(\rho_{11}-1) + K)^2,
\end{aligned}
$$

Thus

$$
\begin{aligned}
A_{ij,kl}\rho_{ij1}\rho_{kl1} =& 2\rho_{111}\rho_{221} - 2\rho_{121}^2 \\
=& -\frac{2f'}{f}\rho_1 K_1(2\rho - |\nabla\rho|^2) - 4K\frac{f'}{f}\rho_1^2(1-\rho_{11}) - \frac{2f'^2}{f^2}\rho_1^2\det - \frac{2f'^2}{f^2}\rho_2^2(\rho_{11}-1)^2 + \\
& \frac{4f'}{f}\rho_2^2 K(\rho_{11}-1) - 2K^2\rho_2^2
\end{aligned}
$$

$$
\begin{aligned}
=& -4K\frac{f'}{f}(1-\rho_{11})|\nabla\rho|^2 - \frac{2f'^2}{f^2}\rho_2^2(\rho_{11}-1)^2 - \frac{2f'}{f}\rho_1K_1(2\rho-|\nabla\rho|^2) - \\
&\frac{2f'^2}{f^2}\rho_1^2K(2\rho-|\nabla\rho|^2) - 2K^2\rho_2^2.
\end{aligned}
$$

Moreover

$$
\begin{aligned}
2\rho_{11} - 2\rho_{k1}\rho_{k1} =& 2\rho_{11} - 2\rho_{11}^2 = 2(1-\rho_{11}) - 2(1-\rho_{11})^2, \\
\rho_k\rho_{k11} =& \rho_k\rho_{1k1} = \rho_k(\rho_{11k} + R_{k112}\rho_2) = \rho_k(-\frac{f'}{f}\rho_k(\rho_{11}-1) + \delta_{k2}K\rho_2) \\
=& -\frac{f'}{f}(\rho_{11}-1)|\nabla\rho|^2 + K\rho_2^2.
\end{aligned}
$$

Plugging the above equations into (4.2.13), we get

$$
\begin{aligned}
0 \leq& -4Kf'(1-\rho_{11})|\nabla\rho|^2 - \frac{2f'^2}{f}\rho_2^2(\rho_{11}-1)^2 - 2f'\rho_1K_1(2\rho-|\nabla\rho|^2) - \frac{2f'^2}{f}\rho_1^2K(2\rho-|\nabla\rho|^2) \\
&- 2K^2f\rho_2^2 - K_{11}f(2\rho-|\nabla\rho|^2) - 4fK_1\rho_1(1-\rho_{11}) - 2fK((1-\rho_{11})-(1-\rho_{11})^2) + \\
&2Kf(-\frac{f'}{f}(\rho_{11}-1)|\nabla\rho|^2 + K\rho_2^2) + f(1-\rho_{11})(K_1\rho_1 - K_2\rho_2) - 2Kf(1-\rho_{11})^2 + 2Kf\det \\
&- (-\frac{2f'^2}{f} + f'')(\det\rho_1^2 + (\rho_{11}-1)^2\rho_2^2) + f'(1-\rho_{11})(2\det + \sum_i A_{ii}) \\
=& -4Kf'(1-\rho_{11})|\nabla\rho|^2 - 2f'\rho_1K_1(2\rho-|\nabla\rho|^2) - 2K^2f\rho_2^2 - K_{11}f(2\rho-|\nabla\rho|^2) \\
&- 4fK_1\rho_1(1-\rho_{11}) - 2fK(1-\rho_{11}) + 2Kf'(1-\rho_{11})|\nabla\rho|^2 + 2fK^2\rho_2^2 + f(1-\rho_{11})(K_1\rho_1 - K_2\rho_2) \\
&+ 2fK\det - f''(\rho_1^2\det + (\rho_{11}-1)^2\rho_2^2) + f'(1-\rho_{11})(2\det + \sum_i A_{ii}) \\
\leq& f'(1-\rho_{11})(2\det + \sum_i A_{ii}) - 2Kf'(1-\rho_{11})|\nabla\rho|^2 - 4fK_1\rho_1(1-\rho_{11}) - \\
&2fK(1-\rho_{11}) + f(1-\rho_{11})(K_1\rho_1 - K_2\rho_2) - f''(\rho_1^2\det + (\rho_{11}-1)^2\rho_2^2) + 2fK\det + C \\
\leq& (-f' - f''\rho_2^2)(1-\rho_{11})^2 + C(1+|f|+|f''|+|f'|)(1-\rho_{11}) + C
\end{aligned}
\tag{4.2.14}
$$

for some constant C depending only on $\|K\|_{C^2}$, $\|\rho\|_{C^1}$, since $\det = K(2\rho - |\nabla\rho|^2)$.

If we take $f(\rho) = 1 + \rho$, (4.2.14) implies that $(1 - \rho_{11})(x_0) \leq C$, for some C depending only on $\|K\|_{C^2}$, $\|\rho\|_{C^1}$. Thus

$$(1-\rho_{11})(x) \leq \frac{(1-\rho_{11})(1+\rho)(x_0)}{1+\rho(x)} \leq (1+\rho(x_0))C \leq C(\|K\|_{C^2}, \|\rho\|_{C^1}), \quad \forall x \in \mathbb{S}^2.$$

This gives the uniform C^2 estimate for ρ. □

Next, we give a short proof of the closedness in the method of continuity to prove Weyl embedding problem by using the warped product metric $\tilde{g}$ and Theorem 4.2.2.

Recall the method of continuity to prove the Weyl embedding problem: Suppose $M = (\mathbb{S}^2, g)$ is a smooth closed surface with positive Gauss curvature K. By uniformization theorem, g is conformal to δ, where δ is the standard metric on $\mathbb{S}^2$. That is, there is a smooth function ϕ on $\mathbb{S}^2$ such that $g = e^{2\phi}\delta$. Define $g_t = e^{2t\phi}\delta$, $0 \leq t \leq 1$. This is a smooth metric on $\mathbb{S}^2$ for any $t \in [0,1]$. Define the set

$$\mathcal{I} = \{t \in [0,1] | (\mathbb{S}^2, g_t) \text{ can be isometrically embedded into } \mathbb{R}^3 \text{ smoothly}\}. \quad (4.2.15)$$

The method of continuity is to prove that $\mathcal{I} = [0,1]$ by proving that $\mathcal{I}$ is non-empty, open and closed. $\mathcal{I}$ is non-empty since when $t = 0$, $(\mathbb{S}^2, g_0) = (\mathbb{S}^2, \delta)$ is the standard sphere and can be embedded into $\mathbb{R}^3$ as the unit sphere. Thus $0 \in \mathcal{I}$. The fact that $\mathcal{I}$ is open ie proved by Nirenberg [48] (see also Theorem 9.2.1 in [38]).

Proof of Theorem 1.3.1. We note that

$$0 < \varepsilon \leq K_{g_t} = e^{-2t\phi}(K_\delta - t\Delta_\delta\phi) = e^{-2t\phi}(te^{2\phi}K_g + (1-t)K_\delta) \leq \frac{1}{\varepsilon} \quad t \in [0,1]. \quad (4.2.16)$$

for some $\varepsilon > 0$ depending on g and independent of k.

Suppose $t_k \in \mathcal{I}$, and $t_k \to t \in [0,1]$, we need prove that $t \in \mathcal{I}$. Since $t_k \in \mathcal{I}$, $(\mathbb{S}^2, g_{t_k})$ can be isometrically embedded into $\mathbb{R}^3$. Let X_{t_k} be the image of the embedding and r_{t_k} be the radial function of X_{t_k}, then $\rho_{t_k} := \frac{1}{2}r_{t_k}^2$ will be an admissible solution of the equation

$$K_{t_k}(2\rho_{t_k} - |\nabla\rho_{t_k}|^2_{g_{t_k}})\det(g_{t_k}) = \det(\rho_{t_k,ij} - g_{t_k,ij}). \tag{4.2.17}$$

Moreover, by Lemma 9.1.1 in [38] and (4.2.16), there is a ball of radius R inside the embeddings of $(\mathbb{S}^2, g_{t_k})$, where R only depends on $\frac{1}{\max K_{g_{t_k}}}(\geq \varepsilon > 0)$. We can choose the origin to be the center of the ball so that

$$\rho_{t_k} \geq \frac{1}{2}R^2 \geq c_1(g) > 0. \tag{4.2.18}$$

On the other hand, for any point $p \in X_{t_k}$, the line through the origin and p will intersect X_{t_k} at another point q. Then by Bonnet-Myers Theorem and (4.2.16)

$$\rho_{t_k}(p) \leq \frac{1}{2}(\bar{pq})^2 \leq \frac{1}{2}dist_{(\mathbb{S}^2,g_{t_k})}(p,q)^2 \leq \frac{1}{2}diam(\mathbb{S}^2, g_{t_k})^2 \leq \frac{C_0}{\min K_{g_{t_k}}} \leq c_2(g), \tag{4.2.19}$$

where C_0 is a universal constant and c_2 depends only on g. By the uniform C^2 estimate (4.2.4), we have

$$\|\rho_{t_k}\|_{C^2} \leq C(g),$$

where C is independent of k.

By Nirenberg's theory for 2-dimensional elliptic equations (see [49] Theorem I, see also [38] Lemma 9.3.4), this implies that

$$\|\rho_{t_k}\|_{C^{2,\alpha}(\mathbb{S}^2)} \leq C(g)$$

for some $\alpha \in (0,1)$, where C is independent of k. Thus $\rho_{t_k} \to \rho_t$ for some $\rho_t \in C^2(\mathbb{S}^2)$ up to a sub-sequence and ρ_t satisfies the equation

$$K_{g_t}(2\rho_t - |\nabla \rho_t|^2_{g_t}) \det(g_t) = \det(\rho_{t,ij} - g_{t,ij}). \tag{4.2.20}$$

Moreover, by the uniform lower bound estimate (4.2.3), we have

$$2\rho_t - |\nabla \rho_t|^2_{g_t} \geq c(g) > 0. \tag{4.2.21}$$

Then (4.2.20) implies that $\det(\rho_{t,ij} - g_{t,ij}) > 0$, and $\rho_{t,ij} - g_{t,ij}$ is negative definite. Thus, ρ_t is an admissible solution of (4.2.20). By using Nirenberg's theory again and the Schauder theory for elliptic equations, we see ρ_t is smooth.

By the discussions in section 4.1, this implies that all the sectional curvatures of $(\mathbb{R}^+ \times \mathbb{S}^2, \tilde{g})$ are zero and $(\mathbb{S}^2, g)$ can be isometrically embedded into $\mathbb{R}^3$ smoothly by the map $\Phi \circ f$, where f is defined by (4.1.1) with $r = \sqrt{2\rho_t}$, and Φ is an isometry $\Phi : (\mathbb{R}^+ \times \mathbb{S}^2, \tilde{g}) \to \mathbb{R}^3$. Thus $t \in \mathcal{I}$, $\mathcal{I}$ is closed.

□

4.3 The stability of HE at critical points

In this section, we discuss the stability of HE at the critical points by calculating the second variation of HE. Izmestiev [42] calculated the first variation of $HE(r)$, and the second variation in the special case $r(t) = r(0) + tr_t$. Here we give a short simple different calculation of the first variation by using integration by parts and then use it to derive the second variation.

We choose a local orthonormal frame of $M = (\mathbb{S}^2, g)$ so that $g_{ij} = \delta_{ij}$, $\det(g) = 1$ and calculate under this frame. Moreover, we use lower index to denote differentiation with respect to the connection on $(\mathbb{S}^2, g)$ for briefness.

First, note that if we let $A_{ij} = \frac{\partial \det(\rho_{ij} - \delta_{ij})}{\partial \rho_{ij}}$,

$$(c_{ij}) := \begin{pmatrix} \frac{\rho_{22}-1}{(2\rho-|\nabla\rho|^2)^\alpha} & -\frac{\rho_{21}}{(2\rho-|\nabla\rho|^2)^\alpha} \\ -\frac{\rho_{12}}{(2\rho-|\nabla\rho|^2)^\alpha} & \frac{\rho_{11}-1}{(2\rho-|\nabla\rho|^2)^\alpha} \end{pmatrix} = \Big(\frac{A_{ij}}{(2\rho - |\nabla\rho|^2)^\alpha}\Big), \tag{4.3.1}$$

where α is a constant, then

$$\begin{aligned} c_{i1,i} =& \frac{\rho_{221}}{(2\rho - |\nabla\rho|^2)^\alpha} - \frac{2\alpha(\rho_{22} - 1)(\rho_1 - \rho_1\rho_{11} - \rho_2\rho_{21})}{(2\rho - |\nabla\rho|^2)^{\alpha+1}} - \frac{\rho_{212}}{(2\rho - |\nabla\rho|^2)^\alpha} \\ &+ \frac{2\alpha\rho_{21}(\rho_2 - \rho_1\rho_{12} - \rho_2\rho_{22})}{(2\rho - |\nabla\rho|^2)^{\alpha+1}} \\ =& \frac{-K\rho_1}{(2\rho - |\nabla\rho|^2)^\alpha} + \frac{2\alpha\rho_1 \det(\rho_{ij} - \delta_{ij})}{(2\rho - |\nabla\rho|^2)^{\alpha+1}} \\ =& \frac{-\rho_1}{(2\rho - |\nabla\rho|^2)^\alpha}(K - 2\alpha\det(B)). \end{aligned}$$

The same calculation shows

$$c_{i2,i} = \frac{-\rho_2}{(2\rho - |\nabla\rho|^2)^\alpha}(K - 2\alpha\det(B)).$$

In summary, we have

$$c_{ij,i} = -\frac{\rho_j}{(2\rho - |\nabla\rho|^2)^\alpha}(K - 2\alpha\det(B)), \quad j = 1, 2. \tag{4.3.2}$$

Now, suppose we have a family of smooth functions $\{\rho(t)\}_{t\geq 0}$ defined on $M=(\mathbb{S}^2,g)$ satisfying $2\rho-|\nabla\rho|^2>0$ on M. By [42] Theorem 4.3.3

$$\begin{aligned} HE &= \int_M h(K+\det(B))darea = \int_M r\cos\alpha(K+\det(B))darea \\ &= \int_M \sqrt{2\rho-|\nabla\rho|^2}(K+\det(B))darea, \end{aligned} \tag{4.3.3}$$

where $h=r\cos\alpha$, $r=\sqrt{2\rho}$, α is the angle between ν and ∂l.

By using integration by parts and (4.3.2), we have

$$\begin{aligned} \dot{HE} &= \frac{dHE(\rho(t))}{dt} \\ &= \int_M (2\rho-|\nabla\rho|^2)^{-\frac{1}{2}}(\rho_t-\rho_i\rho_{it})(K+\det(B)) \\ &\quad + \int_M (2\rho-|\nabla\rho|^2)^{-\frac{1}{2}}A_{ij}\rho_{ijt} - 2(2\rho-|\nabla\rho|^2)^{-\frac{1}{2}}\det(B)(\rho_t-\rho_i\rho_{it})darea \\ &= \int_M (2\rho-|\nabla\rho|^2)^{-\frac{1}{2}}(\rho_t-\rho_i\rho_{it})(K-\det(B)) + (2\rho-|\nabla\rho|^2)^{-\frac{1}{2}}(K-\det(B))\rho_j\rho_{jt}darea \\ &= \int_M (2\rho-|\nabla\rho|^2)^{-\frac{1}{2}}\rho_t(K-\det(B))darea. \end{aligned} \tag{4.3.4}$$

This shows that $\dot{HE}=0$ if

$$K=\det(B)=\frac{\det(\rho_{ij}-g_{ij})}{(2\rho-|\nabla\rho|^2)\det(g)},$$

i.e. $r=\sqrt{2\rho}$ is the radial function of an isometric embedding of$(\mathbb{S}^2,g)$ into $\mathbb{R}^3$. Equivalently, the Euclidean isometric embedding is the critical point of HE in $\{\tilde{g}|\,\tilde{g}$ is of the form (4.0.1)$\}$.

Next, we calculate the second variation of HE. Differentiating (4.3.4) once more shows

$$\begin{aligned}\ddot{HE} =& \int_M \rho_{tt}(2\rho - |\nabla\rho|^2)^{-\frac{1}{2}}(K - \det(B)) - \rho_t(2\rho - |\nabla\rho|^2)^{-\frac{3}{2}}(\rho_t - \rho_i\rho_{it})(K - \det(B)) \\ &- \rho_t(2\rho - |\nabla\rho|^2)^{-\frac{3}{2}}A_{ij}\rho_{ijt} + 2\rho_t(2\rho - |\nabla\rho|^2)^{-\frac{3}{2}}\det(B)(\rho_t - \rho_i\rho_{it})darea \\ =& \int_M \rho_{tt}(2\rho - |\nabla\rho|^2)^{-\frac{1}{2}}(K - \det(B)) + \rho_t(2\rho - |\nabla\rho|^2)^{-\frac{3}{2}}(\rho_t - \rho_i\rho_{it})(-K + 3\det(B)) \\ &+ (2\rho - |\nabla\rho|^2)^{-\frac{3}{2}}A_{ij}\rho_{ti}\rho_{tj} - \rho_t(2\rho - |\nabla\rho|^2)^{-\frac{3}{2}}\rho_j\rho_{jt}(K - 3\det(B))darea \\ =& \int_M \rho_{tt}(2\rho - |\nabla\rho|^2)^{-\frac{1}{2}}(K - \det(B)) + \rho_t^2(2\rho - |\nabla\rho|^2)^{-\frac{3}{2}}(-K + 3\det(B)) \\ &+ (2\rho - |\nabla\rho|^2)^{-\frac{3}{2}}A_{ij}\rho_{ti}\rho_{tj}darea.\end{aligned} \tag{4.3.5}$$

If $\sqrt{2\rho(0)}$ is the radial function of an isometric embedding of $(\mathbb{S}^2, g)$ into $\mathbb{R}^3$, then $\det(B(0)) = \frac{\det(\rho(0)_{ij} - g_{ij})}{(2\rho(0) - |\nabla\rho(0)|^2)\det(g)} = K$, the first term in $\ddot{HE}$ vanishes. Write $\rho_t(x, 0) = \eta(x)$, we have

$$\begin{aligned}\ddot{HE}(0) &= \int_M 2\eta^2(2\rho - |\nabla\rho|^2)^{-\frac{3}{2}}K - (2\rho - |\nabla\rho|^2)^{-1}KB^{ij}\eta_i\eta_j darea \\ &= \int_M (2\rho - |\nabla\rho|^2)^{-1}K(2\eta^2(2\rho - |\nabla\rho|^2)^{-\frac{1}{2}} - B^{ij}\eta_i\eta_j)darea.\end{aligned} \tag{4.3.6}$$

at the critical points.

In the special case when $(\mathbb{S}^2, g) = (\mathbb{S}^2, \delta)$ is the unit sphere, where δ is the standard metric on $\mathbb{S}^2$, then $2\rho - |\nabla\rho|^2 \equiv 1$, $K \equiv 1$, $B^{ij} = \delta_{ij}$, we get

$$\ddot{HE}(0) = \int_{\mathbb{S}^2} 2\eta^2 - |\nabla\eta|^2 darea = \int_{\mathbb{S}^2} 2\eta^2 + \eta\Delta_\delta\eta darea \leq 0 \tag{4.3.7}$$

if $\int_{S^2} \eta darea = 0$ since the maximal nonzero eigenvalue of Δ_δ is -2. Thus, for the unit sphere $(\mathbb{S}^2, \delta)$, the isometric embedding of $(\mathbb{S}^2, \delta)$ into $\mathbb{R}^3$ is a local maximum of

HE for variations s.t. $\int_{\mathbb{S}^2} \eta darea = 0$. In particular, we get the following stability result of HE at the sphere.

Theorem 4.3.1 (Theorem 1.3.2)**.** *Let δ be the standard metric on $\mathbb{S}^2$, $\sqrt{2\rho(0)} \equiv 1$ (be the radial function of the embedding of $(\mathbb{S}^2, \delta)$ into $\mathbb{R}^3$ with origin at the center of the embedding). Let $\eta(x)$ be a smooth function on $(\mathbb{S}^2, \delta)$ s.t. $\int_{\mathbb{S}^2} \eta(x) d\delta = 0$, $\varepsilon > 0$ small, $\mathcal{A}_\eta = \{\rho(t,x) = \rho(0) + t\eta(x) \text{ admissible} | t \in (-\varepsilon, \varepsilon),\ x \in (\mathbb{S}^2, \delta)\}$, then $\rho(0)$ is a local maximum of HE in $\mathcal{A}_\eta$.*

On the other hand, if $\rho_t(x,t)$ is a constant, i.e. when $\rho(t) = \rho_0 + tC$, and $\rho_0 = \rho(0)$ satisfying (4.2.1), we have

$$
\begin{aligned}
HE(t) =& \int_M \sqrt{2\rho_0 - |\nabla \rho_0|^2 + 2tC}\left(K + \frac{\det(\rho_{0ij} - g_{ij})}{(2\rho_0 - |\nabla\rho_0|^2 + 2tC)\det(g)}\right) darea \\
=& \int_M (2\rho_0 - |\nabla\rho_0|^2 + 2tC)^{\frac{1}{2}} K + \frac{\det(\rho_{0ij} - g_{ij})}{\det(g)(2\rho_0 - |\nabla\rho_0|^2 + 2tC)^{\frac{1}{2}}} darea \\
\geq& \int_M 2K(2\rho_0 - |\nabla\rho_0|^2)^{\frac{1}{2}} darea \\
=& HE(0),
\end{aligned}
\tag{4.3.8}
$$

where " = " holds if and only if $(2\rho_0 - |\nabla\rho_0|^2 + 2tC)^{\frac{1}{2}} K = \frac{\det(\rho_{0ij} - g_{ij})}{\det(g)(2\rho_0 - |\nabla\rho_0|^2 + 2tC)^{\frac{1}{2}}}$, i.e. $t = 0$. This means that the Euclidean embedding is the minimum of HE along the variation when $\rho_t(x,t) = constant$.

Having this in mind, we conjecture that for any metric g with $K > 0$, the isometric embedding of $(\mathbb{S}^2, g)$ into $\mathbb{R}^3$ is also a local maximum of HE for variations s.t. $\langle \eta, 1 \rangle_B = 0$, where $\langle \cdot, \cdot \rangle_B$ means the inner product is taken with respect to B.

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