Languages Recognized by Finite Categories

Arkadev Chattopadhyay

School of Computer Science
McGill University, Montreal

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Abstract

The connection between algebra and finite automata theory has been well studied and Eilenberg has shown that the notion of *varieties* in semigroups/monoids can be naturally made to correspond with varieties of languages that they recognize. This has significantly deepened and organized our understanding of finite automata and regular languages. Several researchers have later recognized that the more appropriate algebraic objects to look into are finite categories which generalize monoids in a way that is explained in the work. This point of view not only refines the existing theory but is indispensable when dealing with serial decompositions of automata. In this thesis we try to advance this theory by exploring the connections between the algebraic structure of a finite category and the combinatorial description of languages recognized by it which is a central theme of algebraic theory of automata.

The method of congruence has proved quite successful in the study of languages recognized by finite monoids. In this thesis we show that this method remains powerful and successful even in the categorical setting. Using graph congruences, we obtain some new proofs of old results and some completely new results.

It is known that a finite category can have all its base monoids in a variety $\mathbf{V}$ (i.e. be locally $\mathbf{V}$, denoted by $\ell\mathbf{V}$) , without itself dividing a monoid in $\mathbf{V}$

(i.e. be globally $\mathbf{V}$, denoted $\mathbf{gV}$). This is in particular the case when $\mathbf{V}=\mathbf{Com}$, the variety of commutative monoids. The main result in this work provides a combinatorial characterization of locally commutative categories. This is the first such theorem dealing with a non-trivial variety for which local differs from global. As a consequence, we show that $\ell\mathbf{Com} \subset \mathbf{gV}$ for every variety $\mathbf{V}$ that strictly contains the commutative monoids.

We give new proofs of the locality of the following M-varieties: $\mathbf{R}$ and $\mathbf{L}$, the M-variety of all $\mathcal{R}$ and $\mathcal{L}$-trivial monoids respectively, $\mathbf{R_1}$, $\mathbf{L_1}$ and $\mathbf{R_1} \vee \mathbf{L_1}$, where $\mathbf{R_1}$ is the variety of all $\mathcal{R}$-trivial and idempotent monoids and $\mathbf{L_1}$ is the variety of all $\mathcal{L}$-trivial and idempotent monoids. We provide a simple example illustrating the fact that $\ell\mathbf{J}$ strictly contains the variety $\mathbf{gJ}$ where $\mathbf{J}$ is the variety of all $\mathcal{J}$-trivial monoids. The problem of characterising languages recognized by locally $\mathcal{J}$-trivial categories remains open.

Résumé

Le lien entre l'algèbre et la théorie des automates finis a été étudié par plusieurs chercheurs et Eilenberg a démontré que la notion de variétés dans les semi-groupes ou monoïdes peut correspondre de façon naturelle avec les variétés de langages qu'elles reconnaissent. Cette découverte a approfondi et organisé notre compréhension des automates finis et des langages rationnels, et ce, de façon importante. Plusieurs chercheurs ont plus tard reconnu que les objets algébriques les plus appropriés qui devraient être étudiés sont les catégories finies qui généralisent les monoïdes d'une façon qui est expliquée dans cet ouvrage. Ce point de vue ne fait pas que raffiner la théorie existante; il est également indispensable lorsqu'on traite avec des décompositions sérielles d'automates. Dans ce mémoire, nous tentons de faire progresser cette théorie en explorant les liens entre la structure algébrique des catégories finies et la description combinatoire des langages qu'elle reconnaît, ce qui est un thème central de la théorie algébrique des automates.

La méthode des congruences a fait ses preuve dans l'étude des langages qui peuvent être reconnus par des monoïdes finis. Dans ce mémoire, nous démontrons que cette méthode demeure utile et puissante même dans le contexte des catégories. En utilisant des congruences sur les graphes, nous obtenons de nouvelles preuves de résultats déjà prouvés, ainsi que des résultats complètement

nouveaux.

On sait déjà que pour une catégorie finie, les monoïdes de base peuvent être contenus dans une variété $\mathbf{V}$ (c'est-à-dire, être $\mathbf{V}$ localement, représenté par $\ell\mathbf{V}$) sans qu'elle ne divise un monoïde dans $\mathbf{V}$ (c'est à dire être $\mathbf{V}$ globalement, indiqué par $\mathbf{gV}$). En particulier, c'est le cas lorsque $\mathbf{V} = \mathbf{Com}$, la variété de monoïdes commutatifs. Le résultat principal de cet ouvrage fournit une caractérisation combinatoire des catégories localement commutatives. Il s'agit du premier théorème qui s'applique à une variété non-triviale pour laquelle local diffère de global. En corrolaire, nous démontrons que $\ell\mathbf{Com} \subset \mathbf{gV}$ pour chaque variété $\mathbf{V}$ qui contient strictement les monoïdes commutatifs.

Nous fournissons aussi de nouvelles preuves de la localité des M-variétés suivantes: $\mathbf{R}$ et $\mathbf{L}$, la M-variété de tous les monoïdes $\mathcal{R}$-triviaux et $\mathcal{L}$-triviaux, respectivement; $\mathbf{R_1}$, $\mathbf{L_1}$ et $\mathbf{R_1} \vee \mathbf{L_1}$, ou $\mathbf{R_1}$ est la variété de tous les monoïdes $\mathcal{R}$-triviaux et idempotents et $\mathbf{L_1}$ est la variétés de tous les monoïdes $\mathcal{L}$-triviaux et idempotents. Nous présentons un example simple illustrant le fait que $\ell\mathbf{J}$ contient strictement la variété $\mathbf{gJ}$, où $\mathbf{J}$ est la variétés de tous les monoïdes $\mathcal{J}$-triviaux. Cependant, nous démontrons que $\ell\mathbf{J} = \mathbf{gR} \cap \mathbf{gL}$. Le problème de la caractérisation des langages reconnus par les catégories localement $\mathcal{J}$-triviales demeure ouvert.

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Contents

Chapter 1

Introduction

In algebraic theory of automata, a language $L \subseteq A^*$ is said to be recognized by the finite monoid M if there exists a morphism $\phi : A^* \to M$ and a subset $F \subseteq M$ such that $L = \phi^{-1}(F)$ (see [Pin86] or [Str94] for a very readable introduction of this notion). It is well-known that languages that can be so recognized are precisely the regular languages and that for each regular language there is a unique minimal monoid, called the *syntactic monoid of* L and denoted $M(L)$, that recognizes it. One expects that combinatorial properties of L would be reflected in the algebraic structure of $M(L)$: this intuition is completely valid and a driving theme of the field is to prove theorems of the following form:

"A language L belongs to the combinatorially-defined class $\mathcal{V}$ iff the syntactic monoid $M(L)$ belongs to the algebraically-defined class $\mathbf{V}$."

For technical, but unavoidable, reasons, one sometimes has to deal with subsets of A^+ (instead of A^*) and semigroups (instead of monoids). Most often, "algebraically-defined" means that $\mathbf{V}$ is an M-variety, that is a class of finite monoids which is closed under division (i.e. morphic image and submonoid) and direct product. The notion of S-variety is similarly defined for finite semigroups. Books such as [Alm94, Eil76, Pin86] offer a comprehensive treatment

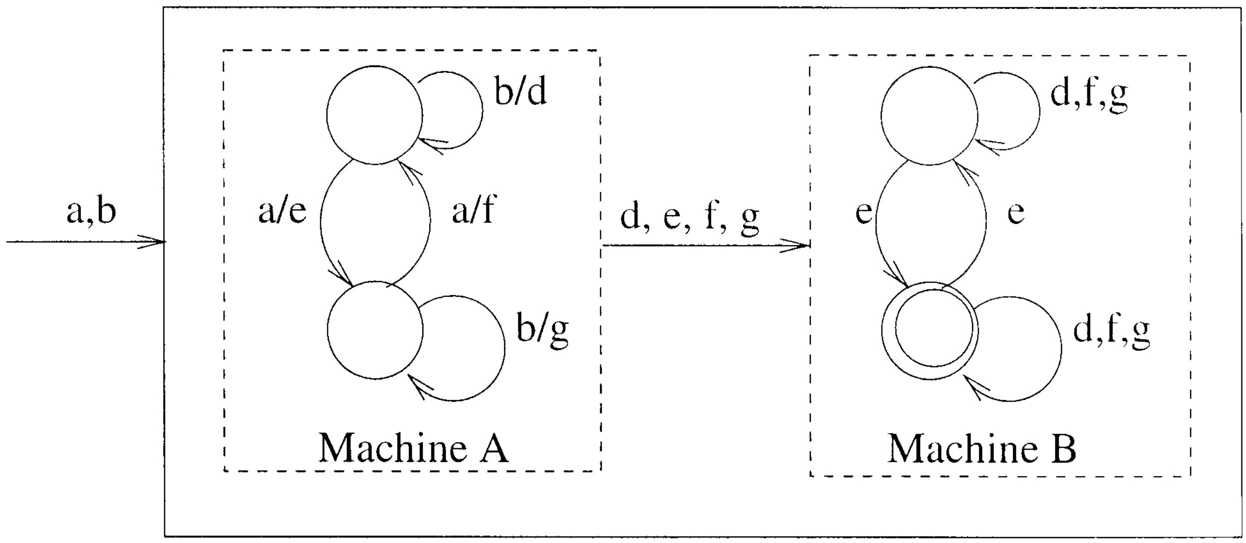


Figure 1.1: Serial connection of automata

of this theory. One interesting by-product of results of the above form is that when membership in **V** is decidable, one gets a decision procedure to test if L is in $\mathcal{V}$, since the monoid $M(L)$ can be effectively computed from any of the common representations used for regular languages (automaton, regular expression, grammar, logical formula). Two classical theorems of that nature are the correspondence between star-free languages and aperiodic monoids [Sch65] and the correspondence between piecewise-testable languages and $\mathcal{J}$-trivial monoids [Sim75].

In automata theory one is often interested in decompositions of automata into simpler units. As an example consider the situation where two automata A and B are connected in series as shown in figure 1.1: for machine B it is no longer the case that the space of inputs it can receive forms a free monoid, since the input sequence is mediated through machine A and some combinations (like 'dg' or 'df') never arise. One simple way of tackling this situation is to view machine B as processing input sequences that are valid paths in the finite directed multi-graph representing the state transition diagram of automaton A. Technically, the right point of view then is to say the input space of automaton B is the free category induced by this graph and we can associate with the machine a congruence of finite index (that we shall henceforth call *graph congruence*) on this free category. The machine thus represents a finite category rather than a monoid. In order to understand the all-important case of serial connection of automata and its algebraic incarnation i.e the wreath product of monoids, it is essential to generalize the above setting to the level of categories. e.g. we shall see in a later section that deciding if a monoid M divides a wreath product of the form $S \circ T$ amounts to decide if a certain category, constructible from M and T, divides S. In this framework, one considers languages as sets of finite-length paths in a directed multi-graph (instead of finite-length sequences over a set) and such languages may be recognized by finite categories (instead of finite monoids). The notion of syntactic monoid for a language over an alphabet naturally generalizes to syntactic category of a language over a graph. In fact any alphabet could be looked at as a one node graph and the free monoid thus becomes the free category induced by such a one node graph. Kleene's theorem about regular languages can also be extended to this framework, as shown in [TSG88]. One can naturally define division and direct product of categories and the notion of M-varieties generalize to C-varieties i.e. a class of finite categories closed under division and direct product. Thus the manipulation and understanding of finite categories as algebraic objects are essential ingredients in manipulation and understanding of regular languages as observed and formalized in the seminal work of [Til87].

Given a C-variety $\mathbf{W}$, it is easily seen that the monoids in $\mathbf{W}$ form an M-variety. It is thus natural to consider the following question: for a fixed M-variety $\mathbf{V}$, what are the C-varieties $\mathbf{W}$ for which the monoids in $\mathbf{W}$ are precisely those of $\mathbf{V}$? Two natural examples emerge readily: the variety $\mathbf{gV} = \{C : C$ divides M for some $M \in \mathbf{V}\}$, and the variety $\ell\mathbf{V} = \{C :$ every base monoid of C is in $\mathbf{V}$ $\}$ are respectively the smallest and the largest C-varieties with that property. It turns out that a combinatorial description of the languages recognized by monoids in $\mathbf{V}$ immediately implies a combinatorial description of the languages recognized by categories in $\mathbf{gV}$; similarly, an algebraic description of the monoids in $\mathbf{V}$ implies an algebraic description of the categories in $\ell\mathbf{V}$. Our understanding is thus complete whenever $\mathbf{gV} = \ell\mathbf{V}$; this happens in a number of interesting cases, e.g. for every non-trivial variety of groups, for semilattices, for aperiodic monoids. But there are also cases where $\mathbf{gV} \not\subseteq \ell\mathbf{V}$, e.g. for the trivial variety, for commutative monoids [TW85], for $\mathcal{J}$-trivial monoids [Kna83]; apart from the case of the trivial variety, it becomes quite a challenge to find an algebraic description of $\mathbf{gV}$ or a combinatorial description of the languages recognized by members of $\ell\mathbf{V}$.

It is a well known fact that combinatorial analysis of congruences on the free monoid generated by a finite set is a powerful tool to describe languages recognized by M-varieties. In this thesis, we exposit the usefulness of this method for deriving results about categories by introducing the notion of graph congruence on a free category. Let γ_V represent a family of congruences (on the free monoid generated by the set of edges of a graph) that describe languages recognized by monoids in a M-variety $\mathbf{V}$. As explained later in this chapter, every such congruence family inudces a graph congruence family denoted by $\overline{\gamma_V}$. We show that for many important M-varieties, languages recognized by a category in $\ell\mathbf{V}$ can be described by a congruence in the family $\overline{\gamma_V}$. For all such cases, we conclude that the C-varieties $\ell\mathbf{V}$ and $\mathbf{gV}$ coincide.

The central result of the thesis provides a combinatorial description of the languages recognized by members of $\ell\mathbf{Com}$, the C-variety of locally commutative categories. This is the first instance of such result for a non-trivial variety $\mathbf{V}$ where $\mathbf{gV} \neq \ell\mathbf{V}$. We give our description via congruences of finite-index and some novel ideas have to be introduced. We also show that $\ell\mathbf{Com}$ is contained in $\mathbf{gV}$ for every M-variety $\mathbf{V}$ that strictly contains all commutative monoids. We then use known techniques to derive results about the S-variety $\mathbf{LCom} = \{S : eSe \in \mathbf{Com} \text{ for every } e = e^2\}$.

The work is organized as follows: the rest of this chapter presents the basic notions that are needed and some elementary theorems. Chapter 2 looks at local C-varieties induced by various varieties of idempotent monoids. chapter 3 reviews some interesting and useful properties of graph congruences that are needed to understand $\ell\mathbf{Com}$, chapter 4 provides a combinatorial description of languages recognized by locally commutative categories and some consequences of that result, chapter 5 presents some elementary combinatorial proofs of theorems about $\ell\mathbf{R}$ and $\ell\mathbf{L}$ and consequently describes the C-variety of locally $\mathcal{J}$-trivial categories in terms of global C-varieties. Finally, chapter 6 discusses some open problems and possible research directions.

1.1 Basic notions

We quickly recall that a monoid M is a set that is closed under an associative binary operation defined on it which has an identity. The monoid is called finite when the underlying set is finite. Given an alphabet Σ, the set of all strings of finite length (including the empty string) with the operation of concatenation forms a monoid that is called the *free monoid* generated by Σ.

A *category* C is given by a finite non-empty set of objects $Obj(C)$ and for every $c_1, c_2 \in Obj(C)$ a set of *arrows* from c_1 to c_2 denoted by $Arr(c_1, c_2)$. For every arrow $x \in Arr(c_1, c_2)$, we say c_1 is the start object of x and c_2 is the end object of x. Arrow x is called a *loop* iff the start object and end object of x coincide. Given an arrow x in $Arr(c_1, c_2)$ and an arrow y in $Arr(c_2, c_3)$, there must exist an arrow $xy \in Arr(c_1, c_3)$. Arrows x and y in this case are called consecutive. Further, if z is an arrow in $Arr(c_3, c_4)$, then $x(yz) = (xy)z$. This defines a partial product on the set of arrows in the category that is associative. With every object c in the category, we associate an arrow $1_c \in Arr(c, c)$ such that for every arrow x, whose start object is c and for every arrow y whose end object is c we have $x = 1_c x$ and $y 1_c = y$. It then becomes easy to see that a monoid is just a category with a single object. A category C is called finite iff it has finite set arrows. This implies that the set of objects is also finite even though the converse is not true. Obviously, the set of loops around any object in a category forms a monoid. Each such monoid is called a base monoid of the category. One can also form a monoid denoted by M_C from a category C by saying that the underlying set of M_C is $A \cup \{0, 1\}$, where A is the set of arrows in the category. The product in M_C of two consecutive arrows in A is their product in C and the product of two non-consecutive arrows is 0. 0 is the zero element of the monoid and 1 is its identity.

A finite directed multi-graph[1] $G = (V, A, \alpha, \omega)$ consists of a set V of vertices, a set A of directed edges and two mappings $\alpha, \omega : A \to V$, which assigns to each edge a the start vertex $\alpha(a)$ and the end vertex $\omega(a)$ of that edge. Two edges a, b are consecutive iff $\omega(a) = \alpha(b)$. A path of length $n > 0$ is a sequence of n consecutive edges[2]; we extend the mappings α and ω to paths in the natural

[1]Note: Henceforth, by graph we will always mean directed multi-graph unless mentioned otherwise explicitly

[2]A path may and will often contain repeated occurrences of one or more edges

way. A path is said to be finite iff it has a finite length. For each vertex v we allow an empty path 1_v of length 0 for which $\alpha(1_v) = \omega(1_v) = v$. The set of all finite paths between any two vertices v, w is denoted by $G^*_{v,w}$. It is easy to see that the set of all finite paths, denoted by G^*, where $G^* = \bigcup_{v,w \in V} G^*_{v,w}$, forms a category called the *free category* generated by the graph G. The length of a path x will be denoted by $|x|$, and the number of occurrences of an edge a in x by $|x|_a$. Often for two paths x and y, we would want to compare the number of occurences of an edge a in them *threshold* t, *modulo* q, denoted by $\equiv_{t,q}$, where $|x|_a \equiv_{t,q} |y|_a$ iff either $|x|_a = |y|_a$ or the following two conditions are satisfied: $|x|_a, |y|_a \geq t$ and $|x|_a \equiv |y|_a (mod) q$, where $(mod) q$ represents modular q counting. We will also be interested in the set of edges (letters) that appear in path (word) x and this would be denoted by $\lambda(x)$, where $\lambda(x) \subseteq A$. Two paths x, y are co-terminal, denoted $x \sim y$ if $\alpha(x) = \alpha(y)$ and $\omega(x) = \omega(y)$. A loop is a path x such that $\alpha(x) = \omega(x)$.

An equivalence β on the set G^* of all paths in G is a graph congruence iff $x \,\beta\, y$ implies $x \sim y$ and $x_1 \,\beta\, y_1$, $x_2 \,\beta\, y_2$, $\omega(x_1) = \alpha(x_2)$ imply $x_1 x_2 \,\beta\, y_1 y_2$. It is easy to see that the set of congruence classes, G^*/β, then forms a category. The objects of this category are the vertices of G and for any $v, w \in V$, the set of arrows is given by $Arr(v, w) \equiv G^*_{v,w}/\beta$. For each path x, we denote the corresponding congruence class containing x by $[x]_\beta$. We note that for every vertex v, the set $\{[x]_\beta : x \text{ is a loop on } v\}$ forms a base monoid of the category. It should be clear that every finite category C can be represented as the quotient of a free category by an appropriate graph congruence. If the category is represented as $C = (N, A)$, where N is its set of objects and A the set of arrows, then one may write $C = G^*/\beta$, where G is the underlying graph and $x \,\beta\, y$ iff the sequence of arrows in x and y multiply out to the same arrow in C. In this work, henceforth whenever we speak of a finite category, we would thus think of an underlying graph and a congruence defined on its paths. Note that if $G = (V, A)$ is any graph, then every congruence γ in A^* induces a graph congruence $\overline{\gamma}$ in G^* in the following way: if x and y are two paths in G^*, then $x \,\overline{\gamma}\, y$ iff x and y are co-terminal and $x \,\gamma\, y$.

A *relational morphism* $< \phi,\, \psi >: C \to D$ between two categories C and D consists of an object function $\phi : Obj(C) \to Obj(D)$ and a morphism relation $\psi : Arr(v,\, w) \to Arr(v\phi,\, w\phi)$ such that

- $x\psi \neq \emptyset$ for each arrow x in C.
- $1_{v\phi} \in 1_v\psi$ for every object v in C.
- $(x\psi)(y\psi) \subseteq (xy)\psi$ for every two arrows x, y in C.

C is a *subcategory* of D if ϕ and ψ are injective functions. We say that C divides D, denoted by $C \prec D$, iff for any two co-terminal arrows x and y in C, $x\psi \cap y\psi \neq \emptyset$ implies $x = y$. It is not hard to see that when C and D are one-object categories i.e. monoids, this definition is equivalent to the standard notion of monoid division that says monoid C is a homomorphic image of a submonoid of D.

We can define the *direct product* of two categories C and D, denoted by $C \times D$, where the objects are given by $Obj(C \times D) = \{(v, w) : v \in Obj(C)$ and $w \in Obj(D)\}$ and the arrows are given by

$$Arr(\,(v,\, w),\, (v',\, w')\,) \;=\; \{(x,\, y) \,:\, x \in Arr(v,\, v'),\, y \in Arr(w,\, w')\}$$

As introduced by [Til87], we define a C-variety to be a class of finite categories which is closed under division and direct product.

1.2 Inducing C-varieties from M-varieties

As monoids can be identified with 1-vertex categories in an obvious way, we can introduce the notion of a category dividing a monoid from the above definitions. We now state two easily verifiable facts below that will be used later:

Fact Given a graph $G = (V, A)$, a graph congruence β for G^* and a finite monoid M , the category $C \equiv G^*/\beta$ divides the monoid M iff there exists a morphism $\gamma : A^* \rightarrow M$ such that the induced graph congruence $\overline{\gamma}$ refines β.

Proof. Consider any path x in G^*. The value of this path in C is given by $[x]_\beta$. Let $[x_1]_{\overline{\gamma}}, \ldots, [x_n]_{\overline{\gamma}}$ be classes of $\overline{\gamma}$ that refine the class $[x]_\beta$. Then each of these classes have a value in the monoid that we denote by $m_1, \ldots, m_n$. We set the arrow relation ψ such that $[x]_\beta\psi = \{m_1^x, \ldots, m_n^x\}$. Similarly, for path y in G^* let $[y_1]_{\overline{\gamma}}, \ldots, [y_m]_{\overline{\gamma}}$ be classes of $\overline{\gamma}$ that refine the class $[y]_\beta$. Then, for any i, j such that $1 \leq i \leq n$ and $1, \leq j \leq m$, we have $x\,\beta\,x_i$ and $y\,\beta\,y_j$, whence $xy\,\beta\,x_iy_j$. Hence for all such i, j, the class $[x_iy_j]_{\overline{\gamma}}$ is contained in the class $[xy]_\beta$. Hence it follows that the monoid element $m_i^x m_j^y$ belongs to the set $[xy]_\beta\psi$, for all such i, j. This establishes $[x]_\beta\psi\,[y]_\beta\,\psi \subseteq [xy]_\beta\,\psi$. If $[x]_\beta\,\psi \cap [y]_\beta\,\psi \neq \emptyset$, then our definition of ψ implies that there exists a $\overline{\gamma}$ class that is contained both in $[x]_\beta$ and $[y]_\beta$ and therefore it must be that x and y are in the same congruence class of β. Thus C divides M.

In order to show that this is a necessary condition, simply observe that we define $\gamma : A^* \rightarrow M$ to be the morphism generated by fixing for every edge a, $a\gamma$ to be any one element in $[a]_\beta\,\psi$. If x and y are two coterminal paths in G^* then clearly $x\gamma$ and $y\gamma$ are contained in $[x]_\beta\psi$ and $[y]_\beta\psi$ respectively. Since C divides M, $x\gamma = y\gamma$ implies $x\,\beta\,y$ and we are done.

□

Fact Every finite category C divides the associated monoid M_C.

Obviously, if we restrict a C-variety to its 1-vertex members, we then get an M-variety. In general, there may exist several C-varieties which coincide on the monoids they contain.

We can then form $\mathbf{gV}$ a set of finite categories containing every monoid in a M-variety $\mathbf{V}$ by defining $\mathbf{gV} = \{C : C \text{ divides } M \text{ for some } M \in \mathbf{V}\}$. It is easily checked that $\mathbf{gV}$ is closed under direct product and division and hence forms a C-variety.

Another set of finite categories, denoted by $\ell\mathbf{V}$ induced from a variety $\mathbf{V}$ of monoids is given by $\ell\mathbf{V} = \{C : \text{every base monoid of } C \text{ is in } \mathbf{V}\ \}$. If D is any finite category in $\ell\mathbf{V}$ and C divides D, then one can easily see that every base monoid of C divides some base monoid of D and hence $\ell\mathbf{V}$ is closed under division. One can also see that if C and D are finite categories then every base monoid of $C \times D$ is a direct product of some base monoid of C and some base monoid of D. This establishes that $\ell\mathbf{V}$ is closed under direct product as well. Hence it follows that $\ell\mathbf{V}$ is a C-variety.

Further, if we consider any category C that divides a monoid M, it is easily seen that every base monoid of C divides M. We will see the converse is not always true. This observation implies that for any M-variety $\mathbf{V}$, the C-variety $\mathbf{gV}$ is always contained in the C-variety $\ell\mathbf{V}$. The C-variety corresponding to $\mathbf{V}$ is unique iff $\mathbf{gV} = \ell\mathbf{V}$ and in this case the variety $\mathbf{V}$ of monoids is said to be *local*. As we shall see, although this holds in several instances, this is not in general true.

1.2.1 Some examples of local M-varieties

We call a monoid M aperiodic if the monoid does not contain any groups. Equivalently, there exists an integer $t > 0$ such that every element x of the monoid satisfies the identity $x^t = x^{t+1}$. The set of all aperiodic monoids forms a variety which is denoted by $\mathbf{A}$. Categories that divide an aperiodic monoid are called *globally aperiodic* and categories all of whose base monoids are aperiodic are called *locally aperiodic*. We are now going to prove the following theorem from [Til87]:

Theorem 1.1 *Every locally aperiodic category is also globally aperiodic. In other words, the C-varieties* $\mathbf{gA}$ *and* $\ell\mathbf{A}$ *are identical.*

Proof. We remind the reader that from facts stated earlier, we know that $\mathbf{gA} \subseteq \ell\mathbf{A}$. For showing the converse, consider the monoid M_C associated with a finite category C where C is assumed to be locally aperiodic. Therefore, there exists a t such that if we take any element x in M_C that is a loop in C, $x^t = x^{t+1}$. If x is not a loop then $x^2 = x^3 = 0$. Thus M_C is aperiodic. But we know that C divides M_C. Hence C must be in $\mathbf{gA}$. □

We will need to prove an interesting property about categories before we can give the next example. Recall that for any directed graph $G = (V, A)$, a subgraph G' is said to be *strongly connected* iff for every two vertices v and w in G', there exists some path in G^* going from v to w and vice-versa. Every maximal strongly connected subgraph is called a *strongly connected component*. Every graph can be uniquely decomposed into its strongly connected components. Given a graph G, by G_i we shall mean its ith strongly connected component. A_i would mean the set of edges contained in the ith component. Let

$B = \{b_1, \ldots, b_t\} \subseteq A$ represent the set of edges which connect vertices in two different components and let t represent the number of such edges. For a category $C \equiv G^*/\beta$, we will denote the subcategory G_i^*/β_i by C_i, where β_i refers to β restricted to paths in G_i^*. We are now in a position to state the following theorem that first appeared in the work of [TW85] and later in [Til87].

Theorem 1.2 *For each non-trivial M-variety* $\mathbf{V}$*, any category* $C = G^*/\beta$ *is in* $\mathbf{gV}$ *iff each of the subcategories* $C_1, \ldots, C_n$ *are in* $\mathbf{gV}$.

Proof. If C divides a monoid M in $\mathbf{V}$, then from fact 1.2 we know that there exists $\gamma : A^* \to M$. We can induce naturally $\gamma_i : A_i^* \to M$, where $a\gamma_i = a\gamma$ for any $a \in A_i$. Clearly $\overline{\gamma_i}$ refines β_i and $M_i = A_i^*/\gamma_i$ divides M and hence is in $\mathbf{V}$. This proves the right to left direction.

For the other direction, we can again use fact 1.2 to assume that there exists $\gamma_i : A_i^* \to M_i$, where M_i is in $\mathbf{V}$ for each i. Let M' be any non-trivial monoid in $\mathbf{V}$ and m' be some element other than identity in M'. We consider the morphism $\gamma : A^* \to M_1 \times M_2 \times \ldots \times M_n \times (M')^t = M$. So M has $n + t$ components, each one of the first n for the corresponding strongly connected component and then t copies of M', one for each edge in B. For any m in M, let $[m]_j$ represent the jth component of m. We fix for every $a \in A_i$, $[a\gamma]_i = a\gamma_i$ and $[a\gamma]_j = 1$ for each $j \neq i$. For the edge b_k in B, we fix $[b_k\gamma]_i = m'$ if $i = n + k$ and 1 otherwise. Let x and y be co-terminal edges in G^* such that $x\gamma = y\gamma$. Then they can be uniquely factorized as $x = x_0 b_{s_0} x_1 \ldots b_{s_p} x_p$ and $y = y_0 b_{s_0} y_1 \ldots b_{s_p} y_p$ where x_i is co-terminal with $x\gamma_i = y\gamma_i$. Since each $\overline{\gamma_i}$ refines β_i, we have $x_i \; \beta_i \; y_i$ for every i and consequently $x \, \beta \, y$. This shows that $\overline{\gamma}$ refines β and we are done. $\square$

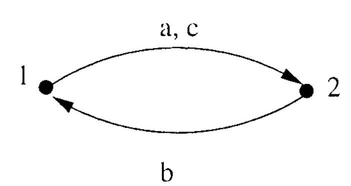
Our next example is taken from [TW85].

Example 1.2. Let $\mathbf{H}$ denote some non-trivial M-variety consisting only of groups. Let $C \equiv G^*/\beta$ be any finite category such that all base monoids of C belong to $\mathbf{H}$. In this case we want to show that C divides a group in $\mathbf{H}$. Using theorem 1.2 we can assume that G is strongly connected. In this case we would show that C in fact divides each base monoid (in this case a group) it contains. We arbitrarily choose a vertex v in G and for every vertex v_i in G we choose paths x_i and y_i such that $\alpha(x_i) = \omega(y_i) = v$, $\omega(x_i) = \alpha(y_i) = v_i$ and $y_i x_i \,\beta\, 1_{v_i}$. Let $\gamma : A^* \to M_v$ be the morphism generated by fixing for every $a \in A$, $a\gamma = [x_i a y_j]_\beta$, where $\alpha(a) = v_i$ and $\omega(a) = v_j$. It is easily seen that for any path x in G^*, we have $x\gamma = [x_i x y_j]_\beta$, where $\alpha(x) = v_i$ and $\omega(x) = v_j$. If x is co-terminal with y and $x\gamma = y\gamma$, then $x_i x y_j \,\beta\, x_i y y_j$ and multiplying on the left by y_i and on the right by x_i we obtain $x \,\beta\, y$. Thus $\overline{\gamma}$ refines β and hence C divides M_v. This implies $\mathbf{gH} = \ell\mathbf{H}$.

1.2.2 Examples of non-local M-varieties

Example 1.2. Let $\mathbf{V} = \mathbf{1}$ be the M-variety consisting of the 1-element monoid only. Then for every graph G, $G^*/\beta \in \mathbf{g1}$ iff β and $\sim$ coincide. On the other hand, let B be the subset of those edges of G for which start and end vertices belong to different strongly connected components. Define $x \,\gamma\, y$ iff $x \sim y$ and, for each $b \in B$, $x = x_0 b x_1$ iff $y = y_0 b y_1$. Clearly, G^*/β is in $\ell\mathbf{1}$ but not in $\mathbf{g1}$ if B is non-empty. Now consider an arbitrary category $C = G^*/\beta \in \ell\mathbf{1}$. Let x and y be any two co-terminal paths not containing any edge $b \in B$. Then there exists a path w from $\omega(x)$ to $\alpha(x)$ since they are in the same strongly connected component. Thus $x \,\beta\, ywx \,\beta\, y$. Hence, γ refines β and we thus observe that $G^*/\beta \in \ell\mathbf{1}$ iff $\gamma \subseteq \beta$. An interesting consequence of this observation is that $\ell\mathbf{1} \subset \mathbf{gV}$ whenever $\mathbf{1} \subset \mathbf{V}$. Indeed an edge b of B can appear in a path zero or one time only: if M is a non-trivial monoid, i.e. M contains an element $m \neq 1$, it can be used to distinguish paths in which b occurs from paths in which it does not, by mapping b to m and every other edge of the graph to 1. Taking a direct product of $|B|$ copies of M insures that we can recover the equivalence class (in γ) of a path from its value in $M^{|B|}$.

Example 1.2. Let $\mathbf{V} = \mathbf{Com}$, the variety of all commutative finite monoids. On any graph G, define $x \,\gamma_{t,q}\, y$ iff $x \sim y$ and for each $a \in A$ either ($|x|_a < t$ and $|x|_a = |y|_a$) or ($|x|_a \geq t, |y|_a \geq t$ and $|x|_a \equiv_q |y|_a$, where $\equiv_q$ denotes modulo q equality). It can be shown that $G^*/\beta \in \mathbf{gCom}$ iff $\gamma_{t,q} \subseteq \beta$ for some $t \geq 0, q \geq 1$. On the other hand, consider the following graph G:



define $x \,\beta\, y$ iff $x \sim y$ and ($|x| \leq 3$ and $x = y$) or ($|x| > 3$ and $x \sim y$). Then $G^*/\beta \in \ell\mathbf{Com}$ but not in $\mathbf{gCom}$.

This example is in some sense generic as [TW85] proves that a category C is in $\mathbf{gCom}$ iff it satisfies $xyz = zyx$ whenever x and z are co-terminal: this result is combinatorially quite delicate to obtain. By definition, a category C is in $\ell\mathbf{Com}$ iff $xy = yx$ for every two loops x, y on the same vertex. The above example shows that knowing the number of occurrences of each edge in a path is not enough information to characterize the value of the path in a locally commutative category.

1.3 Syntactic category and some allied notions

We briefly introduced the notion of finite monoids recognizing languages through homomorphisms from the free monoid generated by an alphabet. We also recalled that every language defined over an alphabet induces a minimal monoid called the *syntactic monoid* of the language. In this section we want to extend these ideas to the notion of finite categories recognizing languages in G^*. Formally, every language $L \subseteq G^*$, induces a congruence that we call the *syntactic graph congruence*, denoted by γ^L, where $x\,\gamma^L\,y$ iff for every u, v in G^*, uxv in L implies uyv is in L and vice-versa. The category G^*/γ^L is called the *syntactic category* of the language L. Conversely, just as in the case of monoids, a finite category C recognizes languages through relational morphism from G^* to C, such that the object function is injective and the morphism relation is a function.

The central theme of traditional algebraic theory of automata is to study relationships between the algebraic structure of monoids and combinatorial descriptions of languages they recognize, and here we would like to connect the algebraic structure of categories and combinatorial descriptions of languages they recognize. The variety of categories we will study in this thesis will be either locally or globally induced from some variety of monoids as explained before.

If γ is any graph congruence, then every language that is a union of some congruence classes of γ will be called a γ-language. For each M-variety that we consider in this thesis, one already knows the existence of a family of congruences, that characterise languages recognized by monoids in that variety. Let $\sim_{V,t}$ denote a congruence from the family such that the syntactic monoid of a language L in Σ^* is in $\mathbf{V}$[3] iff L is a $\sim_{V,t}$-language for some $t > 0$. Then from discussions in the previous section, it is clear that languages recognized by a finite category C in $\mathbf{gV}$ are γ_t^V-languages where $x \, \gamma_t^V \, y$, whenever x and y are co-terminal paths and $x \sim_{V,t} y$. Thus, everytime we show for a M-variety $\mathbf{V}$, $\ell\mathbf{V} = \mathbf{gV}$, we obtain as a corollary a characterization of languages recognized by categories in $\mathbf{gV}$. However to characterize the languages recognized by categories in $\ell\mathbf{V}$, when $\mathbf{V}$ is not local, one has to explicitly work out the combinatorics over graphs. In chapter 4 on $\ell\mathbf{Com}$, we will provide a combinatorial description of that kind.

[3]From Eilenberg's variety theorem one can conclude the existence of one such congruence family for every M-variety

Chapter 2

Sub-varieties of locally idempotent categories

In this chapter we consider local C-varieties induced by some sub-varieties of idempotent monoids. In particular we look at $\ell\mathbf{A_1}$, $\ell\mathbf{J_1}$, $\ell\mathbf{R_1}$ and $\ell(\mathbf{R_1} \vee \mathbf{L_1})$. In each case we show that it coincides with the corresponding global C-variety. It follows from the work of [JS92] that for every M-variety $\mathbf{V}$ of idempotent monoids $\mathbf{gV} = \ell\mathbf{V}$. However, in this chapter we present independent arguments establishing the locality of the following M-varieties : $\mathbf{A_1}$, $\mathbf{J_1}$, $\mathbf{R_1}$ and $\ell(\mathbf{R_1} \vee \mathbf{L_1})$. Our arguments would always crucially use the existing knowledge of algebraic identities characterizing the concerned M-varieties and the congruences that describe the languages recognized by them. We start off by recalling an argument that is due to [BS71] and again appears in the treatise of [Eil76] for the C-varieties induced by $\mathbf{J_1}$.

2.1 The variety $\ell\mathbf{J_1}$

$\mathbf{J_1}$ denotes the variety of all idempotent and commutative monoids. Note that any finite monoid M that is not a group must have an idempotent e other than the identity element. It can be verified that every monoid N in $\mathbf{J_1}$ divides the the direct product $U_1^{|N|}$, where U_1 denotes the monoid $(\{1, e\}, \cdot)$. The variety $\mathbf{J_1}$ is contained in every M-variety that is not a variety of groups. In particular, it is the smallest variety of idempotent monoids. One can also verify quite easily that $\mathbf{J_1}$ recognizes $\sim_{J_1}$-languages where the congruence is generated in Σ^* by the following condition: $x \sim_{J_1} y$ iff $\lambda(x) = \lambda(y)$.

Given a graph $G = (V, A)$, let θ^{J_1} represent the smallest graph congruence generating a category all of whose base monoids are in $\mathbf{J_1}$. Such a congruence is generated by the two following identities: $xy\,\theta^{J_1}\,yx$ and $x\,\theta^{J_1}\,x^2$ whenever x and y are loops in G^*. Clearly, $\ell\mathbf{J_1}$ is contained in $\ell\mathbf{Com}$. From example 1.2.2 , we know that $\mathbf{gCom}$ is not an upper bound for $\ell\mathbf{Com}$. It turns out one can easily show that $\ell\mathbf{J_1} \subseteq \mathbf{gCom}$ modulo the characterization obtained for $\mathbf{gCom}$ in the work of [TW85]. Let xyz be any path in G^*, where x and z are co-terminal paths and hence xy and yz are loops. Thus,

$$(xy)z \;\; \theta^{J_1} \;\; x(yx)(yz) \;\; \theta^{J_1} \;\; (xy)(zy)x \;\; \theta^{J_1} \;\; z(yx)(yx) \;\; \theta^{J_1} \;\; zyx$$

As noted in Chapter 1, $xyz = zyx$ is the generating law for globally commutative categories and hence the upper bound claimed earlier follows. However, we shall give a much sharper bound for $\ell\mathbf{J_1}$ that was known from the work of Simon ([BS71]) long before $\mathbf{gCom}$ was characterized. Before doing so, we shall prove slightly stronger versions of two lemmas that appear in [Eil76].

Consider the graph congruence θ^{R_1} generated by conditions: $xyx\,\theta^{R_1}\,xy$ and $x\,\theta^{R_1}\,x^2$ for every two co-terminal loops x and y. We can now state our lemma:

Lemma 2.1 *If x and y are two consecutive paths (i.e.$\omega(x) = \alpha(y)$) and $\lambda(y) \subseteq \lambda(x)$, then there exists a factorization $x = x_0x_1$, where $\alpha(x_1) = \omega(y)$ and $\omega(x_1) = \alpha(y)$ and $x\,\theta^{R_1}\,xyx_1$.*

Proof. If y is the empty path at the end of x, then the lemma reduces to the trivial statement $x\ \theta^{R_1}\ x$ and we have nothing to prove. We shall prove the lemma now by inducting on the length of y. Let $|y| > 0$ so that we write $y = y'a$, for some edge a that occurs in x. Hence, we can write $x = x_2ax_3$, for some x_2, x_3 in G^*. Also from our induction hypothesis we have

$$x\ \theta^{R_1}\ xy'x_1 \ = \ x_2ax_3y'x_1 \tag{2.1}$$

Duplicating the loop ax_3y', we have from condition 2.1

$$x\ \theta^{R_1}\ x_2ax_3y'ax_3y'x_1 \ = \ xyx_3y'x_1 \tag{2.2}$$

Applying the induction hypothesis $x\ \theta^{R_1}\ xy'x_1$ to condition 2.2 we get

$$x\ \theta^{R_1}\ x(y'x_1)(yx_3)(y'x_1) \tag{2.3}$$

Note the bracketed entities in condition 2.3 are all co-terminal loops and thus applying the generating condition of θ^{R_1}, one finally gets $x\ \theta^{R_1}\ xy'x_1yx_3\ \theta^{R_1}\ xyx_3$. This establishes the lemma. □

We specialize the lemma above to the following result below

Corollary 2.2 *If x is an arbitrary path, then for every loop y at the end of x satisfying $\lambda(y) \subseteq \lambda(x)$ we have $x\ \theta^{R_1}\ xy$ (and $x\ \theta^{J_1}\ y$).*

Proof. In this case from lemma 2.1 we get $x\ \theta^{R_1}\ xyx_1$, where $x = x_0x_1$ and both x_1 and y are loops at the end of x. Thus $x\ \theta^{R_1}\ x_0x_1yx_1\ \theta^{R_1}\ x_0x_1y \ = \ xy$. To show that $x\ \theta^{J_1}\ y$, we simply note that the congruence θ^{R_1} refines θ^{J_1} and we are done. □

Let γ^{J_1} be the congruence in G^* generated by the condition $x\,\gamma^{J_1}\,y$ iff the set of edges that occur in x is the same as the set of edges occurring in y and the paths x and y are co-terminal. We are now in a position to prove the following theorem.

Theorem 2.3 *The syntactic category of a language $L \subseteq G^*$ is in $\ell\mathbf{J_1}$ iff L is a γ^{J_1}-language.*

Proof. The direction from right to left is immediately established by just observing that $x \ \gamma^{J_1} \ x^2$ and $xy \ \gamma^{J_1} \ yx$ whenever x and y are loops.

For the other direction, consider a locally $\mathbf{J_1}$ category, $C = G^*/\theta^{J_1}$. Let two co-terminal paths x, y in G^* be related by $x \ \gamma^{J_1} \ y$. We want to establish $x \ \theta^{J_1} \ y$ by inducting on the length of x. If $|x| = 1$, then y could only contain a single repeated loop-edge and we immediately have the result. Otherwise consider $|x| > 1$. Let u and v be the start and end vertex of x respectively. Consider the subgraph $G_x = (V, A_x)$ of G, where $A_x = \lambda(x) = \lambda(y)$. In this case consider V_v to be the set of vertices that are reachable from v in G_x^*. Let $V' = V$

v. We split our argument into following two cases:

- Let u lie in V_v. In this case, let w be a path in G_x^* from v to u. Then from corollary 2.2 one gets $x \ \theta^{J_1} \ xwy$ and $y \ \theta^{J_1} \ ywx$. Combining them together, one gets

  $$x \ \theta^{J_1} \ xwy \ \theta^{J_1} \ (xw)(yw)xx \ \theta^{J_1} \ y(wx)(wx)x \ \theta^{J_1} \ ywx \ \theta^{J_1} \ y$$

  Note that the bracketed entities in each step are co-terminal loops.

- Let u lie in V' i.e. there are no paths in G_x^* from v to u and hence they lie in two different strongly connected components. On the other hand since both x and y go from u to v, it must be true that there exist unique factorization for x and y, such that $x = x_1 a x_2$ and $y = y_1 a y_2$ with the edge a such that its endpoints $\alpha(a)$ and $\omega(a)$ lie in V' and V_v respectively. Thus we can say that $x_1 \ \gamma^{J_1} \ y_1$ and $x_2 \ \gamma^{J_1} \ y_2$ and from our IH we get $x_1 \ \theta^{J_1} \ y_1$ and $x_2 \ \theta^{R_1} \ y_2$ from which the desired result follows.

This proves the theorem. □

Hence in particular, every category all of whose base monoids are in $\mathbf{J_1}$ divides some monoid in $\mathbf{J_1}$, whence $\ell\mathbf{J_1} = \mathbf{gJ_1}$.

2.2 Categories in $\ell\mathbf{R_1}$ ($\ell\mathbf{L_1}$)

We now want to move up in the lattice of idempotent varieties of monoids and consider categories induced by the M-variety $\mathbf{R_1}$. It is easy to see that the congruence θ^{R_1} introduced in the last section is the smallest to induce a category in $\ell\mathbf{R_1}$ for any graph G. Here we introduce another congruence γ^{R_1} that is used in the theorem below for describing the languages recognized by locally $\mathbf{R_1}$ categories. We say $x \ \gamma^{R_1} \ y$ iff for every prefix x' of x we have a prefix y' of y such that $\lambda(x') = \lambda(y')$ and vice-versa, where x and y are co-terminal paths in G^*. Note if x and y are related by this, then they must start with the same edge. In other words, the longest common prefix of x and y has length at least one, provided x, y are non-empty words.

Theorem 2.4 *The syntactic category of a language $L \subseteq G^*$ is in $\ell\mathbf{R_1}$ iff L is a γ^{R_1}-language.*

Proof. The direction from right to left is easy and left to the reader.

For the other direction, consider $x \ \gamma^{R_1} \ y$. Let $y = y_0$. Our strategy is to keep growing the longest common prefix of x and y_i, by successively changing

y_i to y_{i+1} so that $y_i \;\theta^{R_1}\; y_{i+1}$ and $x \;\gamma^{R_1}\; y_i$ for all i. Thus, for some $n \geq 1$, y_n would have x as a prefix and then applying corollary 2.2 from the last section, we obtain that $x \;\theta^{R_1}\; y_n$ and this would finish the argument. Hence, the only remaining thing is to show that one can always obtain y_{i+1} from y_i, satisfying the condition stated above. Let ρ_i denote the longest common prefix of x and y_i. Let $x = \rho_i a x'$ and $y_i = \rho_i b y'$. If $a = b$, then we have nothing to show. So consider $a \neq b$. We split our argument in following two cases:

- a does not occur in ρ_i. This means that y' can be written as $y' = z_1 a z_2$, where $\lambda(bz_1) \subseteq \lambda(\rho_i)$. Note that bz_1 is a loop at the end of ρ_i and hence $\rho_i \;\gamma^{R_1}\; \rho_i b z_1$ which implies $x = \rho_i a x' \;\gamma^{R_1}\; \rho_i a z_2$. Now applying corrollary 2.2 we get $\rho_i \;\theta^{R_1}\; \rho_i b z_1$ and thus setting $y_{i+1} = \rho_i a z_2$ we are done for this case.

- a occurs in ρ_i. In this case we could write $\rho_i = u_0(au_1) \;\theta^{R_1}\; u_0 a u_1 a u_1 = \rho_i a u_1$. Thus, setting $y_{i+1} = \rho_i a u_1 b y'$ does the job.

This completes the proof. □

The theorem above establishes the equality of C-varieties $\ell\mathbf{R_1}$ and $\mathbf{gR_1}$. Very similar arguments yield the identity $\ell\mathbf{L_1} = \mathbf{gL_1}$.

2.3 $\mathbf{R_1} \vee \mathbf{L_1}$ is local

The M-variety $\mathbf{R_1} \vee \mathbf{L_1}$ is the *join* of the varieties $\mathbf{R_1}$ and $\mathbf{L_1}$. This means that we take the union of the set of monoids in $\mathbf{R_1}$ and the set of monoids in $\mathbf{L_1}$ and then take the closure of this set under direct product and division of monoids. It follows that the congruence corresponding to the variety $\mathbf{R_1} \vee \mathbf{L_1}$, is the intersection of the congruence corresponding to $\mathbf{R_1}$ with the congruence corresponding to $\mathbf{L_1}$. Hence the syntactic monoid of a language L is in $\mathbf{R_1} \bigvee \mathbf{L_1}$ iff L is a $\sim_{R_1 \vee L_1}$-language, where $\sim_{R_1 \vee L_1}$ is the congruence generated by the following condition: for any two words x and y, $x \sim_{R_1 \vee L_1} y$ iff $x \sim_{R_1} y$ and $x \sim_{L_1} y$. In other words, two strings are $\sim_{R_1 \vee L_1}$ related iff the order in which new letters appear in one of them is the same as in the other, irrespective of the way we scan the words. It can be shown that the identities for this M-variety are given by $x = x^2$ and $xyxzx = xyzx$.

We are now interested in looking at the languages recognized by categories in the C-variety $\ell(\mathbf{R_1} \vee \mathbf{L_1})$. Let $\theta^{R_1 \vee L_1}$ be the coarsest graph congruence generated by the following condition on loops: $xyxzx \, \theta^{R_1 \vee L_1} \, xyzx$ and $x \, \theta^{R_1 \vee L_1} \, x^2$, where x, y, z are arbitrary loops.

Let $\gamma^{R_1 \vee L_1}$ denote graph congruence satisfying the following condition: $x \, \gamma^{R_1 \vee L_1} \, y$ iff x and y are co-terminal and $x \sim_{R_1 \vee L_1} y$ when x, y are treated as strings in A^*.

In this section, we want to prove the following result:

Theorem 2.5 *The syntactic category of a language L is in $\ell(\mathbf{R_1} \vee \mathbf{L_1})$ iff L is a $\gamma^{R_1 \vee L_1}$-language.*

The direction from right to left is easily obtained by observing that $\theta^{R_1 \vee L_1}$ refines $\gamma^{R_1 \vee L_1}$. The other direction is somewhat technical and we will have to develop some more results before we can prove it. We will state a result here that easily follows from the theorem given in the next section.

Proposition 2.6 *If $C = G^*/\beta$ is any locally idempotent category (i.e $u \, \beta \, u^2$ for all loops u), then $xvx \, \beta \, x$ if $\lambda(v) \subseteq \lambda(x)$.*

We now want to show a result that is the key lemma of this section and in some sense is a natural generalization of corollary 2.2 given in the last section, to the case of $\ell(\mathbf{R_1} \vee \mathbf{L_1})$.

Lemma 2.7 *Let* usw *be a path such that* s *is a loop satisfying the condition* $\lambda(s) \subseteq \lambda(u), \lambda(w)$. *Then,* $usw\, \theta^{R_1 \vee L_1}\, uw$.

Proof. If s is empty, we have nothing to prove. Otherwise, let a be the last edge of s. Then, there exists some edge b in s such that it can be factorized as $s = p_0 b p_1 a$ and u can be written as $u = u_0 b u'$, where $\lambda(s) \subseteq \lambda(bu')$. Note that $b = a$ is a special case of this for which, we would have written $u = u_0 a u'$. Here we will assume that $b \neq a$, as the other case is much simpler to handle, once we have seen the argument for this case. Symmetrically, one can also write $s = a' q_1 b' q_0$ such that $w = w' b' w_0$ and $\lambda(s) \subseteq \lambda(w'b')$. We also note that both $bu'p_0$ and $q_0 w' b'$ are loops. Hence, by replicating each of them twice, one can write the following:

$$usw\, \theta^{R_1 \vee L_1}\, u_0 bu'p_0 bu'p_0 bu'sw'b'q_0 w'b'q_0 w'b'w_0 \tag{2.4}$$

For compactness, we put $u_s = p_0 bu'$ and $w_s = w'b'q_0$ in equation 2.4 to get the following:

$$usw\, \theta^{R_1 \vee L_1}\, u_0 bu'(u_s)^2 s(w_s)^2 w'b'w_0 \tag{2.5}$$

Since the edge b' occurs in s and we know that $\lambda(s) \subseteq \lambda(bu')$, it follows that b' also occurs in bu' and we can write $bu' = \rho_0 b' \rho_1$. Hence, using equation 2.5 one can write

$$
\begin{aligned}
& usw \quad \theta^{R_1 \vee L_1} \quad u_0\rho_0 b'\rho_1(u_s)^2 s(w_s)^2 w' b' w_0 \\
\Rightarrow \; & usw \quad \theta^{R_1 \vee L_1} \quad u_0\rho_0 b'\rho_1(u_s)^2 s(w_s)^2 w' b'\rho_1(u_s)^2 s(w_s)^2 w' b' w_0 && (2.6)\\
\Rightarrow \; & usw \quad \theta^{R_1 \vee L_1} \quad u_0\rho_0 b'\rho_1(u_s)^2 s w_k b' u_k s(w_s)^2 w' b' w_0 \; = \; x && (2.7)\\
\Rightarrow \; & x \quad = \quad u_0\rho_0 b' x_l b' x_r b' w_0 && (2.8)
\end{aligned}
$$

where, $w_k = (w_s)^2 w'$ and $u_k = \rho_1(u_s)^2$. We will now think of x in equation 2.7 as having two symmetric parts - the left and the right and these are called in equation 2.8 x_l and x_r respectively. We will focus on the segment $(u_s)^2 s$ of x_l on the left and $s(w_s)^2$ of x_r on the right. We will explain here what we do on the left and completely dual arguments would apply on the right. Let us recall $u_s = p_0 bu'$. Since every edge that occurs in s also occurs in bu', we can write $u_s = p_0 bu_1 au_2$, where a is the last edge of s when scanned from left. Hence $(u_s)^2 s$ can be written as $p_0 bu_1 au_2 bu' s_k a$. Observe that $u_2 bu'$ and $s_k a$ are co-terminal loops and since $u_2 bu'$ occurs twice in x, (once in x_l and once in x_r), we can insert another $u_2 bu'$ in the middle. Thus, we can write the following:

$$
x \quad \theta^{R_1 \vee L_1} \quad u_0\rho_0 b'\rho_1 p_0 bu_1 au_2 bu' s_k au_2 bu' w_k b' x_r b' w_0 \tag{2.9}
$$

Since $\lambda(s) \subseteq \lambda(au_2bu')$, we can apply proposition 2.6 to equation 2.9 and get

$$
\begin{aligned}
& x \quad \theta^{R_1 \vee L_1} \quad u_0\rho_0 b'\rho_1 p_0 bu_1 au_2 bu' w_k b' x_r b' w_0 \\
\Rightarrow \; & x \quad \theta^{R_1 \vee L_1} \quad u_0\rho_0 b'\rho_1(u_s)^2 w_k b' x_r b' w_0 && (2.10)
\end{aligned}
$$

Now we can apply very similar arguments for the segment $s(w_s)^2$ in x_r to equation 2.10 and obtain

$$x \quad \theta^{R_1 \vee L_1} \quad u_0 \rho_0 b' \rho_1 (u_s)^2 w_k b' u_k (w_s)^2 w' b' w_0 \tag{2.11}$$

Expanding w_k and u_k and shrinking $(u_s)^2$ and $(w_s)^2$ to u_s and w_s respectively:

$$x \quad \theta^{R_1 \vee L_1} \quad u_0 \rho_0 (b' \rho_1 u_s w_s w')(b' \rho_1 u_s w_s w') b' w_0$$

$$\Rightarrow x \quad \theta^{R_1 \vee L_1} \quad u_0 \rho_0 b' \rho_1 u_s w_s w' b' w_0 \tag{2.12}$$

Recalling $\rho_0 b' \rho_1 = bu'$ and expanding u_s and w_s in equation 2.12

$$usw \quad \theta^{R_1 \vee L_1} \quad u_0 (bu' p_0 bu')(w' b' q_0 w' b') w_0 \tag{2.13}$$

Recall that both bu' and $w'b'$ contained every edge in s. In particular they contain all edges of p_0 and q_0 which we recall to be just segments of s. We can hence apply proposition 2.6 to the bracketed entities in equation 2.13 to obtain

$$usw \quad \theta^{R_1 \vee L_1} \quad u_0 (bu' w' b' w_0) = uw \tag{2.14}$$

This completes the argument. $\square$

We now obtain the following result using the result above:

Lemma 2.8 *If v is a loop at the end of u and $uv\, \gamma^{R_1 \vee L_1}\, u$, then $uv\, \theta^{R_1 \vee L_1}\, u$.*

Proof. Clearly, in this case $\lambda(v) \subseteq \lambda(u)$. Also note that since $uv\, \gamma^{R_1 \vee L_1}\, u$, if u and v are non-empty, then v and u must end with the same letter. Our strategy in this case would be to grow the length of the common suffix of uv and u gradually by changing v until one of the two following things happen:

1. We have transformed v to v' such that $uv\, \theta^{R_1 \vee L_1}\, uv'$ and u is a suffix of v'. In this case $uv' = uwu$ and $\lambda(w) \subseteq \lambda(u)$. Hence applying proposition 2.6 we get $uv'\, \theta^{R_1 \vee L_1}\, u$.

2. The other case is v has been transformed to v' such that v' is a suffix of u. In this case $uv\, \theta^{R_1 \vee L_1}\, uv' = u_0(v')^2\, \theta^{R_1 \vee L_1}\, u_0 v' = u$.

Thus in both cases we get the desired result. Hence the rest of the argument boils down to showing that we can continously go from uv to uv' such that $uv\, \theta^{R_1 \vee L_1}\, uv'$ and the common suffix of v' and u has grown in length. Assume that the common suffix to start with is ρ. Then $uv = uv_0 a\rho$ and $u = u_0 b\rho$. If $b = a$ we are done. Hence, assume $b \neq a$. We now have two cases. If b has already occurred in ρ, then $uv = uv_0 a\rho_1 b\rho_0\, \theta^{R_1 \vee L_1}\, uv_0 a\rho_1 b\rho_1 b\rho_0 = uv_1 b\rho$, where $v_1 = v_0 a\rho_1$. Thus setting the common suffix to $\rho' = b\rho$, we are done. We are thus left with the case in which b does not occur in ρ. In this case there are two possibilities. The first is, b does not occur in v at all. Note in this case, clearly $v = v_0\rho$ and $\lambda(v_0) \subseteq \lambda(\rho)$ since the order in which new edges appear from right to left in uv and u are the same. Thus,we get $uv = u_0\rho v_0\rho$. Hence applying proposition 2.6, we get $uv\, \theta^{R_1 \vee L_1}\, u_0\rho = u$ and we are done. The second possibility is $v = v_0 b v_1 a\rho$. Again, $\lambda(v_1 a) = \lambda(\rho)$ and $v_1 a$ is a loop. Thus we can apply lemma 2.7 to get $uv = uv_0 b v_1 a\rho\, \theta^{R_1 \vee L_1}\, uv_0 b\rho$ and we are done. $\square$

We shall now prove the main theorem of this section.

Proof of Theorem 2.5: Let us assume that $x\, \gamma^{R_1 \vee L_1}\, y$, where x and y are arbitrary paths. Let $y = y_1$. We want to find a series of paths y_i such that $y_i\, \theta^{R_1 \vee L_1}\, y_{i+1}$. Let ρ_i represent the longest common prefix of x and y_i. Then we want $|\rho_i| < |\rho_{i+1}|$. Thus for some n, x itself is a prefix of y_n. Since $\theta^{R_1 \vee L_1}$

refines $\gamma^{R_1 \vee L_1}$, we can apply lemma 2.8 to obtain the result $x\,\theta^{R_1\vee L_1}\,y_n\,\theta^{R_1\vee L_1}\,y$ and we are done. Thus, the whole argument boils down to finding y_{i+1}, given y_i. Let $x = \rho_i b x'$ and $y_i = \rho_i a y'$. If $a = b$, there is nothing to show. Assume $a \neq b$. There are several cases to consider. Assume b occurs in ρ_i. In this case $y_i = wbw'ay'\,\theta^{R_1\vee L_1}\,wbw'bw'ay' = \rho_i bw'ay' = y_{i+1}$. Otherwise, let b not occur in ρ_i. In this case we can write $y_i = \rho_i vby'$, where v is a loop and every edge in v occurs in ρ and v does not contain b. If no edge of v occurs in y', then clearly $\rho_i\,\gamma^{R_1\vee L_1}\,\rho_i v$, since we know $x\,\gamma^{R_1\vee L_1}\,y_i$. Thus applying lemma 2.8 to ρ_i and $\rho_i v$ we get $y_i = \rho_i vby'\,\theta^{R_1\vee L_1}\,\rho_i by' = y_{i+1}$ and we are done. Otherwise, let c be some edge in v that occurs in y'. In this case, c also occurs in ρ_i and let $\rho_i = wcw'$, $y' = v_0cv_1$. Thus, we get

$$\begin{aligned} y_i &= wcw'vbv_0cv_1\,\theta^{R_1\vee L_1}\,wcw'vbv_0cw'vbv_0cv_1 \\ \Rightarrow\ y_i &\ \theta^{R_1\vee L_1}\ \rho_i vbt \end{aligned} \tag{2.15}$$

where $t = v_0cw'vbv_0cv_1$ and clearly every edge that appears in v appears in t. Thus applying lemma 2.7 to equation 2.15 we get $y_i\,\theta^{R_1\vee L_1}\,\rho_i bt = y_{i+1}$ and we are done. $\square$

2.4 $\mathbf{A_1}$ is local

Our arguments are based on the work of [WT86]. For every word x let $x_{\mathcal{P}}$ ($x_{\mathcal{S}}$) represent the longest prefix (longest suffix) of x such that $\lambda(x_{\mathcal{P}}) \neq \lambda(x)$ ($\lambda(x_{\mathcal{S}}) \neq \lambda(x)$). We define a congruence $\sim_{A_1}$ in A^* by saying that $x \sim_{A_1} y$ iff $\lambda(x) = \lambda(y)$, $x_{\mathcal{P}} \sim_{A_1} y_{\mathcal{P}}$ and $x_{\mathcal{S}} \sim_{A_1} y_{\mathcal{S}}$. Note for non-empty words x and y, $x \sim_{A_1} y$ implies that they start and end with the same letter and hence x and y have a common non-empty prefix and suffix. The following remarkable fact can be traced back to McLean and Green-Rees and also appears in the work of [Eil76]:

Theorem 2.9 *The syntactic monoid of a language L is finite and idempotent iff L is a $\sim_{A_1}$ language.*

We can then induce a graph congruence from $\sim_{A_1}$ by imposing conditions of coterminality and denote it by γ^{A_1}. Given a graph $G = (V, A)$, let the coarsest congruence in G^* that induces a locally idempotent category be denoted by θ^{A_1}. The generating identity for such a congruence is given by $x\,\theta^{A_1}\,x^2$ for each loop x in G^*. The first thing to observe is the following easy fact:

Fact θ^{A_1} refines γ^{A_1}.

The non-trivial thing to show however is the other direction:

Lemma 2.10 *γ^{A_1} refines θ^{A_1}*

Proof. We will use nested induction to prove the theorem. We first note that if we restrict ourselves to paths x and y such that $\lambda(x) \leq 1$, then the result is obviously true. Now we will induct on $|\lambda(x)|$. We first make the following claim: If $u\,\gamma^{A_1}\,vw$, then there exists w' in G^*, such that $u\,\theta^{A_1}\,vw'$. We show the claim by again inducting on the size of $\lambda(v)$. Since u and v must begin with the same edge, the desired result immediately follows in the base case of $|\lambda(v)| \leq 1$. Otherwise, let v_0 be the longest common prefix of u and v. If v_0 is the entire path v, then there is nothing to show. If not, then we can write $v = v_0 a v_1$ and $u = v_0 u'$. We split into two cases:

- If a does not occur in v_0, then from the definition of γ^{A_1} it follows that u' can be factorized as $u_1 a u_2$. Then one can write $u = u_0 a u_2$, where

$u_0 = v_0 u_1$ and a does not occur in u_0. Hence $u_0 \, \gamma^{A_1} \, v_0$. Note that $|\lambda(v_0)| < |\lambda(v)|$ and so from our induction hypothesis we obtain $v_0 \, \theta^{A_1} \, u_0$ and hence $u \, \theta^{A_1} \, v_0 a u_2$.

- If a occurs in v_0, then one can write $v_0 = v_2 a v_3$. Thus,

  $u = v_2 a v_3 u' \, \theta^{A_1} \, v_2 a v_3 a v_3 u'$

  as $a v_3$ is a loop.

Hence in all cases we can write $u \, \theta^{A_1} \, u_n$, where the common prefix of u_n and v has grown in length. Since we know that θ^{A_1} refines γ^{A_1}, we can repeat this until the common prefix coincides with v, which establishes the claim. It is easy to see from symmetry that $u \, \gamma^{A_1} \, vw$ also implies $u \, \theta^{A_1} \, u'w$ for some u' in G^*. Consider $x \, \gamma^{A_1} \, y$. From the claim, it follows $y \, \theta^{A_1} \, xz$ for some z. This means $x \, \gamma^{A_1} \, xz$. The symmetric form of the claim then gives $x \, \theta^{A_1} \, wz$ for some w. Since z is a loop, we get $x \;\; \theta^{A_1} \;\; wzz \;\; \theta^{A_1} \;\; xz$, whence finally $x \, \theta^{A_1} \, y$. $\square$

It is interesting to note here that [JS92] have shown that every monoid variety satisfying the identity $x^{n+1} = x$ for some $n \geq 0$, is local if labels appearing in its "Polak ladder" are all local. They have then been able to show as a corollary that any non-trivial M-variety consisting of monoids whose idempotents form a submonoid[1], is local. It then follows that every variety of idempotent monoids is local. Jones and Szendrei use the powerful theory of completely regular semigroups developed in [Pol85], [Pol87] and [Pol88] and conjecture that every non-trivial variety of completely regular monoids[2] is local.

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[1] such monoids are called orthogroups and the variety of all such monoids is denoted by **OG**

[2] monoids that are unions of their subgroups

Chapter 3

Some results on graph congruences

In this chapter, our objective is to present some properties of two graph congruences that help us characterise the C-variety **gCom** in terms of algebraic identities. All of these results first appeared in [TW85] and then in the book of [Str94]. We mildly strengthen lemma 3.8 and 3.9 from their original form in [Str94]. The results in this chapter also prepare us to take on locally commutative categories in the next chapter.

Note that we can form an infinite family of sub-varieties of **Com** by considering variety $\mathbf{Com_{t,q}}$ which is generated by following identities on loops: $x^t = x^{t+q}$ and $xy = yx$. As a byproduct, at the end of this chapter we give another argument for the locality of $\mathbf{J_1}$ by observing that $\mathbf{J_1}$ is identical with $\mathbf{Com_{1,1}}$.

3.1 Free globally commutative categories

Let G be a graph and define on G^* the congruence $x \, \gamma_\infty \, y$ iff $x \sim y$ and $|x|_a = |y|_a$ for every edge a.

Let also θ_∞ denote the coarsest congruence satisfying $xyz\,\theta_\infty\,zyx$ whenever x and z are two co-terminal paths in G^*.

The following lemma shows that θ_∞ characterises free globally commutative categories.

Lemma 3.1 *For two paths x and y, $x\,\gamma_\infty\,y$ iff $x\,\theta_\infty y$.*

Proof. It is straight-forward to see that θ_∞ refines γ_∞. We will show the other direction by induction on the length of paths. For $|x| = |y| = 1$, the base case of the induction, it is obvious. Let $k \geq 2$ be a positive integer with $|x| = |y| = k$. If x and y begin with the same edge a, then we have $x = ax'\,\gamma_\infty\,ay' = y$, which implies that $x'\,\gamma_\infty\,y'$ and applying the induction hypothesis we obtain $x'\,\theta_\infty\,y'$, whence $x\,\theta_\infty\,y$.

So we are left with the case in which $x = ax_1bx_2$ and $y = by_1ay_2$ where a and b are two different edges. If x_1 is empty, then $y\,\theta_\infty\,aby_1y_2 = y'$ as a and by_1 are loops around the same vertex. As x and y' begin with the same edge a, we can apply the argument of the previous paragraph and obtain $x\,\theta_\infty\,y'\,\theta_\infty\,y$. So we assume that x_1 is non-empty. Let $x_1 = u_1u_2\ldots u_s$, where every u_i is an edge in G. If u_1 occurs in y_1, then $y = bw_1u_1w_2ay_2\,\theta_\infty\,au_1w_2bw_1y_2 = y'$ as a and bw_1 are co-terminal paths. Since y' and x begin with a we are done with this case. If u_s occurs in y_2 then $y = by_1aw_1y_sw_2\,\theta_\infty\,aw_1y_sby_1w_2 = y'$ as aw_1y_s and by_1 are loops around the same vertex and we are again done as in the previous cases. Otherwise if there exists an i with $1 < i < s$ such that $u_1, \ldots, u_{i-1}$ occur in y_2 and u_i occurs in y_1, then $y = bw_1u_iw_2av_1u_{i-1}v_2\,\theta_\infty\,av_1u_{i-1}u_iw_2bw_1v_2 = y'$ as bw_1 and av_1u_{i-1} are co-terminal paths and now considering x and y' we are done as before. $\square$

Lemma 3.2 *If x is a loop with $|x|_a \equiv 0(mod)q$ for every edge a, then there exists a loop y such that $x \,\gamma_\infty\, y^q$*

Proof. Every edge a occurs $n_a q$ times in x, where $n_a \geq 0$ is some number. We create a new graph G' having the same vertex set as G by replacing every edge a in G by n_a co-terminal edges that are denoted by $a_1, \ldots, a_{n_a}$. The loop x in G thus gives rise to a circuit in G' such that each edge in G' is repeated exactly q times. This implies that the in-degree of every vertex in G' is the same as its out-degree. So there exists an Euler loop co-terminal with x denoted by y that traverses every edge of G' exactly once. Clearly $x \,\gamma_\infty\, y^q$ $\square$

Lemma 3.3 *If x and y are paths in G^* such that $|x|_a < |y|_a$ for every edge a in G that occurs in x, then $y \,\theta_\infty\, uxv$ for some $u, v \in G^*$.*

Proof. We show this by induction on the length of the path x. For $|x| = 0$, it is clearly true. Consider $|x| \geq 1$. In this case let $x = x'a$ where a is an edge in G. From the induction hypotheses we have $y \,\theta_\infty\, ux'v$ for some u, v such that $|u|_a + |v|_a \geq 2$. The first case is when $|u|_a \geq 2$. Then $y \,\theta_\infty\, u_0 a u_1 a u_2 x' v$. Since au_1 and au_2x' are loops around the same vertex we commute them to get $y \,\theta_\infty\, u_0 a u_2 x' a u_1 v$ and we are done. Next, consider the case $|u|_a = |v|_a = 1$. Then $y \,\theta_\infty\, u_0 a u_1 x' v_0 a v_1$. In this case au_1x' and v_0 are loops around the same vertex and so exchanging them we have $y \,\theta_\infty\, u_0 v_0 a u_1 x' a v_1$ and we are again done. Finally for the case $|v|_a \geq 2$ we get $y \,\theta_\infty ux' v_0 a v_1 a v_2$ and by interchanging the loops v_0 and av_1 we get $y \,\theta_\infty\, ux' a v_1 v_0 a v_2$ and this completes the proof. $\square$

Lemma 3.4 *If x is a loop at the end of y such that every edge a in G occurs at most once in x and for every edge a that occurs in x, $|y|_a > t$ where $t \geq 1$, then $y \,\theta_\infty\, ux^t$ for some u in G^*.*

Proof. We prove this by induction on t. For $t = 1$, lemma 3.3 gives $y\,\theta_\infty\,uxv$. But in this case x and v are loops around the same vertex and so we have $y\,\theta_\infty\,uvx$ and we are done. Let $t \geq 2$. In this case from our induction hypothesis, we get $y\,\theta_\infty\,\rho x^{t-1}$, for some path ρ. For each edge a that occurs in x, we have $|y|_a = |\rho|_a + t - 1 > t$, since a occurs at most once in x. Thus for every such edge a, we have $|x|_a < |\rho|_a$. Applying lemma 3.3 we get $\rho\,\theta_\infty\,\rho_0 x \rho_1$, whence we have $y\,\theta_\infty\,\rho_0 x \rho_1 x^{t-1}$. Since x is a loop, we finally obtain $y\,\theta_\infty\,ux^t$, where $u = \rho_0\rho_1$. □

3.2 Finite globally commutative categories

In a finite category $C = G^*/\beta$, every base monoid is finite. One would thus have from the pigeon hole principle, some $t \geq 0$ and $q \geq 1$ such that $x^t\,\beta\,x^{t+q}$, for every loop x in G^*. Formally, we generalize θ_∞ by saying that $\theta_{t,q}$ represents the coarsest graph congruence satisfying following conditions: $xyz\,\theta_{t,q}\,zyx$, whenever x and z are co-terminal and for every loop x, we have $x^t\,\theta_{t,q}\,x^{t+q}$.

In this section we prove the following theorem that was first proved in [TW85]:

Theorem 3.5 *Category $C = G^*/\beta$ is globally commutative iff β is refined by $\theta_{t,q}$ for some $t \geq 0$ and $q \geq 1$.*

Let $\gamma_{t,q}$ be the coarsest congruence satisfying $x\,\gamma_{t,q}\,y$ iff $|x|_a \equiv_{t,q} |y|_a$ for every edge a and $x \sim y$. Note that in the above theorem if C is globally commutative then by definition, it divides some finite commutative monoid and hence β is clearly refined by $\gamma_{t,q}$ for some t and q. One can easily verify that $\theta_{t,q}$ refines $\gamma_{t,q}$ and this gives the left to right direction of the above theorem.

The other direction is much more delicate to obtain and we first need to prove some lemmas.

Lemma 3.6 *For every graph $G = (V, A)$, given a $t \geq 0$, there exists s dependent on G and t (we write $s = s(G,t)$) such that for each path x in G^*, $|x|_a > s$ implies $x\,\theta_\infty\,\rho(ay)^t w$ for some ρ, y, w in G^*.*

Proof. Let $s = |V| + t(2^{|A|} - 1) + 1$. In this case for any x that has more than s occurrences of edge a, we can write $x = x_0ax_1 \ldots ax_sax_{s+1} = x'ax_{s+1}$, where $x' = x_0a \ldots ax_s$ and no x_i contains any occurrences of a for $i \leq s$. We shall call each ax_i (where $i \leq s$) an a-zone. Since we can commute loops, we can assume w.l.o.g that all the vertices that appear in x' also appear in $u = x_0ax_1a \ldots ax_{|V|}$. Consider any j such that $|V| < j \leq s$ and ax_j has a loop in between. In other words we can write $ax_j = w_0w_1w_2$, where w_1 is a loop. Then from our assumption, we can factorize u as u_0u_1 such that $\alpha(u_1) = \alpha(w_1)$ and hence commuting loops $u_1ax_{|V|+1} \ldots w_0$ and w_1, one gets $u_0w_1u_1ax_{|V|+1} \ldots w_0w_2ax_{j+1} \ldots ax_{s+1}$. Continuing in this way, one can ensure that finally, for all j in the range given by $|V| < j \leq s$, ax_j has at most one occurrence of any edge in A. But then from our choice of s, there must be at least t instances of a-zones which have the same set of edges occurring in them and hence are γ_∞ equivalent to each other. Let them all be equivalent to ay for some y. Using the equivalence of γ_∞ with θ_∞ and commuting loops, one can thus write $x\,\theta_\infty\,\rho(ay)^t w$. $\square$

Lemma 3.7 *Given a graph $G = (V, A)$, and $r \geq 0$, we can find m dependent on G and r (written $m = m(G,r)$) such that for every path x, y in G^*, $x\,\gamma_{m,q}\,y$ implies that there exists a path z in G^* satisfying the following : $|x|_a \leq |z|_a$ for every edge a in A, $x\,\gamma_{r,q}\,z$ and $z\,\theta_{r,q}\,y$.*

Proof. We choose $m(G,r) = s(G,r) + 1$, where $s(G,r)$ comes from lemma 3.6. Let $x\,\gamma_{m,q}\,y$ and let $A' = \{a : |x|_a > |y|_a\}$[1]. Take any edge a from A'. Then a occurs at least m times in y. This means using lemma 3.6, we get $y\,\theta_\infty\,\rho(av)^r w\,\theta_{r,q}\,\rho(av)^{r+kq}w = y_1$. Here we have chosen k such that $r + kq \geq |x|_a$. One can see that any edge occurs in y_1 at least as many time as it occurs in y. Thus, each edge in A' occur in y_1 at least m times and we can repeat the argument, once for each edge in A'. Hence, let $z = y_{|A'|}$ and it can be easily verified that z satisfies all desired conditions as $m > r$ and $\theta_{r,q}$ refines $\gamma_{r,q}$. $\square$

Observe that $\theta_{t,q}$ can be thought of as a rewriting system. If a path y can be obtained from path x using rule for exchanging co-terminal paths (i.e $uwv \to vwu$, where $v \sim u$) and rule for loop replication ($u^t \to u^{t+q}$) without using loop deletion ($u^{t+q} \to u^t$), then we write $x \leq^{\theta}_{t,q} y$. It is a trivial observation that $x \leq^{\theta}_{t,q} y$ implies $x\,\theta_{t,q}\,y$. Clearly $x \leq^{\theta}_{t,q} y$ implies for all $a \in A$, $|x|_a \leq |y|_a$.

Lemma 3.8 *If y is any path in G^* and x is any loop at the end of y, such that for every edge a of G that occurs in x we have $|y|_a > s$ for $s \geq 1$, then $y \leq^{\theta}_{s,q} yx^q$ for every $q > 0$.*

Proof. We will prove this by induction on the length of x. For $|x| = 0$, it is obviously true. For $|x| \geq 1$, if for every edge a, $|x|_a \leq 1$, then lemma 3.4 gives us $y\,\theta_\infty\,ux^s \leq^{\theta}_{s,q} ux^{s+q}\,\theta_\infty\,yx^q$ and the result follows. Now if $|x|_a \geq 2$ for some edge a, then let $x = x_0x_1x_2$, where x_1 is a loop. In this case $|x_1|, |x_0x_2| < |x|$. Hence applying the induction hypothesis to y and x_0x_2, we get $y \leq^{\theta}_{s,q} y(x_0x_2)^q$. But $y(x_0x_2)^q = yx_0x_2(x_0x_2)^{q-1}$. Again applying the IH to yx_0 and x_1 we get $y \leq^{\theta}_{s,q} yx_0x_1^qx_2(x_0x_2)^{q-1}\,\theta_\infty\,y(x_0x_1x_2)x_0x_1^{q-1}x_2(x_0x_2)^{q-2}$. Here we have commuted x_1^{q-1}

[1] If A' is empty, then $z = y$ and we are done

and x_2x_0. We can keep doing this $q-1$ times to get $y \leq^{\theta}_{s,q} y(x_0x_1x_2)^q = yx^q$.
□

Lemma 3.9 *If for two paths x and y, $|x|_a \leq |y|_a$ for every edge a and $x\,\gamma_{t+1,q}\,y$, then $x \leq^{\theta}_{t,q} y$.*

Proof. We will prove this using induction on $d = |y| - |x|$. For $d = 0$, we have $|y|_a = |x|_a$ for every edge a and so from lemma 3.1 we have $x\,\theta_\infty\,y$ and hence $x \leq^{\theta}_{t,q} y$. Consider the case $d \geq 1$. Let u be the longest common prefix of x and y. Then $x = ux_1$ and $y = uy_1$. If x_1 is empty then $|y_1|_a \equiv 0\,(mod\ q)$ for each edge a and so from lemma 3.2, we get $y\,\theta_\infty\,uw^q$ for some loop w in G^*. And so lemma 3.8 gives us $x = u \leq^{\theta}_{t,q} uw^q\,\theta_\infty\,y$ and hence the result.
Now consider the case when x_1 and hence y_1 are non-empty. So we could write $x = uax'$ and $y = uby_0ay_1$, with a and b being edges such that $a \neq b$. If by_0 and y_1 have a vertex in common then by_0ay_1 could be rewritten as $bw_0w_1av_0v_1$ for some w_0, w_1, v_0, v_1 in G^* such that bw_0 and av_0 are co-terminal. Thus $y\,\theta_\infty\,uav_0w_1bw_0v_1 = y'$. Now y' and x have a common prefix that is longer than u and we can repeat this argument until we find a path y^* such that x is a prefix of y^* or we cannot find the common vertex as before. The former case has been treated already and so we consider the latter, which can be stated simply as by_0 and y_1 have no vertices in common.

Claim: If $x = uax'$ and $y = uby_0ay_1$ are two paths with $|x|_c \leq |y|_c$ for every edge c and by_0 and y_1 have no vertex in common, then x' and by_0 contain no edge in common.

If the claim is false then we have a common edge c and so we could write $x = uax_1cx_2$ and $y = uw_1cw_2ay_1$, such that no edge of x_1 occurs in by_0. If x_1

is empty then $\alpha(c) = \alpha(y_1)$ and so this vertex is common to by_0 and y_1. If x_1 is non-empty then the last edge of x_1 must occur in y_1 and again by_0 and y_1 would have a common vertex. So in both cases we have a contradiction. Hence the claim follows.

For each edge c, we have $|by_0|_c + |y_1|_c \equiv_{t,q} |x'|_c$. From the claim above and the condition of the present case by_0 does not have any edge in common with y_1 or with x'. Thus $|by_0|_c \equiv 0\,(mod\ q)$ for each edge c. Applying lemma 3.2 we have $by_0\,\theta_\infty\,(w)^q$, where w is some loop at the end of u. For each edge c that appears in w we have $|x'|_c = 0$ and hence $|u|_c \geq t+1$. Hence applying lemma 3.8 to u and uw^q, we obtain $uay_1 \leq^{\theta}_{t,q} uw^q ay_1\,\theta_\infty\,y$.

Since by_0 and x' have no edges in common, we get $|x|_c = |uax'|_c \leq |uay_1|_c$ for all edges c. If for any edge c we have $|uax'|_c < |uay_1|_c$, then $|uax'|_c < |uby_0ay_1|_c$ and hence $|uax'|_c \geq t+1$ for all such edges c. Also for all edges c in the underlying graph, we have $|uax'|_c \equiv |uby_0ay_1|_c \equiv |uay_1|_c\ (mod\, q)$. Thus, $x\,\gamma_{t+1,q}\,uay_1$ and from our induction hypotheses we obtain $x \leq^{\theta}_{t,q} uay_1 \leq^{\theta}_{t,q} uby_0ay_1 = y$ and we are done. □

Now we are in a position to prove our main theorem 3.5.

Proof of Theorem 3.5. We proved one direction earlier. Here we consider a category $C = G^*/\beta$ such that for some t and q, $\theta_{t,q}$ refines β. We choose $s = m(G, t+1)$. Consider two co-terminal paths x and y such that $x\,\gamma_{s,q}\,y$. Then in this case from lemma 3.7 we get a path z, such that $x\,\gamma_{t+1,q}\,z\,\theta_{t+1,q}\,y$ with $|x|_a \leq |z|_a$ for every edge a in the graph. Then lemma 3.9 gives us $x\,\theta_{t,q}\,z$ and hence $x\,\theta_{t,q}\,y$. Thus, $\gamma_{s,q}$ refines $\theta_{t,q}$, whence it refines β and the theorem is proved. □

3.3 Consequence

We denote by $\theta^{\ell}_{1,q}$ the coarsest graph congruence generated by the following : for all loops x, y $xy = yx$ and $x = x^{q+1}$.

Proposition 3.10 *The graph congruence $\theta_{1,q}$ always refines $\theta^{\ell}_{1,q}$.*

Proof. We will show that $xyz\, \theta^{\ell}_{1,q}\, zyx$, whenever x and z are co-terminal paths and the desired result follows from that.

$$(xy)z\ \theta^{\ell}_{1,q}\ (xy)^{q+1}z = x(yx)^q(yz)\ \theta^{\ell}_{1,q}\ (xy)(zy)x(yx)^{q-1}$$

and thus,

$$xyz\ \theta^{\ell}_{1,q}\ z(yx)^2(yx)^{q-1} = z(yx)^{q+1}\ \theta^{\ell}_{1,q}\ zyx$$

Note that the bracketed entities in each step are loops that are either commuted or replicated/deleted. □

As a consequence of the arguments appearing before, we obtain the following result for the varieties $\mathbf{Com_{1,q}}$.

Lemma 3.11 *For any category $C = G^*/\beta$ in the C-variety $\ell\mathbf{Com_{1,q}}$, the graph congruence $\gamma_{1,q}$ refines β for all $q \geq 1$.*

Proof. We first prove the following

Claim: If $x\, \gamma_{1,q}\, y$, each edge a in G does not lie in any loop in path x iff it does not lie in any loop in path y. . We shall prove the right to left direction and the other direction is symmetric. Consider x in G^*. Starting from the vertex $\alpha(x)$ form a maximal sub-path denoted by x^1_s such that no edge in this sub-path lies in any loop in x. Starting from $\omega(x^1_s)$, form a maximal sub-path such that every edge in this sub-path lies in some loop in x and denote this sub-path by

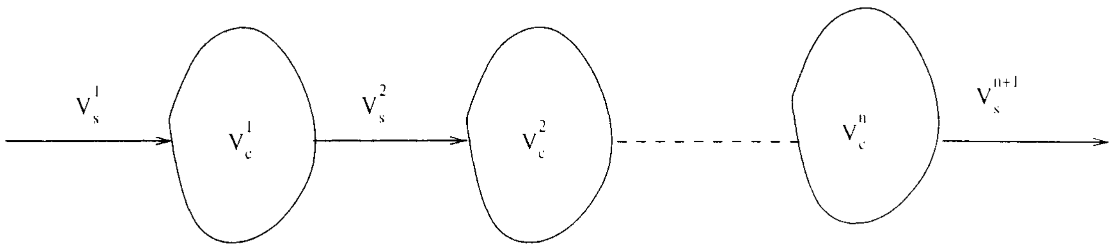


Figure 3.1: Graph G_x induced by path x

x_c^1. We keep doing this such that $\alpha(x_c^i) = \omega(x_s^i)$ and $\alpha(x_s^{i+1}) = \omega(x_c^i)$ for all $i \geq 1$, until all edge occurrences in x are covered. The subgraph induced by the edges appearing in x (or y) is denoted by G_x and looks as shown in figure 3.1. The veritces appearing in x_s^i and x_c^i are denoted by V_s^i and V_c^i respectively. It is not hard to see that no edge occurs in G_x with its endpoints in V_u^j and V_v^k respectively, where $j \neq k$ and $u, v \in \{s, c\}$. Consequently no such edge can occur in path y as graphs G_x and G_y have to be identical. On the other hand, one such edge must exist in y if it has an edge from x_s^i occurring in some loop. This establishes our claim.

Now we can convert x into x_1 and y into y_1 by replicating loops so that every edge in x or y that lies in some loop has at least two occurrences in both x_1 and y_1. For every edge a in G we compare $|x_1|_a$ with $|y_1|_a$. If $|x_1|_a > |y_1|_a$ then from the claim it follows that a lies in a loop in y_1. We replicate this loop enough number of times so that the resulting path y_2 satisfies $|y_2|_a \geq |x_1|_a$. Clearly for some i, $|y_i|_a \geq |x_1|_a$ for all a. Assigning $y' = y_i$ and $x' = x_1$ gives us $x\theta_{1,q}\, x'\, \gamma_{2,q},\, y'\, \theta_{1,q}\, y$ and applying lemma 3.9 we get $x\, \theta_{1,q}\, y$ and from proposition 3.10 above we get $x\, \theta_{1,q}^{\ell}\, y$ and thus $x\, \beta\, y$. $\square$

The arguments above along with the observation that $\mathbf{Com_{0,q}}$ is a subvariety of finite abelian groups and hence the result of example 1.2.1 applies to it, give us the following result:

Theorem 3.12 *The varieties* $\mathbf{Com_{0,q}}$ *and* $\mathbf{Com_{1,q}}$ *are local for each* $q \geq 1$.

The result above appears in [Kna78] and then in more general form in [Alm94]. Note that $\mathbf{Com_{1,1}}$ is exactly the variety $\mathbf{J_1}$ and consequently $\mathbf{J_1}$ is local.

Chapter 4

Locally commutative categories

The work presented here is primarily taken from the author's joint work with Denis Thérien that first appeared in [CT03]. In order to describe the language recognized by categories in $\ell\mathbf{Com}$, we have to introduce some new definitions. In any graph G, we call a loop that consists of a single edge a loop edge; for each path x in G^* we denote by $\overline{x}$ the path obtained from x by removing its loop edges. For a path x and a vertex v, let $x[v]$ stand for the subsequence of x consisting of all edges of the path that are incident on vertex v; note that $x[v]$ is not itself a path, and that when x is a loop $\overline{x}[v]$ has even length for each v.

4.1 Free locally commutative categories

We remind the reader that from last chapter we know that free globally commutative categories are characterised by the congruence θ_∞ , which is the coarsest congruence satisfying the equation $xyz\,\theta_\infty\, zyx$ whenever $x \sim z$.

The free locally commutative congruence on G^*, which we denote θ^ℓ_∞, is the coarsest congruence satisfying $xy\,\theta^\ell_\infty\, yx$ whenever x and y are loops on the same vertex. Obviously, θ^ℓ_∞ refines $\theta_\infty = \gamma_\infty$. We also observe that $x\,\theta^\ell_\infty\, y$ iff $|x|_a = |y|_a$ for every loop edge a and $\overline{x}\,\theta^\ell_\infty\,\overline{y}$, i.e. the presence of loop edges cannot affect the congruence relation provided they are in equal number in both paths. There is another combinatorial property that is preserved by commutation of loops; let v be a vertex such that $|xy|_a = 0$ for each loop edge a on v and such that $|xy[v]|_a \leq 1$ for each a; then the subsequence $xy[v]$ is an even permutation of the subsequence $yx[v]$. We now proceed to show that these combinatorial properties, the last one suitably modified, characterize θ^{ℓ}_{∞}.

In general, it is not the case that every edge appears at most once in a path. Suppose $|x|_a = k$; we make the k occurrences of a in x formally distinct by labelling them, in the order they appear, as $a_{\lambda(1)}, \ldots, a_{\lambda(k)}$, where λ is a permutation of $\{1, \ldots, k\}$. A *labelling* $\Lambda(x)$ of a path x is the result of applying this process to each edge. Thus the edges forming $\Lambda(x)$ can be viewed as being distinct. We will write $I(x)$ when the labelling is based on identity permutations for each edge, i.e. for each a, if $|x|_a = k$, the occurrences of a in x are renamed $a_1, \ldots, a_k$ in that order.

We define on G^* $x \, \gamma^{\ell}_{\infty} \, y$ iff $x \, \gamma_{\infty} \, y$ and there exists a labelling Λ for $\overline{y}$ such that for every vertex v the sequence $\Lambda(\overline{y})[v]$ is an even permutation of the sequence $I(\overline{x})[v]$. It can be checked that γ^{ℓ}_{∞} is a congruence relation.

We state a useful property of γ^{ℓ}_{∞}

Proposition 4.1 *Let $x = x_1 \rho x_2$ and $y = y_1 \rho y_2$ be two paths in a graph such that $x \, \gamma^{\ell}_{\infty} \, y$ and ρ is a loop on some vertex v such that for each edge a in ρ we have $|x|_a = |y|_a = 1$. Then $x_1 x_2 \, \gamma^{\ell}_{\infty} \, y_1 y_2$.*

Proof. Clearly $x_1 x_2 \, \gamma_{\infty} \, y_1 y_2$. For the second property that we need to prove, we can assume that x and y do not contain any loop edge, since this property is dealing with $\overline{x}$ and $\overline{y}$. From the definition of γ^{ℓ}_{∞} there exists a labelling function Λ such that for each vertex v, $\Lambda(y)[v]$ is an even permuation of $I(x)[v]$. But every edge that appears in ρ is unique and so Λ and I must have labelled ρ exactly the same way. Also for every vertex v, $\Lambda(\rho)[v]$ and $I(\rho)[v]$ have the same length, which is even since ρ is a loop. This implies that $\Lambda(y_1 y_2)[v]$ is an even permutation of $I(x_1 x_2)[v]$ for every vertex v, as required. $\square$

An immediate corollary follows

Corollary 4.2 *If two paths x and y satisfy $x \, \gamma_\infty^\ell \, y$ and there are n loops $\rho_1, \ldots, \rho_n$ appearing in x and y where for each edge a in a loop ρ_i we have $|x|_a = |y|_a = 1$ then the paths obtained by deleting these loops from x and y (say x' and y'), satisfy $x' \, \gamma_\infty^\ell \, y'$*

Proof. This follows by repeatedly applying proposition 4.1 once for every loop ρ_i. $\square$

Lemma 4.3 *For two paths x and y, $x \, \gamma_\infty^\ell \, y$ iff $x \, \theta_\infty^\ell \, y$.*

Proof. The implication from right to left is easy and left to the reader.

Now for the other direction we assume $x \, \gamma_\infty^\ell \, y$. Since every loop edge appears the same number of times in the two paths, its suffices to show $\overline{x} \, \theta_\infty^\ell \, \overline{y}$, so we now suppose that x and y have no loop edges. Because of the labelling involved in the definition of γ_∞^ℓ, we can think of x and y as having at most one occurrence of any edge.

We will prove our claim by induction on the length of the paths. For the base case of $|x| = 1$, the lemma is trivially true. Also note that if x and y are two coterminal paths that start with the same edge a and $x = ax'$ and $y = ay'$, then $x \; \gamma_\infty^\ell \, y$ implies $x' \, \gamma_\infty^\ell \, y'$, since the occurrence of a is unique. Thus from the inductive hypothesis we obtain $x' \, \theta_\infty^\ell \, y'$ and this proves $x \, \theta_\infty^\ell \, y$.

Assume next that x and y start with different edges. Let $x = ax_0bx_1, y = by_0ay_1, v = \alpha(x) = \alpha(y)$; If v appears in y_1, i.e. $y_1 = y_{10}y_{11}$ with $\omega(y_{10}) = v$, then we can commute by_0 and ay_{10} and we are back at the previous case. A similar argument holds if x_1 contains v. Otherwise the vertex v, which is the common end vertex of x_0 and y_0, must appear at least once more in those two subpaths because $x[v]$ is an even permutation of $y[v]$. This implies the presence of an edge c in x_0 with start vertex v. This edge also appears in y, hence must appear in y_0. We can thus use loop commuting to bring the c as first edge in each path, and so $x \, \theta^\ell_\infty \, cx' \, \gamma^\ell_\infty \, cy' \, \theta^\ell_\infty \, y$ for some x' and y' (note that this follows from the already proven fact that θ^ℓ_∞ refines γ^ℓ_∞). Now we are back to the case handled before. $\square$

The lemma above combinatorially captures the algebraic congruence θ^ℓ_∞ and so provides a tool for describing the language recognized by free locally commutative categories. But it is impossible to work directly with the congruence γ^ℓ_∞ for the case of finite categories since we have to deal with paths that are equivalent even though their lengths are different and so the concept of even permutations does not work anymore. This motivates us to find another way of characterising θ^ℓ_∞.

Consider the following special case. Let $x = ax_1yx_2zx_3a$ be a path where a is an edge which is coterminal with the subpaths y and z. One verifies that

$$x \; \theta^\ell_\infty zx_3yx_2ax_1a \; \theta^\ell_\infty \; zx_3yx_1ax_2a \; \theta^\ell_\infty \; zx_1ax_3yx_2a \; = \; x' \tag{4.1}$$

and so

$$x' \; \theta^\ell_\infty \; ax_3zx_1yx_2a \; \theta^\ell_\infty \; ax_1yx_3zx_2a \; \theta^\ell_\infty \; ax_1zx_2yx_3a \tag{4.2}$$

Thus we are able to interchange in x the coterminal subpaths y and z by using commutation of loops, because x contains an edge twice which is coterminal with these subpaths. The equivalence between exchange of coterminal paths and commutation of loops holds under a more general condition that we formalize below.

For a path x, define Γ_x^* as the reflexive and transitive closure of the relation Γ_x defined on the vertices by $v_1 \Gamma_x v_2$ whenever there is an edge a such that $|x|_a \geq 2$ and $\alpha(a) = v_1, \omega(a) = v_2$ or $\alpha(a) = v_2, \omega(a) = v_1$.

Lemma 4.4 *For any path $x = x_1x_2x_3x_4x_5$ in G^*, if $\alpha(x_2)\, \Gamma_x^*\, \omega(x_2)$ and $x_2 \sim x_4$ then $x_1x_2x_3x_4x_5\, \theta_\infty^\ell\, x_1x_4x_3x_2x_5$.*

Proof. Let $x = x_1x_2x_3x_4x_5$, $y = x_1x_4x_3x_2x_5$, $v_a = \alpha(x_2) = \alpha(x_4)$, $v_b = \omega(x_2) = \omega(x_4)$. If $v_a = v_b$, the result is immediate. Otherwise we prove the lemma by showing that the hypothesis implies $x\, \gamma_\infty^\ell\, y$. Clearly $|x|_a = |y|_a$ for each a. Consider now $\overline{x}$ and $\overline{y}$, or equivalently assume that x and y contain no loop edges. We have to show that there exists a labelling Λ which will make $\Lambda(y)[v]$ an even permutation of $I(x)[v]$ for every vertex v. Since y is obtained by interchanging subpaths of x, we get naturally from $I(x)$ a first labelling Λ for y. For each vertex $v \neq v_a, v_b$, we have that $I(x_2)[v]$ and $I(x_4)[v]$ have even length. Since $\Lambda(y)[v]$ is obtained from $I(x)[v]$ by interchanging two blocks of even length, it must be an even permutation. The problem is that $I(x_2)[v_a]$ and $I(x_4)[v_a]$ have odd length, hence the permutation $\Lambda(y)[v_a]$ is odd, and the same for v_b. Since $v_a\, \Gamma_x^*\, v_b$ there exists some $n > 0$ such that $v_a = v_0\, \Gamma_x\, v_1\, \Gamma_x\, v_2 \ldots \Gamma_x\, v_{n-1}\, \Gamma_x\, v_n = v_b$. Using the definition of Γ_x let e_i be the edge connecting v_{i-1} and v_i for $i > 0$. Each e_i is directed and its direction is arbitrary. Also there are at least two occurrences of e_i in both x and y. Let us create a new labelling Λ' that switches the labels (as given by Λ) of two arbitrarily chosen instances of e_i for each i. For all other edge occurrences, Λ'

is the same as Λ. For each of $v_1, \ldots, v_{n-1}$, $\Lambda'(y)[v_i]$ differs from $\Lambda(y)[v_i]$ by two transpositions, hence it remains even. For v_0 and for v_n, the difference between Λ and Λ' is one transposition, hence these become even as well. □

An edge e in a path x is called a *special edge for* x iff $\alpha(e)$ and $\omega(e)$ are not related by Γ_x^*. A maximal subpath in a path x that is completely contained inside an equivalence class of Γ_x^* is called a *component of* x. So special edges always connect components that are over different equivalence classes of Γ_x^*. Note that a component could consist of just the identity path in which case two special edges would be adjacent to each other. Clearly every special edge occurs exactly once in a path x. Every path x in G^* is thus now uniquely decomposed as $x_0 e_1 x_1 \ldots e_n x_n$, where the e_i's are the special edges for x and the x_i's are its components. The lemma above then gives the following result

Corollary 4.5 *If a path x has no special edges then for any path y, $x\,\theta_\infty^\ell\,y$ iff $x\,\theta_\infty\,y$ iff $x\,\gamma_\infty\,y$.*

In order to take into account the presence of special edges, we define, for each path x, a *reduced graph* $G_x = (V_x, A_x, \alpha_x, \omega_x)$ where $V_x = V/\Gamma_x^*$, A_x is the set of special edges for x, and α_x, ω_x are defined in the obvious way. The path x induces a path $Red(x)$ in the graph G_x by taking $Red(x)$ to be the sequence of special edges in the order they appear in x. Note that $Red(x)$ is a permutation of A_x and that $x\,\gamma_\infty\,y$ implies that $\Gamma_x^* = \Gamma_y^*$, hence that the graphs G_x and G_y are identical; furthermore we then have that $Red(x) \sim Red(y)$ in this graph.

We now define a congruence on G^* by $x\,\delta_\infty^\ell\,y$ iff $x\,\gamma_\infty\,y$ and $Red(x)\,\gamma_\infty^\ell\,Red(y)$.

Lemma 4.6 *For two paths x and y in G if $x\,\delta_\infty^\ell\,y$ and $Red(x) = Red(y)$ then $x\,\theta_\infty^\ell\,y$.*

Proof. Let $x = x_0e_1x_1 \dots e_nx_n$, $y = y_0e_1y_1 \dots e_ny_n$. Observe that this forces $x_i \sim y_i$ for each i. Fix an equivalence class C in V/Γ_x^* and let $0 \leq i_0 < i_1 < \dots, i_t \leq n$ be the indices for which x_{i_j} is a component of x over C; the same sequence of indices gives the components of y that are over C. For each j replace the subpath of x between $x_{i_{j-1}}$ and x_{i_j} by a "meta-edge" E_j that goes from $\omega(x_{i_{j-1}})$ to $\alpha(x_{i_j})$. Do the same for y. Consider the paths $X = x_{i_0}E_1x_{i_1} \dots E_tx_{i_t}$ and $Y = y_{i_0}E_1y_{i_1} \dots E_ty_{i_t}$. We have that $X \,\gamma_\infty\, Y$ and these two paths now have no special edges since the two endpoints of each E_j are in C. By corollary 4.5, X can be transformed into Y by commuting loops. The corresponding sequence of operations will transform x into a path $x' = x'_0e_1x'_1 \dots e_nx'_n$ where $x'_i = y_i$ for $i \in \{i_0, \dots, i_t\}$ and $x'_i = x_i$ otherwise. Doing this for each class of V/Γ_x^* in turn will transform x into y. □

We are now in a position to prove the equivalence of δ_∞^ℓ and θ_∞^ℓ.

Lemma 4.7 *For any two paths x and y in G $x \,\delta_\infty^\ell\, y$ iff $x \,\theta_\infty^\ell\, y$.*

Proof. The implication from right to left is easy and left to the reader. We prove the second implication. Suppose $x = x_0e_1x_1 \dots e_nx_n$; We fix in each equivalence class C of V/Γ_x^* a vertex v_C, and for each special edge e_i going from vertex v in C to a vertex v' in C', we augment the graph G by introducing four new edges: e_i^C going from v to v_C, f_i^C going from v_C to v, $g_i^{C'}$ going from v' to $v_{C'}$ and $h_i^{C'}$ going from $v_{C'}$ to v'. We create from x a new path x' in the augmented graph by the following process: if e_j is a special edge for x going from vertex v in C to a vertex v' in C', we replace e_j by $e_j^C f_j^C e_j g_j^{C'} h_j^{C'}$. If any loop edges have been added we remove them. We create y' from y similarly. $Red(x') \,\gamma_\infty^\ell\, Red(y')$ comes trivially from the fact that $x \,\delta_\infty^\ell\, y$ (since $Red(x) = Red(x')$ and $Red(y) = Red(y')$). Hence also $Red(x') \,\theta_\infty^\ell\, Red(y')$ by lemma 4.3. By construction, if there is a loop on vertex C appearing in $Red(x')$ in the reduced graph, there is a corresponding loop on vertex v_C appearing in x' in the augmented graph. Thus, corresponding to the sequence of loop commutations that transforms $Red(x')$ to $Red(y')$ in the reduced graph, there is a sequence of loop transformations, in the augmented graph, that transforms x' into a path (say w) in which the special edges appear in the same order as those of y'. Hence using lemma 4.6 it follows $x' \, \theta^\ell_\infty \, w \, \theta^\ell_\infty \, y'$. So $x' \, \gamma^\ell_\infty \, y'$ and by recalling that we obtained x' (y') from x (y) by adding a certain number of loops around every vertex v_C we apply proposition 4.1 and corollary 4.2 to get $x \, \gamma^\ell_\infty \, y$. Hence $x \, \theta^\ell_\infty \, y$. □

Thus δ^ℓ_∞ provides an alternative characterisation of locally commutative free categories. We will see in the next section that this characterisation can be naturally adapted to the case of finite categories.

4.2 Locally commutative finite categories

We recall from chapter 3 that the algebraic description of finite globally commutative categories is given by a path congruence $\theta_{t,q}$ generated by equations: $xyz \, \theta_{t,q} \, zyx$ for $x \sim z$ and $x^t \, \theta_{t,q} \, x^{t+q}$ where x is a loop.

The corresponding combinatorial congruence $\gamma_{r,q}$ is induced by relations: for $x \sim y$ we say $x \, \gamma_{r,q} \, y$ iff for all edge $a \in A$, either ($|x|_a$, $|y|_a < r$ and $|x|_a = |y|_a$) or ($|x|_a$, $|y|_a \geq r$ and $|x|_a \equiv_q |y|_a$)

The main result from the last chapter can obviously be restated in the form below:

Lemma 4.8 *For every $t \geq 0$ and graph G there exists s such that for two paths x and y, $x \, \gamma_{s,q} \, y$ implies $x \, \theta_{t,q} \, y$.*

As an extension of ideas from free locally commutative categories we introduce $\theta^{\ell}_{t,q}$ to be the finite index path congruence generated by the conditions: $xy\,\theta^{\ell}_{t,q}\,yx$ where x and y are loops and $x^t\,\theta^{\ell}_{t,q}\,x^{t+q}$ where x is a loop. Analogous to the global case we write $x \preceq^{\ell\theta}_{t,q} y$, when $x\,\theta^{\ell}_{t,q}\,y$ and y can be obtained from x by just loop commuting and loop replication.

We also extend our combinatorial characterisation from the last section to $\delta^{\ell}_{t,q}$ meaning for two paths x and y $x\,\delta^{\ell}_{t,q}\,y$ iff $x\,\gamma_{t,q}\,y$ and $Red(x)\,\gamma^{\ell}_{\infty}\,Red(y)$, where γ^{ℓ}_{∞} gets defined on the reduced graph G_x. Note that this congruence only depends on the permutation of reduced paths which are of fixed length.

Using the definition of $\theta^{\ell}_{t,q}$ and the lemma 4.4 we can conclude the following:

Corollary 4.9 *For paths with no special edges, $\theta_{t,q}$ and $\theta^{\ell}_{t,q}$ are equivalent.*

This corollary along with lemma 4.8 gives us the intuition to expect the following result

Lemma 4.10 *If $x\,\delta^{\ell}_{s,q}\,y$ and $Red(x) = Red(y)$ then $x\,\theta^{\ell}_{t,q}\,y$ where s and t are related according to lemma 4.8.*

Proof. We direct the attention of the reader to the proof of lemma 4.6. Employing exactly the same technique as in that proof, fixing an equivalence class C in V/Γ^*_x we add "meta-edges" connecting two successive components of that class and obtain paths X and Y respectively from x and y. In our case here, $X\,\gamma_{s,q}\,Y$. Therefore using lemma 4.8 it follows $X\,\theta_{t,q}\,Y$ and since X and Y have no special edges from corollary 4.9 X can be transformed into Y by transformations preserving $\theta^{\ell}_{t,q}$. We apply the same operations on x to get a new path x' and then repeat the procedure with x' for each class of V/Γ^*_x to finally get y. $\square$

We can now combine the proof of lemma 4.10 and lemma 3.9 from the last chapter to obtain the following corollary

Corollary 4.11 *If $x\,\delta^{\ell}_{t+1,q}\,y$ and $Red(x) = Red(y)$ with $|x|_a \leq |y|_a$, then $x \leq^{\ell 0}_{t,q} y$.*

Lemma 4.12 *For every $t \geq 2$ and $q \geq 1$, there exists $R \geq t+1$ such that $x\,\delta^{\ell}_{R,q}\,y$ implies that there exists a path ρ satisfying $x\,\delta^{\ell}_{t+1,q}\,\rho$, where $\rho\,\theta^{\ell}_{t,q}\,y$ and for all edges $a \in A$, $|x|_a \leq |\rho|_a$.*

Proof. We will use lemma 3.6 and 3.7 from the last chapter to prove this. Specifically let $R = m(G, t+1)(|E|+1)+1$ where $m(G, t+1) = |V| + (t+1)(2^{|E|}-1)+2$ as defined in lemma 3.7. So for each edge a such that $|x|_a > |y|_a$ we have $|y|_a \geq R$ and since y can have at most $(|E|+1)$ components there is at least one component that has at least $m(G, t+1)$ occurences of a. We can now straight away apply the argument used to prove lemma 3.7 in the last chapter and obtain the result of the present lemma. □

Lemma 4.13 *If for two paths x and y, $|x|_a \leq |y|_a$ for all $a \in A$ and $x\,\delta^{\ell}_{t+1,q}\,y$, then $x\,\theta^{\ell}_{t,q}\,y$ for $t \geq 2$ and $q \geq 1$.*

Proof. We ask the reader to recall the technique used to prove lemma 4.7. We mimic the steps in that proof to augment the graph G by introducing four new edges for each special edge e_i and then modify the paths x and y to x' and y' respectively as prescribed there. (Note: we are using the same notation as in that proof.) Also let A' represent the set of edges of the augmented graph. The same argumentation of the earlier proof carries over to establish the existence of a path w such that $Red(w) = Red(y')$ and $x'\,\theta^{\ell}_{\infty}\,w\,\delta^{\ell}_{t+1,q}\,y'$. From corollary 4.11

it follows that $w \leq^{\ell\theta}_{t,q} y'$ and hence $x' \leq^{\ell\theta}_{t,q} y'$. This implies that there exists a series of loop commuting and loop duplicating transformations to obtain y' from x'. Let the loops that got duplicated, be called $\rho_1, \ldots, \rho_n$ and let them be around vertices $v_1, \ldots, v_n$ in G respectively. Also let n_i be the number of times ρ_i was duplicated. It is a trivial observation that every vertex v_i occurs somewhere in the path x and every loop ρ_i contains edges strictly from the unaugmented original graph G (since for each edge $a \in A' \backslash A$ we have $|x'|_a = |y'|_a$). Also no loop ρ_i contains any special edges as their count is one in both x' and y'. Hence every loop ρ_i could be added n_i times to path x to obtain a path u in G^* such that $Red(x) = Red(u)$ and hence $u\, \delta^{\ell}_{\infty}\, y$. This implies $x\, \delta^{\ell}_{t+1,q}\, u$ and hence from corollary 4.11 we have $x\, \theta^{\ell}_{t,q}\, u$. Now applying lemma 4.7 to u and y we get $u\, \theta^{\ell}_{\infty}\, y$ and hence $x\, \theta^{\ell}_{t,q}\, y$. □

We now state the main result.

Theorem 4.14 *β is a ℓ**Com**-congruence iff there exists $R \geq 2$, $q \geq 1$ such that $\delta^{\ell}_{R,q} \subseteq \beta$.*

Proof. The direction from right to left is trivial and is left as an exercise for the reader (it can be verified that $\delta^{\ell}_{R,q}$ is a ℓ**Com**-congruence). For $t \geq 2$ we choose $R = m(G, t+1)(|E|+1)+1$ according to lemma 4.12. Then $x\, \delta^{\ell}_{R,q}\, y$ implies there exists a path z with $|x|_a \leq |z|_a$ for each edge $a \in A$ and $x\, \delta^{\ell}_{t+1,q}\, z\, \theta^{\ell}_{t,q}\, y$. Using lemma 4.13 on x and z, we get $x\, \theta^{\ell}_{t,q}\, y$. Recall that for cases $t = 0$ and $t = 1$, we have shown in the last chapter $\ell\mathbf{Com_{0,q}}$ and $\ell\mathbf{Com_{1,q}}$ coincide with $\mathbf{gCom_{0,q}}$ and $\mathbf{gCom_{1,q}}$ respectively. □

4.3 Consequences

In this section, we sketch some consequences of the combinatorial description obtained above. When an M-variety $\mathbf{V}$ is such that the C-varieties $\mathbf{gV}$ and $\ell\mathbf{V}$ differ, then $\ell\mathbf{V}$ cannot be equal to $\mathbf{gW}$ for any M-variety $\mathbf{W}$. This is because if we restrict to one-node categories in $\mathbf{gW}$, we precisely get the monoids of $\mathbf{W}$ and this set is different from the set of monoids in $\mathbf{V}$. How big should $\mathbf{W}$ be to insure $\ell\mathbf{V} \subset \mathbf{gW}$? In example 1.2.2 of chapter 1, we observed that for the trivial M-variety we have $\ell\mathbf{1} \subset \mathbf{gW}$ for every non-trivial $\mathbf{W}$. We now argue that a similar phenomenon occurs for $\mathbf{Com}$.

Theorem 4.15 $\ell\mathbf{Com} \subset \mathbf{gW}$ *for every M-variety* $\mathbf{W}$ *that strictly contains* $\mathbf{Com}$.

Proof. Our main result shows that in every locally commutative category, the value of a path is determined by the number of occurrences of each edge (threshold t, modulo q for some $t \geq 0, q \geq 1$) and the ordering of the so-called "special" edges. The first condition can be determined by using for each edge a cyclic counter of appropriate cardinality. For the second condition, let M be any non-commutative monoid, i.e. M contains two elements m and m' such that $mm' \neq m'm$. Fix two edges of the graph, a and b, map a to m, b to m' and every other edge to 1. If a path x contains at most one occurrence of each of a and b, which is necessarily the case when these two edges are special for x, the value of the path in M is in $\{1, m, m', mm', m'm\}$. In particular if both edges occur once, the order in which they appear can be recovered from the value in the monoid. If the graph has k edges, we can use the direct product of k cyclic counters to count occurrences of each edge, and $O(k^2)$ copies of M, one for each pair of edges. The value of the counters will determine the first condition and also which edges are special for a given path; we can then look up the appropriate copies of M to know in which order the special edges have appeared, hence recover the $\delta^{\ell}_{t,q}$-value of the path. $\Box$. $\Box$

Next, we transfer the last theorem to the S-variety $\mathbf{LCom} = \{S : eSe \in \mathbf{Com}$ for all $e = e^2\}$. For any semigroup S, consider the graph $G = (V, A, \alpha, \omega)$, where V is the set of idempotents of S, $A = V \times S \times V, \alpha(e, s, f) = e, \omega(e, s, f) = f$. Define the congruence β on G^* by identifying co-terminal paths that multiply out to the same element in S. This construction trivially insures that $S \in \mathbf{LCom}$ iff $G^*/\beta \in \ell\mathbf{Com}$. It follows from work of [Str85][1] that $S \in \mathbf{V} * \mathbf{D}$, where $\mathbf{D} = \{S : Se = e$ for all $e = e^2\}$ and $*$ denotes wreath product of varieties, iff $G^*/\beta \in \mathbf{gV}$. We thus get the following

Theorem 4.16 $\mathbf{LCom} \subset \mathbf{V} * \mathbf{D}$ *for every M-variety* $\mathbf{V}$ *that strictly includes the commutative monoids.*

[1]The delay theorem in [Til87] gives this in the language of categories

Chapter 5

Locally $\mathcal{R}, \mathcal{L}, \mathcal{J}$-trivial categories

For any local M-variety $\mathbf{V}$, trivially

$$\ell\mathbf{V} \subset \mathbf{gW} \tag{5.1}$$

where $\mathbf{W}$ is a M-variety that strictly contains $\mathbf{V}$. As we have seen, equation 5.1 is also true when $\mathbf{V}$ is substituted either with the non-local variety $\mathbf{1}$ or $\mathbf{Com}$. A natural question therefore is whether this equation is always true. There is another famous case of an M-variety for which the induced global and local C-varieties are different, namely the variety $\mathbf{J}$ of $\mathcal{J}$-trivial monoids. However, Jorge Almeida has pointed out to us that there exists a C-variety $\mathbf{gV}$ where $\mathbf{V}$ is a M-variety of aperiodic monoids that strictly contains $\mathbf{J}$, such that $\mathbf{gV}$ does not contain $\ell\mathbf{J}$, and hence we conclude that equation 5.1 is not in general true. The main result in this chapter expresses the variety $\ell\mathbf{J}$ in terms of globally defined C-varieties $\mathbf{gR}$ and $\mathbf{gL}$, where $\mathbf{R}$ and $\mathbf{L}$ are the M-varieties of $\mathcal{R}$-trivial and $\mathcal{L}$-trivial monoids respectively. We first show that $\mathbf{R}$ is local. An old result of Stiffler from the 70's (in [Sti73]) along with the application of the Delay Theorem of [Til87] implies this[1]. [Alm96] presents a general argument using

[1] the interested reader may note that Stiffler actually proves the stronger result that each element of $\mathbf{LR}$ divides a semidirect product of semilattices and right zero semigroups, where

profinite techniques that shows both $\mathbf{DA}^2$ and $\mathbf{R}$ to be local. More recently, [Ste] has given an algebraic proof of results of Stiffler using derived category of morphisms between categories[3] . Our method is quite elementary and uses graph congruences.

5.1 Preliminaries about Green's relations

There are excellent books like [Alm94]and [Pin86] giving an extended treatment of these and we recommend the interested reader to go through them. We try to give here a quick overview of these relations that play a central role in understanding the structure of finite semigroups and monoids. We will here assume that we are always dealing with monoids, noting that with very little extra effort, these could be extended to semigroups.

For any monoid M, we say that $x, y \in M$ are $\mathcal{R}$-related to each other, denoted by $x \mathcal{R} y$, iff they generate the same right ideal i.e. $xM \equiv yM$. Dually, we say that $x \mathcal{L} y$ iff $Mx \equiv My$. Clearly, both $\mathcal{R}$ and $\mathcal{L}$ are reflexive, symmetric and transitive and hence are equivalence relations. Their equivalence classes are called $\mathcal{R}$ and $\mathcal{L}$ classes respectively. Similarly we can induce the equivalence relation $\mathcal{J}$ from two-sided ideals, by saying $x \mathcal{J} y$ iff $MxM \equiv MyM$. Note that both $\mathcal{R}$ and $\mathcal{L}$ refine $\mathcal{J}$. Moreover, it can be shown that in a *finite monoid* the following three things are equivalent:

1. $x \mathcal{J} y$

2. there exists an element z in M such that $x \mathcal{R} z$ and $z \mathcal{L} y$

one can first do the right zero semigroups and then the semilattices

[2] Variety of all monoids whose regular $\mathcal{D}$-classes are aperiodic semigroups

[3] we direct the interested reader to [STar] for a description of the derived category of morphisms between categories

3. there exists an element z in M such that $x \mathcal{L} z$ and $z \mathcal{R} y$

If $xM \subseteq yM$, then we write $x \leq_{\mathcal{R}} y$. Obviously, $x \mathcal{R} y$ iff $x \leq_{\mathcal{R}} y$ and $y \leq_{\mathcal{R}} x$. From the definition of the relation $\mathcal{R}$, it should be clear that $x \mathcal{R} y$ iff there exist v and w such that $x = yv$ and $y = xw$. A monoid is said to be $\mathcal{R}$-trivial iff each of its $\mathcal{R}$-class contains a single element. Similarly, one defines $\mathcal{L}$ and $\mathcal{J}$-trivial monoids: they mean respectively that each $\mathcal{L}$ and $\mathcal{J}$-class, contains a single element. It is a simple observation that every $\mathcal{J}$-trivial monoid is both $\mathcal{L}$-trivial and $\mathcal{R}$-trivial. It can be shown that the set of all $\mathcal{R}$-trivial and finite monoids forms a M-variety, denoted by $\mathbf{R}$ and is characterised by the identities $(xy)^n x = (xy)^n$ and $x^n = x^{n+1}$ for some $n \geq 1$. Similarly, the variety of all $\mathcal{J}$-trivial monoids is characterized by the identities : for some $n \geq 1$ we have $(xy)^n = (yx)^n$ and $x^n = x^{n+1}$.

5.2 Languages of $\mathcal{R}$-trivial and $\mathcal{L}$-trivial monoids

We describe the languages recognized by finite $\mathcal{R}$-trivial monoids as explained in [Fic79]. We introduce the notion of *subwords* by saying x is a *subword* of y iff there exists $x_1, \ldots, x_n, u_0, \ldots, u_n$ and y can be factorised as $u_0 x_1 u_1 \ldots x_n u_n$, where $x = x_1 \ldots x_n$. For each word x, let $\mu_n(x)$ denote the set of all subwords of length at most n of x i.e. $\mu_n(x) = \{v : v \text{ is a subword of } x \text{ and } |v| \leq n\}$. We also introduce two related congruences on Σ^*, one denoted by $\sim_n$ and the other $\sim_{n,R}$. For two words x, y, $x \sim_n y$ iff $\mu_n(x) = \mu_n(y)$ i.e. x and y have the same set of subwords upto length n.
$x \sim_{n,R} y$ iff for every prefix x' of x there exists a prefix y' of y such that $x' \sim_n y'$ and vice-versa. In other words, new subwords appear in x and y in the same order, if we scan the words from left to right. Note that we can symmetrically

define the congruence $\sim_{n,L}$ by saying that $x \sim_{n,L} y$ iff for every *suffix* x' of x there exists a *suffix* y' of y such that $x' \sim_n y'$ and vice-versa. In this case we are looking for the order in which subwords appear from right to left. It is a straight-forward exercise to verify that $\sim_n$, $\sim_{n,R}$ and $\sim_{n,L}$ are all congruences of finite index and that both $\sim_{n,R}$ and $\sim_{n,L}$ refine $\sim_n$. We note some useful properties of these congruences[4]:

Lemma 5.1 *For arbitrary two words u and v in Σ^*, $u \sim_n uv$ iff u can be split into n pieces $u_1, \ldots, u_n$ where $\lambda(u_1) \supseteq \lambda(u_2) \supseteq \ldots \supseteq \lambda(u_n) \supseteq \lambda(v)$ and $u = u_1 u_2 \ldots u_n$.*

Lemma 5.2 *$u \sim_{n,R} uv$ iff $u \sim_n uv$, where u and v are arbitrary words.*

We then quote the following result from [Fic79].

Theorem 5.3 *A language L is recognized by a finite $\mathcal{R}$-trivial ($\mathcal{L}$-trivial) monoid iff there exists an integer $n > 0$ such that L is a union of congruence classes of $\sim_{n,R}$ ($\sim_{n,L}$).*

5.3 Languages recognized by locally $\mathcal{R}$-trivial categories

We can now induce a graph congruence γ_n^R on G^* from the congruence $\sim_{n,R}$ on A^*, where $G = (V, A)$. We define $x \, \gamma_n^R \, y$ iff x and y are co-terminal paths in G^* and $x \sim_{n,R} y$ when viewed as words in A^*. Let $\theta_t^{\ell\mathcal{R}}$ denote some graph congruence such that the category $C \equiv G^*/\theta_t^{\ell\mathcal{R}}$ is locally finite, every base monoid of the category is $\mathcal{R}$-trivial and the cardinality of the largest such monoid is t. The result we want to show here is the following:

[4]proofs of these could be found in [Pin86]

Lemma 5.4 *For every graph G, given t and some congruence $\theta_t^{\ell\mathcal{R}}$, there exists s such that γ_s^R refines $\theta_t^{\ell\mathcal{R}}$, where s and t are non-zero positive integers.*

A path e in G^* is called *idempotent* with respect to $\theta_t^{\ell\mathcal{R}}$ iff $e\,\theta_t^{\ell\mathcal{R}}\,ee$. For the sake of brevity we will simply say e is idempotent without mentioning the graph congruence, whenever the associated congruence is clear from the context. e, e', e'' would always denote such idempotent paths unless noted otherwise. Note that idempotent paths are always loops.

Lemma 5.5 *$es\,\theta_t^{\ell\mathcal{R}}\,e$, whenever s is a loop coterminal with e and $\lambda(s) \subseteq \lambda(e)$.*

Proof. We will first consider the case where s is a subword of e. Let $s = a_1a_2...a_n$, where each a_i is an edge in G. Then we can write $e = y_0a_1y_1a_2y_2...y_{n-1}a_ny_n$. Here y_i is a loop around the end vertex of a_i, for $i \geq 1$. Since e is an idempotent we get

$$
\begin{aligned}
e \quad & \theta_t^{\ell\mathcal{R}} \quad y_0a_1y_1a_2y_2\ldots y_{n-1}a_ny_ny_0a_1y_1a_2y_2\ldots y_{n-1}a_ny_n \\
e \quad & \theta_t^{\ell\mathcal{R}} \quad ey_0v_1
\end{aligned}
\tag{5.2}
$$

where $v_1 = a_1y_1a_2\ldots a_ny_n$. Note that e, y_0 and v_1 are all loops around the same vertex $v = \alpha(e)$. Hence using equation 5.2 and the $\mathcal{R}$-triviality of the base monoid M_v one gets

$$e \quad \theta_t^{\ell\mathcal{R}} \quad ey_0 \tag{5.3}$$

Using the idempotence of e and equation 5.3, one sees $es\,\theta_t^{\ell\mathcal{R}}\,eey_0s$ and then expanding

$$
\begin{aligned}
es \quad & \theta_t^{\ell\mathcal{R}} \quad y_0a_1y_1\dots y_{n-1}a_ny_n\,y_0a_1y_1\dots y_{n-1}a_ny_ny_0a_1a_2\dots a_n \\
es \quad & \theta_t^{\ell\mathcal{R}} \quad y_0a_1w^2a_2\dots a_n
\end{aligned} \tag{5.4}
$$

where $w = y_1a_2\dots y_{n-1}a_ny_ny_0a_1$ is a loop around the end vertex of a_1.

Again using the idempotence of e and equation 5.3 one can write

$$
\begin{aligned}
w^2 \quad & = \quad y_1a_2\dots y_{n-1}a_ny_ney_0a_1 \\
w^2 \quad & \theta_t^{\ell\mathcal{R}} \quad y_1a_2\dots y_{n-1}a_ny_ney_0ey_0a_1 \\
w^2 \quad & \theta_t^{\ell\mathcal{R}} \quad w^3
\end{aligned} \tag{5.5}
$$

Using equation 5.5 one can write $w^2\,\theta_t^{\ell\mathcal{R}}\,w^2y_1v_2$, where $v_2 = a_2y_2\dots y_{n-1}a_ny_ny_0a_1$. Note that w^2, y_1 and v_2 are all loops around the end vertex of a_1. Since the base monoid induced by $\theta_t^{\ell\mathcal{R}}$ around this vertex is $\mathcal{R}$-trivial, it follows that

$$
w^2 \quad \theta_t^{\ell\mathcal{R}} \quad w^2y_1 \tag{5.6}
$$

Now considering equation 5.4 and 5.6 one gets

$$
\begin{aligned}
es \quad & \theta_t^{\ell\mathcal{R}} \quad y_0a_1w^2y_1a_2\dots a_n \\
es \quad & = \quad y_0a_1y_1a_2v^2a_3\dots a_n
\end{aligned} \tag{5.7}
$$

where $v = y_2a_3\dots y_{n-1}a_na_1y_1a_2$ and using very similar arguments as above with the $\mathcal{R}$-triviality of the monoid around the start vertex of v, we get $v^2 = v^2y_2$. Applying this to equation 5.7 one obtains

$$
\begin{aligned}
es & \;\theta_t^{\ell\mathcal{R}}\; y_0a_1y_1a_2v^2y_2a_3\ldots a_n \\
es & = y_0a_1y_1a_2y_2a_3u^2a_4\ldots a_n
\end{aligned}
$$

where $u = y_3a_4\ldots a_ny_ny_0\ldots a_3$. We need to apply this argument $|s|+1$ times, using the $\mathcal{R}$-triviality of base monoids around every vertex that appears in path s. This finally yields $es\,\theta_t^{\ell\mathcal{R}}\,eee\,\theta_t^{\ell\mathcal{R}}\,e$ and we are done. □

We use the above result to obtain the following:

Lemma 5.6 *For every two paths u, v in G^* such that v is a loop around $\omega(u)$, $u\,\gamma_s^R\,uv$ implies $u\,\theta_t^{\ell\mathcal{R}}\,uv$ where $s = 2(t+1)$.*

Proof. If v is empty then we have nothing to prove. For any non-empty v, applying lemma 5.1 to the fact $u\,\gamma_s^R\,uv$, we find that u can be split up into s pieces $u_1, \ldots, u_s$ such that $\lambda(u_1) \supseteq \lambda(u_2) \supseteq \ldots \supseteq \lambda(u_s) \supseteq \lambda(v)$. Since v is non-empty, let $v = av'$ for some edge a. Each u_i thus contains at least one occurence of a. We factorise each u_i as $u_{i,0}au_{i,1}$, where $u_{i,0}$ does not contain any a. Let us also denote by v_1 the path $u_{1,1}u_2u_{3,0}$, $v_2 = u_{3,1}u_4u_{5,0}$ and so on. In general $v_i = u_{j,1}u_ku_{l,0}$, where $k = j+1$, $l = j+2$ and $j = (2i-1)$, for $1 \leq i \leq t$. Note that we make $v_{t+1} = u_{2t+1,1}u_{2t+2}$ or in other words $v_{t+1} = u_{s-1,0}u_s$. Thus, we can write

$$
u = u_{1,0}av_1av_2\ldots av_{t+1}
$$

where each av_i is a loop around the start vertex of a and this is also the start vertex of v. Let M denote the base monoid induced by $\theta_t^{\ell\mathcal{R}}$ at this vertex. Also note that $\lambda(av_i) \supseteq \lambda(av_j)$, where $i < j$. Since the cardinality of M is bounded by t, we can apply the pigeon-hole principle to conclude that there exists k, l such that

$$av_1 \dots av_k \quad \theta_t^{\ell\mathcal{R}} \quad av_1 \dots av_k av_{k+1} \dots av_l \tag{5.8}$$

where $k < l$. Let $w = av_{k+1} \dots av_l$. Then using equation 5.8 one can write

$$uv \quad \theta_t^{\ell\mathcal{R}} \quad u_{1,0} av_1 \dots av_k w^\omega av_{l+1} \dots av_s v \tag{5.9}$$

where w^ω is an idempotent and is denoted by e and let $w' = av_{l+1} \dots av_s$. Clearly $\lambda(e) \supseteq \lambda(w'v)$. Note that w, e, w' and v are all loops around the same vertex. Let $u' = u_{1,0} av_1 \dots av_k$. Using lemma 5.5, we conclude

$$uv \, \theta_t^{\ell\mathcal{R}} \, u'ew'v \, \theta_t^{\ell\mathcal{R}} \, u'e \, \theta_t^{\ell\mathcal{R}} \, u'ew' \, \theta_t^{\ell\mathcal{R}} \, u'ww' = u$$

□

Lemma 5.7 *Let ρ, x and y be paths in G^*, such that $\rho x \, \gamma_s^R \, \rho y$ and $\mu_s(\rho) = \mu_s(\rho x) = \mu_s(\rho y)$. Then $\rho x \, \theta_t^{\ell\mathcal{R}} \, \rho y$, where $s = 2(t+1)$.*

Proof. We will show this by induction on $|x|$. Let the base case be $|x| = 0$. In this case y is a loop at the end of ρ with $\rho \, \gamma_s^R \, \rho y$. Using directly lemma 5.6, we have $\rho \, \theta_t^{\ell\mathcal{R}} \, \rho y$. Now as our induction hypothesis (IH) we assume it is true for all x, such that $|x| < k$. Let $|x| = k$. We write $x = ax'$ and $y = by'$. If $a = b$, then applying IH on x' we get our desired result. Assume $a \neq b$. In this case, since $\mu_s(\rho a) = \mu_s(\rho)$, applying lemma 5.1 we split ρ into s paths $\rho_1, \dots, \rho_s$, each of which contains at least one occurence of a. Thus, we can write $\rho_s = \rho_{s,0} a \rho_{s,1}$. Observe that $a\rho_{s,1}$ is a loop at the end of ρ and

$$\mu_s(\rho) \quad = \quad \mu_s(\rho a \rho_{s,1}) \tag{5.10}$$

Hence applying lemma 5.6, we get $\rho\, \theta_t^{\ell\mathcal{R}}\, \rho a \rho_{s,1}$. This means that

$$\rho y \quad \theta_t^{\ell\mathcal{R}} \quad \rho a \rho_{s,1} y \tag{5.11}$$

It follows from equation 5.10 that $\rho y\, \gamma_s^R\, \rho a \rho_{s,1} y$. Assigning $\rho' = \rho a$ and noting that $\rho x = \rho' x'$, we can apply our induction hypothesis to x' obtaining $\rho x\, \theta_t^{\ell\mathcal{R}}\, \rho' \rho_{s,1} y\, \theta_t^{\ell\mathcal{R}}\, \rho y$. This completes the argument. □

We are now in a position to prove our main lemma 5.4.

Proof. Let $s = 2(t+1)$. Assume for x, y in G^*, $x\, \gamma_s^R\, y$. Let ρ_0 be the longest common prefix of x and y. Note that $|\rho_0| \geq s$. Let $y_0 = y$. Our strategy would be to find a series of paths $y_1, \ldots, y_n$ such that $y_i\, \gamma_s^R\, y_{i+1}$ and $y_i\, \theta_t^{\ell\mathcal{R}}\, y_{i+1}$. Let ρ_i denote the longest common prefix of x and y_i. We will satisfy $|\rho_{i+1}| \geq |\rho_i| + 1$. n is thus an integer such that $\rho_n = x$. Thus applying lemma 5.6 at the end to x and y_n we would get our desired result.

Therefore, the argument boils down to showing, given y_i, we can always obtain y_{i+1} satisfying the above conditions. Let $x = \rho_i a x'$ and $y_i = \rho_i b y'$. If $a = b$ we have nothing to show. Assume $a \neq b$. We split up the argument into following cases.

1. a adds new subwords to ρ_i, i.e. $\mu_s(\rho_i a) \supset \mu_s(\rho_i)$. This means b cannot add any subwords to ρ_i. Also in this case y' can be factorised as $\nu_0 a \nu_1$, where $b\nu_0$ is a loop at the end of ρ_i such that

$$\mu_s(\rho_i b\nu_0) = \mu_s(\rho_i) \tag{5.12}$$

Using lemma 5.6 we obtain

$$y_i = \rho_i b\nu_0 a\nu_1 \; \theta_l^{\ell\mathcal{R}} \; \rho_i a\nu_1 = y_{i+1} \tag{5.13}$$

Thus assigning $\rho_{i+1} = \rho_i a$, and using equations 5.12 and 5.13 we see that y_{i+1} satisfies all desired conditions.

2. b adds new subwords to ρ_i. In this case a cannot add any new subwords to ρ_i and x' could be factorised into $\nu_0 b\nu_1$. Mimicking arguments given above, we see that setting $y_{i+1} = \rho_i a\nu_0 by'$ and $\rho_{i+1} = \rho_i a\nu_0 b$ does the job.

3. Neither a nor b adds any new subwords to ρ_i. In this case we take the largest possible prefix of x' denoted by ν_x such that $\mu_s(\rho_i\nu_x) = \mu_s(\rho_i)$. If ν_x is the whole of x', then obviously y' cannot add any new subwords to ρ_i either and using the lemma 5.7 we get $x \; \theta_l^{\ell\mathcal{R}} \; y_i$ and we are done. Otherwise, we write $x' = \nu_x c\nu_x'$. This means y' can be factorised as $y' = \nu_y c\nu_y'$ with $\mu_s(\rho_i\nu_y) = \mu_s(\rho_i)$. We can again apply lemma 5.7 to obtain $\rho_i\nu_x \; \theta_l^{\ell\mathcal{R}} \; \rho_i\nu_y$. Finally we are done by setting $y_{i+1} = \rho_i\nu_x c\nu_y$ and $\rho_{i+1} = \rho_i\nu_x c$.

□

We can state the result given above now in terms of languages as

Corollary 5.8 *Let $G = (V, A)$ be a graph. Then the syntactic category of a language $L \subseteq G^*$ is finite and locally $\mathcal{R}$-trivial iff there exists $s \geq 1$ such that L is a γ_s^R language.*

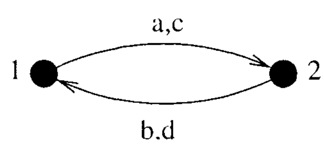
We can alternatively interpret the main lemma as the following theorem:

Theorem 5.9 *The C-varieties* $\ell\mathbf{R}$ *(*$\ell\mathbf{L}$*) and* $\mathbf{gR}$ *(*$\mathbf{gL}$*) coincide.*

Proof. We know from Chapter 1 that for any M-variety $\mathbf{V}$, the C-variety $\mathbf{gV}$ is always contained in the C-variety $\ell\mathbf{V}$. This immediately give us one direction of the theorem. Clearly, for every finite locally $\mathcal{R}$-trivial category C, one can define a graph $G = (V, A)$, such that $C \equiv G^*/\theta_t^{\ell R}$. Consider the monoid $M \equiv A^*/\sim_{s,R}$, where $s = 2(t+1)$. Using lemma 5.4, C divides the monoid M. Finally, M is $\mathcal{R}$-trivial using theorem 5.3. Exactly symmetric arguments exist for the M-variety $\mathbf{L}$. □

5.4 Locally $\mathcal{J}$-trivial categories

We shall first give an example that shows $\mathbf{J}$ is not local. Before doing so we note two things : the beautiful theorem of Simon says that the syntactic monoid of a language L is $\mathcal{J}$-trivial iff L is a $\sim_n$-language for some $n > 0$. Let $\Sigma = \{a, b, c, d\}$. Then, clearly $(ab)^n ad(cd)^n \sim_n (ab)^n(cd)^n$. It is easily verified that no $\mathcal{J}$-trivial monoid whose cardinality is at most n can distinguish these two words. This intuition could be applied to conclude that any category that is in $\mathbf{gJ}$ and hence divides some $\mathcal{J}$-trivial monoid whose cardinality is n cannot distinguish the path $(xy)^n xy'(x'y')^n$ from $(xy)^n(x'y')^n$, where xy and $x'y'$ are co-terminal loops. Using this argument we give the following example of a two node graph:



	$s_1 = *$	$s_2 = (ab)^+$	$s_3 = (cd)^+$	$s_4 = (ab)^+(cd)^+$	$s_5 = \overline{L}$
$s_1 = *$	s_1	s_2	s_3	s_4	s_5
$s_2 = (ab)^+$	s_2	s_2	s_4	s_4	s_5
$s_3 = (cd)^+$	s_3	s_5	s_3	s_5	s_5
$s_4 = (ab)^+(cd)^+$	s_4	s_5	s_4	s_5	s_5
$s_5 = \overline{L}$	s_5	s_5	s_5	s_5	s_5

Figure 5.1: Multiplication table of base monoid S

The graph congruence that we consider is the syntactic congruence γ_L of the language $L \equiv (a+b)^*(c+d)^* \cap \overline{A^*adA^*}$. Obviously, every path of the form $(ab)^m(cd)^m$ is in L for each $m \geq 0$, but the path $(ab)^m ad(cd)^m$ is not in L for any m. Hence, from the observation made in the previous paragraph, it follows that the syntactic category $C \equiv G^*/\gamma_L$ cannot be in **gJ**. Now let us compute base monoids S and T around vertex 1 and 2 respectively. Each element of S (T) corresponds to a congruence class of γ_L that contains loops around vertex 1 (2). Hence, we will represent the elements of S by regular expressions yielding the paths contained in the corresponding congruence class. Thus, $S \equiv \{s_1 = *,\ s_2 = (ab)^+,\ s_3 = (cd)^+,\ s_4 = (ab)^+(cd)^+,\ s_5 = \overline{L}\}$ contains five elements. Similarly, $T \equiv \{t_1 = *, t_2 = (ba)^+, t_3 = (ba)^*bc(dc)^*, t_4 = (dc)^+, t_5 = (ba)^+(dc)^+, t_6 = \overline{L}\}$. Note that $\overline{L}$ represents the complement of the language and for any string w in $\overline{L}$, for all u, v in A^* we have $uwv \in \overline{L}$.

Note from the table given in figure 5.1 that s_1 and s_5 act as the identity and zero element of S respectively. One can verify from the table that $(ss')^2 = (s's)^2$ and $s^2 = s^3$ for each s, s' in S. Similarly from table in figure 5.2, it can be easily verified that $(tt')^2 = (t't)^2$ and $t^2 = t^3$ for each t, t' in T. Thus S and T are both $\mathcal{J}$-trivial.

	$t_1 = *$	$t_2 = (ba)^+$	$t_3 = (ba)^*bc(dc)^*$	$t_4 = (dc)^+$	$t_5 = (ba)^+(dc)^+$	$t_6 = \overline{L}$
$t_1 = *$	t_1	t_2	t_3	t_4	t_5	t_6
$t_2 = (ba)^+$	t_2	t_2	t_3	t_5	t_6	t_6
$t_3 = (ba)^*bc(dc)^*$	t_3	t_6	t_6	t_3	t_6	t_6
$t_4 = (dc)^+$	t_4	t_6	t_6	t_4	t_6	t_6
$t_5 = (ba)^+(dc)^+$	t_5	t_6	t_6	t_5	t_6	t_6
$t_6 = \overline{L}$	t_6	t_6	t_6	t_6	t_6	t_6

Figure 5.2: Multiplication table of base monoid T

It is worthwhile mentioning here that as first shown in [Kna84] and later proved using algebraic methods in [Thé88], the identity characterising **gJ** is indeed given by

$$(uv)^\omega uv'(u'v')^\omega \;=\; (uv)^\omega(u'v')^\omega$$

where x^ω is the unique idempotent that is a power of x. Note in the above identity uv and $u'v'$ are co-terminal loops.

Even though in this section we would have liked to understand languages recognized by categories in $\ell\mathbf{J}$, we are currently not in a position to characterise these languages in terms of any graph congruence. This is ongoing work with Denis Thérien. However using the results of the previous section we can bound the C-variety $\ell\mathbf{J}$ in terms of global varieties rather easily.

Theorem 5.10 $\ell\mathbf{J} \;=\; \mathbf{gR} \cap \mathbf{gL}$.

Proof. We first show that each category in $\mathbf{gR} \cap \mathbf{gL}$ is locally $\mathcal{J}$-trivial. If C is any such category, then every base monoid of C is both $\mathcal{R}$ and $\mathcal{L}$-trivial. This implies from the definition of the relation $\mathcal{J}$, that C is locally $\mathcal{J}$-trivial. For the other direction consider an arbitrary category C in $\ell\mathbf{J}$. This means that C is locally $\mathcal{R}$-trivial and using theorem 5.9 we know that C must be in $\mathbf{gR}$. Similarly C is locally $\mathcal{L}$-trivial and thus is in $\mathbf{gL}$. This completes the argument. $\square$

Chapter 6

Conclusion

In this thesis, we have shown that the method of graph congruence is a useful syntactic technique that can yield important results about finite categories and the languages recognized by them. We have used it for example to prove locality of many M-varieties. In particular, it would be interesting to explore if this technique could be used to show that every variety contained in the variety of all idempotent monoids is local. Note that this is already known from the work of [JS92] using different techniques.

We have given a combinatorial description for the languages that can be recognized by finite locally commutative categories. This is the first result of that kind for a non-trivial M-variety for which the induced global and local C-varieties are different. We derived as a consequence the upper bound that for each M-variety $\mathbf{V}$ properly including the commutative monoids, the inclusion $\ell\mathbf{V} \subset \mathbf{gV}$ holds, which is similar to the situation for the trivial M-variety. It is easily checked that all these results can be proved, mutatis mutandis, for the C-variety of locally aperiodic commutative monoids. An interesting question to explore is: "What are the C-varieties that lie between $\mathbf{gCom}$ and $\ell\mathbf{Com}$?". Are categories defined with a 2-object graph enough to generate these varieties using direct product?

We have given elementary proofs of the locality of **R** and **L**. It would be interesting to see if this could be extended to show the locality of **DA** and other results that appear in [Alm96]. However, the most interesting open question from the point of view of this work is "What is the combinatorial description of languages recognized by locally $\mathcal{J}$-trivial categories?". We recall that both for $\ell\mathbf{1}$ and $\ell\mathbf{Com}$, the crucial first thing to understand was "For what paths the computational power of these local varieties coincide with their global counterpart?". In the case of $\ell\mathbf{1}$, we saw that for paths which were completely contained inside one strongly connected component, $\mathbf{g1}$ and $\ell\mathbf{1}$ behave exactly the same. This directly resulted in the characterisation of languages recognized by $\ell\mathbf{1}$. In the case of $\ell\mathbf{Com}$, the trick was again to see that every path induces an equivalence relation on vertices and for edges between these vertices, we cannot do any more with local than global. We therefore think that the first step for understanding $\ell\mathbf{J}$ would be to discover such a connection with $\mathbf{gJ}$.

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