In Support of Anti-Tori
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Abstract

This thesis provides new examples of compact irreducible *complete square complexes*. This is done by providing an elegant new recipe for producing an *anti-torus* in the universal cover $\widetilde{X} \cong T_V \times T_H$ of a complete square complex X. Additionally, we consider the action of a subgroup $\pi_1 H \subset \pi_1 X$ on the projection $\widetilde{X} \to T_v$ and characterize when the elements have finite order. We also present some evidence that generic complete square complexes contain an anti-torus.

Résumé

Dans cette thèse, nous produisons de nouveaux exemples de *complexes complets carrés irreductibles.* Ceci est accompli par l'entremise d'une nouvelle recette élégante pour produire des *anti-torus* dans le couvert universel $\widetilde{X} \cong T_V \times T_H$ d'un complexe complet carré X. De plus, nous considérons l'action d'un sous-groupe $\pi_1 H \subset \pi_1 X$ sur la projection $\widetilde{X} \to T_v$ et charactérisons lorsque les éléments ont un ordre fini. Nous présentons aussi des résultats qui supportent l'idée que les complexes complets carrés contiennent génériquement des anti-torus.

Contents

Chapter 1

Introduction

1.1 Context

Since Gromov's geometric revolution of the study of infinite groups, a number of classes of groups have risen to prominence. One class are the "word-hyperbolic groups", which generalize free groups and fundamental groups of negatively curved manifolds. Another class are the groups that act properly and cocompactly on CAT(0) spaces, which also generalize the fundamental groups of compact manifolds with nonpositive curvature. During the past 10 years, nonpositively curved spaces arising from cube complexes have become increasingly central, and have led to the solutions of many interesting open problems in geometric group theory and topology. The nonpositively curved spaces that are arguably most remote from hyperbolicity are the products of two or more trees.

A *Complete Square Complex* X is a complex whose universal cover $\widetilde{X}$ is isomorphic to the product of two trees. The simplest example of such a complex X is where $X = A \times B$ where A and B are graphs, so $\widetilde{X} = \widetilde{A} \times \widetilde{B}$ is the product of two trees. However, it turns out that there are much more exotic examples of Complete Square Complexes, and recently there has been a surge of activity in this area.[2, 8, 3, 5, 10, 1, 4, 12]

There are examples of complete square complexes X with the property that X itself is not isomorphic to a produt of two graphs, but X has a finite cover $\widehat{X}$ such that $\widehat{X}$ is isomorphic to the product of two graphs. A complete square complex X is *irreducible* if it does not have a finite cover $\widehat{X}$ that is isomorphic to a product.

There are a number of ways of showing that a complete square complex is irreducible. One comes from arithmetic considerations [10, 9]. A much simpler way comes from Wise [11, 8] who introduced the notion of an "anti-torus", which is a plane $\widetilde{E} \subset \widetilde{X}$ with the property that $\widetilde{E}$ is not periodic, but each vertical and horizontal line of $\widetilde{E}$ is periodic from the viewpoint of the map $\widetilde{E} \to X$. Note that a product $A \times B$ cannot contain an anti-torus, since the periodicity of $\widetilde{E}$ in both the vertical and horizontal directions implies its double-periodicity. Hence if X contains an immersed anti-torus, we see that X must be irreducible. Wise, and later, Janzen-Wise produced complete square complexes that contain an anti-torus. However, their arguments are ad hoc.

The main new contribution of this thesis is a new, more systematic and understandable way of producing an anti-torus. We use this to produce new and transparent examples of irreducible complete square complexes, and moreover, our proofs are much more satisfying than the previous ad hoc constructions in the literature. Our main example can be seen in Figures 1.1, 1.2 and 1.3. We explain this example, and a generalization in Chapter 3.

A second contribution of this thesis is a characterization of torsion of a certain element h in a group that is closely related to a Complete Square Complex. As a complete square complex X has the property that $\widetilde{X} \cong T_V \times T_H$ where T_V and T_H are trees, there is an action of $\pi_1 X$ on this product that projects to an action of $\pi_1(X)$ on T_V. We examine the action of an element stabilizing T_H on T_V and characterize when its action has finite order in $\mathrm{Aut}(T_V)$. This is described in Chapter 4. The relationship between this idea and the anti-torus, is that whereas an anti-torus is about periodicity of a plane $\widetilde{E} = \mathbb{R} \times \mathbb{R}$, the torsion of our element is instead related to periodicity of $T_V \times \mathbb{R} \subset \widetilde{X}$.

We also undertook some computer-aided investigations that lend credence to the belief

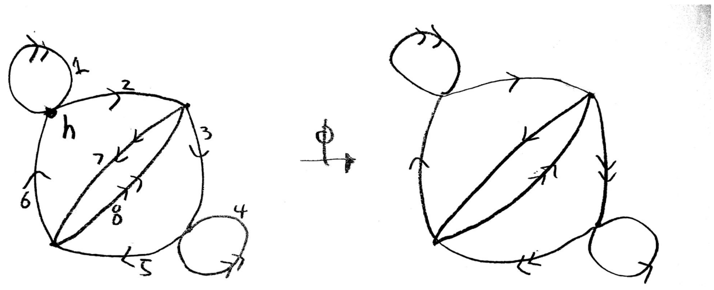


Figure 1.1: The main example: An irreducible complete square complex X.

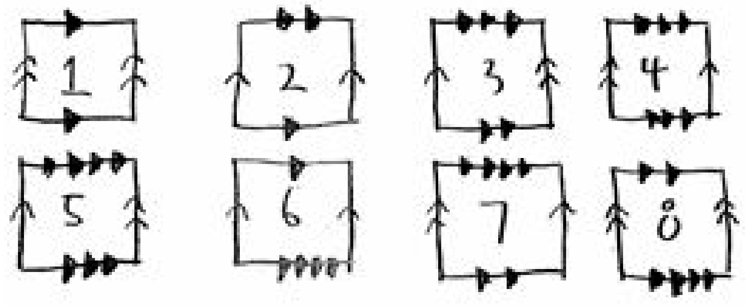


Figure 1.2: The squares of the main example X.

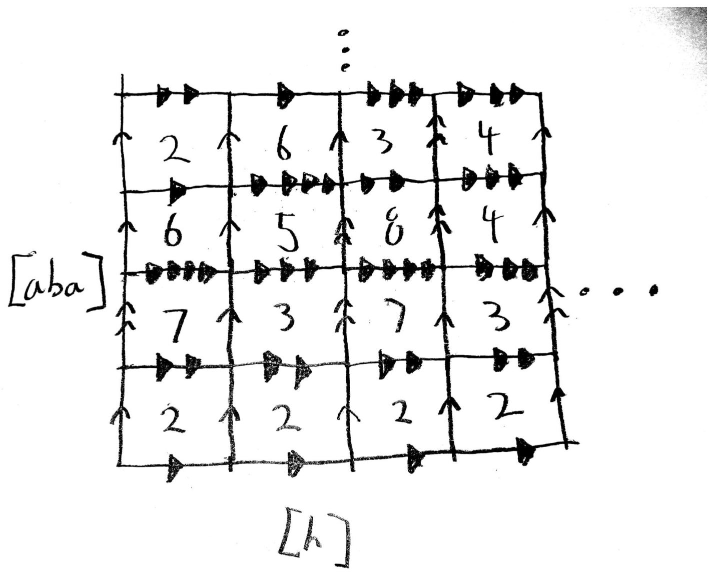


Figure 1.3: Part of the anti-torus in the main example X.

that a random Complete Square Complex always contains an anti-torus. Our results are described briefly in Section 5.

Chapter 2

Definitions

2.1 Standard Definitions

We assume the reader has an understanding of covering spaces and fundamental groups, we use [7] as a reference for the most standard concepts we will be using here.

Definition 2.1 (n-cube)**.** An *n-cube* is a topological space homeomorphic to $[-1,1]^n$ equipped with the standard metric topology.

Definition 2.2 (face)**.** A *face* of a n-cube is the subspace formed by restricting some of the coordinates to ± 1. Observe that using these definitions, faces of cubes can be considered as cubes in their own right.

Definition 2.3 (combinatorial cube complex)**.** A *combinatorial cube complex* is the quotient space obtained by gluing cubes along faces by isometries. Formally, we take a set $\mathcal{C}$ of cubes of various dimensions and a collection $\mathcal{F}$ of isometries and we obtain the cube complex $X = \mathcal{C}/\mathcal{F}$.

Since the identification is done by isometries, we observe that the cube complex can be defined purely combinatorially. The collection of maps $\mathcal{F}$ is therefore more naturally specified by labellings of faces, as can be seen in Figure 2.1.

Definition 2.4 (link of a 0-cube)**.** The *link* of a 0-cube v, denoted $\mathsf{Link}(v)$ is the topological space homeomorphic to the intersection of X with an ϵ-sphere about v. More formally, $\mathsf{Link}(v)$ it is the simplicial complex with vertices corresponding to 1-cubes attached to v and an $n-1$ simplex for every n-cube with a corner at v. (Refer to Figure 2.2)

Definition 2.5 (graph)**.** A graph is a cube complex X consisting of cubes of dimension ≤ 1. This agrees with the usual notion of a graph $X = (X^0, X^1)$, consisting of vertices X^0 and edges X^1. We assume all edges are directed with the convention $e = (i,j) \iff \overline{e} = (j,i)$. Also for $e = (i,j)$, we let $o(e) = i$ and $t(e) = j$.(for origin and target.) Naturally this gives us $o(\overline{e}) = j$ and $t(\overline{e}) = i$.

Definition 2.6 (square complex)**.** A *square complex* is a cube complex consisting of cubes of dimension ≤ 2.

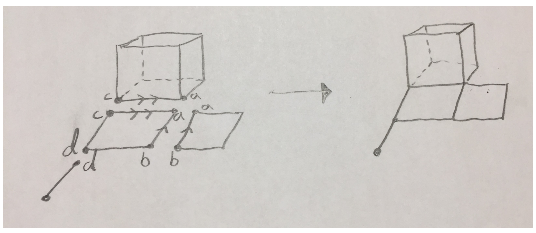


Figure 2.1: A cube complex: a collection of cubes $\mathcal{C}$ and isometries $\mathcal{F}$ indicated by labellings.

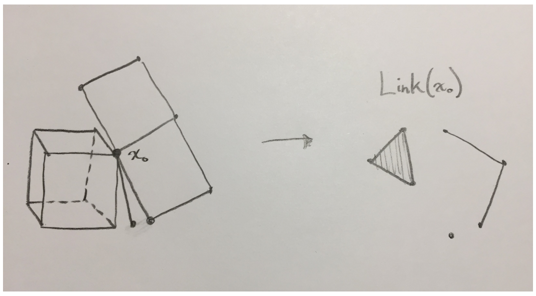


Figure 2.2: On the left we have a cube complex with a marked vertex x_0. On the right we have the link at x_0.

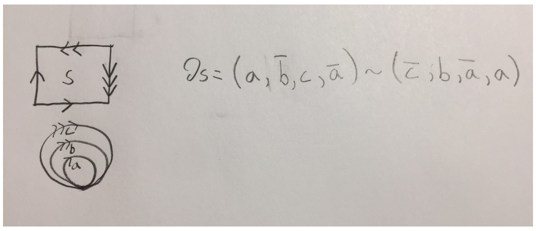
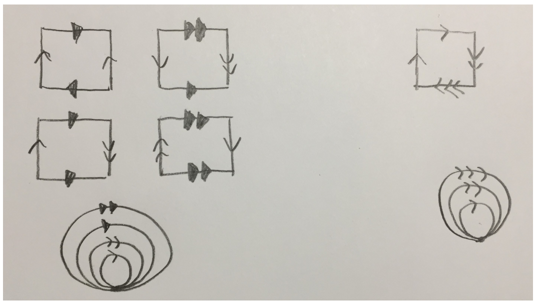


Figure 2.3: A square s and its boundary $\partial_p s$. We give two examples of boundary representatives.

Definition 2.7 ($\mathcal{VH}$-complex)**.** A $\mathcal{VH}$-complex is a square complex X with subgraphs $V, H \subset X^1$ such that $X^1 = V \cup H$ and $X^0 = V \cap H$ such that the bipartition V, H, of its 1-cells induces a bipartition on $Link(v)$, for every $v \in X^0$

Example 2.8. On the left we have an example of a $\mathcal{VH}$-complex with an obvious bipartition of the edges: V and H are both bouquets of two circles intersecting at one point. On the right, the square has an attaching map with two consecutive edges mapping to the same edge, so there is no way to partition the edges to make it a $\mathcal{VH}$-complex.



Definition 2.9 (boundary)**.** Given a square complex X and a square $s \subset X$, we denote the *boundary* of s by $\partial_p s = e_1 e_2 e_3 e_4$. This is just the attaching map of s, where $e_i \subset X^1$. Note the boundary is not unique, as it depends on the choice of basepoint and orientation, but nothing we do is sensitive to this. (Formally, we quotient $\partial_p s$ by D_4.) (Refer to Figure 2.3)

2.2 Complete Square Complexes

The objects we want to work with have more structure than $\mathcal{VH}$-complexes, so we introduce a few more definitions.

Definition 2.10 (Complete Square Complex)**.** A square complex is a complete square complex (CSC) if $\mathsf{Link}(v)$ is a complete bipartite graph for every $v \in X^0$.

The constraint on $\mathsf{Link}(v)$ is quite stringent, which makes building a CSC by hand a little tricky, in particular when the CSC has a large number of squares. We work towards

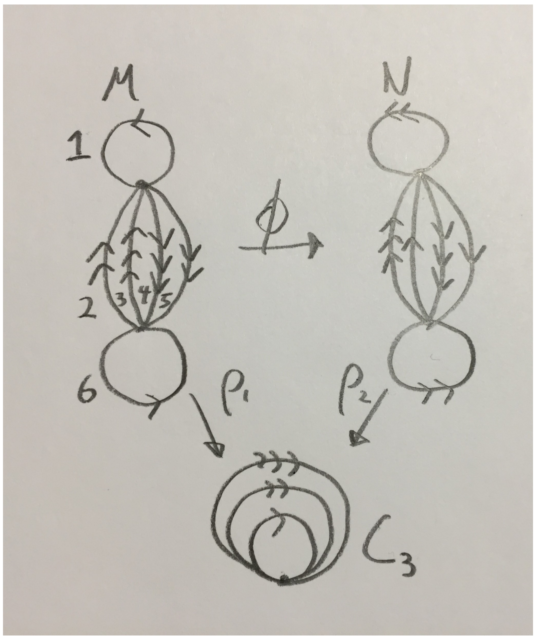


Figure 2.4: The complex square complex $\mathcal{W}$.

outlining an easily understandable and verifiable method of building a CSC. We focus on the case where X has a single 0-cell, but the ideas described here apply more generally.

Construction 2.11. We fix two n-coverings $\rho_M : M \to C_k$, $\rho_N : N \to C_k$ where C_k is a bouquet of k circles, and we assume M, N are isomorphic as unlabelled graphs. Fix a graph isomorphism $\phi : M \to N$ (note that ϕ is *not* in general a covering space isomorphism). We read off the structure of the CSC $X = (M, N, \phi)$ with a $\mathcal{VH}$ structure as follows:

1. X^0 consists of one 0-cube x_0.
2. The subgraph $V \subset X^1$ is a bouquet of k circles and corresponds to C_k.
3. The subgraph $H \subset X^1$ is a bouquet of n circles and corresponds to the vertices of M.
4. We add a square s with labels $(\rho_1(e), t(e), \overline{\rho_2(\phi(e))}, \overline{o(e)})$ for every edge $e \in M$.

The attaching map of every square alternates between edges in V and edges in H, so X is indeed $\mathcal{VH}$. That X is a CSC follows from the fact that we chose M and N to be covers of C_k, so that there will be a unique square with corner (v, h), for every pair of edges $v \in V, h \in H$.

Remark 2.12. More generally, we can construct a CSC as a graph of spaces where all edge spaces and vertex spaces are graphs and all attaching maps are covering maps. See [12].

We illustrate the construction through an example that will be of use later.

Example 2.13. Let $\rho_1 : M \to C_3$ and $\rho_2 : N \to C_3$ be the two degree 2 covers of a bouquet of 3 circles as shown in Figure 2.4. Let ϕ be the obvious graph isomorphism. We obtain the complete square complex $\mathcal{W} = (M, N, \phi)$.

The edges in Figure 2.4 are numbered. We can see the corresponding labelled squares in Figure 2.5 obtained by applying Construction 2.11.

We highlight an equivalent way of understanding Construction 2.11. Let $\rho_M : M \to C_k$ and $\rho_N : N \to C_k$ be covering maps. Assume M, N are isomorphic as unlabelled graphs. Let Γ correspond to the unlabelled graph associated with M and N. Let

$$X = (\Gamma \times [0,1]) \ / \ \{(x,0) \sim \rho_M(x), (x,1) \sim \rho_N(x) \ \mid \ \forall x \in \Gamma\})$$

We henceforth omit the intermediate explanation and denote the CSCs we work with as triples (M, N, ϕ). When it is not specified, we assume that M and N are equipped with covering maps $\rho_M : M \to C_k$, $\rho_N : N \to C_k$. An attractive property of CSCs is:

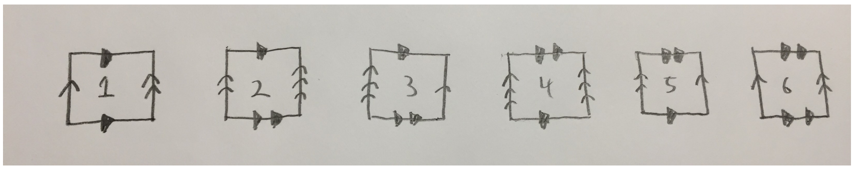


Figure 2.5: The squares in $\mathcal{W}$.

Theorem 2.14. *(X is CSC $\iff \widetilde{X} \cong$ Tree $\times$ Tree)*

Proof. Refer to [11] for a proof. □

For a CSC $X = (M, N, \phi)$ with M an n-cover of a bouquet of k circles, we have $\widetilde{X} \cong T_V \times T_H$, where $T_V \cong \widetilde{V}$ and $T_H \cong \widetilde{H}$ are n-regular and k-regular trees respectively.

In the case a CSC $X = (M, N, \phi)$ has a single 0-cell ($X^0 = \{x_0\}$), $\pi_1(X, x_0)$ is equipped with a canonical cocompact and properly discontinuous action on a product of trees, acting via deck transformations. We can understand the group as having the presentation:

$$\pi_1(X, x_0) = \langle v \in \mathsf{Edges}(\mathrm{V}), h \in \mathsf{Edges}(\mathrm{H}) \mid \partial_p s \ \forall s \in S \rangle$$

That is, the generators correspond to the vertical and horizontal edges and the relations correspond to the attaching maps of the squares.

Definition 2.15 (path)**.** A *path* w in a cube complex is a map $\varphi : I \to X$, with I an interval. We assume paths are combinatorial so $I \to X^1$ and $\varphi(I)$ is the concatenation of edges $e_1 ... e_n$ in X^1. If there is ambiguity we will denote a path with initial point x by w_x. Similar to Definition 2.5, we denote $o(w)$ to be the starting vertex and $t(w)$ to be the endpoint vertex of a path w. When a path is a sequence of edges in X^1 that are directed and labelled, each edge corresponds to an element of $\pi_1 X$ and a path corresponds to a *word* in $\pi_1 X$ so we make use the two terms interchangeably.

From hereon, we assume CSCs have a single 0-cell ($X^0 = \{x_0\}$) and we write $\pi_1 X = \pi_1(X, x_0)$, taking the basepoint to be the single 0-cell in X^0. (Similarly for $\pi_1 V = \pi_1(V, x_0)$ and $\pi_1 H = \pi_1(H, x_0)$).

Lemma 2.16. *Let $\widetilde{X} \to X$ be the universal cover of X. Let $a, b \in \rho^{-1}(x_0)$ be two points in the universal cover of X, where $x_0 \in X^0$. There is a unique embedded path from a to b of the form hv where $h \subset \rho^{-1}(H)$ and $v \subset \rho^{-1}(V)$.*

Proof. Refer to [11] for a proof. □

By symmetry, any two points are joined by a unique path of the form vh where $h \subset \rho^{-1}(H)$ and $v \subset \rho^{-1}(V)$. (Note the only difference between this statement and Lemma 2.16 is the order in which the horizontal and vertical paths appear)

We also note that the subgroup $\pi_1 H < \pi_1 X$ is equipped with a right action on $\pi_1 V$. We now describe the action from the viewpoint of Lemma 2.16. We remark that the action is not by homomorphisms.

Let $a \in \tilde{X}$ be the endpoint of the path $v_x \to \rho^{-1}(V)$ and let b be the endpoint of the path $h_x \to \rho^{-1}(H)$ both with start point x. By Lemma 2.16, there is a unique path of the form $h'v'$ joining a to b with $h' \to \rho^{-1}(H)$ and $v' \to \rho^{-1}(V)$. Thinking of h as an element of $\pi_1 H$ as described in Definition 2.15, we define the action by:

$$h(v) = v'$$

The action is easily understood via the following observation: the paths v and h determine a unique tiling of their convex hull (the tiling is uniquely determined because of the complex is CSC, every corner uniquely determines a square). The image of v under the action is just the label of paths opposite to them in this rectangle, as can be seen in Figure 2.6.

This is indeed a right action: $h_1 h_2(v) = h_2(h_1(v))$, as this corresponds to the tiling generated by h_1 and v concatenated with the tiling generated by h_2 and $h_1(v)$. We naturally

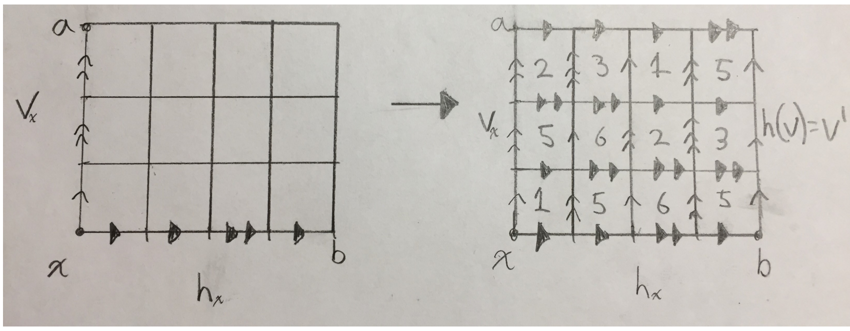


Figure 2.6: The tiling of the convex hull of paths v and h in the CSC $\mathcal{W}$.

extend this to an action of $\pi_1 H$ on rays in T_V. We also have that the action is prefix-preserving, so an element $h \in \pi_1 H$ determines a rooted automorphism of T_V (see Figure 4.1).

Another way to interpret the action of an element of $\pi_1 H$ on T_V is through the following: Given a CSC $X = (M, N, \phi)$, a path $v \in T_V$ and an element $h \in \mathsf{Edges}(\mathrm{H})$, we can interpret $h(v) = v'$ as being the path obtained by tracing the path v inside of M starting at vertex h and looking at its image through ϕ.

The motivation for this paper is that the fundamental groups of these complexes is surprisingly interesting. We introduce some notions we will use to investigate the fundamental groups of CSCs.

Definition 2.17. (Reducible) We say a CSC X is reducible if there exists a finite degree cover $\hat{X} \to X$ such that $\hat{X} \cong A \times B$ where A, B are graphs.

In terms of fundamental groups we have

Theorem 2.18. *Let X be a CSC. Then X is reducible if and only if $\pi(X, x_0)$ contains a finite index subgroup isomorphic to $F_n \times F_m$ for some $m, n \in \mathbb{N}$*

Proof. Refer to [11] for a proof. □

If there does not exist a such a finite cover $\widehat{X} \cong A \times B$, we say X is *irreducible.*

A priori, it is not clear whether there exist any irreducible CSCs. It turns out that they do and we suspect (more on this later) that in fact irreducible CSCs are generic. That some CSCs are irreducible is what makes them alluring. A good portion of this paper is dedicated to detecting and constructing irreducible CSCs. Irreducibility of CSC is being studied by many, refer to [3] for a comprehensive survey. In general it is not known whether there is an algorithm that can decide whether a CSC is irreducible or not. Our main tool for detecting irreducibility of a CSC is through what is called an *anti-torus*:

Definition 2.19. (Anti-Torus and Tiling) Let X be a CSC with a $\mathcal{VH}$-structure. Let $h \to H$ and $v \to V$ be immersed circles based at x_0. The *tiling* generated by h and v is the convex hull of $\tilde{h} \cup \tilde{v}$, where $\tilde{h} \subset \widetilde{X}$ and $\tilde{v} \subset \widetilde{X}$ are lifts of h and v respectively that intersect at a point $\tilde{h} \cap \tilde{v} = x$. It is the product subspace $\tilde{h} \times \tilde{v}$. The plane $\tilde{h} \times \tilde{v} \subset \widetilde{X}$ is an *anti-torus* if is not tiled periodically by preimages of squares of X. In other words, it is an anti-torus if the map $\tilde{h} \times \tilde{v} \mapsto Y$ does not factor through a torus $T^2 \mapsto Y$.

We know the presence of an anti-torus in a CSC X implies X is irreducible because of the following theorem:

Theorem 2.20. *(anti-torus ⇒ irreducible) Let X be a CSC containing an immersed anti-torus. Then X is irreducible.*

Proof. Refer to [11] for a proof. □

It is not known whether the reverse implication is true.

Chapter 3

Main Example

3.1 Explicit construction of an Anti-Torus

We have outlined in Chapter 2 for a CSC X, how $\pi_1 H < \pi_1(X, x_0)$ acts on the factor tree $T_V \cong \widetilde{V}$. We consider here the same action, but focus on the action of $\pi_1 H$ on infinite periodic paths P_V instead of finite paths. Formally, we have

$$P_V = \{w \mid w \in \pi_1 V, w \text{ freely and cyclically reduced}\}/\{u \sim v \text{ if } \exists i, j \in \mathbb{N} \text{ s.t. } u^i = v^j\}$$

A word w is *simple* if $(w = u^i) \Rightarrow (w = u)$. (i.e. w is not a power of a smaller word.) From this we can assign a length function on P_V.

$$\|[w]\| := \text{length of } u, \text{ where } [u] = [w] \text{ and } u \text{ is simple}$$

We can think of $[w]$ as being the infinite periodic ray $www\cdots$ lying in T_V. As in Section 2.2, $\pi_1 H$ acts on T_V and in particular on infinite rays $www\cdots$. We will verify $\pi_1 H$ sends periodic words to periodic words under this action, but we note that the length of the period might change. We introduce more notation before showing this.

Definition 3.1. Let M be a degree n cover of a bouquet of k circles. Define $\sigma_M : \pi_1 V \to \text{Aut}(M^0)$, by the action of $\sigma_M(w)$ on the vertices of M, where $w \in \pi_1 V$:

$$\sigma_M(w)(x) = t(w_x)$$

Where we can also think of w_x as a path in M based at x.

We point out that this is equivalent to the action of $\pi_1 V$ on the cosets $\pi_1 M/\pi_1 V$. We conclude $(w^\ell)_x$ (ℓ repetitions of the path w based at x) is a closed path in M for some positive integer ℓ, so $h([w])$ (i.e. the action of h on $www\cdots$) is a periodic word with period $h((w)^\ell)$. We summarize this in the following lemma:

Lemma 3.2. *Let $[v] \in P_V$ and let $h \in \pi_1 H$. Then $h([v]) = [w]$ for some $[w] \in P_V$.*

For any $h \in \pi(H)$ and periodic word $[v] \in P_v$ we can associate a sequence of rational numbers, $(\alpha_i)_{i\in Z}$, where we define

$$\alpha_i = \frac{\|h^i([v])\|}{\|h^{i-1}([v])\|}$$

We hope to reach some conclusions about the presence of anti-tori in CSCs by studying these sequences. We follow with a step in this direction:

Lemma 3.3. *Let $(\alpha_i)_{i\in Z}$ be the sequence of rational numbers associated to the tiling of the plane generated by $v \in \pi_1 V$ and $h \in \pi_1 H$ in some CSC X. If the tiling factors through a torus, then for an integer ℓ and any $j \in \mathbb{Z}$ we have:*

$$\prod_{i=j+1}^{j+\ell} \alpha_i = 1$$

Proof. If the tiling factors through a torus, then h acts periodically on $[v]$ with some period ℓ. The equality follows immediately from the definition of $(\alpha_i)_{i\in Z}$. $\square$

A consequence of Lemma 3.3 is that if we can show $\prod_{i=j+1}^{j+\ell} \alpha_i$ is unbounded in ℓ for some choice of j we can conclude that the tiling is an anti-torus. This is how an anti-torus was found in [11]. In particular, it was proven that the sequence $(\alpha_i)_{i\in Z}$ associated to a particular tiling had $\alpha_i = 2$ for $i \geq 1$.

We tried showing the presence of unbounded growth by examining the average behaviour of $(\alpha_i)_{i\in Z}$, but this proved unsuccessful.

However, we were able to detect the presence of an anti-torus through a divisibility argument, the remainder of this section is dedicated to this.

Definition 3.4. Let $X = (M, N, \phi)$ be a CSC. We define a useful subset of the rationals:

$$R_X = \left\{ \frac{\|h([v])\|}{\|[v]\|} : v \in P_V, h \in H \right\}$$

We can restrict R_X as follows:

Lemma 3.5. *Let $X = (M, N, \phi)$ be a CSC, with M a degree n cover of a bouquet of circles. Then $R_X \subset \mathbb{Q}_n = \{\frac{a}{b} : 1 \leq a, b \leq n\}$.*

Proof. As noted above, any path $v \subset M$ corresponds to a permutation of the vertices of M. It will take at most n iterations of a permutation before any vertex cycles back to its original position (the largest orbit of a vertex under the action of an element in $\mathrm{Aut}(M^0)$ is n). Let a be the smallest positive integer such that $x = t(v_x^a)$. Hence $a \leq n$ and $[h(v^a)] = h[v]$. It is however possible that $\|[h(v^a)]\| < a\|v\|$ as $h(v^a)$ might not be a simple word. We write $h(v^a) = w^b$ for some simple word $w \in \pi_1 V$, which implies:

$$b\|w\| = a\|v\|$$

Since w also defines a permutation in N, we also note that $o(w^b) = t(w^b)$ must be true for some $b \leq n$. Thus, $\frac{a}{b} \in \mathbb{Q}_n$ and we have:

$$\|w\| = \frac{a}{b}\|v\| \qquad \square$$

Lemma 3.5 restricts the amount by which the length of periodic words can vary under the action of H. We aim to restrict this further. We introduce a bit more notation.

Definition 3.6. Let $X = (M, N, \phi)$ be a CSC. Let $G_M = \sigma_M(\pi_1 V) < \mathrm{Aut}(M^0)$ and $G_N = \sigma_N(\pi_1 V) < \mathrm{Aut}(N^0)$ be as in Definition 3.1.

Definition 3.7. For a subgroup $K < S_n$, define the set of integers associated to cycle lengths of its elements as follows:

$$L(K) = \{\ell : \ell \text{ is the length of a cycle of an element of } K\}$$

Using Definition 3.7 and the same argument as in Lemma 3.5, we obtain:

Lemma 3.8. *Let $X = (M, N, \phi)$ be a CSC and let R_X be as in Definition 3.4. We have:*

$$R_X \subset \left\{ \frac{i}{j} \ : \ i \in L(G_M) \text{ and } j \in L(G_N) \right\}$$

We now construct an explicit example of an anti-torus using Lemma 3.8.

Example 3.9. Consider the CSC $X = (M, N, \phi)$ depicted in Figure 3.1.

Let G_M and G_N be as in Definition 3.6. We have that $G_M = \langle (1, 2, 3, 4), (1, 3) \rangle \cong D_4$, the dihedral group of order 8. In particular, we note that $L(M) = \{1, 2, 4\}$. Consider the periodic path $[aba]$. By observation, aba is a simple word and we have $\|aba\| = 3$. Letting $h \in \pi_1 H$ from Figure 3.1 act on $[aba]$, we get $h([aba]) = [a]$. Clearly, $[a]$ is a simple word and we have $\|a\| = 1$. Refer to Figure 3.2 for an illustration of this process.

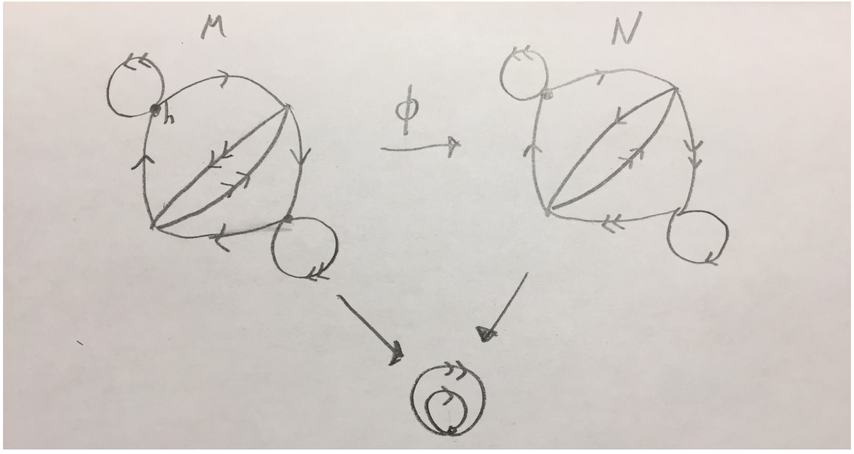


Figure 3.1: The main example: the CSC X. Note the vertex h in the top left part of cover M.

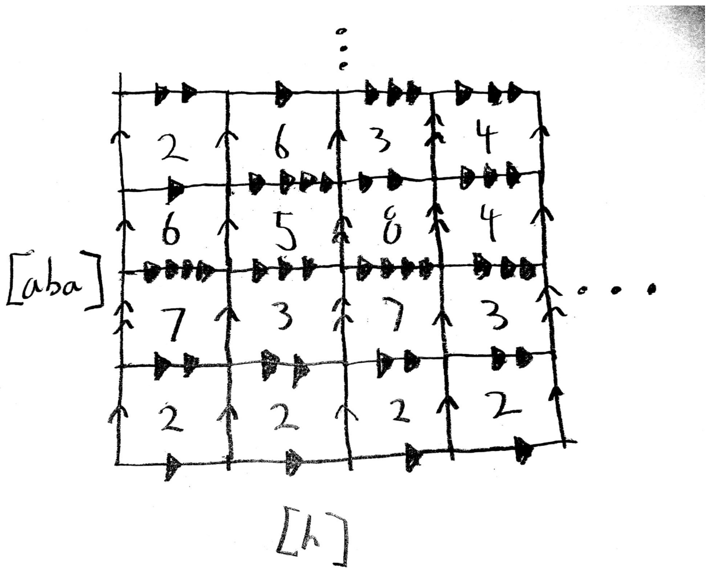


Figure 3.2: Tiling generator by $[h]$ and $[aba]$.

Consider the tiling of the plane generated by aba and by h. If the plane is not an anti-torus then there must be an integer ℓ for which

$$h^{\ell}([aba]) = [aba].$$

Again, having observed the action of h on $[aba]$, we must have for that same integer ℓ:

$$h^{\ell-1}([a]) = [aba]$$

In particular, we have:

$$\prod_{i=1}^{\ell-1} \frac{\|h^i([a])\|}{\|h^{i-1}([a])\|} = \frac{\|[aba]\|}{\|[a]\|} = 3$$

We define $\alpha_i = \frac{\|h^i([a])\|}{\|h^{i-1}([a])\|}$, where $\alpha_i \in R_X$ and observe that this implies:

$$\prod_{i=1}^{\ell-1} \alpha_i = 3$$

Observe this is impossible, as it would imply $3 \in L(M)$.

The following condition guaranteeing an anti-torus is sufficient but not necessary:

Theorem 3.10. *Let $X = (M, N, \phi)$ be a CSC. Let R_X be as in Definition 3.4. Let $\langle R_X \rangle \subset \mathbb{Q}$ be the multiplicative semigroup generated by R_X. If $\langle R_X \rangle$ is not a group, then X contains an anti-torus.*

Proof. If each $r \in R_X$ has $r^{-1} \in \langle R_X \rangle$, then each element $q = r_1 \cdots r_k$ of $\langle R_X \rangle$ (with $r_i \in R_X$) has an inverse $q^{-1} = r_1^{-1} \cdots r_k^{-1}$ (with $r_i^{-1} \in \langle R_X \rangle$). Hence, if $\langle R_X \rangle$ fails to be a group, it is because $\exists\, r \in R_X$ with no inverse in $\langle R_X \rangle$.

Now suppose $\langle R_X \rangle$ fails to be a group. Let r be such that $r^{-1} \notin \langle R_X \rangle$. By Definition 3.4, this implies $\exists v \in P_V$ and $h \in \pi_1 H$ such that $r = \frac{\|h([v])\|}{\|[v]\|}$. By Lemma 3.3, if the tiling generated by v and h factors through a torus then we must have for some integer ℓ:

$$\prod_{i=1}^{\ell} \frac{\|h^i([v])\|}{\|h^{i-1}([v])\|} = 1 \;\Rightarrow\; \prod_{i=2}^{\ell} \frac{\|h^i([v])\|}{\|h^{i-1}([v])\|} = r^{-1}$$

Hence $r^{-1} \in \langle R_X \rangle$, being a finite product of elements in R_X, a contradiction. □

Chapter 4

Characterizing finite order elements in the group of finite synchronous automata

In Section 2.2, we described how the horizontal subgroup $\pi_1 H < \pi_1 X$ of a CSC X, acts on the factor tree T_V, of the universal cover $\widetilde{X} \cong T_V \times T_H$ of X. In Chapter 3 we found a recipe to certify the presence of an anti-torus. Observe that having an anti-torus in the direction of h ensures that h has infinite order in its action on T_V. In this chapter, we focus on the order of the elements h on T_V and characterize when h has finite order.

Let X be given by the data (M, N, ϕ). Note that $\pi_1 H$ is generated by elements $h \in \mathsf{Edges}(H)$. We restrict our attention to the action of a single generator h on T_V. Note that h corresponds to a vertex of M. The rooted automorphism Φ of T_V is determined by the tuple (M, N, ϕ, h), which we hereon denote by $\left[M \xrightarrow[h]{\phi} N\right]$.

The group of all automorphisms of a tree T_V that can be represented in this way is described in [6], and is called the *group of finite synchronous automata* $\mathcal{FGA}(V)$.

Of course, it is possible for the same rooted automorphism of T_V to be denoted in two different representations in the above manner. The following lemma explains that there is a unique minimal representation.

Main Lemma 4.1. *Let Φ be a rooted automorphism of a tree T that is the universal cover of a bouquet of k circles V. There exists a minimal representation $\Phi = \left[A_0 \xrightarrow[h]{\phi} A_1\right]$ in the sense that if $\Phi = \left[M_0 \xrightarrow[h']{\phi'} M_1\right]$ for any covers M_0 and M_1 of V, then there exist covering maps $\rho_0 : M_0 \to A_0$ and $\rho_1 : M_1 \to A_1$ such that the following diagram commutes*

$$\begin{array}{ccc} M_0 & \xrightarrow{\phi'} & M_1 \\ {\scriptstyle \rho_0}\downarrow & & \downarrow{\scriptstyle \rho_1} \\ A_0 & \xrightarrow{\phi} & A_1 \end{array}$$

Proof. Let T_a and T_b be two universal covers of V. Observe that the rooted automorphism of Φ can be represented by:

$$\Phi = \left[T_a \xrightarrow[r]{\Phi} T_b\right]$$

Consider the complex isomorphic to $T \times I$ such that $T \times \{0\}$ and $T \times \{1\}$ are identified with T_a and T_b.

We are interested in the group G of label-preserving automorphisms of this object that stabilizes both $T \times \{0\}$ and $T \times \{1\}$. Let $\rho_a : T_a \to V$ and $\rho_b : T_b \to V$ be covering maps. We can describe G as follows:

$$G_\Phi = \{g \in \pi_1 V \mid \rho_a \circ g = \rho_a \text{ and } \rho_b \circ \Phi(g) = \rho_b\}$$

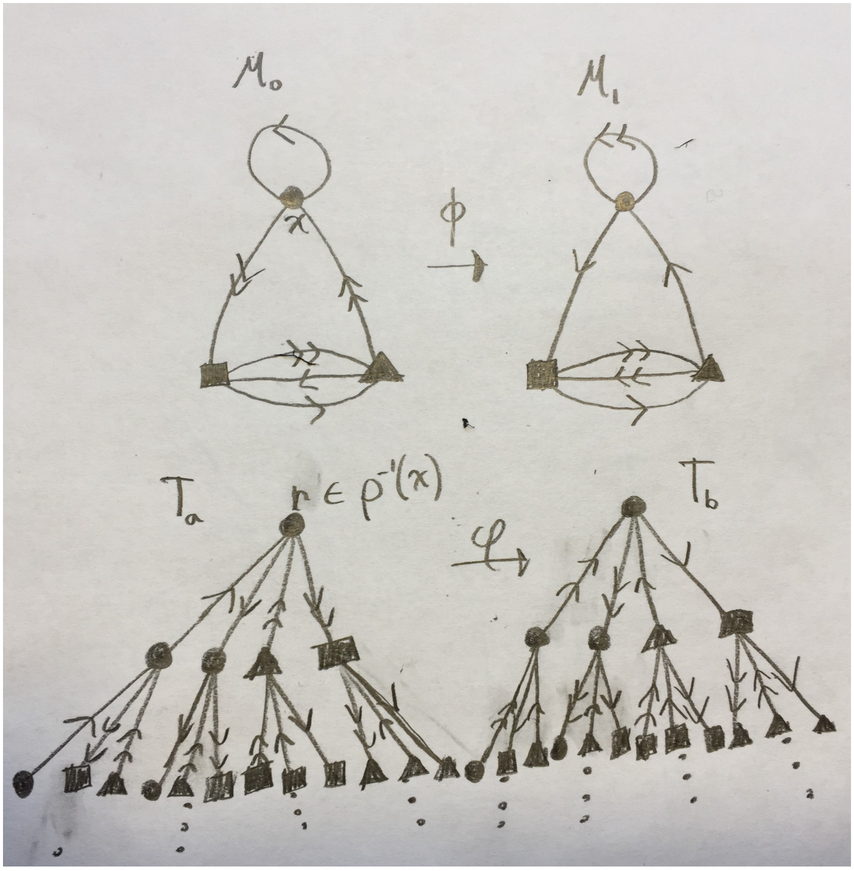


Figure 4.1: The automorphism of T_V determined by M_0, M_1, ϕ and $x \in \pi_1 H$. Quotienting T_a by all deck transformations that permute vertices in $\rho^{-1}(x)$ gives A_0. Similarly for A_1.

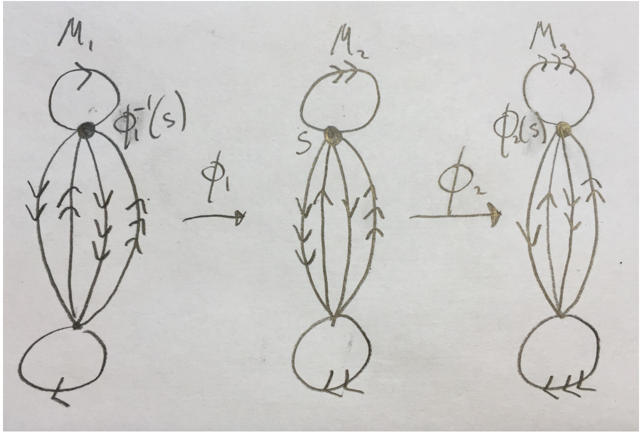


Figure 4.2: Composition of automorphisms

We obtain the minimal representation of Φ by quotienting the trees by the group action. That is, $A_0 = T_a/G_\Phi$ and $A_1 = T_b/\Phi(G_\Phi)$. This is clear, as any other representation $\Phi = \left[M_0 \xrightarrow[s']{\phi'} M_1\right]$ must be obtained as $M_0 = T_a/G'$ and $M_1 = T_b/\Phi(G')$ for some $G' < G_\Phi$. We immediately obtain that $M_0 \to A_0$ is a cover of A_0 and $M_1 \to A_1$ is a cover of A_1. That the diagram commutes follows from the two representations inducing the same tree automorphism Φ. $\square$

We continue with a lemma that has a similar flavor.

Lemma 4.2. *Let $\Phi = \left[M_0 \xrightarrow[s]{\phi} M_1\right]$ be an automorphism of T. Let $\rho_0 : N_0 \to M_0$ be a covering map. Then there exists a graph isomorphism ϕ' and a covering map $\rho_1 : N_1 \to M_1$ to make the following diagram commute:*

$$\begin{array}{ccc} N_0 & \xrightarrow{\phi'} & N_1 \\ \rho_0 \downarrow & & \downarrow \rho_1 \\ M_0 & \xrightarrow{\phi} & M_1 \end{array}$$

Proof. Similarly as in Lemma 4.1, we can realize M_0 as a quotient T/G. Since N_0 is a cover of M_0 we have that $N_0 = T/G'$ for some subgroup $G' < G$. We let $N_1 = T/\Phi(G')$, which makes it a cover of $M_1 = T/\Phi(G)$. We define the covering map $\rho_1 = \phi \circ \rho_0 \circ \phi'$ and obtain the commuting diagram as desired. $\square$

Remark 4.3. Let $\Phi = \left[M \xrightarrow[s]{\phi} M\right]$ be a rooted tree automorphism. If ϕ is the identity automorphism on M, $\Phi = 1_T$ is the trivial automorphism of T.

We continue to develop the notation with a few more technical lemmas

Lemma 4.4. *(Composition Lemma). Let $\Phi_1 = \left[M_1 \xrightarrow[\phi_1^{-1}(s)]{\phi_1} M_2\right]$ and let $\Phi_2 = \left[M_2 \xrightarrow[s]{\phi_2} M_3\right]$. Then $\Phi_2 \circ \Phi_1 = \left[M_1 \xrightarrow[\phi_1^{-1}(s)]{\phi_2 \circ \phi_1} M_3\right]$*

Proof. This is immediate from the definition. Refer to Figure 4.2. $\square$

Corollary 4.5. *Let $\Phi = \left[M_0 \xrightarrow[s]{\phi} M_1\right]$. Then $\Phi^{-1} = \left[M_1 \xrightarrow[\phi(s)]{\phi^{-1}} M_0\right]$.*

Proof. This follows from Lemma 4.4 and Remark 4.3

$$\big[M_0 \xrightarrow[s]{\phi} M_1\big]\big[M_1 \xrightarrow[\phi(s)]{\phi^{-1}} M_0\big] = \big[M_0 \xrightarrow[s]{\phi^{-1}\circ\phi} M_0\big]$$
$$= \big[M_0 \xrightarrow[s]{id} M_0\big]$$
$$= 1_T$$
$$\Rightarrow \big[M_1 \xrightarrow[\phi(s)]{\phi^{-1}} M_0\big] = \big[M_0 \xrightarrow[s]{\phi} M_1\big]^{-1}$$
$$= \Phi^{-1}$$

□

Note that in the previous two examples, the composition was easily realized because the source and target covers were the same. We want a general method of representing the composition of two automorphisms $\Phi_M = \big[M_0 \xrightarrow[s]{\phi_M} M_1\big]$ and $\Phi_N = \big[N_0 \xrightarrow[r]{\phi_N} N_1\big]$:

$$\Phi_N \circ \Phi_M = \Phi_P = \big[P_0 \xrightarrow[t]{\phi_P} P_1\big]$$

Definition 4.6 (Fiber Product)**.** Let C_k be a bouquet of k circles. Let $\rho_A : A \to C_k$ be a degree m cover and let $\rho_B : B \to C_k$ be a degree n cover. The *fiber product* $A \otimes B$ is the graph with vertices $(A \otimes B)^0 = \{(a, b)\ a \in A^0, b \in B^0\}$ and edges:

$$\{\big((a, b), (a', b')\big) \mid \exists (a, a') \in \mathsf{Edges}(A) \text{ and } (b, b') \in \mathsf{Edges}(B) \text{ s.t. } \rho_A(a, a') = \rho_B(b, b')\}$$

Note that $A \otimes B$ is a cover of both A and B. Moreover, $A \otimes B$ contains a connected component Q that is the smallest cover of both A and B. (Q is the *pullback* of A and B.)

Construction 4.7. We construct a representation of $\Phi_N \circ \Phi_M$, where $\Phi_M = \big[M_0 \xrightarrow[s]{\phi_M} M_1\big]$ and $\Phi_N = \big[N_0 \xrightarrow[r]{\phi_N} N_1\big]$. Let P be the connected component of $A \otimes B$ containing the vertex $(\phi(s)\, r)$. From Lemma 4.2, since P covers both M_1 and N_0, we can find $\rho_0 : P_0 \to M_0$ such that $\Phi_M = \big[P_0 \xrightarrow[(\phi')^{-1}(\phi(s),r)]{\phi'} P\big]$ and $\rho_1 : P_1 \to N_1$ such that $\Phi_N = \big[P \xrightarrow[(\phi(s),r)]{\phi''} P_1\big]$. From Lemma 4.4, we have:

$$\Phi_N \circ \Phi_M = \big[P_0 \xrightarrow[(\phi')^{-1}(\phi(s),r)]{\phi''\circ\phi'} P_1\big]$$

Theorem 4.8. *Let $\Phi = \big[M_0 \xrightarrow[s]{\phi} M_1\big]$ be a tree automorphism, where M_0 is a degree n cover of a bouquet of k circles. The following are equivalent:*

1. *$\Phi = \big[M_0 \xrightarrow[s]{\phi} M_1\big]$ is a finite order automorphism*
2. *There exists a finite degree cover $\rho_{M'} : M' \to C_k$ and ϕ' such that $\Phi = \big[M' \xrightarrow[s']{\phi'} M'\big]$ and $\phi'(s') = s'$*

Proof. ((1) ⇐ (2)) By repeated application of Lemma 4.4 we have

$$\Phi^i = \big[M' \xrightarrow[s']{(\phi')^i} M'\big]$$

As ϕ is an automorphism of a finite graph, ϕ must have finite order. There must be a positive integer ℓ satisfying $\Phi^\ell = \big[M' \xrightarrow[s']{id} M'\big]$ where id is the identity automorphism. By Remark 4.3 Φ^ℓ is the identity, so Φ is of finite order.

((1) ⇒ (2)) Let $\Phi = \big[M_0 \xrightarrow[s]{\phi} M_1\big]$ be an automorphism of T of finite order ℓ. Let $\big[A_0^i \xrightarrow[s_i]{\phi_i} A_1^i\big]$ be the minimal representation of Φ^i, for $1 \le i \le \ell$. Let (P, s'') be the smallest based cover of all of the A_0^i with covering maps $\rho_i : P \to A_0^i$ in following sense: if (Q, q)

is a based cover of (A_0^i, s_i) for all $1 \le i \le \ell$, then Q covers P. It is immediate that $P = T/(\bigcap_{i=1}^{\ell} \pi_1(A_0^i, s_i))$. The notion of a *smallest* such cover is well defined as any other based cover of all the (A_0^i, s_i) must be of the form $Q = T/H$ for some $H < \bigcap_{i=1}^{\ell} \pi_1(A_0^i, s_i)$. We note that the based fiber product $(A_0^1 \otimes A_0^2 \otimes \cdots \otimes A_0^{\ell})$ is a based cover of all of the A_0^i, hence a cover of P. This shows P is a finite cover of the bouquet of k circles.

By Lemma 4.2, since P covers A_0^i for $1 \le i \le \ell$, we can find covers $B_1^i \to A_1^i$ so that $\Phi^i = \big[P \xrightarrow[s_i']{\phi_i'} B_1^i\big]$. We note that we can let $s_i' = s''$. Hence, we have $\Phi^i = \big[P \xrightarrow[s'']{\phi_i'} B_1^i\big]$ for every $1 \le i \le \ell$. By Corollary 4.5 we have $\Phi^{-i} = \big[B_1^i \xrightarrow[\phi_i'(s'')]{(\phi_i')^{-1}} P\big]$. By Lemma 4.4 we have:

$$\Phi^{j-i} = \Phi^j \Phi^{-i} = \big[B_1^i \xrightarrow[\phi_i'(s'')]{\phi_j' \circ (\phi_i')^{-1}} B_1^j\big]$$

We complete the proof by recognizing that $B_1^i \cong P$ as covering spaces for $1 \le i \le \ell$. From the last equality, and Lemma 4.1, we know B_1^i must be a cover of A_0^{j-i} (where we consider the indices modulo ℓ). Fixing i and letting j vary, we observe that B_1^i is a cover of A_0^i for $1 \le i \le \ell$. Because P is the smallest cover of $\{A_0^1, A_0^2, \cdots, A_0^{\ell}\}$ it must be that B_1^i covers P for $1 \le i \le \ell$. But B_1^i can only be a degree one cover of P as they are homeomorphic finite graphs. We conclude that B_1^i and P are equal as covering spaces for $1 \le i \le \ell$. In particular, we have $\Phi = \big[P \xrightarrow[s'']{\phi''} P\big]$ for some graph automorphism ϕ''.

All that is left to verify is that ϕ'' can be chosen so that $\phi''(s'') = s''$. Because P is the smallest based cover of all the A_0^i, we have that s'' is the unique vertex that maps to each of the s_i in the covers $\rho_i : P \to A_0^i$. The equality $\Phi^{j-i} = \big[B_1^i \xrightarrow[\phi_i'(s'')]{\phi_j' \circ (\phi_i')^{-1}} B_1^j\big]$ together with Lemma 4.1 gives us covering maps $\tau_{j-i}^0 : B_1^i \to A_0^{j-i}$ and $\tau_{j-i}^1 : B_1^j \to A_1^{j-i}$ as well as the following commuting diagram:

$$\begin{array}{ccc} B_1^i & \xrightarrow{\phi_j' \circ (\phi_i')^{-1}} & B_1^j \\ {\scriptstyle \tau_{j-i}^0}\big\downarrow & & \big\downarrow{\scriptstyle \tau_{j-i}^1} \\ A_0^{j-i} & \xrightarrow{\phi_{j-i}} & A_1^{j-i} \end{array}$$

By minimality of A_0^{j-i}, the vertex s_{j-i} is the unique vertex $v \in A_0^{j-i}$ such that $\Phi^{j-i} = \big[A_0^{j-i} \xrightarrow[v]{\phi_{j-i}} A_1^{j-i}\big]$. Because the diagram commutes, we have $\tau_{j-i}^0(\phi_i'(s'')) = s_{j-i}$. Fixing i and letting j vary, we observe that $\phi_i'(s'')$ also maps to s_k for $1 \le k \le \ell$ in the covers $\tau_{j-i}^0 : B_1^i \to A_0^{j-i}$. By uniqueness, we have $s'' = \phi_i'(s'')$, in particular $\phi''(s'') = s''$. □

Chapter 5

Computer Experiments

We believe that generic CSCs should have irreducible fundamental groups. To this aim we wrote a program in JAVA to produce generic oriented CSC and examined the behaviour of tiling generated by randomly chosen elements in $\phi_1 H$ and $\pi_1 V$. As explained in Chapter 3, we obtain a sequence $(\alpha_i)_{i\in\mathbb{Z}}$ associated to the tiling of the plane generated by any choice of $h \in \pi_1 H$ and $v \in \pi_1 V$ where we defined:

$$\alpha_i = \frac{\|h^i([v])\|}{\|h^{i-1}([v])\|}$$

As explained in Lemma 3.3, anti-tori have unbounded products:

$$\prod_{i=j}^{j} \alpha_i$$

Hence if the typical behaviour of these products $\prod_{i=j} \alpha_i$ grows with j, then this lends credence to the belief that these sequences are unbounded.

Experiment 5.1. We describe the experiment in detail:

1. Fix integers k, i, j, ℓ.
2. Let n vary from 3 to 8.
3. Produce a complete square complex X where V is a bouquet of k circles and H is a bouquet of n circles.
4. We choose a word $h \in \pi_1 H$ of length i and a word $[v] \in \pi_1 V$ of length j.
5. We let h act on $[v]$ as described in Section 3.
6. We observe the average period variation over ℓ iterations of the action of h on $[v]$. That is we compute the value:
$$\left(\frac{\|h^\ell([v])\|}{\|[v]\|}\right)^{1/\ell}$$
7. We repeat this for 10 paths v, h, chosen uniformly at random in each of every 50 complete square complexes X also chosen uniformly at random and compute the geometric mean of every iteration of the experiment and report the value $A(k, n, i, j, \ell)$.

For the first iteration of the experiment, we let $k = 3$, $i = 1$, $j = 1$, $\ell = 10$. The results can be seen in Table 5.1.

For the second iteration of the experiment, we let $k = 5$, $i = 1$, $j = 1$, $\ell = 10$. The results can be seen in Table 5.2.

For the third iteration of the experiment, we let $k = 3$, $i = 4$, $j = 4$, $\ell = 10$. The results can be seen in Table 5.3.

We see from the results reported in the tables that the average period increase $A(k, n, i, j, \ell)$ increases with n and k. This supports the hypothesis that generic complete square complexes (in particular in CSCs with higher n and k) contain anti-tori and hence have irreducible fundamental groups.

n	$A(k, n, i, j, \ell)$
3	1.2371048971298813
4	1.4917452856997475
5	1.7520564802582541
6	2.2051153986908756
7	2.4203332111562417
8	3.0230867656859135

Table 5.1: Experiment Results for $k = 3$, $i = 1$, $j = 1$, $\ell = 10$

n	$A(k, n, i, j, \ell)$
3	1.4953526307299496
4	1.9554697386058972
5	2.341162457888909
6	2.6738257337574542
7	3.037080366243933
8	3.5494533674642295

Table 5.2: Experiment Results for $k = 5$, $i = 1$, $j = 1$, $\ell = 10$

n	$A(k, n, i, j, \ell)$
3	1.342611144322982
4	1.7012106864430632
5	2.2948440979244156
6	2.6253155577091296
7	3.1986077142821525
8	timed out

Table 5.3: Experiment Results for $k = 3$, $i = 4$, $j = 4$, $\ell = 10$

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