PERFECT AND SEMI-PERFECT RINGS

Abstract

This thesis is essentially an exposition of the standard results on perfect and semi-perfect rings. Complete characterizations of perfect and semi-perfect rings are given in Theorems 1.13 and 2.5 respectively. A result on commutative perfect rings is obtained in connection with Conjecture 1.19, which the author considers to be original. Chapter 3 comprises, in part, a characterization of semi-perfect rings in terms of idempotents and a structure theorem for finitely-generated projective modules over semi-perfect rings. Finally, in the Appendix, the author formulates and discusses a conjecture on subrings of perfect rings.

Abdul H. Rahman

Department of Mathematics
McGill University
Montreal, Quebec

March 1973

SUR LES ANNEAUX PARFAITS ET DEMI-PARFAITS

Résumé

Dans cette thèse on développe les résultats fondamentaux de la théorie des anneaux parfaits et demi-parfaits. Une caractérisation est donnée dans les théorèmes 1.13 pour les anneaux parfaits, 2.5 pour les anneaux demi-parfaits. L'auteur a obtenu un résultat en connexion avec la conjecture 1.19sur les anneaux parfaits commutatifs, ce qu'il croit d'être original. Une partie du Chapitre 3 donne une caractérisation des anneaux demi-parfaits en utilisant les idempotents et un théorème de structure des modules projectives avec un nombre fini de générateurs sur les anneaux demi-parfaits. En conclusion, dans l'appendice l'auteur a formulé et discuté une conjecture sur les sous-anneaux des anneaux parfaits.

Abdul H. Rahman

Département de Mathématiques

Université McGill

Montréal, Québec

March 1973

PERFECT AND SEMI-PERFECT RINGS

by

ABDUL HOSEIN RAHMAN

Submitted to the Faculty of Graduate Studies and Research

in partial fulfilment of the requirements

for the degree of Master of Science

Department of Mathematics

McGill University

Montréal, Québec

March 1973

I warmly dedicate this thesis to my Mother, Father, brother Sattaur, and sisters Aisha, Jaitoon, Zohra, Nazmoon and Farida.

PREFACE

The aim of this paper is to give a complete characterization of Perfect and Semi-Perfect rings as defined by H. Bass [2].

In Chapter I, Theorem 1.13 gives a complete characterization of perfect rings which is due to H. Bass. These turn out to be those rings for which every flat R-module is projective. Corollary 1.18 shows that over a perfect ring, a generalized form of Nakayama's Lemma holds. In Theorem 1.29, we prove that a ring R is left perfect ⇔ every completely reducible R-module has a projective cover — a result due to F. Sandomierski [33].

In Chapter II, a characterization of semi-perfect rings also due to H. Bass, is given. A corresponding theorem to 1.29 is proved for semi-perfect rings. As a corollary, we show that a ring R is semi-perfect ⇔ every left simple R-module is of the form $Re|Je$ where $e^2 = e \in R$.

The main theorem in Section A of Chapter III characterizes semi-perfect rings in terms of idempotents. It has been recognized that much of the classical structure theory for Artinian rings can be developed under the weaker hypothesis that R be semi-perfect. Section B contains some results for perfect and semi-perfect rings which are analogous to those characteristic of Artinian rings.

We obtain a decomposition theorem for finitely generated projective modules over semi-perfect rings in Section C. Section D contains two structure theorems for semi-perfect rings. The second, given by E. Behrens [4] is somewhat analogous to the "Splitting Theorem" of A. Zaks [38] for semi-primary rings. Finally in the Appendix, we formulate and discuss a conjecture dealing with subrings of perfect rings.

The author wishes to express his appreciation to his director, Professor J. Lambek for his generous advice and encouragement. Also conversations with Professors M. Barr, J. Beck, I. Connell and J. Golan have been most helpful. In addition, discussions with fellow students - J. Lawrence, E. McKay, C.C. Poon, R. Zeamer and especially D. Handelman, R. McMaster and M. Wright - have been most stimulating. Also to Miss Francine Houle, who has excellently typed this thesis, I offer special thanks.

Abdul H. Rahman

TERMINOLOGY

Ring	an associative ring with identity
Module	a unitary module
$J(R)$	the Jacobson radical of R
$P(R)$	the prime radical of R
$Soc(M)$	the socle of M
R_n	the ring of all $n \times n$ matrices over R
$End_R(M)$	the endomorphism ring of a left R-module M
$Ann(s)$	$\{r \in R \mid rs = 0\}$
$r(S)$	the right annihilator ideal
Mod-R	the category of all right R-modules
$\varinjlim M_i$	direct limit of the R-modules M_i

CONTENTS

CHAPTER I

A: On Small Modules

In order to give the main characterization of left perfect rings, we shall require some results involving the concept of smallness. In this respect, Lemma 1.4 is very useful.

1.1. Definition

Let R be a ring and M a left R-module. A submodule P of M is called small in P if $M = P+Q \Rightarrow M = Q$ for any $Q \subseteq M$.

1.2. Definition

An ideal I of R is called left T-nilpotent if for any sequence $\{a_i\} \subset I$, there exists an integer n such that $\prod_{i=1}^{n} a_i = 0$.

1.3. Remark

Clearly a nilpotent ideal is both left and right T-nilpotent. Also by considering a constant sequence of elements in I, it is clear that a T-nilpotent ideal is nil.

1.4. Lemma

Let R be a ring, J(R) its Jacobson radical and P, Q, M and N left R-modules. Then the following statements are true.

(a) If P is small in M and $f: M \to N$, then f(P) is small in N.

(b) If P is small in M and $Q \subseteq P$, then Q is small in M.

(c) If P and Q are small submodules of M, then so is $P+Q$. (This can be generalized to finite sums.)

(d) If M is finitely-generated, then $J(M)$ is small in M. (In particular, $J(R)$ is small in R.)

(e) $J(M) = \sum_{i\in I} P_i$ such that P_i is small in M. Hence if $f: M\to N$, then $f(J(M)) \subseteq J(N)$.

(f) $J(R)\cdot M \subseteq J(M)$ for any left R-module M.

(g) If P is a projective left R-module, then $J(R)\cdot P = J(P)$.

(h) Let M be a projective left R-module and $E = \mathrm{End}_R(M)$. Then $J(E) = \{f \in E \mid \mathrm{Im}\, f$ is small in $M\}$.

(i) $J(R)$ is left T-nilpotent $\Leftrightarrow$ $J(R)\cdot M$ is small in M, for any left R-module M.

(j) If $R/J(R)$ is completely reducible, then $J(R)\cdot M = J(M)$ for any left R-module M.

Proof

(a) Suppose $N = f(P)+B$ where $B \subseteq N$. Then $M = P+f^{-1}(B) = f^{-1}(B)$ since P is small in M. So $f(P) \subseteq f(M) = B$. Hence $N = f(P)+B \subseteq B$, and so $f(P)$ is small in N.

(b) Trivial.

(c) Suppose $(P+Q)+R = M$ where $R \subseteq N$. Then $P+(Q+R) = M$. So $Q+R = M$ since P is small in M. Hence $R = M$ since Q is also small in M. But then $P+Q$ is small in M. (By induction on the number of small submodules of M, we see that $\sum_{i=1}^{n} P_i$ is small in M if P_i is small in M for each $i=1, \ldots, n$.)

(d) Suppose $J(M)$ is not small in M. Then there exists a proper submodule P of M such that $J(M)+P=M$. Since M is finitely-generated, there exists a maximal submodule Q of M such that $P \subset Q$. So $J(M)+Q=M$. Hence $Q=M$ since $J(M) \subset Q$. But this contradicts the maximality of Q. Hence $J(M)$ is smal' in M.

(e) The proof given here follows that of D.Fieldhouse [12]. It is clear that $J(M) \supseteq \sum_{i \in I} P_i$ where P_i is small in M, since if N is a maximal submodule of M, then $P_i \subset N$. For the reverse inclusion, it suffices to show that $x \in J(M) \Leftrightarrow Rx$ is small in M, since by (c), a finite sum of small submodules of M is small. Equivalently, it suffices to prove that Rx is not small in M $\Leftrightarrow x \notin J(M)$. So suppose $x \notin J(M)$. Then $x \notin N$ for some maximal submodule N of M. So $Rx+N=M$. But $N \neq M$ and so Rx is not small in M. Conversely, suppose Rx is not small in M. Then there exists a proper submodule F of M such that $Rx+F=M$. Define $\mathfrak{F} = \{F \subset M \mid F \text{ is proper and } Rx+F=M\}$. Then $\mathfrak{F} \neq \emptyset$ since $F \in \mathfrak{F}$. Clearly $x \notin F$ for all $F \in \mathfrak{F}$. Observe also that if G is any proper submodule of M and $G \supset F$ where $F \in \mathfrak{F}$, then $G \in \mathfrak{F}$. Now order $\mathfrak{F}$ by set inclusion. Then $\bigcup_{F \in \mathfrak{F}} F$ is a proper submodule of M since $x \notin F$ for all $F \in \mathfrak{F}$. Also since $\bigcup_{F \in \mathfrak{F}} F \supset F$ for all $F \in \mathfrak{F}$, then $\bigcup_{F \in \mathfrak{F}} F \in \mathfrak{F}$. So, by Zorn's Lemma, there exists a maximal element E in $\mathfrak{F}$. Clearly E is a maximal submodule of M, for if $A \supset E$, then $A \in \mathfrak{F}$. But this would contradict that E is maximal in $\mathfrak{F}$. Hence $x \notin J(M)$ since $x \notin E$. This proves (e).

Our remark now follows trivially. Observe that by (a), P_i small in $M \Rightarrow f(P_i)$ is small in N where $f: M \to N$. So

$$\begin{aligned} f(J(M)) &= \sum_{i \in I} f(P_i) && \text{where } P_i \text{ is small in M} \\ &\subseteq \sum_{i \in I} Q_i && \text{where } Q_i \text{ is small in N} \\ &= J(N). \end{aligned}$$

(f) We first observe that $J(M) = \bigcap \operatorname{Ker} f$ such that $f \in \operatorname{Hom}_R(M, S)$ where S is a simple left R-module. Also if we define $g: R \to S$ by $g(r) = r \cdot s$ for $0 \neq s \in S$, then $g(R) \neq 0$. Hence $g(R) = S$ and $S \cong R/\operatorname{Ann}(s)$. So

$$J(R) = \bigcap_{s \in S} \operatorname{Ann}(s) \qquad (*)$$

The above remarks may be found, for example in [29].

So consider $f: M \to S$ where S is a simple left R-module.

Then $f(J(R)\cdot M) \subseteq J(R)\cdot f(M) = J(R)\cdot S = 0$ by (*). So $J(R)\cdot M \subseteq \operatorname{Ker} f$ for every $f: M \to S$. Hence $J(R)\cdot M \subseteq \bigcap \operatorname{Ker} f = J(M)$.

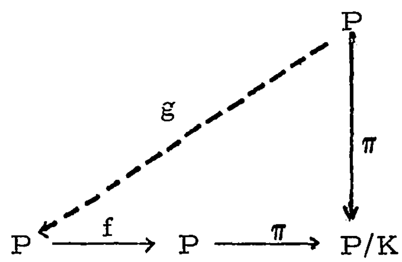
(g) Consider first the case where F is a free left R-module. Then $F = \bigoplus_{i\in I} R_i$ where $R_i \cong R$ for all $i \in I$. Hence $J(F) = J(\bigoplus_{i\in I} R_i) = \bigoplus_{i\in I} J(R_i) = J(R)\cdot F$. Now let P be a projective left R-module. Then P is a direct summand of a free left R-module F. Consider $F \underset{\alpha}{\overset{\pi}{\rightleftarrows}} P$ where π is an epimorphism which splits; i.e. $\pi\alpha = id_P$. Then

$$\begin{aligned} J(P) = \pi\alpha(J(P)) &\subseteq \pi(J(F)) \qquad \text{by (e)} \\ &= \pi(J(R)\cdot F) \\ &\subseteq J(R)\cdot \pi(F) \\ &= J(R)\cdot P. \end{aligned}$$

But $J(R)\cdot P \subseteq J(P)$ by (f). So $J(R)\cdot P = J(P)$.

(h) Our proof essentially follows that of Sandomierski [34]. Let $T = \{f \in E \mid \operatorname{Im} f$ is small in $P\}$. We first show that $T \subset J(E)$. So let $f \in T$. Then $P = \operatorname{Im} f + \operatorname{Im}(1-f) = \operatorname{Im}(1-f)$ since $\operatorname{Im} f$ is small in P. Hence $1-f$ is an epimorphism of P. But P is projective. Then, there exists $g: P \to P$ such that $(1-f)g = id_P$. But then $f \in J(E)$. Now to prove $J(E) \subset T$. So let $f \in J(E)$ and assume $\operatorname{Im} f + K = P$ where $K \subseteq P$. Clearly $\pi f: P \to P \to P/K$ is an epimorphism where $\pi: P \to P/K$ is the canonical map.

Since P is projective, there exists $g: P \to P$ such that



commutes. So $\pi f g = \pi$. Now let $x \in P$. Then $\pi(x - fg(x)) = 0$, and so $x - fg(x) \in \operatorname{Ker} \pi = K$. Hence $\operatorname{Im}(1 - fg) \subseteq K$. But $f \in J(E) \Rightarrow 1 - fg$ is isomorphism of $P \Rightarrow \operatorname{Im}(1 - fg) = P$. So $P \subseteq K$. Hence $P = K$ and $\operatorname{Im} f$ is small in P. This shows $f \in T$, and so $J(E) = T = \{f \in E \mid \operatorname{Im} f$ is small in $P\}$.

(i) We follow the proof given by R. Hamsher [17]. Also, we note that if $J(R) \cdot M$ is small in M, then $J(R) \cdot M = M \Rightarrow M = 0$.

($\Rightarrow$). Assume $J(R) \cdot M$ is not small in M and let $J(R) \cdot M = M$ for some non-zero left R-module M. Then there exists $m_1 \in M$ and $a_1 \in J(R)$ such that $a_1 m_1 \neq 0$. But if $m \in M$, then $m = \sum_{i=1}^{n} a_i m_i$ for $a_i \in J(R)$ and $m_i \in M$. Hence $a_k m_k \neq 0$ for some index k. Hence there exists $m_2 \in M$ and $a_2 \in J(R)$ such that $m_1 = a_2 m_2 \neq 0$, and hence $a_1 a_2 m_2 \neq 0$. In this way, we have a sequence $\{a_i\} \subset J(R)$ and a sequence $\{m_i\} \subset M$ such that $a_1 a_2 \ldots a_k m_k \neq 0$ for all k. This contradicts that J is left T-nilpotent. So, $J(R) \cdot M$ is small in M.

($\Leftarrow$) Let $\{a_i\}$ be a sequence of elements of J(R). Let $F = \bigoplus_{i=1}^{\infty} Rx_i$ be a free left R-module with countable basis $\{x_i\}$. Also define $G = \bigoplus_{i=1}^{\infty} R(x_i - a_i x_{i+1})$. Then, since $J(F) = J(R)\cdot F$ by (g), we have that $F = G + J(R)\cdot F = G$ by hypothesis. Now let $x_1 \in F$. Then $x_1 = \sum_{i=1}^{n} r_i(x_i - a_i x_{i+1}) = r_1x_1 + (r_2 - r_1a_1)x_2 + (r_3 - r_2a_2)x_3 + \ldots$ $\ldots + (r_n - r_{n-1}a_{n-1})x_n - r_na_nx_{n+1}$. Hence, by uniqueness of representation, $r_1 = 1$, $r_2 = r_1a_1$, $r_3 = r_2a_2, \ldots, r_na_n = 0$. So $r_2 = a_1$, $r_3 = a_1a_2$, $r_4 = a_1a_2a_3, \ldots, r_n = a_1 \ldots a_{n-1}$. This shows that $0 = r_na_n = a_1 \ldots a_{n-1}a_n$, and so J(R) is left T-nilpotent.

(We note that this implication only requires that F be a free left R-module.)

(j) Let M be a left R-module and $J = J(R)$. Then M/JM is a left R/J-module, and so is projective since R/J is completely reducible. So by (g), $J(M/JM) = J(R/J)\cdot M/JM$. Hence $J(M/JM) = 0$ since R/J is semi-primitive. But for any left R-module M, $J(M/J(M)) = 0$ and J(M) is the smallest submodule of M with this property. Hence $J(M) \subseteq JM$. So $J(M) = J(R)\cdot M$ since $J(R)\cdot M \subseteq J(M)$ for any left R-module M by (f).

This completes the proof of Lemma 1.4.

1.5. Definition

Let M be a left R-module. An epimorphism $\pi: P \to M$ is a projective cover of M whenever P is a projective left R-module and $\operatorname{Ker} \pi$ is small in P.

From Lemma 1.4, we obtain two interesting corollaries.

1.6. Corollary

Let R be semi-primitive (i.e. $J(R) = 0$). Then a left R-module M is projective $\Leftrightarrow$ it has a projective cover.

Proof

($\Rightarrow$). This implication is clear.

($\Leftarrow$). Suppose $\pi: P \to M$ is an epimorphism where P is projective and $K = \operatorname{Ker} \pi$ is small in P. Then

$$\begin{aligned} K &\subseteq J(P) && \text{by } 1.4(e) \\ &= J(R) \cdot P && \text{by } 1.4(g) \\ &= 0. \end{aligned}$$

So π is an isomorphism and M is projective.

1.7. Corollary

For any ring R, a flat left R-module M is projective $\Leftrightarrow$ it has a projective cover.

Proof

($\Rightarrow$). Trivial.

($\Leftarrow$). Let $\pi: P \to M$ be a projective cover of M and let $K = \operatorname{Ker} \pi$.

Since M is flat, there exists $f: P \to K$ such that $f(k) = k$ for all $k \in K$ [30, p. 61]. So $\text{Im} f \subset K$. But K is small in P. Hence by 1.4(b), Im f is small in P. So $f \in J(E)$ where $E = \text{End}_R(P)$, by 1.4(h). Hence $1-f$ is a unit in E by [23, §3.2, Prop. 5]. But $(1-f)k = k - f(k) = 0$. Hence $k = 0$ for all $k \in K$. This shows that π is an isomorphism and so M is projective.

B: Perfect Rings Characterized

1.8. Definition

H. Bass called a ring R left perfect $\Leftrightarrow$ every left R-module has a projective cover.

1.9. Definition

A submodule A of M is large in M if $A \cap N = 0 \Rightarrow N = 0$ for any $N \subseteq M$.

1.10. Lemma

Let M be a non-zero right R-module. Then M contains a simple R-module $\Leftrightarrow$ Soc(M) is large in M.

Proof.

($\Rightarrow$). Let $N \subseteq M$ such that $N \neq 0$. By hypothesis, there exists a simple right R-module $S \subset N$, $S \neq 0$. But then $S \subset N \cap \text{Soc}(M)$ and hence $N \cap \text{Soc}(M) \neq 0$. So Soc(M) is large in M.

(⇐). Assume Soc(M) is large in M. Let $N \neq 0$ be a submodule of M. Then $\mathrm{Soc}(M) \cap N \neq 0$. But Soc(M) is completely reducible, and hence so is $\mathrm{Soc}(M) \cap N$. So $\mathrm{Soc}(M) \cap N$, and hence N, contains a simple submodule.

1.11. Remark

An equivalent formulation of 1.10 is as follows: A non-zero right R-module has a non-zero socle ⇔ Soc(M) is large in M.

1.12. Definition

Let D be an ideal of R. Then idempotents can be lifted modulo D, if for any idempotent e+D in R/D, there exists an idempotent f in R such that f+D = e+D.

We shall now give the main characteristization of left perfect rings.

1.13. Theorem (Bass [2])

Let R be a ring and $J = J(R)$ its Jacobson radical. Then the following statements are equivalent:

(1) R/J is completely reducible and J is left T-nilpotent.

(2) R is left perfect.

(3) Every flat R-module is projective.

(4) R satisfies the descending chain condition on principal right ideals.

(5) R has no infinite set of orthogonal idempotents, and every non-zero right R-module M contains a simple R-module.

(6) R has no. infinite set of orthogonal idempotents and for every non-zero right R-module M, Soc(M) is large in M.

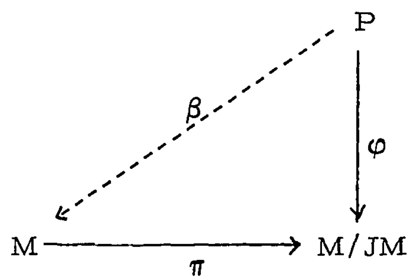
Proof.

The proof follows that of Bass [2]. We also consult [34] extensively.

Now (2) ⇒ (3) follows directly from Corollary 1.7.

Also (5) ⇔ (6) is immediate from Lemma 1.10.

(1) ⇒ (2). Let M be a left R-module. To show that M has a projective cover. Now M/JM is a left R/J-module, and since R/J is completely reducible, then $M/JM = \bigoplus_{i \in I} S_i$ where S_i is a simple left ideal of R/J for each i in some index set I. Hence $S_i = (R/J)\bar{f}_i$ where $\bar{f}_i = f_i + J$ and $\bar{f}_i^{\,2} = \bar{f}_i$. Now J is left T-nilpotent and hence nil. So idempotents can be lifted modulo J by [23, §3.6 Prop. 1]. Hence there exists $e_i^2 = e_i \in R$ such that $\bar{e}_i = \bar{f}_i$. Then $M/JM = \bigoplus_{i \in I} S_i \cong \bigoplus_{i \in I} Re_i/Je_i$. Let $P = \bigoplus_{i \in I} Re_i$ and consider $\varphi: P \to M/JM$, where the restriction of φ to Re_i is the canonical epimorphism $\varphi_i: Re_i \to Re_i/Je_i$. Clearly P is projective, being a direct sum of projective R-modules, and φ is an epimorphism. Also $\mathrm{Ker}\,\varphi = \bigoplus_{i \in I} \mathrm{Ker}\,\varphi_i = \bigoplus_{i \in I} Je_i = J(\bigoplus_{i \in I} Re_i)$ $= J(P) = J(R) \cdot P$ by 1.4(g). But since $J = J(R)$ is left T-nilpotent, then $J(R) \cdot P$ is small in P by 1.4(i). So $\mathrm{Ker}\,\varphi$ is small in P and $\varphi: P \to M/JM$ is a projective cover of M/JM. But P is projective. So there exists $\beta: P \to M$ such that $\pi\beta = \alpha$ where $\pi: M \to M/JM$. That is, the following diagram



commutes. Now $M = \operatorname{Im}\beta + \operatorname{Ker}\pi = \operatorname{Im}\beta + JM = \operatorname{Im}\beta$ since JM is small in M by 1.4(i). So β is an epimorphism. Also $\operatorname{Ker}\beta \subset \operatorname{Ker}\varphi$ which is small in P since $\varphi: P \to M/JM$ is a projective cover. Hence, by 1.4(b), $\operatorname{Ker}\beta$ is small in P.

Hence, by 1.4(b), $\operatorname{Ker}\beta$ is small in P. So $\beta: P \to M$ is a projective cover of M, and R is left perfect.

(3)⇒(4). Let $r_1R \supseteq r_2R \supseteq r_3R \supseteq \ldots$ be a descending chain of principal right ideals of R. Pick a sequence $\{a_i\} \subset R$ such that $r_1 = a_1$ and $r_{k+1} = r_k a_{k+1}$. Then $a_1R \supseteq a_1a_2R \supseteq a_1a_2a_3R \supseteq \ldots$ Hence it suffices to prove $a_1a_2 \ldots a_nR = a_1a_2 \ldots a_{n+k}R$ for $k \geq 1$. Let $F = \overset{\infty}{\underset{i=1}{\oplus}} Rx_i$ be a free left R-module countably generated on $\{x_i\}$. Also let $G = \overset{\infty}{\underset{i=1}{\oplus}} R(x_i - a_ix_{i+1})$ and $G_n = \overset{n}{\underset{i=1}{\oplus}} R(x_i - a_ix_{i+1})$ for each $n = 1, 2, \ldots$. Then clearly F/G_n is a free (hence flat) left R-module with basis $\{\bar{x}_{n+1}, \bar{x}_{n+2}, \ldots\}$ where $\bar{x}_i = x_i + G_n$. But $F/G = \varinjlim F/G_n$. So F/G is flat, since a direct limit of flat R-modules is flat [30, p.33]. So by hypothesis, F/G is projective and hence G is a direct summand of F. The proof would be complete after we prove the following lemma.

Lemma (*)

Suppose G is a direct summand of F where F and G are defined as above. Then the chain $\{a_1 \ldots a_n R\}$ of principal right ideals terminates.

Proof

The proof, based on that of Sandomierski [34], consists of four claims.

Claim 1

For each k, define $J_k = \{r \in R \mid ra_k \ldots ra_{k+n} = 0 \text{ for some } n\}$. Then $J_k = \{r \in R \mid r\bar{x}_k = 0\}$ where $\bar{x}_i = x_i + G$.

Proof of Claim 1

Observe that $\bar{x}_k = x_k + G = (x_k - a_k x_{k+1}) + a_k x_{k+1} + G = a_k x_{k+1} + G$ since $x_k - a_k x_{k+1} \in G$. So $\bar{x}_k = a_k(x_{k+1} + G) = a_k \bar{x}_{k+1}$. By a similar computation, $\bar{x}_{k+1} = a_{k+1}\bar{x}_{k+2}$. So $\bar{x}_{k+1} = a_k a_{k+1} \ldots a_{k+n}\bar{x}_{k+n+1}$ for some n. Now let $r \in J_k$. Then $ra_k \ldots a_{k+n} = 0$ for some n. Hence $r\bar{x}_k = r(a_k a_{k+1} \ldots a_{k+n}\bar{x}_{k+n}) = 0$. So $J_k \subseteq \{r \in R \mid r\bar{x}_k = 0\}$. For the reverse inclusion, let $r\bar{x}_k = 0$. Then $r(a_k \ldots a_{n+1}\bar{x}_{n+2}) = 0$ for $n \geq k$. Hence $ra_k \ldots a_{n+1}x_{n+2} \in G$. But this means that $ra_k \ldots a_{n+1}x_{n+2} = \sum_{i=1}^{n} r_i(x_i - a_i x_{i+1})$. By comparing coefficients, we see that $ra_k \ldots a_{n+1} = 0$, and so $r \in J_k$. Hence $\{r \in R \mid r\bar{x}_k = 0\} \subseteq J_k$, and Claim 1 is proved.

Now, by hypothesis, G is a direct summand of F. So if $\pi: F \to F/G$, then there exists $\alpha: F/G \to F$ such that $\pi\alpha = \mathrm{id}$. So $\bar{x}_n = \pi\alpha(\bar{x}_n) = \alpha(\bar{x}_n) + G$. Hence $x_n - \alpha(\bar{x}_n) \in G$ for all $n = 1, 2, \ldots$. Let $z_n = \alpha(\bar{x}_n)$. Then $x_n = z_n + g_n$ where $g_n \in G$.

Claim 2

$J_k = \{r \in R \mid rz_k = 0\}$ for all k.

Proof of Claim 2

By Claim 1, $J_k = \{r \in R \mid r\bar{x}_k = 0\}$. But $rz_k = r\alpha(\bar{x}_n) = \alpha(r\bar{x}_n)$ since α is an R-module homomorphism. Hence $rz_k = 0 \Leftrightarrow \alpha(r\bar{x}_k) = 0 \Leftrightarrow r\bar{x}_k = 0$ since α is a monomorphism. This proves Claim 2.

Since $z_n \in \operatorname{Im}\alpha$, then $z_n \in F$. So $z_n = \sum_{j=1}^{\infty} c_{n,j}x_j$ where $c_{n,j} \in R$ for $j=1,2,\ldots$. Consider the principal right ideal I of R generated by $\{c_{1,j}\}$ where $j=1,2,\ldots$

Claim 3

$I = \bigcap_{n=1}^{\infty} a_1 \ldots a_n R.$

Proof of Claim 3

Recall we showed in the proof of Claim 1 that $\bar{x}_1 = a_1\bar{x}_2 = \ldots = a_1a_2\ldots a_n\bar{x}_{n+1}$. But since $z_n = \alpha(\bar{x}_n)$ and α is a monomorphism, then $z_1 = a_1z_2 = \ldots = a_1a_2\ldots a_nz_{n+1}$. But $z_{n+1} = \sum_{j=1}^{\infty} c_{n+1,j}x_j$. Hence $z_1 = \sum_{j=1}^{\infty} c_{1,j}x_j = a_1a_2\ldots a_nz_{n+1} = a_1a_2\ldots a_n(\sum_{j=1}^{\infty} c_{n+1,j}x_j)$. So for each $j=1,2,\ldots$ $c_{1,j} = a_1\ldots a_nc_{n+1,j} \in a_1\ldots a_nR$ for all $n=1,2,\ldots$. But I is generated by $\{c_{1,j}\}$ for $j=1,2,\ldots$. So $I \subset \bigcap_{n=1}^{\infty} a_1\ldots a_nR$.

Claim 4

There exists an integer m such that $a_1\ldots a_m \in I$.

<u>Proof of Claim 4</u>

Now $z_1 = \sum_{j=1}^{\infty} c_{1,j}x_j = \sum_{j=1}^{\infty} c_{1,j}(z_j+g_j)$. But $z_1 \in \mathrm{Im}\,\alpha$ where $\alpha: F/G \to F$. So $z_1 = \sum_{j=1}^{\infty} c_{1,j}z_j = \sum_{j=1}^{\infty} c_{1,j}(\sum_{i=1}^{\infty} c_{j,i}x_i) = \sum_{i=1}^{\infty}(\sum_{j=1}^{\infty} c_{1,j}c_{j,i})x_i$. So $c_{1,j} = \sum_{j=1}^{\infty} c_{1,j}c_{j,i}$. In the proof of Claim 3, we showed that $z_j = a_j \ldots a_n z_{n+1}$ for all $n > j$. So $c_{j,k} = a_j \ldots a_n c_{n+1,k}$ for all $n > j$, and all $k=1,2,\ldots$. Hence for some $n = \alpha$, $c_{1,k} = \sum_{j=1}^{\alpha} c_{1,j}(a_j \ldots a_\alpha c_{\alpha+1,k})$ for all $k=1,2,\ldots$. Let $\gamma = \sum_{j=1}^{\alpha} c_{1,j}(a_j \ldots a_\alpha)$. Then $c_{1,k} = \gamma c_{\alpha+1,k}$ and hence $z_1 = \gamma z_{\gamma+1}$. But $z_1 = a_1 a_2 \ldots a_\alpha z_{\alpha+1}$. So $(\gamma - a_1 \ldots a_\alpha)z_{\alpha+1} = 0$. Hence by Claim 2, $\gamma - a_1 \ldots a_\alpha \in J_\alpha$. Let $\rho = \gamma - a_1 \ldots a_\alpha$. Then by Claim 1, $\rho a_1 \ldots a_\beta = 0$ for some β. So $\gamma a_1 \ldots a_\alpha = a_1 a_2 \ldots a_{\alpha+\beta}$. Hence if $m = \alpha+\beta$, then $a_1 a_2 \ldots a_m \in I$. By Claims 3 and 4, Lemma (*) is proved.

(4) ⇒ (5). Suppose there exists an infinite set of non-zero orthogonal idempotents of R. Let $\{e_i\}$ be a countably infinite set. Put $x_n = \sum_{i=1}^{n} e_i$. Then $(1-x_n)(1-x_{n+1}) = 1-x_{n+1} - x_n + x_n x_{n+1} = 1 - x_{n+1}$. So $(1-x_n)R \supseteq (1-x_{n+1})R$ for all $n=1,2,\ldots$. We now show that $(1-x_n)R \neq (1-x_{n+1})R$. For suppose $1-x_n = (1-x_{n+1})r$ for $r \in R$. Then $x_{n+1}(1-x_n) = (x_{n+1} - x_{n+1}^2)r = 0$. Hence $e_{n+1} = 0$, a contradiction. So $(1-x_n)R \supsetneqq (1-x_{n+1})R$ for all n. Let $r_i = 1-x_i$ for all $i=1,2,\ldots$. Then $r_1R \supsetneqq r_2R \supsetneqq r_3R \supsetneqq \ldots$ which contradicts (4). So there cannot exist any infinite set of orthogonal idempotents of R.

Now suppose no non-zero right R-module M contains a simple right R-module. Let $0 \neq x \in M$. Then by assumption, xR is not simple. So there exists $x_1 \neq 0$ such that $x_1 = xr$ where $r \in R$ and $x_1R \subsetneqq xR$. Again, by assumption, x_1R is not simple. Hence there exists $x_2 \neq 0$, $x_2 = x_1r_1$ where $r_1 \in R$ and $x_2R \subsetneqq x_1R$. Continuing in this fashion, we have that $xR \supsetneqq x_1R \supsetneqq x_2R \supsetneqq \dots$ which again contradicts (4). This proves that every non-zero right R-module contains a simple right R-module.

(5) ⇒ (1). If $J = 0$, there is nothing to prove. So assume $J \neq 0$. For each ordinal β, define inductively the right ideal $J_\beta \subset J$ of R by

(1) $\alpha < \beta \Rightarrow J_\alpha \subset J_\beta$.

(2) $J_0 = \{0\}$.

(3)
$$J_\beta = \begin{cases} \bigcup_{\alpha<\beta} J_\alpha & \text{if } \beta \text{ is a limit ordinal} \\ J_{\gamma+1} & \text{where } \beta=\gamma+1 \text{ and where } J_{\gamma+1}/J_\gamma = \mathrm{Soc}(J/J_\gamma). \end{cases}$$

Now there exists β_0 such that $J_{\beta_0} = J_{\beta_0+1}$, and so $\mathrm{Soc}(J/J_{\beta_0}) = J_{\beta_0+1}/J_{\beta_0} = 0$. By hypothesis, a non-zero module has non-zero socle. So $J/J_{\beta_0} = 0$ $\Rightarrow J = J_{\beta_0}$. Now for each $0 \neq a \in J$, define h(a) to be least ordinal β such that $a \in J_\beta$. Then h(a) cannot be a limit ordinal, for if so, then $a \in J_\beta = \bigcup_{\alpha<\beta} J_\alpha$. Hence $a \in J_\alpha$ for some $\alpha < \beta$, which contradicts the minimality of β. So then $h(a) = \gamma+1$ for some γ. But $J_{\gamma+1}/J_{\gamma} = \mathrm{Soc}(J/J_{\gamma})$, and since $\mathrm{Soc}(J/J_{\gamma})$ is completely reducible, then $J_{\gamma+1} \cdot J \subset J_{\gamma}$. So if $a_1, a_2 \in J$, then $h(a_1 a_2) < h(a_1)$. Now let $\{a_i\} \subset J$. If J is not left T-nilpotent, then $\prod_{i=1}^{n} a_i \neq 0$ for all integers n. But then $h(a_1) > h(a_1 a_2) > h(a_1 a_2 a_3) > \ldots$ would be an infinite decreasing chain of ordinals — a contradiction. Hence J is left T-nilpotent. Finally, to show R/J is completely-reducible. We shall prove $R/J = \mathrm{Soc}(R/J)$. Now J is left T-nilpotent and so is nil. Hence, if R has no infinite set of orthogonal idempotents, then so is R/J. So Soc(k/J) is a finite direct sum of minimal ideals each of which is generated by idempotent. Hence Soc(R/J) is a direct summand of R/J. But then the complementary module has zero socle and so is also zero, by hypothesis. Hence $R/J = \mathrm{Soc}(R/J)$.

This completes the proof of Theorem 1.13. We shall now deduce some interesting corollaries.

1.14. Definition

A ring R is called semi-primary ⇔ R/J is completely reducible and $J = J(R)$ is nilpotent.

1.15. Corollary

Let R be right Noetherian. Then R is right Artinian ⇔ R is left perfect.

Proof

(⇒). Let R be right Artinian. Then R has the descending chain condition on right ideals, and hence on principal right ideals. So R is left perfect.

(⇐). Since R is left perfect, then R/J is completely reducible and J is left T-nilpotent. Hence J is nil. But R is right Noetherian, and so by [23, p.70], J is nilpotent. Hence R is semi-primary. But if R is right Noetherian and semi-primary, then R is right Artinian by [6, § 6, Prop. 12].

1.16 Definition

An ideal D of R is called a right annihilator, and denoted by r(D), if $D = \{r \in R \mid Dr = 0\}$.

1.17 Corollary (Faith [11])

Let R satisfy the ascending chain condition on right annihilators. Then R is semi-primary ⇔ R is left perfect.

Proof

(⇒). This implication is clear since a nilpotent ideal is left T-nilpotent.

(⇐). It suffices to prove that J is nilpotent. Consider $r(J) \subseteq r(J^2) \subseteq r(J^3) \subseteq \dots$ By hypothesis, there exists an integer n such that $r(J^n) = r(J^{n+1})$. We claim $r(J^n) = R$. Suppose not. Then $Soc(R/r(J^n)) \neq 0$ since by Theorem 1.13, non-zero modules over a perfect ring have non-zero socles. So $Soc(R/r(J^n)) = I/r(J^n)$ where I is a left ideal such that $I \supsetneq r(J^n)$. But $Soc(R/r(J^n))$ is completely-reducible and hence

$J(I/r(J^n)) = 0$. This implies that $J \cdot I \subset r(J^n)$. Hence $I \subset r(J^{n+1}) = r(J^n)$. So $I = r(J^n)$, a contradiction. So $r(J^n) = R \Rightarrow J^n = 0$.

1.18. Corollary

Let R be a left perfect ring. Then, for all left R-modules M,

(1) $J(R)\cdot M$ is small in M and

(2) $J(R)\cdot M = J(M)$.

Proof

Let R be left perfect. Then R/J is completely reducible and J is left T-nilpotent. Hence conditions (1) and (2) follow from Lemma 1.4(i) and (j).

The author had made the following conjecture in a Student Ring Theory Seminar at McGill University (1973).

1.19. Conjecture

Let R be a ring, J(R) its Jacobson radical and suppose R does not contain an infinite set of orthogonal idempotents. Then R is left perfect $\Leftrightarrow$ for all left R-modules (1) $J(R)\cdot M$ is small in M and (2) $J(R)\cdot M = J(M)$.

Discussion

Now Corollary 1.18 shows that a left-perfect ring satisfies the conditions. So we examine the converse. But Lemma 1.4(i) states that $J(R)\cdot M$ is small in M $\Leftrightarrow$ J(R) is left T-nilpotent. So in view of Theorem 1.13, it suffices to prove that R/J is completely reducible. Since J(R) is left T-nilpotent then it is nil. Hence, since R does not have an infinite set of orthogonal idempotents, neither does R/J. But B.Osofsky [28] has remarked that a regular ring in which there does not exist an infinite set of orthogonal idempotents is completely reducible. So it suffices to prove that R/J is regular. But clearly condition (2) implies that R/J is a V-ring (i.e. every left R/J-module has zero Jacobson radical). However, a theorem of I.Kaplansky [31] states that if a ring S is commutative, then S is regular ⇔ S is a V-ring. Also an example of J.Cozzens [10] shows that the commutativity condition cannot be dropped. So our conjecture is true if R is commutative.

We remark here that H.Bass [2] has conjectured that R is left perfect ⇔ every non-zero R-module has a maximal submodule and there does not exist an infinite set of orthogonal idempotents. R.Hamsher [17] and D.Fieldhouse [13] both independently settled this conjecture in the affirmative, when R is commutative. As above, J. Cozzen's example shows that commutativity is necessary. We do not know if Bass' conjecture is equivalent to Conjecture 1.19.

1.20. Corollary

R is left perfect ⇔ every flat left R-module has a projective cover.

Proof.

(⇒). Follows from the definition of left perfect rings.

(⇐). Suppose M is a flat left R-module which has a projective cover. Then M is projective by Corollary 1.7. So, by Theorem 1.13, R is left perfect.

1.21. Definition

An R-module M is called quasi-projective ⇔ $\mathrm{Hom}_R(M,-)$ preserves the exactness of all short exact sequences of the form $0 \to K \to M \to N \to 0$.

The following result was proved by J. Golan [15] and also by N. Vanaja [36].

1.22. Corollary

R is left perfect ⇔ every flat left R-module is quasi-projective.

Proof.

(⇒). Clear, since a projective module is quasi-projective.

(⇐). Assume every flat left R-module is quasi-projective. Let $F = \bigoplus_{i=1}^{\infty} Rx_i$ be free on the countable set $\{x_i\}$. Also, let $G = \bigoplus_{i=1}^{\infty} R(x_i - a_i x_{i+1})$ and $G_n = \bigoplus_{i=1}^{n} R(x_i - a_i x_{i+1})$ for each $n=1,2,\ldots$. Then as in the proof of Theorem 1.13, F/G_n is a free, hence flat, left R-module for each n. Then $F/G = \varinjlim F/G_n$ is flat. But then $F \oplus F/G$ is also flat, and hence quasi-projective by hypothesis. So by [15, Lemma 1.1], G is a direct summand of F. Hence by Lemma (*) of Theorem 1.13, R satisfies the descending chain condition on principal right ideals.

1.23. Remark

Corollary 1.22 characterizes all rings R for which every flat left R-module is quasi-projective — namely, left perfect rings.

It would be interesting to characterize those rings R for which every quasi-projective R-module is flat. J. Golan has conjectured (in private communication) that every quasi-projective R-module is flat $\Leftrightarrow$ every simple R-module is injective. N. Vanaja [36] has proved that if R is commutative, then every quasi-projective R-module is flat $\Leftrightarrow$ R is regular. Clearly Vanaja's result is a special case of Golan's conjecture since I. Kaplansky [31] has proved that a commutative ring R is regular $\Leftrightarrow$ every simple R-module is injective.

It is well-known that R is left Noetherian $\Leftrightarrow$ a direct limit of injective left R-modules is injective. See, for example [30, p.87]. We now prove a similar result for projective modules over left perfect rings.

1.24. Corollary (Bass [2])

R is left perfect $\Leftrightarrow$ a direct limit of projective left R-modules is projective.

Proof

($\Rightarrow$). Let $\{P_i \mid i \in I\}$ be a family of projective left R-modules and let $P = \varinjlim P_i$. Then P is flat since a direct limit of projective (hence flat) R-modules is flat. Hence P is projective since R is left perfect.

($\Leftarrow$). As in the proof of Corollary 1.22, we have that $F/G = \varinjlim F/G_n$ where $F = \bigoplus_{i=1}^{\infty} Rx_i$, $G = \bigoplus_{i=1}^{\infty} R(x_i - a_i x_{i+1})$ and $G_n = \bigoplus_{i=1}^{n} R(x_i - a_i x_{i+1})$ for each $n = 1, 2, \ldots$. By hypothesis, F/G is projective. Hence G is a direct summand of F. By Lemma (*) of Theorem 1.13, R is left perfect.

1.25. Corollary

Let R be a local ring such that $J = J(R)$ is left T-nilpotent. Then a direct limit of free left R-modules is free.

Proof

Let $\{F_i \mid i \in I\}$ be a family of free left R-modules, and let $F = \varinjlim F_i$. Since R is local, then R/J is completely reducible. Since J is left T-nilpotent, then R is left perfect. So by Corollary 1.24, F is projective. Since R is local, F is free by [19, p.374].

It is well-known that a direct product of projective R-modules need not be projective. R. Baer [20] has shown that if $R = Z$, the ring of integers, then ΠZ is not projective where the product is taken over a countably infinite set. However, we do have the following result.

1.26. Corollary

Let R be left Noetherian and left perfect. Then a direct product of an arbitrary family of projective left R-modules is projective.

Proof.

Let $\{P_i \mid i \in I\}$ be a family of projective left R-modules and let $P = \prod_{i \in I} P_i$. By [7, p. 122], a direct product of flat left R-modules over a left Noetherian ring is flat. Hence P is flat and so projective since R is left perfect.

1.27. Corollary

If R is commutative Artinian, then a direct product of a family of projective R-modules is projective.

Proof

This follows directly from Corollary 1.15 which states that if R is commutative, then a perfect ring which is also Noetherian is Artinian.

1.28. Remarks

(a) A ring R is called right coherent if each of its finitely-generated right ideals is finitely-related (i.e. it is a quotient of a finitely-generated free R-module by a finitely-generated submodule). S. Chase [8] has characterized all rings for which a direct product of projective R-modules is projective. These turn out to be rings which are left perfect and right coherent. It is interesting to note that over a right coherent ring, a direct product of flat left R-modules is flat.

(b) In view of Corollary 1.25, a natural question presents itself: Over what rings is a direct limit of free left R-modules free? Actually, the converse of 1.25 is true, and hence R is local and J(R) is left T-nilpotent ⇔ a direct limit of free left R-modules is free. This follows from a theorem of V. Govorov [14] which states that R is local and J(R) left T-nilpotent ⇔ every flat R-module is free.

We have seen in Corollary 1.20, that R is left perfect $\Leftrightarrow$ every flat left R-module has a projective cover. However, this condition can be weakened further.

<u>1.29. Theorem</u> (Sandomierski [33])

R is left perfect $\Leftrightarrow$ every completely reducible R-module has a projective cover.

<u>Proof</u>.

($\Rightarrow$). Follows from the definition of left perfect rings.

($\Leftarrow$). In view of [23, §4.2, Ex.11], it suffices to prove that $J = J(R)$ is left T-nilpotent. Also by Lemma 1.4(i), it is sufficient to prove $J \cdot F$ is small in F for any free left R-module F. So let F be a free left R-module. Then $F/J\cdot F$ is completely reducible R/J-module. So, being also an R-module, it has a projective cover $\pi: P \to F/JF$. But F, being free, is projective. Hence, there exists $\beta: F \to P$ such that $\pi\beta = \alpha$ where α is the canonical map $F \to F/JF$. That is, the following diagram

$$\begin{array}{ccc} & & F \\ & \swarrow^{\beta} & \downarrow \alpha \\ P & \xrightarrow{\pi} & F/JF \end{array}$$

commutes. Now $P = \operatorname{Im}\beta + \operatorname{Ker}\pi = \operatorname{Im}\beta$ since $\operatorname{Ker}\pi$ is small in P. So β is an epimorphism. But P is projective. Hence $\operatorname{Ker}\beta$ is a direct summand of F. Since F is free, then Ker β is projective. Let $K = \text{Ker}\,\beta$. Then $F \cong P/K$. So by [23, §5.4, Prop.3], $JF \cap K = JK$. But clearly $K = \text{Ker}\,\beta \subseteq \text{Ker}\,\alpha = JF$. Hence $JF \cap K = K$. So $JK = K$. Since K is projective, then by [2, p.474], $K = 0$, and so β is an isomorphism. Hence $J \cdot F$ is small in F.

CHAPTER II

In this chapter, we give a characterization of semi-perfect rings due to H. Bass [2]. The principal result is Theorem 2.5.

2.1. Definition

H. Bass [2] called a ring R left semi-perfect $\Leftrightarrow$ every finitely-generated left R-module has a projective cover.

2.2. Lemma

If R is left semi-perfect, then every finitely-generated flat left R-module is projective.

Proof

Follows directly from Corollary 1.7.

2.3. Remark

The converse of Lemma 2.2 is false. The following example taken from [34] shows this. Let $R = Z$, the ring of integers. Then clearly R satisfies the condition. In fact, it is clear that commutative Noetherian rings have this property. However Z is not semi-perfect. For suppose $Z_n = Z/nZ$, $n > 0$ is a left Z-module which has a projective cover $\pi: P \to Z_n$. Then, since Z is projective as a left Z-module, then there exists $\beta: Z \to Z_n$ such that $\pi\beta = \alpha$ where $\alpha: Z \to Z_n$. Hence $P = \mathrm{Im}\,\beta + \mathrm{Ker}\,\pi = \mathrm{Im}\,\beta$ since $\mathrm{Ker}\,\pi$ is small in P. So β is an epimorphism and hence $P \cong Z_m$ for some integer m. But P is a projective Z-module and hence free.

So $m = 0$ and $P \cong Z$. But

$$\begin{aligned} \operatorname{Ker} \pi &\subseteq J(P) && \text{by } 1.4(e) \\ &= J(Z) && \text{since } P \cong Z \\ &= 0. \end{aligned}$$

Hence π is an isomorphism, and so $P \cong Z_n$ where $n > 0$. This contradicts that $P \cong Z_n$. Hence Z is not semi-perfect.

2.4. Lemma

Let I be a two-sided ideal of R and A an R/I-module. If $\pi: P \to A$ is an R-projective cover of A, when the induced map $\pi': P/PI \to A$ is an R/J-projective cover of A.

Proof

The obvious definition of $\pi': P/PI \to A$ is given by $\pi'(p+PI) = \pi(p)$ for $p \in P$. This is clearly well-defined since I annihilates A. Since $P/PI \cong R/I \otimes_R P$, then P/PI is a projective left R/I-module by [23, §5.3, Prop.3]. Also $\operatorname{Ker} \pi' \subset \operatorname{Ker} \pi/PI$. But $\operatorname{Ker} \pi/PI$ is small in P/PI by 1.4(a). Hence $\operatorname{Ker} \pi'$ is small in P/PI by 1.4(b). This completes the proof.

2.5. Theorem (Bass [2])

Let R be a ring and $J = J(R)$ its Jacobson radical. Then the following are equivalent:

(1) R is left semi-perfect.

(1′) R is right semi-perfect.

(2) Every cyclic left R-module has a projective cover.

(2′) Every cyclic right R-module has a projective cover.

(3) R/J is completely reducible and idempotents can be lifted modulo J(R).

Note that since condition (3) is symmetric, it suffices to prove (1) ⇒ (2) ⇒ (3) ⇒ 1.

<u>Proof</u>

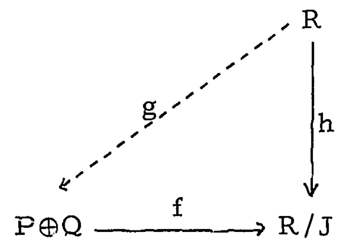
Our proof is based on that of H. Bass.

(1) ⇒ (2). Obvious from Definition 2.1.

(2) ⇒ (3). For this implication, we follow the argument outlined in the exercises of [23, p.93]. So let C be a cyclic left R/J-module. Then C, being also a left R-module, has an R-projective cover. So by Lemma 2.4, C has an R/J-projective cover. But R/J is semi-primitive. Hence by [23, §4.2, Ex. 11], R/J is completely reducible.

We now show that idempotents can be lifted modulo J. So let $r \in R$ such that $\bar{r}^2 = \bar{r}$ where $\bar{r} = r+J$. We shall show that there exists $e \in R$, $e^2 = e$ such that $\bar{e} = \bar{r}$. Let $\bar{s} = \bar{1} - \bar{r}$ and $\bar{R} = R/J$. Also let $\alpha: P \to \bar{R}\bar{r}$ and $\beta: Q \to \bar{R}\bar{s}$ be projective covers for $\bar{R}\bar{r}$ and $\bar{R}\bar{s}$ respectively. (These exist by hypothesis and by Lemma 2.4.) Then by [23, §4.2, Ex. 9], $f: P \oplus Q \to \bar{R}\bar{r} \oplus \bar{R}\bar{s}$ is a projective cover of $\bar{R}$. (Observe that $\bar{R} = \bar{R}\bar{r} \oplus \bar{R}\bar{s}$.) Note that $f(p+q) = \alpha(p) + \beta(q)$ for $p \in P$ and $q \in Q$. But R is projective as an R-module. Hence, there exists

$g: R \to P \oplus Q$ such that $fg = h$ where $h: R \to R/J$. That is, the following diagram



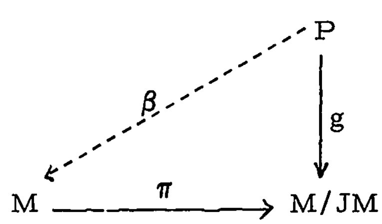
commutes. Hence $P \oplus Q = \mathrm{Im}\, g + \mathrm{Ker}\, f = \mathrm{Im}\, g$ since $\mathrm{Ker}\, f$ is small in $P \oplus Q$. So g is an epimorphism. But $P \oplus Q$ is projective, and so $\mathrm{Ker}\, g$ is a direct summand of R. But $\mathrm{Ker}\, g \subset \mathrm{Ker}\, h = J(R)$ which is small in R by 1.4(d). Hence $\mathrm{Ker}\, g = 0$, and so $R \cong P \oplus Q$. Hence $R = g^{-1}(P \oplus Q) = g^{-1}(P) \oplus g^{-1}(Q) = Re \oplus R(1-e)$ where $e^2 = e \in R$. (This last equality follows from the well-known fact that a left ideal I of R is a direct summand $\Leftrightarrow I = Re$ where $e^2 = e \in R$.) So $h(Re) = fg(Re)$ $= fg(g^{-1}(P)) = f(P) = \alpha(P) = \bar{R}\bar{r}$. But $h(Re) = \bar{R}\bar{e}$. Hence $\bar{R}\bar{e} = \bar{R}\bar{r}$. Similarly $\bar{R}\bar{f} = \bar{R}\bar{s}$ where $\bar{f} = \bar{1} - \bar{e}$. So $\bar{e} + \bar{f} = \bar{1} = \bar{r} + \bar{s}$. Hence $\bar{e} - \bar{r} = \bar{s} - \bar{f}$. But both sides of the last equation are in different summands. So $\bar{e} - \bar{r} = 0 = \bar{s} - \bar{f}$. This shows that $\bar{e} = \bar{r}$, and so idempotents can be lifted modulo $J(R)$.

(3) $\Rightarrow$ (1). Let M be a finitely-generated left R-module. To show that M has a projective cover. Let $\bar{R} = R/J$. Then M/JM is a left $\bar{R}$-module and since $\bar{R}$ is completely reducible, then $M/JM = \bigoplus_{i=1}^{n} S_i$ where S_i is a simple left ideal of $\bar{R}$ for each $i = 1, 2, \ldots, n$.

But since $S_i = \bar{R}\bar{e}_i$ where $\bar{e}_i = e_i + J$ and $e_i^2 = e_i$, then $M/JM = \bigoplus_{i=1}^{n} \bar{R}\bar{e}_i$. But idempotents can be lifted modulo J. So there exists $f_i \in R$, $f_i^2 = f_i$ and $\bar{f}_i = \bar{e}_i$ for each $i = 1, \ldots, n$. Hence $M/JM = \bigoplus_{i=1}^{n} Rf_i/Jf_i$. Define $P = \bigoplus_{i=1}^{n} Rf_i$. Then, clearly, P is a projective left R-module since it is a direct sum of direct summands of R. Consider

$$g: P \to M/JM \qquad (1)$$

We shall show that (1) is a projective cover of P. Note that $g = \bigoplus_{i=1}^{n} \operatorname{Ker} g_i$ where $g_i: Rf_i \to Rf_i/Jf_i$. Clearly $\operatorname{Ker} g_i$ is small in Rf_i, since $\operatorname{Ker} g_i \subset \operatorname{Ker} h$ where $h: R \to R/J$, and $J(R)$ is small in R by 1.4(d). So $\operatorname{Ker} g = \bigoplus_{i=1}^{n} \operatorname{Ker} g_i$ is small in P by 1.4(c). So (1) is a projective cover of M/JM. But P is projective, and hence there exists $\beta: P \to M$ such that $\pi\beta = g$ where π is the canonical map : $M \to M/JM$. That is, the following diagram



commutes. Observe that $\operatorname{Ker} \pi = JM$ is small in M by Nakayama's Lemma. So $M = \operatorname{Im} \beta + \operatorname{Ker} \pi = \operatorname{Im} \beta$. Hence β is an epimorphism. Also $\operatorname{Ker} \beta \subset \operatorname{Ker} g$ which is small in P since (1) is a projective cover of M/JM. Hence $\beta: P \to M$ is a projective cover of M.

This completes the proof.

We shall now obtain a characterization of semi-perfect in terms of a weaker condition than any of the equivalent ones of Theorem 2.5.

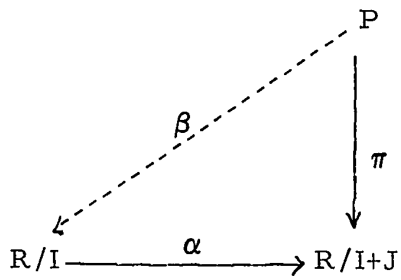
2.6. Theorem (Sandomierski [33])

A ring R is semi-perfect $\Leftrightarrow$ every simple left R-module has a projective cover.

Proof

($\Rightarrow$). R semi-perfect $\Rightarrow$ every cyclic, and hence simple, left R-module has a projective cover.

($\Leftarrow$). First we show that R/J is completely reducible. Let S be a simple left R/J-module. Then S, being also a left R-module, has an R-projective cover. So by Lemma 2.4, S has an R/J-projective cover. Then by [23, p.93], R/J is completely reducible. Now, let C be a cyclic left R-module. We shall show that C has a projective cover. Let $C = R/I$ where I is a left ideal of R. Then $R/I+J$ is a finitely-generated left R/J-module. Since R/J is completely reducible, then $R/I+J = \bigoplus_{i=1}^{n} S_i$ where S_i is a simple left R/J-module for each $i=1, \ldots, n$. As above, each S_i has a projective cover and so does $R/I+J$ by [23, § 4.2, Ex.9]. Let $\pi: P \to R/I+J$ be a projective cover of $R/I+J$. Then since P is projective, there exists $\beta: P \to R/I$ such that $\alpha\beta = \pi$ where α is the canonical map : $P \to P/I$. That is, the following diagram



commutes. Now $\mathrm{Ker}\,\alpha = I+J/I$. But $I+J/I$ is the homomorphic image of $J(R)$ under the canonical map $R \to R/I$. Hence by Lemma 1.4(a), $\mathrm{Ker}\,\alpha$ is small in R/I. But $R/I = \mathrm{Im}\,\beta + \mathrm{Ker}\,\alpha$. Hence $R/I = \mathrm{Im}\,\beta$ and so β is an epimorphism. Also $\mathrm{Ker}\,\beta \subset \mathrm{Ker}\,\pi$ which is small in P. So $\beta: P \to R/I = C$ is a projective cover of C and hence R is semi-perfect.

2.7. Corollary

R is semi-perfect $\Leftrightarrow$ every simple left R-module has the form Re/Je where $e^2 = e \in R$.

Proof

($\Rightarrow$). Let A be a simple left R-module. Then either $JA = 0$ or $JA = A$.

Case 1. Let $JA = 0$. Then A, as an R/J-module is simple, since any submodule of $_{R/J}A$ is also a submodule of $_RA$. So $A = (R/J)\bar{f}$ where $\bar{f} = f+J$ and $\bar{f}^2 = \bar{f}$. But idempotents can be lifted modulo J(R). Hence $A = Re/Je$ where $e^2 = e \in R$ and $\bar{e} = \bar{f}$.

Case 2. Let $A = JA$. Since R is semi-perfect, then every simple left R-module has a projective cover by Theorem 2.6. So let $\pi: P \to A$ be a projective cover of A, and let $K = \text{Ker}\,\pi$. Then $A \cong P/K$ and by assumption, $P/K = J(P/K)$. Hence $P = J(P)+K = J(P)$ since K is small in P. But P is projective, and so by [2, p.474], $P = 0$. Hence $A = 0$.

($\Leftarrow$). Assume every simple left R-module has the form Re/Je where $e^2 = e \in R$. Consider $f: Re \to Re/Je$ where f is the restriction of the canonical epimorphism $g: R \to R/J$. Clearly Re is a projective left R-module and f is an epimorphism. So it remains to show that $\text{Ker}\, f$ is small in Re. But $\text{Ker}\, f = Je = J \cdot Re$ by 1.4(g). Also $J.Re \subseteq J(Re)$ by 1.4(f). Furthermore, $J(Re)$ is small in Re since Re is finitely-generated. Hence $\text{Ker}\, f$ is small in Re. This shows that every simple left R-module has a projective cover, and so, by Theorem 2.6, R is semi-perfect.

CHAPTER III

A: Lifting Idempotents

3.1. Definitions

(a) A set $\{e_i\}$ of idempotents of R is called mutually orthogonal if $e_i e_j = 0$ for $i \neq j$.

(b) An idempotent e of R is called primitive if it cannot be written as the sum of two non-zero orthogonal idempotents.

(c) An idempotent e is called local if eRe is a local ring. (I.e. it has a unique maximal ideal.)

3.2. Lemma

Let R be a ring and J(R) its Jacobson radical. If $x^2 = x \in J(R)$, then $x = 0$.

Proof

Suppose $x^2 = x \in J(R)$. Then $x(1-x) = 0$. But $x \in J(R) \Rightarrow \exists\, r \in R$ such that $(1-x)r = 1$. So $x = x \cdot 1 = x(1-x)r = 0$.

3.3. Lemma

If e and f are idempotents in a ring R such that $ef = fe$ and $e - f \in J(R)$, then $e = f$.

Proof

Since J(R) is a two-sided ideal of R, then $f - ef \in J(R)$. But $(f-ef)^2 =$ $= f^2 - fef - ef^2 + efef = f - ef$. So by Lemma 3.2 $f = ef$. But $(e-f)^2 =$ $= e^2 + f^2 - ef - fe = e + f - 2ef = e - f$. Again by Lemma 3.2, $e = f$.

3.4. Lemma

Let N be a nil ideal of R. Then idempotents can be lifted to idempotents of R/J.

Proof

Our proof follows that of L Levy [24] who attributes it to A.W Goldie. Let $u \in R$ such that $\bar{u}^2 = \bar{u}$ where $\bar{u} = u+N$. Since N is nil, there exists an integer k such that $(u^2-u)^k = 0$. Observe that $(1-x)^k = 1-xf(x)$ where f(x) is some polynomial with integer coefficients. So $0 = (u-u^2)^k =$ $= u^k(1-u)^k = u^k(1-uf(u)) = u^k - u^{k+1}f(u)$. Hence $u^k = u^{k+1}f(u)$. Let $e = u^k f(u)^k$. Then we shall show that $e^2 = e$. But this is clear since $e = u^k f(u)^k =$ $= u^{k+1}f(u)^{k+1} = \ldots = u^{2k}f(u)^{2k} = e^2$, by repeated use of $u^k = u^{k+1}f(u)$. Also $\bar{e} = \bar{u}^k \overline{f(u)}^k = \bar{u}^{k+1}\overline{f(u)}^k = \bar{u}^k \overline{f(u)}^{k-1} = \ldots = \bar{u}^k = \bar{u}$ since $\bar{u}$ is an idempotent. This completes the proof.

3.5. Corollary

R right Artinian $\Rightarrow$ R is semi-perfect.

Proof.

Let R be right Artinian. Then R/J is completely reducible. But J(R) is nil. Hence idempotents can be lifted modulo J(R). So R is semi-perfect.

3.6. Lemma (Kaye [21])

Let R be semi-perfect. Then an idempotent e of R is primitive $\Leftrightarrow$ e+J(R) is primitive in R/J(R).

Proof

Let $\bar{e} = e+J$ where $J = J(R)$, and suppose $\bar{e}$ is not primitive in R/J. Then there exist idempotents f_1 and f_2 in R/J such that $\bar{e} = f_1+f_2$ and $f_1f_2 = 0$. But R is semi-perfect. So there exists idempotents e_i in R where $\bar{e}_i = f_i$ for $i=1,2$. By [23, §3.6, Prop.2], we may assume $e_1e_2 = 0$. But then e and e_1+e_2 are idempotents in R such that $e(e_1+e_2) = (e_1+e_2)e$ and $\bar{e} = \bar{e}_1+\bar{e}_2$. Hence by Lemma 3.3, $e = e_1+e_2$. This contradicts that e is primitive. Hence e is primitive in R $\Rightarrow$ $\bar{e}$ is primitive in R/J.

($\Leftarrow$). This is clear. For suppose e is not primitive in R, then $e = e_1+e_2$ where $e_1e_2 = 0$ and $e_i \neq 0$ for $i=1,2$. But then $\bar{e} = \bar{e}_1+\bar{e}_2$ in R/J where $\bar{e}_i \neq 0$ and $\bar{e}_1\bar{e}_2 = 0$. (Note that $\bar{e} = \pi(e)$ where $\pi: R \to R/J$.) Hence $\bar{e}$ is not primitive in R/J. So $\bar{e}$ primitive in R/J $\Rightarrow$ e is primitive in R.

3.7. Lemma (Lambek [23])

Let R be semi-perfect. Then a primitive idempotent e is local.

Proof

We adopt the proof given in [23]. So let e be a primitive idempotent of R and let $J = J(R)$. Since R is semi-perfect, then R/J is completely reducible and hence regular. So if $u \in eRe$, then there exists $u' \in R$ such that $uu'u \equiv u$ modulo J. We may assume $u' = eue$. Clearly uu' is an idempotent in R/J. Also $e = eu$ and $u' = u'e$, and so $uu'(1-e) = 0$

and $(1-e)uu' = 0$. Since R is semi-perfect, idempotents can be lifted. Hence there exists $f^2 = f \in R$ such that $\bar{f} = \overline{uu'}$. Furthermore, by [23, §3.6, Prop.2], we may assume $f(1-e) = 0$ and $(1-e)f = 0$. So $f = ef = fe$, and hence $f \in eRe$. Clearly f and e-f are orthogonal and $e = f+e-f$. Since e is primitive, $f = 0$ or $e-f = 0$. Now let $\bar{u} \neq 0$. Then $f \neq 0$ since $\overline{fu} = \overline{uu'u} = \bar{u}$. So $e = f$. In this case $\overline{uu'} = \bar{f} = \bar{e}$ and so $\bar{u}$ is right invertible. Similarly, $\bar{u}$ is left invertible. So if $\bar{u} \neq 0$, then it is a unit. This shows that $eRe/J \cap eRe$ is a division ring. But by [23, §3.7, Lemma 1], $J \cap eRe = J(eRe)$. So eRe is a local ring.

3.8. Corollary

A ring R is local ⇔ R is semi-perfect and 1 is a primitive idempotent.

Proof.

(⇒). If R is local, then R/J is a division ring, and hence completely reducible. Also, the two idempotents 0 and 1 are easily lifted.

(⇐). By Lemma 3.7, $R = 1R1$ is local.

3.9. Corollary

Let R be a local ring. Then every finitely-generated flat R-module is free.

Proof.

Since R is local, then it is semi-perfect. So by Lemma 2.2, every finitely-generated flat R-module is projective, and hence free since R is local.

3.10. Theorem (Lambek-Müeller)

Let R be a ring. $J = J(R)$ its Jacobson radical and 1 the identity of R. Then the following statements are equivalent.

(a) R is semi-perfect.

(b) $1 = \sum_{i=1}^{n} e_i$ where e_i is a local orthogonal idempotent for each $i=1, \ldots, n$.

Proof

(a) ⇒ (b). The proof of this implication is given by Lambek [23]. In view of Lemma 3,7, it suffices to prove that $1 = \sum_{i=1}^{n} e_i$ where each e_i is a primitive orthogonal idempotent. Since R is semi-perfect, then R/J is completely reducible. The indecomposable left ideals in R/J are the minimal left ideals. Hence $R/J = \bar{R}\bar{f}_1 + \ldots + \bar{R}\bar{f}_n$ where $\bar{R} = R/J$ and $\{\bar{f}_1 + \ldots + \bar{f}_n\}$ is a set of primitive orthogonal idempotents in $\bar{R}$ such that $\bar{f}_1 + \ldots + \bar{f}_n = 1$. But by hypothesis, idempotents can be lifted modulo J. Hence there exists a set of idempotents $\{e_1, \ldots, e_n\}$ in R such that $\bar{e}_i \bar{f}_i$ for $i=1, \ldots, n$. Also the e_i's are primitive by Lemma 3.6 and mutually orthogonal by [23, §3.6, Prop.2]. Suppose $x = \sum_{i=1}^{n} e_i$. Then $x^2 = x$ and $(1-x)^2 = 1-2x+x^2 = 1-x \equiv 0$ modulo J. Hence $1-x$ is an idempotent in J. By Lemma 3.2, $1-x = 0 \Rightarrow 1 = \sum_{i=1}^{n} e_i$.

(b) ⇒ (a). Our proof essentially follows that of E. Behrens [4]. So assume that $1 = e_1 + \ldots + e_n$ where each e_i is a local idempotent.

First we show that $\bar{R} = R/J$ is completely reducible. Clearly it suffices to prove that $\bar{e}_i\bar{R}$ is irreducible for each i. Consider $\bar{R} = \bar{e}\bar{R} \oplus (\bar{1}-\bar{e})\bar{R}$ where $e = e_i$ for some i. Then $(\bar{1}-\bar{e})\bar{R} \neq \bar{R}$. So there exists a maximal right ideal L of $\bar{R}$ such that $(\bar{1}-\bar{e})\bar{R} \subset L$. Hence $\bar{R} = \bar{e}\bar{R}+L$. We shall now show that this sum is direct. If not, then $L \cap \bar{e}\bar{R} \neq 0$. Hence $(L\cap \bar{e}\bar{R})^2 \neq 0$ for otherwise $L \cap \bar{e}\bar{R} \subset J(\bar{R}) = 0$. Then, there exists $x \in R$ such that $\bar{e}\bar{x} \in L$, and $\bar{e}\bar{x}\bar{e}\bar{R} \neq 0$. In particular $\bar{e}\bar{x}\bar{e} \neq 0$. Moreover, $\bar{e}\bar{x}\bar{e}$ has an inverse $\bar{y}$ in $\bar{e}\bar{R}\bar{e}$ since e local $\Rightarrow eRe/J(eRe) = \bar{e}\bar{R}e$ is a division ring. So $\bar{e}\bar{x}\bar{e}\bar{y} = \bar{e} \in L$. But $(1-\bar{e})R \subset L \Rightarrow \bar{1}-\bar{e} \in L$. Hence $\bar{1} \in L$, which contradicts that L is proper. This proves that $\bar{e}\bar{R}+L = R$ is a direct sum. Since L is maximal, $\bar{e}\bar{R}$ is irreducible.

Applying the above analysis to each e_i, we see that R/J has a finite maximal chain of right ideals $\bar{e}_iR \subset \bar{e}_1R \oplus \bar{e}_2R \subset \ldots \subset \bar{e}_1R \oplus \ldots \oplus \bar{e}_nR$. By the Jordan Hölder Theorem, R/J is completely reducible. We now show that idempotents can be lifted modulo $J = J(R)$. Let $\bar{f} = \bar{f}^2 \in R$. Also let $\bar{f} = \bar{f}_1 + \ldots + \bar{f}_r$ and $\bar{1}-\bar{f} = \bar{f}_{r+1} + \ldots + \bar{f}_m$ where $1 = \bar{f}_1 + \ldots + \bar{f}_m$ is a sum of primitive orthogonal idempotents of $\bar{R}$. Now by [23, p.77], if $1 = \sum_{i=1}^{n} e_i = \sum_{j=1}^{m} f_j$ where $e_i^2 = e_i$ and $f_j^2 = f_j$, then there exists a unit v in R such that $ve_i = f_iv$ and $m = n$. But R/J is Artinian and hence semi-perfect. So by Lemma 3.7, $\bar{e}_i$ and $\bar{f}_j$ are local idempotents for $i=1, \ldots, n$ and $j=1, \ldots, m$. Hence $\exists \bar{v} \in \bar{R}$ such that $\bar{v}\bar{e}_i = \bar{f}_i\bar{v}$ and m=n. Let $\bar{u} = (\bar{v})^{-1}$. Then $\bar{v}(\bar{e}_i + \ldots + \bar{e}_n)\bar{u} = \bar{f}$. Also $\bar{u} = (\bar{v})^{-1}$ $\Rightarrow vu \equiv 1$ modulo J $\Rightarrow vu = 1-w$ for $w \in J$. But $w \in J \Rightarrow 1-w$ is right invertible. Let $(1-w)(1-w')=1$. Then $vu(1-w')=1$. Now let $v_1=v$ and $u_1=u(1-w')$. Then $v_1 \equiv v$ modulo J and $u_1 \equiv u$ modulo J. But $v_1u_1=1$. Hence $u_1v_1u_1v_1=u_1v_1$. So u_1v_1 and hence $1-u_1v_1$ is an idempotent. But $u_1v_1 \equiv u_v \equiv 1$ modulo J. So $1-u_1v_1 \in J$. By Lemma 3.2, $u_1v_1=1$. Finally, $\bar{v}_1(\bar{e}_1+\ldots+\bar{e}_n)\bar{u}_1 = \bar{v}(\bar{e}_1+\ldots+\bar{e}_n)\bar{u} = \bar{f}$. So idempotents can be lifted modulo $J=J(R)$.

3.11. Corollary (Müeller [27])

If (i) every primitive idempotent is local and (ii) there does not exist an infinite set of orthogonal idempotents, then R is semi-perfect.

Proof

It is well-known (see, for example [16, p. 685]) that a sufficient condition for the identity of R to be a sum of orthogonal primitive idempotents, is for condition (ii) to hold. Now by Theorem 3.10 and condition (i), R is semi-perfect.

3.12. Corollary

If R is semi-perfect, then R has a unique decomposition into a finite direct sum of indecomposable left ideals.

Proof

Since $1 = \sum_{i=1}^{n} e_i$ where e_i is a local orthogonal idempotent for each $i=1,\ldots,n$, then $R = \bigoplus_{i=1}^{n} Re_i$. Note the sum is direct by the orthogonality of the e_i's, and Re_i is indecomposable for each i, since the e_i's are local and hence primitive. Since $End_R(Re_i) = e_iRe_i$, which is local for each i, then uniqueness follows from Azumaya's version of the Krull-Remak-Schmidt Theorem, [23, p. 78].

B: Some Artinianly-inspired Results for Perfect and Semi-Perfect Rings

It has been observed that much of the classical structure theory for Artinian rings can be developed under the weaker hypothesis that R be semi-perfect. In this section, we obtain results for Perfect and Semi-perfect rings which are analogous to those characteristic of Artinian rings.

3.13. Lemma

Let R be left perfect, then $P(R) = J(R)$, where $P(R)$ and $J(R)$ denote the prime and Jacobson radicals of R respectively.

Proof

Now $P(R) \subseteq J(R)$ for all rings R. The reverse inclusion is equally trivial, since an element of a left T-nilpotent ideal is clearly, strongly nilpotent, and P(R) is the intersection of all strongly nilpotent elements of R.

3.14. Lemma

Every homomorphic image of a left perfect (resp. semi-perfect) ring R is left perfect (resp. semi-perfect).

Proof

This is an immediate consequence of Lemma 2.4.

3.15. Lemma

Let R be a left perfect ring without zero divisors. Then R is a division ring.

Proof

Let $r \in R$, $r \neq 0$. Consider $rR \supseteq r^2R \supseteq r^3R \supseteq \ldots \supseteq r^nR \supseteq \ldots$. Since R is left perfect, it has the descending chain condition on principal right ideals. So there exists an integer n such that $r^nR = r^{n+1}R$. Hence $r^n = r^{n+1}x$ for some $x \in R$. $\Rightarrow r^n(1-rx) = 0$ $\Rightarrow 1-rx = 0$ since R has no zero divisors. So $1 = rx$ and x is a right inverse of r. Also $x = xrx \Rightarrow (1-xr)x = 0 \Rightarrow 1-xr = 0$ since $x \neq 0$. Hence every non-zero element of R has an inverse, and R is a division ring.

3.16. Corollary

Let R be a commutative perfect ring. Then every prime two-sided ideal is maximal.

Proof.

Let P be a prime two-sided ideal. Then R/P is an integral domain, and so has no zero divisors. By Lemma 3.14, R/P is perfect. By Lemma 3.15, R/P is a field and so P is maximal.

3.17. Remark

The following result has been proved by W. Vasconcelos [37]: Let R be commutative. Then an injective endomorphism of a finitely-generated R-module is an isomorphism ⇔ every prime ideal is maximal. Vasconcelos suggested that this should be true for rings which are close to being Artinian, for instance, perfect rings. Corollary 3.16 shows that perfect rings do satisfy the afore-mentioned result.

3.18. Lemma

The following statements are equivalent for a ring R.

(1) R is completely reducible.

(2) R is left perfect and regular.

(3) R is left perfect and semi-primitive.

Observe that (1) is symmetric and so the word "left" can be replaced by "right" in (2) and (3).

Proof

(1) ⇒ (2). If R is completely reducible, then $R = \bigoplus_{i=1}^{n} D_i$ where D_i is a minimal right ideal of R. So we have a composition series $D_1 \subset D_1+D_2 \subset \ldots \subset D_1+D_2+\ldots+D_n$. Then R is right Artinian and right Noetherian. By Corollary 1.15, R is left perfect. Also, every right ideal, and hence every principal right ideal is a direct summand. So R is regular.

$(2) \Rightarrow (3)$. Let R be regular and let $r \in J(R)$, the Jacobson radical of R. Then $\exists r' \in R$ such that $1-rr'$ is a unit. So there exists $x \in R$ such that $x(1-rr') = 1$. But R regular $\Rightarrow rr'r = r \Rightarrow (1-rr')r = 0$. Hence $r = 1 \cdot r = x(1-rr')r = 0$. So $J(R) = 0$ and R is semi-primitive.

$(3) \Rightarrow (1)$. Let R be left-perfect. Then every left R-module M has a projective cover. But R is semi-primitive. Hence, by Corollary 1.7, M is projective. So we have shown that every left R-module is projective. Hence R is completely reducible.

It is interesting to note that Lemma 3.18 is still valid if the word "perfect" is replaced by "Artinian". See, for example [23, p.68].

We now study some results of Morita Theory which would culminate in Theorem 3.28.

3.19. Definition

Let Mod-R be the category of all right R-modules. Let G and M be objects of Mod-R where G is fixed and M is arbitrary with respect to G. Then G is called a generator of Mod-R provided for all $0 \neq h: M \to X$, there exists $f: G \to M$ such that $hf \neq 0$ for all objects X in Mod-R. A finitely-generated-projective generator is called a pro-generator. It is well-known that if P is a projective right R-module, then P is a generator $\Leftrightarrow P \otimes M = 0 \Rightarrow M = 0$ for any right R-module M.

3.20. Theorem

Let R and T be two rings such that Mod-R and Mod-T are the categories of right R- and right T-modules respectively. Then Mod-R is equivalent to Mod-T $\Leftrightarrow$ Mod-R has a pro-generator P such that $\mathrm{End}_R(P) \cong T$.

Proof

For the proof, we refer to H. Bass [3].

3.21. Definition

Two rings R and T are called Morita invariant $\Leftrightarrow$ Mod-R and Mod-T are equivalent.

3.22. Definition

Let $P^* = \mathrm{Hom}_R(P, R)$. Define $\tau: P^* \otimes P \to R$ by $\tau(\sum_{i=1}^{n} f_i \otimes m_i) = \sum_{i=1}^{n} f_i(m_i)$ where $f_i \in P^*$ and $m_i \in M$ for $i=1, \ldots, n$. Then the image of τ is called the trace ideal of R.

3.23. Lemma

G is a generator $\Leftrightarrow$ $\tau(G) = R$.

Proof

($\Leftarrow$). This implication is clear, from the remark in 3.19.

($\Rightarrow$). Assume $\tau(G) \neq R$, and consider $f: R \to R/\tau(G)$ where $f \neq 0$. Since G is a generator, there exists $h: G \to R$ such that $fh \neq 0$. But $h \in G^* = \mathrm{Hom}_R(G, R)$ and $\mathrm{Ker}\, f = \tau(G)$. Hence $f(\tau(G)) = 0$. Moreover,

$h(g) = \tau(h \otimes g) \in \tau(G)$. Hence $h(G) \subseteq \tau(G)$. This shows that $(fh)(G) = f(h(G)) \subseteq f(\tau(G)) = 0$. Hence $fh = 0$ — a contradiction. So $\tau(G) = R$.

3.24. Lemma

Let R be a ring, and R_n the ring of all $n \times n$ matrices over R. Then (a) R is Morita equivalent to R_n, and (b) R is Morita equivalent to eRe where $e^2 = e \in R$ and $ReR = R$.

Proof

(a). Let $P = nR$ where nR denotes a direct sum of n copies of R $(n > 0)$. Clearly P is finitely-generated and projective. We now show that P is a generator. Suppose $P \otimes_R X = 0$. Then $nR \otimes_R X = 0$ $\Leftrightarrow$ $n(R \otimes_R X) = 0$ $\Leftrightarrow$ $nX = 0$ $\Leftrightarrow$ $X = 0$. So P is a pro-generator. Also $\mathrm{End}_R(P) = R_n$. By Theorem 3.20, R is Morita equivalent to R_n.

(b). Let $P = eR$. Clearly P is finitely-generated and projective. Also $P^* = \mathrm{Hom}_R(eR, R) \cong Re$. So the trace ideal $\tau(P) = ReR = R$ by hypothesis. Hence by Lemma 3.23, P is a pro-generator. Moreover, $\mathrm{End}_R(P) = eRe$. So by Theorem 3.20, R is Morita equivalent to eRe.

3.25. Definition

A property of rings is called Morita invariant if it is preserved under Morita equivalence. We shall show that "perfectness" and "semi-perfectness" are Morita invariant, but first a remark.

3.26. Remarks

(a). E.Mares [25] called a projective R-module M semi-perfect if every factor module of M has a projective cover. Thus this definition shows that semi-perfect modules can be considered as a generalization of semi-perfect rings. Also, she showed [25, Cor.5.3] that a finite direct sum of semi-perfect modules is semi-perfect. Also E.Björk [5] defined an R-module to be perfect if it satisfies the descending chain condition on cyclic submodules. From [5, Thm 2], it is easily deduced that a submodule of a perfect module is perfect, and a direct sum of perfect modules is perfect.

(b). It is well-known that G is a generator $\Leftrightarrow$ R is a direct summand of a finite direct sum of n copies of G. In particular R, as an R-module, is a generator.

3.27. Lemma

The concepts of left (right) perfectness and semi-perfectness on a ring R are Morita invariant.

Proof

To prove that perfectness is Morita invariant it suffices to show that there exists a perfect generator. Clearly if R is perfect, then R is a perfect generator by Remark 3.26(b). Now let G, as an R-module, be a perfect generator. Then by Remark 3.26(b), $nG = R \oplus M$ where nG denotes a direct sum of n copies of G. So R is perfect by Remark 3.26(a).

In a similar way, we see that semi-perfectness is Morita invariant.

3.28. Theorem

(a) R is left perfect (resp. semi-perfect) $\Leftrightarrow$ R_n is.

(b) If $e^2 = e \in R$ where $ReR = R$. Then R is left perfect (resp. semi-perfect) $\Leftrightarrow$ eRe is.

Proof

Immediate from Lemmas 3.24 and 3.27.

We conclude this section by the following remark.

3.29. Remark

I.Connell [9] has proved that the group ring RG is Artinian $\Leftrightarrow$ R is Artinian and G is finite. A corresponding result for perfect rings was established by S.Kaye [21]: RG is left perfect $\Leftrightarrow$ R is left perfect and G is finite. A complete characterization in the semi-perfect case is still an open problem.

C: Structure of Projective Modules over Semi-Perfect Rings

E.Matlis [26] has proved that if R is right Noetherian, then any injective right R-module is a direct sum of indecomposable ones.

U.Shukla [35] has shown that every projective module over a semi-primary ring can be expressed as the direct sum of indecomposable projective modules. We now consider the structure of finitely-generated projective left R-modules over semi-perfect rings.

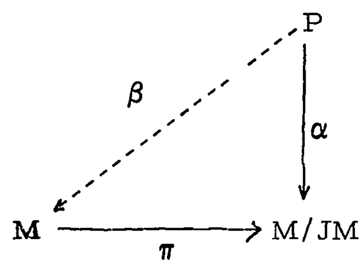
3.30. Theorem

Let R be semi-perfect and M a finitely-generated left R-module. Then M is projective $\Leftrightarrow$ it can be decomposed uniquely into a finite direct sum of indecomposable projective left R-modules.

Proof

($\Leftarrow$). This implication is folk-lore. See for example [23, p. 82].

($\Rightarrow$). Let M be a finitely-generated left R-module and let $J = J(R)$. By hypothesis, R/J is completely reducible, and so $M/JM = \bigoplus_{i=1}^{n} S_i$ where S_i is a simple left ideal of R/J. Hence $S_i = (R/J)\bar{f}_i$ where $\bar{f}_i = f+J$, $\bar{f}_i = \bar{f}_i^2$ and $\bar{f}_i$ is primitive for each $i=1, \ldots, n$. But idempotents can be lifted modulo J. Hence there exist idempotents $e_i \in R$ such that $\bar{e}_i = \bar{f}_i$. So $M/JM = \bigoplus_{i=1}^{n} Re_i/Je_i$. Note also that by Lemma 3.6, e_i is primitive for $i=1, \ldots, n$. Moreover, by Lemma 3.7, the e_i's are local. Let $P = \bigoplus_{i=1}^{n} Re_i$. Then P is a projective left R-module. By the argument used in the proof of Theorem 2.5, we see that $\alpha: P \to M/JM$ is a projective cover. Since P is projective, there exists $\beta: P \to M$ such that $\pi\beta = \alpha$ where π is the canonical map: $M \to M/JM$. That is, the following diagram



commutes. Then $M = \operatorname{Im}\beta + \operatorname{Ker}\pi = \operatorname{Im}\beta$ since $\operatorname{Ker}\pi = J \cdot M$ is small in M by Nakayama's Lemma. So β is an epimorphism. Since M is projective, $\operatorname{Ker}\beta$ is a direct summand of P. But $\operatorname{Ker}\beta \subset \operatorname{Ker}\alpha$ which is small in P. So $\operatorname{Ker}\beta = 0$. And $M \cong P = \bigoplus_{i=1}^{n} Re_i$. Recall that we proved above that the e_i's are local. So uniqueness follows by Azumaya-Krull-Remak-Schmidt Theorem since $\operatorname{End}_R(Re_i) = e_i Re_i$ for $i=1, \ldots, n$. This completes the proof of the theorem.

We now obtain some interesting corollaries. It is well-known that if R is commutative, then R is Artinian $\Leftrightarrow$ R can be decomposed into a finite direct product of Artinian local rings. There is an analogous result for semi-perfect rings.

<u>3.31. Corollary</u>

Let R be commutative. Then R is semi-perfect $\Leftrightarrow$ R can be decomposed uniquely into a finite direct product of local rings.

<u>Proof</u>

By Theorem 3.30, $R \cong \bigoplus_{i=1}^{n} Re_i$ uniquely, since R, considered as a left R-module is finitely-generated and projective. Also $Re_i = e_i Re_i$

since R is commutative. Hence $R \cong \bigoplus_{i=1}^{n} L_i$ where $L_i = e_i R e_i$ is a local ring for each $i = 1, \ldots, n$.

In particular, if R is commutative, then R is left perfect $\Leftrightarrow$ R has a unique decomposition into a finite direct product of local rings, where the unique maximal ideal of each is left T-nilpotent.

<u>3.32. Corollary</u> (Sabbagh [32])

Let R be semi-perfect and P a finitely-generated projective left R-module. Then any surjective endomorphism of P is injective.

<u>Proof</u>

Let $f: P \to P$ be surjective. Since P is projective, then f splits i.e. there exists $g: P \to P$ such that $fg = id_P$. So $P = \operatorname{Ker} f \oplus \operatorname{Im} g$. But g is a monomorphism, and so $P = \operatorname{Ker} f \oplus P_1$ where $P_1 \cong P$. By the uniqueness of decomposition, guaranteed by Theorem 3.30, $\operatorname{Ker} f = 0$. So f is injective.

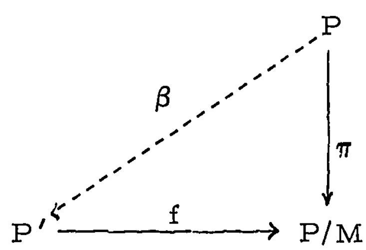
We now obtain some results for projective modules over perfect rings.

<u>3.33. Lemma</u> (Shukla [35])

Let R be a left perfect ring and P a projective left R-module. Then P is indecomposable $\Leftrightarrow$ for every proper submodule M, the canonical epimorphism $\pi: P \to P/M$ is a projective cover of P/M.

Proof

(⇒). Let $M \subsetneqq P$. Then, since R is left perfect, P/M has a projective cover $f: P' \to P/M$. Since P is projective, there exists $\beta: P \to P'$ such that $f\beta = \pi$ where π is the canonical map : $P \to P/M$. That is, the following diagram



commutes. So $P' = \mathrm{Ker} f + \mathrm{Im}\,\beta = \mathrm{Im}\,\beta$ since Ker f is small in P'. Since P' is projective, there exists $\alpha: P' \to P$ such that $\beta\alpha = \mathrm{id}$. Hence $P = \mathrm{Ker}\,\beta \oplus \mathrm{Im}\,\alpha = \mathrm{Ker}\,\beta \oplus P'$. So $\mathrm{Ker}\,\beta = 0$ since P is indecomposable. This shows that $P' \cong P$ and so $\pi: P \to P/M$ is a projective cover.

(⇐). The converse is clear, for if P is decomposable, then Ker π cannot be small in P where $\pi: P \to P/M$.

The following corollaries are also given in [35].

3.34. Corollary

Let R be left-perfect. Then every finitely-generated indecomposable projective left R-module P is principal.

Proof

Suppose $\{x_1, \ldots, x_n\}$ is a minimal set of generators of P and assume $n \gneq 1$. Let Q be generated by $\{x_2, \ldots, x_n\}$ and R generated by $\{x_1\}$. Consider the canonical map $\pi: P \to P/R$. Since P is indecomposable and projective, then by Lemma 3.33, $\operatorname{Ker} \pi = R$ is small in P. But clearly $P = Q+R$. So $P = Q$. This contradicts that $\{x_1, \ldots, x_n\}$ is a minimal set of generators of P. So $n = 1$ and P is principal.

3.35. Corollary

Let R be left Artinian. Then every indecomposable projective left ideal is principal.

Proof

If R is left Artinian, then it is left Noetherian, and so every left ideal is finitely generated. Also R left Artinian $\Rightarrow$ R is semi-primary $\Rightarrow$ R is left (and right) perfect by Corollary 1.15. The result now follows from Corollary 3.34.

D: Structure of Semi-perfect Rings

In this section we prove two structure theorems for semi-perfect rings. Theorem 3.39 is somewhat similar to the "splitting theorem" of A. Zaks [38] for semi-primary rings.

3.36. Theorem

Let R be a semi-perfect ring and $J = J(R)$ its Jacobson radical such that R/J is a simple ring. Then there exists a local ring S such that $R \cong \mathrm{End}_S(F)$ where F is a finitely-generated free S-module.

Proof

We note here that Theorem 3.36 is proved in [23] and also in [4]. Denote R/J by $\bar{R}$. Since R is semi-perfect, there exists a finite set of primitive orthogonal idempotents $\{e_i\}$ such that $\sum_{i=1}^{n} e_i = 1$. Let $\pi(e_i) = \bar{e}_i$ where π is the canonical map $R \to R/J$. Then $\bar{e}\bar{R}$ is a minimal left ideal of $\bar{R}$, since, if R is semi-perfect, the indecomposable left ideals of R correspond to the minimal left ideals of $\bar{R}$ under the canonical map π. Since $\bar{R}$ is completely reducible, these minimal ideals are isomorphic. Let $e = e_1$. Then $\bar{e}\bar{R} \cong \bar{e}_i\bar{R}$. By [23, p.77], there exists $u_i, v_i \in R$ such that $v_i u_i = e$ and $u_i v_i = e_i$. Observe that $u_i e v_i = u_i v_i u_i v_i = (u_i v_i)^2 = e_i^{\,2} = e_i$. We now show that $\mathrm{Hom}_R(R,R) \cong \mathrm{Hom}_{eRe}(Re, Re)$. Let $\varphi \in \mathrm{Hom}_R(R,R)$. Define $\varphi' : Re \to Re$ by $\varphi'(re) = \varphi(re) = \varphi(re^2) = \varphi(re)e$. Also, if $r' \in R$, then $\varphi'(re \cdot er'e) = \varphi(re)er'e = \varphi'(re)er'e$. Hence $\varphi' \in \mathrm{Hom}_{eRe}(Re, Re)$. Now let $r \in R$. Then $r = \sum_{i=1}^{n} e_i r_i = \sum_{i=1}^{n} u_i e v_i r_i$ since $u_i e v_i = e_i$. So $\varphi(r) = \sum_{i=1}^{n} \varphi(u_i e) v_i r_i$ $= \sum_{i=1}^{n} \varphi'(u_i e) v_i r_i$. Clearly, this last equation defines an isomorphism $\eta : \mathrm{Hom}_R(R,R) \to \mathrm{Hom}_{eRe}(Re, Re)$ by $\eta(\varphi) = \varphi'$. Hence $R \cong \mathrm{Hom}_R(R,R)$ $\cong \mathrm{Hom}_{eRe}(Re, Re)$. Since $1 = \sum_{i=1}^{n} e_i$, where e_i's are primitive orthogonal idempotents, then $Re = \bigoplus_{i=1}^{n} e_i Re$. For each i, define $\varphi_i : e_i Re \to eRe$ by $\varphi_i(e_i re) = v_i e_i re$. Note that $v_i e_i re = v_i u_i v_i re$ $= ev_i re$. Then clearly φ_i is an eRe-isomorphism for each i. So Re, as an eRe-module is free on n-generators. Hence $R \cong \mathrm{Hom}_S(F, F)$, where F is a finitely-generated free S-module such that $S = eRe$ and $F = Re$. Moreover S is local, since over a semi-perfect ring, a primitive idempotent is local.

3.37. Remark

Since for any ring R, $J(R_n) = J(R)_n$, then we have an equivalent formulation of 3.36 as follows: If R is semi-perfect such that $R/J \cong D_n$ where D is a division ring, then there exists a local ring S such that $R \cong S_n$ and $S/J(S) \cong D$.

As a corollary of 3.36, we obtain a result of S.Kaye [15].

3.38. Corollary

Let R be semi-perfect and $J = J(R)$ its Jacobson radical. Let $R/J = \bigoplus_{i=1}^{n} (D_i)_{n_i}$ where D_i is a division ring for each i and $(D_i)_{n_i}$ denotes the ring of all $n_i \times n_i$ matrices over D_i. Furthermore, let $A = \sum_{i=1}^{n} e_i R e_i$ where $e_i^2 = e_i \in R$ for $i=1,\ldots,n$. Then there exist local rings L_i such that $L_i/J(L_i) \cong D_i$ and $A = \bigoplus_{i=1}^{n} (L_i)_{n_i}$.

Proof

Since R is semi-perfect, we can choose the e_i's orthogonal and primitive such that $\sum_{i=1}^{n} e_i = 1$. Hence $A = \sum_{i=1}^{n} e_iRe_i$ is direct since $e_ie_j = 0$ if $i \neq j$. Now $e_iRe_i/J(e_iRe_i) = e_iRe_i/e_iJe_i \cong (D_i)_{n_i}$.

Since e_iRe_i is semi-perfect, then by Remark 3.37, there exists a local ring L_i for each $i=1, \ldots, n$ such that $e_iRe_i \cong (L_i)_{n_i}$ and $L_i/J(L_i) \cong D_i$. Hence $A = \bigoplus_{i=1}^{n} e_iRe_i \cong \bigotimes_{i=1}^{n} (L_i)_{n_i}$ where $L_i/J(L_i) \cong D_i$.

This finishes the proof of the Corollary.

3.39. Theorem (Behrens [4])

Let R be semi-perfect and $J = J(R)$ its Jacobson radical. Then the underlying additive group of R admits a decomposition $R = S \oplus N$ where S is a subring of R and N a subgroup of the additive group of J(R) such that,

(a) $S = \bigoplus_{i=1}^{n} S_i$ where $S_i = e_iRe_i$ is semi-perfect for each $i=1, \ldots, n$; $e_ie_j = 0$ for $i \neq j$ and $S_i/J(S_i)$ is simple.

(b) $N = \sum_{i \neq j} e_iRe_j$.

[Note that in the "Splitting Theorem" of Zaks [38], N is a two-sided ideal of R. However, in our case, this is not true in general since $e_iRe_j \cdot e_jRe_i \subseteq e_iRe_i$ for $i \neq j$.]

Proof

R semi-perfect $\Rightarrow R/J$ completely reducible. So $R/J = \bigoplus_{i=1}^{n} \bar{R}_i$ where $\bar{R}_i$ is a simple ring for each $i=1,\ldots,n$. Now R semi-perfect $\Rightarrow R/J$ is also. So $I = \sum_{i=1}^{n} \bar{f}_i$ where $\bar{f}_i$ are primitive orthogonal idempotents of R/J. Since idempotents can be lifted, there exists a set $\{e_i\}_{i=1}^{n}$ of idempotents in R such that $\sum_{i=1}^{n} e_i = 1$. Observe also that the e_i's are also primitive and orthogonal.

So $R = 1R1 = \sum_{i=1}^{n} e_iRe_i + \sum_{i\neq j} e_iRe_j$. Let $S_i = e_iRe_i$. Then S_i is semi-perfect. Also $S_i/J(S_i) = e_iRe_i/e_iJe_i \cong \bar{R}_i$ which is simple. Moreover, if $S = \sum_{i=1}^{n} e_iRe_i$, then this sum is direct since $e_ie_j = 0$ for $i \neq j$. This proves conclusion (a) of the Theorem. Finally, $e_iRe_j \equiv e_ie_jR \equiv 0 \bmod J(R)$, for $i \neq j$. Hence, if $N = \sum_{i\neq j} e_iRe_j$, then N is a subgroup of the additive group of $J(R)$.

APPENDIX

Subrings of Perfect Rings

1. Definition

Let M be a right R-module. M is called faithfully flat $\Leftrightarrow$ M is flat, and, for any left R-module N, $M \otimes_R N = 0 \Rightarrow N = 0$.

2. Lemma

Let M be a right R-module. The following statements are equivalent.

(a) M is faithfully flat.

(b) A sequence $A \to B \to C$ of left R-modules is exact $\Leftrightarrow$

$M \otimes_R A \to M \otimes_R B \to M \otimes_R C$ is exact.

Proof

Evident.

3. Corollary

Let M be a faithfully flat right R-module and $f: A \to B$ a left R-module homomorphism. Then f is injective (resp. surjective, bijective) $\Leftrightarrow$ $id_M \otimes f: M \otimes_R A \to M \otimes_R B$ is also.

Proof

Immediate from condition (b) of Lemma 2.

We remark here that Q is a flat and faithful Z-module but it is not faithfully flat. We now formulate and discuss the following conjecture.

4. Conjecture

Let $R \subset S$ as rings and S a faithfully flat left R-module. Let R-mod denote the category of all left R-modules and let T be the functor $S \otimes - : R\text{-mod} \to S\text{-mod}$. Then T is full and faithful.

Discussion

We prove that T is faithful. Let $f: A \to B$ be a left R-module homomorphism such that $T(f) = 0$. To prove $f = 0$. Now $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact where $C = B/\mathrm{Im}f$. Then $T(A) \xrightarrow{T(f)} T(B) \xrightarrow{T(g)} T(C) \to 0$ is exact since $T = S \otimes -$ is a right exact functor. But $T(f) = 0 \Rightarrow T(g)$ is a monomorphism. By Corollary 3, g is a monomorphism. But then $B \cong B/\mathrm{Im}f$. Hence $\mathrm{Im}f = 0 \Rightarrow f = 0$.

We are unable to prove T is full. However, we shall assume that the conjecture is true, and deduce the following result.

5. Corollary

Assume the hypothesis of the conjecture. If $S \otimes_R M$ is a projective left S-module, then M is a projective left R-module.

Proof

Follows, since $S \otimes -$ is full and faithful by the conjecture.

We now obtain a result on subrings of perfect rings.

6. Theorem

Let $R \subset S$ as rings, and S is a faithfully flat right R-module. If S is a left perfect ring, then so is R.

Proof

Let $\{P_i \mid i \in I\}$ be a family of projective left R-modules. Then $S \otimes_R P_i$ is a projective left S-module for each $i \in I$. Since S is left perfect, $\varinjlim (S \otimes_R P_i) = S \otimes_R \varinjlim P_i$ is a projective left R-module by Corollary 1.24 . By Corollary 5, $\varinjlim P_i$ is a projective left R-module, and so R is left perfect, again by Corollary 1.24.

BIBLIOGRAPHY

1. ARTIN, E., NESBITT, C. and THRALL, R., "Rings with Minimum Condition", University of Michigan Press, Ann Arbour, Michigan, 1955.

2. BASS, H., "Finitistic Homological Dimension and a Homological Generalization of Semi-Primary Rings", Trans.Amer. Math.Soc., 95 (1960), 466-488.

3. ————, "The Morita Theorems", Lecture Notes, University of Oregon, 1962.

4. BEHRENS, E.-A., "Ring Theory", Academic Press, New York and London, 1972.

5. BJÖRK, J., "Rings Satisfying a Minimum Condition on Principal Ideals", J.Reine Angew.Math., 236 (1969), 112-119.

6. BOURBAKI, N., "Algèbre", Paris, Hermann, 1958.

7. CARTAN, H. and EILENBERG, S., "Homological Algebra", Princeton, N.J., Princeton University Press, 1956.

8. CHASE, S., "Direct Products of Modules", Trans.Amer. Math.Soc., 97 (1960), 457-473.

9. CONNELL, I., "On the Group Ring", Can.J.Math., 15 (1963), 650-685.

10. COZZENS, J., "Homological Properties of the Ring of Differential Polynomials", Bull.Amer.Math.Soc., 76 (1970), 75-79.

11. FAITH, C., "Rings with Ascending Condition on Annihilators", Nagoya Math.J., 27 (1966), 179-191.

12. FIELDHOUSE, D., "Pure Simple and Indecomposable Rings", Can.Math.Bull., 13 (1970), 71-78.

13. ——————, "Regular Modules and Rings", Queens' Preprint Papers, Kingston, Ontario, 1970.

14. GOVOROV, V., "Rings over which Flat Modules are Free", Dokl.Akad.Nauk.SSSR, 144 (1962), 965-967.

15. GOLAN, J., "Characterization of Rings Using Quasi-projective Modules", Israel J.Math., 8 (1970), 34-38.

16. GORDON, R., "Rings in which Minimal Left Ideals are Projective", Pacific J.Math., 31 (1969), 679-692.

17. HAMSHER, R., "Commutative Rings over which every Submodule has a Maximal Submodule", Proc.Amer.Math.Soc., 18 (1967), 1133-1137.

18. JONAH, D., "Rings with the Minimum Condition for Principal Right Ideals have the Maximum Condition for Principal Left Ideals", Math.Z., 113 (1970), 106-112.

19. KAPLANSKY, I., "Projective Modules", Ann.Math., 68 (1958), 372-377.

20. ——————, "Infinite Abelian Groups", University of Michigan Press, Ann Arbour, 1954.

21. KAYE, S., "On Perfect and Semi-perfect Group Rings", Ph.D. Thesis, McGill University, 1969.

22. ————, "Ring Theoretic Properties of Matrix Rings", Can. Math.Bull., 10 (1967), 365-375.

23. LAMBEK, J., "Lectures on Rings and Modules", Blaisdell, Waltham, Mass., 1966.

24. LEVY, L., "The Case of the Mis-Lifted Idempotent", (unpublisged).

25. MARES, E., "Semi-Perfect Modules", Math.Z., 82 (1963), 347-360.

26. MATLIS, E., "Injective Modules over Noetherian Rings", Pacific J.Math., 8 (1958), 511-528.

27. MÜELLER, B., "On Semi-Perfect Rings", Ill. J. Math., 14 (1970), 464-467.

28. OSOFSKY, B., "Rings All of Whose Finitely-Generated Modules are Injective", Pacific J.Math., 14 (1964), 645-650.

29. RIBENBOIM, P., "Rings and Modules", Wiley, New York, 1965.

30. ROTMAN, J., "Notes on Homological Algebra", Van Nostrand Reinhold Mathematical Studies No.26.

31. ROSENBERG, A. and ZELINSKY, D., "Finiteness of the Injective Hull", Math.Z., 70 (1959), 372-380.

32. SABBAGH, G., "Endomorphisms of Finitely-Presented Modules", Proc.Amer.Math.Soc., 30 (1971), 75-78.

33. SANDOMIERSKI, F., "On Semi-Perfect and Perfect Rings", Proc.Amer.Math.Soc., 21 (1969), 205-207.

34. ——————, Lecture Notes in Ring Theory, University of Wisconsin, 1966.

35. SHUKLA, U., "On the Projective Cover of a Module and Related Results", Pacific J.Math., 12 (1962), 709-712.

36. VANAJA, N., "Generalized Projectives", Doctoral Dissertation, Madurai University, India.

37. VASCONCELOS, W., "Injective Endomorphisms of Finitely-Generated Modules", Proc. Amer. Math. Soc., 25 (1970), 900-901.

38. ZAKS, A., "Residue Rings of Semi-Primary Hereditary Rings", Nagoya Math. J., 30 (1960), 279-283.