Optimal Control of Deterministic and Stochastic Hybrid Systems: Theory and Applications

Ali Pakniyat

Department of Electrical & Computer Engineering
McGill University
Montreal, Canada

September 2016

A thesis submitted to McGill University in partial fulfillment of the requirements of the degree of Doctor of Philosophy.

Abstract

Deterministic and stochastic hybrid optimal control problems are studied for systems where autonomous and controlled state jumps are allowed at the switching instants and, in addition to running costs, switching between discrete states incurs costs. Features of special interest in this work are the possibility of state space dimension change, and existence of low dimensional switching manifolds. In other words, the hybrid state space is considered as the direct product of a set of discrete state components with finite cardinality and a set of Euclidean spaces whose dimensions depend upon the discrete state components; and Euclidean spaces contain switching manifolds which correspond to autonomous switchings and jumps, and which are allowed to be codimension k submanifolds of the corresponding Euclidean state spaces, where k is greater than or equals to one.

Statements of the Hybrid Minimum Principle (HMP) and Hybrid Dynamic Programming (HDP) are presented and it is shown that under certain assumptions the adjoint process in the HMP and the gradient of the value function in HDP are identical to each other almost everywhere along optimal trajectories. Furthermore, results for stochastic hybrid optimal control problems are established which generalize those of the deterministic case. A key feature of the stochastic hybrid systems framework under consideration is the presence of the hard constraints imposed by switching manifolds on diffusion-driven state trajectories; these constraints influence the boundary conditions in the Stochastic Hybrid Minimum Principle (SHMP).

In addition to analytic examples, an electric vehicle equipped with a dual-stage planetary transmission is modelled in this framework, where, due to the special structure of the transmission, the mechanical degree of freedom changes during the transition period. Hybrid control problems for energy and time optimality of an electric vehicle acceleration task are studied which reveal unanticipated aspects of optimal energy saving strategies for the transmission control.

Résumé

Des problèmes de commande optimal pour systèmes hybrides déterministes et stochastiques sont étudiés dans le cas de systèmes où, aux instants de commutation, des sauts autonomes ainsi que des sauts contrôlés d'état sont permis, les commutations étant soumises à des coût au même titre que l'évolution de la composante continue de la trajectoire d'état. Des caractéristiques d'intérêt particulier dans notre analyse sont la possibilité de changement de dimension de l'espace d'état, ainsi que la possibilité d'existence de variétés de commutation de basse dimension. En d'autres termes, l'espace d'état hybride considéré est formé d'un produit direct d'un ensemble fini de composantes d'état discrètes, et un ensemble d'espaces euclidiens dont les dimensions dépendent des composants d'état discrets. De plus, les variétés de commutation correspondant aux commutations et sauts autonomes sont autorisées à être des sous-variétés de codimension k de l'espace d'état correspondant, où k est égal ou supérieur à l'unité.

Les énoncés du Principe Minimum Hybride (HMP) et de la Programmation Dynamique Hybride (HDP) sont présentés et il est démontré que, sous certaines hypothèses, le processus adjoint dans le HMP et le gradient de la fonction valeur dans HDP sont identiques presque partout sur des trajectoires optimales. De plus, des résultats liés à des problèmes de commande optimale pour systèmes hybrides stochastiques sont établis et généralisent ceux du cas déterministe. Une caractéristique clé du cadre de systèmes hybrides stochastiques considéré est la présence de contraintes strictes imposées sur les processus conduites par la diffusion, et imposées au niveau des variétés de commutation. Ces contraintes influencent les conditions aux limites dans le Principe Minimum Hybride Stochastique (SHMP).

En plus d'exemples purement mathématiques, un véhicule électrique équipé d'une transmission épicycloïdale à deux étages est modélisé dans ce cadre. Le modèle présente la caractéristique que, en raison de la structure particulière de la transmission, le degré de liberté mécanique change au cours de la période de transition. Les problèmes de contrôle hybrides pour l'optimalité d'énergie et la minimisation du temps liés à une tâche d'accélération pour un véhicule électrique sont étudiés, et leur solution révèle des aspects inattendus de stratégies d'économie d'énergie optimales pour le contrôle de la transmission.

Claims of Originality and Published Work

Claims of Originality

The following original contributions are presented in this thesis:

- Presentation of hybrid systems in a unified general framework within which the results of the Hybrid Minimum Principle (HMP) and Hybrid Dynamic Programming (HDP) as well as their relationship (in the form of the adjoint-gradient relation) are valid. This framework permits the study of hybrid systems where autonomous and controlled state jumps are allowed at the switching instants and the associated optimal control problems consisting of a large range of terminal, running, and switching costs.

- The extension of the Hybrid Minimum Principle (HMP) to optimal control problems in the presence of switching costs, and to hybrid systems with possible state space dimension change and existence of low dimensional switching manifolds. Related publications: [J2, P1, C2, C3, C4, C5, C6, C7].

- The formulation and the proof of the Stochastic Hybrid Minimum Principle (SHMP) for the first time. A feature of special interest is the effect of hard constraints imposed by switching manifolds on diffusion-driven state trajectories, that to the best of our knowledge has not been considered in the literature before. Related publications: [P1, C1].

- The extension of Hybrid Dynamic Programming (HDP) within the same unified framework as that in which the HMP results are presented. In particular, in contrast to other versions of HDP, the majority of which study infinite horizon problems where optimal controls are stationary, the results of this thesis are presented for finite horizon problems; furthermore, the assumptions required for the derivation of HDP in this work are less restrictive and, in particular, do not restrict the domains and codomains of the family of jump maps. Related publications: [J1, P2, C2].

- The presentation of two proof methods (one based on variations over optimal trajectories and the other with variations over general, i.e. not necessarily optimal, trajectories) for establishing the relationship between the Minimum Principle and Dynamic Programming for both Classical and Hybrid Systems. Both proofs are different in approach from the classical arguments and require weaker differentiability assumptions. Furthermore, the boundary conditions for the case of hybrid systems have not been proved in the literature before. Related publications: [J1, C5].

- The presentation of three examples with analytical solutions. Related publications: [C2, C4, C6].
- A general Riccatti formalism for optimal tracking problems with quadratic costs for the control of hybrid systems with linear (affine) dynamics. Related publications: [J1].
- The application of hybrid optimal control theory to the industrial problem of vehicle electrification. Related publications: [J2, P3, C3, C8].

Publications

Journal Papers:

[J1] A. Pakniyat, P. E. Caines, "On the Relation between the Minimum Principle and Dynamic Programming for Classical and Hybrid Systems", *arXiv preprint arXiv:1609.03158*, 2016, under review for the IEEE Transactions on Automatic Control.

[J2] A. Pakniyat, P. E. Caines, "Hybrid Optimal Control of an Electric Vehicle with a Dual-Planetary Transmission", *Nonlinear Analysis: Hybrid Systems*, article in press, DOI: 10.1016/j.nahs.2016.08.004.

Conference Papers:

[C1] A. Pakniyat, P. E. Caines, "On the Stochastic Minimum Principle for Hybrid Systems", *Submitted to the 55th IEEE Conference on Decision and Control*, Las Vegas, NV, USA, 2016 (submission number: CDC-16-1629).

[C2] A. Pakniyat, P. E. Caines, "On the Minimum Principle and Dynamic Programming for Hybrid Systems with Low Dimensional Switching Manifolds", *Proceedings of the 54th IEEE Conference on Decision and Control*, Osaka, Japan, 2015, pp. 2567–2573.

[C3] A. Pakniyat, P. E. Caines, "Time Optimal Hybrid Minimum Principle and the Gear Changing Problem for Electric Vehicles", *Proceedings of the 5th IFAC Conference on Analysis and Design of Hybrid Systems*, Atlanta, GA, USA, 2015, pp. 187–192.

[C4] A. Pakniyat, P. E. Caines, "On the Relation between the Hybrid Minimum Principle and Hybrid Dynamic Programming: a Linear Quadratic Example", *Proceedings of the 5th IFAC Conference on Analysis and Design of Hybrid Systems*, Atlanta, GA, USA, 2015, pp. 169–174.

[C5] A. Pakniyat, P. E. Caines, "On the Relation between the Minimum Principle and Dynamic Programming for Hybrid Systems", *Proceedings of the 53rd IEEE Conference on Decision and Control*, Los Angeles, CA, USA, 2014, pp. 19–24.

[C6] A. Pakniyat, P. E. Caines, "On the Minimum Principle and Dynamic Programming for Hybrid Systems", *Proceedings of the 19th IFAC World Congress*, Cape Town, South Africa, 2014, pp. 9629–9634.

[C7] A. Pakniyat, P. E. Caines, "The Hybrid Minimum Principle in the Presence of Switching Costs", *Proceedings of the 52nd IEEE Conference on Decision and Control*, CDC 2013, Florence, Italy, 2013, pp. 3831–3836.

[C8] A. Pakniyat, P. E. Caines, "The Gear Selection Problem for Electric Vehicles: an Optimal Control Formulation", *Proceedings of the 13th International Conference on Control, Automation, Robotics and Vision*, Marina Bay Sands, Singapore, 2014, pp. 1261–1266.

Papers in Preparation:

[P1] A. Pakniyat, P. E. Caines, "The Deterministic and Stochastic Hybrid Minimum Principle", *to be submitted to the SIAM Journal of Control and Optimization*.

[P2] A. Pakniyat, P. E. Caines, "Dynamic Programming for Stochastic Hybrid Systems", *to be submitted to the IEEE Transactions on Automatic Control*.

[P3] A. Pakniyat, P. E. Caines, "Automotive Applications of Hybrid Optimal Control Theory", *to be submitted to Automatica*.

Contribution of Co-authors

Peter E. Caines contributed in the problem formulations and their analyses. These contributions amounted to 25% of the papers cited above, and their corresponding sections of this thesis.

Acknowledgements

It gives me great pleasure to acknowledge a debt of gratitude to my thesis supervisor Professor Peter E. Caines. Throughout my study and research in McGill University, I have benefited a lot from his keen scientific foresight, erudite mathematical guidance, painstaking academic attitude, and solid financial support. I also appreciate the opportunities he has provided to attend conferences and workshops and to interact with leading researchers around the world.

l must also thank the Automotive Partnership Canada (APC) and in particular the APC project leader, Professor Benoit Boulet, for giving me the opportunity to apply the theoretical results of this thesis to the industrial problem of vehicle electrification. His expert guidance and mentorship enriched my studies at McGill University.

I would also like to thank Professors Aditya Mahajan, Hannah Michalska, Ioannis Psaromiligkos and Michael Rabbat from the department of Electrical and Computer Engineering of McGill University, Professors James Forbes and Aditya Paranjape from the department of Mechanical Engineering of McGill University and Professor Roland Malhamé from the department of Electrical Engineering of École Polytechnique de Montréal.

I am grateful for the financial support throughout my studies provided by McGill Engineering Doctoral Award (MEDA), the Natural Sciences and Engineering Research Council of Canada (NSERC) and APC. I am also thankful for the partial financial support from GERAD[1], Geoff Hyland Fellowship in Engineering, and other awards from McGill University.

I would also like to thank the staff and members of the McGill Centre for Intelligent Machines (CIM) and acknowledge the professional services I benefited from CIM, GERAD and REPARTI[2]. I would also like to acknowledge the industrial partners TM4, Linamar, and Infolytica.

Finally, I am extremely thankful to my father, Mr. Ahmad Pakniyat, my mother, Mrs. Zahra Khozaeinejad, my sister, Doctor Nasim Pakniyat, and my brother-in-law, Mr. Hamed Nehrir for all their love, support, and encouragement through the years, which have constituted a crucial part of my life. I present this thesis as a gift to them.

Ali Pakniyat
McGill University, Montreal, Canada
September 1, 2016

[1]Groupe d'Études et de Recherche en Analyse des Décisions (Group for Study and Research in Decision Analysis)

[2]Regroupement pour l'étude des Environnements PARTagés Intelligents (Centre for the Study of Distributed Intelligent Shared Environments)

Contents

List of Figures

List of Tables

Chapter 1

Introduction

There is now an extensive literature on the optimal control of hybrid systems. On one hand, the generalization of the fundamental Pontryagin Maximum Principle (PMP) [1] results in the Hybrid Minimum Principle (HMP). The formulation by Clarke and Vinter [2, 3], referred to by them, as "Optimal Multiprocesses" provides a Minimum Principle for hybrid systems of a very general nature in which switching conditions are regarded as constraints in the form of set inclusions and the dynamics of the constituent processes are governed by (possibly nonsmooth) differential inclusions. A similar philosophy is followed by Sussmann [4, 5] where a nonsmooth Minimum Principle is presented for hybrid systems possessing a general class of switching structures. Due to the generality of the results in [2–5] degeneracy is not precluded and therefore, additional hypotheses need to be imposed to make the HMP results significantly informative (see e.g. Caines, Clarke, Liu, and Vinter [6] for more discussion); such hypotheses (typically of a controllability nature) are usually too restrictive to cover many practical problems of engineering interest. An alternative philosophy, followed by Shaikh and Caines [7], Garavello and Piccoli [8], and Taringoo and Caines [9, 10], is to ensure the validity of the HMP in a non-degenerate form by introducing hypotheses on the dynamics, transitions and switching events. To name a few other versions of the HMP in its appearances within the development of optimal control theory one cites the work of Riedinger, Kratz, Iung and Zanne [11], Xu and Antsaklis [12], Azhmyakov, Boltyanski and Poznyak [13], and Dmitruk and Kaganovich [14].

The generalization of Bellman's Dynamic Programming [15] for hybrid systems, on the other hand, results in the theory of Hybrid Dynamic Programming (HDP). Infinite horizon - HDP formulations have been given by Bensoussan and Menaldi [16], Branicky, Borker and Mitter

[17], Dharmatti and Ramaswamy and later Barles et al. [18, 19], as well as finite horizon HDP formulations appearing in the work of Hedlund and Rantzer [20], Caines, Egerstedt, Malhamé and Schöllig [21, 22] and Shaikh and Caines [23], to name but few of the major publications on the theory of HDP.

The relationship between the Minimum Principle and Dynamic Programming, for classical optimal control problems, was addressed as early as the formal announcement of the Pontryagin Minimum Principle [1], and this relationship has been elaborated by many others since then. In particular, Clarke and Vinter [24, 25], Yong and Zhou [26, 27], Kim [28], Fleming and Rishel [29], Cannarsa and Frankowska [30], and Cernea and Frankowska [31] have established the relationship with rigorous proof methods. The result states that, under technical assumptions, the adjoint process in the MP and the gradient of the value function in DP are equal. In contrast to classical optimal control theory, the relation between the Minimum Principle and Dynamic Programming in the hybrid systems framework has been the subject of a very limited number of studies.

The generalization of classical optimal control theory to stochastic systems results in Stochastic Dynamic Programming (SDP) and Stochastic Minimum Principle (SMP). One of the main differences between the stochastic differential equations appearing in stochastic optimal control problems and deterministic differential equations for deterministic problems is that "time" cannot be reversed and their solvability is interpreted as the existence of solutions adapted solely to the forward filtration (see e.g. Ma and Yong [32]). This requires the introduction of a notion of forward-backward stochastic differential equations (FBSDE), first presented by Bismut [33], and then elaborated more in the optimal control framework by Bensoussan [34], Pardoux and Peng [35], etc., and in the general theory of forward-backward stochastic differential equations by Antonelli, Ma, Protter, Yong, Hu, and Peng (see e.g. [32] and references therein). Stochastic Dynamic Programming (SDP) including the Stochastic Hamilton-Jacobi-Bellman (SHJB) equation are presented by Kushner [36], Krylov [37], Fleming and Soner [38], Fleming and Rishel [29], Yong and Zhou [26], and others. Versions of the Stochastic Minimum Principle (SMP) are presented by Kushner and Schweppe [39, 40], Haussmann [41], Bismut [33], Bensoussan [42], and Peng [43]. The optimal control of stochastic hybrid systems, i.e. control systems that involve the interaction of continuous dynamics, discrete dynamics and stochastic diffusions, has been the subject of a limited number of studies. The SMP formulation in Aghayeva and Abushov [44] considers only controlled switching and jumps, and the Stochastic Dynamic Programming (SDP) formulation in Bensoussan and Menaldi [45] studies infinite horizon problems where optimal controls are stationary.

The primary goal of this thesis is to build a bridge between the above three research fields towards a unified framework for the optimal control of deterministic and stochastic hybrid systems. The secondary goal of the thesis is to illustrate the theoretical results for several analytical and industrial applications, and in particular, the application of hybrid optimal control theory for the control of electric vehicles equipped with multi-stage transmissions. The organisation of the thesis is as follows. Deterministic hybrid optimal control results are presented in Part I as follows:

Chapter 2 introduces a unified general framework for the presentation of hybrid systems and their associated hybrid optimal control problems within which the Hybrid Minimum Principle (HMP), Hybrid Dynamic Programming (HDP) and their mutual relationship can be established. This framework permits the study of hybrid systems with both autonomous and controlled state jumps allowed at the switching instants and their associated optimal control problems with a large range of terminal, running, and switching costs.

The Hybrid Minimum Principle (HMP), presented in Chapter 3, gives necessary conditions for the optimality of the trajectory and the control inputs of a given hybrid system with fixed initial conditions and a predetermined sequence of autonomous and controlled switchings. It should be remarked that the establishment of a central sequence optimization for the determination of the optimal switching sequence for hybrid optimal control problems is an open problem. Distinctive aspects in this work in comparison with other versions of the HMP are the presence of state dependent switching costs, the possibility of state space dimension change, and the existence of low dimensional switching manifolds. The necessary conditions of the HMP are expressed in terms of the minimization of the hybrid system's Hamiltonians defined along the hybrid trajectory corresponding to a sequence of discrete states and continuous valued control inputs on the associated time intervals. A feature of special interest is the boundary conditions on the adjoint processes and the Hamiltonian functions at autonomous and controlled switching times and states; these boundary conditions may be viewed as a generalization of the optimal control case of the Weierstrass–Erdmann conditions of the calculus of variations [46].

Chapter 4 discusses Hybrid Dynamic Programming (HDP) which employs the optimal cost to go (value function) for the hybrid optimal control problem as its fundamental notion. It is proved that on a bounded set in the state space, the cost to go functions are Lipschitz with a common Lipschitz constant which is independent of the control and hence their infimum, i.e. the value function, is Lipschitz with the same Lipschitz constant. The necessary conditions of

HDP are then established in the form of the Hamilton-Jacobi-Bellman (HJB) equation and the corresponding boundary conditions.

The relationship between the HMP and HDP is studied in Chapter 5. We elaborate the classical arguments for the proof of this relationship which are based on the the derivation of the differential equations governing the value function gradient process. However, classical proof methods are based upon the derivation of the gradient dynamics from the Hamilton-Jacobi-Bellman equation, and the employment of the Filippov theorem (see e.g. [47, p. 149-150]), which requires the openness of the set of points from which a transition to the reference trajectory is possible [1, p. 70]. A major difficulty for this approach is the lack of differentiability of the value function, and consequently, the adjoint-gradient relationship is usually expressed within the general framework of nonsmooth analysis that declares the inclusion of the adjoint process in the set of generalized gradients of the value function [24–28, 30, 31]. For classical and hybrid optimal control problems with appropriately smooth vector fields and costs, when the optimal feedback control possesses an admissible set of discontinuities, we establish the adjoint-gradient relationship in the form of an almost everywhere equality. One proof method is based on variations over optimal trajectories and the other one is based upon variations over general trajectories and the study of the gradient of the (not necessarily optimal) cost to go function. For the hybrid case, the arguments are accompanied by the corresponding boundary conditions at the switching instants.

Chapter 6 discusses three examples with analytical solutions and also presents the general Riccatti formalism for tracking problems of hybrid systems with linear / affine vector fields and quadratic costs.

Chapter 7 presents a hybrid systems formulation of an electric vehicle equipped with a dual-stage planetary gearbox and employs hybrid optimal control theory to find the optimal inputs for the gear changing problem for electric vehicles. A feature of special interest is that, due to the perpetual connectedness of the motor to the wheels via the seamless transmission, the mechanical degree of freedom changes during the transition period. Therefore, the modelling of the powertrain requires the consideration of autonomous and controlled state jumps accompanied by changes in the dimension of the state space.

In part II of the thesis in Chapter 8, stochastic hybrid optimal control problems are studied. We extend the framework established in Chapter 2 in order to cover a general class of stochastic hybrid systems with state dependant diffusions which are subject to autonomous and controlled switchings and state jumps. A feature of special interest is the effect of hard constraints imposed by switching manifolds on diffusion-driven state trajectories. A first order variational analysis is performed on the stochastic hybrid optimal control problem via the needle variation methodology and the necessary optimality conditions are then established in the form of the Stochastic Hybrid Minimum Principle (SHMP).

Future research directions are presented in Chapter 9.

Part I

Deterministic Hybrid Optimal Control

Chapter 2

Hybrid Optimal Control Problems

2.1 Hybrid Systems

Hybrid control systems appear in a vast range of natural and artificial settings; examples include multi-tank mixing and fractionating systems in chemical engineering, multi-mode aircraft controls and gain scheduled control laws, space vehicle and satellite control systems, automobile anti-lock breaking systems, multilink and cooperative robotic manipulator systems, and recombinant genetic processes in microbiology.

One of the main difficulties in the discussion of hybrid systems is that the term "hybrid" is not restrictive enough and, inevitably, the domains of definition of hybrid systems in different scientific communities do not necessarily intersect in a general class of systems. For instance, the Computer Science community primarily views hybrid systems as a finite automata computer program interacting with an analogue environment and therefore, the emphasis is often on the discrete event dynamics, whereas the continuous dynamics is frequently of a relatively simple form [48–55]. Even in the Control Systems community, hybrid systems stability theory (see e.g. [55–66]) views hybrid systems differently from hybrid optimal control theory (see e.g. [2–5, 7–9, 11–14, 16–23, 67–73]). Most notably, hybrid control input values in stability analyses have simpler structures compared to the admissible set of input values considered for optimal control purposes.

The definition of hybrid systems in this thesis covers a general class of nonlinear systems with autonomous and controlled state jumps allowed at the switching instants. Features of special interest in this work are the possibility of state space dimension change, and existence of low dimensional switching manifolds. In other words, the hybrid state space is considered as the direct product of a set of discrete state components with finite cardinality and a set of Euclidean spaces whose dimensions depend upon the discrete state components and further, switching manifolds corresponding to autonomous switchings and jumps are allowed to be codimension k submanifolds in $\mathbb{R}^{n_q}$ (the corresponding state space), where $1 \leq k \leq n_q$. Further generalizations such as the lying of the system's vector fields in Riemannian spaces [9, 67], nonsmooth assumptions [2–5,16,18], and state-dependence of the control value sets [8], as well as restrictions to certain subclasses such as those with regional dynamics [21,22], and with specified families of jumps [16–19] are possible throughout minor variations over the framework presented below.

Definition 2.1. A (deterministic) *hybrid system (structure)* $\mathbb{H}$ is a septuple

$$\mathbb{H} = \{H := Q \times M, I := \Sigma \times U, \Gamma, A, F, \Xi, \mathscr{M}\} \tag{2.1}$$

where the symbols in the expression and their governing assumptions are defined as below.

A0: $H := Q \times M$ is called the *(hybrid) state space* of the hybrid system $\mathbb{H}$, where

$Q = \{1, 2, ..., |Q|\} \equiv \{q_1, q_2, ..., q_{|Q|}\}, |Q| < \infty$, is a finite set of *discrete states (components)*, and

$M = \{\mathbb{R}^{n_q}\}_{q \in Q}$ is a family of finite dimensional continuous valued state spaces, where $n_q \leq n < \infty$ for all $q \in Q$.

$I := \Sigma \times U$ is the set of system input values, where

Σ with $|\Sigma| < \infty$ is the set of discrete state transition and continuous state jump events extended with the identity element, and

$U = \{U_q\}_{q \in Q}$ is the set of *admissible input control values*, where each $U_q \subset \mathbb{R}^{m_q}$ is a compact set in $\mathbb{R}^{m_q}$.

The set of admissible (continuous) control inputs $\mathscr{U}(U) := L_\infty([t_0, T_*), U)$, is defined to be the set of all measurable functions that are bounded up to a set of measure zero on $[t_0, T_*), T_* < \infty$. The boundedness property necessarily holds since admissible input functions take values in the compact set U.

$\Gamma : H \times \Sigma \to H$ is a time independent (partially defined) *discrete state transition map*.

$\Xi : H \times \Sigma \to H$ is a time independent (partially defined) *continuous state jump transition map*. All $\xi_\sigma \in \Xi$, $\xi_\sigma : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p}$, $p \in A(q, \sigma)$ are assumed to be continuously differentiable in the continuous state $x \in \mathbb{R}^{n_q}$.

$A : Q \times \Sigma \to Q$ denotes both a deterministic finite automaton and the automaton's associated transition function on the state space Q and event set Σ, such that for a discrete state $q \in Q$ only the discrete controlled and uncontrolled transitions into the q-dependent subset $\{A(q,\sigma), \sigma \in \Sigma\} \subset Q$ occur under the projection of Γ on its Q components: $\Gamma : Q \times \mathbb{R}^n \times \Sigma \to H|_Q$. In other words, Γ can only make a discrete state transition in a hybrid state (q,x) if the automaton A can make the corresponding transition in q.

F is an indexed collection of *vector fields* $\{f_q\}_{q\in Q}$ such that $f_q \in C^{k_{f_q}}\left(\mathbb{R}^{n_q} \times U_q \to \mathbb{R}^{n_q}\right)$, $k_{f_q} \geq 1$, satisfies a joint uniform Lipschitz condition, i.e., there exists $L_f < \infty$ such that $\left\|f_q(x_1,u_1) - f_q(x_2,u_2)\right\| \leq L_f(\|x_1 - x_2\| + \|u_1 - u_2\|)$, for all $x, x_1, x_2 \in \mathbb{R}^{n_q}$, $u, u_1, u_2 \in U_q$, $q \in Q$.

$\mathscr{M} = \{m_\alpha : \alpha \in Q \times Q,\}$ denotes a collection of *switching manifolds* such that, for any ordered pair $\alpha \equiv (\alpha_1, \alpha_2) = (q,r)$, m_α is a smooth, i.e. C^∞ codimension k sub-manifold of $\mathbb{R}^{n_q}$, $k \in \{1, \cdots, n_q\}$, described locally by $m_\alpha = \left\{x : m_\alpha^1(x) = 0 \wedge \cdots \wedge m_\alpha^k(x) = 0\right\}$, and possibly with boundary ∂m_α. It is assumed that $m_\alpha \cap m_\beta = \emptyset$, whenever $\alpha_1 = \beta_1$ but $\alpha_2 \neq \beta_2$, for all $\alpha, \beta \in Q \times Q$. □

We note that the case where m_α is identified with its reverse ordered version $m_{\bar{\alpha}}$ giving $m_\alpha = m_{\bar{\alpha}}$ is not ruled out by this definition, even in the non-trivial case $m_{p,p}$ where $\alpha_1 = \alpha_2 = p$. The former case corresponds to the common situation where the switching of vector fields at the passage of the continuous trajectory in one direction through a switching manifold is reversed if a reverse passage is performed by the continuous trajectory, while the latter case corresponds to the standard example of the bouncing ball.

If not specified explicitly as a *low dimensional switching manifold*, the general use of the term *switching manifold* is for the case where $k = 1$, i.e. switching manifolds are considered to be codimension 1 sub-manifold of $\mathbb{R}^{n_q}$, and those cases with $1 < k \leq n_q$ are referred to as low dimensional switching manifolds.

Switching manifolds will function in such a way that whenever a trajectory governed by the controlled vector field meets the switching manifold transversally there is an autonomous switching to another controlled vector field or there is a jump transition in the continuous state component, or both. A transversal arrival on a switching manifold $m_{q,r}$, at state $x_q \in m_{q,r} = \left\{x \in \mathbb{R}^{n_q} : m_{q,r}(x) = 0\right\}$ occurs whenever

$$\nabla m_{q,r}\left(x_q\right)^T f_q\left(x_q, u_q\right) \neq 0, \tag{2.2}$$

for $u_q \in U_q$, and $q, r \in Q$. It is assumed that:

A1: The initial state $h_0 := (q_0, x(t_0)) \in H$ is such that $m_{q_0,q_j}(x_0) \neq 0$, for all $q_j \in Q$. $\square$

Definition 2.2. A *hybrid input process* is a pair $I_L \equiv I_L^{[t_0,t_f)} := (S_L, u)$ defined on a half open interval $[t_0, t_f)$, $t_f < \infty$, where $u \in \mathscr{U}$ and $S_L = ((t_0,\sigma_0),(t_1,\sigma_1),\cdots,(t_L,\sigma_L))$, $L < \infty$, is a finite *hybrid sequence of switching events* consisting of a strictly increasing sequence of times $\tau_L := \{t_0, t_1, t_2, \ldots, t_L\}$ and a *discrete event sequence* σ with $\sigma_0 = id$ and $\sigma_i \in \Sigma$, $i \in \{1,2,\cdots,L\}$. $\square$

Definition 2.3. A *hybrid state process* (or *trajectory*) is a triple (τ_L, q, x) consisting of the sequence of switching times $\tau_L = \{t_0, t_1, \ldots, t_L\}$, $L < \infty$, the associated sequence of discrete states $q = \{q_0, q_1, \ldots, q_L\}$, and the sequence $x(\cdot) = \{x_{q_0}(\cdot), x_{q_1}(\cdot), \ldots, x_{q_L}(\cdot)\}$ of piece-wise differentiable functions $x_{q_i}(\cdot) : [t_i, t_{i+1}) \to \mathbb{R}^n$. $\square$

Definition 2.4. The *input-state trajectory* for the hybrid system $\mathbb{H}$ satisfying A0 and A1 is a hybrid input $I_L = (S_L, u)$ together with its corresponding hybrid state trajectory (τ_L, q, x) defined over $[t_0, t_f), t_f < \infty$, such that it satisfies:

(i) *Continuous State Dynamics:* The continuous state component $x(\cdot) = \{x_{q_0}(\cdot), x_{q_1}(\cdot), \ldots, x_{q_L}(\cdot)\}$ is a piecewise continuous function which is almost everywhere differentiable and on each time segment specified by τ_L satisfies the dynamics equation

$$\dot{x}_{q_i}(t) = f_{q_i}\left(x_{q_i}(t), u(t)\right), \qquad a.e.\, t \in [t_i, t_{i+1}), \tag{2.3}$$

with the initial conditions

$$x_{q_0}(t_0) = x_0 \tag{2.4}$$

$$x_{q_i}(t_i) = \xi_{\sigma_i}\left(x_{q_{i-1}}(t_i-)\right) := \xi_{\sigma_i}\left(\lim_{t \uparrow t_i} x_{q_{i-1}}(t)\right) \tag{2.5}$$

for $(t_i, \sigma_i) \in S_L$. In other words, $x(\cdot) = \{x_{q_0}(\cdot), x_{q_1}(\cdot), \ldots, x_{q_L}(\cdot)\}$ is a piecewise continuous function which is almost everywhere differentiable and is such that each $x_{q_i}(\cdot)$ satisfies

$$x_{q_i}(t) = x_{q_i}(t_i) + \int_{t_i}^{t} f_{q_i}\left(x_{q_i}(s), u(s)\right) ds \tag{2.6}$$

for $t \in [t_i, t_{i+1})$.

(ii) *Autonomous Discrete Transition Dynamics:* An autonomous (uncontrolled) discrete state transition from q_{i-1} to q_i together with a continuous state jump ξ_{σ_i} occurs at the *autonomous switching time* t_i if $x_{q_{i-1}}(t_i-) := \lim_{t\uparrow t_i} x_{q_{i-1}}(t)$ satisfies a switching manifold condition of the form

$$m_{q_{i-1}q_i}(x(t_i-)) = 0 \tag{2.7}$$

for $q_i \in Q$, where $m_{q_{i-1}q_i}(x) = 0$ defines a (q_{i-1}, q_i) switching manifold and it is not the case that either (i) $x(t_i-) \in \partial m_{q_{i-1}q_i}$ or (ii) $f_{q_i-1}(x(t_i-), x(t_i-)) \perp \nabla m_{q_{i-1}q_i}(x(t_i-))$, i.e. t_i is not a manifold termination instant (see [74]). With the assumptions A0 and A1 in force, such a transition is well defined and labels the event $\sigma_{q_{i-1}q_i} \in \Sigma$, that corresponds to the hybrid state transition

$$h(t_i) \equiv \left(q_i, x_{q_i}(t_i)\right) = \left(\Gamma\left(q_{i-1}, \sigma_{q_{i-1}q_i}\right), \xi_{\sigma_{q_{i-1}q_i}}\left(x_{q_{i-1}}(t_i-)\right)\right) \tag{2.8}$$

(iii) *Controlled Discrete Transition Dynamics:* A controlled discrete state transition together with a controlled continuous state jump ξ_σ occurs at the *controlled discrete event time* t_i if t_i is not an autonomous discrete event time and if there exists a controlled discrete input event $\sigma_{q_{i-1}q_i} \in \Sigma$ for which

$$h(t_i) \equiv \left(q_i, x_{q_i}(t_i)\right) = \left(\Gamma\left(q_{i-1}, \sigma_{q_{i-1}q_i}\right), \xi_{\sigma_{q_{i-1}q_i}}\left(x_{q_{i-1}}(t_i-)\right)\right) \tag{2.9}$$

with $\left(t_i, \sigma_{q_{i-1}q_i}\right) \in S_L$ and $q_i \in A(q_{i-1})$. □

Theorem 2.1. [74] *A hybrid system $\mathbb{H}$ with an initial hybrid state (q_0, x_0) satisfying assumptions A0 and A1 possesses a unique hybrid input-state trajectory on $[t_0, T_{**})$, where T_{**} is the least of*

(i) $T_* \leq \infty$, where $[t_0, T_*)$ is the temporal domain of the definition of the hybrid system,

(ii) a manifold termination instant T_* of the trajectory $h(t) = h(t, (q_0, x_0), (S_L, u))$, $t \geq t_0$, at which either $x(T_*-) \in \partial m_{q(T_*-)q(T_*)}$ or $f_{q(T_*-)}(x(T_*-), u(T_*-)) \perp \nabla m_{q(T_*-)q(T_*)}(x(T_*-))$. □

We note that Zeno times, i.e. accumulation points of discrete transition times are ruled out by Definitions 2, 3 and 4.

2.2 Hybrid Optimal Control Problem

For the class of hybrid systems introduced above, we study hybrid optimal control problems with a large range of running, terminal and switching costs. With the exception of the infinite horizon problems considered in [16–19, 68], this framework is in accordance with the majority of the work on the Hybrid Minimum Principle (HMP) (see [4, 5, 7–9, 11, 12, 67, 75–78]) and a number of publications on Hybrid Dynamic Programming (HDP) (see e.g. [21–23, 69]) defined on finite horizons.

A2: Let $\{l_q\}_{q\in Q}, l_q \in C^{n_l}(\mathbb{R}^n \times U \to \mathbb{R}_+), n_l \geq 1$, be a family of cost functions with $n_l = 2$ unless otherwise stated; $\{c_\sigma\}_{\sigma\in\Sigma} \in C^{n_c}(\mathbb{R}^n \times \Sigma \to \mathbb{R}_+), n_c \geq 1$, be a family of switching cost functions; and $g \in C^{n_g}(\mathbb{R}^n \to \mathbb{R}_+), n_g \geq 1$, be a terminal cost function satisfying the following assumptions:

(i) There exists $K_l < \infty$ and $1 \leq \gamma_l < \infty$ such that $\left|l_q(x,u)\right| \leq K_l\left(1+\|x\|^{\gamma_l}\right)$ and $\left|l_q(x_1,u_1) - l_q(x_2,u_2)\right| \leq K_l(\|x_1-x_2\| + \|u_1-u_2\|)$, for all $x \in \mathbb{R}^n, u \in U, q \in Q$.

(ii) There exists $K_c < \infty$ and $1 \leq \gamma_c < \infty$ such that $|c_\sigma(x)| \leq K_c\left(1+\|x\|^{\gamma_c}\right)$, $x \in \mathbb{R}^n, \sigma \in \Sigma$.

(iii) There exists $K_g < \infty$ and $1 \leq \gamma_g < \infty$ such that $|g(x)| \leq K_g\left(1+\|x\|^{\gamma_g}\right)$, $x \in \mathbb{R}^n$. $\square$

Consider the initial time t_0, final time $t_f < \infty$, and initial hybrid state $h_0 = (q_0, x_0)$. With the number of switchings L held fixed, the set of all hybrid input trajectories in Definition 2.2 with exactly L switchings is denoted by $\boldsymbol{I}_L$, and for all $I_L := (S_L, u) \in \boldsymbol{I}_L$ the hybrid switching sequences take the form $S_L = \left\{(t_0, id), \left(t_1, \sigma_{q_0 q_1}\right), \ldots, \left(t_L, \sigma_{q_{L-1} q_L}\right)\right\} \equiv \{(t_0, q_0), (t_1, q_1), \ldots, (t_L, q_L)\}$ and the corresponding continuous control inputs are of the form $u \in \mathscr{U} = \bigcup_{i=0}^{L} L_\infty([t_i, t_{i+1}), U)$, where $t_{L+1} = t_f$.

Let I_L be a hybrid input trajectory that by Theorem 2.1 results in a unique hybrid state process. Then hybrid performance functions for the corresponding hybrid input-state trajectory are defined as

$$J\left(t_0, t_f, h_0, L; I_L\right) := \sum_{i=0}^{L} \int_{t_i}^{t_{i+1}} l_{q_i}\left(x_{q_i}(s), u(s)\right) ds + \sum_{j=1}^{L} c_{\sigma_{q_{j-1}q_j}}\left(t_j, x_{q_{j-1}}(t_j-)\right) + g\left(x_{q_L}(t_f)\right) \tag{2.10}$$

Definition 2.5. The *Bolza Hybrid Optimal Control Problem* (BHOCP) is defined as the infimization of the hybrid cost (2.10) over the family of hybrid input trajectories $\boldsymbol{I}_L$, i.e.

$$J^o\left(t_0, t_f, h_0, L\right) = \inf_{I_L \in \boldsymbol{I}_L} J\left(t_0, t_f, h_0, L; I_L\right) \tag{2.11}$$

□

Definition 2.6. The *Mayer Hybrid Optimal Control Problem* (MHOCP) is defined as a special case of the BHOCP where $l_q(x,u) = 0$ for all $q \in Q$ and $c_\sigma\left(t_j, x\left(t_j-\right)\right) = 0$ for all $\sigma \in \Sigma$.

□

Remark 2.1. *The Relationship between Bolza and Mayer Hybrid Optimal Control Problems:*

In general, a BHCOP can be converted into an MHCOP with the introduction of the auxiliary state component z and the extension of the continuous valued state to

$$\hat{x}_q := \begin{bmatrix} z_q \\ x_q \end{bmatrix} \tag{2.12}$$

With the definition of the augmented vector fields as

$$\dot{\hat{x}}_q = \hat{f}_q(\hat{x}, u) := \begin{bmatrix} l_q(x,u) \\ f_q(x,u) \end{bmatrix}, \tag{2.13}$$

subject to the initial condition

$$\hat{h}_0 = \left(q_0, \hat{x}_{q_0}(t_0)\right) = \left(q_0, \begin{bmatrix} 0 \\ x_0 \end{bmatrix}\right), \tag{2.14}$$

and with the switching boundary conditions governed by the extended jump function defined as

$$\hat{x}\left(t_j\right) = \hat{\xi}\left(\hat{x}\left(t_j-\right)\right) := \begin{bmatrix} z\left(t_j-\right) + c\left(x\left(t_j-\right)\right) \\ \xi\left(x\left(t_j-\right)\right) \end{bmatrix}, \tag{2.15}$$

the cost (2.10) of the BHOCP turns into the Mayer form with

$$J\left(t_0, t_f, \hat{h}_0, L; I_L\right) := \hat{g}\left(\hat{x}_{q_L}\left(t_f\right)\right), \tag{2.16}$$

where

$$\hat{g}\left(\hat{x}_{q_L}\left(t_f\right)\right) = z\left(t_f\right) + g\left(x\left(t_f\right)\right) . \tag{2.17}$$

□

Definition 2.7. The *cost to go* associated with an instant $t \in \left[t_0, t_f\right]$ which corresponds to some $1 \leq j \leq L+1$, such that $t \in \left(t_{j-1}, t_j\right]$, and for the state $h = (q, x)$ in the hybrid state space is given by

$$J\left(t, t_f, q, x, L-j+1; I_{L-j+1}\right) = \int_t^{t_j} l_q\left(x, u\right) ds + \sum_{i=j}^{L} c_{\sigma_{q_{i-1}q_i}}\left(t_i, x_{q_{i-1}}\left(t_i-\right)\right)$$
$$+ \sum_{i=j}^{L} \int_{t_i}^{t_{i+1}} l_{q_i}\left(x_{q_i}\left(s\right), u\left(s\right)\right) ds + g\left(x_{q_L}\left(t_f\right)\right), \tag{2.18}$$

□

Definition 2.8. The *value function* V is defined as the optimal cost to go over the family of hybrid control inputs, i.e.

$$V\left(t, q, x, L-j+1\right) := \inf_{I_{L-j+1}} J\left(t, t_f, q, x, L-j+1; I_{L-j+1}\right) \tag{2.19}$$

□

Chapter 3

Hybrid Minimum Principle (HMP)

The Minimum Principle (MP), also called the Maximum Principle in the pioneering work of Pontryagin et al. [1], is a milestone of systems and control theory that led to the emergence of optimal control as a distinct field of research. This principle states that any optimal control along with the optimal state trajectory must solve a two-point boundary value problem in the form of an extended Hamiltonian canonical system, as well as an extremization condition of the Hamiltonian function. Whether the extreme value is maximum or minimum depends on the sign convention used for the Hamiltonian definition. The generalization of the Minimum Principle for hybrid systems, i.e., control systems with both continuous and discrete states and dynamics, results in the Hybrid Minimum Principle (HMP). The HMP gives necessary conditions for the optimality of the trajectory and the control inputs of a given hybrid system with fixed initial conditions and a sequence of autonomous and controlled switchings. These conditions are expressed in terms of the minimization of the distinct Hamiltonians indexed by the discrete state sequence of the hybrid trajectory. A feature of special interest is the boundary conditions on the adjoint processes and the Hamiltonian functions at autonomous and controlled switching times and states; these boundary conditions may be viewed as a generalization of the optimal control case of the Weierstrass–Erdmann conditions of the calculus of variations [46].

3.1 The Hybrid Minimum Principle (HMP) for Free Endpoint Problems

Theorem 3.1. *Consider the hybrid system $\mathbb{H}$ subject to assumptions A0-A2, and the HOCP* (2.11) *for the hybrid performance function* (2.10)*. Define the family of system Hamiltonians by*

$$H_q\left(x_q,\lambda_q,u_q\right)=\lambda_q^T f_q\left(x_q,u_q\right)+l_q\left(x_q,u_q\right), \tag{3.1}$$

$x_q,\lambda_q\in\mathbb{R}^{n_q}$, $u_q\in U_q$, $q\in Q$. Then for an optimal switching sequence q^o and along the corresponding optimal trajectory x^o, there exists an adjoint process λ^o such that

$$\dot{x}^o=\frac{\partial H_{q^o}}{\partial\lambda_q}\left(x_q^o,\lambda_q^o,u_q^o\right), \tag{3.2}$$

$$\dot{\lambda}^o=-\frac{\partial H_{q^o}}{\partial x_q}\left(x_q^o,\lambda_q^o,u_q^o\right), \tag{3.3}$$

almost everywhere $t\in\left[t_0,t_f\right]$ with

$$x_{q_0^o}^o\left(t_0\right)=x_0, \tag{3.4}$$

$$x_{q_j^o}^o\left(t_j\right)=\xi_{\sigma_j}\left(x_{q_{j-1}^o}^o\left(t_j-\right)\right), \tag{3.5}$$

$$\lambda_{q_L^o}^o\left(t_f\right)=\nabla g\left(x_{q_L^o}^o\left(t_f\right)\right), \tag{3.6}$$

$$\lambda_{q_{j-1}^o}^o\left(t_j-\right)\equiv\lambda_{q_{j-1}^o}^o\left(t_j\right)=\nabla\xi_{\sigma_j}{}^T\lambda_{q_j^o}^o\left(t_j+\right)+\nabla c_{\sigma_j}+p\nabla m, \tag{3.7}$$

where $p\in\mathbb{R}$ when t_j indicates the time of an autonomous switching, and $p=0$ when t_j indicates the time of a controlled switching. Moreover,

$$H_{q^o}\left(x^o,\lambda^o,u^o\right)\leq H_{q^o}\left(x^o,\lambda^o,u\right), \tag{3.8}$$

for all $u\in U_{q^o}$, that is to say the Hamiltonian is minimized with respect to the control input, and at a switching time t_j the Hamiltonian satisfies

$$H_{q_{j-1}^o}(x^o,\lambda^o,u^o)|_{t_j-}\equiv H_{q_{j-1}^o}\left(t_j\right)=H_{q_j^o}\left(t_j\right)\equiv H_{q_j^o}(x^o,\lambda^o,u^o)|_{t_j+}. \tag{3.9}$$

$\square$

Proof. We first study a needle variation to the optimal input at the last location $u_{q_L}^o$ at a

Lebesgue instant $t \in (t_L, t_{L+1}] \equiv \left(t_L, t_f\right]$ to derive Hamiltonian canonical equations (3.2) and (3.3), the adjoint terminal condition (3.6), and the Hamiltonian minimization condition (3.8) in that location. This part is very similar to the proof of the classical Pontryagin Minimum Principle.

Next, we perform a variation in the penultimate, $L-1^{\text{st}}$, location in order to obtain (i) Hamiltonian canonical equations (3.2) and (3.3), and (ii) the Hamiltonian minimization condition (3.8) at the location q_{L-1}, as well as (iii) the boundary conditions (3.5) and (3.7), and (iv) the Hamiltonian boundary condition (3.9) at time t_L.

Then we extend the analysis for a general switching instant t_j and prove that (i) to (iv) above hold for all locations.

For simplicity of the notation, we assume that the hybrid optimal control problem is presented in the Mayer form (see Remark 2.1).

First, consider a needle variation at a Lebesgue time $t \in (t_L, t_{L+1}] \equiv \left(t_L, t_f\right]$ in the form of

$$u^{\varepsilon}(\tau) = \begin{cases} u^{o}_{q_{j-1}}(\tau) & \text{if} \quad \tau \in \left[t_{j-1}, t_j\right) \quad 1 \le j \le L \\ u^{o}_{q_L}(\tau) & \text{if} \quad \tau \in [t_L, t-\varepsilon) \\ v & \text{if} \quad \tau \in [t-\varepsilon, t) \\ u^{o}_{q_L}(\tau) & \text{if} \quad \tau \in \left[t, t_f\right] \end{cases}. \tag{3.10}$$

This corresponds to a perturbed trajectory $\hat{x}^{\varepsilon}(\tau), \tau \in \left[t_0, t_f\right]$ that necessarily satisfies $\hat{x}^{\varepsilon}(\tau) = \hat{x}^{o}(\tau)$ for $\tau \in [t_0, t)$. Denoting

$$\begin{aligned} \delta \hat{x}^{\varepsilon}_{q_L}(\tau) := \hat{x}^{\varepsilon}_{q_L}(\tau) - \hat{x}^{o}_{q_L}(\tau) = \int_{t-\varepsilon}^{t} \left[\hat{f}_{q_L}\left(\hat{x}^{\varepsilon}_{q_L}(s), v\right) - \hat{f}_{q_L}\left(\hat{x}^{o}_{q_L}(s), u^{o}_{q_L}(s)\right)\right] ds \\ + \int_{t}^{\tau} \left[\hat{f}_{q_L}\left(\hat{x}^{\varepsilon}_{q_L}(s), u^{o}_{q_L}(s)\right) - \hat{f}_{q_L}\left(\hat{x}^{o}_{q_L}(s), u^{o}_{q_L}(s)\right)\right] ds, \end{aligned} \tag{3.11}$$

the first order state variation is defined as

$$y(\tau) := \left.\frac{d}{d\varepsilon}\hat{x}^{\varepsilon}(\tau)\right|_{\varepsilon=0} \equiv \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \delta \hat{x}^{\varepsilon}(\tau), \tag{3.12}$$

and it is shown by Linearization Theory (see e.g. [74, 79] that

$$y_{q_L}\left(t_f\right) = \Phi_{q_L}\left(t_f, t\right)\left[\hat{f}_{q_L}\left(\hat{x}^{\varepsilon}_{q_L}(t), v\right) - \hat{f}_{q_L}\left(\hat{x}^{o}_{q_L}(t), u^{o}_{q_L}(t)\right)\right], \tag{3.13}$$

where Φ_{q_L} is the state transition matrix corresponding to the linearized system, i.e.

$$\frac{d}{d\tau}\Phi_{q_L}(\tau,t) = \frac{\partial}{\partial \hat{x}_{q_L}}\hat{f}_{q_L}\left(\hat{x}^o_{q_L}(\tau), u^o_{q_L}(\tau)\right)\Phi_{q_L}(\tau,t), \tag{3.14}$$

with $\Phi_{q_L}(t,t) = I_{\left(n_{q_L}+1\right)\times\left(n_{q_L}+1\right)}$.

The optimality of $\hat{x}^o$ gives

$$\hat{g}\left(\hat{x}^{\varepsilon}_{q_L}(t_f)\right) \geq \hat{g}\left(\hat{x}^o_{q_L}(t_f)\right), \tag{3.15}$$

which is equivalent to

$$\left.\frac{d}{d\varepsilon}J(u^{\varepsilon})\right|_{\varepsilon=0} = \left[\frac{\partial \hat{g}}{\partial \hat{x}_{q_L}}\left(\hat{x}^o_{q_L}(t_f)\right)\right]^T y_{q_L}(t_f) \geq 0. \tag{3.16}$$

Substitution of (3.13) into (3.16) results in

$$\frac{\partial \hat{g}}{\partial \hat{x}_{q_L}}\left(\hat{x}^o_{q_L}(t_f)\right)^T \Phi_{q_L}(t_f,t)\,\hat{f}_{q_L}\left(\hat{x}^o_{q_L}(t), v\right) \geq \frac{\partial \hat{g}}{\partial \hat{x}_{q_L}}\left(\hat{x}^o_{q_L}(t_f)\right)^T \Phi_{q_1}(t_f,t)\,\hat{f}_{q_1}\left(\hat{x}^o_{q_L}(t), u^o_{q_L}(t)\right). \tag{3.17}$$

Setting

$$\hat{\lambda}^{o\,T}_{q_L}(t) \equiv \left[\lambda^o_{0,q_L}(t), \lambda^{o\,T}_{q_L}(t)\right] = \frac{\partial \hat{g}}{\partial \hat{x}_{q_L}}\left(\hat{x}^o_{q_L}(t_f)\right)^T \Phi_{q_1}(t_f,t), \tag{3.18}$$

for $t \in \left(t_L, t_f\right]$ and evaluating it at $t = t_f$ we obtain

$$\hat{\lambda}^o_{q_L}(t_f) = \frac{\partial \hat{g}}{\partial \hat{x}_{q_L}}\left(\hat{x}^o_{q_L}(t_f)\right), \tag{3.19}$$

where, by the definition (2.17) for $\hat{g}$, it is equivalent to

$$\lambda^o_{0,q_L}(t_f) = 1, \tag{3.20}$$

$$\lambda^o_{q_L}(t_f) = \frac{\partial g}{\partial x_{q_L}}\left(x^o_{q_L}(t_f)\right) \equiv \nabla g\left(x^o_{q_L}(t_f)\right). \tag{3.21}$$

Also by differentiation of (3.40) with respect to t we obtain

$$\frac{d}{dt}\hat{\lambda}^o_{q_L}(t) = -\frac{\partial \hat{f}_{q_L}}{\partial \hat{x}_{q_L}}^T \left[\Phi_{q_L}(t_f,t)\right]^T \frac{\partial \hat{g}}{\partial \hat{x}_{q_L}}\left(\hat{x}^o_{q_L}(t_f)\right) = -\frac{\partial \hat{f}_{q_L}}{\partial \hat{x}_{q_L}}^T \hat{\lambda}^o_{q_L}(t), \tag{3.22}$$

which is equivalent to

$$\frac{d}{dt}\lambda^o_{0,q_L} = 0, \tag{3.23}$$

$$\frac{d}{dt}\lambda^o_{q_L} = -\left(\frac{\partial l_{q_L}\left(x^o_{q_L}(t), u^o_{q_L}(t)\right)}{\partial x_{q_L}}\right)\lambda^o_{0,q_L}(t) - \left(\frac{\partial f_{q_L}\left(x^o_{q_L}(t), u^o_{q_L}(t)\right)}{\partial x_{q_L}}\right)^T \lambda^o_{q_L}(t) \tag{3.24}$$

The zero dynamics (3.23) with the terminal condition (3.20) gives $\lambda^o_{0,q_L}(t) = 1$, for all $t \in \left(t_L, t_f\right)$, and equation (3.24) is equivalent to

$$\dot{\lambda}^o_{q_L} = -\frac{\partial H_{q_L}\left(x^o_{q_L}, \lambda^o_{q_L}, u^o_{q_L}\right)}{\partial x_{q_L}}, \tag{3.25}$$

which is valid on $\left(t_L, t_f\right)$ and where by definition

$$H_{q_L}\left(x_{q_L}, \lambda_{q_L}, u_{q_L}\right) = l_{q_L}\left(x_{q_L}, u_{q_L}\right) + \lambda^T_{q_L} f_{q_L}\left(x_{q_L}, u_{q_L}\right). \tag{3.26}$$

From the definition of Hamiltonian (3.26) and through a simple differentiation, the Hamiltonian canonical equation (3.2) for the state is also verified.

Also from (3.17) and (3.26) the Hamiltonian minimization

$$H_{q_L}\left(x^o_{q_L}, \lambda^o_{q_L}, u^o_{q_L}\right) \leq H_{q_L}\left(x^o_{q_L}, \lambda^o_{q_L}, v\right), \tag{3.27}$$

is obtained for all $v \in U_{q_L}$.

Now consider a needle variation at time $t \in (t_{L-1}, t_L]$ in the form of

$$u^\varepsilon(\tau) = \begin{cases} u^o_{q_{j-1}}(\tau) & \text{if} \quad \tau \in \left[t_{j-1}, t_j\right) \qquad 1 \leq j \leq L-1 \\ u^o_{q_{L-1}}(\tau) & \text{if} \quad \tau \in [t_L, t-\varepsilon) \\ v & \text{if} \quad \tau \in [t-\varepsilon, t) \\ u^o_{q_{L-1}}(\tau) & \text{if} \quad \tau \in [t, t_L - \delta^\varepsilon) \\ u^o_{q_L}(t_L) & \text{if} \quad \tau \in [t_L - \delta^\varepsilon, t_L) \\ u^o_{q_L}(\tau) & \text{if} \quad \tau \in \left[t_L, t_f\right] \end{cases}, \tag{3.28}$$

where $\delta^\varepsilon \geq 0$ corresponds to the case when the perturbed trajectory arrives on the switching manifold $\hat{m}(\hat{x}) := m_{q_{L-1}q_L}(x) = 0$ at an earlier instant (the case with a later arrival time is handled in a similar fashion, and the controlled switching case, i.e. the case with no switching manifold can be studied by setting $\delta^\varepsilon = 0$).

For $\tau \in [t, t_L - \delta^\varepsilon)$ we may write

$$\delta\hat{x}^\varepsilon_{q_{L-1}}(\tau) := \hat{x}^\varepsilon_{q_{L-1}}(\tau) - \hat{x}^o_{q_{L-1}}(\tau) = \int_{t-\varepsilon}^{t} \left[\hat{f}_{q_{L-1}}\left(\hat{x}^\varepsilon_{q_{L-1}}(s), v\right) - \hat{f}_{q_{L-1}}\left(\hat{x}^o_{q_{L-1}}(s), u^o_{q_{L-1}}(s)\right)\right] ds$$
$$+ \int_{t}^{\tau} \left[\hat{f}_{q_{L-1}}\left(\hat{x}^\varepsilon_{q_{L-1}}(s), u^o_{q_{L-1}}(s)\right) - \hat{f}_{q_{L-1}}\left(\hat{x}^o_{q_{L-1}}(s), u^o_{q_{L-1}}(s)\right)\right] ds, \quad (3.29)$$

At t_L the state of the optimal trajectory is determined by

$$\hat{x}^o_{q_L}(t_L) = \hat{\xi}\left(\hat{x}^o_{q_{L-1}}(t_L-)\right) = \hat{\xi}\left(\hat{x}^o_{q_{L-1}}(t_L - \delta^\varepsilon) + \int_{t_L-\delta^\varepsilon}^{t_L} \hat{f}_{q_{L-1}}\left(\hat{x}^o_{q_{L-1}}(\tau), u^o_{q_{L-1}}(\tau)\right) d\tau\right), \quad (3.30)$$

and the state of the perturbed trajectory is calculated as

$$\hat{x}^\varepsilon_{q_L}(t_L) = \hat{\xi}\left(\hat{x}^\varepsilon_{q_{L-1}}(t_L - \delta^\varepsilon -)\right) + \int_{t_L-\delta^\varepsilon}^{t_L} \hat{f}_{q_L}\left(\hat{x}^\varepsilon_{q_L}(\tau), u^o_{q_L}(t_L)\right) d\tau. \quad (3.31)$$

Thus

$$\delta\hat{x}^\varepsilon_{q_L}(t_L) = \hat{x}^\varepsilon_{q_L}(t_L) - \hat{x}^o_{q_L}(t_L) = \hat{\xi}\left(\hat{x}^\varepsilon_{q_{L-1}}(t_L - \delta^\varepsilon -)\right) + \int_{t_L-\delta^\varepsilon}^{t_L} \hat{f}_{q_L}\left(\hat{x}^\varepsilon_{q_L}(\tau), u^o_{q_L}(t_L)\right) d\tau$$
$$- \hat{\xi}\left(\hat{x}^o_{q_{L-1}}(t_L - \delta^\varepsilon) + \int_{t_L-\delta^\varepsilon}^{t_L} \hat{f}_{q_{L-1}}\left(\hat{x}^o_{q_{L-1}}(\tau), u^o_{q_{L-1}}(\tau)\right) d\tau\right), \quad (3.32)$$

and hence, the first order forward state sensitivity at t_L is calculated as

$$y_{q_L}(t_L) = \frac{\partial\hat{\xi}}{\partial\hat{x}_{q_{L-1}}}\left(\hat{x}^o_{q_{L-1}}(t_L-)\right) y_{q_{L-1}}(t_L-) + \lim_{\varepsilon\to 0}\frac{\delta^\varepsilon}{\varepsilon}\left[\hat{f}_{q_L}\left(\hat{\xi}\left(\hat{x}^o_{q_{L-1}}(t_L-)\right), u^o_{q_L}(t_L)\right)\right.$$
$$\left. - \frac{\partial\hat{\xi}}{\partial\hat{x}_{q_{L-1}}}\left(\hat{x}^o_{q_{L-1}}(t_L-)\right) \hat{f}_{q_{L-1}}\left(\hat{x}^o_{q_{L-1}}(t_L-), u^o_{q_{L-1}}(t_L-)\right)\right], \quad (3.33)$$

where

$$\lim_{\varepsilon\to 0}\frac{\delta^{\varepsilon}}{\varepsilon}=\frac{\left[\frac{\partial\hat{m}\left(\hat{x}^{o}_{q_{L-1}}(t_L-)\right)}{\partial\hat{x}_{q_{L-1}}}\right]^{T} y_{q_{L-1}}(t_L-)}{\left[\frac{\partial\hat{m}\left(\hat{x}^{o}_{q_{L-1}}(t_L-)\right)}{\partial\hat{x}_{q_{L-1}}}\right]^{T}\hat{f}_{q_{L-1}}\left(\hat{x}^{o}_{q_{L-1}}(t_L-),u^{o}_{q_{L-1}}(t_L-)\right)}. \tag{3.34}$$

Using the notation

$$\hat{f}^{\hat{\xi},q_{L-1}}_{q_L,\hat{\xi}}:=\hat{f}_{q_L}\left(\hat{\xi}\left(\hat{x}^{o}_{q_{L-1}}(t_L-)\right),u^{o}_{q_L}(t_L)\right)-\frac{\partial\hat{\xi}}{\partial\hat{x}_{q_{L-1}}}\left(\hat{x}^{o}_{q_{L-1}}(t_L-)\right)\hat{f}_{q_{L-1}}\left(\hat{x}^{o}_{q_{L-1}}(t_L-),u^{o}_{q_{L-1}}(t_L-)\right), \tag{3.35}$$

Equation (3.33) is written as

$$\begin{aligned}y_{q_L}(t_L)&=\frac{\partial\hat{\xi}}{\partial\hat{x}_{q_{L-1}}}\left(\hat{x}^{o}_{q_{L-1}}(t_L-)\right)y_{q_{L-1}}(t_L-)\\&\quad+\hat{f}^{\hat{\xi},q_{L-1}}_{q_L,\hat{\xi}}\frac{\left[\frac{\partial\hat{m}\left(\hat{x}^{o}_{q_{L-1}}(t_L-)\right)}{\partial\hat{x}_{q_{L-1}}}\right]^{T} y_{q_{L-1}}(t_L-)}{\left[\frac{\partial\hat{m}\left(\hat{x}^{o}_{q_{L-1}}(t_L-)\right)}{\partial\hat{x}_{q_{L-1}}}\right]^{T}\hat{f}_{q_{L-1}}\left(\hat{x}^{o}_{q_{L-1}}(t_L-),u^{o}_{q_{L-1}}(t_L-)\right)}.\end{aligned} \tag{3.36}$$

Similar to the previous analysis, the first order forward state sensitivity is propagated in $\left[t_L,t_f\right]$ by Φ_{q_L} and therefore

$$\begin{aligned}y_{q_L}(t_f)&=\Phi_{q_L}(t_f,t_L)\frac{\partial\hat{\xi}\left(\hat{x}^{o}_{q_{L-1}}(t_L-)\right)}{\partial\hat{x}_{q_{L-1}}}\Phi_{q_{L-1}}(t_L,t)\left[\hat{f}^{\left(\hat{x}^{o}_{q_{L-1}}(t),v\right)}_{q_{L-1}}-\hat{f}^{\left(\hat{x}^{o}_{q_{L-1}}(t),u^{o}_{q_{L-1}}(t)\right)}_{q_{L-1}}\right]\\&\quad+\Phi_{q_L}(t_f,t_L)\,\hat{f}^{\hat{\xi},q_{L-1}}_{q_L,\hat{\xi}}\frac{\left[\frac{\partial\hat{m}\left(\hat{x}^{o}_{q_{L-1}}(t_L-)\right)}{\partial\hat{x}_{q_{L-1}}}\right]^{T} y_{q_{L-1}}(t_L-)}{\left[\frac{\partial\hat{m}\left(\hat{x}^{o}_{q_{L-1}}(t_L-)\right)}{\partial\hat{x}_{q_{L-1}}}\right]^{T}\hat{f}_{q_{L-1}}\left(\hat{x}^{o}_{q_{L-1}}(t_L-),u^{o}_{q_{L-1}}(t_L-)\right)}.\end{aligned} \tag{3.37}$$

Therefore, the optimality condition (3.16) is expressed as

$$\left[\frac{\partial \hat{g}}{\partial \hat{x}_{q_L}}^T \Phi_{q_L}^{(t_f,t_L)} \frac{\partial \hat{\xi}}{\partial \hat{x}_{q_{L-1}}} + \frac{\frac{\partial \hat{g}}{\partial \hat{x}_{q_L}}^T \Phi_{q_L}^{(t_f,t_L)} \hat{f}_{q_L,\hat{\xi}}^{\hat{\xi},q_{L-1}}}{\frac{\partial \hat{m}}{\partial \hat{x}_{q_{L-1}}}^T \hat{f}_{q_{L-1}}^{\left(\hat{x}^o_{q_{L-1}}(t_L-),u^o_{q_{L-1}}(t_L-)\right)}} \left[\frac{\partial \hat{m}}{\partial \hat{x}_{q_{L-1}}}\right]^T\right] \Phi_{q_{L-1}}^{(t_L,t)} \hat{f}_{q_{L-1}}^{\left(\hat{x}^{\varepsilon}_{q_{L-1}}(t),v\right)}$$
$$\geq \left[\frac{\partial \hat{g}}{\partial \hat{x}_{q_L}}^T \Phi_{q_L}^{(t_f,t_L)} \frac{\partial \hat{\xi}}{\partial \hat{x}_{q_{L-1}}} + \frac{\frac{\partial \hat{g}}{\partial \hat{x}_{q_L}}^T \Phi_{q_L}^{(t_f,t_L)} \hat{f}_{q_L,\hat{\xi}}^{\hat{\xi},q_{L-1}}}{\frac{\partial \hat{m}}{\partial \hat{x}_{q_{L-1}}}^T \hat{f}_{q_{L-1}}^{\left(\hat{x}^o_{q_{L-1}}(t_L-),u^o_{q_{L-1}}(t_L-)\right)}} \left[\frac{\partial \hat{m}}{\partial \hat{x}_{q_{L-1}}}\right]^T\right] \Phi_{q_{L-1}}^{(t_L,t)} \hat{f}_{q_{L-1}}^{\left(\hat{x}^o_{q_{L-1}}(t),u^o_{q_{L-1}}(t)\right)}. \tag{3.38}$$

Denoting

$$p := \frac{\frac{\partial \hat{g}}{\partial \hat{x}_{q_L}}\left(\hat{x}^o_{q_L}(t_f)\right)^T \Phi_{q_L}(t_f,t_L)\, \hat{f}_{q_L,\hat{\xi}}^{\hat{\xi},q_{L-1}}}{\left[\frac{\partial \hat{m}\left(\hat{x}^o_{q_{L-1}}(t_L-)\right)}{\partial \hat{x}_{q_{L-1}}}\right]^T \hat{f}_{q_{L-1}}\left(\hat{x}^o_{q_{L-1}}(t_L-), u^o_{q_{L-1}}(t_L-)\right)}, \tag{3.39}$$

and setting

$$\hat{\lambda}^{o\;T}_{q_{L-1}}(t) = \left[\frac{\partial \hat{g}\left(\hat{x}^o_{q_L}(t_f)\right)}{\partial \hat{x}_{q_L}}^T \Phi_{q_L}(t_f,t_L) \frac{\partial \hat{\xi}\left(\hat{x}^o_{q_{L-1}}(t_L-)\right)}{\partial \hat{x}_{q_{L-1}}} + p\left[\frac{\partial \hat{m}\left(\hat{x}^o_{q_{L-1}}(t_L-)\right)}{\partial \hat{x}_{q_{L-1}}}\right]^T\right] \Phi_{q_{L-1}}(t_L,t), \tag{3.40}$$

for $t \in [t_{L-1}, t_L]$ and evaluating it at $t = t_L$ we obtain

$$\begin{aligned}\hat{\lambda}^{o\;T}_{q_{L-1}}(t_L) &= \frac{\partial \hat{g}\left(\hat{x}^o_{q_L}(t_f)\right)}{\partial \hat{x}_{q_L}}^T \Phi_{q_L}(t_f,t_L) \frac{\partial \hat{\xi}\left(\hat{x}^o_{q_{L-1}}(t_L-)\right)}{\partial \hat{x}_{q_{L-1}}} + p\left[\frac{\partial \hat{m}\left(\hat{x}^o_{q_{L-1}}(t_L-)\right)}{\partial \hat{x}_{q_{L-1}}}\right]^T \\ &= \hat{\lambda}^{o\;T}_{q_L}(t_L+) \frac{\partial \hat{\xi}\left(\hat{x}^o_{q_{L-1}}(t_L-)\right)}{\partial \hat{x}_{q_{L-1}}} + p\left[\frac{\partial \hat{m}\left(\hat{x}^o_{q_{L-1}}(t_L-)\right)}{\partial \hat{x}_{q_{L-1}}}\right]^T\end{aligned} \tag{3.41}$$

By the definition of $\hat{\xi}$ in (2.15), we have

$$\frac{\partial \hat{\xi}\left(\hat{x}^o_{q_{L-1}}(t_L-)\right)}{\partial \hat{x}_{q_{L-1}}} = \begin{bmatrix} \frac{\partial \hat{\xi}}{\partial z} \\ \frac{\partial \hat{\xi}}{\partial x} \end{bmatrix} = \begin{bmatrix} \frac{\partial[z+c]}{\partial z} & \frac{\partial[z+c]}{\partial x_1} & \cdots & \frac{\partial[z+c]}{\partial x_n} \\ \frac{\partial \xi_1}{\partial z} & \frac{\partial \xi_1}{\partial x_1} & \cdots & \frac{\partial \xi_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \xi_n}{\partial z} & \frac{\partial \xi_n}{\partial x_1} & \cdots & \frac{\partial \xi_n}{\partial x_n} \end{bmatrix} = \begin{bmatrix} 1 & \frac{\partial c}{\partial x_1} & \cdots & \frac{\partial c}{\partial x_n} \\ 0 & \frac{\partial \xi_1}{\partial x_1} & \cdots & \frac{\partial \xi_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{\partial \xi_n}{\partial x_1} & \cdots & \frac{\partial \xi_n}{\partial x_n} \end{bmatrix} = \begin{bmatrix} 1 & \nabla c^T \\ \mathbf{0} & \nabla \xi \end{bmatrix}, \tag{3.42}$$

and also since $\frac{\partial m}{\partial z} = 0$ we have

$$\frac{\partial \hat{m}\left(\hat{x}^o_{q_{L-1}}(t_L-)\right)}{\partial \hat{x}_{q_{L-1}}} = \begin{bmatrix} \frac{\partial \hat{m}}{\partial z} \\ \frac{\partial \hat{m}}{\partial x} \end{bmatrix} = \begin{bmatrix} 0 \\ \nabla m \end{bmatrix}. \tag{3.43}$$

Hence, (3.41) is equivalent to

$$\hat{\lambda}^o_{q_{L-1}}(t_L) \equiv \begin{bmatrix} \lambda^o_{q_{L-1},0}(t_L) \\ \lambda^o_{q_{L-1}}(t_L) \end{bmatrix} = \frac{\partial \hat{\xi}\left(\hat{x}^o_{q_{L-1}}(t_L-)\right)}{\partial \hat{x}_{q_{L-1}}}^T \hat{\lambda}^o_{q_L}(t_L+) + p\frac{\partial \hat{m}\left(\hat{x}^o_{q_{L-1}}(t_L-)\right)}{\partial \hat{x}_{q_{L-1}}}$$
$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \frac{\partial c}{\partial x_1} & \frac{\partial \xi_1}{\partial x_1} & \cdots & \frac{\partial \xi_n}{\partial x_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial c}{\partial x_n} & \frac{\partial \xi_1}{\partial x_n} & \cdots & \frac{\partial \xi_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} \lambda^o_{q_L,0}(t_L+) \\ \lambda^o_{q_L}(t_L+) \end{bmatrix} + p \begin{bmatrix} 0 \\ \nabla m \end{bmatrix} = \begin{bmatrix} 1 \\ \nabla \xi^T \lambda^o_{q_L}(t_L+) + \nabla c + p \nabla m \end{bmatrix}, \tag{3.44}$$

i.e.

$$\lambda^o_{q_{L-1},0}(t_L) = 1, \tag{3.45}$$
$$\lambda^o_{q_{L-1}}(t_L) = \nabla \xi^T \lambda^o_{q_L}(t_L+) + \nabla c + p \nabla m. \tag{3.46}$$

Differentiating (3.40) with respect to t leads to

$$\frac{d}{dt}\hat{\lambda}^o_{q_{L-1}}(t) = -\left(\frac{\partial \hat{f}_{q_{L-1}}}{\partial \hat{x}_{q_{L-1}}}\left(\hat{x}^o_{q_{L-1}}(t), u^o_{q_{L-1}}(t)\right)\right)^T \hat{\lambda}^o_{q_{L-1}}(t), \tag{3.47}$$

which is equivalent to

$$\frac{d}{dt}\lambda^{o}_{q_{L-1},0}(t) = 0, \tag{3.48}$$

$$\frac{d}{dt}\lambda^{o}_{q_{L-1}}(t) = -\left(\frac{\partial l_{q_{L-1}}\left(x^{o}_{q_{L-1}}(t), u^{o}_{q_{L-1}}(t)\right)}{\partial x_{q_{L-1}}}\right)\lambda^{o}_{0}(t) - \left(\frac{\partial f_{q_{L-1}}\left(x^{o}_{q_{L-1}}(t), u^{o}_{q_{L-1}}(t)\right)}{\partial x_{q_{L-1}}}\right)^{T}\lambda^{o}_{q_{L-1}}(t). \tag{3.49}$$

Therefore, $\lambda^{o}_{q_{L-1},0}(t) = 1$ for $t \in (t_{L-1}, t_L)$ is obtained as before and

$$\dot{\lambda}^{o}_{q_{L-1}} = -\frac{\partial H_{q_{L-1}}\left(x^{o}_{q_{L-1}}, \lambda^{o}_{q_{L-1}}, u^{o}_{q_{L-1}}\right)}{\partial x_{q_{L-1}}}, \tag{3.50}$$

holds for $t \in (t_{L-1}, t_L)$ with the Hamiltonian defined as

$$H_{q_{L-1}}\left(x_{q_{L-1}}, \lambda_{q_{L-1}}, u_{q_{L-1}}\right) = l_{q_{L-1}}\left(x_{q_{L-1}}, u_{q_{L-1}}\right) + \lambda^{T}_{q_{L-1}} f_{q_{L-1}}\left(x_{q_{L-1}}, u_{q_{L-1}}\right). \tag{3.51}$$

Also from (3.38) the minimization of the Hamiltonian is concluded as

$$H_{q_{L-1}}\left(x^{o}_{q_{L-1}}, \lambda^{o}_{q_{L-1}}, u^{o}_{q_{L-1}}\right) \le H_{q_{L-1}}\left(x^{o}_{q_{L-1}}, \lambda^{o}_{q_{L-1}}, v\right), \tag{3.52}$$

for all $v \in U_{q_{L-1}}$.

Evaluating both $H_{q_{L-1}}$ and H_{q_L} at t_L gives

$$\begin{aligned}
H_{q_{L-1}}(t_L-) &= l_{q_{L-1}}\left(x^{o}_{q_{L-1}}(t_L-), u^{o}_{q_{L-1}}(t_L-)\right) + \lambda^{o}_{q_{L-1}}(t_L-)^{T} f_{q_{L-1}}\left(x^{o}_{q_{L-1}}(t_L-), u^{o}_{q_{L-1}}(t_L-)\right) \\
&= \left[\hat{\lambda}^{o}_{q_{L-1}}(t_L-)\right]^{T} \hat{f}_{q_{L-1}}\left(\hat{x}_{q_{L-1}}(t_L-), u^{o}_{q_{L-1}}(t_L-)\right) \\
&= \left[\frac{\partial \hat{\xi}\left(\hat{x}^{o}_{q_{L-1}}(t_L-)\right)}{\partial \hat{x}_{q_{L-1}}}^{T} \hat{\lambda}^{o}_{q_L}(t_L+) + p\frac{\partial \hat{m}\left(\hat{x}^{o}_{q_{L-1}}(t_L-)\right)}{\partial \hat{x}_{q_{L-1}}}\right]^{T} \hat{f}_{q_{L-1}}\left(\hat{x}_{q_{L-1}}(t_L-), u^{o}_{q_{L-1}}(t_L-)\right) \\
&= \left[\hat{\lambda}^{o}_{q_L}(t_L+)^{T}\frac{\partial \hat{\xi}\left(\hat{x}^{o}_{q_{L-1}}(t_L-)\right)}{\partial \hat{x}_{q_{L-1}}} + \frac{\frac{\partial \hat{g}}{\partial \hat{x}_{q_L}}^{T}\Phi_{q_L}\left(t_f, t_L\right)\hat{f}^{\hat{\xi}, q_{L-1}}_{q_L, \hat{\xi}}}{\frac{\partial \hat{m}}{\partial \hat{x}_{q_{L-1}}}^{T}\hat{f}_{q_{L-1}}^{\left(\hat{x}^{o}_{q_{L-1}}(t_L-), u^{o}_{q_{L-1}}(t_L-)\right)}}\left[\frac{\partial \hat{m}}{\partial \hat{x}_{q_{L-1}}}\right]^{T}\right]\hat{f}^{(t_L-)}_{q_{L-1}}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial \hat{g}}{\partial \hat{x}_{q_L}}^T \Phi_{q_L}^{(t_f,t_L)} \frac{\partial \hat{\xi}}{\partial \hat{x}_{q_{L-1}}} \hat{f}_{q_{L-1}}^{\left(\hat{x}_{q_{L-1}}(t_L-),u^o_{q_{L-1}}(t_L-)\right)} + \frac{\frac{\partial \hat{g}}{\partial \hat{x}_{q_L}}^T \Phi_{q_L}\left(t_f,t_L\right) \hat{f}^{\hat{\xi},q_{L-1}}_{q_L,\hat{\xi}}}{\frac{\partial \hat{m}}{\partial \hat{x}_{q_{L-1}}}^T \hat{f}^{(t_L-)}_{q_{L-1}}} \left[\frac{\partial \hat{m}}{\partial \hat{x}_{q_{L-1}}}\right]^T \hat{f}^{(t_L-)}_{q_{L-1}} \\
&\qquad = \frac{\partial \hat{g}}{\partial \hat{x}_{q_L}}^T \Phi_{q_L}^{(t_f,t_L)} \frac{\partial \hat{\xi}}{\partial \hat{x}_{q_{L-1}}} \hat{f}^{(t_L-)}_{q_{L-1}} + \frac{\partial \hat{g}}{\partial \hat{x}_{q_L}}^T \Phi_{q_L}^{(t_f,t_L)} \left[\hat{f}^{(t_L)}_{q_L} - \frac{\partial \hat{\xi}}{\partial \hat{x}_{q_{L-1}}} \hat{f}^{(t_L-)}_{q_{L-1}}\right] \\
&= \frac{\partial \hat{g}}{\partial \hat{x}_{q_L}}^T \Phi_{q_L}\left(t_f,t_L\right) \hat{f}_{q_L}\left(\hat{\xi}\left(\hat{x}^o_{q_{L-1}}\left(t_L-\right)\right), u^o_{q_L}\left(t_L\right)\right) = \hat{\lambda}^o_{q_L}\left(t_L+\right)^T \hat{f}_{q_L}\left(\hat{x}^o_{q_L}\left(t_L\right), u^o_{q_L}\left(t_L\right)\right) \\
&\qquad = l_{q_L}\left(x^o_{q_L}\left(t_L\right), u^o_{q_L}\left(t_L\right)\right) + \lambda^o_{q_L}\left(t_L+\right)^T f_{q_L}\left(x^o_{q_L}\left(t_L\right), u^o_{q_L}\left(t_L\right)\right) = H_{q_L}\left(t_L+\right),
\end{aligned} \tag{3.53}
$$

which is equivalent to (3.9).

We now consider a needle variation at a Lebesgue time $t \in (t_{n-1}, t_n)$ in the form of

$$
u^{\varepsilon}(\tau) = \begin{cases}
u^o_{q_{j-1}}(\tau) & \text{if} \quad \tau \in \left[t_{j-1}, t_j\right) \qquad 1 \le j \le n-1 \\
u^o_{q_{n-1}}(\tau) & \text{if} \quad \tau \in [t_{n-1}, t-\varepsilon) \\
v & \text{if} \quad \tau \in [t-\varepsilon, t) \\
u^o_{q_{n-1}}(\tau) & \text{if} \quad \tau \in [t, t_n - \delta^{\varepsilon}_n) \\
u^o_{q_n}(t_n) & \text{if} \quad \tau \in [t_n - \delta^{\varepsilon}_n, t_n) \\
u^o_{q_k}(\tau) & \text{if} \quad \tau \in \left[t_k, t_{k+1} - \delta^{\varepsilon}_{k+1}\right) \qquad n \le k \le L \\
u^o_{q_{k+1}}(t_{k+1}) & \text{if} \quad \tau \in \left[t_{k+1} - \delta^{\varepsilon}_{k+1}, t_{k+1}\right) \qquad n \le k < L
\end{cases}. \tag{3.54}
$$

As before,

$$
y_{q_{n-1}}(t_n-) = \Phi_{q_{n-1}}(t_n, t) \left[\hat{f}_{q_{n-1}}\left(\hat{x}^{\varepsilon}_{q_{n-1}}(t), v\right) - \hat{f}_{q_{n-1}}\left(\hat{x}^o_{q_{n-1}}(t), u^o_{q_{n-1}}(t)\right)\right], \tag{3.55}
$$

and

$$
y_{q_n}(t_n) = \left[\frac{\partial \hat{\xi}_{\sigma_n}}{\partial \hat{x}_{q_{n-1}}} + \frac{1}{\left[\frac{\partial \hat{m}_{q_{n-1}q_n}}{\partial \hat{x}_{q_{n-1}}}\right]^T \hat{f}_{q_{n-1}}\left(\hat{x}^o_{q_{n-1}}(t_n-), u^o_{q_{n-1}}(t_n-)\right)} \hat{f}^{\hat{\xi}_{\sigma_n},q_{n-1}}_{q_n,\hat{\xi}_{\sigma_n}} \left[\frac{\partial \hat{m}_{q_{n-1}q_n}}{\partial \hat{x}_{q_{n-1}}}\right]^T\right] y_{q_{n-1}}(t_n-). \tag{3.56}
$$

Therefore,

$$y_{q_{\mathrm{L}}}\left(t_f\right)=\prod_{k=L}^{n}\left[\Phi_{q_k}\left(t_{k+1},t_k\right)\frac{\partial\hat{\xi}_{\sigma_k}}{\partial\hat{x}_{q_{k-1}}}+\gamma_k\hat{f}_{q_k,\hat{\xi}_{\sigma_k}}^{\hat{\xi}_{\sigma_k},q_{k-1}}\left[\frac{\partial\hat{m}_{q_{k-1}q_k}}{\partial\hat{x}_{q_{k-1}}}\right]^T\right] \times\Phi_{q_{n-1}}\left(t_n,t\right)\left[\hat{f}_{q_{n-1}}\left(\hat{x}_{q_{n-1}}^{\varepsilon}\left(t\right),v\right)-\hat{f}_{q_{n-1}}\left(\hat{x}_{q_{n-1}}^{o}\left(t\right),u_{q_{n-1}}^{o}\left(t\right)\right)\right], \quad (3.57)$$

where

$$\hat{f}_{q_k,\hat{\xi}_{\sigma_k}}^{\hat{\xi}_{\sigma_k},q_{k-1}}:=\hat{f}_{q_k}\left(\hat{\xi}_{\sigma_k}\left(\hat{x}_{q_{k-1}}^{o}\left(t_k-\right)\right),u_{q_k}^{o}\left(t_k\right)\right)-\frac{\partial\hat{\xi}_{\sigma_k}}{\partial\hat{x}_{q_{k-1}}}\left(\hat{x}_{q_{k-1}}^{o}\left(t_k-\right)\right)\hat{f}_{q_{k-1}}\left(\hat{x}_{q_{k-1}}^{o}\left(t_k-\right),u_{q_{k-1}}^{o}\left(t_k-\right)\right), \quad (3.58)$$

and

$$\gamma_k:=\begin{cases}0 & \text{controlled switching}\\ \\ \frac{1}{\left[\frac{\partial\hat{m}_{q_{k-1}q_k}}{\partial\hat{x}_{q_{k-1}}}\right]^T\hat{f}_{q_{k-1}}\left(\hat{x}_{q_{k-1}}^{o}\left(t_k-\right),u_{q_{k-1}}^{o}\left(t_k-\right)\right)} & \text{autonomous switching}\end{cases}. \quad (3.59)$$

The optimality condition (3.16) is expressed as

$$\left[\frac{\partial\hat{g}}{\partial\hat{x}_{q_L}}\right]^T\prod_{k=L}^{n}\left[\Phi_{q_k}\left(t_{k+1},t_k\right)\frac{\partial\hat{\xi}_{\sigma_k}}{\partial\hat{x}_{q_{k-1}}}+\gamma_k\hat{f}_{q_k,\hat{\xi}_{\sigma_k}}^{\hat{\xi}_{\sigma_k},q_{k-1}}\left[\frac{\partial\hat{m}_{q_{k-1}q_k}}{\partial\hat{x}_{q_{k-1}}}\right]^T\right] \times\Phi_{q_{n-1}}\left(t_n,t\right)\left[\hat{f}_{q_{n-1}}\left(\hat{x}_{q_{n-1}}^{\varepsilon}\left(t\right),v\right)-\hat{f}_{q_{n-1}}\left(\hat{x}_{q_{n-1}}^{o}\left(t\right),u_{q_{n-1}}^{o}\left(t\right)\right)\right]\geq 0. \quad (3.60)$$

Setting

$$\hat{\lambda}_{q_{n-1}}^{o\;T}\left(t\right)=\left[\frac{\partial\hat{g}}{\partial\hat{x}_{q_L}}\right]^T\prod_{k=L}^{n}\left[\Phi_{q_k}\left(t_{k+1},t_k\right)\frac{\partial\hat{\xi}_{\sigma_k}}{\partial\hat{x}_{q_{k-1}}}+\gamma_k\hat{f}_{q_k,\hat{\xi}_{\sigma_k}}^{\hat{\xi}_{\sigma_k},q_{k-1}}\left[\frac{\partial\hat{m}_{q_{k-1}q_k}}{\partial\hat{x}_{q_{k-1}}}\right]^T\right]\Phi_{q_{n-1}}\left(t_n,t\right), \quad (3.61)$$

for $t \in [t_{n-1}, t_n]$, which is equivalent to

$$\begin{aligned}
\hat{\lambda}^{o}_{q_{n-1}}(t) &\equiv \begin{bmatrix} \lambda^{o}_{q_{n-1},0}(t) \\ \lambda^{o}_{q_{n-1}}(t) \end{bmatrix} \\
&= \left[\Phi_{q_{n-1}}(t_n,t)\right]^T \prod_{k=n}^{L} \left[\left[\frac{\partial \hat{\xi}_{\sigma_k}}{\partial \hat{x}_{q_{k-1}}}\right]^T \left[\Phi_{q_k}(t_{k+1},t_k)\right]^T + \gamma_k \frac{\partial \hat{m}_{q_{k-1}q_k}}{\partial \hat{x}_{q_{k-1}}} \left[\hat{f}^{\hat{\xi}_{\sigma_k},q_{k-1}}_{q_k,\hat{\xi}_{\sigma_k}}\right]^T \right] \frac{\partial \hat{g}}{\partial \hat{x}_{q_L}} \\
&= \left[\Phi_{q_{n-1}}(t_n,t)\right]^T \left[\left[\frac{\partial \hat{\xi}_{\sigma_n}}{\partial \hat{x}_{q_{n-1}}}\right]^T \left[\Phi_{q_n}(t_{n+1},t_n)\right]^T + \gamma_n \frac{\partial \hat{m}_{q_{n-1}q_n}}{\partial \hat{x}_{q_{n-1}}} \left[\hat{f}^{\hat{\xi}_{\sigma_n},q_{n-1}}_{q_n,\hat{\xi}_{\sigma_n}}\right]^T \right] \\
&\quad \times \prod_{k=n+1}^{L} \left[\left[\frac{\partial \hat{\xi}_{\sigma_k}}{\partial \hat{x}_{q_{k-1}}}\right]^T \left[\Phi_{q_k}(t_{k+1},t_k)\right]^T + \gamma_k \frac{\partial \hat{m}_{q_{k-1}q_k}}{\partial \hat{x}_{q_{k-1}}} \left[\hat{f}^{\hat{\xi}_{\sigma_k},q_{k-1}}_{q_k,\hat{\xi}_{\sigma_k}}\right]^T \right] \frac{\partial \hat{g}}{\partial \hat{x}_{q_L}}, \quad (3.62)
\end{aligned}$$

we may evaluate (3.62) at $t = t_n$ to obtain

$$\begin{aligned}
\hat{\lambda}^{o}_{q_{n-1}}(t_n) &= \left[\left[\frac{\partial \hat{\xi}_{\sigma_n}}{\partial \hat{x}_{q_{n-1}}}\right]^T \left[\Phi_{q_n}(t_{n+1},t_n)\right]^T + \gamma_n \frac{\partial \hat{m}_{q_{n-1}q_n}}{\partial \hat{x}_{q_{n-1}}} \left[\hat{f}^{\hat{\xi}_{\sigma_n},q_{n-1}}_{q_n,\hat{\xi}_{\sigma_n}}\right]^T \right] \\
&\quad \times \prod_{k=n+1}^{L} \left[\left[\frac{\partial \hat{\xi}_{\sigma_k}}{\partial \hat{x}_{q_{k-1}}}\right]^T \left[\Phi_{q_k}(t_{k+1},t_k)\right]^T + \gamma_k \frac{\partial \hat{m}_{q_{k-1}q_k}}{\partial \hat{x}_{q_{k-1}}} \left[\hat{f}^{\hat{\xi}_{\sigma_k},q_{k-1}}_{q_k,\hat{\xi}_{\sigma_k}}\right]^T \right] \frac{\partial \hat{g}}{\partial \hat{x}_{q_L}}, \quad (3.63)
\end{aligned}$$

or

$$\begin{aligned}
\hat{\lambda}^{o}_{q_{n-1}}(t_n) &= \left[\frac{\partial \hat{\xi}_{\sigma_n}}{\partial \hat{x}_{q_{n-1}}}\right]^T \left[\Phi_{q_n}(t_{n+1},t_n)\right]^T \prod_{k=n+1}^{L} \left[\left[\frac{\partial \hat{\xi}_{\sigma_k}}{\partial \hat{x}_{q_{k-1}}}\right]^T \left[\Phi_{q_k}(t_{k+1},t_k)\right]^T + \gamma_k \frac{\partial \hat{m}_{q_{k-1}q_k}}{\partial \hat{x}_{q_{k-1}}} \left[\hat{f}^{\hat{\xi}_{\sigma_k},q_{k-1}}_{q_k,\hat{\xi}_{\sigma_k}}\right]^T \right] \frac{\partial \hat{g}}{\partial \hat{x}_{q_L}} \\
&+ \gamma_n \frac{\partial \hat{m}_{q_{n-1}q_n}}{\partial \hat{x}_{q_{n-1}}} \left[\hat{f}^{\hat{\xi}_{\sigma_n},q_{n-1}}_{q_n,\hat{\xi}_{\sigma_n}}\right]^T \prod_{k=n+1}^{L} \left[\left[\frac{\partial \hat{\xi}_{\sigma_k}}{\partial \hat{x}_{q_{k-1}}}\right]^T \left[\Phi_{q_k}(t_{k+1},t_k)\right]^T + \gamma_k \frac{\partial \hat{m}_{q_{k-1}q_k}}{\partial \hat{x}_{q_{k-1}}} \left[\hat{f}^{\hat{\xi}_{\sigma_k},q_{k-1}}_{q_k,\hat{\xi}_{\sigma_k}}\right]^T \right] \frac{\partial \hat{g}}{\partial \hat{x}_{q_L}}.
\end{aligned} \quad (3.64)$$

Having established that

$$\hat{\lambda}^o_{q_n}(\tau) = \left[\Phi_{q_n}(t_{n+1},\tau)\right]^T \prod_{k=n+1}^{L} \left[\left[\frac{\partial \hat{\xi}_{\sigma_k}}{\partial \hat{x}_{q_{k-1}}}\right]^T \left[\Phi_{q_k}(t_{k+1},t_k)\right]^T + \gamma_k \frac{\partial \hat{m}_{q_{k-1}q_k}}{\partial \hat{x}_{q_{k-1}}} \left[\hat{f}^{\hat{\xi}_{\sigma_k},q_{k-1}}_{q_k,\hat{\xi}_{\sigma_k}}\right]^T \right] \frac{\partial \hat{g}}{\partial \hat{x}_{q_L}}, \tag{3.65}$$

and denoting the scalar product

$$p_n := \gamma_n \left[\hat{f}^{\hat{\xi}_{\sigma_n},q_{n-1}}_{q_n,\hat{\xi}_{\sigma_n}}\right]^T \prod_{k=n+1}^{L} \left[\left[\frac{\partial \hat{\xi}_{\sigma_k}}{\partial \hat{x}_{q_{k-1}}}\right]^T \left[\Phi_{q_k}(t_{k+1},t_k)\right]^T + \gamma_k \frac{\partial \hat{m}_{q_{k-1}q_k}}{\partial \hat{x}_{q_{k-1}}} \left[\hat{f}^{\hat{\xi}_{\sigma_k},q_{k-1}}_{q_k,\hat{\xi}_{\sigma_k}}\right]^T \right] \frac{\partial \hat{g}}{\partial \hat{x}_{q_L}}, \tag{3.66}$$

equation (3.64) gives

$$\hat{\lambda}^o_{q_{n-1}}(t_n) = \hat{\lambda}^o_{q_n}(t_n+) + p_n \frac{\partial \hat{m}_{q_{n-1}q_n}}{\partial \hat{x}_{q_{n-1}}}. \tag{3.67}$$

Since the induction hypothesis (3.65) is proved to hold for $n = L-1$ (see (3.41)) and since (3.65) for n implies (3.67), the boundary condition (3.7) is deduced from (3.67) in a similar way as shown in (3.42) to (3.70), i.e. (3.67) is equivalent to

$$\hat{\lambda}^o_{q_{n-1}}(t_n) \equiv \begin{bmatrix} \lambda^o_{q_{n-1},0}(t_n) \\ \lambda^o_{q_{n-1}}(t_n) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \frac{\partial c}{\partial x_1} & \frac{\partial \xi_1}{\partial x_1} & \cdots & \frac{\partial \xi_n}{\partial x_1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial c}{\partial x_n} & \frac{\partial \xi_1}{\partial x_n} & \cdots & \frac{\partial \xi_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} \lambda^o_{q_L,0}(t_L+) \\ \lambda^o_{q_L}(t_L+) \end{bmatrix} + p \begin{bmatrix} 0 \\ \nabla m \end{bmatrix}, \tag{3.68}$$

which gives

$$\lambda^o_{q_{n-1},0}(t_n) = 1, \tag{3.69}$$

$$\lambda^o_{q_{n-1}}(t_n) = \nabla \xi^T \lambda^o_{q_n}(t_n+) + \nabla c_{\sigma_n} + p \nabla m_{q_{n-1}q_n}. \tag{3.70}$$

Differentiating (3.62) with respect to t leads to

$$\frac{d}{dt}\hat{\lambda}^o_{q_{n-1}}(t) = -\left(\frac{\partial \hat{f}_{q_{n-1}}}{\partial \hat{x}_{q_{n-1}}}\left(\hat{x}^o_{q_{n-1}}(t), u^o_{q_{n-1}}(t)\right)\right)^T \hat{\lambda}^o_{q_{n-1}}(t), \tag{3.71}$$

which is equivalent to

$$\frac{d}{dt}\lambda^{o}_{q_{n-1},0}(t) = 0, \tag{3.72}$$

$$\frac{d}{dt}\lambda^{o}_{q_{n-1}}(t) = -\left(\frac{\partial l_{q_{n-1}}\left(x^{o}_{q_{n-1}}(t), u^{o}_{q_{n-1}}(t)\right)}{\partial x_{q_{n-1}}}\right)\lambda^{o}_{0}(t) - \left(\frac{\partial f_{q_{n-1}}\left(x^{o}_{q_{n-1}}(t), u^{o}_{q_{n-1}}(t)\right)}{\partial x_{q_{n-1}}}\right)^{T}\lambda^{o}_{q_{n-1}}(t). \tag{3.73}$$

Therefore, $\lambda^{o}_{q_{n-1},0}(t) = 1$ for $t \in (t_{n-1}, t_n)$ is obtained as before and

$$\dot{\lambda}^{o}_{q_{n-1}} = -\frac{\partial H_{q_{n-1}}\left(x^{o}_{q_{n-1}}, \lambda^{o}_{q_{n-1}}, u^{o}_{q_{n-1}}\right)}{\partial x_{q_{n-1}}}, \tag{3.74}$$

holds for $t \in (t_{n-1}, t_n)$ with the Hamiltonian defined as

$$H_{q_{n-1}}\left(x_{q_{n-1}}, \lambda_{q_{n-1}}, u_{q_{n-1}}\right) = l_{q_{n-1}}\left(x_{q_{n-1}}, u_{q_{n-1}}\right) + \lambda^{T}_{q_{n-1}} f_{q_{n-1}}\left(x_{q_{n-1}}, u_{q_{n-1}}\right). \tag{3.75}$$

Also from (3.60) the minimization of the Hamiltonian is concluded, i.e.

$$H_{q_{n-1}}\left(x_{q_{n-1}}, \lambda_{q_{n-1}}, u_{q_{n-1}}\right) \le H_{q_{n-1}}\left(x_{q_{n-1}}, \lambda_{q_{n-1}}, v\right), \tag{3.76}$$

for all $v \in U_{q_{n-1}}$.

Evaluating both $H_{q_{n-1}}$ and H_{q_n} at t_n gives

$$\begin{aligned}
H_{q_{n-1}}(t_n-) &= l_{q_{n-1}}\left(x_{q_{n-1}}(t_n-), u_{q_{n-1}}(t_n-)\right) + \lambda_{q_{n-1}}(t_n-)^{T} f_{q_{n-1}}\left(x_{q_{n-1}}(t_n-), u_{q_{n-1}}(t_n-)\right) \\
&= \left[\hat{\lambda}^{o}_{q_{n-1}}(t_n-)\right]^{T}\hat{f}_{q_{n-1}}\left(\hat{x}_{q_{n-1}}(t_n-), u^{o}_{q_{n-1}}(t_n-)\right) \\
&= \left[\left(\frac{\partial \hat{\xi}\left(\hat{x}^{o}_{q_{n-1}}(t_n-)\right)}{\partial \hat{x}_{q_{n-1}}}\right)^{T}\hat{\lambda}^{o}_{q_n}(t_n+) + p_n\frac{\partial \hat{m}\left(\hat{x}^{o}_{q_{n-1}}(t_n-)\right)}{\partial \hat{x}_{q_{n-1}}}\right]^{T}\hat{f}_{q_{n-1}}\left(\hat{x}_{q_{n-1}}(t_n-), u^{o}_{q_{n-1}}(t_n-)\right) \\
&= \left[\hat{\lambda}^{o}_{q_n}(t_n+)^{T}\frac{\partial \hat{\xi}\left(\hat{x}^{o}_{q_{n-1}}(t_n-)\right)}{\partial \hat{x}_{q_{n-1}}} + \gamma_n\left[\hat{f}^{\hat{\xi}_{\sigma_n}, q_{n-1}}_{q_n, \hat{\xi}_{\sigma_n}}\right]^{T}\hat{\lambda}^{o}_{q_n}(t_n+)\left[\frac{\partial \hat{m}_{q_{n-1}q_n}}{\partial \hat{x}_{q_{n-1}}}\right]^{T}\right]\hat{f}^{(t_n-)}_{q_{n-1}}
\end{aligned}$$

$$= \hat{\lambda}^{o}_{q_n}(t_n+)^T \frac{\partial \hat{\xi}\left(\hat{x}^{o}_{q_{n-1}}(t_n-)\right)}{\partial \hat{x}_{q_{n-1}}} \hat{f}^{(t_n-)}_{q_{n-1}} + \frac{\left[\hat{f}^{\hat{\xi}_{\sigma_n},q_{n-1}}_{q_n,\hat{\xi}_{\sigma_n}}\right]^T \hat{\lambda}^{o}_{q_n}(t_n+)}{\left[\frac{\partial \hat{m}_{q_{n-1}q_n}}{\partial \hat{x}_{q_{n-1}}}\right]^T \hat{f}^{(t_n-)}_{q_{n-1}}} \left[\frac{\partial \hat{m}_{q_{n-1}q_n}}{\partial \hat{x}_{q_{n-1}}}\right]^T \hat{f}^{(t_n-)}_{q_{n-1}}$$

$$= \hat{\lambda}^{o}_{q_n}(t_n+)^T \frac{\partial \hat{\xi}_{\sigma_n}\left(\hat{x}^{o}_{q_{n-1}}(t_n-)\right)}{\partial \hat{x}_{q_{n-1}}} \hat{f}^{(t_n-)}_{q_{n-1}} + \hat{\lambda}^{o}_{q_n}(t_n+)^T \left[\hat{f}^{(t_n)}_{q_n} - \frac{\partial \hat{\xi}_{\sigma_n}}{\partial \hat{x}_{q_{n-1}}} \hat{f}^{(t_n-)}_{q_{n-1}}\right]$$

$$= \hat{\lambda}^{o}_{q_n}(t_n+)^T \hat{f}_{q_n}\left(\hat{x}^{o}_{q_n}(t_n), u^{o}_{q_n}(t_n)\right)$$

$$= l_{q_n}\left(x^{o}_{q_n}(t_n), u^{o}_{q_n}(t_n)\right) + \lambda^{o}_{q_n}(t_n+)^T f_{q_n}\left(x^{o}_{q_n}(t_n), u^{o}_{q_n}(t_n)\right) = H_{q_n}(t_n+), \quad (3.77)$$

which is equivalent to (3.9). □

3.2 General Endpoint and Boundary Conditions

3.2.1 Time-Varying Vector Fields, Costs, and Switching Manifolds

For simplicity of the notation, in the previous section, the statement of the Hybrid Minimum Principle and its proof was provided for time-invariant vector fields, time-invariant running, switching and terminal costs, and time-invariant switching manifolds. It is, however, no loss of generality as time-varying hybrid optimal control problems can be converted to time-invariant problems by the extension of states, vector fields, etc. as

$$\tilde{x}_q := \begin{bmatrix} \theta \\ \hat{x}_q \end{bmatrix} \equiv \begin{bmatrix} \theta \\ z_q \\ x_q \end{bmatrix}, \quad (3.78)$$

resulting in augmented vector fields as

$$\dot{\tilde{x}}_q = \tilde{f}_q\left(\tilde{x}_q, u_q\right) := \begin{bmatrix} 1 \\ \hat{f}_q\left(\hat{x}_q, u_q, t\right) \end{bmatrix} \equiv \begin{bmatrix} 1 \\ l_q(x,u,\theta) \\ f_q(x,u,\theta) \end{bmatrix}, \quad (3.79)$$

subject to the initial condition

$$\tilde{h}_0 = \left(q_0, \hat{x}_{q_0}(t_0)\right) = \left(q_0, \begin{bmatrix} t_0 \\ 0 \\ x_0 \end{bmatrix}\right), \tag{3.80}$$

with the switching manifold

$$\tilde{m}(\tilde{x}) := m(x, \theta), \tag{3.81}$$

and the extended jump function defined as

$$\tilde{x}_{q_j}(t_j) = \tilde{\xi}_{\sigma_j}\left(\tilde{x}_{q_{j-1}}(t_j-)\right) := \begin{bmatrix} \theta(t_j-) \\ z(t_j-) + c\left(x(t_j-), \theta\right) \\ \xi_{\sigma_j}\left(x(t_j-), \theta\right) \end{bmatrix}. \tag{3.82}$$

3.2.2 Packet of Needle Variations

For free endpoint problems as in the statement of Theorem 3.1, the performance measure (2.10) for the family of trajectories indexed by the needle's width ε must have a minimum at zero, hence its derivative w.r.t. ε must be nonnegative (see (3.16)), and this easily yields the conditions of the HMP. However, if the problem includes constraints on the right endpoint of trajectory, such a variation may generate a trajectory that would not, in general, satisfy them. In order to guarantee that the endpoint of varied trajectories hit the given constraints, the family of variations should be rich enough, and therefore one has to consider not just one, but a finite number of needle variations together, the so-called packet of needle variations, whose parameter is the collection of widths of the needles, independent of each other (see e.g. [80] for more discussion). The packet of needle variations are then used to construct a terminal cone which must be separated from another convex cone generated by the tangent space of the terminal manifold and a ray in the direction of cost decrease. The proof of the Minimum Principle with these terminal conditions are extensively studied in the literature of the classical Pontryagin Minimum Principle (see e.g. [1, 4, 47, 80, 81]).

3.2.3 Abnormal Multiplier

For optimal control problems with general endpoint conditions such as termination at a pre-specified point, another issue that may rise is that the set of admissible input-state trajectories[1] has a single element. While this is no difficulty in determining the optimum which must be the single element, it hinders the development of the theory in that it precludes the construction of mappings based on needle variations of the type (3.10), (3.28), and (3.54). A classical example in which the set of admissible input-state trajectories has a single element is provided in [29] as follows. Consider the optimal control problem with scalar differential equation

$$\dot{x} = u^2, \tag{3.83}$$

with the control set $U = [-1, 1]$, and subject to the initial and terminal conditions

$$x(t_0) = 0, \tag{3.84}$$

$$x\left(t_f\right) = 0, \tag{3.85}$$

and consider any performance criterion as

$$J\left(t_0, t_f, x_0, x_f\right) = \int_{t_0}^{t_f} l\left(x(s), u(s)\right) ds. \tag{3.86}$$

The only control whose corresponding trajectory satisfies the initial and terminal conditions is

$$u(t) \equiv 0, \quad t_0 \leq t \leq t_f, \tag{3.87}$$

and hence this is the only element of the set of admissible input-state trajectories.

Problems like the above example are referred to as abnormal problems (see e.g. [81] for more discussion of abnormal problems) and in order for the necessary optimality conditions to be satisfied, $\lambda^o_{q_k,0}(t)$ in (3.40) and (3.62) must be taken to be 0. The scalar $\lambda^o_{q_k,0}(t)$ is usually referred to as the abnormal multiplier where for abnormal problems it is zero and for normal problems nonzero (and therefore scalable to identity).

[1] The set of input-state trajectories (see Definition 2.4) satisfying the specified initial, boundary and terminal conditions; see e.g. [29, 47] for more details.

3.2.4 Low Dimensional Switching Manifolds

The usual assumption in design, analysis and control of hybrid systems is that switching manifolds corresponding to autonomous switchings and jumps are smooth codimension 1 sub-manifolds of $\mathbb{R}^n$. In some studies like hybrid stability, this assumption reduces the analysis by decoupling the sequence of switching and the uniform convergence of hybrid executions within those with the same switching sequence. However, in the hybrid optimal control context, the assumption of codimension 1 switching manifolds is not a necessity since the optimality conditions are expressed in terms of the admissible controls and their corresponding trajectories that satisfy the desired switching conditions. While numerous hybrid optimal control problems can be considered where the system has switching manifolds with dimensions smaller than $n-1$ (see e.g. Section 6.3), i.e. where switching manifolds are codimension k sub-manifold of $\mathbb{R}^n$ with $k>1$, this class of hybrid systems has been the subject of a limited number of studies in the hybrid optimal control context. The class of hybrid systems under study is further generalized by letting the switching manifolds be codimension k sub-manifold of $\mathbb{R}^n$, with $k \in \{1, \cdots, n\}$.

3.2.5 The General Statement of the Hybrid Minimum Principle

Theorem 3.2. *Define the family of system Hamiltonians by*

$$H_q\left(x_q, \lambda_{q,0}, \lambda_q, u_q, t\right) = \lambda_{q,0}\, l_q\left(x_q, u_q, t\right) + \lambda_q^T f_q\left(x_q, u_q, t\right), \tag{3.88}$$

for $x_q \in \mathbb{R}^{n_q}$, $\lambda_{q,0} \in \mathbb{R}$, $\lambda_q \in \mathbb{R}^{n_q}$, $u_q \in U_q$, $q \in Q$. *Then for the optimal switching sequence* q^o *and along the optimal trajectory* x^o *there exists constants* $\lambda_{q_i,0}^o \geq 0$ *and adjoint processes* $\lambda_{q_i}^o$ *such that* $\left[\lambda_{q_i,0}^o, {\lambda_{q_i}^o}^T\right] \neq 0$ *and*

$$\dot{x}_q^o = \frac{\partial H_{q^o}}{\partial \lambda_q}\left(x_q^o, \lambda_{q,0}^o, \lambda_q^o, u_q^o, t\right), \tag{3.89}$$

$$\dot{\lambda}_q^o = -\frac{\partial H_{q^o}}{\partial x_q}\left(x_q^o, \lambda_{q,0}^o, \lambda_q^o, u_q^o, t\right), \tag{3.90}$$

almost everywhere $t \in [t_0, t_f]$ *with*

$$x^o_{q^o_0}(t_0) = x_0, \tag{3.91}$$

$$x^o_{q^o_{j-1}}(t_j-) \in m_j := \left\{ x \in \mathbb{R}^{n_{q_{j-1}}} : m^1_{q_{j-1}q_j}(x) = 0 \wedge \cdots \wedge m^{k_j}_{q_{j-1}q_j}(x) = 0 \right\}, \tag{3.92}$$

$$x^o_{q^o_j}(t_j) = \xi_{\sigma_j}\left(x^o_{q^o_{j-1}}(t_j-) \right), \tag{3.93}$$

$$x^o_{q^o_L}(t_f) \in m_f \equiv m_{L+1} := \left\{ x \in \mathbb{R}^{n_{q_L}} : m^1_{q_L,stop}(x) = 0 \wedge \cdots \wedge m^{k_{L+1}}_{q_L,stop}(x) = 0 \right\}, \tag{3.94}$$

$$\lambda^o_{q^o_L}(t_f) = \nabla g\left(x^o_{q^o_L}(t_f) \right) + \sum_{i=1}^{k_{L+1}} p^i_{L+1} \nabla m^i_{L+1}\left(x^o_{q^o_L}(t_f) \right), \tag{3.95}$$

$$\lambda^o_{q^o_{j-1}}(t_j-) \equiv \lambda^o_{q^o_{j-1}}(t_j) = \nabla \xi_{\sigma_j}{}^T \lambda^o_{q^o_j}(t_j+) + \nabla c_{\sigma_j}\left(x^o_{q^o_{j-1}}(t_j-) \right) + \sum_{i=1}^{k_j} p^i_j \nabla m^i_j \left(x^o_{q^o_{j-1}}(t_j-) \right), \tag{3.96}$$

where $m_j = \emptyset$ *and* $p^i_j = 0$ *when* t_j *indicates the time of a controlled switching and* $p^i_j \in \mathbb{R}$ *when* t_j *indicates the time of an autonomous switching.*

Moreover, the Hamiltonian is minimized with respect to the control input

$$H_{q^o}\left(x^o_q, \lambda^o_{q,0}, \lambda^o_q, u^o_q, t \right) \leq H_{q^o}\left(x^o_q, \lambda^o_{q,0}, \lambda^o_q, u, t \right), \tag{3.97}$$

for all $u \in U_q$*; and at a switching time* t_j *the Hamiltonian satisfies*

$$\begin{aligned} &H_{q_{j-1}}(t_j-) \equiv H_{q_{j-1}}\left(x^o_{q_{j-1}}, \lambda^o_{q_{j-1},0}, \lambda^o_{q_{j-1}}, u^o_{q_{j-1}}, t^o_j- \right) \\ &= H_{q_j}\left(x^o_{q_j}, \lambda^o_{q_j,0}, \lambda^o_{q_j}, u^o_{q_j}, t^o_j+ \right) + \lambda^o_{q_j,0} \frac{\partial c\left(x^o_{q^o_{j-1}}(t_j-) \right)}{\partial t} + \sum_{i=1}^{k_j} p^i_j \frac{\partial m^i_j\left(x^o_{q^o_{j-1}}(t_j-) \right)}{\partial t} \equiv H_{q_j}(t_j+). \end{aligned} \tag{3.98}$$

For hybrid optimal control problems in which t_f *is not fixed (i.e. not a priori specified), then*

$$H_{q_f}\left(x_q, \lambda_q, u_q, t^o_f \right) + \frac{\partial g\left(x^o_{q^o_L}(t_f) \right)}{\partial t} + \sum_{i=1}^{k_{L+1}} p^i_{L+1} \frac{m^i_{L+1}\left(x^o_{q^o_L}(t_f) \right)}{\partial t} = 0, \tag{3.99}$$

where for the time invariant case becomes

$$H_{q_f}\left(x_q, \lambda_q, u_q, t_f^o\right) = 0. \tag{3.100}$$

□

Corollary 3.3. *For time-invariant low order switching manifolds, the boundary condition* (3.96) *can be stated as*

$$\lambda^o_{q^o_{j-1}}\left(t_j-\right) \equiv \lambda^o_{q^o_{j-1}}\left(t_j\right) = \nabla\xi_{\sigma_j}{}^T \lambda^o_{q^o_j}\left(t_j+\right) + \nabla c_{\sigma_j}\left(x^o_{q^o_{j-1}}\left(t_j-\right)\right) + p\hat{n}_m, \tag{3.101}$$

where

$$\hat{n}_m \parallel \underset{span\{\nabla m^i\}}{\mathrm{PROJ}} f_{q_{j-1}}\left(x^o_{q_{j-1}}\left(t_j-\right), u^o_{q_{j-1}}\left(t_j-\right)\right), \tag{3.102}$$

i.e. $\hat{n}_m$ *is a vector in* $\mathbb{R}^n$ *parallel to the projection of* $f_{q_{j-1}}$ *in the (generally non-orthogonal) vector space generated by the span of* $\left\{\nabla m^i_{q_{j-1}q_j}\right\}$, $i \in \{1, \cdots, k\}$. □

Chapter 4

Hybrid Dynamic Programming (HDP)

The method of Dynamic Programming (DP), originated by R. Bellman [15], is a mathematical technique for making a sequence of interrelated decisions, which can be applied to many optimization problems (including classical discrete and continuous, as well as hybrid optimal control problems). The basic idea of this method applied to continuous optimal controls is to consider a family of optimal control problems with different initial times and states, and to establish relationships among these problems via the so-called Hamilton-jacobi-Bellman (HJB) equation. In contrast to the Minimum Principle, the Dynamic Programming methodology provides solutions to the whole family of problems (with different initial times and states), including the original problem. Similar to the MP, the method of DP constitute necessary conditions for optimality which under certain assumptions become sufficient (see e.g. [26, 29, 47, 82, 83]). However, DP is widely used as a set of sufficient conditions for optimality after the optimal control extension of Carathéodory's sufficient conditions in Calculus of Variations (see e.g. [26, 47, 83]).

The generalization of Dynamic Programming for hybrid systems results in the theory of Hybrid Dynamic Programming (HDP) which, under the assumption of smoothness of the value function, results in the Hamilton-Jacobi-Bellman (HJB) equation of HDP. A major drawback in solving the HJB equation for obtaining the solution(s) of both classical and hybrid optimal control problems is the requirement that the HJB equation admits classical solutions, i.e. the solutions are assumed to be smooth enough to satisfy the HJB equation. Unfortunately, this is not necessarily the case as the differentiability of the value function is violated at certain points for numerous problems (see e.g. [26, 27, 47, 84–86]). To overcome this difficulty, Crandall and Lions [87] introduced the so-called viscosity solutions which is a notion for nonsmooth solutions to partial differential equations. The key feature of viscosity analysis is to replace the conventional derivatives by the (set-valued) super-/sub-differentials while maintaining the uniqueness of solutions under very mild conditions [16, 18, 19, 24–28, 30, 31, 87].

4.1 Properties of the Value Function

Consider the hybrid system $\mathbb{H}$ subject to assumptions A0-A2, and the HOCP (2.11) for the hybrid performance function (2.10). Then for the value function in Definition 2.8 we have the following theorem:

Theorem 4.1. *The value function* (2.19) *is Lipschitz in x uniformly in t for all* $t \in \bigcup_{i=0}^{L}(t_i, t_{i+1})$*, i.e. for* $B_r = \{x \in \mathbb{R}^n : \|x\| < r\}$ *and for all* $t \in (t_i, t_{i+1})$*,* $x \equiv x_t \in B_r$ *there exist a neighbourhood* $N_{r_x}(x_t)$ *and a constant* $0 < K < \infty$ *such that*

$$|V(t,q,x_t,L-j+1)-V(s,q,x_s,L-j+1)|<K\left(\|x_t-x_s\|^2+|s-t|^2\right)^{\frac{1}{2}}, \tag{4.1}$$

for $s \in (t_i, t_{i+1})$ *and* $x_s \in N_{r_x}(x_t)$. □

Proof. For a given hybrid control input $I_{L-j+1} = (S_{L-j+1}, u)$ we use $\hat{x}_\tau \equiv \hat{x}(\tau; t, \hat{x}_t)$ to denote the extended continuous valued state as in (2.12) at the instant τ passing through x_t, where $t \le \tau \le t_f$. We also define

$$K_1 = \sup\left\{\left\|\hat{f}_q(\hat{x}, u)\right\| : (q, \hat{x}, u) \in Q \times \hat{B}_r \times U\right\}, \tag{4.2}$$

where $\hat{B}_r := \left\{\hat{x} = \left[z, x^T\right]^T : |z|^2 + \|x\|^2 < r^2\right\}$.

First, consider the stage where no remaining switching is available and hence $t \in (t_L, t_{L+1}) = (t_L, t_f)$. In this case

$$\hat{x}(t_f; t, \hat{x}_t) = \hat{x}_t + \int_t^{t_f} \hat{f}_{q_L}(\hat{x}_\tau, u_\tau)\, d\tau, \tag{4.3}$$

which gives

$$\left\|\hat{x}(t_f; t, \hat{x}_t) - \hat{x}_t\right\| \le K_1 \left|t_f - t\right| + \int_t^{t_f} \hat{K}_f \left\|\hat{x}(\tau; t, \hat{x}_t) - \hat{x}_t\right\| d\tau, \tag{4.4}$$

where $\hat{K}_f$ depends only on K_f and K_l which are defined in assumptions A0 and A2 respectively. By the Gronwall-Bellman inequality this results in

$$\left\|\hat{x}(t_f; t, \hat{x}_t) - \hat{x}_t\right\| \le K_1 \left|t_f - t\right| + \int_t^{t_f} \hat{K}_f K_1 (\tau - t)\, e^{\hat{K}_f (t_f - \tau)} d\tau \le K_2 \left|t_f - t\right| \le K_2 \left|t_f - t_L\right|, \tag{4.5}$$

where $K_2 = \max\left\{K_1, \hat{K}_f K_1\left(t_f - t_L\right)e^{\hat{K}_f\left(t_f - t_L\right)}\right\}$. Hence, by the semi-group properties of ODE solutions and by use of (4.5), for $s \geq t$ and $\hat{x}_s \in N_{r_{\hat{x}}}\left(\hat{x}_t\right)$ we have

$$\begin{aligned}\left\|\hat{x}\left(t_f;t,\hat{x}_t\right) - \hat{x}\left(t_f;s,\hat{x}_s\right)\right\| &\leq \left\|\hat{x}_t - \hat{x}_s\right\| + \left\|\hat{x}\left(s;t,\hat{x}_t\right) - \hat{x}_t\right\| + \int_s^{t_f}\hat{K}_f\left\|\hat{x}\left(\tau;t,\hat{x}_t\right) - \hat{x}\left(\tau;s,\hat{x}_s\right)\right\|d\tau \\ &\leq \left\|\hat{x}_t - \hat{x}_s\right\| + K_2\left|s-t\right| + \int_s^{t_f}\hat{K}_f\left\|\hat{x}\left(\tau;t,\hat{x}_t\right) - \hat{x}\left(\tau;s,\hat{x}_s\right)\right\|d\tau \quad (4.6)\end{aligned}$$

and therefore, by the Gronwall inequality we have

$$\begin{aligned}\left\|\hat{x}\left(t_f;t,\hat{x}_t\right) - \hat{x}\left(t_f;s,\hat{x}_s\right)\right\| &\leq \left(\left\|\hat{x}_t - \hat{x}_s\right\| + K_2\left|s-t\right|\right)e^{\hat{K}_f\left(t_f - s\right)} \\ &\leq \left(\left\|\hat{x}_t - \hat{x}_s\right\| + K_2\left|s-t\right|\right)e^{\hat{K}_f\left(t_f - t_L\right)} \leq K\left(\left\|\hat{x}_t - \hat{x}_s\right\|^2 + \left|s-t\right|^2\right)^{\frac{1}{2}} \quad (4.7)\end{aligned}$$

for some $K < \infty$ which depends only on $t_f - t_L$, K_1 and $\hat{K}_f$ and not on the control input.

Since $\hat{g}$ is Lipschitz in $\hat{x}$ and $\hat{x}\left(t_f;t,\hat{x}_t\right)$ is Lipschitz in $(t,\hat{x}_t) \equiv \left(t, \left[z_t, x_t^T\right]^T\right)$, the performance function

$$J\left(t,t_f,q,x,0;I_0\right) = \int_t^{t_f} l_q\left(x,u\right)ds + g\left(x_{q_L}\left(t_f\right)\right) \equiv \hat{g}\left(\hat{x}\left(t_f;t,\hat{x}_t\right)\right) \quad (4.8)$$

is Lipschitz in $x \in B_r$ uniformly in $t \in \left(t_L, t_f\right)$ with a Lipschitz constant independent of the control. Further, since the infimum of a family of Lipschitz functions with a common Lipschitz constant is also Lipschitz with the same Lipschitz constant, $V\left(t,t_f,q,x,0\right)$ the value function with no switchs remaining is Lipschitz in $x \in B_r$ uniformly in $t \in \left(t_L, t_f\right)$.

Now consider $t,s \in \left(t_j, t_{j+1}\right)$ where t_{j+1} indicates a time of an autonomous switching for the trajectory $\hat{x}\left(\tau;t,\hat{x}_t\right)$, and consider for definiteness the case where $\hat{x}\left(\tau;s,\hat{x}_s\right)$ arrives on the switching manifold described locally by $m(x) = 0$ at a later time $t_{j+1} + \delta t$ (the case with an earlier arrival time can be handled similarly by considering $\delta t < 0$). It directly follows by replacing $\hat{f}_{q_L}$ and t_f by $\hat{f}_{q_j}$ and $t_{j+1}-$ in the above arguments, that

$$\left\|\hat{x}\left(t_{j+1}-;t,\hat{x}_t\right) - \hat{x}\left(t_{j+1}-;s,\hat{x}_s\right)\right\| \leq K'\left(\left\|\hat{x}_t - \hat{x}_s\right\|^2 + \left|s-t\right|^2\right)^{\frac{1}{2}} \quad (4.9)$$

Now since

$$\left\|\hat{x}\left(t_{j+1} + \delta t-;s,\hat{x}_s\right) - \hat{x}\left(t_{j+1}-;s,\hat{x}_s\right)\right\| \leq K_2\left|t_{j+1} + \delta t - t_{j+1}\right| = K_2\left|\delta t\right| \quad (4.10)$$

and

$$\begin{aligned}&\left\|\hat{x}\left(t_{j+1}+\delta t-;s,\hat{x}_s\right)-\hat{x}\left(t_{j+1}-;t,\hat{x}_t\right)\right\|^2\\&\quad\leq\left\|\hat{x}\left(t_{j+1}+\delta t-;s,\hat{x}_s\right)-\hat{x}\left(t_{j+1}-;s,\hat{x}_s\right)\right\|^2+\left\|\hat{x}\left(t_{j+1}-;t,\hat{x}_t\right)-\hat{x}\left(t_{j+1}-;s,\hat{x}_s\right)\right\|^2\end{aligned}\tag{4.11}$$

it is sufficient to show that the upper bound for $|\delta t|$ is proportional to $\left(\|\hat{x}_t-\hat{x}_s\|^2+|s-t|^2\right)^{\frac{1}{2}}$. This can be shown to hold by considering the fact that

$$\begin{aligned}m\left(x\left(t_{j+1}+\delta t-;s,x_s\right)\right)&=m\left(x\left(t_{j+1}-;s,x_s\right)+\int_{t_j}^{t_j+\delta t}f_{q_j}\left(x\left(\tau;s,x_s\right),u_{t_j-}\right)d\tau\right)\\&=m\left(x\left(t_{j+1}-;t,x_t\right)+\delta x\left(t_{j+1}-\right)+\int_{t_j}^{t_j+\delta t}f_{q_j}^{\left(x(\tau;s,x_s),u_{t_j-}\right)}d\tau\right)=m\left(x\left(t_{j+1}-;t,x_t\right)\right)=0\end{aligned}\tag{4.12}$$

For $\left\|\delta x\left(t_{j+1}-\right)\right\|<\varepsilon_{j+1}$ sufficiently small,

$$\nabla m^T\left(\delta x_{t_{j+1}-}+\int_{t_j}^{t_j+\delta t}f_{q_j}^{\left(x(\tau;s,x_s),u_{t_j-}\right)}d\tau\right)+O\left(\varepsilon_{j+1}^2\right)=0\tag{4.13}$$

which is equivalent to

$$\nabla m^T\delta x\left(t_{j+1}-\right)+\int_{t_j}^{t_j+\delta t}\nabla m^T f_{q_j}^{\left(x(\tau;s,x_s),u_{t_j-}\right)}d\tau+O\left(\varepsilon_{j+1}^2\right)=0\tag{4.14}$$

Due to the transversal arrival of the trajectories with respect to the smooth switching manifold, $\left|\nabla m^T f_{q_j}\right|$ is lower bounded by a strictly positive number $k_{m,f}$ (see (2.2)) and hence,

$$\begin{aligned}\left|\nabla m^T\delta x\left(t_{j+1}-\right)+O\left(\varepsilon_{j+1}^2\right)\right|&=\left|\int_{t_j}^{t_j+\delta t}\nabla m^T f_{q_j}^{\left(x(\tau;s,x_s),u_{t_j-}\right)}d\tau\right|\\&\geq\int_{t_j}^{t_j+\delta t}\left|\nabla m^T f_{q_j}^{\left(x(\tau;s,x_s),u_{t_j-}\right)}\right|d\tau\geq k_{m,f}\left|\delta t\right|,\end{aligned}\tag{4.15}$$

which gives

$$|\delta t| \leq \frac{1}{k_{m,f}} \left(\|\nabla m\| \left\| \delta x \left(t_{j+1}- \right) \right\| + \left| O\left(\varepsilon_{j+1}^{2} \right) \right| \right)$$
$$\leq \frac{1}{k_{m,f}} \|\nabla m\| \varepsilon_{j+1} + \varepsilon_{j+1} \leq \left(\frac{\|\nabla m\|}{k_{m,f}} + 1 \right) \varepsilon_{j+1} = K_{j+1} \varepsilon_{j+1} \quad (4.16)$$

Hence, for $t \in \left(t_j, t_{j+1}\right)$ and $x_t \in B_r$ there exist a neighbourhood $N_{r_x}(x_t)$ such that for $s \in \left(t_j, t_{j+1}\right)$ and $x_s \in \mathcal{N}_{r_x}(x_t)$ we have $\left\|\delta x\left(t_{j+1}-\right)\right\| \leq K'\left(\|\hat{x}_t - \hat{x}_s\|^2 + |s-t|^2\right)^{\frac{1}{2}} < \varepsilon_{j+1}$ in order to ensure that $\delta t \leq K_{j+1}\varepsilon_{j+1}$ and consequently

$$\left\|\hat{x}\left(t_{j+1}+\delta t-; s, \hat{x}_s\right) - \hat{x}\left(t_{j+1}-; t, \hat{x}_t\right)\right\| \leq K\left(\|\hat{x}_t - \hat{x}_s\|^2 + |s-t|^2\right)^{\frac{1}{2}}, \quad (4.17)$$

for K independent of the control. Since $\hat{\xi}$ is smooth and time invariant, it is therefore Lipschitz in $\hat{x}$ uniformly in time. At the switching time t_{j+1} we have

$$J\left(t_{j+1}-, q_j, \hat{x}, L-j; I_{L-j}\right) = J\left(t_{j+1}, q_{j+1}, \hat{\xi}\left(\hat{x}\right), L-j-1; I_{L-j-1}\right) \quad (4.18)$$

the Lipschitz property for the cost to go function $J\left(t_{j+1}-, q_j, \hat{x}, L-j; I_{L-j}\right)$ follows from the smoothness of $\hat{\xi}$ and the Lipschitz property of $J\left(t, q_{j+1}, \hat{x}_t, L-j-1; I_{L-j-1}\right)$. Namely, by backward induction from the Lipschitzness of $J\left(t, q_L, \hat{x}_t, 0; I_o\right)$ proved earlier, it is concluded that $J\left(t, q_{L-1}, \hat{x}_t, 1; I_1\right)$ is Lipschitz, from which $J\left(t, q_{L-2}, \hat{x}_t, 1; I_2\right)$ is concluded to be Lipschitz, etc. Since the Lipschitz constant is independent of the control and because the infimum of a family of Lipschitz functions with a common Lipschitz constant is also Lipschitz with the same Lipschitz constant, (4.1) holds and hence, the value function is Lipschitz. □

Definition 4.1. Let $M_{(i)}$ denote the set of all $(t,x) \in \mathbb{R} \times \bigcup_{i=0}^{L} \mathbb{R}^{n_{q_i}}$ for which the i'th derivatives of V exist and are continuous. □

Note that from Theorem 4.1, it is concluded that $M_{(0)} \supseteq \bigcup_{i=0}^{L} (t_i, t_{i+1}) \times \mathbb{R}^{n_{q_i}}$, i.e. the value function is at most discontinuous at the switching instants with non-zero switching costs and non-identity jump maps.

Corollary 4.2. *From Theorem 4.1 and Rademacher's theorem (see e.g. [29,88,89]), the Lipschitz property of the value function implies the differentiability almost everywhere, and hence the set* $M_{(1)}$ *is dense in* $\bigcup_{i=0}^{L} [t_i, t_{i+1}] \times \mathbb{R}^{n_{q_i}}$. □

4.2 Hybrid Dynamic Programming (HDP)

Theorem 4.3. *Consider the hybrid system* $\mathbb{H}$ *and the HOCP* (2.11) *together with the assumptions A0-A2 as above. Then for all* $(t,x) \in M_{(1)}$ *and* $q \in Q$*, the Hamilton-Jacobi-Bellman (HJB) equation holds, i.e.*

$$-\frac{\partial V}{\partial t} = \inf_{u} \left\{ l_q(x,u) + \left\langle \nabla_x V, f_q(x,u) \right\rangle \right\}, \quad a.e.\ t \in \left[t_0, t_f\right], \tag{4.19}$$

subject to the terminal condition

$$V\left(t_f, q_L, x, 0\right) = g(x), \tag{4.20}$$

and at the switching times $t_i \in \tau_L = \{t_1, \cdots, t_L\}$ *subject to the boundary conditions*

$$V\left(t_j, q, x, L-j+1\right) = \min_{\sigma_j \in \Sigma_j} \left\{ V\left(t_j, \Gamma\left(q, \sigma_j\right), \xi_{\sigma_j}(x), L-j\right) + c_{\sigma_j}\left(t_j, x\right) \right\}, \tag{4.21}$$

and

$$\begin{aligned} l_q\left(x, u^o\left(t_j-, x\right)\right) + \left\langle \nabla_x V, f_q\left(x, u^o\left(t_j-, x\right)\right)\right\rangle &\equiv -\frac{\partial}{\partial t} V\left(t_j-, q, x, L-j+1\right) \\ &= -\frac{\partial}{\partial t} V\left(t_j-, \Gamma\left(q, \sigma_j\right), \xi_{\sigma_j}(x), L-j\right) \\ &\equiv l_{\Gamma\left(q,\sigma_j\right)}\left(x, u^o\left(t_j, \xi_{\sigma_j}(x)\right)\right) + \left\langle \nabla_x V, f_{\Gamma\left(q,\sigma_j\right)}\left(x, u^o\left(t_j, \xi_{\sigma_j}(x)\right)\right)\right\rangle, \end{aligned} \tag{4.22}$$

where if t_j *is a time of a controlled switching then* $\Sigma_j = \Sigma$ *subject to the automaton constraint that* $\Gamma\left(q, \sigma_j\right)$ *is defined; and in the case of an autonomous switching, the set* Σ_j *is reduced to a subset of discrete inputs which are consistent with the switching manifold condition* $m_{q,\Gamma\left(q,\sigma_j\right)}(x) = 0$*.*

□

Proof. In order to derive the HJB equation (4.19), we consider a Lebesgue time t and a hybrid state $\left(q, x_q\right) \in H$, together with its associated number of switchings ahead $L-j+1$, such that $t \in \left(t_{j-1}, t_j\right)$ and $\left(t, x_q\right) \in M_{(1)}$.

The value function (2.19) in Definition 2.8 is defined as the optimal cost to go, i.e.

$$V(t,q,x,L-j+1) := \inf_{I_{L-j+1}} \left\{ \int_t^{t_j} l_q(x,u)\,ds + \sum_{i=j}^{L} c_{\sigma_{q_{i-1}q_i}}\left(t_i, x_{q_{i-1}}(t_i-)\right) \right.$$
$$\left. + \sum_{i=j}^{L} \int_{t_i}^{t_{i+1}} l_{q_i}\left(x_{q_i}(s), u(s)\right) ds + g\left(x_{q_L}(t_f)\right) \right\}. \quad (4.23)$$

For every $t' \in [t, t_j)$, Bellman's *Principle of Optimality* results in

$$V\left(t,q,x_q,L-j+1\right)$$
$$= \inf_{u_q^{[t,t']}} \left\{ \int_t^{t'} l_q\left(x_q\left(s;t,x_q,u_q(\cdot)\right), u_q(s)\right) ds + V\left(t', q, x_q\left(t';t,x_q,u_q(\cdot)\right), L-j+1\right) \right\}, \quad (4.24)$$

where

$$x_q\left(t';t,x_q,u_q(\cdot)\right) = x_q(t) + \int_t^{t'} f_q\left(x_q\left(s;t,x_q,u_q(\cdot)\right), u_q(s)\right) ds\,. \quad (4.25)$$

Fix $v \in U_q$ and let $x_q^{(s;v)} = x_q\left(s;t,x_q,v\right)$ be the state trajectory corresponding to the control $u_q(\tau) \equiv v$. By (4.24) we have

$$V\left(t,q,x_q,L-j+1\right) \leq \int_t^{t'} l_q\left(x_q\left(s;t,x_q,v\right), v\right) ds + V\left(t',q,x_q\left(t';t,x_q,v\right), L-j+1\right), \quad (4.26)$$

and therefore

$$-\frac{V\left(t',q,x_q^{(t';v)},L-j+1\right) - V\left(t,q,x_q,L-j+1\right)}{t'-t} - \frac{1}{t'-t}\int_t^{t'} l_q\left(x_q^{(s;v)}, v\right) ds \leq 0. \quad (4.27)$$

As $t' \downarrow t$

$$-\frac{\partial V\left(t,q,x_q,L-j+1\right)}{\partial t} - \frac{\partial V\left(t,q,x_q,L-j+1\right)}{\partial x_q}^T f_q\left(x_q,v\right) - l_q\left(x_q,v\right) \leq 0, \quad (4.28)$$

which by taking the infimum results in

$$-\frac{\partial V\left(t,q,x_q,L-j+1\right)}{\partial t} \leq \inf_v \left\{ l_q\left(x_q,v\right) + \frac{\partial V\left(t,q,x_q,L-j+1\right)}{\partial x_q}^T f_q\left(x_q,v\right) \right\}. \quad (4.29)$$

On the other hand, for any $\varepsilon > 0$, $t' \in (t, t_j)$ with $0 < t' - t$ small enough, there exists a $u^{\varepsilon,t'}(\cdot) \in \mathscr{U}$ such that

$$\int_t^{t'} l_q\left(x_q\left(s;t,x_q,u^{\varepsilon,t'}(\cdot)\right), u^{\varepsilon,t'}(\cdot)\right) ds + V\left(t',q,x_q\left(t';t,x_q,u^{\varepsilon,t'}(\cdot)\right), L-j+1\right) \leq V\left(t,q,x_q,L-j+1\right) + \varepsilon\left(t'-t\right). \quad (4.30)$$

It should be noted that $u^{\varepsilon,t'}(\cdot)$ may be modified on a set of measure zero in $[t,t']$ so that the instant t is a Lebesgue point with respect to the integral appearing in (4.30) and furthermore, the value of $x_q\left(s;t,x_q,u^{\varepsilon,t'}(\cdot)\right)$ is unchanged (see e.g. [9, 90, 91]) and hence the value of $V\left(t',q,x_q\left(t';t,x_q,u^{\varepsilon,t'}(\cdot)\right), L-j+1\right)$ is unchanged. Noting that $(t,x_q) \in M^{(1)}$, it follows from (4.30) that

$$\begin{aligned}
-\varepsilon &\leq -\frac{V\left(t',q,x_q^{\varepsilon,t'}(t'),L-j+1\right) - V\left(t,q,x_q,L-j+1\right)}{t'-t} - \frac{1}{t'-t}\int_t^{t'} l_q\left(x_q^{\varepsilon,t'}(s), u^{\varepsilon,t'}(s)\right) ds \\
&= \frac{1}{t'-t}\int_t^{t'} -\frac{\partial}{\partial s}V\left(s,q,x_q(s),L-j+1\right) - \left[\frac{\partial}{\partial x}V\left(s,q,x_q^{\varepsilon,t'}(s),L-j+1\right)\right]^T f_q\left(x_q^{\varepsilon,t'}(s),u^{\varepsilon,t'}(s)\right) \\
&\qquad - l_q\left(x_q^{\varepsilon,t'}(s),u^{\varepsilon,t'}(s)\right) ds \\
&\leq \frac{1}{t'-t}\int_t^{t'} -\frac{\partial}{\partial s}V\left(s,q,x_q^{(s)},L-j+1\right) - \inf_{u_q^{(s)}}\left\{l_q\left(x_q^{(s)},u_q^{(s)}\right) + \left[\frac{\partial}{\partial x}V\left(t,q,x_q^{(s)},L-j+1\right)\right]^T f_q\left(x_q^{(s)},u_q^{(s)}\right)\right\} ds.
\end{aligned} \quad (4.31)$$

Due to the uniform continuity of functions f_q and l_q (see A0), we have

$$\lim_{s\downarrow t} \sup_{x\in\mathbb{R}^{n_q}} \inf_{u\in U_q} \left|\psi(s,x,u) - \psi(t,x,u)\right| = 0, \quad (4.32)$$

for $\psi = f_q, l_q$, and therefore as $t' \downarrow t$

$$\inf_{u_q}\left\{l_q\left(x_q,u_q\right) + \left[\frac{\partial}{\partial x}V\left(t,q,x_q,L-j+1\right)\right]^T f_q\left(x_q,u_q\right)\right\} - \varepsilon \leq -\frac{\partial}{\partial t}V\left(t,q,x_q,L-j+1\right). \quad (4.33)$$

Combining (4.29) and (4.33) we obtain

$$-\frac{\partial}{\partial t}V\left(t,q,x_q,L-j+1\right) = \inf_{u_q}\left\{l_q\left(x_q,u_q\right) + \left[\frac{\partial}{\partial x}V\left(t,q,x_q,L-j+1\right)\right]^T f_q\left(x_q,u_q\right)\right\}. \quad (4.34)$$

The terminal condition (4.20) and the switching condition (4.21) are directly deduced from the definition of the value function (2.19). In order to prove (4.22), consider a time and state just before the occurrence of switching $\left(t_j-, h_{j-1}\left(t_j-\right)\right) \equiv \left(t_j-, q_{j-1}, x_{q_{j-1}}\left(t_j-\right)\right)$, or for short, $\left(q, x_q\left(t_j-\right)\right)$. This state is mapped into the state $\left(\Gamma\left(q,\sigma_j\right), \xi_{\sigma_j}\left(x_q\left(t_j-\right)\right)\right)$ after the switching. Consider also a family of adjacent trajectories and their corresponding switching pair $\left(q, x'_q\left(t'_j-\right)\right)$ which experiences the same switching sequence and therefore mapped into $\left(\Gamma\left(q,\sigma_j\right), \xi_{\sigma_j}\left(x'_q\left(t'_j-\right)\right)\right)$. The selection of the adjacent trajectory is such that Cauchy sequences for the difference in the switching instants, $\delta t_j := t'_j - t_j$, and for the distance of the trajectories at min $\left\{t_j, t'_j\right\}$ denoted by δx_q converge independently to zero; the existence of such sequences are guaranteed by the regularity of $\left(q, x_q\left(t_j-\right)\right)$ and the continuous dependence on the initial condition for the corresponding trajectories. Without any loss of generality, it is assumed that $t'_j \geq t_j$, and therefore, $\delta x_q = x'_q\left(t_j-\right) - x_q\left(t_j-\right) \equiv x'_q\left(t_j\right) - x_q\left(t_j-\right)$.

For the reference trajectory

$$x_{\Gamma\left(q,\sigma_j\right)}\left(t'_j\right) = \xi_{\sigma_j}\left(x_q\left(t_j-\right)\right) + \int_{t_j}^{t'_j} f_{\Gamma\left(q,\sigma_j\right)}\left(x_{\Gamma\left(q,\sigma_j\right)}, u_{\Gamma\left(q,\sigma_j\right)}\right) ds, \tag{4.35}$$

and for the adjacent trajectory

$$x'_{\Gamma\left(q,\sigma_j\right)}\left(t'_j\right) = \xi_{\sigma_j}\left(x'_q\left(t_j-\right) + \int_{t_j}^{t'_j} f_q\left(x_q, u_q\right) ds\right), \tag{4.36}$$

By definition, the value function for the reference trajectory satisfies

$$\begin{aligned} V\left(t_j-, q, x_q\left(t_j-\right), L-j+1\right) &= c_{\sigma_j}\left(x_q\left(t_j-\right)\right) \\ &\quad + V\left(t'_j, \Gamma\left(q,\sigma_j\right), x_{\Gamma\left(q,\sigma_j\right)}\left(t'_j\right), L-j\right) + \int_{t_j}^{t'_j} l_{\Gamma\left(q,\sigma_j\right)}\left(x_{\Gamma\left(q,\sigma_j\right)}, u_{\Gamma\left(q,\sigma_j\right)}\right) ds, \end{aligned} \tag{4.37}$$

and for the adjacent trajectory

$$\begin{aligned} V\left(t_j-, q, x'_q\left(t_j-\right), L-j+1\right) &= \int_{t_j}^{t'_j} l_q\left(x_q, u_q\right) ds \\ &\quad + c_{\sigma_j}\left(x'_q\left(t'_j-\right)\right) + V\left(t'_j, \Gamma\left(q,\sigma_j\right), x'_{\Gamma\left(q,\sigma_j\right)}\left(t'_j\right), L-j\right). \end{aligned} \tag{4.38}$$

Subtracting (4.38) from (4.37) and writing the Taylor series expansion of the terms, we have

$$\left[\nabla V\left(t_j-,q,x_q\left(t_j-\right),L-j+1\right)\right]^T\delta x_q=\left\{l_q^{\left(x_q\left(t_j-\right),u_q\left(t_j-\right)\right)}-l_{\Gamma\left(q,\sigma_j\right)}^{\left(x_{\Gamma\left(q,\sigma_j\right)}\left(t'_j\right),u_{\Gamma\left(q,\sigma_j\right)}\left(t'_j\right)\right)}\right\}\delta t$$

$$+\left[\nabla c_{\sigma_j}\left(x_q\left(t_j-\right)\right)\right]^T\left(\delta x_q+f_q\left(x_q\left(t_j-\right),u_q\left(t_j-\right)\right)\delta t\right)+\left[\nabla V\left(t'_j,\Gamma\left(q,\sigma_j\right),x_{\Gamma\left(q,\sigma_j\right)}^{\left(t'_j\right)},L-j\right)\right]^T$$

$$\left(\nabla\xi_{\sigma_j}\left(x_q\left(t_j-\right)\right)\left[\delta x_q+f_q^{\left(x_q\left(t_j-\right),u_q\left(t_j-\right)\right)}\delta t\right]-f_{\Gamma\left(q,\sigma_j\right)}^{\left(x_{\Gamma\left(q,\sigma_j\right)}\left(t'_j\right),u_{\Gamma\left(q,\sigma_j\right)}\left(t'_j\right)\right)}\delta t\right)+o\left(\delta\right),\quad(4.39)$$

or with the drop of arguments of the functions for the clarity of presentation

$$\nabla V_q^T\delta x_q=\left\{l_q-l_{\Gamma\left(q,\sigma_j\right)}\right\}\delta t+\nabla c_{\sigma_j}^T\left(\delta x_q+f_q\delta t\right)$$
$$+\nabla V_{\Gamma\left(q,\sigma_j\right)}^T\left(\nabla\xi_{\sigma_j}\left[\delta x_q+f_q\delta t\right]-f_{\Gamma\left(q,\sigma_j\right)}\delta t\right)+o\left(\delta t,\delta x\right).\quad(4.40)$$

This gives

$$\left[\nabla V_q-\nabla c_{\sigma_j}-\nabla\xi_{\sigma_j}^T\nabla V_{\Gamma\left(q,\sigma_j\right)}\right]^T\delta x_q$$
$$=\left\{l_q-l_{\Gamma\left(q,\sigma_j\right)}+\nabla c_{\sigma_j}^Tf_q+\nabla V_{\Gamma\left(q,\sigma_j\right)}^T\left(\nabla\xi_{\sigma_j}f_q-f_{\Gamma\left(q,\sigma_j\right)}\right)\right\}\delta t+o\left(\delta t,\delta x\right).\quad(4.41)$$

If t_j corresponds to a controlled switching, the choice of $\delta t,\delta x$ can be completely independent. If t_j corresponds to an autonomous switching, then the constraint by the switching manifold requires that

$$m\left(x_q\left(t_j-\right)\right)=0,\quad(4.42)$$

$$m\left(x'_q\left(t'_j-\right)\right)=m\left(x_q\left(t_j-\right)+\delta x_q+\int_{t_j}^{t'_j}f_q\left(x_q,u_q\right)ds\right)=0,\quad(4.43)$$

Using the Taylor series expansion

$$\nabla m\left(x_q\left(t_j-\right)\right)^T\left[\delta x_q+f_q\left(x_q\left(t_j-\right),u_q\left(t_j-\right)\right)\delta t\right]=0,\quad(4.44)$$

or, since by A0, $\nabla m\left(x_q\left(t_j-\right)\right)^T f_q\left(x_q\left(t_j-\right), u_q\left(t_j-\right)\right) \neq 0$,

$$\delta t = \frac{-\nabla m\left(x_q\left(t_j-\right)\right)^T \delta x_q}{\nabla m\left(x_q\left(t_j-\right)\right)^T f_q\left(x_q\left(t_j-\right), u_q\left(t_j-\right)\right)} + o\left(\delta x\right) \tag{4.45}$$

Therefore, the relation (4.41) for the autonomous switching case becomes

$$\begin{aligned}&\left[\nabla V_q - \nabla c_{\sigma_j} - \nabla \xi_{\sigma_j}^T \nabla V_{\Gamma\left(q,\sigma_j\right)}\right]^T \delta x_q \\ &\quad = \left\{l_q - l_{\Gamma\left(q,\sigma_j\right)} + \nabla c_{\sigma_j}^T f_q + \nabla V_{\Gamma\left(q,\sigma_j\right)}^T \left(\nabla \xi_{\sigma_j} f_q - f_{\Gamma\left(q,\sigma_j\right)}\right)\right\} \frac{-\nabla m^T \delta x_q}{\nabla m^T f_q} + o\left(\delta x\right).\end{aligned} \tag{4.46}$$

Since δx is arbitrarily selected in both cases of autonomous and controlled switchings, we can choose $\delta x = -\varepsilon f_q\left(x_q\left(t_j-\right), u_q\left(t_j-\right)\right)$, where $\varepsilon \in \mathbb{R}_+$ is selected to be arbitrarily small. For the controlled switching case where δt is independent of δx, we also select $\delta t = \varepsilon$. Thus, equations (4.41) and (4.46) give

$$0 = \varepsilon\left(l_q + \nabla V_q^T f_q - l_{\Gamma\left(q,\sigma_j\right)} - \nabla V_{\Gamma\left(q,\sigma_j\right)}^T f_{\Gamma\left(q,\sigma_j\right)}\right) + o\left(\varepsilon\right), \tag{4.47}$$

which, in the limit, as $\varepsilon \to 0$ results in (4.22). □

Definition 4.2. A feedback control $I_L\left(t,q,x\right) = \left(S_L, u\left(t,q,x\right)\right)$ is said to have an *admissible set of discontinuities*, if for each $q \in Q$, the discontinuities of the continuous valued feedback control $u\left(t,q,x\right)$ and the discrete valued feedback input $\sigma\left(t,q,x\right)$ are located on lower dimensional manifolds in the time and state space $\mathbb{R} \times \mathbb{R}^n$. □

We note that by A0 an autonomous discrete valued control input σ necessarily satisfies the lower dimensional manifold switching set condition of Definition 4.2 where the sets constitute C^∞ submanifolds.

Remark 4.1. For classical (i.e. non-hybrid) systems, a more detailed definition of a feedback control law with an admissible set of discontinuities can be found in [29, pp. 90–97]. The necessary conditions for the Lipschitz continuity of the optimal feedback control are discussed in [92, 93], and sufficient conditions for continuity with respect to initial conditions are given in [94].

Chapter 5

The Relation between the HMP and HDP

The relationship between the Minimum Principle and Dynamic Programming, which were developed independently in 1950s, was addressed as early as the formal announcement of the Pontryagin Minimum Principle [1]. In the classical optimal control framework, this relationship has been elaborated by many others since then (see e.g. [24–31, 47, 82, 83, 86]). The result states that, under certain assumptions (see e.g. [29]), the adjoint process in the MP and the gradient of the value function in DP are equal, a property which we shall sometimes refer to as the adjoint-gradient relationship. While this relationship has been proved in various forms, the majority of arguments are based on the following two key elements: (i) the assumption of the openness of the set of all points from which an optimal transition to the reference trajectory is possible [1, p. 70] and (ii) the inference of the extremality of the reference optimal state for the corresponding optimal control [1, p. 72]. Then with the assumption of twice continuous differentiability of the value function, the method of characteristics (see e.g. [26, 29]) can be employed to obtain the aforementioned relationship which is analogous to the derivation of the equivalence of the Hamiltonian system and the Hamilton-Jacobi equation. For certain classes of optimal control problems, the assumption of twice differentiability is intrinsically satisfied since the total cost can become arbitrarily large and negative if the second partial derivative ceases to exist (see e.g. [86]). But in general, even once differentiability of the value function is violated at certain points for numerous problems (see e.g. [26, 27, 47, 84–86]). Consequently, the adjoint-gradient relationship is usually expressed within the general framework of nonsmooth analysis that declares the inclusion of the adjoint process in the set of generalized gradients of the value function [24, 25, 27, 28, 30, 31]. However, the general expression of the adjoint-gradient relationship in the framework of nonsmooth analysis is unnecessary for optimal control problems with appropriately smooth vector fields and costs, when the optimal feedback control possesses an admissible set of discontinuities [29].

In contrast to classical optimal control theory, the relation between the Minimum Principle and Dynamic Programming in the hybrid systems framework has been the subject of limited number of studies. One of the main difficulties is that the domains of definition of hybrid systems employed for the derivation of the results of hybrid optimal control theory (see e.g. [2–5, 7–14, 16–19, 21–23, 67–73, 95]) do not necessarily intersect in a general class of systems. This is especially due to the difference in approach and the assumptions required for the derivation of necessary and sufficient optimality conditions in the two key approaches.

In this thesis we provide two different proofs for the relationship between the Minimum Principle and Dynamic Programming in both classical hybrid optimal control frameworks. The first proof is completely different in approach from the classical arguments discussed earlier and in particular, the sequence of proof steps appear in a different order. To be specific, the optimality condition, i.e. the Hamiltonian minimization property (ii) discussed earlier, appears in the last step in order to emphasise the independence of the dynamics of the cost gradient process from the optimality of the control input. Consequently, assumption (i) is used differently here from the classical proof methods and in particular, the optimality of the transitions back to the reference trajectory is relaxed to the existence of (not necessarily optimal) neighbouring trajectories. After the derivation of the dynamics and boundary conditions for the cost gradient, or sensitivity, corresponding to an arbitrary control input, it is shown that an optimal control leads to the same dynamics and boundary conditions for the gradient of the (optimal) cost gradient process as those for the adjoint process. Thus by the existence and uniqueness properties of the governing ODE solutions, it is concluded that the optimal cost gradient, i.e. the gradient of the value function generated by the HJB, is equal to the adjoint process in the corresponding HMP formulation.

The second proof differs from classical proof methods (e.g. [1, 24]) which make use of the Filippov theorem [47, p. 149-150] requiring the openness of the set of points from which a transition to the reference trajectory is possible [1, p. 70]. In particular, the differential equation governing the gradient of the value function and its boundary conditions are derived via suitable variations in the trajectory, and hence the underlying assumptions for the proof of the theorem are less restrictive. Most importantly, this dynamics is shown to hold almost everywhere and hence, the equality of the adjoint process in the HMP and the gradient of the value function in HDP holds almost everywhere for the general class of hybrid optimal control problems whose value function is not necessarily differentiable everywhere, but only differentiable in an open dense set of points in the state space. The derivation of boundary conditions on the sensitivity gradient process, which has not been provided before in the literature, are proved differently in the two proof presented in chapter.

5.1 Evolution of the Cost Sensitivity along a General Trajectory

Theorem 5.1. *Consider the hybrid system $\mathbb{H}$ together with the assumptions A0-A2 and the hybrid cost to go* (2.18)*. Then for a given hybrid feedback control $I_{L-j+1}(t,q,x) = \left(S_{L-j+1}, u(t,x)\right)$ with an admissible set of discontinuities, the sensitivity function $\nabla J \equiv \frac{\partial}{\partial x} J\left(t,t_f,q,x,L-j+1;I_{L-j+1}\right)$ satisfies:*

$$\frac{d}{dt}\nabla J = -\left(\left[\frac{\partial f_q(x,u)}{\partial x}\right]^T \nabla J + \frac{\partial l_q(x,u)}{\partial x}\right) \tag{5.1}$$

subject to the terminal conditions:

$$\nabla J\left(t_f,q_L,x,0\right) = \nabla g(x) \tag{5.2}$$

and the boundary conditions:

$$\nabla J\left(t_j,q_{j-1},x,L-j+1;I_{L-j+1}\right) = \nabla \xi_{\sigma_j}^T \nabla J\left(t_j+,q_j,\xi_{\sigma_j}(x),L-j;I_{L-j}\right) + p\nabla m + \nabla c \tag{5.3}$$

with $p=0$ when t_j indicates the time of a controlled switching and

$$p = \frac{\left[\nabla J\left(t_j,q_j,\xi_{\sigma_j}(x),L-j;I_{L-j}\right)\right]^T f_{q_j,\xi}^{\xi,q_{j-1}} + l_{q_j,\xi}^{q_{j-1}}}{\nabla m^T f_{q_{j-1}}\left(x,u\left(t_j-\right)\right)} \tag{5.4}$$

when t_j indicates the time of an autonomous switching, and where in the above equation $f_{q_j,\xi}^{\xi,q_{j-1}} := f_{q_j}\left(\xi_{\sigma_j}(x),u\left(t_j\right)\right) - \nabla\xi f_{q_{j-1}}\left(x,u\left(t_j-\right)\right)$ and $l_{q_j,\xi}^{q_{j-1}} := l_{q_j}\left(\xi_{\sigma_j}(x),u\left(t_j\right)\right) - l_{q_{j-1}}\left(x,u\left(t_j-\right)\right)$.

□

Proof. We first prove that (5.1) holds for $t \in (t_L,t_{L+1}] \equiv \left(t_L,t_f\right]$ with the terminal condition (5.2). Then by assuming that (5.1) holds for $t \in \left(t_j,t_{j+1}\right]$, $j \le L$ we show that it also holds for $t \in \left(t_{j-1}, t_j\right]$ with the boundary condition (5.3), with $p = 0$ when t_j indicates the time of a controlled switching, and with p given by (5.4) when $t_j \in \tau_L$ indicates the time of an autonomous switching. Hence, by mathematical induction, the relation is proved for all $t \in \left[t_0, t_f\right]$.

(i) **No Switching Ahead:** First, consider a Lebesgue time $t \in [t_L, t_{L+1}] \equiv \left[t_L, t_f\right]$ and the hybrid trajectory passing through (q_L, x), and consider the cost to go (2.18) for $I_0 \equiv I_0^{\left[t,t_f\right]}$ which is

$$J\left(t, q_L, x, 0; I_0\right) = \int_t^{t_f} l_{q_L}\left(x_s, u_s\right) ds + g\left(x_f\right) \tag{5.5}$$

Since by Definition 4.2 the discontinuities in x of $I_0^{\left[t,t_f\right]} \equiv u^{\left[t,t_f\right]}$ lie on lower dimensional sets which are closed in the induced topology of the space, the partial derivative of J with respect to x exists in an open neighbourhood of (t, x), and is derived as

$$\frac{\partial J\left(t, q_L, x, 0; I_0\right)}{\partial x} = \frac{\partial}{\partial x} \int_t^{t_f} l_{q_L}\left(x_s, u_s\right) ds + \frac{\partial}{\partial x} g\left(x_f\right) = \int_t^{t_f} \frac{\partial}{\partial x} l_{q_L}\left(x_s, u_s\right) ds + \frac{\partial}{\partial x} g\left(x_f\right), \tag{5.6}$$

which is equivalent to

$$\frac{\partial J\left(t, q_L, x, 0; I_0\right)}{\partial x} = \int_t^{t_f} \left[\frac{\partial x_s}{\partial x}\right]^T \frac{\partial l_{q_L}\left(x_s, u_s\right)}{\partial x_s} ds + \left[\frac{\partial x_f}{\partial x}\right]^T \frac{\partial g\left(x_f\right)}{\partial x_f} \tag{5.7}$$

Taking $t = t_f$ the terminal condition for $\frac{\partial J}{\partial x}$ is seen to be determined by

$$\frac{\partial J\left(t_f, q_L, x, 0; I_0\right)}{\partial x} = \nabla_{x_f} g\left(x_f\right) \equiv \nabla_x g\left(x\right), \tag{5.8}$$

because $x_f = x$ when $t = t_f$. Hence, (5.2) is proved. With the notation $x_s = \phi_{q_L}\left(s, t, x\right)$ and with the smoothness provided by the assumptions A0-A2 for the given control input with an admissible set of discontinuities, we have

$$\frac{d}{ds}\left(\frac{\partial}{\partial x} x_s\right) = \frac{d}{ds}\left(\frac{\partial}{\partial x} \phi_{q_L}\left(s, t, x\right)\right) = \frac{\partial}{\partial x}\left(\frac{d}{ds} \phi_{q_L}\left(s, t, x\right)\right) = \frac{\partial}{\partial x}\left(f_{q_L}\left(\phi_{q_L}\left(s, t, x\right), u\right)\right), \tag{5.9}$$

from which we obtain

$$\frac{d}{ds}\left(\frac{\partial x_s}{\partial x}\right) = \left[\frac{\partial f_{q_L}}{\partial x_s}\right]^T \frac{\partial \phi_{q_L}\left(s, t, x\right)}{\partial x}, \tag{5.10}$$

with $\frac{\partial \phi_{q_L}(t,t,x)}{\partial x} = I_{n\times n}$, since $\phi_{q_L}(t,t,x) = x$. Let $\Phi^{q_L}_{s,t} \in \mathbb{R}^{n^2}$ denote the solution of

$$\dot{\Phi}^{q_L}_{s,t} = \nabla_{x_s} f_{q_L}(x_s,u_s)^T \Phi^{q_L}_{s,t} \equiv \left[\frac{\partial f_{q_L}(x_s,u_s)}{\partial x_s}\right]^T \Phi^{q_L}_{s,t}, \tag{5.11}$$

with $\Phi^{q_L}_{t,t} = I_{n\times n}$. By the uniqueness of the solutions to (5.10) and (5.11):

$$\frac{\partial}{\partial x}\phi_{q_L}(s,t,x) = \Phi^{q_L}_{s,t}, \tag{5.12}$$

for all $x \in \mathbb{R}^n$. Also by the semi-group property:

$$x = \phi_{q_L}(s,t,x_t) = \phi_{q_L}\left(s,t,\phi_{q_L}(t,s,x)\right), \tag{5.13}$$

and hence by taking the derivative with respect to x we have

$$I_{n\times n} = \left.\frac{\partial \phi_{q_L}(s,t,z)}{\partial z}\right|_{z=\phi(t,s,x_s)} \frac{\partial \phi_{q_L}(t,s,x)}{\partial x}, \tag{5.14}$$

which by (5.12) is equivalent to

$$I_{n\times n} = \Phi^{q_L}_{s,t}\Phi^{q_L}_{t,s}. \tag{5.15}$$

For all $r,s,t \in \left(t_L,t_f\right]$ it is the case that

$$\frac{d}{ds}\Phi^{q_L}_{s,r} = \left[\frac{\partial f_{q_L}(x_s,u_s)}{\partial x_s}\right]^T \Phi^{q_L}_{s,r}, \qquad \Phi^{q_L}_{r,r} = I_{n\times n}, \tag{5.16}$$

and

$$\frac{d}{ds}\left(\Phi^{q_L}_{s,t}\Phi^{q_L}_{t,r}\right) = \left(\left[\frac{\partial f_{q_L}(x_s,u_s)}{\partial x_s}\right]^T \Phi^{q_L}_{s,t}\right)\Phi^{q_L}_{t,r} = \left[\frac{\partial f_{q_L}(x_s,u_s)}{\partial x_s}\right]^T \left(\Phi^{q_L}_{s,t}\Phi^{q_L}_{t,r}\right) \tag{5.17}$$

where for (5.17) at $s = r$ the condition $\Phi^{q_L}_{r,t}\Phi^{q_L}_{t,r} = I_{n\times n}$ holds. Hence, from the uniqueness of the solution for the ODEs (5.16) and (5.17) we obtain $\Phi^{q_L}_{s,t}\Phi^{q_L}_{t,r} = \Phi^{q_L}_{s,r}$. Furthermore, (5.15) gives

$$0 = \frac{d\Phi^{q_L}_{s,t}}{dt}\Phi^{q_L}_{t,s} + \Phi^{q_L}_{s,t}\frac{d\Phi^{q_L}_{t,s}}{dt}, \tag{5.18}$$

and hence

$$\frac{d\Phi_{s,t}^{q_L}}{dt} = -\Phi_{s,t}^{q_L}\frac{d\Phi_{t,s}^{q_L}}{dt}\left[\Phi_{t,s}^{q_L}\right]^{-1} = -\Phi_{s,t}^{q_L}\left[\frac{\partial f_{q_L}(x_t,u_t)}{\partial x_t}\right]^T\Phi_{t,s}^{q_L}\left[\Phi_{t,s}^{q_L}\right]^{-1} = -\Phi_{s,t}^{q_L}\left[\frac{\partial f_{q_L}(x_t,u_t)}{\partial x_t}\right]^T \tag{5.19}$$

Differentiating (5.7) with respect to t along a trajectory (q_L,x) gives

$$\begin{aligned}
\frac{d}{dt}\frac{\partial J(t,q_L,x,0;I_0)}{\partial x} &= \frac{d}{dt}\int_t^{t_f}\left[\frac{\partial \phi_{q_L}(s,t,x)}{\partial x}\right]^T \frac{\partial l_{q_L}(z,u_s)}{\partial z}\bigg|_{z=\phi_{q_L}(s,t,x)} ds \\
&+\frac{d}{dt}\left[\frac{\partial \phi_{q_L}(t_f,t,x)}{\partial x}\right]^T\frac{\partial g(z)}{\partial z}\bigg|_{z=\phi_{q_L}(t_f,t,x)} = -\left\{\left[\frac{\partial \phi_{q_L}(s,t,x)}{\partial x}\right]^T\frac{\partial l_{q_L}(z,u_s)}{\partial z}\bigg|_{z=\phi_{q_L}(s,t,x)}\right\}_{s=t} \\
&+\int_t^{t_f}\frac{d}{dt}\left\{\left[\frac{\partial \phi_{q_L}(s,t,x)}{\partial x}\right]^T\frac{\partial l_{q_L}(z,u_s)}{\partial z}\bigg|_{z=\phi_{q_L}(s,t,x)}\right\}ds+\frac{d}{dt}\left[\frac{\partial \phi_{q_L}(t_f,t,x)}{\partial x}\right]^T\frac{\partial g(z)}{\partial z}\bigg|_{z=\phi_{q_L}(t_f,t,x)} \\
&= -\left\{I_{n\times n}.\frac{\partial l_{q_L}(x_t,u_t)}{\partial x_t}\right\}+\int_t^{t_f}\left\{-\left[\frac{\partial f_{q_L}(x_t,u_t)}{\partial x_t}\right]^T\left[\frac{\partial \phi_{q_L}(s,t,x)}{\partial x}\right]^T\frac{\partial l_{q_L}(z,u_s)}{\partial z}\bigg|_{z=\phi_{q_L}(s,t,x)}+0\right\}ds \\
&+\left\{-\left[\frac{\partial f_{q_L}(x_t,u_t)}{\partial x_t}\right]^T\left[\frac{\partial \phi_{q_L}(t_f,t,x)}{\partial x}\right]^T\frac{\partial g(z)}{\partial z}\bigg|_{z=\phi_{q_L}(t_f,t,x)}+0\right\},
\end{aligned} \tag{5.20}$$

where the zero terms arise from

$$\frac{d}{dt}\nabla_z l_{q_L}(z,u_s)\big|_{z=\phi_{q_L}(s,t,x)} = \frac{d}{dt}\nabla_{x_s} l_{q_L}(x_s,u_s) = 0, \tag{5.21}$$

$$\frac{d}{dt}\nabla_z g(z,u_s)\big|_{z=\phi_{q_L}(t_f,t,x)} = \frac{d}{dt}\nabla_{x_f} g(x_f) = 0. \tag{5.22}$$

Hence,

$$\begin{aligned}
\frac{d}{dt}\frac{\partial J(t,q_L,x,0;I_0)}{\partial x} &= -\frac{\partial l_{q_L}(x_t,u_t)}{\partial x_t} \\
&-\left[\frac{\partial f_{q_L}(x_t,u_t)}{\partial x_t}\right]^T\left\{\int_t^{t_f}\left[\frac{\partial x_s}{\partial x}\right]^T\frac{\partial l_{q_L}(x_s,u_s)}{\partial x_s}ds+\left[\frac{\partial x_f}{\partial x}\right]^T\frac{\partial g(x_f)}{\partial x_f}\right\},
\end{aligned} \tag{5.23}$$

which gives

$$\frac{d}{dt}\frac{\partial J(t,q_L,x,0;I_0)}{\partial x} = -\left[\frac{\partial f_{q_L}(x,u)}{\partial x}\right]^T\frac{\partial J(t,q_L,x,0;I_0)}{\partial x} - \frac{\partial l_{q_L}(x,u)}{\partial x}, \tag{5.24}$$

with

$$\frac{\partial J\left(t_f, q_L, x, 0; I_0^{[t_f,t_f]}\right)}{\partial x} = \nabla_{x_f} g\left(x_f\right) \equiv \nabla_x g\left(x\right) \tag{5.25}$$

(ii) **A Controlled Switching Ahead:** Now assume that (5.1) holds for $\theta \in \left(t_j, t_{j+1}\right]$, $j \le L$ when $t_j \in \tau_L$ indicates a time of a controlled switching. Then for $t \in \left(t_{j-1}, t_j\right]$, and $t_{j-1} < t \le t_j < \theta \le t_{j+1}$

$$J\left(t, q_{j-1}, x, L-j+1; I_{L-j+1}^{[t,t_f]}\right) = \int_t^{t_j} l_{q_{j-1}}(x_s, u_s) ds + c_{\sigma_j}\left(x\left(t_j-\right)\right) + \int_{t_j}^{\theta} l_{q_j}(x_\omega, u_\omega) d\omega + J\left(\theta, q_j, x_\theta, L-j; I_{L-j}^{[\theta,t_f]}\right), \tag{5.26}$$

where

$$x_\theta = \xi\left(x_t + \int_t^{t_j-} f_{q_{j-1}}\left(x_s, u_s\right) ds\right) + \int_{t_j}^{\theta} f_{q_j}\left(x_\omega, u_\omega\right) d\omega \tag{5.27}$$

This gives

$$\frac{\partial J\left(t, q_{j-1}, x, L-j+1; I_{L-j+1}^{[t,t_f]}\right)}{\partial x} = \frac{\partial}{\partial x}\int_t^{t_j} l_{q_{j-1}}(x_s, u_s) ds + \frac{\partial c_{\sigma_j}\left(x\left(t_j-\right)\right)}{\partial x} + \frac{\partial}{\partial x}\int_{t_j}^{\theta} l_{q_j}(x_\omega, u_\omega) d\omega + \frac{\partial J\left(\theta, q_j, x_\theta, L-j; I_{L-j}^{[\theta,t_f]}\right)}{\partial x} \tag{5.28}$$

or

$$\frac{\partial J\left(t, q_{j-1}, x, L-j+1; I_{L-j+1}^{[t,t_f]}\right)}{\partial x} = \int_t^{t_j} \left[\frac{\partial x_s}{\partial x}\right]^T \frac{\partial l_{q_{j-1}}\left(x_s, u_s\right)}{\partial x_s} ds + \left[\frac{\partial x_{t_j-}}{\partial x}\right]^T \frac{\partial c\left(x_{t_j-}\right)}{\partial x_{t_j-}} + \int_{t_j}^{\theta} \left[\frac{\partial x_\omega}{\partial x}\right]^T \frac{\partial l_{q_j}\left(x_\omega, u_\omega\right)}{\partial x_\omega} d\omega + \left[\frac{\partial x_\theta}{\partial x}\right]^T \frac{\partial J\left(\theta, q_j, x_\theta, L-j; I_{L-j}^{[\theta,t_f]}\right)}{\partial x_\theta}, \tag{5.29}$$

with

$$\frac{\partial x_\theta}{\partial x} = \int_{t_j}^{\theta} \frac{\partial f_{q_j}(x_\omega, u_\omega)}{\partial x} d\omega + \frac{\partial \xi\left(x_t + \int_t^{t_j} f_{q_{j-1}}(x_s, u_s)\, ds\right)}{\partial x}$$
$$= \int_{t_j}^{\theta} \frac{\partial f_{q_j}(x_\omega, u_\omega)}{\partial x} d\omega + \frac{\partial \xi\left(x_{t_j-}\right)}{\partial x}, \quad (5.30)$$

which is equivalent to

$$\frac{\partial x_\theta}{\partial x} = \int_{t_j}^{\theta} \left[\frac{\partial x_\omega}{\partial x}\right]^T \frac{\partial f_{q_j}(x_\omega, u_\omega)}{\partial x_\omega} d\omega + \left[\frac{\partial x_{t_j-}}{\partial x}\right]^T \frac{\partial \xi\left(x_{t_j-}\right)}{\partial x_{t_j-}}. \quad (5.31)$$

In particular, for $x = x_t$ as $t \uparrow t_j$ and for x_θ as $\theta \downarrow t_j$ equation (5.29) becomes

$$\frac{\partial J\left(t_j-, q_{j-1}, x_{t_j-}, L-j+1; I_{L-j+1}^{[t_j,t_f]}\right)}{\partial x_{t_j-}} = \int_{t_j-}^{t_j} \left[\frac{\partial x_s}{\partial x}\right]^T \frac{\partial l_{q_{j-1}}(x_s, u_s)}{\partial x_s} ds + \left[\frac{\partial x_{t_j-}}{\partial x_{t_j-}}\right]^T \frac{\partial c\left(x_{t_j-}\right)}{\partial x_{t_j-}}$$
$$+ \int_{t_j}^{t_j+} \left[\frac{\partial x_s}{\partial x}\right]^T \frac{\partial l_{q_j}(x_s, u_s)}{\partial x_s} ds + \left[\frac{\partial x_{t_j+}}{\partial x_{t_j-}}\right]^T \frac{\partial J\left(t_j+, q_j, x_{t_j+}, L-j; I_{L-j}^{[t_j+,t_f]}\right)}{\partial x_{t_j+}}, \quad (5.32)$$

or

$$\frac{\partial J\left(t_j-, q_{j-1}, x_{t_j-}, L-j+1; I_{L-j+1}^{[t_j,t_f]}\right)}{\partial x_{t_j-}}$$
$$= \frac{\partial c\left(x_{t_j-}\right)}{\partial x_{t_j-}} + \left[\frac{\partial x_{t_j+}}{\partial x_{t_j-}}\right]^T \frac{\partial J\left(t_j+, q_j, x_{t_j+}, L-j; I_{L-j}^{[t_j+,t_f]}\right)}{\partial x_{t_j+}}, \quad (5.33)$$

and also (5.31) turns into

$$\frac{\partial x_{t_j+}}{\partial x_{t_j-}} = \int_{t_j}^{t_j+} \left[\frac{\partial x_\omega}{\partial x}\right]^T \frac{\partial f_{q_j}(x_\omega, u_\omega)}{\partial x_\omega} d\omega + \left[\frac{\partial x_{t_j-}}{\partial x_{t_j-}}\right]^T \frac{\partial \xi\left(x_{t_j-}\right)}{\partial x_{t_j-}}, \quad (5.34)$$

or

$$\frac{\partial x_{t_j+}}{\partial x_{t_j-}} = \frac{\partial \xi\left(x_{t_j-}\right)}{\partial x_{t_j-}} = \nabla \xi\big|_{x_{t_j-}}. \quad (5.35)$$

Hence,

$$\frac{\partial J\left(t_j-,q_{j-1},x_{t_j-},L-j+1;I_{L-j+1}^{[t_j,t_f]}\right)}{\partial x_{t_j-}}=\frac{\partial c\left(x_{t_j-}\right)}{\partial x_{t_j-}}+\nabla\xi|_{x_{t_j-}}^T\frac{\partial J\left(t_j+,q_j,x_{t_j+},L-j;I_{L-j}^{[t_j+,t_f]}\right)}{\partial x_{t_j+}} \tag{5.36}$$

and therefore, (5.3) is shown to hold with $p=0$ for the controlled switching case. Writing

$$J\left(t,q_{j-1},x,0;I_{L-j+1}\right)=\int_t^{t_j}l_{q_{j-1}}\left(x_s,u_s\right)ds+J\left(t,q_{j-1},x\left(t_j-\right),L-j+1;I_{L-j+1}\right), \tag{5.37}$$

and following a similar procedure as in part (i) of the proof, equation (5.1) is derived for $t\in\left(t_{j-1},t_j\right]$.

(iii) **An Autonomous Switching Ahead:** Now assume that (5.1) holds for all $\theta\in\left(t_j,t_{j+1}\right]$, $j\le L$ when $t_j\in\tau_L$ indicates a time of an autonomous switching. Then taking the derivative of both sides of the equality (5.26) with respect to x at $t\in\left(t_{j-1},t_j\right]$, with $t_{j-1}<t\le t_j<\theta\le t_{j+1}$, yields

$$\begin{aligned}\frac{\partial J\left(t,q_{j-1},x,L-j+1;I_{L-j+1}^{[t,t_f]}\right)}{\partial x}&=\frac{\partial}{\partial x}\int_t^{t_j}l_{q_{j-1}}\left(x_s,u_s\right)ds+\frac{\partial}{\partial x}c\left(x\left(t_j-\right)\right)\\&\quad+\frac{\partial}{\partial x}\int_{t_j}^{\theta}l_{q_j}\left(x_\omega,u_\omega\right)d\omega+\frac{\partial J\left(\theta,q_j,x_\theta,L-j;I_{L-j}^{[\theta,t_f]}\right)}{\partial x},\end{aligned} \tag{5.38}$$

which gives

$$\begin{aligned}\frac{\partial J\left(t,q_{j-1},x,L-j+1;I_{L-j+1}^{[t,t_f]}\right)}{\partial x}&=\frac{\partial t_j}{\partial x}l_{q_{j-1}}\left(x_s,u_s\right)\big|_{s=t_j-}+\int_t^{t_j}\left[\frac{\partial x_s}{\partial x}\right]^T\frac{\partial l_{q_{j-1}}\left(x_s,u_s\right)}{\partial x_s}ds\\&\quad+\left[\frac{\partial x_{t_j-}}{\partial x}\right]^T\frac{\partial c\left(x_{t_j-}\right)}{\partial x_{t_j-}}-\frac{\partial t_j}{\partial x}l_{q_j}\left(x_\omega,u_\omega\right)\big|_{\omega=t_j}+\int_{t_j}^{\theta}\left[\frac{\partial x_\omega}{\partial x}\right]^T\frac{\partial l_{q_j}\left(x_\omega,u_\omega\right)}{\partial x_\omega}d\omega\\&\quad+\left[\frac{\partial x_\theta}{\partial x}\right]^T\frac{\partial J\left(\theta,q_j,x_\theta,L-j;I_{L-j}^{[\theta,t_f]}\right)}{\partial x_\theta},\end{aligned} \tag{5.39}$$

with the derivative of (5.27) derived as

$$\begin{aligned}\frac{\partial x_\theta}{\partial x} &= \frac{\partial}{\partial x}\xi\left(x_t + \int_t^{t_j} f_{q_{j-1}}(x_s,u_s)ds\right) - \frac{\partial t_j}{\partial x_{t_j-}} f_{q_j}(x_\omega,u_\omega)\big|_{\omega=t_j} + \int_{t_j}^{\theta} \frac{\partial f_{q_j}(x_\omega,u_\omega)}{\partial x} d\omega \\ &= \left[\frac{\partial \xi(z)}{\partial z}\bigg|_{z=x_t+\int_t^{t_j} f_{q_{j-1}}^{(x_s,u_s)}ds}\right]^T \frac{\partial}{\partial x}\left(x_t + \int_t^{t_j} f_{q_{j-1}}(x_s,u_s)ds\right) \\ &\quad - \frac{\partial t_j}{\partial x_{t_j-}} f_{q_j}\left(x_{t_j},u_{t_j}\right) + \int_{t_j}^{\theta} \frac{\partial f_{q_j}(x_\omega,u_\omega)}{\partial x} d\omega, \end{aligned} \tag{5.40}$$

which gives

$$\begin{aligned}\frac{\partial x_\theta}{\partial x} &= -\frac{\partial t_j}{\partial x_{t_j-}} f_{q_j}\left(x_{t_j},u_{t_j}\right) + \int_{t_j}^{\theta} \frac{\partial f_{q_j}(x_\omega,u_\omega)}{\partial x} d\omega \\ &\quad + \nabla\xi|_{x_{t_j^-}}\left(I_{n\times n} + \frac{\partial t_j}{\partial x_{t_j-}} f_{q_{j-1}}\left(x_{t_j-},u_{t_j-}\right) + \int_t^{t_j} \frac{\partial x_s}{\partial x}^T \frac{\partial f_{q_{j-1}}(x_s,u_s)}{\partial x_s} ds\right) \end{aligned} \tag{5.41}$$

Note that in the above equations, the partial derivative $\frac{\partial t_j}{\partial x_{t_j-}}$ is not necessarily zero because for $\delta x_t \in \mathbb{R}^n$ the perturbed trajectory $x_s + \delta x_s$ arrives on the switching manifold m at a different time $t_j'- = \left(t_j + \delta t_j\right)-$, $\delta t_j \in \mathbb{R}$. Consider a locally modified control I_{L-j+1}' of the form

$$I_{L-j+1}' = \left(\left(t_j + \delta t, \sigma_j\right), u'\right), \tag{5.42}$$

with

$$u_s' = \begin{cases} u_s & s \in \left[t, t_j\right) \\ u\left(t_j-\right) & s \in \left[t_j, t_j + \delta t\right) \\ u_s & s \in \left[t_j + \delta t, t_{j+1}\right) \end{cases} \tag{5.43}$$

if $\delta t \geq 0$ and

$$u_s' = \begin{cases} u_s & s \in \left[t, t_j + \delta t\right) \\ u\left(t_j\right) \equiv u\left(t_j+\right) & s \in \left[t_j + \delta t, t_j\right) \\ u_s & s \in \left[t_j, t_{j+1}\right) \end{cases} \tag{5.44}$$

if $\delta t < 0$. Since $I_{L-j+1}' = I_{L-j+1}$ holds everywhere except only on $\left[t_j, t_j + \delta t\right)$ (or $\left[t_j + \delta t, t_j\right)$ if $\delta t < 0$), the measure of the set of modified controls is of the order $|\delta t|$. Evidently the perturbed trajectory arrives on the switching manifold when

$$m\left(x_{t_j+\delta t_j-}+\delta x_{t_j+\delta t_j-}\right)=0. \tag{5.45}$$

For $\delta t\geq 0$ we may write

$$m\left(x_{t_j-}+\delta x_{t_j-}+\int_{t_j}^{t_j+\delta t} f_{q_{j-1}}\left(x_s+\delta x_s,u_{t_j-}\right)\right)=m\left(x_{t_j-}\right)=0, \tag{5.46}$$

that gives

$$\left[\nabla m\left(x_{t_j-}\right)\right]^T\left[\delta x_{t_j-}+f_{q_{j-1}}\left(x_{t_j-},u_{t_j-}\right)\delta t+O\left(\delta t^2\right)\right]=0, \tag{5.47}$$

or

$$\delta t=\frac{-\nabla m^T\delta x_{t_j-}}{\nabla m^T f_{q_{j-1}}\left(x_{t_j-},u_{t_j-}\right)}+O\left(\delta t^2\right). \tag{5.48}$$

Similarly, for $\delta t<0$ the same result is achieved. In particular, as $t\uparrow t_j$ and $\theta\downarrow t_j$ equation (5.39) becomes

$$\begin{aligned}\frac{\partial J\left(t_j-,q_{j-1},x_{t_j-},L-j+1;I_{L-j+1}^{[t_j,t_f]}\right)}{\partial x_{t_j-}}&=\frac{\partial t_j}{\partial x_{t_j-}}l_{q_{j-1}}\left(x_{t_j-},u_{t_j-}\right)+\int_{t_j-}^{t_j}\left[\frac{\partial x_s}{\partial x}\right]^T\frac{\partial l_{q_{j-1}}\left(x_s,u_s\right)}{\partial x_s}ds\\&+\left[\frac{\partial x_{t_j-}}{\partial x_{t_j-}}\right]^T\frac{\partial c\left(x_{t_j-}\right)}{\partial x_{t_j-}}-\frac{\partial t_j}{\partial x_{t_j-}}l_{q_j}\left(x_{t_j},u_{t_j}\right)+\int_{t_j}^{t_j+}\left[\frac{\partial x_s}{\partial x}\right]^T\frac{\partial l_{q_j}\left(x_s,u_s\right)}{\partial x_s}ds\\&+\left[\frac{\partial x_{t_j+}}{\partial x_{t_j-}}\right]^T\frac{\partial J\left(t_j+,q_j,x_{t_j+},L-j;I_{L-j}^{[t_j+,t_f]}\right)}{\partial x_{t_j+}},\end{aligned} \tag{5.49}$$

or

$$\begin{aligned}\frac{\partial J\left(t_j-,q_{j-1},x_{t_j-},L-j+1;I_{L-j+1}^{[t_j,t_f]}\right)}{\partial x_{t_j-}}&=\frac{\partial t_j}{\partial x_{t_j-}}l_{q_{j-1}}\left(x_{t_j-},u_{t_j-}\right)+\frac{\partial c\left(x_{t_j-}\right)}{\partial x_{t_j-}}\\&-\frac{\partial t_j}{\partial x_{t_j-}}l_{q_j}\left(x_{t_j},u_{t_j}\right)+\left[\frac{\partial x_{t_j+}}{\partial x_{t_j-}}\right]^T\frac{\partial J\left(t_j+,q_j,x_{t_j+},L-j;I_{L-j}^{[t_j+,t_f]}\right)}{\partial x_{t_j+}},\end{aligned} \tag{5.50}$$

and also (5.41) turns into

$$\frac{\partial x_{t_j+}}{\partial x_{t_j-}} = \nabla\xi|_{x_{t_j-}} - \frac{\partial t_j}{\partial x_{t_j-}}\left(f_{q_j}\left(x_{t_j},u_{t_j}\right) - \nabla\xi f_{q_{j-1}}\left(x_{t_j-},u_{t_j-}\right)\right). \tag{5.51}$$

Therefore,

$$\begin{aligned}
&\frac{\partial J\left(t_j-,q_{j-1},x_{t_j-},L-j+1;I_{L-j+1}^{[t_j,t_f]}\right)}{\partial x_{t_j-}} = \\
&\qquad \frac{-\partial t_j}{\partial x_{t_j-}}\left(\left(l_{q_j}-l_{q_{j-1}}\right)+\left(f_{q_j}-\nabla\xi f_{q_{j-1}}\right)^T\frac{\partial J\left(t_j+,q_j,x_{t_j+},L-j;I_{L-j}^{[t_j^+,t_f]}\right)}{\partial x_{t_j+}}\right) \\
&\qquad\qquad + \frac{\partial c\left(x_{t_j-}\right)}{\partial x_{t_j-}} + \nabla\xi^T\frac{\partial J\left(t_j+,q_j,x_{t_j+},L-j;I_{L-j}^{[t_j^+,t_f]}\right)}{\partial x_{t_j+}}.
\end{aligned} \tag{5.52}$$

But in the limit as $\delta x_{t_j-} \in \mathbb{R}^n$ becomes sufficiently small, (5.48) gives

$$\frac{\partial t_j}{\partial x_{t_j-}} = \frac{-\nabla m}{\nabla m^T f_{q_{j-1}}\left(x_{t_j-},u_{t_j-}\right)}, \tag{5.53}$$

and hence,

$$\begin{aligned}
&\frac{\partial J\left(t_j-,q_{j-1},x_{t_j-},L-j+1;I_{L-j+1}^{[t_j,t_f]}\right)}{\partial x_{t_j-}} \\
&\qquad = \frac{\left(l_{q_j}-l_{q_{j-1}}\right)+\left(f_{q_j}-\nabla\xi f_{q_{j-1}}\right)^T\frac{\partial J\left(t_j+,q_j,x_{t_j+},L-j;I_{L-j}^{[t_j+,t_f]}\right)}{\partial x_{t_j+}}}{\nabla m^T f_{q_{j-1}}\left(x_{t_j-},u_{t_j-}\right)}\nabla m \\
&\qquad\qquad + \nabla c + \nabla\xi^T\frac{\partial J\left(t_j+,q_j,x_{t_j+},L-j;I_{L-j}^{[t_j+,t_f]}\right)}{\partial x_{t_j+}}
\end{aligned} \tag{5.54}$$

This proves (5.3) with

$$p = \frac{\left(l_{q_j} - l_{q_{j-1}}\right) + \left(f_{q_j} - \nabla\xi f_{q_{j-1}}\right)^T \frac{\partial J\left(t_j+, q_j, x_{t_j+}, L-j; I_{L-j}^{\left[t_j+, t_f\right]}\right)}{\partial x_{t_j+}}}{\nabla m^T f_{q_{j-1}}\left(x_{t_j-}, u_{t_j-}\right)} \tag{5.55}$$

which is the same equation for p as in (5.4).

With the consideration of (5.37) and following a similar procedure as in part (i) of the proof, equation (5.1) is derived for $t \in \left(t_{j-1}, t_j\right]$, and as shown above, it is subject to the terminal and boundary conditions (5.2) and (5.3) respectively. This completes the proof. □

Theorem 5.2. *Consider the hybrid system* $\mathbb{H}$ *together with the assumptions A0-A2 and the HOCP* (2.11) *for the hybrid cost* (2.10)*. If there exists an optimal control input with admissible set of discontinuities, then along each optimal trajectory, the adjoint process* λ *in the HMP and the gradient of the value function* ∇V *in the corresponding HDP satisfy the same family of differential equations, almost everywhere, i.e.*

$$\frac{d}{dt}\nabla V = -\frac{\partial}{\partial x} f_{q^o}\left(x^o, u^o\right)^T \nabla V - \frac{\partial}{\partial x} l_{q^o}\left(x^o, u^o\right), \tag{5.56}$$

and

$$\frac{d}{dt}\lambda^o = -\frac{\partial}{\partial x} f_{q^o}\left(x^o, u^o\right)^T \lambda^o - \frac{\partial}{\partial x} l_{q^o}\left(x^o, u^o\right), \tag{5.57}$$

and undergo the same terminal and boundary conditions, i.e.

$$\nabla V\left(t_f, q^o, x\left(t_f\right), 0\right) = \nabla g\left(x^o\left(t_f\right)\right), \tag{5.58}$$

$$\begin{aligned}\nabla V\left(t_j-, q_{j-1}, x\left(t_j-\right), L-j+1\right) = \nabla\xi|_{x\left(t_j-\right)}^T \nabla V\left(t_j+, q_j, x\left(t_j+\right), L-j\right) \\ + p\,\nabla m|_{x\left(t_j-\right)} + \nabla c|_{x\left(t_j-\right)},\end{aligned} \tag{5.59}$$

for the gradient of the value function, and

$$\lambda^o\left(t_f\right) = \nabla g\left(x^o\left(t_f\right)\right), \tag{5.60}$$

$$\lambda^o\left(t_j-\right) = \nabla\xi|_{x\left(t_j-\right)}^T \lambda^o\left(t_j+\right) + p\,\nabla m|_{x\left(t_j-\right)} + \nabla c|_{x\left(t_j-\right)}, \tag{5.61}$$

for the adjoint process. Hence, the adjoint process and the gradient of the value function are equal almost everywhere, i.e.

$$\lambda^o = \nabla_x V \tag{5.62}$$

almost everywhere in Lebesgue sense on $\left[t_0, t_f\right] \times \mathbb{R}^n$. □

Proof. Equations (5.57), (5.60) and (5.61) are direct results of the Hybrid Minimum Principle in Theorem 3.1, and equations (5.56), (5.58) and (5.59) hold for the optimal feedback control having an admissible set of discontinuities because equations (5.1), (5.2) and (5.3) hold for all feedback controls with admissible sets of discontinuities. Hence, from Theorem 2.1 and the resulting uniqueness of the solutions of (5.56) and (5.57) that are identical almost everywhere on $t \in \left[t_0, t_f\right]$, it is concluded that (5.62) holds almost everywhere in Lebesgue sense on $\left[t_0, t_f\right] \times \mathbb{R}^n$. □

5.2 Variations over Optimal Trajectories

A3: For all $q \in Q$ and $1 \leq k \leq L$, the set $M_{(2)}$ of all points at which the second order derivatives of $V\left(t, q, x, L-k\right)$ exist and are continuous is open dense in $\mathbb{R} \times \mathbb{R}^n$.

Theorem 5.3. *Consider the hybrid system* $\mathbb{H}$ *together with the assumptions A0-A3 and the HOCP* (2.11) *for the hybrid cost* (2.10)*. Then the results of Theorem 5.2 holds, i.e. the adjoint process locally describes the gradient of the value function and* (5.62) *holds almost everywhere in Lebesgue sense on* $\mathbb{R} \times \mathbb{R}^n$. □

Proof. Eq. (5.57) is a direct result of the HMP in Theorem 1. In order to show that Eq. (5.56) holds almost everywhere consider a reference hybrid state trajectory $\left(\tau_L, q, , x\right)$ at a Lebesgue instant $t \in \left(t_k, t_{k+1}\right)$ for some $k \in \{1, 2, \cdots, L\}$. Consider also a family of adjacent optimal trajectories in the form $\left(\tau'_L, q, , x'\right)$ at the same time $t \in \left(t'_k, t'_{k+1}\right)$ as illustrated in Fig. 5.1 such that $\delta x\left(s\right) := x'\left(s\right) - x\left(s\right)$, $s \in \left[t, t_{k+1}\right) \cap \left[t, t'_{k+1}\right)$ and $\delta t_{k+1} := t'_{k+1} - t_{k+1}$ can be selected arbitrarily small and so that $\delta u\left(s\right) := u'\left(s\right) - u\left(s\right)$ lies in a ρ-ball around zero in the metric space defined on $\mathscr{U}$. For simplicity of notation, the proof is presented in Mayer form, i.e. with the running and switching costs embedded respectively in the vector fields and jump maps, as remarked in Section 3.

Consider both of the trajectories in the interval $\left[t, t + \delta t\right] \subset \left(t_k, t_{k+1}\right) \cap \left(t'_k, t'_{k+1}\right)$ where $\delta t \in \mathbb{R}$ can be selected arbitrarily small. Note that $u \in L_\infty\left(\left[t, t + \delta t\right], U\right)$ may be modified on a set of

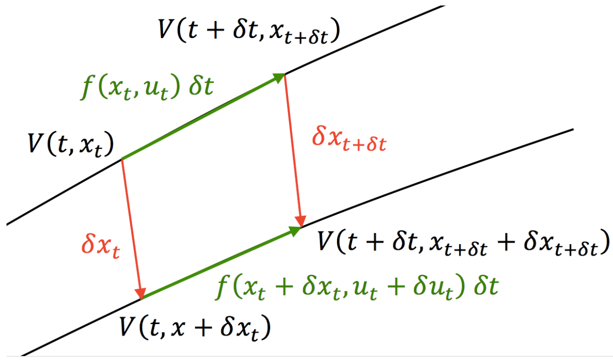

Figure 5.1 The choice of trajectories for the variation of value function to derive ∇V dynamics

measure zero so that all points are Lebesgue points (see e.g. [90, 91]) in which case the value of any cost function is necessarily unchanged [9].

Because of the Mayer representation of the optimal control problem, along the optimal trajectories x and $x' = x + \delta x$, the following equations must hold:

$$V\left(t+\delta t, q_k, x_{t+\delta t}, L-k\right) = V\left(t, q_k, x_t, L-k\right) \tag{5.63}$$

$$V\left(t+\delta t, q_k, [x+\delta x]_{t+\delta t}, L-k\right) = V\left(t, q_k, [x+\delta x]_t, L-k\right) \tag{5.64}$$

Writing the second order Taylor expansion of Eq. (5.63) gives

$$\begin{aligned} V\left(t+\delta t, q_k, x_{t+\delta t}, L-k\right) &= V\left(t, q_k, x_t, L-k\right) + \frac{\partial V}{\partial t}\delta t + \frac{\partial V}{\partial x}^T f_{q_k}\left(x_t, u_t\right)\delta t \\ &+ \frac{1}{2}\left(\frac{\partial^2 V}{\partial t^2}\delta t^2 + 2\frac{\partial^2 V}{\partial t \partial x}^T f_{q_k}\left(x_t, u_t\right)\delta t^2 + f_{q_k}\left(x_t, u_t\right)^T \frac{\partial^2 V}{\partial x^2} f_{q_k}\left(x_t, u_t\right)\delta t\right) + O\left(\delta^3\right), \end{aligned} \tag{5.65}$$

that results in

$$\left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}^T f_{q_k}\left(x_t, u_t\right)\right)\delta t + \frac{1}{2}\left(\frac{\partial^2 V}{\partial t^2} + 2\frac{\partial^2 V}{\partial t \partial x}^T f_{q_k}\left(x_t, u_t\right) + f_{q_k}\left(x_t, u_t\right)^T \frac{\partial^2 V}{\partial x^2} f_{q_k}\left(x_t, u_t\right)\right)\delta t^2 + O\left(\delta^3\right) = 0 \tag{5.66}$$

Expanding $V\left(t,q_k,[x+\delta x]_t,L-k\right)$ in Eq. (5.64) as

$$V\left(t,q_k,[x+\delta x]_t,L-k\right)=V\left(t,q_k,x_t,L-k\right)+\frac{\partial V}{\partial x}^T\delta x_t+\frac{1}{2}\delta x_t^T\frac{\partial^2 V}{\partial x^2}\delta x_t+O\left(\delta^3\right), \tag{5.67}$$

the second order expansion of $V\left(t+\delta t,q_k,[x+\delta x]_{t+\delta t},L-k\right)$ is derived as

$$\begin{aligned}
V\left(t+\delta t,q_k,[x+\delta x]_{t+\delta t}\right) &= \left(V\left(t,q_k,x_t,L-k\right)+\frac{\partial V}{\partial x}^T\delta x_t+\frac{1}{2}\delta x_t^T\frac{\partial^2 V}{\partial x^2}\delta x_t\right)\\
&+\frac{\partial}{\partial t}\left(V\left(t,q_k,x_t,L-k\right)+\frac{\partial V}{\partial x}^T\delta x_t+\frac{1}{2}\delta x_t^T\frac{\partial^2 V}{\partial x^2}\delta x_t\right)\delta t\\
&+\left(f_{q_k}+\frac{\partial f_{q_k}}{\partial x}\delta x_t+\frac{\partial f_{q_k}}{\partial u}\delta u\right)^T\delta t\,\frac{\partial}{\partial x}\left(V\left(t,q_k,x_t,L-k\right)+\frac{\partial V}{\partial x}^T\delta x_t+\frac{1}{2}\delta x_t^T\frac{\partial^2 V}{\partial x^2}\delta x_t\right)\\
&+\frac{1}{2}\frac{\partial^2}{\partial t^2}\left(V\left(t,q_k,x_t,L-k\right)+\frac{\partial V}{\partial x}^T\delta x_t+\frac{1}{2}\delta x_t^T\frac{\partial^2 V}{\partial x^2}\delta x_t\right)\delta t^2\\
&+\frac{1}{2}\Bigg[\left(f_{q_k}+\frac{\partial f_{q_k}}{\partial x}\delta x_t+\frac{\partial f_{q_k}}{\partial u}\delta u\right)^T\\
&\quad\frac{\partial^2}{\partial x^2}\left(V\left(t,q_k,x_t,L-k\right)+\frac{\partial V}{\partial x}^T\delta x_t+\frac{1}{2}\delta x_t^T\frac{\partial^2 V}{\partial x^2}\delta x_t\right)\left(f_{q_k}+\frac{\partial f_{q_k}}{\partial x}\delta x_t+\frac{\partial f_{q_k}}{\partial u}\delta u\right)\Bigg]\delta t^2\\
&+\left(f_{q_k}+\frac{\partial f_{q_k}}{\partial x}\delta x_t+\frac{\partial f_{q_k}}{\partial u}\delta u\right)^T\frac{\partial^2}{\partial t\partial x_t}\left(V\left(t,q_k,x_t,L-k\right)+\frac{\partial V}{\partial x}^T\delta x_t+\frac{1}{2}\delta x_t^T\frac{\partial^2 V}{\partial x^2}\delta x_t\right)\delta t^2+O\left(\delta^3\right),
\end{aligned} \tag{5.68}$$

or

$$\begin{aligned}
V\left(t+\delta t,q_k,[x+\delta x]_{t+\delta t},L-k\right) &= V\left(t,q_k,x_t,L-k\right)+\frac{\partial V}{\partial x}^T\delta x_t+\frac{1}{2}\delta x_t^T\frac{\partial^2 V}{\partial x^2}\delta x_t+\frac{\partial V}{\partial t}\delta t\\
&+\frac{\partial^2 V}{\partial t\partial x}^T\delta x_t\delta t+f_{q_k}^T\frac{\partial V}{\partial x}\delta t+f_{q_k}^T\frac{\partial^2 V}{\partial x^2}\delta x_t\delta t+\delta x_t^T\frac{\partial f_{q_k}}{\partial x}^T\frac{\partial V}{\partial x}\delta t+\delta u^T\frac{\partial f_{q_k}}{\partial u}^T\frac{\partial V}{\partial x}\delta t+\frac{1}{2}\frac{\partial^2 V}{\partial t^2}\delta t^2\\
&+\frac{1}{2}f_{q_k}^T\frac{\partial^2 V}{\partial x^2}f_{q_k}\delta t^2+\frac{\partial^2 V}{\partial t\partial x}^T f_{q_k}\delta t^2+O\left(\delta^3\right),
\end{aligned} \tag{5.69}$$

that from Eq. (5.64) it gives

$$\left(\frac{\partial V}{\partial t}+f_{q_k}^T\frac{\partial V}{\partial x}\right)\delta t+\frac{\partial V}{\partial x}^T\frac{\partial f_{q_k}}{\partial u}\delta u\delta t+\left(\frac{\partial^2 V}{\partial t\partial x}^T+f_{q_k}^T\frac{\partial^2 V}{\partial x^2}+\frac{\partial V}{\partial x}^T\frac{\partial f_{q_k}}{\partial x}\right)\delta x_t\delta t$$
$$+\frac{1}{2}\left(\frac{\partial^2 V}{\partial t^2}+2\frac{\partial^2 V}{\partial t\partial x}^T f_{q_k}+f_{q_k}^T\frac{\partial^2 V}{\partial x^2}f_{q_k}\right)\delta t^2+O\left(\delta^3\right)=0 \quad (5.70)$$

From Eqs. (5.66) and (5.70) it is conluded that

$$\left(\frac{\partial^2 V}{\partial t\partial x}^T+f_{q_k}^T\frac{\partial^2 V}{\partial x^2}+\frac{\partial V}{\partial x}^T\frac{\partial f_{q_k}}{\partial x}\right)\delta x_t+\frac{\partial V}{\partial x}^T\frac{\partial f_{q_k}}{\partial u}\delta u=0 \quad (5.71)$$

Since x is a Lebesgue point at the Lebesgue time t, the HJB equation gives

$$u_x^o=\arg\min_u\left\{\frac{\partial V}{\partial x}^T f_{q_k}\left(x,u\right)\right\} \quad (5.72)$$

and hence the term $\frac{\partial V}{\partial x}^T\frac{\partial f_{q_k}}{\partial u}\delta u$ vanishes in Eq. (5.71) resulting in

$$\left(\frac{\partial^2 V}{\partial t\partial x}^T+f_{q_k}^T\frac{\partial^2 V}{\partial x^2}+\frac{\partial V}{\partial x}^T\frac{\partial f_{q_k}}{\partial x}\right)\delta x_t=0 \quad (5.73)$$

Since δx_t is arbitrarily selected, it is concluded that

$$\frac{\partial^2 V}{\partial t\partial x}^T+f_{q_k}^T\frac{\partial^2 V}{\partial x^2}+\frac{\partial V}{\partial x}^T\frac{\partial f_{q_k}}{\partial x}=0 \quad (5.74)$$

but from the definition of total derative

$$\frac{d}{dt}\left(\frac{\partial V}{\partial x}\right)=\frac{\partial}{\partial t}\left(\frac{\partial V}{\partial x}\right)+f_{q_k}^T\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right)=\frac{\partial^2 V}{\partial t\partial x}^T+f_{q_k}^T\frac{\partial^2 V}{\partial x^2} \quad (5.75)$$

Hence

$$\frac{d}{dt}\left(\frac{\partial V}{\partial x}\right)=-\frac{\partial V}{\partial x}^T\frac{\partial f_{q_k}}{\partial x} \quad (5.76)$$

Since the above arguments were derived for the Mayer presentation of the considered HOCP (see also the remarks in Section 3), the equivalent result for the corresponding Bolza form is concluded as

$$\frac{d}{dt}\left(\frac{\partial V}{\partial x}\right) = -\frac{\partial V}{\partial x}^T \frac{\partial f_{q_k}}{\partial x} - \frac{\partial l_{q_k}}{\partial x} \tag{5.77}$$

which proves Eq. (5.56).

For the terminal and boundary conditions, we note that Eq. (5.60) and Eq. (5.61) are direct results of the HMP and Eq. (5.58) is simply derived by taking the gradient of (4.20).

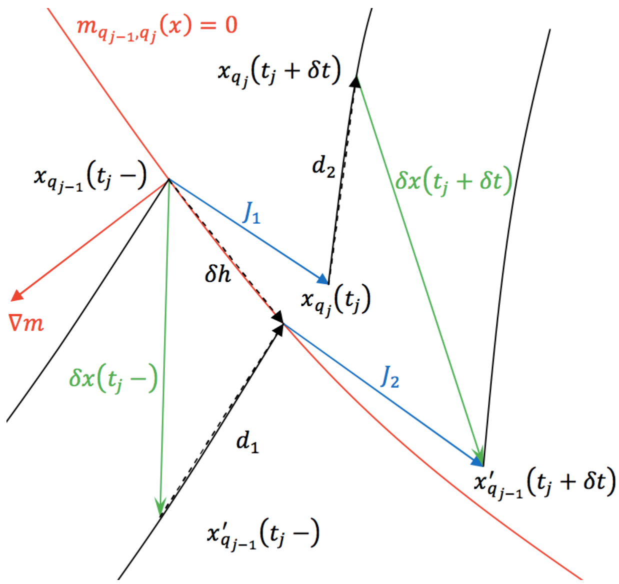

Figure 5.2 The choice of trajectories for variation of the value function

To show the boundary condition (5.59) for the value function, consider again the reference hybrid state trajectory $(\tau_L, q, , x)$ around a switching time $t_j \in \tau_L$ and the same family of adjacent optimal trajectories in the form $(\tau'_L, q, , x')$ around an equivalent switching time $t'_j \in \tau'_L$ as illustrated in Fig. 5.2 such that $\delta h := x'\left(t'_j-\right) - x\left(t_j-\right) \equiv x'\left(t_j + \delta t-\right) - x\left(t_j-\right)$ and $\delta t_j := t'_j - t_j$ can be selected arbitrarily small. Notice that in the autonomous switching case $\delta t \perp\!\!\!\perp \delta h$ but δt depends on $d_1 := x'\left(t'_j-\right) - x'\left(t_j-\right) \equiv \delta h - \delta x\left(t_j-\right)$ for both autonomous and controlled switchings (see Fig. 5.2 for the autonomous switching case).

Consider first the case where t_j and t'_j correspond to an autonomous switching event subject to a switching manifold $m \equiv m_{q_{j-1}q_j}$ with the switching manifold condition $m(x) = 0$. Due to the Mayer representation of the HOCP considered with the switching cost embedded in the jump map, for x and x' at t_j the following equalities must hold:

$$V\left(t_j-,q_{j-1},x\left(t_j-\right),L-j+1\right)=V\left(t_j+,q_j,x\left(t_j+\right),L-j\right), \tag{5.78}$$

$$V\left(t_j+\delta t-,q_{j-1},x'\left(t_j+\delta t-\right),L-j+1\right)=V\left(t_j+\delta t+,q_j,x'\left(t_j+\delta t+\right),L-j\right) \tag{5.79}$$

In addition, since running costs are embedded in vector fields, the following equations hold along x and x'

$$V\left(t_j+,q_j,x\left(t_j+\right),L-j\right)=V\left(t_j+\delta t+,q_j,x\left(t_j+\delta t+\right),L-j\right), \tag{5.80}$$

$$V\left(t_j-,q_{j-1},x'\left(t_j-\right),L-j+1\right)=V\left(t_j+\delta t-,q_{j-1},x'\left(t_j+\delta t-\right),L-j+1\right) \tag{5.81}$$

Eq. (5.78) and (5.80) give

$$V\left(t_j-,q_{j-1},x\left(t_j-\right),L-j+1\right)=V\left(t_j+\delta t+,q_j,x\left(t_j+\delta t+\right),L-j\right) \tag{5.82}$$

and Eq. (5.79) and (5.81) give

$$V\left(t_j-,q_{j-1},x'\left(t_j-\right),L-j+1\right)=V\left(t_j+\delta t+,q_j,x'\left(t_j+\delta t+\right),L-j\right) \tag{5.83}$$

Subtracting (5.82) from (5.83) and an application of Taylor series expansion yields

$$\begin{aligned}&\nabla V\left(t_j-,q_{j-1},x\left(t_j-\right),L-j+1\right)^T\delta x\left(t_j-\right)\\&\qquad=\nabla V\left(t_j+\delta t+,q_j,x\left(t_j+\delta t+\right),L-j\right)^T\delta x\left(t_j+\delta t\right)+O\left(\delta^2\right)\end{aligned} \tag{5.84}$$

The following exact relations hold according to the dynamics and jump maps governing the system's trajectories (see also Fig. 1)

$$x_{q_j}\left(t_j+\delta t+\right)=x_{q_j}\left(t_j\right)+d_2=\xi\left(x_{q_{j-1}}\left(t_j-\right)\right)+d_2 \tag{5.85}$$

$$x'_{q_j}\left(t_j+\delta t+\right)=\xi\left(x'_{q_{j-1}}\left(t_j+\delta t-\right)\right)=\xi\left(x'_{q_{j-1}}\left(t_j-\right)+d_1\right) \tag{5.86}$$

Hence,

$$\delta x\left(t_j+\delta t+\right)=x'_{q_j}\left(t_j+\delta t\right)-x_{q_j}\left(t_j+\delta t\right)=\nabla\xi\left(\delta x\left(t_j-\right)+d_1\right)-d_2+O\left(\delta^2\right) \tag{5.87}$$

where

$$d_1 = \frac{-\nabla m^T \delta x\left(t_j-\right)}{\nabla m^T f_{q_{j-1}}} f_{q_{j-1}} + O\left(\delta^2\right) \tag{5.88}$$

$$d_2 = \frac{-\nabla m^T \delta x\left(t_j-\right)}{\nabla m^T f_{q_{j-1}}} f_{q_j} + O\left(\delta^2\right) \tag{5.89}$$

This gives Eq. (5.87) as

$$\delta x\left(t_j+\delta t+\right) = \nabla\xi\, \delta x\left(t_j-\right) + \frac{\nabla m^T \delta x\left(t_j-\right)}{\nabla m^T f_{q_{j-1}}}\left(f_{q_j} - \nabla\xi\, f_{q_{j-1}}\right) + O\left(\delta^2\right) \tag{5.90}$$

Substituting (5.90) in (5.84) gives

$$\begin{aligned} \nabla V\left(t_j-, q_{j-1}, x\left(t_j-\right), L-j+1\right)^T \delta x\left(t_j-\right) &= \nabla V\left(t_j+\delta t, q_j, x\left(t_j+\delta t\right), L-j\right)^T \nabla\xi\, \delta x\left(t_j-\right) \\ &+ \nabla V\left(t_j+\delta t, q_j, x\left(t_j+\delta t\right), L-j\right)^T \frac{\nabla m^T \delta x\left(t_j-\right)}{\nabla m^T f_{q_{j-1}}}\left(f_{q_j} - \nabla\xi\, f_{q_{j-1}}\right) \end{aligned} \tag{5.91}$$

Noting that

$$\begin{gathered} \nabla V\left(t_j+\delta t, q_j, x\left(t_j+\delta t\right), L-j\right)^T \nabla\xi\, \delta x\left(t_j-\right) = \\ \left[\nabla\xi^T \nabla V\left(t_j+\delta t, q_j, x\left(t_j+\delta t\right), L-j\right)\right]^T \delta x\left(t_j-\right) \end{gathered} \tag{5.92}$$

and

$$\begin{aligned} &\nabla V\left(t_j+\delta t, q_j, x\left(t_j+\delta t\right), L-j\right)^T \frac{\nabla m^T \delta x\left(t_j-\right)}{\nabla m^T f_{q_{j-1}}}\left(f_{q_j} - \nabla\xi\, f_{q_{j-1}}\right) \\ &= \nabla V\left(t_j+\delta t, q_j, x\left(t_j+\delta t\right), L-j\right)^T \frac{\left(f_{q_j} - \nabla\xi\, f_{q_{j-1}}\right)}{\nabla m^T f_{q_{j-1}}} \nabla m^T \delta x\left(t_j-\right) = p \nabla m^T \delta x\left(t_j-\right) \end{aligned} \tag{5.93}$$

with

$$p := \frac{\nabla V\left(t_j+\delta t, q_j, x\left(t_j+\delta t\right), L-j\right)^T \left(f_{q_j} - \nabla\xi\, f_{q_{j-1}}\right)}{\nabla m^T f_{q_{j-1}}} \tag{5.94}$$

Eq. (5.91) can be written as

$$\begin{gathered} \nabla V\left(t_j-, q_{j-1}, x\left(t_j-\right), L-j+1\right)^T \delta x\left(t_j-\right) \\ = \left[\nabla\xi^T \nabla V\left(t_j+\delta t, q_j, x\left(t_j+\delta t\right), L-j\right) + p\nabla m\right]^T \delta x\left(t_j-\right) \end{gathered} \tag{5.95}$$

Since (5.95) holds for every choice of $\delta x\left(t_j-\right)$ arbitrarily small, we must have

$$\begin{gathered}\nabla V\left(t_j-,q_{j-1},[x+\delta x]\left(t_j-\right),L-j+1\right)= \\ \nabla \xi^T \nabla V\left(t_j+\delta t,q_j,x\left(t_j+\delta t\right),L-j\right)+p\nabla m\end{gathered} \tag{5.96}$$

Letting $\delta t \to 0$, Eq. (5.96) becomes

$$\nabla V\left(t_j-,q_{j-1},x\left(t_j-\right),L-j+1\right)=\nabla \xi^T \nabla V\left(t_j+,q_j,x\left(t_j+\right),L-j\right)+p\nabla m \tag{5.97}$$

Noting that $x\left(t_j+\right)=\xi\left(x\left(t_j-\right)\right)$ and with the extraction of the switching cost from the jump as in (2.15), the boundary condition (5.59) for ∇V is derived for the autonomous switching case.

If t_j and t'_j correspond to a controlled switching event, δt is independent of $d_1 = x'\left(t'_j-\right) - x'\left(t_j-\right)$. Similar to the autonomous switching case, (5.84) and (5.87) hold but (5.88) and (5.89) become

$$d_1 = f_{q_{j-1}}\delta t + O\left(\delta^2\right) \tag{5.98}$$

$$d_2 = f_{q_j}\delta t + O\left(\delta^2\right) \tag{5.99}$$

that gives

$$\begin{aligned}\nabla V & \left(t_j-,q_{j-1},x\left(t_j-\right),L-j+1\right)^T \delta x\left(t_j-\right) \\ &= \nabla V\left(t_j+\delta t,q_j,x\left(t_j+\delta t\right),L-j\right)^T \nabla \xi\, \delta x\left(t_j-\right) \\ &\quad - \nabla V\left(t_j+\delta t,q_j,x\left(t_j+\delta t\right),L-j\right)^T \left(f_2-\nabla \xi f_1\right)\delta t\end{aligned} \tag{5.100}$$

or

$$\begin{aligned}&\left[\nabla \xi^T \nabla V\left(t_j+\delta t,q_j,x\left(t_j+\delta t\right),L-j\right)-\nabla V\left(t_j-,q_{j-1},x\left(t_j-\right),L-j+1\right)\right]^T \delta x\left(t_j-\right) \\ &\qquad = \left[\nabla V\left(t_j+\delta t,q_j,x\left(t_j+\delta t\right),L-j\right)^T \left(f_2-\nabla \xi f_1\right)\right]\delta t\end{aligned} \tag{5.101}$$

Letting $\delta t \to 0$ and noting that the above relation must hold for all $\delta x\left(t_j-\right) \in \mathbb{R}^n$, Eq. (5.101) results in

$$\nabla V\left(t_j-,q_{j-1},[x+\delta x]\left(t_j-\right),L-j+1\right)=\nabla \xi^T \nabla V\left(t_j+\delta t,q_j,x\left(t_j+\delta t\right),L-j\right) \tag{5.102}$$

which is the same as Eq. (5.96) with $p = 0$. Thus the boundary condition (5.59) for ∇V is shown to hold in the controlled switching case as well.

In conclusion, the relationship between the Hybrid Minimum Principle and Hybrid Dynamic Programming in the form of Eq. (5.62) is proved from the uniqueness of the solution of (5.56) (and equivalently (5.57)) subject to the boundary conditions (5.58) and (5.59) (equivalently (5.60) and (5.61)). $\square$

Chapter 6

Analytical Examples

6.1 Nonlinear Dynamics and Costs in $\{q_1, q_2\} \times \mathbb{R}$

Consider a hybrid system with the following indexed vector fields

$$\dot{x} = f_1(x,u) = x + xu, \tag{6.1}$$

$$\dot{x} = f_2(x,u) = -x + xu, \tag{6.2}$$

and the hybrid optimal control problem

$$J\left(t_0, t_f, h_0, 1; I_1\right) = \int_{t_0}^{t_s} \frac{1}{2}u^2 dt + \frac{1}{1 + \left[x\left(t_s-\right)\right]^2} + \int_{t_s}^{t_f} \frac{1}{2}u^2 dt + \frac{1}{2}\left[x\left(t_f\right)\right]^2, \tag{6.3}$$

subject to the initial condition $h_0 = (q(t_0), x(t_0)) = (q_1, x_0)$ provided at the initial time $t_0 = 0$.

6.1.1 The HMP Formulation and Results

Writing down the Hybrid Minimum Principle results for the above HOCP, the Hamiltonians are formed as

$$H_{q_1} = \frac{1}{2}u^2 + \lambda x(u+1), \tag{6.4}$$
$$H_{q_2} = \frac{1}{2}u^2 + \lambda x(u-1), \tag{6.5}$$

from which the minimizing control input for both Hamiltonian functions is determined as

$$u^o = -\lambda x \tag{6.6}$$

Therefore, the adjoint process dynamics, determined from (3.3) and with the replacement of the optimal control input (6.6), is written as

$$\dot{\lambda} = \frac{-\partial H_{q_1}}{\partial x} = -\lambda\left(u^o + 1\right) = \lambda\left(\lambda x - 1\right), \qquad t \in \left(t_0, t_s\right) \tag{6.7}$$

$$\dot{\lambda} = \frac{-\partial H_{q_2}}{\partial x} = -\lambda\left(u^o - 1\right) = \lambda\left(\lambda x + 1\right), \qquad t \in \left(t_s, t_f\right) \tag{6.8}$$

which are subject to the terminal and boundary conditions

$$\lambda\left(t_f\right) = \nabla g|_{x\left(t_f\right)} = x\left(t_f\right), \tag{6.9}$$

$$\lambda\left(t_s-\right) \equiv \lambda\left(t_s\right) = \nabla\xi|_{x\left(t_s-\right)}\lambda\left(t_s+\right) + \nabla c|_{x\left(t_s-\right)} = -\lambda\left(t_s+\right) + \frac{-2x\left(t_s-\right)}{\left(1 + \left[x\left(t_s-\right)\right]^2\right)^2} \tag{6.10}$$

The replacement of the optimal control input (6.6) in the continuous state dynamics (3.2) gives

$$\dot{x} = \frac{\partial H_{q_1}}{\partial \lambda} = x\left(1 + u^o\right) = -x\left(\lambda x - 1\right), \qquad t \in \left(t_0, t_s\right) \tag{6.11}$$

$$\dot{x} = \frac{\partial H_{q_2}}{\partial \lambda} = x\left(-1 + u^o\right) = -x\left(\lambda x + 1\right), \qquad t \in \left(t_s, t_f\right) \tag{6.12}$$

which are subject to the initial and boundary conditions

$$x\left(t_0\right) = x\left(0\right) = x_0, \tag{6.13}$$

$$x\left(t_s\right) = \xi\left(x\left(t_s-\right)\right) = -x\left(t_s-\right) \tag{6.14}$$

The Hamiltonian continuity condition (3.9) states that

$$
\begin{aligned}
H_{q_1}(t_s-) = \frac{1}{2}\left[u^o(t_s-)\right]^2 + \lambda(t_s-)\,x(t_s-)\left[u^o(t_s-)+1\right] \\
= \frac{1}{2}\left[-\lambda(t_s-)\,x(t_s-)\right]^2 + \lambda(t_s-)\,x(t_s-)\left[-\lambda(t_s-)\,x(t_s-)+1\right] \\
= H_{q_2}(t_s+) = \frac{1}{2}\left[u^o(t_s+)\right]^2 + \lambda(t_s+)\,x(t_s+)\left[u^o(t_s+)-1\right] \\
= \frac{1}{2}\left[-\lambda(t_s+)\,x(t_s+)\right]^2 + \lambda(t_s+)\,x(t_s+)\left[-\lambda(t_s+)\,x(t_s+)-1\right],
\end{aligned}
\tag{6.15}
$$

which can be written, using (6.14), as

$$
x(t_s-)\left[\lambda(t_s-)-\lambda(t_s+)\right] = \frac{1}{2}\left[x(t_s-)\right]^2\left[\left[\lambda(t_s-)\right]^2-\left[\lambda(t_s+)\right]^2\right]
\tag{6.16}
$$

The solution to the set of ODEs (6.7), (6.8), (6.11), (6.12) together with the initial condition (6.13) expressed at t_0, the terminal condition (6.9) determined at t_f and the boundary conditions (6.14) and (6.10) provided at t_s which is not a priori fixed but determined by the Hamiltonian continuity condition (6.16), results in the determination of the optimal control input and its corresponding optimal trajectory that minimize the cost $J\left(t_0, t_f, h_0, 1; I_1\right)$ over $\boldsymbol{I}_1$, the family of hybrid inputs with one switching.

Analytical Solution to the HMP

In the above arguments, optimal controls and their corresponding optimal trajectories have been identified as the solution of the governing differential equations provided by the HMP. For this particular example, however, we can take further steps in order to reduce the above boundary value ODE problem into a set of algebraic equations by using the special forms of the differential equations under study. A more detailed discussion is presented in [78]. Because of the special dynamics in this example, we can write:

$$
\lambda' := \frac{d\lambda}{dx} = \frac{\dot{\lambda}}{\dot{x}} = \frac{\lambda(\lambda x-1)}{-x(\lambda x-1)} = \frac{-\lambda}{x}, \qquad t\in[0,t_s),
\tag{6.17}
$$

which gives

$$
\lambda = \frac{\alpha}{x}, \qquad t\in[0,t_s).
\tag{6.18}
$$

Similarly,

$$\lambda = \frac{\beta}{x}, \qquad t \in \left[t_s, t_f\right]. \tag{6.19}$$

Substituting (6.18) and (6.19) into (6.6) results in

$$u^o(t) = -\alpha, \qquad t \in [0, t_s), \tag{6.20}$$

$$u^o(t) = -\beta, \qquad t \in \left[t_s, t_f\right]. \tag{6.21}$$

With the substitution of (6.20) into (6.11), and (6.21) into (6.12), we write

$$x(t) = x_0 e^{(1-\alpha)t}, \qquad t \in [0, t_s), \tag{6.22}$$

$$x(t) = x(t_s)\, e^{-(1+\beta)(t-t_s)} = -x_0 e^{(1-\alpha)t_s-(1+\beta)(t-t_s)}, \qquad t \in \left[t_s, t_f\right], \tag{6.23}$$

where in writing of the second equality in (6.23), the boundary condition (6.14) is used. Therefore

$$\lambda(t) = \frac{\alpha}{x(t)} = \frac{\alpha}{x_0 e^{(1-\alpha)t}}, \qquad t \in [0, t_s], \tag{6.24}$$

$$\lambda(t) = \frac{\beta}{x(t)} = \frac{-\beta}{x_0 e^{(1-\alpha)t_s-(1+\beta)(t-t_s)}}, \qquad t \in \left(t_s, t_f\right]. \tag{6.25}$$

Equation (6.16) requires that at least one of the following conditions hold

$$x(t_s-) = 0, \tag{6.26}$$

$$\lambda_1(t_s-) = \lambda_2(t_s+), \tag{6.27}$$

$$x(t_s-)\left[\lambda_1(t_s-) + \lambda_2(t_s+)\right] = 2, \tag{6.28}$$

The first equality (6.26) is ruled out because it is impossible for $x_0 \neq 0$ as the control input cannot steer the trajectories to the origin. Equality (6.28) is also ruled out because it is a contradiction to (6.10) as the sum of the adjoint processes would need to be positive and negative at the same time. Hence,

$$\lambda_1(t_s-) = \lambda_2(t_s+), \tag{6.29}$$

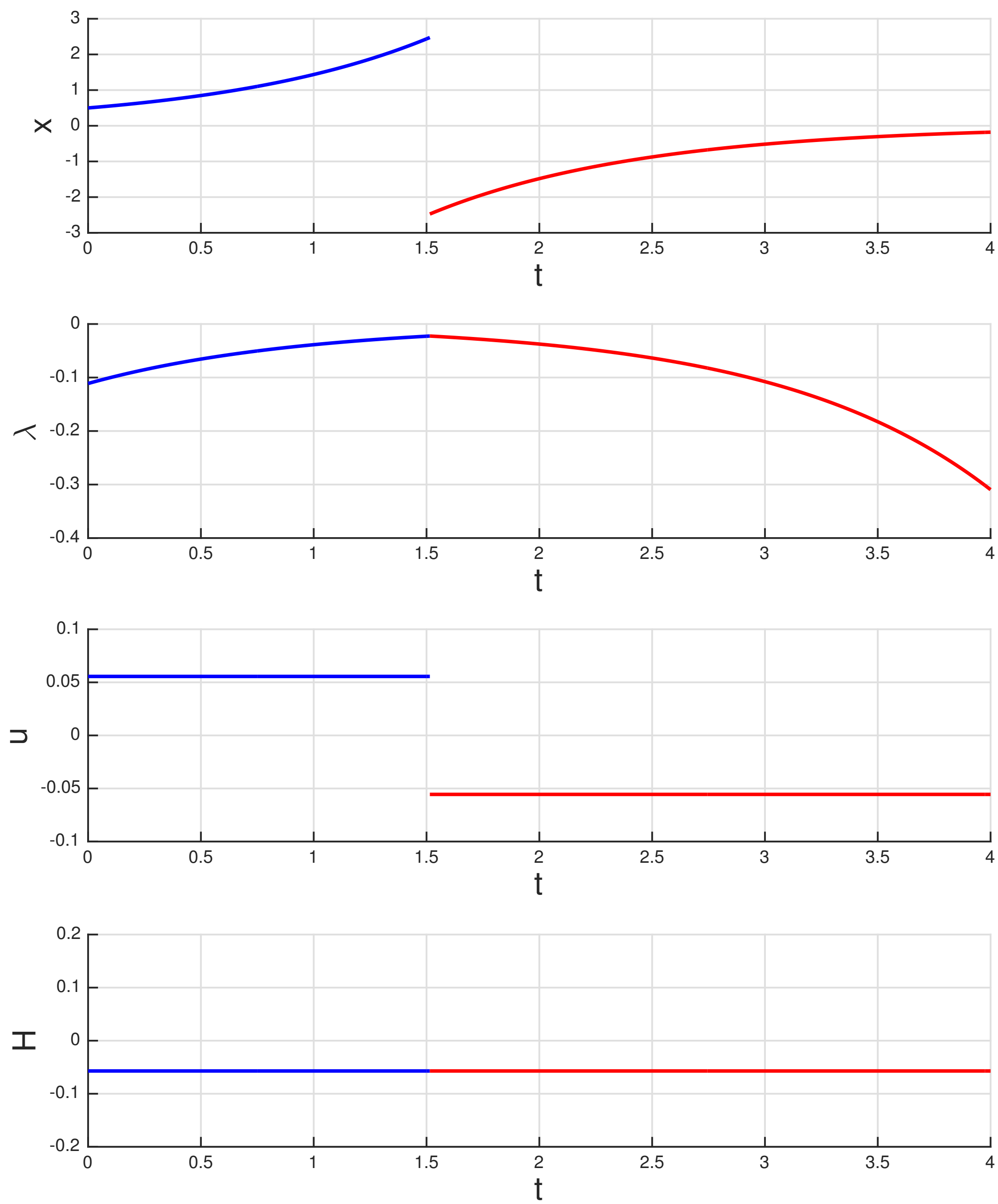

Figure 6.1 The optimal trajectory, the corresponding adjoint processes, optimal inputs and the Hamiltonians for the system in Example 6.1 with $x_0 = 0.5$ and $t_f = 4$

must hold which together with (6.10) result in

$$\lambda_1(t_s-) = \lambda_2(t_s+) = \frac{-x(t_s-)}{\left(1+[x(t_s-)]^2\right)^2}. \tag{6.30}$$

The condition (6.29) gives

$$\alpha = -\beta, \tag{6.31}$$

and (6.30) implies that

$$\frac{\alpha}{x_0 e^{(1-\alpha)t_s}} = \frac{-x_0 e^{(1-\alpha)t_s}}{\left(1+\left[x_0 e^{(1-\alpha)t_s}\right]^2\right)^2}, \tag{6.32}$$

which gives

$$\alpha = \frac{-x_0^2 e^{2(1-\alpha)t_s}}{\left(1+x_0^2 e^{2(1-\alpha)t_s}\right)^2}. \tag{6.33}$$

Furthermore, (6.9) results in

$$\frac{-\beta}{x_0 e^{(1-\alpha)t_s-(1+\beta)(t-t_s)}} = \frac{\alpha}{x_0 e^{(1-\alpha)\left(2t_s-t_f\right)}} = -x_0 e^{(1-\alpha)\left(2t_s-t_f\right)}, \tag{6.34}$$

which gives

$$\alpha = -x_0^2 e^{2(1-\alpha)\left(2t_s-t_f\right)} \tag{6.35}$$

Solving (6.33) and (6.35) determines α (and consequently β) as well as t_s, given that x_0 and t_f are specified. The numerical results for $x_0 = 0.5$ and $t_f = 4$ are illustrated in Figure 6.1.1.

6.1.2 The HDP Formulation and Results

Theorem 4.3 states that the value function satisfies the HJB equation (4.19) almost everywhere. In particular,

$$\begin{aligned}-\frac{\partial V(t,q_2,x,0)}{\partial t} &= \inf_u H_{q_2}\left(x,\frac{\partial V}{\partial x},u\right) = \inf_u\left\{l_{q_2}(x,u)+\frac{\partial V}{\partial x}f_{q_2}(x,u)\right\}\\ &= \inf_u\left\{\frac{1}{2}u^2+\frac{\partial V}{\partial x}[-x+xu]\right\} = \left\{\frac{1}{2}u^2+\frac{\partial V}{\partial x}[-x+xu]\right\}_{u=-x\frac{\partial V}{\partial x}} = \frac{-1}{2}x^2\left(\frac{\partial V}{\partial x}\right)^2 - x\frac{\partial V}{\partial x},\end{aligned} \tag{6.36}$$

and similarly,

$$-\frac{\partial V\left(t,q_1,x,1\right)}{\partial t}=\frac{-1}{2}x^2\left(\frac{\partial V}{\partial x}\right)^2+x\frac{\partial V}{\partial x} \tag{6.37}$$

with the boundary conditions

$$V\left(t_f,q_2,x,0\right)=g\left(x\left(t_f\right)\right)=\frac{1}{2}x^2, \tag{6.38}$$

for $V\left(t,q_2,x,0\right)$, as well as

$$V\left(t_s,q_1,x,1\right)=\min_{\sigma\in\left\{\sigma_{q_1q_2}\right\}}\left\{V\left(t_s,q_2,-x,0\right)+\frac{1}{1+x^2}\right\}, \tag{6.39}$$

and

$$\frac{-1}{2}x^2\left(\frac{\partial V_{q_1}}{\partial x}\right)^2+x\frac{\partial V_{q_1}}{\partial x}=\frac{-1}{2}\left(-x\right)^2\left(\frac{\partial V_{q_2}}{\partial x}\right)^2-\left(-x\right)\frac{\partial V_{q_2}}{\partial x}, \tag{6.40}$$

required for the determination of $V\left(t,q_1,x,1\right)$ and t_s.

6.1.3 The HMP - HDP Relationship

In order to illustrate the the result in Theorem 5.2, we first take the partial derivatives of (6.36) with respect to x to write

$$\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial t}-\frac{1}{2}x^2\left(\frac{\partial V}{\partial x}\right)^2-x\frac{\partial V}{\partial x}\right)=0, \tag{6.41}$$

or

$$\frac{\partial^2 V}{\partial x\partial t}-x\left(\frac{\partial V}{\partial x}\right)^2-x^2\frac{\partial V}{\partial x}\frac{\partial^2 V}{\partial x^2}-\frac{\partial V}{\partial x}-x\frac{\partial^2 V}{\partial x^2}=0 \tag{6.42}$$

It can easily be verified that the set of states with twice differentiability of $V\left(t,q_2,x,0\right)$ is $M_{(2)}=\left(t_s,t_f\right)\times\left(\mathbb{R}-\{0\}\right)$ which is open dense in $\mathbb{R}\times\mathbb{R}$ and therefore,

$$\frac{\partial^2 V}{\partial t\partial x}-x^2\frac{\partial V}{\partial x}\frac{\partial^2 V}{\partial x^2}-x\frac{\partial^2 V}{\partial x^2}=x\left(\frac{\partial V}{\partial x}\right)^2+\frac{\partial V}{\partial x} \tag{6.43}$$

But from the definition of the total derivative, we have

$$\frac{d}{dt}\left(\frac{\partial V}{\partial x}\right) = \frac{\partial^2 V}{\partial t \partial x} + \frac{\partial^2 V}{\partial x^2} f_{q_2(x,u^o)} = \frac{\partial^2 V}{\partial t \partial x} + \frac{\partial^2 V}{\partial x^2}\left(-x^2 \frac{\partial V}{\partial x} - x\right) = \frac{\partial^2 V}{\partial t \partial x} - x^2 \frac{\partial V}{\partial x}\frac{\partial^2 V}{\partial x^2} - x\frac{\partial^2 V}{\partial x^2} \tag{6.44}$$

Therefore, from (6.43) and (6.44), the dynamics for $\nabla V\left(t, q_2, x, 0\right)$ is derived as

$$\frac{d}{dt}\left(\frac{\partial V}{\partial x}\right) = x\left(\frac{\partial V}{\partial x}\right)^2 + \frac{\partial V}{\partial x} = \frac{\partial V}{\partial x}\left(x\frac{\partial V}{\partial x} + 1\right) \tag{6.45}$$

which is the same as the dynamics (6.8) for $\lambda\left(t\right)$, $t \in \left(t_s, t_f\right)$.

Similarly, the differentiation of (6.37) results in

$$\frac{d}{dt}\left(\frac{\partial V}{\partial x}\right) = \frac{\partial V}{\partial x}\left(x\frac{\partial V}{\partial x} - 1\right) \tag{6.46}$$

which is the same as the dynamics (6.7) for $\lambda\left(t\right)$, $t \in \left(t_0, t_s\right)$. The equality of the terminal conditions for $\nabla V\left(t_f, q_2, x, 0\right)$ and $\lambda\left(t_f\right)$ becomes obvious by taking the gradient of (6.38), i.e.

$$\frac{\partial V\left(t_f, q_2, x, 0\right)}{\partial x} = \frac{\partial g\left(x\right)}{\partial x} = x, \tag{6.47}$$

which is equivalent to (6.9).

Moreover, the equality of the boundary conditions for $\nabla V\left(t_f, q_2, x, 0\right)$ and $\lambda\left(t_f\right)$ can be illustrated by taking the gradient of (6.39) and writing

$$\frac{\partial}{\partial x} V\left(t_s, q_1, x, 1\right) = \frac{\partial}{\partial x}\left(V\left(t_s, q_2, -x, 0\right) + \frac{1}{1+x^2}\right), \tag{6.48}$$

that gives

$$\frac{\partial V\left(t_s, q_1, x, 1\right)}{\partial x} = -\left.\frac{\partial V\left(t_s, q_2, y, 0\right)}{\partial y}\right|_{y=-x} + \frac{-2x}{\left(1+x^2\right)^2}, \tag{6.49}$$

which is the same boundary condition as the boundary condition (6.10) for λ. Therefore, by the uniqueness of the results of the set of differential equations (6.45) and (6.46) for ∇V (or equivalently (6.8) and (6.7) for λ) with the terminal and boundary conditions (6.47) and (6.49) for ∇V (or equivalently (6.9) and (6.10) for λ), the gradient of the value function evaluated along every optimal trajectory is equal to the adjoint process corresponding to the same trajectory. Interested readers are referred to [78] for further discussion on this example. □

6.2 Linear Dynamics and Quadratic Costs in $\{q_1, q_2\} \times \mathbb{R}^2$

Consider the hybrid system with the indexed vector fields

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = f_1(x,u) = \begin{bmatrix} x_2 \\ -x_1 + u \end{bmatrix}, \tag{6.50}$$

and

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = f_2(x,u) = \begin{bmatrix} x_2 \\ u \end{bmatrix}, \tag{6.51}$$

where autonomous switchings occur on the switching manifold described by

$$m(x_1(t_s), x_2(t_s-)) \equiv x_2(t_s-) = 0, \tag{6.52}$$

with the continuity of the trajectories at the switching instant. Consider the hybrid optimal control problem defined as the minimization of the total cost functional

$$J = \int_{t_0}^{t_f} \frac{1}{2} u^2 dt + \frac{1}{2}(x_1(t_s))^2 + \frac{1}{2}\left(x_2(t_f) - v_{ref}\right)^2 \tag{6.53}$$

6.2.1 The HMP Formulation and Results

Employing the HMP, the corresponding Hamiltonians are defined as

$$H_1 = \lambda_1 x_2 + \lambda_2(-x_1 + u) + \frac{1}{2}u^2, \tag{6.54}$$

and

$$H_2 = \lambda_1 x_2 + \lambda_2 u + \frac{1}{2}u^2 \tag{6.55}$$

The Hamiltonian minimization with respect to u (Eq. (3.8)) gives

$$u^o = -\lambda_2 \tag{6.56}$$

for both $q = 1$ and $q = 2$.

Therefore the state dynamics (3.2) and the adjoint process dynamics (3.3) become

$$\dot{x}_1 = \frac{\partial H_1}{\partial \lambda_1} = x_2, \tag{6.57}$$

$$\dot{x}_2 = \frac{\partial H_1}{\partial \lambda_2} = -x_1 + u^o = -x_1 - \lambda_2, \tag{6.58}$$

$$\dot{\lambda}_1 = \frac{-\partial H_1}{\partial x_1} = \lambda_2, \tag{6.59}$$

$$\dot{\lambda}_2 = \frac{-\partial H_1}{\partial x_2} = -\lambda_1, \tag{6.60}$$

for $q = 1$, and

$$\dot{x}_1 = \frac{\partial H_2}{\partial \lambda_1} = x_2, \tag{6.61}$$

$$\dot{x}_2 = \frac{\partial H_2}{\partial \lambda_2} = u^o = -\lambda_2, \tag{6.62}$$

$$\dot{\lambda}_1 = \frac{-\partial H_2}{\partial x_1} = 0, \tag{6.63}$$

$$\dot{\lambda}_2 = \frac{-\partial H_2}{\partial x_2} = -\lambda_1, \tag{6.64}$$

for $q = 2$. At the initial time $t = t_0$, the continuous valued states are specified by the initial conditions

$$x_1(t_0) = x_{10}, \tag{6.65}$$

$$x_2(t_0) = x_{20} \tag{6.66}$$

At the switching instant $t = t_s$, the boundary conditions for the states and adjoint processes are determined as

$$x_1(t_s) = x_1(t_s-) \equiv \lim_{t \uparrow t_s} x_1(t), \tag{6.67}$$

$$x_2(t_s) = x_2(t_s-) = 0, \tag{6.68}$$

$$\lambda_1(t_s) = \lambda_1(t_s+) + \frac{\partial c}{\partial x_1} + p\frac{\partial m}{\partial x_1} = \lambda_1(t_s+) + x_1(t_s), \tag{6.69}$$

$$\lambda_2(t_s) = \lambda_2(t_s+) + \frac{\partial c}{\partial x_2} + p\frac{\partial m}{\partial x_2} = \lambda_2(t_s+) + p \tag{6.70}$$

And at the terminal time $t = t_f$, the adjoint processes are determined by (3.6) as

$$\lambda_1\left(t_f\right) = \frac{\partial g}{\partial x_1} = 0, \tag{6.71}$$

$$\lambda_2\left(t_f\right) = \frac{\partial g}{\partial x_2} = x_2\left(t_f\right) - v_{ref} \tag{6.72}$$

Note that unlike t_0 and t_f which are a priori determined, t_s is not fixed and needs to be determined by the Hamiltonian continuity condition (3.9) as

$$\begin{aligned} H_1(t_s-) &= \lambda_1(t_s-)x_2(t_s-) - \lambda_2(t_s-)x_1(t_s-) - \frac{1}{2}\lambda_2(t_s-)^2 = -\lambda_2(t_s)x_1(t_s-) - \frac{1}{2}\lambda_2(t_s)^2 \\ &= H_2(t_s+) = \lambda_1(t_s+)x_2(t_s+) - \frac{1}{2}\lambda_2(t_s+)^2 = -\frac{1}{2}\lambda_2(t_s+)^2, \end{aligned} \tag{6.73}$$

i.e.

$$\lambda_2(t_s)x_1(t_s-) + \frac{1}{2}\lambda_2(t_s)^2 = \frac{1}{2}\lambda_2(t_s+)^2, \tag{6.74}$$

that with the insertion of (6.70), it becomes

$$(\lambda_2(t_s+) + p)x_1(t_s-) + \frac{1}{2}(\lambda_2(t_s+) + p)^2 = \frac{1}{2}\lambda_2(t_s+)^2, \tag{6.75}$$

The set of ODEs (6.57) to (6.64), together with the initial conditions (6.65) and (6.66) expressed at t_0, the boundary conditions (6.67), (6.68), (6.69) and (6.70) provided at t_s, and the terminal conditions (6.71) and (6.72) determined at t_f, with the two unknowns t_s and p determined by the Hamiltonian continuity condition (6.75) and the switching manifold condition (6.52), form an ODE boundary value problem whose solution results in the determination of the optimal control input and its corresponding optimal trajectory that minimize the cost $J\left(t_0, t_f, h_0, 1; I_1\right)$ over $\boldsymbol{I}_1$, the family of hybrid inputs with one switching on the switching manifold (6.52).

Analytical Solution to the HMP

Similar to the previous example, further steps can be taken in order to reduce the above boundary value ODE problem into a set of algebraic equations using the special forms of the differential equations under study. This has been done in detail in [96], and a brief version is provided here.

From (6.63) and (6.69) we may write

$$\lambda_1(t) = 0, \qquad t \in \left(t_s, t_f\right]. \tag{6.76}$$

Therefore, the dynamics of the second component of the adjoint process in $\left(t_s, t_f\right]$ is determined from (6.64) as

$$\dot{\lambda}_2 = 0, \qquad t \in \left(t_s, t_f\right], \tag{6.77}$$

which from (6.72) we conclude that

$$\lambda_2(t) = x_2\left(t_f\right) - v_{ref} \qquad t \in \left(t_s, t_f\right]. \tag{6.78}$$

The boundary conditions (6.69) and (6.70) on adjoint processes at the switchings instant give

$$\lambda_1(t_s) = \lambda_1(t_s+) + x_1(t_s) = x_1(t_s), \tag{6.79}$$

$$\lambda_2(t_s) = \lambda_2(t_s+) + p = x_2\left(t_f\right) - v_{ref} + p, \tag{6.80}$$

The conditions (6.79) and (6.80) serve as terminal conditions for the adjoint processes dynamics (6.59) and (6.59) which have a general solution of the form

$$\lambda_1 = A\sin(t+\alpha), \qquad t \in [t_0, t_s], \tag{6.81}$$

$$\lambda_2 = A\cos(t+\alpha), \qquad t \in [t_0, t_s]. \tag{6.82}$$

Therefore, the state dynamics (6.57) and (6.58) are written as

$$\dot{x}_1 = x_2, \tag{6.83}$$

$$\dot{x}_2 = -x_1 - \lambda_2 = -x_1 - A\cos(t+\alpha), \tag{6.84}$$

for $t \in [t_0, t_s]$, which have a general solution of the form

$$x_1(t) = \frac{-1}{2}At\sin(t+\alpha) + B\sin(t+\beta), \tag{6.85}$$

$$x_2(t) = \frac{-1}{2}At\cos(t+\alpha) - \frac{1}{2}A\sin(t+\alpha) + B\cos(t+\beta), \tag{6.86}$$

for $t \in [t_0, t_s) = [0, t_s)$, subject to the initial conditions

$$x_1(t_0) = B\sin\beta = x_{10}, \tag{6.87}$$

$$x_2(t_0) = -\frac{1}{2}A\sin(\alpha) + B\cos(\beta) = x_{20}. \tag{6.88}$$

At the switching time t_s the continuity condition for x_1 and x_2 are written as

$$x_1(t_s+) \equiv x_1(t_s) = x_1(t_s-), \tag{6.89}$$

$$x_2(t_s+) \equiv x_2(t_s) = x_2(t_s-) = 0, \tag{6.90}$$

which form the initial conditions for the state dynamics in q_2 and $t \in \left[t_s, t_f\right]$, determined from (6.61) and (6.62) as

$$\dot{x}_1 = x_2, \tag{6.91}$$

$$\dot{x}_2 = -\lambda_2 = v_{ref} - x_2\left(t_f\right). \tag{6.92}$$

The above equations have the solution

$$x_1(t) = x_1(t_s) + \frac{1}{2}\left(v_{ref} - x_2\left(t_f\right)\right)(t - t_s)^2, \tag{6.93}$$

$$x_2(t) = \left(v_{ref} - x_2\left(t_f\right)\right)(t - t_s), \tag{6.94}$$

for $t \in \left[t_s, t_f\right]$. Since (6.94) is expressed implicitly in terms of $x_2\left(t_f\right)$, we evaluate (6.94) at t_f to write an explicit form for x_2 as

$$x_2\left(t_f\right) = \left(v_{ref} - x_2\left(t_f\right)\right)\left(t_f - t_s\right), \tag{6.95}$$

which gives

$$x_2\left(t_f\right) = \frac{v_{ref}\left(t_f - t_s\right)}{1 + t_f - t_s}. \tag{6.96}$$

Substitution of (6.96) into (6.93) and (6.94) results in

$$x_1(t) = x_1(t_s) + \frac{v_{ref}}{2\left(1 + t_f - t_s\right)}(t - t_s)^2, \tag{6.97}$$

$$x_2(t) = \frac{v_{ref}}{1 + t_f - t_s}(t - t_s), \tag{6.98}$$

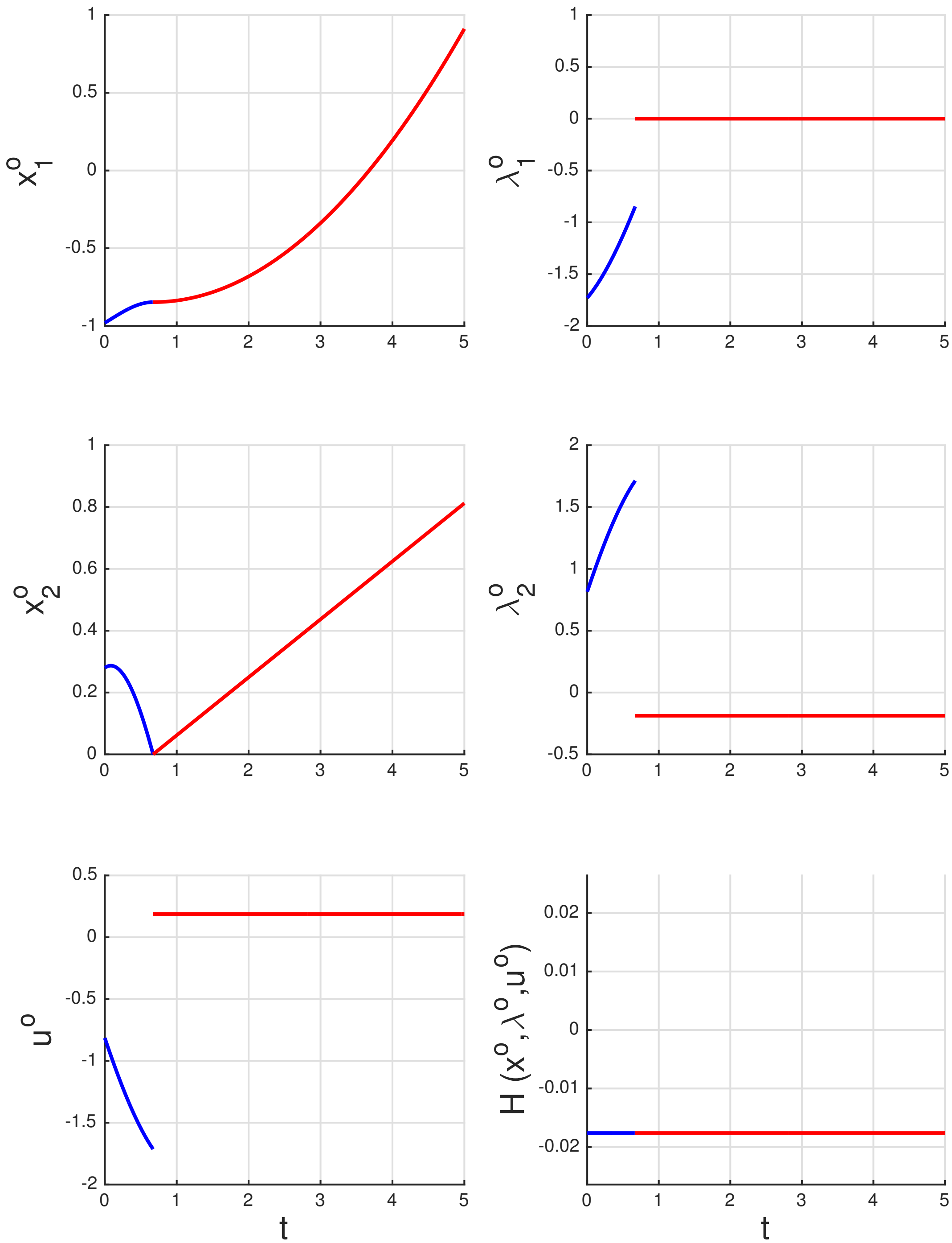

Figure 6.2 The optimal trajectory components x_1^o and x_2^o, the corresponding adjoint process components λ_1^o and λ_2^o, the optimal control input u^o and the corresponding Hamiltonian $H\left(x^o, \lambda^o, u^o\right)$ in Example 6.2 for $t_0 = 0$, $x_{10} = 1$, $x_{20} = -0.5$, $t_f = 5$ and $v_{ref} = 1$

for $t \in \left[t_s, t_f\right]$. This gives the adjoint boundary conditions (6.79) and (6.80) as

$$A\left(1+\frac{t_s}{2}\right)\sin\left(t_s+\alpha\right) = B\sin\left(t_s+\beta\right), \tag{6.99}$$

$$A\cos\left(t_s+\alpha\right) = \frac{v_{ref}}{1+t_f-t_s}+p. \tag{6.100}$$

The switching manifold condition (6.90) states that

$$\frac{-1}{2}At_s\cos\left(t_s+\alpha\right)-\frac{1}{2}A\sin\left(t_s+\alpha\right)+B\cos\left(t_s+\beta\right)=0, \tag{6.101}$$

and the Hamiltonian continuity condition (6.75) gives

$$A\cos\left(t_s+\alpha\right)\left(\frac{-1}{2}At_s\sin\left(t_s+\alpha\right)+B\sin\left(t_s+\beta\right)\right)+\frac{1}{2}A^2\cos^2\left(t_s+\alpha\right)=\frac{1}{2}\left(\frac{v_{ref}}{1+t_f-t_s}\right)^2. \tag{6.102}$$

Hence, by solving simultaneously the set of 6 equations (6.87), (6.88), (6.99), (6.100), (6.101), and (6.102) for the given $t_0 = 0$, $t_f < \infty$, $x(t_0) \equiv [x_{10}, x_{20}]^T$ and v_{ref} the values of the 6 unkown parameters A, α, B, β, t_s and p are determined. For the values of $t_0 = 0$, $x_{10} = 1$, $x_{20} = -0.5$, $t_f = 5$ and $v_{ref} = 1$ the results are demonstrated in Figure 6.2.

6.2.2 The HDP Formulation and Results

For the linear differential equations (6.50) and (6.51), the Hamiltonians for the HJB equation are formed as

$$H_i\left(x, \nabla V, u\right) = \frac{1}{2}u^2 + \nabla V^T\left(A_i x + B_i u\right), \tag{6.103}$$

which have a minimizing control input

$$u^o = -B^T\nabla V = -\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} \frac{\partial V}{\partial x_1} \\ \frac{\partial V}{\partial x_2} \end{bmatrix} = -\frac{\partial V}{\partial x_2}, \tag{6.104}$$

and therefore, the HJB equations are expressed as

$$-\frac{\partial V\left(t, q_2, x, 0\right)}{\partial t} = \frac{-1}{2}\left(\frac{\partial V}{\partial x_2}\right)^2 + x_2\frac{\partial V}{\partial x_1}, \tag{6.105}$$

$$-\frac{\partial V\left(t, q_1, x, 1\right)}{\partial t} = \frac{-1}{2}\left(\frac{\partial V}{\partial x_2}\right)^2 + x_2\frac{\partial V}{\partial x_1} - x_1\frac{\partial V}{\partial x_2}, \tag{6.106}$$

The terminal condition at $t = t_f$ is specified as

$$V\left(t_f, q_2, x, 0\right) = \frac{1}{2}\left(x_2 - v_{ref}\right)^2, \tag{6.107}$$

for $V(t, q_2, x, 0)$, and the boundary condition for $V(t, q_1, x, 1)$ and the switching instant $t = t_s$ are determined by

$$V(t_s, q_1, x, 1) = V(t_s, q_2, x, 0) + \frac{1}{2}x_1^2, \tag{6.108}$$

and

$$\frac{-1}{2}\left(\frac{\partial V_{q_1}}{\partial x_2}\right)^2 + x_2\frac{\partial V_{q_1}}{\partial x_1} - x_1\frac{\partial V_{q_1}}{\partial x_2} = \frac{-1}{2}\left(\frac{\partial V_{q_2}}{\partial x_2}\right)^2 + x_2\frac{\partial V_{q_2}}{\partial x_1}, \tag{6.109}$$

subject to the switching manifold condition (6.52).

6.2.3 The HMP - HDP Relationship

Similar to Example 1, in order to illustrate the result in Theorem 5.2, we take the partial derivatives of (6.105) and (6.106) with respect to x. We note that by the definition of the total derivative,

$$\frac{d}{dt}\left(\frac{\partial V(t, q_i, x, 2-i)}{\partial x}\right) = \frac{\partial^2 V}{\partial t \partial x} + \frac{\partial^2 V}{\partial x^2} f_{q_i}(x, u^o) = \frac{\partial^2 V}{\partial x \partial t} + \frac{\partial^2 V}{\partial x^2} f_{q_i}\left(x, -\frac{\partial V}{\partial x_2}\right), \tag{6.110}$$

which is equivalent to

$$\frac{d}{dt}\begin{bmatrix} \frac{\partial V(t,q_2,x,0)}{\partial x_1} \\ \frac{\partial V(t,q_2,x,0)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 V}{\partial x_1 \partial t} \\ \frac{\partial^2 V}{\partial x_2 \partial t} \end{bmatrix} + \begin{bmatrix} \frac{\partial^2 V}{\partial x_1^2} & \frac{\partial^2 V}{\partial x_1 \partial x_2} \\ \frac{\partial^2 V}{\partial x_2 \partial x_1} & \frac{\partial^2 V}{\partial x_2^2} \end{bmatrix} \begin{bmatrix} x_2 \\ -\frac{\partial V}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 V}{\partial x_1 \partial t} + x_2\frac{\partial^2 V}{\partial x_1^2} - \frac{\partial^2 V}{\partial x_1 \partial x_2}\frac{\partial V}{\partial x_2} \\ \frac{\partial^2 V}{\partial x_2 \partial t} + x_2\frac{\partial^2 V}{\partial x_2 \partial x_1} - \frac{\partial^2 V}{\partial x_2^2}\frac{\partial V}{\partial x_2} \end{bmatrix}, \tag{6.111}$$

for $\nabla V(t, q_2, x, 0)$, and

$$\frac{d}{dt}\begin{bmatrix} \frac{\partial V(t_s,q_1,x,1)}{\partial x_1} \\ \frac{\partial V(t_s,q_1,x,1)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 V}{\partial x_1 \partial t} \\ \frac{\partial^2 V}{\partial x_2 \partial t} \end{bmatrix} + \begin{bmatrix} \frac{\partial^2 V}{\partial x_1^2} & \frac{\partial^2 V}{\partial x_1 \partial x_2} \\ \frac{\partial^2 V}{\partial x_2 \partial x_1} & \frac{\partial^2 V}{\partial x_2^2} \end{bmatrix} \begin{bmatrix} x_2 \\ -x_1 - \frac{\partial V}{\partial x_2} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\partial^2 V}{\partial x_1 \partial t} + x_2\frac{\partial^2 V}{\partial x_1^2} - \frac{\partial^2 V}{\partial x_1 \partial x_2}\frac{\partial V}{\partial x_2} - x_1\frac{\partial^2 V}{\partial x_1 \partial x_2} \\ \frac{\partial^2 V}{\partial x_2 \partial t} + x_2\frac{\partial^2 V}{\partial x_2 \partial x_1} - \frac{\partial^2 V}{\partial x_2^2}\frac{\partial V}{\partial x_2} - x_1\frac{\partial^2 V}{\partial x_2^2} \end{bmatrix}, \tag{6.112}$$

for $\nabla V(t_s, q_1, x, 1)$. Taking the partial derivatives of (6.105) with respect to x and a substitution of the terms introduced by (6.111), gives

$$\frac{d}{dt}\left(\frac{\partial V(t, q_2, x, 0)}{\partial x_1}\right) = 0, \tag{6.113}$$
$$\frac{d}{dt}\left(\frac{\partial V(t, q_2, x, 0)}{\partial x_2}\right) = -\frac{\partial V(t, q_2, x, 0)}{\partial x_1}, \tag{6.114}$$

which are equivalent to the differential equations (6.63) and (6.64) for $\lambda(t), t \in \left(t_s, t_f\right]$. Similarly, the (partial) differentiation of (6.105) with respect to x results in

$$\frac{d}{dt}\left(\frac{\partial V(t, q_1, x, 1)}{\partial x_1}\right) = \frac{\partial V(t, q_1, x, 1)}{\partial x_2}, \tag{6.115}$$
$$\frac{d}{dt}\left(\frac{\partial V(t, q_1, x, 1)}{\partial x_2}\right) = -\frac{\partial V(t, q_1, x, 1)}{\partial x_1}, \tag{6.116}$$

which are equivalent to the differential equations (6.59) and (6.60) for $\lambda(t), t \in (t_0, t_s]$. Moreover, it can easily be verified that the optimal sensitivity process ∇V satisfies the terminal condition

$$\nabla V\left(t_f, q_2, x, 0\right) = \begin{bmatrix} \frac{\partial V\left(t_f, q_2, x, 0\right)}{\partial x_1} \\ \frac{\partial V\left(t_f, q_2, x, 0\right)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 \\ x_2\left(t_f\right) - v_{ref} \end{bmatrix}, \tag{6.117}$$

and the boundary condition

$$\begin{bmatrix} \frac{\partial V(t_s, q_1, x, 1)}{\partial x_1} \\ \frac{\partial V(t_s, q_1, x, 1)}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial V(t_s, q_2, x, 0)}{\partial x_1} \\ \frac{\partial V(t_s, q_2, x, 0)}{\partial x_2} \end{bmatrix} + \begin{bmatrix} \frac{\partial c(x(t_s-))}{\partial x_1} \\ \frac{\partial c(x(t_s-))}{\partial x_2} \end{bmatrix} + p \begin{bmatrix} \frac{\partial m(x(t_s-))}{\partial x_1} \\ \frac{\partial m(x(t_s-))}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial V(t_s, q_2, x, 0)}{\partial x_1} + x_1(t_s-) \\ \frac{\partial V(t_s, q_2, x, 0)}{\partial x_2} + p \end{bmatrix}, \tag{6.118}$$

subject to $x_2(t_s-) = 0$. Therefore, by the uniqueness of the results of the set of governing differential equations for ∇V and λ which are subject to the same terminal and boundary conditions, along any optimal trajectory, the gradient of the value function is equal to the adjoint process corresponding to the same optimal trajectory. Interested readers are referred to [96] for further discussion on this example. □

6.3 Linear Dynamics and Quadratic Costs in $\{q_1, q_2\} \times \mathbb{R}^4$ with Codimension 2 Switching Manifold

Consider the following mechanical system with two point masses m_1 and m_2 each one attached to separate spring and damper with the configuration depicted in Figure 6.3. The spring and the damper attached to the mass m_1 have the stiffness and damping coefficients k_1 and c_1 respectively and apply forces to m_1 in the direction of the x axis and the spring and the damper attached to the mass m_2 have the stiffness and damping coefficients k_2 and c_2 respectively and apply forces to m_2 in the direction of the y axis. The neutral positions for the springs k_1 and k_2 have the coordinates $(d_1, 0)$ and $(0, d_2)$ respectively in the coordinate system shown in Figure 6.3. Denoting $x_1 := x$, $x_2 := \dot{x}$, $x_3 := y$ and $x_4 := \dot{y}$ the dynamics of the system is described as

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{k_1}{m_1}x_1 - \frac{c_1}{m_1}x_2 + \frac{1}{m_1}u_1 + \frac{k_1}{m_1}d_1 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -\frac{k_2}{m_2}x_3 - \frac{c_2}{m_2}x_4 + \frac{1}{m_2}u_2 + \frac{k_2}{m_2}d_2
\end{aligned} \tag{6.119}$$

which has the matrix representation

$$\dot{x} = A_1 x + B_1 u + D_1 \tag{6.120}$$

with

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_1}{m_1} & \frac{-c_1}{m_1} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-k_2}{m_2} & \frac{-c_2}{m_2} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 \\ \frac{k_1}{m_1}d_1 \\ 0 \\ \frac{k_2}{m_2}d_2 \end{bmatrix} \tag{6.121}$$

When both masses pass through the origin at the same time a collision occurs. Denoting the time of the collision by t_s this incident corresponds to a switching manifold in the form of a codimension 2 submanifold of $\mathbb{R}^4$ described by

$$m : \{x_1(t_s-) = 0 \wedge x_3(t_s-) = 0\} \tag{6.122}$$

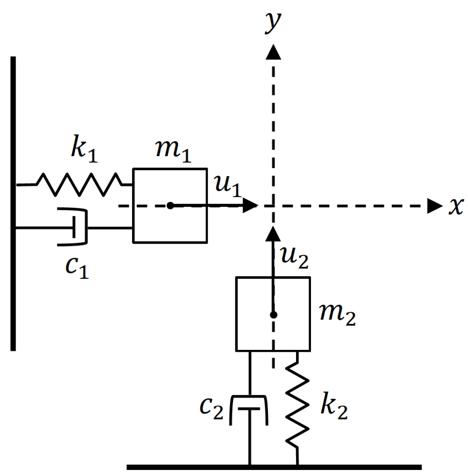

Figure 6.3 The system studied in Example 6.3

Consider a completely plastic collision in which the masses attach to each other and hence, the speeds after the collision determined by the law of conservation of linear momentum are related to speeds before the collision by

$$\begin{aligned} (m_1+m_2)\,v_x(t_s+) &\equiv (m_1+m_2)\,v_x(t_s) = m_1 v_{1x}(t_s-) \\ (m_1+m_2)\,v_y(t_s+) &\equiv (m_1+m_2)\,v_y(t_s) = m_2 v_{2y}(t_s-) \end{aligned} \tag{6.123}$$

that determines the corresponding autonomous jump map as

$$\begin{bmatrix} x_1(t_s) \\ x_2(t_s) \\ x_3(t_s) \\ x_4(t_s) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{m_1}{m_1+m_2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{m_2}{m_1+m_2} \end{bmatrix} \begin{bmatrix} x_1(t_s-) \\ x_2(t_s-) \\ x_3(t_s-) \\ x_4(t_s-) \end{bmatrix} \tag{6.124}$$

Assuming decoupled stiffness and damping in the two directions (see e.g. [97, 98]) the dynamics of the system after the collision is described by

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{k_1}{m_1+m_2}x_1 - \frac{c_1}{m_1+m_2}x_2 + \frac{1}{m_1+m_2}u_1 + \frac{k_1}{m_1+m_2}d_1 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= -\frac{k_2}{m_1+m_2}x_3 - \frac{c_2}{m_1+m_2}x_4 + \frac{1}{m_1+m_2}u_2 + \frac{k_2}{m_1+m_2}d_2
\end{aligned} \tag{6.125}$$

which has the matrix representation

$$\dot{x} = A_2 x + B_2 u + D_2 \tag{6.126}$$

with

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_1}{m_1+m_2} & \frac{-c_1}{m_1+m_2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-k_2}{m_1+m_2} & \frac{-c_2}{m_1+m_2} \end{bmatrix}, \ B_2 = \begin{bmatrix} 0 & 0 \\ \frac{1}{m_1+m_2} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_1+m_2} \end{bmatrix}, \ D_2 = \begin{bmatrix} 0 \\ \frac{k_1}{m_1+m_2}d_1 \\ 0 \\ \frac{k_2}{m_1+m_2}d_2 \end{bmatrix} \tag{6.127}$$

For the hybrid system described above consider the optimal control problem

$$J(x_0, T, u) = \int_0^T l(x,u)\,dt + c(x(t_s-)) + g(x(T)) \tag{6.128}$$

with the running costs

$$l_1(x,u) = l_2(x,u) \equiv l(x,u) = \frac{1}{2}\left(u_1^2 + u_2^2\right) = \frac{1}{2}u^T u \tag{6.129}$$

Take the switching cost as the kinetic energy just before switching (i.e. collision) which is

$$c(x(t_s-)) = \frac{1}{2}m_1 (x_2(t_s-))^2 + \frac{1}{2}m_2 (x_4(t_s-))^2 \tag{6.130}$$

and assume that the terminal cost penalizes the total energy at the final time T, i.e.

$$g(x(T)) = \frac{1}{2}(m_1+m_2)(x_2(T))^2 + \frac{1}{2}(m_1+m_2)(x_4(T))^2 + \frac{1}{2}k_1(x_1(T)-d_1)^2 + \frac{1}{2}k_2(x_3(T)-d_2)^2 \quad (6.131)$$

Consequently, the hybrid optimal control problem is defined as finding the minimum of J in (6.128) and the corresponding minimizing control inputs for the given system.

6.3.1 The HMP Results

Employing Theorem 1, the Hamiltonian is formed as

$$H_i(x,\lambda,u) = \lambda^T(A_i x + B_i u + D_i) + \frac{1}{2}u^T u \quad (6.132)$$

The Hamiltonian minimization condition (3.8) gives

$$\frac{\partial H_i}{\partial u} = 0 \Rightarrow u^o = -B_i^T \lambda^o \quad (6.133)$$

and hence, from (3.2) and (3.3)

$$\dot{x}^o = A_i x^o - B_i B_i^T \lambda^o + D_i \quad (6.134)$$

$$\dot{\lambda}^o = -A_i^T \lambda^o \quad (6.135)$$

with the initial condition for x^o given as

$$x^o(0) = x_0 \quad (6.136)$$

and its boundary condition (3.5) given as

$$x(t_s) = Px(t_s-) \quad (6.137)$$

where P is defined from (6.124) as

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{m_1}{m_1+m_2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{m_2}{m_1+m_2} \end{bmatrix} \tag{6.138}$$

The terminal condition for λ^o is given from (3.6) as

$$\lambda^o(T) = \nabla g(x(T)) = G\left(x - r_f\right) \tag{6.139}$$

with G and r_f determined from (6.131) as

$$G = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ 0 & m_1+m_2 & 0 & 0 \\ 0 & 0 & k_2 & 0 \\ 0 & 0 & 0 & m_1+m_2 \end{bmatrix}, \quad r_f = \begin{bmatrix} d_1 \\ 0 \\ d_2 \\ 0 \end{bmatrix} \tag{6.140}$$

The boundary condition for λ^o is determined by (3.101) as

$$\lambda^o\left(t_j-\right) \equiv \lambda^o\left(t_j\right) = P^T\lambda^o\left(t_j+\right) + p\hat{n}_m + Cx \tag{6.141}$$

with C defined from (6.130) as

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_2 \end{bmatrix} \tag{6.142}$$

and $\hat{n}_m$ determined from (3.102) as

$$\hat{n}_m \parallel \underset{\text{span}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right\}}{\text{PROJ}} \left\{A_1x^o(t_s-) - B_1B_1^T\lambda^o(t_s-) + D_1\right\} = \begin{bmatrix} x_2(t_s-) \\ 0 \\ x_4(t_s-) \\ 0 \end{bmatrix} \tag{6.143}$$

Taking $\hat{n}_m$ equal to its defining vector in (6.143), the boundary condition (6.141) becomes

$$\begin{bmatrix} \lambda_1^o(t_s) \\ \lambda_2^o(t_s) \\ \lambda_3^o(t_s) \\ \lambda_4^o(t_s) \end{bmatrix} = \begin{bmatrix} \lambda_1^o(t_s+) + px_2(t_s-) \\ \frac{m_1}{m_1+m_2}\lambda_2^o(t_s+) + m_1x_2(t_s-) \\ \lambda_3^o(t_s+) + px_4(t_s-) \\ \frac{m_2}{m_1+m_2}\lambda_4^o(t_s+) + m_2x_4(t_s-) \end{bmatrix} \tag{6.144}$$

The scalar parameter p and the switching time t_s together with the optimal trajectory and its corresponding adjoint process are determined by solving the differential equations (6.134) and (6.135) subject to the initial, terminal and boundary conditions (6.136), (6.137), (6.139) and (6.144) together with the Hamiltonian continuity condition from (3.9) as

$$\begin{aligned} &\lambda^{oT}_{(t_s+)}\left[A_2x^o_{(t_s+)} - B_2B_2^T\lambda^o_{(t_s+)} + D_2\right] + \frac{1}{2}\lambda^{oT}_{(t_s+)}B_1B_1^T\lambda^o_{(t_s+)} \\ &\qquad = \lambda^{oT}_{(t_s-)}\left[A_1x^o_{(t_s-)} - B_1B_1^T\lambda^o_{(t_s-)} + D_1\right] + \frac{1}{2}\lambda^{oT}_{(t_s-)}B_1B_1^T\lambda^o_{(t_s-)} \end{aligned} \tag{6.145}$$

or

$$\lambda^{oT}_{(t_s+)}\left[A_2x^o_{(t_s+)} - \frac{1}{2}B_2B_2^T\lambda^o_{(t_s+)} + D_2\right] = \lambda^{oT}_{(t_s-)}\left[A_1x^o_{(t_s-)} - \frac{1}{2}B_1B_1^T\lambda^o_{(t_s-)} + D_1\right] \tag{6.146}$$

The results for the parameter values $m_1 = m_2 = 1$, $k_1 = k_2 = 1$, $c_1 = c_2 = 1$, $d_1 = d_2 = 0.1$, the initial condition $x_0 = \begin{bmatrix} -0.25 & 0 & -0.15 & 0 \end{bmatrix}^T$ and the terminal time $T = 4$ are demonstrated in Figure 6.3.1.

6.3.2 HDP Results from their Relation to the HMP Results

Employing Theorem 3 and the results of Theorem 1 established in the previous part, we find the value function satisfying the necessary conditions in Theorem 2. To this end we rewrite equations (6.134) and (6.135) in the matrix form

$$\begin{bmatrix} \dot{x}^o \\ \dot{\lambda}^o \end{bmatrix} = \begin{bmatrix} A_i & -B_iB_i^T \\ 0 & -A_i^T \end{bmatrix} \begin{bmatrix} x^o \\ \lambda^o \end{bmatrix} + \begin{bmatrix} D_i \\ 0 \end{bmatrix} \tag{6.147}$$

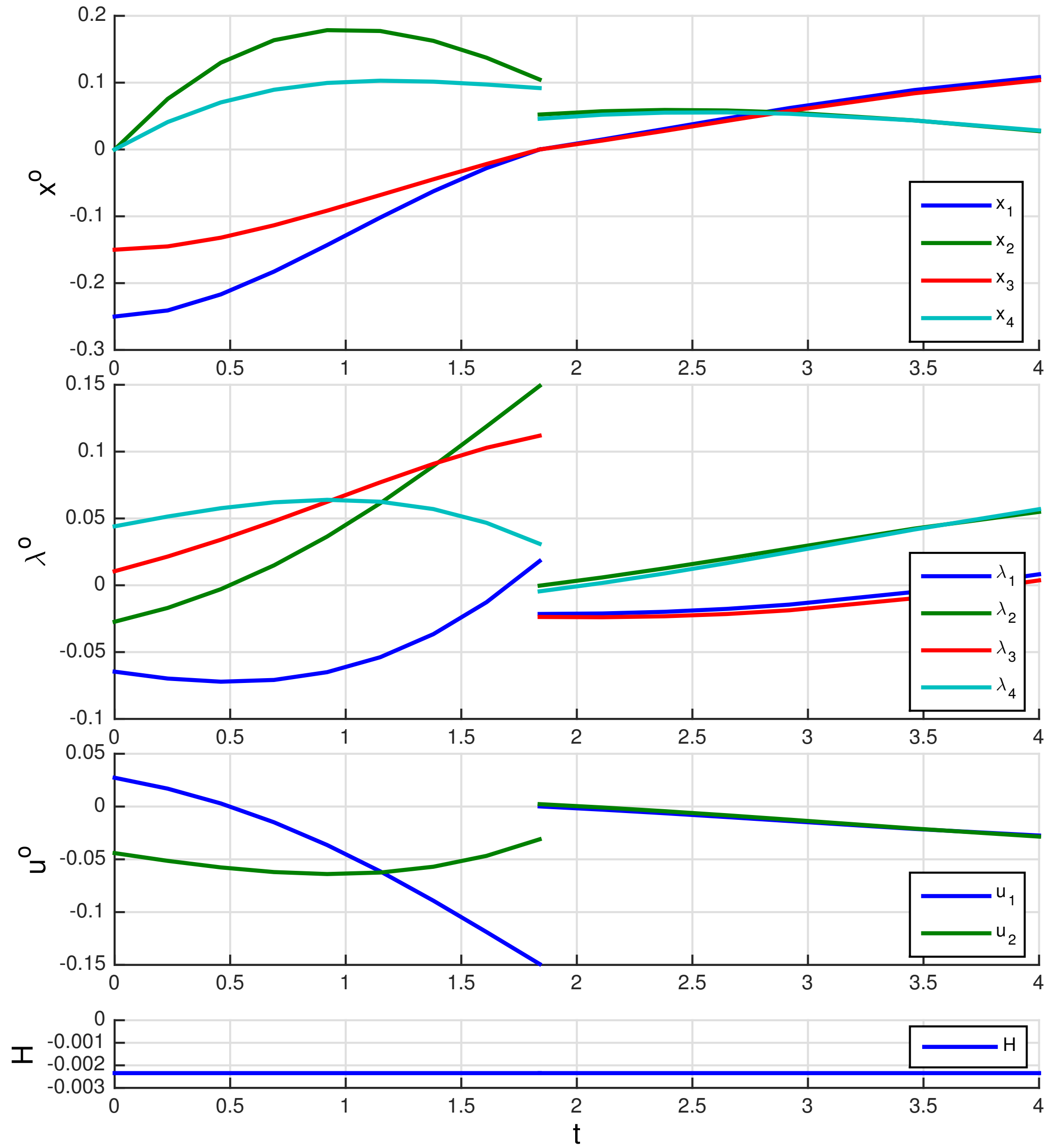

Figure 6.4 The optimal trajectory components x_1^o, x_2^o, x_3^o, and x_4^o, the corresponding adjoint process components λ_1^o, λ_2^o, λ_3^o, and λ_4^o, the optimal control input components u_1^o and u_2^o, and the corresponding Hamiltonian $H\left(x^o, \lambda^o, u^o\right)$ in Example 6.3 for the parameter values $m_1 = m_2 = 1$, $k_1 = k_2 = 1$, $c_1 = c_2 = 1$, $d_1 = d_2 = 0.1$, the initial condition $x_0 = [-0.25, 0, -0.15, 0]^T$ and the terminal time $T = 4$.

and denote its state transition matrix by ϕ_i. Then the solution of (6.147) for $t \in (t_s, T]$ can be written as

$$\begin{bmatrix} x^o(t) \\ \lambda^o(t) \end{bmatrix} = \phi_2(t, t_s) \begin{bmatrix} x^o(t_s) \\ \lambda^o(t_s+) \end{bmatrix} + \int_{t_s}^{t} \phi_2(t, \tau) \begin{bmatrix} D_2(\tau) \\ 0 \end{bmatrix} d\tau \tag{6.148}$$

and also as

$$\begin{bmatrix} x^o(T) \\ \lambda^o(T) \end{bmatrix} = \phi_2(T, t) \begin{bmatrix} x^o(t) \\ \lambda^o(t) \end{bmatrix} + \int_{t}^{T} \phi_2(T, \tau) \begin{bmatrix} D_2(\tau) \\ 0 \end{bmatrix} d\tau \tag{6.149}$$

Partitioning ϕ in the form of

$$\phi_2(T, t) = \begin{bmatrix} \phi_{2,11}(T, t) & \phi_{2,12}(T, t) \\ \phi_{2,21}(T, t) & \phi_{2,22}(T, t) \end{bmatrix} \tag{6.150}$$

and denoting

$$\begin{bmatrix} f_{d2,1}(t) \\ f_{d2,2}(t) \end{bmatrix} := \int_{t}^{T} \begin{bmatrix} \phi_{2,11}(T, t) & \phi_{2,12}(T, t) \\ \phi_{2,21}(T, t) & \phi_{2,22}(T, t) \end{bmatrix} \begin{bmatrix} D_2(\tau) \\ 0 \end{bmatrix} d\tau \tag{6.151}$$

we can rewrite (6.149) as

$$x^o(T) = \phi_{2,11}(T, t)\, x^o(t) + \phi_{2,12}(T, t)\, \lambda^o(t) + f_{d2,1}(t) \tag{6.152}$$

$$\lambda^o(T) = \phi_{2,21}(T, t)\, x^o(t) + \phi_{2,22}(T, t)\, \lambda^o(t) + f_{d2,2}(t) \tag{6.153}$$

Substituting $x^o(T)$ and $\lambda^o(T)$ from (6.152) and (6.153) into (6.139) gives

$$\begin{aligned} &G\left(\phi_{2,11}(T, t)\, x^o(t) + \phi_{2,12}(T, t)\, \lambda^o(t) + f_{d2,1}(t) - r_f\right) \\ &\qquad = \phi_{2,21}(T, t)\, x^o(t) + \phi_{2,22}(T, t)\, \lambda^o(t) + f_{d2,2}(t) \end{aligned} \tag{6.154}$$

or

$$\begin{aligned} &[G\phi_{2,11}(T, t) - \phi_{2,21}(T, t)]\, x^o(t) + G f_{d2,1}(t) - G r_f - f_{d2,2}(t) \\ &\qquad = [\phi_{2,22}(T, t) - G\phi_{2,12}(T, t)]\, \lambda^o(t) \end{aligned} \tag{6.155}$$

that gives

$$\begin{aligned}\lambda^{o}(t) = {} & [\phi_{2,22}(T,t) - G\phi_{2,12}(T,t)]^{-1}[G\phi_{2,11}(T,t) - \phi_{2,21}(T,t)]x^{o}(t) \\ & + [\phi_{2,22}(T,t) - G\phi_{2,12}(T,t)]^{-1}\left[Gf_{d2,1}(t) - Gr_f - f_{d2,2}(t)\right]\end{aligned} \quad (6.156)$$

The existence of the inverse in the previous equation is provided by a theorem of Kalman [99]. Defining

$$K_2(t) := [\phi_{2,22}(T,t) - G\phi_{2,12}(T,t)]^{-1}[G\phi_{2,11}(T,t) - \phi_{2,21}(T,t)] \quad (6.157)$$

and

$$s_2(t) := [\phi_{2,22}(T,t) - G\phi_{2,12}(T,t)]^{-1}\left[Gf_{d2,1}(t) - Gr_f - f_{d2,2}(t)\right] \quad (6.158)$$

the equation (6.156) is expressed as

$$\lambda^{o}(t) = K_2(t)x^{o}(t) + s_2(t), \qquad t \in (t_s, T] \quad (6.159)$$

with

$$K_2(T) = G \quad (6.160)$$

$$s_2(T) = -Gr_f \quad (6.161)$$

In particular, for the right limit at t_s we have

$$\lambda^{o}(t_s+) = K_2(t_s)x^{o}(t_s) + s_2(t_s) \quad (6.162)$$

Similarly, for the solution of (6.147) for $t \in [0, t_s)$ we have

$$x^{o}(t_s-) = \phi_{1,11}(t_s,t)x^{o}(t) + \phi_{1,12}(t_s,t)\lambda^{o}(t) + f_{d1,1}(t) \quad (6.163)$$

$$\lambda^{o}(t_s) = \phi_{1,21}(t_s,t)x^{o}(t) + \phi_{1,22}(t_s,t)\lambda^{o}(t) + f_{d1,2}(t) \quad (6.164)$$

with the definition of $f_{d1,1}(t)$ and $f_{d1,2}(t)$ for $t \in [0, t_s)$ being

$$\begin{bmatrix} f_{d1,1}(t) \\ f_{d1,2}(t) \end{bmatrix} := \int_t^{t_s} \begin{bmatrix} \phi_{1,11}(t_s,\tau) & \phi_{1,12}(t_s,\tau) \\ \phi_{1,21}(t_s,\tau) & \phi_{1,22}(t_s,\tau) \end{bmatrix} \begin{bmatrix} D_1(\tau) \\ 0 \end{bmatrix} d\tau \quad (6.165)$$

Using (6.137) and the boundary condition (6.141) we may write

$$\lambda^o(t_s) = P^T\lambda^o(t_s+) + p\hat{n}_m + Cx^o(t_s-)$$
$$= P^T\left[K_2(t_s)x^o(t_s) + s_2(t_s)\right] + p\hat{n}_m + Cx^o(t_s-)$$
$$= \left[P^T K_2(t_s)P + C\right]x^o(t_s-) + P^T s_2(t_s) + p\hat{n}_m \quad (6.166)$$

Substituting $x^o(t_s-)$ and $\lambda^o(t_s)$ from equations (6.163) and (6.164) we get

$$\phi_{1,21}(t_s,t)x^o(t) + \phi_{1,22}(t_s,t)\lambda^o(t) + f_{d1,2}(t) =$$
$$\left[P^T K_2(t_s)P + C\right]\left[\phi_{1,11}(t_s,t)x^o(t) + \phi_{1,12}(t_s,t)\lambda^o(t) + f_{d1,1}(t)\right] + P^T s_2(t_s) + p\hat{n}_m \quad (6.167)$$

or

$$\left[\phi_{1,22}(t_s,t) - \left[P^T K_2(t_s)P + C\right]\phi_{1,12}(t_s,t)\right]\lambda^o(t)$$
$$= \left(\left[P^T K_2(t_s)P + C\right]\phi_{1,11}(t_s,t) - \phi_{1,21}(t_s,t)\right)x^o(t)$$
$$+ \left[P^T K_2(t_s)P + C\right]f_{d1,1}(t) - f_{d1,2}(t) + P^T s_2(t_s) + p\hat{n}_m \quad (6.168)$$

With the definition of

$$K_1(t) := \left[\phi_{1,22}(t_s,t) - \left[P^T K_2(t_s)P + C\right]\phi_{1,12}(t_s,t)\right]^{-1}\left(\left[P^T K_2(t_s)P + C\right]\phi_{1,11}(t_s,t) - \phi_{1,21}(t_s,t)\right) \quad (6.169)$$

and

$$s_1(t) := \left[\phi_{1,22}(t_s,t) - \left[P^T K_2(t_s)P + C\right]\phi_{1,12}(t_s,t)\right]^{-1}$$
$$\left(\left[P^T K_2(t_s)P + C\right]f_{d1,1}(t) - f_{d1,2}(t) + P^T s_2(t_s) + p\hat{n}_m\right) \quad (6.170)$$

it is concluded that

$$\lambda^o(t) = K_1(t)x^o(t) + s_1(t), \qquad t \in [0, t_s) \quad (6.171)$$

Note that the following relations hold by the definitions of $K_i(t)$ and $s_i(t)$:

$$K_1(t_s) = P^T K_2(t_s)P + C \quad (6.172)$$
$$s_1(t_s) = P^T s_2(t_s) + p\hat{n}_m \quad (6.173)$$

Taking the time derivative of (6.159) and (6.171) it can be shown that

$$\dot{K}_i = K_i B_i B_i^T K_i - K_i A_i - A_i^T K_i \quad (6.174)$$

$$\dot{s}_i = -\left(A_i^T - K_i B_i B_i^T\right) s_i - K_i D_i \quad (6.175)$$

From equation (5.62) and the result of Theorem 3 the gradient of the value function is equal to the adjoint process and hence

$$V(t, q_2, x, 0) = \frac{1}{2} x^T K_2(t) x + s_2(t)^T x + \alpha_2(t) \quad (6.176)$$

where from Theorem 2 and the terminal condition (4.20), $\alpha_2(T)$ should satisfy

$$\alpha_2(T) = \frac{1}{2} D_2^T D_2 \quad (6.177)$$

From Theorem 2 and the HJB equation (4.19) we must have

$$\frac{1}{2} x^T \dot{K}_2 x + \dot{s}_2{}^T x + \dot{\alpha}_2 + \frac{1}{2}(K_2 x + s_2)^T B_2 B_2^T (K_2 x + s_2) + (K_2 x + s_2)^T \left(A_2 x - B_2 B_2^T [K_2 x + s_2] + D_2\right) = 0 \quad (6.178)$$

which results in

$$\frac{1}{2} x^T \left(\dot{K}_2 + K_2 A_2 + A_2^T K_2 - K_2 B_2 B_2^T K_2\right) x + \left(\dot{s}_2 + A_2^T s_2 - K_2 B_2 B_2^T s_2 + K_2 D_2\right)^T x + \dot{\alpha}_2 - \frac{1}{2} s_2^T B_2 B_2^T s_2 + s_2^T D_2 = 0 \quad (6.179)$$

and hence (see also (6.174) and (6.175))

$$\dot{\alpha}_2 = \frac{1}{2} s_2^T B_2 B_2^T s_2 - s_2^T D_2, \qquad t \in (t_s, T] \quad (6.180)$$

Similarly

$$V(t, q_1, x, 1) = \frac{1}{2} x^T K_1(t) x + s_1(t)^T x + \alpha_1(t) \quad (6.181)$$

concludes that

$$\dot{\alpha}_1 = \frac{1}{2} s_1^T B_1 B_1^T s_1 - s_1^T D_1, \qquad t \in [0, t_s) \tag{6.182}$$

which, together with (6.180), gives

$$\dot{\alpha}_i = \frac{1}{2} s_i^T B_i B_i^T s_i - s_i^T D_i \tag{6.183}$$

For determining the boundary condition for $\alpha(t)$ at t_s we consider the boundary condition (4.21) for V that states

$$V(t_s-, q_1, x, 1) = V(t_s+, q_2, Px, 0) + \frac{1}{2} x^T C x \tag{6.184}$$

i.e.

$$\begin{aligned} &\frac{1}{2} x^T K_1(t_s-) x + s_1(t_s-)^T x + \alpha_1(t_s-) \\ &\qquad = \frac{1}{2} x^T \left[P^T K_2(t_s+) P + C \right] x + s_2(t_s+)^T P x + \alpha_2(t_s+) \end{aligned} \tag{6.185}$$

From the boundary conditions for K_i and s_i in (6.172) and (6.173) we get

$$\alpha_1(t_s-) + p \hat{n}_m^T x = \alpha_2(t_s+) \tag{6.186}$$

but since for all $x \in \{x : m(x) = 0\}$

$$\hat{n}_m^T x = \begin{bmatrix} x_2(t_s-) & 0 & x_4(t_s-) & 0 \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = 0 \tag{6.187}$$

the boundary condition for $\alpha(t)$ at t_s becomes

$$\alpha_1(t_s-) \equiv \alpha_1(t_s) = \alpha_2(t_s) \equiv \alpha_2(t_s+) \tag{6.188}$$

Hence, the value function is constructed in the form of equations (6.176) and (6.181) where K_i, s_i and α_i are respectively the solutions of (6.174), (6.175), (6.183) with the terminal conditions (6.160), (6.161), (6.177) and the boundary conditions (6.172), (6.173) and (6.188). □

6.4 Riccati Formalism for Linear – Quadratic Tracking Problems

Consider a hybrid system possessing linear vector fields in the form of

$$\dot{x} = A_{q_i}(t)\,x + B_{q_i}(t)\,u + F_{q_i}(t)\,, \qquad t \in [t_i, t_{i+1}) \tag{6.189}$$

with a given initial condition $(q,x)(t_0) = (q_0, x_0)$ and the jump maps

$$x(t_j) = P_{\sigma_j} x(t_j-) + J_{\sigma_j}, \tag{6.190}$$

provided at the switching instances $t_j, 1 \le j \le L$ which are not a priori fixed. If t_j corresponds to an autonomous switching from q_{j-1} to q_j, the switching manifold constraint $m_{q_{j-1}q_j} x(t_j-) + n_{q_{j-1}q_j} = 0$ is satisfied. A controlled switching instant t_j, in contrast, is a direct consequence of the discrete control input switching command. Consider the HOCP

$$\begin{aligned} J = \sum_{i=0}^{L} \int_{t_i}^{t_{i+1}} \frac{1}{2}\left(x - r_{q_i}(t)\right)^T L_{q_i}(t)\left(x - r_{q_i}(t)\right) + \frac{1}{2} u^T R_{q_i}(t)\, u\, dt \\ + \sum_{j=1}^{L} \frac{1}{2}\left(x(t_j-) - r_{q_{j-1}}(t_j-)\right)^T C_{\sigma_j}\left(x(t_j-) - r_{q_{j-1}}(t_j-)\right) \\ + \frac{1}{2}\left(x(t_f) - r_{q_L}(t_f)\right)^T G_{q_L}\left(x(t_f) - r_{q_L}(t_f)\right), \end{aligned} \tag{6.191}$$

where $L_{q_i}^T = L_{q_i} \ge 0$, $R_{q_i}^T = R_{q_i} > 0$, $C_{\sigma_j}^T = C_{\sigma_j} \ge 0$, $G_{q_L}^T = G_{q_L} \ge 0$. For the ease of notation and unless otherwise states, the time varying, continuously differentiable matrices $A_q(t)$, $B_q(t)$, $C_q(t)$, $L_q(t)$ ans $R_q(t)$ are simply denoted by A_q, B_q, C_q, L_q and R_q.

Starting with the Hybrid Minimum Principle, the Hamiltonians are formed as

$$H_i = \frac{1}{2}\left(x - r_{q_i}(t)\right)^T L_{q_i}(t)\left(x - r_{q_i}(t)\right) + \frac{1}{2} u^T R_{q_i}(t)\, u + \lambda^T \left(A_{q_i}(t)\,x + B_{q_i}(t)\,u + F_{q_i}(t)\right) \tag{6.192}$$

Based on the HMP Theorem, the Hamiltonian minimization gives

$$\frac{\partial H}{\partial u} = 0 \;\Rightarrow\; R_{q_i} u + B_{q_i}^T \lambda = 0 \;\Rightarrow\; u^0 = -R_{q_i}^{-1} B_{q_i}^T \lambda \tag{6.193}$$

and hence

$$\dot{x}^o = \frac{\partial H}{\partial \lambda} = A_{q_i} x^o + B_{q_i} u^o + F_{q_i} = A_{q_i} x^o - B_{q_i} R_{q_i}^{-1} B_{q_i}^T \lambda^o + F_{q_i}, \tag{6.194}$$

$$\dot{\lambda}^o = -\frac{\partial H}{\partial x} = -L_{q_i}(x^o - r(t)) - A_{q_i}^T \lambda^o = -L_{q_i} x^o - A_{q_i}^T \lambda^o + L_{q_i} r_{q_i}(t), \tag{6.195}$$

which has a matrix representation

$$\begin{bmatrix} \dot{x}^o \\ \dot{\lambda}^o \end{bmatrix} = \begin{bmatrix} A_{q_i} & -B_{q_i} R_{q_i}^{-1} B_{q_i}^T \\ -L_{q_i} & -A_{q_i}^T \end{bmatrix} \begin{bmatrix} x^o \\ \lambda^o \end{bmatrix} + \begin{bmatrix} F_{q_i} \\ L_{q_i} r_{q_i}(t) \end{bmatrix} \tag{6.196}$$

for $t \in [t_i, t_{i+1})$, subject to the boundary conditions

$$x^o(t_0) = x_0 \tag{6.197}$$

$$x^o(t_j) = P_{\sigma_j} x^o(t_j-) + J_{\sigma_j} \tag{6.198}$$

$$\lambda^o(t_f) = \nabla g = G_{q_L}\left(x^o(t_f) - r_{q_L}(t_f)\right) \tag{6.199}$$

$$\lambda^o(t_j) = P_{\sigma_j}^T \lambda^o(t_j+) + p m_{q_{j-1} q_j} + C_{\sigma_j}\left(x^o_{(t_j-)} - r_{q_{j-1}(t_j-)}\right) \tag{6.200}$$

Denoting the state transition matrix for the system in Eq. (6.196) by ϕ, the solution of (6.196) can be written as

$$\begin{bmatrix} x^o(t) \\ \lambda^o(t) \end{bmatrix} = \phi(t, t_i) \begin{bmatrix} x^o(t_i) \\ \lambda^o(t_i+) \end{bmatrix} + \int_{t_i}^{t} \phi(t, \tau) \begin{bmatrix} F_{q_i}(\tau) \\ L_{q_i}(\tau) r_{q_i}(\tau) \end{bmatrix} d\tau \tag{6.201}$$

and also as

$$\begin{bmatrix} x^o(t_{i+1}-) \\ \lambda^o(t_{i+1}-) \end{bmatrix} = \phi(t_{i+1}, t) \begin{bmatrix} x^o(t) \\ \lambda^o(t) \end{bmatrix} + \int_{t}^{t_{i+1}} \phi(t_{i+1}, \tau) \begin{bmatrix} F_{q_i}(\tau) \\ L_{q_i}(\tau) r_{q_i}(\tau) \end{bmatrix} d\tau \tag{6.202}$$

Partitioning ϕ in the form of

$$\phi(t_{i+1}, t) = \begin{bmatrix} \phi_{11}(t_{i+1}, t) & \phi_{12}(t_{i+1}, t) \\ \phi_{21}(t_{i+1}, t) & \phi_{22}(t_{i+1}, t) \end{bmatrix} \tag{6.203}$$

and denoting

$$\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix} := \int_{t_i}^{t} \begin{bmatrix} \phi_{11}(t_{i+1}, t) & \phi_{12}(t_{i+1}, t) \\ \phi_{21}(t_{i+1}, t) & \phi_{22}(t_{i+1}, t) \end{bmatrix} \begin{bmatrix} F_{q_i}(\tau) \\ L_{q_i}(\tau) r_{q_i}(\tau) \end{bmatrix} d\tau \tag{6.204}$$

give the Eq. (6.202) as

$$\begin{aligned} x^o(t_{i+1}-) &= \phi_{11}(t_{i+1},t)\,x^o(t) + \phi_{12}(t_{i+1},t)\,\lambda^o(t) + f_1(t) \\ \lambda^o(t_{i+1}-) &= \phi_{21}(t_{i+1},t)\,x^o(t) + \phi_{22}(t_{i+1},t)\,\lambda^o(t) + f_2(t) \end{aligned} \tag{6.205}$$

In the location q_L with $t \in [t_L, t_{L+1}] =: [t_L, t_f]$ the terminal condition for λ^o is provided as

$$\lambda^o(t_f) = G_{q_L}\left(x^o(t_f) - r_{q_L}(t_f)\right) \tag{6.206}$$

Replacing $\lambda^o(t_f)$ from the above equation in the second equation in (6.205) and substituting $x^o(t_f)$ from the first equation in (6.205) result in

$$\begin{aligned} &G_{q_L}\left(\phi_{11}(t_f,t)\,x^o(t) + \phi_{12}(t_f,t)\,\lambda^o(t) + f_1(t) - r_{q_L}(t_f)\right) \\ &\qquad = \phi_{21}(t_f,t)\,x^o(t) + \phi_{22}(t_f,t)\,\lambda^o(t) + f_2(t) \end{aligned} \tag{6.207}$$

or

$$\begin{aligned} &\left[G_{q_L}\phi_{11}(t_f,t) - \phi_{21}(t_f,t)\right] x^o(t) + G_{q_L} f_1(t) - G_{q_L} r_{q_L}(t_f) - f_2(t) \\ &\qquad = \left[\phi_{22}(t_f,t) - G_{q_L}\phi_{12}(t_f,t)\right] \lambda^o(t) \end{aligned} \tag{6.208}$$

From the nonsingularity of the coefficients (see e.g. [99]) we may write

$$\begin{aligned} \lambda^o(t) &= \left[\phi_{22}(t_f,t) - G_{q_L}\phi_{12}(t_f,t)\right]^{-1} \left[G_{q_L}\phi_{11}(t_f,t) - \phi_{21}(t_f,t)\right] x^o(t) \\ &\quad + \left[\phi_{22}(t_f,t) - G_{q_L}\phi_{12}(t_f,t)\right]^{-1} \left[G_{q_L} f_1(t) - G_{q_L} r_{q_L}(t_f) - f_2(t)\right] \end{aligned} \tag{6.209}$$

With the definition of $K_{q_L}(t)$ and $s_{q_L}(t)$ such that

$$\lambda^o(t) = K_{q_L}(t)\,x^o(t) + s_{q_L}(t), \tag{6.210}$$

the optimal control law is given by

$$u^o = -R_{q_L}^{-1} B_{q_L}^T K_{q_L}(t)\,x^o(t) - R_{q_L}^{-1} B_{q_L}^T s_{q_L}(t) =: W_{q_L}(t)\,x(t) + z_{q_L}(t) \tag{6.211}$$

Differentiation of (6.210) gives

$$\dot{\lambda}^o = \dot{K}_{q_L} x^o + K_{q_L} \dot{x}^o + \dot{s}_{q_L} \tag{6.212}$$

Replacing $\dot{\lambda}^o$ and $\dot{x}^o$ from (6.196) and using (6.210) we get

$$\begin{aligned}\left[\dot{K}_{q_L} + L_{q_L} + K_{q_L} A_{q_L} + A_{q_L}^T K_{q_L} - K_{q_L} B_{q_L} R_{q_L}^{-1} B_{q_L}^T K_{q_L}\right] x^o& \\ + \left[\dot{s}_{q_L} + \left(A_{q_L}^T - K_{q_L} B_{q_L} R_{q_L}^{-1} B_{q_L}^T\right) s_{q_L} + K_{q_L} F_{q_L} - L_{q_L} r_{q_L}\right] &= 0\end{aligned} \tag{6.213}$$

Since the equation (6.213) holds for all x^o and $r_{q_L}(t)$, the Riccati equations

$$\dot{K}_{q_L} = -L_{q_L} - K_{q_L} A_{q_L} - A_{q_L}^T K_{q_L} + K_{q_L} B_{q_L} R_{q_L}^{-1} B_{q_L}^T K_{q_L} \tag{6.214}$$

and

$$\dot{s}_{q_L} = -\left(A_{q_L}^T - K_{q_L} B_{q_L} R_{q_L}^{-1} B_{q_L}^T\right) s_{q_L} - K_{q_L} F_{q_L} + L_{q_L} r_{q_L} \tag{6.215}$$

must hold. The terminal conditions can be determined by the evaluation of (6.210) at t_f and the use of (6.206) to get

$$K_{q_L}\left(t_f\right) = G_{q_L} \tag{6.216}$$

and

$$s_{q_L}\left(t_f\right) = -G_{q_L} r_{q_L}\left(t_f\right) \tag{6.217}$$

At the switching instant t_L the adjoint process boundary condition from the HMP is given as

$$\lambda^o(t_L) = P_{\sigma_L}^T \lambda^o(t_L+) + p\, m_{q_{L-1} q_L} + C_{\sigma_L}\left(x^o(t_L-) - r_{q_{L-1}}(t_L-)\right) \tag{6.218}$$

or

$$\begin{aligned}K_{q_{L-1}}(t_L) x^o(t_L-) + s_{q_{L-1}}(t_L) = P_{\sigma_L}^T \left(K_{q_L}(t_L) x^o(t_L) + s_{q_L}(t_L+)\right) + p\, m_{q_{L-1} q_L}& \\ + C_{\sigma_L}\left(x^o(t_L-) - r_{q_{L-1}}(t_L-)\right)&\end{aligned} \tag{6.219}$$

which gives

$$K_{q_{L-1}}(t_L)x^o(t_L-)+s_{q_{L-1}}(t_L)=\left[P_{\sigma_L}^T K_{q_L}(t_L)P_{\sigma_L}+C_{\sigma_L}\right]x^o(t_L-)+P_{\sigma_L}^T s_{q_L}(t_L-)$$
$$+pm_{q_{L-1}q_L}-C_{\sigma_L}r_{q_{L-1}}(t_L-)+P_{\sigma_L}^T K_{q_L}(t_L)J_{\sigma_L} \quad (6.220)$$

Since (6.220) holds for all $x(t_L-)\in\mathscr{M}$ the follow equalities must hold

$$K_{q_{L-1}}(t_L)=P_{\sigma_L}^T K_{q_L}(t_L)P_{\sigma_L}+C_{\sigma_L} \quad (6.221)$$

and

$$s_{q_{L-1}}(t_L)=P_{\sigma_L}^T s_{q_L}(t_L+)+pm_{q_{L-1}q_L}-C_{\sigma_L}r_{q_{L-1}}(t_L-)+P_{\sigma_L}^T K_{q_L}(t_L)J_{\sigma_L} \quad (6.222)$$

For the writing of the Hamiltonian continuity condition (3.9), we note that the Hamiltonian

$$H_i(t)=\frac{1}{2}\left(x-r_{q_i}\right)^T L_{q_i}\left(x-r_{q_i}\right)+\frac{1}{2}u^T R_{q_i}u+\lambda^T\left(A_{q_i}x+B_{q_i}u+F_{q_i}\right), \quad (6.223)$$

for the minimizing control input $u^0=-R_{q_i}^{-1}B_{q_i}^T\lambda$ and the (optimal) adjoint process $\lambda^o(t)=K_{q_L}(t)x^o(t)+s_{q_L}(t)$ (see also (6.210)) is expressed as

$$H_i=\frac{1}{2}\left(x-r_{q_i}\right)^T L_{q_i}\left(x-r_{q_i}\right)+\left(K_{q_i}x+s_{q_i}\right)^T\left(A_{q_i}x-\frac{1}{2}B_{q_i}R_{q_i}^{-1}B_{q_i}^T\left(K_{q_i}x+s_{q_i}\right)+F_{q_i}\right) \quad (6.224)$$

Therefore, the Hamiltonian continuity condition (3.9) at t_L expressed by

$$\frac{1}{2}\left(x_{(t_L-)}-r_{q_{L-1}(t_L-)}\right)^T L_{q_{L-1}}\left(x_{(t_L-)}-r_{q_{L-1}(t_L-)}\right)+\left(K_{q_{L-1}(t_L-)}x_{(t_L-)}+s_{q_{L-1}(t_L-)}\right)^T$$
$$\left\{A_{q_{L-1}}x_{(t_L-)}+F_{q_{L-1}}-\frac{1}{2}B_{q_{L-1}}R_{q_{L-1}}^{-1}B_{q_{L-1}}^T\left(K_{q_{L-1}(t_L-)}x_{(t_L-)}+s_{q_{L-1}(t_L-)}\right)\right\}$$
$$=\frac{1}{2}\left(x_{(t_L)}-r_{q_L(t_L)}\right)^T L_{q_L}\left(x-r_{q_L}\right)$$
$$+\left(K_{q_L(t_L)}x_{(t_L)}+s_{q_L(t_L)}\right)^T\left\{A_{q_L}x_{(t_L)}-\frac{1}{2}B_{q_L}R_{q_L}^{-1}B_{q_L}^T\left(K_{q_L(t_L)}x_{(t_L)}+s_{q_L(t_L)}\right)+F_{q_L}\right\} \quad (6.225)$$

With the substitution of the jump map (6.190) and the relations (6.221) and (6.222) the boundary conditions for the determination of $K_{q_{L-1}}(t)$ and $s_{q_{L-1}}(t)$ for $t\in[t_{L-1},t_L]$ are derived. Using a backward induction and following a similar approach as above, the Riccati formalism for the considered linear - quadratic hybrid optimal tracking problem is established. □

Chapter 7

Hybrid Optimal Control of an Electric Vehicle with a Dual-Planetary Transmission

The goal of this chapter is to present a hybrid systems formulation of an electric vehicle equipped with a dual-stage planetary gearbox and employ hybrid optimal control theory to find the optimal inputs for the gear changing problem for electric vehicles. Due to the special structure of the transmission under study, the mechanical degree of freedom and therefore, the dimension of the (continuous) state space of the system depend on the status of the transmission, i.e. whether a gear number is fixed or the system is undergoing a transition between the two gears. Therefore, the modelling of the powertrain requires the consideration of autonomous and controlled state jumps accompanied by changes in the dimension of the state space.

We formulate the dynamics and energy consumption of gear-equipped electric vehicles by the inclusion of the transmission dynamics, considering the model of a seamless dual break transmission system. After presenting the Kinematic relations in the driveline, the dynamics of the powertrain is derived from the Principle of Virtual Work and the generalized Euler-Lagrange equation. In order to avoid state-dependant input constraints imposed by the maximum torque and maximum power constraints of the electric motor (see also Fig. 7.2) the state-dependant input constraints are converted to state-independent constraints via a change of variables and the introduction of auxiliary discrete states.

7.1 Electric Vehicle with a Dual-Planetary Transmission

The schematic view of the driveline of the electric vehicle under study is illustrated in Fig. 7.1 (see also [100–103]). The power produced by the electric motor is transmitted to the wheels via a dual-stage planetary gear set with common ring and common sun gears. The general configuration of the transmission mechanism has two degrees of freedom, providing different paths for the power flow. Brakes on the common sun gears and the common ring gears direct the power flow by locking the gears and eliminating their corresponding degree of freedom.

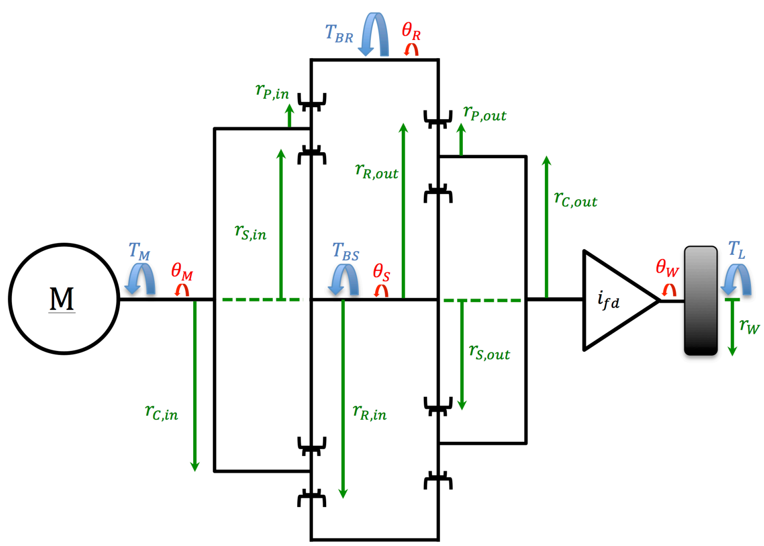

Figure 7.1 A simplified model of the driveline of an EV equipped with the dual planetary transmission

7.1.1 Driveline Kinematics

With the consideration of the longitudinal coordinate z of a car moving on a road with an a priori known grading $\gamma(z)$, and assuming the zero-slippage condition on the wheels, the rotation angle of the wheel θ_W is related to z via

$$r_W\left(\theta_W - \theta_{W,0}\right) = z - z_0, \tag{7.1}$$

where r_W is the wheel radius and $\theta_{W,0}$ and z_0 are the initial values for θ_W and z respectively. Without loss of generality, it is assumed that the car's initial position is zero, i.e. $z_0 = 0$ and also the initial angles in the transmission are zero, i.e. $\theta_{W,0} = \theta_{S,0} = \theta_{R,0} = \theta_{C,in,0} = \theta_{C,out,0} = \theta_{P,in,0} = \theta_{P,out,0} = 0$, for simplicity of the notation. Taking the angle of the common sun gears θ_S and the angle of the common ring gears θ_R as the generalized coordinates of the system, other angles of the components of the dual-stage planetary gear set as well as the car position are determined by the following kinematic relations:

$$\theta_M = \theta_{C,in} = \frac{1}{R_1+1}\theta_S + \frac{R_1}{R_1+1}\theta_R\,, \tag{7.2}$$

$$\theta_{C,out} = \frac{1}{R_2+1}\theta_S + \frac{R_2}{R_2+1}\theta_R\,, \tag{7.3}$$

$$z = \frac{r_W}{i_{fd}}\theta_{C,out} = \frac{r_W}{i_{fd}\,(R_2+1)}\theta_S + \frac{r_W R_2}{i_{fd}\,(R_2+1)}\theta_R\,, \tag{7.4}$$

$$\theta_{P,in} = \frac{-1}{R_1-1}\,\theta_S + \frac{R_1}{R_1-1}\,\theta_R\,, \tag{7.5}$$

$$\theta_{P,out} = \frac{-1}{R_2-1}\,\theta_S + \frac{R_2}{R_2-1}\,\theta_R\,, \tag{7.6}$$

where θ_M is the angle of the electric motor's rotor, $\theta_{C,in}$ and $\theta_{C,out}$ are respectively the angles of input and output carriers and, $\theta_{P,in}$ and $\theta_{P,out}$ are the angles of the planet gears connected to the input and output carriers respectively. In the above equations, i_{fd} is the gear ratio of differential and

$$R_2 := \frac{r_{R,out}}{r_{S,out}} > R_1 := \frac{r_{R,in}}{r_{S,in}} > 1, \tag{7.7}$$

holds with $r_{S,in}$, $r_{S,out}$ denoting the pitch radii of the sun gears in the input and output stages (see also Fig. 7.1), and $r_{R,in}$, $r_{R,out}$ denoting the pitch radii of the ring gears in the input and output stages respectively, whose values are presented in Table 7.1.

It is worth noting that the time derivatives of the above angles, i.e. $v := \dot{z}$ and $\omega_M := \dot{\theta}_M$, etc. can be related to $\omega_S := \dot{\theta}_S$, $\omega_R := \dot{\theta}_R$ via the time differentiation of the above equations. In particular, in the first gear where the common ring gear is held fixed, i.e. $\omega_R = 0$, the time derivatives of (7.2) and (7.3) defines the first gear ratio of the transmission as

$$GR_1 := \left.\frac{\omega_{C,in}}{\omega_{C,out}}\right|_{\omega_R=0} = \frac{R_2+1}{R_1+1}\,. \tag{7.8}$$

Similarly, the second gear corresponds to the configuration where the sun gear is locked, i.e. $\omega_S = 0$ and therefore

$$GR_2 := \left.\frac{\omega_{C,in}}{\omega_{C,out}}\right|_{\omega_S=0} = \frac{(R_2+1)R_1}{(R_1+1)R_2}. \tag{7.9}$$

7.1.2 Dynamics of the Powertrain

By the Principle of Virtual Work, the continuous evolution of the system is governed by the generalized Euler-Lagrange equation, i.e.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = D_i, \tag{7.10}$$

where q_i represents the i^{th} component of the generalized coordinate, $L = T - V$ is the Lagrangian and D_i is the resultant of the generalized dissipative and driving forces acting on the generalized coordinate component q_i (see e.g. [104] for more discussion). In this paper $q \equiv [q_1, q_2]^T = [\theta_S, \theta_R]^T$ is selected as the generalized coordinates for the general configuration of the transmission (i.e. during the gear transition process) and $q = z$ is selected as the generalized coordinate for fixed gear configurations (i.e. the first and the second gears).

The Kinetic Energy T of the system is written as

$$T = \frac{1}{2}mv^2 + \frac{1}{2}J_W\omega_W^2 + \frac{1}{2}J_M\omega_M^2 + \frac{1}{2}J_S\omega_S^2 + \frac{1}{2}J_R\omega_R^2 + \frac{1}{2}J_{P,in}\omega_{P,in}^2 + \frac{1}{2}J_{P,out}\omega_{P,out}^2, \tag{7.11}$$

where m is the total mass of the vehicle, $J_W = 4I_W + I_{shaft} + i_{fd}^2\left(I_{C,out} + 4m_{P,out}r_{C,out}^2\right)$, is the equivalent inertia of the elements directly connected to the wheels, $J_M = I_M + I_{C,in} + 4m_{P,in}r_{C,in}^2$, is the equivalent inertia of the elements directly connected to the electric motor, J_S and J_R are respectively the inertia of the sun and the ring gears, and $J_{P,in} = 4I_{P,in}$ and $J_{P,out} = 4I_{P,out}$ are the total inertia of the input and output planetary sets respectively.

The potential energy V consists only of the gravitational energy which is equivalent to

$$V = mg\Delta h = mg\int_{z_0}^{z}\sin\gamma(z)\,dz. \tag{7.12}$$

The virtual work of the generalized forces consists of the virtual work of the driving motor torque T_M and the brake torques T_{BS} and T_{BR} acting on the common sun and common ring gears respectively, the friction forces D_S and D_R acting on the sun and ring gears as well as the resistance force F_r on the displacement of the vehicle, described by the following variational equation

$$\delta W = \Sigma D_i \delta q_i = T_M \delta \theta_M + (T_{BS} + F_S)\, \delta \theta_S + (T_{BR} + F_R)\, \delta \theta_R + F_r \delta z, \tag{7.13}$$

where $F_r = -\frac{1}{2}\rho C_d A_f v^2 - mgC_r \cos\gamma(z)$ is the sum of the aerodynamic and rolling resistance forces and $F_S = -C_S\omega_S - T_{Sf}\mathrm{sign}\,(\omega_S)$ and $F_R = -C_R\omega_R - T_{Rf}\mathrm{sign}\,(\omega_R)$ are the sum of viscous and Coulomb frictions on the sun and ring gears.

Powertrain Dynamics in the Fixed Gear Configuration

For the first gear ($\omega_R = 0$), the expression (7.11) for the kinetic energy T can be written in terms of the generalized coordinate $q = z$ using the kinematic relations (7.2) to (7.6), giving T as

$$T = \left(m + \frac{J_W}{r_W^2} + \left(\frac{J_M}{(R_1+1)^2} + J_S + \frac{J_{P,in}}{(R_1-1)^2} + \frac{J_{P,out}}{(R_2-1)^2} \right) \frac{i_{fd}^2\,(R_2+1)^2}{r_W^2} \right) \frac{v^2}{2}. \tag{7.14}$$

The virtual work (7.13) is given by the variational equation:

$$\delta W = \left(\frac{i_{fd}\,(R_2+1)}{r_W\,(R_1+1)} T_M - \frac{i_{fd}\,(R_2+1)}{r_W} T_{Sf} - \frac{i_{fd}^2\,(R_2+1)^2}{r_W^2} C_S v - \frac{1}{2}\rho C_d A_f v^2 - mgC_r \cos\gamma(z) \right) \delta z, \tag{7.15}$$

where for simplicity of the notation the sign function is removed assuming that the car is only moving forward, i.e. $v \geq 0 \Rightarrow \omega_S, \omega_R \geq 0$.

Forming the Lagrangian from (7.14) and (7.12) and substitution in the generalized Euler-Lagrange equation (7.10) with the generalized force determined from (7.15), the dynamics in the first gear is given as

$$\begin{aligned} m \left(1 + \frac{J_W}{m r_W^2} + \left(\frac{J_M}{(R_1+1)^2} + J_S + \frac{J_{P,in}}{(R_1-1)^2} + \frac{J_{P,out}}{(R_2-1)^2} \right) \frac{i_{fd}^2\,(R_2+1)^2}{m r_W^2} \right) \dot{v} \\ + mg\sin\gamma(z) = \frac{i_{fd}\,(R_2+1)}{r_W\,(R_1+1)} T_M - \frac{i_{fd}\,(R_2+1)}{r_W} T_{Sf} - \frac{i_{fd}^2\,(R_2+1)^2}{r_W^2} C_S v \\ - \frac{1}{2}\rho C_d A_f v^2 - mgC_r \cos\gamma(z)\,. \end{aligned} \tag{7.16}$$

Similarly, the dynamics in the second gear ($\omega_S = 0$) is found to be

$$
\begin{aligned}
m\left(1+\frac{J_W}{mr_W^2}+\left(\frac{R_1^2 J_M}{(R_1+1)^2}+J_R+\frac{R_1^2 J_{P,in}}{(R_1-1)^2}+\frac{R_2^2 J_{P,out}}{(R_2-1)^2}\right)\frac{i_{fd}^2(R_2+1)^2}{mr_W^2 R_2^2}\right)\dot{v} \\
- mg\sin\gamma(z) = \frac{i_{fd}(R_2+1)R_1}{r_W(R_1+1)R_2}T_M - \frac{i_{fd}(R_2+1)}{r_W R_2}T_{Rf} - \frac{i_{fd}^2(R_2+1)^2}{r_W^2 R_2^2}C_R v \\
- \frac{1}{2}\rho C_d A_f v^2 - mgC_r\cos\gamma(z)\,. \quad (7.17)
\end{aligned}
$$

In general, the above equations are coupled to $\dot{z} = v$ via the coupling term $-mg\sin\gamma(z)$. However, if the road has a negligible slope, i.e. $\sin\gamma(z) \approx 0$, then velocity becomes decoupled from the position, which is the case in the problems studied in this paper.

Powertrain Dynamics in the General Configuration

Using the kinematic relations (7.2) to (7.6), the expression for T in terms of the generalized coordinate $q = [\theta_S, \theta_R]^T$ and its time differential $\dot{q} = [\omega_S, \omega_R]^T$ is written as

$$
\begin{aligned}
T = \frac{1}{2}m\frac{r_W^2(\omega_S+R_2\omega_R)^2}{i_{fd}^2(R_2+1)^2} + J_W\frac{(\omega_S+R_2\omega_R)^2}{i_{fd}^2(R_2+1)^2} + \frac{1}{2}J_M\frac{(\omega_S+R_1\omega_R)^2}{(R_1+1)^2} \\
+ \frac{1}{2}J_S\omega_S^2 + \frac{1}{2}J_R\omega_R^2 + \frac{1}{2}J_{P,in}\frac{(R_1\omega_R-\omega_S)^2}{(R_1-1)^2} + \frac{1}{2}J_{P,out}\frac{(R_2\omega_R-\omega_S)^2}{(R_2-1)^2}\,, \quad (7.18)
\end{aligned}
$$

or

$$
\begin{aligned}
T = \frac{1}{2}\left(\frac{mr_W^2+J_W}{i_{fd}^2(R_2+1)^2}+\frac{J_M}{(R_1+1)^2}+J_S+\frac{J_{P,in}}{(R_1-1)^2}+\frac{J_{P,out}}{(R_2-1)^2}\right)\omega_S^2 \\
+ \frac{1}{2}\left(\frac{(mr_W^2+J_W)R_2^2}{i_{fd}^2(R_2+1)^2}+\frac{J_M R_1^2}{(R_1+1)^2}+J_R+\frac{J_{P,in}R_1^2}{(R_1-1)^2}+\frac{J_{P,out}R_2^2}{(R_2-1)^2}\right)\omega_R^2 \\
+ \left(\frac{(mr_W^2+J_W)R_2}{i_{fd}^2(R_2+1)^2}+\frac{J_M R_1}{(R_1+1)^2}-\frac{J_{P,in}R_1}{(R_1-1)^2}-\frac{J_{P,out}R_2}{(R_2-1)^2}\right)\omega_S\omega_R \\
:= \frac{1}{2}J_{SS}\,\omega_S^2 + \frac{1}{2}J_{RR}\,\omega_R^2 + J_{SR}\,\omega_S\,\omega_R\,. \quad (7.19)
\end{aligned}
$$

In order to find D_i from (7.13), we rewrite the variational argument for the virtual displacements $\delta\theta_M$ and δz in terms of the generalized coordinates virtual displacements $\delta\theta_S$ and $\delta\theta_R$ using (7.2) and (7.4) to get

$$\begin{aligned}\delta W = &\left(\frac{1}{R_1+1}T_M + T_{BS} + F_S + \frac{r_W}{i_{fd}\,(R_2+1)}F_r\right)\delta\theta_S \\ &+ \left(\frac{R_1}{R_1+1}T_M + T_{BR} + F_R + \frac{r_W R_2}{i_{fd}\,(R_2+1)}F_r\right)\delta\theta_R\,. \end{aligned} \tag{7.20}$$

Hence,

$$D_1 = \frac{1}{R_1+1}T_M + T_{BS} + F_S + \frac{r_W}{i_{fd}\,(R_2+1)}F_r\,, \tag{7.21}$$

$$D_2 = \frac{R_1}{R_1+1}T_M + T_{BR} + F_R + \frac{r_W R_2}{i_{fd}\,(R_2+1)}F_r\,, \tag{7.22}$$

since $q_1 = \theta_S$ and $q_2 = \theta_R$ are the selected generalized coordinates.

Forming the Lagrangian $L = T - V$ using (7.19) and (7.12), and substituting the generalized dissipative and driving forces from (7.20) in the Euler-Lagrange equation (7.10), the governing dynamics are derived as

$$J_{SS}\,\dot{\omega}_S + J_{SR}\,\dot{\omega}_R = D_1 + mg\sin\gamma(z)\,\frac{r_W}{i_{fd}\,(R_2+1)}, \tag{7.23}$$

$$J_{SR}\,\dot{\omega}_S + J_{RR}\,\dot{\omega}_R = D_2 + mg\sin\gamma(z)\,\frac{r_W R_2}{i_{fd}\,(R_2+1)}, \tag{7.24}$$

where to obtain (7.23) and (7.24), the relations

$$\frac{\partial L}{\partial\theta_S} = \frac{\partial L}{\partial z}\frac{\partial z}{\partial\theta_S} = \frac{-\partial V}{\partial z}\frac{\partial z}{\partial\theta_S} = -mg\sin\gamma(z)\,\frac{r_W}{i_{fd}\,(R_2+1)}, \tag{7.25}$$

$$\frac{\partial L}{\partial\theta_R} = \frac{\partial L}{\partial z}\frac{\partial z}{\partial\theta_R} = \frac{-\partial V}{\partial z}\frac{\partial z}{\partial\theta_R} = -mg\sin\gamma(z)\,\frac{r_W R_2}{i_{fd}\,(R_2+1)}, \tag{7.26}$$

have been used. Therefore,

$$\dot{\omega}_S = \frac{J_{RR}\left(D_1 + mg\sin\gamma(z)\,\frac{r_W}{i_{fd}(R_2+1)}\right)}{J_{SS}J_{RR} - J_{SR}^2} - \frac{J_{SR}\left(D_2 + mg\sin\gamma(z)\,\frac{r_W R_2}{i_{fd}(R_2+1)}\right)}{J_{SS}J_{RR} - J_{SR}^2}\,, \tag{7.27}$$

$$\dot{\omega}_R = \frac{J_{SS}\left(D_2 + mg\sin\gamma(z)\,\frac{r_W R_2}{i_{fd}(R_2+1)}\right)}{J_{SS}J_{RR} - J_{SR}^2} - \frac{J_{SR}\left(D_1 + mg\sin\gamma(z)\,\frac{r_W}{i_{fd}(R_2+1)}\right)}{J_{SS}J_{RR} - J_{SR}^2}\,. \tag{7.28}$$

Substituting D_1 and D_2 from (7.20) and assuming $\sin\gamma(z) \approx 0$ for the simplicity of the analysis, the dynamics of the powertrain is described by

$$\begin{aligned}\dot{\omega}_S = -A_{SS}\omega_S + A_{SR}\omega_R - A_{SA}\left(\omega_S + R_2\omega_R\right)^2 \\ + B_{SS}T_{BS} - B_{SR}T_{BR} + B_{SM}T_M - D_{SL}\,,\end{aligned} \tag{7.29}$$

$$\begin{aligned}\dot{\omega}_R = A_{RS}\omega_S - A_{RR}\omega_R - A_{RA}\left(\omega_S + R_2\omega_R\right)^2 \\ - B_{RS}T_{BS} - B_{RR}T_{BR} + B_{RM}T_M - D_{RL}\,,\end{aligned} \tag{7.30}$$

where

$$\begin{aligned}
&A_{SS} = \tfrac{J_{RR}C_S}{J_{SS}J_{RR}-J_{SR}^2}\,, \quad A_{SR} = \tfrac{J_{SR}C_R}{J_{SS}J_{RR}-J_{SR}^2}\,, \quad A_{SA} = \tfrac{\rho C_d A_f (J_{RR}-R_2J_{SR}) r_W^3}{2\left(J_{SS}J_{RR}-J_{SR}^2\right) i_{fd}^3 (R_2+1)^3}\,,\\
&B_{SS} = \tfrac{J_{RR}}{J_{SS}J_{RR}-J_{SR}^2}\,, \quad B_{SR} = \tfrac{J_{SR}}{J_{SS}J_{RR}-J_{SR}^2}\,, \quad B_{SM} = \tfrac{J_{RR}-R_1J_{SR}}{J_{SS}J_{RR}-J_{SR}^2}\,,\\
&D_{SL} = \tfrac{(J_{RR}-R_2J_{SR}) r_W mgC_r}{i_{fd}(R_2+1)\left(J_{SS}J_{RR}-J_{SR}^2\right)} + \tfrac{J_{SR}T_{Rf}-J_{RR}T_{Sf}}{J_{SS}J_{RR}-J_{SR}^2}\,,
\end{aligned} \tag{7.31}$$

and

$$\begin{aligned}
&A_{RS} = \tfrac{J_{SR}C_S}{J_{SS}J_{RR}-J_{SR}^2}\,, \quad A_{RR} = \tfrac{J_{SS}C_R}{J_{SS}J_{RR}-J_{SR}^2}\,, \quad A_{RA} = \tfrac{\rho C_d A_f (R_2J_{SS}-J_{SR}) r_W^3}{2\left(J_{SS}J_{RR}-J_{SR}^2\right) i_{fd}^3 (R_2+1)^3}\,,\\
&B_{RS} = \tfrac{J_{SR}}{J_{SS}J_{RR}-J_{SR}^2}\,, \quad B_{RR} = \tfrac{J_{SS}}{J_{SS}J_{RR}-J_{SR}^2}\,, \quad B_{RM} = \tfrac{R_1J_{SS}-J_{SR}}{J_{SS}J_{RR}-J_{SR}^2}\,,\\
&D_{RL} = \tfrac{(R_2J_{SS}-J_{SR}) r_W mgC_r}{i_{fd}(R_2+1)\left(J_{SS}J_{RR}-J_{SR}^2\right)} + \tfrac{J_{SR}T_{Sf}-J_{SS}T_{Rf}}{J_{SS}J_{RR}-J_{SR}^2}\,.
\end{aligned} \tag{7.32}$$

We note that the brake torques T_{BS}, T_{BR} can only be resisting, i.e. $T_{BS} \in \left[-|T_{BS}|^{max}, 0\right]$ and $T_{BR} \in \left[-|T_{BR}|^{max}, 0\right]$.

7.1.3 Electric Motor

The electric motor considered in this paper has specifications similar to the TM4 MOTIVE A® motor whose efficiency map $\eta(T_M, \omega_M)$ is illustrated in Figure 7.2 and whose torque is constrained as a function of speed by

$$|T_M| \leq T_M^{max} \tag{7.33}$$

and

$$|T_M \omega_M| \leq P_M^{max} \tag{7.34}$$

with $T_M^{max} = 200\,N.m$ and $P_M^{max} = 80\,kW$.

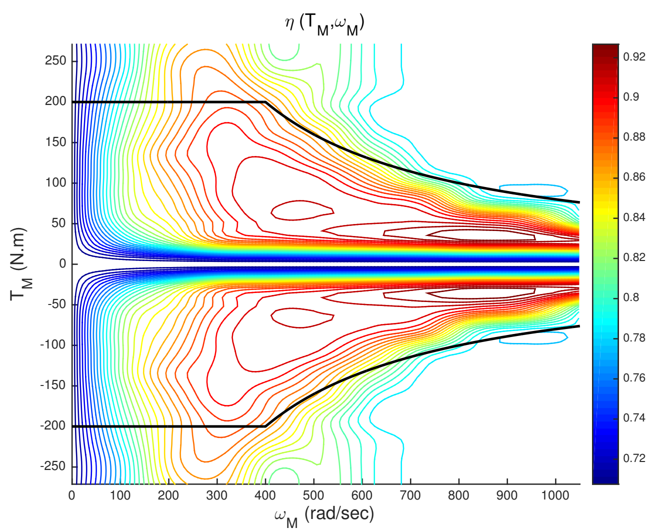

Figure 7.2 Colour Map: The electric motor efficiency map $\eta(T_M, \omega_M)$. Black Curves: Torque constraint $\overline{T}_M$ as a function of the motor speed ω_M.

In order to avoid mixed state and input constraints like (7.34) we define a change of variable by the introduction of

$$u = \frac{T_M}{T_M^{max}}, \qquad \omega_M < \omega^* \tag{7.35}$$

$$u = \frac{T_M \omega_m}{P_M^{max}}, \qquad \omega_M \geq \omega^* \tag{7.36}$$

with $\omega^* = 400\frac{rad}{sec}$. Thus the constraints (7.33) and (7.34) will both become $u \in [-1,1]$ which lies within the assumption A0 requiring U to be an invariant compact set.

The electric power consumed or generated corresponding to a pair (T_M, ω_M) is calculated as

$$P_b(T_M, \omega_M) = \begin{cases} \frac{T_M \cdot \omega_M}{\eta(T_M, \omega_M)} & T_M \omega_M \geq 0 \\ T_M \cdot \omega_M \cdot \eta(T_M, \omega_M) & T_M \omega_M < 0 \end{cases}, \tag{7.37}$$

where $T_M \omega_M \geq 0$ corresponds to power consumption, $T_M \omega_M < 0$ indicates regeneration of power, and $\eta(T_M, \omega_M)$ is illustrated in Figure 7.2.

In the analytical study of optimal control of electric vehicles (see e.g. [105, 106]), it is customary to consider the following expression for the consumption of battery power by the motor

$$P_b(T_M, \omega_M) = L_{T\omega} T_M \omega_M + L_{TT} T_M^2 + L_T T_M + L_\omega \omega_M\,, \tag{7.38}$$

where the values of the parameters $L_{T\omega}, L_{TT}, L_T, L_\omega$ are given in Table 7.1.

7.2 Hybrid System Formulation of the Powertrain

In order to present the system dynamics in the hybrid framework presented in section 8.1, the following discrete states are assigned to each continuous dynamics of the system with the dynamics described by the hybrid automata diagram in Figure 7.3:

Discrete States $\mathbf{q_1}$ and $\mathbf{q_2}$: We assign the discrete state q_1 to the torque constrained region of the first gear where the continuous state $x := v \in \mathbb{R}$ is such that the corresponding motor speed ω_M lies below ω^* and therefore, the motor torque is constrained by the maximum torque value. The vector field corresponding to q_1 is determined from (7.16) and the input is normalized by (7.35), which results in

$$\dot{x} = f_1(x,u) = -A_1 x^2 + B_1 u - C_1 x - D_1\,, \tag{7.39}$$

where

$$A_1 = \frac{\rho_a C_d A_f}{2m_1^{eq}}, \quad B_1 = \frac{i_{fd} G R_1 T_M^{max}}{m_1^{eq} r_W}, \quad C_1 = \frac{i_{fd}^2 (R_2+1)^2 C_S}{m_1^{eq} r_W^2}, \quad D_1 = \frac{mgC_r}{m_1^{eq}}\,, \tag{7.40}$$

and where $m_1^{eq} := m\left(1 + \frac{J_W}{mr_W^2} + \left(\frac{J_M}{(R_1+1)^2} + J_S + \frac{J_{P,in}}{(R_1-1)^2} + \frac{J_{P,out}}{(R_2-1)^2}\right)\frac{i_{fd}^2 (R_2+1)^2}{mr_W^2}\right)$.

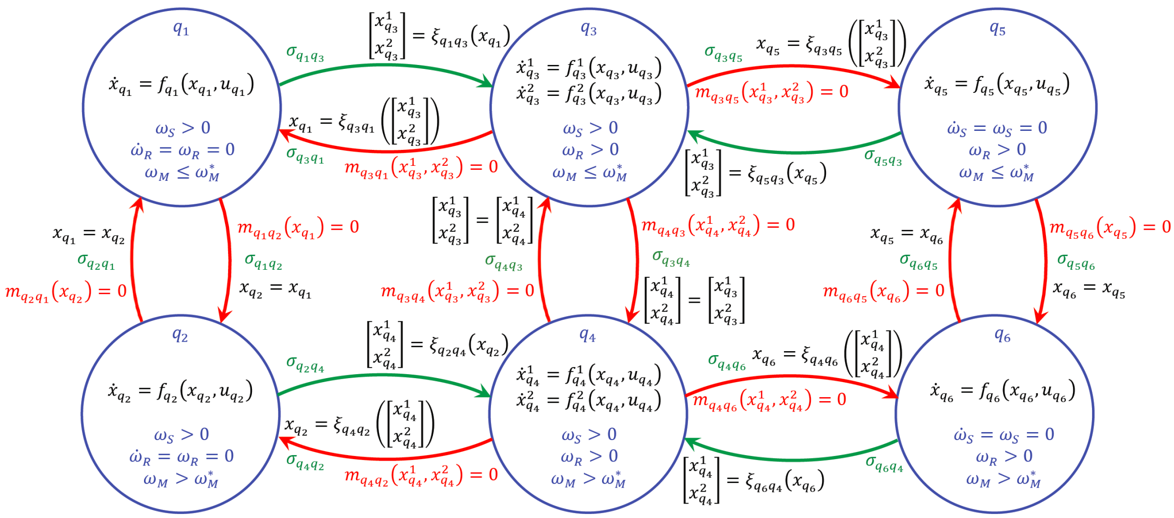

Figure 7.3 Hybrid Automata Diagram for the driveline of an EV equipped with the dual planetary transmission

When the motor speed $\omega_M = \frac{i_{fd} G R_1 v}{R_w}$ reaches $\omega^* = 400 rad/sec$ the system autonomously switches to q_2 with $x = v \in \mathbb{R}$ which corresponds to the dynamics in the power constrained region of the first gear. The vector field in this region is determined from (7.16) and the input is normalized by (7.36), which gives

$$\dot{x} = f_2(x, u) = -A_2 x^2 + B_2 \frac{u}{x} - C_2 x - D_2 \,, \tag{7.41}$$

with

$$A_2 = A_1 \,, \quad B_2 = \frac{P_M^{max}}{m_1^{eq}} \,, \quad C_2 = C_1 \,, \quad D_2 = D_1 \ . \tag{7.42}$$

The switching manifolds $m_{q_1q_2}$ and $m_{q_2q_1}$ are represented by

$$m_{q_1q_2}(x) = m_{q_2q_1}(x) \equiv x - \frac{\omega^* R_w}{i_{fd} G R_1} = 0 \,. \tag{7.43}$$

Discrete States $\mathbf{q_3}$ and $\mathbf{q_4}$: During the gear changing process, if the motor speed is lower than ω^* the input torque is limited by the maximum torque to which we assign the discrete state q_3. The continuous state $x = [\omega_S, \omega_R]^T \in \mathbb{R}^2$ is governed by the powertrain dynamics (7.29) and (7.30) and by the normalization of the motor torque (7.35) and the brake toques $u_2 := T_{BS}/|T_{BS}|^{max}$ and

$u_3 := T_{BR}/\left|T_{BR}\right|^{max}$, the vector field is described by

$$\dot{x} = f_3\left(x,u\right), \tag{7.44}$$

where

$$\begin{aligned}\dot{x}_1 = f_3^{(1)}\left(x,u\right) = &-A_{SS}x_1 + A_{SR}x_2 - A_{SA}\left(x_1 + R_2x_2\right)^2\\ &+ B_{SM}T_M^{max}u_1 + B_{SS}\left|T_{BS}\right|^{max}u_2 - B_{SR}\left|T_{BR}\right|^{max}u_3 - D_{SL},\end{aligned} \tag{7.45}$$

$$\begin{aligned}\dot{x}_2 = f_3^{(2)}\left(x,u\right) = &A_{RS}x_1 - A_{RR}x_2 - A_{RA}\left(x_1 + R_2x_2\right)^2\\ &+ B_{RM}T_M^{max}u_1 - B_{RS}\left|T_{BS}\right|^{max}u_2 - B_{RR}\left|T_{BR}\right|^{max}u_3 - D_{RL},\end{aligned} \tag{7.46}$$

and where $u_1 \in [-1,1]$ is the normalized motor torque in the torque constraint region, and $u_2, u_3 \in [-1,0]$ are the normalized sun brake and the normalized ring brake torques.

We assign q_4 with $x = [\omega_S, \omega_R]^T \in \mathbb{R}^2$ to the dynamics in the power constraint region during the gear changing with the vector field

$$\dot{x} = f_4\left(x,u\right), \tag{7.47}$$

where

$$\begin{aligned}\dot{x}_1 = f_4^{(1)}\left(x,u\right) = &-A_{SS}x_1 + A_{SR}x_2 - A_{SA}\left(x_1 + R_2x_2\right)^2\\ &+ B_{SM}P_M^{max}\left(1+R_1\right)\frac{u_1}{x_1 + R_1x_2} + B_{SS}\left|T_{BS}\right|^{max}u_2 - B_{SR}\left|T_{BR}\right|^{max}u_3 - D_{SL},\end{aligned} \tag{7.48}$$

$$\begin{aligned}\dot{x}_2 = f_4^{(2)}\left(x,u\right) = &A_{RS}x_1 - A_{RR}x_2 - A_{RA}\left(x_1 + R_2x_2\right)^2\\ &+ B_{RM}P_M^{max}\left(1+R_1\right)\frac{u_1}{x_1 + R_1x_2} - B_{RS}\left|T_{BS}\right|^{max}u_2 + B_{RR}\left|T_{BR}\right|^{max}u_3 - D_{RL}.\end{aligned} \tag{7.49}$$

The jump map corresponding to the (controlled) transitions from q_1 to q_3 and from q_2 to q_4 are described by $\xi_{q_1q_3} : \mathbb{R} \to \mathbb{R}^2$ and $\xi_{q_2q_4} : \mathbb{R} \to \mathbb{R}^2$ in the form of

$$x\left(t_s\right) = \xi_{q_1q_3}\left(x\left(t_s-\right)\right) = \frac{i_{fd}\left(1+R_2\right)}{r_W}\begin{bmatrix}1\\0\end{bmatrix}x\left(t_s-\right), \tag{7.50}$$

$$x(t_s) = \xi_{q_2q_4}(x(t_s-)) = \frac{i_{fd}(1+R_2)}{r_W}\begin{bmatrix} 1 \\ 0 \end{bmatrix} x(t_s-), \tag{7.51}$$

Note, however, that the transitions back to q_1 from q_3 and to q_2 from q_4 are autonomous with the switching manifold described by

$$m_{q_3q_1}(x) \equiv x_2 = 0, \tag{7.52}$$

$$m_{q_4q_2}(x) \equiv x_2 = 0, \tag{7.53}$$

i.e. when the ring gear comes to a full stop. The autonomous transition between q_3 and q_4 is constrained to the switching manifold condition

$$m_{q_3q_4}(x) = m_{q_4q_3}(x) \equiv \frac{x_1 + R_1x_2}{R_1+1} - \omega^* = 0, \tag{7.54}$$

with both jump transition maps $\xi_{q_3q_4}, \xi_{q_4q_3} : \mathbb{R}^2 \to \mathbb{R}^2$ being identity.

Discrete States $\mathbf{q_5}$ and $\mathbf{q_6}$: When the speed of the sun gear ω_S becomes zero the system switches to q_5 or q_6 (depending on the corresponding motor speed) with $x = v \in \mathbb{R}$ where q_5 corresponds to the dynamics in the torque constraint region of the second gear and q_6 corresponds to the dynamics in the power constraint region of the second gear. The corresponding vector fields are described by

$$\dot{x} = f_5(x,u) = -A_5x^2 + B_5u - C_5x - D_5, \tag{7.55}$$

and

$$\dot{x} = f_6(x,u) = -A_6x^2 + B_6\frac{u}{x} - C_6x - D_6, \tag{7.56}$$

where

$$A_5 = A_6 = \frac{\rho_a C_d A_f}{2m_2^{eq}}, \quad B_5 = \frac{i_{fd} G R_2 T_M^{max}}{m_2^{eq} r_W} \quad B_6 = \frac{P_M^{max}}{m_2^{eq}}, \tag{7.57}$$

and

$$C_5 = C_6 = \frac{i_{fd}^2 (R_2+1)^2 C_S}{m_2^{eq} r_W^2}, \quad D_5 = D_6 = \frac{mgC_r}{m_1^{eq}}, \tag{7.58}$$

and where $m_2^{eq} := m\left(1 + \frac{J_W}{mr_W^2} + \left(\frac{R_1^2 J_M}{(R_1+1)^2} + J_R + \frac{R_1^2 J_{P,in}}{(R_1-1)^2} + \frac{R_2^2 J_{P,out}}{(R_2-1)^2}\right)\frac{i_{fd}^2(R_2+1)^2}{mr_W^2 R_2^2}\right)$.

The switching manifold corresponding to the transition from q_3 to q_5 and from q_4 to q_6 are described as

$$m_{q_3q_5}(x) \equiv x_1 = 0\,, \tag{7.59}$$

$$m_{q_4q_6}(x) \equiv x_1 = 0\,, \tag{7.60}$$

and the jump map corresponding to these transitions are given by

$$x(t_s) = \xi_{q_3q_5}(x(t_s-)) = \frac{r_W}{i_{fd}(1+R_2)}\begin{bmatrix} 1 & R_2 \end{bmatrix} x(t_s-)\,, \tag{7.61}$$

$$x(t_s) = \xi_{q_4q_6}(x(t_s-)) = \frac{r_W}{i_{fd}(1+R_2)}\begin{bmatrix} 1 & R_2 \end{bmatrix} x(t_s-)\,, \tag{7.62}$$

with $\xi_{q_3q_5} : \mathbb{R}^2 \to \mathbb{R}$ and $\xi_{q_4q_6} : \mathbb{R}^2 \to \mathbb{R}$.

7.3 Acceleration within the Minimum Time Interval

The hybrid optimal control problem considered in this paper is the minimization of the acceleration period required for reaching the top speed of $100\frac{km}{hr} = 27.78\frac{m}{s} \approx 60mph$ starting from the stationary state in the first gear and terminating in the second gear. Hence, the cost to be minimized is

$$J(u, T_{BS}, T_{BR}; t_{s_1}, t_{s_2}, t_{s_3}) = \int_{t_0}^{t_{s_1}} dt + \int_{t_{s_1}}^{t_{s_2}} dt + \int_{t_{s_2}}^{t_{s_3}} dt + \int_{t_{s_3}}^{t_f} dt \tag{7.63}$$

with t_f being the first time that $x(t) = 27.78$ is satisfied.

The family of system Hamiltonians are formed as

$$H_{q_1}(x, \lambda, u) = 1 + \lambda\left(-A_1x^2 + B_1u - C_1x - D_1\right), \tag{7.64}$$

$$H_{q_2}(x, \lambda, u) = 1 + \lambda\left(-A_2x^2 + B_2\frac{u}{x} - C_2x - D_2\right), \tag{7.65}$$

$$
\begin{aligned}
H_{q_4}(x,\lambda,u,T_{BS},T_{BR}) = 1 + \lambda_1 \Bigg(& -A_{SS}x_1 + A_{SR}x_2 - A_{SA}(x_1+R_2x_2)^2 \\
& + B_{SM}P_M^{max}(1+R_1)\frac{u_1}{x_1+R_1x_2} + B_{SS}|T_{BS}|^{max}u_2 - B_{SR}|T_{BR}|^{max}u_3 - D_{SL} \Bigg) \\
+ \lambda_2 \Bigg(& A_{RS}x_1 - A_{RR}x_2 - A_{RA}(x_1+R_2x_2)^2 \\
& + B_{RM}P_M^{max}(1+R_1)\frac{u_1}{x_1+R_1x_2} - B_{RS}|T_{BS}|^{max}u_2 + B_{RR}|T_{BR}|^{max}u_3 - D_{RL} \Bigg)
\end{aligned}
\tag{7.66}
$$

$$
H_{q_6}(x,\lambda,u) = 1 + \lambda\left(-A_6x^2 + B_6\frac{u}{x} - C_6x - D_6\right). \tag{7.67}
$$

Then according to the Hybrid Minimum Principle

$$
\dot{\lambda} = \frac{-\partial H_{q_1}}{\partial x} = -(-2A_1x - C_1)\lambda, \qquad t \in [t_0, t_{s_1}] \tag{7.68}
$$

$$
\dot{\lambda} = \frac{-\partial H_{q_2}}{\partial x} = -\left(-2A_2x - B_2\frac{u^o}{x^2} - C_2\right)\lambda, \qquad t \in (t_{s_1}, t_{s_2}] \tag{7.69}
$$

$$
\dot{\lambda} = \frac{-\partial H_{q_4}}{\partial x}, \qquad t \in (t_{s_2}, t_{s_3}], \tag{7.70}
$$

with

$$
\begin{aligned}
\dot{\lambda}_1 = \frac{-\partial H_{q_4}}{\partial x_1} = -\lambda_1 & \left(-A_{SS} - 2A_{SA}(x_1+R_2x_2) - \frac{B_{SM}P_M^{max}(1+R_1)u_1}{(x_1+R_1x_2)^2}\right) \\
& - \lambda_2\left(A_{RS} - 2A_{RA}(x_1+R_2x_2) - \frac{B_{RM}P_M^{max}(1+R_1)u_1}{(x_1+R_1x_2)^2}\right),
\end{aligned}
\tag{7.71}
$$

$$\dot{\lambda}_2 = \frac{-\partial H_{q_4}}{\partial x_2} = -\lambda_1 \left(A_{SR} - 2R_2A_{SA}\left(x_1 + R_2x_2\right) - \frac{R_1 B_{SM} P_M^{max}\left(1+R_1\right)u_1}{\left(x_1+R_1x_2\right)^2} \right)$$
$$- \lambda_2 \left(-A_{RR} - 2R_2A_{RA}\left(x_1 + R_2x_2\right) - \frac{R_1 B_{RM} P_M^{max}\left(1+R_1\right)u_1}{\left(x_1+R_1x_2\right)^2} \right), \quad (7.72)$$

and also

$$\dot{\lambda} = \frac{-\partial H_{q_6}}{\partial x} = -\left(-2A_6x - B_6\frac{u^o}{x^2} - C_6 \right)\lambda, \qquad t \in \left(t_{s_3}, t_f\right] \quad (7.73)$$

The boundary conditions for λ are determined from Eq. (3.7) as

$$\lambda\left(t_{s_3}\right) = \nabla\xi^T_{q_3q_4}\lambda\left(t_{s_3}+\right) + p_3\nabla m_{q_3q_4} = \frac{R_w}{i_{fd}\left(1+R_2\right)}\begin{bmatrix}1\\R_2\end{bmatrix}\lambda\left(t_{s_3}+\right) + p_3\begin{bmatrix}1\\0\end{bmatrix}, \quad (7.74)$$

$$\lambda\left(t_{s_2}\right) = \nabla\xi^T_{q_2q_3}\lambda\left(t_{s_2}+\right) = \frac{i_{fd}\left(1+R_2\right)}{R_w}\begin{bmatrix}1 & 0\end{bmatrix}\lambda\left(t_{s_2}+\right), \quad (7.75)$$

$$\lambda\left(t_{s_1}\right) = \lambda\left(t_{s_1}+\right) + p_1. \quad (7.76)$$

It can easily be verified that for the above dynamics and boundary conditions, the adjoint process has a negative sign for all $t \in \left[t_0, t_f\right]$ and hence the Hamiltonian minimization condition (3.8) results in $u^o = 1, t \in \left[t_0, t_f\right]$, $T^o_{BS} = -\left|T_{BS}\right|^{max}$ and $T^o_{BR} = 0$ for $t \in \left[t_{s_2}, t_{s_3}\right]$.

The Hamiltonian terminal condition (3.100) gives

$$H_{q_6}\left(x\left(t_f\right), \lambda\left(t_f\right), u\left(t_f\right)\right) = 1 + \lambda\left(t_f\right)\left(-A_6x\left(t_f\right)^2 - B_6\frac{u\left(t_f\right)}{x\left(t_f\right)} - C_6x\left(t_f\right) - D_6 \right) = 0, \quad (7.77)$$

and the Hamiltonian continuity at switching instants is deduced from (3.9) as

$$H_{q_4}\left(x,\lambda,u\right)_{\left(t_{s_3}-\right)} = H_{q_6}\left(x,\lambda,u\right)_{\left(t_{s_3}+\right)}, \quad (7.78)$$

$$H_{q_2}\left(x,\lambda,u\right)_{\left(t_{s_2}-\right)} = H_{q_4}\left(x,\lambda,u\right)_{\left(t_{s_2}+\right)}, \quad (7.79)$$

$$H_{q_1}\left(x,\lambda,u\right)_{\left(t_{s_1}-\right)} = H_{q_2}\left(x,\lambda,u\right)_{\left(t_{s_1}+\right)}. \quad (7.80)$$

The results for the parameter values presented in Table 7.1 are illustrated in Figure 7.4. For better illustration, the speed of the vehicle is shown in km/hr units and in addition, the components λ_1 and λ_2 of the adjoint process in $t \in \left[t_{s_2}, t_{s_3}\right]$ are multiplied by $i_{fd}\left(1+R_2\right)/R_w$

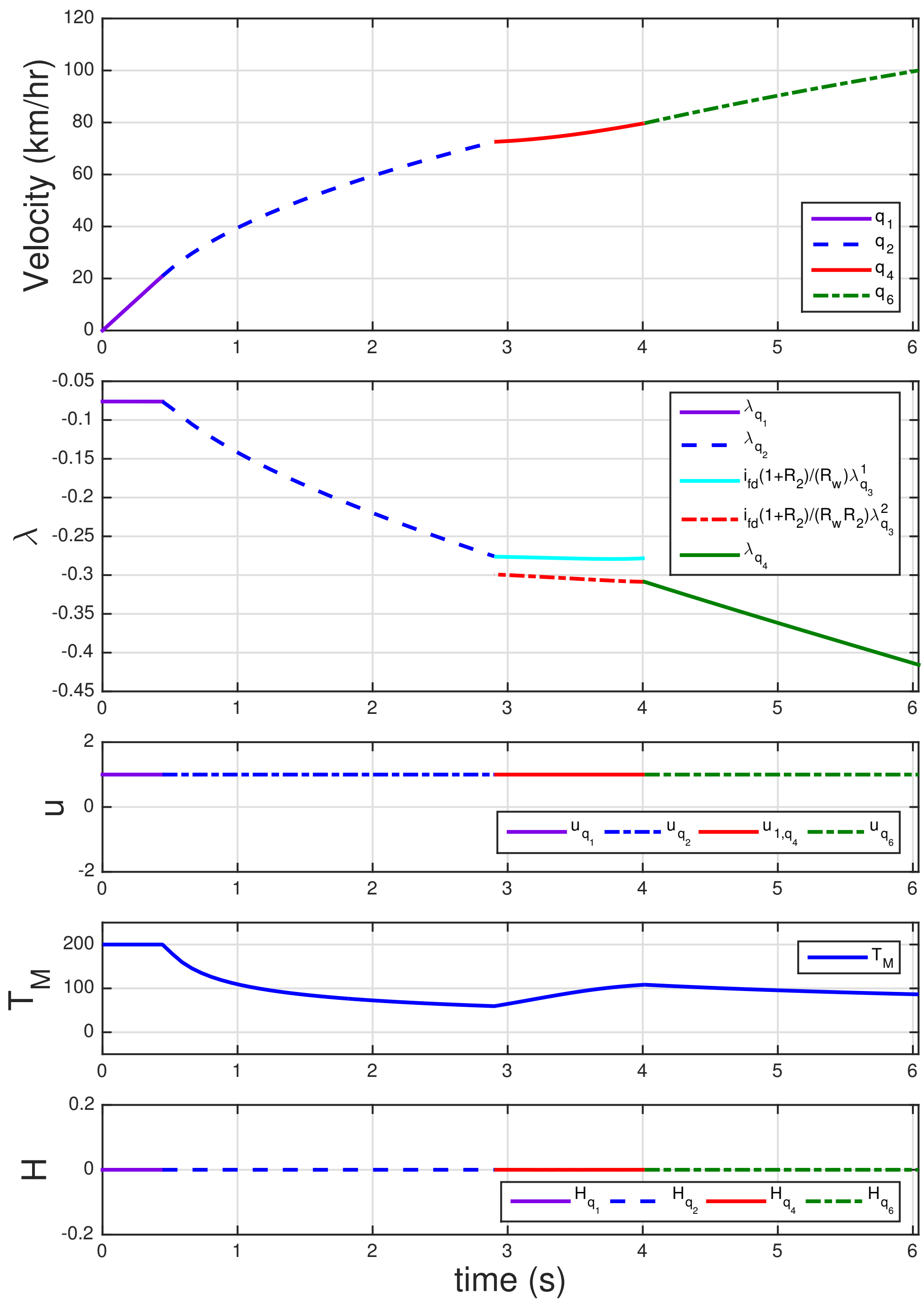

Figure 7.4 The car speed, the adjoint processes and the corresponding Hamiltonians for the minimum acceleration period problem

and $i_{fd}(1+R_2)/(R_w R_2)$ respectively so that the boundary conditions (7.111) and (7.112) can be verified more easily. The optimal values for the switching and final times are $t_{s_1}=0.444, t_{s_2}=2.901, t_{s_3}=4.014, t_f=6.042$.

7.4 Acceleration with the Minimum Energy

The hybrid optimal control problem considered in this paper is the minimization of energy required for reaching of the top speed of $100\frac{km}{hr}=27.78\frac{m}{s}$ at $t_f=6.3$ sec, which is slightly longer than 6.042 sec found in [100] for the minimum time for this task. The vehicle is assumed to start from the stationary state in the first gear which corresponds to q_1, autonomously switch to the torque constraint region q_2, then switch to the gear transition phase initiated by a controlled switching command σ_{q_2,q_4} and finally, reach the terminal state $x_f=27.78\frac{m}{s}$ at $t_f=6.3$ in the power constraint region of the second gear q_6. The cost to be minimized is total the electric energy consumed from the battery, i.e.

$$\begin{aligned} J\left(t_0,t_f,(q_1,0),3;I_3\right) &= \int_{t_0}^{t_f} P_b\left(T_M,\omega_M\right)dt \\ &= \int_{t_0}^{t_{s_1}} l_{q_1}(x,u)\,dt + \int_{t_{s_1}}^{t_{s_2}} l_{q_2}(x,u)\,dt + \int_{t_{s_2}}^{t_{s_3}} l_{q_4}(x,u)\,dt + \int_{t_{s_3}}^{t_f} l_{q_6}(x,u)\,dt\,, \end{aligned} \tag{7.81}$$

where

$$l_{q_1}(x,u) = a_1u^2 + b_1xu + c_1u + d_1x\,, \tag{7.82}$$

$$l_{q_2}(x,u) = a_2\frac{u^2}{x^2} + b_2u + c_2\frac{u}{x} + d_2x\,, \tag{7.83}$$

$$l_{q_4}(x,u) = a_4\frac{u_1^2}{(x_1+R_1x_2)^2} + b_4u_1 + c_4\frac{u_1}{x_1+R_1x_2} + d_4(x_1+R_1x_2)\,, \tag{7.84}$$

$$l_{q_6}(x,u) = a_6\frac{u^2}{x^2} + b_6u + c_6\frac{u}{x} + d_6x\,, \tag{7.85}$$

and where in the above equations

$$a_1 = L_{TT}\left(T_M^{max}\right)^2, \quad b_1 = L_{T\omega}\frac{i_{fd}GR_1T_M^{max}}{r_W}, \quad c_1 = L_TT_M^{max}, \quad d_1 = L_\omega\frac{i_{fd}GR_1}{r_W}\,, \tag{7.86}$$

$$a_2 = L_{TT}\left(\frac{r_WP_M^{max}}{i_{fd}GR_1}\right)^2, \quad b_2 = L_{T\omega}P_M^{max}, \quad c_2 = L_T\left(\frac{r_WP_M^{max}}{i_{fd}GR_1}\right), \quad d_2 = L_\omega\frac{i_{fd}GR_1}{r_W}, \tag{7.87}$$

$$a_4 = L_{TT}\,(P_M^{max})^2(R_1+1)^2,\quad b_4 = L_{T\omega}P_M^{max},\quad c_4 = L_T P_M^{max}(R_1+1),\quad d_4 = \tfrac{L_\omega}{R_1+1}, \tag{7.88}$$

$$a_6 = L_{TT}\left(\tfrac{r_W P_M^{max}}{i_{fd}GR_2}\right)^2,\quad b_6 = L_{T\omega}P_M^{max},\quad c_6 = L_T\left(\tfrac{r_W P_M^{max}}{i_{fd}GR_2}\right),\quad d_6 = L_\omega \tfrac{i_{fd}GR_2}{r_W}, \tag{7.89}$$

Formation of Hamiltonians: The family of system Hamiltonians are formed as

$$H_{q_1}(x,\lambda,u) = a_1u^2 + b_1xu + c_1u + d_1x + \lambda\left(-A_1x^2 + B_1u - C_1x - D_1\right), \tag{7.90}$$

$$H_{q_2}(x,\lambda,u) = a_2\frac{u^2}{x^2} + b_2u + c_2\frac{u}{x} + d_2x + \lambda\left(-A_2x^2 + B_2\frac{u}{x} - C_2x - D_2\right), \tag{7.91}$$

$$\begin{aligned} H_{q_4}(x,\lambda,u) &= a_4\frac{u_1^2}{(x_1+R_1x_2)^2} + b_4u_1 + c_4\frac{u_1}{x_1+R_1x_2} + d_4(x_1+R_1x_2) \\ &+ \lambda_1\Bigg(-A_{SS}x_1 + A_{SR}x_2 - A_{SA}(x_1+R_2x_2)^2 \\ &\qquad + B_{SM}P_M^{max}(1+R_1)\frac{u_1}{x_1+R_1x_2} + B_{SS}|T_{BS}|^{max}u_2 - B_{SR}|T_{BR}|^{max}u_3 - D_{SL}\Bigg) \\ &+ \lambda_2\Bigg(A_{RS}x_1 - A_{RR}x_2 - A_{RA}(x_1+R_2x_2)^2 \\ &\qquad + B_{RM}P_M^{max}(1+R_1)\frac{u_1}{x_1+R_1x_2} - B_{RS}|T_{BS}|^{max}u_2 + B_{RR}|T_{BR}|^{max}u_3 - D_{RL}\Bigg), \end{aligned} \tag{7.92}$$

$$H_{q_6}(x,\lambda,u) = a_6\frac{u^2}{x^2} + b_6u + c_6\frac{u}{x} + d_6x + \lambda\left(-A_6x^2 + B_6\frac{u}{x} - C_6x - D_6\right), \tag{7.93}$$

Hamiltonian Minimization: The Hamiltonian minimization condition (3.8) gives

$$u_{q_1}^o = \mathrm{sat}_{[-1,1]}\left(\frac{-(b_1x + c_1 + B_1\lambda)}{2a_1}\right), \tag{7.94}$$

$$u_{q_2}^o = \mathrm{sat}_{[-1,1]}\left(\frac{-x(b_2x + c_2 + B_2\lambda)}{2a_2}\right), \tag{7.95}$$

$$u^o_{1,q_4} = \mathrm{sat}_{[-1,1]}\left(\frac{-(x_1+R_1x_2)\left[b_4(x_1+R_1x_2)+c_4+B_{SM}P_M^{max}(1+R_1)\lambda_1+B_{RM}P_M^{max}(1+R_1)\lambda_2\right]}{2a_4}\right),$$

$$u^o_{2,q_4} = \begin{cases} -1 & \text{if} \quad B_{SS}\lambda_1 - B_{RS}\lambda_2 \geq 0 \\ 0 & \text{if} \quad B_{SS}\lambda_1 - B_{RS}\lambda_2 < 0 \end{cases}, \tag{7.96}$$

$$u^o_{3,q_4} = \begin{cases} -1 & \text{if} \quad B_{RR}\lambda_2 - B_{SR}\lambda_1 \geq 0 \\ 0 & \text{if} \quad B_{RR}\lambda_2 - B_{SR}\lambda_1 < 0 \end{cases},$$

and

$$u^o_{q_6} = \mathrm{sat}_{[-1,1]}\left(\frac{-x\left(b_6x+c_6+B_6\lambda\right)}{2a_6}\right). \tag{7.97}$$

Continuous State Evolution: The continuous state dynamics (3.2) are equivalent to (7.39), (7.41), (7.47) and (7.56) subject to the stationary initial, boundary and terminal conditions

$$x(t_0) = x(0) = 0, \tag{7.98}$$

$$x(t_{s_1}) = \xi_{q_1q_2}\left(x(t_{s_1}-)\right) = x(t_{s_1}-), \tag{7.99}$$

$$x(t_{s_2}) = \xi_{q_2q_4}\left(x(t_{s_2}-)\right) = \frac{i_{fd}(1+R_2)}{r_W}\begin{bmatrix} 1 \\ 0 \end{bmatrix} x(t_{s_2}-), \tag{7.100}$$

$$x(t_{s_3}) = \xi_{q_4q_6}\left(x(t_{s_3}-)\right) = \frac{r_W}{i_{fd}(1+R_2)}\begin{bmatrix} 1 & R_2 \end{bmatrix} x(t_{s_3}-), \tag{7.101}$$

$$x\left(t_f\right) = x(6.3) = 27.78 \tag{7.102}$$

and where the transitions from q_1 to q_2 and from q_4 to q_6 are subject to the switching manifold conditions

$$m_{q_1q_2}\left(x(t_{s_1}-)\right) \equiv x(t_{s_1}-) - \frac{\omega^* R_w}{i_{fd}GR_1} = 0, \tag{7.103}$$

$$m_{q_4q_6}\left(x(t_{s_3}-)\right) \equiv x_1(t_{s_3}-) = 0. \tag{7.104}$$

Evolution of the Adjoint Process: The adjoint process dynamics (3.3) are governed by

$$\dot{\lambda} = \frac{-\partial H_{q_1}}{\partial x} = -\left(b_1u + d_1 + \lambda\left(-2A_1x - C_1\right)\right), \qquad t \in [t_0, t_{s_1}], \tag{7.105}$$

$$\dot{\lambda} = \frac{-\partial H_{q_2}}{\partial x} = -\left(\frac{-2a_2u^2}{x^3} - \frac{c_2u}{x^2} + d_2 + \lambda\left(-2A_2x - B_2\frac{u^o}{x^2} - C_2\right)\right), \qquad t \in (t_{s_1}, t_{s_2}], \tag{7.106}$$

$$\dot{\lambda} = \frac{-\partial H_{q_4}}{\partial x}, \qquad t \in (t_{s_2}, t_{s_3}], \quad (7.107)$$

with

$$\begin{aligned}\dot{\lambda}_1 = \frac{-\partial H_{q_4}}{\partial x_1} = &-\left(\frac{-2a_4u_1^2}{(x_1+R_1x_2)^3} - \frac{c_4u_1}{(x_1+R_1x_2)^2} + d_4\right) \\ &- \lambda_1\left(-A_{SS} - 2A_{SA}(x_1+R_2x_2) - \frac{B_{SM}P_M^{max}(1+R_1)u_1}{(x_1+R_1x_2)^2}\right) \\ &- \lambda_2\left(A_{RS} - 2A_{RA}(x_1+R_2x_2) - \frac{B_{RM}P_M^{max}(1+R_1)u_1}{(x_1+R_1x_2)^2}\right), \end{aligned} \quad (7.108)$$

and

$$\begin{aligned}\dot{\lambda}_2 = \frac{-\partial H_{q_4}}{\partial x_2} = &-\left(\frac{-2R_1a_4u_1^2}{(x_1+R_1x_2)^3} - \frac{R_1c_4u_1}{(x_1+R_1x_2)^2} + R_1d_4\right) \\ &- \lambda_1\left(A_{SR} - 2R_2A_{SA}(x_1+R_2x_2) - \frac{R_1B_{SM}P_M^{max}(1+R_1)u_1}{(x_1+R_1x_2)^2}\right) \\ &- \lambda_2\left(-A_{RR} - 2R_2A_{RA}(x_1+R_2x_2) - \frac{R_1B_{RM}P_M^{max}(1+R_1)u_1}{(x_1+R_1x_2)^2}\right), \end{aligned} \quad (7.109)$$

as well as

$$\dot{\lambda} = \frac{-\partial H_{q_6}}{\partial x} = -\left(\frac{-2a_6u^2}{x^3} - \frac{c_6u}{x^2} + d_6 + \lambda\left(-2A_6x - B_6\frac{u^o}{x^2} - C_6\right)\right), \quad t \in (t_{s_3}, t_f], \quad (7.110)$$

subject to the boundary condition determined from Eq. (3.7) as

$$\lambda(t_{s_3}) = \nabla\xi_{q_4q_6}^T\lambda(t_{s_3}+) + p_3\nabla m_{q_3q_4} = \frac{R_w}{i_{fd}(1+R_2)}\begin{bmatrix}1\\R_2\end{bmatrix}\lambda(t_{s_3}+) + p_3\begin{bmatrix}1\\0\end{bmatrix}, \quad (7.111)$$

$$\lambda(t_{s_2}) = \nabla\xi_{q_2q_4}^T\lambda(t_{s_2}+) = \frac{i_{fd}(1+R_2)}{R_w}\begin{bmatrix}1 & 0\end{bmatrix}\lambda(t_{s_2}+), \quad (7.112)$$

$$\lambda(t_{s_1}) = \nabla\xi_{q_1q_2}^T\lambda(t_{s_1}+) + p_1 = \lambda(t_{s_1}+) + p_1. \quad (7.113)$$

Boundary Conditions on Hamiltonians: Furthermore, the Hamiltonian continuity at switching instants is deduced from (3.9) as

$$H_{q_4}(x,\lambda,u)_{(t_{s_3}-)} = H_{q_6}(x,\lambda,u)_{(t_{s_3}+)}, \tag{7.114}$$

$$H_{q_2}(x,\lambda,u)_{(t_{s_2}-)} = H_{q_4}(x,\lambda,u)_{(t_{s_2}+)}, \tag{7.115}$$

$$H_{q_1}(x,\lambda,u)_{(t_{s_1}-)} = H_{q_2}(x,\lambda,u)_{(t_{s_1}+)}. \tag{7.116}$$

□

Numerical Results: The results for the parameter values listed in Table 7.1 are illustrated in Figure 7.5, where for better illustration, the speed of the vehicle is shown in km/hr. A phenomenon of special interest that appears in the results is that the optimal control for the minimization of energy consumption coincides with a regeneration of power during the shifting period. This is in contrast with the inputs for the shifting period of the acceleration task in [100] in which the motor produces power at the full rate to reach the top speed in the minimum time possible, and also in contrast with the task of smooth gear changing in [101–103] with (almost) no speed drop. The presence of power regeneration in the currently studied example, not only contributes to the saving of electric energy, but also contributes to the significant decrease of shifting duration from around 1 sec in [100–103] to 0.1058 sec.

In order to illustrate the satisfaction of the adjoint boundary conditions (7.111) and (7.112), the components λ_1 and λ_2 of the adjoint process in $t \in [t_{s_2}, t_{s_3}]$ are multiplied by $i_{fd}(1+R_2)/R_w$ and $i_{fd}(1+R_2)/(R_w R_2)$ respectively in Figure 7.6. The optimal values for the switching times are $t_{s_1} = 0.8570, t_{s_2} = 1.4610, t_{s_3} = 1.5668$ which correspond to the switching states $x(t_{s_1}-) = 6\frac{m}{s} = 21.6\frac{km}{hr}$, $x(t_{s_2}-) = 11.3897\frac{m}{s} = 41.0\frac{km}{hr}$, $x(t_{s_3}-) = 10.6733\frac{m}{s} = 38.4\frac{km}{hr}$ and the terminal state at $t_f = 6.3$ is $x(t_f) = 27.9534\frac{m}{s} = 100.6\frac{km}{hr}$ which is slightly higher than the required speed due to numerical approximations in the solution of the above boundary value differential equations.

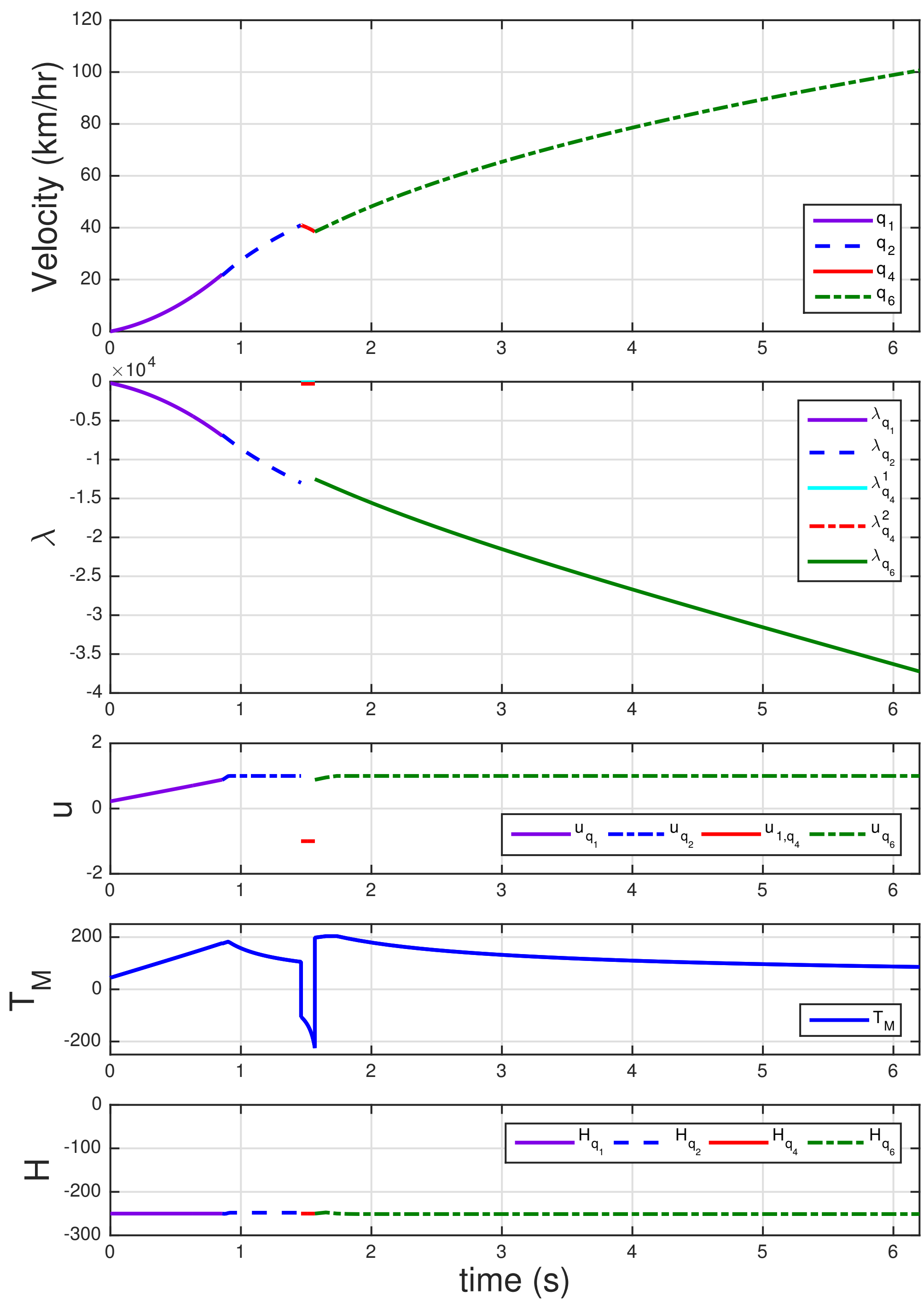

Figure 7.5 The car speed, the adjoint processes and the corresponding Hamiltonians for the minimum energy acceleration problem

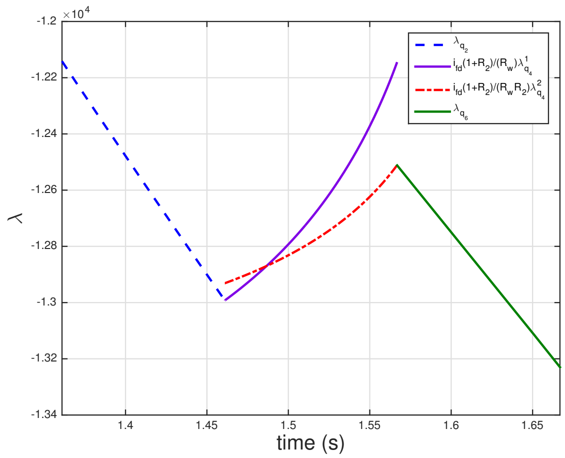

Figure 7.6 Adjoint processes in the vicinity of the shifting process

Parameter	Value	Unit		Parameter	Value	Unit
m	1000	kg		I_S	0.0015	$kg.m^2$
ρ	1.2	$\frac{kg}{m^3}$		I_R	0.009	$kg.m^2$
A_f	2	m^2		$I_{C,in}$	0.0014	$kg.m^2$
C_d	0.3	$-$		$I_{C,out}$	0.1	$kg.m^2$
C_r	0.02	$-$		$I_{P,in}$	6.08×10^{-6}	$kg.m^2$
g	9.81	$\frac{m}{s^2}$		$I_{P,out}$	3.12×10^{-5}	$kg.m^2$
i_{fd}	12	$-$		$m_{P,in}$	0.0512	kg
C_S	0.001	$\frac{N.m.s}{rad}$		$m_{P,out}$	0.12113	kg
C_R	0.001	$\frac{N.m.s}{rad}$		$r_{S,in}$	0.03	m
T_{Sf}	-0.05	$N.m$		$r_{S,out}$	0.015	m
T_{Rf}	-0.05	$N.m$		$r_{R,in}$	0.06	m
L_{TT}	0.1443	$\frac{1}{N.m.s}$		$r_{R,out}$	0.06	m
$L_{T\omega}$	1.014	$-$		$r_{P,in}$	0.015	m
L_T	-0.889	$\frac{1}{s}$		$r_{P,out}$	0.0225	m
L_ω	6.884	$N.m$		r_W	0.3	m

Table 7.1 Parameter Values for the Vehicle, Electric Motor and Transmission

Part II

Stochastic Hybrid Optimal Control

Chapter 8

Optimal Control of Stochastic Hybrid Systems

The generalization of the Minimum Principle for continuous parameter stochastic systems results in the Stochastic Minimum Principle (SMP). When diffusion terms are functions of the system state only, the SMP is derived via similar first-order variational analyses as those employed in the derivation of the deterministic MP. However, unlike the deterministic case for which backward ordinary differential equations for the adjoint process are equivalent to a forward ODE with a reversal of time, the backward stochastic differential equations (BSDE) for the adjoint process coupled to the forward stochastic differential equations (FSDE) for the state must be non-anticipative, and their solutions must be $\mathfrak{I}^t$-adapted, where $\mathfrak{I}^t$ is the natural filtration of the Wiener process. The earliest paper concerning the SMP was published by Kushner [39, 40] where he employed the needle variation and Neustadt's variational principle to derive a SMP. Haussmann [41] investigated the necessary conditions of stochastic optimal state feedback controls based on the Girsanov transformation [26]. However, due to the limitation of the Girsanov transformation, this approach works only for systems with nondegenerate diffusion coefficients [26]. The earliest version of forward-backward stochastic differential equations (FBSDE) were introduced by Bismut [33], with a decoupled form, namely a set of FSDE for the state and a set of (linear) BSDE for the adjoint process in the SMP. The well-posedness of general linear BSDEs was proved by Bensoussan [34] using the martingale representation theorem [32] and then by Pardoux and Peng [35] for the general Pontryagin-type nonlinear BSDEs. FBSDEs with state constraints are studied by Dokuchaev and Zhou [107, 108]. Chighoub, Djehiche, and Mezerdi [109] provided a proof of the Stochastic Minimum Principle in optimal control of degenerate diffusions with non-smooth coefficients. When diffusion terms also depend on the controls, one is required to study both the first-order and second-order variations and derive the SMP in the form of a stochastic Hamiltonian system consisting of two forward-backward stochastic differential equations and a minimization condition with an additional term quadratic in the diffusion coefficient. Peng [43] considered second-order variations and obtained a Stochastic Minimum Principle for systems that are possibly degenerate, with control-dependent diffusions and not necessarily convex control regions. Zhou [110] simplified Peng's proof and also established the relationship between the stochastic maximum principle and dynamic programming via second-order variations.

The optimal control of stochastic hybrid systems, i.e. control systems that involve the interaction of continuous dynamics, discrete dynamics and stochastic diffusions, has been the subject of a limited number of studies. The SMP formulation by Aghayeva and Abushov [44] considers only controlled switching and jumps, and the Stochastic Dynamic Programming (SDP) formulation by Bensoussan and Menaldi [45] studies infinite horizon problems where optimal controls are stationary. Other versions of non-classical stochastic optimal control problems such as those studied by Wu and Zhang [111], Shi and Wu [112], etc. lack many of the key features of hybrid systems (most notably change in dynamics) and are therefore not discussed here.

We extend the framework established in Chapter 2 in order to cover a general class of stochastic hybrid systems with state dependant diffusion fields which are subject to autonomous and controlled switchings and state jumps. A feature of special interest is the effect of hard constraints imposed by switching manifolds on diffusion-driven state trajectories, that to the best of our knowledge has not been considered in the literature before. Furthermore, autonomous and controlled state jumps at switching instants are allowed to be accompanied by changes in the dimension of the state space. Optimal control problems for such stochastic hybrid systems are studied in the presence of a large range of running, terminal and switching costs. First order variational analysis is performed on the stochastic hybrid optimal control problem via the needle variation methodology and the necessary optimality conditions are established in the form of the Stochastic Hybrid Minimum Principle (SHMP).

8.1 Basic Assumptions

Let $(\Omega, \Im, P)$ be a probability space with filtration $\Im^t$, let $w(.)$ be a standard $\mathbb{R}^{n_w}$ valued Wiener process. Consider a *hybrid system* $\mathbb{H}$ as an octuple

$$\mathbb{H} = \{H := Q \times M, I := \Sigma \times U, \Gamma, A, F, G, \Xi, \mathscr{M}\}, \tag{8.1}$$

where the symbols in the expression and their governing assumptions are defined as below.

A0.S: $\Im^t = \boldsymbol{\sigma}\{w(s) : 0 \le s \le t\}$, is an increasing family of sub sigma-algebras of $\Im$ producing a natural filtration generated by $w(t)$, augmented by all the P-null sets in $\Im$.

$H := Q \times M$ is called the *(hybrid) state space* of the hybrid system $\mathbb{H}$, where

$Q = \{1, 2, ..., |Q|\} \equiv \{q_1, q_2, ..., q_{|Q|}\}, |Q| < \infty$, is a finite set of *discrete states (components)*, and

$M = \{\mathbb{R}^{n_q}\}_{q \in Q}$ is a family of finite dimensional continuous valued state spaces, where $n_q \le n < \infty$ for all $q \in Q$.

$I := \Sigma \times U$ is the set of system input values, where

Σ with $|\Sigma| < \infty$ is the set of discrete state transition and continuous state jump events extended with the identity element, and

$U = \{U_q\}_{q \in Q}$ is the set of *admissible input control values*, where each $U_q \subset \mathbb{R}^{m_q}$ is a compact set in $\mathbb{R}^{m_q}$.

The set of admissible (continuous) control inputs $\mathscr{U}(U) := L_\infty([t_0, T_*), U)$, is defined to be the set of $\Im^t$-adapted measurable functions that are bounded up to a set of measure zero on $[t_0, T_*), T_* < \infty$. The boundedness property necessarily holds since admissible input functions take values in the compact set U.

$\Gamma : H \times \Sigma \to H$ is a time independent (partially defined) *discrete state transition map*.

$\Xi : H \times \Sigma \to H$ is a time independent (partially defined) *continuous state jump transition map*. All $\xi_\sigma \in \Xi$, $\xi_\sigma : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p}$, $p \in A(q, \sigma)$ are assumed to be continuously differentiable in the continuous state $x \in \mathbb{R}^{n_q}$. In this chapter, we only consider linear jump maps for which continuous differentiability automatically holds and further, $\xi(c_1 x_1 + c_2 x_2) = c_1 \xi(x_1) + c_2 \xi(x_2) \equiv c_1 \nabla\xi\, x_1 + c_2 \nabla\xi\, x_2$ for $c_1, c_2 \in \mathbb{R}$, $x_1, x_2 \in \mathbb{R}^n$.

$A : Q \times \Sigma \to Q$ denotes both a finite automaton and the automaton's associated transition function on the state space Q and event set Σ, such that for a discrete state $q \in Q$ only the discrete controlled and uncontrolled transitions into the q-dependent subset $\{A(q, \sigma), \sigma \in \Sigma\} \subset Q$ occur under the projection of Γ on its Q components: $\Gamma: Q \times \mathbb{R}^n \times \Sigma \to H|_Q$. In other words, Γ can only make a discrete state transition in a hybrid state (q,x) if the automaton A can make the corresponding transition in q.

F is an indexed collection of Borel measurable *vector fields* $\{f_q\}_{q \in Q}$ such that $f_q \in C^{k_{f_q}}(\mathbb{R}^{n_q} \times U_q \to \mathbb{R}^{n_q})$, $k_{f_q} \geq 1$, satisfies a joint uniform boundedness and Lipschitz condition, i.e. there exists $L_f < \infty$ such that $\|f_q(x,u)\| \leq L_f(1 + \|x\| + \|u\|)$ and $\|f_q(x_1,u_1) - f_q(x_2,u_2)\| \leq L_f(\|x_1 - x_2\| + \|u_1 - u_2\|)$, for all $x, x_1, x_2 \in \mathbb{R}^{n_q}$, $u, u_1, u_2 \in U_q$, $q \in Q$.

G is an indexed collection of Borel measurable *diffusion fields* $\{g_q\}_{q \in Q}$ such that $g_q \in C^{k_{g_q}}(\mathbb{R}^{n_q} \to \mathbb{R}^{n_q \times n_w})$, $k_{g_q} \geq 1$, satisfies a uniform boundedness and Lipschitz condition, i.e. there exists $L_g < \infty$ such that $\|g_q(x)\| \leq L_g(1 + \|x\|)$ and $\|g_q(x_1) - g_q(x_2)\| \leq L_g \|x_1 - x_2\|$, for all $x_1, x_2 \in \mathbb{R}^{n_q}$, $q \in Q$.

$\mathscr{M} = \{m_\alpha : \alpha \in Q \times Q,\}$ denotes a collection of *switching manifolds* such that, for any ordered pair $\alpha \equiv (\alpha_1, \alpha_2) = (q, r)$, m_α is a smooth, i.e. C^∞, codimension 1 sub-manifold of $\mathbb{R}^{n_q}$, described locally by $m_\alpha = \{x \in \mathbb{R}^{n_q} : m_\alpha(x) = 0\}$. It is assumed that $m_\alpha \cap m_\beta = \emptyset$, whenever $\alpha_1 = \beta_1$ but $\alpha_2 \neq \beta_2$, for all $\alpha, \beta \in Q \times Q$.

□

We note that the case where m_α is identified with its reverse ordered version $m_{\bar{\alpha}}$ giving $m_\alpha = m_{\bar{\alpha}}$ is not ruled out by this definition, even in the non-trivial case $m_{p,p}$ where $\alpha_1 = \alpha_2 = p$. The former case corresponds to the common situation where the switching of vector fields at the passage of the continuous trajectory in one direction through a switching manifold is reversed if a reverse passage is performed by the continuous trajectory, while the latter case corresponds to the example of the stochastic motion of a bouncing particle maintained in a turbulent regime due to collisions with a solid plate.

Switching manifolds will function in such a way that whenever a trajectory governed by the controlled vector field and the diffusion field meets the switching manifold transversally there is an autonomous switching to another controlled vector field or there is a jump transition in the continuous state component, or both. A transversal arrival on a switching manifold $m_{q,r}$, at state x occurs whenever

$$\nabla m_{q,r}(x)^T f_q(x,u) \neq 0, \tag{8.2}$$

for $x \in \{x \in \mathbb{R}^{n_q} : m_{q,r}(x) = 0\}$, $u \in U_q$, $q, r \in Q$, and where $\nabla \equiv \frac{\partial}{\partial x_q}$ is used for the simplicity of notation whenever the corresponding differentiation variable is clear.

A1.S: In this chapter, we further assume that

$$g_r\left(\xi_{\sigma_{q,r}}(x)\right) = \xi_{\sigma_{q,r}}\left(g_q(x)\right), \tag{8.3}$$

$q, r \in Q$, $\sigma_{q,r} \in \Sigma$, $r \in A\left(q, \sigma_{q,r}\right)$ and for all $x \in \left\{x \in \mathbb{R}^{n_q} : m_{q,r}(x) = 0\right\}$ we assume that

$$\nabla m_{q,r}(x)^T g_q(x) = 0. \tag{8.4}$$

□

The former condition indicates the equivalence of diffusion fields before and after switching events and the latter corresponds to the absence of transversal diffusion fields on the switching surface. For systems under turbulence-driven diffusion fields and with switching manifolds formed by solid surfaces both (8.3) and (8.4) in A1.S are automatically satisfied. In addition to the basic assumptions in A0.S and A1.S, it is assumed that:

A2.S: The initial state $h_0 := (q_0, x(t_0)) \in H$ is such that $m_{q_0,q}(x_0) \neq 0$, for all $q \in Q$.

□

8.2 Hybrid Optimal Control Problems

A3.S: Let $\left\{l_q\right\}_{q\in Q}, l_q \in C^{n_l}\left(\mathbb{R}^{n_q} \times U_q \to \mathbb{R}_+\right), n_l \geq 1$, be a family of Borel measurable running cost functions; $\{c_\sigma\}_{\sigma\in\Sigma} \in C^{n_c}\left(\mathbb{R}^{n_q} \times \Sigma \to \mathbb{R}_+\right), n_c \geq 1$, be a family of Borel measurable switching cost functions; and $h \in C^{n_h}\left(\mathbb{R}^{n_{q_f}} \to \mathbb{R}_+\right), n_h \geq 1$, be a Borel measurable terminal cost function satisfying the following assumptions:

(i) There exists $K_l < \infty$ and $1 \leq \gamma_l < \infty$ such that $\left|l_q(x_1, u_1) - l_q(x_2, u_2)\right| \leq K_l\left(\|x_1 - x_2\| + \|u_1 - u_2\|\right)$, for all $x_1, x_2 \in \mathbb{R}^{n_q}$, $u_1, u_2 \in U_q$, $q \in Q$.

(ii) There exists $K_c < \infty$ and $1 \leq \gamma_c < \infty$ such that $|c_\sigma(x)| \leq K_c\left(1 + \|x\|^{\gamma_c}\right)$, $\sigma \in \Sigma$, $x \in \mathbb{R}^{n_q}$, $q \in Q$.

(iii) There exists $K_h < \infty$ and $1 \leq \gamma_h < \infty$ such that $|h(x)| \leq K_h\left(1 + \|x\|^{\gamma_h}\right)$, $x \in \mathbb{R}^{n_{q_f}}$, $q_f \in Q$.

□

Consider the initial time t_0, final time $t_f < \infty$, and initial hybrid state $h_0 = (q_0, x_0)$. For a fixed number of switchings $L < \infty$, let $\tau_L := \{t_0, t_1, t_2, \ldots, t_L\}$ be a strictly increasing $\Im^t$-adapted sequence of times and $\sigma_i \in \Sigma$, $i \in \{1, 2, \cdots, L\}$ extended with $\sigma_0 = id$ be a *discrete event sequence* that form a hybrid switching sequence

$$S_L = \left\{(t_0, id), \left(t_1, \sigma_{q_0 q_1}\right), \ldots, \left(t_L, \sigma_{q_{L-1} q_L}\right)\right\} \equiv \{(t_0, q_0), (t_1, q_1), \ldots, (t_L, q_L)\}. \tag{8.5}$$

With the set of admissible continuous control inputs given as $\mathscr{U} = \bigcup_{i=0}^{L} L_\infty\left([t_i, t_{i+1}), U_{q_i}\right)$ with $t_{L+1} = t_f$, a $\Im^t$-adapted *hybrid input process* is denoted by $I_L := (S_L, u)$, $u \in \mathscr{U}$, $u(t) : \Im^t -$ measurable.

Consider the hybrid performance function

$$J\left(t_0, t_f, h_0, L; I_L\right) := \mathbb{E}\left\{\sum_{i=0}^{L} \int_{t_i}^{t_{i+1}} l_{q_i}\left(x_{q_i}(s), u(s)\right) ds + \sum_{j=1}^{L} c_{\sigma_{q_{j-1} q_j}}\left(t_j, x_{q_{j-1}}\left(t_j -\right)\right) + h\left(x_{q_L}\left(t_f\right)\right)\right\}, \tag{8.6}$$

subject to

$$dx_{q_i}(t) = f_{q_i}\left(x_{q_i}(t), u_{q_i}(t)\right) dt + g_{q_i}\left(x_{q_i}(t)\right) dw, \quad t \in [t_i, t_{i+1}), \tag{8.7}$$

$$x_{q_0}(t_0) = x_0, \tag{8.8}$$

$$x_{q_j}\left(t_j\right) = \xi_{\sigma_{q_{j-1} q_j}}\left(x_{q_{j-1}}\left(t_j -\right)\right) \equiv \xi_{\sigma_{q_{j-1} q_j}}\left(\lim_{t \uparrow t_j} x_{q_{j-1}}(t)\right), \tag{8.9}$$

where $0 \le i \le L$, $1 \le j \le L$, $t_{L+1} = t_f < \infty$. If t_j is the time of an autonomous switching, then

$$m_{q_{j-1} q_j}\left(x_{q_{j-1}}\left(t_j -\right)\right) = 0. \tag{8.10}$$

The Hybrid Optimal Control Problem (HOCP) is defined as the infimization of the hybrid cost (8.6) over the family of hybrid input trajectories with L switchings $\boldsymbol{I}_L$, i.e.

$$J^o\left(t_0, t_f, h_0, L\right) = \inf_{I_L \in \boldsymbol{I}_L} J\left(t_0, t_f, h_0, L; I_L\right). \tag{8.11}$$

$\square$

8.3 Stochastic Hybrid Minimum Principle

Theorem 8.1. *Consider the hybrid system $\mathbb{H}$ together with the assumptions A0.S, A1.S, A2.S and A3.S as above and the HOCP* (8.11) *for the hybrid cost* (8.6)*. Define the family of system Hamiltonians by*

$$H_q\left(x_q,u_q,\lambda_q,K_q\right)=l_q\left(x_q,u_q\right)+\lambda_q^T f_q\left(x_q,u_q\right)+tr\left[K_q^T g_q\left(x_q\right)\right], \tag{8.12}$$

with $q\in Q$, $x_q\in\mathbb{R}^{n_q}$, $u_q\in U_q$, $\lambda_q\in\mathbb{R}^{n_q}$, $K_q\in\mathbb{R}^{n_q\times n_w}$. Then for the optimal input u^o and the corresponding trajectory x^o there exists $\lambda^o,K_q^o:\mathfrak{I}^t-$adapted, such that

$$d{x_q}^o=\frac{\partial H_{q^o}}{\partial\lambda_q}\left(x_q^o,u_q^o,\lambda_q^o,K_q^o\right)dt+\frac{\partial H_{q^o}}{\partial K_q}\left(x_q^o,u_q^o,\lambda_q^o,K_q^o\right)dw, \tag{8.13}$$

$$d{\lambda_q}^o=-\frac{\partial H_{q^o}}{\partial x_q}\left(x_q^o,u_q^o,\lambda_q^o,K_q^o\right)dt+K_q^o dw, \tag{8.14}$$

almost everywhere $t\in\left[t_0,t_f\right]$ with

$$x_{q0}^o\left(t_0\right)=x_0, \tag{8.15}$$

$$x_{q_j}^o\left(t_j\right)=\xi_{\sigma_{q_{j-1},q_j}}\left(x_{q_{j-1}}^o\left(t_j-\right)\right), \tag{8.16}$$

$$\lambda_{q_L}^o\left(t_f\right)=\frac{\partial g}{\partial x_{q_L}}\left(x_{q_L}^o\left(t_f\right)\right), \tag{8.17}$$

$$\lambda_{q_{j-1}}^o\left(t_j\right)=\left[\frac{\partial\xi_{\sigma_{q_{j-1},q_j}}}{\partial x_{q_{j-1}}}\right]^T\lambda_{q_j}^o\left(t_j+\right)+p\,\frac{\partial m_{q_{j-1},q_j}}{\partial x_{q_{j-1}}}+\frac{\partial c_{\sigma_{q_{j-1},q_j}}}{\partial x_{q_{j-1}}}, \tag{8.18}$$

where $p\in\mathbb{R}$ when t_j indicates the time of an autonomous switching, and $p=0$ when t_j indicates the time of a controlled switching. Moreover,

$$H_{q^o}\left(x_q^o,u_q^o,\lambda_q^o,K_q^o\right)\le H_{q^o}\left(x_q^o,v,\lambda_q^o,K_q^o\right), \tag{8.19}$$

almost everywhere $t\in\left[t_0,t_f\right]$, almost surely for all v : $\mathfrak{I}^t-$measurable random variables in U_q, that is to say the Hamiltonian is minimized with respect to the control input; and at a switching time t_j the Hamiltonian satisfies

$$H_{q_{j-1}}\left(t_j-\right)\equiv H_{q_{j-1}}\left(t_j\right)=H_{q_j}\left(t_j\right)\equiv H_{q_j}\left(t_j+\right). \tag{8.20}$$

□

Before presenting the proof of Theorem 8.1, we shall present the following fundamental Lemma (taken from [26, Corollary 5.6, p. 37–38]):

Lemma 8.2. *Let Z and $\hat{Z}$ be $\mathbb{R}^n$-valued continuous processes satisfying*

$$dZ(t) = b(t)\,dt + \sigma(t)\,dw(t), \tag{8.21}$$

$$d\hat{Z}(t) = \hat{b}(t)\,dt + \hat{\sigma}(t)\,dw(t), \tag{8.22}$$

where b, $\hat{b}$, σ, $\hat{\sigma}$ are $\mathfrak{I}^t$-adapted measurable processes taking values in $\mathbb{R}^n$, and $w(t)$ is a one dimensional standard Brownian motion. Then for $\tau_2 \geq \tau_1$ we have

$$\begin{aligned}\left\langle Z(\tau_2), \hat{Z}(\tau_2)\right\rangle = \left\langle Z(\tau_1), \hat{Z}(\tau_1)\right\rangle &+ \int_{\tau_1}^{\tau_2} \left\{\left\langle Z(s), \hat{b}(s)\right\rangle + \left\langle b(s), \hat{Z}(s)\right\rangle + \langle \sigma(s), \hat{\sigma}(s)\rangle\right\} ds \\ &+ \int_{\tau_1}^{\tau_2} \left\{\left\langle \sigma(s), \hat{Z}(s)\right\rangle + \langle Z(s), \hat{\sigma}(s)\rangle\right\} dw(s)\end{aligned} \tag{8.23}$$

□

Proof of Theorem 8.1. Consider the case of a hybrid optimal control problem with a single switching, i.e. with $L = 1$, $t_f = t_{L+1} = t_2$ and with the notation $t_s := t_1$. The generalization to several switchings follows a similar approach as in the proof of the deterministic HMP in Chapter 3.

First, consider a needle variation at time $t \in (t_s, t_f)$ in the form of

$$u^{\varepsilon}(\tau) = \begin{cases} u^o_{q_0}(\tau) & \text{if} \quad t_0 \leq \tau < t_s \\ u^o_{q_1}(\tau) & \text{if} \quad t_s \leq \tau < t \\ v & \text{if} \quad t \leq \tau < t+\varepsilon \\ u^o_{q_1}(\tau) & \text{if} \quad t+\varepsilon \leq \tau \leq t_f \end{cases}. \tag{8.24}$$

This corresponds to a perturbed trajectory $\hat{x}^{\varepsilon}(\tau), \tau \in [t_0, t_f]$ for which $x^{\varepsilon}_{q_0}(\tau) = x^o_{q_0}(\tau)$, $t_0 \leq \tau < t_s$ and $x^{\varepsilon}_{q_1}(\tau) = x^o_{q_1}(\tau)$, $t_s \leq \tau \leq t$, and for $t \leq \tau \leq t_f$ we may write:

$$\delta x_{q_1}^{\varepsilon}(\tau) := x_{q_1}^{\varepsilon}(\tau) - x_{q_1}^{o}(\tau) = \int_t^{t+\varepsilon} \left[f_{q_1}\left(x_{q_1}^{\varepsilon}(s), v\right) - f_{q_1}\left(x_{q_1}^{o}(s), u_{q_1}^{o}(s)\right)\right] ds$$
$$+ \int_{t+\varepsilon}^{\tau} \left[f_{q_1}\left(x_{q_1}^{\varepsilon}(s), u_{q_1}^{o}(s)\right) - f_{q_1}\left(x_{q_1}^{o}(s), u_{q_1}^{o}(s)\right)\right] ds + \int_t^{\tau} \left[g_{q_1}\left(x_{q_1}^{\varepsilon}(s)\right) - g_{q_1}\left(x_{q_1}^{o}(s)\right)\right] dw(s). \tag{8.25}$$

Defining the first order state variation as (see also [42])

$$y(\tau) := \frac{d}{d\varepsilon} x^{\varepsilon}(\tau)\bigg|_{\varepsilon=0}, \tag{8.26}$$

the first order dynamics and boundary conditions of the state sensitivity are derived as

$$dy_{q_1}(\tau) = \frac{\partial f_{q_1}}{\partial x_{q_1}}\left(x_{q_1}^{o}(\tau), u_{q_1}^{o}(\tau)\right) y_{q_1}(\tau)\, d\tau + \frac{\partial g_{q_1}}{\partial x_{q_1}}\left(x_{q_1}^{o}(\tau)\right) y_{q_1}(\tau)\, dw(\tau), \tag{8.27}$$

$$y_{q_1}(t) = f_{q_1}\left(x_{q_1}^{o}(t), v\right) - f_{q_1}\left(x_{q_1}^{o}(t), u_{q_1}^{o}(t)\right). \tag{8.28}$$

Similarly, first order (forward) dynamics and boundary conditions of the cost variations are shown to be governed by

$$\frac{d}{d\tau} z_{q_1}(\tau) = \frac{\partial l_{q_1}}{\partial x_{q_1}}\left(x_{q_1}^{o}(\tau), u_{q_1}^{o}(\tau)\right) y_{q_1}(\tau) \tag{8.29}$$

$$z_{q_1}(t) = l_{q_1}\left(x_{q_1}^{o}(t), v\right) - l_{q_1}\left(x_{q_1}^{o}(t), u_{q_1}^{o}(t)\right). \tag{8.30}$$

It is deduced from the optimality conditions that

$$\frac{d}{d\varepsilon} J(u^{\varepsilon})\bigg|_{\varepsilon=0} = \mathbb{E}\left\{ z_{q_1}(t_f) + \left[\frac{\partial h}{\partial x_{q_1}}\left(x_{q_1}^{o}(t_f)\right)\right]^T y_{q_1}(t_f)\right\} \geq 0. \tag{8.31}$$

Define

$$d\lambda_{q_1}^{o} = -\left(\frac{\partial l_{q_1}}{\partial x_{q_1}}\left(x_{q_1}^{o}, u_{q_1}^{o}\right) + \left[\frac{\partial f_{q_1}}{\partial x_{q_1}}\left(x_{q_1}^{o}, u_{q_1}^{o}\right)\right]^T \lambda_{q_1}^{o} + \sum_{k=1}^{n_w} \left[\frac{\partial g_{q_1}}{\partial x_{q_1}}\left(x_{q_1}^{o}\right)\right]^T K_{q_1}^{o(k)} \right) dt + K_{q_1}^{o}(t)\, dw(t), \tag{8.32}$$

$$\lambda_{q_1}^{o}(t_f) = \frac{\partial h}{\partial x_{q_1}}\left(x_{q_1}^{o}(t_f)\right), \tag{8.33}$$

The posited (8.32) with the boundary condition (8.33) is a well-posed BSDE which by [32, Theorem 4.2, p.15] admits a unique adapted solution $\left(\lambda_{q_1}^{o}, K_{q_1}^{o}\right)$, for which Lemma 8.2 can be employed to write

$$
\begin{aligned}
z_{q_1}\left(t_f\right)+\left\langle\lambda_{q_1}\left(t_f\right),y_{q_1}\left(t_f\right)\right\rangle &= z_{q_1}(t)+\left\langle\lambda_{q_1}(t),y_{q_1}(t)\right\rangle+\int_t^{t_f}\left\langle y_{q_1}(s),\frac{\partial l_{q_1}}{\partial x_{q_1}}(s)\right\rangle ds\\
&+\int_t^{t_f}\left\langle y_{q_1}(s),-\left(\frac{\partial l_{q_1}}{\partial x_{q_1}}(s)+\left[\frac{\partial f_{q_1}}{\partial x_{q_1}}\right]^T\lambda_{q_1}^o(s)+\sum_{k=1}^{n_w}\left[\frac{\partial g_{q_1}}{\partial x_{q_1}}\right]^T K_{q_1}^{o(k)}(s)\right)\right\rangle ds\\
&+\int_t^{t_f}\left\{\left\langle\left[\frac{\partial f_{q_1}}{\partial x_{q_1}}\right]^T y_{q_1}(s),\lambda_{q_1}^o(s)\right\rangle+\sum_{k=1}^{n_w}\left\langle\left[\frac{\partial g_{q_1}}{\partial x_{q_1}}\right]^T y_{q_1}(s),K_{q_1}^{o(k)}(s)\right\rangle\right\}ds\\
&+\int_t^{t_f}\left\{\left\langle\left[\frac{\partial g_{q_1}}{\partial x_{q_1}}\right]^T y_{q_1}(s),\lambda_{q_1}^o(s)\right\rangle+\sum_{k=1}^{n_w}\left\langle y_{q_1}(s),K_{q_1}^{o(k)}(s)\right\rangle\right\}dw(s),
\end{aligned}
\tag{8.34}
$$

which after simplification becomes

$$
\begin{aligned}
z_{q_1}\left(t_f\right)+\left\langle\lambda_{q_1}\left(t_f\right),y_{q_1}\left(t_f\right)\right\rangle &= z_{q_1}(t)+\left\langle\lambda_{q_1}(t),y_{q_1}(t)\right\rangle\\
&+\int_t^{t_f}\left\{\left\langle\left[\frac{\partial g_{q_1}}{\partial x_{q_1}}\right]^T y_{q_1}(s),\lambda_{q_1}^o(s)\right\rangle+\sum_{k=1}^{n_w}\left\langle y_{q_1}(s),K_{q_1}^{o(k)}(s)\right\rangle\right\}dw(s)
\end{aligned}
\tag{8.35}
$$

Simply taking expectations at each $t\in\left(t_s,t_f\right)$, gives:

$$
\mathbb{E}\left\{z_{q_1}\left(t_f\right)+\left[\lambda_{q_1}^o\left(t_f\right)\right]^T y_{q_1}\left(t_f\right)\right\}=\mathbb{E}\left\{z_{q_1}(t)+\left[\lambda_{q_1}^o(t)\right]^T y_{q_1}(t)\right\}. \tag{8.36}
$$

Substituting (8.33) into (8.31) and using (8.36) with the substitution of (8.28) and (8.30) give

$$
\begin{aligned}
\mathbb{E}\Big\{ & l_{q_1}\left(x_{q_1}^o(t),v\right)+\left[\lambda_{q_1}^o(t)\right]^T f_{q_1}\left(x_{q_1}^o(t),v\right)\\
&-l_{q_1}\left(x_{q_1}^o(t),u_{q_1}^o(t)\right)-\left[\lambda_{q_1}^o(t)\right]^T f_{q_1}\left(x_{q_1}^o(t),u_{q_1}^o(t)\right)\Big\}\geq 0,
\end{aligned}
\tag{8.37}
$$

which, by [42, Theorem 2.1], results in

$$
\begin{aligned}
& l_{q_1}\left(x_{q_1}^o(t),u_{q_1}^o(t)\right)+\left[\lambda_{q_1}^o(t)\right]^T f_{q_1}\left(x_{q_1}^o(t),u_{q_1}^o(t)\right)\\
&\qquad\leq l_{q_1}\left(x_{q_1}^o(t),v\right)+\left[\lambda_{q_1}^o(t)\right]^T f_{q_1}\left(x_{q_1}^o(t),v\right),
\end{aligned}
\tag{8.38}
$$

a.s. for all $v : \Im^t -$ measurable random variables in U_{q_1}. The Hamiltonian minimization condition (8.19) in location q_1 directly follows (8.38).

Now consider a needle variation at time $t \in (t_0, t_s)$ in the form of

$$u^{\varepsilon}(\tau) = \begin{cases} u^o_{q_0}(\tau) & \text{if} \quad t_0 \le \tau < t \\ v & \text{if} \quad t \le \tau < t+\varepsilon \\ u^o_{q_0}(\tau) & \text{if} \quad t+\varepsilon \le \tau < t_s - \delta^{\varepsilon} \\ u^o_{q_1}(t_s) & \text{if} \quad t_s - \delta^{\varepsilon} \le \tau < t_s \\ u^o_{q_1}(\tau) & \text{if} \quad t_s \le \tau \le t_f \end{cases} . \tag{8.39}$$

where $\delta^{\varepsilon} \ge 0$ corresponds to the case where the perturbed trajectory arrives on the switching manifold $m(x) = 0$ at an earlier instant (the case with a later arrival time is handled in a similar fashion).

For $\tau \in [t_0, t_s - \delta^{\varepsilon})$, we may write:

$$\begin{aligned} \delta x^{\varepsilon}_{q_0}(\tau) := x^{\varepsilon}_{q_0}(\tau) - x^o_{q_0}(\tau) = \int_t^{t+\varepsilon} \left[f_{q_0}\left(x^{\varepsilon}_{q_0}(s), v\right) - f_{q_0}\left(x^o_{q_0}(s), u^o_{q_0}(s)\right)\right] ds \\ + \int_{t+\varepsilon}^{\tau} \left[f_{q_0}\left(x^{\varepsilon}_{q_0}(s), u^o_{q_0}(s)\right) - f_{q_0}\left(x^o_{q_0}(s), u^o_{q_0}(s)\right)\right] ds + \int_t^{\tau} \left[g_{q_0}\left(x^{\varepsilon}_{q_0}(s)\right) - g_{q_0}\left(x^o_{q_0}(s)\right)\right] dw(s), \end{aligned} \tag{8.40}$$

and derive the first order state variation as

$$dy_{q_0}(\tau) = \frac{\partial f_{q_0}}{\partial x_{q_0}}\left(x^o_{q_0}(\tau), u^o_{q_0}(\tau)\right) y_{q_0}(\tau)\, d\tau + \frac{\partial g_{q_0}}{\partial x_{q_0}}\left(x^o_{q_0}(\tau)\right) y_{q_0}(\tau)\, dw(\tau), \tag{8.41}$$

$$y_{q_0}(t) = f_{q_0}\left(x^o_{q_0}(t), v\right) - f_{q_0}\left(x^o_{q_0}(t), u^o_{q_0}(t)\right). \tag{8.42}$$

For $\tau \in [t_s - \delta^{\varepsilon}, t_s)$, the early-switched perturbed trajectory evolves in $\mathbb{R}^{q_1}$ while the original trajectory is still in $\mathbb{R}^{q_0}$. At t_s, both trajectories are in $\mathbb{R}^{q_1}$, and we may write

$$\begin{aligned} \delta x^{\varepsilon}_{q_1}(t_s) &= x^{\varepsilon}_{q_1}(t_s) - x^o_{q_1}(t_s) \\ &= \xi\left(x^{\varepsilon}_{q_1}(t_s - \delta^{\varepsilon} -)\right) + \int_{t_s - \delta^{\varepsilon}}^{t_s} f_{q_1}\left(x^{\varepsilon}_{q_1}(\tau), u^o_{q_1}(t_s)\right) d\tau + \int_{t_s - \delta^{\varepsilon}}^{t_s} g_{q_1}\left(x^{\varepsilon}_{q_1}(\tau)\right) dw(\tau) \\ &- \xi\left(x^o_{q_1}(t_s - \delta^{\varepsilon} -) + \int_{t_s - \delta^{\varepsilon}}^{t_s} f_{q_0}\left(x^o_{q_0}(\tau), u^o_{q_0}(\tau)\right) d\tau + \int_{t_s - \delta^{\varepsilon}}^{t_s} g_{q_0}\left(x^o_{q_0}(\tau)\right) dw(\tau)\right). \end{aligned} \tag{8.43}$$

By invoking (8.3) in A1 we can prove that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\mathbb{E}\left\|\int_{t_s-\delta^\varepsilon}^{t_s} g_{q_1}\left(x_{q_1}^{\varepsilon}(\tau)\right)dw(\tau)-\xi\left(\int_{t_s-\delta^\varepsilon}^{t_s} g_{q_0}\left(x_{q_0}^{o}(\tau)\right)\right)dw(\tau)\right\|^2=0, \tag{8.44}$$

and by employing the Burkholder-Davis-Gundy (BDG) inequality (see e.g. [113, 114] and Appendix A) we deduce

$$y_{q_1}(t_s)=\nabla\xi\, y_{q_0}(t_s-)+\lim_{\varepsilon\to 0}\frac{\delta^\varepsilon}{\varepsilon}\left[f_{q_1}\left(\xi\left(x_{q_0}^{o}(t_s-)\right),u_{q_1}^{o}(t_s)\right)-\nabla\xi\, f_{q_0}\left(x_{q_0}^{o}(t_s-),u_{q_0}^{o}(t_s-)\right)\right], \tag{8.45}$$

almost surely, where by using (8.4) in A1 and the BDG inequality, the limit in (8.45) is determined as

$$\lim_{\varepsilon\to 0}\frac{\delta^\varepsilon}{\varepsilon}=\frac{\nabla m^T y_{q_0}(t_s-)}{\nabla m^T f_{q_0}\left(x_{q_0}^{o}(t_s-),u_{q_0}^{o}(t_s-)\right)}, \tag{8.46}$$

almost surely. Denoting

$$\gamma_s:=\frac{1}{\nabla m^T f_{q_0}\left(x_{q_0}^{o}(t_s-),u_{q_0}^{o}(t_s-)\right)}, \tag{8.47}$$

the first order dynamics of the state sensitivity are

$$y_{q_0}(t)=f_{q_0}\left(x_{q_0}^{o}(t),v\right)-f_{q_0}\left(x_{q_0}^{o}(t),u_{q_0}^{o}(t)\right), \tag{8.48}$$

$$dy_{q_0}(\tau)=\frac{\partial f_{q_0}}{\partial x_{q_0}}\left(x_{q_0}^{o}(\tau),u_{q_0}^{o}(\tau)\right)y_{q_0}(\tau)\,d\tau+\frac{\partial g_{q_0}}{\partial x_{q_0}}\left(x_{q_0}^{o}(\tau)\right)y_{q_0}(\tau)\,dw(\tau), \tag{8.49}$$

$$y_{q_1}(t_s)=\left[\nabla\xi+\gamma_s\left(f_{q_1}^{s}-\nabla\xi\, f_{q_0}^{s}\right)\nabla m^T\right]y_{q_0}(t_s-), \tag{8.50}$$

$$dy_{q_1}(\tau)=\frac{\partial f_{q_1}}{\partial x_{q_1}}\left(x_{q_1}^{o}(\tau),u_{q_1}^{o}(\tau)\right)y_{q_1}(\tau)\,d\tau+\frac{\partial g_{q_1}}{\partial x_{q_1}}\left(x_{q_1}^{o}(\tau)\right)y_{q_1}(\tau)\,dw(\tau), \tag{8.51}$$

where in the above equations $f_{q_0}^{s}:=f_{q_0}\left(x_{q_0}^{o}(t_s-),u_{q_0}^{o}(t_s-)\right)$ and $f_{q_1}^{s}:=f_{q_1}\left(x_{q_1}^{o}(t_s),u_{q_1}^{o}(t_s)\right)$.

Furthermore, the first order dynamics of the (forward) cost sensitivity are determined by

$$z_{q_0}(t)=l_{q_0}\left(x_{q_0}^{o}(t),v\right)-l_{q_0}\left(x_{q_0}^{o}(t),u_{q_0}^{o}(t)\right), \tag{8.52}$$

$$\frac{d}{d\tau}z_{q_0}(\tau)=\frac{\partial l_{q_0}}{\partial x_{q_0}}\left(x_{q_0}^{o}(\tau),u_{q_0}^{o}(\tau)\right)y_{q_0}(\tau), \tag{8.53}$$

$$z_{q_1}(t_s)=z_{q_0}(t_s-)+\left[\nabla c+\gamma_s\left(l_{q_1}^{s}-l_{q_0}^{s}-\nabla c^T f_{q_0}^{s}\right)\nabla m\right]^T y_{q_0}(t_s-), \tag{8.54}$$

$$\frac{d}{d\tau}z_{q_1}(\tau)=\frac{\partial l_{q_1}}{\partial x_{q_1}}\left(x_{q_1}^{o}(\tau),u_{q_1}^{o}(\tau)\right)y_{q_1}(\tau). \tag{8.55}$$

Defining

$$d\lambda_{q_0}^{o} = -\left(\frac{\partial l_{q_0}}{\partial x_{q_0}} \left(x_{q_0}^{o}, u_{q_0}^{o}\right) + \left[\frac{\partial f_{q_0}}{\partial x_{q_0}} \left(x_{q_0}^{o}, u_{q_0}^{o}\right) \right]^T \lambda_{q_0}^{o} + \sum_{k=1}^{n_w} \left[\frac{\partial g_{q_0}}{\partial x_{q_0}} \left(x_{q_0}^{o}\right) \right]^T K_{q_0}^{o(k)} \right) dt + K_{q_0}^{o}(t)\, dw(t), \tag{8.56}$$

and taking similar steps as those in the previous part, Lemma 8.2 is employed to show that there exist $\Im^t$-adapted processes $\lambda_{q_0}^{o}, K_{q_0}^{o}$ such that for $t \in (t_0, t_s)$:

$$\left.\frac{d}{d\varepsilon} J(u^{\varepsilon})\right|_{\varepsilon=0} = \mathbb{E}\left\{ z_{q_1}(t_f) + \left[\lambda_{q_1}^{o}(t_f)\right]^T y_{q_1}(t_f) \right\} = \mathbb{E}\left\{ z_{q_0}(t) + \left[\lambda_{q_0}^{o}(t)\right]^T y_{q_0}(t) \right\} \geq 0, \tag{8.57}$$

Thus,

$$\begin{aligned}\mathbb{E}\Big\{ & l_{q_0}\left(x_{q_0}^{o}(t), v\right) + \left[\lambda_{q_0}^{o}(t)\right]^T f_{q_0}\left(x_{q_0}^{o}(t), v\right) \\ & - l_{q_0}\left(x_{q_0}^{o}(t), u_{q_0}^{o}(t)\right) - \left[\lambda_{q_0}^{o}(t)\right]^T f_{q_0}\left(x_{q_0}^{o}(t), u_{q_0}^{o}(t)\right) \Big\} \geq 0,\end{aligned} \tag{8.58}$$

which, by [42, Theorem 2.1], results in

$$\begin{aligned} & l_{q_0}\left(x_{q_0}^{o}(t), u_{q_0}^{o}(t)\right) + \left[\lambda_{q_0}^{o}(t)\right]^T f_{q_0}\left(x_{q_0}^{o}(t), u_{q_0}^{o}(t)\right) \\ & \leq l_{q_0}\left(x_{q_0}^{o}(t), v\right) + \left[\lambda_{q_0}^{o}(t)\right]^T f_{q_0}\left(x_{q_0}^{o}(t), v\right), \end{aligned} \tag{8.59}$$

a.s. for all $v : \Im^t -$ measurable random variables in U_{q_0}. The Hamiltonian minimization condition (8.19) in location q_0 directly follows (8.59) which together with (8.38) completes the proof of (8.19) for the case under study.

The adjoint process dynamics (8.14) are directly deduced from (8.56) and (8.32) together with the Hamiltonian definition (8.12). In order to derive the adjoint boundary conditions (8.18) we consider (8.36) for $t \downarrow t_s$ and (8.57) for $t \uparrow t_s$ to write

$$\mathbb{E}\left\{ z_{q_1}(t_s) + \left[\lambda_{q_1}^{o}(t_s+)\right]^T y_{q_1}(t_s) \right\} = \mathbb{E}\left\{ z_{q_0}(t_s-) + \left[\lambda_{q_0}^{o}(t_s)\right]^T y_{q_0}(t_s-) \right\}. \tag{8.60}$$

Substitution of $y_{q_1}(t_s)$ and $z_{q_1}(t_s)$ from (8.50) and (8.54) results in

$$\mathbb{E}\Big\{z_{q_0}(t_s-)+\left[\nabla c+\gamma_s\left(l_{q_1}^s-l_{q_0}^s-\nabla c^T f_{q_0}^s\right)\nabla m\right]^T y_{q_0}(t_s-) \\ +\left[\lambda_{q_1}^o(t_s+)\right]^T\left[\nabla\xi+\gamma_s\left(f_{q_1}^s-\nabla\xi\, f_{q_0}^s\right)\nabla m^T\right]y_{q_0}(t_s-)\Big\} \\ =\mathbb{E}\Big\{z_{q_0}(t_s-)+\left[\lambda_{q_0}^o(t_s)\right]^T y_{q_0}(t_s-)\Big\}. \quad (8.61)$$

or

$$\mathbb{E}\Big\{\left[\nabla c+p\nabla m+\nabla\xi^T\lambda_{q_1}^o(t_s+)-\lambda_{q_0}^o(t_s)\right]^T y_{q_0}(t_s-)\Big\}=0, \quad (8.62)$$

in which the notation

$$p:=\gamma_s\Big(l_{q_1}^s-l_{q_0}^s-\nabla c^T f_{q_0}^s+\lambda_{q_1}^o(t_s+)^T\left(f_{q_1}^s-\nabla\xi\, f_{q_0}^s\right)\Big), \quad (8.63)$$

is used. In order to prove the Hamiltonian continuity condition (8.20) we note that on one hand:

$$H_{q_0}(t_s)\equiv H_{q_0}\left(x_{q_0}^o(t_s-),u_{q_0}^o(t_s-),\lambda_{q_0}^o(t_s),K_{q_0}^o(t_s)\right)=l_{q_0}^s+\lambda_{q_0}^{s\,T} f_{q_0}^s+\mathrm{tr}\Big(\left[K_{q_0}^s\right]^T g_{q_0}^s\Big) \\ =l_{q_0}^s+\left[p\nabla m+\nabla c+\nabla\xi^T\lambda_{q_1}^s\right]^T f_{q_0}^s+\mathrm{tr}\Big(\left[K_{q_0}^s\right]^T g_{q_0}^s\Big) \\ =l_{q_0}^s+\gamma_s\nabla m^T f_{q_0}^s\Big(l_{q_1}^s-l_{q_0}^s-\nabla c^T f_{q_0}^s+\lambda_{q_1}^{s\,T}\left(f_{q_1}^s-\nabla\xi\, f_{q_0}^s\right)\Big)+\nabla c^T f_{q_0}^s+\lambda_{q_1}^{s\,T}\nabla\xi\, f_{q_0}^s+\mathrm{tr}\Big(\left[K_{q_0}^s\right]^T g_{q_0}^s\Big) \\ =l_{q_1}^s+\lambda_{q_1}^{s\,T} f_{q_1}^s+\mathrm{tr}\Big(\left[K_{q_0}^s\right]^T g_{q_0}^s\Big), \quad (8.64)$$

where in the derivation of the last equality γ_s is substituted from (8.47). On the other hand,

$$H_{q_1}(t_s)\equiv H_{q_1}\left(x_{q_1}^o(t_s),u_{q_1}^o(t_s),\lambda_{q_1}^o(t_s+),K_{q_1}^o(t_s+)\right)=l_{q_1}^s+\lambda_{q_1}^{s\,T} f_{q_1}^s+\mathrm{tr}\Big(\left[K_{q_1}^s\right]^T g_{q_1}^s\Big) \\ =l_{q_1}^s+\lambda_{q_1}^{s\,T} f_{q_1}^s+\mathrm{tr}\Big(\left[K_{q_1}^s\right]^T\xi\left(g_{q_0}^s\right)\Big)=l_{q_1}^s+\lambda_{q_1}^{s\,T} f_{q_1}^s+\mathrm{tr}\Big(\left[\xi\left(K_{q_1}^s\right)\right]^T g_{q_0}^s\Big) \\ =l_{q_1}^s+\lambda_{q_1}^{s\,T} f_{q_1}^s+\mathrm{tr}\Big(\left[K_{q_0}^s\right]^T g_{q_0}^s\Big). \quad (8.65)$$

In the derivation of the above arguments, we made use the linearity of the mapping ξ provided in A0.S, and we employed the assumption (8.3) in A1.S. This completes the proof of the Stochastic Hybrid Minimum Principle. □

Chapter 9

Future Research Directions

9.1 Stochastic Hybrid Dynamic Programming

In this thesis, the Stochastic Hybrid Minimum Principle (SHMP) has been established for a general class of hybrid systems with both autonomous and controlled switchings and state jumps subject to possible changes in the dimension of the state space. The inevitability of switchings and jumps upon arrival on switching manifolds is of particular importance in the modelling of mechanical impact problems (e.g. [115] as well as the well known example of a bouncing ball in a turbulent environment) and friction-resisted dynamical systems with distinct evolutions under static and dynamic frictions (see e.g. [100]). The SHMP established here generalizes the deterministic HMP presented in Chapter 3. Furthermore, as proved in the case of deterministic hybrid optimal control problems in Chapter 5, the adjoint process in the HMP and the gradient of the value function in Hybrid Dynamic Programming (HDP) are identical to each other almost everywhere. So due to the fact that the same relationship holds for continuous parameter stochastic optimal control problems (see e.g. [26]), it is natural to expect the adjoint process in the SHMP and the gradient of the value function in Stochastic HDP (SHDP) to be identical almost everywhere.

9.2 Sufficient Conditions of Optimality

All the theorems in this thesis hold as necessary conditions of optimality. While the sufficiency of the HMP can be shown to hold for numerous hybrid optimal control problems, for instance when the Hamiltonian is convex, a major drawback in solving the HJB equation for obtaining the solution(s) of both classical and hybrid optimal control problems is the requirement that the HJB equation admits classical solutions, i.e. the solutions are assumed to be smooth enough to satisfy the HJB equation. Unfortunately, this is not necessarily the case and therefore the discussion of the so-called viscosity solutions (e.g. the extension of the results of [16, 18, 19] for the results of Chapter 4) is inevitable for establishing the sufficient optimality conditions based on HDP.

9.3 Algorithms Based on the HMP/SHMP and HDP/SHDP

The HMP-based algorithms give optimal trajectories and optimal control inputs for a given switching sequence based on the Hybrid Minimum Principle [7, 9, 10, 67, 116]. The Optimality Zones algorithms [117–119] partition the Cartesian product of the system's time and state space with itself to give the optimal control law and optimal switching sequence in the state space. The above algorithms are developed based upon earlier versions of the HMP and HDP results, and need to be modified in order to reflect the presence of switching costs, as well as the possibility of state space dimension change, and existence of low dimensional switching manifolds.

Furthermore, the results on the HMP-HDP relationship can be employed to develop HMP-HDP based algorithms that provide the optimal switching sequence together with optimal controls.

The extension of the above suggested algorithms to stochastic optimal control problems is another direction for further research.

9.4 Control Dependent Diffusion Terms and Second Order Adjoint Processes

When diffusion terms are functions of the system state only, as is the case in this thesis, the SHMP is derived via first-order variational analyses as is done in Chapter 8. However, when diffusion terms also depend on the controls, one is required to study both the first-order and second-order variations and derive the SHMP using a stochastic Hamiltonian system consisting of two forward-backward stochastic differential equations and a minimization condition with an additional term quadratic in the diffusion coefficient (see e.g. [26, 42, 43]).

Furthermore, the assumptions in A1.S, as well as the linearity of the jump maps assumed in A0.S, both of which are in force to ensure the validity of first order variations, can be relaxed when second order variations are studied.

9.5 Mean Field Hybrid Games

Mean Field Game (MFG) theory studies the existence of Nash equilibria, together with the individual strategies which generate them, in games involving a large number of agents modelled by controlled stochastic dynamical systems. This is achieved by exploiting the relationship between the finite and corresponding infinite limit population problems. The solution of the infinite population problem is given by the fundamental MFG Hamilton-Jacobi-Bellman (HJB) and Fokker-Planck-Kolmogorov (FPK) equations which are linked by the state distribution of a generic agent, otherwise known as the system's mean field [120].

The study of Mean Field Games that undergo changes in their governing dynamics has not yet been addressed in the literature. However, there are several opportunities for the extension of the results of the current thesis in the MFG framework. In particular

- Autonomous switching of the game dynamics for minor players. An example of which is the introduction of new regulations if the state of a society reaches certain levels of richness or crisis (for instance in the management of water resources).

- Autonomous and controlled switching of the game dynamics when a major player is present. An example of which is the introduction of new products by a company (major player) which results in major change of consumption dynamics for consumers (minor players) as well as the revenue dynamics for the company. Another example is given by electoral dynamics, in particular in the Canadian system where the date of election is also a decision parameter under the control of the sitting government. A Mean Field Hybrid Games framework would enable one to study such elections, not only over a single courses of governance, but also over several elections.

Bibliography

[1] L. Pontryagin, V. Boltyanskii, R. Gamkrelidze, and E. Mishchenko, *The Mathematical Theory of Optimal Processes*, vol. 4. Wiley Interscience, New York, 1962.

[2] F. H. Clarke and R. B. Vinter, "Applications of Optimal Multiprocesses," *SIAM Journal on Control and Optimization*, vol. 27, no. 5, pp. 1048–1071, 1989.

[3] F. H. Clarke and R. B. Vinter, "Optimal Multiprocesses," *SIAM Journal on Control and Optimization*, vol. 27, no. 5, pp. 1072–1091, 1989.

[4] H. J. Sussmann, "A Nonsmooth Hybrid Maximum Principle," *Lecture Notes in Control and Information Sciences, Springer London, Volume 246*, pp. 325–354, 1999.

[5] H. J. Sussmann, "Maximum Principle for Hybrid Optimal Control Problems," in *Proceedings of the 38th IEEE Conference on Decision and Control, CDC*, pp. 425–430, 1999.

[6] P. E. Caines, F. H. Clarke, X. Liu, and R. B. Vinter, "A Maximum Principle for Hybrid Optimal Control Problems with Pathwise State Constraints," in *Proceedings of the 45th IEEE Conference on Decision and Control*, pp. 4821–4825, 2006.

[7] M. S. Shaikh and P. E. Caines, "On the Hybrid Optimal Control Problem: Theory and Algorithms," *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 1587–1603, 2007. Corrigendum: vol. 54, no. 6, pp. 1428, 2009.

[8] M. Garavello and B. Piccoli, "Hybrid Necessary Principle," *SIAM Journal on Control and Optimization*, vol. 43, no. 5, pp. 1867–1887, 2005.

[9] F. Taringoo and P. E. Caines, "On the Optimal Control of Impulsive Hybrid Systems on Riemannian Manifolds," *SIAM Journal on Control and Optimization*, vol. 51, no. 4, pp. 3127–3153, 2013.

[10] F. Taringoo and P. Caines, "Gradient Geodesic and Newton Geodesic HMP Algorithms for the Optimization of Hybrid Systems," *Annual Reviews in Control*, vol. 35, no. 2, pp. 187–198, 2011.

[11] P. Riedinger, F. R. Kratz, C. Iung, and C. Zanne, "Linear Quadratic Optimization for Hybrid Systems," in *Proceedings of the 38th IEEE Conference on Decision and Control, CDC*, pp. 3059–3064, 1999.

[12] X. Xu and P. J. Antsaklis, "Optimal Control of Switched Systems based on Parameterization of the Switching Instants," *IEEE Transactions on Automatic Control*, vol. 49, no. 1, pp. 2–16, 2004.

[13] V. Azhmyakov, V. Boltyanski, and A. Poznyak, "Optimal Control of Impulsive Hybrid Systems," *Nonlinear Analysis: Hybrid Systems*, vol. 2, no. 4, pp. 1089–1097, 2008.

[14] A. V. Dmitruk and A. M. Kaganovich, "The Hybrid Maximum Principle is a consequence of Pontryagin Maximum Principle," *Systems & Control Letters*, vol. 57, no. 11, pp. 964–970, 2008.

[15] R. Bellman, "Dynamic Programming," *Science*, vol. 153, no. 3731, pp. 34–37, 1966.

[16] A. Bensoussan and J. L. Menaldi, "Hybrid Control and Dynamic Programming," *Dynamics of Continuous, Discrete and Impulsive Systems Series B: Application and Algorithm*, vol. 3, no. 4, pp. 395–442, 1997.

[17] M. S. Branicky, V. S. Borkar, and S. K. Mitter, "A Unified Framework for Hybrid Control: Model and Optimal Control Theory," *IEEE Transactions on Automatic Control*, vol. 43, no. 1, pp. 31–45, 1998.

[18] S. Dharmatti and M. Ramaswamy, "Hybrid Control Systems and Viscosity Solutions," *SIAM Journal on Control and Optimization*, vol. 44, no. 4, pp. 1259–1288, 2005.

[19] G. Barles, S. Dharmatti, and M. Ramaswamy, "Unbounded Viscosity Solutions of Hybrid Control Systems," *ESAIM - Control, Optimisation and Calculus of Variations*, vol. 16, no. 1, pp. 176–193, 2010.

[20] S. Hedlund and A. Rantzer, "Convex Dynamic Programming for Hybrid Systems," *IEEE Transactions on Automatic Control*, vol. 47, no. 9, pp. 1536–1540, 2002.

[21] P. E. Caines, M. Egerstedt, R. Malhamé, and A. Schöllig, "A Hybrid Bellman Equation for Bimodal Systems," in *Proceedings of the 10th International Conference on Hybrid Systems: Computation and Control, HSCC*, vol. 4416 LNCS, pp. 656–659, 2007.

[22] A. Schöllig, P. E. Caines, M. Egerstedt, and R. Malhamé, "A hybrid Bellman Equation for Systems with Regional Dynamics," in *Proceedings of the 46th IEEE Conference on Decision and Control, CDC*, pp. 3393–3398, 2007.

[23] M. S. Shaikh and P. E. Caines, “A Verification Theorem for Hybrid Optimal Control Problem,” in *Proceedings of the IEEE 13th International Multitopic Conference, INMIC*, 2009.

[24] F. H. Clarke and R. B. Vinter, “The Relationship between the Maximum Principle and Dynamic Programming,” *SIAM Journal on Control and Optimization*, vol. 25, no. 5, pp. 1291–1311, 1987.

[25] R. B. Vinter, “New Results on the Relationship between Dynamic Programming and the Maximum Principle,” *Journal of Mathematics of Control, Signals and Systems*, vol. 1, no. 1, pp. 97–105, 1988.

[26] J. Yong and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*. Springer-Verlag New York, 1999.

[27] X. Y. Zhou, “Maximum Principle, Dynamic Programming, and their Connection in Deterministic Control,” *Journal of Optimization Theory and Applications*, vol. 65, no. 2, pp. 363–373, 1990.

[28] K. E. Kim, “Relationship between Dynamic Programming and the Maximum Principle under State Constraints,” *Journal of Convex Analysis*, vol. 6, no. 2, pp. 335–348, 1999.

[29] W. Fleming and R. Rishel, *Deterministic and Stochastic Optimal Control*. Springer New York, 1975.

[30] P. Cannarsa and H. Frankowska, “Some Characterizations of Optimal Trajectories in Control Theory,” *SIAM Journal on Control and Optimization*, vol. 29, no. 6, pp. 1322–1347, 1991.

[31] A. Cernea and H. Frankowska, “A Connection between the Maximum Principle and Dynamic Programming for Constrained Control Problems,” *SIAM Journal on Control and Optimization*, vol. 44, no. 2, pp. 673–703, 2005.

[32] J. Ma and J. Yong, *Forward-Backward Stochastic Differential Equations and their Applications*. No. 1702, Springer Science & Business Media, 1999.

[33] J. M. Bismut, “An Introductory Approach to Duality in Optimal Stochastic Control,” *SIAM Review*, vol. 20, no. 1, pp. 62–78, 1978.

[34] A. Bensoussan, “Stochastic Maximum Principle for Distributed Parameter Systems,” *Journal of the Franklin Institute*, vol. 315, no. 5, pp. 387–406, 1983.

[35] E. Pardoux and S. Peng, “Adapted Solution of a Backward Stochastic Differential Equation,” *Systems and Control Letters*, vol. 14, no. 1, pp. 55 – 61, 1990.

[36] H. Kushner, "Optimal Stochastic Control," *IRE Transactions on Automatic Control*, vol. 7, no. 5, pp. 120–122, 1962.

[37] N. V. Krylov, *Controlled Diffusion Processes*, vol. 14. Springer Science & Business Media, 2008.

[38] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, vol. 25. Springer Science & Business Media, 2006.

[39] H. J. Kushner and F. C. Schweppe, "A Maximum Principle for Stochastic Control Systems," *Journal of Mathematical Analysis and Applications*, vol. 8, no. 2, pp. 287–302, 1964.

[40] H. J. Kushner, "Necessary Conditions for Continuous Parameter Stochastic Optimization Problems," *SIAM Journal on Control*, vol. 10, no. 3, pp. 550–565, 1972.

[41] U. G. Haussmann, *A Stochastic Maximum Principle for Optimal Control of Diffusions*. New York, NY, USA: John Wiley & Sons, Inc., 1986.

[42] A. Bensoussan, *Lectures on Stochastic Control*, pp. 1–62. Berlin, Heidelberg: Springer Berlin Heidelberg, 1982.

[43] S. Peng, "A General Stochastic Maximum Principle for Optimal Control Problems," *SIAM J. Control Optim.*, vol. 28, pp. 966–979, June 1990.

[44] C. Aghayeva and Q. Abushov, "The Maximum Principle for the Nonlinear Stochastic Optimal Control Problem of Switching Systems," *Global Optimization*, vol. 56, no. 2, pp. 341–352, 2011.

[45] A. Bensoussan and J. Menaldi, "Stochastic Hybrid Control," *Mathematical Analysis and Applications*, vol. 249, no. 1, pp. 261–288, 2000.

[46] M. S. Shaikh and P. E. Caines, "On Relationships between Weierstrass-Erdmannn Corner Condition, Snell's Law and the Hybrid Minimum Principle," in *Proceedings of International Bhurban Conference on Applied Sciences and Technology, IBCAST*, pp. 117–122, 2007.

[47] D. Liberzon, *Calculus of Variations and Optimal Control Theory: A Concise Introduction*. Princeton University Press, 2011.

[48] A. Puri and P. Varaiya, *Hybrid Systems II*, ch. Verification of Hybrid Systems Using Abstractions, pp. 359–369. Berlin, Heidelberg: Springer Berlin Heidelberg, 1995.

[49] R. Alur, T. A. Henzinger, G. Lafferriere, and G. J. Pappas, "Discrete Abstractions of Hybrid Systems," *Proceedings of the IEEE*, vol. 88, no. 7, pp. 971–984, 2000.

[50] R. Alur, T. Dang, and F. Ivančić, "Progress on Reachability Analysis of Hybrid Systems Using Predicate Abstraction," *Proceedings of the 6th International Workshop on Hybrid Systems: Computation and Control, HSCC, Prague, Czech Republic*, pp. 4–19, 2003.

[51] E. Clarke, A. Fehnker, Z. Han, B. Krogh, J. Ouaknine, O. Stursberg, and M. Theobald, "Abstraction and Counterexample-Guided Refinement in Model Checking of Hybrid Systems," *International Journal of Foundations of Computer Science*, vol. 14, no. 04, pp. 583–604, 2003.

[52] A. Tiwari and G. Khanna, "Series of Abstractions for Hybrid Automata," *Proceedings of the 5th International Workshop on Hybrid Systems: Computation and Control, HSCC, Stanford, CA, USA*, pp. 465–478, 2002.

[53] M. Broucke, "A Geometric Approach to Bisimulation and Verification of Hybrid Systems," in *Hybrid Systems: Computation and Control*, pp. 61–75, Springer, 1999.

[54] M. K. Helwa and P. E. Caines, "Hierarchical Control of Piecewise Affine Hybrid Systems," in *Decision and Control (CDC), 2014 IEEE 53rd Annual Conference on*, pp. 3950–3956, IEEE, 2014.

[55] A. J. Van der Schaft and J. M. Schumacher, "An Introduction to Hybrid Dynamical Systems," *Lecture Notes in Control and Information Sciences, Springer London, Volume 251*, 2000.

[56] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid Dynamical Systems: Modeling, Stability, and Robustness*. Princeton University Press, 2012.

[57] R. Goebel, R. G. Sanfelice, and A. R. Teel, "Hybrid Dynamical Systems," *IEEE Control Systems*, vol. 29, no. 2, pp. 28–93, 2009.

[58] R. G. Sanfelice, R. Goebel, and A. R. Teel, "Invariance Principles for Hybrid Systems With Connections to Detectability and Asymptotic Stability," *IEEE Transactions on Automatic Control*, vol. 52, no. 12, pp. 2282–2297, 2007.

[59] J. Liu and A. R. Teel, "Lyapunov-Based Sufficient Conditions for Stability of Hybrid Systems With Memory," *IEEE Transactions on Automatic Control*, vol. 61, no. 4, pp. 1057–1062, 2016.

[60] D. Liberzon, *Switching in Systems and Control*. Springer Science & Business Media, 2012.

[61] D. Liberzon and A. S. Morse, "Basic Problems in Stability and Design of Switched Systems," *Control Systems, IEEE*, vol. 19, no. 5, pp. 59–70, 1999.

[62] D. Liberzon, J. P. Hespanha, and A. S. Morse, "Stability of Switched Systems: a Lie-Algebraic Condition," *Systems & Control Letters*, vol. 37, no. 3, pp. 117–122, 1999.

[63] J. P. Hespanha, "Uniform Stability of Switched Linear Systems: Extensions of LaSalle's Invariance Principle," *Automatic Control, IEEE Transactions on*, vol. 49, no. 4, pp. 470–482, 2004.

[64] M. S. Branicky, "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Transactions on Automatic Control*, vol. 43, no. 4, pp. 475–482, 1998.

[65] R. A. Decarlo, M. S. Branicky, S. Pettersson, and B. Lennartson, "Perspectives and Results on the Stability and Stabilizability of Hybrid Systems," *Proceedings of the IEEE*, vol. 88, no. 7, pp. 1069–1082, 2000.

[66] M. Johansson and A. Rantzer, "Computation of Piecewise Quadratic Lyapunov Functions for Hybrid Systems," *IEEE transactions on automatic control*, vol. 43, no. 4, pp. 555–559, 1998.

[67] F. Taringoo and P. E. Caines, "Gradient-Geodesic HMP Algorithms for the Optimization of Hybrid Systems Based on the Geometry of Switching Manifolds," in *Proceedings of the 49th IEEE Conference on Decision and Control, CDC*, pp. 1534–1539, 2010.

[68] I. Capuzzo-Dolcetta and L. C. Evans, "Optimal Switching for Ordinary Differential Equations," *SIAM Journal on Control and Optimization*, vol. 22, no. 1, pp. 143–161, 1984.

[69] H. Zhang and M. R. James, "Optimal Control of Hybrid Systems and a System of Quasi-Variational Inequalities," *SIAM Journal on Control and Optimization*, vol. 45, no. 2, pp. 722–761, 2007.

[70] J. Lygeros, "On Reachability and Minimum Cost Optimal Control," *Automatica*, vol. 40, no. 6, pp. 917–927, 2004.

[71] J. Lygeros, C. Tomlin, and S. Sastry, "Multiobjective Hybrid Controller Synthesis," in *Hybrid and Real-Time Systems, International Workshop*, pp. 109–123, 1997.

[72] C. Tomlin, J. Lygeros, and S. Sastry, "A Game Theoretic Approach to Controller Design for Hybrid Systems," *Proceedings of the IEEE*, vol. 88, pp. 949–970, July 2000.

[73] F. Zhu and P. J. Antsaklis, "Optimal Control of Hybrid Switched Systems: A Brief Survey," *Discrete Event Dynamic Systems*, vol. 25, no. 3, pp. 345–364, 2015.

[74] P. E. Caines, "Lecture Notes on Nonlinear and Hybrid Control Systems: Dynamics, Stabilization and Optimal Control," *Department of Electrical and Computer Engineering (ECE), McGill University*, 2013.

[75] B. Passenberg, M. Leibold, O. Stursberg, and M. Buss, “The Minimum Principle for Time-Varying Hybrid Systems with State Switching and Jumps,” in *Proceedings of the 50th IEEE Conference on Decision and Control and European Control Conference, CDC-ECC*, pp. 6723–6729, 2011.

[76] A. Pakniyat and P. E. Caines, “The Hybrid Minimum Principle in the Presence of Switching Costs,” in *Proceedings of the 52nd IEEE Conference on Decision and Control, CDC*, pp. 3831–3836, 2013.

[77] A. Pakniyat and P. E. Caines, “On the Relation between the Minimum Principle and Dynamic Programming for Hybrid Systems,” in *Proceedings of the 53rd IEEE Conference on Decision and Control, CDC*, pp. 19–24, 2014.

[78] A. Pakniyat and P. E. Caines, “On the Minimum Principle and Dynamic Programming for Hybrid Systems,” in *Proceedings of the 19th International Federation of Automatic Control World Congress, IFAC*, pp. 9629–9634, 2014.

[79] E. D. Sontag, *Mathematical Control Theory: Deterministic Finite Dimensional Systems*. Texts in Applied Mathematics, New York, Berlin, Paris: Springer, 1998.

[80] A. Dmitruk and N. Osmolovskii, “On the Proof of Pontryagin’s Maximum Principle by Means of Needle Variations,” *ArXiv e-prints*, Dec. 2014.

[81] A. D. Lewis, “The Maximum Principle of Pontryagin in Control and in Optimal Control,” *Handouts for the course taught at the Universitat Politcnica de Catalunya. Department of Mathematics and Statistics Queen’s University.[online] Available: http://www. mast. queensu. ca/andrew/teaching/courses. shtml (May 5 2006)*, 2006.

[82] S. E. Dreyfus, “Dynamic Programming and the Calculus of Variations,” *Journal of Mathematical Analysis and Applications*, vol. 1, no. 2, pp. 228–239, 1960.

[83] D. Jacobson and D. Mayne, *Differential Dynamic Programming*. American Elsevier Pub. Co., 1970.

[84] R. Vinter, *Optimal Control*. Systems and control, Boston: Birkhäuser, 2000.

[85] L. I. Rozonoer, “Certain Hypotheses in Optimal Control Theory and the Relationship of the Maximum Principle with the Dynamic Programming Method,” *Journal of Automation and Remote Control*, vol. 64, no. 8, pp. 1237–1240, 2003.

[86] J. Speyer and D. Jacobson, *Primer on Optimal Control Theory*. Society for Industrial and Applied Mathematics, 2010.

[87] M. G. Crandall and P. L. Lions, “Viscosity Solutions of Hamilton-Jacobi Equations,” *Transactions of the American Mathematical Society*, vol. 277, no. 1, pp. 1–42, 1983.

[88] S. Jafarpour and A. D. Lewis, *Time-Varying Vector Fields and Their Flows*. SpringerBriefs in Mathematics, Springer International Publishing, 2014.

[89] H. Federer, *Geometric Measure Theory*. Grundlehren der mathematischen Wissenschaften, Springer, 1969.

[90] W. Rudin, *Real and Complex Analysis*. McGraw-Hill, New York, 1987.

[91] I. Segal and R. Kunze, *Integrals and Operators*. Springer-Verlag Berlin Heidelberg, 1978.

[92] V. N. Kolokoltsov, J. Li, and W. Yang, "Mean Field Games and Nonlinear Markov Processes," *ArXiv e-prints*, Dec. 2011.

[93] G. N. Galbraith and R. B. Vinter, "Lipschitz Continuity of Optimal Controls for State Constrained Problems," *SIAM Journal on Control and Optimization*, vol. 42, no. 5, pp. 1727–1744, 2003.

[94] S. Ng Yow Thow and P. E. Caines, "On the Continuity of Optimal Controls and Value Functions with respect to Initial Conditions," *Systems & Control Letters*, vol. 61, no. 12, pp. 1294–1298, 2012.

[95] M. Kamgarpour and C. Tomlin, "On Optimal Control of Non-Autonomous Switched Systems under a Fixed Switching Sequence," *Automatica*, vol. 48, no. 6, pp. 1177–1181, 2012.

[96] A. Pakniyat and P. E. Caines, "On the Relation between the Hybrid Minimum Principle and Hybrid Dynamic Programming: A Linear Quadratic Example," in *Proceedings of the 5th IFAC Conference on Analysis and Design of Hybrid Systems, Atlanta, GA, USA*, pp. 169–174, 2015.

[97] A. Pakniyat and H. Salarieh, "A Parametric Study on Design of a Microrate-Gyroscope with Parametric Resonance," *Measurement: Journal of the International Measurement Confederation*, vol. 46, no. 8, pp. 2661–2671, 2013.

[98] A. Pakniyat, H. Salarieh, and A. Alasty, "Stability Analysis of a New Class of MEMS Gyroscopes with Parametric Resonance," *Acta Mechanica*, vol. 223, no. 6, pp. 1169–1185, 2012.

[99] R. E. Kalman, "Contributions to the Theory of Optimal Control," *Boletin de la Sociedad Matematica Mexicana*, pp. 102–119, 1960.

[100] A. Pakniyat and P. E. Caines, "Time Optimal Hybrid Minimum Principle and the Gear Changing Problem for Electric Vehicles," in *Proceedings of the 5th IFAC Conference on Analysis and Design of Hybrid Systems, Atlanta, GA, USA*, pp. 187–192, 2015.

[101] M. S. Rahimi Mousavi, A. Pakniyat, T. Wang, and B. Boulet, “Seamless Dual Brake Transmission For Electric Vehicles: Design, Control and Experiment,” *Mechanism and Machine Theory*, vol. 94, pp. 96–118, 2015.

[102] M. S. Rahimi Mousavi, A. Pakniyat, and B. Boulet, “Dynamic Modeling and Controller Design for a Seamless Two-Speed Transmission for Electric Vehicles,” in *Proceedings of the 2014 IEEE Conference on Control Applications (CCA)*, pp. 635–640, 2014.

[103] M. S. Rahimi Mousavi, A. Pakniyat, M. K. Helwa, and B. Boulet, “Observer-Based Backstepping Controller Design for Gear Shift Control of a Seamless Clutchless Two-Speed Transmission for Electric Vehicles,” in *Vehicle Power and Propulsion Conference (VPPC), 2015 IEEE*, pp. 1–6, IEEE, 2015.

[104] J. V. José and E. J. Saletan, *Classical Dynamics: A Contemporary Approach*. Cambridge University Press, 1998.

[105] A. Sciarretta, G. De Nunzio, and L. Ojeda, “Optimal Ecodriving Control: Energy-Efficient Driving of Road Vehicles as an Optimal Control Problem,” *Control Systems, IEEE*, vol. 35, pp. 71–90, Oct 2015.

[106] N. Petit and A. Sciarretta, “Optimal Drive of Electric Vehicles Using an Inversion-Based Trajectory Generation Approach,” in *Bittanti Sergio, Angelo Cenedese et Sandro Zampieri, éditeurs: Proceedings of the 18th IFAC World Congress (IFAC’11)*, pp. 14519–14526, 2011.

[107] N. Dokuchaev and X. Y. Zhou, “Stochastic Controls with Terminal Contingent Conditions,” *Mathematical Analysis and Applications*, vol. 238, no. 1, pp. 143–165, 1999.

[108] N. Dokuchaev, “On Transversality Conditions for the Stochastic Maximum Principle,” *(in Russian) Differential Equations*, vol. 24, no. 7, pp. 1266–1269, 1988.

[109] F. Chighoub, B. Djehiche, and B. Mezerdi, “The Stochastic Maximum Principle in Optimal Control of Degenerate Diffusions with Non Smooth Coefficients,” *Random Operators and Stochastic Equations*, vol. 17, no. 1, pp. 37–54, 2009.

[110] X. Y. Zhou, “A Unified Treatment of Maximum Principle and Dynamic Programming in Stochastic Controls,” *Stochastics and Stochastic Reports*, vol. 36, no. 3-4, pp. 137–161, 1991.

[111] Z. Wu and F. Zhang, “Stochastic Maximum Principle for Optimal Control Problems of Forward-Backward Systems Involving Impulse Controls,” *IEEE Transactions on Automatic Control*, vol. 56, no. 6, pp. 1401–1406, 2011.

[112] J. Shi and Z. Wu, "Maximum Principle for Forward-Backward Stochastic Control System with Random Jumps and Applications to Finance," *Journal of Systems Science and Complexity*, vol. 23, no. 2, pp. 219–231, 2010.

[113] D. L. Burkholder, B. J. Davis, and R. F. Gundy, "Integral Inequalities for Convex Functions of Operators on Martingales," in *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Volume 2: Probability Theory*, pp. 223–240, University of California Press, 1972.

[114] I. Karatzas and S. Shreve, *Brownian Motion and Stochastic Calculus*. Springer-Verlag New York, 1998.

[115] A. Pakniyat and P. E. Caines, "On the Minimum Principle and Dynamic Programming for Hybrid Systems with Low Dimensional Switching Manifolds," in *Proceedings of the 54th IEEE Conference on Decision and Control, Osaka, Japan*, pp. 2567–2573, 2015.

[116] F. Taringoo and P. Caines, "The Exponential Gradient HMP Algorithm for the Optimization of Hybrid Systems on Lie Groups," in *IFAC Proceedings Volumes (IFAC-PapersOnline)*, pp. 33–38, 2012.

[117] M. Shaikh and P. Caines, "Optimality Zone Algorithms for Hybrid Systems Computation and Control: From Exponential to Linear Complexity," in *Proceedings of the 44th IEEE Conference on Decision and Control, and the European Control Conference, CDC-ECC '05*, vol. 2005, pp. 1403–1408, 2005.

[118] M. Shaikh and P. Caines, "On the Optimal Control of Hybrid Systems: Analysis and Zonal Algorithms for Trajectory and Schedule Optimization," in *Proceedings of the IEEE Conference on Decision and Control*, vol. 3, pp. 2144–2149, 2003.

[119] M. Shaikh and P. Caines, "On the Optimal Control of Hybrid Systems: Optimization of Trajectories, Switching Times, and Location Schedules," *Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics)*, vol. 2623, pp. 466–481, 2003.

[120] P. E. Caines, *Encyclopedia of Systems and Control*, ch. Mean Field Games, pp. 1–6. London: Springer London, 2013.

Appendix A

Burkholder-Davis-Gundy Inequality

Theorem A.1 (Burkholder-Davis-Gundy Inequality [113, 114])**.** *Consider a continuous martingale M which, along with its quadratic variation process $\langle M \rangle$, is bounded. For every stopping time T, we have then*

$$\mathbb{E}\left\{|M_T|^{2m}\right\} \le C'_m \mathbb{E}\left\{\langle M \rangle_T^m\right\}, \qquad m > 0, \tag{A.1}$$

$$B_m \mathbb{E}\left\{\langle M \rangle_T^m\right\} \le \mathbb{E}\left\{|M_T|^{2m}\right\}, \qquad m > \frac{1}{2}, \tag{A.2}$$

$$B_m \mathbb{E}\left\{\langle M \rangle_T^m\right\} \le \mathbb{E}\left\{(M_T^*)^{2m}\right\} \le C_m \mathbb{E}\left\{\langle M \rangle_T^m\right\}, \qquad m > \frac{1}{2}, \tag{A.3}$$

for suitable positive constants B_m, C_m, C'_m which are universal (i.e., depend only on the number m, not on the martingale M nor the stopping time T). □