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by Gérard Piette
Department of Meteorology, M.Sc.

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The results of few numerical integrations of the adiabatic part, with the static stability kept constant, are given. Then heating and internal viscosity are added. A steady state solution is found and its characteristics are worked out analytically.

A further investigation is made with the inclusion of friction with the boundaries.

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L'investigation se termine par l'addition de la friction aux frontières.

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1. INTRODUCTION

In the search to a better understanding of the atmosphere, scientists constructed different laboratory models. Fultz, Hide and their collaborators worked with a rotating annulus of fluid to which a differential heating was applied; the outer wall of the annulus was kept at a high constant temperature while the inner one was kept at a relatively lower constant temperature. In their experiments they identified four different regimes of flow as depicted schematically in Fig 1. The ordinate is the "forced" thermal Rossby number R_{ot}^*:

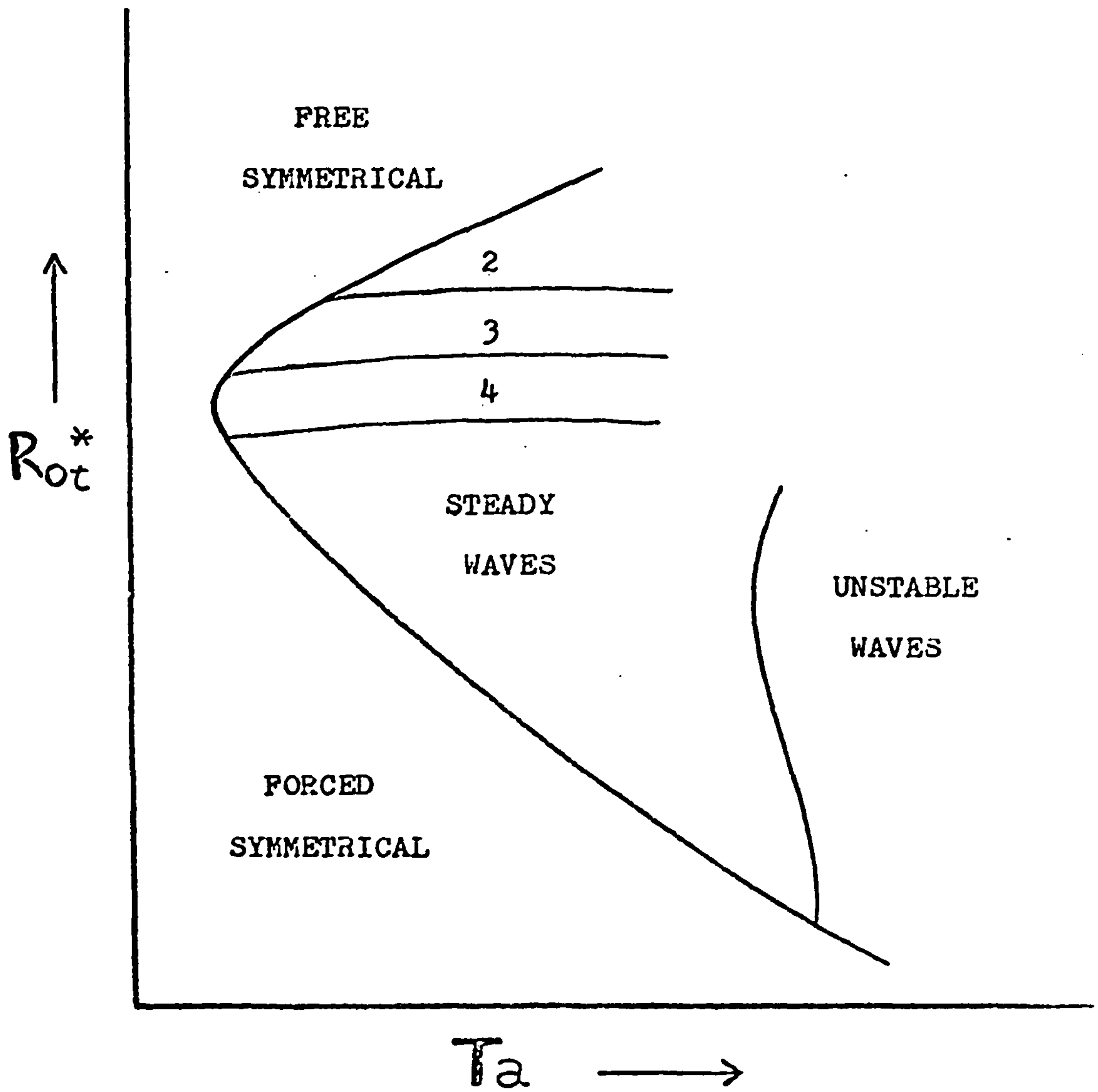
$$R_{ot}^* \equiv \frac{g D \varepsilon (T_b - T_a)}{\Omega^2 (b-a)^2} ,$$

where g is the acceleration of gravity, D is the depth of the fluid, ε is the coefficient of thermal expansion, T_b and T_a are the outer and inner wall temperatures, b and a are the radii of the outer and inner cylindrical walls and Ω is the rotation rate. The abcissa is the Taylor number Ta:

$$Ta \equiv \frac{4 \Omega^2 (b-a)^5}{\nu^2 D} ,$$

where ν is the kinematic viscosity. The steady wave regime is distinguished by the fact that only one wave number at a time dominates the flow, the higher wave numbers occurring

Figure 1. Schematic picture showing the four regimes identified by Fultz et al.



at lower values of R_{ot}*. These waves are quasi-steady and regular in appearance. The unsteady wave regime is characterized by more turbulent, continually changing flow patterns, similar to the large-scale motions of the earth's atmosphere, in which a spectrum of waves of different scales dominates the flow. The axially symmetrical regimes are also called Hadley circulations. In the region between the steady wave regime and the unsteady wave regime, Hide, Fultz and their collaborators have observed a phenomenon which Hide called "vacillation" in which the waves undergo a periodic change first tilting from north-west to south-east and later from north-east to south-west relative to the annulus.

During the winter 62-63, Pfeffer and Chiang observed a different kind of "vacillation" in which the wave pattern expands and contracts with no noticeable change in the tilt of the disturbances. The first type of vacillation was then called "tilted-trough" vacillation and the new one, "amplitude" vacillation.

The main characteristics of the tilted-trough vacillation is the large periodical fluctuation in magnitude and in sign of the rate of conversion of kinetic energy between the eddies and the mean zonal flow. This rate of conversion, $(K_E \rightarrow K_Z)$, depends upon the covariance of $\overline{u'v'}$ and $\partial(\bar{u}/r)/\partial r$. Here u and v are the zonal and meridional components of velocity, respectively. The bar represents a zonal average and the prime represents the departure at a point from this average. This fluctuation was found from

Fultz's data.

In amplitude vacillation, the troughs of the wave are oriented north-south and this implies that $\overline{u'v'}=0$ and $(K_E \rightarrow K_Z)=0$; consequently the energetics of this phenomenon is quite different. There is a striking similarity between this phenomenon and certain index breakdowns in the earth's atmosphere, an example of which is given in Fig 2. Winston and Krueger (1961) did a detailed analysis of the energetics of the index breakdown pictured in Fig 2. They showed it was associated with a rapid adiabatic conversion of potential energy into kinetic energy of the atmosphere.

In order to describe the energy cycle of amplitude vacillation, the simple picture of figure 3 is sufficient. In the figure, the subscripts Z and E refer to the zonal and eddy components respectively; A is the available potential energy, K is the kinetic energy, G is the rate of generation of available potential energy, and F is the rate of dissipation of kinetic energy; the quantities $(A_Z \rightarrow A_E)$, $(A_E \rightarrow K_E)$, $(K_E \rightarrow K_Z)$ and $(K_Z \rightarrow A_Z)$ are the rates of conversion from one form of energy to another. Pfeffer and Chiang's experiments suggest the following cycle. During the first part, where the wave present decays, we have $F_E > (A_E \rightarrow K_E)$ and the authors suggest that $G_Z > (A_Z \rightarrow A_E)$ such that at the end of this first part the strong radial temperature gradient makes conditions favorable for a more rapid release of available potential energy. During the second part, the growth of wave cyclones takes place slowly. During the third

Figure 2. 500 mb charts showing two stages of an atmospheric index cycle. (after Winston and Krueger)

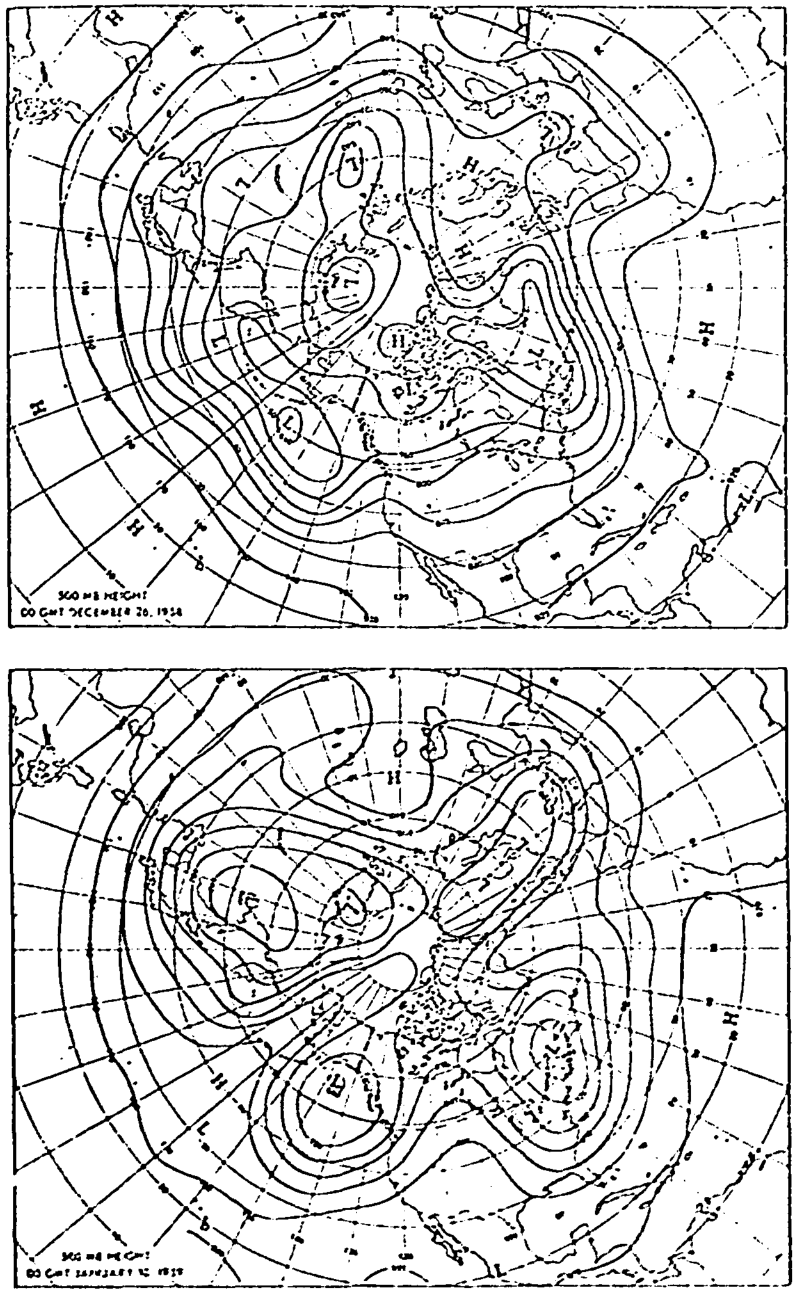
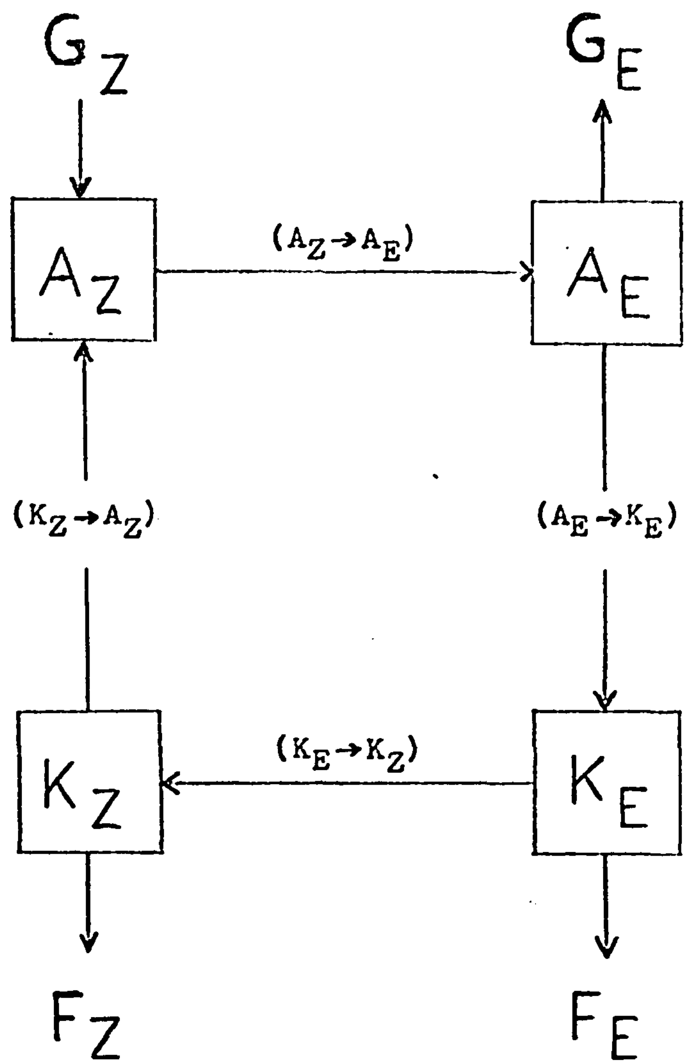


Figure 3. Schematic picture of the energy cycle in a differentially heated, rotating annulus of fluid.



part, the growth becomes more accentuated and it suggests that the adiabatic rates of energy conversion ($A_Z \rightarrow A_E$) and ($A_E \rightarrow K_E$) are the dominant processes affecting A_Z, A_E and K_E. As the eddies grow, the supply of available potential energy diminishes and these conversions must decrease in intensity and we are back to the beginning where the friction takes over. Speculating about the effect of viscosity, Pfeffer and Chiang came to the conclusion that at low Taylor number and high viscosities, only one single two-dimensional wave number can be present and therefore amplitude vacillation can result only from baroclinic interactions between the mean zonal current and the wave. This involves only conversions of the type ($A_Z \rightarrow A_E$) and ($A_E \rightarrow K_E$).

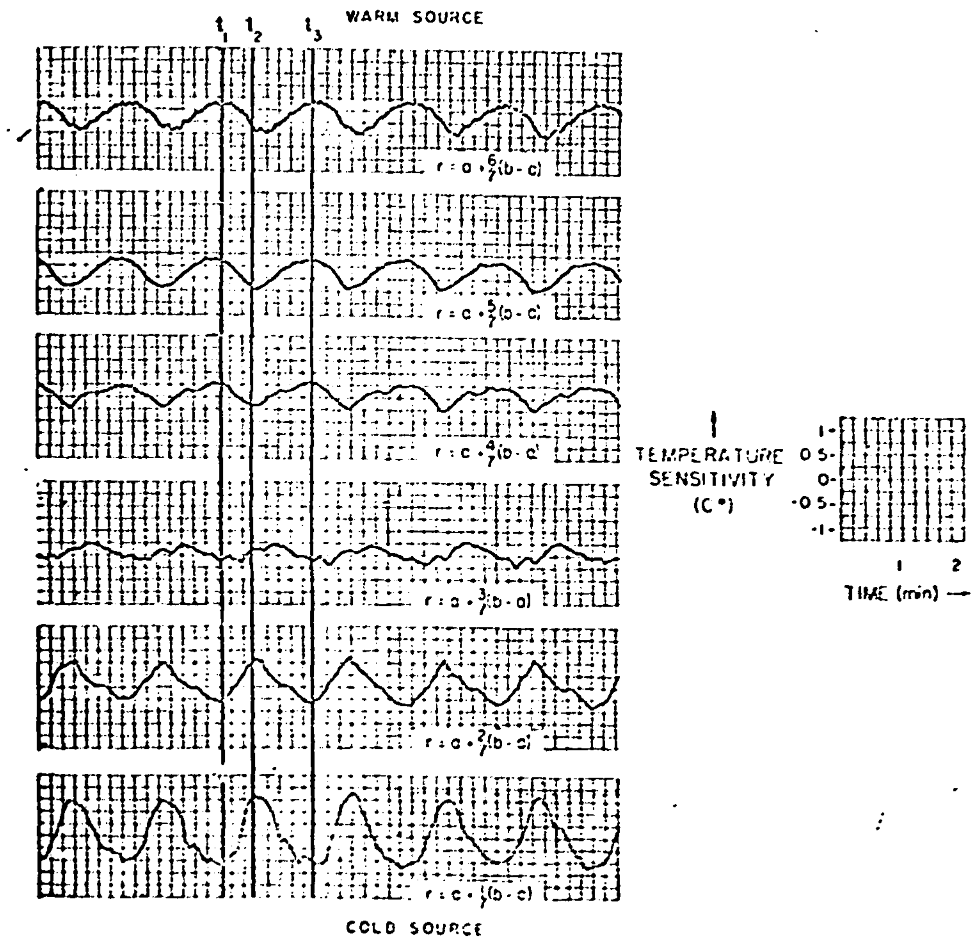
When the Taylor number is increased, a given wave scale in the east-west direction can have more than one wave number in the radial or north-south direction. Under these conditions the resulting wave is free to interact barotropically, as well as baroclinically, with the zonal current, leading to energy conversions of the form ($K_E \rightarrow K_Z$) in addition to other possible energy transformations. In the atmosphere, where many wave scales can be present, an almost pure amplitude vacillation is then sporadic; in laboratory experiments, it is more closely cyclic in nature.

Before coming to the present work, let us mention that Fowlis and Pfeffer (1969) were able to get data of amplitude vacillation of synoptic density. They introduced 50 fine thermocouples in an annulus whose radii were b = 6.03cm and a = 3.49cm. With a clever disposition, they were able to minimize the probe effects to a very small proportion and they got at the same time a good appreciation of the zonally average temperatures at 6 different radii. Figure 4 shows some of their results. They investigated a wave number four pattern. The rapid decrease in the radial temperature gradient between time t_1 and t_2 in figure 4 appears to correspond to the third part of the energy cycle described previously where the wave grows rapidly.

With the aim of knowing something more about vacillation, the following research is done with a mathematical model. It is a spectral model which allows a non-linear interaction between a wave and a zonal flow. In permitting the presence of only one two-dimensional wave number, the tilted-trough vacillation does not occur. Merilees (1968 b) got fairly good results with it in describing the transition between axisymmetric and nonaxisymmetric regimes in a rotating annulus. Merilees also showed that the adiabatic part of the model exibits vacillation and the solutions of the system are elliptic functions of time. (Personnal communication)

In this study we investigate briefly the adiabatic oscillations. The main part of the research includes a simulation of heat exchanges at the boundaries together with diffusion and internal viscosity. Finally, a brief look at the effect of friction with the boundaries completes the work.

Figure 4. Time variations of the zonally averaged temperatures at six different radii over a 10 min. interval. The imposed temperature difference is 10°C.



2. THE MODEL

The following chapter is a partial reproduction of Merilees' article (1968 b); for more detailed information the reader is referred to it.

The model equations, apart from the diabatic terms, are the same as those formulated by Lorenz (1960 b) as the simplest energically consistent model which describes baroclinic flow and considers a variable static stability. If $\psi+\tau$, $\psi-\tau$ denote the stream function for the nondivergent part of the flow in the upper and lower layers, respectively, $\theta+\bar{\sigma}$, $\theta-\bar{\sigma}$ the corresponding temperatures, and $-\chi$ and χ the velocity potentials for the divergent part of the flow in these layers, the governing equations for the model are:

$$\frac{\partial}{\partial t}\nabla^2\psi = -J(\psi,\nabla^2\psi) - J(\tau,\nabla^2\tau) + \left[\frac{\partial}{\partial t}\nabla^2\psi\right]_d , \tag{1}$$

$$\frac{\partial}{\partial t}\nabla^2\tau = -J(\psi,\nabla^2\tau) - J(\tau,\nabla^2\psi) + f\nabla^2\chi + \left[\frac{\partial}{\partial t}\nabla^2\tau\right]_d , \tag{2}$$

$$\frac{\partial\theta}{\partial t} = -J(\psi,\theta) + \bar{\sigma}\nabla^2\chi + \left[\frac{\partial\theta}{\partial t}\right]_d , \tag{3}$$

$$\frac{\partial\bar{\sigma}}{\partial t} = -\overline{\theta\nabla^2\chi} + \left[\frac{d\bar{\sigma}}{dt}\right]_d , \tag{4}$$

where J represents a Jacobian with respect to the horizontal coordinates, f is the (constant) Coriolis parameter and the quantities in brackets are diabatic terms. The bar operator indicates a horizontal average, so that the static stability $\bar{\sigma}$ is function of time alone.

The thermal wind equation for this model is given by

$$\nabla^2 \tau = \frac{1}{4} f^{-1} \epsilon g D \nabla^2 \theta \quad , \tag{5}$$

where ϵ is the coefficient of thermal expansion, g the acceleration due to gravity and D the depth of the fluid. It has been assumed, for the temperature variations anticipated, that the equation of state of the fluid is given by

$$\alpha = \alpha_0 (1 + \epsilon \theta) \quad , \tag{6}$$

where α_0 is a reference value of specific volume.

The frictional effect of the boundaries is modelled in the following way. We suppose that friction with the side boundaries tends to destroy motion relative to the rotating annulus in both the upper and lower layers of the fluid. In addition, the motion in the lower layer is supposed to be retarded by friction with the underlying surface. We also permit an exchange of momentum between the two layers of fluid which depends on the difference of velocities in the two layers. Each of the boundary dissipation terms is modelled in terms of a constant decay rate. This is justified for the bottom boundary as shown by Holton (1965) but is an extremely crude assumption for the side boundary layers. However, as can be seen in Merilees' article, the formulation produced some realistic effects on the transition curves between Hadley and Rossby regimes.

If we further assume that the friction effect of the boundaries depends only on the area of contact between the fluid and the boundary and not on wheter the boundary is a side wall or the bottom, and we include the effect of internal viscosity, we find that

$$\left[\frac{\partial}{\partial t}\nabla^2\psi\right]_d = -k'(1+2\eta)\nabla^2\psi + k'\nabla^2\tau + \nu\nabla^4\psi \,, \qquad (7)$$

$$\left[\frac{\partial}{\partial t}\nabla^2\tau\right]_d = k'\nabla^2\psi - k'(1+2\eta)\nabla^2\tau - 2k''\nabla^2\tau + \nu\nabla^4\tau \,, \qquad (8)$$

where k' is the boundary dissipation rate (time $^{-1}$), k'' the layer exchange dissipation rate, η the ratio of the depth of the fluid annulus to its width, and ν the kinematic viscosity.

The diabatic heating and cooling effects produced by the boundary layers are considered to be Newtonian in form. We imagine a specified temperature distribution toward which the boundary heating effects are continually driving the fluid. The rate of heat exchange with the boundaries is measured by h'. With these assumptions and the inclusion of internal heat conduction, we find that

$$\left[\frac{\partial \Theta}{\partial t}\right]_d = -h'(\Theta - \Theta^*) + K \nabla^2 \Theta \quad , \tag{9}$$

$$\left[\frac{\partial \bar{\sigma}}{\partial t}\right]_d = -h' \bar{\sigma} \quad , \tag{10}$$

where Θ^* is the fixed, preassigned, vertically averaged temperature distribution and K the thermometric conductivity. The preassigned vertical temperature difference is considered to be zero.

The above equations are next simplified by using their spectral forms with a very few number of components. The spectral functions are chosen such that the average vorticity vanishes. A circulation which does not satisfy this condition is necessarily transient in this model as has been shown in Lorenz (1962). The derivation of the spectral equations closely parallels that in Lorenz (1962) and details of the annular equivalent are reported in Merilees (1967). Here follows the resulting nondimensional spectral equations and the definition of the quantities involved.

The expansions of the vertical mean flow Ψ, the vertical shear flow τ, the vertical mean temperature Θ, the vertical temperature difference $\bar{\sigma}$, and the lower layer divergence $\nabla^2 \chi$ (equal and opposite in upper layer) are given by:

$$\Psi = f\, b^2 (\Psi_A F_A + \Psi_K F_K + \Psi_L F_L) \, , \tag{11}$$

$$\tau = f b^2 (\tau_A F_A + \tau_K F_K + \tau_L F_L) , \tag{12}$$

$$\Theta = A f b^2 (\Theta_0 F_0 + \Theta_A F_A + \Theta_K F_K + \Theta_L F_L) , \tag{13}$$

$$\nabla^2 \chi = f (\omega_A F_A + \omega_K F_K + \omega_L F_L) , \tag{14}$$

$$\bar{\sigma} = A f b^2 \sigma_0 F_0 , \tag{15}$$

where $f=2\Omega$, b is the radius of the outer cylinder, $A=4fe^{-1}g^{-1}D^{-1}$, and the F's are the annular spectral functions (Eqs. (31)-(34)). The spectral equations are then:

$$\dot{\psi}_A = -\gamma h (1+2\eta) \psi_A + \gamma h \tau_A - h P_r X \psi_A , \tag{16}$$

$$\begin{aligned} \dot{\psi}_K = -\alpha\beta(\psi_A \psi_L + \tau_A \tau_L) - \gamma h (1+2\eta) \psi_K \\ + \gamma h \tau_K - h P_r X (1-\beta)^{-1} \psi_K , \end{aligned} \tag{17}$$

$$\begin{aligned} \dot{\psi}_L = \alpha\beta(\psi_A \psi_K + \tau_A \tau_K) - \gamma h (1+2\eta) \psi_L \\ + \gamma h \tau_L - h P_r X (1-\beta)^{-1} \psi_L , \end{aligned} \tag{18}$$

$$\dot{\tau}_A = -\lambda^{-2} \omega_A + \gamma h \psi_A - \gamma h (1+\delta+2\eta) \tau_A - h P_r X \tau_A , \tag{19}$$

$$\begin{aligned} \dot{\tau}_K = -\alpha\beta (\psi_A \tau_L + \tau_A \psi_L) - \mu^{-2} \omega_K + \gamma h \psi_K \\ - \gamma h (1+\delta+2\eta) \tau_K - h P_r X (1-\beta)^{-1} \tau_K , \end{aligned} \tag{20}$$

$$\begin{aligned} \dot{\tau}_L = \alpha\beta (\psi_A \tau_K + \tau_A \psi_K) - \mu^{-2} \omega_L + \gamma h \psi_L \\ - \gamma h (1+\delta+2\eta) \tau_L - h P_r X (1-\beta)^{-1} \tau_L , \end{aligned} \tag{21}$$

$$\dot{\theta}_o = -h(\theta_o - \theta_o^*) \quad , \tag{22}$$

$$\dot{\theta}_A = -\alpha(\psi_L\theta_K - \theta_L\psi_K) + \sigma_o\,\omega_A - h(\theta_A - \theta_A^*) - h\chi\theta_A , \tag{23}$$

$$\dot{\theta}_K = \alpha(\psi_L\theta_A - \theta_L\psi_A) + \sigma_o\omega_K - h\theta_K - h\chi(1-\beta)^{-1}\theta_K , \tag{24}$$

$$\dot{\theta}_L = -\alpha(\psi_K\theta_A - \theta_K\psi_A) + \sigma_o\omega_L - h\theta_L - h\chi(1-\beta)^{-1}\theta_L , \tag{25}$$

$$\dot{\sigma}_o = -\omega_A\theta_A - \omega_K\theta_K - \omega_L\theta_L - h\sigma_o , \tag{26}$$

$$\theta_A = \tau_A \;\; ; \;\; \theta_K = \tau_K \;\; ; \;\; \theta_L = \tau_L \quad , \tag{27}$$

where the dot denotes differentiation with respect to nondimensional time t ($t=t^*f$, t^* being dimensional time), $\gamma = k'/h'$, $\delta = k''/k'$, $h=h'f^{-1}$, and the forced temperature contrast is assumed to be zonally symmetric, i.e.,

$$\theta^* = A\,f\,b^2\,(\theta_o^* F_o + \theta_A^* F_A) \quad . \tag{28}$$

h represents the (nondimensional) rate at which heat is exchanged with the boundaries, γh the rate at which momentum is exchanged with the lower boundary, $\gamma h\delta$ the rate at which momentum is exchanged between the layers, and $2\gamma h\zeta$ the rate at which momentum is exchanged with side boundaries. The parameters X and Pr are defined as

$$\text{Pr} = \text{Prandtl number} = \nu\,\kappa^{-1} , \tag{29}$$

$$\chi = \kappa \lambda^2 h^{-1}, \tag{30}$$

where ν, κ are the nondimensional kinematic viscosity and thermal conductivity, respectively (scaled by the factor $b^2/$). The spectral functions are given by:

$$F_0 = 1, \tag{31}$$

$$F_A = N_A\big(J_0'(\lambda R)Y_0(\lambda r) - Y_0'(\lambda R)J_0(\lambda r)\big), \tag{32}$$

$$F_K = N \; f_m(r) \cos m\varphi \quad , \tag{33}$$

$$F_L = N \; f_m(r) \sin m\varphi \quad , \tag{34}$$

where N, N_A are normalizing factors;

$$f_m(r) = Y_m(\mu R)\, J_m(\mu r) - J_m(\mu R)\, Y_m(\mu r) \;;$$

$R = ab^{-1}$; λ is the first root of

$$J_0'(\lambda R)\, Y_0'(\lambda) - Y_0'(\lambda R)\, J_0'(\lambda) = 0 \;;$$

μ is the first root of

$$Y_m(\mu R)\, J_m(\mu) - J_m(\mu R)\, Y_m(\mu) = 0 \;;$$

J_n is the Bessel function of the first kind of order n, and

Y_n is Neumann's Bessel function of the second kind of order n as defined by Abramowitz and Stegun (1965).

The spectral parameters α, β are defined in terms of the spectral functions as

$$\beta = 1 - \lambda^2 \mu^{-2} , \tag{35}$$

$$\alpha = m N^2 (1 - R^2)^{-1} \int_R^1 f_m^2 \frac{d}{dr} F_A \, dr . \tag{36}$$

These parameters depend on n, as well as on the relative horizontal dimensions of the annulus, specifically on R. Their values and the values of μ are given in Tables 1, 2 and 3. As a result of the relationship between J_0' and J_1, and between Y_0' and Y_1, note that μ is given the value of λ when n = 1. Due to the choice of orthogonal functions, wave number one does not develop in this model.

Note that θ_A and θ_A^* are proportional to the interior thermal Rossby number R_{ot} and the "forced" thermal Rossby number R_{ot}^* respectively; σ_o is proportional to the rotational static stability S_z. We have the following definitions:

$$R_{ot} = \frac{g \varepsilon D \Delta_H T}{\Omega^2 (b-a)^2} ,$$

$$S_Z = \frac{g \varepsilon D^2}{\Omega^2 (b-a)^2} \; \frac{\overline{\Delta T}}{\Delta Z} \quad ,$$

where $\Delta_H T$ is the horizontal temperature difference in the interior of the fluid and $\frac{\overline{\Delta T}}{\Delta Z}$ is the mean lapse rate. In terms of spectral amplitudes of the model we have

$$R_{ot} = 16(1-R)^{-2} \Delta F_A \, \Theta_A = B_1 \Theta_A \quad ,$$

$$R_{ot}^* = 16(1-R)^{-2} \Delta F_A \, \Theta_A^* = B_1 \Theta_A^* \quad ,$$

$$S_Z = 64(1-R)^{-2} \, \sigma_0 = B_2 \sigma_0 \quad ,$$

where ΔF_A is the change in the spectral function across the annulus. The parameters B_1, B_2 and ΔF_A are given in Table 4 as function of R.

For the relationship between coefficients representing the boundary layer effects and the physical properties of the fluid, some assumptions must be made. The assumptions used here are the same as the ones in Merilees' article; h^{-2} is set proportional to Ω^2 and the same constant of proportionality is used. It is assumed that

$$h^{-2} = \frac{4 \Omega^2 (b-a)^4}{\gamma^2 \, \nu^{*2}} \quad ,$$

where ν^* is the dimensional kinematic viscosity. This leads to the following useful relations:

$$h = \gamma / \sqrt{Ta} ,$$

$$X P_r = \gamma \lambda^2 (1 - R)^2 .$$

In the experiment described we have made the further assumption that the static stability is constant with respect to time.

Table 1. The parameter β as function of wave number (n) and annular size (R).

R n	1	2	3	4	5	6	7	8	9	10
0.0	0	.443	.639	.745	.809	.851	.881	.902	.918	.930
0.1	0	.413	.618	.730	.798	.842	.874	.896	.913	.926
0.2	0	.342	.561	.689	.767	.818	.854	.880	.899	.914
0.3	0	.260	.475	.619	.713	.776	.820	.852	.876	.894
0.4	0	.183	.371	.521	.630	.708	.764	.806	.837	.861
0.5	0	.120	.265	.402	.516	.606	.676	.730	.772	.805
0.6	0	.071	.169	.276	.378	.469	.547	.612	.666	.710
0.7	0	.037	.092	.159	.233	.306	.377	.443	.502	.554
0.8	0	.015	.039	.070	.107	.149	.194	.240	.286	.331
0.9	0	.003	.009	.017	.026	.038	.051	.066	.082	.100

Table 2. The parameter α as function of wave number (n) and annular size (R).

R n	1	2	3	4	5	6	7	8	9	10
0.0	9.631	16.020	20.763	24.508	27.621	30.249	32.555	34.596	36.455	38.103
0.1	9.350	16.476	21.701	25.815	29.217	32.119	34.650	36.898	38.922	40.765
0.2	9.084	17.145	23.685	28.898	33.184	36.832	40.012	42.834	49.373	47.682
0.3	9.131	17.845	25.766	32.670	38.562	43.605	47.990	51.871	55.356	58.522
0.4	9.552	18.940	27.993	36.540	44.438	51.605	58.043	63.812	69.006	73.717
0.5	10.441	20.819	31.072	41.123	50.895	60.306	69.282	77.764	85.716	93.130
0.6	12.037	24.052	36.023	47.925	59.730	71.407	82.924	94.242	105.324	116.131
0.7	14.946	29.885	44.811	59.717	74.595	89.438	104.235	118.977	133.654	148.252
0.8	21.039	42.076	63.111	84.142	105.166	126.184	147.192	168.188	189.174	210.144
0.9	39.734	79.468	119.202	158.931	198.669	238.398	278.135	317.863	357.594	397.323

Table 3. First roots of $J_n(\mu R)Y_n(\mu) - Y_n(\mu R)J_n(\mu) = 0$.

R \ n	1	2	3	4	5	6	7	8	9	10
0.0	3.832	5.136	6.380	7.588	8.771	9.936	11.086	12.225	13.354	14.476
0.1	3.941	5.142	6.380	7.588	8.771	9.936	11.086	12.225	13.354	14.476
0.2	4.236	5.222	6.395	7.590	8.772	9.936	11.086	12.225	13.354	14.476
0.3	4.706	5.470	6.494	7.623	8.781	9.939	11.087	12.225	13.354	14.476
0.4	5.391	5.966	6.800	7.790	8.863	9.976	11.103	12.232	13.357	14.476
0.5	6.393	6.814	7.458	8.267	9.190	10.189	11.236	12.311	13.403	14.502
0.6	7.930	8.227	8.699	9.317	10.053	10.880	11.777	12.727	13.717	14.735
0.7	10.522	10.720	11.042	11.476	12.012	12.635	13.332	14.092	14.906	15.763
0.8	15.738	15.855	16.050	16.318	16.656	17.061	17.527	18.049	18.623	19.245
0.9	31.429	31.462	31.570	31.693	31.850	32.041	32.265	32.522	32.810	33.130

Table 4. Parameters necessary to define R_{ot}, R_{ot}* and S_Z.

R	B_1	B_2	ΔF_A
0.2	78.5	100	3.139
0.3	98.8	130.6	3.025
0.4	131.1	177.8	2.950
0.5	185.6	256	2.900
0.6	286.5	400	2.865
0.7	506.3	711.1	2.848
0.8	1134.4	1600	2.836
0.9	4528.0	6400	2.830

3. ADIABATIC OSCILLATIONS

When we consider the adiabatic part of the model with a constant static stability, the set of equations (16)-(25) reduces to:

$$\dot{\Theta}_A = \alpha(\Psi_K \Theta_L - \Theta_K \Psi_L)(1 + \lambda^2 \sigma_0)^{-1}, \tag{37}$$

$$\dot{\Psi}_K = -\alpha\beta(\Psi_A \Psi_L + \Theta_A \Theta_L), \tag{38}$$

$$\dot{\Psi}_L = \alpha\beta(\Psi_A \Psi_K + \Theta_A \Theta_K), \tag{39}$$

$$\dot{\Theta}_K = \alpha(1-\beta S)(1+S)^{-1}\Psi_L \Theta_A - \alpha(1+\beta S)(1+S)^{-1}\Psi_A \Theta_L, \tag{40}$$

$$\dot{\Theta}_L = -\alpha(1-\beta S)(1+S)^{-1}\Psi_K \Theta_A + \alpha(1+\beta S)(1+S)^{-1}\Psi_A \Theta_K, \tag{41}$$

Ψ_A = constant; σ_0 = constant; $S = \sigma_0 \mu^2$.

The idea behind the study of such a reduced system is that vacillation occurring here could be similar to the oscillation of a pendulum swinging in a slightly viscous fluid. The fundamental frequency might be affected just a little by viscous forces, although to achieve a steady oscillation, there would have to be an energy source to compensate for the frictional loss of energy.

Equations (37)-(41) are a set of ordinary differential equations. The calculation procedure used to integrate them numerically is the variation of the Runge-Kutta fourth-order process due to Gill. A description of this procedure is given in Ralston & Wilf (1960).

While integrations were performed, the energy of the system was calculated to detect the leaks due to the numerical scheme. All experiments were done in double precision with a time step of one nondimensional time unit (n.t.u.; 4 n.t.u. = period of one annulus revolution). Some integrations were carried up to 2,000 n.t.u. and the energy losses over this whole interval of time were smaller than 1 part in 1,000. Before looking at results, let us recall that the periods of vacillation, T_{vac}, obtained by Fowlis & Pfeffer (loc. cit.) were varying between 320 and 396 n.t.u. and they were increasing with increasing R_{ot}* as shown in Table 5.

From the results of the first three experiments shown in Table 6, a relation between R_{ot} maximum and T_{vac} is easily found to be

$$T_{vac} \propto R_{ot\,max}^{-1}$$

which verifies Merilees' solution. T_{vac} is function of the geometry of the annulus as can be seen from exp. nos 3 and 4; it is also function of the rotational static stability S_Z, exp. nos 1 and 5. Finally exp. nos 6 and 7 indicate that T_{vac} also depends on the intensity of the vertical mean zonal flow .

Note that the maximum value of the internal Rossby number depends on the initial conditions; this is a very

Table 5. Periods of vacillation in Fowlis and Pfeffer's experiments. ($Pr=6.28$, $a=3.49cm$, $b=6.03cm$, $T_b=30^\circ C$, $T_a=20^\circ C$)

Ω sec^{-1}	R_{ot}^*	T_{vac} n.t.u.
2.20	.805	396.
2.30	.737	328.
2.35	.705	320.

Table 6. Periods of vacillation for wave number 4 in the adiabatic model.

EXP. NO.	R	S_Z	Ψ_A/Θ_A	$R_{ot\ max}$	T_{vac} n.t.u.
1	0.5	1.0	2.0	.500	176.
2	0.5	1.0	2.0	.705	125.
3	0.5	1.0	2.0	.805	108.
4	0.6	1.0	2.0	.805	61.
5	0.5	0.1	2.0	.500	158.
6	0.5	0.5	0.5	.500	150.
7	0.5	0.5	0.1	.500	172.

unpleasant result because it implies a considerable degree of arbitrariness in the results. If the periods of vacillation obtained with the model are compared roughly with the ones obtained in laboratory experiments, (roughly, because the amount of energy at initial time in the model influences $R_{ot\ max}$ and because Fowlis & Pfeffer used a slightly different value of R) we see that there is a factor at least 2 of discrepancy between theory and experiments. From this, one may conclude that the diabatic processes play a role which cannot be neglected.

4. SYSTEM WITH HEAT AND INTERNAL VISCOSITY

One logical step forward in the study of vacillation is to add a simple form of heating and a dissipation mechanism. The Newtonian form of heating and the heat conduction process used are described by equations (9) and (10).

The dissipation process considered in this section is internal viscosity; it is represented by terms $\nu\nabla^4\Psi$ and $\nu\nabla^4\tau$ in equations (7) and (8) respectively. When we add the above processes to the adiabatic system we get the following set of equations:

$$\dot{\Theta}_A = c_1(\Psi_K\Theta_L - \Psi_L\Theta_K) - c_8(\Theta_A - c_9) \quad , \qquad (42)$$

$$\dot{\Psi}_K = -c_2\Theta_A\Theta_L - c_3\Psi_L - c_4\Psi_K \quad , \qquad (43)$$

$$\dot{\Psi}_L = c_2\Theta_A\Theta_K + c_3\Psi_K - c_4\Psi_L \quad , \qquad (44)$$

$$\dot{\Theta}_K = c_5\Theta_A\Psi_L - c_6\Theta_L - c_7\Theta_K \quad , \qquad (45)$$

$$\dot{\Theta}_L = -c_5\Theta_A\Psi_K + c_6\Theta_K - c_7\Theta_L \quad , \qquad (46)$$

where $c_1 = \alpha/(1+\lambda^2\sigma_0)$, $c_2 = \alpha\beta$, $c_3 = \alpha\beta\Psi_A$, $c_4 = \nu\mu^2$,

$c_5 = \alpha(1-\beta S)/(1+S)$, $c_6 = \alpha(1+\beta S)\Psi_A/(1+S)$,

$c_7 = \frac{\nu\mu^2}{(1+S)}\left(S + \frac{1}{P_r} + \frac{1-\beta}{\chi P_r}\right)$, $c_8 = \frac{h\ (P_r\chi\lambda^2\sigma_0 + \chi + 1)}{(1+\lambda^2\sigma_0)}$,

$c_9 = \Theta_A^* / (P_r X X^2 \sigma_0 + X + 1)$; σ_0 = constant, $S = \sigma_0 \mu^2$.

This formulation is somewhat inconsistent since we apply dissipation only to the eddy motions and not to the zonal flow, but we are basically looking for an energy cycle which has these properties. Therefore we set Ψ_A to a positive constant.

Equations (42) to (46) were integrated numerically with seven different sets of parametric values. In all cases the evolution of Θ_A consisted of few oscillations followed by a levelling off towards a constant value. The steady state solutions consisted of a wave moving around with a constant amplitude and speed or a Hadley regime.

In order to get some characteristics of the steady state solutions we can do the following. First let us define Ψ and Θ.

$$\Psi \equiv \Psi_K + i\,\Psi_L \quad ,$$

$$\Theta \equiv \Theta_K + i\,\Theta_L \quad , \quad i = \sqrt{-1} \quad .$$

The steady state is characterised by

$$\dot{\Theta}_A = 0 \; , \quad \frac{d}{dt}|\Psi| = 0 \; , \quad \frac{d}{dt}|\Theta| = 0 \quad .$$

We also have

$$\frac{1}{2}\frac{d}{dt}|\Psi|^2 = 0 \quad ,$$

i.e. $$\Psi_K \dot{\Psi}_K + \Psi_L \dot{\Psi}_L = 0 . \tag{47}$$

This can be expanded with the help of equations (43) and (44) to read

$$c_2 \Theta_A (\Psi_K \Theta_L - \Psi_L \Theta_K) + c_4 |\Psi|^2 = 0 . \tag{48}$$

Now combining equation (42) with (48) we get

$$|\Psi|^2 = c_2 c_8 \Theta_A (c_9 - \Theta_A)/c_1 c_4 . \tag{49}$$

We can deduce similarily an equation for the square of the amplitude of the temperature wave.

$$|\Theta|^2 = c_5 c_8 \Theta_A (c_9 - \Theta_A)/c_1 c_7 . \tag{50}$$

In the domain of variation of parameters we are interested in, constants c_1 to c_9 are all positive. If a Hadley regime is present then $|\Psi|^2$ and $|\Theta|^2$ are both zero and we obtain

$$\Theta_A = c_9 = \Theta_A^* /(P_r X \lambda^2 \sigma_0 + X + 1) .$$

In order to get the value of Θ_A as a function of parameters for a Rossby regime we proceed as follows. First note that in a steady state, equation (42) becomes

$$c_8 (\Theta_A - c_9) = c_1 (\Psi_K \Theta_L - \Psi_L \Theta_K) . \tag{51}$$

If we take the time derivative on both sides we obtain

$$\dot{\Psi}_K \Theta_L + \Psi_K \dot{\Theta}_L - \dot{\Psi}_L \Theta_K - \Psi_L \dot{\Theta}_K = 0 \, .$$

We can expand the above equation with (43) to (46); after regrouping we come to

$$-c_2 \Theta_A |\Theta|^2 - c_5 \Theta_A |\Psi|^2 + (c_6 - c_3)(\Psi_K \Theta_K + \Psi_L \Theta_L) - (c_7 + c_4)(\Psi_K \Theta_L - \Psi_L \Theta_K) = 0. \quad (52)$$

A substitution for $|\Psi|^2$, $|\Theta|^2$ and $(\Psi_K \Theta_L - \Psi_L \Theta_K)$ can be done with equations (49), (50) and (51) respectively. Now we want to replace $(\Psi_K \Theta_K + \Psi_L \Theta_L)$ by an expression containing only Θ_A and some of the constants c_1 to c_9:

$$(E) \equiv \Psi_K \Theta_K + \Psi_L \Theta_L \quad ;$$

to do so we define an origin such that:

$$\Theta_K = |\Theta| \cos(\omega t - \varphi_1) \, ,$$

$$\Theta_L = |\Theta| \sin(\omega t - \varphi_1) \, ,$$

$$\Psi_K = |\Psi| \cos(\omega t - \varphi_2) \, ,$$

$$\Psi_L = |\Psi| \sin(\omega t - \varphi_2) \, ,$$

where ω/m is the angular velocity of the Rossby wave with respect to the rotating system of coordinates. It is easy to see that

$$\Theta_L \dot{\Theta}_K - \dot{\Theta}_L \Theta_K = -\omega \, |\Theta|^2 \quad , \tag{53}$$

and also

$$\Psi_L \dot{\Psi}_K - \dot{\Psi}_L \Psi_K = -\omega \, |\Psi|^2 \quad . \tag{54}$$

In a Rossby regime both $|\Theta|^2$ and $|\Psi|^2$ are different from zero and consequently we are allowed to divide (53) by $|\Theta|^2$ and (54) by $|\Psi|^2$ to eliminate ω. The resulting equation can be expanded, again with the help of (43) to (46). In grouping terms we are left with

$$(E) = \frac{(c_6 - c_3)\,|\Theta|^2 |\Psi|^2}{(c_5\, \Theta_A |\Psi|^2 + c_2\, \Theta_A |\Theta|^2)} \quad . \tag{55}$$

With equations (49) and (50) it reduces to

$$(E) = \frac{c_8\,(c_6 - c_3)\,(c_9 - \Theta_A)}{c_1\,(c_4 + c_7)} \quad . \tag{56}$$

In putting this expression back in (52) we finally come to a cubic equation in Θ_A.

$$\frac{c_4 c_7}{c_2 c_5}\left[\frac{(c_4+c_7)^2+(c_6-c_3)^2}{(c_4+c_7)^2}\right](c_9-\Theta_A) - c_9\,\Theta_A^2+\Theta_A^3=0. \quad (57)$$

This equation is of the form

$$x^3-ax^2-bx+ba=0\ ;$$

$$(x-a)(x^2-b) = 0\ .$$

c_9 is a solution, the other two are

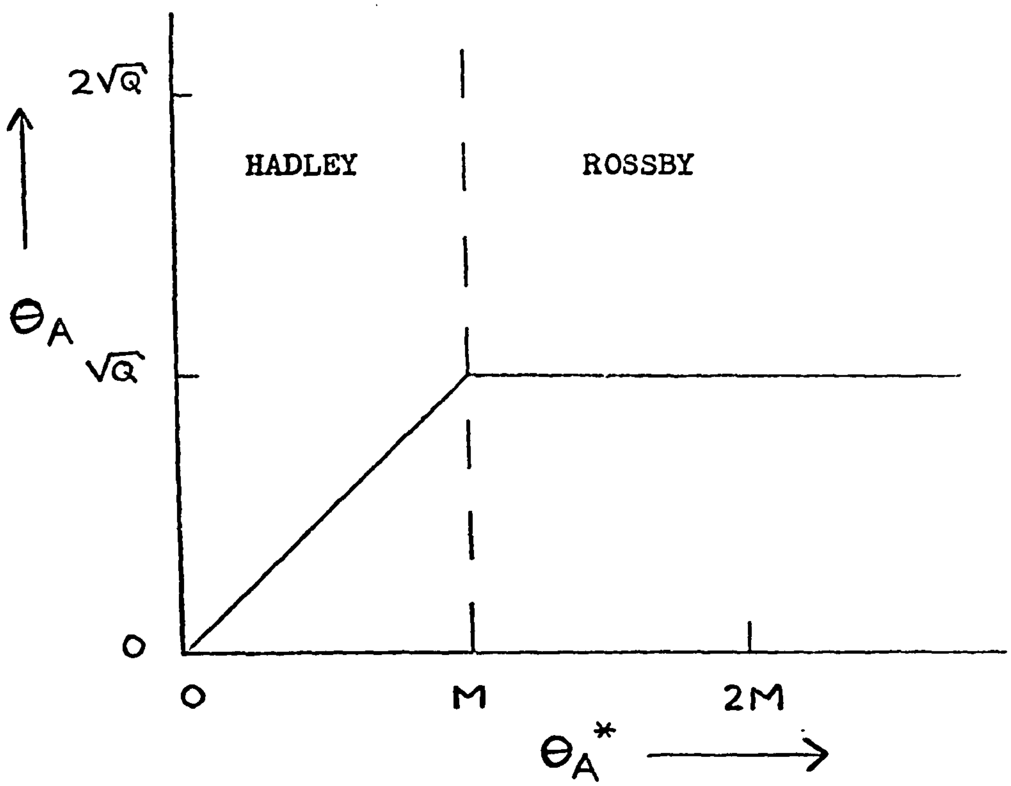
$$\Theta_A = \pm\sqrt{Q} = \pm\left\{\frac{c_4 c_7}{c_2 c_5}\left[\frac{(c_4+c_7)^2+(c_6-c_3)^2}{(c_4+c_7)^2}\right]\right\}^{1/2}.$$

When c_9 is a solution, we are in presence of a Hadley regime; it is easy to see that such a regime is always a solution to the system (42) to (46).

When we pay attention to the $\Theta_A = +\sqrt{Q}$ solution, we note that it is not function of c_9 which is proportional to Θ_A^*. From equations (49) and (50) we can conclude that there will be values of Θ_A^*, or c_9, for which the waves will be nonexistant. We are in a situation, depicted in Figure 5, where for low values of Θ_A^* a Hadley regime is present; this regime is characterized by a linear relationship between Θ_A^* and Θ_A; this relation is easily found by setting the amplitude of wave components equal to zero and $\dot{\Theta}_A = 0$ in (42). As Θ_A^* is increased, it reaches a critical value above which

Figure 5. Properties of the solution of the system.

$$\left[M = \sqrt{Q}\,(\Pr X \lambda^2 \sigma_0 + X + 1)\right]$$



a Rossby regime occurs. The equation for the transition between the Rossby and the Hadley regime is

$$C_q = +\sqrt{Q} \quad .$$

The solution $\Theta_A = -\sqrt{Q}$ implies a situation where the zonal westerlies are decreasing with height; this is not a realistic simulation of mid-latitude tropospheric flow; it is also not the situation in the annulus. Therefore it is disregarded.

We complete our description of the steady state solution by searching expressions for ω and for the angle between the Ψ and Θ waves; the latter is useful in computations of meridional heat flow.

In the derivation of equation (55) we get as a by-product the following two equations

$$-\omega\,|\Theta|^2 = c_5\,\Theta_A\,(E) - c_6\,|\Theta|^2 \quad ,$$

$$-\omega\,|\Psi|^2 = -c_2\,\Theta_A\,(E) - c_3\,|\Psi|^2 \quad ;$$

elimination of (E) leads to

$$\omega\,(c_5\,|\Psi|^2 + c_2\,|\Theta|^2) = c_3\,c_5\,|\Psi|^2 + c_2\,c_6\,|\Theta|^2 \; .$$

In substituting for $|\Psi|^2$ and $|\Theta|^2$ we get after rearrangements

$$\omega = \frac{c_3 c_7 + c_4 c_6}{c_4 + c_7} .$$

If we define x_1 and x_2 such as

$$\tan x_1 = \Theta_L / \Theta_K \qquad \text{and} \qquad \tan x_2 = \Psi_L / \Psi_K ,$$

then the physical angle Φ between the Ψ and the Θ wave is

$$\Phi = (x_2 - x_1)/m ;$$

$$\tan (x_2 - x_1) = (\tan x_2 - \tan x_1)/(1 + \tan x_1 \tan x_2) ;$$

$$\tan (x_2 - x_1) = (\Psi_L \Theta_K - \Theta_L \Psi_K)/(\Psi_K \Theta_K + \Psi_L \Theta_L) .$$

The denominator is the expression (E); the numerator can be substituted for by the use of (51); after simplifications we come to

$$\Phi = \frac{1}{m} \tan^{-1} \left(\frac{c_4 + c_7}{c_6 - c_3} \right) .$$

It is easy to show that the argument of the arctangent is greater than zero; consequently Φ varies from 0 to $\pi/2m$ and there is a net flow of heat advected from equator to pole.

All the above characteristics were checked numerically. The stability of the steady state solution with respect to small perturbations is an important question to consider. Let us consider a solution of the form

$$\Psi = \Psi_S \, e^{i(wt + \Phi)} + \Psi' ,$$

$$\Theta = \Theta_S \, e^{iwt} + \Theta' ,$$

$$\Theta_A = \Theta_{AS} + \Theta_A'$$

where the perturbations are the primed quantities and where $w = \omega/n$. With the change of variable

$$\Psi' = \underset{\sim}{\Psi}' \, e^{iwt} ,$$

$$\Theta = \underset{\sim}{\Theta}' \, e^{iwt} ,$$

and the following definitions

$$\underset{\sim}{\Psi}' \equiv \underset{\sim}{\Psi}_r' + i\,\underset{\sim}{\Psi}_i' , \qquad \underset{\sim}{\Theta}' \equiv \underset{\sim}{\Theta}_r' + \underset{\sim}{\Theta}_i' ,$$

we come to a set of equations for the new perturbations of the form

$$(\dot{P}) = (A)(P) ,$$

where (P) is a column vector whose components are Θ_A', $\underset{\sim}{\Psi}_r'$, $\underset{\sim}{\Psi}_i'$, $\underset{\sim}{\Theta}_r'$, and $\underset{\sim}{\Theta}_i'$. The matrix (A) is the following:

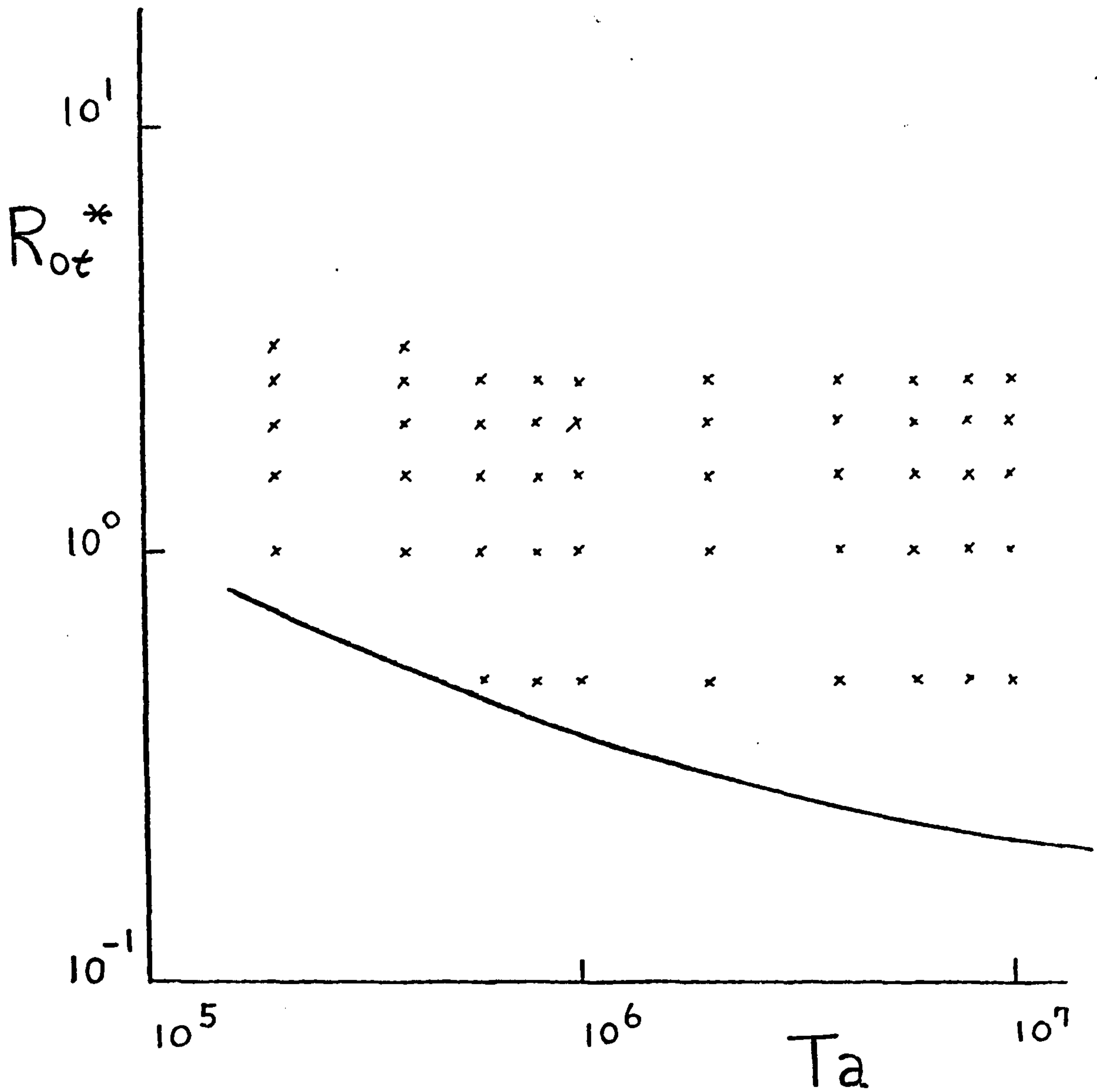
$$\begin{pmatrix} -c_8 & 0 & -c_1\,\Theta_S & -c_1\Psi_S \sin\Phi & c_1\Psi_S\cos\Phi \\ 0 & -c_4 & (\nu - c_3) & 0 & -c_2\,\Theta_{AS} \\ c_2\,\Theta_S & (c_3-\nu) & -c_4 & c_2\,\Theta_{AS} & 0 \\ c_5\Psi_S \sin\Phi & 0 & c_5\,\Theta_{AS} & -c_7 & (\nu - c_6) \\ -c_5\Psi_S\cos\Phi & -c_5\,\Theta_{AS} & 0 & (c_6-\nu) & -c_7 \end{pmatrix}$$

If the real part of the eigenvalues of the above matrix are all negative then perturbations will decay and the regime will be stable. It is difficult to prove analytically that this is the case. As an alternative one can find numerically the roots in a finite number of cases. This was done for 50 cases as shown in Figure 6. The position of crosses indicate the values given to $R_{ot}{}^*$ and Ta in each experiments. Values of the other parameters were: R = 0.5, Pr = 7.0, γ = 3.0, $\Psi_A = 0.5\times10^{-3}$, $\sigma_o = 0.5\times10^{-3}$. Experiments were done with wave number 4. In all cases, two pairs of complex roots with negative real parts and one real negative root were obtained.

To summarize, the system of equations (42) to (46) exibited, in numerical experiments, either a Hadley or a Rossby regime in which the amplitude of the wave was constant. A mathematical analysis shows that the Hadley regime is always a solution and that above a critical value of Θ_A^* a

Figure 6. Experiments with perturbations.

(The solid line is the curve of transition between Hadley and Rossby regimes. Values of other parameters are given in the text)



Rossby regime, in which the amplitude of the wave is constant, is also a solution.

5. EXTENSION OF THE RESEARCH AND DISCUSSION

With the aim of modelling fluctuations between a zonal flow and a single wave and knowing that wind shear favors instability, one is led to think that the insertion of friction with the bottom of the annulus, the walls and between each layer would stress the system in the right direction. When physical processes just mentioned are added to the system of equations (42) to (46), the resulting one is identical to (16) to (25) except for σ_0 which is a constant.

A numerical investigation was done and for the different sets of parameters shown in Table 7, Rossby regimes with no signs of vacillation were observed. It is realized that many more sets of parameters could have been used but from latest laboratory experiments (personnal communication), baroclinic vacillation appears to be a widespread phenomenon and one would like to see in a theory some signs of vacillation at the listed values.

I think one should look back at the approximations used in the modelling of physical processes and try to refine them; the crudest is certainly the heat exchange with boundaries. For further research one should use a different form; one could also look at the effect of letting the static stability vary with time. Let us note that if a new formulation of heating has for sole result a change in constants c_1 to c_9, as long as they remain all positive, the formalism in chapter 4 will still apply.

Table 7. Sets of parameters used in the experiments with friction. ($\gamma = 3.0$, $\delta = 1.0$, $\eta = 3.3$ and $S_Z = .01$ in all experiments)

EXP. NO.	R_{ot}*	Ta	R_{ot} obtained
1	1.0	10.0×10^{6}	.170
2	1.0	20.0	.121
3	1.5	8.0	.189
4	2.0	8.0	.190
5	2.0	40.0	.085
6	4.0	0.7	.638
7	4.0	1.0	.536
8	4.0	5.0	.239
9	4.0	8.0	.190
10	4.0	40.0	.085
11	8.0	8.0	.190
12	8.0	40.0	.085

This study was an attempt to simulate a continuous phenomenon, therefore not much attention was paid to transients. Finally let us mention that as far as atmospheric flow is concerned, in view of results obtained, nothing can be said about the index cycle as whether it is a baroclinic vacillation, a barotropic phenomenon or a specific mixture of both. We are still far from forecasting the period of it. This study suggests that diabatic processes play a very important role in baroclinic vacillation and that a delicate relationship between generation of energy and its dissipation is probably necessary to keep the system from proceeding to a steady state.

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