UNFOLDING THE TWISTED PAIR GROUPOID

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Abstract. In this thesis, we review the classical theory of Stokes phenomenon, and its extension to unfoldings of irregular singularities following the work of J. Hurtubise and C. Rousseau. We then consider the approach to classification of singular systems using groupoids, presented by M. Gualtieri, S. Li, and B. Pym, and how this extends to unfoldings. In particular, we introduce the unfolded twisted pair groupoid, which serves as the universal domain of definition for generic analytic unfoldings of linear differential systems with an irregular singularity at the origin.

Résumé. Dans ce mémoire, nous passons en revue la théorie classique du phénomène de Stokes, et son extension à des déploiements de singularités irrégulières selon les travaux de J. Hurtubise et de C. Rousseau. Nous considérons ensuite l'approche à la classification de systèmes singuliers en utilisant des groupoïdes, telle que présentée dans le travail de M. Gualtieri, S. Li, et B. Pym, et l'extension de cette théorie à des déploiements. En particulier, nous introduisons le déploiement du groupoïde de paires tordu, qui sert de domaine universel de définition pour des déploiements génériques de systèmes différentiels linéaires avec une singularité irrégulière à l'origine.

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"I have been doing what I guess you won't let me do when we are married, sitting up till 3 o'clock in the morning fighting hard against a mathematical difficulty. Some years ago I attacked an integral of Airy's, and after a severe trial reduced it to a readily calculable form. But there was one difficulty about it which, though I tried till I almost made myself ill, I could not get over..."

— Sir George Stokes in a letter to his fiancé, 1857.

Contents

Introduction

The theory of linear systems of meromorphic ODEs on a Riemann surface has a rich history that goes back to the mid-19th century, indeed predating much of our language for the modern theory of linear algebra. This field of study continues to be explored and developed today, and is intimately connected with various incarnations of the Riemann-Hilbert correspondence, the study of moduli spaces of flat connections on bundles over a surface, and modern theories of resurgence, among other topics.

Locally, such systems take the form

$$y' = \frac{A(x)}{x^{k+1}} \cdot y \tag{0.1}$$

where A is a square matrix function that is holomorphic at the origin, and $A(0) \neq 0$. The integer k is often referred to as the Poincaré rank of the singularity. In the case when the singularity is only apparent ($k \leq -1$), classical theorems of linear equations ensure that (0.1) admits a general power series solution that converges in the neighbourhood where A is holomorphic. If the singularity is regular, and in particular whenever the system (0.1) is Fuchsian ($k = 0$), the method of Frobenius series can be employed to find a general solution. (Already small subtleties arise with resonance, when eigenvalues of $A(0)$ are found to be equivalent modulo $\mathbb{N}$, but a generalized version of the Frobenius method will then suffice.) Solutions around regular singularities are generically multivalued. If the singularity is irregular, which is the generic situation when (0.1) is non-Fuchsian ($k \geq 1$), the characterization of solutions becomes a much more difficult and subtle problem. In particular, the search for any series solutions will almost always lead to divergent results. In the prototypical example of Airy's equation, it was rather famously observed by George Stokes, around 1850, that such divergent results could in fact be used to accurately approximate a true analytic solution, namely Airy's integral representation. Apparently it was Poincaré himself who decades later was the first to realize that these strictly formal (divergent) series solutions could be interpreted as asymptotic expansions of actual (holomorphic) solutions. He was able to demonstrate this for a certain class of equations, and indeed was the first to offer a precise meaning of the notion of 'asymptotic series'. By the mid-20th century (see for example [1]), it was well-known that in the neighbourhood of an irregular singularity, one can define a collection of sectors surrounding the pole, on each of which the system (0.1) has a unique solution whose asymptotic expansion is given by the formal series solution. On sectorial overlaps, these solutions are linked by the so-called Stokes matrices (Stokes data, or Stokes factors). The fact that the Stokes data is in general non-trivial is known as the Stokes phenomenon.

For considering the moduli space of germs of systems of the form (0.1), it is typically useful to work in the category where germs are considered equivalent when related by a locally holomorphic gauge transformation, $y \mapsto g(x)y$. With this convention in place, at a regular singularity the local system, generically at least, is characterized by its monodromy representation up to integer shifts in the eigenvalues of the residue matrix. At an irregular singularity, additional formal invariants are needed as well as the Stokes data. One could rephrase these results in terms of the Riemann-Hilbert correspondence, which in its most basic form states that there is natural equivalence between flat connections on a surface (or a manifold) X and representations of its fundamental groupoid $\Pi_1(X)$. Locally then, flat connections with a single simple pole on X will correspond to monodromy representations around the pole, while flat connections with a higher order pole will correspond to monodromy representations plus additional local data related to the asymptotic behaviour of solutions.

At an irregular singularity, one approach to classification taken in [9, 10] is to analytically deform the system (0.1), splitting the single pole of order $k+1$ into multiple generically distinct and hence Fuchsian singular points. The moduli space of this family of so-called *generic unfoldings* is described in [10] as a combination of local monodromy data near each Fuchsian point, plus a collection of transition maps which generalize the Stokes matrices. One particularly appealing aspect of this description of the moduli space is the natural emergence of surrounding sectors which generalize the Stokes sectors around an irregular singularity.

Another approach to classification of singular systems is presented in [17] and makes crucial use of the notion of holomorphic Lie groupoids. For flat meromorphic connections on X with poles bounded by an effective divisor D, one no longer has a simple correspondence to representations of $\Pi_1(X)$ (solutions are generically singular at and multivalued around D). Nor does one have a simple correspondence to monodromy representations of the punctured curve (i.e. representations of $\Pi_1(X \setminus D)$), due to the emergence of the Stokes factors. The main result of [17] is to demonstrate a new correspondence between these meromorphic connections and representations of a novel family of Lie groupoids over X, denoted in [17] by $\Pi_1(X, D)$. The result is succinctly summarized in [17] as follows, "The importance of the Lie groupoid $\Pi_1(X, D)$ is best explained by the fact that the fundamental solution of any meromorphic system with singularities bounded by D, while singular along X, is actually smooth when viewed as a function on the 2-dimensional complex manifold $\Pi_1(X, D)$. In other words, the groupoid is the universal domain of definition for fundamental solutions to such systems." Local parameterizations of $\Pi_1(X, D)$ are explicitly constructed in [17] as the *Stokes groupoids* and the *twisted pair groupoids*. A remarkable implication of the groupoid viewpoint is that one naturally obtains a new method of resummation. The strictly formal series solutions at an irregular singularity can be pulled back to the groupoid, resulting in a holomorphic groupoid representation that defines a convergent universal solution.

In Section 1, we review the classical theory of Stokes phenomenon, and the local analytic classification of linear ordinary differential systems with an irregular singularity. We explore in detail two canonical examples of systems which exhibit Stokes phenomenon. In Section 2, we review the analytic classification of systems with an *unfolded* irregular singularity. This chapter closely follows the results of [9, 10]. In particular, we give a concise review of the emergence of the Douady–Sentenac domains. In Section 3, we begin by briefly reviewing the theory of Lie groupoid representations, and summarizing some of the main results from [17]. We then review the construction of the Stokes groupoids and the twisted pair groupoids using a blowup procedure. We also explore in detail one example of how these groupoids can be applied to the problem of resummation. Finally, we introduce the unfolded twisted pair groupoid, which generalizes the twisted pair groupoid of [17], and show that this groupoid is the universal domain of definition for generic analytic unfoldings of irregular singularities.

1. Classical Theory of Irregular Singularities

1.1. **Gauge Transformations.** Let $A(x)$ be an $N \times N$ matrix of holomorphic germs at the origin in $\mathbb{C}$. Supposing that $A(0) \neq 0$, consider the system of linear differential equations

$$y' = \frac{A(x)}{x^{k+1}} \cdot y, \qquad y \in \mathbb{C}^N, \tag{1.1}$$

for non-negative integer k. For $k = 0$ the singularity at the origin is *Fuchsian* and therefore a regular singularity [1], while for $k > 0$ the singularity generally becomes irregular.

Let us assume that the system is non-resonant i.e. we impose the generic condition that for $k > 0$ the eigenvalues of $A(0)$ are distinct (for $k = 0$ we impose that $A(0)$ is diagonalizable with eigenvalues distinct modulo the non-zero integers) [2]. Non-resonance implies that the singularity at the origin will be irregular for $k > 0$. Without loss of generality we assume that $A(0)$ is diagonal (one performs a change of basis).

This ensures that the system can be taken to a diagonal normal form, through a unique formal gauge transformation $y \mapsto g(x)y$ that is equal to the identity at the origin. That is, there exists a unique formal series of the form $g(x) = I + \mathcal{O}(x)$ bringing (1.1) into a polynomial form

$$y' = \frac{1}{x^{k+1}}(\Lambda_0 + \Lambda_1 x + \ldots + \Lambda_k x^k) \cdot y, \qquad y \in \mathbb{C}^N, \tag{1.2}$$

where the Λ_j are diagonal matrices, and $\Lambda_0 = A(0)$ (see for example [1], [2]). The $N(k+1)$ entries of $\Lambda_0, \ldots, \Lambda_k$ comprise the system's *formal data* and are uniquely determined from the original germ $A(x)$. The normal form (1.2) has a diagonal fundamental matrix solution

$$e^{\Lambda(x)}, \quad \text{where } \Lambda(x) \equiv \Lambda_0 \frac{x^{-k}}{-k} + \ldots + \Lambda_{k-1} \frac{x^{-1}}{-1} + \Lambda_k \log(x). \tag{1.3}$$

The solution (1.3) is holomorphic on the universal cover of the punctured plane $\mathbb{C}^\times$. In practice, one may prefer restricting to a single sheet by specifying a branch of $\log(x)$.

Remark 1.1. Fix a branch of $\log(x)$ and let $Y(x)$ be a germ of fundamental solution to (1.2). There exists a unique $C \in GL_N(\mathbb{C})$ such that $Y(x) = e^{\Lambda(x)} C$, and so the analytic continuation of $Y(x)$ in a loop around the origin (counterclockwise) results in $e^{2\pi i \Lambda_k} Y(x)$. Hence $Y(x)$ extends to a fundamental solution on all of $\mathbb{C}^\times$ if and only if Λ_k is an integer matrix.

As a consequence of formally gauging to normal form, we see that system (1.1) has a *canonical* formal fundamental solution

$$y_f = g^{-1}(x)\, e^{\Lambda(x)} = \Big(\sum_{n=0}^{\infty} F_n x^n\Big) e^{\Lambda(x)}, \qquad F_0 = I. \tag{1.4}$$

Our description of y_f as *canonical* is explained in the following remark.

Lemma 1.2. *Naturally, one could ask for another formal fundamental solution to* (1.1) *of the form*

$$Y_f = \Big(\sum_{n=0}^{\infty} G_n x^n\Big) e^{\Delta(x)}, \quad G_0 = I, \tag{1.5}$$

where $\Delta(x)$ is diagonal, and each diagonal element $\Delta(x)_{ii}$ is a linear combination of $\log(x)$ and negative integer powers of x. It will be shown that $\Delta(x) = \Lambda(x)$ and moreover $Y_f = y_f$.

Proof. As is the case for *actual* fundamental solutions to (1.1), it is likewise true that *formal* fundamental solutions (i.e. of log-exponential type, see [1]) are unique modulo the right action of $GL_N(\mathbb{C})$. Thus we must have $Y_f = y_f C$ for some nonsingular C. This yields componentwise relations

$$\big(I + \mathcal{O}(x)\big)_{ij}\, e^{\Delta(x)_{jj} - \Lambda(x)_{ii}} = C_{ij}, \quad 1 \le i, j \le N. \tag{1.6}$$

The entry C_{ij} may vanish only if $i \neq j$, implying that

$$e^{\Delta(x)_{ii} - \Lambda(x)_{ii}} \to C_{ii} \neq 0\,, \quad \text{as } |x| \to 0\,.$$

This can only be possible if $\Delta(x)_{ii} = \Lambda(x)_{ii}$ and hence $C_{ii} = 1$. It then follows from (1.6) (and our assumption of non-resonance) that $C_{ij} = 0$ for $i \neq j$. Note that this argument holds in the $k = 0$ case just as well as for $k > 0$. □

This also proves our earlier claim that the formal normalizing series $g(x)$ is uniquely determined by setting $g(0) \equiv I$.

Remark 1.3. The general formal series $G(x) = G_0 + \mathcal{O}(x)$ bringing (1.1) to normal form must satisfy $G_0 \Lambda_0 {G_0}^{-1} = \Lambda_0$. For $k > 0$ this implies that $G_0 \equiv D$ is diagonal, and we can write $D^{-1} G(x) = I + \mathcal{O}(x)$. This expression also brings (1.1) to normal form, hence it must be equal to $g(x)$ by uniqueness. Thus the system's *formal data* truly is gauge invariant!

For $k = 0$, it is a well-known classical result that y_f represents an actual fundamental solution, i.e. the sum in (1.4) has a non-zero radius of convergence [1] pg. 117. However, the formal solution y_f will not be convergent in general. In particular, for $k > 0$ the gauge transformation generally turns out to be a divergent series, i.e. the sum in (1.4) has *zero* radius of convergence.

In the Fuchsian case ($k = 0$), one obtains a *true* fundamental solution to (1.1),

$$y_f = \Big(I + \sum_{n=1}^{\infty} F_n x^n\Big) e^{\Lambda_0 \log(x)} = \Big(I + \sum_{n=1}^{\infty} F_n x^n\Big) x^{A(0)},$$

holomorphic on the universal cover of $\big\{x \in \mathbb{C} : \; 0 < |x| < R\big\}$ for some $R > 0$. Notice that the singularity at the origin is regular.

1.2. **Stokes Phenomenon.** Having completely characterized all solutions to (1.1) in the case when $k = 0$, let us assume for the remainder of this chapter that $k > 0$. If the canonical solution y_f should fail to converge, it turns out that this strictly formal object y_f will represent the *asymptotic expansion* of some true solution to (1.1), as $x \to 0$ on appropriately chosen sectors of the form

$$\Omega = \Big\{x \in \mathbb{C} : \; 0 < |x| < \rho, \;\; \theta_1 < \arg(x) < \theta_2\Big\}, \tag{1.7}$$

where ρ is strictly less than the radius of convergence of the germ $A(x)$.

In order to make this statement more precise, first recall that the eigenvalues of $A(0) = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ are distinct. By permuting coordinates of y and performing a rotation $x \mapsto e^{i\theta} x$ if necessary, we may assume that

$$\Re(\lambda_1) > \ldots > \Re(\lambda_N). \tag{1.8}$$

The *Stokes rays* are the half-lines determined by the condition

$$\Re\left[\frac{\lambda_i - \lambda_j}{x^k}\right] = 0 \qquad (i \neq j). \tag{1.9}$$

Hence, there are $2k$ Stokes rays associated with each eigenvalue pair $(\lambda_i, \lambda_j),\ i < j$. For any sector Ω of the form (1.7), we shall call it a *Stokes sector* precisely when it contains exactly one Stokes ray from each eigenvalue pair $(\lambda_i, \lambda_j),\ i < j$, and its closure $\overline{\Omega}$ does not contain any additional Stokes rays. It is not too difficult to see that the punctured disk $D_\rho^\times = \left\{x \in \mathbb{C} :\ 0 < |x| < \rho\right\}$ can be openly covered by $2k$ Stokes sectors, each having an angular opening greater than $\frac{\pi}{k}$. In particular, for any given direction θ, we can find a sufficiently small $\varepsilon > 0$ such that

$$\Omega_1 = \left\{x \in \mathbb{C} :\ 0 < |x| < \rho,\ \ \theta + \varepsilon < \arg(x) < \theta + \frac{\pi}{k} + 2\varepsilon\right\} \tag{1.10}$$

is a Stokes sector, and then so is each of the rotated sectors

$$\Omega_n = e^{i(n-1)\frac{\pi}{k}}\,\Omega_1, \qquad \text{for}\ \ n = 1, \dots, 2k. \tag{1.11}$$

The main importance of Stokes sectors is summarized by the following classical statement [1], [2], [3], [4].

Theorem 1.4. *If Ω is a Stokes sector, then there exists a unique fundamental solution ψ of* (1.1) *such that*

$$\psi \sim y_f \ \ \textit{in}\ \ \Omega \quad (\textit{as } x \to 0). \tag{1.12}$$

The precise meaning of (1.12) is as follows. For any $m \in \mathbb{N}$, there exists a positive real number $C_{m,\Omega}$ such that

$$\left\|\psi(x)e^{-\Lambda(x)} - \sum_{n=0}^{m-1} F_n x^n\right\| < C_{m,\Omega}\,|x|^m \qquad \text{for all } x \in \Omega. \tag{1.13}$$

It is important to note that the theorem *generically fails* on larger sectors, whose closure contains more than $N(N-1)/2$ Stokes rays.

This indicates that, in general, the Stokes sectors represent the "widest possible" sectors on which one is guaranteed to find an actual solution ψ asymptotic to y_f throughout the entire sector. Conversely, on narrower sectors, the uniqueness of ψ may fail. This failure is known as the Stokes phenomenon. To examine this phenomenon further, let us consider the collection of Stokes sectors $\{\Omega_n\}_{n=1}^{2k}$ defined at (1.11). By Theorem 1.4, for each $n = 1, \dots, 2k$ there exists a unique fundamental solution ψ_n of (1.1) such that $\psi_n \sim y_f$ in Ω_n. We shall refer to ψ_n as the *canonical solution* on Ω_n.

Remark 1.5. Due to the multivaluedness of $e^{\Lambda(x)}$, each canonical solution ψ_n is really only unique up to a choice of branch of $\log(x)$. It will be instructive to continue our discussion in the universal covering space,

$$\widetilde{D_\rho^\times} = \bigcup_{n \in \mathbb{Z}} \Omega_n.$$

In particular then $\Omega_{2k+1} \neq \Omega_1$ but rather Ω_{2k+1} is the adjacent sheet over Ω_1. Let ψ_{2k+1} and ψ_1 be the canonical solutions on Stokes sectors Ω_{2k+1} and Ω_1 respectively. Combining Theorem 1.4 with Remark 1.1, we conclude that

$$\psi_{2k+1}(xe^{2\pi i})e^{-2\pi i \Lambda_k} = \psi_1(x), \quad \forall x \in \widetilde{D_\rho^\times}. \tag{1.14}$$

This indicates how canonical solutions on any two sectors Ω_j and Ω_{j+2k} are related via the so-called formal monodromy matrix $e^{2\pi i\Lambda_k}$.

Still working in the universal cover,[1] each sectorial intersection $\Omega_n \cap \Omega_{n+1}$ (for $n = 1, \ldots, 2k$) has an angular opening of ε and contains *no* Stokes rays. On intersections, one has both asymptotic relations:

$$\psi_n \sim y_f, \quad \psi_{n+1} \sim y_f \quad \text{in} \quad \Omega_n \cap \Omega_{n+1}. \tag{1.15}$$

The fundamental solutions must be related by *constant* matrices $S_n \in GL_N(\mathbb{C})$,

$$\psi_{n+1} = \psi_n S_n, \qquad \text{for } n = 1, \ldots, 2k, \tag{1.16}$$

and these $2k$ matrices, $S_1, \ldots, S_{2k}$, are called the *Stokes matrices.*

Each Stokes matrix S_n is completely characterized by $\frac{N(N-1)}{2}$ parameters, as can be deduced from the asymptotic relations of (1.15):

$$e^{\Lambda(x)} S_n e^{-\Lambda(x)} = \left(\psi_n(x)e^{-\Lambda(x)}\right)^{-1} \psi_{n+1}(x)e^{-\Lambda(x)} \to I \quad \text{as } x \to 0,\ x \in \Omega_n \cap \Omega_{n+1},$$

yielding componentwise relations

$$\lim_{x\to 0} e^{\Lambda(x)_{ii}} (S_n)_{ij}\, e^{-\Lambda(x)_{jj}} = \delta_{ij}, \quad x \in \Omega_n \cap \Omega_{n+1}, \quad 1 \le i, j \le N. \tag{1.17}$$

We immediately have that $(S_n)_{ii} = 1$, for $i = 1, \ldots, N$. Fixing any pair of indices $i \neq j$, we have

$$\lim_{x\to 0} (S_n)_{ij} \exp\left(\frac{x^{-k}}{-k}(\lambda_i - \lambda_j)\right) = 0, \quad x \in \Omega_n \cap \Omega_{n+1}. \tag{1.18}$$

Since $\Re\left(x^{-k}(\lambda_i - \lambda_j)\right)$ is bounded away from zero on the intersection $\Omega_n \cap \Omega_{n+1}$, it follows that (at least) one of $(S_n)_{ij}$ or $(S_n)_{ji}$ must vanish. It is conventional [5] to choose Stokes sectors such that the first intersection $\Omega_1 \cap \Omega_2$ contains the ray $\{x : \arg(x) = 0\}$ in the universal cover. This is always possible since $\mathbb{R}_+$ is *not* a Stokes ray by (1.8). Our choice of ordering (1.8) furthermore implies that S_n is *upper triangular* when n is odd, and *lower triangular* when n is even. The remaining $\frac{N(N-1)}{2}$ entries of S_n are uniquely determined by the specific system (1.1) in question — up to a fixing of gauge (see (1.20)).

The $N(N-1)k$ complex parameters which fully characterize $S_1, \ldots, S_{2k}$ are collectively known as the *Stokes data* of system (1.1). As we shall see below, this data is in general nontrivial (non-zero).

1.3. **Monodromy Representation.** The *monodromy* associated with ψ_1 is the constant matrix $M \in GL_N(\mathbb{C})$ such that $\psi_1(xe^{2\pi i}) = \psi_1(x)M$ for all x in the universal cover of $D_\rho^\times$. The existence of such an M follows directly from the fact that both $\psi_1(x)$ and its analytic continuation $\psi_1(xe^{2\pi i})$ are fundamental solutions to (1.1). By (1.14) and (1.16), one may write M in terms of the Stokes matrices as

$$M = e^{2\pi i\Lambda_k} \cdot (S_{2k})^{-1} \cdot \ldots \cdot (S_1)^{-1}. \tag{1.19}$$

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[1]Of course, in practice there is no need to continually work in the universal cover. For example, one can slit the punctured disk $D_\rho^\times$ along some direction $\arg(x) = \theta_0$ such that the slit lies in Ω_1 but *not* in any neighbouring Stokes sector. Then fix the branch cut of $\log(x)$ to be along the slit. The slit disk can be written as the union of sectors:

$$\Omega_1|_{\arg(x)>\theta_0} \cup \Omega_2 \cup \ldots \cup \Omega_{2k} \cup \Omega_{2k+1}|_{\arg(x)<\theta_0+2\pi}.$$

The canonical solutions $\psi_1, \ldots, \psi_{2k+1}$ provide asymptotic solutions all the way around the slit disk, with the Stokes matrices $S_1, \ldots, S_{2k}$ linking solutions on intersections and the formal monodromy matrix $e^{-2\pi i\Lambda_k}$ linking solutions over the branch cut.

It is easily verified that M generates a linear representation of the fundamental group $\pi_1(D_\rho^\times)$. By replacing $\psi_1 \mapsto \psi_1 C$, where $C \in GL_N(\mathbb{C})$, it follows that the monodromy of any arbitrary fundamental solution to (1.1) is *conjugate* to M and hence provides an equivalent representation of $\pi_1(D_\rho^\times)$.

One thus has a well-defined and invariant quantity $\mathcal{M}$ associated to the germ of a system (1.1):

Definition 1.6. The *monodromy group* or *monodromy representation* $\mathcal{M}$ of the system (1.1) (or of its singularity) is the subgroup $\langle M \rangle \subset GL_N(\mathbb{C})$ modulo a global conjugacy in $GL_N(\mathbb{C})$.

This definition involves some notational abuse as $\mathcal{M}$ is neither a group nor a representation, but rather an entire conjugacy class of subgroups.

The monodromy representation $\mathcal{M}$ is *inherently* gauge invariant, by definition. To see this more explicitly, any valid change of gauge must take the form $g(x) \mapsto Dg(x)$ where D is an invertible diagonal matrix (see Remark 1.3). Correspondingly, canonical solutions are transformed according to $\psi_n \mapsto \psi_n D^{-1}$ and so the Stokes matrices are transformed according to the simultaneous conjugation

$$S_n \mapsto D S_n D^{-1}, \quad n = 1, \dots, 2k, \tag{1.20}$$

leaving $\mathcal{M}$ unchanged.

1.4. **Analytic Classification.** We have already seen (by Remark 1.3) that our system (1.1) is *completely characterized* by its formal data $\{\Lambda_0, \dots, \Lambda_k\}$ up to a formal gauge equivalence. That is to say, given any other holomorphic germ $B(x)$ defining the meromorphic system

$$z' = \frac{B(x)}{x^{k+1}} \cdot z, \qquad z \in \mathbb{C}^N, \qquad B(0) = A(0), \tag{1.21}$$

the formal data of (1.1) and (1.21) will coincide *precisely when* there exists a formal gauge transformation $G(x)$ between the two systems

$$y \mapsto G(x) y = z. \tag{1.22}$$

This gauge transformation is unique up to its leading term $G(0)$ which must be an invertible diagonal matrix. Recall that the condition on $G(x)$ to formally gauge between the two systems is equivalent to asking that the following equality hold formally

$$\frac{B(x)}{x^{k+1}} = G(x) \frac{A(x)}{x^{k+1}} G^{-1}(x) + \frac{dG(x)}{dx} G^{-1}(x). \tag{1.23}$$

To seek a local *holomorphic* gauge equivalence between the two systems (1.1) and (1.21), the Stokes matrices will play a crucial role. *Recall that in the Fuchsian case, we have already seen that any formal gauge equivalence* (1.22) *is necessarily holomorphic at the origin.*

In the presence of irregular singularities, the relationship between holomorphic and formal gauge equivalence becomes more subtle.

Definition 1.7. The Stokes data $\{S_1, \dots, S_{2k}\}$ of system (1.1) and the Stokes data $\{\mathcal{S}_1, \dots, \mathcal{S}_{2k}\}$ of system (1.21) are said to be *equivalent* when related by a simultaneous diagonal conjugation of the form (1.20) i.e.

$$\mathcal{S}_n = D S_n D^{-1}, \quad n = 1, \dots, 2k. \tag{1.24}$$

Theorem 1.8. *The two systems* (1.1) *and* (1.21) *are locally holomorphically gauge equivalent precisely when they have identical formal data* and *equivalent Stokes data.*

That is to say, given that (1.1) *and* (1.21) *are formally gauge equivalent, then the gauge transformation is convergent if and only if the respective sets of Stokes matrices are equivalent.*

Proof. Let $g_A(x)$ and $g_B(x)$ be the unique formal gauge transformations of the form $I + \mathcal{O}(x)$ carrying (1.1) and (1.21) to their respective normal forms. The normal forms are identical since the formal data is identical. Assume there exists some $G(x)$ which obeys (1.23) and is holomorphic at the origin. Then both $g_A(x)$ and the composition $g_B(x)G(x)$ are formal gauge transformations which take (1.1) to normal form. Hence, by Remark 1.3 there exists some invertible diagonal matrix D such that $Dg_A(x) = g_B(x)G(x)$.

Let ψ_n and ϕ_n denote the (unique) canonical solutions on sector Ω_n for the systems (1.1) and (1.21) respectively. One has that

$$\psi_n \sim g_A^{-1}(x)e^{\Lambda(x)} = G^{-1}(x)g_B^{-1}(x)e^{\Lambda(x)}D \quad \text{in } \Omega_n.$$

By the uniqueness of the canonical solutions, it follows that

$$\phi_n(x) = G(x)\psi_n(x)D^{-1}, \quad n = 1, \dots, 2k+1.$$

The Stokes matrices $\mathcal{S}_n$ of (1.21) are then easily calculated to be

$$\mathcal{S}_n = \phi_n^{-1}\phi_{n+1} = D\psi_n^{-1}\psi_{n+1}D^{-1} = DS_nD^{-1}, \quad n = 1, \dots, 2k. \tag{1.25}$$

To prove the converse direction, let us assume that (1.25) holds for some diagonal matrix D. Then let

$$h(x) := \phi_1(x)D\psi_1^{-1}(x). \tag{1.26}$$

Observe that $h(x)$ is both holomorphic and invertible on the universal cover $\widetilde{D_\rho^\times}$. By its construction, $h(x)$ provides a gauge transformation from system (1.1) to system (1.21). Indeed, a quick computation reveals that

$$h(x)\frac{A(x)}{x^{k+1}}h^{-1}(x) + \frac{dh(x)}{dx}h^{-1}(x) = \frac{B(x)}{x^{k+1}}, \quad x \in \widetilde{D_\rho^\times}. \tag{1.27}$$

It remains to be shown that $h(x)$ is holomorphic and invertible at the origin. Our first step will be to demonstrate that $h(x)$ descends to a holomorphic function on the punctured disk $D_\rho^\times$ by verifying that its monodromy is trivial. The monodromy matrix of $h(x)$ is calculated with the help of equations (1.19) and (1.25) to be

$$\begin{aligned} h^{-1}(x)h(xe^{2\pi i}) &= \psi_1(x)D^{-1}\phi_1^{-1}(x)\phi_1(xe^{2\pi i})D\psi_1^{-1}(xe^{2\pi i}) \\ &= \psi_1(x)D^{-1}\left(e^{2\pi i\Lambda_k}\mathcal{S}_{2k}^{-1}\dots\mathcal{S}_1^{-1}\right)D\psi_1^{-1}(xe^{2\pi i}) \\ &= \psi_1(x)\left(e^{2\pi i\Lambda_k}S_{2k}^{-1}\dots S_1^{-1}\right)\psi_1^{-1}(xe^{2\pi i}) \\ &= \psi_1(xe^{2\pi i})\psi_1^{-1}(xe^{2\pi i}) \\ &= I. \end{aligned}$$

Next, we shall demonstrate that the singularity of $h(x)$ at the origin is removable. Notice that

$$\begin{aligned} h(x) &= \phi_1(x)DS_1D^{-1}DS_1^{-1}\psi_1^{-1}(x) \\ &= \phi_1(x)\mathcal{S}_1D\left(\psi_1(x)S_1\right)^{-1} \\ &= \phi_2(x)D\psi_2^{-1}(x) \end{aligned}$$

and similarly,

$$h(x) = \phi_n(x) D \psi_n^{-1}(x), \quad n = 1, \ldots, 2k, \quad x \in D_\rho^\times.$$

By (1.13), as well as the fact that systems (1.1) and (1.21) share formal data, it follows that

$$h(x) = \phi_n(x) e^{-\Lambda(x)} D \left(\psi_n(x) e^{-\Lambda(x)}\right)^{-1} \to D \quad \text{as } x \to 0,\ x \in \Omega_n,$$

for each $n = 1, \ldots, 2k$. Hence,

$$h(x) \to D \quad \text{as } x \to 0,\ x \in D_\rho^\times.$$

By applying the Riemann removable singularity theorem, the proof is completed by setting $h(0) \equiv D$ and noting that D is invertible. □

Corollary 1.9. Our original (non-resonant) system (1.1) is locally holomorphically gauge equivalent to its normal form, if and only if the Stokes data is trivial i.e. $S_1 = \ldots = S_{2k} = I$. Equivalently, the canonical formal solution y_f is an *actual* fundamental solution to (1.1) precisely when the Stokes data is trivial.

Theorem 1.8 provides us with a notion of the *moduli space* of the family of systems of the form (1.1) (non-resonant and for fixed $k > 0$). Namely, that up to holomorphic equivalence, the system (1.1) is completely characterized by the $N(k+1)$ parameters of $\Lambda_0, \ldots, \Lambda_k$ (the formal data) and the $N(N-1)k$ parameters of $S_1, \ldots, S_{2k}$ (the analytic or Stokes data), modulo a simultaneous diagonal conjugacy on the Stokes data which reduces the total number of parameters by $N - 1$. (Of course, we have already seen that when $k = 0$ the moduli space is simply the set of possible residue matrices $A(0) = \Lambda_0$.) A natural question arises, as to what sorts of Stokes matrices might actually be realized as the Stokes data set of some system (1.1). In other words, we wish to identify which points in the moduli space can be realized. The answer turns out to be that each point is realized. This is the content of the following theorem due to Birkhoff [5].

Theorem 1.10. *For $k > 0$, specify any desired set of (non-resonant) formal data $\Lambda_0, \ldots, \Lambda_k$. Then for any set of unipotent and alternating upper/lower triangular matrices $S_1, \ldots, S_{2k}$, there exists some system of the form (1.1) with the prescribed formal data and having $S_1, \ldots, S_{2k}$ as its Stokes matrices.*

Thus, we see that the triangularity properties derived above are not only necessary, but sufficient conditions for characterizing Stokes data of the generic system. Theorem 1.10 seems to imply that the generic non-resonant system (1.1) for $k > 0$ is *practically guaranteed* to exhibit Stokes phenomenon and have a divergent formal normalizing series.

Example 1.11. Let us attempt to find an extremely simple system of the form (1.1) exhibiting Stokes phenomenon. If $A(x) = \sum_{n=0}^{\infty} x^n A_n$ is diagonal, then (1.1) is trivially integrable, and the canonical solution $y_f = \exp\left(\int \frac{A(x)}{x^{k+1}}\right)$ is convergent. Hence we set $N = 2$ and $k = 1$ for the present example. Next we must specify some non-diagonal germ $A(x)$ such that A_0 is diagonal and obeys (1.8). An obvious and simple choice would be

$$A(x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + x \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

hence defining the system

$$y' = \frac{1}{x^2}\begin{bmatrix}1 & x\\ 0 & 0\end{bmatrix} y, \qquad y \in \mathbb{C}^2. \tag{1.28}$$

In order to determine the formal normalizing series $g(x)$ of system (1.28), let us first consider the simpler gauge transformation $y \mapsto (I + xM)y$ which carries the system (1.28) to

$$\begin{aligned} y' &= \left((I + xM)\frac{A_0 + xA_1}{x^2}(I - xM) + \mathcal{O}(1)\right) y \\ &= \left(\frac{A_0}{x^2} + \frac{A_1 + MA_0 - A_0M}{x} + \mathcal{O}(1)\right) y. \end{aligned} \tag{1.29}$$

Demanding that $A_1 + MA_0 - A_0M$ be a diagonal matrix will lead to the following componentwise relations

$$M_{ij} = \frac{A_{1ij}}{\lambda_i - \lambda_j}, \quad i \neq j. \tag{1.30}$$

The diagonal entries of M remain undetermined, and we may freely set them equal to zero. Hence we have

$$M = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},$$

and one may verify that (1.29) reduces *precisely* to

$$y' = \left(\frac{A_0}{x^2} + M\right) y.$$

This process may be repeated indefinitely, and the following statement can be proven by direct calculation. For any $n \geq 0$ the system

$$y' = \left(\frac{A_0}{x^2} + n!\, x^{n-1} M\right) y$$

will be carried to the system

$$y' = \left(\frac{A_0}{x^2} + (n+1)!\, x^{n} M\right) y$$

by applying the gauge transformation $y \mapsto \big(I + n!\, x^{n+1} M\big)\, y$. Composing these gauge transformations in sequence yields the desired formal normalizing series

$$\begin{aligned} g(x) &= \lim_{n\to\infty} (I + n!\, x^{n+1}M)\dots(I + x^2M)(I + xM) \\ &= I + \sum_{n=0}^{\infty} n!\, x^{n+1} M. \end{aligned} \tag{1.31}$$

This is the *unique* formal series $I + \mathcal{O}(x)$ which carries (1.28) to its gauge-invariant normal form

$$y' = \frac{A_0}{x^2}\, y. \tag{1.32}$$

Remark 1.12. For the generic non-resonant system (1.1), the proof of *existence* of a formal normalizing series closely resembles our methods above, see [1], [2], [5]. In the resonant case, when eigenvalues of $A(0)$ are not all distinct, the formal normalizing series $g(x)$ (and the canonical formal solution y_f) must be modified to include fractional powers of x. This occurs for example in the famous case of Airy's equation.

The normal form (1.32) indicates that the formal data of system (1.28) is

$$\Lambda_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \Lambda_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence the canonical formal fundamental solution to (1.28) is

$$\begin{aligned} y_f &= g^{-1}(x)\, e^{\Lambda(x)} = \Big(I - \sum_{n=0}^{\infty} n!\, x^{n+1} M \Big) e^{-\Lambda_0/x} \\ &= \begin{bmatrix} e^{-1/x} & -\sum_{n=1}^{\infty} (n-1)!\, x^n \\ 0 & 1 \end{bmatrix}. \end{aligned} \tag{1.33}$$

Since y_f is clearly divergent, we expect that Stokes phenomenon will occur, as predicted by Corollary 1.9.

The two Stokes rays are defined by $\Re(\pm 1/x) = 0$ and coincide with the positive and negative imaginary half-lines. In accordance with our convention that $\Omega_1 \cap \Omega_2$ should contain the ray $\{\arg(x) = 0\}$, we let

$$\Omega_1 = \Big\{ x \in \widetilde{\mathbb{C}^\times} \; : \; -3\pi/2 + \varepsilon < \arg(x) < \pi/2 - \varepsilon \Big\}$$

for some fixed $\varepsilon \in (0, \pi/2)$. Rotating the sector by π we obtain

$$\Omega_2 = \Big\{ x \in \widetilde{\mathbb{C}^\times} \; : \; -\pi/2 + \varepsilon < \arg(x) < 3\pi/2 - \varepsilon \Big\}$$

and an additional rotation by π produces

$$\Omega_3 = \Big\{ x \in \widetilde{\mathbb{C}^\times} \; : \; \pi/2 + \varepsilon < \arg(x) < 5\pi/2 - \varepsilon \Big\}.$$

On each of these Stokes sectors, Theorem 1.4 predicts the existence of a *unique* fundamental solution ψ_n such that $\psi_n \sim y_f$ in Ω_n. Let us attempt to find these canonical solutions and verify their uniqueness explicitly. The following solution to (1.28) is easily obtained by using an integrating factor. At any $x_0 \neq 0$, the matrix function

$$\Psi_{x_0}(x) = \begin{bmatrix} e^{-1/x} & e^{-1/x} \int_{x_0}^{x} \frac{e^{1/z}}{z} dz \\ 0 & 1 \end{bmatrix}$$

is a germ of fundamental solution to (1.28), extending uniquely to a fundamental solution on the universal cover $\widetilde{\mathbb{C}^\times}$. The result can be generalized slightly as follows. Consider taking the limit of $\Psi_{x_0}(x)$ as $|x_0| \to 0$ along a path which becomes tangent to the ray $\{x : \arg(x) = -\pi\} \subset \Omega_1$. The resulting limit function, denoted by $\psi_1(x)$, is a well-defined fundamental solution to (1.28) on $\widetilde{\mathbb{C}^\times}$. As proven in Appendix A, this solution $\psi_1(x)$ is indeed the desired canonical solution on Stokes sector Ω_1.

Remark 1.13. The asymptotic relationship $\psi_1 \sim y_f$ does *not* continue to hold in the larger sector $\{-3\pi/2 < \arg(x) < \pi/2\}$, although it does remain valid in $\{-3\pi/2 < \arg(x) < \pi/2 - \varepsilon\}$ (see Appendix A for details). In particular, this would imply by the residue theorem that $e^{-1/x}\, 2\pi i \to 0$ along the purely imaginary ray $i\mathbb{R}_+$.

If we instead take the limit of $\Psi_{x_0}(x)$ as $|x_0| \to 0$ along a path which becomes tangent to the ray $\{x : \arg(x) = \pi\} \subset \Omega_2 \cap \Omega_3$, then we obtain another fundamental solution which we denote by $\psi_2(x)$. By the same arguments (as given in Appendix A), we see that $\psi_2(x)$ is the desired canonical solution on *both* of the Stokes sectors Ω_2 and Ω_3. That is to say that $\psi_2 \sim y_f$ in $\Omega_2 \cup \Omega_3$. This immediately implies that the second Stokes matrix is trivial, i.e. $S_2 = I$. The remaining Stokes matrix S_1 can be calculated with the help of the residue theorem, and we find that

$$S_1 = \psi_1^{-1}(x)\psi_2(x) = \begin{bmatrix} 1 & -2\pi i \ \mathrm{Res}_{z=0}\left(\frac{e^{1/z}}{z}\right) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2\pi i \\ 0 & 1 \end{bmatrix}.$$

The Stokes matrices have the anticipated triangular structure. Furthermore, as predicted by (1.19) the monodromy matrix of solution $\psi_1(x)$ is indeed given by

$$M = e^{2\pi i \Lambda_1} \cdot (S_2)^{-1} \cdot (S_1)^{-1} = (S_1)^{-1} = \begin{bmatrix} 1 & 2\pi i \\ 0 & 1 \end{bmatrix}.$$

Thus we have completely characterized the Stokes phenomenon of system (1.28). The quintessential Stokes behaviour is summarized by the observation that any asymptotic solutions to (1.28) cannot be analytically extended indefinitely around the origin, without breaking the asymptotics. One has that $\psi_2 \sim y_f$ in any compact subsector of $\{-\pi/2 < \arg(x) < 5\pi/2\}$, but not in the entire open sector. Equivalently, $\psi_1 \sim y_f$ in any compact subsector of $\{-5\pi/2 < \arg(x) < \pi/2\}$, but fails on the entire open set. We reiterate that this result is equivalent to the well-known asymptotic expansion of the special function $E_1(x)$, as noted in Appendix A.

Example 1.14. One of the most famous examples of Stokes phenomenon arises from Airy's equation

$$\frac{d^2y}{dx^2} = xy. \tag{1.34}$$

Indeed, this is the original problem studied by Stokes himself over 150 years ago [13]. He observed that a certain solution to (1.34) (essentially what is now called $\mathrm{Ai}(x)$) could be well-approximated for $|x| \gg 0$ through the strategic truncation of divergent series, *but that divergent series of two different forms were needed* in sectorial neighbourhoods of $\mathbb{R}_+$ and $\mathbb{R}_-$ respectively.[2]

Finding solutions to Airy's equation is equivalent to solving the coupled system

$$y' = \begin{bmatrix} 0 & x \\ 1 & 0 \end{bmatrix} y, \qquad y \in \mathbb{C}^2. \tag{1.35}$$

The singularity at $x = \infty$ can be moved to the origin by substituting $z = 1/x$, leading to the system

$$\frac{dy}{dz} = \frac{1}{z^3}\begin{bmatrix} 0 & -1 \\ -z & 0 \end{bmatrix} y. \tag{1.36}$$

[2] In his own words [13], "When [$\Re(x)$ or $\Re(-x)$] is at all large, the [asymptotic] series are at first rapidly convergent, but they are ultimately in all cases hypergeometrically divergent. Notwithstanding this divergence, we may employ the series in numerical calculation, provided we do not take in the divergent terms." He further states that, "The integral [asymptotic series] will have different forms according as [the real part of x] is positive or negative."

This system (1.36) is in the form of (1.1), but importantly it is *resonant*, as the characteristic equation of $A(0)$ has a double root at $z = 0$. The resonance can be dealt with, and in fact removed completely, by means of a so-called *shearing transformation*, a diagonal gauge transformation of the form

$$y \mapsto Z(z)\, y = \begin{bmatrix} 1 & 0 \\ 0 & z^{-\beta} \end{bmatrix} y, \quad \beta \in \mathbb{Q}^{+}, \tag{1.37}$$

at the cost of increasing the order of the singularity at $z = 0$. This strategy, hinted at in Remark 1.12, can always be used to remove any resonances occurring in the generic linear system. Further details can be found in [5] and a complete analysis in [3]. In the present case, the unique value to eliminate resonance is $\beta = 1/2$, leading to the gauge-transformed system

$$\frac{dy}{dz} = \begin{bmatrix} 0 & -z^{-5/2} \\ -z^{-5/2} & -\frac{1}{2}z^{-1} \end{bmatrix} y. \tag{1.38}$$

Introducing another change of variables $s = \sqrt{z}$ (arbitrarily choosing the principal branch) the system becomes

$$\frac{dy}{ds} = \frac{1}{s^4} \begin{bmatrix} 0 & -2 \\ -2 & -s^3 \end{bmatrix} y. \tag{1.39}$$

This system is non-resonant, having leading order eigenvalues $\lambda = \pm 2$. Performing a change of basis

$$y \mapsto P^{-1} y, \text{ where } P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \tag{1.40}$$

we finally obtain

$$\frac{dy}{ds} = \frac{1}{s^4} \left(\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} - \frac{s^3}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right) y, \tag{1.41}$$

which is in the originally assumed form of (1.1) with $k = 3$.

The Canonical Formal Solution. In order to obtain asymptotic solutions to system (1.41), we should like to calculate its formal normalizing series, which is uniquely of the form

$$g(s) = I + s^3 M + \mathcal{O}(s^6). \tag{1.42}$$

Following the same methods as outlined in (1.29) to (1.30), one can determine the off-diagonal entries of M (while the diagonal entries remain unknown).

$$M = \begin{bmatrix} \cdot & -\frac{1}{8} \\ \frac{1}{8} & \cdot \end{bmatrix} \tag{1.43}$$

This is already sufficient to see that the (diagonal) normal form of (1.41) is

$$\frac{dy}{ds} = \frac{1}{s^4} \left(\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} - \frac{s^3}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) y. \tag{1.44}$$

A more detailed analysis, provided in Appendix B, will solve for the missing diagonal entries of M, allowing us to write down the canonical formal fundamental solution to (1.41), out to one term beyond the leading order,

(1.45)

$$y_f(s) = \left(I - s^3 \begin{bmatrix} -\frac{1}{48} & -\frac{1}{8} \\ \frac{1}{8} & \frac{1}{48} \end{bmatrix} + \mathcal{O}(s^6) \right) \begin{bmatrix} s^{-1/2} \exp\left(-\frac{2}{3}s^{-3}\right) & 0 \\ 0 & s^{-1/2} \exp\left(\frac{2}{3}s^{-3}\right) \end{bmatrix}.$$

The Stokes Sectors. According to (1.9), the Stokes rays of system (1.41) are defined by $\Re(\pm 4s^{-3}) = 0$ and hence occur along the rays $\left\{\arg(s) = \frac{(2n+1)}{6}\pi \,:\, n \in \mathbb{Z}\right\}$. There are $2k = 6$ Stokes rays in the s-plane. Recall that a Stokes sector in the s-plane is one whose closure contains *precisely one* of these rays. On any such sector, the system (1.41) has a unique fundamental solution $\psi(s)$ that is asymptotic to $y_f(s)$ (by Theorem 1.4).

Let us rephrase this in terms of the variable z. On any sectorial region of the z-plane, adherent to the origin and whose closure contains precisely one of the three rays $\left\{\arg(z) = \pi, \pm\frac{\pi}{3}\right\}$, there exists a unique fundamental solution $\psi(\sqrt{z})$ that is asymptotic to $y_f(\sqrt{z})$ as $z \to 0$. Rephrasing in terms of the original variable x, on any sectorial region of the x-plane, adherent to the point at ∞ and whose closure contains precisely one of the three rays $\left\{\arg(x) = \pi, \pm\frac{\pi}{3}\right\}$, there exists a unique fundamental solution $\psi(1/\sqrt{x})$ that is asymptotic to $y_f(1/\sqrt{x})$ as $x \to \infty$.

Going back to the original system (1.35), let us denote one of its arbitrary fundamental solutions by

$$Y(x) = \begin{bmatrix} y_1' & y_2' \\ y_1 & y_2 \end{bmatrix}. \tag{1.46}$$

Observe that $y_1(x)$ and $y_2(x)$ are linearly independent scalar solutions to Airy's equation (1.34). Tracing back our steps through the various gauge transformations, we see that $\psi(s)$ is a fundamental solution to (1.41) if and only if

$$\psi(s) = P^{-1} Z(z) Y(x),$$

for some fundamental solution $Y(x)$ of the form (1.46). Finally, by combining this result with our statement from the preceding paragraph, we conclude that on any sectorial region of the x-plane, whose closure contains precisely one of the three rays $\left\{\arg(x) = \pi, \pm\frac{\pi}{3}\right\}$, there exists a unique solution $Y(x)$ of the form (1.46) such that

$$P^{-1} Z\left(1/x\right) Y(x) \sim y_f\big(1/\sqrt{x}\big) \quad (\text{as } x \to \infty). \tag{1.47}$$

Referring to (1.45), this is equivalent to

$$\begin{bmatrix} y_1'(x) & y_2'(x) \\ \sqrt{x}\, y_1(x) & \sqrt{x}\, y_2(x) \end{bmatrix} \sim P\, y_f\big(1/\sqrt{x}\big) \tag{1.48}$$

$$= \left(\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{x^{-3/2}}{48} \begin{bmatrix} -7 & -7 \\ -5 & 5 \end{bmatrix} + \mathcal{O}(x^{-3}) \right) \begin{bmatrix} \exp\left(-\frac{2}{3}x^{3/2}\right) & 0 \\ 0 & \exp\left(\frac{2}{3}x^{3/2}\right) \end{bmatrix} x^{1/4}.$$

One particular consequence is that on the sector $\{-\pi + \varepsilon < \arg(x) < \frac{\pi}{3} - \varepsilon\}$ in the x-plane, for any $\varepsilon > 0$, there exists a unique solution $y_1(x)$ to Airy's equation such that

$$y_1(x) \sim \frac{e^{-\frac{2}{3}x^{3/2}}}{x^{1/4}} \left(1 - \frac{5}{48} x^{-3/2} + \mathcal{O}(x^{-3}) \right) \quad \text{as } |x| \to \infty. \tag{1.49}$$

Moreover, the same can be said for the sector $\{-\frac{\pi}{3} + \varepsilon < \arg(x) < \pi - \varepsilon\}$. Since the Stokes matrix S_1 linking solutions over $\mathbb{R}_+$ is upper triangular by (1.8), it follows that the asymptotic relation (1.49) remains valid on *all of* $\{-\pi + \varepsilon < \arg(x) < \pi - \varepsilon\}$. Furthermore, any other linearly independent solution to Airy's equation must blow up as $x \to \infty$, following from (1.48). In other words, we have found the asymptotic expansion (away from $\mathbb{R}_-$) of the *unique* solution to Airy's equation (up to a rescaling) which remains finite as $x \to \infty$. The function $y_1(x)$ is typically denoted by $2\sqrt{\pi}\,\mathrm{Ai}(x)$.[3]

Notice that (1.49) cannot remain valid on all of $\{-\pi < \arg(x) < \pi\}$. The multivalued terms would result in two different asymptotic expansions for $y_1(x)$ along the negative real axis, contradicting the fact that solutions to (1.35) are entire functions. This observation implies (through Corollary 1.9) that the gauge series (1.42) is divergent.

As a bonus, we can use these results to obtain the asymptotics of $y_1(x)$ in a sectorial neighbourhood of $\mathbb{R}_-$. One can verify that both of $y_1(xe^{\pm 2\pi i/3})$ are additional solutions to Airy's equation. Their asymptotic expansions are obtained directly from (1.49), on sectors which avoid the rays $\frac{\pi}{3}$ and $-\frac{\pi}{3}$, respectively. The asymptotics will show that the two new solutions are independent. Hence $y_1(x)$ is some linear combination of $y_1(xe^{2\pi i/3})$ and $y_1(xe^{-2\pi i/3})$. Appealing to the asymptotics as $x \to \infty$, as well as the fact that $y_1(x)$ is analytic at the origin, one obtains

$$y_1(x) = -e^{-2\pi i/3}\, y_1(xe^{-2\pi i/3}) - e^{2\pi i/3}\, y_1(xe^{2\pi i/3}).$$

Both terms on the right-hand side have their asymptotic expansions valid on the common sector $\{|\arg(-x)| < \frac{2\pi}{3} - \varepsilon\}$, but care must be taken in that their arguments differ by 2π. This leads to the following relation valid on $\{|\arg(x)| < \frac{2\pi}{3} - \varepsilon\}$.

$$\begin{aligned} y_1(-x) &\sim -e^{-2\pi i/3} \frac{e^{\frac{2}{3}(e^{\pi i}x)^{3/2}}}{(e^{\pi i}x)^{1/4}\, e^{-\pi i/6}} - e^{2\pi i/3} \frac{e^{\frac{2}{3}(e^{-\pi i}x)^{3/2}}}{(e^{-\pi i}x)^{1/4}\, e^{\pi i/6}} \\ &= \frac{e^{\pi i/4}\, e^{-\frac{2}{3}ix^{3/2}}}{x^{1/4}} + \frac{e^{-\pi i/4}\, e^{\frac{2}{3}ix^{3/2}}}{x^{1/4}} \\ &= \frac{2\sin\left(\frac{2}{3}x^{3/2} + \frac{\pi}{4}\right)}{x^{1/4}}. \end{aligned} \tag{1.50}$$

Precisely on the ray $\arg(x) = 0$, this is no longer a bona fide asymptotic relationship, as the ratio between sides of (1.50) is unbounded near zeros of the oscillating approximating function. However, on this ray the *absolute error* is readily seen to be bounded by $\frac{5}{24}x^{-7/4}$, which follows from (1.49).

2. Unfolding the Singularity

Let us consider a generic unfolding of the non-resonant system (1.1) having an irregular singularity ($k > 0$). Up to a translation in x, the generic splitting of the repeated root of x^{k+1} is parameterized by $\epsilon \in \mathbb{C}^k$, with the monomial x^{k+1} in (1.1) being replaced by the polynomial

$$p_\epsilon(x) = x^{k+1} + \epsilon_{k-1}x^{k-1} + \ldots + \epsilon_1 x + \epsilon_0. \tag{2.1}$$

The roots of $p_\epsilon(x)$ are generically distinct, with repetition of roots occurring along the polynomial discriminantal locus $\{\epsilon : \Delta(\epsilon) = 0\}$, where $\Delta(\epsilon)$ is the discriminant

[3] By the same reasoning, we may obtain the asymptotic expansion of solution $y_2(x)$ valid on all of $\{-\frac{\pi}{3} + \varepsilon < \arg(x) < \frac{5\pi}{3} - \varepsilon\}$. This linearly independent solution is not proportional to $\mathrm{Bi}(x)$, but rather to $\mathrm{Ai}(xe^{-2\pi i/3})$.

of $p_\epsilon(x)$. Note that the discriminantal locus is of real codimension 2. This leads us to consider the so-called deformed system

$$y' = \frac{A(\epsilon, x)}{p_\epsilon(x)} \cdot y, \tag{2.2}$$

where $A(\epsilon, x)$ is an analytic germ at the origin in $\mathbb{C}^{k+1}$ and we will impose that $A(0, x) = A(x)$. The deformed system reduces to (1.1) in the limit that $\epsilon \to 0$, and we say that (2.2) defines a *generic analytic unfolding* of system (1.1). This family of unfoldings has been studied for example in [7] and more recently in [8, 9, 10]. As we shall see, the formal invariants of the unfolded system are relatively easy to identify, and reduce analytically to the formal data of the original system (1.1) in the limit that $\epsilon \to 0$. A complete set of analytic invariants of the unfolded system can also be obtained. In [7], these analytic invariants are described as generalizations of the Stokes matrices, and are shown to converge to the usual Stokes data as $\epsilon \to 0$, at least within certain proper subdomains of the parameter space i.e. along some (but not all) paths in the parameter space which approach $\epsilon = 0$. In [8, 9] these results are extended and proven to hold on a full neighbourhood of $\epsilon = 0$, thus establishing the complete moduli space for the family of generically unfolded systems (2.2). In [10], the question of which moduli can be realized is answered, thus providing an analog of Theorem 1.10 for the generically unfolded system.

In [7], [8], [9], [10], the unfolded system is investigated by introducing an additional complex parameter t and rewriting (2.2) as the following pair of coupled equations

$$\dot{y} = A(\epsilon, x) \cdot y \tag{2.3}$$

$$\dot{x} = p_\epsilon(x). \tag{2.4}$$

By investigating these two equations separately, we will outline how the complete set of formal and analytic invariants for the unfolded system can be obtained.

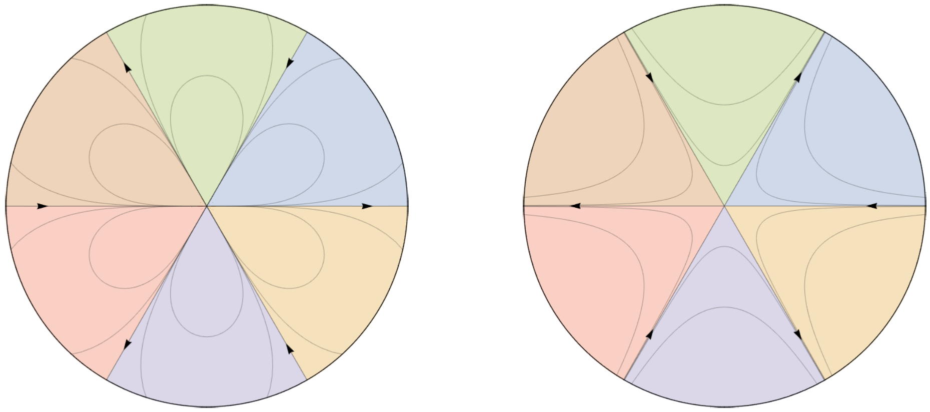
2.1. **The Scalar Equation.** We begin by studying integral curves of the scalar equation

$$\dot{x} = \frac{dx}{dt} = x^{k+1} + \epsilon_{k-1}x^{k-1} + \ldots + \epsilon_1 x + \epsilon_0 \tag{2.5}$$

on $\mathbb{CP}^1$. The trajectories of interest will be those $x(t)$ which are images of Im(t) = constant. That is, we consider the complex flow in x-space as foliated by real flow lines. The trajectories are of course implicitly solvable by

$$t(x) = \int \frac{dx}{p_\epsilon(x)}. \tag{2.6}$$

For $\epsilon = 0$ one finds that $t - t_0 = -\frac{1}{kx^k}$ and the real trajectories converge to the origin in x-space as $\mathrm{Re}(t) \to \pm\infty$. In particular, the real flow along $\mathrm{Im}(t) = \mathrm{Im}(t_0)$ will reach ∞ in x-space at the finite time t_0. The multivaluedness of $x(t)$ implies that this real flow line through $\mathrm{Im}(t_0)$ naturally divides x-space into $2k$ congruent sectors, as pictured in Figure 1 for $k = 3$. As a function on $\mathbb{CP}^1$ the mapping $x(t)$ is multivalued with branch points at 0 and ∞. (The inverse mapping $t(x)$ is a degree k covering of $\mathbb{CP}^1$ ramified at 0 and ∞.)



(a) Centred at the origin in x-space

(b) Centred at infinity in x-space

FIGURE 1. Real flow lines of the scalar equation (2.5) ($\epsilon = 0$). At $\mathrm{Im}(t) = \mathrm{Im}(t_0)$ the real trajectories are *rays* emanating from 0 and ∞.

Remark 2.1. One can already see how the real flow lines of the scalar equation (2.5) naturally reproduce the $2k$ Stokes sectors of Section 1.2. This behaviour will persist for generic values of ϵ, and the corresponding *generalized Stokes sectors* will be discussed in further detail below in Section 2.2.

Next we consider the phase portrait for generic $\epsilon \neq 0$. We shall restrict ourselves to the complement of the discriminantal locus, namely

$$\Sigma_0 = \{\epsilon : \Delta(\epsilon) \neq 0\} \tag{2.7}$$

where the $k+1$ singular points of (2.2) are Fuchsian. Near infinity in x-space, the real trajectories of (2.5) will be asymptotically similar to those of Figure 1(b) since $p_\epsilon(x) \sim x^{k+1}$.[4] One has $2k$ distinguished trajectories called *separatrices* alternately emanating from and reaching ∞. Following each separatrix either forwards or backwards from infinity, it must eventually land at one of the roots x_l of $p_\epsilon(x)$ *or* return to the point at infinity. (The phase portrait contains no limit cycles.) The former case is the generic one, while the latter case describes a *homoclinic orbit* i.e. a real flow line emerging from ∞ in x-space and flowing back to itself in finite time. In the absence of homoclinic orbits, each root x_l is either the *source* of one or more separatrices ($\mathrm{Re}(t) \to -\infty$) or the *sink* of one or more separatrices ($\mathrm{Re}(t) \to \infty$). No root x_l can simultaneously be a source and a sink (since $\epsilon \in \Sigma_0$).

Remark 2.2. In the absence of homoclinic orbits, the fact that each root x_l *must* be tied to some separatrix can be proved combinatorially by induction on k. Make use of the observation that separatrices do not cross (real flow lines do not intersect)

[4] A full proof of the stability of the phase portrait near $x = \infty$ can be found in [11]. Heuristically, for large $|x|$ one has that

$$dt = \frac{dx}{p_\epsilon(x)} = \frac{dx}{x^{k+1}} \frac{1}{\left(1 + \epsilon_{k-1}x^{-2} + \ldots + \epsilon_0 x^{-k-1}\right)} = \frac{dx}{x^{k+1}} \left(1 + \mathcal{O}(x^{-2})\right).$$

and that each x_l may be a source or a sink but not both. For an analytic proof of this fact, see Propositions 1.4.1 and 1.5.1 of [11].

Classifying roots x_l as either attracting or repelling (the nearby real flow lines) is rather simple:

(i) x_l is attracting (i.e. is a sink) precisely when $\mathrm{Re}(p'_\epsilon(x_l)) < 0$.
(ii) x_l is repelling (i.e. is a source) precisely when $\mathrm{Re}(p'_\epsilon(x_l)) > 0$.
(iii) x_l is a *center* (neither attracting nor repelling) precisely when $p'_\epsilon(x_l) \in i\mathbb{R}$.

Note that $p'_\epsilon(x_l) \neq 0$ by distinctness of the roots. The case of x_l being a center implies the existence of a homoclinic orbit (by Remark 2.2), as will be discussed in further detail below. The above claims are easily verified by considering the linearized equation near x_l

$$\dot{x} = p_\epsilon(x) = p'_\epsilon(x_l)(x - x_l) + \mathcal{O}\left((x - x_l)^2\right). \tag{2.8}$$

Integrating (2.8) implies that for some non-zero constant C we have

$$x - x_l \sim C \exp\left(p'_\epsilon(x_l)\, t\right) \quad \text{as } x \to x_l. \tag{2.9}$$

Generic real flow lines are thus asymptotic to *logarithmic spirals* near each singular point x_l, with trajectories spiraling inwards when x_l is a sink and outwards when x_l is a source. The direction of the spiraling is determined by the sign of $\mathrm{Im}(p'_\epsilon(x_l)$, with nearby trajectories becoming straight lines when $p'_\epsilon(x_l) \in \mathbb{R}$ (one has a radial node). In the case when $p'_\epsilon(x_l) \in i\mathbb{R}$ the spiraling phase portrait near x_l bifurcates into periodic orbits, asymptotically similar to concentric circles centred about x_l.

Homoclinic orbits will occur precisely when the parameter $\epsilon \in \Sigma_0$ lies on a certain real codimension 1 set known as the *bifurcation locus*. On the bifurcation locus, the roots $x_1, \dots, x_{k+1}$ can be partitioned into two non-empty sets I_1 and I_2 such that

$$\sum_{l \in I_1} \frac{1}{p'_\epsilon(x_l)} \in i\mathbb{R}. \tag{2.10}$$

The case when $|I_1| = 1$ corresponds to x_l being a center. Notice also that one *always* has

$$\sum_{l=1}^{k+1} \frac{1}{p'_\epsilon(x_l)} = -\mathrm{Res}_{\,x=\infty}\left(\frac{1}{p_\epsilon(x)}\right) = 0.$$

The necessity of condition (2.10) can then be verified by integrating $\frac{dx}{p_\epsilon(x)}$ along any homoclinic orbit and applying the residue theorem.[5] Following from the work of Douady and Sentenac [11], the bifurcation locus partitions Σ_0 into a certain number of connected components called *DS domains*. (The closure of the bifurcation locus partitions the entire parameter space $\mathbb{C}^k$ into the same number of connected components.) Each DS domain $S_s \subset \Sigma_0$ is simply connected, and the total number of DS domains is given by the k-th Catalan number

$$C_k = \frac{1}{k+1}\binom{2k}{k}. \tag{2.11}$$

[5] The condition is sufficient when $k = 1$ or $k = 2$ (there is a center and thus a homoclinic orbit) but *not* when $k \geq 3$. That is to say, for any $k \geq 3$ there exist partitionings of the form (2.10) when *no* homoclinic orbits appear.

The nontrivial observation that S_s is simply connected is proven by Douady and Sentenac [11] where they construct an explicit biholomorphism from S_s to $\mathbb{H}^k$, where $\mathbb{H}$ is the upper half-plane.

Remark 2.3. For $k = 1$ the bifurcation locus is precisely $\mathbb{R}_{>0}$. For $k = 2$ the situation is already significantly more complicated. The bifurcation diagram for $k = 2$ is described in detail in [12].

2.2. **Generalized Sectors.** The following construction is outlined in [10] and is largely due to the work of Douady and Sentenac [11]. Further outlines of the construction can be found in [14], [15], [16]. Throughout this discussion, it may be helpful for readers to refer to Figure 2 below. Away from the bifurcation locus, let Γ_ϵ denote the collection of all $2k$ separatrices (including their endpoints). $\mathbb{CP}^1 \setminus \Gamma_\epsilon$ consists of k connected components or *zones*. Each zone is adherent to two roots of $p_\epsilon(x)$ (one sink x_ω and one source x_α) and each separatrix lies on the boundary of one or two zones. All real trajectories within a given zone emerge from x_α and terminate at x_ω. By arbitrarily selecting any one such trajectory per zone, we form a *tree graph* G whose $k+1$ vertices are the roots of $p_\epsilon(x)$. The complement of $G \cup \Gamma_\epsilon$ in $\mathbb{CP}^1$ consists of $2k$ connected components. That is, the tree graph along with the collection of separatrices serves to divide x-space into $2k$ open sectors, cyclically labelled clockwise around infinity, $\Omega^+_{1,\epsilon}$, $\Omega^-_{1,\epsilon}$, $\ldots$, $\Omega^+_{k,\epsilon}$, $\Omega^-_{k,\epsilon}$, as pictured in Figure 2. Topologically, each sector $\Omega^\pm_{j,\epsilon}$ is a triangle whose three sides are formed by: an *incoming* separatrix (real trajectory from x_α to ∞; mapped to from $[-\infty, t_0]$ in t-space), an *outgoing* separatrix (real trajectory from ∞ to x_ω; mapped to from $[t_0, \infty]$ in t-space), and a real trajectory from x_α to x_ω (mapped to from a horizontal line in t-space). Looking outwards from the origin, sectors of the form $\Omega^+_{j,\epsilon}$ have their incoming separatrix on the right-hand side and their outgoing separatrix on the left. (For sectors of the form $\Omega^-_{j,\epsilon}$ this orientation is reversed.)

This construction generalizes the previous decomposition of $\mathbb{CP}^1$ into $2k$ congruent sectors, which naturally arose when $\epsilon = 0$ (see Figure 1). Indeed, the congruent sectors (of Figure 1) are recovered (and the tree graph vanishes i.e. collapses to a single point) in the limit that $\epsilon \to 0$ within any given DS domain S_s. Due to the alternating incoming/outgoing nature of the separatrices around infinity, there is a natural *sectorial pairing* that uniquely matches each $\Omega^+_{j,\epsilon}$ with some $\Omega^-_{\sigma(j),\epsilon}$ (i.e. the closure of any zone is equal to the closure of $\Omega^+_{j,\epsilon} \cup \Omega^-_{\sigma(j),\epsilon}$ for some $j \in \{1, \ldots, k\}$.) Keeping in mind that zones are disjoint, the number of valid sectorial pairings can be enumerated by taking $2k$ equally spaced points on a circle (representing the 'ends' of zones) and counting the number of ways to connect them pairwise using k *non-intersecting* chords. This enumeration bijects to the set of valid parenthesis expressions that consist of k pairs of parentheses. The number of possibilities is given by the k-th Catalan number (2.11). Each DS domain S_s is hence uniquely associated with some sectorial pairing. The sectorial pairing σ is a locally constant function of the roots, i.e. locally constant on Σ_0 away from the bifurcation locus, and so we say that σ is a combinatorial invariant of $p_\epsilon(x)$.[6]

[6] A currently unresolved problem is noted by K. Pilgrim in [16], "A. Douady has asked whether there exists an algorithm which, given σ, produces some explicit polynomial whose combinatorial invariant is σ." The topology of the bifurcation locus becomes sufficiently twisted such that finding an explicit representative point from any given DS domain is not at all obvious.

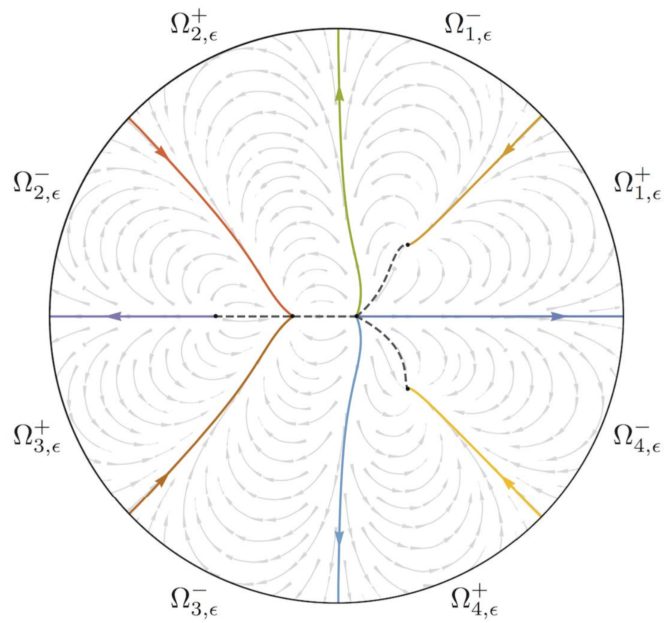
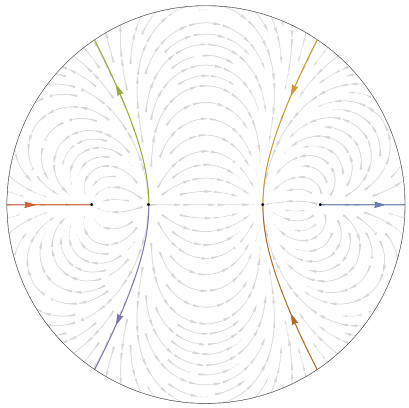


FIGURE 2. Real flow lines of the scalar equation (2.5) with $\epsilon \in \Sigma_0$ and $k = 4$ (centred at the origin in x-space). The 8 separatrices are pictured as directed coloured arrows, while edges of the tree graph G are pictured as dotted lines.

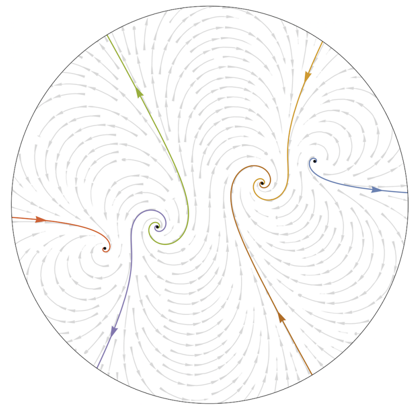
Passing through the bifurcation locus (from one DS domain into another), the points of attachment of separatrices to the roots of $p_\epsilon(x)$ change, and a new sectorial pairing emerges. This process can be observed in the following example.

Example 2.4. Here we consider the case when $k = 3$ and the four roots of $p_\epsilon(x)$ are colinear. The line containing the roots necessarily passes through the origin, and forms an angle θ with the real axis. One can show analytically that bifurcation occurs if and only if $\theta = \pm\frac{\pi}{6}$ or $\frac{\pi}{2}$. The angular regions $(-\frac{\pi}{2}, -\frac{\pi}{6})$, $(-\frac{\pi}{6}, \frac{\pi}{6})$, and $(\frac{\pi}{6}, \frac{\pi}{2})$ correspond to 3 distinct DS domains.

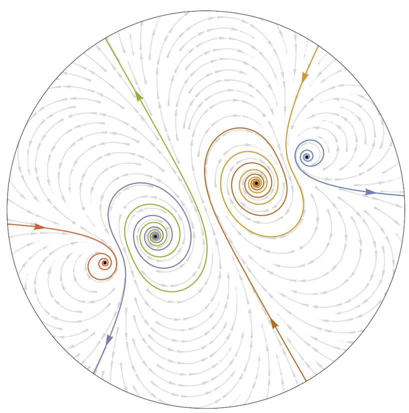
The particular case of colinear roots $\pm e^{i\theta}$, $\pm 2e^{i\theta}$ is pictured in Figure 3 for various values of $\theta \in [0, \frac{\pi}{3}]$. The six separatrices are depicted as coloured arrows, while generic real flow lines are shown in grey. At $\theta = 0$ each root is a radial node (Figure 3(a)). Increasing the angle, real trajectories begin to exhibit spiraling behaviour (Figures 3(b)(c)) until at $\theta = \frac{\pi}{6}$ (on the bifurcation locus) three homoclinic orbits appear (Figure 3(d)) and every root is a center. Following the bifurcation, a new sectorial pairing emerges (see Figure 3(e)). Each root that was previously a *sink* is now a *source*, and vice versa. Finally at $\theta = \frac{\pi}{3}$ each root is again a radial node (Figure 3(f)). The interested reader can visit youtube.com/watch?v=7xjrkKaHBZk to see a video of this bifurcation occurring (and links to other examples).



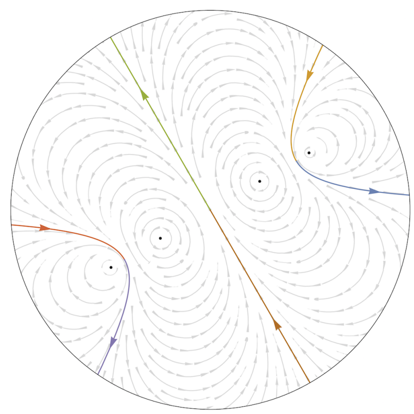
(a) $\theta = 0,$ each root is a radial node.



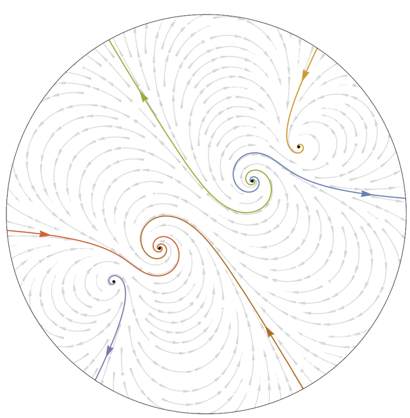
(b) $\theta = \pi/8$



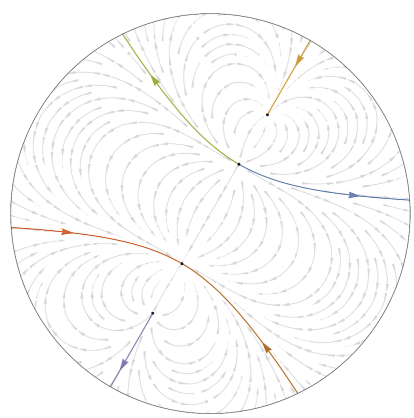
(c) $\theta = 2\pi/13,$ approaching the bifurcation locus.



(d) $\theta = \pi/6,$ three homoclinic orbits appear. Each root is a center.



(e) $\theta = \pi/5,$ the sectorial pairing has changed.



(f) $\theta = \pi/3,$ each root is a radial node.

FIGURE 3. Passing through the bifurcation locus (see Example 2.4).

As a final remark for this section, we note that bifurcation (the existence of homoclinic orbits) does not necessarily imply the existence of *centers*. As an example, consider the case when $p_\epsilon(x) = x^4 + 1$. There is a homoclinic orbit running along the real axis, but none of the four roots of $p_\epsilon(x)$ are centers. Consequently, singular points (roots of $p_\epsilon(x)$) maintain their respective attracting/repelling behaviour as one passes through the bifurcation locus. This type of bifurcation can only occur for $k \geq 3$. The interested reader can visit youtube.com/watch?v=GZ0jWqqBCVs to see a video of this bifurcation occurring.

2.3. **The Vector Equation.** We now turn our attention to the vector equation (2.3)

$$\dot{y} = A(\epsilon, x) \cdot y.$$

Without loss of generality, we may assume that

$$A(\epsilon, x) = \Lambda(\epsilon, x) + p_\epsilon(x) R(\epsilon, x), \tag{2.12}$$

where

$$\Lambda(\epsilon, x) = \sum_{n=0}^{k} \Lambda_n(\epsilon) x^n, \tag{2.13}$$

the matrices $\Lambda_0(\epsilon), \ldots, \Lambda_k(\epsilon)$ are diagonal and analytic in ϵ at the origin, and $R(\epsilon, x)$ is a matrix of holomorphic germs at the origin in $\mathbb{C}^{k+1}$. This is achieved by holomorphically gauging the deformed system (2.2) to its so-called *prenormal form* as outlined in the following remark.

Remark 2.5. Recall that we have by assumption that $A(0,0)$ is diagonal with distinct eigenvalues $\lambda_1, \ldots, \lambda_N$ whose real parts are ordered according to (1.8). Thus within a sufficiently small neighbourhood of $(\epsilon, x) = (0, 0)$, the eigenvalues of $A(\epsilon, x)$ remain distinct and there exists a change of basis matrix $P(\epsilon, x)$ depending holomorphically on (ϵ, x) that will diagonalize $A(\epsilon, x)$. The matrix function $P(\epsilon, x)$ can be found with the help of the implicit function theorem, as described in [9]. We may impose that $P(0,0) = I$. Performing the locally analytic gauge transformation $y \mapsto P(\epsilon, x)^{-1} y$ takes the deformed system (2.2) to

$$y' = \left(\frac{D(\epsilon, x)}{p_\epsilon(x)} + \frac{\partial P(\epsilon, x)^{-1}}{\partial x} P(\epsilon, x) \right) \cdot y,$$

where $D(\epsilon, x)$ is diagonal and holomorphic in ϵ and x at the origin. Dividing by the degree $k+1$ polynomial $p_\epsilon(x)$ yields

$$D(\epsilon, x) = p_\epsilon(x) Q(\epsilon, x) + \Lambda(\epsilon, x),$$

where $Q(\epsilon, x)$ is holomorphic at the origin and the remainder term $\Lambda(\epsilon, x)$ is uniquely determined and in the form of (2.13). Notice that $\Lambda(0,0) = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$. Redefining terms, the system is written in its prenormal form

$$y' = \left(\frac{\Lambda(\epsilon, x)}{p_\epsilon(x)} + R(\epsilon, x) \right) \cdot y, \tag{2.14}$$

proving the assertion of (2.12). Note that the diagonalizing matrix function $P(\epsilon, x)$ is defined up to the right action of any diagonal matrix $K(\epsilon, x)$ holomorphic and invertible at the origin, while the polynomial term $\Lambda(\epsilon, x)$ remains invariant.

Before we look for solutions to the vector equation (2.3), let us see how the prenormal form determines a complete set of formal invariants for the generically unfolded system (2.2). In [9] it is shown (using an extension of the methods applied in Appendix B) that there exists a formal gauge transformation $y \mapsto (I + g(\epsilon, x))y$, that is, $g(\epsilon, x)$ is a formal power series in $\epsilon_0, \ldots, \epsilon_{k-1}, x$ with zero constant term, carrying the prenormal form (2.14) to

$$y' = \frac{\Lambda(\epsilon, x)}{p_\epsilon(x)} \cdot y. \tag{2.15}$$

This diagonal system (2.15) is the *formal normal form* of the generic unfolding (2.2). Moreover, the formal normal form is uniquely determined from the unfolding. This generalizes the previous normal form (1.2) which is recovered in the limit $\epsilon \to 0$. Indeed, our previous formal classification of system (1.1) is nothing but a special case of the following theorem, fully proven in [9].

Theorem 2.6. *Two systems* (2.2) *are formally gauge equivalent if and only if they share the same* $\Lambda(\epsilon, x)$ *in prenormal form. Hence, the* $N(k+1)$ *entries of* $\Lambda_0(\epsilon), \ldots, \Lambda_k(\epsilon)$*, each of which is holomorphic in* ϵ *at the origin, constitute a complete set of formal invariants for the unfolded system* (2.2).

Remark 2.7. If we additionally assume that $\epsilon \in \Sigma_0$, then (2.15) can be expanded in terms of its residue matrices

$$y' = \sum_{l=1}^{k+1} \frac{\mathcal{U}_l}{(x - x_l)} \cdot y, \tag{2.16}$$

where

$$\mathcal{U}_l = \frac{\Lambda(\epsilon, x_l)}{p'_\epsilon(x_l)}, \qquad \text{for } \ l = 1, \ldots, k+1.$$

The system (2.16) has a fundamental solution

$$y = \prod_{l=1}^{k+1} (x - x_l)^{\mathcal{U}_l},$$

where the monodromy matrix around each singular point x_l is given by $e^{2\pi i \mathcal{U}_l}$. We have already seen that near any singular point x_l (for fixed $\epsilon \in \Sigma_0$), the local formal and analytic data of (2.14) is given by the diagonal matrix $\mathcal{U}_l$, assuming there are no resonances. That is, assuming the eigenvalues of $\mathcal{U}_l$ are distinct modulo $\mathbb{N}$. In other words, one can use the Frobenius method to find a convergent series $y_l = (I + \mathcal{O}(x - x_l))\,(x - x_l)^{\mathcal{U}_l}$ that solves (2.14) holomorphically in a slit neighbourhood of x_l. Glutsuk has shown in [7] that certain[7] Stokes matrices of the unperturbed system (1.1) are recovered as the limit ($\epsilon \to 0$) of transition operators between the various $\{y_l\}_{l=1}^{k+1}$. Importantly, care is taken to avoid values of ϵ which lead to any resonances. In the case of resonances, the fundamental solution y_l is no longer valid (not even formally) and must be modified so that $\mathcal{U}_l$ contains additional off-diagonal entries, leading to extra logarithmic terms in the solution [1], [2].

The present approach to finding a basis of solutions to (2.3) (and hence (2.2)) deals with this difficulty of resonant values in parameter space, by appealing to the

[7]With only $k+1$ independent transition operators among the fundamental solutions $\{y_l\}_{l=1}^{k+1}$, only $k+1$ Stokes matrices (or Stokes matrix products) are recovered [7]. This in fact turns out to be equivalent to recovering *all* the Stokes data of (1.1).

following theorem of Levinson (see [1], pg. 92). This is the approach utilized in [9] where it is a key step in determining the analytic modulus of the generic unfolding.

2.4. **Solving the Vector Equation.**

Theorem 2.8. *As stated in* [9], [10]*. Let a system of linear differential equations of the form*

$$\dot{y} = \left(\widetilde{\Lambda}_0(\epsilon) + \widetilde{\Lambda}(\epsilon, t) + P(\epsilon, t)\right) y \tag{2.17}$$

be given on the real line, for which $\widetilde{\Lambda}_0$ is diagonal, with distinct real parts of the eigenvalues, $\widetilde{\Lambda}(\epsilon, t)$ is also diagonal, with limit zero at $t = \infty$ and

$$\int_0^\infty \left| \frac{d}{dt}\big(\widetilde{\Lambda}(\epsilon, t)\big) \right| dt < \infty, \qquad \int_0^\infty |P(\epsilon, t)|\, dt < \infty. \tag{2.18}$$

Then, setting $\lambda_i(\epsilon, t)$, $i = 1, \ldots, N$, to be the successive eigenvalues of $\widetilde{\Lambda}_0(\epsilon)+\widetilde{\Lambda}(\epsilon, t)$, there exist $t_0 \in (0, \infty)$ and solutions $\phi_{i,\epsilon}(t)$ of the system for $t \in (t_0, \infty)$ with

$$\lim_{t\to\infty} \phi_{i,\epsilon}(t) \cdot \exp\left(- \int_{t_0}^t \lambda_i(\tau) d\tau \right) = v_i(\epsilon), \tag{2.19}$$

for $v_i(\epsilon)$ a non-zero eigenvector of $\widetilde{\Lambda}_0(\epsilon)$ corresponding to $\lambda_i(\epsilon, \infty)$, proportional to the standard basis vector e_i.

Let x_ω be a sink of $p_\epsilon(x)$, with $\epsilon \in \Sigma_0$ fixed away from the bifurcation locus. We can apply the above theorem by setting

$$\widetilde{\Lambda}_0(\epsilon) = \Lambda(\epsilon, x_\omega), \quad \widetilde{\Lambda}(\epsilon, t) = \Lambda(\epsilon, x(t)) - \Lambda(\epsilon, x_\omega), \quad P(\epsilon, t) = p_\epsilon(x) R(\epsilon, x(t)),$$

and solving (2.17) along a separatrix (or any real trajectory) which terminates at x_ω ($t = \infty$). Indeed, for sufficiently small ϵ the eigenvalues of

$$\Lambda(\epsilon, x_\omega) = \Lambda_0(\epsilon) + \ldots + \Lambda_k(\epsilon)(x_\omega)^k$$

maintain the ordering on their real parts,

$$\Re(\lambda_1(\epsilon, \infty)) > \ldots > \Re(\lambda_N(\epsilon, \infty)). \tag{2.20}$$

Integrating along the separatrix, the convergence of both integrals (2.18) follows from the fact that trajectories are logarithmic spirals (2.9) whose tail length

$$\int_{x_0}^{x_\omega} |dx| = |C p'_\epsilon(x_\omega)| \int_{t_0}^{\infty} \left|\exp\left(p'_\epsilon(x_\omega)\, t\right)\right| dt$$

is finite precisely when $\mathrm{Re}(p'_\epsilon(x_\omega)) \neq 0$. This is the case since x_ω is not a center.

Along (t_0, ∞) we obtain solutions $\{\phi_{i,\epsilon}(t)\}_{i=1}^N$ to the vector equation (2.3), thus providing solutions $\phi_{i,\epsilon}|_{t(x)}$ to (2.2) along the chosen separatrix in x-space. The matrix $\Phi_\epsilon(t)$ with columns $\phi_{1,\epsilon}(t), \ldots, \phi_{N,\epsilon}(t)$ is a fundamental solution, and the columns are ordered in the sense that $\phi_{i,\epsilon}(t)$ is exponentially dominant over $\phi_{i+1,\epsilon}(t)$ as $t \to \infty$.[8] Of course, multiplying time by -1, the theorem also yields asymptotic solutions $\vartheta_{i.\epsilon}(t)$ along any separatrix reaching a *source* x_α (as $t \to -\infty$). The asymptotic ordering is then reversed, with $\vartheta_{i+1,\epsilon}(t)$ exponentially dominant over

[8] For any $1 \leq j \leq N$, the componentwise ratio $\dfrac{\Phi_\epsilon(t)_{j,i+1}}{\Phi_\epsilon(t)_{i,i}} = \mathcal{O}(|t|^{-m})$ for each $m \in \mathbb{N}$.

$\vartheta_{i,\epsilon}(t)$ (as $t \to -\infty$). Let $\Theta_\epsilon(t)$ denote the matrix with columns $\vartheta_{1,\epsilon}(t), \ldots, \vartheta_{N,\epsilon}(t)$. We may then define two complete flags of subspaces within the space of solutions:

$$\begin{aligned} W_i(x_\omega) &= < \phi_{N-i+1}, \ldots, \phi_N >, \\ W_i(x_\alpha) &= < \vartheta_1, \ldots, \vartheta_i > . \end{aligned} \tag{2.21}$$

It is clear that

$$W_1(x_\omega) \subset W_2(x_\omega) \subset \ldots \subset W_N(x_\omega),$$

and

$$W_1(x_\alpha) \subset W_2(x_\alpha) \subset \ldots \subset W_N(x_\alpha),$$

where the nesting order corresponds to the increasing asymptotic growth rates of solutions.

Recall that each generalized sector $\Omega^{\pm}_{j,\epsilon}$ has two bounding separatrices: one incoming and one outgoing (see Figure 2). Along each bounding separatrix, we have a complete flag of solutions (either $W_i(x_\omega)$ or $W_i(x_\alpha)$). Both flags extend analytically into the entire generalized sector (at least within some disk $D_\rho = \{|x| < \rho\}$, containing the roots of $p_\epsilon(x)$, where $A(\epsilon, x)$ remains analytic). Furthermore, the flags are transverse for sufficiently small ϵ (see Remark 2.9). In particular, one has that the i-th intersection

$$W_{N-i+1}(x_\omega) \cap W_i(x_\alpha) \tag{2.22}$$

is of dimension one.

Remark 2.9. The solutions $\phi_{i,\epsilon}$ of Theorem 2.8 are analytic in ϵ within any DS domain S_s, and continuous up to the boundary ∂S_s away from the bifurcation locus; in particular, solutions have a continuous limit at $\epsilon = 0$. This follows directly from Levinson's proof of the theorem, as explained in [9]. Continuity in ϵ ensures that the flag structure is preserved locally. Hence, to see that the flags are transverse in a neighbourhood of $\epsilon = 0$ (within any DS domain), it suffices to show that they are transverse at $\epsilon = 0$. This will follow from our construction in Section 1.2 of the canonical solutions ψ_n on Stokes sectors Ω_n for $n = 1, \ldots, 2k$. It will suffice to provide the argument for $\Omega^+_{1,0}$ (which is contained in Ω_2 by our convention that $\Omega_1 \cap \Omega_2 \supset \mathbb{R}_+$). At $\epsilon = 0$, the expression (2.19) reduces to

$$\lim_{x \to 0} \Phi_0(x) \cdot e^{-\Lambda(x)} = D,$$

where $\Lambda(x)$ is defined as in (1.3), the matrix D is diagonal and invertible, and the limit is taken along the ray $\{\arg x = \pi i/k\}$. Similarly, for $t \to -\infty$ one obtains

$$\lim_{x \to 0} \Theta_0(x) \cdot e^{-\Lambda(x)} = D',$$

where the limit is taken along $\mathbb{R}_+$. An analogous argument to the one given at (1.18) demonstrates that Φ_0 and the canonical solution ψ_2 are related by a *lower triangular* matrix, while Θ_0 and ψ_2 are related by an *upper triangular* matrix.

$$\Phi_0(x) = \psi_2(x) \cdot C^L \qquad \Theta_0(x) = \psi_2(x) \cdot C^U$$

Hence, at $\epsilon = 0$ the flags (2.21) are transverse — they are simply the *standard* and *opposite* flags in terms of columns of ψ_2. (In particular, the i-th column of ψ_2 is a basis for the i-th intersection (2.22).)

Let v_i be a non-zero element of (2.22), then the matrix $V^{\pm}_{j,\epsilon,S_s}$ with columns $v_1,\dots,v_N$ is a fundamental solution to (2.2) on the generalized sectorial region $\Omega^{\pm}_{j,\epsilon}\cap D_\rho$, and uniquely defined up to the right action of diagonal invertible matrices. The $2k$ fundamental solutions $V^{+}_{1,\epsilon,S_s}, V^{-}_{1,\epsilon,S_s},\dots,V^{+}_{k,\epsilon,S_s}, V^{-}_{k,\epsilon,S_s}$ depend analytically on ϵ within the relevant DS domain S_s. Since these solutions extend analytically beyond their domain's boundary, i.e. can be made to overlap on separatrices as well as on edges of the tree graph,[9] one has constant transition matrices linking solutions as follows (indices are defined modulo k).

$$V^{+}_{j,\epsilon,S_s} = V^{-}_{j-1,\epsilon,S_s}\cdot C^{U}_{j,\epsilon,S_s}$$
$$V^{-}_{j,\epsilon,S_s} = V^{+}_{j,\epsilon,S_s}\cdot C^{L}_{j,\epsilon,S_s}$$

The matrices $C^{U}_{j,\epsilon,S_s}\in GL_N(\mathbb{C})$ (respectively C^{L}_{j,ϵ,S_s}), $j=1,\dots,k$, compare solutions along separatrices attached to repelling (respectively attracting) singular points, and are upper triangular (respectively lower triangular) as can be shown using (2.19), (2.20) and (2.22). These matrices are called the *generalized Stokes matrices* [10]. There are $2k$ of them associated with each DS domain, in which they will depend analytically on ϵ and have a continuous invertible limit at $\epsilon=0$ (see Remark 2.9).

The transition matrices relating solutions along edges of the tree graph G are called *gate matrices* and defined as follows:

$$V^{-}_{\sigma(j),\epsilon,S_s} = V^{+}_{j,\epsilon,S_s}\cdot C^{G}_{j,\sigma(j),\epsilon},$$

where the permutation σ is the sectorial pairing described previously. Since any edge of G has its endpoints located at one sink and one source, the matrices $C^{G}_{j,\sigma(j),\epsilon}$, $j=1,\dots,k$, are both upper *and* lower triangular, i.e. they are diagonal.[10]

2.5. **Normalization.** Unlike the Stokes matrices of Chapter 1, the diagonal entries of the generalized Stokes matrices remain undetermined, and one is required to provide an appropriate normalization. One could ask that each generalized Stokes matrix be equal to I along the diagonal, in hopes of recovering the Stokes data of the unperturbed system (1.1) as $\epsilon\to 0$. This motivates the following choice for a normalization, as in [9, 10]. Recall that each solution $V^{\pm}_{j,\epsilon,S_s}$ is defined up to a

[9]For simplicity, consider $\Omega^{+}_{1,\epsilon}$ which is mapped to biholomorphically from a strip $\{c_0<\mathrm{Im}(t)<c_1\}$ in t-space via (2.6). Without loss of generality, the lower boundary coincides with the real axis ($c_0=0$), and $t=0$ maps to ∞ in x-space. Consider the expanded strip $\{-\delta<\mathrm{Im}(t)<c_1+\delta\}\setminus\{i\mathbb{R}_{\le 0}\}$, i.e. avoiding the negative imaginary axis, and observe that the biholomorphism extends to this new simply connected region (with horizontal paths mapping to real trajectories). The fundamental solution V^{+}_{1,ϵ,S_s} will extend to this expanded domain accordingly (within D_ρ).

[10]As noted in [10], within any *zone*, namely biholomorphic to a horizontal strip in t-space, the flags maintain their asymptotics i.e. are "independent of the real flow line chosen." Relating $\Phi_\epsilon(t)$ along $\mathbb{R}$ to its analytic continuation along the parallel line $\mathrm{Im}(t)=\beta$, one can conclude from (2.17) that $\Phi_\epsilon(t+i\beta)\Phi_\epsilon^{-1}(t)\to\exp\big(i\beta\Lambda(\epsilon,x_\omega)\big)$ as $t\to\infty$. Thus, (2.19) continues to hold as $\mathrm{Re}(t)\to\infty$ along lines in t-space parallel to and neighbouring the real axis: the limit remains finite, non-zero and proportional to $v_i(\epsilon)$. (Similarly, for x_α as $t\to-\infty$.) The gate matrices are then seen to be diagonal, as they compare solutions on an overlap region in x-space containing real trajectories adherent to both x_ω and x_α, where solutions must respect the asymptotics of *both* flags $W_i(x_\omega)$ and $W_i(x_\alpha)$.

diagonal scaling

$$V^{\pm}_{j,\epsilon,S_s} \mapsto V^{\pm}_{j,\epsilon,S_s} \cdot D_{\epsilon,S_s}. \tag{2.23}$$

One asks that D_{ϵ,S_s} be analytic in ϵ within S_s and have a continuous invertible limit at $\epsilon = 0$. Fix any such scaling for the columns of V^{+}_{1,ϵ,S_s}. Each remaining $V^{\pm}_{j,\epsilon,S_s}$ is then uniquely determined by imposing that the diagonal part of every generalized Stokes matrix be equal to the identity, with the exception of C^{U}_{1,ϵ,S_s} whose diagonal part is to be determined by monodromy conditions, and must equal $e^{-2\pi i\Lambda_k(\epsilon)}$, as explained below. Rather than considering solutions over a ramified domain (as with the canonical solutions of Chapter 1), here we are specifying a determination of the logarithm by taking branch cuts along the entire tree graph G as well as along the single separatrix bordering $\Omega^{+}_{1,\epsilon}$ and $\Omega^{-}_{k,\epsilon}$. In particular, one has that

$$V^{-}_{k,\epsilon,S_s}(xe^{2\pi i}) \cdot C^{U}_{1,\epsilon,S_s} = V^{+}_{1,\epsilon,S_s}(x),$$

relating the two solutions over the branch cut which lies along the separatrix. The monodromy around all $k+1$ singular points is immediately seen to be

$$M = (C^{L}_{k,\epsilon,S_s})^{-1} \cdot (C^{U}_{k,\epsilon,S_s})^{-1} \cdot \dots \cdot (C^{L}_{1,\epsilon,S_s})^{-1} \cdot (C^{U}_{1,\epsilon,S_s})^{-1}. \tag{2.24}$$

Around a *single* singular point x_l (adherent to a given generalized sector $\Omega^{\pm}_{j,\epsilon}$), the monodromy of $V^{\pm}_{j,\epsilon,S_s}$ is equal to the product of various C^{U}_{j,ϵ,S_s} and $C^{G}_{j,\sigma(j),\epsilon}$ matrices or their inverses when x_l is a source, and is equal to the product of various C^{L}_{j,ϵ,S_s} and $C^{G}_{j,\sigma(j),\epsilon}$ matrices or their inverses when x_l is a sink. (Refer to Figure 2.) Hence the *diagonal part* of the monodromy is equal to the product of the gate matrices surrounding x_l when it is a sink (respectively the inverse gate matrices when x_l is a source). On the other hand, the standard theory of Fuchsian singularities reveals that the diagonal part of the monodromy is equal to the *formal monodromy* around x_l, namely $e^{2\pi i \mathcal{U}_l}$ (defined in Remark 2.7). This result holds true regardless of whether or not the Fuchsian singular point is resonant, as proved in Appendix C. It follows by inductively moving through the tree graph that

$$\operatorname{diag}(C^{U}_{1,\epsilon,S_s}) = \prod_{n=1}^{k+1} e^{-2\pi i \mathcal{U}_n},$$

which by the residue theorem is equal to

$$\exp\left(2\pi i \operatorname{Res}_{x=\infty}\left(\frac{\Lambda(\epsilon,x)}{p_\epsilon(x)}\right)\right) = \exp\big(-2\pi i\Lambda_k(\epsilon)\big),$$

as claimed above. Lastly, we note that any change in scale (2.23) of the columns of V^{+}_{1,ϵ,S_s} (for any fixed DS domain) results in a global conjugation of the normalized generalized Stokes matrices

$$C^{L,U}_{j,\epsilon,S_s} \mapsto D^{-1}_{\epsilon,S_s} \cdot C^{L,U}_{j,\epsilon,S_s} \cdot D_{\epsilon,S_s}.$$

The gate matrices remain unaffected.

With the above convention for normalization in place, it follows by Remark 2.9 that the Stokes data $S_1, \dots, S_{2k}$ of the unperturbed system (1.1) — modulo a simultaneous diagonal conjugacy as in (1.24) — is recovered as the limit of the (normalized) generalized Stokes matrices when $\epsilon \to 0$ within any DS domain. Furthermore, we have the following theorem establishing a complete set of analytic invariants for the generically unfolded system.

Theorem 2.10. [9] *Two generic unfoldings* (2.2)*, i.e. of the system* (1.1) *with a non-resonant irregular singularity, are analytically gauge equivalent if and only if they share the same* $\Lambda(\epsilon, x)$ *in prenormal form (i.e. have the same formal data)* *<u>and</u> satisfy the following condition:*

For each DS domain S_s*,* $s = 1, \dots, C_k$*, there exists a diagonal invertible matrix* $K_s(\epsilon)$*, depending analytically on* $\epsilon \in S_s$*, with a continuous invertible limit at* $\epsilon = 0$*, such that the two collections* $\{C^{L,U}_{j,\epsilon,S_s}\}$ *and* $\{\mathcal{C}^{L,U}_{j,\epsilon,S_s}\}$ *of normalized generalized Stokes matrices (corresponding to the two unfoldings) satisfies*

$$K_s(\epsilon)\, C^{L,U}_{j,\epsilon,S_s} = \mathcal{C}^{L,U}_{j,\epsilon,S_s} K_s(\epsilon). \tag{2.25}$$

In particular, a generic unfolding is analytically gauge equivalent to its formal normal form (2.15) *if and only if all the generalized Stokes matrices are diagonal. (Notice the analogy to Theorem 1.8 and its Corollary 1.9.)*

The necessary direction of the proof follows directly from Theorem 2.6 plus our above construction of the normalized generalized Stokes matrices. Proof of sufficiency is much more involved, and this is the main result of [9]. We briefly comment on the general strategy. One is required to construct an analytic equivalence between the two unfoldings. This can be done on each DS domain S_s, but then must be extended to 'fill in' the surrounding bifurcation locus, as well as along the discriminantal locus $\Delta(\epsilon) = 0$. The idea is to allow DS domains to become ramified and intersect one another, by considering the complex flow of (2.4) to be foliated along lines which are slanted, i.e. no longer restricted to be parallel to $\mathbb{R}$. The analytic equivalence is further extended to a generic subset of $\Delta(\epsilon) = 0$ (where precisely two singular points coincide) by the results of [8]. Lastly, the remaining points in $\Delta(\epsilon) = 0$ are filled in using Hartogs' theorem. Hence, one obtains an analytic equivalence between the two unfoldings on an entire neighbourhood of $\epsilon = 0$ in the parameter space, and within some disk D_ρ in x-space.

We conclude this section with a final note on the moduli space. An analytic modulus for the family of generic unfoldings (2.2) is provided by Theorem 2.10, but the natural question remains as to which points in the moduli space can be realized. That is, given a collection $\{C^{L,U}_{j,\epsilon,S_s}\}$ (of normalized alternating–lower/upper triangular matrices, analytic in $\epsilon \in S_s$, with continuous invertible limits at $\epsilon = 0$, and continuous limits up to the boundary ∂S_s away from the bifurcation locus), does this collection correspond to the set of normalized generalized Stokes matrices of some unfolding (2.2)? The answer[11] is in the affirmative, as shown by the results of [10], provided that the collection $\{C^{L,U}_{j,\epsilon,S_s}\}$ satisfy certain conditions on the monodromy representations which they define (via taking products of C^U_{j,ϵ,S_s}, C^L_{j,ϵ,S_s}, $C^G_{j,\sigma(j),\epsilon}$ as described at (2.24)). Essentially the monodromy representations over intersecting DS domains must be the same, that is, conjugate to one another, and this conjugacy *must become trivial* as singular points coalesce $\epsilon \to 0$ (see [10] for details).

3. The Stokes Groupoids

Recall from Chapter 1 that the locally meromorphic system (1.1) is taken to an integrable, diagonal form by a formal gauge transformation $y \mapsto g(x)y$. Although the gauge series is generically divergent for positive Poincaré rank ($k > 0$), it does

[11]This can be viewed as the 'unfolded version' of Birkhoff's classical result of Theorem 1.10.

provide a complete asymptotic expansion of actual fundamental solutions to (1.1), valid on sectors surrounding and adherent to the origin. It is well-known that certain techniques do exist by which the purely formal solution can be transformed to an actual one, by *resumming* the series (e.g. Borel resummation). Here in this chapter, we shall explore a more recent technique, developed in [17] (2013), by which the divergent gauge series $g(x)$ is rather simply and directly manipulated to yield actual solutions to a given system (1.1). The main technique from [17] involves redefining the problem of solving (1.1), and viewing solutions as lying within a higher-dimensional universal domain that is endowed with a particular analytical groupoid structure.

Many of the particular groupoid domains of interest are defined and explicitly constructed in [17]. These groupoids serve as the *universal domains of definition* for systems, such as (1.1), with singularities bounded by an effective divisor on a curve. In particular, the groupoids have representations which serve as *universal solutions* to the system, in that they provide parallel transport isomorphisms to solve the system along arbitrary paths in the punctured curve. Furthermore, the representations remain smooth over points where the system becomes singular, *and* they can be constructed from strictly formal solutions. Adding to the collection of groupoids studied in [17], here we define the groupoid that is relevant for analyzing the generically unfolded system of Chapter 2. Lastly, we briefly comment on possible connections between this new perspective and analytic invariants of the unfolding, as well as its monodromy representation.

We begin by reformulating the original problem using a more modern terminology, both for simplicity and to be consistent with [17]. Let X be a Riemann surface, and D an effective divisor on X. Then let $\mathcal{E}$ be a vector bundle over X of rank N. A connection ∇ on the bundle $\mathcal{E}$ is a $\mathbb{C}$-linear map between sections

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$$

such that the Leibniz rule is satisfied. Notice that the connection is necessarily flat since X is a Riemann surface (one can locally solve ODEs uniquely). We will be interested in the case when ∇ is meromorphic with poles bounded by D, i.e. we replace Ω^1_X above with $\Omega^1_X(D)$.[12]

Setting $X = \mathbb{C}$ and $D = (k+1) \cdot 0$, the problem of finding fundamental solutions to (1.1) is equivalent to finding a basis of $\mathcal{E}$ that is flat with respect to the meromorphic connection defined by

$$\nabla := d - \frac{A(x)}{x^{k+1}}\, dx. \tag{3.1}$$

Any trivialization of $\mathcal{E}$ will generically be singular along D, and multivalued on the punctured space $X \setminus D$. The Leibniz rule is indeed satisfied, since for any section $s \in \mathcal{E}$ and any $f \in \mathcal{O}_X$ one has that $\nabla(fs) = f(\nabla s) + s \otimes df$ by the product rule for the exterior derivative. Conversely, by selecting a local smooth frame for $\mathcal{E}$, it can be shown that any meromorphic connection on $\mathcal{E}$ has the local form of (3.1).

3.1. Lie Groupoid Representations.

Definition 3.1. A (*holomorphic*) *Lie groupoid* (*on* X) is a groupoid whose set of objects X and set of arrows G are complex manifolds. The source and target

[12] That is to say, $\mathcal{E}$ is a vector bundle over X whose generic smooth sections are allowed to become singular along D.

maps $s, t : G \to X$ are holomorphic submersions, and the composition of arrows is smooth.

A simple example of a Lie groupoid on X is the *pair groupoid* $\mathrm{Pair}(X)$ whose set of arrows is $\mathrm{Pair}(X) = X \times X$. The source and target maps are given by the canonical projections, and the set of identity arrows is the diagonal embedding. The composition of arrows is defined by

$$(x, y) \cdot (y, z) = (x, z).$$

Definition 3.2. A Lie groupoid G is *source-simply connected* if each fiber of the source map $s : G \to X$ is connected and simply connected.

The canonical example of a source-simply connected Lie groupoid is the *fundamental groupoid* $\Pi_1(X)$ which is defined to be the set of all paths in X up to homotopy and with fixed endpoints; it is the covering space of $X \times X$ corresponding to the diagonal subgroup of $\pi_1(X)$ in $\pi_1(X \times X)$. Source-simply connectedness follows from the fact that any source fiber $s^{-1}(\{x\})$ of $\Pi_1(X)$ is isomorphic to $\{x\} \times \widetilde{X}$ with the target map given by

$$t : (x, y) \mapsto \pi(y),$$

where $\pi : \widetilde{X} \to X$ is the universal covering map. For example, the fundamental groupoid of the circle is the infinite cylinder, corresponding to the diagonal copy of $\mathbb{Z}$ in $\pi_1(S^1 \times S^1) = \mathbb{Z} \times \mathbb{Z}$, and the source (or target) fibers are embedded lines.

$$\Pi_1(S^1) \cong S^1 \times \mathbb{R}$$

Definition 3.3. Let G be a Lie groupoid on X, and let $\mathcal{E}$ be a vector bundle over X. A *representation of* G is a smooth homomorphism $\Psi : G \to \mathrm{Hom}(s^*\mathcal{E}, t^*\mathcal{E})$ that respects the composition of groupoid arrows.[13] That is, for any groupoid element $g \in G$, the map

$$\Psi|_g : s^*\mathcal{E} \to t^*\mathcal{E} \tag{3.2}$$

is an isomorphism between source and target fibers of $\mathcal{E}$, and for any pair of composable arrows $g_1, g_2 \in G$, one has

$$\Psi|_{g_1} \circ \Psi|_{g_2} = \Psi|_{g_1 \circ g_2}.$$

The representation Ψ can be viewed as a parallel transport between fibers of $\mathcal{E}$, with zero curvature, i.e. the transport along a path $\gamma : I \to X$ depends only on the homotopy class of γ. Hence, as long as the underlying groupoid is *connected*, i.e. any two points in X are connected by a groupoid arrow, then Ψ is automatically also a representation of the fundamental groupoid $\Pi_1(X)$. This is equivalent to one direction of the Riemann-Hilbert correspondence; that flat connections on X correspond to representations of $\Pi_1(X)$.

For the remainder of this chapter, we fix the notation that $(\mathcal{E}, \nabla) \to (X, D)$ denotes a vector bundle $\mathcal{E}$ over a Riemann surface X, along with a meromorphic connection ∇ whose poles are bounded by the divisor D. Unlike the usual parallel transport defined by integrating ∇, which generically becomes singular over D, there exist certain representations Ψ which will extend *smoothly* over points of D, *provided that we choose the appropriate groupoid* G. The explicit construction

[13] In the case that X is a single point, this definition naturally reduces to that of a representation of a Lie group.

of such groupoids, for various cases of punctured surface (X, D) is given in [17]. In the case that D is a single point on $\mathbb{C}$ (connections of the form (3.1)), the appropriate groupoids will be described explicitly in the following section.

Proposition 3.4. Let ψ be a flat basis of $(\mathcal{E}, \nabla) \to (X, D)$. That is, $\nabla\psi = 0$ on the universal cover of $X \setminus D$. Then the expression

$$\Psi := t^*\psi \cdot (s^*\psi)^{-1} \tag{3.3}$$

gives a representation of the fundamental groupoid $\Pi_1(X \setminus D)$.

Generically, the expression (3.3) is multivalued on $X \setminus D$, and singular at points of D, and thus does *not* define a representation of any groupoid on X. The following theorem assures that, for certain groupoids, it does.

Theorem 3.5. [17] *Let (X, D) be a Riemann surface equipped with effective divisor. Then there exists a source-simply connected Lie groupoid G on X such that:*

(i) $G|_{X\setminus D} \cong \Pi_1(X \setminus D)$,

(ii) For any flat basis ψ of $(\mathcal{E}, \nabla) \to (X, D)$, the expression $\Psi = t^*\psi \cdot (s^*\psi)^{-1}$ is a representation of G.

By definition, then Ψ is single-valued, smooth and invertible on G. We say that $(\mathcal{E}, \nabla) \to (X, D)$ integrates to a representation of G.

The groupoid G from the above theorem is described heuristically in [17] as consisting of all homotopy classes of paths in $X \setminus D$, together with a space of limiting paths (i.e. a Lie group of loops) over each point of D.[14] More specifically, each point of D constitutes an individual groupoid orbit (see Lemma 3.12 of [17]), as must be the case in order for (3.3) to be non-singular.

The remarkable point of Theorem 3.5 is that $\Psi = t^*\psi \cdot (s^*\psi)^{-1}$ extends holomorphically over D, when written on the appropriate groupoid. Even more remarkable is that the representation Ψ can be computed using a *strictly formal* flat basis ψ; it need not be an actual (analytic) solution. This will be discussed further below and applied in Example 3.14.

3.2. **The Stokes Groupoids.** Here we shall construct the groupoids predicted by Theorem 3.5, in the case when D is a single point on $X = \mathbb{C}$, bounding a pole of degree k at the origin (i.e. $D = k \cdot 0$). These constructions are due to [17].

Definition 3.6. Sto_1 is the Lie groupoid on $\mathbb{C}$ whose set of arrows is given in terms of coordinates $\{(x, u) \in \mathbb{C} \times \mathbb{C}\}$, with source and target maps defined by

$$s : (x, u) \mapsto x$$
$$t : (x, u) \mapsto \exp(u)x.$$

The composition of arrows is given by

$$(x_2, u_2) \cdot (x_1, u_1) = (x_1, u_1 + u_2).$$

Note that arrows are compatible precisely when $s(x_2, u_2) = t(x_1, u_1)$.

[14]This is hinting that a blowup procedure may be useful for parameterizing G. Indeed this is one of the strategies adopted in [17].

It is straightforward to verify that Sto_1 is indeed a Lie groupoid on $\mathbb{C}$. Note that Sto_1 is source-simply connected, with each source fiber being isomorphic to $\mathbb{C}$. There are two distinct groupoid orbits: the point at the origin, and its complement in $\mathbb{C}$. Also note that the restriction of Sto_1 to $\mathbb{C}^\times$ gives a parameterization of the fundamental groupoid $\Pi_1(\mathbb{C}^\times)$.

One can obtain another Lie groupoid on $\mathbb{C}$ by performing a blowup of Sto_1 along its singleton orbit.[15] The new groupoid Sto_2 will have coordinates $\{(x', u') \in \mathbb{C} \times \mathbb{C}\}$ such that $x' = x$ and $u = x'u'$ (in relation to the coordinates of Sto_1). The source and target maps are then given by

$$s : (x', u') \mapsto x'$$
$$t : (x', u') \mapsto \exp(x'u')x'.$$

Note that Sto_2 is source-simply connected, has the same orbits as Sto_1, and its restriction to $\mathbb{C}^\times$ is again isomorphic to $\Pi_1(\mathbb{C}^\times)$.

By iteratively performing this blowup procedure, we obtain the following family of *Stokes groupoids*:

Definition 3.7. For $k \in \mathbb{N}$, Sto_k is the Lie groupoid on $\mathbb{C}$ whose set of arrows is given in terms of coordinates $\{(x, u) \in \mathbb{C} \times \mathbb{C}\}$, with source and target maps defined by

$$s : (x, u) \mapsto x$$
$$t : (x, u) \mapsto \exp(ux^{k-1})x.$$

The composition of arrows is given by

$$(x_2, u_2) \cdot (x_1, u_1) = (x_1, u_2 \exp((k-1)u_1 x_1^{k-1}) + u_1).$$ [16]

Each groupoid Sto_k is source-simply connected, and has the same orbits as Sto_1, and for each $k \in \mathbb{N}$ one has that $Sto_k|_{\mathbb{C}^\times} \cong \Pi_1(\mathbb{C}^\times)$. Indeed the results of [17] (see Proposition 5.1, and Theorem 4.1) assert that $(\mathcal{E}, \nabla) \to (\mathbb{C}, k \cdot 0)$ integrates to a representation of Sto_k.

Hence, given any flat basis ψ of $(\mathcal{E}, \nabla) \to (\mathbb{C}, k \cdot 0)$ (even a strictly formal one!) the expression $\Psi = t^*\psi \cdot (s^*\psi)^{-1}$ is a representation of Sto_k. Note that Ψ is independent of our choice of basis ψ. The representation Ψ is then holomorphic at each point $(x, u) \in \mathbb{C} \times \mathbb{C}$ of Sto_k, and invertible everywhere. Although ψ is generically a multivalued function, the single-valuedness of Ψ is ensured by imposing that $\Psi = I$ along the identity bisection; $\Psi|_{u=0} = I$.

Example 3.8. For fixed $a \in \mathbb{C}$ and fixed $k \in \mathbb{N}$, consider the rank 1 bundle $(\mathcal{E}, \nabla) \to (\mathbb{C}, k \cdot 0)$ with connection

$$\nabla = d + ax^{-k}dx.$$

Choosing a flat basis amounts to fixing any non-zero multiple of

$$\psi_1 = x^{-a}, \quad \text{for } k = 1,$$
$$\psi_k = \exp\left(\frac{ax^{-(k-1)}}{k-1}\right), \quad \text{for } k > 1.$$

[15] This idea is generalized in Theorem 3.14 of [17].

[16] This follows from the necessary condition for composing arrows that $s(x_2, u_2) = t(x_1, u_1)$.

The corresponding representations of Sto_k are then computed to be

$$\begin{aligned} \Psi_1 &:= t^*\psi_1 \cdot (s^*\psi_1)^{-1} = e^{-au}, \quad \text{for } k = 1, \\ \Psi_k &:= t^*\psi_k \cdot (s^*\psi_k)^{-1} = e^{-aS_k}, \quad \text{for } k > 1, \end{aligned} \tag{3.4}$$

where the function S_k is given by

$$S_k(x, u) = \frac{1 - e^{-(k-1)ux^{k-1}}}{(k-1)x^{k-1}}.$$

Notice that the singularity of S_k at $x = 0$ is removable. Hence, for each $k \in \mathbb{N}$, the corresponding representation Ψ_k is indeed holomorphic, single-valued, and invertible (non-zero) on Sto_k, i.e. at each point $(x, u) \in \mathbb{C} \times \mathbb{C}$.

Remark 3.9. We emphasize that the smoothness of $t^*\psi \cdot (s^*\psi)^{-1}$ over D is not guaranteed, and one must compute over the appropriate groupoid. Consider the connection $\nabla = d + x^{-2}dx$ whose kernel is spanned by $\psi = e^{1/x}$. Over Sto_1, the expression

$$t^*\psi \cdot (s^*\psi)^{-1} = \exp\left(\frac{e^{-u} - 1}{x}\right)$$

is singular at $x = 0$ (and thus is not a representation of Sto_1). On the other hand, $t^*\psi \cdot (s^*\psi)^{-1}$ gives a representation of $Sto_{k'}$ for each integer $k' \geq 2$. This is a consequence of the fact that Sto_1 is naturally covered by $Sto_{k'}$:

Proposition 3.10. If $(\mathcal{E}, \nabla) \to (\mathbb{C}, D)$ integrates to a representation of Sto_k, then it also integrates to a representation of Sto_{k+1}. This follows from the fact that there is a canonical smooth groupoid homomorphism $Sto_{k+1} \to Sto_k$ given by $(x, u) \mapsto (x, ux)$.

3.3. The Twisted Pair Groupoids.

As an alternative to the Stokes groupoids, we present another family of groupoids having the property that $(\mathcal{E}, \nabla) \to (\mathbb{C}, k \cdot 0)$ integrates to representations of them. The following definition is taken from [17].

Definition 3.11. For $k \in \mathbb{N}$, the *twisted pair groupoid* $\mathrm{Pair}(\mathbb{C}, k \cdot 0)$ is the Lie groupoid on $\mathbb{C}$ whose set of arrows is the complement of the curve $1 + ux^{k-1} = 0$ in $\mathbb{C} \times \mathbb{C}$, with source and target maps defined by

$$\begin{aligned} s &: (x, u) \mapsto x \\ t &: (x, u) \mapsto (1 + ux^{k-1})x. \end{aligned}$$

The composition of arrows is given by

$$(x_2, u_2) \cdot (x_1, u_1) = \left(x_1, u_1 + u_2(1 + u_1x_1^{k-1})^k\right).$$

Example 3.12. The first twisted pair groupoid $\mathrm{Pair}(\mathbb{C}, 1 \cdot 0)$ is simply the action groupoid $\mathbb{C}^\times \ltimes \mathbb{C}$ (objects are acted on by the torus).

The relationship between $\mathrm{Pair}(\mathbb{C}, k \cdot 0)$ and Sto_k is as follows. While both groupoids have the same orbits, namely the point at the origin and its complement in $\mathbb{C}$, the twisted pair groupoid fails to be source-simply connected, and generic source fibers are isomorphic to $\mathbb{C}^\times$. The twisted pair groupoid is canonically covered by the Stokes groupoid, via the smooth groupoid homomorphism $E : Sto_k \to \mathrm{Pair}(\mathbb{C}, k \cdot 0)$ defined by

$$E(x, u) = \left(x, \frac{e^{ux^{k-1}} - 1}{x^{k-1}}\right). \tag{3.5}$$

Notice that E is smooth over $x = 0$ since the singularity of (3.5) is removable. Thus if $(\mathcal{E}, \nabla) \to (\mathbb{C}, k \cdot 0)$ integrates to a representation Ψ_{Pair} of $\text{Pair}(\mathbb{C}, k \cdot 0)$, then it also integrates to a representation of Sto_k — the representation of Sto_k is simply the pullback

$$\Psi_{Sto_k} := E^* \Psi_{\text{Pair}}.$$

Since E is not invertible, the converse statement is false in general. However, (3.5) is of course locally invertible, with a smooth multivalued inverse given by

$$(x, u) \mapsto \left(x, \frac{\log(1 + ux^{k-1})}{x^{k-1}} \right), \tag{3.6}$$

branching around the locus $\{1 + ux^{k-1} = 0\}$ on the boundary of $\text{Pair}(\mathbb{C}, k \cdot 0)$. Thus, given a representation Ψ_{Sto_k} of Sto_k, one can pull back via (3.6) to obtain a *(possibly) multivalued* expression Ψ_{Pair} which integrates $(\mathcal{E}, \nabla) \to (\mathbb{C}, k \cdot 0)$ *locally.*

In particular, while $(\mathcal{E}, \nabla) \to (\mathbb{C}, k \cdot 0)$ may fail to integrate to a representation of $\text{Pair}(\mathbb{C}, k \cdot 0)$ (due to $t^*\psi \cdot (s^*\psi)^{-1}$ possessing some non-trivial monodromy around the locus $1 + ux^{k-1} = 0$), it does integrate to a *local* representation of $\text{Pair}(\mathbb{C}, k \cdot 0)$ in the following sense: *at any point $(x, u) \in Pair(\mathbb{C}, k \cdot 0)$, the expression $\Psi_{Pair} = t^*\psi \cdot (s^*\psi)^{-1}$ is holomorphic and invertible, and thus extends smoothly to a neighbourhood of (x, u) on which it is single-valued and invertible.*

Indeed, this result is expected since $Sto_k \sim \text{Pair}(\mathbb{C}, k \cdot 0)$ for $|u| \ll 1$. The two groupoids are locally isomorphic to one another along their identity bisections.

Example 3.13. For the rank 1 bundle defined in Example 3.8, the corresponding representations of $\text{Pair}(\mathbb{C}, k \cdot 0)$ are

$$\begin{aligned} \Psi_1 &:= t^*\psi_1 \cdot (s^*\psi_1)^{-1} = (1 + u)^{-a}, \quad \text{for } k = 1, \\ \Psi_k &:= t^*\psi_k \cdot (s^*\psi_k)^{-1} = e^{-aT_k}, \quad \text{for } k > 1, \end{aligned} \tag{3.7}$$

where the function T_k is given by

$$T_k(x, u) = \frac{1 - (1 + ux^{k-1})^{-(k-1)}}{(k-1)x^{k-1}},$$

and is holomorphic on $\text{Pair}(\mathbb{C}, k \cdot 0)$. The expression Ψ_1 is really a representation of $\text{Pair}(\mathbb{C}, 1 \cdot 0)$ only when $a \in \mathbb{Z}$ (when the monodromy is trivial). Otherwise, Ψ_1 is a *local* representation of $\text{Pair}(\mathbb{C}, 1 \cdot 0)$. For each $k > 1$, however, Ψ_k is a true representation of $\text{Pair}(\mathbb{C}, k \cdot 0)$.

3.4. **Resummation.** Consider two bundles $(\mathcal{E}_1, \nabla_1)$ and $(\mathcal{E}_2, \nabla_2) \to (\mathbb{C}, k \cdot 0)$, with respectively flat bases ψ_1 and ψ_2. In the language of Chapter 1, these are two locally defined singular systems of Poincaré rank $k - 1$, having respective fundamental solution matrices ψ_1 and ψ_2. We have already seen that the expressions

$$\Psi_1 := t^*\psi_1 \cdot (s^*\psi_1)^{-1}, \quad \Psi_2 := t^*\psi_2 \cdot (s^*\psi_2)^{-1}$$

are representations of Sto_k (smooth, invertible and single-valued on the groupoid), and are independent of the choices of bases. *As mentioned previously, the flat bases ψ_j need not be convergent solutions; if ψ_j is a strictly formal solution, i.e. $\nabla_j \psi_j = 0$, then one obtains the same representation Ψ_j* (see Theorem 5.3 of [17]). The implication for resummation is immediate. Assume that the two systems are related by a formal gauge transformation $\psi_1 \mapsto g(x)\psi_1$ (as in Chapter 1). That is, $\nabla_1 \psi_1 = 0$ and $\nabla_2(g\psi_1) = 0$, where the latter equality holds formally and g is an invertible formal power series centered at the origin. Then the above representation Ψ_2 can alternatively be computed using the *formal solution* as follows

$$\begin{aligned}\Psi_2 &= t^*(g\psi_1)\cdot(s^*(g\psi_1))^{-1}\\ &= t^*g\cdot t^*\psi_1\cdot(s^*\psi_1)^{-1}\cdot(s^*g)^{-1}\\ &= t^*g\cdot\Psi_1\cdot(s^*g)^{-1}.\end{aligned}\tag{3.8}$$

The content of Theorem 5.3 ([17]) is that one may compute Ψ_2 using an *actual* flat basis ψ_2, or the *purely formal* flat basis $g\psi_1$; the resulting representation will be the same. Hence (3.8) is holomorphic, invertible and single-valued on Sto_k despite the divergent nature of g. If for example ∇_1 is a diagonal connection, then Ψ_1 is easily computable, and then so is Ψ_2 via (3.8). With the representation Ψ_2 in hand, one can reconstruct an (actual) flat basis ψ_2 of $(\mathcal{E}_2,\nabla_2)\to(\mathbb{C},k\cdot 0)$ from the parallel transport isomorphisms (3.2). Namely, from the condition that for any $x_0\in\mathbb{C}^\times$, one has

$$\Psi_2(x_0,u)\cdot\psi_2(x_0)=\psi_2(t(x_0,u))\tag{3.9}$$

over the groupoid Sto_k.

Example 3.14. Consider the pair of meromorphic rank 2 connections

$$\nabla_1 = d+\frac{1}{x^2}\begin{bmatrix}-1 & 0\\ 0 & 0\end{bmatrix}dx,\quad \nabla_2 = d+\frac{1}{x^2}\begin{bmatrix}-1 & -x\\ 0 & 0\end{bmatrix}dx.$$

A flat basis trivializing the diagonal connection ∇_1 is readily seen to be given by

$$\psi_1=\begin{bmatrix}e^{-1/x} & 0\\ 0 & 1\end{bmatrix}.$$

As we saw in Example 1.11, a flat basis trivializing ∇_2 is given in terms of the exponential integral (see Appendix A). Here we recover this basis of solutions, using the resummation method outlined above.

From Example 1.11, a formal gauge transformation carrying the diagonal system $\nabla_1\psi=0$ to the non-diagonal system $\nabla_2\psi=0$ is provided by

$$\psi\mapsto\begin{bmatrix}1 & f(x)\\ 0 & 1\end{bmatrix}\psi,$$

where $f(x)=-\sum_{n=0}^{\infty}n!\,x^{n+1}$. Hence, a *formal* flat basis trivializing ∇_2 is given by

$$\begin{bmatrix}1 & f\\ 0 & 1\end{bmatrix}\psi_1$$

(see (1.31) and (1.33)). The corresponding groupoid representations Ψ_1,Ψ_2 are defined on Sto_2, but for ease of computation it is preferable to compute over the twisted pair groupoid $\mathrm{Pair}(\mathbb{C},2\cdot 0)$. This choice is justified by our discussion following Example 3.12 above.[17]

$$\Psi_1=t^*\psi_1\cdot(s^*\psi_1)^{-1}=\begin{bmatrix}e^{-1/(x+ux^2)} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}e^{1/x} & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}e^{u/(1+ux)} & 0\\ 0 & 1\end{bmatrix}$$

This corresponds to the calculation at (3.7) for $k=2$. Observe that Ψ_1 is holomorphic, invertible and single-valued on $\mathrm{Pair}(\mathbb{C},2\cdot 0)$ (i.e. at all points (x,u) such

[17]In particular, the representations will be holomorphic at the point $(0,0)\in\mathrm{Pair}(\mathbb{C},2\cdot 0)$, but may fail to be single-valued when extended to the entire (*non*-source-simply connected) groupoid.

that $1 + xu \neq 0$). Following [17] (see Example 5.5), it is convenient to introduce an (invertible) change of variables on the groupoid, defined by setting $\mu = u(1+ux)^{-1}$. Then, by applying (3.8) the representation Ψ_2 is computed to be

$$\Psi_2 = \begin{bmatrix} 1 & t^*f \\ 0 & 1 \end{bmatrix} \cdot \Psi_1 \cdot \begin{bmatrix} 1 & -s^*f \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e^\mu & t^*f - e^\mu s^*f \\ 0 & 1 \end{bmatrix}.$$

While this expression is ostensibly a purely formal sum in the two variables (x, μ), centered at (0,0), we expect from the above theory that it in fact has some *non-zero* radius of convergence; hence yielding a germ Ψ_2 of some holomorphic function at (0,0).[18] Let us show that this is indeed the case. The target map in the new coordinates is $t(x, \mu) = x(1 - \mu x)^{-1}$ and thus

$$t^*f = -\sum_{n=0}^{\infty} n!\, \frac{x^{n+1}}{(1-\mu x)^{n+1}}. \tag{3.10}$$

Next, consider the geometric expansion $(1 - \mu x)^{-1} = \sum_{k=0}^{\infty} (\mu x)^k$ and differentiate on both sides, n times with respect to μ, to obtain the following formal equality of series

$$n!\, x^n (1 - \mu x)^{-n-1} = \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} (\mu x)^k x^n.$$

Inserting the result into (3.10) yields

$$t^*f = -\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(k+n)!}{k!} (\mu x)^k x^{n+1} = -\sum_{j=0}^{\infty}\sum_{k=0}^{j} \frac{j!}{k!} \mu^k x^{j+1}$$

which is of course divergent for $x \neq 0$. But while t^*f and $e^\mu s^*f$ are both purely formal, their difference

$$\begin{aligned} t^*f - e^\mu s^*f &= \sum_{j=0}^{\infty} j!\, x^{j+1} \left(e^\mu - \sum_{k=0}^{j} \frac{\mu^k}{k!} \right) \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{x^{j+1} \mu^{j+k+1}}{(j+1)(j+2)\dots(j+k+1)} \end{aligned} \tag{3.11}$$

is easily seen to be convergent for $|\mu x| < 1$ by the comparison test. Furthermore, it can be shown (see Appendix D) that (3.11) is the Laurent series expansion of

$$\rho(x, \mu) = e^{\frac{x\mu - 1}{x}} \left(E_1(\tfrac{x\mu - 1}{x}) - E_1(-\tfrac{1}{x}) \right), \tag{3.12}$$

centered at the point $(0, 0)$, where E_1 is the standard *exponential integral*

$$E_1(x) := \int_x^\infty \frac{e^{-t}}{t}\, dt.$$

That is to say, $\rho(x, \mu)$ is the unique analytic extension of the germ (3.11) to the entire groupoid $\mathrm{Pair}(\mathbb{C}, 2 \cdot 0)$. Although $E_1(x)$ is multivalued on $\mathbb{C}^\times$, with branch points at 0 and ∞, the combination (3.12) is indeed holomorphic (and single-valued) on $|\mu x| < 1$ (see Appendix D for details). The apparent singularity of $\rho(x, \mu)$ at $x = 0$ is thus removable. When extended to all of $\mathrm{Pair}(\mathbb{C}, 2 \cdot 0)$, the function

[18] Again, we note that the use of the particular groupoid Sto_2 (or any other groupoid locally isomorphic to it near the origin (0,0)) is essential. Computing Ψ_2 over a generic Lie groupoid (e.g. Sto_1), the expression $t^*f - e^\mu s^*f$ remains non-convergent.

$\rho(x,\mu)$ becomes multivalued, branching around the set of deleted groupoid arrows, $\{(x,\mu) : x\mu - 1 = 0\}$, as anticipated in footnote 17.

We note that this re-expression of (3.11) in terms of 'special functions' is not of particular relevance to the present resummation theory, although it will allow for us to easily check that our final basis of solutions below corresponds to that which was found in Example 1.11.

Finally, we are ready to construct an *actual* flat basis ψ_2 that trivializes ∇_2. It is easy to check that, up to the right action of $GL_2(\mathbb{C})$, the basis can be written as

$$\psi_2 = \begin{bmatrix} h & \eta \\ 0 & 1 \end{bmatrix}, \tag{3.13}$$

where $h(x)$ and $\eta(x)$ are holomorphic functions on $\widetilde{\mathbb{C}^\times}$, which remain to be determined. Fixing any $x_0 \in \mathbb{C}^\times$ and then performing the parallel transport (3.9) will solve for the two unknown functions in terms of their initial conditions at x_0.

$$\begin{aligned} h(x) &= h(x_0)\, e^{\frac{1}{x_0} - \frac{1}{x}} \\ \eta(x) &= \eta(x_0)\, e^{\frac{1}{x_0} - \frac{1}{x}} + \rho\big(x_0, \tfrac{1}{x_0} - \tfrac{1}{x}\big) \\ &= \eta(x_0)\, e^{\frac{1}{x_0} - \frac{1}{x}} + e^{-\frac{1}{x}} \Big(E_1(-\tfrac{1}{x}) - E_1(-\tfrac{1}{x_0})\Big) \end{aligned} \tag{3.14}$$

One then has a germ $\psi_2(x)$ at x_0 of fundamental solution trivializing ∇_2, which extends analytically to a solution on $\widetilde{\mathbb{C}^\times}$. Right-multiplying (3.13) by the appropriate element of $GL_2(\mathbb{C})$ (or equivalently by specifying initial conditions), it follows that one particular flat basis trivializing ∇_2 is given by

$$\psi_2(x) = \begin{bmatrix} e^{-\frac{1}{x}} & e^{-\frac{1}{x}} E_1(-\frac{1}{x}) \\ 0 & 1 \end{bmatrix}, \tag{3.15}$$

which is precisely the basis of solutions found in Example 1.11.

This resummation method is entirely general, and can be used to construct explicit fundamental solutions to the generic *non-Fuchsian* system (1.1), in terms of a Laurent series in powers of x, convergent on a full neighbourhood of $x_0 \neq 0$ and hence holomorphically extendable to a solution on the universal cover of the punctured disk. In some sense then, this resummation method resembles the *Frobenius method* for determining power series solutions to the *Fuchsian* system. The trade-off here is that the germ of fundamental solution produced by resumming is a local object at x_0, and it may be difficult to explicitly determine its analytic continuation. (Of course in the present example, this was not the case.) The resummation process is illustrated again in Example 5.6 of [17] for the case of Airy's equation.

Remark 3.15. As a final remark, we note that the equivalence of (3.11) and (3.12), proved explicitly by the complicated calculations of Appendix D, can instead be viewed as a direct consequence of the resummation theory of [17]. (Our work in Appendix D then serves to *verify* the theory.) On the one hand, (3.15) is a fundamental solution matrix following from Example 1.11 (solving with an integrating factor). On the other hand, following from Example 3.14, so is

$$\begin{bmatrix} e^{-\frac{1}{x}} & \rho\big(x_0, \frac{1}{x_0} - \frac{1}{x}\big) \\ 0 & 1 \end{bmatrix},$$

where $\rho(x, \mu)$ is defined to be the convergent double series (3.11). Since $\rho(x, 0) = 0$ and any two fundamental solutions are related by the right action of $GL_2(\mathbb{C})$, it will follow that

$$\rho\big(x_0, \tfrac{1}{x_0} - \tfrac{1}{x}\big) = e^{-\frac{1}{x}} \Big(E_1(-\tfrac{1}{x}) - E_1(-\tfrac{1}{x_0})\Big)\,.$$

The identities applied in Appendix D (e.g. Lemma 4.2) can then be seen as a consequence of the general resummation theory. One might conjecture that further identities (combinatorial, hypergeometric, etc.) may find novel new proofs by applying Theorem 5.3 from [17] to solve various non-Fuchsian systems.

3.5. **Unfolding the Twisted Pair Groupoid.** Recall in Chapter 2 we examined the deformed system (2.2),

$$y' = \frac{A(\epsilon, x)}{p_\epsilon(x)} \cdot y, \qquad y \in \mathbb{C}^N,$$

which we referred to as the *generic unfolding* of (1.1). Our investigations culminated in the statement of Theorem 2.10 (proved in [9]) which gave an analytic modulus for the entire family of generic unfoldings, in terms of formal data plus the generalized Stokes data. Also recall that

$$p_\epsilon(x) = x^{k+1} + \epsilon_{k-1} x^{k-1} + \ldots + \epsilon_1 x + \epsilon_0 = \prod_{l=1}^{k+1} (x - x_l)$$

has $k+1$ generically distinct roots, $x_1, \ldots, x_{k+1}$, with repetition of roots occurring along the discriminantal locus $\Delta(\epsilon) = 0$. Let us now define the effective divisor

$$D_\epsilon := 1 \cdot x_1 + \ldots + 1 \cdot x_{k+1} = \sum_{l=1}^{k+1} 1 \cdot x_l$$

on $\mathbb{C}$, and the meromorphic connection

$$\nabla_\epsilon := d - \frac{A(\epsilon, x)}{p_\epsilon(x)}\, dx,$$

whose poles are bounded by D_ϵ. In Chapter 2 we used Theorem 2.8 to construct various flat bases $V^{\pm}_{j,\epsilon,S_s}(x)$ of the bundle $(\mathcal{E}, \nabla_\epsilon) \to (\mathbb{C}, D_\epsilon)$. These solution bases were related via the so-called generalized Stokes matrices and the gate matrices.

For the purpose of analyzing solutions by means of a universal parallel transport that extends smoothly over D_ϵ, Theorem 3.5 tells us that there exists some source-simply connected Lie groupoid G_ϵ on $\mathbb{C}$ such that $(\mathcal{E}, \nabla_\epsilon) \to (\mathbb{C}, D_\epsilon)$ integrates to a representation of G_ϵ. Then for any flat basis ψ (e.g. any solution $V^{\pm}_{j,\epsilon,S_s}$), the expression $\Psi = t^*\psi \cdot (s^*\psi)^{-1}$ is single-valued, holomorphic and invertible on G_ϵ. Furthermore, the orbits of G_ϵ will necessarily consist of the individual points of D_ϵ (the roots of $p_\epsilon(x)$) as well as their complement $\mathbb{C} \setminus D_\epsilon$.

Just as the source-simply connected Stokes groupoids Sto_k are parameterized by means of the universal covering

$$\exp : \mathbb{C} \to \mathbb{C}^\times,$$

any explicit description of the source-simply connected groupoid G_ϵ will necessitate uniformizing the $(k+1)$-punctured space $\mathbb{C} \setminus \{x_1, \ldots, x_{k+1}\}$. In the case when $k = 1$, this can be accomplished by means of elliptic modular functions,

$$\lambda : \mathbb{H} \to \mathbb{C} \setminus \{x_1, x_2\},$$

and indeed this is the strategy adopted in Sections 4.2 and 4.3 of [17]. For $k \geq 2$, one requires a more general class of uniformizing functions of the form

$$\pi : \mathbb{H} \to \mathbb{C} \setminus \{x_1, \dots, x_{k+1}\}.$$

An explicit description of G_ϵ is thus rather complicated. Here we shall introduce a different approach to obtaining a smooth parallel transport of solutions. Let us instead modify the twisted pair groupoid $\mathrm{Pair}\,(\mathbb{C}, (k+1)\cdot 0)$ by *unfolding* it. We shall obtain a new groupoid $\mathrm{Pair}(\mathbb{C}, D_\epsilon)$ on $\mathbb{C}$ which, although is no longer source-simply connected (and hence condition *(i)* of Theorem 3.5 fails, i.e. we only obtain 'local paths'), it will have the property that $(\mathcal{E}, \nabla_\epsilon) \to (\mathbb{C}, D_\epsilon)$ integrates to a *local* representation of $\mathrm{Pair}(\mathbb{C}, D_\epsilon)$. As defined earlier in Section 3, this implies that at any $g \in \mathrm{Pair}(\mathbb{C}, D_\epsilon)$ the expression $\Psi = t^*\psi \cdot (s^*\psi)^{-1}$ is holomorphic and invertible, and thus extends smoothly to a neighbourhood of g on which it is single-valued and invertible. In other words, we have a *locally defined* holomorphic parallel transport that extends to the divisor D_ϵ.

Definition 3.16. For $k \in \mathbb{N}$ and any fixed $\epsilon \in \mathbb{C}^k$, the *unfolded twisted pair groupoid* $\mathrm{Pair}(\mathbb{C}, D_\epsilon)$ is the Lie groupoid on $\mathbb{C}$ whose set of arrows is given in terms of coordinates

$$\big\{(x,u) \in \mathbb{C} \times \mathbb{C} : f_l(x,u) \neq 0,\ \ l = 1, \dots, k+1\big\},$$

where

$$f_l(x,u) = 1 + u \prod_{\substack{j=1 \\ j \neq l}}^{k+1} (x - x_{j,\epsilon}), \quad \text{for } \ l = 1, \dots, k+1,$$

with $\{x_{j,\epsilon} = x_j \ : \ j = 1, \dots, k+1\}$ being the (possibly repeated) roots of $p_\epsilon(x)$, and with source and target maps defined by

$$\begin{aligned} s &: (x,u) \mapsto x \\ t &: (x,u) \mapsto x + p_\epsilon(x)u. \end{aligned} \tag{3.16}$$

The composition of arrows is then given by

$$(x', u') \cdot (x, u) = \left(x,\, u + \frac{p_\epsilon\big(x + p_\epsilon(x)u\big)}{p_\epsilon(x)}\, u' \right). \tag{3.17}$$

Remark 3.17. Just as the twisted pair groupoids were constructed by iteratively blowing up the pair groupoid along the origin, the unfolded twisted pair groupoid can be viewed as a blowup of the pair groupoid along $p_\epsilon(x) = 0$.

It is straightforward to verify that $\mathrm{Pair}(\mathbb{C}, D_\epsilon)$ is indeed a Lie groupoid on $\mathbb{C}$, except perhaps to check that the arrow composition is smooth. This is obviously the case away from any zeros of $p_\epsilon(x)$, and at any given zero x_l the singularity is removable:

$$\lim_{x \to x_l} \frac{p_\epsilon\big(x + p_\epsilon(x)u\big)}{p_\epsilon(x)} = f_l(x_l, u) = 1 + p_\epsilon'(x_l)u.$$

The limit may be computed by exploiting the known factorization of $p_\epsilon(\cdot)$.

Defining $Z(f_l) = \{(x,u) : f_l(x,u) = 0\}$ to be the zero-locus of f_l, one can describe the coordinate space of $\mathrm{Pair}(\mathbb{C}, D_\epsilon)$ as the complement of $\cup_{l=1}^{k+1} Z(f_l)$ in $\mathbb{C}^2$. Each zero-locus $Z(f_l)$ is an affine variety. The various zero-loci are generically pairwise disjoint, and will coincide precisely when corresponding roots of $p_\epsilon(x)$ coalesce:

Proposition 3.18. Let $l, m \in \{1, \dots, k+1\}$ and so x_l, x_m are roots of $p_\epsilon(x)$. Then

$$\begin{cases} x_l \neq x_m \implies Z(f_l) \cap Z(f_m) = \varnothing, \\ x_l = x_m \implies Z(f_l) = Z(f_m). \end{cases}$$

The second statement is obvious, while the contrapositive of the first statement is easily proven by using the fact that

$$f_l(x,u) = 1 + \frac{p_\epsilon(x)}{x - x_l} u \quad \text{for } x \neq x_l.$$

At the confluence of all roots, i.e. when $\epsilon \to 0$ and hence $D_\epsilon \to (k+1) \cdot 0$, the twisted pair groupoid of Definition 3.11 is recovered,

$$\mathrm{Pair}(\mathbb{C}, D_\epsilon) \longrightarrow \mathrm{Pair}\left(\mathbb{C}, (k+1) \cdot 0\right).$$

That is to say that as $\epsilon \to 0$, the generically disjoint loci $Z(f_1), \dots, Z(f_{k+1})$ all coalesce to form the single curve $\{1 + ux^k = 0\}$, and the groupoid target map (3.16) and corresponding arrow composition law (3.17) *smoothly* reduce to that of the twisted pair groupoid $\mathrm{Pair}\left(\mathbb{C}, (k+1) \cdot 0\right)$ (see Definition 3.11).

Remark 3.19. In order for $(\mathcal{E}, \nabla_\epsilon) \to (\mathbb{C}, D_\epsilon)$ to integrate to a local representation of $\mathrm{Pair}(\mathbb{C}, D_\epsilon)$, it is a necessary condition that the orbits of $\mathrm{Pair}(\mathbb{C}, D_\epsilon)$ consist of each individual point of D_ϵ as well as their complement $\mathbb{C} \setminus D_\epsilon$. This is indeed the case, and is a consequence of our imposition that $f_l(x,u) \neq 0$ on $\mathrm{Pair}(\mathbb{C}, D_\epsilon)$ for each $l = 1, \dots, k+1$.[19] In particular, let us assume that x_l is a zero of $p_\epsilon(x)$ of order n. Then

$$s^{-1}(\{x_l\}) = t^{-1}(\{x_l\}) = \begin{cases} \left\{(x_l, u) : u \neq \frac{-1}{p'_\epsilon(x_l)}\right\} & \text{if } n = 1, \\ \{(x_l, u) : u \in \mathbb{C}\} & \text{if } n > 1. \end{cases}$$

Moreover, any two points in $\mathbb{C} \setminus D_\epsilon$ are joined by a unique groupoid arrow.

Let us reiterate that the unfolded twisted pair groupoid $\mathrm{Pair}(\mathbb{C}, D_\epsilon)$ is defined over the entire parameter space $\epsilon \in \mathbb{C}^k$ including along $\Delta(\epsilon) = 0$, and we have already observed that at $\epsilon = 0$ one recovers the familiar twisted pair groupoid. In fact, a more general statement is true regarding the local structure of the unfolded twisted pair groupoid.

Proposition 3.20. Fix $k \in \mathbb{N}$ and $\epsilon \in \mathbb{C}^k$. Let x_l be a zero of $p_\epsilon(x)$ of order $n \in \{1, \dots, k+1\}$. Then at the identity arrow $(x_l, 0) \in \mathrm{Pair}(\mathbb{C}, D_\epsilon)$, we have a *local* isomorphism

$$\mathrm{Pair}(\mathbb{C}, D_\epsilon) \xrightarrow{\cong} \mathrm{Pair}\left(\mathbb{C}, n \cdot 0\right)$$

such that $(x_l, 0) \mapsto (0, 0)$. In other words, near a zero of $p_\epsilon(x)$ of order n, $\mathrm{Pair}(\mathbb{C}, D_\epsilon)$ 'looks like' the twisted pair groupoid of order n, locally.

Proof. Let us temporarily adopt a new notation and use t_ϵ and t_0 to denote the respective target maps of $\mathrm{Pair}(\mathbb{C}, D_\epsilon)$ and $\mathrm{Pair}\left(\mathbb{C}, n \cdot 0\right)$. It will suffice to demonstrate that t_ϵ and t_0 have the same local behaviour.

[19] Incidentally, the condition $f_l(x_l, u) \neq 0$ also ensures that the target map (3.16) is indeed a submersion at x_l, as required by Definition 3.1.

First observe that there exists a neighbourhood $U \subset \mathrm{Pair}(\mathbb{C}, D_\epsilon)$ of $(x_l, 0)$ such that the map $\phi : U \to \mathrm{Pair}\,(\mathbb{C}, n \cdot 0)$ defined by

$$\phi : (x, u) \mapsto \left(x - x_l,\, \sum_{j=n}^{k+1} \frac{p_\epsilon^{(j)}(x_l)}{j!} (x - x_l)^{j-n} u \right)$$

is a biholomorphism from $U \to \phi(U)$. This follows from the fact that the Jacobian determinant

$$\det J_\phi|_{(x_l,0)} = \frac{p_\epsilon^{(n)}(x_l)}{n!} \neq 0.$$

Next, let φ be the automorphism of $\mathbb{C}$ defined by

$$\varphi : x \mapsto x - x_l,$$

and then note that

$$t_0 = \varphi \circ t_\epsilon \circ \phi^{-1} : \phi(U) \to \mathbb{C}.$$

In other words, we have explicitly found local charts ϕ and φ to show that the groupoid target maps t_ϵ and t_0 are *equivalent* holomorphic mappings, i.e. they look the same locally. Lastly, we remark that in the case $n = k + 1$ one has $\phi = id$. □

Recall from our previous discussions on the twisted pair groupoid that $(\mathcal{E}, \nabla) \to (\mathbb{C}, k \cdot 0)$ integrates to a local representation of $\mathrm{Pair}(\mathbb{C}, k \cdot 0)$. One recalls here that the poles of ∇ are assumed to be bounded by the divisor $k \cdot 0$. An immediate consequence of Proposition 3.20 is then the following result:

Theorem 3.21. *Fix $k \in \mathbb{N}$ and $\epsilon \in \mathbb{C}^k$. Then $(\mathcal{E}, \nabla_\epsilon) \to (\mathbb{C}, D_\epsilon)$ integrates to a local representation of $Pair(\mathbb{C}, D_\epsilon)$. That is to say, given any flat basis ψ of $(\mathcal{E}, \nabla_\epsilon) \to (\mathbb{C}, D_\epsilon)$ the parallel transport isomorphisms*

$$\Psi|_g : s^*\mathcal{E} \to t^*\mathcal{E}, \quad g \in Pair(\mathbb{C}, D_\epsilon)$$

defined by the expression $\Psi = t^*\psi \cdot (s^*\psi)^{-1}$ are locally holomorphic and invertible on the unfolded twisted pair groupoid $Pair(\mathbb{C}, D_\epsilon)$.

We can explicitly verify Theorem 3.21 in the case when $\Delta(\epsilon) \neq 0$, that is when D_ϵ consists of $k + 1$ distinct points. When $s(g) \notin D_\epsilon$ then $\Psi = t^*\psi \cdot (s^*\psi)^{-1}$ is obviously holomorphic and invertible in a neighbourhood of g, since ψ is a flat basis of ∇_ϵ. As usual, single-valuedness of Ψ near g is ensured by imposing that $\Psi = I$ along the identity bisection; $\Psi|_{u=0} = I$. When $s(g) \in D_\epsilon$ (i.e. $s(g) = x_l$ is a root of $p_\epsilon(x)$) then the claim that Ψ is holomorphic and invertible at g (or even well-defined) must be verified. Recall that x_l is a singular point of the corresponding Fuchsian system ($\nabla_\epsilon \psi = 0$) and so there are two cases to consider: whether $s(g) = x_l$ is non-resonant or resonant.

In the former case, and as discussed in Remark 2.7, a flat basis ψ is provided in terms of a Frobenius series

$$\psi(x) = (I + \mathcal{O}(x - x_l))\,(x - x_l)^{\mathcal{U}_l},$$

where $\mathcal{U}_l$ is the diagonal matrix given by $\frac{\Lambda(\epsilon, x_l)}{p_\epsilon'(x_l)}$. The convergent series ψ is a fundamental solution on a slit neighbourhood of x_l. We may now easily compute the local representation Ψ as follows:

$$\begin{aligned}\Psi(x, u) &= (I + \mathcal{O}(x - x_l + p_\epsilon(x)u))\,(x - x_l + p_\epsilon(x)u)^{\mathcal{U}_l}\,(x - x_l)^{-\mathcal{U}_l}\,(I + \mathcal{O}(x - x_l)) \\ &= (I + \mathcal{O}(x - x_l + p_\epsilon(x)u))\,(f_l(x, u))^{\mathcal{U}_l}\,(I + \mathcal{O}(x - x_l))\,.\end{aligned}$$

We should like to verify that Ψ is holomorphic and invertible at any fixed $(x_l, u) \in \mathrm{Pair}(\mathbb{C}, D_\epsilon)$. Indeed this is the case since $\Psi(x_l, u) = (f_l(x_l, u))^{\mathcal{U}_l}$ and $f_l \neq 0$ on $\mathrm{Pair}(\mathbb{C}, D_\epsilon)$. (The series converges in a neighbourhood of (x_l, u) and $\log(f_l)$ is holomorphic there.)

In the latter case when x_l is resonant, a flat basis ψ takes the form described in Appendix C. Namely, by applying a change of basis if necessary, we can assume that the real parts of the eigenvalues α_i of the diagonal residue matrix $\mathcal{U}_l$ are ordered

$$\Re(\alpha_1) \geq \ldots \geq \Re(\alpha_N).$$

Then a flat basis ψ can be written as

$$\psi(x) = (I + \mathcal{O}(x - x_l))\,(x - x_l)^{\mathcal{U}_l}(x - x_l)^{\mathcal{N}},$$

where $\mathcal{N}$ is strictly upper triangular (hence nilpotent) and satisfies the condition

$$\mathcal{N}_{ij} = 0 \quad \text{unless} \quad \alpha_i - \alpha_j \in \mathbb{N}. \tag{3.18}$$

Again the convergent series ψ is a fundamental solution on a slit neighbourhood of x_l. The local representation Ψ can then be computed:

$$\begin{aligned}\Psi(x, u) &= (I + \mathcal{O}(x - x_l + p_\epsilon(x)u))\,(x - x_l + p_\epsilon(x)u)^{\mathcal{U}_l}\,(x - x_l + p_\epsilon(x)u)^{\mathcal{N}} \\ &\qquad (x - x_l)^{-\mathcal{N}}(x - x_l)^{-\mathcal{U}_l}\,(I + \mathcal{O}(x - x_l)) \\ &= (I + \mathcal{O}(x - x_l + p_\epsilon(x)u))\,(x - x_l)^{\mathcal{U}_l}\,(f_l(x, u))^{\mathcal{U}_l}\,(f_l(x, u))^{\mathcal{N}} \\ &\qquad (x - x_l)^{-\mathcal{U}_l}\,(I + \mathcal{O}(x - x_l)).\end{aligned}$$

Again, we should like to verify that Ψ is holomorphic and invertible at any fixed $(x_l, u) \in \mathrm{Pair}(\mathbb{C}, D_\epsilon)$. It suffices to show that

$$\Phi(x, u) := (x - x_l)^{\mathcal{U}_l}\,(f_l(x, u))^{\mathcal{U}_l}\,(f_l(x, u))^{\mathcal{N}}\,(x - x_l)^{-\mathcal{U}_l}$$

is holomorphic and invertible at (x_l, u). The matrix entries of $\Phi(x, u)$ are computed to be

$$\Phi(x, u)_{ij} = (x - x_l)^{\alpha_i - \alpha_j}\,(f_l(x, u))^{\alpha_i}\,(\delta_{ij} + \xi(x, u)_{ij}), \quad i, j = 1, \ldots, N,$$

where

$$\xi(x, u) = \mathcal{N} \log f_l(x, u) + \tfrac{1}{2!}\mathcal{N}^2 \log^2 f_l(x, u) + \ldots$$

is a finite sum and strictly upper triangular. Recalling that $f_l \neq 0$ on $\mathrm{Pair}(\mathbb{C}, D_\epsilon)$, we immediately see that $\Phi(x, u)$ is invertible at (x_l, u) and that each matrix entry *not* lying above the diagonal is holomorphic at (x_l, u). Thus it remains to show that each matrix entry $\Phi(x, u)_{ij}$, with $i < j$, is holomorphic at (x_l, u). It suffices to demonstrate this for the single matrix element $\Phi(x, u)_{1N}$ (the remaining cases are proved analogously). If $\alpha_1 - \alpha_N \in \mathbb{N}$, then we are done and $\Phi(x, u)_{1N}$ is holomorphic at (x_l, u). Hence, let us assume that $\alpha_1 - \alpha_N \notin \mathbb{N}$. Our goal is then to show that

$$\mathcal{N}_{1N} = (\mathcal{N}^2)_{1N} = \ldots = (\mathcal{N}^{N-1})_{1N} = 0.$$

Let $m \in \{1, 2, \ldots, N-1\}$. The matrix entry $(\mathcal{N}^m)_{1N}$ can be expressed as

$$\sum_{1 = i_0 < i_1 < \ldots < i_m = N} \mathcal{N}_{i_0 i_1}\mathcal{N}_{i_1 i_2} \ldots \mathcal{N}_{i_{m-1} i_m},$$

summing over all appropriate tuples. Therefore, if $(\mathcal{N}^m)_{1N} \neq 0$ then there exists *some* fixed tuple, $1 = i_0 < i_1 < \ldots < i_m = N$, such that $\mathcal{N}_{i_0 i_1}\mathcal{N}_{i_1 i_2} \ldots \mathcal{N}_{i_{m-1} i_m} \neq 0$. But this would imply that *none* of $\mathcal{N}_{i_0 i_1},\ \mathcal{N}_{i_1 i_2},\ \ldots,\ \mathcal{N}_{i_{m-1} i_m}$ vanish and hence

$\alpha_1 - \alpha_N \in \mathbb{N}$ by (3.18), which is a contradiction. In other words, either $\alpha_1 - \alpha_N \in \mathbb{N}$ (and $\Phi(x,u)_{1N}$ is holomorphic at (x_l, u)) or $\Phi(x,u)_{1N} = 0$.

———

To summarize the novel results of this section, we have explicitly parameterized the so-called unfolded twisted pair groupoid $\text{Pair}(\mathbb{C}, D_\epsilon)$, and have shown that it serves as the universal domain of definition for *any system* that is meromorphic on $\mathbb{C}$ with a finite number of poles. The source-simply connected extension of $\text{Pair}(\mathbb{C}, D_\epsilon)$ is described rather abstractly in Theorem 4.1 of [17], in terms of a collection of charts which individually cover each singular point of the system (where the groupoid locally looks like Sto_n). The presently described groupoid $\text{Pair}(\mathbb{C}, D_\epsilon)$, on the other hand, is given in terms of *one global chart*. However, while $\text{Pair}(\mathbb{C}, D_\epsilon)$ does integrate the corresponding system, i.e. the canonical representation $\Psi = t^*\psi \cdot (s^*\psi)^{-1}$ remains locally holomorphic and invertible over the groupoid, the unfolded twisted pair groupoid (like the twisted pair groupoid) is a local object over $\mathbb{C}$; it fails to be source-simply connected and hence only parameterizes local paths in the punctured plane.

Recall that the Stokes groupoid Sto_{k+1} serves as the universal domain of definition for locally meromorphic systems of the form (1.1),

$$y' = \frac{A(x)}{x^{k+1}} \cdot y.$$

Among other implications, this means that universal solutions to (1.1) can be constructed from strictly formal (divergent) solutions. It has been remarked by M. Gualtieri that the Stokes groupoids can be used to recover the Stokes data of system (1.1), and that indeed this *must* be the case since the groupoid representation Ψ captures the full solution of the singular system. This suggests that the unfolded twisted pair groupoid $\text{Pair}(\mathbb{C}, D_\epsilon)$ may prove useful for extracting analytic invariants of the generic unfolding of (1.1),

$$y' = \frac{A(\epsilon, x)}{p_\epsilon(x)} \cdot y.$$

In other words, that the generalized Stokes matrices of Chapter 2 can be extracted from the representations of $\text{Pair}(\mathbb{C}, D_\epsilon)$. It is currently unknown to the author by what means this may be accomplished.

4. Appendices

Appendix A. In regards to Example 1.11, in order to demonstrate that $\psi_1 \sim y_f$ in Ω_1, it suffices to show that

$$\left| e^{-1/x} \int_{0^-}^{x} \frac{e^{1/z}}{z} dz + \sum_{n=1}^{m-1} (n-1)!\, x^n \right| = \mathcal{O}\left(|x|^m\right),$$

$$\text{as } x \to 0 \text{ in } \Omega_1, \ \text{ for all } \ m \in \mathbb{N}.$$

For any $m \in \mathbb{N}$, repeated integration by parts yields

$$\int_{0^-}^{x} \frac{e^{1/z}}{z} dz = -e^{1/x}\big(x + x^2 + \ldots + (m-1)!\, x^m\big) + \int_{0^-}^{x} m!\, z^{m-1} e^{1/z}\, dz,$$

and so it remains to be shown that

$$\left| e^{-1/x} \int_{0^-}^{x} m!\, z^{m-1} e^{1/z}\, dz \right| = \mathcal{O}\left(|x|^m\right),$$

$$\text{as } x \to 0 \text{ in } \Omega_1, \ \text{ for all } \ m \in \mathbb{N}.$$

Consider evaluating the integral along a contour γ in Ω_1, where γ is the concatenation of a straight line path in $\mathbb{R}^-$ from 0^- to $-|x|$, followed by a circular path in $\{z \in \Omega_1 : |z| = |x|\}$ from $-|x|$ to x. For $x \in \Omega_1$ such that $-3\pi/2 + \varepsilon < \arg(x) \le 0$, the modulus $\left|e^{1/z}\right|$ is an increasing function along the contour γ, and so we obtain the following bound

$$\left| e^{-1/x} \int_{0^-}^{x} m!\, z^{m-1} e^{1/z}\, dz \right| \le m!\, |x|^{m-1} \int_{0^-}^{x} |dz| \le m!\, |x|^m (1+\pi).$$

Otherwise $0 < \arg(x) < \pi/2 - \varepsilon$ and the modulus $\left|e^{1/z}\right|$ begins to decrease after γ crosses the positive real axis. In this case, apply the residue theorem to obtain the following bound

$$\left| e^{-1/x} \int_{0^-}^{x} m!\, z^{m-1} e^{1/z}\, dz \right| \le \left| e^{-1/x}\, 2\pi i \right| + m!\, |x|^m (1+\pi).$$

In the sectorial region of $\arg(x) \in (0, \frac{\pi}{2} - \varepsilon)$, it is clear that $\left|e^{-1/x}\right| = \mathcal{O}(|x|^m)$ for each $m \in \mathbb{N}$, and so the proof that $\psi_1 \sim y_f$ in Ω_1 is complete. Moreover, these same estimates demonstrate that the asymptotic relationship $\psi_1 \sim y_f$ continues to hold in $\{-3\pi/2 < \arg(x) < \pi/2 - \varepsilon\}$, but fails in $\{-3\pi/2 < \arg(x) < \pi/2\}$.

Remark 4.1. We note that the above numerical estimates on error terms are rather crude and can be improved, see for example [6]. Furthermore,

$$\int_{0^-}^{x} \frac{e^{1/z}}{z} dz = E_1(-1/x), \quad x \in \Omega_1,$$

where E_1 is the standard *exponential integral*

$$E_1(x) := \int_{x}^{\infty} \frac{e^{-t}}{t} dt,$$

albeit with its branch cut defined along the positive imaginary axis rather than the more commonly chosen principal branch. Our claim that $\psi_1 \sim y_f$ in Ω_1 is confirmed by citing the standard asymptotic expansion of the analytic extension of $E_1(x)$ to $\widetilde{\mathbb{C}^\times}$ (see [6])

$$E_1(x) \sim \frac{e^{-x}}{x} \sum_{n=0}^{\infty} \frac{(-1)^n\, n!}{x^n} \quad \text{as } x \to \infty, \quad |\arg(x)| \leq \frac{3\pi}{2} - \delta \; (< \frac{3\pi}{2}).$$

Lastly, the uniqueness of the canonical solution ψ_1 is proven as follows. Any fundamental solution to (1.28) must be equal to $\psi_1(x)C$ for some nonsingular

$$C = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}.$$

If $\psi_1(x)C \sim y_f(x)$, then the asymptotic condition (1.13) necessitates that

$$\psi_1(x)Ce^{-\Lambda(x)} = \begin{bmatrix} c_1 + c_3E_1(-1/x) & e^{-1/x}\,(c_2 + c_4E_1(-1/x)) \\ c_3e^{1/x} & c_4 \end{bmatrix} \longrightarrow I,$$

as $x \to 0$ in Ω_1. It follows immediately that $C = I$.

Appendix B. In regards to Example 1.14, let us rewrite the system (1.41) as

$$\frac{dy}{ds} = \frac{\Lambda_0 + s^3A_3}{s^4}\, y, \tag{4.1}$$

where

$$\Lambda_0 = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}, \quad A_3 = -\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

A gauge transformation of the form $y \mapsto (I + s^3G)y$ carries the system to

$$\frac{dy}{ds} = \left((I + s^3G)\, \frac{\Lambda_0 + s^3A_3}{s^4}\, (I - s^3G + s^6G^2) + 3s^2G + \mathcal{O}(s^5) \right) y. \tag{4.2}$$

As previously mentioned in the example, system (4.2) is diagonalized up to order s^{-1} precisely when

$$G = \begin{bmatrix} a & -\frac{1}{8} \\ \frac{1}{8} & b \end{bmatrix}, \quad a, b \in \mathbb{C}.$$

This leads to

$$\frac{dy}{ds} = \left(\frac{\Lambda_0 + s^3\Lambda_3}{s^4} + s^2\left(GA_3 - A_3G - G\Lambda_0G + \Lambda_0G^2 + 3G \right) + \mathcal{O}(s^5) \right) y,$$

where Λ_3 is the diagonal part of A_3. Computation of the s^2 coefficient yields

$$\begin{bmatrix} 3a + \frac{1}{16} & -\frac{a}{2} - \frac{3}{8} \\ -\frac{b}{2} + \frac{3}{8} & 3b - \frac{1}{16} \end{bmatrix}, \tag{4.3}$$

which is diagonal precisely when $a = -\frac{3}{4}$ and $b = \frac{3}{4}$. A further gauge transformation of the form $y \mapsto (I + s^9F)y$ will suffice to diagonalize the s^5 term, without affecting terms of lower order. Proceeding by induction, there exists a gauge series $I + s^3G + \sum_{n=0}^{\infty} s^{9+3n}F_n$ carrying the system (4.1) to

$$\frac{dy}{ds} = \left(\frac{\Lambda_0 + s^3\Lambda_3}{s^4} + H(s) \right) y,$$

where $H(s) = \sum_{n=0}^{\infty} s^{2+3n}H_n$ and each H_j is diagonal. (H_0 is given by (4.3).) A final gauge transformation $y \mapsto e^{-\int H(s)}\, y$ will take us to the normal form (1.44).

To leading order, the complete gauge series $g(s)$ bringing (4.1) to normal form is thus calculated to be

$$I + s^3\Big(G - \frac{1}{3}H_0\Big) + \mathcal{O}(s^6) = I + s^3\begin{bmatrix} -\frac{1}{48} & -\frac{1}{8} \\ \frac{1}{8} & \frac{1}{48} \end{bmatrix} + \mathcal{O}(s^6).$$

Notice how this process can be used to compute the formal normalizing series $g(s)$ out to arbitrary order, using nothing but basic matrix algebra.

Appendix C. As mentioned in Remark 2.7, in a neighbourhood of the non-resonant Fuchsian singular point x_l (adherent to the generalized sector $\Omega^{\pm}_{j,\epsilon}$) the corresponding solution $V^{\pm}_{j,\epsilon,S_s}$ must take the form

$$(I + \mathcal{O}(x - x_l))\,(x - x_l)^{\mathcal{U}_l} \cdot T, \quad T \in GL_N(\mathbb{C}).$$

Applying a change of basis if necessary, it suffices to assume that the real parts of the eigenvalues α_i of the diagonal residue matrix $\mathcal{U}_l$ are ordered

$$\Re(\alpha_1) \geq \ldots \geq \Re(\alpha_N).$$

In the case when x_l is *resonant*, the solution $V^{\pm}_{j,\epsilon,S_s}$ takes the form

$$(I + \mathcal{O}(x - x_l))\,(x - x_l)^{\mathcal{U}_l}(x - x_l)^{\mathcal{N}} \cdot T, \quad T \in GL_N(\mathbb{C}),$$

where $\mathcal{N}$ is strictly upper triangular (hence nilpotent) and satisfies the condition

$$\mathcal{N}_{ij} = 0 \ \text{ unless } \ \alpha_i - \alpha_j \in \mathbb{N},$$

see for example [2], [5]. Next, the asymptotics (2.19) of Theorem 2.8 indicate that the limit

$$\lim_{x \to x_l} V^{\pm}_{j,\epsilon,S_s}(x) \cdot \exp\left(-\int_{x_0}^{x} \frac{\Lambda(\epsilon, x)}{p_\epsilon(x)}\,dx\right)$$

is diagonal and invertible, with the limit being taken along the separatrix which borders $\Omega^{\pm}_{j,\epsilon}$ and reaches x_l. Taking the residue expansion of the integrand (see (2.16)), it follows that

$$\lim_{x \to x_l} V^{\pm}_{j,\epsilon,S_s}(x) \cdot (x - x_l)^{-\mathcal{U}_l}$$

is diagonal and invertible. This limit can in fact be taken along *any ray of constant local argument* (as per footnotes 9 and 10) and the result remains diagonal and invertible. Thus in either case, whether x_l is resonant or non-resonant, the ordering on eigenvalues of $\mathcal{U}_l$ implies that T is upper triangular. It remains to compute the diagonal part of the monodromy matrix M of solution $V^{\pm}_{j,\epsilon,S_s}$. In the non-resonant case, one has

$$M = T^{-1}\, e^{2\pi i\,\mathcal{U}_l}\, T$$

and hence

$$\mathrm{diag}(M) = e^{2\pi i\,\mathcal{U}_l}.$$

In the resonant case, note that

$$(x - x_l)^{\mathcal{N}} = I + \mathcal{N}\log(x - x_l) + \frac{1}{2}\mathcal{N}^2\log^2(x - x_l) + \ldots$$

is a finite sum, and observe that the commutator $\left[e^{2\pi i\,\mathcal{U}_l},\,\mathcal{N}\right]=0$. The monodromy around x_l is then computed to be

$$\begin{aligned}M &= T^{-1}\,e^{2\pi i\,\mathcal{U}_l}\,e^{2\pi i\,\mathcal{N}}\,T\\ &= T^{-1}\,e^{2\pi i\,\mathcal{U}_l}\Big(I+2\pi i\,\mathcal{N}+\ldots+\frac{(2\pi i)^{N-1}}{(N-1)!}\,\mathcal{N}^{N-1}\Big)\,T\\ &= T^{-1}\,e^{2\pi i\,\mathcal{U}_l}\,T,\end{aligned}$$

where the last equality follows since $\mathcal{N}\,T=0$. Again we find that

$$\mathrm{diag}(M)=e^{2\pi i\,\mathcal{U}_l},$$

precisely as claimed.

Appendix D. Here it will be shown that the double series (3.11),

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{x^{j+1}\mu^{j+k+1}}{(j+1)(j+2)\ldots(j+k+1)},$$

is indeed the Laurent series expansion of the function

$$\rho(x,\mu)=e^{\frac{x\mu-1}{x}}\,\left(E_1(\tfrac{x\mu-1}{x})-E_1(-\tfrac{1}{x})\right),$$

about the point $(0,0)$. The exponential integral

$$E_1(-\tfrac{1}{x})=\int_{0^-}^{x}\frac{e^{1/z}}{z}dz$$

is of course multivalued, branching around 0 and ∞, and so it is not obvious that $\rho(x,\mu)$ is holomorphic at the origin. Let us demonstrate that this is indeed the case. Consider the restricted domain $S=\{\Re(x)>0\}\cap\{0<|x\mu|<1\}$ which is a subset of $\mathrm{Pair}(\mathbb{C},2\cdot 0)$. On this region, one has that

$$\left|\arg(x)-\arg\left(\tfrac{x}{1-x\mu}\right)\right|<\frac{\pi}{2}.$$

The integral representation of the function

$$\rho(x,\mu)=e^{\frac{x\mu-1}{x}}\int_{x}^{x(1-x\mu)^{-1}}\frac{e^{1/z}}{z}\,dz$$

is in general not well-defined, and depends on the homotopy class of the contour of integration. However, this discrepancy is resolved for $\rho(x,\mu)|_S$ since the contour then never crosses $\mathbb{R}_-$, by the previous inequality. This implies that one may integrate term by term, to obtain the following series representation, valid on S.

$$\rho(x,\mu)|_S=e^{\frac{x\mu-1}{x}}\left(-\mathrm{Log}(1-x\mu)+\sum_{k=1}^{\infty}\frac{1-(1-x\mu)^k}{x^k\,k\,k!}\right)\tag{4.4}$$

Since the right-hand side is holomorphic on $\{0<|x\mu|<1\}$, it follows by the identity theorem that (4.4) extends from S to remain valid on all of $\{0<|x\mu|<1\}$.

It is now clear that $\rho(x,\mu)$ is multivalued on $\mathrm{Pair}(\mathbb{C},2\cdot 0)$, branching around the locus $\{x\mu-1=0\}$ of deleted groupoid arrows. It is also clear that $\rho(x,0)=0$, but one notices a singularity at $x=0$. This singularity is in fact removable, as will be made clear by proceeding with finding the Laurent series expansion of $\rho(x,\mu)$.

By naively applying Taylor series and binomial expansions, we obtain the following series representation valid on $\{0 < |x\mu| < 1\}$.

$$\begin{aligned}\rho(x,\mu) &= \left(\sum_{j=0}^{\infty}\frac{(x\mu-1)^j}{x^j j!}\right)\left(\sum_{i=1}^{\infty}\frac{(x\mu)^i}{i}+\sum_{k=1}^{\infty}\frac{1-(1-x\mu)^k}{x^k k\, k!}\right)\\ &= \underbrace{\sum_{j=0}^{\infty}\sum_{i=1}^{\infty}\sum_{n=0}^{j}\frac{(x\mu)^{n+i-j}\,(-1)^{j-n}}{i\, j!}\binom{j}{n}\mu^j}_{\text{(i)}}\\ &\qquad -\underbrace{\sum_{j=0}^{\infty}\sum_{k=1}^{\infty}\sum_{n=0}^{j}\sum_{m=1}^{k}\frac{(x\mu)^{n+m}\,(-1)^{j-n+m}}{x^{j+k}\, j!\, k\, k!}\binom{j}{n}\binom{k}{m}}_{\text{(ii)}}\end{aligned}$$

The second expression (ii) can be rearranged by noting that all powers of x are non-positive. One may define a new index $\lambda = j+k-n-m \in \mathbb{Z}_{\geq 0}$. Then for any *fixed* λ, one has that $l := j-n \in \{0,1,\dots,\lambda\}$, allowing us to rewrite (ii) as

$$\begin{aligned}&\sum_{\lambda=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{l=0}^{\lambda}\frac{(x\mu)^{-\lambda}\,(-1)^{l+m}\,\mu^{\lambda+n+m}}{(n+l)!\,(\lambda+m-l)\,(\lambda+m-l)!}\binom{n+l}{n}\binom{\lambda+m-l}{m}\\ &= \sum_{\lambda=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{l=0}^{\lambda}\frac{(x\mu)^{-\lambda}\,(-1)^{l+m}\,\mu^{\lambda+n+m}}{(\lambda+m-l)\,n!\,m!\,l!\,(\lambda-l)!}\\ &= \sum_{\lambda=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{(x\mu)^{-\lambda}\,(-1)^{m+\lambda}\,\mu^{\lambda+n+m}}{n!\,(m+\lambda)!\,m}.\end{aligned}$$

This last equality follows by applying the following identity.

Lemma 4.2. *Let n be a non-negative integer, and let $m \in \mathbb{N}$. Then*

$$\sum_{k=0}^{n}\frac{(-1)^k}{k!\,(n-k)!\,(m+k)} = \frac{1}{m(m+1)(m+2)\dots(m+n)}.$$ [20]

Proof. Consider the following meromorphic function on $\mathbb{C}$

$$F_n(x) := \frac{1}{x(x+1)(x+2)\dots(x+n)}.$$

The function has a partial fraction decomposition (residue expansion) of the form

$$F_n(x) = \frac{a_0}{x}+\frac{a_1}{x+1}+\dots+\frac{a_n}{x+n}.$$

On the other hand, for $x \notin \{0,-1,-2,\dots\}$ one has that

$$F_n(x) = \frac{\Gamma(x)}{\Gamma(x+n+1)}.$$

Hence, for each $k = 0,1,\dots,n$, we can compute

$$a_k = \lim_{x\to -k}(x+k)F_n(x) = \frac{\operatorname{Res}_{x=-k}\Gamma(x)}{\Gamma(n-k+1)} = \frac{(-1)^k}{k!\,(n-k)!}.$$

[20] This is a special case of the Chu-Vandermonde identity. It can also be restated as a special case of Gauss' hypergeometric theorem, namely that ${}_2F_1(-n,m;m+1;1) = \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+1+n)\Gamma(1)}$.

Evaluating the function at $x = m$ completes the proof. □

Meanwhile, expression (i) can be broken into the sum of two parts, according to whether powers of x are positive or non-positive. Expression (i) is then equal to

$$\sum_{\lambda=1}^{\infty}\sum_{j=0}^{\infty}\sum_{n=0}^{j}\frac{(x\mu)^{\lambda}\,(-1)^{j-n}}{(\lambda+j-n)\,j!}\binom{j}{n}\mu^{j}+\sum_{\lambda=0}^{\infty}\sum_{i=1}^{\infty}\sum_{n=0}^{\infty}\frac{(x\mu)^{-\lambda}\,(-1)^{\lambda+i}}{i\,(\lambda+n+i)!}\binom{\lambda+n+i}{n}\mu^{\lambda+n+i}.$$

All non-positive integer powers of x from expressions (i) and (ii) perfectly cancel, and we are left with the result that

$$\rho(x,\mu)=\sum_{\lambda=1}^{\infty}\sum_{j=0}^{\infty}\sum_{n=0}^{j}\frac{x^{\lambda}\mu^{\lambda+j}\,(-1)^{j-n}}{(\lambda+j-n)\,(j-n)!\,n!}\,,\qquad |x\mu|<1.$$

Another application of Lemma 4.2 then yields that

$$\rho(x,\mu)=\sum_{\lambda=1}^{\infty}\sum_{j=0}^{\infty}\frac{x^{\lambda}\mu^{\lambda+j}}{\lambda(\lambda+1)(\lambda+2)\ldots(\lambda+j)}.$$

Indeed this is the same Laurent series expansion written at (3.11), precisely as claimed.

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